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**Geometric recursion and volumes of moduli spaces :  
Oriented ribbon graphs, acyclic decomposition,  
“Cut-and-Join” operators**

Simon Barazer

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# Geometric recursion and volumes of moduli spaces: Oriented ribbon graphs, acyclic decomposition, "Cut-and-Join" operators

*Réursion géométrique et volumes des espaces de  
modules : Graphes en rubans orientés, décomposition  
acyclique et opérateurs de "Cut-and-Join"*

**Thèse de doctorat de l'université Paris-Saclay**

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Thèse préparée dans le **Laboratoire Alexander Grothendieck** (Université Paris-Saclay, IHES, CNRS), sous la direction de **Maxim KONTSEVICH**, Professeur et la co-direction de **Anton ZORICH**, Professeur.

**Thèse soutenue à Paris-Saclay, le 19 janvier 2024, par**

**Simon BARAZER**

## Composition du jury

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**Titre :** Récursion géométrique et volumes des espaces de modules : graphes en rubans orientés, décomposition acyclique et opérateurs de "Cut-and-Join"

**Mots clés :** Topologie, Géométrie, Espaces de modules, récursion topologique, graphes en rubans, différentielles quadratiques.

**Résumé :** Dans cette thèse on étudie les liens entre les récurrences topologique et géométrique et les volumes de Masur-Veech des espaces des modules des différentielles quadratiques et Abéliennes. On a choisi de s'intéresser aux graphes en rubans car ils peuvent être utilisés pour calculer les volumes de Masur-Veech. Dans le cas des graphes trivalents on propose une formule de récurrence géométrique, qui a été donnée indépendamment dans "On the Kontsevich geometry of the combinatorial Teichmüller space". On étudie ensuite les graphes en rubans orientés, dans ce cas, on propose une décomposition des graphes que l'on a choisi d'appeler "La décomposition acyclique". Cette décomposition permet de décomposer un graphe en ruban orienté général en une famille de graphes à un sommet. Le caractère "acyclique" provient d'une condition sur les recollements qui permet d'obtenir l'unicité de la décomposition. Ici, les graphes stables qui encodent la topologie

des multicourbes utilisées pour faire les découpages sont dirigés et acycliques. En utilisant ce théorème on peut maintenant calculer les volumes des espaces des modules. Il est possible d'obtenir une récurrence en considérant toutes les façon d'extraire un graphe à un sommet, on donne explicitement la formule de récurrence dans le cas des graphes quadrivalents.

On relie ensuite la décomposition acyclique aux opérateurs de "Cut-and-Join" dans le cas des graphes quadrivalents on obtient aussi des liens avec la récurrence topologique. Pour les graphes ayant des sommets d'ordres supérieurs à 4, on obtient des opérateurs de "Cut-and-Join" d'ordre supérieur aussi appelés  $W$  opérateurs.

A la fin du mémoire on étudie les dégénérescences de graphes en rubans et on montre que les volumes des espaces des modules admettent un prolongement par continuité.

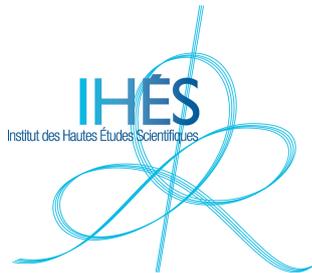
**Title:** Geometric recursion and volumes of moduli spaces: oriented ribbon graphs, acyclic decomposition, “Cut-and-Join” operators

**Keywords:** Topology, Geometry, Moduli spaces, topological recursion, ribbon graphs, quadratic differentials.

**Abstract:** In this thesis we study the relations between topological and geometric recursions and Masur-Veech volumes of moduli spaces of quadratic and Abelian differentials. We chose to study ribbon graphs because they can be used to compute these volumes. In the case of trivalent ribbon graphs we give a geometric recursion formula that was also independently found in “On the Kontsevich geometry of the combinatorial Teichmüller space”. We also study oriented ribbon graphs, in this case we found decomposition of graphs that we call “The acyclic decomposition”. This decomposition allow to decompose general oriented ribbon graphs into graphs with only one vertex. The “acyclic” part of the name comes from a condition on surgeries, this allow to obtain the uniqueness of the decomposition. Stables

graphs that determine topology of multicurves used for the surgeries are directed and acyclic. Using this we are able to compute volumes of their moduli spaces, we obtain a recursion by considering all the possible ways to remove a vertex by surgeries and make it explicit in the case of graphs with quadrivalent vertices only. We relate the acyclic decomposition to “Cut-and-Join” operators. In the case of quadrivalent graphs we also relate it to topological recursion. In the case of graphs with higher order vertices we obtain higher order Cut-and-Join equations, also called  $W$  operators.

At the end of the memoir we study degeneration’s of ribbon graphs and show that volumes of moduli spaces of oriented ribbon graphs admit continuous extensions.



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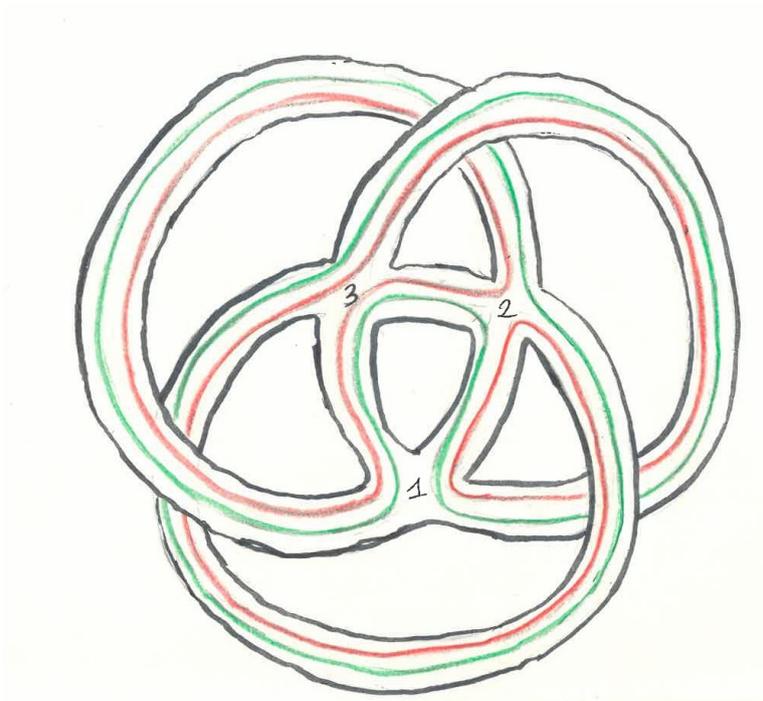
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# Chapter 1

## Introduction (En français )

Le but de cette thèse était d'étudier les interactions possibles entre les récurrences géométriques et topologiques introduites dans [Miro7] et [ABO17],[EO07]. Avec d'une part l'évolution de Shramm-Lowner [?] qui peut être vue comme une famille de processus stochastiques qui produisent des courbes aléatoires fractales. Ces processus sont universels et apparaissent comme limites d'échelle de nombreux systèmes issus de la physique statistique et de la combinatoire. D'autre part cette thèse était aussi portée sur l'étude des volumes de Masur-Veech des espaces de modules des différentielles abéliennes et quadratiques, ces volumes contiennent de nombreuses informations sur la géométrie des surfaces plates et ont fait l'objet de nombreuses recherches. Dans cette introduction nous rappellerons brièvement les acteurs en jeu avant de présenter les travaux effectués durant cette thèse, l'accent a été mis principalement sur les volumes de Masur-Veech.

### 1.1 Brève introduction, état de l'art

#### 1.1.1 Récurrence topologique

La récurrence topologique est apparue au cours des dernières années en géométrie énumérative et en physique. Le principe de base est le suivant, on dispose d'une suite  $(V_{g,n})$  indexée par deux entiers, le genre  $g$  et le nombre  $n$  de bords ou de points marqués. Généralement on suppose la condition de stabilité sur la caractéristique d'Euler

$$2g - 2 + n > 0.$$

Les quantités  $V_{g,n}$  sont souvent des séries génératrices définies de façon géométrique qui "comptent" des structures sur une surface de type  $(g, n)$ . Ce sont généralement des polynômes ou des séries formelles à  $n$  variables. On cherche alors à déterminer les quantités  $V_{g,n}$  par récurrence, la caractéristique d'Euler mesure la complexité topologique d'une surface et il est alors naturel de chercher à exprimer  $V_{g,n}$  en fonction des  $V_{g',n'}$  tels que

$$2g - 2 + n > 2g' - 2 + n'.$$

Une telle récurrence peut être appelée récurrence topologique car elle porte sur la topologie des surfaces. Un exemple bien connu est la récurrence obtenue par M.Mirzakhani pour les volumes de Weil-Pertersson des espaces des modules des surfaces hyperboliques. A titre

d'exemple on rappelle brièvement ce résultat. Soit  $\mathcal{M}_{g,n}^{hyp}$  l'espace des modules des surfaces hyperboliques de genre  $g$  avec  $n$  bords géodésiques [Miro7]. On dispose d'une application qui mesure la longueur des bords pour la métrique hyperbolique

$$L_{\partial} : \mathcal{M}_{g,n}^{hyp} \longrightarrow \mathbb{R}_{\geq 0}^n.$$

Chaque ligne de niveau  $\mathcal{M}_{g,n}^{hyp}(L) = L_{\partial}^{-1}(\{L\})$  est munie d'une forme de volume obtenue à partir de la forme symplectique de Weil-Petersson  $\omega_{WP}$  et les volumes de Weil-Petersson sont définis par

$$V_{g,n}^{hyp}(L) = \int_{\mathcal{M}_{g,n}^{hyp}(L)} \frac{\omega_{WP}^{3g-3+n}}{(3g-3+n)!}.$$

M.Mirzakhani a obtenu une formule de récurrence pour ces volumes dans [Miro7]. Pour cela elle a inventé la formule de Mirzakhani-McShane en généralisant les travaux de G. McShane [McS98].

**Théorème 1.1** ([Miro7]). *Pour  $(g, n) \neq (1, 1), (0, 3)$ , les volumes  $V_{g,n}^{hyp}$  sont calculés par la récurrence*

$$\begin{aligned} L_1 V_{g,n+1}^{hyp}(L) &= \sum_{i \neq 1} \int_x \mathcal{D}(L_1, L_i, x) V_{g,n}^{hyp}(x, L_{\{1,j\}^c}) dx \\ &+ \frac{1}{2} \int_{x_1, x_2} \mathcal{R}(L_1, x_1, x_2) V_{g-1, n+2}^{hyp}(x_1, x_2, L_{\{1,j\}^c}) x_1 x_2 dx_1 dx_2 \\ &+ \frac{1}{2} \sum_{g_i, n_i, I_i} \int_{x_1, x_2} \mathcal{R}(L_1, x_1, x_2) V_{g_1, n_1+1}^{hyp}(x_1, L_{I_1}) V_{g_2, n_2+1}^{hyp}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2 \end{aligned}$$

La somme porte sur toutes les paires de triplés  $(g_i, n_i, I_i)$  telles que

$$g_1 + g_2 = g, \quad I_1 \sqcup I_2 = \{2, \dots, n+1\} \text{ et où } n_i = \#I_i$$

Et la récurrence est initialisée par

$$V_{0,3}^{hyp} = 1 \quad V_{1,1}^{hyp}(L) = \frac{L^2}{24} + \frac{\pi^2}{6}$$

Les notations sur les indices sont données dans la partie 3.1, en réalité  $V_{1,1}^{hyp}(L)$  se déduit de  $V_{0,3}^{hyp} = 1$  par une récurrence légèrement différente du cas général. Les fonctions  $\mathcal{D}$  et  $\mathcal{R}$  sont explicites et proviennent de la géométrie hyperbolique. Le membre de droite contient des surfaces (potentiellement non connexes) de caractéristique d'Euler strictement supérieure au membre de gauche, la différence est de 1. Chaque terme dans la formule correspond à une opération de chirurgie sur les surfaces hyperboliques et la signification des différents termes est donnée dans la figure 1.1. La formule de récurrence revient à réduire la topologie en sommant sur toute les façons d'extraire un pantalon (sphère à trois trous) qui contient le bord 1. On peut dire que cette récursion est d'ordre 1 car on réduit l'opposée de la caractéristique d'Euler de 1 à chaque étape.

Ce type de formule apparaît dans de nombreux domaines sous des formes diverses. Une formulation particulière a été donnée par B.Eynard et N.Orantin qui étudiaient initialement les modèles de matrices aléatoires [EO07]. Elle a ensuite trouvé de nombreuses applications et a été baptisée la récurrence topologique. Cette reformulation apparemment compliquée est

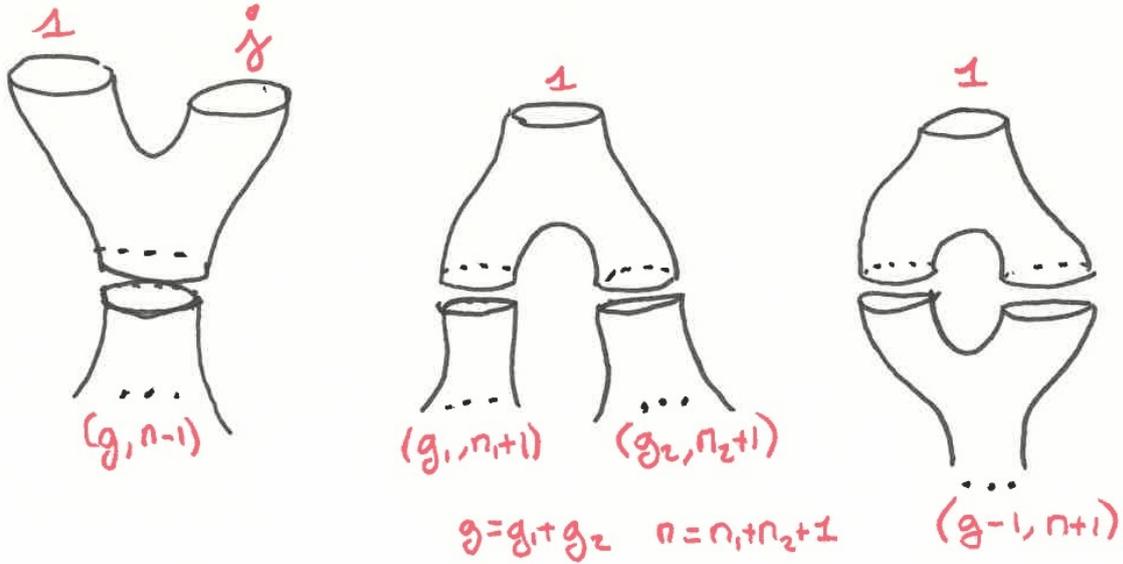


Figure 1.1: Différents types de recollements apparaissant dans la récurrence de M. Mirzakhani.

cependant universelle et possède de très nombreuses propriétés. On rappelle brièvement le principe général. On dispose d'une surface de Riemann  $X$  et on cherche à calculer des polydifférentielles  $\omega_{g,n}$  sur  $X^n$ . Localement elles s'écrivent sous la forme

$$\omega_{g,n} = \omega_{g,n}(z_1, \dots, z_n) dz_1 \otimes \dots \otimes dz_n.$$

On dispose aussi d'une application méromorphe  $x$  définie sur  $X$  qui a des ramifications simples sur un sous-ensemble fini  $\text{Br}(x) \subset X$ , on suppose aussi que l'on dispose d'une famille de noyaux

$$K = K_\alpha(z_1, z_2) \frac{dz_1}{dz_2}$$

pour tout point  $z_\alpha \in \text{Br}(x)$ . Enfin au voisinage de chaque point  $z_\alpha \in \text{Br}(x)$  on dispose de l'involution locale  $\sigma_\alpha$  qui préserve  $x$ . La récurrence topologique exprime  $\omega_{g,n}$  de la manière suivante :

$$\begin{aligned} \omega_{g,n} = & \sum_{\alpha} \text{Res}_{\alpha} K_{\alpha}(z_1, z) (\omega_{g-1, n+1}(z, \sigma_{\alpha}(z), z_{\{1\}^c}) \\ & + \sum_{g_i, n_i} \omega_{g_1, n_1+1}(z, z_{I_1}) \otimes \omega_{g_2+1, n_2+1}(\sigma_{\alpha}(z), z_{I_2})) \end{aligned}$$

$K$  est construit à partir de  $\omega_{0,1}$  et  $\omega_{0,2}$ , généralement  $\omega_{0,2}$  est universelle et est donnée par un choix de normalisation du noyau de Bergmann [EO07]. La forme  $\omega_{0,1}$  s'écrit  $\omega_{0,1} = ydx$  où  $y$  est une fonction méromorphe, holomorphe au niveau des points de branchement. Dans de nombreux cas les fonctions  $(x, y)$  vérifient une relation de la forme

$$P(x, y) = 0$$

que l'on appelle courbe spectrale (la courbe spectrale correspond plutôt à  $(X, x, y, \omega_{0,2})$ ). La récurrence topologique de E-O a su s'imposer et a trouvé des applications dans de nombreux

domaines. Un exemple très connu est celui des produits d'intersections des classes tautologiques sur les espaces des modules  $\overline{\mathcal{M}}_{g,n}$ . Les espaces des modules paramétrisent les surfaces de Riemann  $(C, z_1, \dots, z_n)$  de genre  $g$  avec  $n$  points marqués. Le fibré  $\mathcal{L}_i$  est le fibré en droite dont la fibre au dessus de  $(C, z)$  est l'espace cotangent  $T_{z_i}^*C$  (voir chapitre 6.3). La première classe de Chern de ce fibré définit une classe de cohomologie  $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ . On considère alors [Eyn14b]

$$\omega_{g,n} = \sum_{\alpha} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{z_i^{2d_i+1}} \otimes_{i=1}^n dz_i$$

Dans ce cas on dispose du résultat suivant qui est équivalent à la conjecture de Witten [Wit90] qui a été prouvée à de nombreuses reprises sous diverses formes et pour la première fois dans [Kon92].

**Théorème 1.2.** *Les différentielles  $\omega_{g,n}$  sont calculées par la récurrence topologique gouvernée par la courbe spectrale  $x = z^2, y = z$  au voisinage de  $z = 0$*

Ce résultat se généralise dans le cadre de la théorie des champs cohomologiques, cette théorie a été développée par M.Kontsevich et Y.Manin pour étudier les invariants de Gromov-Witten [KM94]. La donnée de base est une algèbre de Frobenius et la sortie est une famille de classes de cohomologie  $\Omega_{g,n}(\alpha_1, \dots, \alpha_n)$  sur l'espace des modules  $\overline{\mathcal{M}}_{g,n}$ . Grâce à la correspondance de Givental-Teleman [Giv01],[Tel11] dans le cas où l'algèbre de Frobenius est semi-simple on sait alors que les produits d'intersection des classes  $\Omega_{g,n}$  avec les classes  $\psi$  sont calculables par la récurrence topologique et sont exactement les courbes spectrales à ramifications simples. La récurrence topologique de Eynard-Orantin a su trouver de multiples applications :

- Calcul des moments des matrices aléatoires [Eyn05].
- Volumes de Masur-Veech de la strate principale des espaces des modules des différentielles quadratiques [ABC<sup>+</sup>23].
- Géométrie énumérative : cartes [Eyn05], nombres d'Hurwitz [BMo8],[EMS11], points entiers dans les espaces de modules [Nor13].
- Topologie algébrique des espaces de modules, théorie des champs cohomologiques, reconstruction de Givental-Teleman [Eyn14a], [DBOSS14].
- Géométrie non commutative,
- Approximation WKB [EGFEN21].
- Et bien d'autres encore.

La récurrence topologique n'est pas la seule récurrence possible. En effet sa formulation est un peu obscure et plus récemment M.Kontsevich et I.Sobeilman ont introduit dans [KS17] une formulation alternative et plus générale qu'ils ont appelée Structures d'Airy dans le but d'apporter une interprétation géométrique. On se place dans un espace  $\mathbb{C}[[\hbar]][[(x_\alpha)_\alpha]]$  où  $(x_\alpha)$  est une famille possiblement infinie de variables. On cherche à calculer une série formelle de la forme

$$\phi(\hbar) = \exp\left(\sum_{g,n} \hbar^{2g-2} S_{g,n}\right)$$

où la somme porte sur tous les couples d'entiers positifs  $(g, n)$  tels que  $2g - 2 + n > 0$ . Les éléments  $S_{g,n}$  sont dans  $\mathbb{C}[[x_\alpha]]$  et sont directement reliés aux différentielles  $\omega_{g,n}$ . Les contraintes vérifiées par  $\phi$  sont alors de la forme

$$H_i \phi = 0$$

Où les  $H_i$  sont des opérateurs différentiels de la forme

$$H_i = -\hbar \partial_i + \sum_{j,k} a_{i,j,k} x_j x_k + \hbar \sum_{j,k} b_{i,j}^k x_j \partial_k + \hbar^2 \sum_{j,k} c_k^{i,j} \partial_i \partial_j + \hbar \epsilon_i.$$

On dispose aussi des contraintes sur ces opérateurs données dans [KS17] qui assurent l'unicité de la solution. Les relations entre la récurrence topologique et les structures d'Airy ont été données dans [KS17] et aussi présentées dans [Eyn19] sous une autre forme.

Une troisième approche est celle donnée par les opérateurs de Cut-and-Join, ils sont définis eux aussi sur un modèle tensoriel mais on dispose d'un opérateur :

$$P = \sum_{i,j,k} a_{i,j}^k t_i t_j \partial_k + \sum_{i,j,k} b_i^{j,k} t_i \partial_j \partial_k + \sum_{i,j,k} c_{i,j,k} t_i t_j t_k$$

$P$  agit sur un espace  $\mathbb{C}[[t_0, t_1, \dots]]$  de séries formelles en une infinité de variables, dans ce cas on cherche à déterminer une série formelle  $Z(q) \in \mathbb{C}[[t_0, t_1, \dots]]$  qui satisfait l'équation linéaire,

$$\frac{\partial Z}{\partial q} = PZ$$

le problème étudié fournit généralement la condition initiale. Les opérateurs de Cut-and-Join apparaissent aussi dans de nombreux domaines et sont souvent reliés aux nombres d'Hurwitz [Zhu12] et dans la théorie des matrices aléatoires.

Ces différentes approches interagissent entre elles mais ne sont pas équivalentes. Dans [KS17] il est prouvé que

$$\text{TR avec points de ramifications simples} \longrightarrow \text{Structures d'Airy}$$

Et suite aux travaux de A.Alexandrov [Ale22] utilisant la correspondance de Givental-Teleman on sait que

$$\text{TR avec ramifications simples} \longrightarrow \text{Cut-and-Join}$$

Mais ce n'est pas le cas des inclusions inverses.

Lors de la récurrence topologique on réduit l'opposée de la caractéristique d'Euler de 1 à chaque étape, mais il existe de possibles généralisations dans lesquels la caractéristique décroît plus rapidement. Dans la formulation de Eynard-Orantin cela correspond aux cas où la fonction  $x$  posséderait des points de ramifications non simples. Une approche possible a été donnée dans [?] par V.Bouchard et B.Eynard. Une étude de généralisations possibles des structures d'Airy est donnée dans [BBC<sup>+</sup>21] puis dans [BKS23], ces généralisations utilisent les représentations de  $W$  algèbres. Dans le cas de la théorie cohomologique des champs cela correspondrait aux cas où l'algèbre de Frobenius n'est plus semi-simple. Mais de manière générale ces récurrences sont peu comprises, on verra par la suite des cas de récurrences supérieures mais qui peuvent à priori se ramener à une récurrence topologique d'ordre 1.

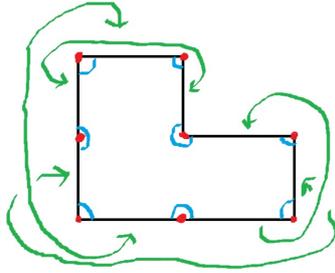


Figure 1.2: Surface de translation de genre 2 avec une singularité conique d'angle  $6\pi$ .

### 1.1.2 Surfaces plates et volumes de Masur-Veech

Une structure de surface de translation  $X$  sur une surface topologique est un atlas de cartes à valeurs dans  $\mathbb{C}$  dont les fonctions de transition sont des translations. De manière informelle une surface de translation s'obtient en recollant des feuilles de papier entre elles, par exemple en recollant un polygone le long de bords parallèles comme sur la figure 1.2. Les surfaces de translation sont naturellement reliées aux différentielles abéliennes sur les surfaces de Riemann. Les translations permettent de tirer en arrière la structure complexe de  $\mathbb{C}$  et préservent la forme locale  $dz$  qui est alors bien définie sur  $X$ . Plus généralement les surfaces de demi-translation sont associées à des différentielles quadratiques qui sont localement le carré de différentielles abéliennes. Dans ce cas on dispose de changements de cartes de la forme

$$z \longrightarrow \pm z + c.$$

Les surfaces de translation et de demi-translation sont fréquemment appelées par abus de langage "surfaces plates" car la métrique  $|dz|$  est bien définie. En vertu du théorème de Gauss-Bonnet une métrique plate ne peut exister en tout point sur les surfaces de genre  $\geq 2$ , une telle surface doit avoir des singularités qui correspondent aux zéros de la différentielle abélienne ou quadratique qui lui est associée. On appelle ces singularités des singularités coniques, autour d'une singularité conique l'angle mesurée n'est pas égal à  $2\pi$ . Pour une différentielle abélienne un zéro d'ordre  $k$  correspond à un angle de  $2(k+1)\pi$  et pour une différentielle quadratique à un angle de  $(2+k)\pi$  (voir figure 1.2).

L'étude des surfaces plates a été initiée par Veech, H.Masur A.Zorich A.Eskin. Un des sujets d'étude est le flot géodésique sur les surfaces plates qui a des applications en dynamique notamment dans l'étude des billards. Les géodésiques sont localement très simples, ce sont des lignes droites lorsque l'on se place dans les cartes locales, mais leur comportement global est beaucoup moins trivial. Le flot géodésique dans une direction fixée est généralement chaotique. En effet pour toute surface de translation ou de demi-translation et presque toute direction le flot géodésique est uniquement ergodique. Le flot géodésique est notamment relié à la théorie des échanges d'intervalles étudiée en particulier par P.Rauzy, H.Masur J.C.Yoccoz, A.Avila [Mas82]. Dans une série d'articles [Zor96],[Zor97] A.Zorich a étudié le comportement des grandes géodésiques : en utilisant la théorie ergodique il a mis en évidence l'existence d'un cycle asymptotique dans l'homologie de la surface sur lequel les grandes trajectoires "s'enroulent". Il a ensuite étudié les déviations de ce cycle en utilisant le théorème d'Osseldet appliqué au cocycle de Kontsevich-Zorich du flot géodésique de Teichmüller sur les espaces des modules des surfaces plates. Une autre question importante concerne la distribution des

trajectoires périodiques et plus généralement des trajectoires singulières qui relient deux singularités, on appelle ces trajectoires des liens de selle. Les trajectoires périodiques ne sont pas isolées et définissent des cylindres plats feuilletés par des trajectoires périodiques parallèles. Le comptage des liens de selles et des cylindres de longueur  $\leq L$  est un sujet qui a fait couler beaucoup d'encre et a donné lieu à de nombreuses publications dans le domaine. Un théorème remarquable a été prouvé par H.Masur et A.Eskin.

**Théorème 1.3.** *Pour presque toute surface de translation (ou de demi-translation)  $S$  soit  $N_{sd}(S, L)$  le nombre de liens de selle de longueur  $\leq L$ . Il existe une constante  $C_{sd}(S)$  telle que*

$$\lim_{L \rightarrow \infty} \frac{N_{sd}(S, L)}{L^2} = C_{sd}(S)$$

Il faut apporter des précisions à ce théorème. La constante  $C_{sd}(S)$  apparaissant dans ce théorème est la constante de Siegel-Veech. Et cette constante est... constante. Il apparaît que  $C_{sd}(S)$  ne dépend pas vraiment de  $S$  mais seulement de sa topologie et de la structure des singularités  $S$ . Afin d'obtenir des résultats valables pour des familles de surfaces de translation il est naturel de considérer les espaces de modules associés. Pour étudier les questions sur les surfaces de translation ou plus généralement les structures géométriques, il est apparu qu'il était très prolifique de considérer non pas une surface de translation mais l'ensemble des classes d'équivalence de surfaces de translation modulo homéomorphismes. Ces espaces s'appellent les espaces de modules, généralement les espaces de modules sont des "orbifolds" : ils possèdent des points présentant des symétries. A de nombreuses reprises les espaces de modules ont montré qu'ils recèlent une quantité phénoménale d'informations sur les objets qu'ils contiennent. Étudier la topologie ou la dynamique sur les espaces de modules a permis de démontrer de nombreux théorèmes majeurs au cours des dernières décennies. Le fait de considérer les objets en famille permet de s'intéresser à leurs déformations, il apparaît que de nombreuses quantités sont constantes ou constantes presque partout, comme c'est le cas des constantes de Siegel-Veech. Les espaces des modules regroupent l'ensemble des classes d'isomorphismes de surfaces de translation (ou de demi-translation) ayant une topologie fixée. Comme il a été mentionné précédemment les surfaces de translation possèdent des singularités coniques. Pour une partition  $\nu = (0^{\nu(0)}, 1^{\nu(1)}, \dots)$  il est alors naturel de considérer les strates  $\mathcal{H}(\nu)$  (resp  $\mathcal{Q}(\nu)$ ) regroupant l'ensemble des surfaces de translation connexes (resp de demi translation) ayant  $\nu(i)$  singularités d'angle  $2\pi(i + 1)$  (respectivement  $\pi(2 + i)$  pour  $\mathcal{Q}(\nu)$ ) . La partition  $\nu$  détermine aussi la topologie des surfaces.

Pour revenir à nos liens de selle, pour une partition fixée il apparaît que la quantité  $C_{sd}(S)$  est constante égale à  $C_{sd}(\nu)$  pour presque toutes les surfaces de l'espace  $\mathcal{H}(\nu)$  (idem pour  $\mathcal{Q}(\nu)$ ). Le presque provient du fait que certaines surfaces admettent des symétries ou ont des propriétés arithmétiques particulières et sont des exceptions à la règle. Il est alors naturel de se demander si il est possible de calculer ces constantes et de relier  $C_{sd}(\nu)$  à la géométrie de  $\mathcal{H}(\nu)$ . Un exemple d'application est un ensemble de formules remarquables prouvées par A.Eskin et A.Zorich dans [?] pour le cas des différentielles abéliennes puis dans [Gou15] pour les différentielles quadratiques. Ces formules relient les constantes de Siegel-Veech et les volumes des espaces de modules des différentielles abéliennes ou quadratiques. Comme nous le verrons plus en détails dans la partie suivante, les espaces de modules possèdent une forme de volume naturelle construite à partir de la mesure de Lebesgue. Les volumes associés à ces mesures sont appelés volumes de Masur-Veech et nous les dénoterons généralement  $\vartheta(\nu)$  dans le cas

des différentielles quadratiques et  $\vartheta^\circ(\nu)$  pour les différentielles abéliennes. On rappelle une définition de ces volumes dans le chapitre 7.4.2.

**Théorème 1.4.** *La constante  $C_{i \leftrightarrow j}(\nu)$  comptant le nombre asymptotique de liens de selle reliant des zéros d'ordres  $i, j$  avec  $i \neq j$  est donnée par*

$$C_{i \leftrightarrow j}(\nu) = c_{i,j} \frac{\vartheta^\circ(\nu - (i) - (j) + (i + j))}{\vartheta^\circ(\nu)}$$

où la constante  $c_{i,j}$  est explicite.

La formule précédente est un cas particulier correspondant à une configuration très simple de liens de selle. Le cas général est donné dans [?] et [Gou15] et met en lumière l'importance des volumes de Masur-Veech dans l'étude de la géométrie des surfaces plates. Une des questions qui a motivé les recherches effectuées durant cette thèse est la suivante.

**Question 1.1.** *Peut-on calculer les volumes de Masur-Veech ?*

Mentionnons que durant ces dernières années de nombreux chercheurs se sont intéressés aux propriétés des surfaces de grand genre notamment le comportement asymptotique des volumes de Masur-Veech lorsque  $d(\nu) \rightarrow \infty$ . Des conjectures ont été proposées par A.Zorich et ses collaborateurs dans [EZ15] pour les différentielles abéliennes et dans [ADG<sup>+</sup>20] pour les différentielles quadratiques. Ces résultats ont été prouvés dans [CMSZ20],[Agg20],[ADG<sup>+</sup>20],[Agg21] et [DGZZ22] en utilisant notamment des méthodes développées par S.Bloch et A.Okounkov se basant sur les formes quasi-modulaires.

## 1.2 Position du problème

### 1.2.1 Volumes de Masur-Veech et graphes en rubans : cas de la strate principale des différentielles quadratiques

**Principe général :** Le calcul des volumes de Masur-Veech des espaces des modules des surfaces plates est aussi un sujet qui a fait couler beaucoup d'encre. Un moyen naturel de calculer les volumes d'un sous ensemble ouvert  $U$  de  $\mathbb{R}^m$  pour la mesure de Lebesgue est de compter le nombre de points entiers qu'il contient, c'est à dire le nombre de points dans  $U \cap \mathbb{Z}^m$ . Cela donne une idée du volume; et pour obtenir une valeur exacte, il faut considérer le comportement asymptotique du nombre de points dans  $\frac{1}{N}\mathbb{Z}^m$  lorsque  $N$  tend vers l'infini. Ce principe de discrétisation est similaire au calcul d'une intégrale en utilisant la sommation de Riemann. On remarque que généralement, un espace de modules admet une structure similaire à celle des objets qu'il contient. Typiquement les espaces des surfaces hyperboliques admettent des métriques naturelles et les espaces de modules des surfaces de Riemann une structure de variété complexe. Dans le cas des surfaces plates, on peut voir que les espaces de modules sont "presque" plats. Ils admettent une structure linéaire dont le groupe structural est  $GL_m(\mathbb{Z})$ . On dispose d'un atlas de cartes à valeurs dans  $\mathbb{C}^m$  et les changements de cartes sont des transformations linéaires qui préservent le réseau  $(\mathbb{Z} \oplus i\mathbb{Z})^m$ . Les volumes de Masur-Veech sont calculés à l'aide de la mesure de Lebesgue sur  $\mathbb{C}^m$ , tirée en arrière et normalisée par ces points entiers. On peut alors estimer un volume de Masur-Veech en comptant le nombre de points entiers dans l'espace des modules correspondant. Les détails de cette méthode sont exprimés

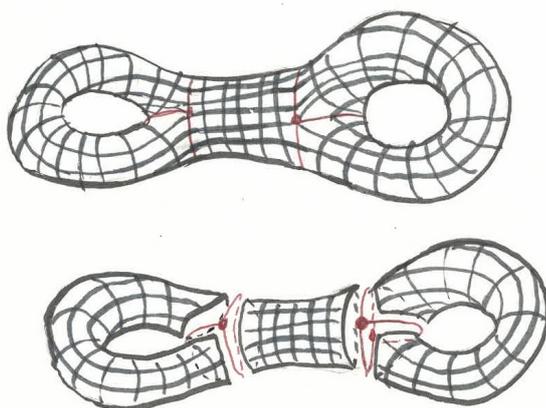


Figure 1.3: Décomposition en cylindres d'une surface à petits carreaux, les graphes en rubans sont en rouge.

dans le chapitre 7.4.2 dans le cas des différentielles abéliennes. L'espace des modules n'est pas un espace ordinaire et l'interprétation géométrique des points entiers est tout aussi extraordinaire. Les points entiers correspondent aux "surfaces à petits carreaux", dans le cas des différentielles abéliennes, ce sont des revêtements du tore  $\mathbb{C}/\mathbb{Z}[i]$  ramifiés uniquement au dessus du point 0. Une surface à petits carreaux peut-être obtenue à partir de carrés  $[0, 1]^2$  recollés le long de leurs bords par des translations. Les surfaces à petits carreaux sont des objets à la fois très simples à construire mais ayant des propriétés arithmétiques très profondes. Il existe plusieurs façons de compter les surfaces à petits carreaux, l'une d'elle utilise le comptage des revêtements via la formule de Frobenius qui exprime le nombre de surfaces à petits carreaux à l'aide des caractères du groupe symétrique. Ce chemin a été suivi par A.Eskin et A.Okounkov [EO01],[AO06] et Agarwall [Agg20]. La seconde a été développée par A.Zorich et ses collaborateurs dans [Zoro2],[DGZZ21] et plus récemment par E.Goujard et E.Duriev, c'est cette approche que l'on a suivie. Le flot géodésique des surfaces à petits carreaux dans des directions rationnelles est totalement périodique. Si on se fixe une surface à petits carreaux et une direction périodique du flot géodésique, par exemple la direction horizontale. Les trajectoires périodiques dans cette direction définissent des cylindres qui recouvrent la surface. Ces cylindres sont recollés entre eux le long de leurs bords et la plus grande partie de l'information est donc contenue dans la façon de les recoller. Ces recollements ont lieu sur des configurations de liens de selles qui tracent un graphe sur la surface (voir figure 1.3. Ces graphes sont appelés graphes en rubans et ont fait plusieurs apparitions dans les mathématiques modernes. Ils sont en relation avec de nombreux objets apparaissant en combinatoire, dans la topologie en basse dimension et dans la théorie des cordes. De manière générale on peut reconstruire les surfaces à petits carreaux en recollant des graphes en rubans le long de cylindres. Dans le cas des différentielles abéliennes ces graphes en rubans possèdent une propriété supplémentaire : ils sont orientables. Cette propriété signifie que l'on peut orienter les arêtes de façon cohérente. L'étude des graphes en rubans orientables est l'objet d'une grande partie du présent mémoire. Mais avant d'aborder ce point nous allons d'abord nous intéresser dans le paragraphe suivant au cas bien moins dégénéré des strates principales des espaces de modules des différentielles quadratiques.

**Espace de modules des graphes en rubans et conjecture de Witten :** Soit  $M$  une surface de genre  $g$  avec  $n$  bords. Un graphe en rubans  $R$  sur  $M$  est un graphe tracé sur  $M$  et sur lequel  $M$  se rétracte. Les graphes en rubans peuvent être définis de façon purement combinatoire. Une métrique sur le graphe est la donnée d'un nombre réel strictement positif associé à chaque arête du graphe, la métrique indique la longueur de cette arête. Comme précédemment il est possible de définir l'espace des modules  $\mathcal{M}_{g,n}^{comb}$  des graphes en rubans métrisés sur  $M$ , on l'appelle l'espace des modules combinatoire. Pour appuyer la philosophie générale sur les espaces des modules, un graphe en rubans définit un 1-complexe cellulaire plongé dans  $M$  et l'espace des modules  $\mathcal{M}_{g,n}^{comb}$  des graphes en rubans métrisés admet aussi une structure de complexe cellulaire tout à fait naturelle. Suite aux travaux de K.Strebel [Str84] on sait que l'espace  $\mathcal{M}_{g,n}^{comb}$  s'identifie à  $\mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n$  et cela fournit des décompositions cellulaires des espaces des modules usuels  $\mathcal{M}_{g,n}$ . Comme dans le cas des surfaces hyperboliques, une métrique sur un graphe en rubans permet de mesurer la longueurs des bords et cela fournit une application

$$L_\partial : \mathcal{M}_{g,n}^{comb} \longrightarrow \mathbb{R}_{>0}^n.$$

Afin de calculer les volumes de Masur-Veech il est alors nécessaire de calculer les volumes  $V_{g,n}^{comb}(L)$  des lignes de niveau  $\mathcal{M}_{g,n}^{comb}(L) = L_\partial^{-1}(L)$ . Ces volumes ont été étudiés pour la première fois par M.Kontsevich lors de la preuve de la conjecture de Witten. Les volumes sont calculés à l'aide d'une forme symplectique naturelle  $\omega_{g,n}^{comb}(L)$  introduite par M.Kontsevich. On dispose alors du résultat suivant démontré dans [Kong2] sous une autre forme.

**Théorème 1.5** (Kontsevich [Kong2]). *Les volumes sont donnés par la formule suivante :*

$$V_{g,n}^{comb}(L) = \sum_{\alpha} \langle \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \rangle \prod_i \frac{L_i^{2\alpha_i}}{2^{\alpha_i} \alpha_i!}$$

Ces polynômes sont généralement appelés polynômes de Kontsevich et sont directement reliés aux différentielles données précédemment.

**Calcul des volumes de Masur-Veech, cas de la strate principale des différentielles quadratiques :** En utilisant le contenu des deux paragraphes précédents, il a alors été possible à V.Delecroix E.Goujard P.Zograph et A.Zorich [DGZZ21] de donner une formule explicite pour les volumes de Masur-Veech dans le cas de la strate principale des différentielles quadratiques. Cette formule avait cependant été démontrée quelques années plus tôt par M.Mirzakhani dans [Miro8a], mais est passée inaperçue.

**Théorème 1.6.** *Les volumes de Masur-Veech de  $\mathcal{Q}^{(1)}((-1)^n, 1^{4g-4+n})$  (i.e de la strate principale) sont donnés par*

$$\vartheta_{g,n} = \sum_{\mathcal{G}} \prod_{\gamma} \zeta(2\alpha_{\gamma} + 2) \int_{\overline{\mathcal{M}}_{g,n}} \Delta_{\mathcal{G}} \prod_{\gamma} \psi_{\gamma}^{\alpha_{\gamma}}.$$

où la somme porte sur toutes les composantes de bord de l'espace  $\overline{\mathcal{M}}_{g,n}$ .

La somme porte sur tous les graphes stables et  $\Delta_{\mathcal{G}}$  correspond au sous espace  $\mathcal{M}_{g,n}(\mathcal{G})$  de la compactification de Deligne-Mumford  $\overline{\mathcal{M}}_{g,n}$  qui est la fermeture de l'ensemble des courbes nodales ayant  $\mathcal{G}$  pour graphe stable (voir chapitre 6.3 pour des précisions). Chaque graphe stable correspond à une configuration possible de cylindres dans les surfaces à petits carreaux. Les éléments de la formule sont tous calculables par récurrence. Mais la formule de récurrence

pour  $V_{g,n}^{comb}(L)$  vérifie une propriété forte; en un certain sens elle commute avec la somme sur les graphes stables. Ainsi le terme général de la formule vérifie lui aussi une formule de récurrence topologique. Cette approche a été développée pour la première fois dans [ABC<sup>+</sup>23] et utilise certaines statistiques sur les longueurs des multi-géodésiques sur les surfaces hyperboliques.

**Théorème 1.7** ([ABC<sup>+</sup>23]). *On dispose d'un polynôme  $V_{g,n}^{MV}$  tel que  $V_{g,n}^{MV}(0) = \frac{2^{4g-4+n}(4g-4+n)!}{(6g-7+2n)!} \vartheta_{g,n}$  de plus  $V_{g,n}^{MV}$  peut être calculé à partir de la récurrence topologique.*

On ne donne pas la courbe spectrale car elle est peu parlante, le coefficient devant  $\vartheta_{g,n}$  dépend du choix de la normalisation pour les volumes de Masur-Veech. Pour être plus précis, les statistiques des multi-géodésiques commutent avec la formule de Mirzakhani-McShane et la récurrence géométrique en général. Suite à cet énoncé nous pouvons énoncer la question suivante :

**Question 1.2.** *Est-il possible de généraliser ces résultats aux cas d'autres strates ?*

Comme souvent nous verrons que la réponse est mitigée. L'approche précédente utilise de façon cruciale les volumes des espaces des modules des graphes en rubans. Dans le cas de la strate principale on utilise les volumes  $V_{g,n}^{comb}(L)$  de  $\mathcal{M}_{g,n}^{comb}(L)$  qui correspondent aux graphes trivalents. L'espace  $\mathcal{M}_{g,n}^{comb}$  admet une stratification naturelle  $\mathcal{M}_{g,n}^{comb}(\nu)$  indexée par des partitions  $\nu$  et  $\mathcal{M}_{g,n}^{comb}(\nu)$  correspond à la fermeture de l'ensemble des graphes en rubans ayant  $\nu(i)$  sommets de degré  $i + 2$ . Dans ce cas il est toujours possible de définir les volumes  $V_{g,n}^\nu(L)$  de  $\mathcal{M}_{g,n}^{comb}(\nu, L)$  mais pour appliquer la stratégie précédente on doit se poser la question suivante :

**Question 1.3.** *Est-il possible de calculer les volumes des espaces des modules des graphes en rubans plus généraux ?*

Dans le cas des différentielles abéliennes comme on l'a mentionné précédemment les graphes sont orientés, et nous verrons plus tard ce que l'on entend par "plus général".

## 1.3 Travaux effectués durant la thèse

### 1.3.1 Volumes des espaces des modules des graphes en rubans

Dans la partie précédente, on a vu que dans le cas de la strate principale des différentielles quadratiques on peut exprimer les volumes de Masur-Veech en fonction de ceux des espaces des modules des graphes en rubans  $V_{g,n}^{comb}$ . On peut se demander si cette approche se généralise. L'espace  $\mathcal{M}_{g,n}^{comb}$  admet une stratification naturelle  $\mathcal{M}_{g,n}^{comb}(\nu)$  indexée par des partitions  $\nu$ .  $\mathcal{M}_{g,n}^{comb}(\nu)$  correspond à la fermeture de l'ensemble des graphes en rubans ayant  $\nu(i)$  sommets de degrés  $i + 2$ . Dans ce cas, il est toujours possible de définir les volumes  $V_{g,n}^\nu(L)$  de  $\mathcal{M}_{g,n}^{comb}(\nu, L)$  mais pour appliquer la stratégie précédente il faut d'abord répondre à la question suivante :

**Question 1.4.** *Est-il possible de calculer les volumes  $V_{g,n}^\nu(L)$  des espaces des modules des graphes en rubans plus généraux ?*

En dehors du cas des graphes trivalents, très peu de choses étaient connues sur ces volumes. La façon naïve est d'énumérer les graphes et de calculer la contribution de chacun. Mais c'est une tâche extrêmement fastidieuse qui ne peut fournir que des résultats ponctuels. Pour obtenir des résultats plus généraux on peut chercher une possible relation de récurrence topologique pour ces volumes. Dans le cas des différentielles abéliennes on a mentionné précédemment que les graphes en rubans sont orientés, ces graphes avaient déjà été étudiés dans différents domaines de mathématiques [DM18],[KZ03] et plus récemment dans [Yak22]. Ces graphes ont de nombreuses propriétés, en particulier l'orientation des arêtes induit une orientation des bords de la surface. Certains bords sont orientés conformément au sens induit par l'orientation de la surface et on leur attribue un signe  $+$ , d'autres dans le sens contraire et on leur attribue un signe  $-$ . On peut alors définir l'espace  $\mathcal{M}_{g,n^+,n^-}^{comb}$  des graphes en rubans orientés avec  $n^+$  bords  $+$  et  $n^-$  bords  $-$ . Dans ce cas si on note

$$L^\pm : \mathcal{M}_{g,n^+,n^-}^{comb} \longrightarrow \mathbb{R}^{n^\pm}$$

le vecteur qui contient les longueurs des bords  $\pm$ , alors les volumes  $V_{g,n^+,n^-}(L^+|L^-)$  de  $\mathcal{M}_{g,n^+,n^-}^{comb}(L^+|L^-)$  définissent des fonctions supportées par l'hyperplan d'équation

$$\sum_i L_i^+ = \sum_i L_i^-.$$

Plus généralement  $V_{g,n^+,n^-}^\nu(L)$  est défini comme le volume de la strate  $\mathcal{M}_{g,n^+,n^-}^{comb}(\nu, L)$  correspondant aux graphes avec  $\nu(i)$  sommets de degré  $2i + 2$ . Il est alors aussi naturel de se poser la question suivante :

**Question 1.5.** *Peut-on calculer les fonctions  $V_{g,n^+,n^-}^\nu$  ?*

### 1.3.2 Premier pas vers une généralisation

Avant de chercher à répondre aux questions précédentes on s'est d'abord posé la question suivante :

**Question 1.6.** *Peut-on trouver une preuve de la récurrence pour  $V_{g,n}^{comb}$  qui utilise les chirurgies sur les graphes en rubans ?*

Car initialement les preuves de la conjecture de Witten se basaient sur des outils extérieurs comme les intégrales de matrices [Kong2] ou encore la théorie de l'intersection des espaces des modules [KLo7]. Une récurrence utilisant les graphes en rubans trivalents pourrait alors se généraliser à une classe plus générale de graphes. Le premier résultat de la thèse a donc été de répondre à cette question. Cette récurrence utilise la récurrence géométrique formulée dans [ABO17] et utilisée pour la première fois dans [Miro7] pour calculer les volumes de Weil-Petersson. Une première formulation de la récurrence que nous allons donner dans ce paragraphe avait été donnée dans une première version de l'article [ABO17] en utilisant le fait que  $V_{g,n}^{comb}$  est le terme de plus haut degré de  $V_{g,n}^{hyp}$  et la formule de Mirzakhani-McShane pour les surfaces hyperboliques [Miro7]. De façon indépendante de l'auteur, les auteurs de [ABO17] ont donné dans [ABC<sup>+</sup>20] une interprétation de la formule comme une formule de Mirzakhani-McShane pour les graphes en rubans. Pour démontrer cette formule nous étudierons en quelque sorte la géométrie des graphes en rubans métrisés. C'est-à-dire que nous allons

considérer des courbes, nous intéresser à leur longueur, effectuer des découpages et des recolllements. Dans l'esprit des travaux précédemment réalisés pour les surfaces hyperboliques, beaucoup d'opérations se transposent aux graphes trivalents. Il y avait donc un espoir que ces analogies permettent de démontrer une formule de Mirzakhani-McShane et d'en déduire des relations de récurrence par intégration sur les espaces des modules. La preuve utilise la récurrence géométrique introduite dans [ABO17] et qui a pour but de capturer les propriétés de la formule éponyme de Mirzakhani-McShane. Cette formule a été découverte par Mac-Shane dans [McS98] puis Mirzakhani lui a donnée toute sa puissance en la généralisant au cas des surfaces hyperboliques à bords. On fixe  $M$  une surface de genre  $g$  avec  $n$  bords numérotés de 1 à  $n$ . Pour tout  $i$  on note

- $\text{Irr}_{1,i}(M)$  l'ensemble de toute les classes d'isotopies  $P$  de pantalons plongés dans  $M$  tel que l'image de  $P$  contienne les bords 1,  $i$ .
- $\text{Irr}_{1,1}(M)$  les pantalons  $P$  contenant le bord 1 et aucun autre bord.

Pour chaque pantalon notons  $L_P(S)$  la longueur des bords de  $P$  qui ne sont pas dans  $\partial M$  on montrera alors le théorème suivant

**Théorème 1.8** (théorème 6.2). *On dispose de la formule de Mirzakhani-McShane suivante, pour tout graphe en rubans métrisé générique  $S$*

$$L_1(S) = \sum_{i \neq 1} \sum_{P \in \text{Irr}_{1,i}(M)} F^+(L_1(S), L_i(S) | L_P(S)) \\ + \sum_{P \in \text{Irr}_{1,1}(M)} F^-(L_1(S) | L_P(S)).$$

La somme est infinie mais ne contient qu'un nombre fini de termes non nuls, les fonctions  $F^\pm$  sont explicites et linéaires par morceau. La formule de Mirzakhani-McShane est très pratique car on peut l'intégrer sur l'espace des modules. En généralisant des techniques développées dans [Miro7] pour calculer ce type d'intégrale on peut obtenir la récurrence suivante, qui est très similaire à celle du théorème 1.1 et qui est aussi obtenue dans [ABC<sup>+</sup>20].

**Proposition 1.1** (proposition 6.3). *Les volumes  $V_{g,n}^{\text{comb}}(L)$  sont des polynômes symétriques solutions de la récurrence*

$$L_1 V_{g,n}^{\text{comb}}(L) = \sum_{j \neq 1} \int_{\mathbb{R}_{\geq 0}} F^+(L_1, L_j | x) V_{g,n-1}^{\text{comb}}(x, L_{\{1,j\}^c}) dx \\ + \frac{1}{2} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g-1,n+1}^{\text{comb}}(x_1, x_2, L_{\{1\}^c}) x_1 x_2 dx_1 dx_2 \\ + \frac{1}{2} \sum_{g_1+g_2=g, I_1 \sqcup I_2 = \{2, \dots, n\}} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g_1, n_1+1}^{\text{comb}}(x_1, L_{I_1}) V_{g_2, n_2+1}^{\text{comb}}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2$$

avec les conditions initiales

$$V_{0,3}^{\text{comb}}(L_1, L_2, L_3) = 1 \quad V_{1,1}^{\text{comb}}(L) = \frac{L^2}{24}$$

On remarque qu'en utilisant des méthodes similaires on peut aussi obtenir la formule suivante

$$(6g - 6 + 3n) = \frac{1}{2} \sum_{i,j} \sum_{P \in \text{Irr}_{i,j}(M)} H_P(L_{\{i,j\}}(S), L_P(S))$$

qui est vraie pour tout graphe en rubans métrisé trivalent  $S$ ;  $L_{\{i,j\}}(S)$  est le vecteur indiquant la longueur des bords  $\{i, j\}$  et  $H_P$  est explicite. Ce corollaire permet alors de montrer que la fonction de partition  $Z^K$  de Kontsevich-Witten est solution d'une équation de Cut-and-Join.

**Proposition 1.2** (proposition 6.5 et [Ale11]). *La fonction de partition de Kontsevich-Witten  $Z^K$  est solution de l'équation de Cut-and-Join*

$$\frac{\partial Z^K}{\partial q} = \sum_{k+l=n+1} (2k+1)(2l+1)t_{2k+1}t_{2l+1}\partial_{2n+1}Z^K + \sum_{k,l} (2k+2l+5)t_{2k+2l+5}\partial_{2k+1}\partial_{2l+1}Z^K + t_1^3Z^K + \frac{t_3Z^K}{24}$$

avec pour condition initiale

$$Z^K(0) = 1.$$

### 1.3.3 Cas des sommets d'ordre impair

La première direction envisagée pour généraliser la formule de Mirzakhani-McShane à une classe plus générale de graphes en rubans est celle des graphes ayant uniquement des sommets de degrés impairs. En effet de nombreux résultats valables dans le cas des graphes trivalent se généralisent (ou sont censés se généraliser) dans ce cas. Les volumes  $V_{g,n}^\nu$  sont calculés à l'aide d'une forme symplectique qui est une généralisation directe de la forme de Kontsevich. Il est aussi pressenti que ces volumes sont des "polynômes" et que leurs coefficients sont reliés à la théorie de l'intersection des espaces  $\overline{\mathcal{M}}_{g,n}$  [Kon92], [AC94]. Un second axe de travail a donc été le suivant :

**Question 1.7.** *Peut-on généraliser la formules de Mirzakhani-McShane aux cas des graphes avec uniquement des sommets impairs ?*

Après des investigations poussées sur les graphes en rubans il est apparu pour l'auteur qu'il n'était pas sans espoir de chercher à répondre à cette question. Il y a cependant quelques particularités. Tout d'abord l'approche précédente utilise de façon cruciale les chirurgies et le flot de twist le long d'une courbe. Le flot de twist s'utilise notamment pour intégrer la formule de Mirzakhani-McShane. Si on sort du cadre des graphes trivalents, il est possible qu'une courbe sépare un sommet du graphe en plusieurs sommets de degrés inférieurs. Ceci est problématique car dans ce cas le flot de twist n'est plus défini et il est alors difficile de calculer les intégrales. De plus cela fait apparaître des graphes dont les sommets ont de nouveaux degrés. En d'autres termes si on veut calculer le volumes des strates des graphes ayant des sommets de degré 7 il est préférable de ne pas faire apparaître des graphes avec des sommets de degrés 3 ou 5. Lors des chirurgies il est alors nécessaire de se restreindre aux courbes qui ne séparent pas les sommets du graphe, on introduit ces courbes dans la section 4.2 et on étudie leurs propriétés (voir figure 1.4). On a choisi d'appeler ces courbes admissibles car ce sont celles que l'on autorise lors des chirurgies. On voit donc que dans le cas général tous les découpages ne sont pas légaux; en particulier, il n'est pas toujours possible d'extraire des pantalons. Cependant il existe une classe de graphes que l'on peut appeler irréductibles et qui n'admettent pas de courbes admissibles, il n'est donc pas possible de les découper en graphes plus simples. Ils sont de genre 0 avec seulement deux sommets, ces graphes sont des candidats potentiels pour généraliser les pantalons utilisés précédemment. Et il semblait possible de généraliser la formule de Mirzakhani-McShane; dans ce cas on n'extrait plus seulement des pantalons mais des

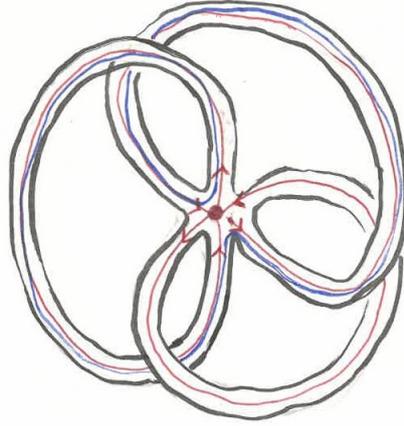


Figure 1.4: Courbe admissible (en bleu ) sur un graphe en rubbans orienté.

sous-surfaces de topologie plus complexe et cela pourrait être relié à des formules de récurrence topologique supérieures. Néanmoins des cas particuliers sont apparus qui n'existaient pas dans le cas des graphes trivalents et actuellement aucune solution n'a encore été trouvée pour écrire la formule de Mirzakhani-McShane dans ce cas. Cette question reste donc ouverte et sera peut-être l'objet de futures recherches.

### 1.3.4 Graphe orientable, décomposition acyclique et récurrence pour les volumes $V_{g,n^+,n^-}^\nu$

**Graphes orientés et décompositions acycliques :** Il s'avère que les tentatives de prouver la formules de Mirzakhani-McShane au cas des graphes en rubans avec des sommets d'ordres impairs ont permis d'obtenir des relations de récurrence dans un autre cas. Il s'agit de celui des graphes orientables: comme on l'a mentionné précédemment, ils apparaissent dans l'étude des différentielles abéliennes. Dans ce cas les sommets sont tous d'ordre pair, on peut toujours considérer les espaces des modules mais les 2-formes de M.Kontsevich ne sont généralement plus du tout symplectiques mais dégénérées et les volumes sont reliés au dénombrement des points entiers. Le cas des graphes en rubans orientés diffère radicalement de celui des graphes trivalents. Comme on l'a dit précédemment, l'orientation des arêtes du graphe induit une orientation pour les composantes de bords (voir figure 1.5). Dans ces graphes chaque arête est limitrophe d'un bord  $-$  et d'un bord  $+$  et cela induit une structure bipartite sur le graphe dual (des précisions seront données dans la partie 4.1.1). Cette condition implique notamment que la somme des longueurs des bords négatifs est égale à la somme des longueurs des bords positifs. Le volume de l'espace des module des graphes en rubans orientés de genre  $g$  avec  $n^+$  bords positif (resp  $n^-$  bords négatifs) est donc une fonction  $V_{g,n^+,n^-}(L^+|L^-)$  de deux jeux de variables. De plus les  $L^+, L^-$  appartiennent au sous-espace

$$\Lambda_{n^+,n^-} = \{(L^+, L^-) \in \mathbb{R}_{\geq 0}^{n^+} \times \mathbb{R}_{\geq 0}^{n^-} \mid \sum_{i=1}^{n^+} L_i^+ = \sum_{i=1}^{n^-} L_i^-\}.$$

. On note  $R^\circ$  les graphes en rubans orientés, comme auparavant il est possible de considérer les courbes admissibles sur les graphes en rubans orientés. Dans ce cas après avoir découpé

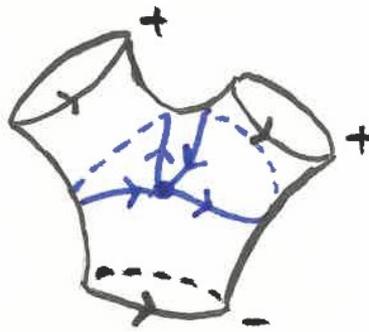


Figure 1.5: Un graphe en rubans dirigé sur un pantalon.

le graphe selon une courbe admissible, le résultat obtenu est toujours une famille de graphes orientés, de plus deux bords recollés le long d'une courbe ont des signes opposés. Cette condition peut être familière à ceux qui ont étudié les différentielles abéliennes et les feuilletages orientables. Cette condition sur les recollements induit une orientation des arêtes des graphes stables qui sont alors des graphes dirigés (voir définition 3.5 et figure ??). L'étude des courbes sur les graphes orientés a permis de démontrer le résultat suivant qui est le deuxième résultat de cette thèse :

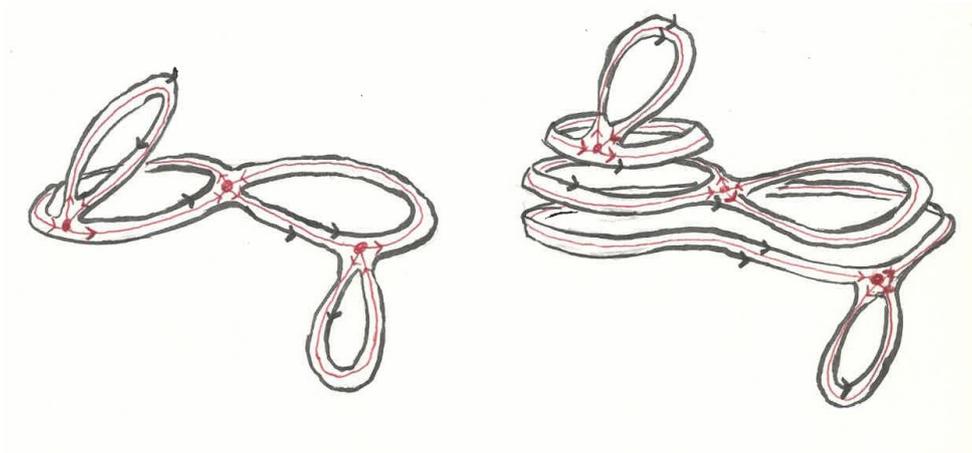


Figure 1.6: Décomposition acyclique d'un graphe de type  $(0, 3, 2)$ .

**Théorème 1.9** (théorème 4.2). *Soit  $R^\circ$  un graphe en rubans orienté connexe avec au moins deux sommets. Pour chaque sommet  $v$  il existe une unique multi-courbe admissible primitive  $\Gamma_v^+$  telle que*

- *Le graphe dirigé  $\mathcal{G}_v^\circ$  de  $\Gamma_v^+$  contient une composante  $c_v$  qui sépare  $v$  du reste de la surface.*
- *Toutes les courbes de  $\Gamma_v^+$  sont des bords de  $c_v$ .*
- *$c_v$  est recollée selon ses bords  $-$ .*

Ce résultat peut paraître surprenant par sa rigidité: en effet ce théorème stipule que si on se fixe un graphe orienté  $R^\circ$  et un sommet  $v$  il existe une unique façon d'extraire  $v$  en effectuant des chirurgies le long de courbes admissibles. Cela est vrai si l'on impose une condition supplémentaire sur les recollements. Sans cette condition il peut y avoir une infinité de façons d'extraire ce sommet. Un point important est que ces opérations utilisent des courbes admissibles et préservent la structure des sommets. La preuve de ce théorème sera donnée dans le chapitre 4.5.2 dans la section 4.3.2 et utilise des outils mathématiques relativement simples comme l'homologie des surfaces compacte à bord et les nombreux outils qu'offre cette théorie. En itérant le théorème 4.2 on peut obtenir une décomposition d'un graphe en ruban orienté général en une famille de graphes en ruban à un sommet, ce qui est le contenu du théorème 4.3. De plus les conditions du théorème 4.2 impliquent une structure particulière pour les recollements. Les graphes stables dirigés sont acycliques dans le sens où il n'y a pas de cycle orienté sur le graphe. Nous avons choisi d'appeler une telle décomposition une décomposition acyclique. On a choisi de nommer les graphes à un sommet graphes minimaux, par analogie avec la terminologie utilisée pour les différentielles abéliennes. Minimal n'est pas la même chose qu'irréductible: ici les graphes peuvent avoir un genre non trivial, mais ces graphes n'admettent pas de décomposition acyclique (non triviale) et sont donc des briques élémentaires pour étudier les graphes orientés généraux. Une décomposition acyclique est maximale si toutes ces composantes sont minimales. Pour résumer cette discussion, on donne le théorème suivant :

**Théorème 1.10** (Théorème 4.3). *Un graphe en ruban orienté avec une énumération des sommets admet une unique décomposition acyclique maximale qui respecte cette énumération.*

Encore une fois l'unicité est surprenante: on définira les termes de cet énoncé plus précisément dans ce mémoire.

**Récurrence pour les volumes  $V_{g,n^+,n^-}$  :** La structure des récollements et des graphes acycliques fait qu'il est possible de déduire du théorème 4.3 une relation de récurrence pour les volumes  $V_{g,n^+,n^-}$ . Cette récurrence est une conséquence du théorème 4.2. Le fait qu'il existe une unique courbe permet d'utiliser le théorème en "famille" et d'en déduire des relations de récurrence pour les volumes. Ces relations font l'objet du théorème suivant que nous donnons dans le cas de la strate principale, c'est-à-dire des graphes ayant uniquement des sommets d'ordre 4.

**Théorème 1.11** (Théorème 4.6). *Pour toutes les valeurs de  $L^+$  les volumes  $V_{g,n^+,n^-}(L^+|L^-)$  sat-*

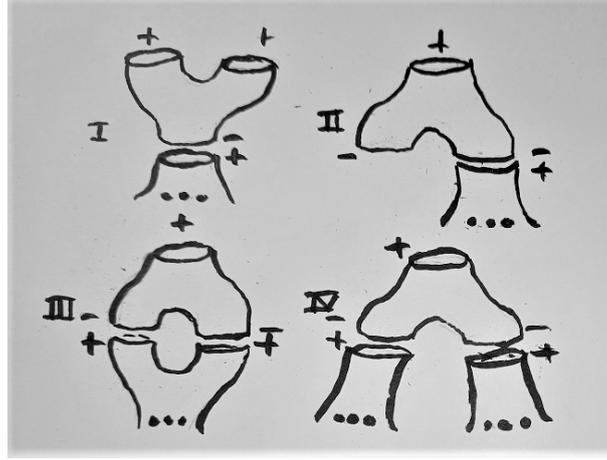


Figure 1.7: Différents recollements apparaissant dans le théorème 4.6

isfont la récurrence

$$\begin{aligned}
(2g - 2 + n^+ + n^-)V_{g,n^+,n^-}(L^+|L^-) &= \sum_i \sum_j [L_i^+ - L_j^-]_+ V_{g,n^+,n^- - 1}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) \\
&+ \frac{1}{2} \sum_{i \neq j} (L_i^+ + L_j^+) V_{g,n^+ - 1, n^-} (L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-) \\
&+ \frac{1}{2} \sum_i \int_0^{L_i^+} V_{g-1, n^+ + 1, n^-} (x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) x (L_i^+ - x) dx \\
&+ \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^\pm \sqcup I_2^\pm = I^\pm}} x_1 x_2 V_{g_1, n_1^+, n_1^-} (x_1, L_{I_1^+}^+ | L_{I_1^-}^-) V_{g_2, n_2^+, n_2^-} (x_2, L_{I_2^+}^+ | L_{I_2^-}^-).
\end{aligned}$$

Où on utilise la notation

$$x_l = \sum_{i \in I_l^-} L_i^- - \sum_{i \in I_l^+} L_i^+.$$

Et on initialise par

$$V_{0,2,1} = 1 \quad V_{0,1,2} = 1.$$

Cette récurrence a une structure similaire à celle de la proposition 6.3. Dans le cas des graphes orientés il existe cependant plusieurs façons d'extraire un pantalon de la surface. Notons que dans ce cas il existe deux types de pantalons  $P_+$ ,  $P_-$  qui sont de type  $(0, 2, 1)$  et  $(0, 1, 2)$  et les différents recollements possibles sont listés dans la figure 1.7. Chaque type de recollements correspond à une ligne de la formule de récurrence. Nous verrons plus tard comment exploiter cette relation de récurrence. Mentionnons un autre résultat qui montre que les volumes diffèrent fortement du cas des graphes trivalents :

**Théorème 1.12.** *Les volumes  $V_{g,n^+,n^-}(L^+|L^-)$  sont des polynômes par morceaux dont les murs sont déterminés.*

Ce théorème peut se montrer par récurrence mais nous le montrerons en utilisant le théorème 4.3 et des résultats généraux sur les graphes acycliques qui utilisent la théorie de Ehrhart [Baro8] (voir théorème 3.5 et proposition 4.26).

**Cas des sommets de degrés supérieurs :** Ces résultats montrent que les volumes  $V_{g,n^+,n^-}$  des espaces des modules des graphes dirigés sont calculables. Qu'en est-il du cas des graphes ayant des sommets d'ordre supérieurs? On peut déjà voir que le cas des sommets d'ordre inférieur i.e d'ordre 2 est presque trivial. En effet si on note  $V_{g,n^+,n^-,m}$  le volume associé aux graphes orientables génériques avec en plus  $m$  sommets bivalents, alors  $V_{g,n^+,n^-,m}$  est donné par

$$V_{g,n^+,n^-,m} = \frac{E^m}{m!} V_{g,n^+,n^-}$$

où  $E$  est la fonction définie sur  $\Lambda_{n^+,n^-}$  par

$$E(L) = \sum_i L_i^+ = \sum_i L_i^-.$$

Ce résultat est aussi une conséquence du théorème 4.2. Dans le cas le plus général on peut aussi utiliser ce théorème pour obtenir un analogue de la formule du théorème 4.6. Dans ce cas on ne somme plus sur toutes les façons d'extraire des pantalons mais sur toutes les façons d'extraire des surfaces minimales de caractéristique d'Euler  $-i$ , où  $2i + 2$  est le degré du sommet que l'on extrait. La formule de récurrence utilise alors les volumes associés aux graphes minimaux. Ces volumes sont calculables en toute généralité et ont par exemple été étudiés par I.Yakovlev dans [Yak22]. Combinant ses résultats et la récurrence du théorème 4.3 il est alors possible de calculer en toute généralité les volumes des espaces des modules des graphes en rubans orientés. On peut alors conclure cette partie par le résultat suivant :

**Théorème 1.13.** *On peut calculer les volumes  $V_{g,n^+,n^-}^\nu$ , de plus il existe un polynôme par morceaux continu qui coïncide avec  $V_{g,n^+,n^-}^\nu$  pour presque toutes les valeurs de  $L$ .*

Les murs définissant les polynômes sont explicites et sont les mêmes que pour  $V_{g,n^+,n^-}$ . Le dernier point du théorème peut paraître obscur: nous reviendrons dessus plus tard dans l'introduction.

**Peut-on calculer les volumes de Masur-Veech ?** La formule de Mirzakhani-McShane a permis de démontrer dans [?] une formule similaire pour certaines statistiques multiplicatives concernant les longueurs des multi-géodésiques sur les surfaces hyperboliques et aussi pour les courbes sur les graphes en rubans trivalents. Malheureusement la formule de Mirzakhani-McShane et la décomposition en graphes acycliques sont deux formules différentes. Dans la première, on réduit la taille d'un bord; dans la seconde, on retire un sommet. Ces différences font que la décomposition acyclique ne commute pas avec les statistiques des multi-courbes et il est donc difficile de généraliser directement les résultats de [?]. Il est cependant possible d'obtenir des formules ponctuelles en calculant l'amplitude associée à chaque graphe stable dirigé. Néanmoins la décomposition en graphes acycliques possède de nombreuses propriétés intéressantes que nous verrons dans la section suivante.

### 1.3.5 Volumes de graphes en rubans orientables et opérateurs de Cut-and-Join :

**Cas particulier de  $V_{g,1}^\circ$  :** Il est frustrant de constater que les volumes  $V_{g,n^+,n^-}$  ne sont seulement que des polynômes par morceaux. Cela réduit fortement les possibles applications aux

calculs des volumes de Masur-Veech et il n'est pas direct d'en extraire des informations. Il y a cependant un cas particulier où ces volumes sont des polynômes, il s'agit de celui des surfaces à un bord négatif. Dans ce cas  $\Lambda_{n^+, n^-} \simeq \mathbb{R}_{\geq 0}^{n^+}$  et les volumes  $V_{g, n, 1}(L^+|L^-)$  ne dépendent seulement que des variables  $L^+$ , ils peuvent s'écrire sous la forme :

$$V_{g, n}^\circ(L^+) = V_{g, n, 1}(L^+|L^-).$$

Dans la partie 4.5.2 on se propose de démontrer le résultat suivant qui est une conséquence directe du théorème 4.6.

**Théorème 1.14** (Théorème 4.11).  $V_{g, n}^\circ$  est un polynôme symétrique homogène de degrés  $4g - 4 + n$  et satisfait la relation de récurrence.

$$(2g + n - 1)V_{g, n}^\circ(L) = \frac{1}{2} \sum_{i \neq j} (L_i + L_j) V_{g, n-1}^\circ(L_i + L_j, L_{\{i, j\}^c}) \\ + \frac{1}{2} \sum_i \int_0^{L_i} V_{g-1, n+1}^\circ(x, L_i - x, L_{\{i\}^c}) x(L_i - x) dx$$

Initialisée par  $V_{0, 2}^\circ(L) = 1$

Les coefficients de  $V_{g, n}^\circ$  ont plusieurs interprétations combinatoires et sont reliés aux graphes orientés et aux nombres d'Hurwitz associés à certains types de dessins d'enfants (revêtements de la sphère ramifiés au dessus de trois points). Notons  $c_{g, n}(\alpha)$  le coefficient de  $V_{g, n}^\circ$  devant  $L^\alpha$  où  $\alpha = (\alpha_1, \dots, \alpha_n)$  est un multi-indice. Les coefficients  $c_{g, n}$  sont symétriques et peuvent être indexés par des partitions  $\mu = (\mu(0), \mu(1), \dots)$ . On peut se passer des indices  $(g, n)$  car ils sont déterminés par  $\mu$ . On considère alors la série génératrice suivante :

$$Z^\circ(q, \mathbf{t}) = \sum_{\mu} \frac{q^{\frac{d(\mu) + n(\mu)}{2}} \prod_i (i!)^{\mu(i)} t_i^{\mu(i)}}{\prod_i \mu(i)!} c(\mu)$$

On obtient alors que la récurrence du théorème précédent est équivalente au corollaire suivant :

**Corollaire 1.1** (corollaire 4.16). La série  $Z^\circ(q, \mathbf{t})$  est solution de l'équation linéaire

$$\frac{\partial Z^\circ}{\partial q} = \frac{1}{2} \sum_{i, j} (i + j) t_i t_j \partial_{i+j-1} Z^\circ + \frac{1}{2} \sum_{i, j} (i + 1)(j + 1) t_{i+j+3} \partial_i \partial_j Z^\circ + \frac{t_0^2}{2}$$

avec comme initialisation  $Z^\circ(0, \mathbf{t}) = 0$ .

L'opérateur apparaissant dans l'équation précédente est un opérateur de Cut-and-Join. Ces opérateurs sont notoirement connus et apparaissent dans différents domaines de la topologie énumérative et de la physique.

**Généralisation, opérateurs associés aux volumes  $V_{g, n^+, n^-}$  :** Le cas particulier des surfaces à un bord – est intéressant mais ce n'est qu'un cas particulier. Afin de généraliser ce résultat, j'ai adopté le point de vue suivant. On fixe  $(g, n^+, n^-)$  soit  $f$  une fonction continue sur  $\mathbb{R}_{\geq 0}^{n^-}$  et

$$K_{g, n^+, n^-}(L^+|L^-) = \prod_i L_i^+ V_{g, n^+, n^-}(L^+|L^-)$$

On peut alors considérer l'intégrale

$$K_{g,n^+,n^-} \cdot f(L^+) = \frac{1}{n^-!} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} K_{g,n^+,n^-}(L^+|L^-) f(L^-) d\sigma_{n^-}^{|L^+|}.$$

L'intégrale est bien définie car le domaine d'intégration est borné et la fonction est continue, les quelques notations apparaissant dans cette formule sont introduites dans la section 3.1. La définition de  $K_{g,n^+,n^-}$  n'est pas due au hasard. La composition de deux opérateurs correspond à l'opérateur associé au noyau suivant :

$$\prod_i L_i^+ \int_{x \in |L^+| \cdot \Delta_k} \frac{V_{g_1,n^+,k}(L^+|x) V_{g_2,k,n^-}(x|L^-)}{k!} \prod_j x_j d\sigma_k^{|L^+|}.$$

La mesure  $d\sigma_k^{|L^+|}$  est précisée dans la partie 3.1. Le lecteur familier avec les travaux de M.Mirzakhani peut reconnaître dans le membre de droite des similarités avec les formules obtenues lorsque l'on effectue des chirurgies sur les surfaces hyperboliques par exemple. Composer des opérateurs revient donc à recoller des surfaces. Tout aurait pu s'arrêter là mais le théorème 4.11 admet la généralisation suivante qui provient d'un lemme que l'on a appelé lemme de transfert (lemme 5.7). La généralisation naturelle de la première partie du théorème 4.11 est alors donnée par la proposition suivante :

**Proposition 1.3** (cas particulier de la proposition 5.7). *Pour tout  $(g, n^+, n^-)$  et pour tout  $P \in \mathbb{Q}[L^+]$  homogène de degré  $d$ ,  $K_{g,n^+,n^-} \cdot P$  est un polynôme symétrique homogène de degré  $d + 4g - 4 + 2n^+ + 2n^-$ .*

On exploite alors le résultat de la proposition 5.7 de la façon suivante. Soit  $V = \mathbb{Q}[L]$  on considère alors  $S(V)$  l'espace des polynômes symétriques: il s'agit d'une algèbre commutative pour le produit symétrique  $\sqcup$  qui est de plus graduée par le degré et le nombre de variables.  $K_{g,n^+,n^-}$  définit alors un endomorphisme

$$K_{g,n^+,n^-} : S(V) \longrightarrow S(V)$$

L'espace des endomorphismes sur  $S(V)$  admet aussi une structure d'algèbre commutative pour une généralisation de  $\sqcup$ . Si  $M^\circ = \sqcup_c M^\circ(c)$  est une surface dirigée qui n'est pas forcément connexe on peut définir les volumes  $V_{M^\circ}$  et  $K_{M^\circ}$  l'opérateur associé. Dans ce cas aussi  $K_{M^\circ}$  définit un endomorphisme de  $S(V)$ . L'espace des surfaces dirigées est muni d'une structure de monoïde pour l'union disjointe  $\sqcup$  et on dispose de la formule

$$K_{M_1^\circ \sqcup M_2^\circ} = K_{M_1^\circ} \sqcup K_{M_2^\circ}.$$

Suite à cette propriété, il est naturel de considérer les opérateurs tous ensemble. On peut former l'algèbre commutative  $S(\mathcal{M})$  engendrée par des vecteurs  $e_{M^\circ}$  pour chaque surface dirigée. Alors l'opérateur  $\mathbf{K}(e_{M^\circ}) = K_{M^\circ}$  se prolonge en un morphisme d'algèbres commutatives

$$\mathbf{K} : S(\mathcal{M}) \longrightarrow \text{End}(S(V)).$$

Un détail reste cependant à régler: pour faire fonctionner les formules, il faut inclure dans  $\mathcal{M}$  le cylindre de type  $(0, 1, 1)$  et la surface vide. Dans les deux cas le bon choix est  $id$  respectivement sur  $S_1(V)$  et  $S_0(V)$ . Dans ce cas l'opérateur  $\exp_{\sqcup}(K_{0,1,1})$  est égal à  $id$  de  $S(V)$ , ce qui est bon signe car recoller des cylindres ne change rien.

**Opérateur total :** La récurrence 4.6 peut être vue comme toutes les façons d'extraire un pantalon de type  $(0, 2, 1)$  ou de type  $(0, 1, 2)$ . Cette récurrence utilise des intégrales et on peut se demander si elle peut se traduire en terme des opérateurs  $K_{g,n^+,n^-}$ . Afin de répondre à cette question on considère la série suivante

$$K = \sum_{M^\circ} q^{d(M^\circ)} K_{M^\circ} = \exp_{\sqcup} \left( \sum_{g,n^+,n^-} q^{2g-2+n^++n^-} K_{g,n^+,n^-} \right)$$

où l'on somme sur toutes les surfaces dirigées non nécessairement connexes (y compris les cylindres). Cette somme est infinie; pour lui donner sens on utilise une complétion naturelle  $\hat{S}(V)$  de  $S(V)$  à l'aide des graduations et on montre que  $K$  définit un endomorphisme de  $\hat{S}(V)$  pour tout  $q$ . Ensuite on considère les deux opérateurs

$$P^+ = K_{0,2,1} \sqcup \exp_{\sqcup}(K_{0,1,1}) \quad P^- = K_{0,1,2} \sqcup \exp_{\sqcup}(K_{0,1,1})$$

qui représentent toutes les façons de recoller des pantalons de types  $(0, 2, 1)$  ou  $(0, 1, 2)$ . Le terme  $\exp_{\sqcup}(K_{0,1,1})$  est essentiel car on veut pouvoir recoller un pantalon sur certains bords tout en laissant les autres intacts (comme c'est le cas du théorème 4.6). De façon surprenante, la récurrence du théorème 4.6, qui est en apparence fort compliquée se traduit très simplement dans ce formalisme. Soit

$$P = P_+ + P_-,$$

l'opérateur qui correspond à toutes les façons de recoller un pantalon dirigé.

**Théorème 1.15** (théorème 5.1). *La série  $K(q)$  est solution de l'équation d'évolution*

$$\frac{dK}{dq} = PK$$

avec la condition initiale suivante  $K(0) = id$  on a donc

$$K(q) = \exp(qP).$$

Cette équation est l'équation de Cut-and-Join. On verra que l'espace  $\hat{S}(V)$  s'identifie à l'espace des séries  $\mathbb{Q}[[t_0, t_1, \dots]]$  en une infinité de variables. On verra aussi (dans la section 5.1) que des considérations générales sur les opérateurs de création et d'annihilation sur les espaces de Fock permettent de montrer que  $K, P^+, P^-$  sont des opérateurs différentiels sur  $\mathbb{Q}[[t_0, t_1, \dots]]$ . On a en particulier la formule

$$P = \frac{1}{2} \sum_{i,j} (i+1)(j+1)t_{i+1}t_{j+1}\partial_{i+j} + \frac{1}{2} \sum_{i,j} (i+j+2)t_{i+j+2}\partial_i\partial_j.$$

Cette approche généralise donc directement les résultats du théorème 4.11 et du corollaire 4.16.

**Fonction de partition :** Les fonctions  $V_{g,n^+,n^-}$  sont des polynômes mais  $K$  est un opérateur: on peut se demander quelle est l'information contenue dans ses coefficients. En général la réponse n'est pas claire mais dans un cas particulier les coefficients ont une interprétation combinatoire. On considère les polynômes  $G_{g,n^+,n^-}$  définies par

$$G_{g,n^+,n^-} = \frac{1}{n^-!} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} K_{g,n^+,n^-}(L|L^-) d\sigma_n^{L^+}.$$

On peut voir que  $G_{g,n^+,n^-} = K_{g,n^+,n^-} \cdot e_0$  donc  $G_{g,n^+,n^-}$  est en quelque sorte l'image du vide (vacuum) dans l'espace de Fock  $\hat{S}(V)$ . Les coefficients des polynômes  $G_{g,n^+,n^-}$  sont intéressants et comptent plusieurs quantités

- Graphes en rubans orientés avec des sommets quadrivalents
- Dessin d'enfant avec des ramifications simples au dessus de  $x_0$ .

On peut généraliser la définition aux surfaces non connexes et définir de façon similaire  $G = K \cdot e_\emptyset$ ; notons  $Z$  l'image de  $G$  dans  $\mathbb{Q}[[q, t]]$ . On a alors l'équation suivante qui est une conséquence triviale du théorème 5.1.

**Corollaire 1.2.**  $Z$  est solution de l'équation différentielle

$$\frac{\partial Z}{\partial q}(q) = \frac{1}{2} \sum_{k,l} (k+l+2)t_{k+l+2} \partial_k \partial_l Z + \frac{1}{2} \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1} \partial_{k+l} Z.$$

On verra que cette équation permet de retrouver les résultats précédents pour  $Z^1$  comme premier terme d'un développement.

**Contraintes de Virasoro et récurrence topologique :** On peut se demander si on peut obtenir des liens entre les récurrences précédentes et la récurrence topologique qui est reliée aux structures d'Airy introduites par M.Kontsevich et I.Sobeilman. Comme on l'a mentionné plus tôt dans l'introduction, il n'est pas automatique que les équations de Cut-and-Join produisent des structures d'Airy. Il y a un écueil: l'équation de Cut-and-Join et les structure d'Airy sont deux équations de natures différentes. Pour obtenir ces relations, on utilise d'une façon différente le théorème 4.2 pour en déduire une relation légèrement différente pour les volumes (donnée par le lemme 5.14). Cette relation ne permet a priori de calculer ni les volumes  $V_{g,n^+,n^-}$  ni les opérateurs  $K$ . En revanche elle se traduit par une récurrence pour les coefficients de  $Z$  qui produisent une série de contraintes

**Théorème 1.16** (lemme 5.17 et proposition 5.21). *La série génératrice  $Z$  est solution des contraintes de Virasoro*

$$L_i(Z) = 0 \quad \forall i \geq -2.$$

Avec

$$L_i = -\partial_{i+2} + \sum_j (j+1)t_{j+1} \partial_{i+j+1} + \sum_{k+l=i} \partial_k \partial_l + \delta_{i,-2}$$

Les relations de commutation des opérateurs  $L_k$  sont bien connues des physiciens, les  $L_k$  sont des éléments d'une algèbre de Lie que l'on appelle l'algèbre de Virasoro (ou plutôt de Witt dans notre cas). On a les relations

$$[L_i, L_j] = (i-j)L_{i+j} \quad i, j \geq -1$$

Il est alors d'usage de considérer la transformée de Laplace des polynômes  $G_{g,n^+,n^-}$

$$W_{g,n^+,n^-}(x) = \int_L G_{g,n^+,n^-}(L) e^{-\sum_i x_i L_i} dL.$$

On peut alors sommer sur la variable  $n^-$ . On obtient ainsi la série

$$W_{g,n} = \sum_{n^-} W_{g,n,n^-}.$$

Les  $W_{g,n}$  sont des germes de fonctions analytiques au voisinage de  $\infty$ . En utilisant les résultats précédents, on peut obtenir la formule de récurrence suivante :

**Théorème 1.17** (théorème 5.3). *Les transformées de Laplace sont solutions de la récurrence suivante*

$$x_1 W_{g,n} = \sum_{i \neq 1} \frac{\partial}{\partial x_i} \left( \frac{W_{g,n-1}(x_1, x_{\{1,i\}^c}) - W_{g,n-1}(x_i, x_{\{1,i\}^c})}{x_1 - x_i} \right) + W_{g-1,n+1}(x_1, x_1, x_{\{1\}^c}) + \sum_{I_1, I_2, g_1+g_2=g} W_{g_1-1, n_1+1}(x_1, x_{I_1}) W_{g_2-1, n_2+1}(x_1, x_{I_2}) + \delta_{g,0} \delta_{n,1}.$$

Cette formule est très similaire aux formules obtenues par les équations de boucles dans la théorie des matrices aléatoires [Eyn05]. Cette récurrence permet de calculer les  $W_{g,n}$  en fonction de  $W_{0,1}$ . Ce dernier satisfait la contrainte

$$xW_{0,1} = W_{0,1}^2 + 1$$

avec la condition  $W_{0,1}(x) =_{\infty} \frac{1}{x} + o(\frac{1}{x})$ . Notons que la solution de cette équation n'est pas univaluée en tant que fonction de  $x$ . Il est alors d'usage dans la théorie des matrices aléatoires de considérer le tiré en arrière par la fonction

$$z \longrightarrow z + \frac{1}{z}.$$

On obtient ainsi un germe de fonction analytique qui se prolonge sur  $\mathbb{C}P_1$  en une fonction méromorphe. En reproduisant des arguments désormais classiques, on montre que la récurrence du théorème implique qu'il est possible de montrer que les fonctions  $W_{g,n}$  sont déterminées par la récurrence topologique de Eynard-Orantin. En suivant les lignes de leur raisonnement on obtient le corollaire suivant :

**Corollaire 1.3.** *La récurrence du théorème 5.3 implique la récurrence topologique de Eynard-Orantin pour la courbe spectrale*

$$xy = y^2 + 1$$

Cette courbe spectrale est bien connue des chercheurs issus de la géométrie énumérative car elle est la courbe spectrale associée aux nombres d'Hurwitz des dessins d'enfants [?]. Elle apparaît aussi dans la théorie des matrices aléatoire et est reliée à la série génératrice des moments du modèle aléatoire des matrices hermetiennes. Suite aux travaux de P.Norbury [Noro8] [Nor13], cette courbe spectrale est aussi associée aux comptage des points entiers dans l'espace des modules des graphes en ruban métrisés. On propose une démonstration combinatoire des liens entre  $W_{g,n}$  et les fonctions étudiés par P.Norbury dans la partie 5.7.1 qui n'était pas connue de l'auteur.

**Remarques sur  $V_{g,n^+,n^-,m}$  :** Comme on a vu précédemment on peut considérer les volumes  $V_{g,n^+,n^-,m}$  correspondant aux graphes ayant en plus  $m$  sommet bivalents. Dans ce cas on peut définir les opérateurs  $K_{g,n^+,n^-,m}$  et la série

$$K^\bullet(q_0, q_1) = \exp_{\square} \left( \sum_{g,n^+,n^-,m} q_0^m q_1^{2g-2+n^++n^-} K_{g,n^+,n^-,m} \right).$$

On considère aussi l'opérateur  $E$  qui agit sur  $S(V)$  par multiplication par la fonction  $E$ , en terme d'opérateur différentiel on a

$$E = \sum_i (i+1) t_{i+1} \partial_i.$$

On dispose alors du résultat suivant

**Théorème 1.18.** Les opérateurs  $P, E$  commutent et on a les équations de Cut-and-Join

$$\frac{dK^\bullet}{dq_0} = EK^\bullet \quad \frac{dK^\bullet}{dq_1} = PK^\bullet$$

donc

$$K^\bullet(q_0, q_1) = \exp(q_0E + q_1P)$$

On va voir dans la suite comment se généralise ce résultat.

**Sommet de degré supérieur à 4 :** On cherche ensuite à généraliser les résultats précédents en dehors du cas générique. Pour chaque  $(g, n^+, n^-, \nu)$  on peut définir des opérateurs  $K_{g, n^+, n^-}^\nu$  à partir des volumes  $V_{g, n^+, n^-}^\nu$ . On dispose toujours d'une formule de récurrence pour les volumes  $V_{g, n^+, n^-}^\nu$  et si on note

$$\bar{K}(q) = \exp_{\sqcup} \left( \sum_{g, n^+, n^-, \nu} q^\nu K_{g, n^+, n^-}^\nu \right)$$

Alors cette série converge pour tout  $q = (q_0, q_1, \dots)$  et on retrouve

$$\bar{K}(0, q_1, 0, \dots) = K(q_1) \quad \bar{K}(q_0, q_1, 0, \dots) = K^\bullet(q_0, q_1)$$

On peut alors généraliser les opérateurs  $P_+, P_-$  de la façon suivante: on note

$$\mathcal{W}_{g, n^+, n^-} = K_{g, n^+, n^-}^{(2g-2+n^++n^-)} \sqcup \exp_{\sqcup}(K_{0,1,1})$$

et on forme

$$\mathcal{W}_i = \sum_{2g-2+n^++n^-=i} \mathcal{W}_{g, n^+, n^-}$$

Par analogie l'opérateur  $\mathcal{W}_i$  doit être vu comme toutes les façons de recoller une surface de caractéristique d'Euler  $-i$  i.e toutes les façons de rajouter un sommet de degré  $2i + 2$  dans nos graphes. On sait que les  $\mathcal{W}_i$  sont des opérateurs différentiels et on retrouve les cas précédents

$$\mathcal{W}_0 = E \quad \mathcal{W}_1 = P.$$

Le théorème 5.1 se généralise donc de la façon suivante :

**Théorème 1.19** (Théorème 5.6). La série  $\bar{K}$  est solution du système suivant

$$\frac{\partial \bar{K}}{\partial q_i}(q) = \mathcal{W}_i \bar{K}(q)$$

Dans ce théorème chaque opérateur  $\mathcal{W}_i$  est homogène de degré  $i + 1$ . De plus, de par leur définition et par analogie avec la définition de  $P$  on voit que l'opérateur  $\mathcal{W}_i$  correspond à toutes les façons d'extraire une surface de caractéristique d'Euler donnée par  $-i$ . Ces remarques justifient le fait que les opérateurs  $\mathcal{W}_i$  forment une instance d'opérateurs de Cut-and-Join d'ordre supérieur.

Pour pouvoir utiliser le résultat du théorème précédent en pratique il est important de connaître les relations de commutations entre les opérateurs  $\mathcal{W}_i$  pour pouvoir calculer l'exponentielle. En utilisant le théorème 4.2 énoncé dans les paragraphes précédents on peut alors en déduire le théorème suivant

**Théorème 1.20** (théorème 5.8). *Les opérateurs  $\mathcal{W}_i$  commutent deux à deux, on a donc la formule*

$$\overline{K}(\mathbf{q}) = \exp\left(\sum_i q_i \mathcal{W}_i\right)$$

Cette relation est surprenante: en effet, le fait que deux opérateurs commutent signifie qu'ils sont indépendants. Ce théorème dit ni plus ni moins que deux sommets de degrés différents peuvent être extraits indépendamment.

Dans des cas simples on donne une formule explicite pour  $\mathcal{W}_i$ . On a déjà vu le cas particulier de  $i = 0, 1$ , on obtient aussi sans peine la formule suivante.

**Proposition 1.4** (Proposition 5.7). *L'opérateur  $\mathcal{W}_2$  est donné par :*

$$\begin{aligned} \mathcal{W}_2 = & \frac{1}{6} \sum_i (i+1)i(i-1)t_{i+1}\partial_{i-2} \\ & + \frac{1}{2} \sum_{i+j=k+l+1} (i+1)(j+1)t_{i+1}t_{j+1}\partial_k\partial_l \\ & + \frac{1}{3} \sum_{i,j,k} (i+1)(k+1)(l+1)t_{i+1}t_{j+1}t_{k+1}\partial_{i+j+k} \\ & + \frac{1}{3} \sum_{i,j,k} (i+j+k+3)t_{i+j+k+3}\partial_i\partial_j\partial_k. \end{aligned}$$

De manière générale, il est naturel de chercher une expression pour les opérateurs  $\mathcal{W}_i$ . Un fait surprenant a été découvert dans [WLZZ22] [MA23] et apparaît dans l'étude de certains modèles d'intégrales de matrices. Par analogie avec leur travaux on considère l'opérateur suivant

$$\mathcal{W}_{-1} = \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1}\partial_{k+l+1} + \sum_{k,l} (l+k+1)t_{k+l+1}\partial_k\partial_l.$$

On propose alors la conjecture suivante.

**Conjecture 1.1** (conjecture 5.1). *Les opérateurs  $\mathcal{W}_i$  sont générés à partir de  $\mathcal{W}_{-1}$  et  $\mathcal{W}_0$  par la récurrence suivante*

$$(i+2)\mathcal{W}_{i+1} = [\mathcal{W}_{-1}, \mathcal{W}_i]$$

Comme on l'a vu précédemment l'opérateur  $\mathcal{W}_0$  est très simple, donné par

$$\mathcal{W}_0 = \sum_i (i+1)t_{i+1}\partial_i.$$

On remarque que cette conjecture n'en n'est à priori pas une. Des bijections combinatoires devraient permettre de relier notre problème à celui de [WLZZ22]. Mais il serait intéressant d'interpréter cette formule comme une récurrence pour les volumes associés aux graphes en rubans minimaux. Une façon de réécrire cette formule est la suivante. On pose  $\mathcal{W}(q)$

$$\mathcal{W}(q) = \sum_i (i+1)q^i \mathcal{W}_i$$

On obtient alors que  $\mathcal{W}(q)$  est la solution de l'équation de Lax

$$\frac{\partial \mathcal{W}}{\partial q} = [\mathcal{W}_{-1}, \mathcal{W}(q)].$$

Avec la condition initiale  $\mathcal{W}(0) = \mathcal{W}_0$  et on a donc  $\mathcal{W}(q) = \exp(q\mathcal{W}_{-1})\mathcal{W}_0 \exp(-q\mathcal{W}_{-1})$ . On remarque qu'il est aussi possible de généraliser les contraintes de Virasoro mais nos résultats sont vérifiés que pour de petites valeurs de  $i$ . On peut trouver des opérateurs  $\tilde{L}_{k,i}$  tels que

$$\mathcal{W}_i = \sum_k (k+1)t_{k+1}\tilde{L}_{k,i}.$$

On peut alors former

$$L_i(\mathbf{q}) = \partial_{i+1} - \sum_k q_k \tilde{L}_{i,k}.$$

Soit  $\bar{Z}$  la fonction de partition

$$\bar{Z}(\mathbf{q}) = \bar{K}(\mathbf{q}) \cdot \mathbf{e}_\emptyset$$

on conjecture alors que pour tout  $q$  la fonction de partition est solution des contraintes

$$L_i(\mathbf{q})\bar{Z}(\mathbf{q}) = 0.$$

### 1.3.6 Discontinuités des volumes et applications :

Suite à plusieurs discussions avec Elise Goujard, il est apparu qu'en dehors du cas des strates principales, la formule utilisée pour prouver le théorème 2.5 ne permet pas de calculer les volumes de Masur-Veech. Il y a un surplus qui correspond à des graphes en rubans dégénérés. En réalité, les volumes  $V_{g,n^+,n^-}^\nu$  ne sont pas nécessairement continus, sur certaines sous-variétés ils sautent. Ces sauts ont lieu sur des sous-espaces  $\Lambda_W \subset \Lambda_{n^+,n^-}$  (ce sont les mêmes que les murs des polynômes par morceaux  $V_{g,n^+,n^-}$ ). Le but du chapitre 7.4.2 de ce mémoire est de montrer qu'ils admettent une extension continue  $\bar{V}_{g,n^+,n^-}^\nu$ . Cette extension est polynomiale par morceaux et en pratique plus facile à manipuler. On traite le cas des graphes orientés car le mémoire est plutôt orienté dans ce sens, mais des résultats similaires sont vrais pour les volumes  $V_{g,n}^\nu$  avec quelques modifications. Les sauts sont associés à des graphes en rubans orientés  $R^\circ$  qui dégénèrent pour certaines valeurs de  $L$  mais les volumes  $V_{R^\circ}(L')$  des cellules associées ne tendent pas vers 0 lorsque  $L'$  tend vers  $L$ ; il y a une perte de masse. Pour comprendre ce phénomène, on étudie la compactification  $\bar{\mathcal{M}}_{g,n^+,n^-}^{comb}$  de  $\mathcal{M}_{g,n^+,n^-}^{comb}$  via les graphes en rubans nodaux. On identifie les dégénérescences responsables de discontinuités dans les volumes que l'on nomme bords apparents de  $\bar{\mathcal{M}}_{g,n^+,n^-}^{comb}$ . Pour simplifier un peu certaines notations, notons  $\bar{M}^\circ = (M^\circ, \nu)$ . Si  $R^\circ$  est un bord apparent de  $\bar{\mathcal{M}}^{comb}(\bar{M}^\circ)$ , pour étudier la continuité il faut connaître combien de graphes en rubans dégénèrent vers un  $R^\circ$  et cela dans une direction donnée du fibré normal du sous-espace  $\Lambda_W$ . En d'autres termes l'application  $L_\partial$  est ramifiée au voisinage d'un bord apparent et nous devons en calculer le degré. Il faut d'abord montrer qu'il est bien défini; pour cela nous utilisons le cône normal d'un complexe cellulaire. On peut alors définir le degré  $\deg_{R^\circ}(\bar{M}^\circ)$  d'un bord apparent de  $\bar{\mathcal{M}}^{comb}(\bar{M}^\circ)$ . On considère alors le volume augmenté

$$\bar{V}_{\bar{M}^\circ}(L) = V_{g,n^+,n^-}^\nu + \sum_{R^\circ} \deg_{R^\circ}(\bar{M}^\circ) V_{R^\circ}(L)$$

où l'on somme sur toutes les dégénérescences apparentes, pondérées par le degré de ramification. Le but de la première partie du chapitre 7.4.2 est donc de prouver le résultat suivant.

**Proposition 1.5** (proposition 7.11). *La fonction  $\bar{V}_{\bar{M}^\circ}(L)$  est continue sur  $\Lambda_{n^+,n^-}$  et coïncide avec  $V_{\bar{M}^\circ}(L)$  sur  $\Lambda_{n^+,n^-}^*$ .*

Le sous-ensemble  $\Lambda_{n^+,n^-}^*$  correspond au complémentaire de l'ensemble des murs. L'espace  $\bar{\mathcal{M}}^{comb}(\bar{M}^\circ)$  admet une stratification naturelle donnée par des surfaces nodales décorées  $\bar{\mathcal{D}}^\circ = (\mathcal{D}^\circ, \nu)$ . On peut écrire

$$\bar{\mathcal{M}}^{comb}(\bar{M}^\circ) = \sqcup_{\bar{\mathcal{D}}^\circ} \mathcal{M}^{comb}(\bar{\mathcal{D}}^\circ).$$

Les dégénérescences apparentes correspondent à des surfaces nodales maximales dans l'espace des modules. Il est possible de décrire l'ensemble de ces surfaces par leur topologie. Si  $\bar{\mathcal{D}}^\circ$  est maximale et  $L$  est dans l'image de  $L_\partial : \bar{\mathcal{M}}^{comb}(\bar{\mathcal{D}}^\circ) \rightarrow \Lambda_{n^+,n^-}$  alors  $\bar{\mathcal{M}}^{comb}(\bar{\mathcal{D}}^\circ, L)$  et  $\bar{\mathcal{M}}^{comb}(\bar{M}^\circ, L)$  sont de même dimension, donc  $\bar{\mathcal{M}}^{comb}(\bar{\mathcal{D}}^\circ, L)$  n'est plus négligeable lorsque l'on calcule les volumes. On dénote  $V_{\bar{\mathcal{D}}^\circ}(L)$  le volume de  $\bar{\mathcal{M}}^{comb}(\bar{\mathcal{D}}^\circ, L)$ . La désingularisation  $\bar{\mathcal{D}}^\circ$  peut s'exprimer comme une union de surface décorées  $(\bar{\mathcal{D}}^\circ(c))_c$  avec des points marqués qui correspondent aux noeuds. On peut écrire

$$V_{\bar{\mathcal{D}}^\circ}(L) = \prod_c V_{\bar{\mathcal{D}}^\circ(c)}(L).$$

Le degré d'un graphe en rubans nodal dans  $\bar{\mathcal{M}}^{comb}(\bar{\mathcal{D}}^\circ)$  ne dépend que de  $\bar{\mathcal{D}}^\circ$  et  $\bar{M}^\circ$  et peut être dénoté  $\deg_{\bar{\mathcal{D}}^\circ}(\bar{M}^\circ)$ . On a alors la proposition suivante :

**Proposition 1.6.** *On a la relation*

$$\bar{V}_{\bar{M}^\circ}(L) = \sum_{\bar{\mathcal{D}}^\circ} \deg_{\bar{\mathcal{D}}^\circ}(\bar{M}^\circ) V_{\bar{\mathcal{D}}^\circ}(L)$$

où l'on somme sur les surfaces nodales maximales ( $V_{\bar{M}^\circ}$  fait bien sûr partie de la somme).

Cette formule permet de relier les volumes augmentés aux volumes usuels. Pour ce qui est des degrés, on conjecture une formule explicite qui a été vérifiée dans de nombreux cas.

**Formule pour les volumes de Masur Veech augmentés :** Les fonctions  $V_{\bar{M}^\circ}$  et  $\bar{V}_{\bar{M}^\circ}$  sont égales presque partout et il est plus commode d'utiliser les secondes car elles sont continues et polynomiales par morceaux. Lorsque l'on cherche à calculer les volumes de Masur-Veech on s'intéresse aux intégrales de la forme

$$\vartheta^\circ(\bar{\mathcal{G}}^\circ) = \frac{1}{\#\text{Aut}(\bar{\mathcal{G}}^\circ)} \int_{\Lambda_{\bar{\mathcal{G}}^\circ}} \prod_\gamma \frac{l_\gamma e^{-l_\gamma}}{1 - e^{-l_\gamma}} \prod_c V_{\bar{\mathcal{G}}^\circ(c)}(L(c)) d\sigma_{\bar{\mathcal{G}}^\circ}$$

où  $\bar{\mathcal{G}}^\circ$  est un graphe stable dirigé et décoré i.e chaque composante  $\mathcal{G}^\circ(c)$  de  $\bar{\mathcal{G}}^\circ$  est décorée d'une partition  $\nu_c$  et  $\Lambda_{\bar{\mathcal{G}}^\circ}$  est un polytope explicite construit à partir de  $\bar{\mathcal{G}}^\circ$ . On a alors le résultat suivant, mais dans ce cas on ne dispose pas de formule explicite pour les coefficients des polynômes par morceaux .

**Proposition 1.7.** *Les volumes de Masur-Veech sont donnés par*

$$\vartheta^\circ(\nu) = \sum_{\bar{\mathcal{G}}^\circ} \vartheta^\circ(\bar{\mathcal{G}}^\circ).$$

Comme les fonctions  $\bar{V}_{\bar{M}^\circ}$  coïncident avec  $V_{\bar{M}^\circ}$  presque partout, on peut se demander si on peut les substituer dans les formules précédentes. La réponse est non. En effet les contraintes imposées par les graphes dirigés font que généralement l'image de

$$\Lambda_{\mathcal{G}^\circ} \longrightarrow \Lambda_{\mathcal{G}^\circ(c)}$$

est contenue dans un sous-espace de la forme  $\Lambda_{W_{\mathcal{G}^\circ(c)}}$  et est de mesure non nulle dans ce sous-espace. En conséquence lorsque l'on calcule les intégrales on doit restreindre les fonctions  $\bar{V}_{\bar{\mathcal{G}}^\circ(c)}$  sur des murs sur lesquelles elles diffèrent des fonctions  $V_{\bar{\mathcal{G}}^\circ(c)}$ . Si on dénote par analogie  $\bar{\vartheta}_{\bar{\mathcal{G}}^\circ}^\circ$  et

$$\bar{\vartheta}^\circ(\nu) = \sum_{\bar{\mathcal{G}}^\circ} \bar{\vartheta}^\circ(\bar{\mathcal{G}}^\circ).$$

Alors en dehors du cas de la strate principale les quantités  $\bar{\vartheta}^\circ(\nu)$  et  $\vartheta^\circ(\nu)$  ne sont pas égales.  $\bar{\vartheta}^\circ(\nu)$  contient aussi des termes correspondants à des surfaces nodales. On considère  $\bar{\mathcal{T}}$  un certain sous ensemble (explicite) de surfaces nodales (dont le graphe stable est un arbre). Le volume  $\vartheta^\circ(\bar{\mathcal{T}})$  est donné en fonction des volumes de Masur-Veech par la formule

$$\vartheta^\circ(\bar{\mathcal{T}}) = \prod_c \vartheta^\circ(\bar{\mathcal{T}}(c))$$

où on réalise le produit sur les composantes connexes de la désingularisation. On obtient alors le théorème suivant qui a été conjecturé par Elise Goujard et Adrien Sauvaget dans le cas des différentielles quadratiques.

**Théorème 1.21.** *On a la relation*

$$\bar{\vartheta}^\circ(\nu) = \sum_{\bar{\mathcal{T}}} \deg_{\bar{\mathcal{T}}}(\nu) \vartheta^\circ(\bar{\mathcal{T}}).$$

On remarque qu'il est possible d'inverser la formule et d'obtenir les volumes de Masur-Veech en fonction des volumes  $\bar{\vartheta}^\circ(\nu)$  par un principe d'inclusion-exclusion. On remarque aussi que les surfaces nodales décorées que l'on utilise sont similaires à certains bords dans la compactification des espaces des modules des différentielles abéliennes [BCG<sup>+</sup>22].

**Remark 1.1.** *Les résultats de ce chapitre se généralisent aux graphes plus généraux notamment ceux avec des sommets d'ordre impair, avec certaines modifications. Dans un graphe non orientable il est possible que certaines composantes deviennent orientables et cela crée un nouveau type de dégénérescences apparentes.*

## 1.4 Organisation du mémoire

Ce mémoire se compose de 7 chapitres. Le chapitre 3 contient des définitions et notations utiles pour les chapitres 4, 5, 6, 7. Le chapitre ' peut se lire indépendamment des chapitres suivants. Le chapitre 5 utilise les deux précédents et le chapitre 6 utilise les résultats des chapitre 3, 4, 5. Chaque chapitre correspond plus ou moins à un travail différent (hormis le 3).

**Chapitre 3** Dans ce premier chapitre on rappelle de nombreuses notions de topologie des surfaces et on introduit diverses notations concernant les surfaces. On définit en particulier les surfaces dirigées et décorées qui seront utilisées tout au long du mémoire pour simplifier certaines notations. On rappelle des résultats concernant les courbes sur les surfaces, les arcs, les feuilletages mesurables et les différentielles quadatriques et abéliennes. Ces résultats seront utiles afin d'effectuer les découpages sur les graphes. On introduit la notion de courbes orientées et de graphes stables dirigés et on étudie ces notions; on insiste particulièrement sur les graphes acycliques qui seront utilisés tout au long du mémoire et on démontre des résultats préliminaires qui serviront à établir la polynomialité par morceaux.

**Chapitre 4** Dans ce chapitre on étudie les graphes en rubans et leur géométrie. On démontre en particulier le théorème 4.2 et la récurrence 4.6. On rappelle de nombreuses propriétés relatives aux graphes en rubans, on définit les graphes en rubans dirigés et on étudie leurs propriétés. On définit ensuite les courbes admissibles, les opérations de chirurgie et le flot de twist. En étudiant les dégénérescences de la forme symplectique, on démontre le théorème 4.2.

**Chapitre 5** Le chapitre 5 est consacré aux opérateurs de Cut-and-Join. Après avoir rappelé les notions d'algèbre symétrique utiles pour la suite, on définit les opérateurs  $K$  associés au volumes. On démontre notamment le théorème 5.1 et les liens avec la récurrence topologique. On donne aussi des interprétations combinatoires des séries génératrices obtenues.

**Chapitre 6** Dans le chapitre 6 on établit la formule de Mirzakhani-McShane pour les graphes en rubans trivalents. Cette formule est aussi donnée dans [ABC<sup>+</sup>20]. On donne aussi une preuve de la formule de Cut-and-Join qui utilise une autre version de la formule de Mirzakhani-McShane.

**Chapitre 7** Dans ce chapitre on étudie la continuité des volumes  $V_{g,n^+,n^-}^\nu$  en général et on démontre les résultats précédemment énoncés.

# Chapter 2

## Introduction (In english )

The goal of this thesis was to study the possible interactions between geometric and topological recursions, introduced in [Miro7] and [ABO17],[EO07]. With the Schramm-Lowner evolution in one hand [?], which is a family of stochastic processes that produce random fractal curves on Riemann surfaces. These processes are universal and appear as the scaling limits of several systems related to statistical physics and combinatorics. On the other hand, with Masur-Veech volumes of quadratic and Abelian differentials, these volumes contain numerous informations on the geometry of flat surfaces. In this introduction, we briefly recall the actors before explaining the content of the present thesis. We mainly focus on Masur-Veech volumes.

### 2.1 Brief introduction and state of the art

#### 2.1.1 Topological recursion

Topological recursion has appeared in the last few decades in physics and enumerative geometry. The principle is the following: We start with the data of a sequence  $(V_{g,n})$  that depends on two parameters, the genus  $g$  and the number of boundaries  $n$ . We usually assume the stability condition of the Euler characteristic:

$$2g - 2 + n > 0.$$

The objects  $V_{g,n}$  often have a geometric meaning; they count structures on a surface of genus  $g$  and  $n$  boundaries. They are usually polynomials or generating series in  $n$  variables. The goal is to compute these quantities recursively. The Euler characteristic is a way to measure the topological complexity of a surface, and then it's natural to try to express  $V_{g,n}$  in terms of the  $V_{g',n'}$  with

$$2g - 2 + n > 2g' - 2 + n'$$

Such recursion can be called a topological recursion because we reduce the topology of the surface. A well-known example was given by M. Mirzakhani for the Weil Pertersson volumes of the moduli space of hyperbolic surfaces. To illustrate the method, we briefly give her results. Let  $\mathcal{M}_{g,n}^{hyp}$  the moduli space of hyperbolic surfaces of genus  $g$  with  $n$  geodesic boundaries [Miro7]. We have an application that measures the length of the boundaries for the hyperbolic metric.

$$L_{\partial} : \mathcal{M}_{g,n}^{hyp} \longrightarrow \mathbb{R}_{\geq 0}^n$$

The level sets  $\mathcal{M}_{g,n}^{hyp}(L) = L_{\partial}^{-1}(\{L\})$  are diffeomorphic to  $\mathcal{M}_{g,n}$ , the usual moduli space. Moreover, each level set has a symplectic structure given by the Weil Perterson form  $\omega_{WP}^{3g-3+n}$ , and the volumes are equal to

$$V_{g,n}^{hyp}(L) = \int_{\mathcal{M}_{g,n}^{hyp}(L)} \frac{\omega_{WP}^{3g-3+n}}{(3g-3+n)!}.$$

Maryam Mirzakhani discovers that it's possible to compute these volumes recursively in [Miro7], and to do that, she invents the Mirzakhani-McShane formula by using the earlier works of G.McShane [McS98]. Notations about indices are given in section 3.1.

**Theorem 2.1.** *The volumes  $V_{g,n}^{hyp}$  are given by the recursion:*

$$\begin{aligned} L_1 V_{g,n+1}^{hyp}(L) &= \sum_{i \neq 1} \int_x \mathcal{D}(L_1, L_i, x) V_{g,n}^{hyp}(x, L_{\{1,j\}^c}) dx \\ &+ \frac{1}{2} \int_{x_1, x_2} \mathcal{R}(L_1, x_1, x_2) V_{g-1, n+2}^{hyp}(x_1, x_2, L_{\{1,j\}^c}) dx \\ &+ \frac{1}{2} \sum_{g_i, n_i, I_i} \int_{x_1, x_2} \mathcal{R}(L_1, x_1, x_2) V_{g_1, n_1+1}^{hyp}(x_1, L_{I_1}) V_{g_2, n_2+1}^{hyp}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2 \end{aligned}$$

The sum is over all the 3-upplets  $(g_i, n_i, I_i)$  such as

$$g_1 + g_2 = g \quad I_1 \sqcup I_2 = \{2, \dots, n+1\} \quad n_i = \#I_i.$$

The recursion can be initialized by

$$V_{0,3}^{hyp} = 1 \quad V_{1,1}^{hyp}(L) = \frac{L^2}{24} + \frac{\pi^2}{6}.$$

The functions  $\mathcal{D}$  and  $\mathcal{R}$  are explicit and come from hyperbolic geometry. The RHS contains only surfaces with an Euler characteristic strictly bigger than the LHS (the difference is 1). Each term in this formula corresponds to a different surgery on the surface, and all these surgeries are pictured in figure 2.1. The recurrence formula is a sum over all the ways to extract an embedded pant's (sphere with three boundaries removed), which contains the first boundary. We can say that this recursion is of order 1 because we reduce the opposite of the Euler characteristic by 1 at each step. Similar formulas appear in numerous fields and in different shapes. A possible formulation is given by B.Eynard and N.Orantin, who were studying large random matrices [EO07]. This form found numerous applications and was later called "The Topological Recursion". This is an apparently more complicated way to formulate the recursion, but it appears that this form is universal and possesses many enjoyable properties. We recall briefly what it is about. We start with a Riemann surface  $X$ , and our goal is to compute poly-differentials  $\omega_{g,n}$  defined on  $X^n$ . They can be written locally as

$$\omega_{g,n} = \omega_{g,n}(z_1, \dots, z_n) dz_1 \otimes \dots \otimes dz_n$$

We have a meromorphic map  $x$  on  $X$  with simple ramifications on a finite set  $\text{Br}(x) \subset X$ , and we also have a kernel:

$$K = K_{\alpha}(z_1, z_2) \frac{dz_1}{dz_2}$$

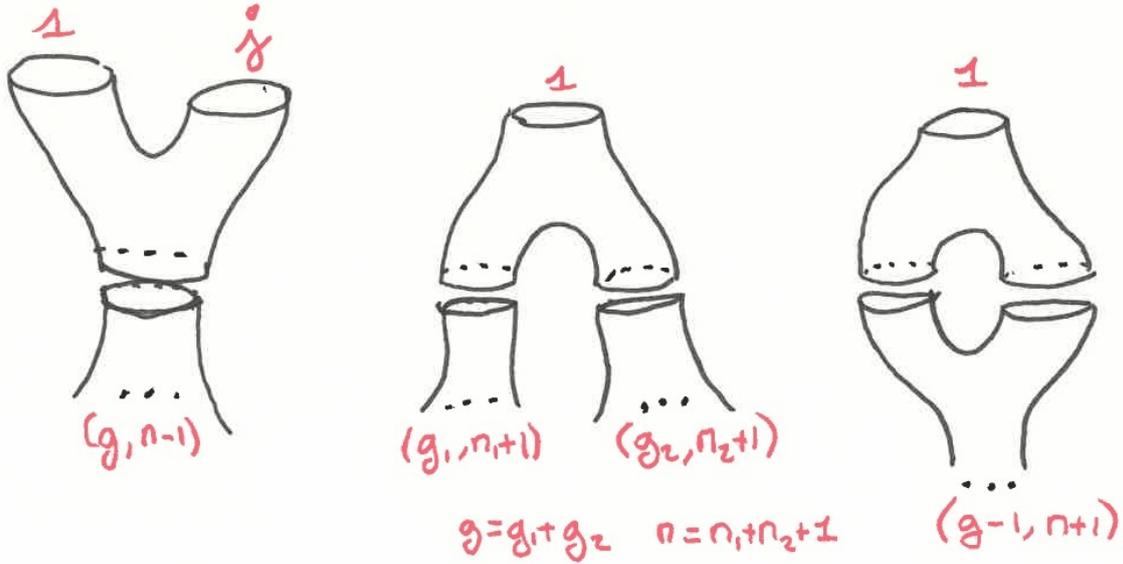


Figure 2.1: Different gluings appearing in the topological recursion.

for all  $z_\alpha \in \text{Br}(x)$ . At the neighborhood of a point  $\zeta_\alpha \in \text{Br}(x)$ , we have a local Galois involution  $\sigma_\alpha$  that preserves  $x$ . The topological recursion expresses  $\omega_{g,n}$  by the recursive formula:

$$\begin{aligned} \omega_{g,n} = & \sum_{\alpha} \text{Res}_{\alpha} K_{\alpha}(z_1, z) (\omega_{g-1, n+1}(z, \sigma_{\alpha}(z), z_{\{1\}^c}) \\ & + \sum_{g_i, n_i} \omega_{g_1, n_1+1}(z, z_{I_1}) \otimes \omega_{g_2+1, n_2+1}(\sigma_{\alpha}(z), z_{I_2})). \end{aligned}$$

The kernels  $K_{\alpha}$  are constructed from  $\omega_{0,1}, \omega_{0,2}$  and  $\sigma$ . Usually  $\omega_{0,2}$  is universal and equal to a suitable choice of Bergmann Kernel of  $X$  [EO07]. The form  $\omega_{0,1}$  can be written as  $\omega_{0,1} = y dx$  where  $y$  is a meromorphic function, holomorphic at the branch points. In many cases, the pair  $(x, y)$  is solution of an algebraic equation

$$P(x, y) = 0,$$

and the curve defined by this equation is called the spectral curve<sup>1</sup>. Topological recursion has many applications; a famous example is the computation of intersection pairings between tau-topological classes on  $\mathcal{M}_{g,n}$ . The moduli space  $\mathcal{M}_{g,n}$  parametrizes Riemann surfaces  $(C, z_1, \dots, z_n)$  of genus  $g$  and with  $n$  marked points. Let  $\mathcal{L}_i$  be the line bundle over  $\mathcal{M}_{g,n}$  with stalk over  $(C, z)$  is given by the cotangent space  $T_{z_i}^* C$  (see chapter 6.3).  $\mathcal{L}_i$  extends to the compactification  $\overline{\mathcal{M}}_{g,n}$ , and the first Chern class of  $\mathcal{L}_i$  defines a cohomology class  $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ . In [Eyn14b] the authors consider

$$\omega_{g,n} = \sum_{\alpha} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{z_i^{2d_i+1}} \otimes_{i=1}^n dz_i$$

The next result is equivalent to the Witten conjecture [Wit90]; it has been proved in many different ways, but the first proof was given in [Kon92].

<sup>1</sup>The spectral curve is ended  $(X, x, y, \omega_{0,2})$

**Theorem 2.2.** *The differentials  $\omega_{g,n}$  can be computed by the topological recursion using the spectral curve  $x = z^2, y = z, \omega_{0,2} = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}$  at the neighborhood of  $z = 0$*

This result can be generalized within the framework of cohomological field theories. It has been developed by M. Kontsevich and Y. Manin [KM94] to study Gromov-Witten invariants. The input is a Frobenius algebra, and the output is a family of cohomological classes  $\Omega_{g,n}(\alpha_1, \dots, \alpha_n)$  on the moduli space  $\overline{\mathcal{M}}_{g,n}$ . Thanks to the Givental-Teleman correspondence [Giv01],[Tel11] we know that the intersection pairings of  $\Omega_{g,n}$  and  $\psi$ -classes are computable by using topological recursion, and these cases correspond exactly to spectral curves with simple ramifications. In general, the topological recursion of Eynard-Orantin finds numerous applications:

- Large random matrices [Eyn05].
- Masur-Veech volumes for the principal stratum of quadratic differentials [DGZZ21].
- Enumerative geometry: maps [Eyn05], Hurwitz numbers [BMo8] [EMS11], integral points in moduli spaces [Nor13].
- Algebraic topology of moduli spaces, cohomological field theory, Givental-Teleman correspondence [Eyn14a],[DBOSS14].
- Non-commutative geometry
- WKB approximation [EGFEN21].
- And so much more.

Topological recursion is not the only possible formalism. Recently, M. Kontsevich and I. Sobeilman introduced in [KS17] an alternative formulation that is more general; they call it Airy structures. It takes place in  $\mathbb{C}[[\hbar]][[(x_i)_i]]$  where  $(x_i)$  is a possibly infinite family of variables. We are looking for a formal series.

$$\phi(\hbar) = \exp\left(\sum_{g,n} \hbar^{2g-2} S_{g,n}\right).$$

Where we sum over all the  $(g, n)$  with  $2g - 2 + n > 0$ , the elements  $S_{g,n}$  are in  $\mathbb{C}[[\hbar]][[(x_i)_i]]$  and are related to the differentials  $\omega_{g,n}$ . The constraints on  $\phi$  take the following form:

$$H_i \phi = 0.$$

Where the  $H_i$  are differentials operators of the form

$$H_i = -\hbar \partial_i + \sum_{j,k} a_{i,j,k} x_j x_k + \hbar \sum_{j,k} b_{i,j}^k x_j \partial_k + \hbar^2 \sum_{j,k} c_k^{i,j} \partial_i \partial_j + \hbar \epsilon_i.$$

Moreover, these operators also satisfy constraints given in [KS17] that ensure the uniqueness of the solution. The relation between Airy structures and TR is also given in [Eyn19].

A third approach is given by cut and join operators; they are also constructed from a tensorial model and take the form

$$P = \sum_{i,j,k} a_{i,j}^k t_i t_j \partial_k + \sum_{i,j,k} b_i^{j,k} t_i \partial_j \partial_k + \sum_{i,j,k} c_{i,j,k} t_i t_j t_k$$

$P$  acts on a space  $\mathbb{C}[[t_0, t_1, \dots]]$  of formal series, in this case we are interested in some formal serie  $Z(q) \in \mathbb{C}[[t_0, t_1, \dots]]$  that satisfies the linear evolution

$$\frac{\partial Z}{\partial q} = PZ$$

The initial condition is given by the "physics" of the problem.

These different approaches interact but they are not equivalent. In [KS17] it's proved that

TR simple ramifications  $\longrightarrow$  Airy structures

After the works of A.Alexandrov [Ale22] using the Givental Teleman correspondence, we know

TR simple ramifications  $\longrightarrow$  Cut and Join

But we don't know anything about the converse arrows.

During the topological recursion, we reduce the opposite of the Euler characteristic by 1 at each step, but there exist generalizations in which the Euler characteristic decreases faster. In the formulation of Eynard-Orantin, it corresponds to the case when  $x$  possesses ramification points of higher degree. A possible formulation was given in [?] by V. Bouchard and B. Eynard. A possible generalization of Airy structures was given in [BBC<sup>+</sup>21], and in [BKS23], these generalizations use, for instance, representations of  $W$ -algebra. In general, these recursions are not well understood; we see later instances of higher-order recursions, but that can be factorized in some sense to obtain the usual topological recursion of order 1.

### 2.1.2 Flat surfaces and Masur-Veech volumes

A structure of translation surface  $X$  on a topological surface is an atlas of charts with values in  $\mathbb{C}$  and such that the transition functions are given by translations. In an informal way, translation surfaces are obtained by gluing sheets of paper together, for instance, by gluing a polygon along pairs of parallel sides (see figure 2.2). Translation surfaces are naturally related to Abelian differentials on Riemann surfaces. Translations allow to pull back the one form  $dz$  on each chart, which is then well defined on  $X$ . More generally, we can also consider half-translation surfaces, which are associated with quadratic differentials. In this case, the transition functions take the form:

$$z \longrightarrow \pm z + c.$$

Translation surfaces and half translation surfaces are flat surfaces; the metric  $|dz|$  is well defined. In virtue of the Gauss-Bonnet theorem, a flat metric cannot exist at every point of a surface of genus  $g \geq 2$ . Flat surfaces must have singularities that are related to zeros of the corresponding Abelian or quadratic differential. These singularities are called conical singularities; the angle measured around a conical singularity is not equal to  $2\pi$ . For an Abelian differential, a zero of order  $k$  corresponds to an angle of  $2\pi(k + 1)$ , and for a quadratic differential, it is  $(k + 2)\pi$ .

The study of flat surfaces has been initiated by Veech, H. Masur, A. Zorich, and A. Eskin. An important question is the behavior of the geodesic flow for the flat metric; it has application to the study of billiards, for instance. Geodesics are locally straight lines in the local charts,

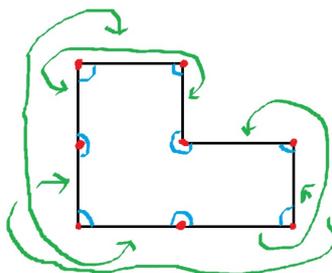


Figure 2.2: A translation surface of genus 2 with one singularity of angle  $6\pi$ .

but the global behavior of trajectories is not trivial at all. The geodesic flow in a fixed direction is often chaotic. For all translation surfaces, or half translation surfaces, and almost all directions, the flow is uniquely ergodic. The geodesic flow is related to the theory of interval exchanges studied by P. Rauzy, H. Masur, J.C. Yoccoz, and A. Avila [Mas82]. In a sequence of articles [Zor96],[Zor97] A. Zorich studied the behavior of large geodesics, and using the ergodic theory, he emphasized the existence of an asymptotic cycle on which the large trajectories “are close”. He also studied the deviations from this cycle by using Ossendet’s theorem applied to the Kontsevich-Zorich cocycle for the Teichmüller flow. Another important question was the study of periodic orbits, and more generally, saddle connections on flat surfaces. The count of saddle connections of length smaller than  $L$  is a subject widely studied in the last decades. An important theorem is the following:

**Theorem 2.3.** *For almost all translation surfaces (or half translation surfaces)  $S$ , let  $N_{sd}(S, L)$  be the number of saddle connections of length  $\leq L$ . There is a constant  $C_{sd}(S)$  such as*

$$\lim_{L \rightarrow \infty} \frac{N_{sd}(S, L)}{L^2} = C_{sd}(S).$$

The constant  $C_{sd}(S)$  in this theorem is the Siegel-Veech constant. And this constant is, constant.  $C(S)$  does not really depend on  $S$  but only on its topology and the structure of the singularities. To obtain results that are true for a family of surfaces, it’s very efficient to consider not only a given surface but the full set of equivalence classes of surfaces modulo homeomorphisms. These spaces are called moduli spaces; generally, moduli spaces are orbifolds, i.e., manifolds with symmetries. In many examples, moduli spaces prove that they contain a huge amount of information about the structures they parameterize. Studying the topology or the dynamics of the moduli space has allowed to prove several important theorems during the last decades. By considering objects in families, we can study their deformations, and it appears that many quantities are constant or almost surely constant, such as the Siegel-Veech constant, for instance. We see that translation surfaces have conical singularities. For a partition  $\nu = (0^{\nu(0)}, 1^{\nu(1)}, \dots)$  it’s natural to consider the moduli space  $\mathcal{H}(\nu)$  (resp  $\mathcal{Q}(\nu)$ ) that parameterizes the translation surfaces (resp half translation surfaces) with a profile of singularities given by  $\nu$ , i.e.,  $\mathcal{H}(\nu)$  parameterizes the translation surfaces that are connected and with  $\nu(i)$  singularities of angle  $2\pi(i + 1)$  (respectively  $\pi(2 + i)$  for  $\mathcal{Q}(\nu)$ ). The partition  $\nu$  also determines the topology of the surface (by the Gauss-Bonnet theorem, for instance).

## 2.2 Position of the problem

### 2.2.1 Masur-Veech volumes and ribbon graphs, case of the principal stratum

**Principal stratum:** The computation of the Masur-Veech volumes of moduli spaces of flat surfaces is also a subject of first importance. A natural way to compute the volume of an open subset  $U$  of  $\mathbb{R}^m$  is to count the number of points in  $U \cap \mathbb{Z}^m$ . It gives a good approximation of the volume, and to have the exact value, we have to study the asymptotic behavior of  $\#U \cap \frac{1}{N}\mathbb{Z}^m$  when  $N$  goes to infinity. This method is in some sense similar to computing an integral by using Riemann sums; in our case, the function to integrate is constant.

We remark that a moduli space usually admits a structure similar to the one on the objects it contains. Typically, moduli spaces of hyperbolic surfaces admit a natural metric; moduli spaces of Riemann surfaces have a natural structure of complex manifolds... In the case of translation of half-translation surfaces, the moduli spaces are almost flat. They admit a natural linear structure with the structural group  $GL_m(\mathbb{Z})$ . We have local charts with values in  $\mathbb{C}^m$ , and the changes of charts are linear transformations that preserve the lattice  $(\mathbb{Z} \oplus i\mathbb{Z})^m$ . The Masur-Veech volumes are calculated by using the Lebesgue measure on  $\mathbb{C}^m$ . We can estimate a Masur-Veech volume by counting the integral points in the corresponding moduli space. Details of this method are given in 7.4.2 in the case of Abelian differentials. The moduli spaces are not ordinary spaces, and the geometric interpretation of integral points is also extraordinary. The integral points correspond to "square tiled surfaces", In the case of the Abelian differential, they are coverings over the torus  $\mathbb{C}/\mathbb{Z}[i]$  ramified only above 0. A square-tilted surface can be constructed by gluing little squares  $[0, 1]^2$  along their boundaries. There are several ways to count square-tilted surfaces. One of them uses the count of coverings using the Frobenius formulas, which express the number of square tiled surfaces by using characters of the symmetric group. This method has been developed by A.Eskin and A.Okounkov [EO01],[AO06] and A. Aggarwal [Agg20]. A second way has been developed by A.Zorich and his collaborators in [Zor02],[DGZZ21] and also more recently by E.Goujard and E.Duriev. We follow this path in this work. The geodesic flow of a square tiled surface is totally periodic in all the rational directions. If we fix one of them, for instance, the horizontal direction, the periodic trajectories in this direction define maximal cylinders that fill the surface. These cylinders are glued along their boundaries, and all the information is contained in the way to glue them (see figure 2.3 ). These gluings draw a graph on the surface formed by configurations of horizontal saddle connections. These graphs are usually called ribbon graphs, and they appear in several places in modern literature. They interact with numerous structures in combinatorics and in low-dimensional topology. The general principle is the following: we can recover the square-tiled surfaces by gluing ribbon graphs along their boundaries. In the case of Abelian differentials, the ribbon graphs have an additional property: they are oriented. This property means that we can orient the edges in a coherent way. The study of oriented ribbon graphs is the subject of a large part of this thesis. But before studying oriented ribbon graphs, we consider the case of generic graphs.

**Moduli space of ribbon graphs and Witten conjecture:** Let  $M$  be a surface of genus  $g$  with  $n$  boundaries. A ribbon graph  $R$  on  $M$  is a graph drawn on  $M$  and on which  $M$  retracts (see figure ??). Ribbon graphs can be defined in a purely combinatorial way. A metric on the graph

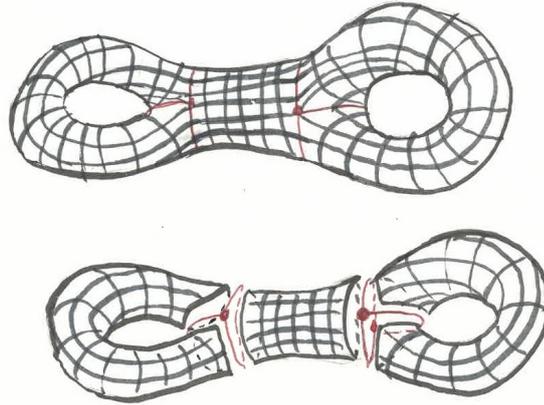


Figure 2.3: A cylinder decomposition of a square tiled surface, ribbon graphs are in red.

is then the assignment of a positive real number for each edge of the graph; the metric is the length of this edge. As before, it's possible to define the moduli space  $\mathcal{M}_{g,n}^{comb}$  of metric ribbon graphs on  $M$ . To emphasize the general principle on moduli spaces, a ribbon graph defines an embedded 1-cell complex in  $M$ , and  $\mathcal{M}_{g,n}^{comb}$  also admits a natural structure of cell complex. By using the works of K.Strebel [Str84], we know that  $\mathcal{M}_{g,n}^{comb}$  can be identified with  $\mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n$ , and it gives a cellular decomposition of this second space. As in the case of hyperbolic surfaces, the metrics on the edges of the graphs allow us to measure the length of the boundaries. We have a function

$$L_{\partial} : \mathcal{M}_{g,n}^{comb} \longrightarrow \mathbb{R}_{>0}^n.$$

In this approach, in order to compute the Masur-Veech volumes, it's necessary to compute the volumes  $V_{g,n}^{comb}(L)$  of the level sets  $\mathcal{M}_{g,n}^{comb}(L) = L_{\partial}^{-1}(L)$ . These volumes have been computed for the first time by M. Kontsevich when he proves the Witten conjecture in [Kon92]. The volumes are measured using a natural symplectic form  $\omega_{g,n}^{comb}(L)$  introduced by M. Kontsevich. He proves the following result in [Kon92], but in a different form.

**Theorem 2.4** (Kontsevich [Kon92]). *The volumes are expressed by the following formula:*

$$V_{g,n}^{comb}(L) = \sum_{\alpha} \langle \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \rangle \prod_i \frac{L_i^{2\alpha_i}}{\alpha_i!}$$

These polynomials are generally called the Kontsevich polynomials and are directly related to the differentials given earlier.

**Computation of Masur-Veech volumes:** By using the content of the last two paragraphs, it was possible for V.Delecroix, E.Goujard, P.Zograph, and A.Zorich [DGZZ21]. To give an explicit formula for the Masur-Veech in the case of the principal stratum's of the moduli spaced of quadratic differentials. This formula was given by M. Mirzakhani in [Miro8a], but was not fully understood by the community.

**Theorem 2.5.** *The Masur-Veech volume  $\vartheta_{g,n}$  is given by*

$$\vartheta_{g,n} = \sum_{\mathcal{G}} \prod_{\gamma} \zeta(2\alpha_{\gamma} + 2) \int_{\overline{\mathcal{M}}_{g,n}} \Delta_{\mathcal{G}} \prod_{\gamma} \psi_{\gamma}^{\alpha_{\gamma}}.$$

where we sum over all the boundary components of  $\overline{\mathcal{M}}_{g,n}$ .

The sum is over all the stable graphs  $\mathcal{G}$ ,  $\Delta_{\mathcal{G}}$  corresponds to the subspace  $\mathcal{M}_{g,n}(\mathcal{G})$  in the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{g,n}$ , which is the closure of the set of stable nodal curves with a stable graph given by  $\mathcal{G}$  (see chapter 6.3). All the elements in this formula can be computed recursively. But the recursion for  $V_{g,n}^{comb}(L)$  satisfies a strong property; in some sense, it commutes with the formula for the Masur-Veech volumes. Then the whole term of the formula also satisfies a topological recursion formula. This approach has been developed for the first time in [ABC<sup>+</sup>23] and uses some multiplicative statistics for the length of multi-geodesics on hyperbolic Riemann surfaces.

**Theorem 2.6** ([ABC<sup>+</sup>23]). *There is a polynomial  $V_{g,n}^{MV}$  such as  $V_{g,n}^{MV}(0) = \frac{2^{4g-4+n}(4g-4+n)!}{(6g-7+2n)!} \vartheta_{g,n}$ ; moreover,  $V_{g,n}^{MV}$  can be computed using the topological recursion.*

We don't give the spectral curve because it's not very eloquent; the coefficient in front of  $\vartheta_{g,n}$  depends on the choice of the normalization of the Masur-Veech volumes. Then we can ask the following first question:

**Question 2.1.** *Is it possible to generalize these results to the cases of other strata ?*

As often, the answer is "Yes, but...". The precedent approach uses, in a crucial way, the volumes of moduli spaces of metric ribbon graphs. In the case of the principal stratum, the volumes  $V_{g,n}^{comb}(L)$  correspond to trivalent ribbon graphs. The space  $\mathcal{M}_{g,n}^{comb}$  admits a natural stratification  $\mathcal{M}_{g,n}^{comb}(\nu)$  indexed by partitions  $\nu$ , and  $\mathcal{M}_{g,n}^{comb}(\nu)$  corresponds to the closure of the set of ribbon graphs with  $\nu(i)$  vertices of degree  $i + 2$ . In this case, it's always possible to define the volumes  $V_{g,n}^{\nu}(L)$  of  $\mathcal{M}_{g,n}^{comb}(\nu, L)$ , but to apply the last strategy, we need to answer the following:

**Question 2.2.** *Is it possible to compute the volumes of moduli spaces of more general ribbon graphs ?*

As we see before, in the case of Abelian differentials, the ribbon graphs are oriented, and we will see later what we mean by more general.

## 2.3 Results of the thesis

### 2.3.1 Volumes of moduli spaces of metric ribbon graphs

In the last part, we see that in the cases of the principal stratum's of quadratic differentials, the Masur-Veech volumes are expressed in terms of the volumes  $V_{g,n}^{comb}$  of moduli space of metric ribbon graphs. We can wonder if it's possible to generalize these results. As we see before, the moduli space  $\mathcal{M}_{g,n}^{comb}$  admits a natural stratification  $\mathcal{M}_{g,n}^{comb}(\nu)$  indexed by partitions  $\nu$ .  $\mathcal{M}_{g,n}^{comb}(\nu)$  corresponds to the closure of the subset of ribbon graphs with  $\nu(i)$  vertices of degree  $i + 2$ . In this case, it's still possible to define the volumes  $V_{g,n}^{\nu}(L)$  of  $\mathcal{M}_{g,n}^{comb}(\nu, L)$ , but before trying to apply the precedent strategy, we had to give an answer to the following question:

**Question 2.3.** *Is it possible to compute the volumes  $V_{g,n}^\nu(L)$  in general?*

Outside the case of trivalent ribbon graphs, nothing was known about these volumes in general. The naive way to compute them is to enumerate all the possible ribbon graphs and sum the contribution of each graph. This is extremely fastidious, and this approach can only give sporadic results. In order to obtain more general results, it's possible to try to find a recursion for these functions. In the case of Abelian differential, the situation is different. As we said before, the graphs are oriented in this case (we denote them  $R^\circ$ ), these graphs appear in several domains of mathematics [DM18],[KZo3] and more recently in [Yak22]. This case is radically different from the case of trivalent graphs. The orientation of the edges induces an orientation of the boundary components. Some of them are oriented according to the orientation given by the surface and then labeled by  $+$ , while others are oriented in the opposite direction and are labeled by  $-$ . We call such a surface with labels  $\pm$  on the boundaries a directed surface (and denoted  $M^\circ$ ). In an oriented ribbon graph, each edge is "parallel" to a positive boundary and a negative boundary. This induces a bipartite structure on the dual (see section 4.1.1 for details). This condition implies that the sum of the lengths of the positive boundaries is equal to the sum of the lengths of the negative boundaries. Let  $\mathcal{M}_{g,n^+,n^-}^{comb}$  be the moduli space of oriented ribbon graphs of genus  $g$  and  $n^+$  positive (resp  $n^-$  negative) boundaries. There are two maps.

$$L^\pm : \mathcal{M}_{g,n^+,n^-}^{comb} \longrightarrow \mathbb{R}^{n^\pm}$$

that measure the length of the positive and negative boundaries. The volume  $V_{g,n^+,n^-}(L^+|L^-)$  of the level set  $\mathcal{M}_{g,n^+,n^-}^{comb}(L^+|L^-)$  is then a function of two sets of variables  $L^+, L^-$ , which are defined on the hyperplane.

$$\Lambda_{n^+,n^-} = \{(L^+, L^-) \in \mathbb{R}_{\geq 0}^{n^+} \times \mathbb{R}_{\geq 0}^{n^-} \mid \sum_{i=1}^{n^+} L_i^+ = \sum_{i=1}^{n^-} L_i^-\}.$$

More generally, for each partition  $\nu$ , we can also consider the function  $V_{g,n^+,n^-}^\nu$  that corresponds to the volume of the stratum  $\mathcal{M}_{g,n^+,n^-}^{comb}(\nu)$ , which is the closure of the set of oriented ribbon graphs with  $\nu(i)$  vertices of degree  $2i + 2$ . Then another natural question is:

**Question 2.4.** *Can we compute the volumes  $V_{g,n^+,n^-}$  and  $V_{g,n^+,n^-}^\nu$  ?*

### 2.3.2 First step to a generalization

Before trying to answer the precedent question, we can ask the following:

**Question 2.5.** *Can we find a proof of the recursion for  $V_{g,n}^{comb}$  that uses surgeries on ribbon graphs?*

Proofs of the Witten conjecture rely on other notions such as matrix integrals [Kong2] or the intersection theory on moduli spaces [KLo7]. The first result of this thesis was to answer this question. This recursion uses geometric recursion formulated in [ABO17] and invented in [Miro7] in order to compute Weil-Petersson volumes. A first formulation of the Mirzakhani-McShane identity for trivalent ribbon graphs was given in [ABO17] by using the fact that  $V_{g,n}^{comb}$  is the term of higher degree in  $V_{g,n}^{hyp}$  and using the Mirzakhani Mac Shane identity for hyperbolic surfaces [Miro7]. Independently from the author of the thesis, the authors of [ABO17] give

in [ABC<sup>+</sup>20] the interpretation of this formula as a Mirzakhani-McShane formula for ribbon graphs. To prove this formula, we study, in some sense, the geometry of metric ribbon graphs. We consider curves, compute their length, and perform surgeries and gluings on ribbon graphs. In the spirit of what was done before for hyperbolic surfaces, many operations can be transposed to ribbon graphs. There was an hope that these analogies would allow us to prove a Mirzakhani-McShane formula and then deduce a recurrence relation for the volumes by taking the integral of the formula over the moduli space  $\mathcal{M}_{g,n}^{comb}(L)$ . To state the formula, we fix  $M$  as a surface of genus  $g$  with  $n$  boundaries enumerated from 1 to  $n$ . We denote:

- For  $i \neq 1$  let  $Irr_{1,i}(M)$  be the set of all the isotopy classes of a pair of pant's  $P$  embedded in  $M$  such that the image of  $P$  contains the boundaries 1,  $i$ .
- $Irr_{1,1}(M)$  is the set of pant's  $P$  that contain the boundary 1 and no other boundaries.

For each pant's  $P$  let  $L_P(S)$  be the length of the boundaries of  $P$  that are not in  $\partial M$ . We prove the following theorem:

**Theorem 2.7** (theorem 6.2). *We have the following Mirzakhani-McShane formula for all trivalent metric ribbon graphs  $S$ :*

$$L_1(S) = \sum_{i \neq 1} \sum_{P \in Irr_{1,i}(M)} F^+(L_1(S), L_i(S) | L_P(S)) \\ + \sum_{P \in Irr_{1,1}(M)} F^-(L_1(S) | L_P(S))$$

The sum is infinite but contains only a finite number of non-vanishing terms. The functions  $F^\pm$  are explicit and piece-wise linear. The Mirzakhani-McShane formula is very useful; we can integrate it over the moduli space. Generalizing techniques developed in [Miro7] to compute these kinds of integrals, we can obtain the next recursion. It's very similar to theorem 2.1 and was also obtained in [ABC<sup>+</sup>20].

**Proposition 2.1** (proposition 6.3). *The volumes  $V_{g,n}^{comb}(L)$  are symmetric polynomial solutions to the recursion:*

$$L_1 V_{g,n}^{comb}(L) = \sum_{j \neq 1} \int_{\mathbb{R}_{\geq 0}} F^+(L_1, L_j | x) V_{g,n-1}(x, L_{\{1,j\}^c}) dx \\ + \frac{1}{2} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g-1,n+1}(x_1, x_2, L_{\{1\}^c}) x_1 x_2 dx_1 dx_2 \\ + \frac{1}{2} \sum_{g_1+g_2=g, I_1 \sqcup I_2 = \{2, \dots, n\}} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g_1, n_1+1}(x_1, L_{I_1}) V_{g_2, n_2+1}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2$$

With initial data's

$$V_{0,3}(L_1, L_2, L_3) = 1 \quad V_{1,1}(L) = \frac{L^2}{24}$$

We remark that, using similar methods, we can also obtain

$$(6g - 6 + 3n) = \frac{1}{2} \sum_{i,j} \sum_{P \in Irr_{i,j}(M)} H_P(L_{\{i,j\}}(S), L_P(S)),$$

for an explicit function  $H_P$  and  $L_{\{i,j\}}(S)$  is the vector that contains the length of the boundaries  $\{i, j\}$ . This formula allows us to prove that the Kontsevich Witten partition function is a solution to a cut and join equation.

**Proposition 2.2** (proposition 6.5 and [Ale11]). *The Kontsevich-Witten partition function  $Z^K$  is the solution of the following cut and join equation:*

$$\frac{\partial Z^K}{\partial q} = \sum_{k+l=n+1} (2k+1)(2l+1)t_{2k+1}t_{2l+1}\partial_{2n+1}Z^K + \sum_{k,l} (2k+2l+5)t_{2k+2l+5}\partial_{2k+1}\partial_{2l+1}Z^K + t_1^3 Z^K + \frac{t_3 Z^K}{24}.$$

With initial condition

$$Z^K(0) = 1.$$

### 2.3.3 Case of graphs with vertices of odd degree only

The first direction to generalize the Mirzakhani-McShane formula was the one of ribbon graphs with only vertices of odd degree. Many results about trivalent ribbon graphs transpose to this case. The volumes  $V_{g,n}^\nu$  are measured by using a symplectic form, which is a natural generalization of the Kontsevich form. It's expected that the volumes are then polynomials, and their coefficients should be related to intersection theory on the space  $\overline{\mathcal{M}}_{g,n}$  [Kon92], [AC94]. The first direction was then the following:

**Question 2.6.** *Can we generalize the Mirzakhani-McShane formula to the case of ribbon graphs with only vertices of odd degree?*

It was not hopeless to try to answer this question. But there are some specificities. The case of trivalent ribbon graphs uses in a crucial way the twist flow along a curve drawn on the graphs; it's used, for instance, to integrate the Mirzakhani-McShane formula over the moduli space. Outside the generic case, it might happen that a curve splits a vertex into several vertices of smaller degrees. This is indeed an issue; in this case, the twist flow is not defined, and then it's problematic to compute the integrals. When we do surgeries, we must restrict them to curves that do not split any vertex of the ribbon graph (see figure 2.4. We call such curves admissible; we introduce them in section 4.2 and study their properties. We can see that, in general, all the surgeries are not legal; in particular, we cannot extract a pair of pants. However, there is a more general class of graphs that we can call irreducible because there is no admissible curve on them. These graphs are natural generalizations of pairs of pants; they are of genus 0 with only two vertices and are good candidates to write a Mirzakhani-McShane formula. In this case, we do not extract a pair of pants but surfaces of higher topology, and this could be related to a higher topological recursion relation. Nevertheless, particular cases appear, and today we do not have a solution to write a Mirzakhani-McShane formula in general. Answering this question could be the subject of future research.

### 2.3.4 Orientable graph, acyclic decomposition, and recursion for the volumes $V_{g,n^+,n^-}^\nu$

The attempts to prove the Mirzakhani-McShane formula in the case of ribbon graphs with odd vertices lead to a recurrence relation for the case of oriented ribbon graphs; this was quite

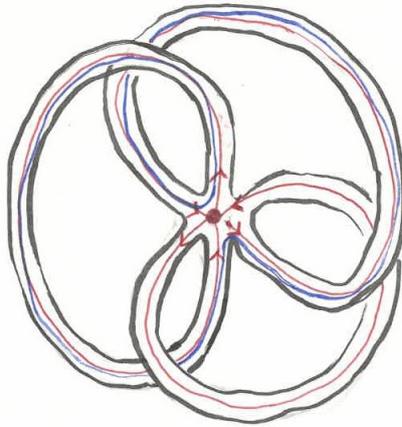


Figure 2.4: Admissible curve (in blue) on an oriented ribbon graph.



Figure 2.5: An oriented ribbon graph on a pair of pants.

unexpected. These graphs appear in the study of Abelian differentials. In this case, all the vertices are even, and the Kontsevich 2–forms are often degenerate. The volumes  $V_{g,n^+,n^-}^\nu$  are computed by using a Lebesgue measure. In this case, we can also consider admissible curves and perform surgeries on the graphs. The result of a surgery on an oriented ribbon graph along an admissible curve is always a family of oriented graphs. Then the boundaries of these graphs are labeled by  $\pm$ , and two boundaries glued along a curve have opposite signs. This condition might be familiar to researchers that study Abelian differentials or oriented foliations. This condition on the gluings induces an orientation of the edges of the stable graphs, which are called directed stable graphs (see definition 3.5 and figure ??). The study of curves on oriented graphs leads to the following theorem, which is the second result of this thesis.

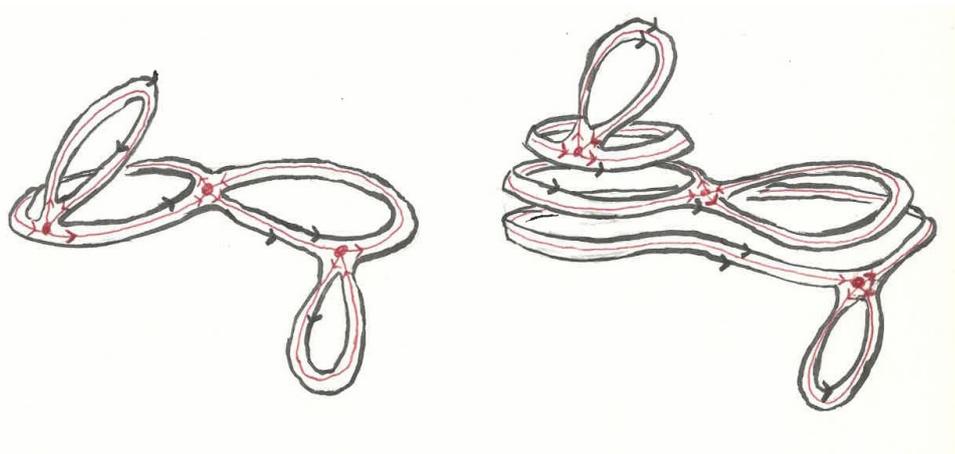


Figure 2.6: Acyclic decomposition of a surface of type  $(0, 3, 2)$ .

**Theorem 2.8** (Theorem 4.2). *Let  $R^\circ$  be a connected, oriented ribbon graph with at least two vertices. For each vertex  $v$ , there is a unique admissible primitive multi-curve  $\Gamma_v^+$  such that*

- *The directed stable graph  $\mathcal{G}_v^\circ$  associated to  $\Gamma_v^+$  contains a component  $c_v$  that spares  $v$  from the rest of the surface.*
- *All the curves  $\Gamma_v^+$  are in the boundary of  $c_v$ .*
- *$c_v$  is glued along negative boundaries only.*

This result is quite unclear; let us explain it. This theorem says that if we fix a graph and a vertex, there is a unique way to extract this vertex by performing admissible surgeries. There is a connected component that contains  $v$  and no other vertex. This is true if we impose a condition on the signs of the gluing. Without this condition, we could find an infinite number of ways to spare this vertex. The condition on the gluings means that the graph  $\mathcal{G}_v^\circ$  is acyclic. An important point is that the curves are admissible, and then the decomposition preserves the degree of the vertices. The proof of this theorem is given in chapter 4.5.2 in the section 4.3.2. It uses relatively simple mathematical tools, such as the homology of compact surfaces. By iterating the theorem 4.2, we can obtain a decomposition of an oriented ribbon graph into graphs with only one vertex, which is the content of the theorem 4.3. Moreover, the condition of theorem 4.2 implies a special structure for the gluings allowed. The directed stable graphs are

acyclic, in the sense that there is no oriented cycle in the stable graph. We chose to call such a decomposition an acyclic decomposition. We call graphs with one vertex “minimal ribbon graph” by analogy with the terminology for Abelian differentials. Minimal is not the same as irreducible; here the graphs can have a non-trivial genus, but these graphs admit no non-trivial acyclic decomposition, and they are minimal bricks to build general oriented ribbon graphs (see figure 2.6). An acyclic decomposition is maximal if all the components are minimal. We can summarize this discussion by using the following theorem:

**Theorem 2.9** (Theorem 4.3). *An oriented ribbon graph with an enumeration of the vertices admits a unique maximal acyclic decomposition that respects this enumeration.*

As before, the uniqueness is surprising. We define the terms that appear in this sentence later in the memoir. An acyclic graph induces a partial order relation on its vertices, and “respect” means that the enumeration gives a linear order on this partial order.

**Recursion for the volumes**  $V_{g,n^+,n^-}$  The structure of the gluings in acyclic stable graphs makes it possible to deduce from theorem 4.2 a recurrence relation for the volumes  $V_{g,n^+,n^-}$ . As there is a unique curve that satisfies the assumption of the theorem, it allows us to use this result's in family and deduce a recursion for the volumes.

**Theorem 2.10** (Theorem 4.6). *For all the values of  $L^\pm$ , the functions  $V_{g,n^+,n^-}(L^+|L^-)$  satisfy the recursion:*

$$\begin{aligned}
(2g - 2 + n)V_{g,n^+,n^-}(L^+|L^-) &= \sum_i \sum_j [L_i^+ - L_j^-]_+ V_{g,n^+,n^- - 1}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) \\
&+ \frac{1}{2} \sum_{i \neq j} (L_i^+ + L_j^+) V_{g,n^+ - 1, n^-} (L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-) \\
&+ \frac{1}{2} \sum_i \int_0^{L_i^+} V_{g-1, n^+ + 1, n^-} (x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) x (L_i^+ - x) dx \\
&+ \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^\pm \sqcup I_2^\pm = I^\pm}} x_1 x_2 V_{g_1, n_1^+, n_1^-} (x_1, L_{I_1^+}^+ | L_{I_1^-}^-) V_{g_2, n_2^+, n_2^-} (x_2, L_{I_2^+}^+ | L_{I_2^-}^-).
\end{aligned}$$

Where we use

$$x_l = \sum_{i \in I_l^-} L_i^- - \sum_{i \in I_l^+} L_i^+.$$

And the initialization is:

$$V_{0,2,1} = 1 \quad V_{0,1,2} = 1.$$

This recursion is similar to the one in 6.3. However, in the case of oriented graphs, there are more ways to extract a pair of pants. Indeed, in this case, there are two types of pairs of pants:  $P_+$  and  $P_-$ ; they are of type  $(0, 2, 1)$ ,  $(0, 1, 2)$ , and the different possible gluings are given in figure 2.7. Each type corresponds to a line in the recurrence formula. We will see later how to use this recursion. Let us mention another result that shows that the volumes  $V_{g,n^+,n^-}$  are radically different to the case of trivalent graphs:

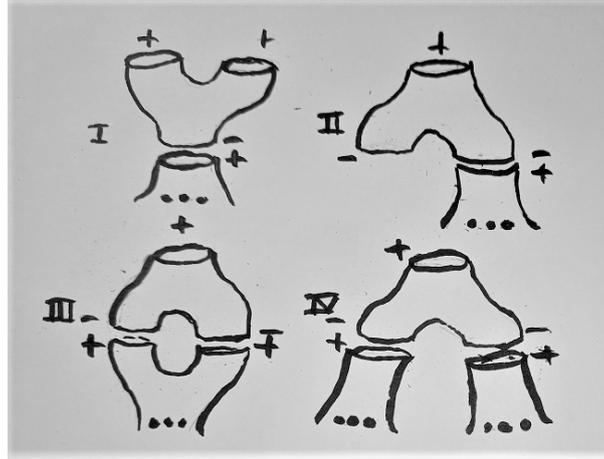


Figure 2.7: Different gluings that appear in 4.6

**Theorem 2.11.** *The functions  $V_{g,n^+,n^-}(L^+|L^-)$  are only piece-wise polynomials, and the walls are known.*

We can prove this theorem by recursion, but we prefer to use 4.3 and general results on acyclic graphs that use the Ehrhart theorem [Bar08] (see theorem 3.5 and proposition 4.26)

**Case of higher vertices:** The last results show that the volumes  $V_{g,n^+,n^-}$  of the moduli spaces of oriented fourvalent ribbon graphs are computable. What's about the case of oriented graphs with higher vertices? We can already see that the cases of graphs with vertices of degree 2 are almost trivial. Indeed, if we denote  $V_{g,n^+,n^-,m}$  the volumes associated with fourvalent graphs with  $m$  bivalent vertices, it is given by

$$V_{g,n^+,n^-,m} = \frac{E^m}{m!} V_{g,n^+,n^-,m}.$$

Where  $E$  is the function on  $\Lambda_{n^+,n^-}$  defined by

$$E(L) = \sum_i L_i^+ = \sum_i L_i^-.$$

This result is still a consequence of theorem 4.2. In the general case, we can also use this theorem to obtain formulas similar to the ones in theorem 4.6. In this case, we don't sum over all the ways to extract a pair of pants but over all the ways to extract a minimal surface of Euler characteristic  $-i$ , where  $2i + 2$  is the degree of the vertex that we choose to extract. The recursion uses the volumes associated to minimal graphs. These volumes are polynomials computed by I. Yakovlev in [Yak22]. Combining his results and the recursion of theorem 4.3, it's then possible to compute these volumes in full generality. We can conclude this part by:

**Theorem 2.12.** *We can compute the volumes  $V_{g,n^+,n^-}^\nu$ ; moreover, there is a continuous piece-wise polynomial that coincides with  $V_{g,n^+,n^-}^\nu$  for almost all values of  $L$ .*

The walls that define the piece-wise polynomials are explicit and are the same as the ones of  $V_{g,n^+,n^-}$ . The last point of this theorem might seem obscure; we will come back on it later in this introduction.

**Can we compute the Masur-Veech volumes?** The Mirzakhani-McShane formula allows to prove in [?] a similar formula for the multiplicative statistics for length of multi-geodesics on hyperbolic surfaces; it generalizes in a straightforward way for curves on trivalent graphs. Then, from these results, it's possible to compute recursively the Masur-Veech volumes of the principal stratum of moduli spaces of quadratic differential. Unfortunately, the situation is more complicated in the case of oriented ribbon graphs. The Mirzakhani-McShane formula and the acyclic decomposition are two different recursions. In the first one, we extract a boundary, and in the second, we extract a vertex. In consequence, the acyclic decomposition does not commute with the statistic of multi-curves, and then it's difficult to obtain a recurrence relation for the volumes of Abelian differentials and generalize the results of [?]. However, the acyclic decomposition possesses many enjoyable properties that we will explore in the next section.

### 2.3.5 Volumes of the moduli space of oriented metric ribbon graphs and cut and join operators

**Special case of  $V_{g,1}^\circ$ :** It's a bit frustrating that the volumes  $V_{g,n^+,n^-}$  are only piece-wise polynomials. It reduces the possible applications to compute the Masur-Veech volumes, and it's quite indirect to extract any information. There is a particular case where these functions are polynomials; it's the case of surfaces with only one negative boundary. In this case,  $\Lambda_{n^+,n^-} \simeq \mathbb{R}_{\geq 0}^{n^+}$  and  $V_{g,n,1}(L^+|L^-)$  depend only on variables  $L^+$ . We can write it as

$$V_{g,n}^\circ(L^+) = V_{g,n,1}(L^+|L^-).$$

In part 4.5.2 we give the following result, which is a consequence of 4.6

**Theorem 2.13** (theorem 4.11).  $V_{g,n}^\circ$  is a symmetric homogeneous polynomial of degree  $4g - 4 + n$  and satisfies the recursion.

$$(2g + n - 1)V_{g,n}^\circ(L) = \frac{1}{2} \sum_{i \neq j} (L_i + L_j) V_{g,n-1}^\circ(L_i + L_j, L_{\{i,j\}^c}) \\ + \frac{1}{2} \sum_i \int_0^{L_i} V_{g-1,n+1}^\circ(x, L_i - x, L_{\{i\}^c}) x(L_i - x) dx$$

with the initial condition  $V_{0,2}^\circ(L) = 1$ .

The coefficients of  $V_{g,n}^\circ$  have several combinatorial interpretations and are related to the count of oriented ribbon graphs and also the Hurwitz numbers of the Grotendieck dessins d'enfants ( coverings of the Riemann sphere ramified over three points). Let  $c_{g,n}(\alpha)$  be the coefficient of  $V_{g,n}^\circ$  in front of  $L^\alpha$ , where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-indices. The coefficients  $c_{g,n}$  are symmetric and can be indexed by partitions  $\mu = (\mu(0), \mu(1), \dots)$ . The lowerscript  $(g, n)$  is not essential as it can be determined by the partition. We can form the following generating series:

$$Z^\circ(q, \mathbf{t}) = \sum_{\mu} q^{\frac{d(\mu)+n(\mu)}{2}} \frac{\prod_i (i!)^{\mu(i)} t_i^{\mu(i)}}{\prod_i \mu(i)!} c(\mu)$$

with  $\mathbf{t} = (t_0, t_1, \dots)$ . We then obtain the following corollary:

**Corollary 2.1** (corollary 4.16). *The series  $Z^1(q, \mathbf{t})$  is a solution of the linear equation*

$$\frac{\partial Z^\circ}{\partial q} = \frac{1}{2} \sum_{i,j} (i+j)t_i t_j \partial_{i+j-1} Z^\circ + \frac{1}{2} \sum_{i,j} (i+1)(j+1)t_{i+j+3} \partial_i \partial_j Z^\circ + \frac{t_0^2}{2}$$

with the initialization  $Z^\circ(0, \mathbf{t}) = 0$ .

The operator in the RHS of the equation is a cut and join operator. As we saw in the introduction, these operators are notably known and appear in different domains of enumerative geometry and physics.

**Generalization, operators associated to the volumes  $V_{g,n^+,n^-}$ :** The particular case of the surface with one negative boundary is interesting, but it's only a particular case. In order to generalize these results, I adopt the following point of view: We fix  $(g, n^+, n^-)$ , and we consider the slightly different functions.

$$K_{g,n^+,n^-}(L^+|L^-) = \prod_i L_i^+ V_{g,n^+,n^-}(L^+|L^-).$$

For all  $f$  continuous on  $\mathbb{R}_{\geq 0}^{n^-}$ , we can consider the integral

$$K_{g,n^+,n^-} \cdot f(L^+) = \frac{1}{n^-!} \int_{L^- \in \Delta_{n^-}(|L^+|)} K_{g,n^+,n^-}(L^+|L^-) f(L^-) d\sigma_{n^-}^{|L^+|}.$$

It's well defined, the domain of integration is bounded, and the function is continuous. The notations appearing in this formula are given in section 3.1. The definition of  $K_{g,n^+,n^-}$  is not random; the composition of two operators is equal to the integral operator associated to the following kernel:

$$\prod_i L_i^+ \int_{x \in |L^+| \cdot \Delta_k} \frac{V_{g_1,n^+,k}(L^+|x) V_{g_2,k,n^-}(x|L^-)}{k!} \prod_j x_j d\sigma_k^{|L^+|}$$

The measure  $d\sigma_k^{|L^+|}$  is also given in section 3.1. The reader who is familiar with the works of M. Mirzakhani can recognize in this formula similarities with formulas that are obtained when we perform surgeries on hyperbolic surfaces, for instance. Theorem 4.11 admits a generalization that we can call the transfert lemma (lemma 5.7). The natural generalization of the first part of theorem 4.11 is given by the following proposition:

**Proposition 2.3** (particular case of 5.7). *For all  $(g, n^+, n^-)$  and all  $P \in \mathbb{Q}[L^-]$  a homogeneous polynomial of degree  $d$ ,  $K_{g,n^+,n^-} \cdot P$  is a symmetric homogeneous polynomial of degree  $d + 4g - 4 + 2n^+ + 2n^-$*

We use this proposition in the following ways: Let  $V = \mathbb{Q}[L]$ , we consider  $S(V)$  the space of symmetric polynomials; this is a graded commutative algebra for the symmetric product  $\sqcup$ . Then  $K_{g,n^+,n^-}$  defines an endomorphism:

$$K_{g,n^+,n^-} : S(V) \longrightarrow (V).$$

The space of endomorphisms on  $S(V)$  also admits a structure of commutative algebra for a generalization of  $\sqcup$ . If  $M^\circ = \sqcup_c M^\circ(c)$  is a disconnected directed surface, we can still define the

volume  $V_{M^\circ}$  and associate an operator  $K_{M^\circ}$ , which also defines an endomorphism of  $S(V)$ . The set of directed surfaces is endowed with a monoid structure given by the disjoint union  $\sqcup$ , and we have the natural formula,

$$K_{M_1^\circ \sqcup M_2^\circ} = K_{M_1^\circ} \sqcup K_{M_2^\circ}.$$

With this property, it's natural to consider the commutative algebra  $S(\mathcal{M})$  generated by a vector  $e_{M^\circ}$  for each directed surface. Then the operator  $\mathbf{K}$  can be extended to a morphism,

$$\mathbf{K} : S(\mathcal{M}) \longrightarrow \text{End}(S(V)).$$

There is still an issue. To make formulas work in general, we have to include in  $\mathcal{M}$  the cylinder of type  $(0, 1, 1)$  and the empty surface. In both cases, the good choice for the operators is the identity, respectively on  $S_1(V)$  and  $S_0(V)$ . With these assumptions, the operator  $\exp_{\sqcup}(K_{0,1,1})$  is equal to  $id$  on  $S(V)$ , which is coherent because in practice, adding cylinders does not change anything.

**Total operator:** The recursion of theorem 4.6 can be seen as all the ways to extract a pant's of type  $(0, 2, 1)$  or  $(0, 1, 2)$ . This recursion uses integrals, and we can wonder if it's possible to write it in terms of the operators  $K_{g,n^+,n^-}$ . To do that, we consider the following series:

$$K = \sum_{M^\circ} q^{d(M^\circ)} K_{M^\circ} = \exp_{\sqcup} \left( \sum_{g,n^+,n^-} q^{2g-2+n^++n^-} K_{g,n^+,n^-} \right).$$

We sum over all the possible directed surfaces, not necessarily connected (we also include cylinders). This sum is infinite; to make sense of this expression, we use the natural completion  $\hat{S}(V)$  of  $S(V)$  by using graduation. We then show that  $K$  defines an endomorphism of  $\hat{S}(V)$  for all  $q$ . We consider the two operators

$$P^+ = K_{0,2,1} \sqcup \exp_{\sqcup}(K_{0,1,1}) \quad P^- = K_{0,1,2} \sqcup \exp_{\sqcup}(K_{0,1,1}).$$

They represent all the ways to glue a pant's of types  $(0, 2, 1)$  and  $(0, 1, 2)$ . The term  $\exp_{\sqcup}(K_{0,1,1})$  is essential in order to glue pant's on some boundaries and let the other remain unchanged. In a surprising way, the recursion of theorem 4.6, which is pretty complicated, can be written in the following simpler way in this formalism. Let

$$P = P_+ + P_-,$$

the operator corresponding to all the ways to glue a directed pant.

**Theorem 2.14** (Theorem 5.1). *The series  $K(q)$  is a solution to the evolution equation:*

$$\frac{dK}{dq} = PK.$$

*With the initial condition  $K(0) = id$ , and then*

$$K(q) = \exp(qP).$$

This equation is a cut and join equation. As we see, the space  $\hat{S}(V)$  can be identified with the space  $\mathbb{Q}[[\mathbf{t}]]$  of formal series in an infinite number of variables  $\mathbf{t} = (t_0, t_1, \dots)$ . We see also (in section 5.1) that general considerations on creation and annihilation operators on Fock spaces allow to show that  $K, P^+, P^-$  are differential operators on  $\mathbb{Q}[[\mathbf{t}]]$ . In particular, we have the formula

$$P = \frac{1}{2} \sum_{i,j} (i+1)(j+1)t_{i+1}t_{j+1}\partial_{i+j} + \frac{1}{2} \sum_{i,j} (i+j+2)t_{i+j+2}\partial_i\partial_j.$$

This approach generalizes directly the results of the theorem 4.11 and the corollary 4.16.

**Partition function:** The functions  $V_{g,n^+,n^-}$  are polynomials, but since  $K$  is an operator, we can wonder what information is encoded in its coefficients. In general, the answer is unclear, but in a particular case, the coefficients have a combinatorial interpretation. We consider the polynomials

$$G_{g,n^+,n^-} = \frac{1}{n^-!} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} K_{g,n^+,n^-}(L|L^-) d\sigma_{n^-}^{|L^+|}.$$

We have  $G_{g,n^+,n^-} = K_{g,n^+,n^-} \cdot \mathbf{e}_\emptyset$ , and  $G_{g,n^+,n^-}$  is then the image of the vacuum  $\mathbf{e}_\emptyset \in \hat{S}(V)$ . The coefficients of the polynomial  $G_{g,n^+,n^-}$  are interesting and count several quantities.

- Oriented ribbon graphs with quadrivalent vertices,
- Dessins d'enfants with simple ramifications over one point.

We can generalize the definition to non-connected surfaces and define  $G = K \cdot \mathbf{e}_\emptyset$ , let  $Z$  be the image of  $G$  in  $\mathbb{Q}[[q, \mathbf{t}]]$ . We then have the following equation, which is a trivial consequence of the cut and join equation (theorem 5.1).

**Corollary 2.2.**  $Z$  is a solution of the differential equation.

$$\frac{\partial Z}{\partial q}(q) = \frac{1}{2} \sum_{k,l} (k+l+2)t_{k+l+2}\partial_k\partial_l Z + \frac{1}{2} \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1}\partial_{k+l} Z.$$

As we see, this equation allows us to recover the special case of  $Z^1$  as the first term of a development.

**Virasoro constraints and topological recursion:** We can ask if there is any link with the topological recursion or the Airy structures introduced by M. Kontsevich and I. Sobeilman [KS17]. As we mentioned before in the introduction, it's not automatic that a cut and join equation produces Airy structures. To obtain such a relation, we use theorem 4.2 in a different way and obtain a slightly different recursion for the volumes given by lemma 5.14. This relation can't be used to compute the volumes  $V_{g,n^+,n^-}$  or the operator  $K_{g,n^+,n^-}$ . But it leads to the following recursion for  $Z$ .

**Theorem 2.15** (lemma 5.17 and proposition 5.21). *The generating series  $Z$  is the solution of the Virasoro constraint's,*

$$L_i(Z) = 0 \quad \forall i \geq -2.$$

with

$$L_i = -\partial_{i+2} + \sum_j (j+1)t_{j+1}\partial_{i+j+1} + \sum_{k+l=i} \partial_k\partial_l + \delta_{i,-2}$$

The commutation relations of the operators  $L_k$  are well known by physicists; the  $L_k$  are elements of the Virasoro algebra. We have the relations

$$[L_i, L_j] = (i - j)L_{i+j} \quad i, j \geq -1.$$

We can then consider the Laplace transform of the polynomials  $G_{g,n^+,n^-}$

$$W_{g,n^+,n^-}(x) = \int_L G_{g,n^+,n^-}(L) e^{-\sum_i x_i L_i} dL.$$

We can sum over  $n^-$  and obtain the series:

$$W_{g,n} = \sum_{n^-} W_{g,n,n^-}.$$

The  $W_{g,n}$  are germs of analytic functions near  $\infty$ . Using the result of the last part, we can obtain the following recurrence relation:

**Theorem 2.16** (Theorem 5.3). *The Laplace transforms are solutions of the following equations:*

$$\begin{aligned} x_1 W_{g,n} = & \sum_{i \neq 1} \frac{\partial}{\partial x_i} \left( \frac{W_{g,n-1}(x_1, x_{\{1,i\}^c}) - W_{g,n-1}(x_i, x_{\{1,i\}^c})}{x_1 - x_i} \right) \\ & + W_{g-1,n+1}(x_1, x_1, x_{\{1\}^c}) + \sum_{I_1, I_2, g_1+g_2=g} W_{g_1-1, n_1+1}(x_1, x_{I_1}) W_{g_2-1, n_2+1}(x_1, x_{I_2}) + \delta_{g,0} \delta_{n,1} \end{aligned}$$

This formula is closely similar to formula obtained from the loop equation in random matrix theory [Eyn05]. This recursion allows us to compute the  $W_{g,n}$  in terms of  $W_{0,1}$ . The function  $W_{0,1}$  is then a solution to the equation:

$$x W_{0,1} = W_{0,1}^2 + 1.$$

With the normalization  $W_{0,1}(x) = \infty \frac{1}{x} + o(\frac{1}{x})$ . This solution is not univalued, it's customary in the theory of random matrices to take the pullback by the map

$$z \longrightarrow z + \frac{1}{z}.$$

We obtain a germ of convergent series that admits an extension to  $\mathbb{C}P_1$ . By following arguments that are now classical, we can show that this recursion implies that the  $W_{g,n}$  are computed by the topological recursion of Eynard-Orantin. Following their arguments, we obtain the following:

**Corollary 2.3.** *The recurrence of theorem 5.3 implies the topological recurrence with spectral curve*

$$xy = y^2 + 1$$

This spectral curve is well known to researchers that work in enumerative geometry; it's the spectral curve that counts Grotendieck's dessin d'enfants. It also appears in random matrix theory and is related to the generating series of moments of the Gaussian unitary ensemble. Thanks to the works of P. Norbury [Nor08] [Nor13], this spectral curve is also associated to counting integral points in the moduli space  $\mathcal{M}_{g,n}^{comb}$ . We give a combinatorial proof that relies on  $W_{g,n}$  to the functions studied by P. Norbury in section 5.7.1.

**Remarks on  $V_{g,n^+,n^-,m}$  :** As we see before, we can consider the volumes  $V_{g,n^+,n^-,m}$  corresponding to graphs with  $m$  bivalent vertices. In this case, we can define the operators  $K_{g,n^+,n^-,m}$  and the series

$$K^\bullet(q_0, q_1) = \exp_{\square} \left( \sum_{g,n^+,n^-,m} q_0^m q_1^{2g-2+n^++n^-} K_{g,n^+,n^-,m} \right).$$

We also consider the operator  $E$  that acts on  $S(V)$  by multiplication by the function  $E(L) = \sum_i L_i$ . In terms of differential operators, we have

$$E = \sum_i (i+1)t_{i+1}\partial_i.$$

Then we can obtain the following result:

**Théorème 2.1.** *The operators  $P, E$  are commuting, and we have the cut and join equations.*

$$\frac{dK^\bullet}{dq_0} = EK^\bullet \quad \frac{dK^\bullet}{dq_1} = PK$$

then

$$K^\bullet(q_0, q_1) = \exp(q_0E + q_1P)$$

We will see next how to make this theorem more general.

**Vertices of degree  $\geq 4$  :** To conclude this part, we generalize the last results to the cases of oriented ribbon graphs with vertices of higher degrees. For each  $(g, n^+, n^-, \nu)$ , we can define operators  $K_{g,n^+,n^-, \nu}^\nu$  using the volumes  $V_{g,n^+,n^-, \nu}^\nu$  instead of  $V_{g,n^+,n^-}^\nu$ . We still have a recursion recursion for the functions  $V_{g,n^+,n^-, \nu}^\nu$ , and if

$$\bar{K}(\mathbf{q}) = \exp_{\square} \left( \sum_{g,n^+,n^-, \nu} \mathbf{q}^\nu K_{g,n^+,n^-, \nu}^\nu \right).$$

Then this series converges for all  $\mathbf{q} = (q_0, q_1, \dots)$ , and we can see by specialization

$$\bar{K}(0, q_1, 0, \dots) = K(q_1) \quad \bar{K}(q_0, q_1, 0, \dots) = K^\bullet(q_0, q_1)$$

We can then generalize the operators  $P_+, P_-, E$  in the following way: Let

$$\mathcal{W}_{g,n^+,n^-} = K_{g,n^+,n^-}^{(2g-2+n^++n^-)} \square \exp_{\square}(K_{0,1,1})$$

and we form

$$\mathcal{W}_i = \sum_{2g-2+n^++n^-=i} \mathcal{W}_{g,n^+,n^-}.$$

Similarly to  $P$ , the operator  $\mathcal{W}_i$  can be seen as all the ways to glue a surface with Euler characteristic given by  $-i$  i.e., all the ways to add a vertex of degree  $2i + 2$  in the ribbon graphs. We know that the  $\mathcal{W}_i$  are differential operators, and we can see that

$$\mathcal{W}_0 = E \quad \mathcal{W}_1 = P.$$

Theorems 5.1 and ?? can be generalized in the following way:

**Theorem 2.17** (Theorem 5.6). *The series  $\overline{K}$  is a solution of the following system:*

$$\frac{\partial \overline{K}}{\partial q_i}(\mathbf{q}) = \mathcal{W}_i \overline{K}(\mathbf{q})$$

To use this theorem, it's important to know the commutative structure of the operator  $\mathcal{W}_i$  in order to compute the exponential. By using the theorem 4.2 stated in the last paragraph, we can deduce the following theorem:

**Theorem 2.18** (theorem 5.8). *The operators  $\mathcal{W}_i$  are pairwise commuting, and we have*

$$\overline{K}(\mathbf{q}) = \exp\left(\sum_i q_i \mathcal{W}_i\right)$$

This relation is surprising because it means that the operators are independent. Then two vertices of different orders can be extracted independently.

In some low-degree cases, we give an explicit formula for  $\mathcal{W}_i$ . We already know them for  $i = 0, 1$ , and we obtain without pain the formula.

**Proposition 2.4** (proposition 5.7). *The differential operator  $\mathcal{W}_2$  is given by*

$$\begin{aligned} \mathcal{W}_2 = & \frac{1}{6} \sum_i (i+1)i(i-1)t_{i+1}\partial_{i-2} \\ & + \frac{1}{2} \sum_{i+j=k+l+1} (i+1)(j+1)t_{i+1}t_{j+1}\partial_k\partial_l \\ & + \frac{1}{3} \sum_{i,j,k} (i+1)(k+1)(l+1)t_{i+1}t_{j+1}t_{k+1}\partial_{i+j+k} \\ & + \frac{1}{3} \sum_{i,j,k} (i+j+k+3)t_{i+j+k+3}\partial_i\partial_j\partial_k. \end{aligned}$$

We list in figure ?? the topological interpretation of the terms in this formula. It's natural to try to find a formula for the operator  $\mathcal{W}_i$ ; it's painful to try to compute them by hand. In low order, we can count the graphs with vertices of degree  $2i + 2$  and sum all the terms, but we cannot have a general formula. A surprising fact was discovered in [WLZZ22], [MA23] it's called  $\mathcal{W}$ -representation or super integrability. It appears in the study of random matrix models. By analogy with their work, we consider the following operator:

$$\mathcal{W}_{-1} = \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1}\partial_{k+l+1} + \sum_{k,l} (l+k+1)t_{k+l+1}\partial_k\partial_l.$$

We guess the following conjecture:

**Conjecture 2.1** (conjecture 5.1). *The operators  $\mathcal{W}_i$  are generated from  $\mathcal{W}_{-1}$  and  $\mathcal{W}_0$  by the following recursion:*

$$(i+2)\mathcal{W}_{i+1} = [\mathcal{W}_{-1}, \mathcal{W}_i].$$

As we see, the operator  $\mathcal{W}_0$  is simple and given by

$$\mathcal{W}_0 = \sum_i (i+1)t_{i+1}\partial_i.$$

We remark that this conjecture is not really a conjecture. Bijection should directly relate our model to the one of [WLZZ22]. But it should be interesting to interpret this formula as a recursion for volumes corresponding to minimal graphs, that are used to define  $\mathcal{W}_i$ . A way to rewrite this recursion is the following: Let  $\mathcal{W}(q)$  the series defined by

$$\mathcal{W}(q) = \sum_i (i+1)q^i \mathcal{W}_i.$$

We obtain that  $\mathcal{W}(q)$  is the solution of the Lax equation:

$$\frac{\partial \mathcal{W}}{\partial q} = [\mathcal{W}_{-1}, \mathcal{W}(q)]$$

with  $\mathcal{W}(0) = \mathcal{W}_0$  and then  $\mathcal{W}(q) = \exp(q\mathcal{W}_{-1})\mathcal{W}_0 \exp(-q\mathcal{W}_{-1})$ . We also mention that the Virasoro constraints seem to generalize, but our results are verified for only small values of  $i$ . We can find operators  $\tilde{L}_{k,i}$  such as

$$\mathcal{W}_i = \sum_k (k+1)t_{k+1} \tilde{L}_{k,i}.$$

We can then form

$$L_i(\mathbf{q}) = \partial_{i+1} - \sum_k q_k \tilde{L}_{i,k}.$$

Let  $\bar{Z}$  be the partition function

$$\bar{Z}(\mathbf{q}) = \bar{K}(\mathbf{q}) \cdot \mathbf{e}_0$$

We conjecture that for all values of  $\mathbf{q}$  the partition function is solution of the constraints

$$L_i(\mathbf{q})\bar{Z}(\mathbf{q}) = 0.$$

## 2.3.6 Discontinuities of the volumes and applications

A surprising fact is that, outside the case of the principal stratum, the formula used to prove theorem ?? does not allow to compute the Masur-Veech volumes. There is an excess that corresponds to degenerated ribbon graphs. Actually, the functions  $V_{g,n^+,n^-}^\nu$  (and  $V_{g,n}^\nu$ ) are not continuous in general; on some affine subspaces, they can jump. These jumps appear on subspaces  $\Lambda_W \subset \Lambda_{n^+,n^-}$  (the same as the walls of the piece-wise polynomials  $V_{g,n^+,n^-}$ ). The goal of chapter ?? of this memoir is to show that  $V_{g,n^+,n^-}^\nu$  admits a continuous extension  $\bar{V}_{g,n^+,n^-}^\nu$ . These extensions are piece-wise polynomials, and they're easier to work with in practice. This text only covers the case of oriented ribbon graphs, but similar results are true for the  $V_{g,n}^\nu$  with some modifications. The jumps correspond to oriented ribbon graphs  $R^\circ$  that degenerate for some values of  $L$ , but the volumes  $V_{R^\circ}(L')$  of the corresponding cell in the moduli space do not tend to 0 when  $L'$  tends to  $L$ ; then we lose some mass. To understand this phenomenon, we study the compactification  $\bar{\mathcal{M}}_{g,n^+,n^-}^{comb}$  of  $\mathcal{M}_{g,n^+,n^-}^{comb}$  by using nodal ribbon graphs. We identify the degenerations that are responsible of discontinuities, and we call them apparent boundaries of  $\bar{\mathcal{M}}_{g,n^+,n^-}^{comb}$ . To simplify some notations, we denote  $\bar{M}^\circ = (M^\circ, \nu)$ , to study the continuity, we need to know how many ribbon graphs degenerate to  $R^\circ$  and in a given direction of the normal bundle to the subspace  $\Lambda_W$ . In other words, the map  $L_\partial$  is ramified at the neighborhood of an apparent boundary, and we had to compute the degree. To do that, we need to show that

the degree is well defined; we use the normal cone of a cell complex. We can then define the degree  $\deg_{R^\circ}(\overline{M}^\circ)$  of an apparent boundary in  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ)$ . We then consider the augmented volume.

$$\overline{V}_{\overline{M}^\circ}(L) = V_{g,n^+,n^-}^\nu + \sum_{R^\circ} \deg_{R^\circ}(\overline{M}^\circ) V_{R^\circ}(L).$$

We sum up all the apparent degeneration's, weighted by the degree of ramification. The goal of the first part of chapter ?? is to give a proof of the following result:

**Proposition 2.5** (proposition 7.11). *The function  $\overline{V}_{\overline{M}^\circ}(L)$  is continuous on  $\Lambda_{n^+,n^-}$  and coincides with  $V_{\overline{M}^\circ}(L)$  on  $\Lambda_{n^+,n^-}^*$ .*

The subset  $\Lambda_{n^+,n^-}^*$  corresponds to the complement of the union of the walls. The space  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ)$  admits a natural stratification given by nodal decorated surfaces  $\overline{\mathcal{D}}^\circ = (\mathcal{D}^\circ, \nu)$ . We can write

$$\overline{\mathcal{M}}^{comb}(\overline{M}^\circ) = \sqcup_{\overline{\mathcal{D}}^\circ} \mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ).$$

The apparent degenerations correspond to maximal nodal decorated surfaces. It's possible to describe these nodal surfaces by their topology. If  $\overline{\mathcal{D}}^\circ$  is maximal and  $L$  is in the image of  $L_{\partial} : \overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ) \rightarrow \Lambda_{n^+,n^-}$ , then  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ, L)$  and  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ, L)$  have the same dimension, and then  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ, L)$  is not a null set when we compute the volumes. We denote  $V_{\overline{\mathcal{D}}^\circ}(L)$  the volumes of  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ, L)$ . The desingularization  $\overline{\mathcal{N}}^\circ$  can be expressed as a union of decorated surfaces  $(\overline{\mathcal{D}}^\circ(c))_c$  with marked points at the nodes. We can write

$$V_{\overline{\mathcal{D}}^\circ}(L) = \prod_c V_{\overline{\mathcal{D}}^\circ(c)}(L).$$

The degree of a ribbon graph in  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ)$  only depends on  $\overline{\mathcal{D}}^\circ$  and  $\overline{M}^\circ$  and can be denoted  $\deg_{\overline{\mathcal{D}}^\circ}(\overline{M}^\circ)$ . We can then prove the following:

**Proposition 2.6.** *We have the formula*

$$\overline{V}_{\overline{M}^\circ}(L) = \sum_{\overline{\mathcal{D}}^\circ} \deg_{\overline{\mathcal{D}}^\circ}(\overline{M}^\circ) V_{\overline{\mathcal{D}}^\circ}(L)$$

where we sum over all the maximal decorated nodal surfaces ( $V_{\overline{M}^\circ}$  is of course included in the sum).

This formula relates the augmented volumes to the usual ones. For the degree, we conjecture an explicit formula.

**Formula for the augmented Masur-Veech volumes:** The functions  $V_{\overline{M}^\circ}$  and  $\overline{V}_{\overline{M}^\circ}$  are equal almost everywhere, and it's easier to use the second ones because they are continuous piece-wise polynomials. When we want to compute the Masur-Veech volumes, we are interested in the integrals of the form.

$$\vartheta^\circ(\overline{\mathcal{G}}^\circ) = \frac{1}{\#\text{Aut}(\overline{\mathcal{G}}^\circ)} \int_{\Lambda_{\overline{\mathcal{G}}^\circ}} \prod_{\gamma} \frac{l_{\gamma} e^{-l_{\gamma}}}{1 - e^{-l_{\gamma}}} \prod_c V_{\overline{\mathcal{G}}^\circ(c)}(L(c)) d\sigma_{\overline{\mathcal{G}}^\circ}$$

Where  $\overline{\mathcal{G}}^\circ$  is a directed decorated stable graph i.e., each component  $\mathcal{G}^\circ(c)$  of  $\overline{\mathcal{G}}^\circ$  is decorated with a partition  $\nu_c$ , and  $\Lambda_{\overline{\mathcal{G}}^\circ}$  is an explicit polytope constructed by using  $\overline{\mathcal{G}}^\circ$ . We then have the following formula:

**Proposition 2.7.** *The Masur-Veech volumes are given by*

$$\vartheta^\circ(\nu) = \sum_{\bar{\mathcal{G}}^\circ} \vartheta^\circ(\bar{\mathcal{G}}^\circ).$$

As the functions  $\bar{V}_{\bar{M}^\circ}$  coincide with  $V_{\bar{M}^\circ}$  almost everywhere, we can wonder if it's possible to substitute them in the last formulas. The answer is no. Actually, constraints imposed by directed graphs imply that the image of the projection

$$\Lambda_{\mathcal{G}^\circ} \longrightarrow \Lambda_{\mathcal{G}^\circ(c)}.$$

is contained in a subspace of the form  $\Lambda_{W_{\mathcal{G}^\circ(c)}}$  and is of positive measure in this subspace. Consecutively, when we compute the integrals, we restrict the functions  $\bar{V}_{\bar{\mathcal{G}}^\circ(c)}$  on walls on which they differ from the function  $V_{\bar{\mathcal{G}}^\circ(c)}$ . If we denote by analogy  $\bar{\vartheta}_{\bar{\mathcal{G}}^\circ}^\circ$  and

$$\bar{\vartheta}^\circ(\nu) = \sum_{\bar{\mathcal{G}}^\circ} \bar{\vartheta}^\circ(\bar{\mathcal{G}}^\circ).$$

Then, outside the cases of principal strata, the two quantities  $\bar{\vartheta}^\circ(\nu)$  and  $\vartheta^\circ(\nu)$  are not equal.  $\bar{\vartheta}^\circ(\nu)$  also contains terms that correspond to nodal surfaces. We consider  $\bar{\mathcal{T}}$  nodal surfaces that belong to an explicit subset of nodal decorated surfaces (stable graph is a tree). The volume  $\vartheta^\circ(\bar{\mathcal{T}})$  is given in terms of the Masur-Veech volumes by the formula

$$\vartheta^\circ(\bar{\mathcal{T}}) = \prod_c \vartheta^\circ(\bar{\mathcal{T}}(c))$$

where the product is over the set of connected components of the desingularization. We obtain the following result that was conjectured before by E. Goujard and A. Sauvaget in the case of quadratic differentials:

**Théorème 2.2.** *We have the formula*

$$\bar{\vartheta}^\circ(\nu) = \sum_{\bar{\mathcal{T}}} \deg_{\bar{\mathcal{T}}}(\nu) \vartheta^\circ(\bar{\mathcal{T}}).$$

We remark that it's possible to inverse the formula and obtain the Masur-Veech volumes by using the volumes  $\bar{\vartheta}^\circ(\nu)$  by the exclusion-inclusion principle. We remark that the nodal surfaces that appear in the formula are similar to boundary components in the compactification of moduli spaces of Abelian differentials [BCG<sup>+</sup>22].

**Remark 2.1.** *The results of this chapter can be generalized to more general graphs, such as graphs with vertices of odd degree, with some modifications. In this case, it's possible that a non-orientable component degenerates into an orientable graph, and this causes a new kind of degeneration.*

## 2.4 Organisation of the memoir

This memoir contains 7 chapters. Chapter 3 collects definitions and notations for all the chapters 4, 5, 6, 7. Chapter 4 can be read independently of the next chapters. Chapter 5 uses the two previous chapters, and chapter 6 uses the results of chapters 3, 4, 5, 7. Each chapter corresponds more or less to a different work (except the third).

**Chapter 3 :** In this first chapter we recall notions about the topology of surfaces and introduce several notations. We define directed and decorated surfaces that are used throughout this memoir to simplify some notations. We recall results about curves on surfaces, arcs, measured foliations, quadratic, and Abelian differentials. These results have been useful throughout this thesis. We also define directed stable graphs and study these notions; we insist on acyclic graphs that are used in the next chapter. We also prove results useful to state the piece-wise polynomiality.

**Chapter 4:** In this chapter, we study ribbon graphs and their geometry. We prove the theorem 4.2 and the recursion of the theorem 4.6. We recall several properties relative to ribbon graphs, define directed ribbon graphs, and study their properties. We give the definition of admissible curves, surgeries, and twist flow. By studying the degeneration's of the symplectic form, we prove the theorem 4.2.

**Chapter 5:** This chapter is devoted to cut and join operators. After recalling useful results on symmetric algebra, we define the operator  $K$  associated with the volumes. We show the theorem 5.1 and the relation with the topological recursion. We also give combinatorial interpretations of generating series.

**Chapter 6:** In chapter 6 we state the Mirzakhani-McShane formula for trivalent ribbon graphs. This formula is also given in [ABC<sup>+</sup>20]. We also give a proof of the cut and join equation that uses a slightly different version of the Mirzakhani-McShane formula.

**Chapter 7** In this chapter, we study the continuity of the volumes  $V_{g,n^+,n^-}^\nu$  and we prove the results given in the introduction.

# Chapter 3

## Background on surfaces

### 3.1 Generalities and notations

We introduce various notations that are used throughout this text, we collect them for the reader reference in this first section. We denote  $\mathbb{R}_{\geq 0} = [0, \infty[$  and  $\mathbb{R}_{> 0} = ]0, \infty[$ .

**Multi-graphs:** In this text, we consider multi-graphs (see [Bol12]), but we generally call them graphs by abuse of language. A multi-graph can be defined by:

$$G = (X_0G, XG, (XG(c))_{c \in X_0G}, s_1).$$

Where  $XG$  is the set of half edges,  $X_0G$  is the set of vertices,  $(XG(c))_{c \in X_0G}$  is a partition of  $XG$ , indexed by the vertices, and  $s_1$  is an involution that encodes how to glue two half edges together. A multi-graph can have multiple edges, loops, and also legs, which are the fixed points of  $s_1$ . The quotient  $X_1G = XG/\langle s_1 \rangle$  corresponds to the set of edges of the graph. We denote  $\partial G$  the legs and  $X^{int}G$  is the set of internal edges, which are the elements of order 2. For each  $e \in XG$  we denote  $[e]_0$  and  $[e]_1$  the projections on  $X_0G$  and  $X_1G$ . Two graphs  $G, G'$  are isomorphic iff there is a bijection  $\phi : XG \rightarrow XG'$  that preserves the two partitions and satisfies  $s'_1 \circ \phi = \phi \circ s_1$ . We always assume that an automorphism fix the legs of the graph.

If  $G$  is a graph and  $E$  a subset of  $X_1^{int}G$ , we can define  $G_{\langle E \rangle}$  the quotient. It's the graph obtained by identifying two vertices that are joined by an edge in  $E$  and removing these edges.

For  $E \subset X_1G$  we can also define the graph  $G_E$  by removing the edges in  $E$  and deleting the vertices with no edges or half edges.

---

<sup>1</sup>Formally, let  $\tilde{E} \subset XG$  the subset such that  $s_1(\tilde{E}) = \tilde{E}$  and  $\tilde{E}/\langle s_1 \rangle = E$ . We consider the equivalence relation  $\sim_E$  on  $X_0G$  generated by the following symmetric relation:

$$c_1 \sim'_E c_2 \Leftrightarrow \exists e_1, e_2 \in \tilde{E} \text{ such as } e_i \in XG(c_i) \text{ and } e_1 = s_1(e_2).$$

Then we denote  $X_0G_{\langle E \rangle} = X_0G/\sim_E$  the quotient. For each  $c \in X_0G_{\langle E \rangle}$  we consider the subset  $XG_{\langle E \rangle}(c) = \sqcup_{c' \sim_E c} XG(c') \setminus \tilde{E}$ , and we take the restriction of  $s_1$  to  $XG_{\langle E \rangle} = XG \setminus \tilde{E}$ .

**Measure on affine spaces:** We generally consider a vector space  $V$ , of finite dimension  $n$ , and a convex polytope  $X \subset V$ . A polytope is a subspace defined by a finite number of linear inequalities.

$$X = \bigcap_{i=1}^r \{x \in V, l_i(x) \geq b_i\}.$$

With  $l_i \in V^*$  and  $b_i \in \mathbb{R}$  for all  $i$ , a polytope is open if the inequalities are strict. We recommend [Baro8] for an introduction to the subject. We often have a lattice  $V(\mathbb{Z})$  in  $V$  that we call “integer points” (sometimes we can also denote it  $V_{\mathbb{Z}}$ ). Then we can define the Lebesgue measure<sup>2</sup> on  $V$  normalized by  $V(\mathbb{Z})$ . This measure is given by the pullback of the usual Lebesgue measure under any linear isomorphism  $\phi : V \simeq \mathbb{R}^n$ , such that  $\phi(V(\mathbb{Z})) = \mathbb{Z}^n$ .

We use fibrations which are locally given by linear maps between convex polytopes. The following lemma is useful at many places in the text and is a basic result of linear algebra.

**Lemma 3.1.** *Assume that  $V_i$  with  $i = 1, 2, 3$  are vector spaces with lattices of integer points  $V_i(\mathbb{Z})$ . Let  $d\sigma_i$  the Lebesgue measure normalised by  $V_i(\mathbb{Z})$ . If  $A : V_2 \rightarrow V_1$  is a linear map, which induces an exact sequence:*

$$\{0\} \rightarrow V_3(\mathbb{Z}) \rightarrow V_2(\mathbb{Z}) \rightarrow V_1(\mathbb{Z}) \rightarrow \{0\}.$$

*Then the measure  $d\sigma_3$  is the conditional measure of  $d\sigma_2$  with respect to  $d\sigma_1$*

This lemma implies the following fact: If  $X_2 \subset V_2$  and  $X_1 \subset V_1$  are polytopes, and  $A : X_2 \rightarrow X_1$  is linear such that:

- $A(V_2(\mathbb{Z})) = V_1(\mathbb{Z})$ ,
- $\ker(A) \cap V_2(\mathbb{Z})$  is a lattice in  $\ker(A)$ .

For  $y \in X_1$ , let  $X_2(y) = A^{-1}(\{y\})$  the fiber over  $y$ , and  $d\sigma_3(y)$  the measure on  $X_2(y)$  normalized by  $\ker(A) \cap V_2(\mathbb{Z})$ . Then for all integrable functions  $f$  we have

$$\int_{X_2} f d\sigma_2 = \int_{X_1} \int_{X_2(y)} f d\sigma_3(y) d\sigma_1.$$

This formula also works for maps between spaces that are cells complexes, such that cells are polytopes.

**Notations on indices:** We also use several notations relative to indices. Let  $I$  be a totally ordered set and  $L \in \mathbb{R}^I$  be a vector. For each subset  $J \subset I$  we use the notation  $L_J = (L_i)_{i \in J}$ , ordered according to the order induced on  $J$  by  $I$  (in many cases, we are considering symmetric functions, so the order is sometimes irrelevant). When there is no possible confusion on  $I$ , for each  $J \subset I$ , we denote  $L_{J^c}$  the vector  $L_{I \setminus J}$ . The notation

$$|L| = \sum_i |L_i|,$$

is also used, and denote by  $E$ , the function  $E(L) = |L|$ . Additionally we denote:

$$\Delta_n = \{L \in \mathbb{R}_{\geq 0}^n, |L| = 1\},$$

---

<sup>2</sup>If  $X$  is relatively compact in  $V$ . The volume of  $X$  is related to the asymptotic behavior of the number of integral points in  $t \cdot X$  when  $t$  tends to  $\infty$

the standard simplex. For  $t \in \mathbb{R}_{>0}$ , the set  $t \cdot \Delta_n$  is the dilatation of  $\Delta_n$  by  $t$ . These simplices have a natural Lebesgue measure  $d\sigma_n^t$  normalized by

$$\int_{t \cdot \Delta_n} d\sigma_n^t = \frac{t^{n-1}}{(n-1)!}.$$

For each  $n^+, n^- \in \mathbb{N}_{>0}$  we consider

$$\Lambda_{n^+, n^-} = \{(L^+, L^-) \in \mathbb{R}_{\geq 0}^{n^+} \times \mathbb{R}_{\geq 0}^{n^-}, |L^+| = |L^-|\}, \quad (3.1)$$

and denote  $d\sigma_{n^+, n^-}$  the measure on  $\Lambda_{n^+, n^-}$ , proportional to the Lebesgue measure and normalized by the set of integer points. We have two projections:

$$L_{\pm} : \Lambda_{n^+, n^-} \longrightarrow \mathbb{R}_{\geq 0}^{n^+}.$$

And we still denote

$$E : \Lambda_{n^+, n^-} \rightarrow \mathbb{R}_{\geq 0},$$

the function

$$E(L) = |L^+| = |L^-|.$$

**Partitions:** We use partitions and give some notations. In this text, a partition  $\nu \in \mathcal{P}(\mathbb{N})^3$  is a sequence  $(\nu(0), \nu(1), \dots)$  of positive integers, with only a finite number of non-zero coefficients ( $\mathcal{P}(\mathbb{N}) = \mathbb{N}^{(\mathbb{N})}$ ). We denote by  $\delta_i$  the partition with a single bloc of size  $i$  and  $+$  the natural law given by:

$$(\nu_1 + \nu_2)(i) = \nu_1(i) + \nu_2(i).$$

We also use notations:

$$n(\nu) = \sum_i \nu(i), \quad d(\nu) = \sum_i \nu(i)i, \quad \text{and} \quad \mu! = \prod_i \mu(i)!.$$

Partitions appear in stratifications of various moduli spaces and also when we consider formal series. Let  $\mathbb{Q}[[\mathbf{t}]]$  the set of formal series in the variables  $\mathbf{t} = (t_0, t_1, \dots)$ . We denote

$$\mathbf{t}^\nu = \prod_i t_i^{\mu(i)}, \quad \text{and} \quad \partial_\nu = \prod_i \frac{\partial^{\nu(i)}}{\partial t_i^{\nu(i)}}.$$

## 3.2 Surfaces and directed surfaces

**Topological surfaces:** In this text, structures we consider live on a compact, oriented, topological surfaces  $M$ , with boundary and possible marked points. We mostly work in the category of topological surfaces but according to standard results on surfaces, each topological surface admits also a unique smooth structure<sup>4</sup>. By abuse of notations we denote  $\partial M$  the set of boundary components  $\pi_0(\partial M)$ . It follow from the assumption that each boundary component of is homeomorphic to a circle.

<sup>3</sup>We can also generalize this to any set  $A$  instead of  $\mathbb{N}$ .

<sup>4</sup>It can be useful it to define the notion of transversality, to work with foliations or when we use cohomology



Figure 3.1: A pair of pants.

According to fundamental results in topology of surfaces, a connected oriented compact surface is characterized, up to homeomorphism by a triple  $(g, n, m)$ . With  $g$  it's genus,  $n$  is the number of boundary components and  $m$  is the number of marked points. Let

$$d_{g,n}^m = 2g - 2 + n + m,$$

the opposite of the Euler characteristic. We always assume that  $d \geq 0$ , a connected surface of type  $(g, n, m)$  is called stable iff it satisfies  $d_{g,n}^m > 0$ . We denote  $\mathbf{bord}^c$  the category of connected surfaces with boundaries (i.e.,  $n > 0$ ) and morphisms given by homeomorphisms,  $\mathbf{bord}^{5,c}$  is the subcategory of stable surfaces. More generally,  $\mathbf{bord}$  and  $\mathbf{bord}^s$  correspond to the categories of non-connected surfaces such that each connected component is respectively in  $\mathbf{bord}^c$  or  $\mathbf{bord}^{5,c}$ . When surfaces possess marked points, we use the notation  $\mathbf{bord}^\bullet$ ,  $\mathbf{bord}^{5,\bullet,c}$ .... As we see later, a surface of first importance in  $\mathbf{bord}^{5,c}$  is the pair of pants; it corresponds to a sphere with three boundaries, i.e., of type  $(0, 3)$  (see figure 3.1). The disjoint union  $\sqcup$  defines a monoidal structure on  $\mathbf{bord}^\bullet$ . The opposite of the Euler characteristic  $d$  is well defined on  $\mathbf{bord}^\bullet$  and is additive

$$d(M_1 \sqcup M_2) = d(M_1) + d(M_2).$$

If  $M \in \mathbf{bord}^\bullet$  and  $c \in \pi_0(M)$ , we denote  $M(c)$  the corresponding surface and write

$$M = \sqcup_c M(c).$$

Finally, we use sporadically the notation  $\mathbf{top}$ ,  $\mathbf{top}^\bullet$ , ..., for surfaces without boundary. To a surface  $M \in \mathbf{bord}^\bullet$ , we can associate two different surfaces in  $\mathbf{top}^\bullet$ :

- The surface  $M^{cap}$  is obtained by gluing a disc on each boundary component of  $M$ .
- The surface  $M^\bullet$  is obtained by gluing a marked disc on each boundary component of  $M$ .

These constructions define two functors. For  $\beta \in \partial M$  we use the notation  $\beta^\bullet$  for the puncture in  $M^\bullet$  that corresponds to  $\beta$ .

It's sometimes convenient to have an enumeration of the boundary components; it's given by a bijection

$$\sigma : \partial M \longrightarrow \llbracket 1, n \rrbracket.$$

A pair  $(M, \sigma)$  is called a labeled surface, and we denote  $\mathbf{bord}_t$ ,  $\mathbf{bord}_t^\bullet$ , ... the different categories of labeled surfaces.

**Remark 3.1** (orientation of the boundary components). *We choose to orient the boundaries of the surface such that  $(t, n^{ext})$  is a direct basis of tangent space, where  $t$  is a vector to the boundaries and  $n^{ext}$  is the normal vector pointing outside the surface (this can be done by using the smooth structure but also topologically).*

**Mapping class groups:** Groups of first importance in the theory of surfaces are mapping class groups. For  $M \in \mathbf{bord}$ , we denote  $\text{Mod}(M)$  the group of isotopy classes of homeomorphisms of  $M$  that preserve the orientation. An homeomorphism induces an action on the set  $\partial M$  by permutation. We assume that the elements of  $\text{Mod}(M)$  do not permute the boundary components. For all  $g \in \text{Mod}(M)$  and  $\beta \in \partial M$  we have  $g(\beta) = \beta$ . An element of the mapping class group can permute the marked points.

**Directed surfaces:** We give the following definition:

**Definition 3.1.** A directed surface  $M^\circ$  is a pair  $(M, \epsilon)$ , with  $M \in \mathbf{bord}^\bullet$  and

$$\epsilon : \partial M \longrightarrow \{\pm 1\}$$

is a map, non-constant on each connected component of  $M$ .

As before, we use the notation  $\mathbf{bord}^\circ$ ,  $\mathbf{bord}^{\circ,\bullet}$ ... for the different categories of directed surfaces. By assumption, a direction divides the boundary into two non-empty sets,  $\partial^\pm M^\circ$ . The positive boundary components  $\partial^+ M^\circ$  are in some sense the outputs, and the negative boundary components  $\partial^- M^\circ$  are the inputs of the surface. If we denote

$$n^\pm = \#\partial^\pm M^\circ \in \mathbb{N}^*.$$

A connected directed surface is characterized by a triple  $(g, n^+, n^-)$  (resp.  $(g, n^+, n^-, m)$  for surfaces with marked points). In the case of directed surfaces, there are three surfaces of first importance. They are the two pairs of pants  $P_+$  and  $P_-$ , they are respectively of type  $(0, 2, 1)$  and  $(0, 1, 2)$ ; and also the unstable cylinder of type  $(0, 1, 1)$  (see figure 3.2). As before, we can consider labeled directed surfaces by using two bijections

$$\sigma_\pm : \partial^\pm M^\circ \rightarrow \llbracket 1, n^\pm \rrbracket.$$

We denote  $\mathbf{bord}_l^\circ$  as the category of labeled directed surfaces.

**Remark 3.2** (Orientation of the boundary components.). *A direction defines an orientation of the boundary components. We choose to orient the positive boundaries according to the orientation induced by the orientation of the surface and the negative boundaries in the opposite direction.*

**Remark 3.3** (General consideration on oriented / directed objects). *First of all we are a bit struggling with terminology. Sometimes it's coherent to use directed and sometimes oriented (ex: directed surfaces, directed graphs, oriented curves, oriented ribbon graphs). For instance, we cannot use "oriented surfaces" because this leads to a confusion and we cannot use directed foliations because the term oriented is preferred in this context. The second is the fact that an oriented or directed object is defined by a pair  $A^\circ = (A, \epsilon)$ , sometimes some quantity  $E$  related to  $A^\circ$  is independent of  $\epsilon$  and we denote it  $E(A)$  instead of  $E(A^\circ)$ .*

**Boundary lengths:** Let  $M^\circ$  a connected directed surface. We define the convex cone  $\Lambda_{M^\circ}$  associated to  $M^\circ$  by

$$\Lambda_{M^\circ} = T_{M^\circ} \cap \mathbb{R}_{\geq 0}^{\partial M}, \quad \text{and} \quad T_{M^\circ} = \left\{ L \in \mathbb{R}^{\partial M}, \sum_{\beta} \epsilon(\beta) L_\beta = 0 \right\}. \quad (3.2)$$

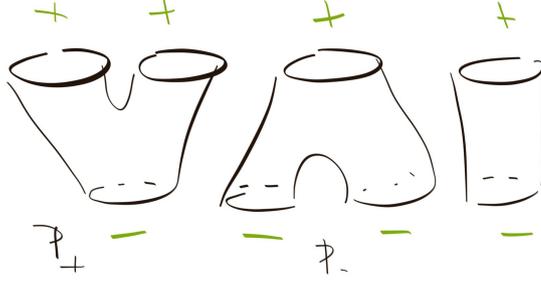


Figure 3.2: The three directed surfaces of type  $(0, 2, 1)$ ,  $(0, 1, 2)$  and  $(0, 1, 1)$ . We use vertical notation, the positive boundary components are at the top.

In general if  $M^\circ = \sqcup_c M^\circ(c)$  we set

$$\Lambda_{M^\circ} = \prod_c \Lambda_{M^\circ(c)}$$

and define  $T_{M^\circ}$  similarly.  $\Lambda_{M^\circ}$  is a convex cone supported by  $T_{M^\circ}$ . In this definition, to each boundary component we associate a positive real number, and we assume that the total weight of the positive boundary is equal to the total weight of the negative boundary. It's in some sense the Kirchhoff law. If the boundary components are labelled and the surface is connected of type  $(g, n^+, n^-)$ ,  $\Lambda_{M^\circ}$  is identified with  $\Lambda_{n^+, n^-}$  defined in formula 3.1.

**Decorations:** We introduce a notation useful to define stratification's of moduli spaces. As in subsection 3.1, we consider partition  $\nu = (\nu(i))_{i \geq 1}$ . A decorated surface  $\overline{M}$  is a pair  $(M, \nu)$  where  $M \in \mathbf{bord}$ ,  $\nu = (\nu_c)_{c \in \pi_0(M)}$  is the data of a partition for each connected component of  $M$ . Moreover, we impose the following constraints:

$$d(\nu_c) = 2d(M(c)) + 2n(\nu_c), \quad \forall c \in \pi_0(M).$$

If  $M^\circ = (M, \epsilon)$  is directed, then  $\overline{M}^\circ = (M^\circ, \nu)$  is a decorated surface if we have

$$d(\nu_c) = d(M^\circ(c)) + n(\nu_c), \quad \forall c \in \pi_0(M^\circ).$$

In the case of marked surfaces, for each marked point  $x$  we assume that we have the additional data of an integer  $\kappa_x$ . In general, we assume  $\kappa_x \geq -1$  and  $\kappa_x \geq 0$  in the directed case. As before, we denote  $\mathbf{bord}$ ,  $\mathbf{bord}^\circ$ , ... the different categories of decorated surfaces.

**Remark 3.4** (Degeneration of decorations ). *It's possible to collapse two blocs of a partition by the transformation  $\nu' = \nu - \delta_i - \delta_j + \delta_{i+j}$ , when  $\nu(i), \nu(j) \geq 1$ . We denote  $\nu_1 < \nu_2$  if it's possible to obtain  $\nu_1$  by a succession of collapsing on  $\nu_2$ . This relation defines a partial order relation on the space of partitions, and then also on decorated surfaces.*

**Subsurfaces:** For  $M$  any oriented compact surface, we denote  $\text{Sub}(M)$  the set of isotopy classes of embedding

$$f : N \rightarrow M$$

with  $N$  an oriented compact surface. We only impose the following constraints:

- If  $\beta \in \partial N$  and  $f(\beta) \cap \partial M \neq \emptyset$  then  $f(\beta) \subset \partial M$ .

- $N$  is stable.

**Remark 3.5.** We remark that we can consider the category where objects are surfaces and morphisms are given by subsurfaces.

$$\text{Hom}(N, M) = \{f : N \rightarrow M\} / \sim .$$

As we see later, functors from this category are interesting. A covariant functor is in some sense compatible with gluings, and a contravariant functor with cuttings.

## 3.3 Curves, stable graphs and arcs on surfaces

### 3.3.1 Curves and multi-curves

**Definitions:** Let  $M \in \mathbf{bord}$ , an essential simple curve is a closed curve that does not self-intersect, is non contractible, non oriented and does not retract on a boundary component of  $M$ . When  $M$  has marked points, we assume that the curve does not pass through these points and does not retract to a marked point. We denote  $\mathcal{S}(M)$  the set of isotopy classes of essential simple curves [FLP21], the isotopies are taken relatively to the marked points if any. If  $M$  has boundary, we also denote

$$\tilde{\mathcal{S}}(M) = \mathcal{S}(M) \sqcup \partial M.$$

The union of  $\mathcal{S}(M)$  and the set of boundary curves.

**Remark 3.6.** For compact surfaces, homotopy and isotopy are equivalent notions. For simple curves,  $\mathcal{S}(M)$  injects into the space  $\mathcal{C}(M)$  of homotopy classes of closed curves [Eps66].

A primitive multi-curve  $\Gamma \in \mathcal{MS}(M)$  is a family of disjoint, pair-wise non-isotopic essential curves (see figure 3.3 ). Multi-curves are also considered up to isotopies. Following the same line, we can consider integral multi-curves  $\Gamma \in \mathcal{MS}_{\mathbb{Z}}(M)$ , which can be represented formally by a sum over  $\mathcal{S}(M)$ ,

$$\Gamma = \sum_{\gamma} m_{\gamma} \gamma.$$

Weights  $(m_{\gamma})$  are positive integers, and two curves in the support of the sum are non-intersecting. This condition implies that the sum is finite; the number of curves in a multi-curve is bounded by  $3g - 3 + n + m$  for a surface of type  $(g, n, m)$ . We also define  $\widetilde{\mathcal{MS}}(M)$  and  $\widetilde{\mathcal{MS}}_{\mathbb{Z}}(M)$  in the same way as  $\tilde{\mathcal{S}}(M)$ .

**Thurston intersection pairing:** W.Thurston introduces the geometric intersection pairing  $\iota$  between isotopy classes of essential curves,[FLP21].

$$\iota : \mathcal{S}(M) \times \mathcal{S}(M) \longrightarrow \mathbb{N}.$$

To construct the pairing, Thurston takes the infimum of the number of intersection points over the set of all representatives of the two curves. This is well defined up to isotopies; Thurston proves that the map

$$\iota : \mathcal{S}(M) \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}^{\mathcal{S}(M)} \setminus \{0\}.$$

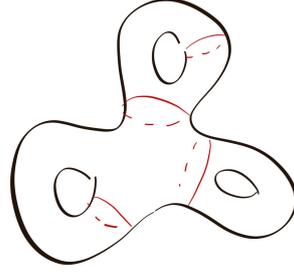


Figure 3.3: Multi-curve on a surface of genus 3.

is injective. Given two non-isotopic curves, there is always a curve that intersects the first and not the second. Thurston endows  $\mathcal{S}(M) \times \mathbb{R}_{>0}$  with the topology given by the product topology on the target. If  $M$  has non empty boundary, this fails to be true; a boundary curve cannot "intersect" a simple curve in the surface. We say that two families of disjoint essential curves,  $\Gamma_1$  and  $\Gamma_2$  are in minimal position if they minimize their intersection, i.e.,

$$\#\Gamma_1 \cap \Gamma_2 = \iota(\Gamma_1, \Gamma_2).$$

The following criterion characterizes this condition: Consider two families of curves that intersect transversally, let  $M_{\Gamma_1 \cup \Gamma_2}$  be the closure of the surface obtained by cutting  $M$  along  $\Gamma_1 \sqcup \Gamma_2$ . The set  $\Gamma_1 \cap \Gamma_2$  defines marked points on the boundary of  $M_{\Gamma_1 \cup \Gamma_2}$ . We call  $n$ -gone a topological disc with  $n$  marked points on the boundary. We can see that, under our assumptions, each boundary component must contain an even number of marked points.

**Lemma 3.2** (Bigon criteriom 1).  $\Gamma_1, \Gamma_2$  are in minimal position iff  $M_{\Gamma_1 \cup \Gamma_2}$  does not contain components that are 2-gones.

### 3.3.2 Stable graphs

**Surgery along multi-curves and stable graphs:** If  $\Gamma$  is a family of disjoint essential curves on  $M$ , it's always possible to cut  $M$  along  $\Gamma$ . We obtain a topological surface with some "open boundaries", it's possible to compactify it, and we denote  $M_\Gamma$  the resulting surface. For a multi-curve, the procedure depends on the choice of the representative, but all the possible surfaces obtained by surgeries are canonically identified up to isotopies. Then for  $\Gamma \in \mathcal{MS}(M)$  we denote  $M_\Gamma$  a possible representative. Surgeries along multi-curves are encoded by stable graphs; we give a possible definition.

**Definition 3.2.** A stable graph is given by  $\mathcal{G} = (X_0\mathcal{G}, (\mathcal{G}(c))_{c \in X_0\mathcal{G}}, s_1)$  where

- $X_0\mathcal{G}$  is the set of vertices.
- $(\mathcal{G}(c))_{c \in X_0\mathcal{G}}$  is a family of topologically stable surfaces with boundary.
- If  $X\mathcal{G} = \sqcup_c \pi_0(\partial\mathcal{G}(c))$  the set of boundaries of the surfaces, then  $s_1 : X\mathcal{G} \rightarrow X\mathcal{G}$  is an involution.

If  $X\mathcal{G}(c) = \partial\mathcal{G}(c)$ , then a stable graph defines a graph  $(X_0\mathcal{G}, X\mathcal{G}, (X\mathcal{G}(c))_{c \in X_0\mathcal{G}}, s_1)$ . But it also contains information about the topology of the "vertices", it is encoded in a genus map

$g : X_0\mathcal{G} \rightarrow \mathbb{N}$  with the stability condition

$$2g(c) - 2 + \#\partial X\mathcal{G}(c) > 0, \quad \forall c \in X_0\mathcal{G}.$$

If we only assume that the  $d(\mathcal{G}(c)) \geq 0$ , we call it a pre-stable graph. In this case, we allow components to be homeomorphic to cylinders. We say that two graphs are isomorphic if there is a bijection  $\phi : X\mathcal{G} \rightarrow X\mathcal{G}$ , which is an isomorphism of graphs and preserves the genus map. The involution divides  $X_1\mathcal{G}$  into two subsets:  $X^{int}\mathcal{G}$  is the set of elements of order two;  $\partial\mathcal{G}$  are the fixed points. We denote by  $\text{Aut}(\mathcal{G})$  the automorphism group of the stable graph, as before we assume that all the elements of  $\text{Aut}(\mathcal{G})$  act trivially on the set  $\partial\mathcal{G}$ .

A stable graph  $\mathcal{G}$  encodes how to glue a family of stable surfaces together.  $\mathcal{G}$  defines a surface  $M_{\mathcal{G}}$  by gluing the pairs of boundary components in  $X^{int}\mathcal{G}$  that are in the same orbit of  $s_1$ . There is an identification  $\partial M_{\mathcal{G}} = \partial\mathcal{G}$  and the image of an element  $\gamma \in X_1^{int}\mathcal{G}$  defines an essential curve  $\gamma \in \mathcal{S}(M_{\mathcal{G}})$ . The union of all these curves is a primitive multi-curve  $\Gamma_{\mathcal{G}}$  on  $M_{\mathcal{G}}$ , the connected components are stable, and two distinct curves are non-isotopic<sup>5</sup>. Conversely, if  $M$  is a topological surface, each multi-curve  $\Gamma \in \mathcal{MS}(M)$  defines a stable graph  $\mathcal{G}_{\Gamma}$ . It's defined by  $X_0\mathcal{G}_{\Gamma} = \pi_0(M_{\Gamma})$  and for each  $c \in \pi_0(M_{\Gamma})$ ,  $\mathcal{G}_{\Gamma}(c)$  is the connected component  $M_{\Gamma}(c)$ . The involution exchanges two boundary components glued along the same curve in  $\Gamma$ , and fixes the boundaries of  $M$ . The mapping class group acts on the set  $\mathcal{MS}(M)$ ; moreover, we can see that two multi-curves are isomorphic iff their stable graphs are isomorphic. Then we have the identification:

$$\text{st}(M) = \mathcal{MS}(M)/\text{Mod}(M).$$

Here,  $\text{st}(M)$  is the set of stable graphs  $\mathcal{G}$  such that  $M_{\mathcal{G}} \simeq M$ , and with a bijection  $\partial\mathcal{G} \simeq \partial M$ .  $\text{st}(M)$  is defined by "finite" combinatorial data, then it's a finite set. There are only a finite number of orbits of primitive multi-curves. Two multi-curves are in the same orbit iff they have the same topology, which is exactly the information contained in a stable graph.

For each multi-curve  $\Gamma \in \mathcal{MS}(M)$ , we can consider  $\text{Stab}(\Gamma) \subset \text{Mod}(M)$ , the stabilizer of  $\Gamma$ . For each  $\gamma \in \mathcal{S}(M)$ , we denote  $\delta_{\gamma} \in \text{Mod}(M)$ , the dehn twist along  $\gamma$ . If  $D_{\Gamma} = \langle \delta_{\gamma}, \gamma \in \Gamma \rangle$ , this is an abelian normal subgroup of  $\text{Stab}(\Gamma)$ . Moreover, we have the following exact sequence of surgery:

$$\{0\} \rightarrow \text{Mod}(M_{\Gamma}) \rightarrow \text{Stab}(\Gamma)/D_{\Gamma} \rightarrow \text{Aut}(\mathcal{G}_{\Gamma}) \rightarrow \{0\}.$$

**Remark 3.7** (Quotient). *As for usual graphs, it's possible to take the quotient of a stable graph. For a set  $E \subset X^{int}\mathcal{G}$  we can define the stable graph  $\mathcal{G}_{\langle E \rangle}$  by gluing the boundary components in  $\tilde{E}$  that are identified by  $s_1$ . This operation is compatible with the quotient of graph.*

**Product along a stable graph:** We introduce a notation; it's not essential but appears at several places in this text. Let  $\mathcal{G}$  be a stable graph and

$$T_{\mathcal{G}} = \{(l_{\beta}) \in \mathbb{R}^{X\mathcal{G}} \mid l_{\beta} = l_{s_1(\beta)}\}.$$

---

<sup>5</sup>The surface  $M_{\mathcal{G}}$  is not canonical; to construct it, we need gluing maps. But two such surfaces are canonically identified, up to the action of Dehn twist along the curves in  $\Gamma_{\mathcal{G}}$



Figure 3.4: A stable graph.

$T_{\mathcal{G}}$  is a linear subspace of  $\mathbb{R}^{X_{\mathcal{G}}}$  that is canonically isomorphic to  $\mathbb{R}^{X_1\mathcal{G}}$ . For each edge  $\gamma \in X_1\mathcal{G}$ , we denote:

$$l_{\gamma} : \Lambda_{\mathcal{G}} \longrightarrow \mathbb{R}^?$$

the projection. For a family of sets  $(E(c))_{c \in X_0\mathcal{G}}$ , with maps  $L_c : E_c \rightarrow \mathbb{R}^{\partial\mathcal{G}(c)}$ . We can consider the fiber product  $\prod_{\mathcal{G}} E_c$  defined by the diagram.

$$\begin{array}{ccc} \prod_{\mathcal{G}} E_c & \longrightarrow & \prod_c E_c \\ \downarrow & & \downarrow \\ \Lambda_{\mathcal{G}} & \longrightarrow & \mathbb{R}^{X_{\mathcal{G}}} \end{array}$$

It corresponds to the elements of the product that equalize the lengths of two elements of  $X_{\mathcal{G}}$  in the same orbit under the involution. We have a projection

$$l_{\gamma} : \prod_{\mathcal{G}} E_c \rightarrow \mathbb{R},$$

by using the projection  $\prod_{\mathcal{G}} E_c \rightarrow T_{\mathcal{G}}$ .

### 3.3.3 Multi-arcs on a surface

Let  $M \in \mathbf{bord}^{\bullet}$  a surface with boundary, we call "essential arc", isotopy classes of unoriented simple arcs, with extremities in  $\partial M$ . Such arcs are assumed to be non trivial, in the sense that they do not retract to a portion of a boundary. A family of simple arcs that are disjoint and pairwise non isotopic defines a multi-arc (see figure 3.5).

If  $A$  is a family of disjoint simple arcs, we can cut  $M$  along  $A$  and obtain a surface  $M_A$  with marked points at the boundary (they correspond to  $A \cap \partial M$ ). As in the case of lemma 3.2, the following result characterizes multi-arcs.

**Lemma 3.3.** *A family of disjoint simple arcs  $A$  defines a multi-arc iff the surface  $M_A$  do not contain 2-gones or 4-gones.*

Similarly to weighted multi-curves, a weighted multi-arc is defined as a formal sum

$$\sum_a m_a a.$$

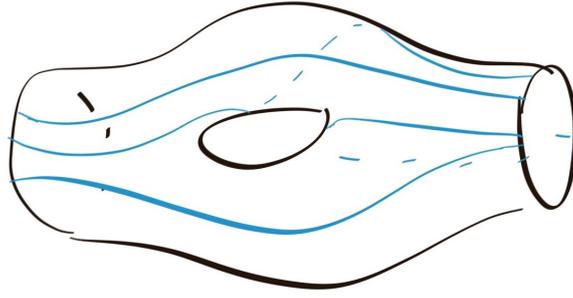


Figure 3.5: Multi-arc on a torus with one boundary.

The arcs in the sum are pairwise distinct and non-intersecting. This implies that the number of arcs is bounded by  $6g - 6 + 3n + 2m$  if the surface is of type  $(g, n, m)$ . We denote  $\mathcal{A}(M)$  the set of arcs,  $\mathcal{MA}(M)$  the set of multi-arcs and  $\mathcal{MA}_{\mathbb{R}}(M)$  the space of weighted multi-arcs. The intersection pairing between a multi-arc and an element of  $\tilde{\mathcal{S}}(M)$  is well defined. It provides a map

$$I : \mathcal{MA}(M) \longrightarrow \mathbb{R}_+^{\tilde{\mathcal{S}}(M)} \setminus \{0\}.$$

A combinatorial argument allow to prove the following result.

**Proposition 3.1.** *The map  $I$  is injective and continuous<sup>6</sup>.*

We give sketch of the proof.

*Proof.* Let  $A \in \mathcal{MA}_{\mathbb{R}}$ , as we see in 7.1, we can associate to an arc  $a$  a “boundary multi-curve”  $\Gamma_a$ . In proposition 6.2, we see that linear combinations of the  $(\iota(\gamma, A))_{\gamma \in \Gamma_a}$  and the intersection with boundary components of  $M$  allow to recover the coefficient  $m_a$ . Then the map is injective. As we have  $\iota(A, \gamma) = \sum_a m_a \iota(a, \gamma)$  this map is continuous.  $\square$

**Remark 3.8** (Bigon criterion 2). *As in the case of two multi-curves. By a doubling argument, we can see that : a multi-arc and a multi-curve are in minimal position iff  $M_{A \cup \Gamma}$  does not contain 2-gones.*

## 3.4 Foliation and quadratic differentials

### 3.4.1 Case of surfaces with empty boundary

**Measured foliations:** Let  $M$  be a compact, oriented surface (without boundaries), a measured foliation  $\lambda$  on  $M$  is a foliation with a transverse measure. It can be defined by an atlas of charts  $\{(U_i, \phi_i), i\}$  of  $M \setminus Z$ , where  $Z$  is a finite set of singularities of the foliation. We assume that the transition functions are of the form

$$\phi_i \circ \phi_j^{-1}(x, y) = (f_{i,j}(x, y), \pm y + c_{i,j}). \quad (3.3)$$

<sup>6</sup>As we see later, the space  $\mathcal{MA}_{\mathbb{R}}(M)$  is a cell complex and can be endowed with the topology given by this structure.

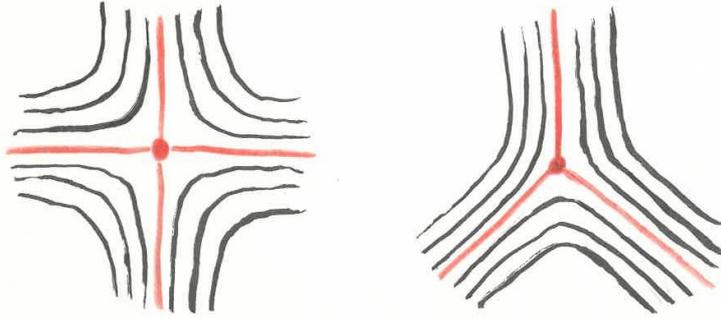


Figure 3.6: Singularities of orders 2 and 1.

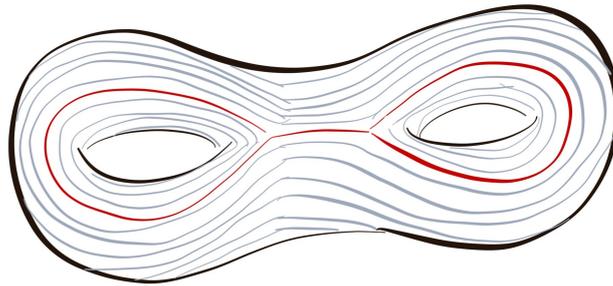


Figure 3.7: Foliation with 4 singularities of order 1 (2 on this side 2 on the other).

Then they preserve the foliation with horizontal lines. The transverse measure of an arc  $\alpha : [0, 1] \rightarrow M \setminus \Sigma$  is defined locally by the absolute variation of the function  $y_i \circ \alpha$  in coordinates. Alternatively, a measured foliation can be defined locally by an exact one form  $dy_i$  on each chart; at the overlap of two charts, we have  $dy_i = \pm dy_j$ . The singularity of the foliation at the neighborhood of a point  $x \in Z$  is given by the multi-valued one form:

$$\text{Im}(z^{\frac{k}{2}} dz).$$

Such a singularity is pictured in figure 3.6, the number  $k \geq 1$  is then the order of the singularity. A foliation defines an equivalence relation on the surface  $M \setminus Z$ . Two points  $x_1, x_2$  are equivalent iff there is a continuous path that starts at  $x_1$ , ends at  $x_2$  and such that the vertical coordinates  $y$  are locally constant along the path. A leaf of the foliation is then an equivalence class for this relation. Generally, the leaves have chaotic behavior. According to figure 3.7 at a singularity of degree  $k$  there is  $k + 2$  leaves that are "based" at this singularity. Such a leaf is called a singular leaf, and a saddle connection is a leaf that connects two singularities, possibly the same. Another particular kind of leaves are periodic leaves, they are homeomorphic to circles.

The space of measured foliations on  $M$  is too big; they are considered up to equivalences. Two foliations are equivalent if they are related by isotopies and Whitehead moves; these moves are obtained by collapsing a saddle connection that connects two different singularities (see [FLP21]). We denote  $\mathcal{MF}(M)$  the set of equivalence classes of measured foliations on  $M$ . Thurston characterizes these relations by using intersection pairing. He generalizes the intersection pairing between a measured foliation  $\lambda \in \mathcal{MF}(M)$  and a simple curve  $\gamma \in \mathcal{S}(M)$

and shows that the map

$$\begin{aligned} I : \mathcal{MF}(M) &\longrightarrow (\mathbb{R}_{\geq 0})^{\mathcal{S}(M)} \\ \lambda &\longrightarrow (\iota(\lambda, \gamma))_{\gamma \in \mathcal{S}(M)} \end{aligned}$$

is well defined and injective. Indeed, two isotopy classes of foliations are Whitehead equivalent iff they have the same intersection pairing. The topology on the space of measured foliations is then the topology induced by intersection pairings with essential curves. A famous result of Thurston on measured foliations is the following:

**Theorem 3.1.**  *$\mathcal{MF}(M)$  is identified with the closure of  $\mathcal{S}(M) \times \mathbb{R}_{>0}$  in  $\mathbb{R}_{\geq 0}^{\mathcal{S}(M)} \setminus \{0\}$  for the product topology.*

**Remark 3.9** (marked points and foliations). *When the surface has marked points, we allow marked singularities of order  $k \geq -1$  at these points. In this case, we assume that isotopies are relative to the marked points, and Whitehead moves between two marked singularities are not allowed. The results of Thurston remain true under these assumptions.*

**Quadratic differentials:** If  $M$  is a compact topological surface with empty boundary, a structure of Riemann surface is an atlas of charts with values in  $\mathbb{C}$  and such transition functions are holomorphic maps. A quadratic differential  $q$  is then a section of the lined bundle  $K_X^{\otimes 2}$  over  $X$ . In other words,  $q$  is locally given by

$$q = f(z)(dz)^2,$$

in local coordinates, with  $f$  a holomorphic function. A quadratic differential might have zeros or poles; indeed, if we denote  $k_x$  the order of the zeros at  $x$  and  $M$  is of genus  $g$ , we must have

$$\sum_x k_x = 4g - 4.$$

A quadratic differential also defines the structure of a half-translation surface. Outside the set of zeros, it's possible to find coordinates in which the differential is given by  $(dz)^2$ . The change of flat coordinates is of the form  $z \rightarrow \pm z + c$ , where  $c \in \mathbb{C}$ . Using this, a quadratic differential  $q$  defines two foliations: the vertical and horizontal foliations. They are given by

$$\lambda_v(q) = \operatorname{Re}\sqrt{q} \quad \text{and} \quad \lambda_h(q) = \operatorname{Im}\sqrt{q}.$$

Where  $\sqrt{q}$  is a local square root of  $q$ , which is well defined up to a sign and is closed. We denote  $\mathcal{QT}(M)$  the Teichmüller space of quadratic differentials. The Teichmüller space is the space of triples  $(\phi, X, q)$  where:

- $X$  is a Riemann surface,
- $\phi : M \rightarrow X$  is a homeomorphism,
- $q$  is a holomorphic quadratic differential on  $X$ .

Two triples are equivalent if there is  $h : X' \rightarrow X$  biholomorphic such that  $h^*q = q'$  and  $\phi^{-1} \circ h \circ \phi'$  is isotopic to the identity. A quadratic differential  $q$  defines a metric and then a volume form

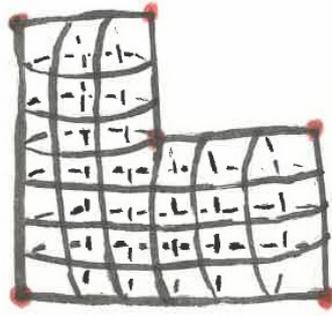


Figure 3.8: A quadratic differential (pillow case cover).

given by  $\operatorname{Re}\sqrt{q} \wedge \operatorname{Im}\sqrt{q}$ . It's defined without sign ambiguities. The area  $A(X, q)$  of  $(X, q)$  is then given by:

$$A(X, q) = \int_X \operatorname{Re}\sqrt{q} \wedge \operatorname{Im}\sqrt{q}.$$

We give more results on quadratic differentials later in the text. Let us mention that the spaces of the quadratic differentials are stratified. For each  $q$  we denote  $\nu_q$  the partition given by

$$\nu_q(i) = \#\{x, k_x = i\}, \quad \forall i > 0$$

the number of zeros of order  $i$ . If  $\overline{M} = (M, \nu)$ , we denote  $\mathcal{QT}(\overline{M})$  as the subset of quadratic differentials  $q \in \mathcal{QT}(M)$  with  $\nu_q = \nu$ .

**Remark 3.10** (Marked point's). *When  $M$  has marked points, we consider quadratic differentials that are holomorphic outside the set of marked points and with  $k_x \geq -1$  for all marked points, i.e., sections of  $K_X^{\otimes 2} \otimes \mathcal{O}(\sum_x x)$ , where we sum over the marked points.*

**Pair of transverse foliations:** Two measurable foliations  $\lambda_1, \lambda_2$  on a surface without boundaries are transverse iff for all essential simple curve  $\gamma$  they satisfy

$$\iota(\lambda_1, \gamma) + \iota(\lambda_2, \gamma) > 0.$$

A quadratic differential  $q$  defines a pair of foliations  $(\lambda_v(q), \lambda_h(q))$ , and it is possible to see that these two foliations are necessarily transverse [HM79]. In fact, the converse statement is true and is part of the Hubbard-Masur theorem.

**Theorem 3.2** (Hubbard-Masur [HM79]). *If  $M$  is compact and stable, the map  $q \rightarrow (\lambda_v(q), \lambda_h(q))$  induces a homeomorphism:*

$$\mathcal{QT}(M) \rightarrow \mathcal{MF}(M)^2 \setminus \Delta.$$

Where  $\Delta$  is the set of pairs of non-transverse foliations.

Moreover, we also have the following relation:

$$A(q) = \iota(\lambda_v(q), \lambda_h(q)).$$

As we will see later, the Hubbard-Masur theorem has many interesting consequences.

**Jenkin-Strebel differentials:** A leaf of a foliation is periodic if it is closed and does not contain singularities. Such a leaf cannot be isolated and is contained in a maximal cylinder foliated by parallel periodic leaves. A cylinder is a map.

$$\varphi : \mathbb{R}/\mathbb{Z} \times ]0, h[ \rightarrow M.$$

Such that in this chart, the foliation is given by the one form  $dy$ . The real  $h$  is its height. Two cylinders are equivalent if there is a homotopy that preserves the leaves of the foliation and there is an obvious partial order relation on cylinders; moreover, every cylinder is contained in a unique maximal cylinder. A cylinder admits a unique isotopy class of essential curves, which is its core curve. If  $\lambda$  is a foliation and  $C$  is a maximal cylinder, the core curve  $\gamma_C$  of  $C$  is the image of the core curve of the cylinder in  $\mathcal{S}(M)$ . The height of  $C$  is then denoted  $h_C$ . It's possible to see that the core curves are necessarily non-contractible; moreover, if two maximal cylinders are distinct, then the core curves are non-isotopic. The multi-curve associated with  $\lambda$  is

$$\Gamma(\lambda) = \sum_C h_C \gamma_C,$$

where we sum over all the maximal cylinders. A foliation is periodic if all the leaves are either periodic or are saddle connections; in this case, the closure of the union of all the maximal cylinders covers the surface. Outside the cylinders, the leaves are saddle connections. In this case, for all  $\gamma \in \mathcal{S}(M)$ , we have the equality

$$I(\Gamma(\lambda)) = I(\lambda).$$

Conversely, for each multi-curve  $\Gamma \in \mathcal{MS}_{\mathbb{R}}(M)$ , it's possible to associate a periodic foliation  $\lambda(\Gamma)$  that is unique up to Whitehead moves. Then intersection pairing allows us to identify periodic foliations with the space of weighted multi-curves  $\mathcal{MS}_{\mathbb{R}}(M)$ .

In this text, a Jenkin-Strebel differential is a quadratic differential with a periodic horizontal foliation (see [Str84]). Then, by [HM79], the Hubbard-Masur map identifies the subspace of Jenkin-Strebel differentials with

$$\mathcal{MF}(M) \times \mathcal{MS}_{\mathbb{R}}(M) \setminus \Delta.$$

If  $\Gamma \in \mathcal{MS}(M)$  is a primitive multi-curve, we denote  $\mathcal{MF}_{\Gamma}(M)$  the set of foliations transverse to  $\Gamma$ . The Hubbard-Masur map defines a map.

$$q_{\Gamma} : \mathcal{MF}_{\Gamma}(M) \longrightarrow \mathcal{QT}(M).$$

The quadratic differential  $q_{\Gamma}(\lambda)$  is Jenkin-Strebel and

$$\lambda_h(q_{\Gamma}(\lambda)) = \lambda(\Gamma), \quad \text{and} \quad \lambda_v(q_{\Gamma}(\lambda)) = \lambda.$$

The image of  $q_{\Gamma}$  is the space

$$\mathcal{QT}_{\Gamma}(M) = \{q \in \mathcal{QT}_0(M) \mid \lambda_h(q) = \lambda(\Gamma)\}.$$

### 3.4.2 Foliations and quadratic differentials with double poles

**Foliations with boundaries and double poles:** Let  $M \in \mathbf{bord}$ ; we need to consider measured foliation on  $M$ . A possible way is to take foliations on  $M$  such that, for all boundary  $\beta$ , all the leaves of the foliation that cross  $\beta$  are either transverse to  $\beta$ , or  $\beta$  is a non-singular periodic leaf of the foliation. As before, we consider measured foliation up to isotopies and Whitehead moves; moreover, we do not assume that isotopies are relative to the boundaries. We denote  $\widetilde{\mathcal{MF}}(M)$  the space of these foliations.

Nevertheless, this is not completely satisfactory; it's somewhat better to consider foliations with poles rather than foliations with boundaries. We introduce the space  $\mathcal{MF}(M)$  of measured foliations on the punctured surface  $M^\bullet$ , with singularities of order  $-2$  (double poles) at punctures that correspond to boundary components of  $M$ . It means that each  $\beta^\bullet$  admits a neighborhood  $U_\beta$  that is isomorphic to a disc with a local chart:

$$U_\beta \longrightarrow \mathbb{R}/l_\beta\mathbb{Z} \times \mathbb{R}_{\geq 0}.$$

The leaves of the foliation are either the vertical or the horizontal lines in the half-infinite cylinder  $\mathbb{R}_+ \times \mathbb{R}/l_\beta\mathbb{Z}$ . When the leaves are horizontal, the value of  $l_\beta$  is not relevant, and we take  $l_\beta = 1$ . Otherwise, the number  $l_\beta$  corresponds to the absolute value of the residue at the double pole  $\beta^\bullet$ . As below, these foliations are considered up to isotopies and Whitehead moves. When the surface  $M$  has marked points, we allow simple poles, regular points, or zeros at these marked points. It is possible to extend the Thurston intersection pairing in this case.

$$\iota : \widetilde{\mathcal{S}}(M) \times \mathcal{MF}(M) \longrightarrow \mathbb{R}_{\geq 0}.$$

The absolute value of the residue of an element in  $\mathcal{MF}(M)$  at a pole  $\beta^\bullet$  defines a map  $l_\beta : \mathcal{MF}(M) \longrightarrow \mathbb{R}_{\geq 0}$ <sup>7</sup>, and we denote  $L_\partial$

$$L_\partial : \mathcal{MF}(M) \longrightarrow \mathbb{R}_{\geq 0}^{\partial M}.$$

With  $L_\partial(\lambda) = (l_\beta(\lambda))_\beta$ .

An element of  $\widetilde{\mathcal{MF}}(M)$  can be extended uniquely to a foliation on  $M^\bullet$ , and at the punctures the foliation has a double pole. Then there is a surjective, but not injective, map:

$$\widetilde{\mathcal{MF}}(M) \longrightarrow \mathcal{MF}(M).$$

If we restrict the map to the subset of foliation with non-vanishing residues, i.e the leaves are transverse to the boundary, then the map is injective.

**Multi-arcs and foliations:** As for multi-curves, a weighted multi-arc defines a partial foliation of the surface which can be extended by the Thurston enlargement procedure [FLP21]. Then it's possible to construct a unique map that preserves the intersection product:

$$\mathcal{MA}_\mathbb{R}(M) \longrightarrow \mathcal{MF}(M).$$

---

<sup>7</sup>It's the intersection pairing with the circle  $\beta$  around this pole (which corresponds to a boundary of  $M$ )

Moreover, this map is continuous. We give this map explicitly in the case of filling multi-arcs later in the text. A fact that is important in the sequel is that the converse map exists. A foliation on a surface with boundary defines in a natural way a weighted multi-arc by considering the leaves that connect two poles. This procedure is the exploration of the surface from the boundary.

**Proposition 3.2.** *There is a continuous map:*

$$A : \mathcal{MF}(M) \longrightarrow \mathcal{MA}_{\mathbb{R}}(M) \cup \{0\}.$$

*Which coincides with the identity on  $\mathcal{MA}_{\mathbb{R}}(M)$ .*

*Proof.* For the construction of the map, consider for each pole with non-vanishing residue a circle around it. Assume that these circles are pairwise disjoint, and each of them bounds a disc that contains a double pole and no other singularity. By using local coordinates around the pole, we can assume that the foliation intersects transversely these circles. Each circle defines a contractible neighborhood  $U_{\beta}$  of a pole  $\beta^{\bullet}$  and if a leaf enters such a neighborhood, it cannot escape. Then the intersection of the singular leaves of the foliation and the circles  $C_{\beta}$  is a finite set  $X_0\lambda$  and each circle  $C_{\beta}$  is divided into a finite number of intervals. We denote  $X\lambda$  the set of intervals. If  $x \in C_{\beta}$  is a point in one of these intervals, it's possible to consider the half leaf starting at  $x$  in the direction opposite to  $U_{\beta}$ . By assumption, this leaf does not hit any singularities. By the Thurston recurrence lemma [FLP21] such a leaf must intersect another circle  $C_{\beta'}$  at a point  $T(x)$ . The map  $T$  is well defined on the union of the intervals and induces a map:

$$s_1 : X\lambda \longrightarrow X\lambda,$$

such that  $T$  maps  $I$  to  $s_1(I)$ . The map  $s_1$  is an involution. A leaf of the foliation that joins  $I$  and  $s_1(I)$  defines an arc  $a_I$  on the surface. By using the bigon criterium (remark 3.8) it's possible to see that two arcs are necessarily non-homotopic. And then the foliation defines a multi-arc. Moreover, the transverse measure on  $\lambda$  induces a measure on each interval  $I$ , and the total mass gives a weight  $m_I(\lambda)$ . The map  $T$  preserves these measures and then  $m_I(\lambda)$  defines weight on the arcs  $a_I$ , and this construction gives the desired map. The continuity is due to the fact that the weights on the edges can be computed by using intersection pairings with appropriate curves.  $\square$

**Topology on  $\mathcal{MF}(M)$ :** Let  $M$  be a surface with boundaries. We can make the statement of proposition 3.2 more precise and describe the entire foliation. From this, it's possible to derive the following theorem, which is similar to the results of Thurston.

**Theorem 3.3.** *Let  $M$  a surface with boundary; the map:*

$$I : \mathcal{MF}(M) \longrightarrow \mathbb{R}_{\geq 0}^{\tilde{\mathcal{S}}(M)},$$

*is injective. Moreover, the space  $\mathcal{MF}(M)$  is the closure of  $\mathcal{MA}_{\mathbb{R}}(M)$  for the topology given by the intersection pairing.*

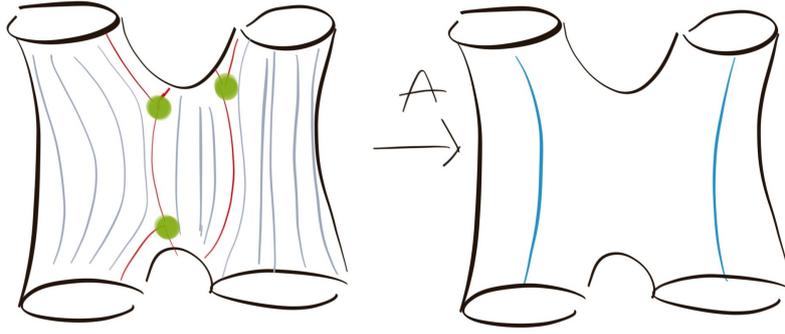


Figure 3.9: Illustration of the map  $A$ .

*Proof.* We already know the injectivity for weighted multi-arcs. In general, we need a finer result than the proposition 3.2 (the precise statement is a bit complicated, and we chose not to include it in these notes). We need to study the foliation outside the surface filled by the multi-arc. Indeed, the multi-arc  $A_\lambda$  spans a subsurface delimited by configurations of saddle connections. On the complement of this subsurface, the restriction of  $\lambda$  defines a possibly trivial foliation  $\lambda^0$  that does not intersect the boundary components. Then, using proposition 3.2 and the theorem 3.1, we can show that the map is injective. To prove the second statement, we need to show that the foliation  $\lambda^0$  can be approximated by arcs, and we can do it by using the theorem 3.1. We just need to take care that these new arcs do not intersect  $A_\lambda$ .  $\square$

**Foliations with vanishing residues:** In this paragraph, we consider  $\mathcal{MF}_0(M)$  the subset of  $\mathcal{MF}(M)$  formed by foliations with vanishing residues.  $\mathcal{MF}_0(M)$  contains a trivial element, which is the trivial Jenkin-Strebel foliation; it is periodic, and all the non-singular trajectories retract to a pole. We denote this foliation as  $\lambda_0$ ; it's unique up to Whitehead moves and isotopies (not completely trivial to prove). A nontrivial foliation contains leaves that are not homotopic to boundary curves. It's then possible to pinch the boundary components and obtain a foliation on the punctured surface  $M^\bullet$ , with no double poles and marked conical singularities on each puncture of  $M^\bullet$ .

**Proposition 3.3.** *The contraction defines a bijection:*

$$C : \mathcal{MF}_0(M) \setminus \{\lambda_0\} \longrightarrow \mathcal{MF}(M^\bullet).$$

*Moreover, the bijection is characterized by:*

$$\iota(\lambda, \gamma) = \iota(C(\lambda), \gamma)$$

*for all curves  $\gamma \in \mathcal{S}(M^\bullet)$ .*

In a similar way, we can consider the space  $\widetilde{\mathcal{MF}}_0(M)$  of foliations in  $\widetilde{\mathcal{MF}}(M)$  with vanishing residues. It corresponds to elements of  $\mathcal{MF}_0(M)$  marked by a choice of non-singular trajectories around each pole. Then we have an identification

$$\widetilde{\mathcal{MF}}_0(M) \simeq \mathcal{MF}_0(M) \times \mathbb{R}_{>0}^{\partial M}.$$

**Quadratic differentials on a surface with boundary:** For a surface  $M \in \mathbf{bord}$ , we consider quadratic differentials on the punctured surface  $M^\bullet$  with double poles at the marked points  $\beta^\bullet$ . As before, we can consider the Teichmüller space  $\mathcal{QT}(M)$ , this does not depend on the choice of  $M^\bullet$ . It's possible to take the residue around a double pole of  $q$  by taking a local square root; the result is an element of  $\mathbb{C}/\{\pm 1\}$ , but we can fix a representative by assuming that the real part is positive (and the imaginary part when the real part is zero). We denote  $\mathcal{QT}_0(M)$  the subspace of quadratic differentials with real residues at the poles. These Teichmüller spaces are also stratified by the order of the zeros, and we denote  $\mathcal{QT}(\overline{M})$ ,  $\mathcal{QT}_0(\overline{M})$  the stratum.

**Pairs of transverse foliations:** Transversality has a straightforward generalization for foliations with double poles, but we need to replace  $\mathcal{S}(M)$  by  $\tilde{\mathcal{S}}(M)$  in the characterization. Which means that we include the boundary curves. As before, a quadratic differential with double poles defines a pair of transverse foliations with poles. We have a map:

$$\mathcal{QT}(M) \rightarrow \mathcal{MF}(M)^2 \setminus \Delta.$$

We do not give the most general statement of the theorem, but only the part that is useful in this text. A doubling argument allows us to prove the following result by using the Hubbard-Masur theorem 3.2.

**Theorem 3.4.** *Let  $M$  a surface with boundary, and let  $\Delta$  be the subspace of pairs of non-transverse foliations in  $\mathcal{MF}(M) \times \mathcal{MF}_0(M)$ . Then the map:*

$$\mathcal{QT}_0(M) \longrightarrow \mathcal{MF}(M) \times \mathcal{MF}_0(M) / \Delta,$$

*is a bijection.*

*Proof.* The map is given by considering the horizontal and vertical foliations of a quadratic differential in  $\mathcal{QT}_0(M)$ . We construct the converse map. Let  $(\lambda_1, \lambda_2)$  be a pair of transverse foliations in  $\mathcal{MF}(M) \times \mathcal{MF}_0(M) / \Delta$ . We can double these foliations to define foliations  $(\lambda_1^{db}, \lambda_2^{db})$  on  $M^{db}$  the surface obtained by doubling  $M$  along it's boundary. In the case of  $\mathcal{MF}_0(M)$ , we add a cylinder of height one on each boundary curve  $\partial M$ . These two foliations are transverse; if  $\gamma$  is a curve, either it's in  $M_{\partial M}^{db}$  and then we use the fact that  $(\lambda_1, \lambda_2)$  are transverse, or  $\gamma$  intersects  $\partial M$  and then  $\iota(\gamma, \lambda_2^{db}) > 0$ . When we apply theorem 3.2 to the pair  $(\lambda_1^{db}, \lambda_2^{db})$ , we obtain a quadratic differential  $q^{db}$ . By construction, it has a horizontal cylinder of height one on each curve in  $\partial M$ . We can cut this differential along these cylinders and add a half-infinite cylinder on each side created by these cuttings. We obtain a quadratic differential on each connected component of  $M_{\partial M}^{db}$ . The uniqueness in the Hubbard-Masur theorem says that the two components are interchanged by the involution, and then we have a unique quadratic differential  $q$  on  $M$ . By computing the intersection pairing with a simple closed curve, we see that the horizontal and vertical foliations of  $q$  are  $(\lambda_1, \lambda_2)$ .  $\square$

**Periodic foliations:** We can generalize the notion of a cylinder in a straightforward way. There is only one difference: a cylinder can have an infinite height. It happens for a foliation that has a pole with vanishing residue. In this case, there is a maximal neighborhood of the poles of the form  $\mathbb{S}_1 \times \mathbb{R}_{>0}$  in which  $\lambda$  is given by  $dy$ . We can also generalize the notion of

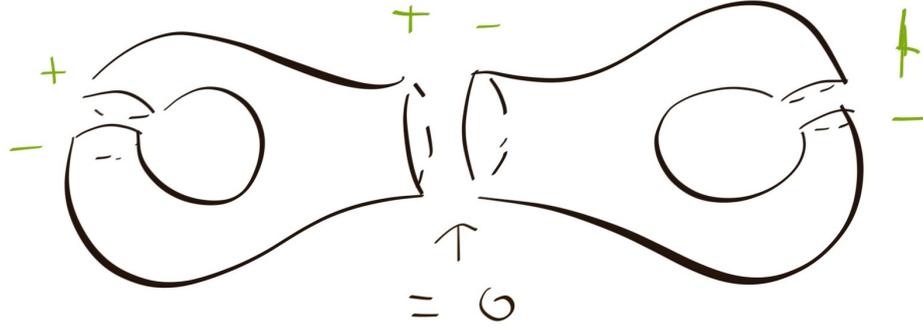


Figure 3.10: Degenerate surface

periodic foliations, but we must add the Jenkin-Strebel foliation given by the empty multi-curve 0. The space of periodic foliations is then identified with

$$(\mathcal{MS}_{\mathbb{R}}(M) \sqcup \{0\}) \times \mathcal{MF}(M) \setminus \Delta.$$

## 3.5 Directed stable graphs

### 3.5.1 Oriented curves, multi-arcs, and foliations

**Oriented multi-curves:** In the context of directed surfaces, it's natural to introduce oriented multi-curves.

**Definition 3.3.** Let  $M^\circ = (M, \epsilon)$ , an oriented multi-curve is a pair  $\Gamma^\circ = (\Gamma, \epsilon_\Gamma)$  such that:

1.  $\Gamma$  is a primitive multi-curve.
2.  $\epsilon_\Gamma$  is a map,  $\epsilon_\Gamma : \partial M_\Gamma \rightarrow \{\pm 1\}$  such that:

$$\epsilon_\Gamma(\beta) = \epsilon(\beta) \quad \forall \beta \in \partial M \quad \text{and} \quad \epsilon_\Gamma(s_1(\beta)) = -\epsilon_\Gamma(\beta) \quad \forall \beta \in X^{int}\Gamma.$$

3. For each component  $M_\Gamma(c)$ , let  $\epsilon_{\Gamma,c}$  be the restriction of  $\epsilon_\Gamma$  to  $\partial M_\Gamma(c)$ , then we assume that  $M_{\Gamma^\circ}^\circ(c) = (M_\Gamma(c), \epsilon_{\Gamma,c})$  is a directed surface.

An oriented multi-curve is non-degenerate iff

4. For each sub multi-curve  $\Gamma' \subset \Gamma$ , if  $\epsilon_{\Gamma'}$  is the restriction of  $\epsilon_\Gamma$  to  $\partial M_{\Gamma'}$ , then  $(\Gamma', \epsilon_{\Gamma'})$  is an oriented multi-curve.

We denote  $\mathcal{MS}(M^\circ)$  the set of non-degenerate oriented multi-curves on  $M^\circ$ .

**Remark 3.11** (Degenerations). The point (4) in the definition is more technical but necessary; otherwise, it can happen that the multi-curve contains a sub-multi-curve  $\Gamma'$  such that the direction is constant on some connected components of  $M_{\Gamma'}$ , and such multi-curves are not realizable in practice (see figure 3.10).

**Oriented multi-arcs:** If  $M^\circ$  is a directed surface, we consider the following definition.

**Definition 3.4.** *A multi-arc  $A$  is oriented iff each arc  $a \in A$  joins a positive and a negative boundary.*

We denote  $\mathcal{MA}(M^\circ)$  (resp  $\mathcal{MA}_\mathbb{R}(M^\circ)$ ) the subset  $\mathcal{MA}(M)$  (resp  $\mathcal{MA}_\mathbb{R}(M)$ ) of oriented multi-arcs (resp oriented weighted multi-arcs).

**Oriented foliations and Abelian differentials:** A measured foliation  $\lambda$  is orientable if it can be globally defined by a closed one form. It's equivalent to the fact that the changes of local charts take the form

$$(x, y) \longrightarrow (f(x, y), y + c).$$

When the surface is connected, there is at most two possible orientations, and an oriented foliation will be denoted by  $\lambda^\circ$ . Oriented foliations are naturally related to Abelian differentials. An Abelian differential is the data of a structure of Riemann surface  $X$ , with a holomorphic one form  $\alpha$  on  $X$  (i.e., a section of  $K_X$ ).  $\alpha$  might have zeros, and in this case, we have:

$$\sum_x k_x = 2g - 2.$$

The square of an Abelian differential is a quadratic differential. According to this, an Abelian differential defines a pair of transverse measured foliations. In this case, outside the set of zeros, we can find local coordinates in which  $\alpha$  is of the form  $dz$ . Moreover, the transition functions are translations; there is no sign ambiguities. Then the two foliations  $\lambda_h(\alpha^{\otimes 2}), \lambda_v(\alpha^{\otimes 2})$  are naturally oriented, and we denote them  $\lambda_h^\circ(\alpha), \lambda_v^\circ(\alpha)$ . As before, we can consider the Teichmüller  $\mathcal{HT}(M)$  of Abelian differentials on  $M$ ; this space is naturally stratified. If  $\nu_\alpha(i)$  is the number of zeros of order  $i$ , we denote  $\mathcal{HT}(M, \nu)$  the stratum of Abelian differentials with  $\nu_\alpha(i) = \nu(i)$ .

**Remark 3.12.** *If  $M$  has marked points, we assume that these points correspond to marked regular points or marked zeros of the Abelian differentials.*

For a surface with boundary, the definition is similar. We consider oriented foliation  $\lambda^\circ$  with double poles. In this case, when we compute the residues, there are no sign ambiguities. The residues define real numbers by taking the period of the close one form. The signs of these numbers define a sign for each boundary component. If the foliation is defined by a closed one form, then, by virtue of the Stokes theorem, the sum of the residues is zero. If the residues are non-vanishing, there is at least one positive and one negative boundary components. According to this, we see that an oriented foliation with non-vanishing residues on a surface  $M$  defines a direction on this surface, according to paragraph 3.2. Then, for each directed surface  $M^\circ$  we can consider the space  $\mathcal{MF}(M^\circ)$  of oriented foliations that give a direction compatible with  $M^\circ$ . Moreover, by Stokes theorem, the sum of the residues of the positive boundaries is equal to the sum of the residues of the negative boundaries. And then the map  $L_\partial$  induces a map:

$$L_\partial : \mathcal{MF}(M^\circ) \longrightarrow \Lambda_{M^\circ}.$$

There is an equivalent version of the theorem 3.2 in this case.

**Proposition 3.4.** *Let  $M^\circ$  be a directed surface. The restriction of  $A : \mathcal{MF}(M) \longrightarrow \mathcal{MA}_\mathbb{R}(M)$  induces a map:*

$$A : \mathcal{MF}(M^\circ) \longrightarrow \mathcal{MA}_\mathbb{R}(M^\circ).$$

The second is:

**Proposition 3.5.** *Let  $M^\circ$  be a directed surface, then each element  $A \in \mathcal{MA}_\mathbb{R}(M^\circ)$  defines an oriented foliation by enlarging procedure, there is a unique map:*

$$\mathcal{MA}_\mathbb{R}(M^\circ) \longrightarrow \mathcal{MF}(M^\circ).$$

*That preserves the intersection pairings.*

**Cylinders of an oriented foliation:** If  $M^\circ$  is a directed surface and  $\lambda^\circ$  is an oriented foliation, it's possible to orient the core curve of a cylinder in a natural way. Indeed, for each cylinder, we can find a map:

$$\varphi : \mathbb{R}/\mathbb{Z} \times ]0, h[ \rightarrow M,$$

such that the foliation is given by  $dy$  in these coordinates. We chose to label the upper boundary of the cylinder by  $+$  and the lower boundary by  $-$ . This defines the orientation of the curves. Similarly to the general case, we can associate with  $\lambda$  a multi-curve  $\Gamma(\lambda)$ , moreover, the orientation  $\lambda^\circ$  induces an orientation  $\Gamma^\circ(\lambda^\circ)$  on  $\Gamma$ . Then, by the Stokes theorem, we can prove the following:

**Lemma 3.4.**

- *If  $\lambda^\circ \in \mathcal{MF}(M^\circ)$ , the multi-curve  $\Gamma^\circ(\lambda^\circ) \in \mathcal{MS}_\mathbb{R}(M^\circ)$  is non-degenerate.*
- *For each non-degenerate multi-curve  $\Gamma^\circ \in \mathcal{MS}_\mathbb{R}(M^\circ)$ , there is an oriented foliation  $\lambda^\circ = \lambda^\circ(\Gamma^\circ) \in \mathcal{MF}(M^\circ)$  such that*

$$\Gamma^\circ = \Gamma^\circ(\lambda^\circ),$$

*and this foliation is unique up to Whitehead moves.*

### 3.5.2 Directed stable graphs:

The topology of an oriented multi-curve is encoded by a directed stable graph; we just translate the definition into the language of stable graphs.

**Definition 3.5.** *A directed, stable graph  $\mathcal{G}^\circ$  is a pair  $(\mathcal{G}, \epsilon)$  such that:*

1.  *$\mathcal{G}$  is a stable graph, and  $\epsilon : X\mathcal{G} \rightarrow \{\pm 1\}$  is a map.*
2. *For each  $c \in X_0\mathcal{G}$ , the restriction  $\epsilon^c$  of  $\epsilon$  to  $X\mathcal{G}(c)$  defines a directed surface  $\mathcal{G}^\circ(c)$ .*
3. *The involution reverses the orientation of the elements of  $X^{int}\mathcal{G}$ .*

*A directed stable graph is non-degenerate iff*

4. *For all  $E \in X_1^{int}\mathcal{G}$ , the direction restricted to  $\mathcal{G}_{\langle E \rangle}$  defines a directed stable graph.*

In other words, all the components are directed surfaces, and two boundary components glued along a curve have opposite signs. An important point is that the edges and the half edges of a directed stable graph are directed. We assume that the edges are directed from the  $+$  to the  $-$  (see figure 3.11). Then a directed stable graph defines a directed graph by forgetting the topology of the components.

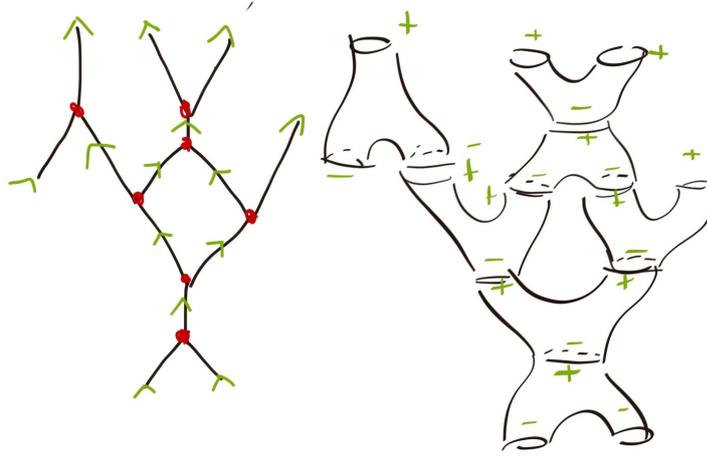


Figure 3.11: An acyclic directed pants decomposition of a directed surface of type  $(1, 4, 3)$ .

As in the case of stable graphs, a non-degenerate directed stable graph defines a directed surface  $M_{\mathcal{G}^\circ}^\circ$  and an oriented primitive multi-curve  $\Gamma_{\mathcal{G}^\circ}^\circ$  on  $M_{\mathcal{G}^\circ}^\circ$ . Moreover, it's straightforward to see that this curve is non-degenerate iff the stable graphs are non-degenerate. If  $M^\circ$  is a fixed directed surface, we denote  $\text{st}(M^\circ)$  the subset of non-degenerate directed stable graphs up to homeomorphisms. Then, as in the usual case, we have the identification:

$$\text{st}(M^\circ) \simeq \mathcal{MS}(M^\circ)/\text{Mod}(M).$$

The group of automorphism's  $\text{Aut}(\mathcal{G}^\circ)$  of a directed stable graph  $\mathcal{G}^\circ$  is then the subgroup of  $\text{Aut}(\mathcal{G})$  that preserves the direction.

**Cone of relative cycles:** If  $\mathcal{G}^\circ = (\mathcal{G}, \epsilon)$  is a directed stable graph, we can construct the cone  $\Lambda_{\mathcal{G}^\circ}$  of directed cycles on  $\mathcal{G}^\circ$ ,

$$\Lambda_{\mathcal{G}^\circ} = \prod_{\mathcal{G}} \Lambda_{\mathcal{G}^\circ(c)} = \left\{ L \in \prod_c \Lambda_{\mathcal{G}^\circ(c)}, L_\beta = L_{s_1(\beta)} = \quad \forall \beta \in X\mathcal{G} \right\}. \quad (3.4)$$

This definition means that at each node of the graph, we impose the Kirchoff law on the boundaries of this component. We denote  $\mathring{\Lambda}_{\mathcal{G}^\circ} = \Lambda_{\mathcal{G}^\circ} \cap \mathbb{R}_{>0}^{X_1\mathcal{G}}$  the interior of the cone. As before, for each  $\gamma \in X_1\mathcal{G}$  we can define the length  $l_\gamma$  of  $\gamma$ :

$$l_\gamma : \Lambda_{\mathcal{G}^\circ} \longrightarrow \mathbb{R}_{\geq 0}.$$

If  $\mathcal{G}^\circ \in \text{st}(M^\circ)$ , we can also consider the map  $L_\partial = (l_\beta)_{\beta \in \partial M}$ .

$$L_\partial : \Lambda_{\mathcal{G}^\circ} \longrightarrow \Lambda_{M^\circ}.$$

For all  $L \in \Lambda_{M^\circ}$  we denote  $\Lambda_{\mathcal{G}^\circ}(L)$  the level set  $L_\partial^{-1}(\{L\})$ .

A trail in a directed graph is a directed path (see [Bol12]); it is a sequence of positive half edges  $(\beta_1, \dots, \beta_r)$  with  $[s_1\beta_k]_0 = [\beta_{k+1}]_0$  for all  $k \in \{1, \dots, r-1\}$ . We call an absolute cycle a closed trail, and a relative cycle is a trail that goes from a negative boundary to a positive boundary. An absolute or relative cycle  $c$  defines an element  $[c]$  of  $\Lambda_{\mathcal{G}^\circ}$ , which motivates the terminology. We call a primitive cycle, a cycle that can't be written as a union of two distinct cycles.

**Proposition 3.6.** *Each element of  $\Lambda_{\mathcal{G}^\circ}$  is a linear combination of primitive cycles with positive coefficients.*

A different way to write this proposition is the following: the primitive cycles are the extremal rays of  $\Lambda_{\mathcal{G}^\circ}$ .

*Proof.* If  $u$  is an element of  $\Lambda_{\mathcal{G}^\circ}$  and  $\gamma$  is an edge in the support of  $u$ , i.e.,  $u_\gamma > 0$ . The fact that the sums of the inputs and the outputs are equal at each vertex of the graph implies that we can find a primitive cycle  $c$  that passes through this edge with support contained in the one of  $u$ . This is a consequence of an exploration process in the directed graph. Moreover, by multiplying  $c$  by a real number, we can assume that  $u \geq c$  and there is an edge  $\gamma'$  with  $u_{\gamma'} = c_{\gamma'} > 0$ . Then the support of  $u - c$  is strictly contained in the one of  $u$ . By induction on the cardinal of the support of  $u$ , we obtain the claim.  $\square$

**Tangent space  $T_{\mathcal{G}^\circ}$**  The tangent space  $T_{\mathcal{G}^\circ}$  of  $\Lambda_{\mathcal{G}^\circ}$  at a point in  $\mathring{\Lambda}_{\mathcal{G}^\circ}$  is the product

$$T_{\mathcal{G}^\circ} = \prod_{\mathcal{G}} T_{\mathcal{G}^\circ(c)}.$$

$T_{\mathcal{G}^\circ}$  can be identified with the homology  $H_1(\mathcal{G}, \partial\mathcal{G})$  of the graph relatively to the boundary by using the complex

$$0 \rightarrow \mathbb{R}^{X_1\mathcal{G}} \rightarrow \mathbb{R}^{X_0\mathcal{G}} \rightarrow 0.$$

A directed edge  $\gamma$  has a positive  $\gamma_+$  and a negative  $\gamma_-$  extremity, and we set  $\partial\gamma = \gamma_- - \gamma_+$ . This space is independent of the choice of the direction  $\mathcal{G}^\circ$  on  $\mathcal{G}$  up to isomorphism. For completeness, we give the following lemma:

**Lemma 3.5.** *Let  $r(\mathcal{G})$  be the number of connected components of  $M_{\mathcal{G}}$  with empty boundary, then*

$$\dim H_1(\mathcal{G}, \partial\mathcal{G}) = \#X_1\mathcal{G} - \#X_0\mathcal{G} + r(\mathcal{G}).$$

*Proof.* By computing the Euler characteristic of the complex, we have

$$\#X_1\mathcal{G} - \#X_0\mathcal{G} = \dim H_1(\mathcal{G}, \partial\mathcal{G}) - \dim H_0(\mathcal{G}, \partial\mathcal{G}).$$

We can see that  $H_0(\mathcal{G}, \partial\mathcal{G}) \simeq H_0(M_{\mathcal{G}}, \partial M_{\mathcal{G}})$ , we conclude by using  $r(\mathcal{G}) = \dim H_0(M_{\mathcal{G}}, \partial M_{\mathcal{G}})$ .  $\square$

An exact sequence in homology allows us to prove the following lemma:

**Lemma 3.6.** *If  $E \in X_1^{int}\mathcal{G}^\circ$ , the projection*

$$T_{\mathcal{G}^\circ} \longrightarrow T_{\mathcal{G}^\circ(E)}$$

*is surjective.*

Let  $TL_\partial$  be the tangent map of  $L_\partial$  and  $K_{\mathcal{G}^\circ}$  be the kernel of  $TL_\partial$ . Using the last lemma, we obtain the following corollary:

**Corollary 3.1.** *The map  $T$  fits in the following exact sequence:*

$$0 \longrightarrow K_{\mathcal{G}^\circ} \longrightarrow T_{\mathcal{G}^\circ} \xrightarrow{TL_\partial} T_{M^\circ} \longrightarrow 0.$$

**Remark 3.13.** *Indeed, the space  $K_{\mathcal{G}^\circ}$  is isomorphic to the homology  $H_1(\mathcal{G})$ , and the last sequence corresponds to the long exact sequence of relative homology;  $TL_\partial$  is the boundary morphism.*

**Behavior of  $l_\gamma$  on  $\Lambda_{\mathcal{G}^\circ}$ :** Let  $\mathcal{G}^\circ$  be a directed graph, and  $\gamma$  is an edge in  $X_1\mathcal{G}^\circ$ . The length of  $\gamma$  is given by the projection:

$$l_\gamma : \Lambda_{\mathcal{G}^\circ} \longrightarrow \mathbb{R}_{\geq 0}.$$

**Proposition 3.7.** *According to the topology of the graph, we have the following dichotomy:*

- *The length function  $l_\gamma$  is unbounded on  $\Lambda_{\mathcal{G}^\circ}(L)$  for each  $L$  in the image of  $L_\partial$ ;*
- *The length satisfies*

$$l_\gamma \leq \sum_{\beta \in \partial^+\mathcal{G}^\circ} l_\beta.$$

We call the first edges the unbounded edges and the second the bounded edges<sup>8</sup>. To prove proposition 3.7, we use the following lemma, and the proof also gives the bound.

**Lemma 3.7.** *An edge is bounded iff it's not contained in the support of any absolute cycle.*

*Proof.* We use proposition 3.6. Let an element  $x \in \Lambda_{\mathcal{G}^\circ}$  which is a linear combination of irreducible relative cycles  $x = \sum_i x_i \gamma_i$ . Each cycle  $\gamma_i$  connects a positive and a negative boundary, then we have  $\sum_i x_i = \sum_{\beta \in \partial^+\mathcal{G}^\circ} l_\beta(x)$ . Each cycle crosses the edge  $\gamma$  at most once because they are primitive, then  $l_\gamma(x) \leq \sum_i x_i$ . We can conclude that the edge is bounded if it's not contained in the support of any absolute cycle and  $l_\gamma \leq \sum_{\beta \in \partial^+\mathcal{G}^\circ} l_\beta$ . Conversely, if an edge is crossed by an absolute cycle  $c$  in the graph, we have  $l_\gamma(c) > 0$ . Let  $L$  be in the image of  $L_\partial$  and  $x \in \Lambda_{\mathcal{G}^\circ}(L)$ . For each  $t > 0$ , we can see that  $x + tc$  defines an element of  $\Lambda_{\mathcal{G}^\circ}(L)$ , and then the edge is unbounded because  $l_\gamma(x + tc) \geq tl_\gamma(c)$ .  $\square$

**Acyclic directed stable graphs:** We use a particular kind of directed stable graphs, which are acyclic stable graphs. A directed, stable graph induces a relation on the set of vertices. We say that  $x \geq y$  iff there is a path oriented positively from  $y$  to  $x$  (see figure 3.11).

**Definition 3.6.** *A directed graph is acyclic iff it satisfies one of the following equivalent conditions:*

- *The graph contains no directed absolute cycle.*

---

<sup>8</sup>because for each  $L$  the function  $l_\gamma$  is bounded in  $\Lambda_{\mathcal{G}^\circ}(L)$

- The relation on the vertices of the graph is a strict partial order.

*Proof.* It's straightforward to see that the graph admits a cycle iff the relation is not anti-symmetric.  $\square$

In what follows, we need to label the component of an acyclic graph, and then we use the following definition:

**Definition 3.7.** A linear order on a directed acyclic graph is an enumeration of the vertices (elements of  $X_0\mathcal{G}^\circ$ ), which is increasing for the order relation.

An important property of acyclic stable graphs is the following:

**Corollary 3.2.** A directed stable graph is acyclic iff all the edges are bounded (proposition 3.7).

*Proof.* Using proposition 3.7, if a graph is bounded, an edge can't lie in the support of an absolute cycle, and then the graph is acyclic. Conversely, if the graph is acyclic, there is no absolute cycle, and then all the edges are bounded according to proposition 3.7.  $\square$

The following proposition is used to prove lemma 4.26. If  $\mathcal{G}^\circ$  is a directed stable graph, and for each  $c \in X_0\mathcal{G}^\circ$ , let  $\mathcal{G}^{\circ,c}$  be a directed stable graph on  $\mathcal{G}^\circ(c)$ . We can consider the directed stable graph  $\tilde{\mathcal{G}}^\circ$  obtained by gluing the stable graphs  $(\mathcal{G}^{\circ,c})_{c \in X_0}$  according to  $\mathcal{G}^\circ$ .

**Proposition 3.8.** If the stable graphs  $\mathcal{G}^\circ$  and  $(\mathcal{G}^{\circ,c})_{c \in X_0}$  are acyclic, then  $\tilde{\mathcal{G}}^\circ$  is also acyclic.

*Proof.* A cycle on  $\tilde{\mathcal{G}}^\circ$  induces a possibly trivial cycle on  $\mathcal{G}^\circ$ . If  $\tilde{\mathcal{G}}^\circ$  is not acyclic, we can consider an absolute primitive cycle in  $\tilde{\mathcal{G}}^\circ$ . Then either this cycle induces a non trivial cycle in  $\mathcal{G}^\circ$  and then  $\mathcal{G}^\circ$  is not acyclic; or the cycle is trivial, it's contained in a component  $\mathcal{G}^{\circ,c}$  for some  $c$  and then this component is not acyclic. By contraposition, this gives the proposition.  $\square$

**Directed stable trees:** Another particular kind of directed graphs are directed trees; we give the following characterization: An edge  $\gamma \in X_1\mathcal{G}^\circ$  is constant if the function  $l_\gamma$  is constant on  $\Lambda_{\mathcal{G}^\circ}(L)$  for each  $L$  in the image of  $L_\partial$ .

**Proposition 3.9.** A directed, stable graph  $\mathcal{G}^\circ$  is a directed stable tree iff all the edges in  $X_1^{int}\mathcal{G}^\circ$  are constant.

*Proof.* As  $l_\gamma$  is linear, it's constant on  $\Lambda_{\mathcal{G}^\circ}(L)$  iff  $dl_\gamma$  is zero on  $K_{\mathcal{G}^\circ}$ . Moreover,  $\mathcal{G}^\circ$  is a tree iff  $H_1(\mathcal{G}) = 0$ . We see before that  $H_1(\mathcal{G}) \simeq K_{\mathcal{G}^\circ}$ , which is the kernel of  $TL_\partial$ , and then all the edges are constant if  $\mathcal{G}^\circ$  is a tree. Conversely, if all the edges are constant, then the map  $T_{\mathcal{G}^\circ} \rightarrow \mathbb{R}^{X_1\mathcal{G}}$  is zero on  $K_{\mathcal{G}^\circ}$ . But this map is injective, and then  $K_{\mathcal{G}^\circ} = \{0\}$ .  $\square$

We also give the following proposition:

**Proposition 3.10.** An edge is constant iff it splits a connected component of the graph into two connected components.

*Proof.* In an equivalent way, an edge is constant if the tangent map  $dl_\gamma$  vanishes on the space  $K_{\mathcal{G}^\circ}$ . It's possible to identify this space with the homology group  $H_1(\mathcal{G})$ . Let  $\mathcal{G}_\gamma$  (not  $\mathcal{G}_{\langle\gamma\rangle}$ ) be the stable graph obtained after removing  $\gamma$ . We have a natural map,

$$H_1(\mathcal{G}_\gamma) \longrightarrow H_1(\mathcal{G}).$$

The LHS is also the kernel of  $dl_\gamma$ . Computing the dimension we obtain,

$$\dim H_1(\mathcal{G}) - \dim H_1(\mathcal{G}_\gamma) = 1 + \dim H_0(\mathcal{G}) - \dim H_0(\mathcal{G}_\gamma).$$

Then the kernel is equal to the full space iff the edge splits a component into two components.  $\square$

If  $\gamma$  spares  $\mathcal{G}^\circ$  in two connected components, the length  $l_\gamma$  of  $\gamma$  factors through  $L_\partial$  and is the restriction of a linear function.

$$l_{\mathcal{G}^\circ, \gamma} : \Lambda_{M^\circ} \longrightarrow \mathbb{R}.$$

If  $I_1, I_2$  are the boundaries of these components, then

$$l_{\mathcal{G}^\circ, \gamma} = \left| \sum_{\beta \in I_1} \epsilon(\beta) l_\beta \right| = \left| \sum_{\beta \in I_2} \epsilon(\beta) l_\beta \right|.$$

which is linear on the cell.

**Degenerate graphs:** As we see, a directed stable graph or an oriented multi-curve can be degenerate. We give the following definition:

- We say that an edge  $\gamma \in X_1 \mathcal{G}^\circ$  is degenerate if  $l_\gamma$  is constant equal to zeros on  $\Lambda_{\mathcal{G}^\circ}$ .

The next proposition relies on degenerate directed graphs and degenerate edges.

**Proposition 3.11.** *A directed stable graph is degenerate iff it has degenerate edges.*

To prove the proposition, we need the following lemmas:

**Lemma 3.8.** *An edge is degenerate iff it's not contained in the support of any cycles.*

*Proof.* A cycle defines an element of  $\Lambda_{\mathcal{G}^\circ}$ ; then if an edge is degenerate, it can't lie in the support of any cycle. According to proposition 3.6, the cone  $\Lambda_{\mathcal{G}^\circ}$  is generated by cycles; if the edge is not contained in any cycle, it's degenerate.  $\square$

**Lemma 3.9.** *An acyclic directed stable graph is necessarily non-degenerate.*

*Proof.* Indeed, if  $\mathcal{G}^\circ$  is an acyclic stable graph, it is degenerate because it contains a component that does not satisfy the point 3 of definition 3.5.  $\square$

To prove proposition 3.11, we use the following construction: Let  $\mathcal{G}^\circ$  be a directed stable graph and  $E$  be the subset of unbounded edges. According to proposition 3.7,  $E$  corresponds to the subset of edges that belong to the support of an absolute cycle in  $\mathcal{G}^\circ$ . We define  $\mathcal{G}_{acycl}^\circ = \mathcal{G}_{\langle E \rangle}^\circ$  as the quotient of  $\mathcal{G}^\circ$  by  $E$ . The graph  $\mathcal{G}_{acycl}^\circ$  is not a directed stable graph because it does not necessarily satisfy the third point of definition 3.5.

**Proposition 3.12.** *The graph  $\mathcal{G}_{acycl}^\circ$  is acyclic, and an acyclic quotient of  $\mathcal{G}^\circ$  is a quotient of  $\mathcal{G}_{acycl}^\circ$ .*

Then, in some sense, it's the bigger acyclic directed sub-graph.

*Proof.* This proposition relies on the following lemma:

**Lemma 3.10.** *Each cycle in  $\mathcal{G}_{acycl}^\circ$  lifts to a cycle in  $\mathcal{G}^\circ$ .*

By using this lemma, if there is an absolute cycle in  $\mathcal{G}_{acycl}^\circ$ , we obtain an absolute cycle in  $\mathcal{G}^\circ$ . But all the edges that are in the support of an absolute cycle have been removed. Then the graph  $\mathcal{G}_{acycl}^\circ$  must be acyclic.  $\square$

This lemma gives a proof of proposition 3.12. And we can prove proposition 3.11.

*Proof.* If a graph is degenerate, it's straightforward to see that it has degenerate edges. For the converse, if the graph has a degenerate edge, then the image of the edge in  $\mathcal{G}_{acycl}^\circ$  is still degenerate in  $\mathcal{G}_{acycl}^\circ$ . Then, by lemma 3.9 and proposition 3.12, the graph  $\mathcal{G}_{acycl}^\circ$  has a degenerate component, and then the directed stable graph  $\mathcal{G}^\circ$  is degenerate.  $\square$

**Remark 3.14** (Orientation of the curves). *An oriented multi-curve  $\Gamma^\circ$  that satisfies only (1), (2) defines the orientation of each curve  $\gamma \in \Gamma$ ;  $\gamma$  belongs to two boundaries  $\gamma_+, \gamma_-$ , which are oriented according to the orientation of  $M_\Gamma$ . We choose to orient  $\gamma$  according to  $\gamma_+$ . By using this convention, we see that the multi-curve defines an element  $[\Gamma^\circ]$  in the homology  $H_1(M^\bullet, \mathbb{Z})$ . We assume that the boundaries of  $\partial M$  are curves in  $[\Gamma^\circ]$ . Then we can give the following characterization of degenerate multi-curves:*

**Proposition 3.13.** *An oriented multi-curve  $\Gamma^\circ$  is non-degenerate iff for all sub-multi-curve  $\Gamma_0^\circ \subset \Gamma^\circ$ , the homology class  $[\Gamma_0^\circ]$  is non-zero.*

### 3.5.3 Measure on the cone of relative cycles and volumes:

**Measures on the convex cone:** Let  $\mathcal{G}^\circ$  be a directed stable graph marked by  $M^\circ$ . The length of the boundary components defines a map  $L_\partial$ . The tangent space  $K_{\mathcal{G}^\circ}^\circ$  of the polytope  $\Lambda_{\mathcal{G}^\circ}(L)$  is defined by the exact sequence:

$$0 \longrightarrow K_{\mathcal{G}^\circ} \longrightarrow T_{\mathcal{G}^\circ} \xrightarrow{TL_\partial} T_{M^\circ} \longrightarrow 0.$$

Each of these sets contains a natural lattice of integer points  $K_{\mathcal{G}^\circ}(\mathbb{Z}), T_{\mathcal{G}^\circ}(\mathbb{Z}), T_{M^\circ}(\mathbb{Z})$ . On an affine space, a lattice in the tangent space defines a measure by assuming that the covolume of the lattice is one (see subsection 3.1 for more details). We denote  $d\sigma_{\mathcal{G}^\circ}, d\sigma_{\mathcal{G}^\circ}(L)$  and  $d\sigma_{M^\circ}$  these measures on  $\Lambda_{\mathcal{G}^\circ}, \Lambda_{\mathcal{G}^\circ}(L)$  and  $\Lambda_{M^\circ}$ . Then the following exact sequence and proposition ?? allow to decompose the measures.

**Proposition 3.14.** *The sequence*

$$0 \longrightarrow K_{\mathcal{G}^\circ}(\mathbb{Z}) \longrightarrow T_{\mathcal{G}^\circ}(\mathbb{Z}) \longrightarrow T_{M^\circ}(\mathbb{Z}) \longrightarrow 0,$$

*is also exact. Then for each  $L$ , the measure  $d\sigma_{\mathcal{G}^\circ}(L)$  is the conditional measure of  $d\sigma_{\mathcal{G}^\circ}$  with respect to  $d\sigma_{M^\circ}$ .*

*Proof.* The exact sequence of corollary 3.1 is given by a long exact sequence of relative homology of graphs. The same sequence remains true for integral homology, which allows us to prove the first part of proposition 3.14. For the second part, we can use proposition 3.1.  $\square$

By using the corollary 3.2, if the graph  $\mathcal{G}^\circ$  is acyclic, the edges are bounded, and then the subset  $\Lambda_{\mathcal{G}^\circ}(L)$  is bounded and has a finite measure. If  $F = (F_c)_{c \in X_0 \mathcal{G}^\circ}$  is a family of functions such that  $F_c$  is continuous on  $\Lambda_{\mathcal{G}^\circ(c)}$ , it makes sense to compute the integral.

$$V_{\mathcal{G}^\circ}(F)(L) = \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} \prod_{c \in X_0 \mathcal{G}^\circ} F_c(L_c(x)) \prod_{\gamma \in X_1^{\text{int}} \mathcal{G}^\circ} l_\gamma(x) d\sigma_{\mathcal{G}^\circ}(L) :$$

where  $L_c : \Lambda_{\mathcal{G}^\circ} \rightarrow \Lambda_{\mathcal{G}^\circ(c)}$  is the projection. In particular, we are interested in the function.

$$V_{\mathcal{G}^\circ}(L) = \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} \prod_{\gamma \in X_1^{\text{int}} \mathcal{G}^\circ} l_\gamma(x) d\sigma_{\mathcal{G}^\circ}(L).$$

**Boundary lengths and wall configurations:** For each directed connected surface  $M^\circ$ , a wall  $W \in \mathbf{Wall}(M^\circ)$  is the data of:

- a set  $X_0 W$ ,
- for each  $c \in X_0 W$  a pair of subsets  $I_W^+(c), I_W^-(c)$  with  $I_W^\pm(c) \subset \partial^\pm M^\circ$ .

We assume two things:

- $I_W^\pm = (I_W^\pm(c))_c$  defines two partitions of  $\partial^\pm M^\circ$ ,
- If  $I^\pm(c)$  is empty, the set  $I^\mp(c)$  must contain exactly one element.

We denote  $\Lambda_W$  the subset.

$$\Lambda_W = \prod_c \left( \Lambda_{M^\circ} \cap (\mathbb{R}^{I^-(c)} \times \mathbb{R}^{I^-(c)}) \right).$$

Its a subspace of  $\Lambda_{M^\circ}$  of codimension  $\#X_0 W - 1$ . For instance:

$$W = \{(\{1, 2\}, \{1\}), (\{3\}, \{2\}), (\emptyset, \{3\})\}, \text{ and } \Lambda_W = \{(L^+, L^-) \in (\mathbb{R}_{\geq 0}^3)^2 \mid L_1^+ + L_2^+ = L_1^-, L_3^+ = L_2^-, L_3^- = 0\}$$

There is a natural strict partial order relation on the set of walls,  $W' \leq W$ , iff the partition  $I_{W'}^\pm$  is a sub-partition of  $I_W^\pm$ . We can use the notations,

$$\mathbf{Wall}(W) = \{W' \mid W' \leq W\}, \text{ and } \mathbf{Wall}^*(W) = \{W' \mid W' < W\}.$$

If we have  $W \geq W'$ , we then have a natural inclusion  $\Lambda_{W'} \rightarrow \Lambda_W$ , and the map  $W \rightarrow \Lambda_W$  is strictly increasing. We denote  $\Lambda_{M^\circ}^*$  the complement of the union of all the  $\Lambda_W$ ,

$$\Lambda_{M^\circ}^* = \Lambda_{M^\circ} \setminus (\bigsqcup_{W' \in \mathbf{Wall}^*(M^\circ)} \Lambda_{W'}).$$

Similarly, we can also denote  $\Lambda_W^*$  the complement of all the walls  $W'$  with  $W' < W$ . We define the rank of a wall  $W$  as the codimension of  $\Lambda_W$  inside  $\Lambda_{M^\circ}$ ; the rank also corresponds to  $\#X_0 W - 1$ .

**Remark 3.15.** We can also define  $\mathbf{Wall}_{n^+, n^-}$ , as the set of walls in  $\Lambda_{n^+, n^-}$ . These walls were used in several places; they were used in the study of double Hurwitz numbers in [HKL18] and also in [Yak22].

**Piece-wise polynomials:** To this configuration of walls, we can associate the space  $\mathcal{P}_{M^\circ}$  (or  $\mathcal{P}_{n^+, n^-}$ ) of continuous piece-wise polynomials with walls in this configuration. For each wall  $W$  of rank one, we can write  $W = (I_1^\pm, I_2^\pm)$  and define

$$L_W = ||L_{I_1^+}| - |L_{I_1^-}|| = ||L_{I_2^+}| - |L_{I_2^-}||.$$

Then  $\mathcal{P}_{M^\circ}$  is the space of functions on  $\Lambda_{M^\circ}$ , polynomials in the variables  $L_W$ ,

$$\mathcal{P}_{M^\circ} = \mathbb{R}[\Lambda_{M^\circ}][L_W, W].$$

**Remark 3.16.** We denote by abuse of notation  $\mathbb{R}[\Lambda_{M^\circ}]$  the polynomial functions on  $\Lambda_{M^\circ}$ . Using  $L_W^2 \in \mathbb{R}[\Lambda_{M^\circ}]$ , we can see that the space  $\mathcal{P}_{M^\circ}$  is a finite extension of the space  $\mathbb{R}[\Lambda_{M^\circ}]$ .

We denote by abuse of notation  $\mathbb{R}[\Lambda_{M^\circ}]$  the polynomial functions on  $\Lambda_{M^\circ}$ . Using  $L_W^2 \in \mathbb{R}[\Lambda_{M^\circ}]$ , we can see that the space  $\mathcal{P}_{M^\circ}$  is a finite extension of the space  $\mathbb{R}[\Lambda_{M^\circ}]$ .

### Polynomial behavior of the volumes:

**Theorem 3.5.** Assuming that  $\mathcal{G}^\circ$  is acyclic, the function  $V_{\mathcal{G}^\circ}(L)$  is a homogeneous polynomial of degree  $d(\mathcal{G}^\circ)$  on the cone  $L_{\partial}(\Lambda_{\mathcal{G}^\circ})$ , which vanishes on the boundary of the cone. Moreover, this remains true for  $V_{\mathcal{G}^\circ}(F)(L)$  for every family of polynomials  $F$ .

From this, we can derive the corollary.

**Corollary 3.3.** If  $\mathcal{G}^\circ$  is acyclic, the functions  $V_{\mathcal{G}^\circ}(F)(L)$  belong to  $\mathcal{P}_{n^+, n^-}$ , for any family of polynomials  $F$

**Remark 3.17** (Case of trees). If  $\mathcal{G}^\circ$  is a tree, then the space  $\Lambda_{\mathcal{G}^\circ}(L)$  contains at most one element. By using the results of paragraph 3.5.2, we can see that:

$$V_{\mathcal{G}^\circ}(L) = \prod_{\gamma} l_{\mathcal{G}^\circ, \gamma}(L).$$

Ending each edge  $\gamma$  spare the graph in two connected components, then it defines a wall  $W_\gamma(\mathcal{G}^\circ)$  and we have  $l_{\mathcal{G}^\circ, \gamma} = L_{W_\gamma(\mathcal{G}^\circ)}$ . According to this, we see that we already proved theorem 3.5 and corollary 3.3 in this case. Moreover, the piece-wise polynomial is rather explicit.

The proof of this theorem which relies on the Ehrhart theory, is an application of the following theorem, which is a continuous version of the theorem 18.1 given by Barvinok in [Baro8].

**Theorem 3.6.** Let  $(v_i(\alpha))_i$  vectors in  $\mathbb{R}^n$  that depend on a parameter  $\alpha$ . Let  $P_\alpha$  be a family of polytopes in  $\mathbb{R}^n$  defined by

$$P_\alpha = \text{conv}(v_1(\alpha), \dots, v_n(\alpha)).$$

Assume that the cone of feasible directions of  $P_\alpha$  at  $v_i(\alpha)$  is independent of  $\alpha$  for all  $i$ . Then there is a polynomial  $P$  such as

$$\text{vol}(P_\alpha) = P(v_1(\alpha), \dots, v_n(\alpha)).$$

Then, to prove theorem 3.5, it remains to study the structure of the convex polytope  $\Lambda_{\mathcal{G}^\circ}(L)$ . In our case, the set  $V(\Lambda_{\mathcal{G}^\circ}(L))$  of extremal points is related to spanning trees in  $\mathcal{G}^\circ$ .

**Definition 3.8.** A spanning tree  $\mathcal{T}^\circ$  of  $\mathcal{G}^\circ$  is a subset  $E_{\mathcal{T}^\circ} \in X_1^{int}\mathcal{G}^\circ$  such that  $\mathcal{G}_E^\circ$  is a connected directed tree.

If  $\mathcal{T}^\circ$  is a spanning tree, according to prop 3.9 for each edge  $\gamma \in X_1\mathcal{T}^\circ$ , there is a function

$$l_{\mathcal{T}^\circ, \gamma} : \Lambda_{M^\circ} \longrightarrow \mathbb{R}_{\geq 0}.$$

It expresses the length of  $\gamma$  in terms of the boundary lengths  $L_\partial$ . There is a natural inclusion  $\Lambda_{\mathcal{T}^\circ} \rightarrow \Lambda_{\mathcal{G}^\circ}$ , and this gives a map:

$$x_{\mathcal{T}^\circ} : L_\partial \Lambda_{\mathcal{T}^\circ} \rightarrow \Lambda_{\mathcal{G}^\circ}.$$

It's given explicitly by

$$l_\gamma(x_{\mathcal{T}^\circ}(L)) = \begin{cases} l_{\mathcal{T}^\circ, \gamma}(L) & \text{if } \gamma \in X_1\mathcal{T}^\circ \\ 0 & \text{else} \end{cases}.$$

**Lemma 3.11.** For each  $L \in L_\partial \left( \Lambda_{\mathcal{G}^\circ}^\circ(L) \right)$ , the vectors  $x_{\mathcal{T}^\circ}(L)$  are well defined and are the extremal points of  $\Lambda_{\mathcal{G}^\circ}(L)$ .

*Proof.* Consider an element  $u \in K_{\mathcal{G}^\circ}$  such that the ray  $x_{\mathcal{T}^\circ}(L) + tu$  belongs to  $\Lambda_{\mathcal{G}^\circ}(L)$  for  $t$  small enough, then  $l_\gamma(u) = 0$  for all  $\gamma \in E_{\mathcal{T}^\circ}$ . And  $u$  is in the image of  $H_1(\mathcal{T}^\circ)$ . As  $\mathcal{T}^\circ$  is a tree,  $u = 0$  and  $x_{\mathcal{T}^\circ}(L)$  is an extremal point. Conversely, given an extremal point  $x$  and  $E = \{\gamma | l_\gamma(x) = 0\}$ , we have a map  $H_1(\mathcal{G}_E^\circ) \rightarrow H_1(\mathcal{G}^\circ)$ . For each  $u$  in the image of this map,  $x + tu$  belongs to  $\Lambda_{\mathcal{G}^\circ}(L)$  for  $t$  small enough. Then we must have  $H_1(\mathcal{G}_E^\circ) = 0$ , and then  $\mathcal{G}_E^\circ$  is a tree.  $\square$

The second ingredient is the following:

**Lemma 3.12.** For each  $\mathcal{T}^\circ$  spanning tree in  $\mathcal{G}^\circ$ , the cone of feasible directions  $\text{fcone}(\Lambda_{\mathcal{G}^\circ}(L), x_{\mathcal{T}^\circ}(L))$  does not depend on  $L$ .

*Proof.* This is trivial indeed if  $x_{\mathcal{T}^\circ}(L) + tu$  is in  $\Lambda_{\mathcal{G}^\circ}(L)$  iff  $l_\gamma(u) \geq 0, \forall \gamma \in E_{\mathcal{T}^\circ}$  and then

$$\text{fcone}(\Lambda_{\mathcal{G}^\circ}(L), x_{\mathcal{T}^\circ}(L)) \simeq K_{\mathcal{G}^\circ} \cap \{l_\gamma(u) \geq 0, \forall \gamma \in E_{\mathcal{T}^\circ}\}$$

$\square$

*Proof.* With these two lemmas, we can apply the theorem 3.6 and prove that there is a polynomial  $P_{\mathcal{G}^\circ}$  in vectorial variables  $X_{\mathcal{T}^\circ}$  indexed by the spanning trees. The  $X_{\mathcal{T}^\circ}$  take their arguments in  $\mathbb{R}^{X_1^{int}\mathcal{G}^\circ}$ , and we have

$$V_{\mathcal{G}^\circ}(1)(L) = P_{\mathcal{G}^\circ}(((x_{\mathcal{T}^\circ}(L))_{\mathcal{T}^\circ}).$$

The RHS is a polynomial in  $(l_{\mathcal{T}^\circ, \gamma})_\gamma$  and then is in  $\mathcal{P}_{M^\circ}$ . To prove the general statement and deal with the factors  $\prod_\gamma l_\gamma$ , we consider:

$$\Lambda'_{\mathcal{G}^\circ}(L) = \{(x, y) \in \Lambda_{\mathcal{G}^\circ}(L) \times \mathbb{R}_+^{X_1^{int}\mathcal{G}^\circ} | y_\gamma \leq x_\gamma, \forall \gamma\}.$$

By direct computation, the volume of  $\Lambda'_{\mathcal{G}^\circ}(L)$  is equal to  $V_{\mathcal{G}^\circ}(L)$ . To treat this case, we just need to slightly change the last lemmas. The set of extremal points in this case is bigger; each spanning tree is associated with  $2^{\#X_1^{int}\mathcal{T}^\circ}$  extremal points, and at each of these extremal points, the cone of feasible directions remains constant, which gives the proof by applying theorem 3.6. The case of  $F$  polynomials is similar.  $\square$

## Chapter 4

# Surgeries on oriented ribbon graphs and acyclic decomposition

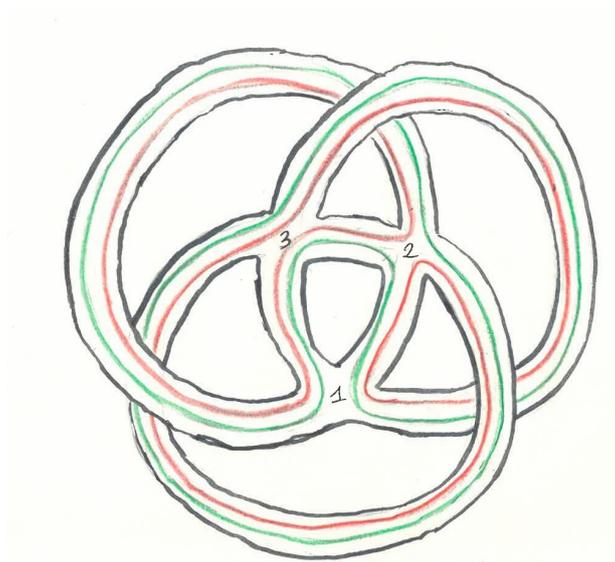


Figure 4.1: An acyclic decomposition.

### 4.1 Ribbon graphs and their moduli spaces

In this section, we discuss ribbon graphs, metric ribbon graphs, and their relations with multi-arcs and measured foliations. We define oriented ribbon graphs and list some of their properties. We construct cohomologies of ribbon graphs that are related to deformations of metric ribbon graphs, they are useful to make some statements in the next sections more general. We also discuss the moduli spaces of metric ribbon graphs and their stratification. We define natural measures on each stratum and volumes  $V_{g,n^+,n^-}^\nu$  that will be studied all along this memoir.

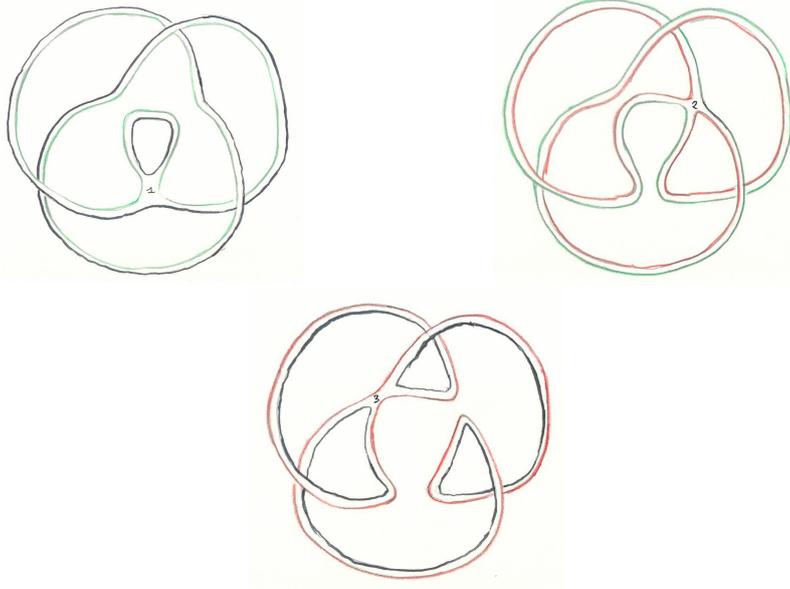


Figure 4.2: The three components of the decomposition in figure 4.1.

#### 4.1.1 Ribbon graphs and related constructions

**Combinatorial ribbon graphs:** Following M. Kontsevich [Kong2], we give the following formal definition:

**Definition 4.1.** A combinatorial ribbon graph  $R$  is defined by a quadruplet  $(XR, s_2, s_1, s_0)$  with:

- $XR$  a set of directed edges (or equivalently half edges).
- Two permutations  $s_0, s_2 : XR \rightarrow XR$ , which define respectively the vertices and boundary components (or faces) of the graph.
- An involution  $s_1 : XR \rightarrow XR$  without fixed points.
- The datas satisfy the condition:  $s_2 s_1 s_0 = id$ .

An isomorphism  $\phi : R \rightarrow R'$  is a bijection  $\phi : XR \rightarrow XR$  that preserves these datas. The group  $\widetilde{Aut}(R)$  is then the subgroup of  $\mathfrak{S}(XR)$  that preserves  $(s_2, s_1, s_0)$ .

As in [Kong2], we use the notation  $X_i R$  for the set of  $i$ -cycles in the graph, i.e., the orbit of  $s_i$

$$X_i R = XR / \langle s_i \rangle.$$

For all  $e \in XR$ , we denote  $[e]_i \in X_i R$  the projection and  $\#[e]_i$  the cardinal of the orbit of  $e$  under  $s_i$ . A ribbon graph defines a graph in a natural way. The orbits of  $s_0$  define a partition of  $XR$  indexed by  $X_0 R$ , and this partition defines the vertices. The involution  $s_1$  encodes how to glue two half edges together to obtain an edge.  $X_1 R$  is then the set of edges of the ribbon graph, and the triple  $(XR, X_0 R, s_1)$  defines the graph (according to paragraph 3.1). The permutation  $s_0$  gives an additional structure, a cyclic order on the edges around each vertex (see figure 4.3). As we see later, the set  $X_2 R$  represents the boundary components of a tubular neighborhood of the graph, and  $s_2 e$  is the successor of  $e$  in the boundary (see figure 4.3 and paragraph 4.1.1).

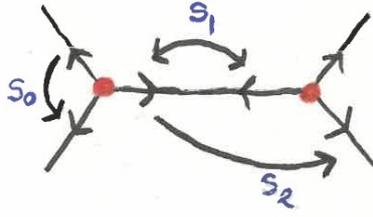


Figure 4.3: The three permutations  $(s_0, s_1, s_2)$ .

In what follows, we generally assume that the automorphisms preserve the boundaries; in general, the action of  $\widetilde{\text{Aut}}(R)$  on  $XR$  induces an action on  $X_2R$ , and then we restrict to the subgroup

$$\text{Aut}(R) \subset \widetilde{\text{Aut}}(R),$$

that act trivially on  $X_2R$ .

**Remark 4.1. (Duality)** A ribbon graph admits a dual. If  $R = (XR, s_2, s_1, s_0)$  is a ribbon graph, then  $R^* = (XR, s_0^{-1}, s_1, s_2^{-1})$  is also a ribbon graph, and we have

$$X_i R^* = X_{2-i} R.$$

**Oriented ribbon graphs:** We give the following definition for oriented (or Abelian) ribbon graphs (see figure 4.4). These objects also appear in [KZo3], where M. Kontsevich and A. Zorich use them to classify the connected components of stratum's in the moduli space of Abelian differentials. More recently, they also appear in the work of O. Dumitrescu and M. Mulase [DMSS13].

**Definition 4.2.** An orientation  $\epsilon$  on a ribbon graph  $R$  is a map  $\epsilon : XR \rightarrow \{\pm 1\}$ , such that

$$\epsilon \circ s_2 = \epsilon \quad \text{and} \quad \epsilon \circ s_1 = -\epsilon.$$

A ribbon graph is orientable if it admits an orientation, and an oriented ribbon graph  $R^\circ$  is a couple  $(R, \epsilon)$ .

When  $R^\circ$  is oriented, we denote:

$$X^\pm R^\circ = \{e \in XR, \epsilon(e) = \pm 1\}.$$

It's sometimes convenient to use the two "zig-zags" permutations (see figure 4.4 and 4.6),

$$s_+ = s_2, \quad \text{and} \quad s_- = s_1 s_2^{-1} s_1.$$

**Remark 4.2.** If the graph is oriented, we can see that these two permutations preserve  $\epsilon$  and then induce two permutations on  $X^+ R^\circ$ . Then we can show that the triple  $(X^+ R^\circ, s_+, s_-)$  characterizes  $R^\circ$  up to isomorphism and give an alternative definition of an oriented ribbon graph.

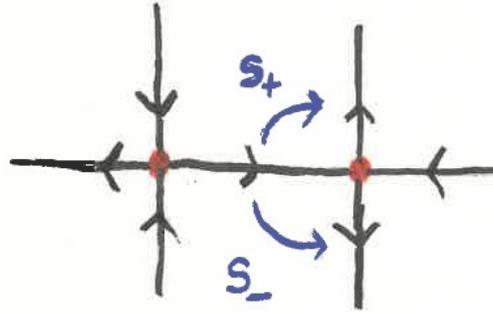


Figure 4.4: Combinatorix of oriented ribbon graphs

We summarize some elementary properties of orientable ribbon graphs in the following proposition:

**Proposition 4.1.** 1. *If  $R$  is connected, it admits at most two orientations.*

2. *If  $R$  is orientable, then  $R$  has only vertices of even degree.*

3. *An orientation of the graph induces a non constant map*

$$\epsilon : X_2R \longrightarrow \{\pm 1\}.$$

*Which defines a partition of the set of boundary components into two non-empty sets  $X_2R = X_2^+R \sqcup X_2^-R$ .*

4. *If  $R$  is not orientable, there is a canonical double cover  $\tilde{R}$  that is orientable and ramified over the vertices of odd degree.*

5. *A ribbon graph is orientable iff his dual is bipartite, i.e., there is a map  $\epsilon : X_0R^* \rightarrow \{\pm 1\}$  and two vertices joined by an edge have opposite signs.*

*Proof.* 1. When the graph is connected, the group generated by  $s_1, s_2$  acts transitively on the set of half edges. Then an orientation is determined by its value on a single edge.

2. If  $R$  is orientable, we have  $\epsilon \circ s_0 = -\epsilon$ , then by iterating this, for all  $v \in X_0R$ , we have  $(-1)^{\deg(v)} = 1$ . Then  $\deg(v)$  is even.

3. An orientation defines a map from the set of half edges to  $\{\pm 1\}$ , which is constant on each boundary component because we have  $\epsilon \circ s_2 = \epsilon$ . Then, for each  $\beta \in X_2R$ , pick a half edge with  $[e]_2 = \beta$  and set  $\epsilon(\beta) = \epsilon(e)$ , this does not depend on the choice of the edge because an other edge is given by applying  $s_2$  several times.

4. Consider the set  $X\tilde{R} = XR \times \{\pm 1\}$  with the three permutations.

$$\tilde{s}_0(e, x) = (s_0(e), -x), \quad \tilde{s}_1(e, x) = (s_1(e), -x) \quad \text{and} \quad \tilde{s}_2(e, x) = (s_2(e), x).$$

These data define a ribbon graph  $\tilde{R} = (X\tilde{R}, \tilde{s}_0, \tilde{s}_1, \tilde{s}_2)$  and the second projection  $\epsilon$ , defines an orientation. If  $p$  is the first projection, we have  $p \circ \tilde{s}_i = s_i$  for all  $i$ , then  $p$  defines a covering of the ribbon graph. From the expression of  $\tilde{s}_0$ , we see that the covering is ramified over the vertices of odd degree.

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<sup>1</sup>The terminology zig-zag come from the article [GK21]

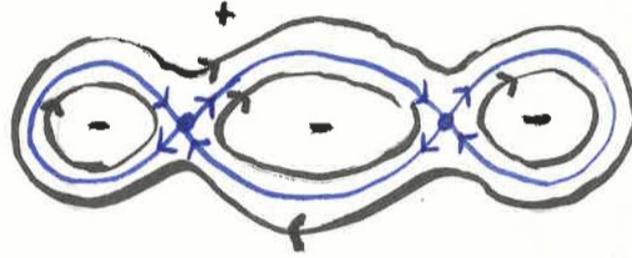


Figure 4.5: An oriented ribbon graph

5. If  $R$  is orientable, by definition of the dual, an orientation  $\epsilon$  induces a map  $\epsilon^* : XR^* \rightarrow \{\pm 1\}$  with  $\epsilon^* \circ s_1^* = -\epsilon^*$  and  $\epsilon^* \circ s_0^* = \epsilon^*$ , and then  $R^*$  is bipartite. The converse statement is similar. □

**Zippered rectangles:** A ribbon graph naturally defines a surface with boundary; it's his topological realization. We give a construction by using a zippered rectangles construction <sup>2</sup>. We start with the oriented case; let  $R^\circ$  be an oriented combinatorial ribbon graph. We consider for each  $e \in X^+R^\circ$  a rectangle

$$R_e = [0, 1] \times [-1, 1].$$

And we glue these rectangles using  $s_+, s_-$  by identifying their sides (see figure 4.6),

$$\{1\} \times [0, 1] \subset R_e \rightarrow \{0\} \times [0, 1] \subset R_{s_+e}, \quad \text{and} \quad \{1\} \times [-1, 0] \subset R_e \rightarrow \{0\} \times [-1, 0] \subset R_{s_-e}.$$

We denote  $M_R$  this topological space. It's possible to make these gluings by using translations. There is singularities at the points  $(0, 0), (1, 0) \in R_e$ , we can send neighborhoods of these points to the unit disc and then  $M_R$  admits a structure of topological surface with boundary <sup>3</sup>. Moreover, images of the lines  $[0, 1] \times \{0\}$  define a "topological" graph embedded in  $M_R$ . The edges are indexed by  $X_1R$  and the set of vertices corresponds to  $X_0R$ . If  $\phi : R_1^\circ \rightarrow R_2^\circ$  is an isomorphism of oriented ribbon graphs, it induces a natural homeomorphism  $\phi : M_{R_1} \rightarrow M_{R_2}$  by identifying  $R_e \simeq R_{\phi(e)}$ , the operation is compatible with gluings and the map  $R^\circ \rightarrow M_R$  is functorial.

We can see that the two horizontal boundary components of a rectangle  $R_e$  lie in the boundary of the surface  $M_R$ . For each  $e \in X_1R$  there is two half edges  $e_\pm$  with  $[e_\pm]_1 = e$  and  $e_\pm \in X^\pm R^\circ$ , we identify these two edges with the two horizontal boundary components  $e_\pm = [0, 1] \times \{\pm 1\} \subset$

<sup>2</sup>The name "zippered rectangles" is borrowed from the theory of quadratic and Abelian differentials by analogy with the Veech zippered rectangles construction (see [Vee82] instance)

<sup>3</sup>We use the notation  $M_R$  instead of  $M_{R^\circ}$  because the surface does not depend on the choice of the orientation; even we use it to construct it. We also remark that when there are bivalent vertices, we consider them as marked points, and in this case,  $M_R \in \mathbf{bord}^\bullet$ .

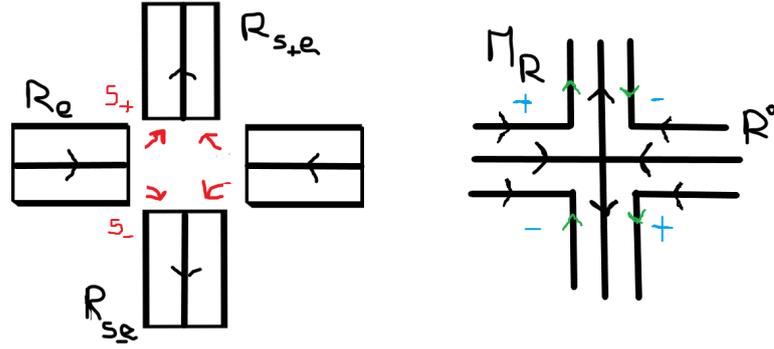


Figure 4.6: Zippered rectangles for an oriented ribbon graph  $R^\circ$ , orientation of the boundary induced by the surface  $M_R$  in green, orientation given by  $R^\circ$  in black, sign's in blue.

$R_{e_\pm}$ . Then we see that  $s_2 e_\pm$  is the half edge that follows  $e_\pm$  according to the orientation of the boundary. By using this, we can identify

$$X_2 R = \partial M_R.$$

Moreover, we can say that  $e_+$  is the top boundary and  $e_-$  is the bottom boundary of  $R_{e_+}$ ; this convention is preserved by  $s_2$ , and then this defines a map

$$\epsilon : \partial M_R \rightarrow \{\pm 1\}.$$

Then, we can see that an oriented ribbon graph  $R^\circ$  defines a directed surface  $M_{R^\circ} = (M_R, \epsilon)$ <sup>4</sup>.

In general, if  $R$  is a ribbon graph, it is not necessarily orientable. From proposition 4.1, we can consider the oriented cover  $\tilde{R}^\circ$  and construct the surface  $\tilde{M}_R = M_{\tilde{R}^\circ}$ . As it has been mentioned in proposition 4.1, there is an involution  $\sigma$  on  $\tilde{R}$ , and the quotient is  $R$ . In particular, by identifying  $X^\pm \tilde{R}^\circ \simeq X_1 \tilde{R}$ ,  $\sigma$  induces an involution

$$\sigma : X^+ \tilde{R}^\circ \rightarrow X^+ \tilde{R}^\circ.$$

Then, for each  $e \in X^+ \tilde{R}^\circ$ , we can consider the central symmetry

$$\begin{aligned} \sigma_e : R_e &\rightarrow R_{\sigma(e)} \\ (x, y) &\rightarrow (1 - x, -y). \end{aligned}$$

We can see that these maps are compatible with the regluing's and induce an involution

$$\sigma : \tilde{M}_R \rightarrow \tilde{M}_R.$$

The quotient is the desired surface, denoted  $M_R$ <sup>5</sup>. In the case where  $R^\circ = (R, \epsilon)$  is oriented, we can see that there is a canonical homeomorphism with the surface constructed in the oriented case.

<sup>4</sup>This time, the directed surface  $M_{R^\circ}$  depends on the orientation  $R^\circ$

<sup>5</sup>Our way to construct  $M_R$  is not the most straightforward, but it's convenient for us. In the unoriented case, we cannot fix the top and bottom of the rectangles  $(R_e)_e$ , which makes gluings more delicate; we need to flip the rectangles. To avoid this, we choose to use the oriented cover, but there are also different ways.

**Remark 4.3.** (Surface  $M_R^\bullet$ ) We can also construct the surface  $M_R^\bullet$  by gluing infinite strips  $R_e^\bullet = [0, 1] \times \mathbb{R}$  in the same way as  $M_R$  and the surface  $M_R^{cap}$  obtained by capping the boundaries of  $M_R$ . The graph  $R$  induces a cell decomposition of  $M_R^{cap}$ , which is generally called a map. The orientability is then equivalent to the fact that this map is bipartite. It means that faces are labeled by  $\pm 1$ , and two adjacent faces have opposite signs.

**Embedded ribbon graphs:** If  $M \in \mathbf{bord}$  is a surface and  $R$  a ribbon graph with no vertices of degree one or two. An embedding of  $R$  is a homeomorphism

$$\phi : M_R \rightarrow M.$$

Two embeddings  $(R, \phi), (R', \phi')$  are equivalent iff there is an isomorphism  $f : R \rightarrow R'$  such that  $\phi' \circ f \circ \phi^{-1}$  is isotopic to the identity. An embedding of  $R$  in  $M$  is then a way to draw  $R$  on  $M$  such that  $M$  retracts on the image of  $R$ . For a directed surface  $M^\circ$ , we assume that the homeomorphism preserves the orientation of the boundary of  $M^\circ$  and  $M_{R^\circ}^\circ$ . We denote  $\text{Rib}(M^\circ), \text{Rib}(M)$  these sets of embedded ribbon graphs. The mapping class group  $\text{Mod}(M)$  of  $M$  acts on these sets, and we denote the quotients  $\text{rib}(M^\circ), \text{rib}(M)$ . The set  $\text{rib}(M^\circ)$  corresponds to all the combinatorial ribbon graphs with the same topology as  $M^\circ$  and with a map  $X_2^\pm R^\circ \rightarrow \partial^\pm M^\circ$ . Then the stabilizer of a graph under the mapping class group action is finite and equal to the automorphism group of the graph  $\text{Aut}(R)$ , then an embedding of  $R$  gives an inclusion

$$\text{Aut}(R) \subset \text{Mod}(M).$$

**Remark 4.4** (Marked points). When the surface  $M$  (resp  $M^\circ$ ) has marked points, we assume these points correspond to marked vertices of degree at least one (resp two in the oriented case). This convention is useful when we introduce surgeries on ribbon graphs.

**Cohomology of a ribbon graph:** We construct the homology and cohomology groups of a ribbon graph. We start with the case of oriented graphs because the construction is straightforward. We generalize to the general case by using the orientation cover, and finally, we identify the two notions for oriented ribbon graphs.

Let  $R^\circ$  be an oriented ribbon graph, and consider the complex of chains  $C_*(R^\circ)$  given by

$$C_0(R^\circ) = \mathbb{R}^{X_0 R}, \quad C_1(R^\circ) = \mathbb{R}^{X_1^+ R^\circ}, \quad \text{and} \quad C_2(R^\circ) = \mathbb{R}^{X_2 R}.$$

We define boundary operators, for each  $\beta \in X_2 R$  and  $e \in X_1^+ R$

$$\partial\beta = - \sum_{e', [e']_2 = \beta} \epsilon(e') [e']_1^+, \quad \text{and} \quad \partial e = [s_1 e]_0 - [e]_0.$$

The complex of cochain's  $C^*(R^\circ)$  is defined by duality. Similarly, we can consider relative homology and cohomology. Let  $C_*(X_0 R)$  be the complex with only one non trivial element given by  $C_0(X_0 R) = C_0(R^\circ)$ . The relative complex is then

$$C_*(R^\circ, X_0 R) = C_*(R^\circ) / C_*(X_0 R),$$

and  $C^*(R^\circ, X_0 R)$  the dual. We can also define the complex of cochains  $C^*(X_2 R)$  with only one non trivial element  $C_2(X_2 R) = \mathbb{R}^{X_2 R}$ , and we form

$$C^*(R^\circ, X_0 R, X_2 R) = C^*(R^\circ, X_0 R) / C^*(X_2 R),$$

$C_*(R^\circ, X_0R, X_2R)$  is defined by duality. We denote

$$H^*(R^\circ), \quad H^*(R^\circ, X_0R), \quad \text{and} \quad H^*(R^\circ, X_0R, X_2R),$$

the cohomology groups.

**Lemma 4.1.** *The cohomologies of an oriented ribbon graph compute the cohomologies of  $M_R^{cap}$  and  $M_R$  we have*

$$\begin{aligned} H^*(R^\circ) &\simeq H^*(M_R^{cap}, \mathbb{R}) & H^*(R^\circ, X_0R) &\simeq H^*(M_R^{cap}, X_0R, \mathbb{R}) \\ H^*(R^\circ, X_0R, X_2R) &\simeq H^*(M_R, X_0R, \mathbb{R}). \end{aligned}$$

*Proof.* We consider the case of  $H_*(R^\circ)$ ; the ribbon graph  $R^\circ$  defines a cell decomposition of the surface  $M_R^{cap}$  and the complex  $C_*(R^\circ)$  is the complex of chains associated with this decomposition. It's well known that the homology of a cell complex computes the homology of the underlying topological space, so  $H_*(R^\circ) = H_*(M_R^{cap}, \mathbb{R})$ . A similar statement is also valid for relative homology. The complex  $C_*(R^\circ, X_0R, X_2R)$  is trivial with a single element  $C_1(R^\circ, X_0R, X_2R) = \mathbb{R}^{X_1R}$  and is the complex associated with the relative homology of the ribbon graph view as a CW complex. The surface  $M_R$  retracts on  $R$ , and then the complex  $C_*(R^\circ, X_0R, X_2R)$  computes the relative homology  $H_*(M_R, X_0R, \mathbb{R})$ .  $\square$

We generalize the construction to unoriented ribbon graphs. Let  $R$  be a ribbon graph and  $\tilde{R}^\circ$  be the oriented cover. The last construction allows us to define the homology  $H_*(\tilde{R}^\circ)$ . The involution  $\sigma$  on  $\tilde{R}^\circ$  induces a linear involution  $\sigma_*$  on  $C_*(\tilde{R}^\circ)$  (resp.  $\sigma^*$  on  $C^*(\tilde{R}^\circ)$ ), which is given by

$$\sigma_*(e) = -\tilde{s}_1(\sigma(e)) \quad \sigma^*(\beta) = \sigma(\beta), \quad \text{and} \quad \sigma^*(v) = \sigma(v).$$

These relations force to use anti-invariant cohomology. We can diagonalize  $\sigma^*$  and decompose  $C^*(\tilde{R}^\circ)$  into invariant and anti-invariant cocycles

$$C^*(\tilde{R}^\circ) = C^*(\tilde{R}^\circ)^+ \oplus C^*(\tilde{R}^\circ)^-.$$

The involution is compatible with boundary operators, then the direct sum is valid in the category of cochain complexes. This induces a decomposition

$$H^*(\tilde{R}^\circ) = H^*(\tilde{R}^\circ)^+ \oplus H^*(\tilde{R}^\circ)^-.$$

Where  $H^*(\tilde{R}^\circ)^\pm$  is the cohomology of  $C^*(\tilde{R}^\circ)^\pm$ , it's also the space of anti-invariant elements in  $H^*(\tilde{R}^\circ)$ . We denote  $H^*(R)$  the anti-invariant cohomology<sup>6</sup>

$$H^*(R) = H^*(\tilde{R}^\circ)^-.$$

We can define the relative anti-invariant cohomology  $H^*(R, X_0R)$ ,  $H^*(R, X_0R, X_2R)$  as well; these groups satisfy the following proposition, similar to lemma 4.1.

<sup>6</sup>The use of anti-invariant can be familiar to readers who are working on quadratic differential. This approach is similar to the one used to define the tangent space of the moduli space of quadratic differential [AEZ15]

**Proposition 4.2.** *The cohomology of the ribbon graph computes the anti-invariant cohomology of the oriented cover:*

$$\begin{aligned} H^*(R) &\simeq H^*(\tilde{M}_R^{cap}, \mathbb{R})^- & H^*(R, X_0R) &\simeq H^*(\tilde{M}_R^{cap}, X_0\tilde{R}, \mathbb{R})^- \\ & & H^*(R, X_0R, X_2R) &\simeq H^*(\tilde{M}_R, X_0\tilde{R}, \mathbb{R})^- \end{aligned}$$

Moreover, if the graph is oriented, the cohomologies of  $R$  and  $R^\circ$  are identified<sup>7</sup>.

*Proof.* We consider the case of  $H^*(R)$ , as we see in lemma 4.1, we have  $H^*(\tilde{R}^\circ) = H^*(\tilde{M}_R^{cap}, \mathbb{R})$ ; moreover, under this identification, the involution corresponds to the Galois involution, and then  $H^*(\tilde{R}^\circ)^- = H^*(\tilde{M}_R^{cap}, \mathbb{R})^-$ .  $\square$

### 4.1.2 Metric ribbon graphs

**Definition:** A metric on a ribbon graph  $R$  is a map  $m$

$$m : X_1R \longrightarrow \mathbb{R}_{>0}.$$

And according to the precedent subsection, an embedded metric ribbon graph on  $M$  is a pair  $S = (R, m)$  where  $R \in \text{Rib}(M)$  and  $m$  is a metric on  $R$ . We denote, respectively

$$\text{Met}(R) = \mathbb{R}_{>0}^{X_1R}, \quad \text{and} \quad T_R = \mathbb{R}^{X_1R},$$

the cone of metrics on  $R$  and the tangent space to a point in  $\text{Met}(R)$ . We also use the notation

$$m_e : \text{Met}(R) \rightarrow \mathbb{R},$$

for the canonical coordinates and denote  $(\partial_e)_{e \in X_1R}, (dm_e)_{e \in X_1R}$  the associated basis of  $T_R$  and  $T_R^*$ .

**Tangent space and cohomology:** Let  $R$  be a ribbon graph and  $\tilde{R}^\circ$  be the oriented double cover. For all  $e \in XR$ , there is a unique lift  $e_+ \in X\tilde{R}$  of  $e$  that is oriented positively. For each edge  $e \in X_1R$ , let  $e_1, e_2 \in XR$  be the two "extremities" of  $e$ . The element

$$[e] = \frac{[e_1^+] + [e_2^+]}{2}$$

defines a vector in  $C_1(\tilde{R}^\circ)$ . We can see that it's anti-invariant, and then in  $C_1(R)$ , moreover, the family  $([e])_{e \in X_1R}$  forms a basis of  $C_1(R)$ .

**Lemma 4.2.** *There is a linear isomorphism*

$$f_R : T_R \longrightarrow H^1(R, X_0R, X_2R).$$

*Such that  $\langle f_R(x), [e] \rangle = dm_e(x) = x_e$  for all  $x \in T_R$ . There is also a dual morphism*

$$f_R^* : T_R^* \longrightarrow H_1(R, X_0R, X_2R),$$

*such that  $f_R^*(dm_e) = [e]$  and the bracket between  $T_R$  and  $T_R^*$  corresponds to the pairing between homology and cohomology.*

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<sup>7</sup>In what follows, we use the second definition, which is more general. But when we work on an oriented ribbon graph, we don't refer to the anti-invariant cohomology.

By identifying the homology/cohomology of ribbon graphs with the one of  $M_{\tilde{R}}$ , the pairing between  $T_R^*$  and  $T_R$  corresponds to the one between  $H^1(M_{\tilde{R}}, X_0\tilde{R}, \mathbb{R})^-$  and  $H_1(M_{\tilde{R}}, X_0\tilde{R}, \mathbb{R})^-$ .

*Proof.* The complex  $C^*(R, X_0R, X_2R)$  is very simple; the only non trivial cohomology group is  $H^1(R, X_0R, X_2R) = C^1(R)$ , then there is a unique isomorphism

$$f_R : T_R \longrightarrow H^1(R, X_0R, X_2R),$$

such that

$$\langle f_R(x), [e] \rangle = dm_e(x) = x_e.$$

The rest of the proof is also straightforward.  $\square$

**Remark 4.5.** *If  $R^\circ$  is oriented, an oriented edge  $e \in X^+R$  defines a cycle  $[e]$  in the relative homology  $H_1(M_R^\bullet, X_0R, \mathbb{R})$  and using the result of the last section, they define a basis of this homology space. Then we can also define an isomorphism*

$$f_{R^\circ} : T_R \longrightarrow H^1(M_R, X_0R, \mathbb{R}).$$

*The opposite orientation gives the opposite morphism  $f_{-R^\circ} = -f_{R^\circ}$ ; similarly, we can also construct an isomorphism  $f_{R^\circ}^* : T_R^* \longrightarrow H_1(M_R, X_0R, \mathbb{R})$ . In this case, the pairing between  $T_R$  and  $T_R^*$  corresponds to the natural pairing between  $H^1(M_R, X_0R, \mathbb{R})$  and  $H_1(M_R, X_0R, \mathbb{R})$ .*

**Remark 4.6.** *It's possible and sometimes more convenient to work in homology,*

$$H^1(M_R, X_0R, \mathbb{R}) \simeq \text{Hom}(H_1(M_R, X_0R, \mathbb{R}), \mathbb{R}) \simeq H_1(M_R^{\text{cap}} \setminus X_0R, X_2R, \mathbb{R}).$$

*By using the non degenerate intersection form*

$$H_1(M_R^{\text{cap}} \setminus X_0R, X_2R, \mathbb{R}) \times H_1(M_R^{\text{cap}} \setminus X_2R, X_0R, \mathbb{R}) \rightarrow \mathbb{R}.$$

*If  $T_R^*$  is still identified with  $H_1(M^{\text{cap}} \setminus X_2R, X_0R, \mathbb{R})$ , the bracket is preserved.*

**Lengths of boundary components and orientability:** Let  $M$  and  $\beta \in \pi_0(\partial M)$  for each embedded metric ribbon graph  $S = (R, m)$  on  $M$ . It's possible to define the length of  $\beta$  as

$$l_\beta(S) = \sum_{e \in XR, [e]_2 = \beta} m_{[e]_1}(S).$$

This defines a linear function  $l_\beta : \text{Met}(R) \rightarrow \mathbb{R}_{>0}$ . Let  $M^\circ$  be a connected directed surface and  $R^\circ = (R, \epsilon) \in \text{Rib}(M^\circ)$ . The fact that the dual of an oriented ribbon graph is bipartite implies the following lemma:

**Lemma 4.3.** *Let  $R^\circ$  be as before, we have on  $\text{Met}(R)$*

$$\sum_{\beta} \epsilon(\beta) l_\beta = 0.$$

Then the image of the application  $L_\partial = (l_\beta)_{\beta \in \partial M}$  lies in the hyperplane  $\Lambda_{M^\circ}$  (equation 3.2).

*Proof.* By proposition 4.1 each edge  $e \in X_1R$  is contained in exactly one positive and one negative boundary, then for each  $m \in \text{Met}(R)$  we have

$$\sum_{\beta, \epsilon(\beta)=1} l_\beta = \sum_e m_e = \sum_{\beta, \epsilon(\beta)=-1} l_\beta$$

which give the claim.  $\square$

The differential  $dl_\beta$  of  $l_\beta$  is an element in  $T_R^*$ . Let  $H_R$  be the span of  $(dl_\beta)_{\beta \in X_2R}$  and  $K_R$  be the annihilator of  $H_R$ , i.e., the subspace of  $T_R$  defined by the exact sequence:

$$0 \longrightarrow K_R \longrightarrow T_R \longrightarrow H_R^* \longrightarrow 0.$$

Each  $\beta$  defines an element  $[\beta] \in H_1(R, X_0R, X_2R)$ ; moreover, we can see that

$$dl_\beta(x) = \langle f_R(x), [\beta] \rangle.$$

The next proposition characterizes orientability, and despite his simplicity, it's very useful for us.

**Proposition 4.3.** *Let  $R$  be a connected ribbon graph not necessarily orientable with  $n$  boundaries. The dimension of  $H_R$  is*

- $n$  if  $R$  is not orientable,
- $n - 1$  if  $R$  is orientable and the only relation is given by the orientation

$$\sum_{\beta} \epsilon(\beta) dl_\beta = 0.$$

*Proof.* We give a cohomological proof, but it can be done in a more straightforward way. From paragraph 4.2,  $T_R$  is identified with  $H^1(R, X_0R, X_2R)$ . The short, exact sequence of complexes

$$0 \longrightarrow C^*(X_2R) \longrightarrow C^*(R, X_1R) \longrightarrow C^*(R, X_1R, X_2R) \longrightarrow 0,$$

leads to a long exact sequence:

$$0 \longrightarrow H^1(R, X_0R) \longrightarrow H^1(R, X_0R, X_2R) \longrightarrow H^2(X_2R) \longrightarrow H^2(R, X_0R) \longrightarrow 0.$$

We can identify  $H^2(X_2R) \simeq \mathbb{R}^{\partial M}$ ,  $H^1(R, X_0R, X_2R) \simeq T_R$ ,  $H^2(R, X_0R) \simeq H^2(R)$ . The second nontrivial map corresponds to the evaluation of  $(dl_\beta)_{\beta \in \partial M}$ . Then we can rewrite the sequence in the following way:

$$0 \longrightarrow K_R \longrightarrow T_R \xrightarrow{TL\partial} \mathbb{R}^{\partial M} \longrightarrow H^2(R) \longrightarrow 0.$$

Finally,  $H^2(R)$  can be identified with  $H^2(M_{\tilde{R}}^{cap}, \mathbb{R})^-$ ; the surface  $M_{\tilde{R}}^{cap}$  is connected if  $R$  is unorientable, otherwise it has two connected components exchanged by the Galois involution. Then we have

$$H^2(R) = \begin{cases} \mathbb{R} & \text{if } R \text{ is orientable} \\ 0 & \text{else.} \end{cases}$$

The obstruction to being orientable is then in  $H^2(R)$ . To conclude, from the exact sequence we obtain

$$\dim H_R = \#\partial M - \dim H^2(R),$$

which gives the claim. According to lemma 4.3, the only possible relation is given by the orientation.  $\square$

According to the proof of proposition 4.3, we obtain the following corollary:

**Corollary 4.1.** *The map  $f_R$  provides the following identifications:*

$$T_R \simeq H^1(R, X_0R, X_2R), \quad \text{and} \quad K_R \simeq H^1(R, X_0R).$$

### 4.1.3 Relations between metric ribbon graphs, weighted multi-arcs, and measured foliations

**Ribbon graphs and filling multi-arcs:** In this part, we study the relation between multi-arcs and ribbon graphs; similar results are also presented in [ACG11].

**Lemma 4.4.** *For each embedded ribbon graph  $R$  in  $M$ , we can associate a multi-arc  $A(R)$ .*

We give an illustration of this construction in figure 4.7.

*Proof.* Let  $R$  be an embedded ribbon graph on  $M$  with no vertices of degree one or two. For each edge  $e \in X_1R$ , there is a unique arc  $e^*$  in  $R_e$  that joins the two horizontal boundaries of the rectangle  $R_e$  (namely  $\{1/2\} \times [0, 1]$ ). The arc  $e^*$  intersects  $e$  and no other edge of the graph. Then the union of all the arcs  $e^*$  forms a multi-arc  $A(R)$  (figure 4.7). The fact that the arcs are non trivial and pairwise non homotopic is a consequence of the Bigon criterium (lemma 3.2). When there are vertices of degree one or two,  $A(R)$  is still a multi-arc, but on the surface obtained by removing the corresponding points (We make this choice to be consistent with the non triviality of the arcs lemma 3.2). We can see that the construction is compatible with isotopies, and the map  $A : \text{Rib}(M) \rightarrow \mathcal{MA}(M)$  is well defined.  $\square$

A ribbon graph defines a multi-arc, but the converse is not always true; the map  $A$  is not surjective. We give the following definition:

**Definition 4.3.** *A multi-arc  $A \in \mathcal{MA}(M)$  is filling<sup>8</sup> if  $\iota(A, \gamma) > 0$  for all  $\gamma \in \tilde{\mathcal{S}}(M)$ . We denote  $\mathcal{MA}^0(M)$ ,  $\mathcal{MA}_{\mathbb{R}}^0(M)$  the subset of filling multi-arcs and weighted multi-arcs.*

We give the following proposition that relies on filling multi-arcs and ribbon graphs:

**Proposition 4.4.** *A multi-arc  $A \in \mathcal{MA}(M)$  is filling iff  $A = A(R)$  for an embedded ribbon graph  $R$ , then the map*

$$A : \text{Rib}(M) \rightarrow \mathcal{MA}^0(M),$$

*is a bijection, and we denote  $R$  the inverse.*

To prove this proposition, we use the following criterion to check if a multi-arc is filling or not. We recall that for all multi-arc  $A$  on  $M$ ,  $M_A$  is the surface obtained after surgeries along the arcs in  $A$  (see lemma 3.2).

**Lemma 4.5.** *A multi-arc is filling iff all the components of  $M_A$  are topological discs with at most one interior marked point.*

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<sup>8</sup>The terminology is borrowed from the theory of measured foliations [LM08] but such multi-arcs are also called proper [ACG11]

*Proof.* If  $M_A$  is topologically a union of discs, then an essential curve in  $\tilde{\mathcal{S}}(M)$  must intersect an arc in  $A$ . Conversely, if  $M_A$  contains a component that is not a disc with at most one puncture, then this component contains at least an essential curve that intersects no arc of  $A$ .  $\square$

We now prove proposition 4.4.

*Proof.* Let  $A$  be a filling multi-arc on  $M$ ; assume for simplicity that  $M$  has no marked point; and consider  $A^\bullet$  on  $M^\bullet$  as in lemma 3.2. Each connected component of  $M_A$  is a polygone. We place a vertex at the center of each polygone and rely by an edge two vertices that belong to two polygones glued along an arc in  $A^\bullet$ . In this way, we obtain an embedded ribbon graph  $R = R(A)$  in  $M$ . We can see that this ribbon graph satisfies  $A(R) = A$ . Moreover, if  $R$  is a ribbon graph, we can see according to lemma 4.5 that the multi-arc  $A(R)$  is necessarily filling, which concludes the proof of proposition 4.4.  $\square$

**Metric ribbon graphs and filling foliations:** According to results of the last section, a ribbon graph defines a filling multi-arc, and similarly, a metric ribbon graph  $S$  defines a weighted multi-arc  $A(S)$ :

$$A(S) = \sum_{e \in X_1 R} m_e(S) e^*.$$

Then the map  $A : S \rightarrow A(S)$  induces a bijection between embedded metric ribbon graphs and filling weighted multi-arcs. In paragraph 3.4.2, we explain that we can associate a measured foliation to a weighted multi-arc by the enlargement procedure. We make this explicit in this section; a similar approach is also done in [ABC<sup>+</sup>20].

**Definition 4.4.** A foliation  $\lambda \in \mathcal{MF}(M)$  is filling if  $\iota(\lambda, \gamma) > 0$  for all  $\gamma \in \tilde{\mathcal{S}}(M)$ , we denote  $\mathcal{MF}^0(M)$  the subset of filling foliations.

An alternative definition of filling foliations is the following: A foliation is filling iff its graph of singular leaves is contractible, i.e., iff it's equivalent to a foliation without saddle connections.

**Lemma 4.6.** To each embedded metric ribbon graph  $S$  in  $M$ , we can associate a foliation  $\lambda(S) \in \mathcal{MF}(M)$  such that

$$\iota(\lambda(S), \gamma) = \iota(A_S, \gamma), \quad \forall \gamma \in \tilde{\mathcal{S}}(M).$$

Moreover, a foliation  $\lambda \in \mathcal{MF}(M)$  is filling iff  $\lambda = \lambda(S)$  for a metric ribbon graph  $S$ .<sup>9</sup>

*Proof.* We consider the zippered rectangle construction of section 4.1.1. Let  $(R, m)$  be an embedded oriented metric ribbon graph. For each  $e \in X^+ R$ . We endow  $R_e^\bullet$  with the measured foliation given by  $m_e dx$ . This defines a topological measured foliation in the sense of section 3.4.1. At a vertex of degree  $k$ , the foliation has a singularity of order  $k - 2$ . Moreover, we can see that this foliation is oriented. In general, we can use the oriented cover and do the same construction; the involution preserves the foliation but changes the orientation. Then it's possible

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<sup>9</sup>This construction gives the map  $\mathcal{MA}^0(M) \rightarrow \mathcal{MF}(M)$  that preserves the intersection pairing. A generalization of this construction allows to prove the statement of paragraph 3.4.2 in general. The converse map is then given by the map  $A$  defined in proposition 3.2, which gives a bijection between filling weighted multi-arcs and filling foliations,

$$A : \mathcal{MF}^0(M) \rightarrow \mathcal{MA}_{\mathbb{R}}^0(M).$$

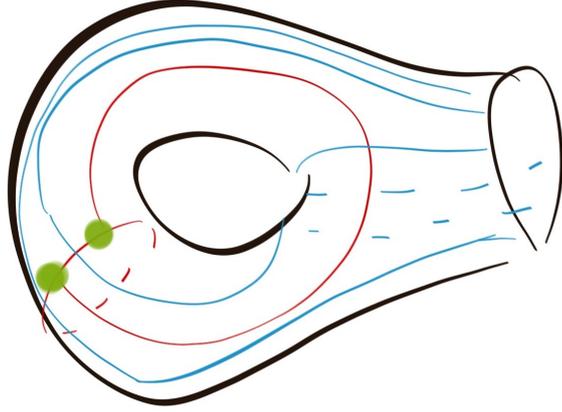


Figure 4.7: Ribbon graph and the multi-arc on a torus

to take the quotient, and this defines a foliation on  $M$ , which is not necessarily orientable. The other points are more technical; to prove that each filling foliation is given by a ribbon graph, we can use a stronger version of proposition 3.2. To compute the intersection pairing, we can use the notion of quasi-transverse curves [FLP21].  $\square$

The foliation  $\lambda_S$  is also given by the real part of a quadratic differential  $q_S(0) \in \mathcal{QT}(M)$ . In each rectangle  $R_e^\bullet = [0, 1] \times \mathbb{R} \subset \mathbb{C}$ , the quadratic differential is defined by

$$q_S(0)|_{R_e^\bullet} = m_e dz^2.$$

It has a double pole at each marked point  $\beta^\bullet$  for  $\beta \in \partial M$ . The form  $q_S(0)$  is Jenkin-Strebel in the sense that leaves of the horizontal foliation are periodic; moreover, all non-singular trajectories are circles around double poles. The horizontal foliation is the trivial foliation  $\lambda(0)$ , and  $q_S(0)$  corresponds to the image of  $(\lambda(S), \lambda(0)) \in \mathcal{MF}(M) \times \mathcal{MF}_0(M)$  under the Hubbard-Masur map.

**Case of oriented ribbon graphs:** In the case of oriented metric ribbon graphs  $S^\circ = (R^\circ, m)$ , we can see that, according to definition 3.4, the multi-arc  $A(R)$  is also oriented (figure 4.8), it's a consequence of proposition 4.1. Similarly, the foliation  $\lambda(S)$  is naturally oriented; we can see it in the proof of lemma 4.6 that it's given by a globally defined one form. Moreover, the differential  $q_S(0)$  admits a square root  $\alpha_{S^\circ}(0)$ , and  $\lambda(S)$  is the real part of  $\alpha_{S^\circ}(0)$ . From this, we can obtain the following corollary:

**Corollary 4.2.** *The subset of filling oriented foliations  $\mathcal{MF}^0(M^\circ)$  is identified with the subset of filling-weighted-oriented multi-arcs  $\mathcal{MA}_{\mathbb{R}}^0(M^\circ)$ .*

#### 4.1.4 Moduli spaces of metric ribbon graphs and their volumes

**Teichmüller spaces:** Let  $M \in \mathbf{bord}$ . The Teichmüller space  $\mathcal{T}^{comb}(M)$  of metric ribbon graphs on  $M$  is the space of all embedded metric ribbon graphs in  $M$ . It is a disjoint union of

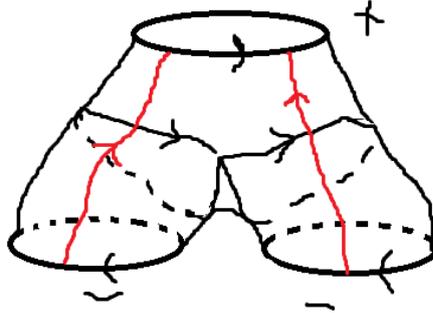


Figure 4.8: An oriented ribbon graph  $R^\circ$  and the oriented multi-arc  $A^\circ(R^\circ)$ .

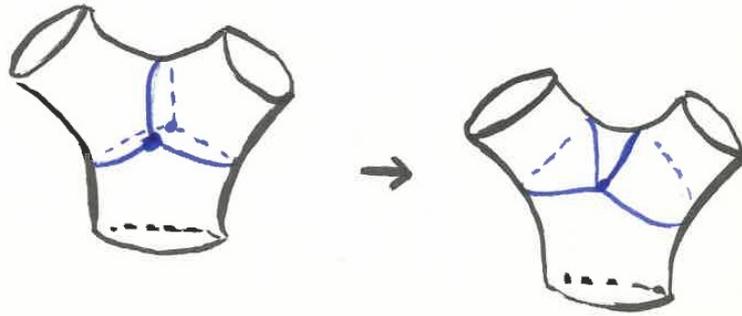


Figure 4.9: Degeneration of a ribbon graph on a pair of pants.

cells:

$$\mathcal{T}^{comb}(M) = \bigsqcup_{R \in \text{Rib}(M)} \text{Met}(R).$$

A ribbon graph  $R$  can degenerate to another ribbon graph  $R_{\langle E \rangle}$  by contracting a set of edges  $E$  that form a disjoint union of sub-trees  $E$  (see figure 4.9). In this case, the construction of the quotient of graphs preserves the structure of ribbon graphs (see paragraph 3.1). This also induces a map

$$\text{Met}(R_{\langle E \rangle}) \longrightarrow \overline{\text{Met}}(R),$$

with  $\overline{\text{Met}}(R) = \mathbb{R}_{\geq 0}^{X_1 R} \setminus \{0\}$ . Degeneration's of ribbon graphs induce a structure of cell complex on  $\mathcal{T}^{comb}(M)$  (but it's not closed in some sense). If  $M \in \mathbf{bord}^\bullet$ , we do not allow to contract an edge that joins two different marked vertices. The dimension of a cell  $\text{Met}(R)$  is denoted  $\bar{d}(R)$  and given by the following lemma:

**Lemma 4.7.** *The dimension  $\bar{d}(R)$  is equal to  $d(M) + \#X_0 R$ .*

*Proof.* If the graph is connected, We have  $\bar{d}(R) = \#X_1 R$ ,  $R$  defines a cell decomposition of  $M_R^{cap}$  with  $n$  faces, by computing the Euler characteristic  $2g - 2 = n - \#X_1 R + \#X_0 R$ . In

general, the dimension  $\bar{d}(R)$  and the quantity  $d(M) + \#X_0R$  are both additives for the disjoint union. Then we deduce the result from the connected case.  $\square$

The top cells of  $\mathcal{T}^{comb}(M)$  correspond to the subset  $\text{Rib}^*(M)$  of embedded ribbon graphs that are trivalent with univalent vertices at the marked points. If  $M$  is of type  $(g, n, m)$ , we obtain

$$\dim(T_R) = 6g - 6 + 3n + 2m \quad \text{and} \quad \dim K_R = 6g - 6 + 2n + 2m.$$

if  $R$  in  $\text{Rib}^*(M)$ . Then, in this case, the combinatorial moduli space is a cell complex of dimension  $6g - 6 + 3n + 2m$ .

If  $M^\circ$  is a directed surface, we denote  $\mathcal{T}^{comb}(M^\circ)$  the Teichmüller space of oriented metric ribbon graphs on  $M^\circ$ :

$$\mathcal{T}^{comb}(M^\circ) = \bigsqcup_{R^\circ \in \text{Rib}(M^\circ)} \text{Met}(R).$$

We can see that a degeneration of an oriented graph is also oriented and this defines a structure of cell complex on  $\mathcal{T}^{comb}(M^\circ)$  and the projection

$$\mathcal{T}^{comb}(M^\circ) \rightarrow \mathcal{T}^{comb}(M)$$

defines a subcomplex of  $\mathcal{T}^{comb}(M)$ . In the case of oriented ribbon graphs, the top cells correspond to quadrivalent oriented graphs with bivalent vertices at the marked points. In this case, the dimension of a top cell is given by:

$$\dim T_R = 4g - 4 + 2n^+ + 2n^- + m \quad \dim K_R = 4g - 3 + 2n^+ + 2n^- + m.$$

And then  $\mathcal{T}^{comb}(M^\circ)$  is of dimension  $4g - 4 + 2n^+ + 2n^- + m$ .

**Combinatorial Teichmüller spaces and measured foliations:** This part is just a reformulation of the results of proposition 4.4, lemma 4.6, and corollary 4.2. There are two injective maps

$$\mathcal{T}^{comb}(M) \longrightarrow \mathcal{MA}_{\mathbb{R}}(M) \longrightarrow \mathcal{MF}(M).$$

Moreover, by using proposition 4.4 and lemma 4.6, we have seen that the Teichmüller space is identified with the space of filling foliations and also with the space of filling multi-arcs

$$\mathcal{T}^{comb}(M) = \mathcal{MA}_{\mathbb{R}}^0(M) = \mathcal{MF}^0(M).$$

In this picture, the top cells  $\mathcal{T}^{comb,*}(M)$  of  $\mathcal{T}^{comb}(M)$  correspond to the subspace  $\mathcal{MF}^*(M)$  of foliations with simple zeros and no saddle connection (and simple poles at the marked points).

In a similar way, we have the inclusions

$$\mathcal{T}^{comb}(M^\circ) \longrightarrow \mathcal{MA}_{\mathbb{R}}(M^\circ) \longrightarrow \mathcal{MF}(M^\circ).$$

Corrolary 4.2 gives the identifications

$$\mathcal{T}^{comb}(M^\circ) = \mathcal{MA}_{\mathbb{R}}^0(M^\circ) = \mathcal{MF}^0(M^\circ).$$

**Remark 4.7.** As before,  $\mathcal{T}^{comb,*}(M^\circ)$  corresponds to the space  $\mathcal{MF}^*(M^\circ)$  of oriented foliations with singularities of order 2 and no saddle connection.

**Moduli spaces:** The mapping class group  $\text{Mod}(M)$  acts on  $\mathcal{T}^{comb}(M)$ ,  $\mathcal{T}^{comb}(M^\circ)$ , and the moduli spaces  $\mathcal{M}^{comb}(M)$ ,  $\mathcal{M}^{comb}(M^\circ)$  are the quotients under these actions. The moduli spaces are orbifold cell complexes; the set of cells is  $\text{rib}(M) = \text{Rib}(M)/\text{Mod}(M)$  which is the set of combinatorial ribbon graphs (see definition 4.1). This set is finite, and we have

$$\mathcal{M}^{comb}(M) = \bigsqcup_{R \in \text{rib}(M)} \text{Met}(R)/\text{Aut}(R),$$

where  $\text{Aut}(R)$  is a finite and acts freely by linear transformations. A similar statement is true for  $\mathcal{M}^{comb}(M^\circ)$ .

$$\mathcal{M}^{comb}(M^\circ) = \bigsqcup_{R^\circ \in \text{rib}(M^\circ)} \text{Met}(R)/\text{Aut}(R).$$

**Level sets:** For a metric ribbon graph embedded in  $M$ , the length  $l_\beta$  of the boundary components  $\beta \in \partial M$  is defined in 4.1.2. This defines a function

$$l_\beta : \mathcal{T}^{comb}(M) \rightarrow \mathbb{R}.$$

We also denote  $L_\partial = (l_\beta)_{\beta \in \partial M}$ , which takes values in  $\Lambda_M = \mathbb{R}_{\geq 0}^{\partial M}$ . If  $M^\circ$  is an oriented surface, then by using the results of proposition 4.3, the map  $L_\partial$  defined on  $\mathcal{T}^{comb}(M^\circ)$  takes values in  $\Lambda_{M^\circ}$ :

$$L_\partial : \mathcal{T}^{comb}(M^\circ) \rightarrow \Lambda_{M^\circ}.$$

The map  $L_\partial$  is invariant under the action of  $\text{Mod}(M)$ , and it descends to a map on the moduli spaces  $\mathcal{M}^{comb}(M)$ ,  $\mathcal{M}^{comb}(M^\circ)$ . In what is next, we study the level sets:

$$\mathcal{T}^{comb}(M^\circ, L) = \{S \in \mathcal{T}^{comb}(M^\circ) \mid L_\partial(S) = L\} \text{ and } \mathcal{M}^{comb}(M^\circ, L) = \mathcal{T}^{comb}(M^\circ, L)/\text{Mod}(M),$$

for each  $L \in \Lambda_{M^\circ}$ . These spaces are also cell complexes, and according to proposition 4.3,  $\mathcal{M}^{comb}(M^\circ, L)$  are locally affine sub-manifolds of codimension  $n-1$  in the moduli space  $\mathcal{M}^{comb}(M^\circ)$ . We have

$$\dim \mathcal{M}^{comb}(M^\circ, L) = 4g - 3 + n^+ + n^- + m,$$

if  $M^\circ$  is of type  $(g, n^+, n^-, m)$ .

**Stratification and decorations:** A ribbon graph  $R$  defines a decoration  $\nu_R$  in the sense of paragraph 3.2:

$$\nu_R(i) = \text{number of vertices of degree } i + 2.^{10}$$

Decorations are used to stratify the Teichmüller space  $\mathcal{T}^{comb}(M)$ . Let  $\overline{M} = (M, \nu)$  be a decoration on  $M$ . We denote  $\text{Rib}^*(\overline{M})$  the ribbon graphs  $R \in \text{Rib}(M)$  with  $\nu_R = \nu$  and  $\text{Rib}(\overline{M})$  the ones with  $\nu_R \leq \nu$  (the meaning of  $\leq$  is given in paragraph 3.2).

$$\mathcal{T}^{comb}(\overline{M}) = \bigsqcup_{R \in \text{Rib}(\overline{M})} \text{Met}(R), \text{ and } \mathcal{T}^{comb,*}(\overline{M}) = \bigsqcup_{R \in \text{Rib}^*(\overline{M})} \text{Met}(R).$$

<sup>10</sup>If  $A$  is filling,  $M_A$  is a union of polygons, we set  $\nu_A(i)$  the number of polygons with  $i$  sides, then according to the co

$$\nu_R = \nu_{A_R}.$$

We can see that if  $R' \leq R$  is a degeneration of  $R$ , then  $\nu_{R'} \leq \nu_R$  is a degeneration of  $\nu_R$  (see paragraph 3.2). Then  $\mathcal{T}^{comb}(\overline{M})$  is a closed sub-complex of  $\mathcal{T}^{comb}(M)$ , and it's the closure of  $\mathcal{T}^{comb,*}(\overline{M})$  in  $\mathcal{T}^{comb}(M)$ . The principal stratum corresponds to the decoration  $\nu_0 = (1^m, 3^{4g-4+2n+m})$ ; in this case, we have  $\mathcal{T}^{comb}(M, \nu_0) = \mathcal{T}^{comb}(M)$ . In general, lemma 4.7 shows that

$$\dim(\mathcal{T}^{comb}(\overline{M})) = d(M) + n(\nu).$$

In a similar way, we can define the decoration of an oriented ribbon graph  $R^\circ$ . But in this case, we define

$$\nu_{R^\circ}(i) = \text{number of vertices of degree } 2i-2.$$

Then decoration defines a stratification  $\mathcal{T}^{comb}(\overline{M}^\circ)$  of  $\mathcal{T}^{comb}(M^\circ)$ . In this case, the principal stratum corresponds to  $\nu_0^\circ = (0^m, 1^{2g-2+n+m})$ . The dimension is still given by:

$$\dim(\mathcal{T}^{comb}(\overline{M}^\circ)) = d(M) + n(\nu).$$

These two stratifications are compatible with the action of  $\text{Mod}(M)$ , and then moduli spaces are still stratified, and we denote  $\mathcal{M}^{comb}(\overline{M}), \mathcal{M}^{comb}(\overline{M}^\circ)$  the strata in the moduli spaces.

**Remark 4.8.** *When we are working with a connected surfaces of type  $\overline{M} = (g, n, \nu)$ , it's convenient to use  $\mathcal{T}_{g,n}^{comb}(\nu), \mathcal{M}_{g,n}^{comb}(\nu)$  instead of  $\mathcal{T}^{comb}(\overline{M}), \mathcal{M}^{comb}(\overline{M})$ .*

**Measures on the Teichmüller and moduli spaces of metric ribbon graphs:** If  $R$  is a ribbon graph, it's natural to endow  $\text{Met}(R)$  with the Lebesgue measure  $\prod_e dm_e$  by using the identification  $\text{Met}(R) = \mathbb{R}_{>0}^{X_1^R}$ . The group  $\text{Aut}(R)$  acts by permutation on  $\text{Met}(R)$ , and then the measure is well defined on the quotient  $\text{Met}(R)/\text{Aut}(R)$ ; we denote  $d\mu_R$  the quotient measure (We remark that the action of  $\text{Aut}(R)$  preserves the measure but a priori not the orientation).

Then each stratum  $\mathcal{T}^{comb}(\overline{M})$  (resp  $\mathcal{T}^{comb}(\overline{M}^\circ)$ ) carries a measure supported on  $\mathcal{T}^{comb,*}(\overline{M}^\circ)$  (resp  $\mathcal{T}^{comb,*}(\overline{M}^\circ)$ ) the set of cells of maximal dimension. This measure descends to a measure  $d\mu_{\overline{M}}$  on  $\mathcal{M}^{comb}(\overline{M})$  and  $d\mu_{\overline{M}^\circ}$  on  $\mathcal{M}^{comb}(\overline{M}^\circ)$ . In the case of a principal stratum, we denote simply  $d\mu_M$  and  $d\mu_{M^\circ}$  the measures on  $\mathcal{M}^{comb}(M)$  and  $\mathcal{M}^{comb}(M^\circ)$ .

We can also define measures on the level sets of  $L_\partial$ . For each ribbon graph  $R$  and each  $L$ , we can consider the Lebesgue measure on  $\text{Met}(R, L)$  (similarly to paragraph 3.1). The tangent space of  $\text{Met}(R, L)$  is the space  $K_R$  defined in 4.1.2. To normalize the Lebesgue measure on  $K_R$ , we use the lattice of integral points:  $K_R(\mathbb{Z}) = K_R \cap \mathbb{Z}^{X_1^R}$ . The space  $\text{Met}(R, L)$  is contained in an affine subspace directed by  $K_R$ . A choice of origin  $m \in \text{Met}(R, L)$  allows us to identify it with an open polytope in  $K_R$ . Different choices of base points produce a change of coordinates given by a translation and then preserves the Lebesgue measure on  $K_R$  normalized by  $K_R(\mathbb{Z})$ . Then for each  $L$  in  $L_\partial(\text{Met}(R))$  there is a measure  $d\tilde{\mu}_R(L)$  on the space  $\text{Met}(R, L)$  normalized by  $K_R(\mathbb{Z})$ . The lattice  $K_R(\mathbb{Z})$  is invariant by  $\text{Aut}(R)$  (because it's defined using  $TL_\partial$ ), and then the Lebesgue measure induces a measure  $d\mu_R(L)$  on  $\text{Met}(R, L)/\text{Aut}(R)$ . Then, for each  $\overline{M}^\circ$  (or  $\overline{M}$ ), these measures define a measure  $d\mu_{\overline{M}^\circ}(L)$  on the level set  $\mathcal{M}^{comb}(\overline{M}^\circ, L)$  supported by the top cells (resp.  $d\mu_{\overline{M}^\circ}(L)$  on  $\mathcal{M}^{comb}(\overline{M}, L)$ ).

**Volumes of the moduli space:** Each stratum  $\mathcal{M}^{comb}(\overline{M}^\circ)$  (resp.  $\mathcal{M}^{comb}(\overline{M})$ ), possesses a natural measure supported on the top cells, but its volume is infinite. Nevertheless, it's possible to consider the measure  $dV_{\overline{M}^\circ}$  on  $\Lambda_{M^\circ}$  (resp.  $dV_{\overline{M}}$  on  $\Lambda_M$ ) defined as the push forward of  $d\mu_{\overline{M}^\circ}$  under the map  $L_\partial$ .

**Lemma 4.8.** *The measures  $dV_{\overline{M}^\circ}$  (resp.  $dV_{\overline{M}}$ ) are sigma finite.*

*Proof.* We have

$$dV_{\overline{M}^\circ} = L_{\partial*} d\mu_{\overline{M}^\circ} = \sum_{R^\circ \in \text{rib}^*(\overline{M}^\circ)} dV_{R^\circ}.$$

Where the sum is finite and for  $R^\circ$  we denote  $dV_{R^\circ} = L_{\partial*} d\mu_{R^\circ}$ . Each edge of  $R^\circ$  is contained in a boundary, then we have

$$m_e \leq \|L_\partial(m)\|_\infty, \quad \forall e \in X_1 R. \quad (4.1)$$

Then we can see that the measures  $dV_{\overline{M}^\circ}$  are sigma-finite because the preimage of a bounded set by  $L_\partial$  is still bounded (this is also true for  $dV_{\overline{M}}$ ).  $\square$

These measures are characterized by the relation

$$\int_{\Lambda_{M^\circ}} f(L) dV_{\overline{M}^\circ} = \int_{\mathcal{M}^{comb}(\overline{M}^\circ)} f(L_\partial(S)) d\mu_{\overline{M}^\circ},$$

for  $f$  a measurable and positive function on  $\Lambda_{M^\circ}$ . As we see in the last paragraph, for each  $L \in \Lambda_{M^\circ}$ , the level set  $\mathcal{M}^{comb}(\overline{M}^\circ, L)$  is equipped with its Lebesgue measure  $d\mu_{\overline{M}^\circ}(L)$ . In what follows, we consider the volume of  $\mathcal{M}^{comb,*}(\overline{M}^\circ, L)$  of the subset of generic oriented ribbon graphs. It makes sense to compute the volume of  $\text{Met}(R^\circ, L)$  by using the last inequality,  $\text{Met}(R^\circ, L)$  is a relatively compact convex polytope, and then

$$V_{R^\circ}(L) = \int_{\text{Met}(R^\circ, L)} d\mu_{R^\circ}(L)$$

is finite for all  $R^\circ$  and  $L$ . Then the total volume of  $\mathcal{M}^{comb,*}(\overline{M}^\circ, L)$  is equal to the finite sum

$$V_{\overline{M}^\circ}(L) = \int_{\mathcal{M}^{comb,*}(\overline{M}^\circ, L)} d\mu_{\overline{M}^\circ}(L) = \sum_{R^\circ \in \text{rib}^*(\overline{M}^\circ)} V_{R^\circ}(L),$$

then, it's also finite. In the case of a connected surface, we can also use the notation  $V_{g,n^+,n^-}^\nu(L^+|L^-)$ . The three objects  $dV_{\overline{M}^\circ}$ ,  $V_{\overline{M}^\circ}$ , and  $d\sigma_{M^\circ}$  (the measure on  $\Lambda_{M^\circ}$  see paragraph 3.5.3) are related by the following proposition.

**Proposition 4.5.** *The measure  $dV_{\overline{M}^\circ}$  is absolutely continuous with respect to  $d\sigma_{M^\circ}$ . On  $\Lambda_{M^\circ}$ , we have the relation*

$$\frac{dV_{\overline{M}^\circ}}{d\sigma_{M^\circ}} = V_{\overline{M}^\circ}(L), \quad a.s.$$

*And then for a measurable function*

$$\int_{\mathcal{M}^{comb}(\overline{M}^\circ)} f(L_\partial(S)) d\mu_{\overline{M}^\circ} = \int_{\Lambda_{M^\circ}} f(L) V_{\overline{M}^\circ}(L) d\sigma_{M^\circ}.$$

This proposition this proposition is a consequence of the following lemma and from results of paragraph 3.1.

**Lemma 4.9.** *For all oriented ribbon graphs  $R^\circ$  embedded in  $M^\circ$ , the map  $TL_\partial$  fits into the following exact sequence:*

$$\{0\} \rightarrow K_R(\mathbb{Z}) \rightarrow T_R(\mathbb{Z}) \xrightarrow{TL_\partial} T_{M^\circ}(\mathbb{Z}) \rightarrow \{0\}.$$

When the graph is non-orientable, we have the following lemma; in this case, there is an extra factor  $\frac{1}{2}$ .

**Proposition 4.6.** *If  $R$  is non-orientable, embedded in  $M$  and  $d\sigma_M$  is the Lebesgue measure on  $\Lambda_M$ . We have:*

$$\frac{dV_R}{d\sigma_M} = \frac{V_R(L)}{2}.$$

*Proof.* This is due to the fact that  $TL_\partial(T_R(\mathbb{Z}))$  is equal to the subgroup of  $\mathbb{Z}^{\partial M}$  of vectors  $(l_\beta)$  with  $\sum_\beta l_\beta \in 2\mathbb{Z}$ . But it's not straightforward to prove that the image is exactly this lattice.  $\square$

## 4.2 Curves on ribbon graphs

### 4.2.1 Definition and surgeries

**Combinatorial representation:** Let  $R$  be a ribbon graph. A combinatorial curve is a path in the graph, i.e., a sequence of half edges  $(e_1, \dots, e_r)$ , with  $e_i \in XR \forall i$ , such that  $[e_{i+1}]_0 = [s_1 e_i]_0 \forall i \in \mathbb{Z}/r\mathbb{Z}$ . Such curves are defined modulo the action of  $\mathbb{Z} \times \{\pm 1\}$  by shifting the sequence and reversing the order. Nevertheless, there is still several representations of an isotopy class of curves on  $M_R$  as a combinatorial curve on the graph. To get around this problem, we say that  $\gamma_1 \geq \gamma_2$  if we can obtain  $\gamma_2$  by removing successively subsequences of the form  $(e, s_1 e)$  in  $\gamma_1$ . And we have  $\gamma_1 \sim \gamma_2$  iff there is  $\gamma$  such that  $\gamma_i \geq \gamma, i = 1, 2$ . We have the following lemma:

**Lemma 4.10.** *The relation  $\sim$  defines an equivalence relation on the set of combinatorial curves.*

This equivalence corresponds to the combinatorial notion of homotopy, and we denote  $\mathcal{C}^{comb}(R)$  the set of equivalence classes of combinatorial curves. We say that a combinatorial curve is minimal if it's a minimal element for the partial order relation  $\geq$ . Finally, if the graph has bivalent or univalent vertices, sub-sequences of the form  $(e, s_1 e)$  are allowed iff the degree of  $[s_1 e]_0$  is one or two.

**Proposition 4.7.** *Let  $R$  be a ribbon graph. Each equivalence class of combinatorial curves contains a unique minimal curve. Moreover, if  $R$  is embedded in  $M$ , the natural map:*

$$\mathcal{C}^{comb}(R) \longrightarrow \mathcal{C}(M),$$

*is a bijection.*

Where  $\mathcal{C}(M)$  is the set of homotopy classes of non contractible curves in  $M$

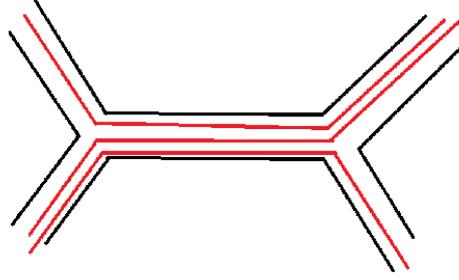


Figure 4.10: Curve on a metric ribbon graph.

*Proof.* We can obtain a minimal representation of a curve by removing successively subsequences of the form  $(e, s_1e)$ . The fact that  $\sim$  is an equivalence relation implies that there is a unique minimal element in each equivalence class, which implies the first part of the proposition. For the second, assume that  $M$  is connected. We can find a spanning tree  $T$  in the ribbon graph  $R$  and consider the quotient  $R_{\langle T \rangle}$ . This graph has a single vertex  $v$ . There is a contraction map that associates to each minimal combinatorial curve in  $R$  a minimal combinatorial curve in  $R_{\langle T \rangle}$ . Using the fact that  $T$  is a tree, we can define the converse map and identify  $\mathcal{C}^{comb}(R) \simeq \mathcal{C}^{comb}(R_{\langle T \rangle})$ . The ribbon graph  $R_{\langle T \rangle}$  is a bucket of circles, and the surface  $M$  retracts on it. It's well known that the fundamental group of a bucket of  $n$  circles is the free group with  $n$  generator. If we fix an orientation for the edges of  $R$ , we identify  $\pi_1(M, v)$  with the free group generated by  $X_1R_{\langle T \rangle}$ . Using this, we can see that each element in  $\mathcal{C}(M)$  admits a unique minimal combinatorial representation and show that the map  $\mathcal{C}^{comb}(R) \rightarrow \mathcal{C}(M)$  is a bijection.  $\square$

**Remark 4.9.** *When the graph has univalent or bivalent vertices, the last proposition remains true if we consider them as marked points (punctures) in  $M$  and exclude in  $\mathcal{C}(M)$  curves that retract on marked point.*

**Length of a curve on a metric ribbon graph:** Let  $S = (R, m)$  be a metric ribbon graph, then the length  $l_\gamma(S)$  of a combinatorial curve  $\gamma$  is defined by summing the metric of each edge:

$$l_\gamma(S) = \sum_{i=1}^r m_{[e_i]}(S).$$

By using proposition 4.7, we can define the combinatorial length of a curve  $\gamma \in \mathcal{C}(M)$  as the length of the unique minimal combinatorial representation. It's the smallest length among all possible representations of  $\gamma$ . There is also a topological way to compute the length of a curve. As we see in (lemma 4.4), each edge  $e \in X_1R$  is associated with an arc  $e^*$  that relies on the two boundaries  $[e]_2, [s_1e]_2$ . Let  $y_e(\gamma)$  be the intersection pairing of a curve  $\gamma \in \tilde{\mathcal{S}}(M)$  with the arc  $e^*$

$$y_e(\gamma) = \iota(\gamma, e^*). \quad (4.2)$$

**Lemma 4.11.** *Let  $M \in \mathbf{bord}^\bullet$  and  $S = (R, m) \in \mathcal{T}^{comb}(M)$  a metric ribbon graph. The length of  $\gamma \in \tilde{\mathcal{S}}(M)$  is given by the intersection pairing with  $A(S)$ :*

$$l_\gamma(S) = \iota(A(S), \gamma) = \sum_{e \in X_1R} m_e(S) y_e(\gamma).$$

The reals  $(y_e(\lambda))_{e \in X_1 R}$  are well defined for a foliation  $\lambda \in \widetilde{\mathcal{MF}}_0(M)$ . Then the second definition of the combinatorial length is more general and still makes sense for foliations in  $\widetilde{\mathcal{MF}}_0(M)$ . The length  $l_\lambda$  for  $\lambda \in \mathcal{MF}_0(M)$  defines a continuous function on  $\mathcal{T}^{comb}(M)$  and  $l_\lambda$  is linear on each cell.

*Proof.* Let  $\gamma \in \mathcal{C}(M)$ , the minimal combinatorial representation  $\gamma_R^{comb}$  defines an element in homotopy class  $\gamma$  (see figure 4.10 ). Using the bigon criterium (remark 3.8)  $\gamma_R^{comb}$  and  $A(S)$  are in minimal position. We can see that the intersection pairing  $\iota(\gamma_R^{comb}, A(S))$  is equal to  $l_\gamma(S)$  and then we obtain the lemma.  $\square$

**Surgery along a curve:** Let  $R$  be a ribbon graph embedded in  $M$ , and  $\Gamma \in \mathcal{MS}(M)$  a multi-curve. We recall that  $M_\Gamma$  is the surgery of  $M$  along  $\Gamma$ . In this part, we show that it's always possible to cut  $R$  along  $\Gamma$  and obtain a ribbon graph  $R_\Gamma$  on  $M_\Gamma$ . We can define the ribbon graph  $R_\Gamma$  using combinatorics (see figure (4.12)), but it's not easy to workout for ribbon graphs and more straightforward for multi-arcs.

**Lemma 4.12.** *If  $\Gamma$  is a multi-curve, there is a map:*

$$cut_\Gamma : \mathcal{MA}_R(M) \longrightarrow \mathcal{MA}_R(M_\Gamma).$$

*Which is linear on each cell. Moreover,  $cut_\Gamma$  is the unique map such that*

$$\iota(cut_\Gamma(A), \gamma) = \iota(A, \gamma), \quad \forall \gamma \in \tilde{\mathcal{S}}(M_\Gamma).$$

We give a sketch of the proof of this lemma.

*Proof.* If  $A$  is a multi-arc and  $\Gamma$  is a multi-curve, then, up to homotopy, we can assume that  $A, \Gamma$  are in minimal position. This means that the intersections are transverse, and then the multi-arc and the multi-curve minimise the number of intersection points up to homotopy. The fact that they intersect transversely makes it possible to cut the surface and the arcs along  $\Gamma$ . The result is a family  $\tilde{A}_\Gamma$  of arcs on each connected component of  $M_\Gamma$ . If  $A$  and  $\Gamma$  are in minimal position, then by using the Bigon criterion (remark 3.8), we can see that the arcs of  $\tilde{A}_\Gamma$  are non trivial. There are possibly families of homotopic arcs in  $\tilde{A}_\Gamma$ ; we identify the homotopic arcs and obtain a smaller family of arcs  $A_\Gamma$  on  $M_\Gamma$ , which is now a multi-arc. If  $m \in \text{Met}(A)$  is a weight on  $A$ , it induces a weight on each arc of  $\tilde{A}_\Gamma$ . We sum the weights of the arcs that are identified and then define weights on the arcs of  $A_\Gamma$ . This construction defines a map:

$$cut_\Gamma : \text{Met}(A) \longrightarrow \text{Met}(A_\Gamma),$$

which is linear. Then we can see that this map preserves the intersection pairing; if  $\gamma \in \mathcal{S}(M_\Gamma)$ , there is  $c$  such that  $\gamma \in \mathcal{S}(M_\Gamma(c))$ . Let  $A_\Gamma(c)$  be the arcs of  $cut_\Gamma(A)$  that are in  $M_\Gamma$ . Then, if  $A$  and  $\Gamma \sqcup \{\gamma\}$  are in minimal position by remark 3.8, we can see that  $A_\Gamma(c)$  and  $\gamma$  are also in minimal position. Then we have

$$\iota(A, \gamma) = \iota(A_\Gamma(c), \gamma) = \iota(cut_\Gamma(A), \gamma).$$

When  $\gamma \in \Gamma \sqcup \partial M$ , the situation is similar. The uniqueness of the construction follows from proposition 3.1.  $\square$

This lemma has a generalization for foliations. We don't give a complete proof of this proposition here.

**Proposition 4.8.** *If  $\Gamma$  is a multi-curve, there is a map:*

$$\text{cut}_\Gamma : \mathcal{MF}(M) \longrightarrow \mathcal{MF}(M_\Gamma)$$

*which coincides with  $\text{cut}_\Gamma$  on  $\mathcal{MA}_\mathbb{R}(M)$ . Moreover, this map is characterized by*

$$\iota(\text{cut}_\Gamma(\lambda), \gamma) = \iota(\lambda, \gamma), \quad \gamma \in \tilde{\mathcal{S}}(M_\Gamma).$$

**Remark 4.10.** *From the last property, the map  $\text{cut}_\Gamma$  is continuous for the topology given by the intersection pairing.*

*Proof.* The existence of the map is based on the results of W.Thurston collected in [FLP21] and uses the notion of quasi-transverse curves to a foliation and the enlargement procedure. The uniqueness is a consequence of ?? □

Using these results and results of the last section connecting filling multi-arcs and ribbon graphs, we derive the following corollary for ribbon graphs and metric ribbon graphs:

**Corollary 4.3.** *If  $M \in \mathbf{bord}^\bullet$  and  $A \in \mathcal{MA}^0(M)$  is a filling multi-arc, then for all  $\Gamma \in \mathcal{MS}(M)$ , the multi-arc  $A_\Gamma$  is also filling on  $M_\Gamma$ . For all  $R$  ribbon graphs, there is a ribbon graph  $R_\Gamma$  obtained after cutting  $R$  along  $\Gamma$ , and this induces a linear map:*

$$\text{cut}_\Gamma : \text{Met}(R) \longrightarrow \text{Met}(R_\Gamma).$$

*It defines a continuous map  $\text{cut}_\Gamma : \mathcal{T}^{\text{comb}}(M) \longrightarrow \mathcal{T}^{\text{comb}}(M_\Gamma)$ ; moreover, this is the unique map that satisfies:*

$$l_\gamma(\text{cut}_\Gamma(S)) = l_\gamma(S), \quad \forall \gamma \in \tilde{\mathcal{S}}(M_\Gamma).$$

**Remark 4.11.** *We also use the notation  $S_\Gamma$  for  $\text{cut}_\Gamma(S)$  to be consistent with the other notations.*

*Proof.* According to the definition 4.4,  $A_\Gamma$  is filling if for all  $\gamma' \in \tilde{\mathcal{S}}(M_\Gamma)$  we have  $\iota(A_\Gamma, \gamma') \neq 0$ . Then, if  $A$  is filling,  $A_\Gamma$  is also filling. By using the second part of lemma 4.12, we have  $\iota(A_\Gamma, \gamma) = \iota(A, \gamma) \neq 0$  for all  $\gamma \in \tilde{\mathcal{S}}(M_\Gamma) \subset \tilde{\mathcal{S}}(M)$ . Then, for each ribbon graph  $R$ , the dual multi-arc  $A(R)$  is filling, and then also  $A(R)_\Gamma$  on  $M_\Gamma$ . By using 4.4, there is a unique ribbon graph  $R_\Gamma$  on  $M_\Gamma$  such that  $A(R_\Gamma) = A(R)_\Gamma$ . Then the map  $\text{cut}_\Gamma$  defined in lemma 4.12 induces a map:

$$\text{cut}_\Gamma : \text{Met}(R) \rightarrow \text{Met}(R_\Gamma)$$

which is linear. Using 4.11, the length of a curve is given by

$$l_\gamma(S) = \iota(A(S), \gamma).$$

Finally, using lemma 4.12 again, we have

$$l_\gamma(S) = \iota(A(S), \gamma) = \iota(\text{cut}_\Gamma(A(S)), \gamma) = l_\gamma(\text{cut}_\Gamma(S))$$

if  $\gamma \in \tilde{\mathcal{S}}(M_\Gamma)$ . □

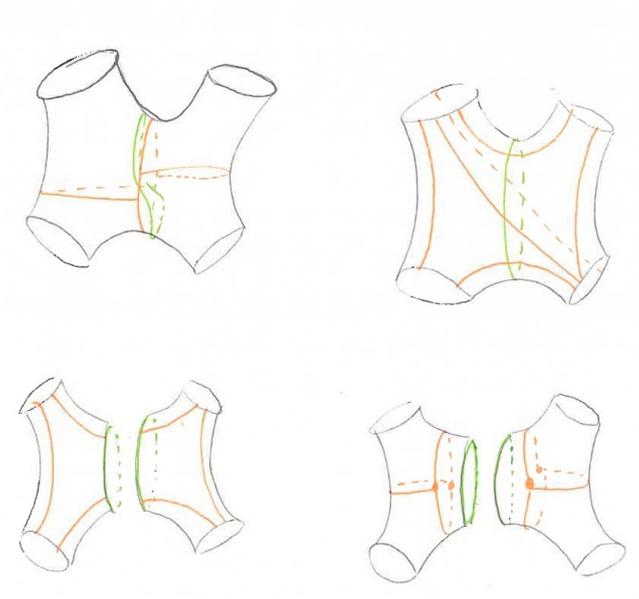


Figure 4.11: Procedure to cut a ribbon graph by using the dual multi-arc.

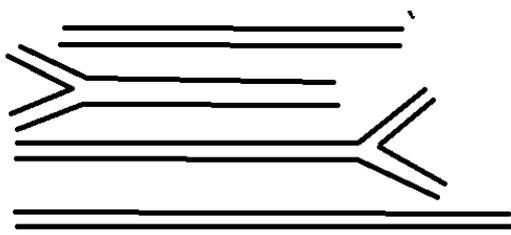


Figure 4.12: Ribbon graph cut along the curve of figure (4.10)

**Multi-curves on an oriented ribbon graph:** In the case of oriented ribbon graphs, it's natural to consider oriented multi-curves. We give the following definition:

**Definition 4.5.** *On an oriented ribbon graph, a curve is orientable if the orientation  $\epsilon$  is constant along the curve. In this case, it is possible to orient the curve such that  $\epsilon = 1$ .*

The following fact can be used to characterize the minimal representation. But the converse is not true in general.

**Lemma 4.13.** *Let  $R^\circ$  be an oriented graph:*

- *Let  $\gamma$  be a combinatorial curve; if it's orientable, then it's minimal.*
- *If  $\Gamma$  is an orientable multi-curve, then the orientation of the curves induces a non-degenerate orientation  $\Gamma^\circ$  on  $\Gamma$ .*

*Proof.* For the first point, obviously, if there is a sequence  $(e, s_1 e)$ , the curve is not orientable because  $\epsilon(s_1 e) = -\epsilon(e)$ , then an orientable curve is minimal. For the second point, for all  $S^\circ$  and  $\Gamma$ , we must have  $l_\gamma(S) > 0$  for all  $\gamma \in \Gamma$  because  $A_S$  is filling; moreover, from proposition 3.11, we can deduce that if  $\Gamma^\circ$  is degenerate, there is  $\gamma \in \Gamma$  with  $l_\gamma(S) = 0$ , which is impossible.  $\square$

For a more general oriented multi-arc  $A^\circ$ , a curve is orientable if it admits a representation that crosses arcs in  $A^\circ$  in a direct way (according to figure 4.13). For each oriented non-degenerate multi-curve  $\Gamma^\circ \in \mathcal{MS}(M^\circ)$ , there is a subset of weighted multi-arcs  $\mathcal{MA}_{\Gamma^\circ}(M^\circ) \subset \mathcal{MA}(M)$  such that  $\Gamma$  is orientable on each element of  $\mathcal{MA}_{\Gamma^\circ}(M^\circ)$  and the orientation induced on  $\Gamma$  is  $\Gamma^\circ$ .

**Corollary 4.4.** *For an oriented, non-degenerate multi-curve  $\Gamma^\circ$ , the restriction of  $cut_\Gamma$  induces a map  $cut_{\Gamma^\circ}$ :*

$$cut_\Gamma : \mathcal{MA}_{\Gamma^\circ}(M^\circ) \longrightarrow \mathcal{MA}(M_{\Gamma^\circ}^\circ).$$

*Proof.* This is straightforward because we have a map  $\mathcal{MA}(M) \longrightarrow \mathcal{MA}(M_\Gamma)$ , and by assumption we restrict to the preimage of  $\mathcal{MA}(M_{\Gamma^\circ}^\circ)$ .  $\square$

In a more precise way, the following lemma asserts that the orientable ribbon graphs are stable under surgeries along orientable multi-curves. Conversely, if we glue oriented graphs by identifying positive and negative boundaries, the result is still oriented.

**Corollary 4.5.** *Let  $\Gamma$  be a multi-curve:*

1. *If  $R^\circ$  is an oriented ribbon graph such that  $\Gamma$  is orientable, then the ribbon graph  $R_\Gamma$  possesses a natural orientation  $\epsilon_{\Gamma, R^\circ}$ , which induces an orientation  $\Gamma^\circ$  on  $\Gamma$ .*
2. *If  $R$  is a metric ribbon graph such that  $R_\Gamma$  is oriented and this orientation induces an orientation on  $\Gamma$ , Then the graph  $R$  is also oriented, and this orientation is compatible with the orientation on  $\Gamma$ .*

## 4.2.2 Admissible curves

In this part, we introduce admissible curves and foliations, give a combinatorial definition, a definition by surgeries, and a more topological definition.

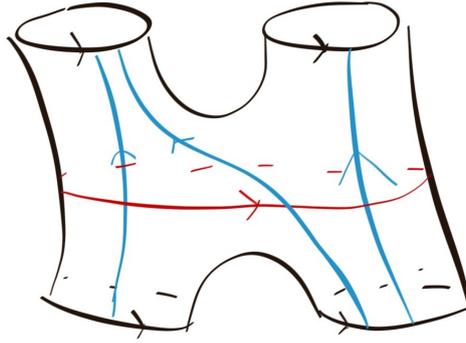


Figure 4.13: Oriented curves on an oriented multi-arc.

**Combinatorial definition:** To define admissibility, we start with combinatorial curves. Let  $R$  be a ribbon graph. We introduce in paragraph 4.1.1 the two "zig-zag" permutations:

$$s_2^+ = s_2 \quad \text{and} \quad s_2^- = s_1 s_2^{-1} s_1.$$

These two permutations are represented in figure 4.6.

**Definition 4.6.** A combinatorial curve is admissible<sup>11</sup> iff it admits a representation  $(e_i)$  of the form

$$e_{i+1} = s_{\pm} e_i.$$

From this definition, we obtain the following basic facts.

**Proposition 4.9.** An admissible combinatorial curve is necessarily minimal.

*Proof.* The only case when we have  $s_{\pm} e = s_1 e$  is when  $[s_1 e]_0$  is a univalent vertex. □

There are two cases where the set of admissible curves is particularly simple. We recall that an unorientable ribbon graph is generic iff it has vertices of degree one or three only.

**Lemma 4.14.** On a generic, unorientable ribbon graph, all the minimal curves are admissible.

The converse is also true, but use proposition 3.2. This lemma implies that if we fix a generic ribbon graph, every isotopy class of curves admits a unique admissible representation on the ribbon graph.

*Proof.* Assume that all the vertices are trivalent. Then, if  $[s_1 e_i]_0 = [e_{i+1}]_0$  and  $s_1 e_i \neq e_{i+1}$ , we must have  $e_{i+1} = s^{\pm} e_i$ . In general, if  $s_1 e_i = e_{i+1}$ , then  $[s_1 e_i]_0$  is univalent, and then  $s^+ e_i = s^- e_i = e_{i+1}$ . □

In the case of oriented ribbon graphs, we can obtain the next lemma. In this case, an oriented ribbon graph is generic iff it has only vertices of degree two or four.

**Lemma 4.15.** On an oriented ribbon graph:

- An admissible curve is orientable.

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<sup>11</sup>The admissible curve can be seen as a generalization of zig-zag curves [GK21]

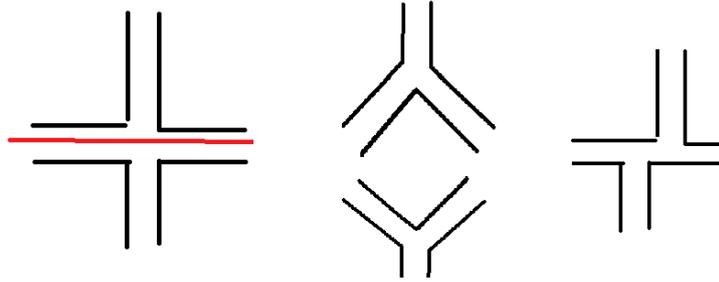


Figure 4.14: A non admissible curve.

- If the graph is generic, a combinatorial curve is orientable iff it's admissible.

As before, the second statement also characterizes generic orientable ribbon graphs.

*Proof.* This is a consequence of the representation (lemma 4.16) because the two permutations  $s^\pm$  preserve  $\epsilon$ . If there is only vertices of degree 4 it's easy to see that an orientable curve is admissible.  $\square$

**Definition by surgeries:** When the graph  $R$  is not generic, it might happen that a curve splits a vertex into several vertices of lower degrees (see figure 4.14). Then an admissible curve is a curve that does not split any vertex. To make this statement more formal, we can associate to a ribbon graph  $R$  a decoration  $\nu_R = (\nu_R(i))_i$  such that  $\nu_R(i)$  is a number of vertices of degree  $i + 2$  (see paragraph 3.2). For all multi-curves  $\Gamma$ , there is a second decoration  $\nu_{R_\Gamma}$  that corresponds to vertices of the graph  $R_\Gamma$  on  $M_\Gamma$  (see corollary 4.3). This defines a decorated multi-curve  $(\Gamma, \nu_{R_\Gamma})$ , and we must have

$$n(\nu_{R_\Gamma}) \geq n(\nu_R).$$

**Definition 4.7.** A multi-curve  $\Gamma$  is admissible on  $R$  iff  $\Gamma$  does not split any vertex of the graph, which is equivalent to  $\nu_{R_\Gamma} = \nu_R$ .

We denote  $\mathcal{S}(R)$  the set of admissible curves,  $\mathcal{MS}(R)$  the admissible (primitive) multi-curves, and  $\mathcal{MF}_\mathbb{Z}(R)$  the admissible integral multi-curves.

**Lemma 4.16.** A curve  $\gamma \in \mathcal{S}(M)$  is admissible on  $R$  iff  $\gamma_R^{comb}$  is admissible.

*Proof.* Let  $\gamma \in \mathcal{S}(M)$  already see that  $\gamma_R^{comb}$  defines a curve in minimal position with  $A = A(R)$ . Then we can see that surgeries along the curve won't change the decoration  $\nu_A = \nu_{A_\Gamma}$  iff the curve is admissible. The converse also works.  $\square$

**Admissible foliations:** Admissible curves on metric ribbon graphs are intimately related to quadratic differentials with double poles and prescribed singularities. For a quadratic differential  $q \in \mathcal{QT}_0(M)$ , it's possible to consider the decoration  $\nu_q$ :

$$\nu_q(i) = \#\{\text{singularities of } q \text{ of order } i\}.$$

The following definition is a generalization of admissibility for foliations.



Figure 4.15: Admissible curves on an oriented ribbon graph are oriented.

**Definition 4.8.** For all foliation  $\lambda \in \mathcal{MF}(M)$  and  $S$  a metric ribbon graph, let  $q_\lambda(S) \in \mathcal{QT}(M)$  the quadratic differential with double poles such that

$$\operatorname{Re}(q_\lambda(S)) = \lambda(S), \text{ and } \operatorname{Im}(q_\lambda(S)) = \lambda.$$

Then  $\lambda$  is admissible on  $S$  iff it satisfies  $\nu_{q_\lambda(S)} = \nu_S$ .

This definition coincides with the first one for multi-curves, according to the next lemma.

**Lemma 4.17.** The admissible integral foliations are the admissible integral multi-curves:

$$\mathcal{MS}_{\mathbb{Z}}(R) = \mathcal{MF}_0(R) \cap \mathcal{MS}_{\mathbb{Z}}(M).$$

*Proof.* Fix  $R$  and let  $\Gamma = \sum_{\gamma} m_{\gamma} \gamma \in \mathcal{MS}_{\mathbb{Z}}(R)$ . For each  $S \in \operatorname{Met}(R)$  we can build a quadratic differential in the following way: Using 4.3 we can cut  $S$  along  $\Gamma$ ; the result is a family of metric ribbon graphs  $S_{\Gamma} = (S_{\Gamma}(c))$  on  $M_{\Gamma}$ . On  $M_{\Gamma}$ , there is a Jenkin-Strebel differential  $q_{\lambda_0}(S_{\Gamma}(c))$  on each of its connected components  $c$ . Every non-singular horizontal trajectories of  $q_{\lambda_0}(S_{\Gamma})$  are periodic and surround a double pole. Let  $\gamma$  be in the support of  $\Gamma$  and  $(\gamma^1, \gamma^2)$  be the two boundaries of  $M_{\Gamma}$  that correspond to  $\gamma$ . It's possible to glue a horizontal cylinder of height  $m_{\gamma}$  to the two boundaries  $\gamma^1, \gamma^2$  of  $M_{\Gamma}$  associated to  $\gamma \in \Gamma$ . The result is a Jenkin-Strebel differential  $q$  on  $M$  such that

$$\operatorname{Re}(q) = \lambda(S) \text{ and } \operatorname{Im}(q) = \lambda(\Gamma).$$

By uniqueness in theorem 3.4  $q = q_{\Gamma}(S)$ , and by construction we have

$$\nu_{S_{\Gamma}} = \nu_q = \nu_{q_{\Gamma}(S)}.$$

Then the two definitions coincide. □

### 4.2.3 Zippered rectangles and coordinates for admissible foliations

**Coordinates for  $\mathcal{MF}(R)$  and  $\mathcal{MF}_0(R)$ :** Let  $M$  be a surface with boundary and  $R$  be an embedded ribbon graph. We define:

$$\mathcal{Q}(R) = \{q \in \mathcal{QT}(M) \mid \operatorname{Re}(q) \in \operatorname{Met}(R) \text{ and } \nu_q = \nu_R\} \text{ and } \mathcal{Q}_0(R) = \mathcal{Q}(R) \cap \mathcal{QT}_0(M).$$

From definition 4.8, they correspond to subsets of admissible foliations on metric ribbon graphs in  $\operatorname{Met}(R)$ .

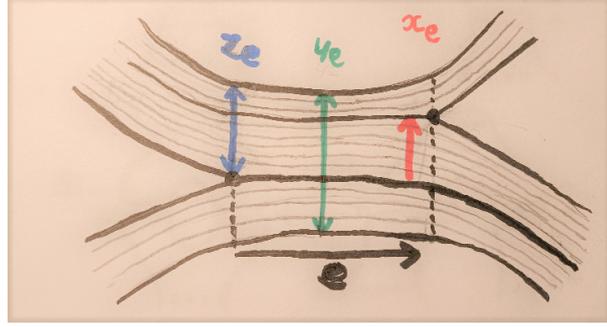


Figure 4.16: coordinates  $x, y, z$

**Proposition 4.10.** For all  $S = (R, m)$  a metric ribbon graph, the spaces  $\mathcal{MF}(S)$ ,  $\mathcal{MF}_0(S)$  depend only on the ribbon graph  $R$  and we have:

$$\mathcal{Q}(R) = \text{Met}(R) \times \mathcal{MF}(R), \quad \text{and} \quad \mathcal{Q}_0(R) = \text{Met}(R) \times \mathcal{MF}_0(R).$$

There is bijections that preserve the integral points:

$$\mathcal{Q}(R) \longrightarrow \text{Met}(R) \times T_R, \quad \text{and} \quad \mathcal{Q}_0(R) \longrightarrow \text{Met}(R) \times K_R.$$

In particular, they induce bijections:

$$\mathcal{MF}(R) \simeq T_R, \quad \text{and} \quad \mathcal{MF}_0(R) \simeq K_R.$$

**Remark 4.12.** We recall that  $T_R$  is the tangent space of  $\text{Met}(R)$  and  $K_R$  is the subspace of tangent vectors that preserve the boundary lengths (see paragraph 4.1.2). Using corollary 4.1, we can identify these spaces with the cohomologies  $H^1(R, X_0R, X_2R)$  and  $H^1(R, X_0R)$ .

**Remark 4.13** (Thurston theorem). A consequence of proposition 4.10 is the following theorem proved by Thurston for surfaces without boundaries:

**Proposition 4.11.** The projective space of measured foliations  $\mathcal{PMF}_0(M) = (\mathcal{MF}_0(M) \setminus \{0\}) / \mathbb{R}_+^*$  is homeomorphic to a sphere of dimension  $6g - 5 + 2n$ .

*Proof.* Using proposition 3.3, the space  $\mathcal{MF}_0(M) \setminus \{0\}$  is identified with the space  $\mathcal{MF}(M^\bullet)$ . Moreover, on a generic ribbon graph, all the foliations are admissible. If we fix  $R$ , theorem 4.10 gives an isomorphism  $\mathcal{MF}_0(M) \simeq K_R$ . According to 4.3, the space  $K_R$  is of dimension  $6g - 6 + 2n$ . Then we see that the projective space is a sphere of dimension  $6g - 5 + 2n$ .  $\square$

Now we give a proof of proposition 4.10.

*Proof.* We first construct the map, which is the period map of the imaginary foliation along the edges of the embedded graph  $R$

$$\mathcal{Q}(S) \longrightarrow \text{Met}(R) \times T_R.$$

We use the zippered rectangle construction (see proposition 3.2), which is a decomposition of foliations with poles. For all  $q$  and all  $e \in XR$ , there is a maximal embedded infinite rectangle  $R_e^\bullet \rightarrow M^\bullet$  such that  $\text{Re}(q)$  is locally given by  $|dx|$  on  $R_e^\bullet$ . Moreover, we assume that the direction

of  $[0, 1]$  corresponds with the direction of  $e$ ;  $q$  has no singularities on the interior of  $R_e^\bullet$ , but the maximality of  $R_e^\bullet$  implies that there is at least one singularity on each boundary component. As  $\nu_q = \nu_R$ , then  $q$  has no vertical saddle connections, then there is only one singularity on each boundary of  $R_e^\bullet$ . It's possible to choose a square root's  $\alpha_e$  of  $q$  on  $R_e^\bullet$  such that  $\text{Re}(\alpha_e) = dx$ . After this choice, we denote  $x_e^-$  the singularity on the left boundary and  $x_e^+$  the one on the right. Let  $I_e \subset R_e^\bullet$  be the horizontal maximal open interval directed according to  $e$  such that the left extremity is  $x_e^-$ . There is an isometry  $I_e = [0, m_e(S_q)]$ , and we can consider for all  $x \in I_e$  the vertical flow  $v_t$  such that  $\text{Im}(\alpha)(\partial v_t) = 1$ . It defines a map:

$$\begin{aligned} \phi_e : I_e \times \mathbb{R} &\longrightarrow R_e^\bullet \\ (x, y) &\longrightarrow v_y(x). \end{aligned}$$

If  $(x, y)$  are complex coordinates, then the pullback of  $\alpha$  under this map is equal to  $dz$ . In the local coordinates given by  $\phi_e$ , we can define

$$x_e^+ = m_e(\alpha) + ix_e(\alpha).$$

This is the relative period of  $\sqrt{q}$  along the edge  $e$ . There are no sign ambiguities because we assume that the real part of the period is positive, and then  $x_{s_1e} = x_e$ . Then this defines an element of the tangent space  $T_R$ .

By the zippered rectangle construction, the data  $(m, x)$  are enough to recover  $q$ . We simply glue the rectangles  $R_e^\bullet = [0, m_e] \times \mathbb{R}$  by performing a shear of parameter  $x_e$  on the right boundary. There is no constraint on  $(m, x)$  to perform the construction. We obtain in this way a Riemann surface with an Abelian differential given by  $dz$  on each rectangle. The surface is  $\tilde{M}_R^\bullet$  the oriented cover of  $R$ . Then the square of the Abelian differential defines a quadratic differential  $q_S(x)$  on  $M^\bullet$ , which is locally given by  $(dz)^2$  on each rectangle. The two constructions are the inverse of each other, and then there is a bijection:

$$\mathcal{Q}(R) \longrightarrow \text{Met}(R) \times T_R.$$

The imaginary part of the quadratic differential  $q_S(x)$  defines a foliation  $\lambda_R(x)$ , which does not depend on  $m$ , so the space  $\mathcal{MF}(S)$  depends only on  $R$ . Moreover

$$l_\beta(\lambda) = \sum_{e \in X_1 R} y_e(\beta) x_e(\lambda) = dl_\beta \left( \sum_e x_e(\lambda) \partial_e \right).$$

Then the elements of  $\mathcal{MF}_0(S)$  correspond exactly to the vectors in  $K_R$ . □

**Case of oriented ribbon graphs:** If  $R^\circ$  is an oriented ribbon graph on  $M^\circ$ , we consider in a similar way the spaces of Abelian differentials  $\mathcal{H}(R^\circ), \mathcal{H}_0(R^\circ)$ . As a corollary of 4.10, we obtain the following result:

**Corollary 4.6.** *All the admissible foliations on  $R^\circ$  are orientable,*

$$\mathcal{MF}(R) \subset \mathcal{MF}(M^\circ).$$

*The quadratic differential  $q_S(x)$  is the square of an Abelian differential  $\alpha_S(x)$ , and we have bijection*

$$\mathcal{H}(R^\circ) \simeq \text{Met}(R) \times T_R, \quad \text{and} \quad \mathcal{H}_0(R^\circ) \simeq \text{Met}(R) \times K_R.$$

**Remark 4.14** (Relation with cohomology). *If  $R^\circ$  is embedded in  $M$  we see that we have isomorphism*

$$K_R = H^1(M^{cap}, X_0R, \mathbb{R}) \quad T_R = H^1(M^\bullet, X_0R, \mathbb{R}).$$

*Then the map corresponds to the period coordinates.*

**Coordinate for integral multi-curves:** An important consequence of proposition 4.10 is the following proposition that characterizes the set of admissible integral multi-curves:

**Proposition 4.12.** *The set of admissible integral multi-curves  $\mathcal{MF}_{\mathbb{Z}}(R)$  is identified with the lattice of non-zero integral points  $K_R(\mathbb{Z}) \setminus \{0\}$  under the period coordinates  $x$ .*

*Proof.* This is mainly straightforward by using the construction. For an element in  $K_R(\mathbb{Z}) \setminus \{0\}$ , we can see that the horizontal foliation is necessarily periodic with cylinders of integral height. Conversely, if we take such foliation, we can see that the periods are integers.  $\square$

**Remark 4.15.** *The imaginary part of  $\alpha_S(x)$  then defines an element  $x_\lambda$  that belongs to the space  $T_R \simeq H^1(M, X_0R, \mathbb{R})$ . In a dual way, this element can be seen as a cycle in  $H_1(M^{cap} \setminus X_0R, X_2R, \mathbb{R})$ . The period coordinates are then  $x_\lambda(e) = \int_e x_\lambda = \langle x_\lambda, e \rangle$ . As we see an admissible curve  $\gamma$  is parametrized by the zig-zag permutation  $s^\pm$ , this can be used to define a cycle  $[\gamma] \in H_1(M^{top} \setminus X_0R, X_2R, \mathbb{Z})$ , we have the equality  $[\gamma] = x_\gamma$ .*

**Irreducible ribbon graphs and Fenchel-Nielsen decomposition:** Irreducible ribbon graphs generalize pairs of pants outside the generic case (vertices of order one or two). They form an interesting class of ribbon graphs, but we won't use them so much here.

**Definition 4.9.** *A ribbon graph is irreducible iff it's connected and satisfies  $\mathcal{MS}_{\mathbb{Z}}(R) = \emptyset$*

We call these graphs irreducible because we cannot reduce their topology by admissible surgeries. In some sense, they are minimal objects in the family of decorated surfaces. From the results of the last section, we can derive the following important characterisation:

**Proposition 4.13.** *A ribbon graph is irreducible iff it's connected and satisfies*

$$H^1(R, X_0R) = \{0\}.$$

Or, in an equivalent way, iff it satisfies  $K_R = \{0\}$  (and is connected).

*Proof.* The proof uses proposition 4.10. We have  $\mathcal{MF}_{\mathbb{Z}}(R) = K_R(\mathbb{Z}) \setminus \{0\}$  and then  $\mathcal{MF}(R)$  is empty iff  $K_R(\mathbb{Z})$  is zero and then iff  $K_R = \{0\}$ .  $\square$

**Remark 4.16.** *This proposition says that the only moduli for irreducible ribbon graphs are the boundary lengths. Then stratum in  $\mathcal{M}^{comb}(M, L)$  associated with irreducible ribbon graphs are finite sets with explicit cardinals. Moreover, for such graphs, if  $\Lambda_R = L_\partial(\text{Met}(R))$ , then the map:*

$$L_\partial : \text{Met}(R) \longrightarrow \Lambda_R$$

*is injective because the kernel is  $K_R$ . Then it's possible to define (explicitly) the inverse map. This generalizes result known for ribbon graphs on a pair of pants  $[ABC^+20]$  and see proposition 6.2.*

The irreducible ribbon graphs can be classified by the following interesting proposition, which is given by the computation of the dimension of  $K_R$ .

**Proposition 4.14.** *A ribbon graph is irreducible iff it's of genus zeros and it satisfies one of the two following conditions:*

- *It has only two vertices, and they are of odd degree.*
- *It's orientable and has only one vertex.*

*Proof.* In the oriented case, the dimension of  $K_R$  is given by  $2g-1+\#X_0R$ , and then the graph is necessarily a sphere with one vertex. In the un-orientable case, the dimension is  $2g-2+\#X_0R$ , and then it's a sphere with two vertices.  $\square$

To conclude this part, irreducible graphs can be used to obtain Fenchel-Nielsen decompositions of a non-generic graph. But outside the generic case, there is no hope that these coordinates are global coordinates on a given stratum of metric ribbon graphs.

**Corollary 4.7.** *For each ribbon graph  $R$ , there exists an admissible multi-curve  $\Gamma$  such that : all the components of  $R_\Gamma$  are irreducible. Such a multi-curve can be called a maximal admissible multi-curve.*

*Proof.* This is immediate by induction; using 4.13, if a graph is not irreducible, then  $\mathcal{MS}(R)$  is non-empty. By picking some element, we can cut along the admissible multi-curve and then reduce the topology. By doing this, we must reduce the dimension of  $K_R$  by at least one, and then, by induction, we obtain a maximal admissible multi-curve.  $\square$

**Dual coordinates on  $\widetilde{\mathcal{MF}}_0(R)$  and  $\mathcal{MF}_0(R)$ :** For each  $R$ , we define in paragraph 3.4.2 the space  $\widetilde{\mathcal{MF}}_0(M)$ . An element of this space can be represented by a foliation in  $\mathcal{MF}_0(M)$  marked by a choice of a periodic trajectory around each pole. Such foliation can be represented by a formal sum:

$$\tilde{\lambda} = \lambda + \sum_{\beta} h_{\beta}\beta; \quad \text{with } \lambda \in \mathcal{MF}_0(R), \text{ and } h \in \mathbb{R}_{\geq 0}^{\partial M}.$$

Let  $R$  be an embedded ribbon graph. It's possible to define the subspace  $\widetilde{\mathcal{MF}}_0(R)$  of foliations in  $\widetilde{\mathcal{MF}}_0(M)$  that are admissible on  $R$ . There is a map

$$\widetilde{\mathcal{MF}}_0(R) \longrightarrow \mathcal{MF}_0(R).$$

Quantities  $(y_e)_e$  of equation 4.2 are well defined for the elements of  $\widetilde{\mathcal{MF}}_0(R)$ . But in general, the map

$$y : \widetilde{\mathcal{MF}}_0(R) \longrightarrow \mathbb{R}_{\geq 0}^{X_1R}$$

is neither injective nor surjective. We define other parameters in the following way: For each  $e \in XR$ , let  $\gamma_e$  be the undirected arc that joins  $[e]_2$  and  $[e]_0$ , and we denote

$$|z_e| = \iota(\lambda, \gamma_e) \in \mathbb{R}_{>0}.$$

Which is the distance between the singularity  $[e]_0$  and the boundary curve  $[e]_2$ . They satisfy the relation (see figure 4.16)

$$x_e(\lambda) = |z_e(\lambda)| - |z_{s_2e}(\lambda)|, \quad \text{and} \quad y_e(\lambda) = |z_{s_0^{-1}e}(\lambda)| + |z_e(\lambda)|.$$

And the vector  $(|z_e|)$  also satisfies the constraints:

$$|z_{s_2s_1e}(\lambda)| + |z_e(\lambda)| = |z_{s_2e}(\lambda)| + |z_{s_1e}(\lambda)|, \quad \forall e \in XR. \quad (4.3)$$

Let  $W_R^+$  be the set of  $(|z_e|) \in \mathbb{R}_{>0}^{XR}$  that satisfy the last relations, and  $W_R = TW_R^+$  the tangent space.

When  $R^\circ$  is oriented according to proposition 4.6, all the admissible foliations are also oriented. It's possible to consider the cycles  $[\gamma_e]$  oriented from  $[e]_0$  to  $[e]_2$ ; they satisfy the relations

$$[\gamma_e] - [\gamma_{s_0^{-1}e}] = e^* \quad \text{and} \quad [\gamma_e] - [\gamma_{s_2e}] = e.$$

The family  $([\gamma_e])_{e \in XR}$  generates the relative homology, but it's not free; the cycles satisfy the boundary condition

$$[\gamma_e] - [\gamma_{s_0^{-1}e}] + [\gamma_{s_1e}] - [\gamma_{s_2e}] = 0.$$

Then, in the oriented case,  $W_R$  is identified with the cohomology

$$W_R = H^1(M_R^{cap}, X_0R \cup X_2R, \mathbb{R}).$$

When the graph is not oriented, the space  $W_R$  is given by the anti-invariant cohomology of the two covers  $\tilde{R}$

$$W_R = H^1(\tilde{M}_R^{cap}, X_0\tilde{R} \cup X_2\tilde{R}, \mathbb{R})^-.$$

**Proposition 4.15.** *The map*

$$\begin{aligned} |z| : \widetilde{\mathcal{MF}}_0(R) &\longrightarrow W_R^+ \\ \lambda &\longrightarrow |z\lambda| \end{aligned}$$

*is a bijection and identifies  $W_R^+(\mathbb{Z})$  with  $\widetilde{\mathcal{MF}}_{\mathbb{Z}}(M)$ . Moreover, in the oriented case, this map lifts to a map*

$$z : \widetilde{\mathcal{MF}}_0(R) \longrightarrow H^1(M_R^{cap}, X_0R \cup X_2R, \mathbb{R}).$$

*Such that*

$$z_e(\lambda) = \epsilon(e)|z_e(\lambda)| = \int_{[\gamma_e]} z\lambda.$$

We now prove the proposition 4.15; we restrict to oriented graphs for simplicity.

*Proof.* We use zippered rectangles for all  $(S, z) \in \text{Met}(R) \times W_R^+$  and let  $x(z)$  be the x-coordinates given by the last relation. From proposition 4.10 we can construct an Abelian differential  $\alpha_S(x)$  on  $M^\bullet$ . As we have

$$x_e = z_{s_2e} - z_e,$$

the sum along a boundary is zero, and then  $x(z)$  is in  $K_R$ , so  $\text{Im}(\alpha_S(x))$  is in  $\mathcal{MF}_0(R)$ . For each  $e$ , we can consider the horizontal trajectory along  $[e]_2$  that passes through the point  $(0, z_e) \in R_e$ ; this is well defined due to the constraints 4.3. The horizontal foliation and the trajectory do not depend on the choice of  $S \in \text{Met}(R)$ , and then we have the inverse map:

$$W_R^+ \longrightarrow \widetilde{\mathcal{MF}}_0(R).$$

□

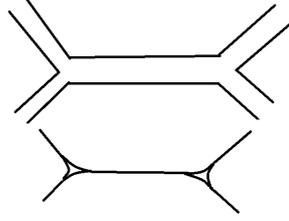


Figure 4.17: The train track.

**Remark 4.17.** *Relation with train tracks:*

**Lemma 4.18.** *There is a train track  $\tau_R$  such that  $W_R^+$  is the set of weights on  $\tau_R$ .*

*We construct the train track in the following way (see figure ??). For each oriented edge, we associate a vertex ( $v_e$ ). The edges of  $\tau_R$  are of two types:*

- *There is an edge  $s_e$  for all  $e \in X_1R$ ,  $s_e$  joins the two vertices labeled by the two directions of  $e$ .*
- *There is an edge  $s'_e$  for all  $e \in XR$ ; it joins  $v_{[e]_1}$  and  $v_{[s_0e]_1}$ .*

*Then  $W_R^+$  is the set of positive weights on the train track  $\tau_R$ . The train track is always non-degenerate; if  $W_R$  is the same space with real weights, then  $\dim W_R = \dim W_R^+ = \#X_1R + 1$ .*

**Length of curves:** From the proposition 4.15, it's possible to derive the following corollary. It is false if we restrict  $dl_\lambda$  to the space  $K_R$ .

**Corollary 4.8.** *For all  $\lambda$ , the one form  $dl_\lambda$  is equal to zeros on  $T_R$  iff the foliation  $\lambda$  is trivial.*

*Proof.* If  $dl_\lambda$  is zero on  $T_R$ , then using

$$dl_\lambda = \sum_e y_e(\lambda) dm_e$$

and the fact that  $(dm_e)_{e \in X_1R}$  is free implies that all the  $y_e$  are equal to 0. Then all the  $|z_e|(\lambda)$  are also vanishing, and then using proposition 4.15  $\lambda$  is trivial.  $\square$

## 4.2.4 Deformations of metric ribbon graphs, twist and horocyclic flows

We use curves and foliations to study the deformations of metric ribbon graphs. As in [ABC<sup>+</sup>20], we consider twist flows along admissible curves and relate these flows to the horocyclic flow on the space of quadratic differentials with double poles. It's much easier to understand them. This approach is inspired by the works of Mirzakhani for hyperbolic surfaces [Miro7], [Miro8b].

**Combinatorial twist:** We give the first intuitive definition of the twist flow, which is the same as the definition of the twist flow along a geodesic on a hyperbolic Riemann surface. For ribbon graphs, the situation is more complicated because gluing ribbon graphs does not necessarily creates a ribbon graph. Let  $S \in \mathcal{T}^{comb}(M)$  be an embedded metric ribbon graph and  $\gamma \in \mathcal{S}(M)$ . After cutting  $S$  according to  $\gamma$ , there is two new boundaries  $\gamma^1, \gamma^2$  in  $M_\gamma$  that

correspond to  $\gamma$ . The metric on the new graph identifies the two boundaries with  $\mathbb{R}/l_\gamma\mathbb{Z}$ , and there is a gluing map  $\phi : \mathbb{R}/l_\gamma\mathbb{Z} \rightarrow \mathbb{R}/l_\gamma\mathbb{Z}$  that identifies two points that coincide in  $S$ . The gluing map is an isometry and takes the form  $x \rightarrow -x + u$ . If we fix a base point  $x_0 \in \gamma_1$ , it's possible to recover  $S$  from  $S_\gamma$  by gluing the two boundaries  $\gamma^1, \gamma^2$  so that the two points  $x_0$  and  $\phi(x_0)$  coincide. For  $t$  small enough, it's also possible to do a translation  $x_t$  of  $x_0$  by  $-t$  in the direction of the boundary  $\gamma_1$ . This gives a new point  $\phi(x_t)$  on  $\gamma_2$ , and then we can perform the gluing by identifying  $x_0$  and  $\phi(x_t)$ . This corresponds to the choice of a new gluing map  $\phi_t = \phi + t$ . It should be somehow possible to glue the two surfaces together by using this new map. Indeed, by identifying  $S_\gamma$  to a foliation with leaves transverse to the boundaries, the operation makes sense. Nevertheless, it's not obvious that this process produces a ribbon graph or even a multi-arc. It could be a more general foliation transverse to the boundaries of  $M$ . We will give later a proof of the following proposition:

**Lemma 4.19.**

- For all metric ribbon graph  $S \in \text{Met}(R)$  and all curve  $\gamma$  on it, there is  $\delta > 0$  small enough and  $R^+ \geq R$  (resp  $R^- \geq R$ ) such that  $\phi_\gamma^t(S) \in \text{Met}(R^+)$  (resp  $\phi_\gamma^{-t}(S) \in \text{Met}(R^-)$ ) for  $t \in ]0, \delta[$ .
- Moreover, if  $S \in \text{Met}(R)$ , then  $\phi_\gamma^t(S) \in \text{Met}(R)$  for  $|t|$  small enough iff  $\gamma$  is admissible on  $R$ .

For two disjoint curves  $\gamma_1, \gamma_2$  we have  $(S_{\gamma_1})_{\gamma_2} = (S_{\gamma_2})_{\gamma_1} = S_{\gamma_1 \cup \gamma_2}$  and then the twist flows along disjoint curves commute

$$\phi_{\gamma_1}^t \circ \phi_{\gamma_2}^t = \phi_{\gamma_2}^t \circ \phi_{\gamma_1}^t.$$

Using that, it's possible to define for all  $\Gamma \in \mathcal{MS}_\mathbb{R}(M)$

$$\phi_\Gamma^t = \prod_{\gamma} \phi_\gamma^{m_\gamma t},$$

for  $t$  small enough.

**Remark 4.18** (Twist flow in coordinates:). *Let  $\gamma$  be an admissible curve on  $R$ . From lemma 4.16 there is a combinatorial representation  $(e_i)$  with  $e_{i+1} = s^\pm e_i$ . Then it's possible to define the algebraic intersection number  $\iota_\gamma(i)$  by*

$$\iota_\gamma(i) = \begin{cases} 1 & \text{if } e_i = s^+ e_{i-1} \quad e_{i+1} = s^- e_i \\ -1 & \text{if } e_i = s^- e_{i-1} \quad e_{i+1} = s^+ e_i \\ 0 & \text{else.} \end{cases}$$

Then  $\iota_\gamma(e)$  for  $e \in X_1 R$  is defined as the sum of the  $\iota_\gamma(i)$  over the  $i$  such that  $[e_i]_1 = e$ . These coefficients are independent of the combinatorial representation; they are well defined for integral multi-curves by linearity and satisfy the following lemma:

**Lemma 4.20.** *If  $\Gamma$  is admissible on  $R$ , the twist flow  $\Gamma$  is given locally by*

$$m_e(\phi_\Gamma^t(S)) = m_e(S) + t \iota_e(\Gamma).$$

Then, if  $\xi_\Gamma$  is the tangent vector of the twist flow with respect to  $\Gamma$ , we have

$$\xi_\Gamma = \sum_e \iota_e(\Gamma) \partial_e.$$

**Twist flows and horocyclic flow:** Another way to define the twist flows is by using the horocyclic flow for quadratic differentials. If  $M$  is in **bord\*** and  $q \in \mathcal{QT}(M)$ , locally we can find local coordinates  $z = x + iy$  such that  $q = (dz)^2$ . Then  $U_t q$  is defined locally by  $(dx + tdy + idy)^2$ ;  $U_t q$  then defines a new quadratic differential on a different Riemann surface. We can see that the flow is complete; for all  $t \in \mathbb{R}$ , it induces a homeomorphism<sup>12</sup>

$$U_t : \mathcal{QT}(M) \longrightarrow \mathcal{QT}(M).$$

The action on residues is of the form  $x + iy \longrightarrow x + ty + iy$ , then the horocyclic flow preserves the subspace  $\mathcal{QT}_0(M)$  of quadratic differentials with real residues, and it also preserves the level sets  $\mathcal{QT}_0(M, L)$ . By using the identification (cf. theorem 3.4),

$$\mathcal{QT}_0(M) \simeq \mathcal{MF}(M) \times \mathcal{MF}_0(M) \setminus \Delta.$$

For a pair of transverse foliations  $(\lambda_1, \lambda_2)$  in the RHS we can see that there is a foliation  $\phi_{\lambda_2}^t(\lambda_1)$  transverse to  $\lambda_2$  and such that

$$U_t(\lambda_1, \lambda_2) = (\phi_{\lambda_2}^t(\lambda_1), \lambda_2).$$

For all multi-curve  $\Gamma \in \mathcal{MS}_{\mathbb{R}}(M)$ , it's possible to consider the subspace  $\mathcal{MF}_{\Gamma}(M)$  in  $\mathcal{MF}(M)$  formed by foliations transverse to  $\Gamma$ . We have a natural map

$$q_{\Gamma} : \mathcal{MF}_{\Gamma}(M) \longrightarrow \mathcal{QT}(M).$$

Moreover the image is stable under the horocyclic flow. This defines a flow

$$\phi_{\Gamma}^t : \mathcal{MF}_{\Gamma}(M) \longrightarrow \mathcal{MF}_{\Gamma}(M),$$

we claim the following:

**Lemma 4.21.** *The flow  $\phi_{\Gamma}^t$  defined in this section corresponds to the twist flow defined in the last section.*

*Proof.* Assuming  $\gamma \in \mathcal{S}(M)$  and  $S$  a metric ribbon graph, we can consider  $q_{\gamma}(S)$ , which is Jenkin Strebel with a unique cylinder. Moreover, as we saw before, a way to construct this quadratic differential is to cut  $S$  along  $\gamma$  and glue the boundaries by adding a cylinder of height one. Then the intuitive notion of twist corresponds to the shear of this cylinder.  $\square$

We can sketch a proof of lemma 4.19.

*Proof.* We can see that the real part of the period of  $q_{\lambda}(S)$  along saddle connections belongs to the set  $\{\sum_e m_e(S) n_e | n \in \mathbb{N}^{X_1 R}\}$  which is discrete in  $\mathbb{R}$ . And then, for  $t \neq 0$  small enough, we can see that  $U_t q_{\lambda}(S)$  has no vertical saddle connection. Then we can deduce that for  $t > 0$  (resp  $t < 0$ ) small enough, there is  $R^+$  (resp  $R^-$ ) with  $\phi_{\gamma}^t(S)$  in  $\text{Met}(R^+)$  (resp  $\text{Met}(R^-)$ ). It's straightforward that if  $\lambda$  is admissible, we must have  $U_t(q_{\lambda}(S)) \in \mathcal{Q}_0(R)$  for  $t$  small enough, and then  $\phi_{\lambda}^t(S) \in \text{Met}(R)$  for  $t$  small enough. Conversely, if  $\phi_{\lambda}^t(S)$  is in  $\text{Met}(R)$  for small  $t$ , we can see that  $q_{\lambda}(S)$  has no vertical saddle connection, and then  $\nu_{q_{\lambda}(S)} = \nu_S$ .  $\square$

The following proposition is essential because it allows us to compute the twist flow in coordinates.

---

<sup>12</sup>For the topology given by the period coordinates.

**Proposition 4.16.** *If  $\lambda$  is admissible, the twist flow is locally a translation generated by the locally constant vector field  $\xi_\lambda$ .*

$$\xi_\lambda = \sum_e x_e(\lambda) \partial_e.$$

*Coordinates  $(x_e(\lambda))_{e \in X_1 R}$  are defined in proposition 4.10.*

Then, by using propositions 4.10 and 4.16, we obtain a proof of the following result:

**Corollary 4.9.** *Each vector in  $K_R$  is the tangent vector of a unique trajectory of the twist flow.*

**Tangent of cut $_\Gamma$ :** The following proposition is useful to decompose the measures on the moduli space; we restrict it to the case of oriented ribbon graphs. Let  $R^\circ$  be an oriented ribbon graph,  $\Gamma^\circ$  an admissible multi-curve, and  $\mathcal{G}^\circ$  the associated directed stable graph. We denote  $T_{R^\circ, \Gamma^\circ}$  the subspace of

$$T_{R^\circ, \Gamma^\circ} = \prod_c T_{R^\circ, \Gamma^\circ(c)}$$

defined by (see paragraph 3.3.2 for notation)

$$T_{R^\circ, \Gamma^\circ} = \prod_{\mathcal{G}} T_{R^\circ, \Gamma^\circ(c)}.$$

And let

$$T_{R^\circ, \Gamma^\circ}(\mathbb{Z}) = T_{R^\circ, \Gamma^\circ} \cap T_{R^\circ, \Gamma^\circ}(\mathbb{Z}),$$

which is a lattice in  $T_{R^\circ, \Gamma^\circ}$ .

**Proposition 4.17.** *If  $R^\circ$  is an oriented ribbon graph and  $\Gamma^\circ$  is admissible on  $R^\circ$ , there is an exact sequence:*

$$0 \longrightarrow \mathbb{Z}^{X_1 \Gamma} \longrightarrow T_{R^\circ}(\mathbb{Z}) \xrightarrow{T_{\text{cut}_\Gamma}} T_{R^\circ, \Gamma^\circ}(\mathbb{Z}) \longrightarrow 0.$$

*Moreover, the first map is defined by  $\gamma \rightarrow \xi_\gamma$ .*

*Proof.* It's the result of a long, exact sequence of relative integral homology; we have

$$0 \rightarrow H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(M^{\text{cap}} \setminus X_0 R, X_2 R, \mathbb{Z}) \rightarrow H_1(M^{\text{cap}} \setminus X_0 R, X_2 R \sqcup \Gamma, \mathbb{Z}) \rightarrow H_0(\Gamma, \mathbb{Z}) \rightarrow 0.$$

The kernel of

$$H_1(M^{\text{cap}} \setminus X_0 R, X_2 R \sqcup \Gamma, \mathbb{Z}) \rightarrow H_0(\Gamma, \mathbb{Z}),$$

is the space  $T_{R^\circ, \Gamma^\circ}(\mathbb{Z})$ ; the first non-trivial arrow is the inclusion, the second space is identified with the tangent space using  $f_R$  and Poincare duality. By excision, the last space is identified with

$$H_1(M_\Gamma^{\text{cap}} \setminus X_0 R_\Gamma, X_2 R_\Gamma, \mathbb{Z}),$$

which is also identified with  $T_{R_\Gamma}$ . It remains to prove that the last arrow is the tangent of the cutting map. We have for all  $\gamma \in \tilde{\mathcal{S}}(M_\Gamma)$

$$l_\gamma(S_\Gamma) = l_\gamma(S).$$

And then

$$\text{cut}_\Gamma^* dl_\gamma = dl_\gamma.$$

Moreover, we have  $f_R^*(dl_\gamma) = y_\gamma$ . Then, for all cocycles  $\langle \text{cut}_\Gamma(x), \gamma \rangle = \langle x, \gamma \rangle$ , we can now conclude because the only map that satisfies this property is the natural inclusion.  $\square$

**Gluing and bundles for directed surfaces:** In this part, we study in more detail the map  $\text{cut}_\Gamma$  in a particular case. Let  $M^\circ$  be a directed surface and  $\Gamma^\circ$  be a directed multi-curve on it. We consider the subset  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  of oriented foliations on  $M^\circ$  transverse to  $\Gamma$ , which are represented by an Abelian differential. Such differential induces a direction on the curve of  $\Gamma$ , and we assume that it corresponds to  $\Gamma^\circ$ . For all stable graphs  $\mathcal{G}^\circ$ , let  $\mathcal{T}^{\text{comb}}(\mathcal{G}^\circ)$  be the subset of  $\prod_c \mathcal{T}^{\text{comb}}(\mathcal{G}^\circ(c))$ :

$$\mathcal{T}^{\text{comb}}(\mathcal{G}^\circ) = \prod_{\mathcal{G}} \mathcal{T}^{\text{comb}}(\mathcal{G}^\circ(c)).$$

Let  $\Gamma^\circ$  be an oriented multi-curve with a directed stable graph  $\mathcal{G}^\circ$ , then we have the following proposition:

**Proposition 4.18.**  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  is stable under the twist flow along each curve in  $\Gamma$ ; moreover, the map

$$\text{cut}_\Gamma : \mathcal{MF}_{\Gamma^\circ}(M^\circ) \longrightarrow \mathcal{T}^{\text{comb}}(\mathcal{G}^\circ).$$

is surjective, and it's an affine  $\mathbb{R}^\Gamma$  bundle compatible with the integral structure.

**Remark 4.19.** This proposition remains true for each stratum. If  $\overline{M}^\circ$  is directed,  $\overline{\Gamma}^\circ \in \mathcal{MS}(\overline{M}^\circ)$ , and  $\overline{\mathcal{G}}^\circ$  is the corresponding decorated directed stable graph. Then we can consider  $\mathcal{MF}_{\overline{\Gamma}^\circ}(\overline{M}^\circ)$  the oriented foliations  $\lambda^\circ$  such that  $\lambda$  is transverse to  $\Gamma$ , and the orientation induced on  $\Gamma$  corresponds to  $\Gamma^\circ$ . Moreover, the square root of  $q_\Gamma(\lambda)$  defines a decoration on each connected component of  $M_{\Gamma^\circ}^\circ$  and then on  $\Gamma^\circ$ . We assume that this decoration coincides with  $\overline{\Gamma}^\circ$ . Then we still have

$$\text{cut}_\Gamma : \mathcal{MF}_{\overline{\Gamma}^\circ}(\overline{M}^\circ) \rightarrow \mathcal{T}^{\text{comb}}(\overline{\mathcal{G}}^\circ),$$

and the statement of the last proposition remains true.

We prove the following lemma, which gives the existence of local twist coordinates and also gives the proposition.

**Lemma 4.22.** For each  $S \in \mathcal{T}^{\text{comb}}(\mathcal{G}^\circ)$ , it exists  $V \subset \mathcal{T}^{\text{comb}}(\mathcal{G}^\circ)$  and  $U \subset \mathcal{MF}_{\Gamma^\circ}(M^\circ)$  such that

- $V$  is an open neighborhood of  $S$  invariant by dilatation, and  $U = \text{cut}_{\Gamma^\circ}^{-1}(V)$ .
- For all  $\gamma \in \Gamma$ , we can find a map  $t_\gamma : U \rightarrow \mathbb{R}$  such that

$$t_\gamma(\phi_{\gamma'}^t) = t_\gamma + t\delta_{\gamma,\gamma'}$$

for all  $\gamma' \in \Gamma$ .

- There is a piecewise linear isomorphism:

$$U \longrightarrow V \times \mathbb{R}^\Gamma,$$

which induces a bijection

$$U_{\mathbb{Z}} \longrightarrow V_{\mathbb{Z}} \times \mathbb{Z}^\Gamma.$$

Where  $U_{\mathbb{Z}}, V_{\mathbb{Z}}$  are the sets of integer points

**Remark 4.20.** In this lemma, another choice for the parameter  $(t'_\gamma)$  is related to  $(t_\gamma)$  by a relation of the form

$$t'_\gamma = t_\gamma + h_\gamma(S_\Gamma),$$

with  $h_\gamma$  linear and corresponds to another choice of saddle connection.

*Proof.* Let  $S \in \mathcal{T}_{\Gamma^\circ}^{comb,*}(M^\circ)$  and let  $(R_\Gamma(c))_{c \in X_0 \mathcal{G}}$  such that  $S_\Gamma \in \prod_c \text{Met}(R_\Gamma(c))$ , we define  $U$  as the set of foliation  $\lambda \in \mathcal{MF}_{\Gamma^\circ}(M^\circ)$  such that  $\text{cut}_\Gamma(\lambda) \in \prod_c \text{Met}(R_\Gamma(c))$  and we take  $V$  as the set  $S' \in \prod_c \text{Met}(R_\Gamma(c))$  with  $l_\beta(S') = l_{s_1(\beta)}(S')$ . For each  $\gamma \in \Gamma$  we have

$$\phi_\gamma^t(\lambda) = \text{Re}(U_t q_\gamma(\lambda)).$$

So  $\text{cut}_\Gamma(\phi_\gamma^t(\lambda)) = \text{cut}_\Gamma(\lambda)$  and then  $U$  is invariant under the twist flow. For all  $\gamma \in \Gamma$ , we denote  $C_\gamma^\Gamma(\lambda)$ , the cylinder in  $q_\Gamma(\lambda)$  associated with  $\gamma$ . It's possible to fix two singularities  $s_\gamma^1, s_\gamma^2$  one on each boundary of  $C_\gamma^\Gamma(\lambda)$  and an arc  $a_\gamma$  contained in  $C_\gamma^\Gamma(\lambda)$  that joins these two singularities. Then we can consider

$$t_\gamma(\lambda) = \frac{\text{Re}\langle \sqrt{q_\Gamma(\lambda)}, a_\gamma \rangle}{\text{Im}\langle \sqrt{q_\Gamma(\lambda)}, a_\gamma \rangle}.$$

Which does not depend of the choice of the roots of the quadratic differential. We have the relation

$$t_\gamma(\phi_{\gamma'}^t \lambda) = t_\gamma(\lambda) + t\delta_{\gamma,\gamma'}$$

We have  $U_{\mathbb{Z}} = U \cap \mathcal{MF}_{\mathbb{Z}}(M^\circ)$ , and then if  $\lambda \in U_{\mathbb{Z}}$ , we have  $t_\gamma(\lambda) \in \mathbb{Z}$  (because the denominator is in  $\{1, -1\}$ ) and  $\text{cut}_\Gamma(\lambda) \in V_{\mathbb{Z}}$  because by definition the periods are integers. By gluing, it's possible to construct the inverse map

$$gl_{\Gamma,a} : V \times \mathbb{R}^\Gamma \longrightarrow U$$

By construction, the maps  $\text{cut}_\Gamma, t_\gamma, gl_\Gamma$  preserve the set of integral points. □

**Covering of admissible curves:** Let  $\Gamma^\circ$  be a directed curve on  $M^\circ$ , and let  $\mathcal{T}_{\Gamma^\circ}^{comb,*}(M^\circ)$  be the set of generic metric ribbon graphs  $S$  such that  $\Gamma$  is admissible and the directions are compatible  $\Gamma_S^\circ = \Gamma^\circ$ . There is a natural inclusion:

$$\mathcal{T}_{\Gamma^\circ}^{comb,*}(M^\circ) \longrightarrow \mathcal{MF}_{\Gamma^\circ}(M^\circ),$$

and the restriction of the cutting map defines a map:

$$\mathcal{T}_{\Gamma^\circ}^{comb,*}(M^\circ) \longrightarrow \mathcal{T}^{comb,*}(M_{\Gamma^\circ}^\circ).$$

Let  $\mathcal{MF}_{\Gamma^\circ}^*(M^\circ)$  be the subset of foliations with no saddle connection at all. This subset corresponds to the subset of foliation represented by an Abelian differential with simple zeros. The foliations in  $\mathcal{MF}_{\Gamma^\circ}^*(M^\circ)$  are necessarily represented by a quadrivalent ribbon graph. And then there is a bijection:

$$\mathcal{MF}_{\Gamma^\circ}^*(M^\circ) \longrightarrow \mathcal{T}_{\Gamma^\circ}^{comb,*}(M^\circ).$$

**Remark 4.21.** A similar statement is also valid for  $\overline{M}^\circ, \overline{\Gamma}^\circ, \overline{\mathcal{G}}^\circ$ ; we have an identification

$$\mathcal{MF}_{\overline{\Gamma}^\circ}^*(\overline{M}^\circ) \longrightarrow \mathcal{T}_{\overline{\Gamma}^\circ}^{comb,*}(\overline{M}^\circ)$$

Where  $\mathcal{T}_{\overline{\Gamma}^\circ}^{comb,*}(\overline{M}^\circ)$  is the set of oriented metric ribbon graphs such that  $\Gamma$  is admissible, the decoration given by  $\Gamma$  corresponds to  $\overline{\Gamma}$ , and the orientation is the one of  $\overline{\Gamma}^\circ$ .

## 4.3 Acyclic decomposition

### 4.3.1 Symplectic geometry on the space of metric ribbon graphs

**The anti-symmetric pairing on  $K_R$ :** Let  $R$  be a ribbon graph. As we see in paragraph 4.1 there are natural identifications between the tangent space and the cohomology of the ribbon graph:

$$T_R \simeq^{f_R} H^1(R, X_0R, X_2R) \quad \text{and} \quad K_R \simeq^{f_R} H^1(R, X_0R).$$

The space  $H^1(R)$  can be identified with the anti-invariant cohomology of the two covers  $\tilde{M}_R^{cap}$  (and the cohomology of  $M_R^{cap}$  when the graph is oriented). There is a natural anti-symmetric pairing given by the cup product on the cohomology of  $\tilde{M}_R^{cap}$ ,

$$\langle \omega_1, \omega_2 \rangle = \frac{1}{2} \int_{\tilde{M}_R^{cap}} \omega_1 \wedge \omega_2.$$

We can drop the  $\frac{1}{2}$  and integrate over  $M_R^{cap}$  in the oriented case. This defines an anti-symmetric pairing  $\Omega_R$  on  $H^1(R)$ .

**Lemma 4.23.** *The space  $(H^1(R), \Omega_R)$  is a symplectic vector space.*

*Proof.* It's well known that the intersection pairing is non-degenerate on the cohomology of a compact surface; moreover, the decomposition into anti-invariant and invariant cohomologies respects the two forms, and then it's orthogonal. Then the pairing is non-degenerate on the anti-invariant cohomology. The space  $(H^1(R), \Omega_R)$  is then a symplectic vector space.  $\square$

There is a natural map from the relative cohomology to the absolute cohomology:

$$K_R \simeq H^1(R, X_0R) \longrightarrow H^1(R),$$

We also denote  $\Omega_R$ , the anti-symmetric pairing induced on  $K_R$  by taking the pull back of  $\Omega_R$  on  $H^1(R)$ .

**Degeneration of the symplectic structure:** The long exact sequence in the relative cohomology is useful to study the degeneration of the symplectic structure. We have the following lemma:

**Lemma 4.24.** *For each ribbon graph  $R$ , we have an exact sequence of relative cohomology:*

$$0 \rightarrow H^0(R) \rightarrow H^0(X_0R) \rightarrow H^1(R, X_0R) \rightarrow H^1(R) \rightarrow 0.$$

*Moreover, the dimension of  $H^0(X_0R)$  is the number of vertices with an even degree, and the dimension of  $H^0(R)$  is the number of orientable connected components of the graph  $R$ .*

*Proof.* We have a short exact sequence

$$0 \rightarrow C^*(R, X_0R) \rightarrow C^*(R) \rightarrow C^*(X_0R) \rightarrow 0,$$

and it gives the long exact sequence of cohomology. We have  $H^0(R, X_0R) = 0$  because  $C^*(R, X_0R) = 0$ ; and  $H^1(X_0R) = 0$  because  $C^1(X_0R) = 0$ . We also have

$$H^0(X_0R) \simeq C^0(X_0R)^- \simeq (\mathbb{R}^{X_0\tilde{R}})^-.$$

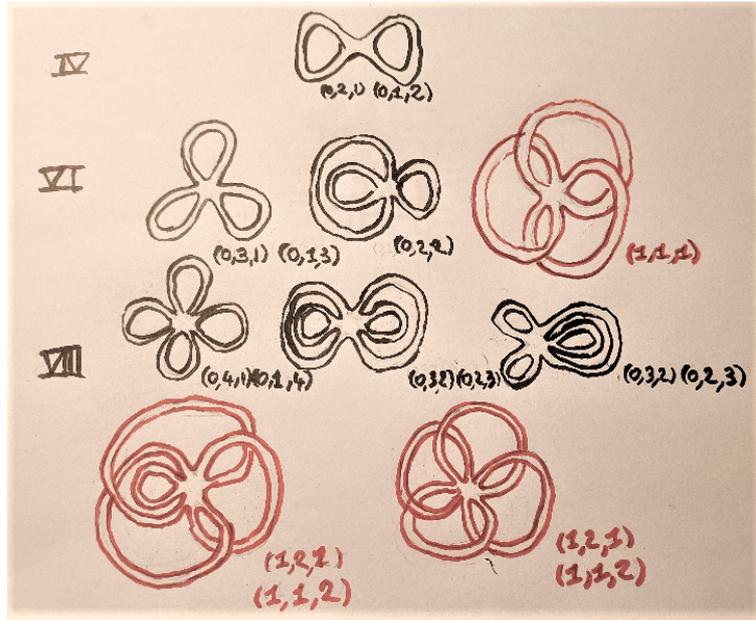


Figure 4.18: List of minimal oriented graphs with a vertex of degree 4, 6, 8; the red ones are not irreducible

The involution acts on  $X_0\tilde{R}$ . For each  $v \in X_0R$ , either  $v$  is odd, and then it has only one pre-image fixed by the involution. Either it's even and it has two pre-images exchanged by the involution. Then we see that the space of anti-invariant elements is generated by  $X_0^{\text{even}}R$ .  $\square$

As we see in corollary 4.1, we can identify  $H^1(R, X_0R)$  with  $K_R$ , and the image  $\hat{H}_R$  of  $H^0(X_0R)$  in  $K_R$  measures how the pairing is degenerate. In particular, we can see that the pairing is non-degenerate when the graph has only vertices of odd degrees. When the ribbon graph is orientable, the dimension of this space is the number of vertices of the graph minus one. The pairing is then non-degenerate on the quotient  $K_R/\hat{H}_R$ . We remark that there is no natural pairing on the space  $T_R$ .

**Minimal graphs:** From lemma 4.24, when the graph is oriented, the form  $\Omega_R$  is non-degenerate iff the graph has only one vertex. In a dual way, such graphs are also unicellular maps, and as we see later, they are building bricks to construct more complicated oriented ribbon graphs.

**Definition 4.10.** An oriented ribbon graph  $R^\circ$  is minimal iff it has only one vertex, i.e., iff  $(K_R, \Omega_R)$  is a symplectic vector space.

In figure 4.18, we give minimal graphs for low degrees. In general, a minimal ribbon graph is not necessarily irreducible; it can have a nontrivial genus (see 4.9 for a definition of an irreducible graph).

**The dual pairing:** To work with  $dl_\lambda$ , it's somewhat useful to consider the dual picture; according to paragraph 4.1.2 there is also a map

$$T_R^* \xrightarrow{f_R^*} H_1(R, X_0R, X_2R).$$

Let  $\hat{K}_R$  be the subspace  $\hat{K}_R = H_1(R, X_2R)$ . In this case, we can write the exact sequence for the relative homology:

$$0 \leftarrow H_0(R) \leftarrow H_0(X_0R) \leftarrow H_1(R, X_0R, X_2R) \leftarrow H_1(R, X_2R) \leftarrow 0.$$

The space  $\hat{K}_R$  is the kernel of the map  $T_R^* \rightarrow \hat{H}_R^*$  and then is equal to  $(T_R/\hat{H}_R)^*$ . As in the case of the tangent space, the intersection pairing induces an anti-symmetric pairing  $\hat{\Omega}_R$  on  $\hat{K}_R$ . Instead of the relative sequence, we use the Mayer-Vietoris sequence for the homology to study the degenerations of the pairing. The sequence takes the form

$$0 \leftarrow H_1(R) \leftarrow H_1(R, X_2R) \leftarrow H_2(X_2R) \leftarrow H_2(R) \leftarrow 0.$$

The image of  $H^0(X_2R, \mathbb{R})$  corresponds to the space  $H_R$ , and  $\hat{\Omega}_R$  induces a symplectic structure on  $\hat{K}_R/H_R$ .

**Pairing on  $W_R$ :** In section 4.2.3, we consider the spaces

$$W_R = H^1(\tilde{M}_R^{cap}, X_2\tilde{R} \cup X_0\tilde{R}, \mathbb{R})^- = H_1(\tilde{M}_R^{cap} \setminus X_2\tilde{R} \cup X_0\tilde{R}, \mathbb{R})^-.$$

There are two canonical, forgetful maps:

$$\begin{aligned} p_R &: H^1(\tilde{M}_R^{cap}, X_2\tilde{R} \cup X_0\tilde{R}, \mathbb{R})^- \longrightarrow H^1(\tilde{M}_R^{cap}, X_0\tilde{R}, \mathbb{R})^-, \\ p_R^* &: H^1(\tilde{M}_R^{cap}, X_2\tilde{R} \cup X_0\tilde{R}, \mathbb{R})^- \longrightarrow H^1(\tilde{M}_R^{cap}, X_2\tilde{R}, \mathbb{R})^-. \end{aligned}$$

Which are surjectives. On each of these cohomology spaces, there is a canonical intersection pairing  $\langle \cdot, \cdot \rangle$ , and they satisfy:

$$\langle \alpha_1, \alpha_2 \rangle = \langle p_R(\alpha_1), p_R(\alpha_2) \rangle = \langle p_R^*(\alpha_1), p_R^*(\alpha_2) \rangle.$$

Let  $\bar{\Omega}_R$  be the pairing induced on  $W_R$ , then we have the following lemma:

**Lemma 4.25.** *The three pairings  $\Omega_R, \hat{\Omega}_R$  and  $\bar{\Omega}_R$  satisfy*

$$\bar{\Omega}_R(z_1, z_2) = \Omega_R(p_R(z_1), p_R(z_2)) = \hat{\Omega}_R(p_R^*(z_1), p_R^*(z_2)) = \langle p_R^*(z_2), p_R(z_1) \rangle \quad \forall z_1, z_2 \in W_R$$

*The last bracket corresponds to the one between  $T_R^*$  and  $T_R$ .*

**Hamiltonian of the horocyclic flow:** Here we give a sketch of the proof of the following theorem, which was also proved in [ABO17] for the principal stratum.

**Theorem 4.1.** *Let  $R$  be any ribbon graph and  $\lambda \in \mathcal{MF}_0(R)$ . We have the following identity on  $K_R$ :*

$$\Omega_R(\xi_\lambda, \cdot) = -dl_\lambda.$$

*Proof.* The proof is immediate, for all  $\lambda \in \mathcal{MF}_0(R)$ , we construct in proposition 4.15 an element  $z(\lambda) \in H^1(\tilde{M}_R^{cap}, X_0\tilde{R} \cup X_2\tilde{R}, \mathbb{R})^-$  and by definition of  $p_R$ .

$$p_R(z(\lambda)) = x(\lambda) \quad \text{and} \quad p_R^*(z(\lambda)) = y(\lambda)$$

Where  $x(\lambda)$  and  $y(\lambda)$  are defined in 4.10. Moreover, from proposition 4.16, we have  $f_R(\xi_\lambda) = x(\lambda)$  and  $\hat{f}_R(dl_\lambda) = y(\lambda)$ . And finally, from 4.25 we have:

$$\Omega_R(\xi_\lambda, \xi_{\lambda'}) = \langle x(\lambda), x(\lambda') \rangle = -\langle y(\lambda), x(\lambda') \rangle = -\sum_e y_e(\lambda) x_e(\lambda') = -dl_\lambda(\xi_{\lambda'})$$

The map  $\lambda' \rightarrow \xi_{\lambda'}$  is surjective on  $K_R$ ; we deduce that

$$\Omega_R(\xi_\lambda, \cdot) = -dl_\lambda(\cdot).$$

□

**Computation of the pairing in coordinates:** In this section, we give several expressions for the pairing. The elements  $(x_e)_{e \in X_1 R}$  and  $(y_e)_{e \in X_1 R}$  define 1-forms on  $W_R$ ; using them, it's possible to express the 2-form  $\bar{\Omega}_R$ :

$$\bar{\Omega}_R = \frac{1}{2} \sum_{e \in X_1 R} x_e \wedge y_e.$$

And then,  $\bar{\Omega}_R$  is one half the pullback of the canonical symplectic form under a map:

$$\begin{aligned} W_R &\longrightarrow T_R \times T_R^* \\ z &\longrightarrow (p_R(z), p_R^*(z)) \end{aligned}$$

In term of the  $z$  coordinates, we also have the following expression:

$$\bar{\Omega}_R = \frac{1}{2} \sum_{e \in X R} z_{s_0 e} \wedge z_e,$$

which is in some sense the Thurston form due to similarity with the Thurston two form on the train track  $\tau_R$ . There is a dual expression of this formula. The roles of  $s_0$  and  $s_2$  are in some sense symmetric, and we have

$$\bar{\Omega}_R = \frac{1}{2} \sum_{e \in X R} z_e \wedge z_{s_2 e}.$$

These two expressions come from the formula

$$x_e \wedge y_e = z_e \wedge z_{s_0^{-1} e} + z_{s_1 e} \wedge z_{s_0^{-1} s_1 e} = z_e \wedge z_{s_2 e} + z_{s_1 e} \wedge z_{s_2 s_1 e}.$$

It's also possible to give expressions in terms of the forms  $x$  and  $y$ . In [Kong2], M. Kontsevich introduces for each boundary  $\beta$  a two-form on  $K_R$  defined in the following way: Let an edge  $e$  with  $[e]_2 = \beta$  and assume that the boundary contains  $r$  edges,

$$\omega_\beta = \sum_{0 \leq i < j < r} x_{s_2^i e} \wedge x_{s_2^j e} = \sum_{1 \leq j \leq r} z_{s_2^{i-1} e} \wedge z_{s_2^i e} = \sum_{e, [e]_2 = \beta} z_e \wedge z_{s_2 e}.$$

Then, by summing over the boundaries, we recover the second Thurston form and deduce

$$\bar{\Omega}_R = \frac{1}{2} \sum_{\beta} \omega_\beta.$$

In a similar way, for each vertex  $v$ , we can fix an edge  $e$  with  $[e]_0 = v$ . If the vertex is of degree  $r$ , then we can set

$$\hat{\omega}_v = \sum_{0 \leq i < j < r} (-1)^{i+j} y_{s_0^i e} \wedge y_{s_0^j e} = - \sum_{1 \leq j \leq r} z_{s_0^i e} \wedge z_{s_0^{i-1} e}$$

and then

$$\bar{\Omega}_R = \frac{-1}{2} \sum_{\beta} \hat{\omega}_v.$$

**Degenerations of the structure:** From the results of the last section, the structure is degenerated on the space  $\hat{H}_R$ , which is the image of

$$H^0(X_0R) \longrightarrow K_R.$$

The derivative of the combinatorial length defines a map:

$$dl_{\bullet} : \mathcal{MF}_0(R) \longrightarrow K_R^*.$$

Moreover, we have the following corollary:

**Corollary 4.10.** *An element  $\xi$  is in  $\hat{H}_R$  iff  $dl_{\lambda}(\xi) = 0$  for all  $\lambda \in \mathcal{MF}_0(R)$*

**Remark 4.22.** *This result means that two metric ribbon graphs in  $\text{Met}(R)$  are on the same leaf of the kernel foliation  $\hat{H}_R$  iff they have the same "geometry". In the sense that they have the same length spectrum on admissible closed curves. This is not true for non-admissible curves.*

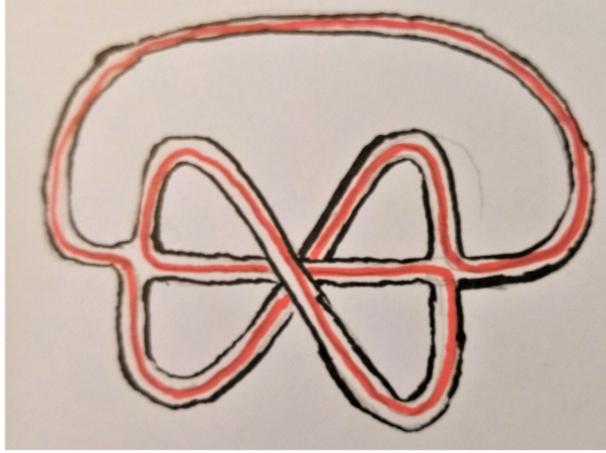


Figure 4.19: Acyclic decomposition of a graph with two vertices

### 4.3.2 Acyclic decomposition

In this section, we state the main theorem of the chapter theorem 4.2. We construct canonical curves that lie in the kernel foliation, and we use them to decompose ribbon graphs with vertices of even degree.

**Statement of results:** Let  $R$  be a ribbon graph. We say that an admissible multi-curve spares a vertex  $v$  in  $R$  from the rest of the graph if the component that contains  $v$  in  $R_\Gamma$  is minimal (see definition 4.10).

**Theorem 4.2.** Let  $R^\circ$  be an oriented metric ribbon graph with at least two vertices. For each vertex  $v$ , there exists a unique admissible primitive multi-curve  $\Gamma_v^+$  such that:

- The stable graph  $\mathcal{G}_v^\circ$  of  $\Gamma_v^+$  contains a component  $c_v$  that spares  $v$  from the rest of the graph.
- All the curves in  $\Gamma_v^+$  are boundaries of  $c_v$ .
- $c_v$  is glued along negative boundaries only.

These multi-curves are intimately related to degenerations of the symplectic structure  $\Omega_R$ . In other words, the last point means that the directed stable graph associated with  $\Gamma_v^+$  is acyclic and the component  $c_v$  is a maximal element for the partial order. The multi-curve  $\Gamma_v^+$  satisfies several elementary properties.

- The multi-curve is functorial in the sense that if  $g : (R_1^\circ, v_1) \rightarrow (R_2^\circ, v_2)$  is a morphism that preserves the orientation, then  $g \cdot \Gamma_{v_1}^+(R_1^\circ) = \Gamma_{v_2}^+(R_2^\circ)$ .
- There is a dual theorem for negative boundaries:  $\Gamma_v^-(R^\circ)$  is defined to be  $\Gamma_v^+(-R^\circ)$ , where  $-R^\circ$  is obtained by reversing the orientation of  $R^\circ$ . If  $\xi_v^\pm$  is the twist flow along  $\Gamma_v^\pm(R^\circ)$ , then we have  $\xi_v^- = -\xi_v^+$ .
- The tangent vectors of the twist flow  $\xi_v^+$  are in  $\hat{H}_R$ .

**Remark 4.23.** *Theorem 4.2 is quite surprising; it means that the local structure of the graph around an even vertex admits some model that corresponds to a minimal ribbon graph. Moreover, it's possible to cut the graph around the vertex in a canonical way. This process allows us to recover the structure of the graph inductively on the number of vertices, and the recursion scheme is very similar to an oriented version of the topological recursion, as we will see later.*

**Acyclic decomposition:** We start with the following three definitions:

**Definition 4.11.**

1. *An acyclic decomposition of an oriented ribbon graph  $R^\circ$  is an admissible multi-curve  $\Gamma^\circ$  such that the directed stable graph  $\mathcal{G}^\circ$  associated to  $\Gamma^\circ$  is acyclic.*
2. *An acyclic decomposition is maximal if it's not contained in another acyclic decomposition.*
3. *A linear order<sup>13</sup> on  $R^\circ$  is an enumeration of the vertices of the ribbon graph.*

In this section, we prove the following theorem given in the introduction:

**Theorem 4.3.** *Let  $R^\circ$  be an oriented ribbon graph with a linear order; then there is a unique admissible primitive multi-curve  $\Gamma^\circ$  such that*

1. *The components of  $R_{\Gamma^\circ}^\circ$  are minimal.*
2. *The directed stable graph  $\mathcal{G}^\circ$  associated to  $\Gamma^\circ$  is acyclic.*
3. *The linear order on the vertices induces a linear order on the acyclic stable graph.*

With the next lemma, we can rephrase this result in the following way: Given a linear order on the ribbon graph  $R^\circ$ , there is a unique maximal acyclic decomposition that is compatible with this linear order. We remark that different linear orders can produce the same decomposition; a given acyclic stable graph can have several linear orders.

**Lemma 4.26.** *An acyclic decomposition  $\Gamma^\circ$  of an oriented graph  $R^\circ$  is maximal iff all the components of  $R_{\Gamma^\circ}^\circ$  are minimal (see definition 4.10).*

*Proof.* Assume that some components are not minimal. Pick one of them. Using the theorem 4.2, we can decompose this component along a multi-curve  $\Gamma'$  by removing a vertex. Moreover, the directed stable graph associated to this new decomposition is acyclic. Using proposition 3.8, the stable graph associated with  $\Gamma \sqcup \Gamma'$  is still acyclic, and then  $\Gamma$  is not maximal. Assuming that all the components are minimal, it's easy to see that an admissible multi-curve on a minimal graph is necessarily associated with a directed stable graph with a non-trivial cycle, and then a finer decomposition won't be acyclic.  $\square$

We now prove the theorem 4.3.

---

<sup>13</sup>We use this terminology about linear order to be coherent with the terminology on directed graphs and order relations in general.

*Proof.* We can proceed by induction on the number of vertices. It's, of course, trivial when the graph has only one vertex; the multi-curves are empty in this case. If we assume the property is true for ribbon graphs with  $n$  vertices and let  $R^\circ$  be an oriented graph with  $n + 1$  vertices and a linear order, Let  $v_{n+1}$  be the vertex labeled by  $n + 1$ . According to theorem 4.2, we can extract  $v_{n+1}$  by using  $\Gamma_{v_{n+1}}^+$ . We obtain a ribbon graph  $R_{n+1}^\circ$  that contains  $v_{n+1}$  and a family of ribbon graphs glued to  $R_{n+1}^\circ$ . The linear order on  $R^\circ$  induces a linear order on each of these graphs, and then, by assumption, we can find an acyclic decomposition of each graph compatible with the linear order. Then, using proposition 3.8, the concatenation of these decompositions is still acyclic, and the linear order on  $R^\circ$  induces a linear order on the directed stable graph. Uniqueness is a consequence of uniqueness in theorem 4.2. Assuming that we have such decomposition, then the component that contains  $v_{n+1}$  is necessarily a local maximum of the acyclic stable graph, and this component is necessarily glued along negative boundaries only. If  $\Gamma_{n+1}$  are the curves in the decomposition that are in the boundary of this component, then  $\Gamma_{n+1}$  satisfies the hypothesis of theorem 4.2 and then must be  $\Gamma_{v_{n+1}}^+$ . By induction, we obtain uniqueness.  $\square$

**Remark 4.24.** *This result is different in general to a decomposition into Fenchel-Nielsen coordinates. In this case, the graph is not necessarily acyclic, and the components are not supposed to be irreducible. With theorem 4.3, we are not allowed to cut the surface along a curve, which is a handle; some components of the decomposition can have non trivial genus.*

In the case of the sphere, the minimal ribbon graphs are irreducible, so we have the following corollary:

**Corollary 4.11.** *Let  $R^\circ$  be an oriented ribbon graph on the sphere with a linear order, then there is a unique admissible multi-curve  $\Gamma^\circ$  such that:*

1. *The directed stable graph  $\mathcal{G}^\circ$  associated with  $\Gamma^\circ$  is acyclic.*
2. *The linear order on the vertices is compatible with the order on the graph.*
3. *All the components of  $R_{\Gamma^\circ}^\circ$  are irreducible.*

We give the proof of theorem 4.2, but this proof can be improved by using more cohomology. It might be possible to find a combinatorial proof; our proof is between combinatorix and cohomology.

**Proof of the theorem 4.2:** Let  $R^\circ = (R, \epsilon)$  be an oriented ribbon graph, and let  $v$  be a vertex. We construct the tangent vector associated with the twist flow along  $\Gamma_v^+$ . We consider

$$\xi_v^+(R^\circ) = - \sum_{e, [e]_0=v} \epsilon(e) \partial_{[e]_1}.$$

By using corollary 4.9, we have the following corollary:

**Corollary 4.12.** *There is a unique admissible multi-curve  $\Gamma_v^+(R^\circ) \in \mathcal{MS}_{\mathbb{Z}}(R)$  such that:*

$$\xi_{\Gamma_v^+(R^\circ)} = \xi_v^+(R^\circ).$$

We prove that the multi-curve  $\Gamma_v^+$  satisfies all the desired properties. We start by proving two lemmas.

**Lemma 4.27.** *The vector  $\xi_v^+(R^\circ)$  belongs to  $\hat{H}_R(\mathbb{Z})$  (see paragraph 4.3.1 for definitions) and the only relations between the  $(\xi_v^+(R^\circ))_{v \in X_0 R}$  are proportional to*

$$\sum_{v \in X_0 R} \xi_v^+(R^\circ) = 0.$$

Moreover, the vectors  $(\xi_v^+)_{v \in X_0 R}$  span  $\hat{H}_R$ .

*Proof.* We have the exact sequence for the relative homology:

$$0 \longleftarrow H_0(R) \longleftarrow H_0(X_0 R) \xleftarrow{\partial} H_1(R, X_0 R) \longleftarrow H_1(R)$$

Moreover, the boundary map is just

$$\partial[e] = [s_1 e]_0 - [e]_0.$$

Then, by duality, the boundary map in the cohomology

$$d : H^0(X_0 R) \longrightarrow H^1(R, X_0 R),$$

is defined by

$$\langle d[v], [e] \rangle = \langle [v], d[e] \rangle = \delta_{v, [s_1 e]_0} - \delta_{v, [e]_0}.$$

By using the identification between  $H^1(R, X_0 R)$  and  $K_R$ , we obtain

$$\partial[v] = \sum_{[e], \epsilon(e)=1} \langle \partial[v], [e] \rangle \partial_e = \xi_v^+.$$

Then, by using the dual sequence in relative cohomology, we can finish the proof of the lemma.  $\square$

From this lemma, we can obtain the following proposition:

**Proposition 4.19.** *Assuming  $R^\circ$  is oriented and connected, we have  $\xi_v^+(R^\circ) = 0$  iff the graph  $R^\circ$  is minimal.*

The following lemma is also elementary but very useful.

**Lemma 4.28.** *Let  $\Gamma$  be an admissible curve,  $v \in X_0 R$ , and  $v' \in X_0 R_\Gamma$  be the image of  $v$  in the graph  $R_\Gamma$ . Then we have the following relation*

$$Tcut_\Gamma(\xi_v^+(R^\circ)) = \xi_{v'}^+(R_{\Gamma^\circ}^\circ).$$

*Proof.* To prove this lemma, we use the following identification, which is valid for oriented ribbon graphs:

$$T_R \simeq H_1(M_R^{cap} \setminus X_0 R, X_2 R, \mathbb{R}).$$

The vector  $\xi_v^+(R^\circ)$  corresponds to a circle around the vertex  $v$  in the homology. The tangent map of  $cut_\Gamma$  corresponds to the map in homology.

$$H_1(M_R^{cap} \setminus X_0 R, X_2 R, \mathbb{R}) \longrightarrow H_1(M_R^{cap} \setminus X_0 R, X_2 R \sqcup \Gamma, \mathbb{R}) \simeq H_1(M_{R_\Gamma}^{cap} \setminus X_0 R_\Gamma, \partial X_2 R_\Gamma, \mathbb{R}).$$

This map is the inclusion of the homology into the relative homology, and then it maps the circle around  $v$  to the circle around  $v'$ .  $\square$

Let  $R_v^+$  be the component of  $R_{\Gamma_v^+}^\circ$  that contains  $v$ . Then, by using the last lemma, we obtain the following first statement:

**Lemma 4.29.** *The graph  $R_v^+$  is minimal.*

*Proof.* By using 4.28, we have the relation

$$T\text{cut}_v(\xi_v^+(R^\circ)) = \xi_v^+(R_v^+).$$

But as we seen in proposition 4.17, the twist flow along  $\Gamma_v^+$  is in the kernel of  $T\text{cut}_{\Gamma_v^+}$ , so we have  $T\text{cut}_v(\xi_v^+(R^\circ)) = 0$ . and then  $\xi_v^+(R_v^+) = 0$  by using lemma 4.28. Moreover, by using proposition 4.19, we can conclude that  $R_v^+$  is minimal.  $\square$

Let  $\mathcal{G}_{v,+}^\circ$  be the stable graph associated with  $\Gamma_v^+$ . We can decompose  $\Gamma_v^+$  into three sets of curves  $A_i$ ,  $i = 1\dots 3$ . Where

- $A$  are the curves that join a boundary of  $R_v^+$  to an other vertex of  $\mathcal{G}_{v,+}^\circ$ .
- $A_1$  are the curves that join two boundaries of  $R_v^+$
- $A_2$  are the curves that are not connected to  $R_v^+$

Then we prove the following step:

**Lemma 4.30.** *The sets  $A_1, A_2$  are empties.*

*Proof.* We consider the graph  $R_A^\circ = R_{v,1}^\circ \sqcup R_{v,2}^\circ$ , where  $R_{v,1}^\circ$  is the connected component that contains  $v$  and  $R_{v,2}^\circ$  is the union of all the other components. By the lemma 4.28 we have

$$T\text{cut}_A(\xi_v^+(R^\circ)) = \xi_v^+(R_A^\circ) = \xi_v^+(R_{v,1}^\circ).$$

The graph  $R_{v,1}^\circ$  is minimal because  $(R_{v,1}^\circ)_{A_3} = R_{v,+}^\circ$  then we have

$$\xi_v^+ \in \text{Ker}(T\text{cut}_A).$$

By using proposition 4.17, the kernel of  $T\text{cut}_{A_2}$  is generated by twist flows along  $A$ , and then

$$\xi_v^+ \in \bigoplus_{\gamma \in A} \mathbb{Z} \xi_\gamma.$$

By using proposition 4.17 again, the tangent vectors of twist flow along disjoint curves are independent. Then the vectors  $\xi_\gamma, \gamma \in \Gamma_v^+$  are free. By writing

$$\xi_v^+ = \xi + \xi_1 + \xi_2,$$

with  $\xi_i \in \text{Span}(\xi_\gamma, \gamma \in A_i)$ , we see that  $\xi_1, \xi_2$  must vanish, which implies that  $A_1, A_2$  must be empty.  $\square$

We give an algorithmic construction of the curve  $\Gamma_v^+$ :

- Start with and edge  $e_0$  with  $[e_0]_0 = v$  and  $\epsilon(e_0) = 1$ .
- If  $[s^+e_k]_0 = v$  then  $e_{k+1} = s^-e_k$ .
- Else, if  $[s^+e_k]_0 \neq v$ , then  $e_{k+1} = s^+e_k$ .

After some time, the process ends at  $e_0$ , and the result is a combinatorial curve that is independent of the starting edge  $e_0$ . If the curve does not contain all the oriented edges  $e$  with  $\epsilon(e) = 1$  and  $[e]_0 = v$ , then we restart the procedure on another edge. In other words, to construct the curve, we take all the positive boundaries that meet  $v$  and perform cutting and gluing at  $v$ , which are described in figure 4.20. At the end, the procedure creates the minimal representation of a family of curves  $\tilde{\Gamma}_v^+$ . From the construction and result of 4.2.2, we can derive

- The intersection coordinates  $y_e(\tilde{\Gamma}_v^+)$  are in  $\{0, 1\}$ ,
- Each curve in  $\tilde{\Gamma}_v^+$  is simple, and  $\tilde{\Gamma}_v^+$  is a multi-curve.
- Each curve in  $\tilde{\Gamma}_v^+$  is admissible.

The multi-curve  $\tilde{\Gamma}_v^+$  is in  $\widetilde{\mathcal{MS}}_{\mathbb{Z}}(R)$ ; all connected components are either in  $\mathcal{S}(M)$  or are homotopic to a negative boundary by using the construction. Let  $B^+$  be the positive boundaries of  $R^\circ$  that are adjacent to  $v$  and  $B^-$  be the negative boundaries of  $R^\circ$  that are adjacent to  $v$  only. From the construction used in Section 4.18, we have the following lemma:

**Lemma 4.31.** *The curve constructed in this way is  $\Gamma_v^+ + \sum_{\beta \in B^-} \beta$ , and we have on  $T_R$*

$$dl_{\Gamma_v^+} = \sum_{\beta \in B^+} dl_\beta - \sum_{\beta \in B^-} dl_\beta$$

where we sum over the boundaries that contain  $v$  and are oriented positively.

We prove the last part. Let  $M_v \subset M$  be the component that contains  $v$ , then we have the relative sequence:

$$H^1(M, X_0R, \mathbb{R}) \longrightarrow H^1(M_v, \mathbb{R}) \longrightarrow H^2(M, M_v, \mathbb{R}) \longrightarrow 0,$$

where the first map is the projection  $\text{cut}_v$  of  $\text{cut}_{\Gamma_v^+}$

$$T_R \longrightarrow T_{R_{\Gamma_v^+}} \longrightarrow T_{R_v^+}$$

Then the obstruction to the map being surjective is in  $H^2(M, M_v, \mathbb{R})$ . Then the next step is

**Lemma 4.32.** *The space  $H^2(M, M_v, \mathbb{R})$  is 0.*

Let  $R_v(c)$  be a component of  $R_{\Gamma_v^+}$  that doesn't contain  $v$ , and  $\Gamma_v^+(c)$  be the boundaries of  $R_v(c)$  that are in  $\Gamma_v^+$ . Let  $\epsilon_v$  be the orientation on  $\Gamma_v^+$  induced by  $R_v^+$ . Then we have the following fact:

**Lemma 4.33.** *The orientation  $\epsilon_v$  is constant on  $\Gamma_v^+(c)$ .*

*Proof.* Assume that there is  $c$  such that the orientation is not constant on  $\Gamma_v^+(c)$ . We see that we can find a simple curve in the homology such that

$$\langle [\Gamma_v^+(c)], [\gamma'] \rangle \neq 0; \quad \text{and} \quad \langle [\Gamma_v^+(c')], [\gamma'] \rangle = 0, \quad \text{if } c \neq c'.$$

Then

$$dl_{\Gamma_v^+}(\xi_{\gamma'}) = \langle [\Gamma_v^+], [\gamma'] \rangle \neq 0$$

but by assumption,  $dl_{\Gamma_v^+} = 0$  on  $K_R$ , and then the orientation must be constant.  $\square$

Then we prove lemma 4.32

*Proof.* The dimension of the space  $H^2(M, M_v, \mathbb{R})$  is equal to the number of components of  $M_{\Gamma_v^+}$  such that all the boundaries of the component are in  $\Gamma_v^+$ . Then, using lemma 4.33, this is impossible because the orientation is constant on a given component.  $\square$

From this lemma, we obtain

**Lemma 4.34.** *On  $T_{R_v}$ , we have the relation*

$$dl_{\Gamma_v^+} + dl_{B^-} - dl_{B^+} = 0$$

From proposition 4.3, the only relation on  $T_{R_v}$  is given by the orientation  $\epsilon_v$  of the graph. Then, by writing

$$dl_{\Gamma_v^+} = \sum_{\gamma \in \Gamma_v^+} m_\gamma dl_\gamma.$$

The relation of lemma 4.34 is proportional to

$$\sum_{\gamma \in \Gamma_v^+} \epsilon_v(\gamma) dl_\gamma + dl_{B^+} - dl_{B^-}$$

then we obtain that

$$\epsilon_v(\gamma) = -m_\gamma$$

Then we get that  $m_\gamma = 1$  and  $\epsilon_v(\gamma) = -1$ . Then we see that the multi-curve is primitive and satisfies the desired properties.

To prove the converse statement of theorem 4.2, we give the following generalization of the last result: For each subset  $I \subset X_0 R$ , let

$$\xi_I^+ = \sum_{v \in I} \xi_v^+,$$

and  $\xi_I^-$  defined in a similar way. Then  $\xi_I^+$  satisfies the following properties:

**Proposition 4.20.** *There is a primitive multi-curve  $\Gamma_I^+$  such that*

$$\xi_I^+ = \xi_{\Gamma_I^+}$$

Moreover, the multi-curves satisfy

- We have  $R_{\Gamma_I^+}^\circ = R_I^{\circ,+} \sqcup R_{I^c}^{\circ,-}$  where  $X_0 R_I^+ = I$ ,
- All the curves in  $\Gamma_I^+$  rely on  $R_I^{\circ,+}$  to  $R_{I^c}^{\circ,-}$ ,
- $R_I^{\circ,+}$  is glued along negative boundaries.

We can prove the converse statement of theorem 4.2.

**Proposition 4.21.** *If  $\Gamma^\circ$  satisfies the properties of theorem 4.2, it's  $\Gamma_v^+$ .*

**Remark 4.25.** *By using the same argument, it's possible to prove that  $\Gamma_I^+$  is the unique primitive multi-curve that fills the conditions of proposition 4.20.*

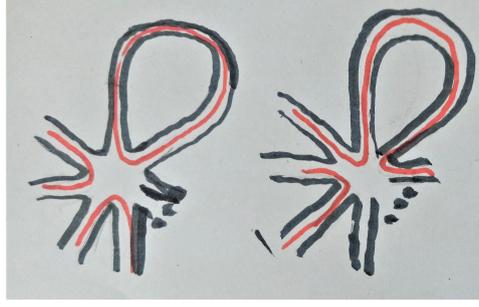


Figure 4.20: Illustration of the construction.

*Proof.* Let  $\Gamma^\circ$  be a multi-curve that fills the condition of theorem 4.2. Then we have the relation on  $T_R$ .

$$dl_\Gamma = dl_{B^+} - dl_{B^-}$$

Which means that  $\xi_\Gamma$  belongs to  $H_R$ , and then from lemma 4.27

$$\xi_\Gamma = \sum_{u \neq v} \lambda_u \xi_u.$$

Let  $R_{\Gamma^\circ}^\circ(c)$  be a connected component of  $R_{\Gamma^\circ}^\circ$  that doesn't contain  $v$ ;  $\Gamma_c^\circ$  are the curves in  $\Gamma^\circ$  that are in the boundary of  $R_{\Gamma^\circ}^\circ(c)$ ; and  $B_+(c), B_-(c)$  are the other boundaries of  $R_{\Gamma^\circ}^\circ(c)$ . We have the relation on  $T_R$ .

$$dl_{\Gamma_c^\circ} = dl_{B_-(c)} - dl_{B_+(c)}$$

and then  $\xi_{\Gamma_c^\circ}$  is in  $H_R$ . Moreover,  $\xi_{\Gamma^\circ(c)}$  is in the kernel of  $T\text{cut}_{\Gamma_c^\circ}$ , the graph  $R_{\Gamma^\circ}^\circ(c)$  has only two connected components, and then the space  $\text{Ker}(T\text{cut}_{\Gamma_c^\circ}) \cap H_R$  is a one-dimensional vector space generated by  $\xi_{X_0 R_{\Gamma(c)}}^+$ . Then

$$\xi_{\Gamma(c)} = \lambda_c \xi_{X_0 R_{\Gamma(c)}}^+.$$

If  $\Gamma(c)$  is primitive, then  $\lambda_c \in \{\pm 1\}$ . We have  $-\xi_{X_0 R_{\Gamma(c)}}^+ = \xi_{X_0 R_{\Gamma(c)}}^-$  and  $R_{\Gamma(c)}$  is glued along positive boundaries, then we conclude from proposition 4.20 that  $\lambda_c = -1$ . To conclude, we have

$$\xi_\Gamma = \sum_c \xi_{\Gamma(c)} = - \sum_c \xi_{X_0 R_{\Gamma(c)}}^+$$

because the curves are disjoint. From lemma 4.27, we have

$$- \sum_c \xi_{X_0 R_{\Gamma(c)}}^+ = \xi_v^+,$$

And then we conclude the proof:  $\xi_\Gamma = \xi_v^+$ .

□

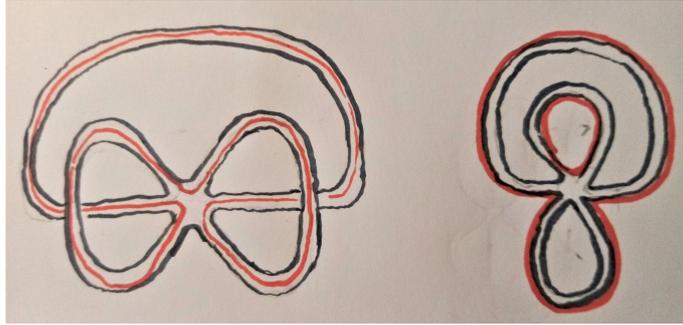


Figure 4.21: Acyclic curve on a non orientable graph

**Case of un-orientable ribbon graphs:** When we consider un-orientable surface with at least one vertex of even degree, we still have degeneration of the symplectic structure, and we can also find canonical curves that spare this vertex from the rest of the surface.

**Theorem 4.4.** *If  $R$  is any ribbon graph and  $v$  a vertex of even degree, there are exactly two admissible multi-curves  $\Gamma_v^\pm$  such that:*

- $\Gamma_v^\pm$  spares  $v$  from the rest of the surface.
- The component  $R_v^\pm$  is orientable and admits an orientation such that it's glued along it's negative boundaries.
- All curves in  $\Gamma_v^\pm$  are boundaries of  $R_v^\pm$ .

This theorem contains the case of the theorem 4.2.

**Remark 4.26.** *In this case, the group of automorphisms of the surface can eventually exchange the two multi-curves  $\Gamma_v^\pm$ . This happens, for instance, for the torus with one boundary.*

**Extracting a pair of pants on an oriented surface:** In the generic case, when there are only vertices of degree 4, the only minimal ribbon graphs are topological pairs of pants. In this case, minimal and irreducible ribbon graphs coincide, and then the last theorem gives a particular family of canonical Fenchel-Nielsen decompositions of a ribbon graph. Let  $R^\circ$  be a generic oriented ribbon graph, an embedded bounded pair of pants<sup>14</sup> in  $R^\circ$  is an admissible curve  $\Gamma^\circ$  such that:

- There is a component  $c_v$ <sup>15</sup> of  $\mathcal{G}^\circ$ , which is a pair of pants.
- All the curves in  $\Gamma$  are in the boundary of  $c_v$ .

<sup>14</sup>The terminology is motivated by the third condition; it implies that the length of such a curve is necessarily bounded by a function of the lengths of the boundaries. This implies by 4.10 that there is only a finite number of bounded embedded pairs of pants.

<sup>15</sup>These three conditions also imply that the choice of the marked component  $c_v$  is canonical. Moreover, there is only one oriented ribbon graph on a pair of pants and it contains a unique vertex of order four. Then a bounded pair of pants on a generic oriented metric ribbon graph spares a vertex from the rest of the graph.

- $c_v$  is glued along its negative boundaries.

**Theorem 4.5.** *For each generic oriented metric ribbon graph  $S$  and each vertex  $v$ , there exists a unique bounded pair of pants  $\Gamma_v^+$  that spares  $v$  from the rest of the surface.*

A reformulation of theorem 4.3 gives the following result:

**Corollary 4.13.** *Let  $R^\circ$  be a generic oriented ribbon graph with a linear order. There is a unique oriented multi-curve  $\Gamma^\circ$  such that*

1.  $\Gamma^\circ$  is maximal, i.e., components of  $R_{\Gamma^\circ}^\circ$  are pairs of pants.
2. The oriented stable graph  $\mathcal{G}^\circ$  associated with  $\Gamma^\circ$  is acyclic.
3. The linear order on  $R^\circ$  induces a linear order on  $\mathcal{G}^\circ$ .

## 4.4 Recurrence for the volumes of moduli spaces of oriented fourvalent ribbon graphs

In this part, we give the recursion for the volumes of moduli spaces  $\mathcal{M}^{comb}(M^\circ, L)$ , where  $M^\circ$  is a connected directed surface. As we see in section 4.1.1, for each directed surface  $M^\circ$ , the moduli space  $\mathcal{M}^{comb}(M^\circ, L)$  of oriented metric ribbon graphs with fixed boundary lengths possesses a measure  $d\mu_{M^\circ}(L)$ . We recall that  $V_{M^\circ}(L)$  denotes the volume of  $\mathcal{M}^{comb}(M^\circ, L)$  for this measure. It's a function defined on  $\Lambda_{M^\circ}$ .

$$V_{M^\circ}(L) = \int_{\mathcal{M}^{comb}(M^\circ, L)} d\mu_{M^\circ}(L).$$

According to section 4.1.1, the volume is finite for each  $L$  and  $M^\circ$ . In general, the boundaries are labeled, and then we choose to separate the positive and negative variables and use the notation:

$$V_{g, n^+, n^-}(L^+ | L^-).$$

Where  $g$  is the genus and  $n^\pm$  are the numbers of positive and negative boundaries.

### 4.4.1 Surgery for the volumes and directed stable graphs:

In this section, we give results that are similar to the ones of [ABC<sup>+</sup>20]. We cover the case of oriented metric ribbon graphs. We state the results in the generic case, but they are also valid for other strata of the moduli spaces of oriented metric ribbon graphs.

**Covering and decomposition of the measures :** Let  $M^\circ$  be a directed surface and  $\mathcal{G}^\circ$  be a directed stable graph in  $\text{st}(M^\circ)$ . There is a natural bundle over  $\mathcal{M}^{comb}(\mathcal{G}^\circ)$  that corresponds to all the possible gluing's of surfaces in  $\mathcal{M}^{comb}(M^\circ)$ . If  $\Gamma^\circ$  is a multi-curve that represents  $\mathcal{G}^\circ$ , we consider the space  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  of oriented foliations transverse to  $\Gamma^\circ$  (see paragraph 4.2.4). This space carries an action of  $\text{Stab}(\Gamma^\circ)$  the stabilizer of  $\Gamma^\circ$ , by the action of the mapping class group  $\text{Mod}(M)$ , and it's possible to consider the quotient.

$$B\mathcal{M}^{comb}(\mathcal{G}^\circ) = \mathcal{MF}_{\Gamma^\circ}(M^\circ) / \text{Stab}(\Gamma^\circ).$$

It does not depend on the choice of  $\Gamma^\circ$  only of  $\mathcal{G}^\circ$ . The reasons for this choice are the following:

- The space  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  contains, in a natural way  $\mathcal{T}_{\Gamma^\circ}^{comb}(M^\circ)$ .
- It's possible to cut an element of  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  along the curves in  $\Gamma$ , and the result is an element of  $\mathcal{T}^{comb}(\mathcal{G}^\circ)$ , and this induces a map.

$$\text{cut}_\Gamma : B\mathcal{M}^{comb}(\mathcal{G}^\circ) \longrightarrow \mathcal{M}^{comb}(\mathcal{G}^\circ).$$

- Moreover, the twist flow along the curves in  $\Gamma$  is well defined on  $\mathcal{MF}_\Gamma(M^\circ)$ , which was not the case of  $\mathcal{T}^{comb}(M^\circ)$ , and this induces an orbifold torus action on  $B\mathcal{M}^{comb}(\mathcal{G}^\circ)$ .

This space is a piece-wise linear orbifold with a natural measure  $d\tilde{\mu}_{\mathcal{G}^\circ}$  normalized by the set of integer points in the tangent space. Using the results of 4.18, the map

$$\text{cut}_{\mathcal{G}^\circ} : B\mathcal{T}^{comb}(\mathcal{G}^\circ) \longrightarrow \mathcal{M}^{comb}(\mathcal{G}^\circ),$$

is an  $\mathbb{R}^{\#X_1\mathcal{G}}$  bundle, the Dehn twist  $\delta_\gamma$  acts on the twist parameter by  $t_\gamma(\delta_\gamma(S)) = t_\gamma(S) + l_\gamma(S)$ . And then  $B\mathcal{T}^{comb}(\mathcal{G}^\circ)$  is a torus bundle over  $\mathcal{M}^{comb}(\mathcal{G}^\circ)$ .

**Lemma 4.35.** *The measures  $d\tilde{\mu}_{\mathcal{G}^\circ}$  and  $d\mu_{\mathcal{G}^\circ}$  satisfy the relation*

$$\text{cut}_{\mathcal{G}^\circ, *} d\tilde{\mu}_{\mathcal{G}^\circ} = \prod_{\gamma \in X_1 \mathcal{G}} l_\gamma d\mu_{\mathcal{G}^\circ}$$

*Proof.* This lemma is a consequence of the propositions 4.17, 3.1, and 4.18. From the two first propositions, we have

$$\text{cut}_{\mathcal{G}^\circ, *} d\tilde{\mu}_{\mathcal{G}^\circ} = \text{Vol}(\text{cut}_{\mathcal{G}^\circ}^{-1} S) d\mu_{\mathcal{G}^\circ}$$

The Dehn twist along a curve acts on the fiber as

$$t_\gamma \longrightarrow t_\gamma + l_\gamma(S)$$

then the fiber over a point is  $\prod_\gamma \mathbb{R}/l_\gamma \mathbb{Z}$  with the Haar measure  $\prod_\gamma dt_\gamma$ . We conclude that

$$\text{Vol}(\text{cut}_{\Gamma^\circ}^{-1} S) = \prod_\gamma l_\gamma(S).$$

□

For any stable graph and any positive symmetric function

$$F : \mathbb{R}_+^{X_1 \mathcal{G}} \longrightarrow \mathbb{R}_+$$

It's possible to consider the integral

$$V_{\mathcal{G}^\circ}^{\text{comb}}(F)(L) = \int_{B\mathcal{M}^{\text{comb}}(\mathcal{G}^\circ, L)} F(L_{\mathcal{G}}(S)) d\tilde{\mu}_{\mathcal{G}^\circ}(L).$$

Where  $L_{\mathcal{G}}$  is the length of the curves of the stable graph and  $L$  is a variable indexed by the boundaries.

**Proposition 4.22.** *For all  $\mathcal{G}^\circ$  stable graphs, the function  $V_{\mathcal{G}^\circ}^{\text{comb}}(F)(L)$  is given by*

$$V_{\mathcal{G}^\circ}^{\text{comb}}(F)(L) = \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} F(x) \prod_c V_{\mathcal{G}^\circ(c)}(L_c(x)) \prod_\gamma l_\gamma(x) d\sigma_{\mathcal{G}^\circ}(L).$$

*Proof.* To prove this proposition, we use the lemma 3.1 and also the proposition 4.35. □

**Statistics for multi-curves covering and integration formula** If  $\Gamma^\circ$  is an oriented multi-curve on  $M^\circ$ , we denote

$$\mathcal{T}_{\Gamma^\circ}^{\text{comb},*}(M^\circ) = \{S \in \mathcal{T}^{\text{comb},*}(M^\circ) \mid \Gamma \in \mathcal{MF}_0(S), \text{ and } \Gamma^\circ = \Gamma_S^\circ\}.$$

The space of generic oriented metric ribbon graphs  $S$  such that  $\Gamma$  is admissible on  $S$  and the orientation on it, induced by  $S$ , corresponds to  $\Gamma^\circ$ . Let  $\mathcal{G}^\circ$  be the stable graph of  $\Gamma^\circ$ . There is an action of  $\text{Stab}(\Gamma^\circ)$  on  $\mathcal{T}_{\Gamma^\circ}^{\text{comb},*}(M^\circ)$ , and we denote  $\mathcal{M}_{\mathcal{G}^\circ}^{\text{comb},*}(M^\circ)$  the quotient

$$\mathcal{M}_{\mathcal{G}^\circ}^{\text{comb},*}(M^\circ) = \mathcal{T}_{\Gamma^\circ}^{\text{comb},*}(M^\circ) / \text{Stab}(\Gamma^\circ).$$

It corresponds to the oriented metric ribbon graphs marked by an admissible multi-curve such that the orientation induced on the curves corresponds to  $\Gamma^\circ$ . There is a canonical map

$$\pi_{\mathcal{G}^\circ} : \mathcal{M}_{\mathcal{G}^\circ}^{\text{comb},*}(M^\circ) \longrightarrow \mathcal{M}^{\text{comb},*}(M^\circ).$$

The fiber  $\pi_{\mathcal{G}^\circ}^{-1}(S)$  over  $S$  is the set of admissible multi-curves on  $S$  that are in the orbit of  $\Gamma^\circ$ , or equivalently, admissible multi-curves with a stable graph given by  $\mathcal{G}^\circ$ . The map  $\pi_{\mathcal{G}^\circ}$  is a covering over each cell, and the space  $\mathcal{M}_{\mathcal{G}^\circ}^{comb,*}(M^\circ)$  is equipped by the pullback of the measure on  $\mathcal{M}^{comb,*}(M^\circ)$ . As in [?],[ABC<sup>+</sup>20] we consider statistics for the distribution of the lengths of curves in a multi-curve. We define  $N_{\mathcal{G}^\circ}F(S)$  as the sum of  $F(L_\Gamma(S))$  over all the admissible curves with stable graph  $\mathcal{G}^\circ$ .

$$N_{\mathcal{G}^\circ}F(S) = \sum_{\Gamma \in \pi_{\mathcal{G}^\circ}^{-1}(S)} F(L_\Gamma(S)).$$

This is, by definition, the push forward of  $F \circ L_\Gamma$  under  $\pi_{\mathcal{G}^\circ}$ . The function  $N_{\mathcal{G}^\circ}F$  is well defined on the moduli space  $\mathcal{M}^{comb}(M^\circ)$  because of the relation

$$F(L_{g \cdot \Gamma}(g \cdot S)) = F(L_\Gamma(S)),$$

and because the map

$$g : \mathcal{MS}(R) \longrightarrow \mathcal{MS}(g \cdot R)$$

preserves the direction on the stable graphs (for  $g \in \text{Mod}(M)$ ). Then it satisfies the following relation, which is the formula for a push forward under a covering.

$$\int_{\mathcal{M}^{comb,*}(M^\circ,L)} N_{\mathcal{G}^\circ}F(S) d\mu_{M^\circ}(L) = \int_{\mathcal{M}_{\mathcal{G}^\circ}^{comb,*}(M,L)} F(L_\Gamma(S)) d\mu_{M^\circ}(L).$$

There is a canonical map:

$$\mathcal{M}_{\Gamma^\circ}^{comb,*}(M^\circ) \longrightarrow B\mathcal{M}^{comb,*}(\mathcal{G}^\circ).$$

This map is not surjective, but the following lemma allows us to avoid this problem.

**Lemma 4.36.** *The subset  $\mathcal{M}_{\mathcal{G}^\circ}^{comb,*}(M^\circ)$  is of full measure in  $B\mathcal{M}^{comb,*}(\mathcal{G}^\circ)$ , and this is also true for  $\mathcal{M}_{\mathcal{G}^\circ}^{comb,*}(M^\circ, L)$  in  $B\mathcal{M}^{comb,*}(\mathcal{G}^\circ, L)$  for all  $L$ .*

*Proof.* We don't give a detailed proof of this lemma; it's based on the fact that the complementary set is formed by foliations that contain saddle connections. In the orientable case, such saddle connections can be generic. But this phenomenon cannot happen if the foliation is transverse to a multi-curve. Then the only configurations of saddle connections that are possible are not generic, and then the problem is in some codimension-one sub-spaces. There is only a countable number of such sub-spaces, and then the issues are in a null set. But writing this argument correctly is not very interesting here and could take a lot of space. In the case of the acyclic gluing that we use, the situation is indeed much simpler because the gluing's always create multi-arcs; the space  $\mathcal{MF}_{\Gamma^\circ}(M^\circ)$  is included in  $\mathcal{MA}_\mathbb{R}(M^\circ)$ , because gluing cannot create a cycle.  $\square$

Using this lemma, we obtain the relation

$$V_{\mathcal{G}^\circ}^{comb}(F)(L) = \int_{\mathcal{M}^{comb,*}(M^\circ,L)} N_{\mathcal{G}^\circ}F(S) d\mu_{M^\circ}(L)$$

Then it gives the following proposition, which was first proved by M. Mirzakhani in the case of hyperbolic surfaces:

**Proposition 4.23.** *The integral of  $N_{\mathcal{G}^\circ}F(S)$  satisfies the Mirzakhani integral formula.*

$$\int_{\mathcal{M}^{comb}(M^\circ,L)} N_{\mathcal{G}^\circ}F(S) d\mu_{M^\circ}(L) = \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} F(x) \prod_c V_{\mathcal{G}^\circ(c)}(L_c(x)) \prod_\gamma l_\gamma(x) d\sigma_{\mathcal{G}^\circ}(L).$$

## 4.4.2 Recursion for the volumes

In this part, we use the theorem 4.2 and the results of the last part to obtain recursion for the volumes  $V_{g,n^+,n^-}(L^+|L^-)$ .

**A degenerated geometric recursion formula:** We denote  $P(M^\circ)$  the set of stable graphs on  $M^\circ$  that correspond to bounded pairs of pants (see theorem 4.5). There are four families of such stable graphs, which are represented in figure 4.22 and are also given in the introduction:

1. We remove a pants that contains two positive boundaries  $(i, j)$ .
2. We remove a pants that contains a positive boundary  $i$  and a negative boundary  $j$ .
3. We remove a pants with one positive boundary  $i$  which is connected to the surface by two negative boundaries, and do not disconnect the surface.
4. We remove a pants with one positive boundary  $i$ , which is connected to the surface by two negative boundaries, and disconnect the surface into two connected components.

Then we can rewrite the theorem 4.5 in the following way, which is a kind of degeneration of the geometric recursion formula [ABO17]. This formulation is straightforward to integrate over the moduli space  $\mathcal{M}^{comb}(M^\circ)$  by using results of the last section. The proposition means that the covering associated with the bounded pants is of degree  $2g - 2 + n^+ + n^-$ .

**Proposition 4.24.** *For all  $S \in \mathcal{M}^{comb,*}(M^\circ, L)$ , we have*

$$2g - 2 + n^+ + n^- = \sum_{\mathcal{G}^\circ \in P(M^\circ)} N_{\mathcal{G}^\circ}(1)(S).$$

*Proof.* This is just a reformulation of theorem 4.5 by using the definition of the functions  $N_{\mathcal{G}^\circ}(1)(S)$ . □

**Recursion for volumes:** In this section, we make effective theorem 4.5 in the case of graphs with vertices of degree four only.

**Theorem 4.6.** *For all values of the boundary lengths, the volumes satisfy the recursion:*

$$\begin{aligned} (2g - 2 + n)V_{g,n^+,n^-}(L^+|L^-) &= \sum_i \sum_j [L_i^+ - L_j^-]_+ V_{g,n^+,n^- - 1}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) \\ &+ \frac{1}{2} \sum_{i \neq j} (L_i^+ + L_j^+) V_{g,n^+ - 1, n^-} (L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-) \\ &+ \frac{1}{2} \sum_i \int_0^{L_i^+} V_{g-1, n^+ + 1, n^-} (x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) x (L_i^+ - x) dx \\ &+ \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^\pm \sqcup I_2^\pm = I^\pm}} x_1 x_2 V_{g_1, n_1^+ + 1, n_1^-} (x_1, L_{I_1^+}^+ | L_{I_1^-}^-) V_{g_2, n_2^+ + 1, n_2^-} (x_2, L_{I_2^+}^+ | L_{I_2^-}^-); \end{aligned}$$

where we use the notation

$$x_l = \sum_{i \in I_l^-} L_i^- - \sum_{i \in I_l^+} L_i^+.$$

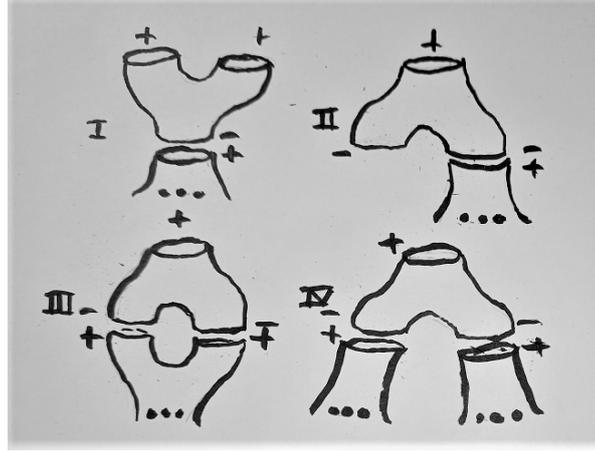


Figure 4.22: Different gluings appear in the recursion.

The index  $'$  above the sum means that we sum over all the stable pairs  $(g_i, n_i^+, n_i^-)$  with  $2g_i + n_i^+ + n_i^- - 2 > 0$ .

**Remark 4.27.** For each  $L \in \mathbb{R}_+^E$  and  $I \subset E$ , we use the notation  $L_I = (L_x)_{x \in I} \in \mathbb{R}_+^I$ .

We give a proof of the theorem 4.6.

*Proof.* To obtain the theorem, we multiply the formula of proposition 4.24 by the measure  $d\mu_{M^\circ}(L)$  and integrate over the moduli space. By using the results of Section 4.4.1, we obtain the following formula:

$$(2g - 2 + n)V_{M^\circ} = \sum_{\mathcal{G}^\circ \in P(M^\circ)} V_{\mathcal{G}^\circ}^{comb}(1).$$

The covering of bounded pairs of pants splits into several coverings, which correspond to the four types of gluing with all the possible choices of boundaries. Now, from the results of the last section, we have

$$V_{\mathcal{G}^\circ}^{comb}(1)(L) = \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} \prod_c V_{\mathcal{G}^\circ(c)}(L_c(x)) \prod_\gamma l_\gamma(x) d\sigma_{\mathcal{G}^\circ}(L).$$

Then there is a unique component in our graph, which is a pair of pants glued along its negative boundaries. The volume associated with this component is constant, equal to one. It's because there is only one oriented graph on an oriented pair of pants. Then this term disappears in the formulas. To finish the proof, it remains to compute the domain of integration in each case. All the multi-curves used are rigid in the sense that  $\Lambda_{\mathcal{G}^\circ}(L)$  is reduced to a point with one exception when the genus is reduced by one, which corresponds to type *III* in figure 4.22.  $\square$

**Case of marked points:** If  $M^\circ$  has  $m$  marked points that are not labeled, as we explain, a generic directed ribbon graph is then a quadrivalent graph with bivalent vertices at the marked points. In the case where the boundaries are labeled, we denote

$$V_{g, n^+, n^-, m}(L^+ | L^-), \tag{4.4}$$

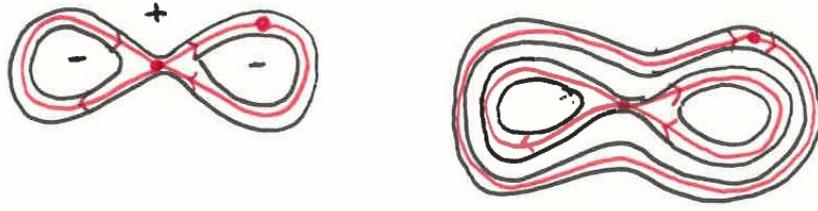


Figure 4.23: The trivial recursion for vertex of degree two.

the volume of the moduli space. Then, using the theorem 4.2 applied to the bivalent vertices, we can derive the following proposition: We denote  $E$  the energy function, which is defined on  $\Lambda_{n^+,n^-}$  by

$$E(L) = \sum_i L_i^+ = \sum_i L_i^-.$$

**Proposition 4.25.** *The volumes satisfy*

$$mV_{g,n^+,n^-,m} = E V_{g,n^+,n^-,m-1}.$$

And then

$$V_{g,n^+,n^-,m} = \frac{E^m}{m!} V_{g,n^+,n^-}.$$

*Proof.* This is also a consequence of theorem 4.2. We can apply it to vertices of degree two. In this case, there is only one minimal surface; it corresponds to a cylinder with one vertex of degree two and only one edge. The recursion is the following:

$$mV_{g,n^+,n^-,m} = \sum_i L_i^+ V_{g,n^+,n^-,m-1}.$$

And then give a proof of the proposition. □

### 4.4.3 Graphical expansion and structure of the volumes

**Graphical expansion:** As in the case of the topological recursion, the formula 4.6 admits a graphical expansion obtained by iterating the recursion. Proposition 4.13 gives canonical maximal acyclic multi-curves that decompose the surface. In the case of a generic directed ribbon graph, the decomposition is a pants decomposition.

**Proposition 4.26.** *The volume  $V_{g,n^+,n^-}$  satisfies:*

$$(2g - 2 + n)! V_{g,n^+,n^-}(L) = \sum_{\mathcal{G}^\circ} \frac{1}{\#\text{Aut}(\mathcal{G}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}(L)} \prod_{\gamma} l_{\gamma}(x) d\sigma_{\mathcal{G}^\circ}(L) = \sum_{\mathcal{G}^\circ} V_{\mathcal{G}^\circ}(L).$$

*The sum over all the directed acyclic pants decompositions with a linear order, of genus  $g$  and with  $n^+$  positives and  $n^-$  negatives labeled boundaries.*

*Proof.* We can use the theorem 4.13, which means that a linear order on the ribbon graphs corresponds to a pants decomposition with a linear order. According to proposition 4.24, we can derive the following formula for each surface  $S \in \mathcal{M}^{comb,*}(M^\circ)$  :

$$(2g - 2 + n^+ + n^-)! = \sum_{\mathcal{G}^\circ} N_{\mathcal{G}^\circ}(1)(S).$$

Then, by taking the integral over  $\mathcal{M}^{comb}(M^\circ, L)$  and applying proposition 4.22, we obtain the formula:

$$(2g - 2 + n^+ + n^-)! V_{g,n^+,n^-}(L^+|L^-) = \sum_{\mathcal{G}^\circ} V_{\mathcal{G}^\circ}(L^+|L^-).$$

□

**Piece-wise polynomiality:** In this part, we prove the following theorem:

**Theorem 4.7.** *The volume  $V_{g,n^+,n^-}(L^+|L^-)$  is an element of  $\mathcal{P}_{n^+,n^-}$ . It's a continuous piece-wise polynomial of degree  $4g - 3 + n^+ + n^-$ .*

*Proof.* We have already got all the ingredients for the proof. From theorem 3.5, we know that the volumes  $V_{\mathcal{G}^\circ}$  are continuous piece-wise polynomials in  $\mathcal{P}_{n^+,n^-}$  homogeneous of degree  $4g - 3 + n^+ + n^-$ . Moreover, by using proposition 4.26, we deduce the theorem by linearity. □

**String equation for the volumes:** The functions  $V_{g,n^+,n^-}(L^+|L^-)$  satisfy very simple relations when a variable goes to zero, which is an analogous of the string equation. We will investigate a special case of this recursion later and give a dilaton equation in these cases.

**Proposition 4.27.** *The volumes satisfy the following relation when  $L_1^+ = 0$*

$$V_{g,n^++1,n^-}(0, L^+|L^-) = V_{g,n^+,n^-,1}(L^+|L^-) = EV_{g,n^+,n^-}(L^+|L^-)$$

**Laplace transform of the recursion:** In this section, we compute the Laplace transform of the last recursion:

$$\mathcal{V}_{g,n^+,n^-}(\lambda^+|\lambda^-) = \int e^{-\lambda^+ \cdot L^+ - \lambda^- \cdot L^-} dV_{g,n^+,n^-}(L^+|L^-).$$

The fact that the support of the measure is contained in  $\{\sum_i L_i^+ = \sum_i L_i^-\}$  implies the following symmetry for this function:

$$\mathcal{V}_{g,n^+,n^-}(\lambda^+ + t|\lambda^- - t) = \mathcal{V}_{g,n^+,n^-}(\lambda^+|\lambda^-).$$

**Theorem 4.8.** *The functions  $\mathcal{V}_{g,n^+,n^-}$  satisfy the recursion:*

$$\begin{aligned} \mathcal{V}_{g,n^+,n^-}(\lambda^+|\lambda^-) &= - \sum_{i,j} \frac{\partial_1^+ \mathcal{V}_{g,n^+,n^- - 1}(\lambda_i^+, \lambda_{\{i\}^c}^+ | \lambda_{\{j\}^c}^-)}{\lambda_i^+ + \lambda_j^-} \\ &+ \sum_{i \neq j} \frac{1}{2} \frac{\partial_1^+ \mathcal{V}_{g,n^+ - 1, n^-}(\lambda_i^+, \lambda_{\{i,j\}^c}^+ | \lambda^-) - \partial_1^+ \mathcal{V}_{g,n^+, n^-}(\lambda_j^+, \lambda_{\{i,j\}^c}^+ | \lambda^-)}{\lambda_i^+ - \lambda_j^+} \\ &+ \frac{1}{2} \sum_i \partial_1^+ \partial_2^+ \mathcal{V}_{g-1, n^+ + 1, n^-}(\lambda_i^+, \lambda_i^+, \lambda_{\{i\}^c}^+ | \lambda^-) \\ &+ \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^\pm \cup I_2^\pm = I^\pm}} \partial_1^+ \mathcal{V}_{g_1, n_1^+ + 1, n_1^-}(\lambda_i^+, \lambda_{I_1^+}^+ | \lambda_{I_1^-}^-) \partial_1^+ \mathcal{V}_{g_2, n_2^+ + 1, n_2^-}(\lambda_i^+, \lambda_{I_2^+}^+ | \lambda_{I_2^-}^-). \end{aligned}$$

Where we denote  $\partial_i^\pm$  the derivative with respect to  $\lambda_i^\pm$ .

*Proof.* To prove this theorem, we only need to compute the Laplace transform of the terms in the recursion. For instance, if  $f$  is a continuous function and  $\mathcal{L}f$  is the Laplace transform, we have

$$\int_{x_1 > x_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} (x_1 - x_2) f(x_1 - x_2) dx_1 dx_2 = \frac{1}{\lambda_1 + \lambda_2} \int_x e^{-\lambda_1 x} f(x) x dx = -\frac{\partial_1 \mathcal{L}f(\lambda_1)}{\lambda_1 + \lambda_2}$$

and then

$$\int e^{-\lambda^+ \cdot L^+ - \lambda^- \cdot L^-} [L_i^+ - L_j^-]_+ V_{g, n^+, n^-}([L_i^+ - l_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) d\sigma = -\frac{\partial_1^+ \mathcal{V}_{g, n^+, n^-}(\lambda_i^+, \lambda_{\{i\}^c}^+ | \lambda_{\{j\}^c}^-)}{\lambda_i^+ + \lambda_j^-}$$

For the other terms of the recursion, we use the formulas:

$$\begin{aligned} \int_{x_1, x_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} x_1 x_2 f(x_1 + x_2) dx_1 dx_2 &= \frac{\partial \mathcal{L}f(\lambda_1) - \partial \mathcal{L}f(\lambda_2)}{\lambda_1 - \lambda_2} \\ \int_{x_1, x_2} e^{-\lambda x_1 - \lambda x_2} x_1 x_2 f(x_1, x_2) dx_1 dx_2 &= \partial_1 \partial_2 \mathcal{L}f(\lambda, \lambda). \end{aligned}$$

Here, the function  $f$  is continuous with polynomial growth. □

#### 4.4.4 General formula for higher volumes

Assumes that  $\overline{M}^\circ = (M^\circ, \nu)$  is a decorated directed surface. We consider the volumes

$$V_{\overline{M}^\circ}(L) = \int_{\mathcal{M}^{comb, *}(\overline{M}^\circ, L)} d\mu_{\overline{M}^\circ}(L).$$

In this part, we give the recursion of theorem 4.2. We plan to make it more explicit in future work. We define  $\text{Bi}_i(\overline{M}^\circ)$  as the set of decorated acyclic stable graphs  $\overline{\mathcal{G}}^\circ$  with a height

$$h : X_0 \mathcal{G} \longrightarrow \{0, 1\}.$$

A height is a function that preserves the order of the acyclic graph. We assume that  $h$  satisfies the following conditions:

- $\overline{\mathcal{G}}^\circ \in \text{st}(\overline{M}^\circ)$ .
- $h$  is strictly increasing for the order relation on the graph.
- There is a unique component  $c_1$  with  $l(c_1) = 1$  and  $\overline{\mathcal{G}}^\circ(c_1)$  is minimal and  $\nu(c_1) = (i)$ .

The last condition says that there is a unique component at the top of the graph, and it's minimal with a vertex of degree  $2i + 2$ . It's glued along its negative boundaries (because it's at the top of an acyclic graph). And then we can rewrite theorem 4.2 in the following way: For each  $S^\circ = (R^\circ, m)$  and each  $v \in X_0 R^\circ$  of degree  $2i + 2$ , there is a unique admissible curve  $\overline{\Gamma}^\circ$  such that  $\overline{\mathcal{G}}_{\overline{\Gamma}^\circ}^\circ \in \text{Bi}_i(\overline{M}^\circ)$ . And  $\overline{\Gamma}^\circ$  spares  $v$  from the rest of the surface. Then we can derive the following corollary that generalizes proposition 4.24.

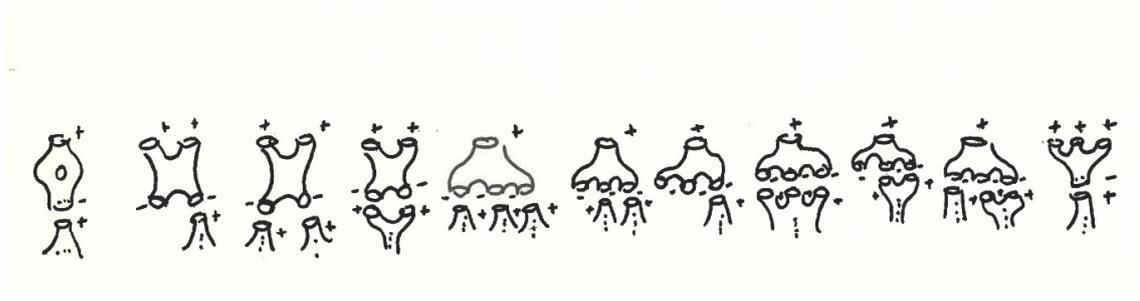


Figure 4.24: Different gluing's appearing when we remove a vertex of degree 6.

**Proposition 4.28.** For each  $\overline{M}^\circ$  and each  $i$ , we have for all  $S \in \mathcal{M}^{comb,*}(\overline{M}^\circ)$ :

$$\nu(i) = \sum_{\overline{\mathcal{G}}^\circ \in Bi_i(\overline{M}^\circ)} N_{\overline{\mathcal{G}}^\circ}(1).$$

And from this, using integration over the moduli space, we can obtain the following:

**Theorem 4.9.** The volumes satisfy the recursion:

$$\nu(i)V_{\overline{M}^\circ}(L^+|L^-) = \sum_{\overline{\mathcal{G}}^\circ \in Bi_i(\overline{M}^\circ)} \frac{1}{\#\text{Aut}(\overline{\mathcal{G}}^\circ)} \int_{\Lambda_{\mathcal{G}^\circ}(L)} V_{\overline{\mathcal{G}}^\circ(1)}(L_1)V_{\overline{\mathcal{G}}^\circ(0)}(L_0) \prod_{\gamma} l_{\gamma} d\sigma_{\mathcal{G}^\circ}(L).$$

In this formula,  $V_{\overline{\mathcal{G}}^\circ(1)}$  corresponds to the volume associated to the minimal component and  $V_{\overline{\mathcal{G}}^\circ(0)}$  to the volume of the other components. Then the formula gives a recursion that computes  $V_{\overline{M}^\circ}$  by removing vertices of degree  $2i - 2$ . As before, we can also obtain an expansion in terms of acyclic stable graphs. Let  $\overline{\text{acycl}}^*(\overline{M}^\circ)$  be the set of decorated acyclic stable graphs on  $\overline{M}^\circ$ , such that all components are minimal. This is also the set of maximal acyclic decompositions of  $\overline{M}^\circ$ . An before  $n_{\mathcal{G}^\circ}$  denotes the number of linear orders. Let

$$V_{\overline{\mathcal{G}}^\circ}(L) = \frac{1}{\#\text{Aut}(\overline{\mathcal{G}}^\circ)} \int_{x \in \Lambda_{\mathcal{G}^\circ}} \prod_c V_{\overline{\mathcal{G}}^\circ(c)}(L_c(x)) \prod_{\gamma} l_{\gamma} d\sigma_{\mathcal{G}^\circ}(L).$$

Then we have the following lemma:

**Proposition 4.29.** We have the relation :

$$V_{\overline{M}^\circ}(L) = \sum_{\overline{\mathcal{G}}^\circ \in \overline{\text{acycl}}^*(\overline{M}^\circ)} n_{\overline{\mathcal{G}}^\circ} V_{\overline{\mathcal{G}}^\circ}(L).$$

Using this, we can derive the following theorem:

**Theorem 4.10.** The volume  $V_{\overline{M}^\circ}(L)$  coincides with a piecewise polynomial for almost all values of  $L$

## 4.5 Surfaces with one negative boundary

In this section, we study in more details the case of surfaces with only one negative boundary. The recursion in this case takes a much simpler form, and it's possible to rely it to Cut-and-Join equations. This section is the starting point of investigations in the next chapter.

### 4.5.1 Recursion for the volumes

First of all, if  $n^- = 1$ , the first projection allows us to identify

$$\Lambda_{n^+,1} \simeq (\mathbb{R}_{\geq 0})^{n^+},$$

and then it's possible to drop negative boundaries and write  $V_{g,n^+,1}(L^+|L^-)$  as

$$V_{g,n^+}^\circ(L) = V_{g,n^+,1}(L||L),$$

where  $V_{g,n^+}^\circ(L)$  is now a function on  $\mathbb{R}_{\geq 0}^{n^+}$ . We can see that the recursion of theorem 4.6 preserves the family of surfaces with one negative boundary component. Extracting a bounded pair of pants on a surface with only one negative boundary creates only surfaces of this type. Moreover, these gluings are necessarily non-separating and can't be of type *II*. Then the recursion takes the following form:

**Theorem 4.11.** *Functions  $V_{g,n}^\circ$  are homogeneous polynomials of degree  $4g - 2 + n$  and satisfy the following recursion:*

$$\begin{aligned} (2g + n - 1)V_{g,n}^\circ(L) &= \frac{1}{2} \sum_{i \neq j} (L_i + L_j) V_{g,n-1}^\circ(L_i + L_j, L_{\{i,j\}^c}) \\ &+ \frac{1}{2} \sum_i \int_0^{L_i} V_{g-1,n+1}^\circ(x, L_i - x, L_{\{i\}^c}) x(L_i - x) dx \end{aligned}$$

For this proposition, we can deduce  $V_{g,n}^\circ(L)$  from the case  $V_{0,2}^\circ(L_1, L_2) = 1$ .

*Proof.* To prove this proposition, we remark that when we apply theorem 4.5 to a surface with only one boundary, surgeries that are allowed are necessarily of type *I* or *III*; in other cases, a component of the surface must contain only positive boundary components, which is impossible. Reciprocally performing gluing of type *I* or *III* preserves the sub-family of surfaces with only one negative boundary, then we can deduce the formula. The form of the recursion preserves the space of polynomials, and the first line increases the degree by one, and then

$$4g - 2 + n - 1 + 1 = 4g - 2 + n$$

the second by 3, and

$$4(g - 1) - 2 + n + 1 + 3 = 4g - 2 + n.$$

By induction, this achieves the proof. □

**String and dilaton equations:** Functions  $V_{g,n}^\circ$  satisfy two series of equations that are similar to string and dilaton equations. The string equation is given by rewriting the formula of proposition 4.27. We obtain the formula:

$$V_{g,n+1}^\circ(0, L) = |L|V_{g,n}^\circ(L).$$

This formula is obtained by computing the volume of the space of a ribbon graph with only one edge in the first boundary component ( $\alpha_1^+ = 1$ ). By looking at the contribution of ribbon graphs with at most two edges in the first boundary component, it's possible to compute the first order of  $V_{g,n+1}$  at  $L_1 = 0$ . This gives the following relation, which is an analogous of dilaton equation.

**Proposition 4.30.** *The volume  $V_{g,n+1}^\circ$  satisfies the relation:*

$$\frac{\partial V_{g,n+1}^\circ}{\partial L_1}(0, L) = (2g + n - 1)V_{g,n}^\circ(L).$$

**Case of genus 0:** A particular case is one of the spheres. Rewriting the last formula, we obtain the relation:

**Corollary 4.14.** *The volumes  $V_{0,n}^\circ(L)$  satisfy the recursion,*

$$(n - 1)V_{0,n}^\circ(L) = \sum_{i < j} (L_i + L_j)V_{0,n-1}^\circ(L_i + L_j, L_{\{i,j\}}).$$

Then they are given by:  $V_{0,n}^\circ(L) = |L|^{n-2}$ .

*Proof.* The recursion is a consequence of corollary 4.11; in the case of the sphere, gluings of type three are not allowed. To obtain the explicit formula, we can see that the recursion determines the  $V_{0,n}^\circ(L)$  for all  $n$  with the initial condition  $V_{0,2}^\circ = 1$ . Then, by a direct computation, we can see that the functions  $|L|^{n-2}$  satisfy the recursion and also the initial condition.  $\square$

But in the case of genus zero, the surface with only one positive boundary is also preserved by the recursion along positive boundaries:

$$(n - 1)V_{0,1,n}^\circ(L_1^+ | L^-) = \frac{1}{2} \sum_{I_1^-, I_2^-} |L_{I_1^-}^-| |L_{I_2^-}^-| V_{0,1,n_1}^\circ(|L_{I_1^-}^-| |L_{I_1^-}^-|) V_{0,1,n_2}^\circ(|L_{I_2^-}^-| |L_{I_2^-}^-|).$$

Then, we have  $V_{0,1,n}^\circ(|L| | L) = V_{0,n,1}^\circ(L | |L) = V_{0,n}^\circ(L)$ , and it's possible to obtain another recurrence relation for the function  $V_{0,n}^\circ(L)$ :

$$(n - 1)V_{0,n}^\circ(L) = \frac{1}{2} \sum_{I_1, I_2} |L_{I_1}| |L_{I_2}| V_{0,n_1+1}^\circ(L_{I_1}) V_{0,n_1+1}^\circ(L_{I_2}).$$

These two recurrence relations are in fact directly related to planted numbered trees and correspond to remove a pair of leaves or the root.

**Recursion for the coefficients and Cut-and-Join equation:** In this section, we investigate the recursion obtained for coefficients of the polynomials  $V_{g,n}^\circ(L)$ . We can write

$$V_{g,n}^\circ(L) = \sum_{\alpha} c(\alpha) L^\alpha, \quad \text{with } L^\alpha = \prod_i L_i^{\alpha_i},$$

with:

$$d(\alpha) = 4g - 2 + n \quad \text{and} \quad n(\alpha) = n.$$

So it's possible to drop indices  $(g, n)$  in the notation. By expending the last formula, we derive the following relation for these coefficients:

**Corollary 4.15.** *Coefficients  $(c(\alpha))$  satisfy the following recursion:*

$$\begin{aligned} (2g - 1 + n)c(\alpha) &= \frac{1}{2} \sum_{i \neq j} \frac{(\alpha_i + \alpha_j)!}{\alpha_i! \alpha_j!} c(\alpha_i + \alpha_j - 1, \alpha_{\{i,j\}^c}) \\ &+ \frac{1}{2} \sum_{i, x_1 + x_2 = \alpha_i - 3} \frac{(x_1 + 1)!(x_2 + 1)!}{\alpha_i!} c(x_1, x_2, \alpha_{\{i\}^c}). \end{aligned}$$

Coefficients  $c(\alpha)$  satisfy an important symmetry; they are invariant under permutations. Then, as for the intersection numbers of the tautological class over the moduli space, it's possible to see  $c$  as a function on the set of generalized partitions; we write  $c(\mu)$  where  $\mu = (\mu(0), \mu(1), \dots) \in \mathcal{P}(\mathbb{N})$ . Then we can consider the following formal series with infinitely many variables:

$$Z^\circ(q, \mathbf{t}) = \sum_{\mu} \frac{q^{\frac{d(\mu) + n(\mu)}{2}} \prod_i (i!)^{\mu(i)} t_i^{\mu(i)}}{\prod_i \mu(i)!} c(\mu).$$

From corollary 4.15, we obtain the following result:

**Corollary 4.16.** *The series  $Z^\circ(q, \mathbf{t})$  satisfies the following equation:*

$$\frac{\partial Z^\circ}{\partial q} = \frac{1}{2} \sum_{i,j} (i+j) t_i t_j \partial_{i+j-1} Z^\circ + \frac{1}{2} \sum_{i,j} (i+1)(j+1) t_{i+j+3} \partial_i \partial_j Z^\circ + \frac{t_0^2}{2},$$

with  $Z^\circ(0, \mathbf{t}) = 0$ .

In fact, variables  $q$  and  $\mathbf{t}$  are not independent; we have the relation

$$Z^\circ(q, \mathbf{t}) = Z^\circ(1, t(q)) = Z^\circ(t(q)), \quad \text{with } t_i(q) = q^{\frac{i+1}{2}} t_i.$$

And then we have, by taking the derivative and evaluating at  $q = 1$ ,

$$\frac{\partial Z^\circ}{\partial q} = \sum_i \frac{i+1}{2} t_i \partial_i Z^\circ.$$

Then we can obtain

$$\sum_i (i+1) t_i Z^\circ = \sum_{i,j} (i+j) t_i t_j \partial_{i+j-1} Z^\circ + \sum_{i+j} (i+1)(j+1) t_{i+j-3} \partial_i \partial_j Z^\circ + \frac{t_0^2}{2}.$$

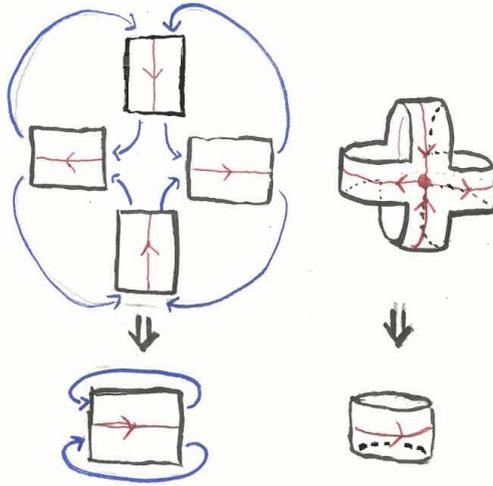


Figure 4.25: From oriented ribbon graphs to coverings.

### 4.5.2 Dual problem and Hurwitz number:

Let  $R_{g,n^+,n^-}^{\circ\nu}(\alpha^+|\alpha^-)$  be the number of oriented ribbon graphs of genus  $g$  such that:

- The vertices are prescribed by  $\nu$ .
- The perimeter of the positive (resp. negative) boundary component is given by  $\alpha^+$  (resp.  $\alpha^-$ ).

We denote simply  $R_{g,n^+,n^-}^{\circ}(\alpha^+|\alpha^-)$  the generic graph with only quadrivalent vertices. On the other hand, we denote  $h_{g,n^+,n^-}^{\nu}(\alpha^+|\alpha^-)$  the Hurwitz number of coverings of the sphere ramified over three points  $(x_0, x_+, x_-)$  such that:

- There is  $\nu(i)$  ramification of order  $i$  over  $x_0$ .
- There are  $n^-$  ramifications. over  $x_-$  and  $n^+$  over  $x_+$ , which are both labeled and of order  $\alpha_i^{\pm} - 1$ .

These coverings are called dessins d'enfants and were studied in many places. And similarly,  $h_{g,n^+,n^-}(\alpha^+|\alpha^-)$  represents case where all preimages of  $x_0$  have simple ramifications.

**Lemma 4.37.** *We have the following equality:*

$$R_{g,n^+,n^-}^{\circ\nu}(\alpha^+|\alpha^-) = h_{g,n^+,n^-}^{\nu}(\alpha^+|\alpha^-).$$

We give a pictorial explanation in figure 4.25.

*Proof.* Let  $R^{\circ}$  be an oriented ribbon graph made by gluing rectangles  $R_e^{\bullet}$  where  $e \in X^+R^{\circ}$  is oriented positively according to construction 4.1.1. From 4.10 we have a canonical one form  $\omega_R$  given locally by  $dz$  on each  $R_e$ . If we fix a vertex  $v \in X_0R$  of the graph, then the period map

$$z \longrightarrow \int_v^z \omega_R,$$

is well defined on the universal cover of  $M_R^\bullet$  and the image of the fundamental group is contained in  $\mathbb{Z} \subset \mathbb{C}$ , then by taking the quotient we have a well defined map:

$$M_R^\bullet \longrightarrow \mathbb{C}/\mathbb{Z}.$$

The target is an infinite cylinder (sphere with two marked points). Under this map,  $\omega_R$  is given by the pullback of  $\omega_0 = dz$  on  $\mathbb{C}/\mathbb{Z}$ . When we pull back a differential  $z^l dz$  by the map  $\phi = z^k$ , we obtain  $\phi^* w = k z^{(l+1)k-1} dz$ ; moreover, if  $l = -1$ , we also have  $\text{Res}_0 \phi^* w = k \text{Res}_0 w$ . According to this, a vertex of degree  $2i + 2$  is mapped to  $0 \in \mathbb{C}/\mathbb{Z}$  and corresponds to a zero of degree  $i$  of  $\omega_R$ . Then we see that the map  $\phi_R$  is ramified of degree  $i$  at this point. We can see  $\mathbb{C}/\mathbb{Z}$  as a sphere with two removed points at  $x_+, x_-$  and  $\omega_0$  as two simple poles at these points; under our assumption, the positive boundary components are mapped to the pole at  $x_+$  (resp.  $x_-$ ). Their preimages are simple poles of  $w_R$ , and the absolute value of the residue at a pole corresponds to the length of the boundary component; it is also equal to the number of edges that it contains. On the other hand, for all coverings ramified over  $x_0, x_-, x_+$ , it's possible to obtain a ribbon graph on the surface by looking at the pre-image of the circle based at  $x_0$ ; the orientation on the circle induces an orientation on the graph (see figure 4.25).  $\square$

Then oriented generic ribbon graphs with one negative boundary correspond to dessins d'enfants with a maximal ramification over  $x^-$  and  $2g - 2 + n$  simple ramifications over 0. Let  $h_{g,n+1}(\alpha^+)$  be the corresponding Hurwitz number. We assume that the ramifications over the first point are labeled. The last lemma and an explicit computation of the volumes in this case allow us to write the following formula:

**Lemma 4.38.** *The volumes  $V_{g,n}^\circ$  are polynomials that are naturally related to Hurwitz numbers in the following way:*

$$V_{g,n}^\circ(x_1, \dots, x_n) = \sum_{\alpha} h_{g,n+1}(\alpha) \prod_i \frac{x_i^{\alpha_i-1}}{(\alpha_i-1)!}$$

*Proof.* To prove this result, we compute the volume  $V_{R^\circ}(L)$  associated with an oriented ribbon graph with a single negative boundary component. This is an integral over some affine subspace in  $\text{Met}(R)$ . The relation on  $\text{Met}(R)$  is given by  $L_i^+(m) = L_i$  for each  $i = 1 \dots n^+$ . As the graph is directed, the dual is bipartite, and then an edge of the graph appears in exactly one of these equations with a weight equal to 1. In other words, there is an identification.

$$\text{Met}(R^\circ, L, |L|) = \prod_i \{(m_e)_{[e]_2 = \beta_i^+} \mid \sum_e m_e = L_i\}$$

Then the volume is given by the volume of a product of simplices. Each factor is equipped with the affine measure and

$$\int_{\sum_1^n y_j = x} d\sigma = \frac{x^{n-1}}{(n-1)!}.$$

Then the volume of  $\text{Met}(R^\circ, L, |L|)$  is

$$\prod_i \frac{L_i^{\alpha_i^+(R^\circ)-1}}{(\alpha_i^+(R^\circ)-1)!},$$

where  $\alpha_i^+(R^\circ)$  is the number of edges in the boundary  $\beta_i^+$ . By summing the contributions of all the graphs

$$V_{g,n}^\circ = \sum_{R^\circ} V_{R^\circ} = \sum_{R^\circ} \frac{1}{\#\text{Aut}(R^\circ)} \frac{L_i^{\alpha_i^+(R^\circ)-1}}{(\alpha_i^+(R^\circ)-1)!}.$$

The coefficient in front of  $\prod_i \frac{L_i^{\alpha_i-1}}{(\alpha_i-1)!}$  is the number of quadrivalent ribbon graphs with  $\alpha_i$  edges on the  $i$ -ième positive boundary components, counted with automorphisms. By using lemma 4.37, we can conclude the proof.  $\square$

# Chapter 5

## Volumes of moduli spaces of oriented ribbon graphs and Cut-and-Join operators

In this chapter, we investigate the algebraic aspects of the recursion of chapter 4.5.2. Using the functions  $V_{g,n^+,n^-}$  by the formula

These operators map polynomials to polynomials and we show that the recursion of theorem ?? is related to Cut-and-Join equation. To do this, we use the formalism of formal Fock space in order to define the series

$$K(q) = \exp_{\sqcup} \left( \sum_{g,n^+,n^-} K_{g,n^+,n^-} \right)$$

Where  $\sqcup$  is the symmetric product that corresponds topologically to the disjoint union of two surfaces. The operator  $K(q)$  acts on the space  $\mathbb{Q}[[t_0, t_1, \dots]]$  and it satisfies the Cut-and-Join equation with  $\frac{dK}{dq} = PK(q)$  with

$$P = \frac{1}{2} \sum_{i,j} (i+1)(j+1)t_{i+1}t_{j+1}\partial_{i+j} + \frac{1}{2} \sum_{i,j} (i+j+2)t_{i+j+2}\partial_i\partial_j.$$

We also consider the generating series  $Z = K \cdot \mathbf{e}$  where  $\mathbf{e} = \exp(t_0)$  is the vacuum. We show that this series is a generating series for Grothendieck dessins d'enfants and obtain topological recursion with the spectral curve  $xy = y^2 + 1$ . We also investigate the case of graphs with vertices of higher degrees.

### 5.1 Fock spaces

In this part, we recall construction of the bosonic Fock spaces and consider linear operators on them. A large part of what we do is more or less classical.

### 5.1.1 Generalities on bosonic Fock spaces

**Tensor algebras:** Let  $A$  be a countable set (usually  $\mathbb{N}$  or  $\mathbb{N}^*$ ) and  $V$  be the free vector space over  $\mathbb{Q}$  generated by  $A$ :

$$V = \bigoplus_{\alpha \in A} \mathbb{Q} e_{\alpha}.$$

For each integer  $n \geq 0$ , we denote

$$T_n(V) = V^{\otimes n},$$

the  $n$ -th tensor power of  $V$ . The space  $T_n(V)$  admits a basis given by the vectors

$$e_{\alpha} = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n).$$

They are indexed by multi-indices  $\alpha \in A^n$ . By convention, the space  $T_0(V)$  is one-dimensional and generated by a vector  $e_{\emptyset} = 1$ , which is represented by the empty multi-index. The tensor algebra is then the direct sum

$$T(V) = \bigoplus_n T_n(V)$$

which admits a basis  $(e_{\alpha})_{\alpha \in \mathcal{I}(A)}$  indexed by  $\mathcal{I}(A) = \bigsqcup_{n \geq 0} A^n$ . The space  $V$  can be endowed with the scalar product  $\langle \cdot, \cdot \rangle$ , which makes the vectors  $e_{\alpha}$  orthonormal. This scalar product can be extended to each  $T_n(V)$  by setting

$$\langle e_{\alpha}, e_{\alpha'} \rangle = \frac{\delta_{\alpha, \alpha'}}{n!}.$$

Finally, we define a scalar product on the full space  $T(V)$  by assuming that the spaces  $(T_n(V))_{n \geq 0}$  are pairwise orthogonal.

**Polynomial Fock space:** The symmetric group  $\mathfrak{S}_n$  acts on the set  $A^n$  of multi-indices of length  $n$ . Orbits in  $A^n$  under this action are indexed by "generalized" partitions  $\mathcal{P}_n(A)$  (see also subsection 3.1). They are maps

$$\mu : A \longrightarrow \mathbb{N},$$

with

$$n(\mu) = \sum_{\alpha \in A} \mu(\alpha) = n.$$

For each multi-index  $\alpha \in \mathcal{I}_n(A)$ , we denote  $\mu_{\alpha}(a) = \#\{i | \alpha_i = a\}$ , which defines an element  $\mu_{\alpha} \in \mathcal{P}_n(A)$ . This induces a bijection,

$$A^n / \mathfrak{S}_n \rightarrow \mathcal{P}_n(A).$$

The orbit of  $\alpha$  under this action is then the set  $C(\mu_{\alpha})$  of all  $\alpha'$  with  $\mu_{\alpha'} = \mu_{\alpha}$ . We have the relation

$$\#C(\mu) = \frac{n(\mu)!}{\mu!} \quad \text{with} \quad \mu! = \prod_{\alpha} \mu(\alpha)!.$$

The group  $\mathfrak{S}_n$  acts on  $T_n(V)$  by permuting variables, this action is compatible with the one on multi-indices. Let  $S_n(V)$  be the space of invariant elements under this action.

$$S_n(V) = T_n(V)^{\mathfrak{S}_n}.$$

$S_n(V)$  admits a basis  $e_\mu$  indexed by generalized partitions with  $\mu \in \mathcal{P}_n(A)$ ; these vectors are defined by

$$e_\mu = \sum_{\alpha \in C(\mu)} e_\alpha.$$

If we denote

$$s_n : T_n(V) \longrightarrow S_n(V),$$

the symmetrisation operator, we have the relation  $s_n(e_\alpha) = \mu! e_{\mu\alpha}$ . The "polynomial" Bosonic Fock space (or symmetric algebra) is then the direct sum:

$$S(V) = \bigoplus_n S_n(V) = \bigoplus_{\mu \in \mathcal{P}(A)} \mathbb{Q} e_\mu,$$

with  $\mathcal{P}(A) = \bigsqcup_n \mathcal{P}_n(A) = \mathbb{N}^{(A)}$ . The basis  $(e_\mu)_{\mu \in \mathcal{P}(A)}$  is orthogonal for the restriction of  $\langle \cdot, \cdot \rangle$  to  $S(V)$ , with the normalization  $\mu! \langle e_\mu, e_\mu \rangle = 1$ .

**Formal Fock spaces:** Generally, to deal with infinite dimensions, it's interesting to introduce grading. Let  $d$  be any function

$$d : A \longrightarrow \mathbb{N},$$

$d$  admits a natural extension by setting

$$d(\alpha) = \sum_i d(\alpha_i), \quad d(\mu) = \sum_\alpha \mu(\alpha) d(\alpha).$$

We assume that for each integer  $d$  the space

$$V^d = \bigoplus_{d(\alpha)=d} \mathbb{Q} e_\alpha,$$

spanned by homogeneous elements of degree  $d$  is finite dimensional. The tensor and the symmetric algebra's then admit two gradings, which are the degree  $d$  and the number  $n$ :

$$T(V) = \bigoplus_{n,d} T_n^d(V), \quad S(V) = \bigoplus_{n,d} S_n^d(V)$$

In a fancy way, the grading  $d$  defines a topology, the adic topology. The completion of  $V$  for this topology will be denoted  $\hat{V}$ . As we assume that the sub-spaces  $V^d$  are finite dimensional, then  $\hat{V}$  is the space of all formal sums

$$\sum_\alpha a(\alpha) e_\alpha \in \prod_d V^d.$$

Similarly, we can take the completion of  $T(V)$  and  $S(V)$  for the topology given by the two gradings  $n, d$ . We denote these spaces as  $\hat{T}(V)$  and  $\hat{S}(V)$ , we call the space  $\hat{S}(V)$  the formal bosonic Fock space:

$$\hat{T}(V) = \prod_{d,n} T_n^d(V) \quad \text{and} \quad \hat{S}(V) = \prod_{d,n} S_n^d(V).$$

The formal bosonic Fock space is identified with the space  $\mathbb{Q}^{\mathcal{P}(A)}$ , and then we represent the elements of  $\hat{S}(V)$  as "formal sums"

$$\sum_{\mu \in \mathcal{P}(A)} a(\mu) e_\mu.$$

**Remark 5.1.** These two gradings  $d, n$  are naturally associated to two operators  $D, N$  (that could be denoted  $\hat{d}, \hat{n}$ ), which commute together and are diagonal in the canonical basis  $(e_\mu)_\mu$ . We have

$$Ne_\mu = n(\mu)e_\mu \quad \text{and} \quad De_\mu = d(\mu)e_\mu.$$

The decomposition

$$S(V) = \bigoplus_{n,d} S_n^d(V),$$

is a decomposition in eigenspaces. The grading  $D$  generates a flow on  $V$ , which is defined by the equation  $\partial_t x_t = Dx_t$ , then  $x_t = e^{Dt}x_0$  with  $e^{Dt}e_\mu = e^{d(\mu)t}e_\mu$ .

**Symmetric product (disjoint union):** The space  $T(V)$  comes with a structure of algebra defined by the tensor product:

$$T_{n_1}^{d_1}(V) \otimes T_{n_2}^{d_2}(V) \longrightarrow T_{n_1+n_2}^{d_1+d_2}(V),$$

It's given in coordinates by the concatenation of indices and it's related to the monoid structure on  $\mathcal{I} = \sqcup_n A^n$ . The tensor product is additive for  $d, n$ ; this implies that it's still well defined on the completion  $\hat{T}(V)$  of  $T(V)$ .

The symmetrization of the tensor product defines the symmetric product (disjoint union):

$$\sqcup : S_{n_1}(V) \otimes S_{n_2}(V) \longrightarrow S_{n_1+n_2}(V).$$

It's defined by

$$x \sqcup y = \frac{s_{n_1+n_2}(x \otimes y)}{n_1!n_2!} \quad \forall (x, y) \in S_{n_1}(V) \times S_{n_2}(V).$$

As before, the disjoint union  $\sqcup$  is still additive for  $d, n$ , then it's well defined on the completion of the space

$$\sqcup : \hat{S}(V) \otimes \hat{S}(V) \longrightarrow \hat{S}(V).$$

On the canonical basis of vectors  $e_\mu$ , the product is characterized by:

$$e_\mu \sqcup e_\nu = \binom{\mu + \nu}{\mu \ \nu} e_{\nu+\mu}, \tag{5.1}$$

with

$$\binom{\mu + \nu}{\mu \ \nu} = \frac{(\mu + \nu)!}{\mu!\nu!} = \prod_{\alpha} \frac{(\mu(\alpha) + \nu(\alpha))!}{\mu(\alpha)!\nu(\alpha)!}.$$

From formula 5.1, we see that the disjoint union defines a commutative product; moreover, it's linear, then  $\hat{S}(V)$  is a commutative algebra. If two partitions have disjoint supports, according to formula 5.1, they satisfy

$$e_\mu \sqcup e_\nu = e_{\mu+\nu}.$$

Then we can also deduce,

$$e_\alpha^{\sqcup k} = e_{\delta_\alpha}^{\sqcup k} = k!e_{k\delta_\alpha} \quad \alpha \in A.$$

These relations give the factorization formula

$$\mu!e_\mu = \bigsqcup_{\alpha} e_\alpha^{\mu(\alpha)}. \tag{5.2}$$

**Series with an infinite number of variables:** From formula 5.2, it is natural to associate to a vector  $a \in S(V)$  the polynomial

$$Z_a(\mathbf{t}) = \sum_{\mu} \frac{a(\mu) \mathbf{t}^{\mu}}{\mu!}, \quad \mathbf{t}^{\mu} = \prod_{\alpha \in A} t_{\alpha}^{\mu(\alpha)}.$$

$Z_a(\mathbf{t})$  belongs to  $\mathbb{C}[\mathbf{t}]$ , the space of polynomials in the variables  $\mathbf{t} = (t_{\alpha})_{\alpha \in A}$ , which depend on only a finite number of variables. According to the formulas 5.1, we can see that this application satisfies the following multiplicative property:

$$Z_{a_1 \sqcup a_2}(\mathbf{t}) = Z_{a_1}(\mathbf{t}) Z_{a_2}(\mathbf{t}).$$

Then  $Z$  defines a morphism of algebras,

$$Z : S(V) \longrightarrow \mathbb{C}[\mathbf{t}].$$

It's the exponential generating series (or partition function). According to the formula 5.2,  $Z$  is an isomorphism. In this sense,  $S(V)$  is the free commutative algebra generated by the space  $V$ . The map  $Z$  preserves the two degrees and then extends to the completion of the space  $\hat{S}(V)$  and induces an isomorphism,

$$Z : \hat{S}(V) \longrightarrow \mathbb{C}[[\mathbf{t}]].$$

Where  $\mathbb{C}[[\mathbf{t}]]$  is the space of formal series in the variables  $\mathbf{t} = (t_{\alpha})_{\alpha \in A}$

**Remark 5.2.** This identification is also given in [KS17] in the converse direction,

$$\mathbb{C}[[\mathbf{t}]] \longrightarrow \hat{S}(V).$$

This map is given by the Jacobian tensor

$$\text{Jac}(F)(\mu) = \left( \prod_{\alpha} \partial_{\alpha}^{\mu(\alpha)} \right) F$$

**Remark 5.3.** The choice of the map is not canonical, indeed, for a morphism  $s : \mathcal{P} \longrightarrow \mathbb{R}_{>0}$ , if we set  $s \cdot a(\mu) = s(\mu) a(\mu)$ , then the map

$$Z_a^s(\mathbf{t}) = Z_{s \cdot a}(\mathbf{t}) = Z_a(s \cdot \mathbf{t}),$$

is still a morphism that corresponds to a change of variables  $(s \cdot \mathbf{t})_{\alpha} = s(\alpha) t_{\alpha}$  and then a diagonal change of basis  $e_{\alpha} \mapsto s(\alpha)^{-1} e_{\alpha}$ . More generally, a change of basis induces a change of variables in the partition function.

## 5.1.2 Operators on Fock spaces

**Operators:** In this part, we consider linear operators on Fock spaces. We start with  $A \in \text{End}(T(V))$

$$A : T(V) \longrightarrow T(V).$$

By using restrictions and projections  $A$  defines a family of operators  $A_{n^+, n^-} \in \text{Hom}(T_{n^-}(V), T_{n^+}(V))$

$$A_{n^+, n^-} : T_{n^-}(V) \longrightarrow T_{n^+}(V).$$

It's possible to consider the matrix coefficients of  $A$  in the basis  $(e_\alpha)_{\alpha \in \mathcal{I}(A)}$ , and we denote  $(A[\alpha^+ | \alpha^-])_{(\alpha^+, \alpha^-)}$  the infinite matrix indexed by  $\mathcal{I}(A) \times \mathcal{I}(A)$ . To be well defined as an operator on  $T(V)$ , each column of this matrix must contain only a finite number of non-vanishing coefficients.

These considerations generalize to symmetric operators. But this time  $A \in \text{End}(S(V))$  is a matrix indexed by pairs of partitions  $(A[\mu^+ | \mu^-])_{(\mu^+, \mu^-)}$ .

We can also consider operators on the formal completions  $\hat{T}(V)$  and  $\hat{S}(V)$ ; they are matrices with finite rows. In other words, we have  $A \in \text{End}(\hat{T}(V))$  iff for each  $\alpha^+$  there is only a finite number of  $\alpha^-$  such that  $A[\alpha^+ | \alpha^-]$  is non-zero. This is natural because the dual of the space  $T(V)$  is identified with its formal completion  $\hat{T}(V)$ , the transpose of a matrix with finite columns has finite rows.

An operator is of degree  $(d, n)$  if it satisfies the following commutation rules: <sup>1</sup>

$$[D, A] = dA, \quad [N, A] = nA,$$

and then it induces an operator

$$A : T_{n_1}^{d_1}(V) \longrightarrow T_{n_1+n}^{d_1+d}(V),$$

for each  $(d_1, n_1)$ . An homogeneous operator of degree  $(d, n)$  always defines an operator in  $\text{End}(\hat{T}(V))$  <sup>1</sup>

**Coproduct and union of operators:** The product defines a map

$$\theta : S(V) \otimes S(V) \longrightarrow S(V),$$

then the coproduct is the dual operator

$$\theta^* : S(V)^* \longrightarrow S(V)^* \otimes S(V)^*.$$

To avoid any problems, we restrict ourselves to homogeneous elements of degree  $d$ . Using the scalar product, the coproduct defines a map

$$\theta^* : S^d(V) \longrightarrow (S(V) \otimes S(V))^d.$$

The target is the subspace of elements of degree  $d$  in the tensor product. By using duality, we can obtain the following formula:

$$\theta^* e_\mu = \sum_{\mu_1 + \mu_2 = \mu} e_{\mu_1} \otimes e_{\mu_2}. \quad (5.3)$$

Which defines an operator on both  $S(V)$  and  $\hat{S}(V)$ .

<sup>1</sup>A sum of the form  $A = \sum_{n,d} A_n^d$  does not always converge because  $(d, n)$  could take negative values. In the case of operators in  $\text{Hom}(\hat{T}_{n^-}(V), \hat{T}_{n^+}(V))$  an expression of the form

$$\sum_{d \geq k} A_{n^+, n^-}^d \quad k \in \mathbb{Z}.$$

is always well defined.

**Remark 5.4.** From the last formula, we can see  $\theta^*$  as the splitting operator and  $\theta$  as the union operator.

With the coproduct and the product, it's then natural to define the product of two operators  $A_1, A_2$  by the formula

$$A_1 \sqcup A_2 = \theta \circ (A_1 \otimes A_2) \circ \theta^*.$$

This definition leads to the quite natural formula:

$$(A_1 \sqcup A_2)e_\mu = \sum_{\mu_1 + \mu_2 = \mu} (A_1 e_{\mu_1}) \sqcup (A_2 e_{\mu_2}) \quad \forall \mu \in \mathcal{P}(A). \quad (5.4)$$

In terms of the coefficients of the matrices, this leads to the following formula:

$$(A_1 \sqcup A_2)[\mu^+ | \mu^-] = \sum_{\mu^+ = \mu_1^+ + \mu_2^+, \mu^- = \mu_1^- + \mu_2^-} \binom{\mu^+}{\mu_1^+ \mu_2^+} A_1[\mu_1^+ | \mu_1^-] A_2[\mu_2^+ | \mu_2^-]. \quad (5.5)$$

From formulas 5.4 and 5.5, we see that the union is well defined for operators on  $S(V)$  and  $\hat{S}(V)$ , and the sums that are involved are always finite. We can summarize this discussion by the following proposition.

**Proposition 5.1.** *The disjoint union of operators defines a commutative product*

$$End(\hat{S}(V)) \otimes End(\hat{S}(V)) \longrightarrow End(\hat{S}(V)).$$

*Which preserves the two gradings, and this statement remains true if we replace  $\hat{S}(V)$  by  $S(V)$ .*

We can define the projection  $pr_k$

$$pr_k : S(V) \longrightarrow S_k(V).$$

**Lemma 5.1.** *The projection  $pr_0$  is the unit for  $\sqcup$ , i.e.,*

$$pr_0 \sqcup A = A$$

*The power  $pr_1^{\sqcup k}$  is the symmetrisation on  $T_k(V)^2$  and then on  $S(V)$*

$$\frac{pr_1^{\sqcup k}}{k!} = pr_k$$

*We can deduce the identity*

$$id = \exp_{\sqcup}(pr_1)$$

*on  $S(V)$ .*

**Remark 5.5.** *We can associate to an operator  $A$  the formal series*

$$Z_A = \sum_{\mu_+, \mu_-} A[\mu_+ | \mu_-] \frac{\mathbf{t}_+^{\mu_+} \mathbf{t}_-^{\mu_-}}{\mu_+!}.$$

---

<sup>2</sup>The operation  $\sqcup$  can also be defined on  $End(T(V))$

It is an element of  $\mathbb{Q}[\mathbf{t}_-][[\mathbf{t}_+]]$  if  $A \in \text{End}(\hat{S}(V))$  (resp  $\mathbb{Q}[\mathbf{t}_+][[\mathbf{t}_-]]$  if  $A \in \text{End}(S(V))$ ). In this formula, we denote

$$\mathbf{t}_+^{\mu^+} = \prod_{\alpha} \mathbf{t}_{\alpha,+}^{\mu^+(\alpha)} \quad \mathbf{t}_-^{\mu^-} = \prod_{\alpha} \mathbf{t}_{\alpha,-}^{\mu^-(\alpha)}.$$

Thus, similarly to the case of  $\hat{S}(V)$ , by using formula 5.4, these series satisfy the following multiplicative property

$$Z_{A_1 \sqcup A_2} = Z_{A_1} Z_{A_2}.$$

Indeed, the elements  $t_{\alpha,+}$  and  $t_{\alpha,-}$  correspond, respectively to

$$e_{\alpha,+} = e_{\alpha} \otimes e_{\emptyset}^*, \quad \text{and} \quad e_{\alpha,-} = e_{\emptyset} \otimes e_{\alpha}^*.$$

If

$$E_{\mu^+,\mu^-} = e_{\mu^+} \otimes e_{\mu^-}^*,$$

then we have

$$E_{\mu_1^+,\mu_1^-} \sqcup E_{\mu_2^+,\mu_2^-} = \begin{pmatrix} \mu_1^+ + \mu_2^+ \\ \mu_1^+ \mu_2^+ \end{pmatrix} E_{\mu_1^+ + \mu_2^+,\mu_1^- + \mu_2^-}$$

and then we have the factorization

$$E_{\mu^+,\mu^-} = E_{\mu^+,\emptyset} \sqcup E_{\emptyset,\mu^-}$$

The  $E_{\mu^+,\mu^-}$  span a subalgebra of  $\text{End}(S(V))$ , which is then isomorphic to

$$S(V \oplus V^*) = S(V) \otimes S(V^*)$$

which justifies the identification with  $\mathbb{Q}[[\mathbf{t}_+, \mathbf{t}_-]]$

**Creation and annihilation operator:** In the theory of Fock spaces, natural operators are the creation operators

$$\varphi_{\alpha}(x) = e_{\alpha} \sqcup x,$$

which satisfies

$$\varphi_{\alpha} e_{\mu} = (\mu(\alpha) + 1) e_{\mu + \delta_{\alpha}}.$$

The annihilation operator  $\varphi_{\alpha}^*$  is defined by duality; we have

$$\varphi_{\alpha}^* e_{\mu}^* = \mu(\alpha) e_{\mu - \delta_{\alpha}}^*, \quad \varphi_{\alpha}^* e_{\mu} = \delta_{\mu(\alpha) > 0} \Leftrightarrow e_{\mu - \delta_{\alpha}}.$$

Creation operators belong to  $\text{End}(\hat{S}(V)) \cap \text{End}(S(V))$ , then by duality, annihilation operators are also in this space. In other words, both are well defined on the polynomial and the formal Fock spaces. The two operators  $\varphi_{\alpha}, \varphi_{\alpha'}^*$  satisfy the usual commutation relation

$$[\varphi_{\alpha}^*, \varphi_{\alpha'}] = \delta_{\alpha,\alpha'}.$$

By using the morphism  $S(V) \rightarrow \mathbb{Q}[\mathbf{t}]$ , it's straightforward to see that the operators  $\varphi_{\alpha}, \varphi_{\alpha}^*$  are given by the following differential operators:

$$\varphi_{\alpha} = t_{\alpha}, \quad \text{and} \quad \varphi_{\alpha}^* = \partial_{\alpha} \quad \partial_{\alpha} = \frac{\partial}{\partial t_{\alpha}}. \quad (5.6)$$

In other words, for all  $\alpha \in A$

$$Z_{\varphi_\alpha(x)} = t_\alpha Z_x \quad Z_{\varphi_\alpha(x)} = \partial_\alpha Z_x \quad \forall x \in \hat{S}(V)$$

Then it's possible to consider the algebra generated by these operators. Let

$$\varphi_\mu = \prod_{\alpha} \varphi_\alpha^{\mu(\alpha)}$$

and  $\varphi_\mu^*$  the dual operator. It's straightforward to see that  $\varphi_\mu$  and  $\varphi_\mu^*$  act on  $\mathbb{Q}[[\mathbf{t}]]$  as

$$Z_{\varphi_\mu(x)} = \mathbf{t}^\mu Z_x \quad Z_{\varphi_\mu^*(x)} = \partial_\mu Z_x \quad \partial_\mu = \prod_i \partial_i^{\mu(i)}$$

Then we consider operators of the form

$$\sum_{\mu^+, \mu^-} D[\mu^+, \mu^-] \varphi_{\mu^+} \varphi_{\mu^-}^*.$$

The degree of  $\varphi_{\mu^+} \varphi_{\mu^-}^*$  is  $d(\mu^+) - d(\mu^-)$ . Then such an operator is well defined on  $S(V)$  iff the matrix  $D[., .]$  has finite columns and on  $\hat{S}(V)$  iff it has finite rows. We denote  $\mathcal{D}(V)$  and  $\hat{\mathcal{D}}(V)$  these subspaces of  $\text{End}(S(V))$  and  $\text{End}(\hat{S}(V))$ .

### 5.1.3 Projective limit's

**Forgetting the number of variables and projective limit:** In what follows, there is a zero element  $0 \in A$  such that  $d(0) = 0$ . We assume that such an element is unique and denote  $A^* = A \setminus \{0\}$ . Then we can consider the annihilation operator  $\varphi_0^*$ ; it induces a family of operators

$$\varphi_0^* : S_{n+1}(V) \rightarrow S_n(V)$$

which define a projective system.

$$S_0(V) \longleftarrow S_1(V) \longleftarrow S_2(V) \longleftarrow \dots$$

We denote

$$S_\infty^d(V) = \varprojlim_n S_n^d(V), \quad \text{and} \quad \hat{S}_\infty(V) = \prod_d S_\infty^d(V). \quad (5.7)$$

$\hat{S}_\infty(V)$  is the formal completion (for the adic topology). We still have a projection

$$\hat{S}_\infty(V) \longrightarrow \hat{S}(V).$$

An element  $a \in \hat{S}_\infty(V)$  is an element of  $\hat{S}(V)$  that satisfies

$$\varphi_0^* a = a.$$

By using the correspondence with the set  $\mathbb{Q}[[\mathbf{t}]]$ , the annihilation operator  $\varphi_0^*$  corresponds to the derivative  $\partial_0$ . Then an element of  $a \in \hat{S}_\infty(V)$  corresponds to a formal series  $\phi_a \in \mathbb{Q}[[\mathbf{t}]]$  that satisfies

$$\partial_0 Z_a = Z_a.$$

Hence,  $Z_a$  takes the form

$$Z_a(\mathbf{t}) = \exp(t_0)Z_a(0, \mathbf{t}^*). \quad (5.8)$$

Where we denote  $\mathbf{t}^* = (t_\alpha)_{\alpha \in A^*}$ . In particular, for each  $\mu$ , the vectors  $\mathbf{e}_\mu = e_\mu \sqcup \exp_{\sqcup}(e_0)$  for  $\mu \in \mathcal{P}(A^*)$  form a basis of  $S_\infty(V)$ . They correspond to series  $\mathbf{t}^\mu \exp(t_0)$ . In particular, the vacuum is

$$\mathbf{e}_\emptyset = \exp_{\sqcup}(e_0)$$

and corresponds to  $\exp(t_0)$ .

**Remark 5.6** (Symmetric functions). *If  $V = \mathbb{Q}[L]$  is the space of polynomials,  $e_0$  corresponds to 1 the constant polynomial, and the map  $\varphi_0^*$  corresponds to the evaluation of the first variable at 0, we have*

$$\varphi_0^*P(L) = P(0, L).$$

The space  $S_\infty(V)$  is, in this case, the space of symmetric functions with an arbitrary number of variables;  $\hat{S}_\infty(V)$  is then the formal completion of this space for the graduation given by the degree.

**Projective limit and differential operators:** Let  $A^s$  be an operator on  $\hat{S}(V)$ , we can define the following operator:

$$A^u = A^s \sqcup id.$$

**Proposition 5.2.** *Let  $A^s$  and  $(A^s[\mu^+, \mu^-])$  be its matrices. We have the following expression in terms of creation and annihilation operators:*

$$A^u = \sum_{\mu^+, \mu^-} A^s[\mu^+ | \mu^-] \frac{\varphi_{\mu^+} \varphi_{\mu^-}^*}{\mu^+!}.$$

Then we see that the operator  $A^u$  acts on  $\mathbb{Q}[[\mathbf{t}]]$  as a differential operator

$$A^u = \sum A^s[\mu^+ | \mu^-] \frac{\mathbf{t}^{\mu^+} \partial_{\mu^-}}{\mu^+!}. \quad (5.9)$$

If  $\hat{\mathcal{D}}(V)$  is the algebra of such operators, an element  $A^u$  is well defined on  $\mathbb{Q}[[\mathbf{t}]]$  iff  $(A^s[\mu^+ | \mu^-])$  as finite rows, then iff  $A^s$  is in  $\text{End}(\hat{S}(V))$ . Then  $A^s \rightarrow A^s \sqcup id$  identifies  $\text{End}(\hat{S}(V))$  and  $\hat{\mathcal{D}}(V)$  (and also  $\mathcal{D}(V)$  with  $\text{End}(S(V))$ ).

*Proof.* By using the formula 5.4, we obtain

$$A^u e_\mu = \sum_{\mu_1 + \mu_2 = \mu} (A^s e_{\mu_1}) \sqcup e_{\mu_2} = \sum_{\mu^-, \mu^+} \delta_{\mu \geq \mu^-} A^s[\mu^+ | \mu^-] e_{\mu^+} \sqcup e_{\mu - \mu^-}.$$

We have for all  $\mu^+, \mu^-, \mu$

$$\varphi_{\mu^+} e_\mu = \mu^+! e_{\mu^+} \sqcup e_\mu, \quad \varphi_{\mu^-}^* e_\mu = \delta_{\mu \geq \mu^-} e_{\mu - \mu^-},$$

and then we can rewrite

$$e_{\mu^+} \sqcup e_{\mu - \mu^-} = \frac{\varphi_{\mu^+} e_{\mu - \mu^-}}{\mu^+!} = \frac{\varphi_{\mu^+} \varphi_{\mu^-}^* e_\mu}{\mu^+!}.$$

It allows us to prove the proposition. □

**Remark 5.7.** We can see that

$$E_{\mu^+, \emptyset} \sqcup id = \frac{\varphi_{\mu^+}}{\mu^+!} \quad \text{and} \quad E_{\emptyset, \mu^-} \sqcup id = \varphi_{\mu^-}^*.$$

Then we have

$$E_{\mu^+, \mu^-} \sqcup id = \frac{\varphi_{\mu^+} \varphi_{\mu^-}^*}{\mu^+!},$$

which also allows to recover the last formula.

A direct computation gives the following proposition:

**Proposition 5.3.** There is a unique operator  $A_1^{\natural} * A_2^{\natural} \in \text{End}(\hat{S}(V))$  that satisfies

$$(A_1^{\natural} \sqcup id) \circ (A_2^{\natural} \sqcup id) = (A_1^{\natural} * A_2^{\natural}) \sqcup id$$

*Proof.* This is the consequence of the last proposition and the fact that  $\hat{D}(V)$  is stable under composition  $\square$

**Remark 5.8** (Coefficients). In terms of matrices, by using proposition 5.2, the expression of  $*$  is encoded in the product of differential operators.

The spaces  $\mathcal{D}(V)$ ,  $\hat{D}(V)$  are not stable under the disjoint union. The time-ordered product  $::$  is defined on differential operators by

$$: \mathbf{t}^{\mu_1} \mathbf{t}^{\mu_2} : = \mathbf{t}^{\mu_1 + \mu_2}, \quad : \partial_{\mu_1} \partial_{\mu_2} : = \partial_{\mu_1 + \mu_2} \quad \text{and} \quad : \mathbf{t}^{\mu_1} \partial_{\mu_2} : = : \partial_{\mu_1} \mathbf{t}^{\mu_2} : = \mathbf{t}^{\mu_1} \partial_{\mu_2}.$$

**Proposition 5.4.** If we have  $A_i = A_i^{\natural} \sqcup id$ , then

$$(A_1^{\natural} \sqcup A_2^{\natural}) \sqcup id = : A_1 A_2 : .$$

Finally, if we assume that

$$\varphi_0^* A^{\natural} = 0.$$

From the formula

$$A^u(e_{\mu} \sqcup e_{k\delta_0}) = \sum_{\mu_1 + \mu_2 = \mu, r \leq k} (A^{\natural} e_{\mu_1 + r\delta_0}) \sqcup e_{\mu_2} \sqcup e_{(k-r)\delta_0}.$$

We can show that

$$A^u(e_{\mu} \sqcup e_{\emptyset}) = \sum_{\mu_1 + \mu_2 = \mu, r \leq k} (A^{\natural} e_{\mu_1 + r\delta_0}) \sqcup e_{\mu_2} \sqcup e_{\emptyset}.$$

And then  $A^u$  defines an operator on the projective limit  $\hat{S}_{\infty}(V)$ . Moreover, if  $A_1^{\natural}, A_2^{\natural}$  satisfy the last assumption, then we have

$$\varphi_0^* \circ (A_1^{\natural} * A_2^{\natural}) = 0.$$

### 5.1.4 Space of directed surfaces and acyclic stable graphs

**Symmetric algebra and directed surfaces:** Let  $\mathcal{M}$  be the vector space over  $\mathbb{Q}$  generated by isomorphism classes  $M_1^\circ$  of directed surfaces with labeled boundaries (they are not necessarily connected or stable), we also introduce the empty surface  $\emptyset$ . For each  $M_1^\circ$ , we denote  $e_{M_1^\circ} \in \mathcal{M}$  the corresponding vector in  $\mathcal{M}$ .  $\mathcal{M}$  contains the subspace  $\mathcal{M}^s$  generated by stable surfaces, and  $\mathcal{M}^c, \mathcal{M}^{cs} = \mathcal{M}^c \cap \mathcal{M}^s$  the spaces of connected surfaces. We assume that  $\emptyset$  is not in these two spaces. There are three natural gradings on  $\mathcal{M}$ ; the first is the opposite of the Euler characteristic denoted by  $d$ . If  $M_1^\circ$  is connected of type  $(g, n^+, n^-)$ , we have

$$d(M_1^\circ) = 2g - 2 + n^+ + n^-.$$

The two other graduations  $n^+, n^-$  are the numbers of positive and negative boundary components. Using this, we have a direct sum

$$\mathcal{M} = \bigoplus_d \mathcal{M}^d = \bigoplus_{n^+, n^-} \mathcal{M}_{n^+, n^-} = \bigoplus_{d, n^+, n^-} \mathcal{M}_{n^+, n^-}^d.$$

For each  $d, n^+, n^-$ , the spaces  $\mathcal{M}^d, \mathcal{M}_{n^+, n^-}$  are infinite dimensional, but  $\mathcal{M}_{n^+, n^-}^d$  is finite dimensional. We can consider the completion  $\hat{\mathcal{M}}$  of  $\mathcal{M}$  for these three gradings; it's given by:

$$\hat{\mathcal{M}} = \prod_{d, n^+, n^-} \mathcal{M}_{n^+, n^-}^d = \mathbb{Q}^{\mathbf{bord}_1^\circ}.$$

The action of  $\mathfrak{S}_{n^+, n^-}$  on labels induces an action on  $\mathcal{M}_{n^+, n^-}$ . As we explain in 3.2, a disconnected directed surface defines a pair of partitions of sets of positive and negative boundary components  $(I^+, I^-)$ . The group  $\mathfrak{S}_{n^+, n^-}$  acts on the pair of partitions  $(I^+, I^-)$ . We denote  $S_{n^+, n^-}(\mathcal{M})$  the subspace of invariant elements under this action, and we define the symmetric space,

$$S(\mathcal{M}) = \bigoplus_{n^+, n^-} S_{n^+, n^-}(\mathcal{M}).$$

$S(\mathcal{M})$  corresponds to the free module generated by surfaces with unlabeled boundaries (i.e., elements in  $\mathbf{bord}^\circ$ ). If  $M^\circ \in \mathbf{bord}^\circ$ , we denote  $e_{M^\circ}$  as the corresponding element of  $S(\mathcal{M})$ , which is the sum of all possible ways  $M_1^\circ$  to label boundary components of  $M^\circ$ . Connected surfaces are fixed by the action of the symmetric group, then  $\mathcal{M}^c \subset S(\mathcal{M})$ . It's still possible to consider the completion of the space for gradings  $d, n^+, n^-$ , we denote  $\hat{S}(\mathcal{M})$  this space. It corresponds to the space of "formal sum"

$$x = \sum_{M^\circ \in \mathbf{bord}^\circ} x(M^\circ) e_{M^\circ} \quad \hat{S}(\mathcal{M}) \simeq \mathbb{Q}^{\mathbf{bord}^\circ}.$$

**Remark 5.9** (Time inversion). *There is a natural involution that acts by reversing the sign of the boundaries*

$$\iota : \mathcal{M}_{n^+, n^-} \longrightarrow \mathcal{M}_{n^-, n^+},$$

*it preserves the Euler characteristic. This induces an involution on  $\mathcal{M}, S(\mathcal{M}), \hat{S}(\mathcal{M}) \dots$*

**Disjoint union:** As in the case of the operators on the Fock space, there is a first algebraic structure on  $\mathcal{M}$  given by the disjoint union of directed surfaces. Let  $M_{i_1}^\circ, M_{i_2}^\circ \in \mathbf{bord}_i^\circ$  two directed surfaces. By abuse of notation (and analogy with the tensor algebra), we denote  $M_{i_1}^\circ \otimes M_{i_2}^\circ \in \mathbf{bord}_i^\circ$  the surface obtained by taking the disjoint union and enumerating the boundaries by starting with the ones of  $M_1$ . We define the symmetric union by

$$e_{M_{i_1}^\circ} \sqcup e_{M_{i_2}^\circ} = \frac{1}{n_1^+! n_1^-! n_2^+! n_2^-!} \sum_{\sigma=(\sigma^+, \sigma^-)} \sigma \cdot (e_{M_{i_1}^\circ \otimes M_{i_2}^\circ})$$

where we sum over the group  $\mathfrak{S}_{n_1^++n_2^+, n_1^-+n_2^-}$ . As in the case of symmetric algebra, the disjoint union is associative and commutative and then defines a commutative product,

$$S(\mathcal{M}) \otimes S(\mathcal{M}) \longrightarrow S(\mathcal{M}). \quad (5.10)$$

This product is additive for graduations and then extends to the completion. The space  $S_{0,0}(\mathcal{M})$  is generated by one element, which is the empty surface  $e_\emptyset$  and corresponds to the unit of the disjoint union. We can see  $S(\mathcal{M})$  as the vector space generated by the elements of  $\mathbf{bord}^\circ$ . The disjoint union  $\sqcup$  defines also a monoidal structure on  $\mathbf{bord}^\circ$  moreover, it's compatible with the union on  $S(\mathcal{M})$ . If for all  $M^\circ \in \mathbf{bord}^\circ$  and for all  $M_1^\circ \in \mathbf{bord}^{\circ,c}$  we denote  $\nu_{M^\circ}(M_1^\circ)$  the number of connected components of  $M^\circ$  isomorphic to  $M_1^\circ$ , this defines an element in  $\mathcal{P}(\mathbf{bord}^{\circ,c})$ . For all  $M_1^\circ, M_2^\circ \in \mathbf{bord}^\circ$ , we then have

$$e_{M_1^\circ} \sqcup e_{M_2^\circ} = \begin{pmatrix} \nu_{M_1^\circ} + \nu_{M_2^\circ} \\ \nu_{M_1^\circ} \nu_{M_2^\circ} \end{pmatrix} e_{M_1^\circ \sqcup M_2^\circ}.$$

Then, similarly to the case of the Fock space, the algebra  $(S(\mathcal{M}), \sqcup, +)$  is the free commutative algebra generated by the connected surfaces in  $\mathbf{bord}^{\circ,c}$ . In particular, we have the following formula that relates connected and disconnected surfaces:

$$\sum_{M^\circ \in \mathbf{bord}^\circ} e_{M^\circ} = \exp_{\sqcup} \left( \sum_{M^\circ \in \mathbf{bord}^{\circ,c}} e_{M^\circ} \right). \quad (5.11)$$

**Gluing of surfaces and acyclic stable graph:** It's tempting to define a second structure on  $S(\mathcal{M})$  (or  $\hat{S}(\mathcal{M})$ ) by gluing surfaces along their boundaries. Let  $M_{i_1}^\circ, M_{i_2}^\circ \in \mathbf{bord}_i^\circ$  such that  $k = n^-(M_{i_1}^\circ) = n^+(M_{i_2}^\circ)$ . We define  $M_{i_1}^\circ \cdot' M_{i_2}^\circ$  as the surface obtained after gluing the boundaries of  $\partial^+ M_{i_2}^\circ$  with the ones of  $\partial^- M_{i_1}^\circ$  by identifying labels. On  $\mathcal{M}$ , we define the product by

$$e_{M_{i_1}^\circ} \cdot e_{M_{i_2}^\circ} = \begin{cases} \frac{e_{M_{i_1}^\circ \cdot' M_{i_2}^\circ}}{n^-(M_{i_1}^\circ)!} & \text{if } n^-(M_{i_1}^\circ) = n^+(M_{i_2}^\circ), \\ 0 & \text{else} \end{cases}$$

It's straightforward to see that this operation is associative and defines a product on  $\mathcal{M}$ , which is not commutative. But in what follows, we are interested in representations that associate to each  $e_{M_i^\circ} \in S(\mathcal{M})$  an operator  $K(e_{M_i^\circ})$  acting on some Fock space  $S(V)$ . In general, we have

$$K(e_{M_{i_1}^\circ}) \circ K(e_{M_{i_2}^\circ}) \neq K(e_{M_{i_1}^\circ} \cdot e_{M_{i_2}^\circ}).$$

To get around this problem, we introduce the vector space  $\mathcal{A}$  generated by all the acyclic stable graphs with labeled boundaries. We assume that these directed graphs can have components isomorphic to cylinders (see figure 5.1), and we denote  $\mathbf{acycl}_{i,+}$  these graphs. We have

$$\mathcal{A} = \bigoplus_{\mathcal{G}_i^\circ \in \mathbf{acycl}_{i,+}} \mathbb{Q} e_{\mathcal{G}_i^\circ}.$$

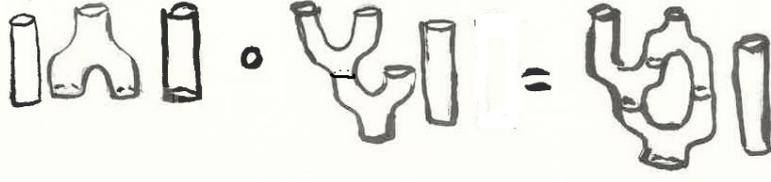


Figure 5.1: Composition of two acyclic stable graphs.

The symmetric group also acts on the labels, and we denote  $S(\mathcal{A})$  as the subspace of symmetric elements. As before, it is the vector space generated by the vectors  $e_{\mathcal{G}^\circ}$  for  $\mathcal{G}^\circ \in \mathbf{acycl}$ . The disjoint union is still well defined, and  $S(\mathcal{A})$  is the free commutative algebra generated by the connected acyclic directed stable graphs.

The composition  $\mathcal{G}_{t_1}^\circ \cdot \mathcal{G}_{t_2}^\circ$  of two directed stable graphs is defined by gluing the negative boundary components of  $\mathcal{G}_{t_1}^\circ$  to the positive components of  $\mathcal{G}_{t_2}^\circ$ . We forget curves in this new graph that are boundaries of cylinders (see figure 5.1). As we see in proposition 3.8, the result is always an acyclic stable graph, then we have an internal law

$$\mathbf{acycl}_{t,+} \times \mathbf{acycl}_{t,+} \longrightarrow \mathbf{acycl}_{t,+}.$$

This operation is associative and defines a structure of algebra on  $\mathcal{A}$  by

$$e_{\mathcal{G}_{t_1}^\circ} \cdot e_{\mathcal{G}_{t_2}^\circ} = \frac{e_{\mathcal{G}_{t_1}^\circ \cdot \mathcal{G}_{t_2}^\circ}}{k!},$$

where  $k = n^-(\mathcal{G}_{t_1}^\circ) = n^+(\mathcal{G}_{t_2}^\circ)$  is the number of boundaries glued. Contrary to the disjoint union, the composition is not commutative; moreover, it's not distributive for  $\sqcup$ , and then  $(S(\mathcal{A}), \cdot, \sqcup)$  is not a ring.

**Projective limit:** We can forget the number of cylinders by using projective limit, as in the case of Fock space. Let  $\varphi_{0,1,1}^*$  be the annihilation operator that removes a cylinder  $e_{0,1,1}$ , defined by analogy to the Fock space. This forms a projective system

$$\varphi_{0,1,1}^* : S_{n^+,n^-+1}(\mathcal{A}) \longrightarrow S_{n^+,n^-}(\mathcal{A}), \quad \forall n^+, n^- \geq 1.$$

We denote  $S_\infty(\mathcal{A})$  the projective limit. As a vector space,  $S_\infty(\mathcal{A})$  is isomorphic to the space  $S(\mathcal{A}^\natural)$  generated by elements in  $\mathbf{acycl}$ . If  $\mathcal{G}^\circ \in \mathbf{acycl}$ , it's represented by vector

$$\mathbf{e}_{\mathcal{G}^\circ} = e_{\mathcal{G}^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1}).$$

The interpretation is the following: we add to a graph an arbitrary number of cylinders, and as we see later, this allows more possibilities of gluing's. As in the case of operators on Fock spaces, we have the following proposition:

**Proposition 5.5.** *For all  $\mathcal{G}_1^\circ, \mathcal{G}_2^\circ \in \mathbf{acycl}$ , there is a unique  $e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}$  in  $S(\mathcal{A}^\natural)$  such that*

$$(e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}) \sqcup \exp_{\sqcup}(e_{0,1,1}) = \mathbf{e}_{\mathcal{G}_1^\circ} \circ \mathbf{e}_{\mathcal{G}_2^\circ}.$$

Then the composition induces a product structure on  $S_\infty(\mathcal{A})$ . The result of the composition of two graphs is given in figure 5.2 to contrast with the usual composition. If we add an arbitrary number of cylinders, we allow all possible gluing.



Figure 5.2: Composition of two graphs in  $S_\infty(\mathcal{A})$

**Exponential structure for the generating series of acyclic graphs:** We give some formulas relating exponentials of elements in  $S_\infty(\mathcal{A})$  to generating series of acyclic stable graphs. For each  $\mathcal{G}^\circ$ , we denote  $n_{\mathcal{G}^\circ}$  as the number of linear orders on  $\mathcal{G}^\circ$ .

**Proposition 5.6.** *In  $\hat{S}_\infty(\mathcal{A})$ , we have the following formula:*

$$\exp\left(\sum_{g,n^+,n^-} e_{g,n^+,n^-} q^{d(M^\circ)}\right) = \sum_{(\mathcal{G}^\circ) \in \mathbf{acycl}} \frac{n_{\mathcal{G}^\circ} q^{d(\mathcal{G}^\circ)}}{\#X_0 \mathcal{G}^\circ! \#Aut(\mathcal{G}^\circ)} e_{\mathcal{G}^\circ}.$$

$e_{g,n^+,n^-}$  corresponds to the surface of type  $(g, n^+, n^-)$ .

The exponential means that we consider all the ways to compose directed surfaces together. It's then quite natural that we obtain, in this way, all the possible acyclic directed stable graphs; the composition of surfaces cannot create cycles in the graph. The proof of these formulas uses the fact that directed graphs are encoded by a family of surfaces with permutations to define the gluings. We don't give the details here. In particular, the exponential of

$$P = e_{0,1,2} + e_{0,2,1},$$

corresponds to acyclic pants decompositions:

$$\exp(qP) = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}^*} \frac{n_{\mathcal{G}^\circ} q^{d(\mathcal{G}^\circ)}}{d(\mathcal{G}^\circ)! \#Aut(\mathcal{G}^\circ)} e_{\mathcal{G}^\circ}. \quad (5.12)$$

Where  $\mathbf{acycl}^*$  is the set of directed stable graphs such that each component is a pair of pants. We illustrate in figure 5.3 the first terms of this formula.

**Representations:** Let  $V$ , in what follow we consider a family of linear operators

$$K_{g,n^+,n^-} \in \text{Hom}(\hat{S}_{n^-}(V), \hat{S}_{n^+}(V)).$$

As  $S(\mathcal{M})$  is the free algebra generated by connected surfaces, we can define a unique morphism

$$K : S(\mathcal{M}) \longrightarrow \text{End}(\hat{S}(V)).$$

If we assume that  $K_{0,1,1} = \text{pr}_1$  is the identity of  $V$ , then we can extend  $K$  to  $S_\infty(\mathcal{M})$ . We can ask if it is possible to obtain a representation of  $S_\infty(\mathcal{A})$ .

Figure 5.3: First terms of the expansion of  $\exp(qP)$  in the variable  $q$ .

**Lemma 5.2.** For each  $K$ , there exists a unique linear map

$$K : S_\infty(\mathcal{A}) \longrightarrow \hat{\mathcal{D}}(V)$$

that extends  $K$  on  $\mathcal{M}$  with

$$K(e_{\mathcal{G}_1^\circ} \sqcup e_{\mathcal{G}_2^\circ}) = K(e_{\mathcal{G}_1^\circ}) \sqcup K(e_{\mathcal{G}_2^\circ}), \quad \text{and} \quad K(e_{\mathcal{G}_1^\circ} \cdot e_{\mathcal{G}_2^\circ}) = K(e_{\mathcal{G}_1^\circ}) \circ K(e_{\mathcal{G}_2^\circ})$$

In particular, according to the precedent results, the space  $\hat{\mathcal{D}}(V)$  corresponds to formal differential operators.

*Proof.* We prove that we can extend the representation to a morphism

$$\mathcal{A} \longrightarrow \text{End}(\hat{\mathcal{S}}(V))$$

To do this, we give the explicit formula for the operators  $K_{\mathcal{G}^\circ}$ . If we fix  $\mathcal{G}^\circ$  and consider the set  $A^{X_1 \mathcal{G}^\circ}$  for each  $c \in X_0 \mathcal{G}^\circ$ , we can find a projection

$$L_c^\pm : A^{X_1 \mathcal{G}^\circ} \longrightarrow A^{n^\pm(c)}$$

And also have projection  $L^\pm : A^{X_1 \mathcal{G}^\circ} \longrightarrow A^{\partial^\pm \mathcal{G}^\circ}$ . For each  $\alpha^\pm \in A^{\partial^\pm \mathcal{G}^\circ}$  let

$$\Lambda_{\mathcal{G}^\circ, \mathcal{A}}(\alpha^+, \alpha^-) = \{\alpha \in A^{X_1 \mathcal{G}^\circ} \mid L^+(\alpha) = \alpha^+, L^-(\alpha) = \alpha^-\}.$$

Then we set

$$K_{\mathcal{G}^\circ}[\alpha^+, \alpha^-] = \frac{1}{n - (\mathcal{G}^\circ)!} \sum_{\alpha \in \Lambda_{\mathcal{G}^\circ, \mathcal{A}}[\alpha^+, \alpha^-]} \prod_c K'_{\mathcal{G}^\circ(c)}[L_c^+(\alpha), L_c^-(\alpha)]$$

with  $K'_{\mathcal{G}^\circ(c)} = n^-! K_{g(c), n^-(c), n^-(c)}$ . According to the fact that the operators  $K_{g, n^+, n^-}$  are symmetric and are defined on  $\hat{\mathcal{S}}(V)$ , which shows that the expression is well defined, we can show that the last sum is indeed finite. More precisely, we can see that for each  $\alpha^+$ , the number of  $\alpha^-$  such that  $K_{\mathcal{G}^\circ}[\alpha^+, \alpha^-]$  is non-zero is actually finite. This is due to the fact that this property is true for each  $K_{g(c), n^-(c), n^-(c)}$  and the graph is acyclic. Then  $K_{\mathcal{G}^\circ}$  is a matrix with finite rows and then defines an operator in  $\text{End}(\hat{\mathcal{T}}(V))$ . By construction, we can see that

$$K_{\mathcal{G}_1^\circ \cdot \mathcal{G}_2^\circ}[\alpha^+, \alpha^-] = \sum_\beta K_{\mathcal{G}_1^\circ}[\alpha^+, \beta] K_{\mathcal{G}_2^\circ}[\beta, \alpha^-].$$

And then we obtain a morphism for the composition. Now if  $\mathcal{G}_i^\circ$  is a graph, a labeled graph with a linear order for each  $i$ , we can add a sufficient number of cylinders to  $\mathcal{G}^\circ(i)$  to obtain a new graph  $\tilde{\mathcal{G}}^\circ(i)$  and find a label  $\tilde{\mathcal{G}}_i^\circ(r)$  of  $\tilde{\mathcal{G}}^\circ(i)$  to obtain the factorization:

$$\mathcal{G}_i^\circ = \tilde{\mathcal{G}}_i^\circ(r) \cdot \dots \cdot \tilde{\mathcal{G}}_i^\circ(1),$$

with  $r = \#X_0\mathcal{G}^\circ$ . Then the algebra  $\mathcal{A}$  is generated by  $\mathcal{M}$ , and this allows to obtain the uniqueness.  $\square$

## 5.2 Operators associated to directed ribbon graphs

In this part, we construct a family of integral operators  $K_{g,n^+,n^-}$  from the volumes  $V_{g,n^+,n^-}$  and study their elementary properties. If we set  $V = \mathbb{Q}[L]$  the space of polynomials, these operators are defined on the formal Fock space  $\hat{S}(V)$ , which is the space of series in arbitrary many variables. We start by giving a lemma that proves the consistency of the action.

### 5.2.1 Transfert lemma

The functions  $V_{g,n^+,n^-}$  introduced in chapter 4.5.2 are only piecewise polynomials. It's possible to know the walls and express them using the elements

$$\| |L_{I^+}| - |L_{I^-}| \|,$$

for  $I_\pm \subset \llbracket 1, n^\pm \rrbracket$ , but we do not know any explicit formula. Nevertheless, things get more explicit when we consider them as integral operators, and we follow this path in next sections. The starting point is the following proposition, which is central to what follows.

**Proposition 5.7.** *Let  $(g, n^+, n^-, \nu)$  and  $\alpha \in \mathbb{N}^{n^-}$  be a multi-index, then the formula*

$$\int_{L^- \in |L^+| \cdot \Delta_{n^-}} V_{g,n^+,n^-}^\nu(L^+|L^-) \prod_i (L_i^-)^{\alpha_i^-} d\sigma_{n^-}^{|L^+|},$$

is well defined for  $L^+ \in (\mathbb{R}_{\geq 0})^{n^+}$  and is a homogeneous symmetric polynomial of degree  $2g - 2 + n^- + n(\nu) + d(\alpha)$ .

The proof of this proposition is a consequence of the following lemma, which can be called the transfer lemma. Let  $R^\circ$  be an oriented ribbon graph and consider the quantity

$$\bar{P}_{R^\circ}(\alpha^+|\alpha^-) = \# \{x \in \mathbb{N}^{X_1 R} | L_i^\pm(x) = \alpha_i^\pm, \quad \forall i \in \llbracket 1, n^\pm \rrbracket\}.$$

It's the number of degenerate integer points.

**Lemma 5.3.** *Let  $R^\circ$  be an oriented ribbon graph and  $\alpha^- \in \mathbb{N}^{n^-}$  a multi-index, the integral*

$$\int_{L^- \in |L^+| \cdot \Delta_{n^+}} V_{R^\circ}(L^+|L^-) \prod_i (L_i^-)^{\alpha_i^-} d\sigma_{n^-}^{|L^+|},$$

is a homogeneous polynomial in  $L^+$  given by

$$\int_{L^- \in |L^+| \cdot \Delta_{n^+}} V_{R^\circ}(L^+|L^-) \prod_i (L_i^-)^{\alpha_i^-} d\sigma_{n^-}^{|L^+|} = \sum_{\alpha^+} P_{R^\circ}(\alpha^+ - \alpha^+(R^\circ) + 1|\alpha^-) \frac{\alpha^-!}{\alpha^+!} \prod_i (L_i^+)^{\alpha_i^+}.$$

*Proof.* The space  $\text{Met}(R^\circ, L^+, L^-)$  is the subspace defined by relations  $L_i^\pm(x) = L_i^\pm$  for all  $i$ . Let  $\text{Met}_+(R^\circ, L^+)$  be the subspace defined by only  $L_i^+(x) = L_i^+$ . As before, we have a measure  $d\mu_{R^\circ, +}(L^+)$  on  $\text{Met}_+(R^\circ, L^+)$  that satisfies

$$\int_{\text{Met}_+(R^\circ, L^+)} (L^-)^{\alpha^-} d\mu_{R^\circ, +}(L^+) = \int_{L^- \in |L^+| \cdot \Delta_{n^-}} V_{R^\circ}(L^+|L^-) (L^-)^{\alpha^-} d\sigma_{n^-}^{|L^+|} \quad \forall L^+ \in \mathbb{R}_{\geq 0}^{n^+}.$$

Each edge  $e$  belongs to a unique positive boundary component; this forms a partition of the set of edges. Then we need to compute the integral of  $\prod_i L_i^-(x)^{\alpha_i^+}$  over the set

$$\{x | L_i^+(x) = L_i^+\} = \prod_i \left\{ (x_e)_{e, [e]^+ = i} \in \mathbb{R}_+^{\alpha_i^+(R)}, \sum_{e, [e]^+ = i} x_e = L_i^+ \right\},$$

which is a product of simplices. The decomposition preserves the lattice of integer points, and then the measure on the RHS is the product of the Lebesgue measures on each factor. Moreover, by Newton binomial formula, we have

$$\prod_j L_j^-(x)^{\alpha_j^-} = \prod_j \alpha_j^-! \sum_{m \in \mathbb{N}^{X_1 R}, L_j^-(m) = \alpha_j^-} \prod_e \frac{x_e^{m_e}}{m_e!},$$

then when we integrate over the product of simplices, it leads to

$$\int_{\{x | L_i^+(x) = L_i^+\}} \prod_e \frac{x_e^{m_e}}{m_e!} dx = \prod_i \int_{(x_e)_{e, [e]^+ = i}, \sum_{e, [e]^+ = i} x_e = L_i^+} \prod_{e, [e]^+ = i} \frac{x_e^{m_e}}{m_e!} d\sigma_i = \prod_i \frac{(L_i^+)^{\alpha_i^+ + \alpha_i^+(R^\circ) - 1}}{(\alpha_i^+ + \alpha_i^+(R^\circ) - 1)!},$$

because

$$\int_{(x_e)_{e, [e]^+ = i}, \sum_{e, [e]^+ = i} x_e = L_i^+} \prod_{e, [e]^+ = i} \frac{x_e^{m_e}}{m_e!} d\sigma_i = \frac{(L_i^+)^{\sum_e (m_e + 1) - 1}}{(\sum_e (m_e + 1) - 1)!}.$$

Then we finally obtain the formula,

$$\int_{L^- \in |L^+| \cdot \Delta_{n^-}} V_{R^\circ}(L^+|L^-) \prod_i (L_i^-)^{\alpha^-} d\sigma_{n^-}^{|L^+|} = \sum_{\alpha^+} \bar{P}_{R^\circ}(\alpha^+ | \alpha^-) \frac{\alpha^-! \prod_i (L_i^+)^{\alpha_i^+ + \alpha_i^+(R^\circ) - 1}}{(\alpha^+ + \alpha^+(R^\circ) - 1)!}.$$

□

*Proof.* To prove proposition 5.7, we just need to sum over all possible ribbon graphs with a profile given by  $(g, n^+, n^-, \nu)$ :

$$\begin{aligned} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} V_{g, n^+, n^-}^\nu(L^+|L^-) \prod_i (L_i^-)^{\alpha_i} d\sigma_{n^-}^{|L^+|} &= \sum_{R^\circ \in \text{rib}_{g, n^+, n^-}^\nu} \frac{1}{\#\text{Aut}(R^\circ)} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} V_{R^\circ}(L^+|L^-) \prod_i (L_i^-)^{\alpha_i} d\sigma_{n^-}^{|L^+|} \\ &= \sum_{\alpha^+ \geq \alpha^+(R^\circ)} \prod_i (L_i^+)^{\alpha_i^+} \sum_{R^\circ \in \text{rib}_{g, n^+, n^-}^\nu} \frac{\bar{P}_{R^\circ}(\alpha^+ - \alpha^+(R^\circ) + 1 | \alpha^-) \alpha^-!}{\#\text{Aut}(R^\circ)} \frac{\alpha^-!}{\alpha^-!} \end{aligned}$$

We can see that if  $\bar{P}_{R^\circ}(\alpha^+ - \alpha^+(R^\circ) + 1 | \alpha^-)$  is non-zero, we must have

$$d(\alpha^+) - d(\alpha^+(R^\circ)) + n^+(R^\circ) = d(\alpha^-).$$

And then

$$d(\alpha^+) - d(\alpha^-) = d(\alpha^+(R^\circ)) - n^+(R^\circ) = \#X_1 R - n^+,$$

by using lemma 4.7, we have  $\#X_1 R = d(\nu) + n(\nu) = 2g - 2 + n^+ + n^-$

$$d(\alpha^+) = d(\alpha^-) + 2g - 2 + n^- + n(\nu).$$

Then the polynomial is of degree  $d(\alpha^-) + 2g - 2 + n^- + n(\nu)$ .

□

## 5.2.2 Operators associated with directed surfaces

**Definition of the operators:** Let  $(g, n^+, n^-)$  be such that  $2g - 2 + n^+ + n^- > 0$ . The volume  $V_{g,n^+,n^-}$  is a function of two sets of variables, and then it can defines integration kernel. It's slightly better to use the piece-wise polynomials defined by

$$K_{g,n^+,n^-}(L^+|L^-) = \prod_i L_i^+ V_{g,n^+,n^-}(L^+|L^-).$$

For each  $P$  homogeneous polynomial, we consider the integral

$$K_{g,n^+,n^-} \cdot P(L^+) = \frac{1}{n^-!} \int_{L^- \in |L^+| \cdot \Delta_{n^-}} K_{g,n^+,n^-}(L^+|L^-) P(L^-) d\sigma_{n^-}^{|L^+|}. \quad (5.13)$$

The factor  $\frac{1}{n^-!}$  is to avoid over-counting when composing the operators. According to proposition 5.7, the integral is well defined, and  $K_{g,n^+,n^-} \cdot P$  is a homogeneous symmetric polynomial of degree  $2d_{g,n^+,n^-} + d(P)$ .

Using the formalism of section 5.1, let  $V = \mathbb{Q}[L]$  be the space of polynomials, then  $T_n(V) = \mathbb{Q}[L_1, \dots, L_n]$  is the space of polynomials in  $n$  variables. In this space, the natural grading  $d$  is given by the degree. The completion  $\hat{T}_n(V) = \mathbb{Q}[[L_1, \dots, L_n]]$  is the space of formal series in  $n$  variables. The symmetric space  $S(V)$  is then the space of symmetric polynomials, and  $\hat{S}(V)$  is its completion. Then, with proposition 5.7, we obtain the following corollary:

**Corollary 5.1.** *Functions  $K_{g,n^+,n^-}$  define linear operators*

$$K_{g,n^+,n^-} : S_{n^-}(V) \longrightarrow S_{n^+}(V).$$

*which are homogeneous of degree  $2d_{g,n^+,n^-}$ .*

The operators  $K_{g,n^+,n^-}$  naturally extend to  $S(V)$  (by assuming that they act by zeros on  $S_k(V)$  with  $k \neq n^-$ ). Moreover, as we see in section 5.1, a homogeneous operator always extends to the formal formal Fock space, then the  $K_{g,n^+,n^-}$  induces endomorphism's

$$K_{g,n^+,n^-} : \hat{S}(V) \longrightarrow \hat{S}(V).$$

**Remark 5.10** (Twists). *We change  $V_{g,n^+,n^-}$  to  $K_{g,n^+,n^-}$  for two reasons: first, the degree is twice the Euler characteristic in the second case, which is more natural and additive under composition. Moreover, the composition of two operators is given by the formula*

$$\begin{aligned} K_{g_1,n^+,k} \circ K_{g_2,k,n^-} &= \int_{L \in |L^+| \cdot \Delta_k} \frac{K_{g_1,n^+,k}(L^+|L) K_{g_2,k,n^-}(L|L^-)}{k!} d\sigma_k^{|L^+|} \\ &= \prod_i L_i^+ \int_{x \in |L^+| \cdot \Delta_k} \frac{V_{g_1,n^+,k}(L^+|x) V_{g_2,k,n^-}(x|L^-)}{k!} \prod_j L_j d\sigma_k^{|L^+|}. \end{aligned}$$

*Then we see that the "twist's" naturally appear in this formula; they correspond to the factor  $\prod_j L_j$ .*

**Remark 5.11** (Times inversion, duality, and scalar product). *Functions  $V_{g,n^+,n^-}$  have an additional symmetry that we can call times inversion. We have*

$$V_{g,n^+,n^-}(L^+|L^-) = V_{g,n^-,n^+}(L^-|L^+).$$

This is interesting for the following reason: we can consider, for instance, the scalar product on  $S_n(V)$  given by

$$(f, g) = \frac{1}{n!} \int_{\mathbb{R}_{\geq 0}^{n^+}} f(x)g(x)e^{-|x|} dx.$$

We then have the formula

$$(f, V_{g, n^+, n^-} \cdot g) = \int_{\Lambda_{n^+, n^-}} f(L^+) V_{g, n^+, n^-}(L^+ | L^-) g(L^-) e^{-|L^+|} \frac{d\sigma_{n^+, n^-}}{n^+! n^-!}.$$

On  $\Lambda_{n^+, n^-}$ , we also have the relation  $|L^+| = |L^-|$  by definition, and this implies that

$$\begin{aligned} \int_{\Lambda_{n^+, n^-}} f(L^+) V_{g, n^+, n^-}(L^+ | L^-) g(L^-) e^{-|L^+|} \frac{d\sigma_{n^+, n^-}}{n^+! n^-!} &= \int_{\Lambda_{n^+, n^-}} f(L^+) V_{g, n^-, n^+}(L^- | L^+) g(L^-) e^{-|L^+|} \frac{d\sigma_{n^+, n^-}}{n^+! n^-!} \\ &= \int_{\Lambda_{n^+, n^-}} g(L^-) V_{g, n^-, n^+}(L^- | L^+) f(L^+) e^{-|L^+|} \frac{d\sigma_{n^+, n^-}}{n^+! n^-!} \end{aligned}$$

And then we obtain the following nice formula:

$$(f, V_{g, n^+, n^-} \cdot g) = (V_{g, n^-, n^+} \cdot f, g).$$

In particular, the operators  $V_{g, n, n}$  are self-adjoint operators, and it might be possible to carry out their analysis. But we do not go deeper in this direction in this text. The time inversion then defines a structure of involutive algebra.

**Disconnected surfaces and disjoint union:** It's natural to consider the volumes associated to disconnected surfaces. For  $M_1^\circ$  an oriented surface of type  $(g, n^+, n^-)$  with labeled boundaries, we can use the notation  $K_{M_1^\circ}$  instead of  $K_{g, n^+, n^-}$ . If  $M^\circ = \sqcup_{c \in \pi_0(M)} M^\circ(c)$  is a disconnected surface, the volume  $K_{M_1^\circ}$  satisfies the formula

$$K_{M_1^\circ}(L^+ | L^-) = \prod_c K_{M^\circ(c)}(L_{I^+(c)}^+ | L_{I^-(c)}^-).$$

Where  $I^\pm(c)$  are the two partitions of the positive and negative boundaries, defined by  $\partial^\pm M_1^\circ(c) = I^\pm(c)$ . The function  $K_{M_1^\circ}$  associated with an union of  $k$  connected surfaces is defined on the codimension  $k$  affine subspace  $\Lambda_{M_1^\circ} = \prod \Lambda_{M^\circ(c)}$  of  $\mathbb{R}^{n^+ + n^-}$ . To define operators, we consider the set

$$\Delta_{M_1^\circ}(L^+) = \prod_c |L^+(c)| \cdot \Delta_{I^-(c)} = \{L^- \in \mathbb{R}_{\geq 0}^{n^-} \mid |L^+(c)| = |L^-(c)| \ \forall c \in \pi_0(M^\circ)\}$$

with the measure

$$d\sigma_{M_1^\circ}^{L^+} = \prod_c d\sigma_{I^-(c)}^{|L^+(c)|}.$$

And we define the operator  $K_{M_1^\circ}$

$$K_{M_1^\circ} \cdot P = \frac{1}{n^-!} \int_{L^- \in \Delta_{M_1^\circ}(L^+)} K_{M_1^\circ}(L^+ | L^-) P(L^-) d\sigma_{M_1^\circ}^{L^+}.$$

By Fubini theorem if we can factorize  $P(L^-) = \prod_c P_c(L_{I^-(c)}^-)$ , we can write

$$\begin{aligned} \int_{\Delta_{M_1^\circ}(L^+)} K_{M_1^\circ}(L^+|L^-) P d\sigma_{M_1^\circ}^{L^+} &= \prod_c \int_{L^- \in |L^+(c)| \cdot \Delta_{n^-(c)}} K_{M_1^\circ(c)}(L_{I^+(c)}^+|L^-) P_c(L^-) d\sigma_{n^-(c)}^{|L^-(c)|} \\ &= \prod_c (K_{M_1^\circ(c)} \cdot P_c)(L_{I^+(c)}). \end{aligned}$$

By these formulas,  $K_{M_1^\circ}$  defines a linear operator

$$K_{M_1^\circ} : T(V) \longrightarrow T(V).$$

Which is homogeneous of degree  $2d(M^\circ)$ . However, when  $M_1^\circ$  is not connected, then  $K_{M_1^\circ}$  is not symmetric; we only have  $K_{M_1^\circ} \in \text{End}(\hat{T}(V))$ . By using linearity, we define an operator,

$$\mathbf{K} : \mathcal{M}^s \longrightarrow \text{End}(\hat{T}(V)),$$

by  $\mathbf{K}(e_{M_1^\circ}) = K_{M_1^\circ}$ . In part 5.1.4, we consider the action of the symmetric group by permuting the label and define the set of symmetric elements  $S(\mathcal{M}^s)$ . It admits a basis indexed by vectors  $e_{M^\circ}$ , for  $M^\circ \in \mathbf{bord}^{\circ,s}$  the set of unlabeled directed stable surfaces we set

$$K_{M^\circ} = \mathbf{K}(e_{M^\circ})$$

$K_{M^\circ}$  is then, in some sense, the symmetrisation of the operator  $K_{M_1^\circ}$  for any choice of label and corresponds to the sum of all the possible ways to label the boundaries of  $M^\circ$  up to homeomorphism's. This time,  $K_{M^\circ}$  is a symmetric operator and extends to a linear operator on the formal Fock space,

$$K_{M^\circ} : \hat{S}(V) \longrightarrow \hat{S}(V).$$

**Remark 5.12** (Connected surfaces). *If  $M^\circ$  is connected of type  $(g, n^+, n^-)$ , there is only one possible way of labeling the boundaries, and then  $K_{g, n^+, n^-}$  and  $K_{M^\circ}$  coincide in this case.*

In section 5.1.4, we define the disjoint union  $\sqcup$  on  $S(\mathcal{M}^s)$  (and  $\hat{S}(\mathcal{M}^s)$ ), and we have the following natural proposition:

**Proposition 5.8.** *The operators satisfy the following rule:*

$$K_{M_1^\circ \sqcup M_2^\circ} = K_{M_1^\circ} \sqcup K_{M_2^\circ}$$

for each  $M_1^\circ, M_2^\circ$ .

*Proof.* To prove proposition 5.8, we can start with the following equality that follows from the definition:

$$K_{M_1^\circ \sqcup M_2^\circ}(L^+|L^-) = \sum_{I_i^\pm} K_{M_1^\circ}(L_{I_1^+}^+|L_{I_1^-}^-) K_{M_2^\circ}(L_{I_2^+}^+|L_{I_2^-}^-).$$

where the sum is over all pairs of partitions  $I_1^\pm \sqcup I_2^\pm = \{1, \dots, n_1^\pm + n_2^\pm\}$  in two sets of respective cardinals  $n_1^\pm, n_2^\pm$ . To prove the proposition, we can apply it to a vector  $e_\mu \in S(V)$  and use the formula for the union of operators.  $\square$

From this, we can summarize the results of this section by the following corollary:

**Corollary 5.2.** *The operator*

$$\mathbf{K} : \hat{S}(\mathcal{M}^s) \longrightarrow \text{End}(\hat{S}(V)),$$

*defines a morphism of commutative algebras for the disjoint union.*

*Proof.* To prove this proposition, we only need to prove that the operator extends to  $\hat{S}(\mathcal{M}^s)$ . This is the consequence of the fact that  $\mathbf{K}_{M^\circ}$  is of degree  $2d(M^\circ)$  and the fact that  $S^d(\mathcal{M}^s)$  is finite dimensional; there are only a finite number of stable surfaces with a fixed Euler characteristic. Then, from the results of section 5.1 for  $x = (x_d) \in \hat{S}(\mathcal{M}^s)$ , the series  $\sum_d \mathbf{K}(x_d)$  converges to an operator in  $\text{End}(\hat{S}(V))$ .  $\square$

**Unstable surfaces:** The case of the cylinder (surface of type  $(0, 1, 1)$ ) is quite fundamental. However, the volume  $V_{0,1,1}$  of "quadrivalent" ribbon graphs on the cylinder is not defined, there is no ribbon graph. But there is one ribbon graph on the cylinder associated with the partition  $(0^1)$ , with one bivalent vertex. The volume is given by

$$V_{0,1,1}^{(0)}(L^+|L^-) = \delta(L^+ - L^-)$$

And then, according to proposition 4.25, we shall have

$$V_{0,1,1} = \frac{1}{L_+}$$

which is coherent with the fact that the cylinder has a continuous group of automorphisms. Therefore, the correct definition for the function  $K_{0,1,1}$  is

$$K_{0,1,1} = 1$$

on  $\Lambda_{1,1}$ . The operator  $K_{0,1,1}$  corresponds to the identity on  $V$  and corresponds to the projection.

$$pr_1 : \hat{S}(V) \longrightarrow \hat{V}.$$

**Lemma 5.4.** *We have the following relation in  $\text{End}(\hat{S}(V))$*

$$\exp_{\square}(K_{0,1,1}) = id.$$

**Remark 5.13.** *This identity is indeed quite natural; in practice, gluing cylinders on the boundaries does not change anything.*

*Proof.* It's straight forward by using the results of lemma 5.1. As we prove that

$$\exp(pr_1) = id.$$

$\square$

As in corollary 5.2, we have the following statement:

**Corollary 5.3.** *The operator  $\mathbf{K}$  is well defined on  $\hat{S}(\mathcal{M})$  and defines a morphism of commutative algebras*

$$\mathbf{K} : \hat{S}(\mathcal{M}) \longrightarrow \text{End}(\hat{S}(V)).$$

*Proof.* In this case, we cannot apply the argument on the degree in a straightforward way. Because  $K_{0,1,1}$  is of degree 0 and then the spaces  $S^d(\mathcal{M})$  are infinite dimensional, the expression  $\mathbf{K}(x)$  can contain an infinite number of terms of degree zeros. Nevertheless, adding cylinders increases the number of boundary components, and this ensures convergence for the adic topology. Then the linear map  $\mathbf{K}$  is well defined on the completion  $\hat{S}(\mathcal{M})$ .  $\square$

**Ancestor potential:** By using the results of the last section, we can consider several operators. The first one,  $K^{s,c}$ , is associated with connected and stable surfaces. It's given by the sum

$$K^{s,c} = \sum'_{g,n^+,n^-} K_{g,n^+,n^-} = \sum_{M^\circ \in \mathbf{bord}^{o,s,c}} K_{M^\circ}.$$

Where in the first sum, ' means that we impose  $2g - 2 + n^+ + n^- > 0$ . This operator is well defined by using the discussion in the last section and is in  $\text{End}(\hat{S}(V))$ . To this operator, we can associate  $K^s$ , which is the formal sum over stable and non-necessarily connected surfaces.

$$K^s = \sum_{M^\circ \in \mathbf{bord}^{o,s}} K_{M^\circ}.$$

These two operators are naturally related in the following lemma. It's no more than the fact that the generating series of disconnected objects is the exponential of connected ones for  $\sqcup$ .

**Lemma 5.5.** *The operators  $K^s$  and  $K^{c,s}$  are related by the formula*

$$K^s = \exp_{\sqcup}(K^{s,c}).$$

*Proof.* By using proposition 5.8, we have

$$K_{M_1^\circ \sqcup M_2^\circ} = K_{M_1^\circ} \sqcup K_{M_2^\circ}.$$

Moreover, by formula 5.11, we have

$$\sum_{M^\circ \in \mathbf{bord}^{o,s}} e_{M^\circ} = \exp_{\sqcup} \left( \sum_{M^\circ \in \mathbf{bord}^{o,s,c}} e_{M^\circ} \right).$$

Then we have

$$K^s = \mathbf{K} \left( \exp_{\sqcup} \left( \sum_{M^\circ \in \mathbf{bord}^{o,s,c}} e_{M^\circ} \right) \right) = \exp_{\sqcup} \left( \sum_{M^\circ \in \mathbf{bord}^{o,s,c}} \mathbf{K}(e_{M^\circ}) \right) = \exp_{\sqcup}(K^{s,c}).$$

□

We can also consider the unstable operators defined by

$$K^c = \sum_{M^\circ \in \mathbf{bord}^{o,c}} K_{M^\circ} \quad K = \sum_{M^\circ \in \mathbf{bord}^o} K_{M^\circ}$$

We still have the relation,

$$K = \exp_{\sqcup}(K^c).$$

Moreover, we can also write

$$K^c = K_{0,1,1} + K^{c,s}$$

Then, lemma 5.4, when we take the exponential, we obtain the following important relation

$$K = id \sqcup K^s. \tag{5.14}$$

As we see in section 5.1, the formal Fock space can be identified with the space of formal serie  $\mathbb{Q}[[\mathbf{t}]] = \mathbb{Q}[[t_0, t_1, \dots]]$ . Then the operator  $K$  defines a linear operator on this space. Using the last result and also results of proposition 5.2, we obtain the following corollary:

**Corollary 5.4.** *The operator  $K$  is a differential operator on  $\mathbb{Q}[[t]]$ , it admits an expression*

$$K = \sum_{\mu^+, \mu^-} K^s[\mu^+ | \mu^-] \frac{t^{\mu^+}}{\mu^+!} \partial_{\mu^-} \quad (5.15)$$

Where  $(K^s[\mu^+ | \mu^-])$  are the matrix coefficients of  $K^s$ .

In a similar way, we can consider for each surface  $M^\circ \in \mathbf{bord}^{0,5}$

$$K_{M^\circ}^u = \mathbf{K}(e_{M^\circ}) = \mathbf{K}(e_{M^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1})) = K_{M^\circ} \sqcup id$$

As we see in paragraph 5.1.4, the element  $e_{M^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1})$  corresponds to  $M^\circ$  with an arbitrary number of cylinders. By using the formalism of 5.1, we know that all the operators  $K_{M^\circ}^u$  are also differential operators. Moreover, if  $: AB :$  is the "time-ordered" product of two operators  $A, B$ , we have the following:

**Corollary 5.5.** *The differential operators satisfy*

$$K_{M_1^\circ \sqcup M_2^\circ}^u =: K_{M_1^\circ}^u K_{M_2^\circ}^u :$$

And the series  $K^{c,u} = K^{c,s} \sqcup id$  and  $K$  are related by  $K =: \exp(K^{c,u}) :$  .

### 5.2.3 Product of operators

In this part, we extend the result of the last subsection to acyclic stable graphs.

**Operators associated with a directed acyclic graph:** The composition of kernels corresponds to gluing of surfaces. But the map

$$\mathbf{K} : S(\mathcal{M}) \longrightarrow \text{End}(S(V))$$

is not a morphism for the composition. To obtain such properties, as we see in part 5.1.4, we need to introduce the operator  $K_{\mathcal{G}_i^\circ}$  for  $\mathcal{G}_i^\circ \in \mathbf{acycl}_{i,+}$ , a labeled acyclic stable graph. To do that, we use the cone  $\Lambda_{\mathcal{G}_i^\circ}$  of directed cycles in  $\mathcal{G}_i^\circ$ ; see 3.5.2. For a fixed  $L = (L^+, L^-) \in \Lambda_{\partial \mathcal{G}_i^\circ}$ , we consider the convex set  $\Lambda_{\mathcal{G}_i^\circ}(L)$ , it as a "Lebesgue" measure  $d\sigma_{\mathcal{G}_i^\circ}(L)$ . Then we can define the kernels

$$K_{\mathcal{G}_i^\circ}(L^+ | L^-) = \int_{\Lambda_{\mathcal{G}_i^\circ}(L)} \prod_c K_{\mathcal{G}_i^\circ(c)}(L^+(c) | L^-(c)) d\sigma_{\mathcal{G}_i^\circ}(L).$$

Similarly to the case of  $K_{M_i^\circ}$ , we can consider operator  $K_{\mathcal{G}_i^\circ}$  on continuous functions by using  $K_{\mathcal{G}_i^\circ}(L^+ | L^-)$ . We also denote  $K_{\mathcal{G}^\circ}$  the symmetrisation for  $\mathcal{G}^\circ \in \mathbf{acycl}$ , as in the precedent case we have the following:

**Lemma 5.6.** *The operator  $K_{\mathcal{G}^\circ}$  preserves the space of polynomials and then induces a homogeneous linear operator of degree  $2d(\mathcal{G}^\circ)$ .*

$$K_{\mathcal{G}^\circ} : \hat{S}(V) \longrightarrow \hat{S}(V).$$

We can extend  $\mathbf{K}$  to the free algebra  $\hat{S}(\mathcal{A})$  and define a linear map:

$$\mathbf{K} : \hat{S}(\mathcal{A}) \longrightarrow \text{End}(\hat{S}(V)).$$

The space  $S(\mathcal{A})$  has a second algebraic structure  $\cdot$  given by the composition of two acyclic graphs. The following proposition shows that  $K_{\mathcal{G}^\circ}$  defines the representation of  $\hat{S}(\mathcal{A})$ .

**Proposition 5.9.** *The morphism  $\mathbf{K}$  defines a representation of the algebra  $\hat{S}(\mathcal{A})$ , and for each  $\mathcal{G}_i^\circ$  we have*

$$\mathbf{K}(e_{\mathcal{G}_1^\circ} \cdot e_{\mathcal{G}_2^\circ}) = \mathbf{K}(e_{\mathcal{G}_1^\circ}) \circ \mathbf{K}(e_{\mathcal{G}_2^\circ}), \quad \text{and} \quad \mathbf{K}(e_{\mathcal{G}_1^\circ} \sqcup e_{\mathcal{G}_2^\circ}) = \mathbf{K}(e_{\mathcal{G}_1^\circ}) \sqcup \mathbf{K}(e_{\mathcal{G}_2^\circ}).$$

**Projective limit and acyclic stable graphs:** For two stable surfaces  $M_1^\circ, M_2^\circ$  the composition

$$M_1^\circ \cdot M_2^\circ,$$

corresponds to the stable graphs obtained by gluing all negative boundary components of  $M_1^\circ$  to all positive boundary components of  $M_2^\circ$ . In particular, we need the same number of boundaries. These gluings are too restrictive; we don't want to glue  $M_1^\circ$  along all its negative boundaries, but consider all the ways to glue it to  $M_2^\circ$  (see figure 5.1). A way to do that is to add an arbitrary number of cylinders to the two surfaces and remove the cylinders that are glued to other cylinders. This is what we do in paragraph 5.1.4, by defining

$$e_{\mathcal{G}^\circ} = e_{\mathcal{G}^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1}).$$

Then, as we see before, the product  $e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}$  of two symmetric acyclic directed stable graphs  $e_{\mathcal{G}_1^\circ}, e_{\mathcal{G}_2^\circ}$  is defined by the relation

$$e_{\mathcal{G}_1^\circ} \cdot e_{\mathcal{G}_2^\circ} = (e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}) \sqcup \exp_{\sqcup}(e_{0,1,1}).$$

In a similar way, in 5.1.3 we also define the  $*$  product of two operators acting in  $\hat{S}(V)$ . The following proposition relies on these two notions.

**Proposition 5.10.** *For  $e_{\mathcal{G}_1^\circ}, e_{\mathcal{G}_2^\circ}$ , we have the relation*

$$\mathbf{K}(e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}) = \mathbf{K}(e_{\mathcal{G}_1^\circ}) * \mathbf{K}(e_{\mathcal{G}_2^\circ})$$

*and then the restriction of  $\mathbf{K}$  to  $\hat{S}(\mathcal{A}^s)$  defines a morphism compatible with  $*$ .*

Then, in other words, the operator  $\mathbf{K}$  defines a representation

$$\mathbf{K} : \hat{S}_\infty(\mathcal{A}) \longrightarrow \hat{\mathcal{D}}(V).$$

**Remark 5.14.** *The interesting point of this proposition is the fact that the LHS gives a graphical expression of powers of differential operators in the RHS.*

*Proof.* This result is a consequence of paragraph 5.1.4 we have

$$(e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}) \sqcup \exp_{\sqcup}(e_{0,1,1}) = (e_{\mathcal{G}_1^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1})) \circ (e_{\mathcal{G}_2^\circ} \sqcup \exp_{\sqcup}(e_{0,1,1})).$$

As  $\mathbf{K}(\exp_{\sqcup}(e_{0,1,1})) = id$ , and using the fact that  $\mathbf{K}$  is a morphism for  $\circ$  and  $\sqcup$ , we have

$$\mathbf{K}(e_{\mathcal{G}_1^\circ} * e_{\mathcal{G}_2^\circ}) \sqcup id = (\mathbf{K}(e_{\mathcal{G}_1^\circ}) \sqcup id) \circ (\mathbf{K}(e_{\mathcal{G}_2^\circ}) \sqcup id) = (\mathbf{K}(e_{\mathcal{G}_1^\circ}) * \mathbf{K}(e_{\mathcal{G}_2^\circ})) \sqcup id.$$

□

**Remark 5.15.** In the simplest case, when we consider two connected surfaces  $M_1^\circ, M_2^\circ$ , we can define the  $*_k$  product on the kernels by

$$(K_{M_1^\circ} *_k K_{M_2^\circ})(L^+|L^-) = \frac{1}{k!} \sum_{I_i^\pm} \int_{L \in (|L_{I_1^+}^+| - |L_{I_1^-}^-|) \cdot \Delta_k} K_{M_1^\circ}(L_{I_1^+}^+|L, L_{I_1^-}^-) K_{M_2^\circ}(L, L_{I_2^+}^+|L_{I_2^-}^-) d\sigma_k^{|L_{I_1^+}^+| - |L_{I_1^-}^-|}.$$

Where we integrate over  $k$  variables and sum over all ways to partition positive and negative boundary components into two sets. The domain of integration is a simplex. Then we have the relation for the kernels

$$\mathbf{K}(e_{M_1^\circ} * e_{M_2^\circ}) = \sum_k K_{M_1^\circ} *_k K_{M_2^\circ}.$$

In particular, we have

$$\mathbf{K}(e_{M_1^\circ} * e_{M_2^\circ}) = K_{M_1^\circ} \sqcup K_{M_2^\circ} + \cdots + K_{M_1^\circ} \circ K_{M_2^\circ}$$

And then  $*$  contains both the union and the composition. These considerations are valid for more general acyclic-directed stable graphs, but we need to be a bit more careful when taking integrals.

## 5.2.4 Surface with marked points

We can generalize all of this to surfaces with marked points by using the kernel

$$K_{g,n^+,n^-,m}(L^+|L^-) = \prod_i L_i^+ V_{g,n^+,n^-,m}.$$

$K_{g,n^+,n^-}$  represents graphs with bivalent vertices at the marked points. We can generalize this definition for all  $M_i^\circ \in \mathbf{bord}_i^{\circ,\bullet}$  and define operators  $K_{M_i^\circ}$

$$K_{M_i^\circ} : \hat{T}(V) \longrightarrow \hat{T}(V).$$

Which are homogeneous of degree  $2d_{g,n^+,n^-,m}$ . If we consider  $\mathcal{M}^\bullet$  the vector space generated by  $\mathbf{bord}_i^{\circ,\bullet}$ , we can similarly define the algebra  $\mathcal{A}^\bullet$  generated by acyclic stable graphs with marked points. Then, similarly to the precedent case, we can extend  $\mathbf{K}$  to obtain a representation

$$\mathbf{K} : \hat{S}(\mathcal{A}^\bullet) \longrightarrow \text{End}(\hat{S}(V)).$$

And we can define

$$K^\bullet = \sum_{M^\circ \in \mathbf{bord}^{\circ,\bullet}} K_{M^\circ} \quad \text{and} \quad K^\bullet(q_0, q_1) = \exp_{\sqcup} \left( \sum_{g,n^+,n^-,m} q_0^m q_1^{2g-2+n^++n^-} K_{g,n^+,n^-,m} \right).$$

We have the specialization

$$K^\bullet(1, 1) = K^\bullet, \quad \text{and} \quad K^\bullet(0, q) = K(q).$$

## 5.3 Cut-and-Join equations

In this part, we use Theorem 4.6 to obtain Theorem 5.1 stated in the introduction. We also study the case of marked surfaces (ribbon graphs with possible bivalent vertices).

### 5.3.1 Recursion for the volumes

From Theorem 4.6, we obtain the following recursion for the kernels  $K_{g,n^+,n^-}(L^+|L^-)$ . In the next formula, we denote  $[x]_+ = \max\{x, 0\}$ .

**Corollary 5.6.** *Assume that  $2g - 2 + n^+ + n^- > 1$ , then for all values of boundary lengths,  $K_{g,n^+,n^-}$  satisfies the recursion:*

$$\begin{aligned} (2g - 2 + n^+ + n^-)K_{g,n^+,n^-}(L^+|L^-) &= \frac{1}{2} \sum_{i \neq j} L_i^+ L_j^+ K_{g,n^+-1,n^-}(L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-) \\ &\quad + \sum_i \sum_j L_i^+ K_{g,n^+,n^- - 1}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) \\ &\quad + \frac{1}{2} \sum_i L_i^+ \int_0^{L_i^+} K_{g-1,n^++1,n^-}(x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) dx \\ &\quad + \frac{1}{2} \sum_i \sum_{\substack{g_1+g_2=g \\ I_1^\pm \sqcup I_2^\pm = I^\pm}} L_i^+ K_{g_1,n_1^++1,n_1^-}([x_1]_+, L_{I_1^+}^+ | L_{I_1^-}^-) K_{g_2,n_2^++1,n_2^-}([x_2]_+, L_{I_2^+}^+ | L_{I_2^-}^-) \end{aligned}$$

Where the sum in the RHS contains only stable surfaces, and we use the notations:

$$x_l = \sum_{i \in I_l^-} L_i^- - \sum_{i \in I_l^+} L_i^+,$$

the sets  $I^\pm$  correspond to  $\{1, \dots, n^\pm\}$  minus the positive boundary  $i$ .

In what follows, we work on this relation and rewrite it by using the Cut-and-Join operator. The formalism works perfectly well in this case.

**Remark 5.16.** *The RHS of the formula in corollary 5.6 corresponds to all the ways to glue a directed pair of pants; the different gluings are named I, II, III, IV and drawn on figure 1.7.*

### 5.3.2 Cut-and-join equation for the ‘‘ancestor potential’’

We consider two pants gluing operators given by

$$P_- = K_{0,1,2} \sqcup id \quad \text{and} \quad P_+ = K_{0,2,1} \sqcup id.$$

These operators correspond to adding a pair of pants with an arbitrary number of cylinders. We set

$$P = P_+ + P_-.$$

Using the formula, we can compute the action of these operators.

**Lemma 5.7.** *The action of the operators  $P_\pm$  on  $\hat{S}(V)$  is given by*

$$P_+ F(L) = \frac{1}{2} \sum_{i \neq j} L_i L_j F(L_i + L_j, L_{\{i,j\}^c}) \quad \text{and} \quad P_- F(L) = \frac{1}{2} \sum_i L_i \int_0^{L_i} F(x, L_i - x, L_{\{i\}}) dx.$$

*Proof.* We have

$$K_{0,2,1}(L_1^+, L_2^+ | L_1^-) = L_1^+ L_2^+ \delta_{L_1^+ + L_2^+ = L_1^-} \quad \text{and} \quad K_{0,1,2}(L_1^+ | L_1^-, L_2^-) = L_1^+ \delta_{L_1^- + L_2^- = L_1^+}.$$

By using the formula for the union of two operators, we obtain the proposition.  $\square$

Let  $K(q)$  be the operator defined by

$$K(q) = q^{\frac{D}{2}} K q^{-\frac{D}{2}} = \sum_d q^d K^d.$$

As we explained in the last part, the sum is convergent for all  $q$  because  $K^d$  is of degree  $2d$ . The first order in  $q$  is given by the pant gluing operator :

$$K(q) = id + qP + o(q).$$

The recursion 5.15 can be written in the following very simple way:

**Theorem 5.1** (Cut-and-Join equation). *The operator  $K(q)$  satisfies the following linear evolution called the Cut-and-Join equation*

$$\frac{dK}{dq} = PK.$$

With the initial condition  $K(0) = id$ , and then

$$K(q) = \exp(qP).$$

**Remark 5.17.** *The variable  $q$  is not essential here because it's half of the degree. Indeed, the last equation can be written as*

$$[D, K] = 2PK, \quad \text{or} \quad [D, K] = 2P_+K + 2P_-K.$$

We first propose a straightforward proof.

*Proof.* There are several transformations to do in order to go from the formula in corollary 5.6 to the Cut-and-Join equation. First of all, we put together terms of type  $I, IV$  in the recursion; the kernel  $K_{0,1,1}$  is constant equal to 1, and then we can write

$$\sum_i \sum_j L_i^+ K_{g,n^+,n^-}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) = \sum_i \sum_j L_i^+ K_{g,n^+,n^-}(x_1, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) K_{0,1,1}(x_2 | L_j^-).$$

With the notations  $x_1 = [|L_{I_1^-}^-| - |L_{I_1^+}^+|]_+ = [L_i^+ - L_j^-]_+$  and  $x_2 = L_j^- = [|L_{I_2^-}^-| - |L_{I_2^+}^+|]_+$ , the partitions are given by  $I_1^+ = \{i\}^c, I_1^- = \{j\}^c$  and  $I_2^+ = \emptyset, I_2^- = \{j\}$ . Using these notations, the second and fourth lines fit together and lead to the new recursion:

$$\begin{aligned} (2g - 2 + n^+ + n^-)K_{g,n^+,n^-}(L^+ | L^-) &= \frac{1}{2} \sum_{i \neq j} L_i^+ L_j^+ K_{g,n^+ - 1, n^-}(L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-) \\ &+ \frac{1}{2} \sum_i L_i^+ \int_0^{L_i^+} K_{g-1, n^+ + 1, n^-}(x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) dx \\ &+ \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^+ \sqcup I_2^+ = I^+}} L_i^+ K_{g_1, n_1^+ + 1, n_1^-}(x_1, L_{I_1^+}^+ | L_{I_1^-}^-) K_{g_2, n_2^+ + 1, n_2^-}(x_2, L_{I_2^+}^+ | L_{I_2^-}^-) \end{aligned}$$

This time last term also includes unstable terms of type  $(0, 1, 1)$ , and this is one of the reasons why we introduced the cylinder. To make things more explicit, we apply to a vector  $e_\mu \in S(V)$  both sides of the equation, divide by  $n^-!$ , and integrate. Of course, the LHS is

$$(2g - 2 + n^+ + n^-)K_{g,n^+,n^-} \cdot e_\mu.$$

Moreover, by using lemma 5.7, the first line becomes

$$\begin{aligned}
& \frac{1}{2} \frac{1}{n^-!} \sum_{i \neq j} (L_i^+ L_j^+) \int_{L^- \in |L^+| \Delta_{n^-}} K_{g, n^+ - 1, n^-} (L_i^+ + L_j^+, L_{\{i, j\}^c}^+ | L^-) e_\mu(L^-) d\sigma_{n^-}^{|L^-|} \\
&= \frac{1}{2} \sum_{i \neq j} (L_i^+ L_j^+) (K_{g, n^+ - 1, n^-} \cdot e_\mu) (L_i^+ + L_j^+, L_{\{i, j\}^c}^+) \\
&= (P_+ \circ K_{g, n^+ - 1, n^-}) \cdot e_\mu
\end{aligned}$$

Similarly, we can treat the third line using the Fubini theorem.

$$\begin{aligned}
& \frac{1}{2} \frac{1}{n^-!} \sum_i L_i^+ \int_{L^- \in |L^+| \Delta_{n^-}} \int_0^{L_i^+} K_{g-1, n^+ + 1, n^-} (x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) e_\mu(L^+) dx d\sigma_{n^-}^{|L^-|} \\
&= \frac{1}{2} \sum_i L_i^+ \int_0^{L_i^+} (K_{g-1, n^+ + 1, n^-} \cdot e_\mu) (x, L_i^+ - x, L_{\{i\}^c}^+) dx \\
&= (P_- \circ K_{g-1, n^+ + 1, n^-}) \cdot e_\mu.
\end{aligned}$$

The last step is to recover the disconnected term. We first consider the contraction operator:

$$C : \hat{S}(V) \otimes \hat{S}(V) \longrightarrow \hat{S}(V)$$

defined by

$$C(f \otimes g) = \frac{1}{2} \sum_i \sum_{I_1^\pm \sqcup I_2^\pm = I^\pm} L_i^+ \int_0^{L_i^+} f(x, L_{I_1^+}^+) g(L_i^+ - x, L_{I_2^+}^+) dx.$$

Secondly, for each partition  $(I_1^-, I_2^-)$  we can write

$$e_\mu(L^-) = \sum_{\mu_1 + \mu_2 = \mu} e_{\mu_1}(L_{I_1^-}^-) e_{\mu_2}(L_{I_2^-}^-).$$

Now, rewriting the last line of the recursion, we need to carefully compute the new domain of integration, and we obtain:

$$\begin{aligned}
& \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^\pm \sqcup I_2^\pm = I^\pm}} \sum_{\mu_1 + \mu_2 = \mu} L_i^+ \frac{n_1^-! n_2^-!}{n^-!} \int_0^{L_i^+} (K_{g_1, n_1^+ + 1, n_1^-} \cdot e_{\mu_1})(x, L_{I_1^+}^+) (K_{g_2, n_2^+ + 1, n_2^-} \cdot e_{\mu_2})(x - L_i^+, L_{I_1^+}^+) dx \\
&= \frac{1}{2} \sum_i \sum_{\substack{g_1 + g_2 = g \\ I_1^+ \sqcup I_2^+ = \{i\}^c}} \sum_{\mu_1 + \mu_2 = \mu} L_i^+ \int_0^{L_i^+} (K_{g_1, n_1^+ + 1, n_1^-} \cdot e_{\mu_1})(x, L_{I_1^+}^+) (K_{g_2, n_2^+ + 1, n_2^-} \cdot e_{\mu_2})(x - L_i^+, L_{I_1^+}^+) dx \\
&= \sum_{\mu_1 + \mu_2 = \mu} C((K_{g_1, n_1^+ + 1, n_1^-} \cdot e_{\mu_1}) \otimes (K_{g_2, n_2^+ + 1, n_2^-} \cdot e_{\mu_2}))
\end{aligned}$$

The first equality is the result of integration; in the second, the inverse of the binomial coefficient absorbs the sum over  $(I_1^-, I_2^-)$ . To finish the proof, we use the following equality that shows how  $P$  acts on disconnected objects:

**Lemma 5.8.** *If  $x^c = \sum_{g, n^+, n^-} x_{g, n^+, n^-} \in \hat{S}(V)$  and  $x = \exp_{\sqcup}(x^c)$ , then we have*

$$P_- x = \left( \sum_{g, n^+, n^-} P_- x_{g, n^+, n^-} + \sum_{g_i, n_i^+, n_i^-} C(x_{g_1, n_1^+, n_1^-} \otimes x_{g_2, n_2^+, n_2^-}) \right) \sqcup x$$

The proof of this lemma uses the fact that  $K_{0,2,1}$  is a tensor of type (2, 1) and  $K_{0,1,2}$  is of type (1, 2).  $\square$

**Remark 5.18** (Proof by using acyclic stable graphs). *We give a second proof that uses the language of acyclic stable graphs and is a bit more conceptual, but it's related directly to the structure of the recursion. By using results of paragraph 4.4.3, we derive the following:*

**Proposition 5.11.** *The volumes  $K_{g,n^+,n^-}$  are given by:*

$$K_{g,n^+,n^-}(L^+|L^-) = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}_{g,n^+,n^-}^*} \frac{n_{\mathcal{G}^\circ}}{d_{g,n^+,n^-}! \# \text{Aut}(\mathcal{G}^\circ)} K_{\mathcal{G}^\circ}(L^+|L^-)$$

Where  $\mathbf{acycl}_{g,n^+,n^-}$  is the set of directed acyclic pants decomposition's with genus  $g$  and  $n^\pm$  boundaries.

This proposition remains true for operators we still have

$$K_{g,n^+,n^-} = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}_{g,n^+,n^-}^*} \frac{n_{\mathcal{G}^\circ} K_{\mathcal{G}^\circ}}{d_{g,n^+,n^-}! \# \text{Aut}(\mathcal{G}^\circ)} K_{\mathcal{G}^\circ}.$$

And then, by using  $\exp_{\sqcup}$ , we have

$$K = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}^*} \frac{n_{\mathcal{G}^\circ} K_{\mathcal{G}^\circ}}{d(\mathcal{G}^\circ)! \# \text{Aut}(\mathcal{G}^\circ)}.$$

Then we can use the following formula valid in the algebra  $\hat{S}(\mathcal{A})$ , which is given in proposition 5.6. We have in  $\hat{S}_\infty(\mathcal{A})$

$$\exp(\mathbf{e}_{0,2,1} + \mathbf{e}_{0,1,2}) = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}} \frac{n_{\mathcal{G}^\circ}}{d(\mathcal{G}^\circ)! \# \text{Aut}(\mathcal{G}^\circ)} \mathbf{e}_{\mathcal{G}^\circ}$$

Note that we can apply  $\mathbf{K}$  on both sides of this equation by using the fact that it's a morphism for the composition we have

$$\begin{aligned} \mathbf{K}(\mathbf{e}_{0,2,1}) &= \mathbf{K}(e_{0,2,1}) \sqcup \mathbf{K}(\exp_{\sqcup}(e_{0,1,1})) = K_{0,2,1} \sqcup id = P_+ \\ \mathbf{K}(\mathbf{e}_{0,1,2}) &= \mathbf{K}(e_{0,1,2}) \sqcup \mathbf{K}(\exp_{\sqcup}(e_{0,1,1})) = K_{0,1,2} \sqcup id = P_-. \end{aligned}$$

And then

$$\mathbf{K}(\mathbf{e}_{0,2,1} + \mathbf{e}_{0,1,2}) = \exp(P).$$

Similarly, we also have

$$\mathbf{K}\left(\sum_{\mathcal{G}^\circ \in \mathbf{acycl}} \frac{n_{\mathcal{G}^\circ}}{d(\mathcal{G}^\circ)! \# \text{Aut}(\mathcal{G}^\circ)} \mathbf{e}_{\mathcal{G}^\circ}\right) = \sum_{\mathcal{G}^\circ \in \mathbf{acycl}} \frac{n_{\mathcal{G}^\circ}}{d(\mathcal{G}^\circ)! \# \text{Aut}(\mathcal{G}^\circ)} \mathbf{K}(e_{\mathcal{G}^\circ}) = K.$$

Then, we finally obtain

$$K = \exp(P).$$

We recover the theorem by using  $K(q) = q^{\frac{D}{2}} K q^{-\frac{D}{2}}$

**Remark 5.19** (Proof by using only stable surfaces and  $*$ ). *We can also use the formalism of the product  $*$  given paragraph 5.1.4. By using the formulas, we have the following lemma:*

**Lemma 5.9.** *The  $*$ -product of  $K_{0,2,1}$ ,  $K_{0,1,2}$  and  $K_{g,n^+,n^-}$  are given by*

$$K_{0,1,2} * K_{g,n^+,n^-} = \sum_i \sum_j L_i^+ K_{g,n^+,n^-}([L_i^+ - L_j^-]_+, L_{\{i\}^c}^+ | L_{\{j\}^c}^-) \\ + \frac{1}{2} \sum_i L_i^+ \int_0^{L_i^+} K_{g,n^+,n^-}(x, L_i^+ - x, L_{\{i\}^c}^+ | L^-) dx,$$

and

$$K_{0,2,1} * K_{g,n^+,n^-} = \frac{1}{2} \sum_{i \neq j} (L_i^+ + L_j^+) K_{g,n^+,n^-}(L_i^+ + L_j^+, L_{\{i,j\}^c}^+ | L^-).$$

Applying this to disconnected surfaces, if we denote  $K^s = \exp(\sum'_{g,n^+,n^-} K_{g,n^+,n^-}) = \sum_d K^{s,d}$  we obtain:

**Lemma 5.10.** *We have*

$$dK^{s,d} = K_{0,1,2} * K^{s,d-1} + K_{0,2,1} * K^{s,d-1}. \quad (5.16)$$

We know that we have

$$(K_{0,1,2} * K^{d-1,s}) \sqcup id = (K_{0,1,2} \sqcup id) \circ (K^{d-1,s} \sqcup id) = P^+ \circ K^{d-1},$$

and similarly for the other term. Then, if we take the union with  $id$  in equation 5.16, we finally obtain the recursive relation

$$dK^d = PK^{d-1}.$$

Multiplying by  $q^{d-1}$  and summing over  $d$  gives the statement of the theorem.

**Ancestor potential and differential operators:** In this part, we rewrite the recursion by using the isomorphism

$$\hat{S}(V) \longrightarrow \mathbb{Q}[[\mathbf{t}]].$$

Which was presented in Section 5.1. Using the fact that

$$K = K^s \sqcup id,$$

and the result of Section 5.1.3, we already know the operator  $K$  is expressed in terms of annihilation and creation operators. In this part, we give an explicit expression of the Cut-and-Join operators.

**Lemma 5.11.** *Operators  $P_-$  and  $P_+$  are given by the formula*

$$P_- = \frac{1}{2} \sum_{k,l} \frac{k!l!}{(k+l+1)!} t_{k+l+2} \partial_k \partial_l, \quad \text{and} \quad P_+ = \frac{1}{2} \sum_{k,l} \frac{(k+l)!}{k!l!} t_{k+1} t_{l+1} \partial_{k+l}.$$

*Proof.* By definition,  $P_{\pm}$  can be written as  $P_{\pm}^s \sqcup id$  and then can be expressed in terms of creation and annihilation operators. Under the map to  $\mathbb{Q}[[\mathbf{t}]]$ , they are formal differential operators. Moreover, using proposition 5.2, we know that the coefficient in front of  $t^{\mu^+} \partial_{\mu^-}$  in  $P_+$  ( resp  $P_-$ ) is the matrix coefficient of  $K_{0,2,1}$  (resp  $K_{0,1,2}$ ). By using

$$K_{0,2,1} \cdot f = L_1 L_2 f(L_1 + L_2).$$

We obtain

$$K_{0,2,1} \cdot e_k = \sum_{i+j=k} \frac{k!}{i!j!} e_{i+1} \otimes e_{j+1}.$$

Which gives us the formula

$$P_+ = K_{0,2,1} \sqcup id = \sum_{i,j,k} \frac{k!}{i!j!} t_{i+1} t_{j+1} \partial_k.$$

Similarly, using

$$K_{0,1,2} \cdot f = \frac{L_1}{2} \int_0^{L_1} f(x, L_1 - x) dx,$$

we obtain

$$K_{0,2,1} \cdot e_i \otimes e_j = \frac{1}{2} \frac{i!j!}{(i+j+1)!} e_{i+j=1}.$$

And then

$$P_- = \frac{1}{2} \sum_{k,l} \frac{k!l!}{(k+l+1)!} t_{k+l+2} \partial_k \partial_l.$$

□

**Remark 5.20** (Change of coordinates). *It's better to see these operators in the basis given by*

$$\tilde{e}_\mu = \frac{e_\mu}{\prod_i (i!)^{\mu(i)}}.$$

*It corresponds to the basis of polynomials  $\prod_i \frac{L_i^{\alpha_i}}{\alpha_i!}$  instead of  $\prod_i L_i^{\alpha_i}$ . This diagonal change of basis corresponds to a change of  $t$  coordinates*

$$\frac{t_i}{i!} \rightarrow t_i \quad i! \partial_i \rightarrow \partial_i.$$

*After this change, the operator becomes*

$$P = \frac{1}{2} \sum_{k,l} (k+l+2) t_{k+l+2} \partial_k \partial_l + \frac{1}{2} \sum_{k,l} (k+1)(l+1) t_{k+1} t_{l+1} \partial_{k+l}.$$

*We choose to keep the notation  $t_i$  for the new variables. This new basis simplifies some expressions but also makes sense when we deal with the Laplace transform.*

### 5.3.3 Case of surfaces with marked points

**Recurrence for the volumes:** Another consequence of chapter 4.5.2 is given by proposition 4.25. This leads to the following proposition, which concerns the volumes of moduli space of oriented metric ribbon graphs with marked points  $K_{g,n^+,n^-,m}$ .

**Corollary 5.7.** *Functions  $K_{g,n^+,n^-,m}$  satisfy the recursive formula on  $m$ :*

$$mK_{g,n^+,n^-,m} = E K_{g,n^+,n^-,m-1}.$$

From this proposition, we can obtain the following formula:

$$\sum t^m K_{g,n^+,n^-,m} = \exp(tE) K_{g,n^+,n^-}.$$

**Cut-and-join equation:** We can immediately rephrase the results of the last paragraph in terms of operators. Let

$$E : \hat{S}(V) \longrightarrow \hat{S}(V)$$

the multiplication by the function  $E$ , we have

$$E(P) = \sum_i L_i P(L).$$

A direct computation allows us to give the expression of the action on  $\mathbb{Q}[[t]]$ . In the basis  $\tilde{e}_\mu$ , we have

$$E = \sum_i (i+1)t_{i+1}\partial_i.$$

Then, from the last recursion, we obtain the following:

**Theorem 5.2.** *The series  $K^\bullet(q_0, q_1)$  satisfies the two equations.*

$$\frac{\partial K^\bullet}{\partial q_0} = EK^\bullet \quad \text{and} \quad \frac{\partial K^\bullet}{\partial q_1} = PK^\bullet;$$

with the initialization  $K^\bullet(0, 0) = id$ . Moreover,  $P$  and  $E$  commute, then

$$K^\bullet(q_0, q_1) = K^\bullet(q_0, 0)K^\bullet(0, q_1) = \exp(q_0 E)K(q_1).$$

*Proof.* According to corollary 5.7 in terms of operators, we also have

$$mK_{g,n^+,n^-,m} = E \circ K_{g,n^+,n^-,m-1}.$$

And then we can obtain the relation

$$\frac{\partial K^\bullet}{\partial q_0} = EK^\bullet.$$

The volumes  $K_{g,n^+,n^-,m}$  also satisfy the recursion of corollary 5.15, and then we still have

$$\frac{\partial K^\bullet}{\partial q_1} = PK^\bullet.$$

The fact that  $E$  and  $P$  are commuting can be done by hand, using formulas. But let  $K$  be any positive continuous function on  $\Lambda_{n^+,n^-}$  that satisfies the statement of the transfer lemma, then according to remark 5.11 on  $\Lambda_{n^+,n^-}$  we have  $E(L^+) = E(L^-)$ , then for any  $f$  we have

$$\int_{|L^+| \cdot \Delta_n^+} K(L^+|L^-)E(L^-)f(L^-)d\sigma_n^{|L^+|} = E(L^+) \int_{|L^+| \cdot \Delta_n^+} K(L^+|L^-)f(L^-)d\sigma_n^{|L^+|}.$$

We see that  $K$  commutes with  $E$ . According to this, all the  $K_{M_i^\circ}$  commutes with  $E$  and so does  $P$ . Using commutativity, we can obtain the factorization.  $\square$

## 5.4 Partition function and combinatorial interpretations

### 5.4.1 The partition function

**Definitions:** Let  $(g, n^+, n^-)$ , in this part, we consider the following integrals:

$$G_{g,n^+,n^-}(L) = \frac{1}{n^-!} \int_{L^- \in |L| \cdot \Delta_{n^-}} K_{g,n^+,n^-}(L|L^-) d\sigma_{n^-}^{|L|}. \quad (5.17)$$

According to proposition 5.7, these functions are homogeneous symmetric polynomials of degree  $4g - 4 + 2n^+ + 2n^- = 2d_{g,n^+,n^-}$ . In the case  $(0, 1, 1)$  we have  $G_{0,1,1} = 1$ , which corresponds to the amplitude of the cylinder. More generally, for each disconnected surface  $M^\circ$  it's also possible to consider:

$$G_{M^\circ}(L) = \int_{L^- \in \Delta_{M^\circ}(L)} K_{M^\circ}(L|L^-) \frac{d\sigma_{M^\circ}^L}{n^-!}.$$

It's also a homogeneous polynomial of degree  $d(M^\circ)$ . As in 5.1.3 the vacuum is defined by:

$$\mathbf{e}_\emptyset = \exp_{\sqcup}(e_\emptyset).$$

It corresponds to the constant polynomial "in arbitrary many variables". Then, applying  $K_{M^\circ}$  to  $\mathbf{e}_\emptyset$ , we recover  $G_{M^\circ}$ ,

$$G_{M^\circ} = K_{M^\circ} \cdot \mathbf{e}_\emptyset.$$

We can generalize this by linearity using the operator  $\mathbf{K}$  defined in the last section. We can set

$$\mathbf{G}(x) = \mathbf{K}(x) \cdot \mathbf{e}_\emptyset \quad \forall x \in \mathcal{M}.$$

As  $\mathbf{K}$  is compatible with the grading, it's also true for the operator  $\mathbf{G}$ , and then  $\mathbf{G}$  extends to the formal completion  $\hat{\mathcal{M}}$ . Moreover, the image of a symmetric element  $e_{M^\circ} \in S(\mathcal{M})$  is a symmetric "polynomial". Putting that together, we see that the operator  $\mathbf{G}$  restricts to a linear map:

$$\begin{aligned} \mathbf{G} : \hat{S}(\mathcal{M}) &\longrightarrow \hat{S}(V) \\ x &\longrightarrow \mathbf{K}(x) \cdot \mathbf{e}_\emptyset. \end{aligned}$$

Moreover, using properties of  $\mathbf{e}$ , we obtain the following.

**Proposition 5.12.** *The linear map  $\mathbf{G} \in \text{End}(\hat{S}(\mathcal{M}))$  is a morphism of commutative algebras, we have  $\mathbf{G}(x \sqcup y) = \mathbf{G}(x) \sqcup \mathbf{G}(y)$ .*

*Proof.* Using the results of the last section (see formula 5.4), we have for each  $e_{M_1^\circ}, e_{M_2^\circ} \in S(\mathcal{M})$

$$G_{M_1^\circ \sqcup M_2^\circ} = K_{M_1^\circ \sqcup M_2^\circ} \cdot \mathbf{e}_\emptyset = (K_{M_1^\circ} \sqcup K_{M_2^\circ}) \cdot \mathbf{e}_\emptyset = (K_{M_1^\circ} \cdot \mathbf{e}_\emptyset) \sqcup (K_{M_2^\circ} \cdot \mathbf{e}_\emptyset) = G_{M_1^\circ} \sqcup G_{M_2^\circ}$$

□

**Partition function:** The partition function is the image of the vacuum by  $K$ :

$$G = K \cdot \mathbf{e}_\emptyset \in \hat{S}(V).$$

We also define  $G(q)$  as the image of  $\mathbf{e}_\emptyset$  by  $K(q)$ , which is the series

$$G(q) = \sum_{M^\circ} q^{d(M^\circ)} G_{M^\circ} = q^{\frac{D}{2}} G.$$

The last relation comes from the fact that  $q^{-\frac{D}{2}} \mathbf{e}_\emptyset = \mathbf{e}_\emptyset$  and also  $K(q) = q^{\frac{D}{2}} G q^{-\frac{D}{2}}$ . As before, we can also consider  $G^s, G^c$  and  $G^{c,s}$ , which are associated with stable, connected surfaces and connected stable surfaces. We derive the relation  $K = \exp_{\sqcup}(K^c)$  in lemma 5.5 and using proposition 5.12 it immediately implies the same relation for  $G$

$$G = \exp_{\sqcup}(G^c).$$

Moreover, a stable operator is related to  $K$  by the formula:

$$K = K^s \sqcup id.$$

And then we have the relation

$$G = G^s \sqcup \mathbf{e}_\emptyset.$$

then  $G$  belongs to  $\hat{S}_\infty(V)$ .

**Formal series:** By abuse of notation, we denote  $Z$  the image of  $G$  in  $\mathbb{Q}[[t]]$ , and then in the coordinates given by  $\tilde{e}_\mu$  we have an expression of the form

$$Z(q, \mathbf{t}) = \sum_{\mu} c(\mu) q^{\frac{d(\mu)}{2}} \prod_i t_i^{\mu(i)}.$$

**Remark 5.21.** *As before, the variable  $q$  is not essential.*

Using formula 5.4.1, the non-connected partition function is then the exponential of the connected partition function,

$$Z(q, \mathbf{t}) = \exp(Z^c(q, \mathbf{t})).$$

This time the exponential is the one on the space of formal series.

**Remark 5.22** (Dependence in  $t_0$ ). *As we see before,  $G$  is in the subspace  $S_\infty(V)$ , and then it satisfies  $G = G^s \sqcup \mathbf{e}_\emptyset$ . In terms of formal series, by using the results of Section 5.1, it's equivalent to*

$$Z(q, \mathbf{t}) = \exp(t_0) Z^s(q, \mathbf{t}^*).$$

Where  $\mathbf{t}^* = (t_1, t_2, \dots)$  is independent of  $t_0$ .

## 5.4.2 Combinatorial interpretations

**First combinatorial interpretation:** Using proposition 5.7, it's possible to give a combinatorial interpretation to the coefficients of the polynomial  $G_{g,n^+,n^-}$ . For each  $\alpha = (\alpha_1, \dots, \alpha_{n^+})$ , let  $R_{g,n^+,n^-}^\circ(\alpha)$  be the number of oriented ribbon graphs  $R^\circ$  counted with a weight equal to  $\frac{1}{\#\text{Aut}(R^\circ)}$  and such that:

- the graph is quadrivalent,
- there are  $n^-$  negative unlabelled boundary components,
- there are  $n^+$  positive labelled boundary components, and the perimeter of the  $i$ -th boundary is  $\alpha_i$ .

We assume that the elements of  $\text{Aut}(R^\circ)$  do not permute the positive boundaries.

**Proposition 5.13.** *The polynomial  $G_{g,n^+,n^-}$  is a generating function for the numbers  $R_{g,n^+,n^-}^\circ(\alpha)$ , it satisfies*

$$G_{g,n^+,n^-}(L) = \sum_{\alpha} R_{g,n^+,n^-}^\circ(\alpha) \prod \frac{L_i^{\alpha_i}}{(\alpha_i - 1)!}.$$

*Proof.* The polynomial  $G_{g,n^+,n^-}(L)$  is given by

$$\sum_{R^\circ \in \text{rib}_{g,n^+,n^-}} \sum_{\alpha^+} K_{g,n^+,n^-}[\alpha^+ | 0, \dots, 0] \prod_{i=1}^{n^+} L_i^{\alpha_i^+}.$$

To prove this proposition, we use proposition 5.7. In this lemma, we obtain the following expression for any oriented ribbon graph:

$$K_{R^\circ}[\alpha^+ | \alpha^-] = \frac{\overline{P}_{R^\circ}(\alpha^+ - \alpha^+(R^\circ) | \alpha^-)}{n^-! \#\text{Aut}(R^\circ)} \frac{\alpha^-!}{(\alpha^+ - 1)!}.$$

In our case, entries of  $\alpha^-$  are zeros. The graph is oriented, then if  $\overline{P}_{R^\circ}(\alpha^+ | \alpha^-)$  is non-zero; we must have

$$d(\alpha^+ - \alpha^+(R^\circ)) = d(\alpha^-) = 0.$$

And then  $\alpha^+ = \alpha^+(R^\circ)$ . Therefore, the value of  $\overline{P}_{R^\circ}(0 | 0)$  is 1 because there is only one integral point; it's given by  $m_e = 0$  for all  $e \in X_1 R$ . Then we have

$$K_{R^\circ}[\alpha^+ | 0, \dots, 0] = \begin{cases} \frac{1}{\#\text{Aut}(R^\circ) (\alpha^+(R^\circ) - 1)!} & \text{if } \alpha^+ = \alpha^+(R^\circ) \\ 0 & \text{else.} \end{cases}$$

By summing over  $\text{rib}_{g,n^+,n^-}$ , we obtain the claim. □

**Oriented ribbon graphs and dessins d'enfants:** There are numerous bijections between combinatorial structures on surfaces; the subject is very rich; the literature is very wide; and the knowledge of the author is quite poor. We present here some bijections that are meaningful in the context of oriented ribbon graphs. The first one relates the oriented ribbon graphs to the Grothendieck dessins d'enfants. We recall that Grothendieck dessins d'enfants are coverings over the sphere ramified over three points  $(x_0, x_+, x_-)$ . In particular, we can relate count of oriented quadrivalent ribbon graphs to Hurwitz numbers of Grothendieck dessins d'enfants. Let  $h_{g,n^+,n^-}(\alpha)$  the Hurwitz number of dessins d'enfants with:

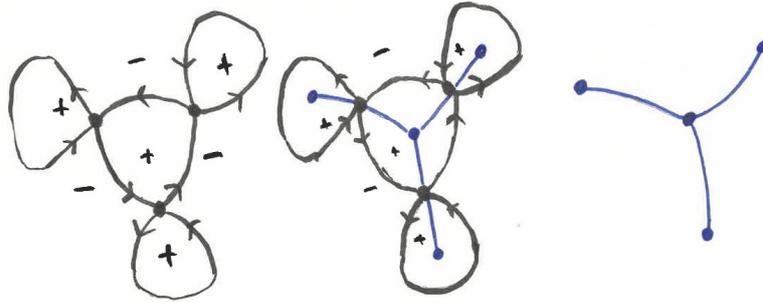


Figure 5.4: Description of the bijection.

- Unlabelled simple ramifications over one point  $x_0$ .
- $n^-$  unlabelled ramifications over  $x_-$ .
- The ramifications over the third point  $x_+$  are labelled and specified by  $\alpha$ .

Then bijection described in lemma 4.37 provides the following formula:

$$R_{g,n^+,n^-}^\circ(\alpha) = h_{g,n^+,n^-}(\alpha).$$

**Dessin d'enfant to undirected ribbon graph:** There is a second bijection that identifies dessins d'enfants and usual ribbon graphs. If a dessin d'enfant counted by  $h_{g,n^+,n^-}(\alpha)$  is given, we can obtain a second graph by looking at preimages of the segment that connects  $x_0$  and  $x_+$ . It's not necessarily an orientable ribbon graph, and it satisfies the following properties:

- There's  $n^+$  labelled boundary components of perimeters given by  $2\alpha$ .
- The graph is bipartite; there are  $n^-$  white vertices and  $2g - 2 + n^+ + n^-$  bivalent black vertices.

Then we can forget black vertices because they are bivalent and the graph is bipartite, then we obtain an usual ribbon graph with prescribed perimeters  $\alpha$  for his boundary components and no restriction on the degree of his vertices (we allow univalent and bivalent vertices). Let  $R_{g,n}(\alpha)$  be the number of such ribbon graphs weighted by  $\frac{1}{\#\text{Aut}(R)}$ .

**Proposition 5.14.** *We have the relation*

$$R_{g,n^+,n^-}^\circ(\alpha) = R_{g,n^-}(\alpha)$$

**Remark 5.23** (Description). *In this picture, the positive boundaries are the boundaries of the new graph; the quadrivalent vertices are mapped to bivalent vertices and then deleted; and finally, the negative boundary components become the vertices of the new graph. This process is described in figure 5.4.*

### 5.4.3 Cut-and-Join operators

**The generic case:** In this part, we use theorem 5.1 to derive recursion for  $Z$ . Results of this section are a reformulation of theorem 5.1.

**Proposition 5.15.** *As an element of  $\mathbb{Q}[[\mathbf{t}]]$ , the partition function  $Z$  satisfies the following linear equation:*

$$\frac{\partial Z}{\partial q}(q) = \frac{1}{2} \sum_{k,l} (k+l+2)t_{k+l+2} \partial_i \partial_j Z(q) + \frac{1}{2} \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1} \partial_{k+l} Z(q)$$

*Proof.* This is straightforward using the cut and join equation for  $K$

$$\frac{\partial K}{\partial q} = PK(q).$$

Applying this to the vacuum, we obtain

$$\frac{\partial Z}{\partial q} = PZ.$$

□

**Remark 5.24.** *As before, the variable  $q$  is not necessary in this equation, as we have*

$$\frac{\partial Z}{\partial q}(1) = \sum_i it_i \partial_i Z(1).$$

In a similar way, the series  $Z^c$  associated with connected surfaces also satisfies a recursion; the following fact is classical. The use of exponentials makes this new equation non-linear.

**Corollary 5.8.** *The series associated with connected-oriented ribbon graphs satisfies the following nonlinear equation:*

$$\frac{\partial Z^c}{\partial q} = \frac{1}{2} \sum_{i,j} (i+1)(j+1)t_{i+1}t_{j+1} \partial_{i+j} Z^c + \frac{1}{2} \sum_{i,j} (i+j+2)t_{i+j+2} \partial_i \partial_j Z^c + \frac{1}{2} \sum_{i,j} (i+j+2)t_{i+j+2} \partial_i Z^c \partial_j Z^c.$$

**Surfaces with one negative boundary component:** In section 4.5.1, we see that the kernels  $K_{g,n+,1}(L^+|L^-)$  are polynomials in  $L^+$ , and we derive from theorem 4.9 a recursion for them. In this part, we recover these formulas by using the formalism of Fock space. We introduce a formal variable  $t_-$ , and the modified vacuum:

$$\mathbf{e}_\emptyset(t_-) = \exp(t_- t_0) = t_-^N \cdot \mathbf{e}_\emptyset.$$

Then we consider

$$G(q, t_-) = K(q) \cdot \mathbf{e}_\emptyset(t_-).$$

And  $Z(q, \mathbf{t}, t_-)$  admits a development in the variable  $t_-$ ,

$$Z(q, \mathbf{t}, t_-) = \sum_{n^-} Z^{n^-}(q, \mathbf{t}) \frac{t_-^{n^-}}{n^-!}.$$

We can obtain the following result

**Corollary 5.9.** *The generating series  $Z^1$  also satisfies the Cut-and-Join equation*

$$\frac{\partial Z^1}{\partial q} = PZ^1.$$

*But with the initial condition  $Z^1(0, t) = t_0$ .*

The result is slightly different from the one given in corollary 4.16, but we can easily make the bridge.

*Proof.* Applying the vacuum  $\mathbf{e}_\emptyset(t_-)$  to the Cut-and-Join equation leads to

$$\frac{\partial Z}{\partial q}(q) = PZ.$$

If we expand this equation to powers of  $t_-$ , we obtain the claim by collecting the coefficient of  $t_-$ , and we do the same for the initialization.  $\square$

**Removing  $t_0$  from the equation:** Sometimes a Cut-and-Join equation can be presented without  $t_0$ .

$$\frac{\partial Z}{\partial t_0} = Z \quad \text{and} \quad Z(\mathbf{t}) = \exp(t_0)Z(0, \mathbf{t}^*) = \exp(t_0)Z^*(\mathbf{t}^*).$$

Then it's natural to consider the operators  $P_0 = \exp(-t_0)P \exp(t_0)$  acting on  $\mathbb{Q}[[\mathbf{t}^*]]$ , we have the expression

$$P_0 = \frac{1}{2} \sum_{i+j>0} (i+1)(j+1)t_{i+1}t_{j+1}\partial_{i+j} + \frac{1}{2} \sum_{i,j>0} (i+j+2)t_{i+j+2}\partial_i\partial_j + \sum_i (i+1)t_{i+1}\partial_i Z + \frac{t_1^2}{2} + t_2$$

**Corollary 5.10.**  *$Z^*(q, \mathbf{t}^*)$  is the solution of the linear homogeneous equation*

$$\frac{\partial Z^*}{\partial q} = P_0 Z^*,$$

*with the initial condition  $Z^*(0, \mathbf{t}^*) = 1$ .*

#### 5.4.4 Case of marked surfaces

We can generalize all of this to the case of marked surfaces; for each  $M^\circ \in \mathbf{bord}^{\circ, \bullet}$  we have

$$G_{M^\circ} = K_{M^\circ} \cdot \mathbf{e}_\emptyset,$$

we can also define a morphism of commutative algebra that extends  $\mathbf{G}$ :

$$\mathbf{G} : \hat{S}(\mathcal{A}^\bullet) \longrightarrow \hat{S}(V).$$

The image defines a module over  $\hat{S}(\mathcal{A}^\bullet)$ . We also denote  $G^\bullet(q_0, q_1) = K^\bullet(q_0, q_1)\mathbf{e}_\emptyset$  and  $Z^\bullet(q_0, q_1) \in \mathbb{Q}[[q_0, q_1]][[\mathbf{t}]]$ . We still have the specialization,

$$Z^\bullet(0, q) = Z(q)$$

**Cut-and-join equation:** According to the results of the last part, the series  $Z^\bullet$  satisfies the two equations:

$$\frac{\partial Z^\bullet}{\partial q_0} = EZ^\bullet, \quad \text{and} \quad \frac{\partial Z^\bullet}{\partial q_1} = PZ^\bullet.$$

We can also obtain the following corollary:

**Corollary 5.11.** *The generating series  $Z^\bullet$  is a solution of the Cut-and-Join equation  $\frac{\partial Z^\bullet}{\partial q_1} = PZ^\bullet$  with the initial condition*

$$Z^\bullet(q_0, 0) = \exp\left(\sum_i q_0^i t_i\right).$$

*Proof.* We have the following lemma:

**Lemma 5.12.** *We have the relation*

$$\exp(q_0 E) \mathbf{e}_\emptyset = \exp\left(\sum_i q_0^i t_i\right).$$

*Proof.* We can prove this by taking the derivative of the two sides by  $q_0$ . We obtain that the RHS is solution of  $Y' = EY$ ; the two sides are then solutions that coincide at  $q_0 = 0$ , which gives the equality.  $\square$

Then, using the Cut-and-Join equations, we can write

$$Z^\bullet(q_0, q_1) = K(0, q_1) K(q_0, 0) \mathbf{e}_\emptyset = \exp(q_1 P) \exp(q_0 E) \mathbf{e}_\emptyset = \exp(q_1 P) \exp\left(\sum_i q_0^i t_i\right).$$

Which gives the proof.  $\square$

**Ribbon graph with marked points and integral points:** The polynomials  $G_{g,n^+,n^-,m}$  is a generating series for oriented ribbon graphs with quadrivalent vertices and  $m$  bivalent vertices. Let  $R_{g,n^+,n^-,m}^\circ$  be the number of such graphs weighted by the group of automorphisms. Then we have

$$G_{g,n^+,n^-,m}(L) = \sum_{\alpha} R_{g,n^+,n^-,m}^\circ(\alpha) \prod \frac{L_i^{\alpha_i}}{(\alpha(i) - 1)!}.$$

We also give a last combinatorial bijection that concerns integral points in the moduli space  $\mathcal{M}_{g,n^+,n^-}^{comb}$  of oriented combinatorial ribbon graphs. Let  $P_{g,n^+,n^-}^\circ(\alpha^+|\alpha^-)$  the number of integral points in the moduli space  $\mathcal{M}_{g,n^+,n^-}^{comb,*}(\alpha^+|\alpha^-)$ , which is the number of integral points that are supported by oriented ribbon graphs with vertices of degree four and

$$P_{g,n^+,n^-}^\circ(\alpha^+) = \frac{1}{n^-!} \sum_{\alpha^-} P_{g,n^+,n^-}^\circ(\alpha^+|\alpha^-)$$

**Lemma 5.13.** *We have the relation*

$$P_{g,n^+,n^-}^\circ(\alpha) = R_{g,n^+,n^-,r}^\circ(\alpha)$$

with  $r = d(\alpha) - 4g + 4 - n^+ - n^-$ .

*Proof.* This relies on the fact that bivalent vertices define metric on the edges.  $\square$

This result allows us to rely generating series of integral points to  $G_{g,n^+,n^-,r}$ . Let

$$\mathbf{G}_{g,n^+,n^-}(L) = \frac{1}{n^-!} \sum_{\alpha} P_{g,n^+,n^-}^{\circ}(\alpha) \frac{L^{\alpha}}{(\alpha-1)!}$$

Then we can derive the following surprising formula, where  $E(L) = \sum_i L_i$ .

**Proposition 5.16.** *The generating series  $\mathbf{G}_{g,n^+,n^-}$  is given by the following formula:*

$$\mathbf{G}_{g,n^+,n^-}(L) = \exp(E(L))G_{g,n^+,n^-}(L).$$

*Proof.* Using lemma 5.13, we can obtain:

$$\sum_{\alpha} P_{g,n^+,n^-}^{\circ}(\alpha) \frac{L^{\alpha}}{(\alpha-1)!} = \sum_m G_{g,n^+,n^-,m}(L).$$

Using proposition 5.13, we know

$$G_{g,n^+,n^-,m}(L) = \frac{E(L)^m}{m!} G_{g,n^+,n^-}(L).$$

Then, summing over  $m$ , we obtain the formula that relies integral points and usual graphs.

$$\sum_{\alpha} P_{g,n^+,n^-}^{\circ}(\alpha) \frac{L^{\alpha}}{\alpha!} = \exp(E(L))G_{g,n^+,n^-}(L).$$

□

**Remark 5.25** (Negative boundary components). *Unfortunately, we do not have any control on the length of negative boundary components.*

Indeed, we can write the following relation

$$\mathbf{G}(q) = G^{\bullet}(q, q), \quad \text{and} \quad \mathbf{Z}(q) = Z^{\bullet}(q, q).$$

and obtain the following immediate corollary:

**Corollary 5.12.** *The series  $\phi$  is solution of the evolution equation*

$$\frac{\partial \mathbf{Z}}{\partial q} = (E + P)\mathbf{Z}$$

*with the initial condition  $\mathbf{Z}(0) = \exp(t_0)$*

## 5.5 Laplace transforms, Tutte equation and topological recursion

### 5.5.1 Tutte equations

In this part, we give applications of theorem 4.2. We found different recursion for  $G_{g,n}$ , and this leads to an instance of the famous Tutte equation, proved in [Tut63] and studied in a more general picture in [Eyn16]. Using this recursion, we derive the Eynard-Orantin topological recursion. All these results are classical, we recover them here to completeness.

**Refined recursion:** We recall that the function  $K_{g,n^+,n^-}$  is defined by the following formula:

$$K_{g,n^+,n^-} = \sum_{R^\circ \in \text{rib}_{g,n^+,n^-}} \frac{K_{R^\circ}}{\#\text{Aut}(R^\circ)}.$$

**Lemma 5.14.** *The functions  $K_{g,n^+,n^-}$  satisfy the following “equation”*

$$\begin{aligned} \sum_{R^\circ \in \text{rib}_{g,n^+,n^-}} \frac{\alpha_1^+(R^\circ) K_{R^\circ}}{\#\text{Aut}(R^\circ)} &= \sum_{i \neq 1} L_1^+ L_i^+ K_{g,n^+ - 1, n^-} (L_1^+ + L_i^+, L_{\{1,i\}^c}^+ | L^-) \\ &+ 2 \sum_i L_1^+ K_{g,n^+, n^- - 1} ([L_1^+ - L_i^-]_+, L_{\{1\}^c}^+ | L_{\{i\}^c}^-) \\ &+ L_1^+ \int_0^{L_1^+} K_{g-1, n^+ + 1, n^-} (x, L_1^+ - x, L_{\{1,i\}^c}^+ | L^-) dx \\ &+ \sum'_{g_1, n_i^\pm, I_i^\pm} L_1^+ K_{g_1, n_1^+ + 1, n_1^-} (x_1, L_{I_1^+}^+ | L_{I_1^+}^-) K_{g_2, n_2^+ + 1, n_2^-} (x_2, L_{I_2^+}^+ | L_{I_2^+}^-). \end{aligned}$$

The notation  $x_i$  was defined in theorem 4.6. And the upper script' means that we exclude the unstable term  $(g, n^+, n^-) = (0, 1, 1)$  in the sum.

**Remark 5.26** (corollary 5.15). *We remark that this formula implies the corollary 5.15. Using symmetry, lemma 5.14 is true for all boundary component  $\beta_i^+$ , then taking the sum over  $i$  leads to corollary 5.15. We just need the formula:*

$$d(\alpha^+(R^\circ)) = \sum_i \alpha_i^+(R^\circ) = 2d_{g,n^+,n^-},$$

for each  $R^\circ \in \text{rib}_{g,n^+,n^-}$  and divide the two sides by 2. However, it doesn't seem for the author that the last formula leads to a recursion for piece-wise polynomials  $K_{g,n^+,n^-}$ . We don't know any simple way to express the LHS of the formula in terms of functions  $K_{g,n^+,n^-}$ . Nevertheless, this formula allows us to compute recursively coefficients of  $G$ .

*Proof.* To prove this proposition, we use theorem 4.2. If we fix  $(g, n^+, n^-)$ , it's possible to consider moduli space of generic oriented metric ribbon graphs of type  $(g, n^+, n^-)$  with a marked quadrivalent vertex on the first boundary  $\beta_1^+$ . If a vertex appears two times in the boundary  $\beta_1^+$ , we count it twice. This space admits a measure by taking the pullback of the lebesgue measure by the covering map; moreover, the volume of this space is the LHS of the equation. This space is mapped to the set of oriented ribbon graphs with a marked vertex, but the map is neither surjective nor injective. Nevertheless, by applying theorem 4.2, this vertex is contained in a unique admissible curve, which spares it from the rest of the graph, and  $[1]^+$  is contained in the component that is removed. In the case where the gluing's are of type  $I$ , there are only two possible vertices in the boundary of the corresponding pair of pant's, and then the contribution is

$$2 \sum_j K_{g,n^+, n^- - 1} ([L_1^+ - L_i^-]_+, L_{\{1,i\}^c}^+ | L^-).$$

In the case of type  $II$ , there are two vertices in the boundary, then the choice of the vertex removes the symmetry which exchanges the two other boundaries of the pant's, and the contribution is

$$\int_0^{L_1^+} K_{g-1, n^+ + 1, n^-} (x, L_1^+ - x, L_{\{1,i\}^c}^+ | L^-) dx.$$

We lose the factor  $\frac{1}{2}$ . In a similar way, we can obtain two other terms. □

**The integra ‘‘Mirzakhani’’ recursion for  $G$ :** We consider the formal series in  $\hat{S}_n(V)$ :

$$G_{g,n}(L) = \sum_{n^-} G_{g,n,n^-}(L).$$

Additionally, we also denote  $G_n^d$  as the series associated with non-connected surfaces with opposite Euler characteristic equal to  $d$ , and with  $n$  positive boundaries. Using lemma 5.14, we can derive the following proposition:

**Proposition 5.17.** *Functions  $(G_n^d)_{n,d}$  satisfy the following recursion*

$$\partial_1 G_n^d(L) = \sum_{j \neq 1} L_j G_{n-1}^{d-1}(L_1 + L_j, L_{\{1,j\}^c}) + \int_0^{L_1} G_{n+1}^{d-1}(x, L_1 - x, L_{\{1\}^c}) dx.$$

Moreover, connected functions satisfy for all  $(g, n)$  with  $2g - 2 + n \geq -1$ :

$$\begin{aligned} \partial_1 G_{g,n}(L) &= \sum_{j \neq 1} L_j G_{g,n-1}(L_1 + L_j, L_{\{1,j\}^c}) + \int_0^{L_1} G_{g-1,n+1}(x, L_1 - x, L_{\{1\}^c}) dx \\ &+ \sum_{g_i, I_i, n_i} \int_0^{L_1} G_{g_1, n_1+1}(x, L_{I_1}) G_{g_2, n_2+1}(L_1 - x, L_{I_2}) dx. \end{aligned}$$

**Remark 5.27.** *As we have  $G_n^d(0, L_1^c) = 0$  if  $d > 0$  and  $G_n^0 = 1$ , the last equation determines  $G_n^d$  with these two conditions.*

**Tutte equations:** We consider the rational numbers  $R_{g,n}^\circ(\alpha)$  defined for all  $(g, n) \neq (0, 0)$  and all  $\alpha \in \mathbb{N}^n$  by

$$R_{g,n}^\circ(\alpha) = \sum_{n^-} R_{g,n,n^-}^\circ(\alpha) = R_{g,n^+, d(\alpha) - 2d_{g,n}}^\circ(\alpha),$$

with  $d_{g,n} = 2g - 2 + n$ . Proposition 5.17 can provide a recursion for these numbers, which is similar to the Tutte formula and is, in some sense, the Tutte formula for oriented graphs. See [Eyn16] for instance to compare with the Tutte formula. We rather use

$$\tilde{R}_{g,n}^\circ(\alpha) = \prod_i \alpha_i R_{g,n}^\circ(\alpha).$$

**Proposition 5.18.** *The coefficients  $\tilde{R}_{g,n}^\circ(\alpha)$  satisfy the equation:*

$$\begin{aligned} \tilde{R}_{g,n}^\circ(\alpha) &= \sum_{i \neq 1} \alpha_i \tilde{R}_{g,n-1}^\circ(\alpha_i + \alpha_1 - 2, \alpha_{\{1,i\}^c}) \\ &+ \sum_{k+l=\alpha_1-2} \tilde{R}_{g-1,n+1}^\circ(k, l, \alpha_{\{1\}^c}) + \sum_{I_1, I_2, g_1, g_2} \tilde{R}_{g_1, n_1+1}^\circ(k, \alpha_{I_1}) \tilde{R}_{g_2, n_2+1}^\circ(l, \alpha_{I_2}). \end{aligned}$$

With the initialization  $\tilde{R}_{0,1}^\circ(0) = 1$ .

*Proof.* By using proposition 5.17 that gives the recursion for  $G_{g,n}$  and also the formula:

$$G_{g,n}(L) = \sum_{\alpha} \tilde{R}_{g,n}^\circ(\alpha) \frac{L^\alpha}{\alpha!}.$$

We can derive the formula given in proposition 5.18 by extraction of the coefficient  $\alpha$ . □

**Remark 5.28** (Combinatorial proof). *We remark that proposition 5.18 can be obtained in a combinatorial way by using bijection in a way similar to usual Tutte formulas and techniques described in [Eyn16].*

**Laplace transform:** We consider the Laplace transform  $W_{g,n^+,n^-}$  of  $G_{g,n^+,n^-}$ .

$$W_{g,n^+,n^-}(x) = \int_L G_{g,n^+,n^-}(L) e^{-\sum_i x_i L_i} dL = \sum_{\alpha} \tilde{R}_{g,n^+,n^-}^{\circ} \prod_i \frac{1}{x_i^{\alpha_i+1}}.$$

We can also consider the formal Laplace transform of  $G_{g,n}$  denoted by  $W_{g,n}$  and defined by

$$W_{g,n} = \sum_{n^-} W_{g,n,n^-}.$$

The formal series  $W_{g,n}$  is well defined and admits the following expansion:

$$W_{g,n} = \sum_{\alpha} \frac{\tilde{R}_{g,n}^{\circ}(\alpha)}{x^{\alpha+1}}.$$

**Loop equation for Laplace transforms:** We translate proposition 5.17 in a recursion for  $W_{g,n}$  by applying the Laplace transform. This kind of recursion also arises in the theory of formal matrices integrals and is called loop equation in this context [EKR15],[Eyn16].

**Proposition 5.19.** *The Laplace transform satisfies the recursion:*

$$\begin{aligned} x_1 W_{g,n} &= \sum_{i \neq 1} \frac{\partial}{\partial x_i} \left( \frac{W_{g,n-1}(x_1, x_{\{1,i\}^c}) - W_{g,n-1}(x_i, x_{\{1,i\}^c})}{x_1 - x_i} \right) \\ &+ W_{g-1,n+1}(x_1, x_1, x_{\{1\}^c}) + \sum_{I_1, I_2, g_1+g_2=g} W_{g_1-1, n_1+1}(x_1, x_{I_1}) W_{g_2-1, n_2+1}(x_1, x_{I_2}) + \delta_{g,0} \delta_{n,1} \end{aligned}$$

*Proof.* To prove the result, it suffices to compute the Laplace transform of the equation in proposition 5.17. We can proceed as in the case of  $\mathcal{V}_{g,n^+,n^-}$  in chapter 4.5.2. The only difference is the LHS. We have

$$\int_0^{\infty} \partial_1 G_{g,n}(L) e^{-\sum_i x_i L_i} = x_1 W_{g,n}(x) + G_{g,n}(0, L_{\{i\}^c}) = x_1 W_{g,n}(x) + \delta_{g,0} \delta_{n,1}.$$

□

**Remark 5.29** (Unstable term). *It's better to extract two terms of the sum over  $(g, n)$ , the ones that contain  $W_{0,1}$ . We can write*

$$\begin{aligned} (x_1 - 2W_{0,1}(x_1)) W_{g,n} &= \sum_{i \neq 1} \frac{\partial}{\partial x_i} \left( \frac{W_{g,n-1}(x_1, x_{\{1,i\}^c}) - W_{g,n-1}(x_i, x_{\{1,i\}^c})}{x_1 - x_i} \right) \\ &+ W_{g-1,n+1}(x_1, x_1, x_{\{1\}^c}) + \sum'_{I_1, I_2, g_1+g_2=g} W_{g_1-1, n_1+1}(x_1, x_{I_1}) W_{g_2-1, n_2+1}(x_1, x_{I_2}) + \delta_{g,0} \delta_{n,1} \end{aligned}$$

*For  $(g, n) \neq (0, 1)$ , this time we do not include  $(0, 1)$  in the sum.*

## 5.5.2 Topological recursion

**The amplitude  $W_{0,1}$  of the disc:** The series  $W_{0,1}$  is associated to counting problems for oriented ribbon graphs with only one positive boundary and drawn on the sphere. The only nontrivial term of the sequence of coefficients  $(\tilde{R}_{0,1,n}^{\circ}(\alpha))_{\alpha}$  is  $\tilde{R}_{0,1,n}^{\circ}(2n-2)$  and corresponds

to graphs with  $n$  labeled negative faces.

The recursion directly gives the following closed equation for  $W_{0,1}$ , which is classical [Eyn16] in the theory of tree enumeration and in the theory of topological recursion for matrix integrals:

$$W_{0,1}^2 - xW_{0,1} + 1 = 0.$$

**Remark 5.30** (Trees). *This equation (and the series  $W_{0,2}$ ) is very close to the one that characterizes the generating series of Catalan numbers and is related to counting trees.*

Solving this equation leads to the formula for  $x > 2$

$$W_{0,1} = \frac{1}{2} \left( x \pm \sqrt{x^2 - 4} \right).$$

This defines a germ of a convergent series at  $\infty$ . To fix the solution, we can use the fact that the expansion of  $W_{0,1}$  for large  $x$  is in  $\mathbb{C}[[x^{-1}]]$  and then

$$W_{0,1} = \frac{1}{2} \left( x - \sqrt{x^2 - 4} \right).$$

Expanding  $W_{0,1}$  in the variable  $\frac{1}{x}$  leads to the formula

$$W_{0,1} = \sum_{n \geq 1} \frac{2^n (2n-3)!!}{2 n!} \frac{1}{x^{2n-1}} = \sum_{n \geq 1} \frac{1}{2(2n-1)} \binom{2n}{n} \frac{1}{x^{2n-1}},$$

which gives the formula for the coefficients

$$\tilde{R}_{0,1,n}^\circ(2n-2) = \frac{1}{2(2n-1)} \binom{2n}{n}.$$

**Remark 5.31** (Expression for  $G_{0,1}$ ). *It's straightforward to compute the inverse Laplace transform of  $W_{0,1}(x)$ . We obtain the following formula for  $G_{0,1}(L)$*

$$G_{0,1}(L) = \sum_{k \geq 1} \frac{L^{2k-2}}{k!(k-1)!} = \frac{I_1(2L)}{L}$$

where  $I_1$  is the modified Bessel function of the first kind.

**Zukowsky variables and the universal expression for the cylinder amplitude:** We give a classical derivation of the formula for  $W_{0,2}$ ; this involves a change of variables by using the Zukowsky map; our main references are [Eyn14b] or [Eyn16]. These techniques have been developed by B.Eynard to solve the Loop/Tute equation.

The germ of  $W_{0,1}$  that we studied in the last paragraph is convergent in the neighborhood of  $\infty$  in  $\mathbb{C}P_1$ , but due to the square root, it does not extend to a single-valued function on  $\mathbb{C}P_1$ . Nevertheless, we can consider the smallest cover of  $\mathbb{C}P_1$  on which the function is indeed single valued. In our case, there is two branch points at  $\pm 2$ , and then we might use a two-cover. The covering is explicitly given by the Zukowsky map, which has been introduced by Zukowsky for application to aeronautics.

$$x : \mathbb{C}P_1 \rightarrow \mathbb{C}P_1$$

It is given explicitly by

$$x(z) = \left( z + \frac{1}{z} \right).$$

The map  $x$  is ramified at the two points  $1, -1$ , and the Galois involution is the inversion  $\sigma(z) = \frac{1}{z}$ . By abuse of notation, we still denote  $W_{0,1}(z)$  the pullback of the germ  $W_{0,1}(x)$  at  $\infty$ . From the formula for  $W_{0,1}$ , we derive

$$W_{0,1}(z) = \frac{1}{z},$$

at the neighborhood of  $\infty$  and then on  $\mathbb{C}P_1$  by analytic continuation. Then  $W_{0,1}$  is meromorphic on  $\mathbb{C}P_1$ .

In a similar way, it's possible to define the series  $W_{g,n}(z) = W_{g,n}(x(z))$ , the composition by  $x(z)$  makes sense in a neighborhood of  $\infty$ .

What is surprising in the Zulkowsky coordinates is the fact that the formula for  $W_{0,2}(z_1, z_2)$  is universal (see [Eyn16] for a more general and sophisticated case).

**Proposition 5.20.** *The function  $W_{0,2}(z_1, z_2)x'(z_1)x'(z_2)$  admits a meromorphic extension to  $\mathbb{C}P_1^2$ . Moreover, it satisfies*

$$W_{0,2}(z_1, z_2)x'(z_1)x'(z_2) = \frac{1}{(z_1z_2 - 1)^2} = \frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}$$

*Proof.* A detailed proof is given in [Eyn16] in general, and it uses the fact that as a function of  $z_1$ , the quantity  $W_{0,2}(z_1, z_2)x'(z_1)x'(z_2)$  decreases like  $z_1^2$  for large  $z_1$  and has only one pole at  $1/z_2$  of order two, and the leading term is  $1/z_2^2$  with no residue. The starting point is the recursive formula for  $W_{0,2}(x_1, x_2)$ , by using proposition 5.19, we have

$$x_1W_{0,2}(x_1, x_2) = \frac{\partial}{\partial x_2} \frac{W_{0,1}(x_1) - W_{0,1}(x_2)}{x_1 - x_2} + 2W_{0,1}(x_1)W_{0,2}(x_1, x_2).$$

And then

$$W_{0,2}(x_1, x_2) = \frac{1}{x_1 - 2W_{0,1}} \frac{\partial}{\partial x_2} \frac{W_{0,1}(x_1) - W_{0,1}(x_2)}{x_1 - x_2}$$

Then, using the change of variables, we obtain

$$W_{0,2}(z_1, z_2)x'(z_1)x'(z_2) = \frac{x'(z_1)}{z_1 - \frac{1}{z_1}} \frac{\partial}{\partial z_2} \frac{\frac{1}{z_1} - \frac{1}{z_2}}{z_1 - z_2 + \frac{1}{z_1} - \frac{1}{z_2}} = \frac{1}{(z_1z_2 - 1)^2}.$$

On the other hand, we have

$$\frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2} = \frac{1}{(z_1z_2 - 1)^2}$$

which give the claim. □

**Differentials**  $\omega_{g,n}$  Walking in the steps of B.Eynard and N.Orantin, we define differentials  $\omega_{g,n}$  by

$$\omega_{g,n}^0 = W_{g,n}(z) \bigotimes_i x'(z_i) dz_i.$$

In the case when  $(g, n) \in \{(0, 1), (0, 2)\}$ , the formula is

$$\omega_{g,n} = \omega_{g,n}^0 - \delta_{(g,n),(0,1)} \frac{x(z_1)x'(z_1)dz_1}{2} + \delta_{(g,n),(0,2)} \frac{x'(z_1)x'(z_2)dz_1 \otimes dz_2}{(x(z_1) - x(z_2))^2}.$$

In particular, if we set

$$y(z) = \frac{1}{2} \left( z - \frac{1}{z} \right),$$

then we have the formula for  $\omega_{0,1}$

$$\omega_{0,1} = ydx.$$

For  $\omega_{0,2}$ , we already see the universal expression using the Bergman Kernel

$$\omega_{0,2} = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}$$

Moreover, we can see that we have the algebraic relation

$$4y^2 - x^2 = 4.$$

In the case  $(0, 2)$

$$\omega_{0,2} = W_{0,2}(z_1, z_2)dz_1 \otimes dz_2 + \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2} = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2},$$

is the Bergman Kernel, or the fundamental second-kind differential of  $\mathbb{C}P_1$ . Each of these differentials defines a germ of multi-differentials near  $\infty \in \mathbb{C}P_1$ .

**Topological recursion:** We derive the topological recursion formula for  $\omega_{g,n}$ . We denote  $\sigma_i^* \omega_{*,*}$  the pullback of  $\omega_{*,*}$  by  $\sigma_i$  acting in the  $i$ -th variable and  $d_i$  the differential of a function with respect to the  $i$ -th variable.

**Theorem 5.3.** *The  $\omega_{g,n}$  are given by the following recursion:*

$$\begin{aligned} \omega_{g,n} = \sum_{\epsilon=\pm 1} \oint_{z=\epsilon} \frac{dz_1}{2y(z)x'(z)dz} \left( \frac{1}{z - z_1} - \frac{1}{z^{-1} - z_1} \right) \cdot (\sigma_2^* \omega_{g-1,n+1}(z, z, z_{\{1\}^c}) \\ + \sum_{g_i, n_i, I_i} \omega_{g_1, n_1+1}(z, z_{I_1}) \otimes \sigma_1^* \omega_{g_2, n_2+1}(z, z_{I_2})). \end{aligned}$$

**Remark 5.32.** *The last formula is apparently complicated, but it's in reality simpler than 4.6. There is a magic phenomenon: the contribution of type 3 gluing fits exactly with the one of type 2 and disappears in the final formula. This is an explanation of why we change  $\omega_{0,2}^0$  to  $\omega_{0,2}$ .*

To prove this proposition, we need two lemma; they are classical's and given in [Eyn16].

**Lemma 5.15.** *The differentials  $\omega_{g,n}$  are anti-invariant under the Galois involution*

$$\sigma_i^* \omega_{g,n} = -\omega_{g,n}.$$

And in this lemma, we insist on the fact that we consider the differentials. The second lemma is about the singularities of the differentials.

**Lemma 5.16.** For  $(g, n)$  with  $2g - 2 + n > 0$ , the differential  $\omega_{g,n}$  has no poles on  $\mathbb{C}P_1 \setminus \{\pm 1\}$ .

Using these two lemmas, we give a proof of 5.3.

*Proof.* We start with the following classical formula of complex analysis:

$$\omega_{g,n} = \oint_{z=z_1} \frac{\omega_{g,n}(z, z_{\{1\}^c})}{(z - z_1)} \otimes dz_1$$

Then we use a trick that we can call the Eynard trick and rely on the Stokes theorem. The sum of the residues of a meromorphic differential is zero. Using lemma 5.16, the only poles of the integrand are at  $z_1, 1, -1$ , and then we have

$$\omega_{g,n} = - \sum_{\epsilon=\pm 1} \oint_{z=\epsilon} \frac{\omega_{g,n}(z, z_{\{1\}^c})}{(z - z_1)} \otimes dz_1$$

The second trick is the use of the involution  $\sigma$ , it fixes the integration contour, and by lemma 5.15 we have

$$\oint_{z=\epsilon} \frac{\omega_{g,n}(z, z_{\{1\}^c})}{(z - z_1)} = \oint_{z=\epsilon} \frac{\sigma_1^* \omega_{g,n}(z, z_{\{1\}^c})}{(z^{-1} - z_1)} = - \oint_{z=\epsilon} \frac{\omega_{g,n}(z, z_{\{1\}^c})}{(z^{-1} - z_1)}.$$

Then we have

$$\oint_{z=\epsilon} \frac{\omega_{g,n}(z, z_{\{1\}^c})}{(z - z_1)} = \frac{1}{2} \oint_{z=\epsilon} \omega_{g,n}(z, z_{\{1\}^c}) \left( \frac{1}{(z - z_1)} - \frac{1}{(z^{-1} - z_1)} \right).$$

Then we replace  $\omega_{g,n}$  by using the recursive formula for  $W_{g,n}$ . Let's start with  $\omega_{g,n}^0$  we can multiply the formula by  $\otimes_i x'(z_i) dz_i$  and write

$$\begin{aligned} 2y(z)\omega_{g,n}^0(z, z_{\{1\}^c}) &= \frac{\omega_{g-1,n+1}^0(z, z_{\{1\}^c})}{x'(z)dz} \\ &+ \sum_{g_i, n_i, I_i} \frac{\omega_{g_1, n_1+1}^0(z, z_{I_1}) \otimes \omega_{g_2, n_2+1}^0(z, z_{I_2})}{x'(z)dz} \\ &+ \sum_{i \neq 1} d_i \left( \frac{\omega_{g,n-1}^0(z, z_{\{1,i\}^c}) - \omega_{g,n-1}^0(z, z_{\{1,i\}^c}) \otimes (x'(z_i) dz_i)^{\otimes -1} \otimes x'(z) dz}{x(z) - x(z_i)} \right) \end{aligned}$$

To replace  $\omega_{g,n}^0$  by  $\omega_{g,n}$ , we need to take care of the case  $(0, 2)$ . But we can split the last term in the formula into two parts. The first is

$$\sum_{i \neq 1} d_i \frac{\omega_{g,n-1}^0(z, z_{\{1,i\}^c})}{x(z) - x(z_i)} = \sum_{i \neq 1} \omega_{g,n-1}^0(z, z_{\{1,i\}^c}) \otimes \frac{x'(z_i) dz_i}{(x(z) - x(z_i))^2}$$

On the other hand, the second line contains two terms with  $\omega_{0,2}^0$ . Combining these terms with the last one, we obtain

$$\sum_{i \neq 1} \omega_{g,n-1}^0(z, z_{\{1,i\}^c}) \otimes \frac{x'(z_i) dz_i}{(x(z) - x(z_i))^2} + 2 \sum_{i \neq 1} \frac{\omega_{g,n-1}^0(z, z_{\{1,i\}^c}) \otimes \omega_{0,2}^0(z, z_i)}{x'(z_1) dz_1} = 2 \sum_{i \neq 1} \frac{\omega_{g,n-1}^0(z, z_{\{1,i\}^c}) \otimes \omega_{0,2}(z, z_i)}{x'(z_1) dz_1}$$

We have almost the good form indeed we have

$$\begin{aligned}
2y(z)\omega_{g,n}(z, z_{\{1\}^c}) &= \frac{\omega_{g-1,n+1}(z, z, z_{\{1\}^c})}{x'(z)dz} \\
&+ \sum_{g_i, n_i, I_i} \frac{\omega_{g_1, n_1+1}(z, z_{I_1}) \otimes \omega_{g_2, n_2+1}^0(z, z_{I_2})}{x'(z)dz} \\
&+ \sum_{i \neq 1} d_i \left( \frac{-\omega_{g, n-1}^0(z, z_{\{1, i\}^c}) \otimes (x'(z_i)dz_i)^{\otimes -1} \otimes x'(z)dz}{x(z) - x(z_i)} \right)
\end{aligned}$$

To fit with the formula of Eynard Orantin, we apply the Galois involution by using the invariance we have

$$\frac{\omega_{g-1,n+1}(z, z, z_{\{1\}^c})}{x'(z)dz} = \frac{-\sigma_2^* \omega_{g-1,n+1}(z, z, z_{\{1\}^c})}{x'(z^{-1})d(z^{-1})}.$$

Similar considerations are also valid for the sum in the second line. Then the next and last step is the observation that the remaining term does not contribute to the contour integral because it has no poles at the branch points.

$$\oint_{\epsilon} \left( \frac{1}{(z - z_1)} - \frac{1}{(z^{-1} - z_1)} \right) \frac{x'(z)dz}{2y(z)} \sum_{i \neq 1} d_i \left( \frac{-\omega_{g, n-1}^0(z, z_{\{1, i\}^c}) \otimes (x'(z_i)dz_i)^{\otimes -1}}{x(z) - x(z_i)} \right) = 0.$$

We refer to [Eyn16] for a proof. □

The kernel in the computation of the residue is then

$$\mathbb{K}(z_1, z) = \frac{1}{2y(z)x'(z)dz} \left( \frac{1}{(z - z_1)} - \frac{1}{(z^{-1} - z_1)} \right).$$

By using

$$\omega_{0,1} - \sigma^* \omega_{0,1} = 2y(z)x'(z)dz$$

and

$$\int_{z'=\sigma(z)}^z \omega_{0,2}(z_1, z') = \frac{dz_1}{(z - z_1)} - \frac{dz_1}{(z^{-1} - z_1)}$$

We obtain the Eynard Orantin formula for the kernel

$$\mathbb{K}(z_1, z) = \frac{1}{\omega_{0,1} - \sigma^* \omega_{0,1}} \int_{z'=\sigma(z)}^z \omega_{0,2}(z_1, z').$$

**The spectral curve:** We can conclude this part on TR with the following theorem.

**Theorem 5.4.** *The recursion is equivalent to the Eynard-Orantin topological recursion with spectral curve given by the data's  $(x, y, \omega_{0,2})$ . In other words, the spectral curve of the recursion is given by the following equation:*

$$4y^2 = x^2 - 4$$

In this theorem, the curve is defined by the equation:

$$4y^2 = x^2 - 4$$

is called the spectral curve of the recursion. Nevertheless, the spectral curve is not unique... Indeed, by looking at the expression of the Eynard Orantin kernel, the only term that contains  $y$  is

$$\omega_{0,1} - \sigma^* \omega_{0,1} = (y - y \circ \sigma) dx = 2y^- dx.$$

Where  $y^-$  is the anti-invariant part of  $y$  under the involution. Then adding an invariant function does not change the differential  $\omega_{g,n}$  for  $(g, n) \neq (0, 1)$ . An invariant function is necessarily the pullback of a meromorphic function by  $x$ . Then changing the  $y$  by a term of the form  $P(x)$  does not affect the recursion. There is a canonical choice given by taking an anti-invariant  $y$ , and in our case, it corresponds to

$$y = z - \frac{1}{z}.$$

In the case of Grotendieck dessins d'enfants, the spectral curve is

$$xy = y^2 + 1.$$

And correspond to the choice

$$y = W_{0,1}(z) = \frac{1}{z}.$$

But the two recursions are the same.

### 5.5.3 Virasoro constraints

In this part, we derive Virasoro constraints associated to the recursion 5.19. Indeed, these constraints are called Virasoro constraints, but we only consider the upper part of the Lie algebra, and then we shall call them "Witt constraints". These constraints are satisfied by the potential  $Z$ . Let  $L_i$  be the differential operators given by

$$L_i = -\partial_{i+2} + \sum_j (j+1)t_{j+1}\partial_{i+j+1} + \sum_{k+l=i} \partial_k \partial_l \quad \forall i \geq -1.$$

**Lemma 5.17.** *These operators satisfy*

$$[L_i, L_j] = (i-j)L_{i+j}.$$

*Proof.* It's a direct computation; we just need to expand the lie bracket. □

Then the operators  $L_i$  form a representation of the well-known Witt algebra

$$\text{Witt} = \mathbb{C}[z] \frac{d}{dz} \quad L_i = z^{i+1} \partial_z.$$

The operator  $L_{-1} = -\partial_1 + \sum_j (j+1)t_{j+1}\partial_j$  corresponds to the string equation.

**Proposition 5.21.** *The generating series  $Z$  satisfies the "Virasoro" constraints,*

$$L_i(Z) = 0 \quad \forall i \geq -1.$$

To prove this proposition, we start with a reformulation of proposition 5.18. The non-connected coefficients  $\tilde{R}_n^{\circ,d}(\alpha)$  are symmetric and can then be denoted  $\tilde{R}^\circ(\mu)$  for  $\mu$  a partition. We can drop indices  $d, n$  as we have  $d = d(\mu)$  and  $n = n(\mu)$ . Then, by using proposition 5.18, we can obtain the following lemma:

**Lemma 5.18.** *The coefficients  $\tilde{R}_n^{\circ,d}(\alpha)$  satisfy the recursion*

$$\begin{aligned} i\mu(i) \frac{\tilde{R}^\circ(\mu)}{\mu!} &= \sum_j ij(\mu(i+j-2)+1) \frac{\tilde{R}^\circ(\mu-\delta_i-\delta_j+\delta_{i+j-2})}{(\mu-\delta_i-\delta_j+\delta_{i+j-2})!} \\ &+ \sum_{k+l=i-2} i(\mu(k)+1)(\mu(l)+1) \frac{\tilde{R}^\circ(\mu-\delta_i+\delta_k+\delta_l)}{(\mu-\delta_i+\delta_k+\delta_l)!}. \end{aligned}$$

*Proof.* Proposition 5.18 admits a generalization for non-connected graphs, which is

$$\begin{aligned} \tilde{R}_n^{\circ,d}(\alpha) &= \sum_{i \neq 1} \alpha_i \tilde{R}_{n-1}^{\circ,d+1}(\alpha_i + \alpha_1 - 2, \alpha_{\{1,i\}^c}) \\ &+ \sum_{k+l=\alpha_1-2} \tilde{R}_{n+1}^{\circ,d-1}(k, l, \alpha_{\{1\}^c}) \end{aligned}$$

Let  $\mu$  and  $\alpha$  with  $\mu(\alpha) = \mu$ . For a fixed  $i$ , we can apply the last formula for all the boundary  $j$  with  $\alpha_j = i$ . It leads to

$$\begin{aligned} \mu(i) \tilde{R}^\circ(\mu) &= \sum_j j\mu(i)\mu(j) \tilde{R}^\circ(\mu-\delta_i-\delta_j+\delta_{i+j-2}) \\ &+ \sum_{k+l=i-2} \tilde{R}^\circ(\mu-\delta_i+\delta_k+\delta_l) \end{aligned}$$

Then dividing by  $\mu! = \prod_j \mu(j)!$ , we obtain the claim. □

Then we prove proposition 5.21.

*Proof.* We start with the formula

$$Z = \sum_{\mu} \frac{\tilde{R}_n^{\circ,d}(\mu)}{\mu!} \mathbf{t}^{\mu}$$

then we plug the recursion of lemma 5.18 in this generating series, and this leads to the desired equation for  $Z$ ,

$$it_i \partial_i Z = \sum_j ijt_j \partial_{i+j-2} Z + \sum_{k+l=i-2} it_i \partial_k \partial_l Z.$$

□

## 5.6 Higher orders operators

### 5.6.1 Higher operators

In this section, we generalize results of the last sections to the case of graphs with higher-order vertices. These works are actually in progress.

**Definition of the ancestor potential:** In general, we define operators  $K_{g,n^+,n^-}^\nu$  for all  $\nu \in \mathcal{P}(\mathbb{N})$  with  $d(\nu) = d_{g,n^+,n^-} + n(\nu)$ , by using the integration kernel:

$$K_{g,n^+,n^-}^\nu(L^+|L^-) = \prod_i L_i^+ V_{g,n^+,n^-}^\nu(L^+|L^-).$$

Where  $V_{g,n^+,n^-}^\nu$  are the volumes associated to oriented ribbon graphs with vertices of degree prescribed by  $\nu$  (defined in 4.1.4). We recover the case of quadrivalent oriented ribbon graphs and the case of marked graphs with bivalent vertices by the relation:

$$K_{g,n^+,n^-}^\nu = K_{g,n^+,n^-}^{(1^{2g-2+n^++n^-})} \quad K_{g,n^+,n^-,m} = K_{g,n^+,n^-}^{(0^m, 1^{2g-2+n^++n^-})}.$$

By using the lemma 5.7 and results of section 5.2, we have the following:

**Lemma 5.19.** *For each  $(g, n^+, n^-, \nu)$ , the function  $K_{g,n^+,n^-}^\nu$  defines a linear operator:*

$$K_{g,n^+,n^-}^\nu : \hat{S}(V) \longrightarrow \hat{S}(V),$$

which is homogeneous of degree  $d_{g,n^+,n^-} + n(\nu)^3$ .

We can generalize these operators to the case of non-connected decorated surface

$$\overline{M}^\circ = (M^\circ, \nu) = \sqcup_c (M^\circ(c), \nu_c).$$

In this case, the operator  $K_{\overline{M}^\circ}$  is also homogeneous of degree  $d(M^\circ) + n(\nu)$ . We can also consider  $\overline{\mathcal{M}}$  the space generated by decorated surfaces,  $\overline{\mathcal{A}}$  the algebra of acyclic decorated stable graphs, and  $S(\overline{\mathcal{M}}), S(\overline{\mathcal{A}})$  their symmetrisation. There are natural injective morphisms of algebras:

$$S(\mathcal{A}) \rightarrow S(\mathcal{A}^\bullet) \rightarrow S(\overline{\mathcal{A}}).$$

As before,  $\mathbf{K}$  extends naturally to an operator

$$\mathbf{K} : \hat{S}(\overline{\mathcal{A}}) \longrightarrow \hat{S}(V).$$

And we give the following natural generalization of proposition 5.9.

**Corollary 5.13.**  *$\overline{K}$  is a morphism for both  $\sqcup$  and  $\cdot$ , and the restriction of  $\overline{K}$  to  $\hat{S}_\infty(\overline{\mathcal{A}})$  defines a morphism*

$$\overline{\mathbf{K}} : \hat{S}_\infty(\overline{\mathcal{A}}) \longrightarrow \hat{\mathcal{D}}(V)$$

In other words, for each element  $x \in \hat{S}_\infty(\overline{\mathcal{A}})$ , the operator  $\overline{\mathbf{K}}(x)$  is a formal differential operator. It's then natural to consider series in  $\hat{\mathcal{D}}(V)[[\mathbf{q}]]$  with  $\mathbf{q} = (q_0, q_1, \dots)$ ,

$$\overline{K}(q) = \sum_{\overline{M}^\circ} \mathbf{q}^\nu K_{\overline{M}^\circ}^\nu.$$

But according to the fact that the operators are graded, we can evaluate variables  $\mathbf{q}$ . In particular, we have

$$\overline{K}(0, q_1, 0, 0, \dots) = K(q_1) \quad \overline{K}(q_0, q_1, 0, \dots) = K^\bullet(q_0, q_1).$$

In general, we denote

$$\overline{K}(0, \dots, q_i, 0, 0, \dots) = \overline{K}_i(q_i).$$

---

<sup>3</sup>According to results of chapter 7.4.2, the function  $K_{g,n^+,n^-}^\nu(L^+|L^-)$  is not continuous on  $\Lambda_{n^+,n^-}$ , but these discontinuities fall into sets of zeros measure when we perform the integral, and then  $K_{g,n^+,n^-}^\nu$  and it's continuous extension defines the same operator.

**Partition function:** We give the following generalization for  $G_{g,n^+,n^-}^\nu$ ; similarly, we define  $R_{g,n^+,n^-}^{\circ,\nu}(\alpha)$  as the count of oriented ribbon graphs with:

- Unlabelled vertices prescribed by  $\nu$ .
- Labeled positive boundaries of perimeter given by  $\alpha$ .
- Arbitrary unlabelled negative boundaries.

**Corollary 5.14.** *The polynomial  $G_{g,n^+,n^-}^\nu$  is a generating function for the numbers  $R_{g,n^+,n^-}^{\circ,\nu}(\alpha)$ , we have*

$$G_{g,n^+,n^-}^\nu(L) = \sum_{\alpha} R_{g,n^+,n^-}^{\circ,\nu}(\alpha) \prod \frac{L_i^{\alpha_i}}{(\alpha_i - 1)!}.$$

By lemma 4.37  $G_{g,n^+,n^-}^\nu$  is also a generating series for the count of dessins d'enfants. We have

$$R_{g,n^+,n^-}^{\circ,\nu}(\alpha) = h_{g,n^-}^\nu(\alpha).$$

Where  $h_{g,n^-}^\nu(\alpha)$  is the Hurwitz number of dessins d'enfants with a profile of ramification given by  $\nu$  over  $x_0$  and  $\alpha$  over  $x_+$ . The partition function is defined by the relation

$$\bar{Z}(\mathbf{q}) = \bar{K}(\mathbf{q}) \cdot \mathbf{e}_\emptyset.$$

## 5.6.2 Recursion and graphical expansion

**Higher order recursions for volumes:** In general, by using theorem 4.9 in chapter 4.5.2, we can obtain the following corollary:

**Corollary 5.15.** *The kernels  $K_{M^\circ}^{\nu+(i)}$  satisfy the following recursion:*

$$(\nu(i)+1) K_{M^\circ}^{\nu+(i)}(L^+|L^-) = \sum_{\bar{\mathcal{G}}^\circ \in Bi_i(\bar{M}^\circ)} \int_{\Lambda_{\mathcal{G}^\circ}(L^+|L^-)} K_{\bar{\mathcal{G}}^\circ(1)}(L^+(1)|L^-(1), x) K_{\bar{\mathcal{G}}^\circ(0)}(L^+(0), x|L^-(0)) d\sigma_{\bar{\mathcal{G}}^\circ}^L.$$

As in theorem 4.9 the sum is over all the set  $Bi_i(\bar{M}^\circ)$  of decorated acyclic stable graphs such as

- There is a height function  $h : X_0\mathcal{G}^\circ \rightarrow \{0, 1\}$
- There is only one component at the level 1 denoted  $\bar{\mathcal{G}}^\circ(1)$  and its decoration is  $(i)$ .
- $\bar{\mathcal{G}}^\circ(0)$  denotes the union of components at the level zero.

**Decomposition in acyclic graphs:** We denote  $\bar{\mathcal{M}}^{min} \subset \bar{\mathcal{M}}$  the vector space generated by surfaces  $\bar{M}^\circ = (M^\circ, \nu)$  with  $n(\nu) = 1$ ; it's naturally isomorphic to  $\mathcal{M}$ . We also denote  $\bar{\mathcal{A}}^{min}$  as the algebra generated by  $\bar{\mathcal{M}}^{min}$ , which corresponds to acyclic graphs with only minimal components, and  $\bar{\mathcal{A}}^{min}$  is isomorphic to  $\mathcal{A}$ . The following theorem is a result of iteration of theorem 4.2 and is proved in theorem 4.3.

**Theorem 5.5.** *The function  $K_{\overline{M}^\circ}$  satisfies the identity*

$$K_{\overline{M}^\circ} = \sum_{\overline{\mathcal{G}}^\circ \in \overline{\mathbf{acycl}}^*(\overline{M}^\circ)} \frac{n_{\mathcal{G}^\circ} K_{\overline{\mathcal{G}}^\circ}(L^+|L^-)}{n(\nu)! \# \text{Aut}(\overline{\mathcal{G}}^\circ)}.$$

With

$$K_{\overline{\mathcal{G}}^\circ}(L^+|L^-) = \int_{\Lambda_{\mathcal{G}^\circ}(L^+|L^-)} \prod_c K_{\overline{\mathcal{G}}^\circ(c)}(L^+(c)|L^-(c)) d\sigma_{\mathcal{G}^\circ}(L^+|L^-).$$

And the sum is over acyclic decompositions in minimal pieces, i.e., all the maximal acyclic stable graphs.

The functions  $K_{\overline{\mathcal{G}}^\circ}$  define linear operators, and the theorem can be rewritten in the form:

$$\overline{K} = \sum_{\overline{\mathcal{G}}^\circ \in \overline{\mathbf{acycl}}^*} \frac{n_{\mathcal{G}^\circ} K_{\overline{\mathcal{G}}^\circ}}{n(\nu)! \# \text{Aut}(\overline{\mathcal{G}}^\circ)}.$$

We have the following corollary of proposition 5.6:

**Corollary 5.16.** *The operator  $\overline{K}$  satisfies*

$$\overline{K} = \exp\left(\sum_{\overline{M}^\circ \in \overline{\mathbf{bord}}_{min}^\circ} q^\nu K_{\overline{M}^\circ} \sqcup id\right),$$

where the sum of the RHS is over minimal decorated surfaces.

### 5.6.3 Higher orders Cut-and-Joins:

We generalize the statement of theorem 5.1; our approach gives results similar to the ones of [WLZZ22] and [MA23]. We define the higher Cut-and-Join generators  $\mathcal{W}_i$  by the formula

$$\mathcal{W}_i^s = \sum_{g, n^+, n^-} K_{g, n^+, n^-}^{(i)}, \quad \text{and} \quad \mathcal{W}_i = \mathcal{W}_i^s \sqcup id.$$

We sum over the  $(g, n^+, n^-)$  with

$$d_{g, n^+, n^-} = 2g - 2 + n^+ + n^- = i,$$

which is the set of minimal surfaces of Euler characteristic  $i$ , and it corresponds to ribbon graphs with only one vertex of degree  $2i + 2$ . In the case  $i = 0, 1$ , we recover the expression

$$\mathcal{W}_0 = E \quad \mathcal{W}_1 = P.$$

The statement of theorem 4.9 can be rewritten in the following form, which can be called a higher Cut-and-Join equation.

**Theorem 5.6.** *The operator  $\overline{K}(\mathbf{q})$  satisfies the equations*

$$\frac{\partial \overline{K}}{\partial q_i} = \mathcal{W}_i \overline{K}(\mathbf{q}), \quad \forall i \geq 0.$$

**Remark 5.33.** We have the following relations

$$\overline{K}_i(q_i) = q_i^{\frac{D}{i+1}} \overline{K}_i q_i^{-\frac{D}{i+1}} \quad \text{and} \quad \frac{\partial \overline{K}_i}{\partial q_i} = \frac{[D, \overline{K}_i]}{i+1}.$$

The Cut-and-Join equation can be formulated independently of  $q_i$  by,

$$[D, \overline{K}_i] = (i+1)\mathcal{W}_i \overline{K}_i.$$

From the proposition 5.16, we already have

$$\overline{K}(\mathbf{q}) = \exp\left(\sum_i q_i \mathcal{W}_i\right).$$

But to obtain the next theorem, we need the commutativity of operators, which is another statement; we show later that it's a consequence of 4.9. We can also obtain the following specialization when only one variable is involved:

$$\frac{\partial \overline{K}_i}{\partial q_i} = \mathcal{W}_i \overline{K}_i(q_i), \quad \forall i \geq 0.$$

And then also

$$\overline{K}_i(q_i) = \exp(q_i \mathcal{W}_i).$$

*Proof.* This is a consequence of theorem 4.9. We can write

$$\overline{K}(\mathbf{q}) = \sum_{\nu} \mathbf{q}^{\nu} K^{\nu}.$$

Then the statement of theorem 4.9 is equivalent to

$$\nu(i) K^{\nu} = \delta_{\nu(i)>0} \mathcal{W}_i K^{\nu-(i)}.$$

To obtain this result, we use arguments similar to remark 5.19. We can rewrite the formula:

$$K^{\nu} = \sum_{2g-2+n^++n^- = i} K_{g,n^+,n^-}^{(i)} * K^{\nu-(i)}.$$

Indeed, the equation 4.9 corresponds to all the ways to remove a surface of type  $(g, n^+, n^-, (i))$  with  $2g - 2 + n^+ + n^- = i$ . This is also what is computed by the RHS of the last equation.  $\square$

**Case of vertices of degree 6:** We make the last formula explicit; in the case of  $\overline{K}_2$ , we have the following:

**Theorem 5.7.** The operator  $\mathcal{W}_2$  is given by the formula

$$\begin{aligned} \mathcal{W}_2 = & \frac{1}{6} \sum_i (i+1)i(i-1)t_{i+1}\partial_{i-2} + \frac{1}{2} \sum_{i+j=k+l+1} (i+1)(j+1)t_{i+1}t_{j+1}\partial_k\partial_l \\ & + \frac{1}{3} \sum_{i,j,k} (i+1)(j+1)(k+1)t_{i+1}t_{j+1}t_{k+1}\partial_{i+j+k} + \frac{1}{3} \sum_{i,j,k} (i+j+k+3)t_{i+j+k+3}\partial_i\partial_j\partial_k. \end{aligned}$$

We don't give a detailed proof but we give the expression of the  $K_{g,n^+,n^-}^{(2)}$  that is used to prove this proposition. After that, we can compute the action similarly to the case of  $P^+, P^-$  in 5.7.

**Lemma 5.20.** *In genus 0, the volumes  $K_{0,3,1}^{(2)}, K_{0,2,2}^{(2)}, K_{0,1,3}^{(2)}$  are constant and equal to 2. The only non-constant volumes  $K_{1,1,1}^{(2)}$  is given by*

$$K_{1,1,1}^{(2)}(l|l) = \frac{l^3}{6}.$$

*Proof.* This lemma is straightforward by counting all the possible oriented graphs with one vertex and with a given topology. The first point comes from the following more general lemma:

**Lemma 5.21.** *If  $i = n^+ + n^- - 2$ , then  $K_{0,n^+,n^-}^{(i)}$  is constant and equal to  $i!$ .*

It can be proven inductively. For the second, we can see that the only ribbon graph  $R^\circ$  that contributes to the volume. It has a group of automorphisms given by  $\mathbb{Z}/3\mathbb{Z}$ , and  $\text{Met}(R^\circ, l, l)$  is the simplex  $l \cdot \Delta_3$  and has a volume equal to  $\frac{1}{2}$ .  $\square$

**Remark 5.34.** *We are lying a bit; indeed,  $K_{0,n^+,n^-}^{(i)}$  are only constant almost everywhere, but these discontinuities are not relevant in the integrals that we take.*

## 5.6.4 Commutativity of $\mathcal{W}$ operators :

We see that the operator  $E$  commutes with  $K$ , and so with  $P$ , this can be rewritten as

$$[\mathcal{W}_0, \mathcal{W}_1] = 0.$$

This statement remains true in general, and it's the content of the following proposition:

**Theorem 5.8.** *The operators  $(\mathcal{W}_i)_i$  are pairwise commuting:*

$$[\mathcal{W}_i, \mathcal{W}_j] = 0, \quad \forall i, j \geq 0.$$

And then the  $\mathcal{W}_i$  spans a commutative subalgebra of  $\hat{\mathcal{D}}(V)$ , which is generated by the  $K^\nu$ . But it's not true that  $K_{\overline{M}^\circ}$  belong to this subalgebra; they are not commuting.

*Proof.* The fact that the operators are commuting comes from theorem 4.9. Let  $i \neq j$ , then the product  $\mathcal{W}_i \mathcal{W}_j$  is indeed equal to the operators  $K^{\delta_i + \delta_j} \sqcup id$  associated with oriented ribbon graphs with only two vertices of degree  $i, j$ . According to theorem 4.9, the order in which we apply the recursion is not important; we can choose to remove the vertices of degree  $j$  first and  $i$  in second, then we also obtain

$$\mathcal{W}_i \mathcal{W}_j = K^{\delta_i + \delta_j} \sqcup id = \mathcal{W}_j \mathcal{W}_i$$

and give the claim.  $\square$

We can consider

$$\bar{K}_i(q_i) = \bar{K}(0, \dots, 0, q_i, 0, \dots).$$

According to theorem 5.6, we have the formula

$$\bar{K}_i(q_i) = \exp(q_i \mathcal{W}_i)$$

and then the last formula gives the factorization:

$$\bar{K}_i(q_0, q_1, \dots, q_r, 0, \dots) = \prod_{i=0}^r \exp(q_i \mathcal{W}_i).$$

It's possible to take the limit as  $r$  tends to infinity for any sequence  $(q_i)$ , and then we can write formally

$$\bar{K} = \prod_i \bar{K}_i.$$

Which is very curious, vertices of different orders are in some sense independent.

### 5.6.5 Conjecture on super-integrability

The last formalism is very similar to the one developed in [WLZZ22] by using integral matrices model. It's seems that our results are directly related to their work, and using their approach, we can state the following operator:

$$\mathcal{W}_{-1} = \sum_{k,l} (k+1)(l+1)t_{k+1}t_{l+1}\partial_{k+l+1} + \sum_{k,l} (l+k+1)t_{k+l+1}\partial_k\partial_l.$$

Then, by direct computation, we can check the following:

**Proposition 5.22.** *The operators satisfy the following identities:*

$$\mathcal{W}_1 = \frac{1}{2}[\mathcal{W}_{-1}, \mathcal{W}_0], \quad \text{and} \quad \mathcal{W}_2 = \frac{1}{3}[\mathcal{W}_{-1}, \mathcal{W}_1].$$

In general, we have the following guesses:

**Conjecture 5.1.** *The operators  $\mathcal{W}_i$  are recursively generated by the equation*

$$\mathcal{W}_{i+1} = \frac{1}{i+2}[\mathcal{W}_{-1}, \mathcal{W}_i] \quad \forall i \geq 0$$

*and the initialization  $\mathcal{W}_0 = \sum_i (i+1)t_{i+1}\partial_i$ .*

We make several remarks:

- From this proposition, all the  $\mathcal{W}_i$  and then also  $\bar{K}$  belong to the algebra generated by  $\mathcal{W}_0, \mathcal{W}_{-1}$ , which is rather surprising. The operators  $\mathcal{W}_i$  then measure the lack of commutativity between  $[\mathcal{W}_{-1}, \mathcal{W}_0]$ .
- It should be interesting to have a proof of this result that uses combinatorix of graphs, which shall be obtained by a recursion for minimal graphs and could be a consequence of [Yak22].
- For now, we do not know an interpretation of the operator  $\mathcal{W}_{-1}$ .

**Relation to Lax pairs:** We remark that commutation relations can be encoded in the following more compact way: Let  $\mathcal{W}(q)$  be the operator

$$\mathcal{W}(q) = \sum_i (i+1) \mathcal{W}_i q^i.$$

It converges for all  $q$ . Then we have the following:

**Lemma 5.22.** *The commutation relations of conjecture 5.1 are equivalent to the following Lax equation:*

$$\frac{\partial \mathcal{W}(q)}{\partial q} = [\mathcal{W}_{-1}, \mathcal{W}(q)],$$

which determines  $\mathcal{W}(q)$  with the initial data

$$\mathcal{W}(0) = \mathcal{W}_0.$$

**The trivial part:** We can then reduce the theory to genus zeros in the following way: We consider the operators

$$\overline{K}_i^- = \exp_{\square} \left( \sum_r K_{0,1,1+ri}^{\tau \delta_i} \right)$$

which corresponds to surfaces of genus zeros, such as each connected component has one positive boundary. And similarly,  $\overline{K}_i^+$ . According to the structure of  $\mathcal{W}_{-1}$ , we can write

$$\mathcal{W}_{-1} = \mathcal{W}_{-1}^+ + \mathcal{W}_{-1}^-,$$

and also defines

$$\mathcal{W}_i^- = \mathcal{W}_{0,1,i+1} \quad \text{and} \quad \mathcal{W}_i^+ = \mathcal{W}_{0,i+1,1}.$$

Then we have the following lemma :

**Lemma 5.23.** *The operator  $\mathcal{W}_i^-$  is given by*

$$\begin{aligned} \mathcal{W}_i^- &= \frac{1}{i} \sum_{k_1, \dots, k_i} (k_1 + \dots + k_i + i) t_{k_1 + \dots + k_i + i} \partial_{k_1} \dots \partial_{k_i} \\ \mathcal{W}_i^+ &= \frac{1}{i} \sum_{k_1, \dots, k_i} (k_1 + 1) \dots (k_i + 1) t_{k_1 + 1} \dots t_{k_i + 1} \partial_{k_1 + \dots + k_i}. \end{aligned}$$

And then we have the following:

**Proposition 5.23.** *We have the relation*

$$\frac{\partial \overline{K}_i^{\pm}}{\partial q_i} = \mathcal{W}_i^{\pm} \overline{K}_i^{\pm}.$$

Moreover, we have

$$\mathcal{W}_{i+1}^{\pm} = \frac{[\mathcal{W}_{-1}^{\pm}, \mathcal{W}_i^{\pm}]}{i+2}$$

### 5.6.6 Higher Virasoro for the vacuum amplitude

As we see in the case of quadrivalent graphs, the series  $\phi$  satisfies an additional equation called the Virasoro constraints. To obtain these constraints, we use theorem 4.3 to obtain a more general version of lemma 5.14. We generalize this approach to case of graphs with vertices of higher degree. We denote

$$\bar{Z}_i = \bar{K}_i \cdot \mathbf{e}_\emptyset.$$

The main purpose of this part is to convince the reader of the following conjecture. There is a family of differential operators  $(L_{k,0})$  such as

$$D - \mathcal{W}_i = \sum (k+1)t_{k+1}L_{k,i}.$$

And with the convention  $L_{k,-1} = -\partial_0 + 1$ . For instance, we have for  $k \geq 0$

$$L_{k,0} = -\partial_{k+1} + \partial_k, \quad \text{and} \quad L_{k,1} = L_k.$$

Our guess is the following: for all  $i$  and all  $k \geq -1$ , we have the formula

$$L_{k,i}\bar{Z}_i = 0.$$

We remark on the following things:

- We do not know the relations satisfied by these operators.
- According to the proof of lemma 5.14, the chance that we have

$$L_{k,i}\bar{Z} = 0$$

are very low. But we can do the following: we can write

$$L_{i,k} = \partial_{k+1} - L'_{i,k}$$

and consider the operator

$$L_k(\mathbf{q}) = \partial_{k+1} - \sum_i q_i L'_{i,k}$$

we can ask if

$$L_k(\mathbf{q}) = 0, \quad \forall k \geq 0.$$

Indeed, by generalizing our results, we can obtain the results for all  $\mathbf{q}$  of the form  $\mathbf{q} = (q_0, q_1, q_2, 0, \dots)$ .

- As before, these relations concern only the series  $\bar{Z}$  and are not satisfied by  $\mathbf{K}$ .

**Higher tute equations:** We give the following generalization of results in part 5.5. The proof follows the same lines, but the situation is complicated by the fact that there are more minimal graphs to consider. The proof for  $i = 3$  is also doable by hand, but the number of graphs increases quickly, and this direction is challenging to pursue. We then obtain, using by direct integration, the following formula for the vacuum expectation:  $G_{g,n^+,n^-}^\nu$

**Proposition 5.24.** *The polynomials  $G_n^{r\delta_2}(L)$  are given by the following recursion:*

$$\begin{aligned}\partial_1 G_n^{r\delta_2}(L) &= \sum_{i \neq j \neq 1} L_i L_j G_{n-2}^{(r-1)\delta_2}(L_1 + L_i + L_j, L_{\{1,i,j\}^c}) \\ &\quad + \frac{3}{2} \sum_{i \neq 1} L_j \int_0^{L_1+L_j} G_n^{(r-1)\delta_2}(x, L_1 + L_j - x, L_{\{1,i\}^c}) dx \\ &\quad + \int_{x_1+x_2 \leq L_1} G_{n+2}^{(r-1)\delta_2}(x_1, x_2, L_1 - x_1 - x_2, L_{\{1\}^c}) dx_1 dx_2 \\ &\quad + \frac{L_1^2}{2} G_n^{(r-1)\delta_2}(L)\end{aligned}$$

As a corollary, we give the recursive integral formula for the connected function  $G_{g,n}^{(2*)}$ . But as we can see in the formula, the number of topological gluing is growing pretty fast.

**Proposition 5.25.** *The series  $G_{g,n}^{(2*)}(L)$  is given by the following recursion:*

$$\begin{aligned}\partial_1 G_{g,n}^{(2*)}(L) &= \sum_{i \neq j \neq 1} L_i L_j G_{g,n-2}^{(2*)}(L_1 + L_i + L_j, L_{\{1,i,j\}^c}) \\ &\quad + \frac{3}{2} \sum_{i \neq 1} L_j \int_0^{L_1+L_j} G_{g-1,n}^{(2*)}(x, L_1 + L_j - x, L_{\{1,i\}^c}) \\ &\quad + \sum_{g_i, n_i, L_i} G_{g_1, n_1+1}^{(2*)}(x, L_{I_1}) G_{g_2, n_2+1}^{(2*)}(L_1 + L_j - x, L_{I_2}) dx \\ &\quad + \int_{x_1+x_2 \leq L_1} G_{n+2}^{(2*)}(x, L_1 - x, L_1 - x_1 - x_2, L_{\{1\}^c}) \\ &\quad + \sum_{g_i, n_i, L_i} G_{g_1, n_1+1}^{(2*)}(x_1, x_2, L_{I_1}) G_{g_2, n_2+1}^{(2*)}(L_1 - x_1 - x_2, L_{I_2}) \\ &\quad + \sum_{g_i, n_i, L_i} G_{g_1, n_1+1}^{(2*)}(x_1, L_{I_1}) G_{g_2, n_2+1}^{(2*)}(x_2, L_{I_2}) G_{g_3, n_3+1}^{(2*)}(L_1 - x_1 - x_2, L_{I_3}) dx_1 dx_2 \\ &\quad + \frac{L_1^2}{2} G_{g-1+n}^{(2*)}(L).\end{aligned}$$

**Higher Virasoro:** From the last relation, we can obtain the following formula:

**Theorem 5.9.** *Let for  $j \geq 0$*

$$\begin{aligned}L_{j,2} &= -\partial_{j+1} + \frac{j(j-1)}{2} \partial_{j-2} + \frac{3}{2} \sum_{j+k=l+m+1} (k+1) t_{k+1} \partial_l \partial_m + \sum_{k,l} (k+1)(l+1) t_{k+1} t_{l+1} \partial_{k+j+l} \\ &\quad + \sum_{k+m+l=j-2} \partial_k \partial_m \partial_l\end{aligned}$$

*then we have for all  $k \geq 0$*

$$L_{k,2} \overline{Z}^2 = 0$$

But we do not know anything about the lie algebra generated by these operators.

*Proof.* Indeed, this equation is just a consequence of the last proposition and writing the recursion in terms of generating series.  $\square$

**Higher topological recursion:** In this part, we give a formulation of the recursion of proposition 5.24 in terms of the fraction  $W_{g,n^+,n^-}^{(2^*)}$ . This expression can be obtained by direct application of the Laplace transform.

**Theorem 5.10.** *The series  $(W_{g,n}^2)$  is given by the recursion*

$$\begin{aligned}
x_1 W_{g,n}^2 &= \sum_{i \neq j \neq 1} \partial_i \partial_j \left( \frac{W_{g,n-2}^2(x_1, x_{\{1,i,j\}^c})}{(x_i - x_1)(x_j - x_1)} + \frac{W_{g,n-2}^2(x_i, x_{\{1,i,j\}^c})}{(x_1 - x_i)(x_j - x_i)} + \frac{W_{g,n-2}^2(x_j, x_{\{1,i,j\}^c})}{(x_i - x_j)(x_1 - x_j)} \right) \\
&+ \frac{3}{2} \sum_{i \neq 1} \partial_i \left( \frac{W_{g-1,n+2}^2(x_1, x_1, x_{\{1,i\}^c}) - W_{g-1,n+2}^2(x_i, x_i, x_{\{1,i\}^c})}{x_1 - x_i} \right) \\
&+ \sum_{g_1+g_2=g,i} \frac{W_{g_1,n_1+1}^2(x_1, x_{I_1}) W_{g_2,n_2+1}^2(x_1, x_{I_2}) - W_{g_1,n_1+1}^2(x_i, x_{I_1}) W_{g_2,n_2+1}^2(x_i, x_{I_2})}{x_1 - x_i} \\
&+ \sum_{g_1+g_2=g-1} W_{g_1,n_1+2}^2(x_1, x_1, x_{I_1}) W_{g_2,n_2+1}^2(x_1, x_{I_2}) \\
&+ \sum_{g_1+g_2+g_3=g} W_{g_1,n_1+1}^2(x_1, x_{I_1}) W_{g_2,n_2+1}^2(x_2, x_{I_2}) W_{g_3,n_3+1}^2(x_3, x_{I_3}) \\
&+ 2W_{g-2,n+2}^2(x_1, x_1, x_1, x_{\{1\}^c}) \\
&+ \frac{\partial_1^2 W_{g-1,n}^2}{2}
\end{aligned}$$

The  $W_{g,n}^2$  defined by

$$W_{g,n}^2 = \sum_{n^-} W_{g,n,n^-}^{(2^*)}$$

And the term  $W_{0,1}^2$  is the solution of the equation

$$xW_{0,1}^2(x) = W_{0,1}^2(x)^3 + 1,$$

and at the neighborhood of  $\infty$ , it's normalized to be in  $x^{-1}\mathbb{Q}[[x^{-1}]]$ . More generally, we have the relation

$$xW_{0,1}^i(x) = W_{0,1}^i(x)^{i+1} + 1.$$

Then it's seems natural to consider the spectral curve

$$xy = y^{i+1} + 1.$$

Which is well known in the theory of counting Grotendieck dessins d'enfants. According to [?] [DM14] after a change of variables, the  $W_{g,n^+,n^-}^2$  are computed by topological recursion using the last spectral curve. It should be interesting to know how to go from the last recursion to the Eynard-Orantin topological recursion. An interesting fact is that the last equations look like higher-order recursions, but indeed they are not because the spectral curve has only simple ramifications.

## 5.7 Appendix

### 5.7.1 Relation to Norbury polynomials:

The Norbury polynomials are Ehrhart quasi-polynomials that count the number of integral metric ribbon graphs. They have been introduced by P. Norbury in [Noro8] and studied also

in [Nor13]. Let  $\mathcal{M}_{g,n,\mathbb{Z}}^{comb}$  be the integral points in the combinatorial moduli spaces  $\mathcal{M}_{g,n}^{comb}$ . For each  $\alpha$ , we denote  $\mathcal{M}_{g,n,\mathbb{Z}}^{comb}(\alpha)$  the subset of metric ribbon graphs such that  $L_i(S) = \alpha_i$  for all  $i \in \llbracket 1, n \rrbracket$ . Then the Norbury polynomial is given by

$$N_{g,n}(\alpha) = \sum_{S \in \mathcal{M}_{g,n,\mathbb{Z}}^{comb}(\alpha)} \frac{1}{\#\text{Aut}(S)}.$$

The generating function for these polynomials is the Ehrhart series; it's given by

$$F_{g,n}^{comb}(y) = \sum_{g,n} N_{g,n}(\alpha) y^\alpha$$

and defined in [Nor13]. The aim of this paragraph is to give a proof of a surprising and beautiful formula that relies on the series  $F_{g,n}^{comb}$  and  $W_{g,n}^*$ . As in [Nor13], we also denote

$$\Omega_{g,n} = d_1 \dots d_n N_{g,n}.$$

Where  $d_i$  is the differential with respect to the  $i$ -th variable.

**Theorem 5.11.** *The generating series  $F_{g,n}^{comb}$  and  $W^*$  are related by the following formula:*

$$F_{g,n}^{comb}(u(x)) = W_{g,n}^*(x).$$

Where  $u$  is given by

$$u(x) = \frac{x - \sqrt{x^2 - 4}}{2}.$$

Consecutively, we have the relation

$$W_{g,n} \otimes_i dx_i = u^* \Omega_{g,n}. \quad (5.18)$$

*Proof.* Let  $\text{rib}_{g,n,*}$  be the set of ribbon graphs with  $n$  faces and possibly univalent vertices; let also  $\text{rib}'_{g,n}$  be the set of ribbon graphs with no univalent vertices. There is a contraction map

$$\text{Trunc} : \text{rib}_{g,n,*} \longrightarrow \text{rib}_{g,n},$$

which is, for instance, described in [Oko], or in figure 5.5. This map is a fibration, fiber consists of trees planted around vertices of the graphs. We can choose to conserve the bivalent vertices that are in the trunc of the graph. These bivalent vertices define a metric on the ribbon graph, and then this induces a surjection

$$\text{rib}_{g,n,*} \longrightarrow \mathcal{M}_{g,n,\mathbb{Z}}^{comb}.$$

Where  $\mathcal{M}_{g,n,\mathbb{Z}}^{comb}$  is the set of all integral metric ribbon graphs and the set of integral points in the combinatorial moduli space. This defines a new map

$$\text{Trunc}' : \text{rib}_{g,n} \longrightarrow \mathcal{M}_{g,n,\mathbb{Z}}^{comb}.$$

For each  $S$  in the image, we can consider the formal series

$$T_S = \sum_{R, \text{Trunc}'(R)=S} \frac{1}{\#\text{Aut}(R)} \prod_i \frac{1}{x_i^{\alpha_i(R)}},$$

and then we have the push-forward formula:

$$\sum_{R, \text{Trunc}'(R)=S} \frac{1}{\#\text{Aut}(R)} \prod_i \frac{1}{x_i^{\alpha_i(R)}} = \sum_S \frac{T_S}{\#\text{Aut}(S)}. \quad (5.19)$$

Now we prove the following proposition:

**Lemma 5.24.** Near  $\infty$ , we have the formula

$$T_S = \prod_i u(x_i)^{L_i(S)},$$

where  $L_i$  is the combinatorial length of the  $i$ -th boundary.

*Proof.* Using figure 5.5, we see that the fiber corresponds to a planar tree planted at each "corner" of the boundary. In the generating series, we add to remember the perimeter of the boundary, and then each tree  $T$  glued to the boundary  $i$  is weighted by  $\frac{1}{x_i^{\#X_1T+1}}$  where  $X_1T$  is the set of edges in the boundary. Because each edge in the tree contributes twice to the boundary length, we also include the edge on the "left" of the tree. Then, to compute the preimage, we start by computing the generating series associated with a single corner in a given boundary  $i$ . Let  $u_0$  be the series

$$u_0(x) = \sum_T \frac{1}{x^{2\#X_1T}}.$$

The sum is over all the planar tree with a marked vertex. This generating series is easily computable; indeed, by removing the roots, we can derive the following recurrence relation:

$$u_0(x) = 1 + \frac{u_0(x)}{x^2} + \frac{u_0(x)^2}{x^4} + \dots = \frac{x^2}{x^2 - u_0(x)}.$$

And then

$$u_0(x)^2 - x^2u_0(x) + x^2 = 0.$$

Solving this equation leads to the formula

$$u_0(x) = \frac{x^2 - \sqrt{x^4 - 4x^2}}{2},$$

where we use  $u_0(x) = 1 + o(1)$ . Then we obtain

$$u(x) = \frac{u_0(x)}{x} = \frac{x - \sqrt{x^2 - 4}}{2}.$$

Then, to prove the lemma, the contribution of each corner is given by  $u(x)$ , and the contribution of the boundary  $i$  in  $S$  is

$$u(x)^{L_i(S)}.$$

Because there are exactly  $L_i(S)$  corners in the boundary  $i$ . As the contribution of the boundaries is independent, we obtain

$$T_S(x) = \prod_i u(x)^{L_i(S)}.$$

□

Using this lemma, we finish the proof of 5.11. It suffices to substitute the expression of  $T_S(x)$  in the formula 5.19 to find  $F_{g,n}^{comb}(u(x))$ . We have in one hand

$$W_{g,n^+,n^-}^*(x) = \int_{\Lambda_{n^+,n^-}} V_{g,n^+,n^-}, (L^+|L^-) \exp\left(\sum_i x_i L_i^+\right) d\sigma_{n^+,n^-}$$

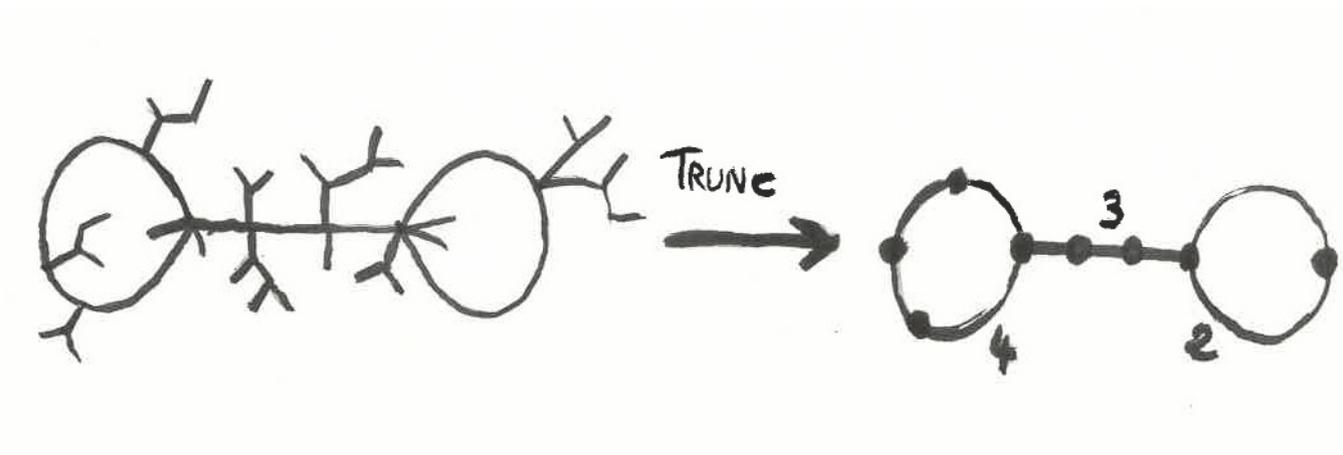


Figure 5.5: Effect of the map, Trunc.

and, on the other hand

$$K_{g,n^+,n^-}(L^+|L^-) = \prod_i L_i^+ V_{g,n^+,n^-}(L^+|L^-).$$

Which leads to

$$W_{g,n^+,n^-}(x) = \partial_1 \dots \partial_{n^+} W_{g,n^+,n^-}^*(x).$$

Summing over  $n^-$  gives the second claim of the theorem 5.11. □

# Chapter 6

## Mirzakhani-McShane formula for trivalent ribbon graphs

In this part, we focus on the case of trivalent ribbon graphs (with possible univalent vertices). We recall some classical results about the volumes of the combinatorial moduli spaces; in this case, since the works of M. Kontsevich [Kon92] we know that the volumes are polynomials with coefficients given by intersection numbers of tautological classes. These polynomials are solutions to several recursions; in particular, they are related to Kdv hierarchy, Virasoro constraints, topological recursion, and Cut-and-Join equations. Here we give a recursion that uses a Mirzakhani-McShane formula for trivalent ribbon graphs (a geometric recursion formula according to [ABO17]). By using integrations over the combinatorial moduli spaces, we obtain a recursion for functions  $V_{g,n}^{comb}$ . These results were also presented in [ABC<sup>+</sup>20]. We also give in proposition 6.4 a relation that is in some sense similar to the one in proposition 4.24 and use it to derive in proposition 6.5 the Cut-and-Join formula for the Kontsevich Witten partition function (which was obtained in [Ale11]).

### 6.1 Kontsevich volumes of combinatorial moduli spaces

#### 6.1.1 Cohomology of moduli spaces

**Deligne Munford compactification:** Let  $\mathcal{M}_{g,n}$  be the moduli space of compact stable Riemann surfaces of genus  $g$  with  $n$  marked points. We denote  $\overline{\mathcal{M}}_{g,n}$  the Deligne-Munford compactification introduced by Deligne and Mumford in [DM69] by adding stable nodal surfaces to  $\mathcal{M}_{g,n}$  (see [Zv012] or [ACG11] for presentations). There is a projection.

$$\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}^{comb}.$$

But the second space is a quotient of the first one, as we see in chapter 7.4.2. The universal curve  $\mathcal{CM}_{g,n}$  is the fibration over  $\mathcal{M}_{g,n}$  with fiber over  $(X, P)$  given by the surface  $X$ . We denote  $\overline{\mathcal{CM}}_{g,n}$  its compactification, we have a projection  $\overline{\mathcal{CM}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ .

**Tautological classes:** For all  $i \in \{1, \dots, n\}$ , tautological bundle  $\mathcal{L}_i$  is the line bundle over  $\overline{\mathcal{M}}_{g,n}$  such that the stalk over  $(X, P)$  is the cotangent space  $T_{p_i}^* X$ . Tautological classes are the

first Chern class of these line bundles:

$$\psi_i = c_1(\mathcal{L}_i).$$

That are also called  $\psi$ -classes, we have  $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$  It's natural to consider intersection pairings

$$\langle \prod_i \psi_i^{\alpha_i} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \prod_i \psi_i^{\alpha_i}.$$

These are rational numbers that appear in modern physics such as string theory and quantum gravity and have been at the center of extensive research over the last three decades. They also arise in the enumerative geometry of surfaces, based on the Givental-Teleman action on semisimple cohomological field theory. The group  $\mathfrak{S}_n$  acts on  $\overline{\mathcal{M}}_{g,n}$ , we have  $\sigma^* \mathcal{L}_i = \mathcal{L}_{\sigma(i)}$ , and then  $\sigma^* \psi_i = \psi_{\sigma(i)}$ . It implies that for each multi-index  $\alpha$

$$\langle \prod_i \psi_i^{\alpha_i} \rangle = \langle \sigma^* \prod_i \psi_i^{\alpha_i} \rangle = \langle \prod_i \psi_{\sigma(i)}^{\alpha_i} \rangle.$$

Then the intersection numbers are invariant by permutation and only depend on the partition  $\mu \in \mathcal{P}(\mathbb{N})$  associated with  $\alpha$ . The use is to denote

$$\langle \prod_k \tau_k^{\mu(k)} \rangle := \langle \prod_i \psi_i^{\alpha_i} \rangle$$

these pairings.

## 6.1.2 Volumes of combinatorial moduli spaces and Kontsevich polynomials

**A Kontsevich formula:** As we see in section 4.3.2, if  $R$  is a ribbon graph with only odd vertices, the two form  $\Omega_R$  is non-degenerate and hence is a symplectic form on  $K_R$ . Applying this to the case of trivalent graphs, we see that top cells of the moduli space  $\mathcal{M}_{g,n}^{comb}$  admit a symplectic structure. We denote  $\Omega_K$  this symplectic form, it's called the Kontsevich symplectic form [Kong2]. For each  $L$ , we see in 4.1.4 that the volume of  $\mathcal{M}_{g,n}^{comb}(L)$  can be computed using the Lebesgue measure.

**Lemma 6.1.** *We have the relation:*

$$d\mu_{g,n}^{comb}(L) = \frac{|\Omega_K^{3g-3+n}|}{2^{2g-1+n}(3g-3+n)!}.$$

Then we define

$$V_{g,n}^{comb}(L) = \int_{\mathcal{M}_{g,n}^{comb}(L)} \frac{\Omega_K^{3g-3+n}}{(3g-3+n)!}.$$

We remark that our normalization for the symplectic form differs from the one of [Kong2] by a factor  $\frac{1}{2}$ .

**Theorem 6.1** (Kontsevich [Kong2]). *The volumes are expressed by the following formula:*

$$V_{g,n}^{comb}(L) = \frac{1}{2^{3g-3+n}} \sum_{\alpha} \langle \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \rangle \prod_i \frac{L_i^{2\alpha_i}}{\alpha_i!}.$$

The proof of Kontsevich formula uses:

$$\Omega_K = \frac{1}{2} \sum_i \omega_i.$$

Where the  $\omega_i$  and the formula are given in section 4.3.2. The second point is the fact that, for each  $i$ , the one form  $\omega_i$  is the curvature of the circle bundle associated with  $\mathcal{L}_i$ . Proof of these results can be found in [Kong2], details have been fixed later and given in [Zvoo2], and an exposition can be found in [LZZo4]. We do not explain the matrix integral representation by using the Airy matrix function, which was originally found by M. Kontsevich.

**Some results on the Kontsevich form:** We give a result that is similar to the one of [ABC<sup>+</sup>20], this proposition allows to decompose the symplectic structure. Let  $R$  be a ribbon graph with only odd vertices. As we see in 4.2.3, it's possible to find an admissible curve  $\Gamma \in \mathcal{MS}(R)$  such that all the components of  $R_\Gamma$  are irreducible. Let  $(t_\gamma)$  be a choice of twist parameters, then according to proposition 4.18,  $(l_\gamma, t_\gamma)_{\gamma \in \Gamma}$  defines local coordinates, and we have the following proposition.

**Proposition 6.1.** *The elements  $dl_\gamma, dt_\gamma$  for  $\gamma \in \Gamma$  form a basis of  $K_R$ ; furthermore, we have*

$$\frac{\Omega_R^{3g-3+n}}{(3g-3+n)!} = \bigwedge_{\gamma} dl_\gamma \wedge dt_\gamma.$$

This proposition is a bit weaker than the one of [ABC<sup>+</sup>20]; we do not show that these coordinates are Darboux coordinates, and we are not sure about that (outside the case of trivalent graphs). Secondly, in this case, a maximal admissible multi-curve does not define coordinates on a full stratum of the Teichmuller space. We might use several sets of coordinates. Nevertheless, the last proposition allows us to decompose the measure and prove the Mirzakhani formula for coverings over the moduli space, similarly to paragraph 4.4.1.

**Covering and integration:** Let  $\mathcal{G}$  be a stable graph, as in 4.4.1. There is a natural bundle  $B\mathcal{M}^{comb}(\mathcal{G})$  over  $\mathcal{M}^{comb}(\mathcal{G})$  given by  $\mathcal{MF}_\Gamma(M)/\text{Stab}(\Gamma)$  where  $\Gamma \in \mathcal{MS}(M)$  represents  $\mathcal{G}$  (see paragraph 4.2.4). On this space, there is a natural volume form  $d\tilde{\mu}_{\mathcal{G}}$ , and if  $(t_\gamma)_{\gamma \in \Gamma}$  are twist parameters, we can decompose it as

$$\bigwedge_c \frac{\text{pr}_c^* \Omega_K^{3g(c)-3+n(c)}}{(3g(c)-3+n(c))!} \wedge \bigwedge_{\gamma} dl_\gamma \wedge dt_\gamma.$$

Where  $\text{pr}_c : B\mathcal{M}^{comb}(\mathcal{G}) \rightarrow \mathcal{M}^{comb}(\mathcal{G}(c))$  is the projection given by the cutting map. As before, the quotient

$$\mathcal{M}_\Gamma^{comb}(M) = \mathcal{T}_\Gamma^{comb,*}(M)/\text{Stab}(\Gamma).$$

is a covering over  $\mathcal{M}^{comb}(M)$  that corresponds to metric ribbon graphs marked by a multi-curve with stable graph  $\mathcal{G}$ . Then, for a symmetric function  $F$  on  $\mathbb{R}_{\geq 0}^\Gamma$ , we can consider statistics

$$N_{\mathcal{G}} F(S) = \sum_{\Gamma' \simeq \mathcal{G}} F(L_{\Gamma'}(S)).$$

This is by definition the push forward of  $F \circ L_\Gamma$  under  $\pi_G : \mathcal{M}_\Gamma^{comb}(M) \rightarrow \mathcal{M}^{comb}(M)$ . Then it satisfies the following relation, which is the formula for a push forward under a covering

$$\int_{\mathcal{M}^{comb,*}(M)} N_G F(S) d\tilde{\mu}_M^{comb}(L) = \int_{\mathcal{M}_\Gamma^{comb,*}(M,L)} F(L_\Gamma(S)) d\mu_M^{comb}(L).$$

There is a canonical map

$$\mathcal{M}_\Gamma^{comb,*}(M) \longrightarrow B\mathcal{M}^{comb,*}(\mathcal{G})$$

And as before,  $B\mathcal{M}^{comb,*}(\mathcal{G})$  is of full measure in  $B\mathcal{M}^{comb}(\mathcal{G})$ , and then, by using the formula for the measures, we obtain the relation

$$\int_{\mathcal{M}^{comb}(M,L)} N_G F(S) d\tilde{\mu}_M(L) = \frac{1}{\#\text{Aut}(\mathcal{G})} \int_{l \in (\mathbb{R}_{\geq 0})^{X_1 \mathcal{G}}} F(l) \prod_c V_{\mathcal{G}(c)}^{comb}(l(c), L(c)) \prod_\gamma l_\gamma dl_\gamma$$

## 6.2 Mirzakhani-McShane formula for trivalent ribbon graphs

In this part, we give a proof of the Mirzakhani-McShane formula for generic (trivalent) metric ribbon graphs, and give a straightforward generalization to the case of vertices of degree 1. This formula was found independently in [ABC<sup>+</sup>20] and stated in a first version [ABO17] but without the interpretation using ribbon graphs.

### Flux between two boundaries

Let  $S = (R, m)$  a metric ribbon graph and  $(\beta, \beta')$  a pair of possibly identical boundaries. The flux between  $\beta, \beta'$  is the sum of the weights of the edges adjacent to these two boundaries (if an edge appears twice in a boundary, we count it twice)

$$m_{\beta, \beta'}(S) = \sum_{e \in XR, [e]_2 = \beta, [s_1 e]_2 = \beta'} m_e(S).$$

In an equivalent way, if we see  $S$  as a weighted multi-arc  $A(S)$  (see 4.4), the flux is the sum of the (oriented) arcs  $e^* \in A(S)$  that connect the two boundaries  $\beta, \beta'$ . In order to prove the Mirzakhani-McShane formula, we start by giving classical results on pairs of pants. Similar results can be found in [FLP21] or in [ABC<sup>+</sup>20].

**Proposition 6.2.** *Let  $S$  a metric ribbon graph on a pair of pants and  $\{\beta, \beta', \beta''\}$  the three boundaries. We have the relation*

$$m_{\beta, \beta'}(S) = \frac{[L_\beta(S) + L_{\beta'}(S) - L_{\beta''}(S)]_+ - [L_\beta(S) - L_{\beta'}(S) - L_{\beta''}(S)]_+ - [L_{\beta'}(S) - L_\beta(S) - L_{\beta''}(S)]_+}{2}$$

$$m_{\beta, \beta}(S) = [L_\beta(S) - L_{\beta'}(S) - L_{\beta''}(S)]_+$$

To prove that, we use the classification of metric ribbon graphs on a pair of pants. The moduli space  $\mathcal{M}_{0,3}^{comb}$  is isomorphic to  $(\mathbb{R}_{>0})^3$  (And the compactification  $\mathcal{M}_{0,3}^{comb}$  is  $\mathbb{R}_{\geq 0}^3 \setminus \{0\}$ ). As a cell complex, the space contains four top cells, and they fall into two categories.

- The first type corresponds to a unique cell supported by the graph with two vertices of degree three; each boundary of the graph is connected to the two other by an edge. The cell is characterized by the relations

$$L_\beta < L_{\beta'} + L_{\beta''} \quad L_{\beta'} < L_\beta + L_{\beta''} \quad L_{\beta''} < L_{\beta'} + L_\beta$$

And it's located at the center of the figure 6.1.

- The second type is related to graphs with two vertices of degree three and the following condition: There is one big boundary  $\beta$  and two small boundaries with the constraint:

$$L_\beta > L_{\beta'} + L_{\beta''}.$$

These ribbon graphs look like a pair of glasses, and there is three top cells according to the choice of the big boundary component. And these graphs are located at the corners of figure 6.1.

The full decomposition of the compactification contains 19 cells, as shown in figure (6.1), but such figures can also be found in [ABO17] or in [FLP21].

**Remark 6.1.** *If we consider a cylinder with one marked point, there are 6 top cells given by ribbon graphs with one vertex of degree three and one of degree one at the marked point. If we have two marked points, we have only one ribbon graph, which has two vertices of degree one and is called a pocket.*

*proposition 6.2.* It remains to check that the formulas give the expected numbers in all cases. For type one we can see by seeing the graph that we have

$$m_{\beta,\beta'}(S) = \frac{L_\beta + L_{\beta'} - L_{\beta''}}{2} \quad m_{\beta,\beta}(S) = 0$$

Furthermore, in the formula for  $m_{\beta,\beta'}$  of proposition 6.2, we can see that the only non-vanishing term of the sum is the first one. This is due to the constraints  $L_\beta < L_{\beta'} + L_{\beta''}$   $L_{\beta'} < L_\beta + L_{\beta''}$ . Then we have the good count (also for  $m_{\beta,\beta}$ ). For type two, if  $\beta$  is the big boundary, then for all  $\beta'$ , we have  $m_{\beta,\beta'}(S) = L_{\beta'}$ ; moreover

$$L_\beta > L_{\beta''} + L_{\beta'} \quad L_{\beta'} < L_{\beta''} + L_\beta \quad L_{\beta''} < L_{\beta'} + L_\beta$$

Then only the first two terms in the formula are non-vanishing, and we obtain the good count. Similarly

$$m_{\beta,\beta}(S) = L_\beta - L_{\beta'} - L_{\beta''},$$

which is also ok. Else if  $\beta$  is a small boundary  $m_{\beta,\beta}(S) = 0$  and if  $\beta'$  is a small boundary  $m_{\beta,\beta'}(S) = 0$  else if  $\beta'$  is the big boundary we have  $m_{\beta,\beta'}(S) = L_\beta$ . In all cases, it agrees with the formula by using the constraints on the boundary length. In general, we can see that  $m_{\beta,\beta'}$  and the RHS of proposition 6.2 are both continuous, and then agree on  $\mathcal{M}_{0,3}^{comb}$ , because they coincide on the top cells.  $\square$

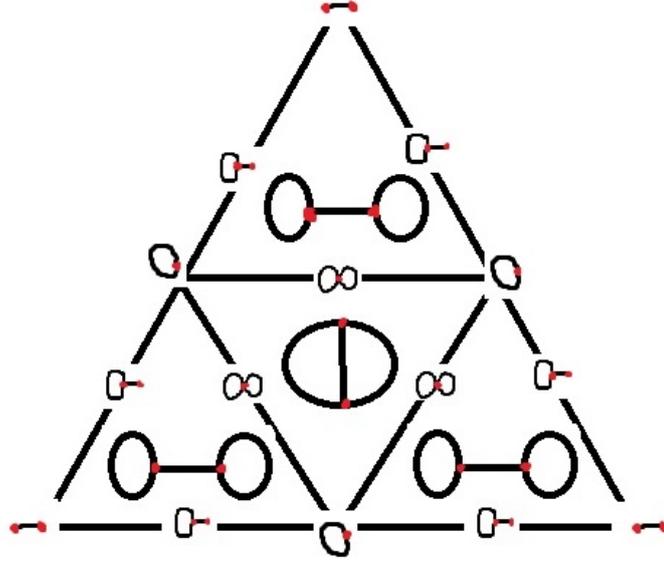


Figure 6.1: Projection of the cell decomposition of  $\overline{\mathcal{M}}_{0,3}^{comb}$  on the simplex  $\{L_\beta + L_{\beta'} + L_{\beta''} = 1\}$

### Mirzakhani-McShane formula for combinatorial surfaces

To state the Mirzakhani-McShane formula, we introduce two functions  $F^+, F^-$  also given in [ABC<sup>+</sup>20] denoted differently:

$$F^+(L_1, L_2 | L_3) = \frac{[L_1 + L_2 - L_3]_+ + [L_1 - L_2 - L_3]_+ - [L_2 - L_1 - L_3]_+}{2}$$

$$F^-(L_1 | L_2, L_3) = [L_1 - L_2 - L_3]_+.$$

### Geometric recursion

We consider embedded pants in the surface; let  $M$  and  $\beta$  be a boundary of  $M$ . We can consider all the isotopy classes of subsurfaces

$$f : P \longrightarrow M$$

Such that  $P$  is a pair of pants, and if  $\partial P = \{\beta_1, \beta_2, \beta_3\}$  then  $f$  map  $\beta_1$  on  $\beta$  and the two other boundaries are either map to a curve in  $\mathcal{S}(M)$  or a boundary. We denote  $\text{Irr}_\beta(M)$  this set. For each  $P$  we can consider  $\partial P \cap \partial M$ , then either this set is the singleton  $\{\beta\}$  or it contains a second boundary  $\beta'$ . Then we can write

$$\text{Irr}_\beta(M) = \bigsqcup_{\beta'} \text{Irr}_{\beta, \beta'}(M).$$

The set  $\gamma \in \partial P \setminus \partial M$  defines a multi-curve, and we denote  $L_P = (l_{\beta''})_{\gamma \in \partial P \setminus \partial M}$  its length.

**Theorem 6.2.** For any generic connected metric ribbon graph  $S$  that satisfies  $d(S) > 1$  and for all boundary  $\beta$  of  $S$ , we have

$$L_\beta(S) = \sum_{\beta'} \sum_{P \in \text{Irr}_{\beta, \beta'}(M)} F^+(L_\beta(S), L_{\beta'}(S) | L_P(S)) \\ + \sum_{\Sigma \in \text{Irr}_{\beta, \beta}(M)} F^-(L_\beta(S) | L_P(S)).$$

In the case of the torus with one boundary, we have

$$L_\beta(S) = \sum_{\gamma \in \mathcal{S}(M)} [L_\beta(S) - 2L_\gamma(S)]_+.$$

**Remark 6.2.** The last formula is different from the original Mirzakhani-McShane. In the formula for ribbon graphs, the sum is finite, almost all the terms are zero on a given cell. But when we cross walls in the Teichmüller space, some terms can vanish and new terms can appear.

**Remark 6.3** (relation to hyperbolic surfaces). The formula of theorem 6.2 is related to hyperbolic surfaces with large boundaries. Indeed, by the collar lemma, such surface can be very thin and look like a ribbon graph. Such an approach can be written in a rigorous way, and this has been done in [ABC<sup>+</sup>20], but we do not explore this direction.

We use the following lemma for the proof of theorem 6.2, and we remark that it can also be derived from theorem 4.2.

**Lemma 6.2.** Let  $R$  be a trivalent metric ribbon graph. For each  $e$ , there is a unique  $P_e \in \text{Irr}(M)$  such that  $e^*$  is supported by  $P_e$ .

*Proof.* Let  $P_e$  be the support of  $e^*$  (see proposition 7.1). According to the same proposition,  $e^*$  is filling on  $P_e^1$  and then  $P_e^1$  must be a pair of pants. If  $P \in \text{Irr}(M)$  such that  $e^*$  is supported by  $P$ , then  $P_e \subset P$ . As the two surfaces are pairs of pants, we have  $P_e = P$ .  $\square$

*Theorem 6.2.* We have the relation

$$L_\beta(S) = m_{\beta, \beta}(S) + \sum_{\beta' \neq \beta} m_{\beta, \beta'}(S).$$

By using lemma 6.2, an edge belongs to a unique embedded pair of pants, which implies that

$$m_{\beta, \beta'}(S) = \sum_{P \in \text{Irr}_{\beta, \beta'}(M)} m_{\beta, \beta'}^P(S),$$

where  $m_{\beta, \beta'}^P(S)$  is the contribution of the edges supported by  $P$ . Let  $S_P$  be the ribbon graph restricted to  $P$ . As  $\beta, \beta'$  are both contained in  $S_P$  then we have

$$m_{\beta, \beta'}^P(S) = m_{\beta, \beta'}(S_P).$$

Indeed, the arcs in  $S_P$  that connect  $\beta, \beta'$  are exactly the arcs in  $A_S$  that connect  $\beta$  and  $\beta'$  and are supported by  $P$ . Each  $P \in \text{Irr}_{\beta, \beta'}(M)$  defines a curve  $\gamma_P \in \mathcal{S}(M)$ , and we have  $L_P = l_{\gamma_P}$ , by using proposition 6.2, we obtain

$$m_{\beta, \beta'}^P(S) = \frac{[L_\beta(S) + L_{\beta'}(S) - L_{\gamma_P}(S)]_+ - [L_\beta(S) - L_{\beta'}(S) - L_{\gamma_P}(S)]_+ - [L_{\beta'}(S) - L_\beta(S) - L_{\gamma_P}(S)]_+}{2}.$$

For a single boundary  $\beta$ , we have using proposition 6.2

$$m_{\beta,\beta}(S) = \sum_{\beta'} \sum_{P \in \text{Irr}_{\beta,\beta'}(M)} m_{\beta,\beta}^P(S) + \sum_{P \in \text{Irr}_{\beta,\beta}(M)} m_{\beta,\beta}^P(S).$$

If  $P \in \text{Irr}_{\beta,\beta'}(M)$  and using proposition 6.2, we obtain

$$\begin{aligned} m_{\beta,\beta}^P(S) &= [L_\beta - L_{\beta'} - L_{\gamma_P}]_+ \quad \forall P \in \text{Irr}_{\beta,\beta'}(M), \\ m_{\beta,\beta}^P(S) &= [L_\beta - L_{\gamma_P^1} - L_{\gamma_P^2}]_+ \quad \forall P \in \text{Irr}_{\beta,\beta}(M). \end{aligned}$$

Then we can write

$$m_{\beta,\beta}(S) = \sum_{\beta'} \sum_{P \in \text{Irr}_{\beta,\beta'}(M)} \frac{[L_\beta - L_{\beta'} - L_{\gamma_P}]_+}{2} + \sum_{P \in \text{Irr}_{\beta,\beta}(M)} [L_\beta - L_{\gamma_P^1} - L_{\gamma_P^2}]_+.$$

Finally, by summing all the different terms, we obtain the desired formula.

$$\begin{aligned} L_\beta(S) &= \sum_{\beta'} m_{\beta,\beta'}(S) + m_{\beta,\beta}(S) \\ &= \sum_{\beta'} \sum_{P \in \text{Irr}_{\beta,\beta'}(M)} F^+(L_\beta, L_{\beta'} | L_{\gamma_P}) \\ &\quad + \sum_{P \in \text{Irr}_{\beta,\beta}(M)} F^-(L_\beta | L_{\gamma_P^1}, L_{\gamma_P^2}). \end{aligned}$$

We remark that the boundary components  $L_{\gamma_P^1}, L_{\gamma_P^2}$  of the elements in  $\text{Irr}_{\beta,\beta}$  are not labeled, but the function  $F^-$  is symmetric. The case of the torus is similar, but in this case we only consider the pants in  $\text{Irr}_{\beta,\beta}(M)$ . The two boundary curves  $\gamma_P^1, \gamma_P^2$  are equal, and for each curve in  $\gamma \in \mathcal{S}(M)$ , the surface  $M_\gamma$  is a pair of pants, then we have  $\text{Irr}_{\beta,\beta'} \simeq \mathcal{S}(M)$ .  $\square$

### Case of punctured surfaces

For a surface  $(M, X) \in \mathbf{bord}^\bullet$ , the situation is similar, but we need to consider the contribution of subsurfaces that are cylinders with one marked point. We denote then  $\text{Irr}_{\beta,x}(M)$  for  $\beta \in \partial M$  and  $x \in X$ . For such a pair of pants, we have

$$m_{\beta,\beta}^P(S) = [L_\beta(S) - L_{\gamma_P}(S)]_+.$$

Then we get the Mirzakhani-McShane formula in this case, which is a degeneration of the original MMS of theorem 6.2.

**Corollary 6.1.** *We have the relation*

$$L_\beta(S) = \sum_{\beta'} \sum_{P \in \text{Irr}_{\beta,\beta'}(M)} F^+(L_\beta(S), L_{\beta'}(S) | L_P(S)) \quad (6.1)$$

$$+ \sum_{P \in \text{Irr}_{\beta,\beta}(M)} F^-(L_\beta(S) | L_P(S)) \quad (6.2)$$

$$+ \sum_x \sum_{P \in \text{Irr}_{\beta,x}(M)} F^+(L_\beta(S), 0 | L_P). \quad (6.3)$$

## Topological recursion relation for volumes

Using techniques introduced by Mirzakhani in [Miro7] and applied also in [ABC<sup>+</sup>20], we can integrate the Mirzakhani-McShane formula over the moduli space  $\mathcal{M}_{g,n}^{comb}(L)$  for the Kontsevich measure.

**Proposition 6.3.** *The volumes  $V_{g,n}^{comb}(L)$  are the symmetric polynomials solution of the recursion:*

$$\begin{aligned} L_1 V_{g,n}^{comb}(L) = & \sum_{j \neq 1} \int_{\mathbb{R}_{\geq 0}^3} F^+(L_1, L_j | x) V_{g,n-1}^{comb}(x, L_{\{1,j\}^c}) x dx \\ & + \frac{1}{2} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g-1, n+1}^{comb}(x_1, x_2, L_{\{1\}^c}) x_1 x_2 dx_1 dx_2 \\ & + \frac{1}{2} \sum_{g_1+g_2=g, I_1 \sqcup I_2 = \{2, \dots, n\}} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_1 | x_1, x_2) V_{g_1, n_1+1}^{comb}(x_1, L_{I_1}) V_{g_2, n_2+1}^{comb}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2 \end{aligned}$$

With initial data

$$V_{0,3}^{comb}(L_1, L_2; L_3) = 1 \quad V_{1,1}^{comb}(L) = \frac{L^2}{24}$$

This recursion leads to a recursion for the coefficients, which can be used to re-demonstrate the Witten conjecture.

*Proof.* Let  $M \in \mathbf{bord}$  a surface of type  $(g, n)$ . First, we fix  $\beta \neq \beta'$ , and we consider the quantity

$$\int_{\mathcal{M}_{g,n}^{comb}(L)} \sum_{P \in \text{Irr}_{\beta, \beta'}(M)} F^+(L_\beta(S), L_{\beta'}(S), L_{\gamma_P}(S)) \frac{\Omega_K^{3g-3+n}}{(3g-3+n)!}.$$

The series is a sum over a single orbit of embedded pairs of pants (multi-curves) under the action of the mapping class group, we denote this orbit as  $[P]$ . As we see  $F^+$  is positive, then using the Mirzakhani integral formula, we have

$$\begin{aligned} \int_{\mathcal{M}_{g,n}^{comb}(L)} \sum_{P \in \text{Irr}_{\beta, \beta'}(M)} F^+(L_\beta(S), L_{\beta'}(S) | L_{\gamma_P}(S)) \frac{\Omega_K^{3g-3+n}}{(3g-3+n)!} &= \int_{\mathcal{M}^{comb}([P], L)} F^+(L_\beta, L_{\beta'} | L_{\gamma_P}(S)) \frac{\Omega_K^{3g-3+n}}{(3g-3+n)!} \\ &= \int_{\mathbb{R}_{>0}} F^+(L_\beta, L_{\beta'} | l) V_{g,n-1}^{comb}(l, L_{\{\beta, \beta'\}^c}) l dl \end{aligned}$$

Where  $\mathcal{M}^{comb}([P], L) \simeq \mathcal{T}_{g,n}^{comb}(L) / \text{Stab}(P)$  is the covering associated with the orbit  $[\gamma_P]$ .

In the case of  $\text{Irr}_{\beta, \beta'}$  there are more orbits. The case  $M \setminus P$  connected corresponds to only one orbit in  $\text{Irr}_{\beta, \beta'}(M)$ , we denote it  $[P_{con}]$ . We have  $\text{Aut}([P_{con}]) \simeq \mathbb{Z}_2$ , and then we obtain

$$\int_{\mathcal{M}_{g,n}^{comb}(L)} \sum_{P' \in [P]} F^-(L_\beta | L_{\gamma_1}(S), L_{\gamma_2}(S)) \frac{\Omega_K^{3g-3+n}}{(3g-3+n)!} = \frac{1}{4} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_\beta | x_1, x_2) V_{g-1, n+1}^{comb}(x_1, x_2, L_{\{\beta\}^c}) x_1 dx_1 x_2 dx_2$$

For  $\Sigma^0$  disconnected, the orbits of the possible subsurfaces are determined by their stable graphs. In this case, they correspond to graphs with three vertices: a vertex  $c_0$  of type  $(0, 3)$ , and the two others  $c_1, c_2$  are of type  $(g_1, n_1), (g_2, n_2)$ . They contain the set of boundaries  $I_1$  and  $I_2$ , with  $I_1 \sqcup I_2 = \partial M_{g,n} \setminus \{\beta\}$ . The only case when there is a non-trivial element of  $\text{Aut}(\Gamma)$  is when  $n = 1$  and  $g_1 = g_2$ . And then the contribution for a given orbit is

$$\frac{1}{2^{\delta_{n,1} \delta_{g_1, g_2}}} \int_{\mathbb{R}_{\geq 0}^2} F^-(l_\beta | x_1, x_2) V_{g_1, n_1+1}^{comb}(x_1, L_{\{I_1\}^c}) V_{g_2, n_2+1}^{comb}(x_2, L_{\{I_2\}^c}) x_1 x_2 dx_1 dx_2$$

We cannot make a distinction between the two vertices  $c_1, c_2$ , the total contribution is

$$\frac{1}{2} \sum_{g_1+g_2=g, I_1 \sqcup I_2 = \partial M_{g,n} \setminus \{\beta\}} \int_{\mathbb{R}_{\geq 0}^2} F^-(L_\beta | x_1, x_2) V_{g_1, n_1+1}^{comb}(x_1, L_{\{I_1\}^c}) V_{g_2, n_2+1}^{comb}(x_2, L_{\{I_2\}^c}) x_1 x_2 dx_1 dx_2.$$

□

### 6.3 Cut-and-Join equation for the Kontsevich-Witten partition function

We give a proof of the cut and join formula exposed in [Ale11]. We use the formalism of metric ribbon graphs and ideas similar to those of proposition 4.24. We consider  $H = \mathbb{1}_{\mathbb{R}_{>0}}$  the Heaveside function, and for  $P \in \text{Irr}(M)$  we denote

$$\begin{aligned} H_P(S) &= H(L_\beta + L_{\beta'} - L_{\gamma_P}) + H(L_\beta - L_{\beta'} - L_{\gamma_P}) + H(L_{\beta'} - L_\beta - L_{\gamma_P}) \quad P \in \text{Irr}_{\beta, \beta'}(M) \text{ for } \beta \neq \beta' \\ H_P(S) &= H(L_\beta - L_{\gamma_P^1} - L_{\gamma_P^2}) \quad \text{else.} \end{aligned}$$

The direct analog of proposition 4.24 in this case is the following identity:

**Proposition 6.4.** *We have the following formula true on  $\mathcal{M}^{comb,*}(M)$ :*

$$6g - 6 + 3n = \sum_{P \in \text{Irr}(M)} H_P(S) \quad (6.4)$$

**Corollary 6.2.** *The volumes also satisfy the recurrence relation.*

$$\begin{aligned} (6g - 6 + 3n) V^{comb}(L) &= \frac{1}{2} \sum_{i \neq j} \int_0^{L_i+L_j} V_{g,n-1}^{comb}(x, L_{\{i,j\}^c}) x dx + \int_0^{|L_i-L_j|} V_{g,n-1}^{comb}(x, L_{\{i,j\}^c}) x dx \\ &+ \frac{1}{2} \sum_i \int_{x_1+x_2 \leq L_i} V_{g-1, n+1}^{comb}(x_1, x_2, L_{\{i\}^c}) x_1 x_2 dx_1 dx_2 \\ &+ \frac{1}{2} \sum_i \sum_{g_i, I_i, n_i} \int_{x_1+x_2 \leq L_i} V_{g_1-1, n_1+1}^{comb}(x_1, L_{I_1}) V_{g_2-1, n_2+1}^{comb}(x_2, L_{I_2}) x_1 x_2 dx_1 dx_2 \end{aligned}$$

As we do in section 5.2, we consider the polynomials.

$$G_{K,g,n}(L) = \prod_i L_i V_{g,n}^{comb}(L). \quad (6.5)$$

According to the Kontsevich formula (proposition 6.1) we can see that the  $G_{K,g,n}$  are elements of the Fock space  $S(V_{odd})$ , where  $V_{odd} = L\mathbb{Q}[L^2]$ . Moreover, according to the same proposition, they are homogeneous of degree  $3d_{g,n} = 6g_6 + 3n$ . We can consider

$$G_K(q) = \exp_{\square} \left( \sum_{g,n} q^{d_{g,n}} G_{g,n} \right),$$

and  $G_K^d = [q^d] G_K$ .

Let  $Z_K$  be the image in  $\mathbb{Q}[[t_1, t_3, \dots]]$  of  $G_K$  with the choice of basis of  $V_{\text{odd}}$  given by

$$\frac{L^{2k+1}}{(2k+1)!} = t_{2k+1}.$$

The last recursion can be rewritten in the following way by using general consideration about disconnected objects (see section 4.15). For  $d \geq 1$ , we have

$$\begin{aligned} 3dG_K^d &= \frac{1}{2} \sum_{i \neq j} \int_0^{L_i+L_j} G_K^{d-1}(x, L_{\{i,j\}^c}) dx + \int_0^{|L_i-L_j|} G_K^{d-1}(x, L_{\{i,j\}^c}) dx \\ &+ \frac{1}{2} \sum_i \int_{x_1+x_2 \leq L_i} G_K^{d-1}(x_1, x_2, L_{\{i\}^c}) dx_1 dx_2. \end{aligned}$$

**Proposition 6.5.** *The Kontsevich-Witten partition function  $\phi^K$  satisfies the following Cut-and-Join equation:*

$$\frac{\partial Z^K}{\partial q} = \sum_{k+l=n+1} (2k+1)(2l+1)t_{2k+1}t_{2l+1}\partial_{2n+1}Z^K + \sum_{k,l} (2k+2l+5)t_{2k+2l+5}\partial_{2k+1}\partial_{2l+1}Z^K + t_1^3Z^K + \frac{t_3Z^K}{24}.$$

With the initial condition

$$Z^K(0) = 1.$$

If we denote by  $P_{K,\pm}$  the operators defined on  $\hat{S}(V_{\text{odd}})$  by the formulas

$$\begin{aligned} P_{K,+}F &= \frac{1}{2} \sum_{i \neq j} L_i L_j \int_0^{L_i+L_j} F(x, L_{\{i,j\}^c}) dx + \int_0^{|L_i-L_j|} F(x, L_{\{i,j\}^c}) dx \\ P_{K,-}F &= \frac{1}{2} \sum_i \int_{x_1+x_2 \leq L_i} F(x_1, x_2, L_{\{i\}^c}) dx_1 dx_2. \end{aligned}$$

We can expand

$$G_K(q) = \sum_{d \geq 0} q^d G_K^d.$$

Using techniques similar to the ones in the last chapter, we can rewrite the recursion in the following way:

$$dG_K^d = \frac{P_K G_K^{d-1}}{3} + G_K^1 \sqcup G_K^{d-1}.$$

Then we obtain the Cut-and-Join equation.

$$\frac{\partial G_K}{\partial q} = \frac{P_K G_K}{3} + G_K^1 \sqcup G_K.$$

To obtain the equation in terms of differential operators, we use the following lemma:

**Lemma 6.3.** *We have*

$$\begin{aligned} P_{K,+}F &= \sum_{k+l=n+1} (2k+1)(2l+1)t_{2k+1}t_{2l+1}\partial_{2n+1}F \\ P_{K,-}F &= \frac{1}{2} \sum_{k,l} (2k+2l+5)t_{2k+2l+5}\partial_{2k+1}\partial_{2l+1}F. \end{aligned}$$

*Proof.* We can write  $P_{K,\pm} = P_{K,\pm}^5 \sqcup id$  with  $P_{K,\pm}^5$  being operators of type  $(2, 1)$  and  $(1, 2)$ . We have the formulas.

$$L_1 L_2 \int_0^{L_1+L_2} \frac{x^{2n+1}}{(2n+1)!} dx = \sum_{k+l=2n+2} (k+1)(l+1) \frac{L_1^{k+1}}{(2k+1)!} \frac{L_2^{l+1}}{(2l+1)!}$$

$$L_1 L_2 \int_0^{|L_1-L_2|} \frac{x^{2n+1}}{(2n+1)!} dx = \sum_{k+l=2n+2} (-1)^k (k+1)(l+1) \frac{L_1^{k+1}}{(2k+1)!} \frac{L_2^{l+1}}{(2l+1)!}.$$

Summing the two lines, we get the action of  $P_{K,+}^5$ .

$$P_{K,+}^5 \cdot e_{2n+1} = 2 \sum_{k+l=n+1} (2k+1)(2l+1) e_{2k+1} \otimes e_{2l+1}.$$

Then according to the formulas in section 5.1, an operator of the form  $\dots \sqcup id$  is a differential operator; moreover, we have

$$P_{K,+}^5[(2k+1, 2l+1)|2n+1] = \delta_{k+l=n+1} 2(2k+1)(2l+1).$$

Finally, according to proposition 5.2, we have

$$P_{K,+} = \sum_{k+l=n+1} (2k+1)(2l+1)(2k+1)(2l+1) t_{2k+1} t_{2l+1} \partial_{2n+1}.$$

We derive the formula for  $P_{K,-}$  by using

$$\begin{aligned} P_{K,-}^5 e_{2k+1} \otimes e_{2l+1} &= L \int_{x_1+x_2 \leq L} \frac{x_1^{2k+1} x_2^{2l+1}}{(2k+1)!(2l+1)!} dx_1 dx_2 \\ &= L \int_{x_1+x_2 \leq L} \frac{x_1^{2k+1} x_2^{2l+1}}{(2k+1)!(2l+1)!} dx_1 dx_2 \\ &= (2k+2l+5) \frac{L^{2l+2k+5}}{(2l+2k+5)!} \\ &= (2k+2l+5) e_{2k+2l+5}. \end{aligned}$$

□

*proposition 6.5.* By using the formula 6.2 and the lemma 6.3 we have almost all the ingredients. We have

$$G_K^1 = G_{K,0,3} + G_{K,1,1} = L_1 L_2 L_3 + \frac{L_1^3}{24},$$

and then

$$Z_K^1 = t_1^3 + \frac{t_3}{24}.$$

Finally we obtain

$$\frac{\partial Z_K}{\partial q} = P_K \phi_K + Z_K^1 Z_K.$$

# Chapter 7

## Continuous extension for the piece-wise polynomials

In this part, we study in more details the regularity of the functions:

$$V_{\overline{M}^\circ} : \Lambda_{M^\circ} \longrightarrow \mathbb{R}.$$

We focus on the case of oriented graphs, but similar statements are also valid for general ribbon graphs with some modifications. It happens that the volumes  $V_{\overline{M}^\circ}$  do not define continuous functions; they are only continuous on  $\Lambda_{M^\circ}^*$  and can jump on the wall in  $\mathbf{Wall}(M^\circ)$  (see subsection 3.5.3). These jumps are due to degenerations of ribbon graphs; on some walls, some ribbon graphs are forced to degenerate into nodal ribbon graphs; these looses are responsible for the discontinuities. Several ribbons can degenerate into the same nodal graph, and we need to consider a weighted sum of possible degenerations.

- We study the degenerations of ribbon graphs and identify the apparent degenerations that are responsible for the jumps.
- We study the structure of the stratum around an apparent degeneration.
- We prove the continuity theorem.
- We apply this to express the volumes of the stratum of moduli space of abelian differentials.

### 7.1 Nodal ribbon graphs and compactification

#### 7.1.1 Nodal ribbon graphs

**Nodal surface:** In this text, a nodal surface  $N$  is defined by a family of surfaces  $(N(c))_{c \in X_0 N}$  in  $\mathbf{bord}^\bullet$  with a negative Euler characteristic and with an involution  $s_1$  on the set of punctures  $XN$ .  $N$  is the quotient of  $\sqcup_c N(c)$  by  $s_1$ . A nodal surface defines naturally a pre-stable graph  $\mathcal{G}_N$ , and the two notions are identical.  $N$  defines a surface  $M_N$  and a family of curves such that  $M_N/\langle \Gamma \rangle$ . We denote  $\mathcal{N}$  a pair  $(\mathcal{N}, \vartheta)$  with  $\varsigma : X_0 N \longrightarrow \{0, 1\}$  is a map that satisfies:

- $(\mathcal{G}_N, \varsigma)$  is a bipartite graph.

- All the components at level 1 are stable.
- If  $M_N$  has a non-empty boundary, we assume that all the components at level 1 have at least one boundary component.

**Subsurfaces:** For each  $M$  a topological surface, we denote  $\text{Sub}(M)$  the set of isotopy classes of maps:

$$f : N \longrightarrow M.$$

With  $N$  a stable surface in **bord\***. We use the notation  $\mathcal{D} = (f, N, M)$ , and we assume:

- Each connected component of  $N$  is stable and contains at least one boundary component of  $M$ . Moreover, if  $\beta \in \partial M$  and  $\beta \cap f(N) \neq \emptyset$ , then  $\beta \in \pi_0(\partial f(N))$ .
- $\mathcal{D}^0 = M \setminus f(N)$  is open in  $M$  and has no component isomorphic to a disc or a disc with one marked point.

We also denote  $\mathcal{D}^1 = f(N)$ . The surface  $\mathcal{D}^0$  is a stable subsurface with an union of disjoint cylinders; moreover, the core curves of the cylinders are pairwise non-homotopic. We denote  $\Gamma_{\mathcal{D}} = \partial \mathcal{D}^1 \setminus \partial M$ . This family of curves defines a "pre-stable" graph  $\mathcal{G}_{\mathcal{D}}$ , which can have components isomorphic to cylinders. From the fact that all components of  $\mathcal{D}^1$  are stable, a curve in  $\Gamma_{\mathcal{D}}$  must connect one boundary of  $\mathcal{D}^1$  and one of  $\mathcal{D}^0$ . Then the decomposition  $X_0 \mathcal{G}_{\mathcal{D}} = \pi_0(\mathcal{D}^0) \sqcup \pi_0(\mathcal{D}^1)$  defines a level map:

$$\varsigma : X_0 \mathcal{G}_{\mathcal{D}} \longrightarrow \{0, 1\}.$$

From the definition, the pair  $(\mathcal{G}_{\mathcal{D}}, \varsigma)$  is a bipartite graph. Moreover, all the components at level 1 are stable and contain at least one boundary component of  $M$ . We can define a surface  $M_{\mathcal{D}}^1$ , which is in **bord\***, by

$$M_{\mathcal{D}}^1 = \mathcal{D}_{(\Gamma_{\mathcal{D}})}^1.$$

We collapse all the boundary components connected to a component at level 0.

In the case of an oriented surface  $M^\circ$ , the definition is similar, but we add the following condition:

- $\mathcal{D}^\circ \in \text{Sub}(M^\circ)$  if  $\mathcal{D} \in \text{Sub}(M)$  and the restriction of the direction of  $M^\circ$  induces a direction on  $M_{\mathcal{D}}^1$ .

In both cases, the mapping class group acts on  $\text{Sub}(M)$  and  $\text{Sub}(M^\circ)$ ; two elements are in the same orbit iff they have the same pre-stable graph. Then we define

$$\text{sub}(M) = \text{Sub}(M)/\text{Mod}(M)$$

and  $\text{sub}(M^\circ)$  by analogy.

**Support of a multi-arc:** Let  $A$  be a multi-arc. We construct a subsurface  $\mathcal{D}_A$ , which is in some sense the support of  $A$  when  $A$  does not fill the surface. We say that a subsurface  $\mathcal{D}$  supports  $A$  if:

- For all arcs  $a$  in  $A$ , up to isotopies,  $a$  is contained by  $\mathcal{D}^1$ .

We give a picture of the construction in figure 7.1.

**Proposition 7.1.** *Let  $A$  be a multi-arc in  $\mathcal{MA}(M)$ . There is a unique subsurface  $\mathcal{D} = \mathcal{D}_A$  such that:*

- *For all  $\gamma \in \mathcal{S}(\mathcal{D}^0)$ , the curve  $\gamma$  does not intersect  $A$ .*
- *$A$  induces a filling multi-arc on  $M_{\mathcal{D}}^1$ .*

*When  $M^\circ$  is directed and  $A^\circ$  is oriented, the direction on  $M^\circ$  induces a direction  $\mathcal{D}_{A^\circ}^\circ$  on  $\mathcal{D}_A$ .*

The subsurface  $\mathcal{D}_A$  can also be characterized by the following property:

- If  $\mathcal{D}$  supports  $A$ , then we must have  $\mathcal{D}_A \leq \mathcal{D}$  (in the sense that  $\mathcal{D}_A^0 \subset \mathcal{D}^0$ ).

From this proposition, we can derive the following corollary that provides a characterization for filling multi-arcs.

**Corollary 7.1.** *A multi-arc is filling iff it satisfies one of the two equivalent conditions:*

- *$\iota(A, \gamma) > 0$  for all  $\gamma \in \tilde{\mathcal{S}}(M)$ .*
- *$M_{\mathcal{D}_A}^1 = M$ .*

*Proof.* (Proposition 7.1) Let  $A \in \mathcal{MA}(M)$ , we can consider a tubular neighborhood  $U_A$  of  $A \cup \partial M$ . Let  $\gamma$  be a connected component of  $\partial U_A$ ; then  $\gamma$  is a simple closed curve, and several cases can occur:

- $\gamma$  is homotopic to a boundary component.
- $\gamma$  retracts to a puncture.
- $\gamma$  is contractible.
- $\gamma$  is in  $\mathcal{S}(M)$ .

Let  $\Gamma_A \in \mathcal{MS}(M) \cup \{\emptyset\}$  be the union of the curves that fall in the first and fourth cases. Let  $I_A$  be the set of boundary components  $\beta$  such that  $\iota(A, \beta) \neq 0$ , then  $\Gamma_A \sqcup I_A$  bounds a subsurface  $\mathcal{D}$ . Moreover, we can see that  $A$  induces a filling multi-arc on  $\mathcal{D}^1$ . By construction, a curve in  $\mathcal{D}^1$  that does not intersect  $A$  either retracts on a point or a curve in  $\Gamma_A$ . By assumption, for all curves  $\gamma \in \mathcal{D}^0$ , we must have  $\iota(A, \gamma) = 0$ . □

**Nodal ribbon graphs:** A nodal ribbon graph  $R$  on  $M$  can be defined as a subsurface  $\mathcal{D} \in \text{Sub}(M)$  and a ribbon graph on each connected component of  $M_{\mathcal{D}}^1$ , with marked vertices at the marked points. In the case of directed surfaces, the definition is similar; we only assume that each connected component of the ribbon graph is oriented. Using the duality between filling multi-arcs and ribbon graphs and also the proposition 7.1 on the support of a multi-arc, we can derive the following lemma:

**Proposition 7.2.** *The set  $\overline{\text{Rib}}(M)$  (resp  $\overline{\text{Rib}}(M^\circ)$ ) of nodal ribbon graphs on  $M$  (resp  $M^\circ$ ) is identified with the set of multi-arcs  $\mathcal{MA}(M)$  (resp  $\mathcal{MA}(M^\circ)$ ).*

Nodal ribbon graphs are related to degenerations of ribbon graphs by using proposition 7.1. It asserts that we can always obtain a nodal ribbon graph by collapsing the edges of a usual (or nodal) ribbon graph.

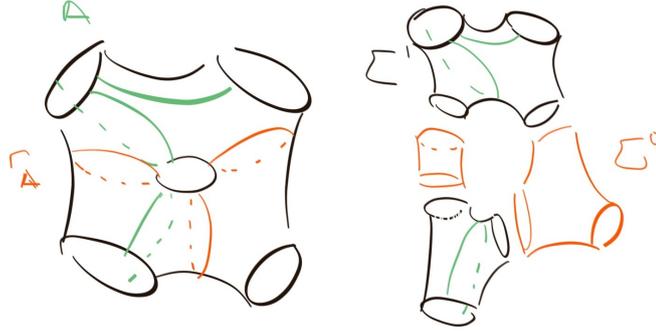


Figure 7.1: Support of a multi-arc (in black) and the complement (in orange)

**Lemma 7.1.** *If  $R$  (resp.  $R^\circ$ ) is a nodal ribbon graph and  $E \in X_1R$  is a set of edges that is not equal to  $X_1R$ , there is a nodal ribbon graph  $R_{\langle E \rangle}$  (resp. oriented nodal ribbon graph) obtained by collapsing the edges in  $E$ .*

*Proof.* We apply 7.1. To  $R$ , using proposition 7.2, we can associate a multi-arc  $A(R)$ . Let  $E \subset X_1R$ ; we still denote  $E$  the set of arcs in  $A(R)$ , then  $A' = A(R) \setminus E$  is still a multi-arc. Let  $\mathcal{D} = \mathcal{D}_{A'}$ , then by lemma 7.1 the multi-arc  $A'$  induces a filling multi-arc on  $\mathcal{D}^1$  and then, by using proposition 4.4, a ribbon graph on  $M_{\mathcal{D}}^1$ .  $\square$

**Decorated subsurface:** As we see, a nodal ribbon graph defines a subsurface. We restrict ourselves to the case of oriented ribbon graphs. Let  $R^\circ$  on  $M^\circ$  and  $\mathcal{D}^\circ$  the support of  $R^\circ$ .  $R^\circ$  defines several decorations on  $\mathcal{D}^\circ$ :

- For each  $c \in X_0^1\mathcal{D} = \pi_0(\mathcal{D}^1)$ , the component  $R^\circ(c)$  defines a partition  $\nu_c = \nu_{R^\circ(c)}$  that corresponds to the unmarked vertices only.
- For each  $\gamma \in \Gamma_{\mathcal{D}} = X_1\mathcal{G}_{\mathcal{D}}$ , we have an integer  $\kappa_\gamma$ , where  $2\kappa_\gamma + 2$  is the degree of the marked vertex located at the marked points corresponding to  $\gamma$  in  $M_{\mathcal{D}}^1$ .

These data satisfy the following conditions:

- For each  $c$  with  $\varsigma(c) = 1$ , let  $\kappa(c)$  be the vector whose components are the  $\kappa_\gamma$  such that  $\gamma^\bullet$  is a marked point of the component  $M_{\mathcal{D}}^1(c)$ . The pair  $(\nu_c, \kappa(c))$  must satisfy the Euler Poincaré formula:

$$d(\nu(c)) + d(\kappa(c)) = 2g(c) - 2 + n^+(c) + n^-(c).$$

These conditions can be used to define a decorated subsurface.

**Definition 7.1.** *A decorated subsurface  $\overline{\mathcal{D}}^\circ$  is the data of  $(\mathcal{D}^\circ, (\nu(c))_{c, \varsigma(c)=1}, (\kappa_\gamma))$  that satisfies the last conditions.*

We denote  $\overline{\text{Sub}}(M^\circ)$  the set of decorated subsurfaces; the mapping class group acts on this set; and we denote  $\overline{\text{sub}}(M^\circ)$  the quotient. Then, for a nodal ribbon graph  $R^\circ$ , we can associate a decorated subsurface  $\overline{\mathcal{D}}_{R^\circ}^\circ$ . For each  $c$  with  $\varsigma(c) = 0$  let  $\kappa(c)$  the vector of the  $\kappa_\gamma$

such that  $\gamma^\bullet$  is a puncture in  $M_D^0$  we denote  $\nu(c)$  the partition with a single block of size  $\nu_c = d(\kappa(c)) + 2g(c) - 2 + n^+(c) + n^-(c)$

$$\nu(c) = \delta_{\nu_c}.$$

In we have  $n(\nu(c)) = 1$ . There is a natural order relation on degeneration's. We say that  $\overline{\mathcal{D}}^\circ$  is a degeneration of  $(M^\circ, \nu)$  if:

- $\mathcal{D}^\circ \in \text{Sub}(M^\circ)$ .
- The decoration defined by

$$\sum_{\varsigma(c)=1} \nu(c) + \sum_{\varsigma(c)=0} \nu(c)$$

is a degeneration of  $\nu$ , see 3.2 for precision on the terminology.

In a similar way, we can define degenerations of decorated subsurfaces. This defines a partial order relation on the set  $\overline{\text{Sub}}(M^\circ)$  (and also on the quotient  $\overline{\text{sub}}(M^\circ)$ ). If  $\overline{\mathcal{D}}^\circ$  is fixed, we denote  $\overline{\text{Sub}}(\overline{\mathcal{D}}^\circ)$  and  $\overline{\text{sub}}(\overline{\mathcal{D}}^\circ)$  the degeneration's of  $\overline{\mathcal{D}}^\circ$ .

### 7.1.2 Combinatorial moduli spaces, compactification and volumes

**Compactification:** The construction of the compactification of the moduli spaces of metric ribbon graphs  $\mathcal{M}_{g,n}^{\text{comb}}$  was given for the first time in [Kong2]. It uses nodal ribbon graphs, and in an equivalent way, it can be defined by using weighted multi-arcs. For each  $M \in \mathbf{bord}$

$$\overline{\mathcal{T}}^{\text{comb}}(M) = \mathcal{MA}_{\mathbb{R}}(M)$$

By the result of proposition 4.4, the space  $\mathcal{T}^{\text{comb}}(M) = \mathcal{MA}_{\mathbb{R}}^0(M)$  corresponds to filling multi-arcs, and then  $\overline{\mathcal{T}}^{\text{comb}}(M) \setminus \mathcal{T}^{\text{comb}}(M)$  is the space of nodal ribbon graphs. The space  $\overline{\mathcal{T}}^{\text{comb}}(M)$  is a cell complex, but it's not locally finite. We can fix this issue, the mapping class group acts on  $\overline{\mathcal{T}}^{\text{comb}}(M)$  and the quotient

$$\overline{\mathcal{M}}^{\text{comb}}(M) = \overline{\mathcal{T}}^{\text{comb}}(M) / \text{Mod}(M).$$

is now a finite orbifold cell complex. The fact that  $\overline{\mathcal{M}}^{\text{comb}}(M)$  is the compactification of  $\mathcal{M}^{\text{comb}}(M)$  comes from the following lemma:

**Proposition 7.3.** *The quotient  $\mathbb{P}\overline{\mathcal{M}}^{\text{comb}}(M) = \overline{\mathcal{M}}^{\text{comb}}(M) / \mathbb{R}_{>0}$  is compact. And for each  $L \in (\mathbb{R}_{>0})^n$ , the space  $\overline{\mathcal{M}}^{\text{comb}}(M, L)$  is also compact.*

*Proof.* To prove this result, we can use the fact that the quotient  $\mathcal{MA}(M) / \text{Mod}(M)$  is a finite set. There is only a finite number of combinatorial ribbon graphs, and  $\mathcal{MA}^0(M) / \text{Mod}(M)$  is in bijection with  $\text{rib}(M)$ . For a non-filling multi-arc  $A$ , we can associate a subsurface  $\mathcal{D}$  such that  $A$  is filling on  $M_{\mathcal{D}}^1$ . Up to the action of the mapping class group, there is a finite number of subsurfaces, and then  $\mathcal{MA}(M) / \text{Mod}(M)$  is finite. Note that each closed simplex  $\mathbb{R}_+^A \setminus \{0\}$  is compact in  $\mathbb{P}\mathcal{MA}_{\mathbb{R}}(M)$ , and then  $\mathbb{P}\mathcal{MA}(M) / \text{Mod}(M)$  is compact because we can cover the space with a finite number of cells isomorphic to  $\mathbb{R}_{\geq 0}^A \setminus \{0\} / \text{Aut}(A)$ .  $\square$

In a similar way, we can define  $\overline{\mathcal{M}}^{comb}(M^\circ)$  as the compactification of the space of oriented ribbon graphs. We have

$$\overline{\mathcal{M}}^{comb}(M^\circ) = \overline{\mathcal{T}}^{comb}(M^\circ)/\text{Mod}(M),$$

where  $\overline{\mathcal{T}}^{comb}(M^\circ) = \mathcal{MA}_{\mathbb{R}}(M^\circ)$  is the space of oriented multi-arcs; it's mapped to a closed subcomplex of  $\overline{\mathcal{M}}^{comb}(M)$ .

**Boundary components and subsurfaces:** We mainly focus on the case of directed surfaces for simplicity. As we see in 4.1.4, the moduli space  $\mathcal{M}^{comb}(M^\circ)$  is stratified by specifying the vertices of the graphs. Stratum's are indexed by decorated surfaces  $\overline{M}^\circ = (M^\circ, \nu)$ ; each stratum admits a compactification  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ)$ , which is given by taking the closure of  $\mathcal{M}^{comb}(\overline{M}^\circ)$  in  $\overline{\mathcal{M}}^{comb}(M^\circ)$  for the topology given by the structure of the cell complex. The space  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ)$  also admits stratification, given this time by decorated sub-surfaces. Let  $\overline{\mathcal{D}}^\circ$  we denote  $\mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ)$  the set of nodal metric ribbon graphs  $(R^\circ, m)$  with  $\overline{\mathcal{D}}_{R^\circ}^\circ = \overline{\mathcal{D}}^\circ$  and as before we consider  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ)$  it's closure in  $\overline{\mathcal{M}}^{comb}(M^\circ)$ . In this way, we obtain a stratification of  $\overline{\mathcal{M}}^{comb}(M^\circ)$ . If  $\overline{\mathcal{D}}_1^\circ \leq \overline{\mathcal{D}}_2^\circ$  (see section 7.1.1), we have the natural inclusion  $\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}_1) \subset \overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}_2)$ ; moreover, the stratification is explicitly given by

$$\overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ) = \bigsqcup_{\overline{\mathcal{D}}_1^\circ \leq \overline{\mathcal{D}}^\circ} \mathcal{M}^{comb}(\overline{\mathcal{D}}_1^\circ).$$

**Image of  $L_\partial$  and volumes:** If  $R^\circ$  is an oriented nodal ribbon graph supported by  $\overline{\mathcal{D}}^\circ$  we have

$$\text{Met}(R^\circ) = \prod_{c, s(c)=1} \text{Met}(R^\circ(c)).$$

Then the image of the map  $L_\partial$  is an open set in

$$\prod_{c, s(c)=1} \Lambda_{M_{\overline{\mathcal{D}}^\circ}(c)}.$$

This subspace defines a wall  $W(R^\circ) \in \mathbf{Wall}(M^\circ)$ , according to subsection 3.5.3. The wall  $W(R^\circ)$  is indeed entirely determined by  $\overline{\mathcal{D}}^\circ$ ; we denote it  $W(\overline{\mathcal{D}}^\circ)$ , then  $W(R^\circ) = W(\overline{\mathcal{D}}^\circ)$ . Then we have a map:

$$L_\partial : \overline{\mathcal{M}}^{comb}(\overline{\mathcal{D}}^\circ) \longrightarrow \Lambda_{W(\overline{\mathcal{D}}^\circ)}.$$

There is a natural measure  $d\mu_{\overline{\mathcal{D}}^\circ}$  supported on the top cell of  $\mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ)$  and  $d\mu_{\overline{\mathcal{D}}^\circ}(L)$  on  $\mathcal{M}^{comb,*}(\overline{\mathcal{D}}^\circ, L)$  for each  $L \in \Lambda_{W(\overline{\mathcal{D}}^\circ)}$ . We denote

$$V_{\overline{\mathcal{D}}^\circ}(L) = \int_{\mathcal{M}^{comb,*}(\overline{\mathcal{D}}^\circ, L)} d\mu_{\overline{\mathcal{D}}^\circ}(L),$$

the volume of the moduli space. Then, as in the usual case, we have the following relation:

$$L_{\partial*} d\mu_{\overline{\mathcal{D}}^\circ} = V_{\overline{\mathcal{D}}^\circ} d\sigma_{W(\overline{\mathcal{D}}^\circ)}.$$

## 7.2 Apparent degenerations

In this part, we study in more detail the degenerescence of ribbon graphs. We start by introducing apparent degenerations that are related to discontinuities in the volumes. We characterize these degenerations in terms of their topology and also characterize the associated boundary component in the moduli space. We study the normal cone near apparent degeneration.

### 7.2.1 Apparent degenerations

We start with the following definition:

**Definition 7.2.** A nodal ribbon graph  $R_1^\circ < R^\circ$  is an apparent degeneration of  $R^\circ$  of order  $k \geq 1$  if:

- $\dim(K_{R_1}) = \dim(K_R)$ ,
- $\dim(T_{R_1}) + k = \dim(T_R)$ .

Notice that according to the definition, a ribbon graph  $R_1^\circ$  is a apparent degeneration relatively to a given ribbon graph  $R^\circ$ . We say that  $R_1^\circ$  is a apparent boundary in  $\mathcal{M}^{comb}(\overline{M}^\circ)$  if there is some  $R^\circ \in \text{rib}^*(\overline{M}^\circ)$  such that  $R_1^\circ$  is a apparent degeneration of  $R^\circ$ . Apparent boundaries cause trouble because the space  $\text{Met}(R_1^\circ, L)$  is then a top cell in the stratum  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ, L)$ , then apparent boundaries contribute to the volume of the moduli space. We denote  $\text{Rib}^{ap}(\overline{M}^\circ)$  and  $\text{rib}^{ap}(\overline{M}^\circ)$  the set of apparent boundaries.

The following lemma says that an apparent degeneration can happen only on walls in  $\Lambda_{M^\circ}$ .

**Lemma 7.2.** Let  $R^\circ$  be a apparent boundary of order  $k$ , the wall  $W(R^\circ)$  is of codimension  $k$  in  $\Lambda_{M^\circ}$ .

*Proof.* Assume  $R^\circ \leq R_1^\circ$  with  $R_1^\circ$  an usual graph, we have

$$\dim T_{M^\circ} - \dim TL_\partial(T_{R^\circ}) = \dim T_{R_1^\circ} - \dim K_{R_1^\circ} - (\dim T_{R^\circ} - \dim K_{R^\circ}) = k.$$

□

In other words, the image of  $L_\partial$  is an open cone in a codimension  $k$  subspace of  $\Lambda_{M^\circ}$ . A direct consequence of the last proposition is the following corollary, which fails to be true when the graph is non-oriented.

**Lemma 7.3.** If  $R^\circ$  is an apparent boundary, it's a nodal graph.

*Proof.* Indeed, according to proposition 4.3, if the rank of the degeneration corresponds to  $\#\pi_0(R^\circ) = k + 1$ , then  $R_1^\circ$  is not connected and hence is a nodal ribbon graph. □

**Topology of apparent degeneration:** The following proposition describes the different possibilities of apparent degenerations in the oriented case.

**Proposition 7.4.** If  $R^\circ$  is in the apparent boundary of order  $k$  of  $\mathcal{M}^{comb}(\overline{M}^\circ)$ , then:

- The “pre-stable” graph associated with  $R^\circ$  is a bipartite tree.

- For each vertex  $c$  at level 1, we have  $\nu(c)$  a partition and an integer  $k_\gamma \geq 1$  for each curve in the pre-stable graph.
- All the components at level 0 are spheres; moreover, there is a vertex  $v_c$  of degree  $\nu_c$  such that:

$$\nu_c = \sum_{\gamma} (k_\gamma + 2) - 2,$$

where we sum over all the nodal points connected to  $c$  and

$$\nu = \sum_{\varsigma(c)=1} \nu(c) + \sum_{\varsigma(c)=0} \nu(c).$$

with  $\nu(c) = \delta_{\nu_c}$  if  $\vartheta(c) = 0$

**Maximal sub-surface:** Consider  $\overline{\mathcal{D}}^\circ$  a decorated subsurface. We define  $\dim(\overline{\mathcal{D}}^\circ)$  as the dimension of  $K_R$  for  $R^\circ$ , a generic graph in  $\mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ)$ . We said that a subsurface  $\overline{\mathcal{D}}^\circ$  is maximal in  $\overline{\text{sub}}(\overline{\mathcal{M}}^\circ)$  if

$$\dim(\overline{\mathcal{D}}^\circ) = \dim(\overline{\mathcal{M}}^\circ).$$

We have the following elementary fact:

**Lemma 7.4.** *A nodal ribbon graph  $R^\circ$  is an apparent boundary iff  $\overline{\mathcal{D}}_{R^\circ}^\circ$  is maximal.*

In this part, we study the topology of maximal decorated subsurfaces.

**Proposition 7.5.** *A decorated subsurface  $\overline{\mathcal{D}}^\circ = (\mathcal{D}^\circ, \nu, \kappa)$  in  $\overline{\text{sub}}(\overline{\mathcal{M}}^\circ)$  is maximal iff:*

- The graph  $\mathcal{G}_{\overline{\mathcal{D}}^\circ}$  is a bipartite tree.
- Each component at level 0 is a sphere.
- The decoration  $\nu_{\overline{\mathcal{D}}^\circ}$  satisfies

$$\nu_{\overline{\mathcal{D}}^\circ} = \nu_{\overline{\mathcal{M}}^\circ}.$$

We can prove this by using the following formula. We denote  $g_{\overline{\mathcal{D}}^\circ}^0$  the genus of the surface  $\mathcal{D}_0$ .

**Lemma 7.5.** *The dimension  $\dim(\overline{\mathcal{D}}^\circ)$  is given by:*

$$\dim(\overline{\mathcal{D}}^\circ) = \dim(\nu_{\overline{\mathcal{D}}^\circ}) - 2g_{\overline{\mathcal{D}}^\circ}^0 + \chi(\mathcal{G}_{\overline{\mathcal{D}}^\circ}) - n + 1.$$

With  $\dim(\nu) = d(\nu) + n(\nu)$ .

*Proof.* Starting from the RHS by definition of  $\nu_{\overline{\mathcal{D}}^\circ}$

$$\dim(\nu_{\overline{\mathcal{D}}^\circ}) = \sum_{c, \varsigma(c)=1} (d(\nu(c)) + n(\nu(c))) + \sum_{c, \varsigma(c)=0} (\nu_c + 1).$$

We have

$$\nu_c + 1 = 2g(c) - 1 + n(c) + \sum_{\gamma \sim c} (k_\gamma + 1).$$

Then

$$\sum_{c, \varsigma(c)=0} (\nu_c + 1) = 2g_{\overline{\mathcal{D}}^\circ}^0 - \#X_0^0 \mathcal{G}_{\overline{\mathcal{D}}^\circ} + n_0 + \#X_1 \mathcal{G}_{\overline{\mathcal{D}}^\circ} + \sum_{\gamma} \kappa_\gamma,$$

with  $\#X_0^0\mathcal{G}_D$  the number of components at level 0, and  $n_0$  is the number of boundaries of these components. On the other side,

$$\dim(\overline{\mathcal{D}}^\circ) = \sum_{c, \varsigma(c)=1} (\dim(\nu(c)) + d(\kappa(c)) + n(\kappa(c)) - n(c) + 1) = \dim(\nu) + \#X_0^1\mathcal{G}_D + \sum_{\gamma} (\kappa_{\gamma} + 1).$$

Then, by eliminating  $\sum_{\gamma} \kappa_{\gamma}$ , we deduce

$$\dim(\overline{\mathcal{D}}^\circ) = \dim(\nu_{\overline{\mathcal{D}}^\circ}) - 2g_D^0 + \#X_0^0\mathcal{G}_D + \#X_0^1\mathcal{G}_D - \#X_1\mathcal{G}_D - n.$$

Finally, using  $\chi(\mathcal{G}_D) = \#X_0\mathcal{G}_D - \#X_1\mathcal{G}_D - 1$ , we obtain the formula.  $\square$

We now give a proof of the proposition.

*Proof.* The codimension of  $\mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ, L)$  inside  $\mathcal{M}^{comb}(\overline{M}^\circ, L)$  is given by  $\dim(\nu) - \dim(\overline{\mathcal{D}}^\circ)$ . By the formula of the lemma 7.5, we have:

$$\dim(\mathcal{M}^{comb}(\overline{M}^\circ, L)) - \dim(\mathcal{M}^{comb}(\overline{\mathcal{D}}^\circ, L)) = (\dim(\nu) - \dim(\nu_{\overline{\mathcal{D}}^\circ})) + 2g_D - \chi(\mathcal{G}_D).$$

Notice that all the elements in the RHS are positive, and then the sum vanishes iff

$$\dim(\nu) - \dim(\nu_{\overline{\mathcal{D}}^\circ}) = 0, \quad g_D = 0, \quad \text{and} \quad \chi(\mathcal{G}_D) = 0.$$

From the first equality, as we have

$$\dim(\nu) - \dim(\nu_D) = n(\nu) - n(\nu_D),$$

then  $n(\nu) = n(\nu_D)$ , but  $\nu \geq \nu_D$ , which implies that  $\nu = \nu_D$ . From the second, we deduce that all the components at level zero are spheres; from the third, we deduce that the graph  $\mathcal{G}_D^\circ$  is a tree.  $\square$

## 7.2.2 Normal cone

**Definition of the normal cone:** We define the normal cone of a ribbon graph and an oriented ribbon graph. Let  $R$  be a ribbon graph, and let  $R_1 \leq R$  be a degeneration of  $R$ . We denote  $T_{R_1}(R)$  the subset of vectors in  $T_R$  such that  $m_e(x) \geq 0$  for all  $e \in X_1R \setminus X_1R_1$ . The tangent space  $T_{R_1}$  acts on  $T_{R_1}(R)$ , and we consider  $N_{R_1}(R)$  the quotient that is canonically identified with  $\mathbb{R}_{\geq 0}^{X_1R \setminus X_1R_1}$ . Let  $R$  be a possibly nodal ribbon graph on  $M$ . If we consider two ribbon graphs, and  $R_1, R_2$  with  $R \leq R_1 \leq R_2$ , we have a linear map

$$T_R(R_1) \longrightarrow T_R(R_2), \quad \text{and} \quad N_R(R_1) \longrightarrow N_R(R_2).$$

We can see that the collection of the  $T_R(R_1)$  (resp.  $N_R(R_1)$ ) with  $R_1 \geq R$  forms an inductive system. Then the tangent and normal cone to  $R$  are:

$$T_R(M) = \varinjlim_{R_1 \geq R} T_R(R_1), \quad \text{and} \quad N_R(M) = \varinjlim_{R_1 \geq R} N_R(R_1).$$

The tangent space  $T_R$  still acts on  $T_R(M)$  and  $N_R(M)$  is the quotient. The space  $N_R(M)$  has a structure of cell complex, but when the graph  $R$  is nodal, it's neither finite nor locally finite.

Nevertheless, as in the case of moduli space, the stabilizer  $\text{Stab}(R)$  is infinite, and we can see that the problem disappears when we take the quotient. Then

$$\mathcal{N}_R(M) = N_R(M)/\text{Stab}(R)$$

is a finite orbifold cell complex.

If  $M^\circ = (M, \epsilon)$  is a directed surface and  $R^\circ = (R, \epsilon)$  is an oriented ribbon graph, we can also define the cones  $T_{R^\circ}(M^\circ)$  and  $\mathcal{N}_{R^\circ}(M^\circ)$ . These cones can be mapped to  $T_R(M)$  and  $\mathcal{N}_R(M)$  and define sub-cones.

If we fix  $\overline{M}^\circ$  (resp  $\overline{M}$ ) a decorated directed surface and  $R^\circ \in \overline{\text{rib}}(\overline{M}^\circ)$ , we can also consider the normal cone  $\mathcal{N}_{R^\circ}(\overline{M}^\circ)$  by restricting the inductive system to ribbon graph  $R_1^\circ$  in  $\overline{\text{rib}}(\overline{M}^\circ)$  (resp  $\mathcal{N}_R(M)$ ),  $\mathcal{N}_{R^\circ}(\overline{M}^\circ)$  is the normal cone of  $R^\circ$  in the stratum  $\overline{\mathcal{M}}^{\text{comb}}(\overline{M}^\circ)$ .

**Normal map:** Let  $M^\circ$  and  $W \in \mathbf{Wall}(M^\circ)$ . We can also define  $N_W(M^\circ) = T_{M^\circ}/T_W$  as the normal cone of the subspace  $\Lambda_W$ . This vector space is identified with

$$\{(x_c) \in \mathbb{R}^{X_0 W} \mid \sum_{c \in X_0 W} x_c = 0\},$$

by using the maps

$$N_{W,c}(L) = \sum_{i \in I_W^+(c)} L_i^+ - \sum_{i \in I_W^-(c)} L_i^- = |L_{I_W^+}^+| - |L_{I_W^-}^-|.$$

The following lemma is straightforward.

**Lemma 7.6.** *Let  $W \in \mathbf{Wall}(M^\circ)$  and  $R^\circ \in \overline{\text{rib}}(M^\circ)$  such that  $W(R^\circ) = W$ , then  $TL_\partial$  induces*

$$TL_\partial : N_{R^\circ}(\overline{M}^\circ) \longrightarrow N_W.$$

*Proof.* Under these assumptions, the tangent map

$$T_{R^\circ}(R_1^\circ) \longrightarrow T_{M^\circ},$$

is compatible with the inductive limit and then induces

$$T_{R^\circ}(\overline{M}^\circ) \longrightarrow T_{M^\circ}.$$

Moreover, the image of  $T_{R^\circ}$  belongs to  $T_W$  because  $W(R^\circ) = W$ , and then we have the normal map

$$N_{R^\circ}(\overline{M}^\circ) \longrightarrow N_W.$$

□

The subset

$$\bigsqcup_{M^\circ > W' \geq W} T_{W'}$$

is invariant by the action of  $T_W$  and also its complement in  $T_{M^\circ}$ . Then we can consider  $N_W^*$  the quotient

$$N_W^* = (T_{M^\circ} \setminus \bigsqcup_{M^\circ > W' \geq W} T_{W'})/T_W,$$

which is the subset of  $N_W$  given by

$$\{(x_c) | x_c \neq 0 \sum_c x_c = 0\}.$$

$N_W$  has a cell structure, and  $N_W^*$  corresponds to the top cells.

If  $R^\circ$  is a nodal ribbon graph in  $\overline{\text{rib}}(\overline{M}^\circ)$  such that  $W(R^\circ) = W$ , we denote  $N_{R^\circ}^*(\overline{M}^\circ)$  the pre-image of  $N_W^*$  by  $TL_\partial$ .  $N_{R^\circ}(\overline{M}^\circ)$  is a cell complex, but the map  $TL_\partial$  does not preserve the structure of the cell complex. To obtain a morphism of cell complexes, we need to take the pre-image of the cell decomposition in  $N_W$ . Then  $N_{R^\circ}^*(\overline{M}^\circ)$  corresponds to the top cells of this new decomposition of  $N_{R^\circ}(\overline{M}^\circ)$ .

### 7.2.3 Apparent boundaries and normal cones

**Normal map of an apparent boundary:** Let  $\overline{M}^\circ, W$  and  $R^\circ$  such that  $R^\circ \in \overline{\text{rib}}(\overline{M}^\circ)$  and  $W(R^\circ) = W$ .

**Lemma 7.7.** *Assume that  $R^\circ \in \overline{\text{rib}}^{ap}(\overline{M}^\circ)$ , then the normal map:*

$$TL_\partial : N_{R^\circ}^*(\text{Mod}) \longrightarrow N_W^*,$$

*is a local covering.*

*Proof.* The map is linear on each cell of  $N_{R^\circ}^*$  and locally surjective by using proposition 4.3. By computing dimensions, we can see that the kernel is zero, and then the map is a local covering.  $\square$

**Degree of an apparent boundary:** In this part, we give results on the degree.

**Proposition 7.6.** *The degree of the map*

$$TL_\partial : N_{R^\circ}^*(\overline{M}^\circ) \longrightarrow N_W^*,$$

*is constant over  $N_W^*$  and is denoted  $\text{deg}_{R^\circ}(\overline{M}^\circ)$ .*

To prove this proposition, it seems that it suffices to prove that it's actually true in codimension one. In this case, the situation is much easier. Either the degeneration is a ringlet and there is only one sector in  $N_W(M^\circ)^*$ , and then there is nothing to prove. In the second case, there are only two sectors in  $N_W^*(M^\circ)$ , and we only need to check that the counts agree on the two sides. In this case, we also have the order of the ramification on a codimension one apparent boundary.

**Lemma 7.8.** *If  $R^\circ$  is an apparent ribbon graph of order one and  $\overline{\mathcal{D}}_{R^\circ}^\circ$  is a bipartite tree with two marked vertices (nodes) of degree  $2k_1, 2k_2$ , then, on each simplex of  $N_{M^\circ, W}^*$ , the degree of  $TL_\partial$  is  $k_1 k_2$ .*

*Proof.* This proposition is obtained by counting the number of ways to glue two orientable nodal ribbon graphs by identifying two vertices of degree  $2k_1, 2k_2$  (see figure 7.2). To do that, we add a simple loop around each vertex, and we glue the two ribbon graphs along this loop. There are  $2k_1 k_2$  ways to do that because we want an oriented graphs. But according to the

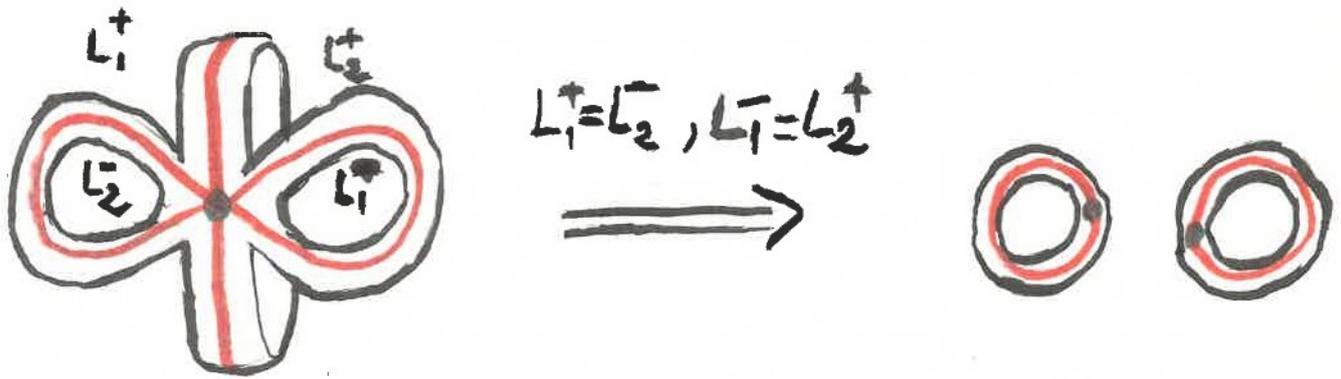


Figure 7.2: An apparent degeneration of order 1

position of the loop modulo two around the vertices, the boundary associated with the loop can have two different orientations. This parameter determines the sector in the normal bundle, and moving the loops around the vertices changes the sector. Then we have an involution on the normal bundle which is a leaf of  $x \rightarrow -x$ . Then we can conclude that the degree is equal on each side and given by  $k_1 k_2$ .  $\square$

*proof proposition 7.6.* To prove the proposition in general, we use the lemma 7.8 to see that the count is constant when we cross a wall of codimension one in  $N_W \setminus N_W^*$ . For the points of the fiber that degenerate into a nodal ribbon graph, the lemma ensures that the count is equal on each side of the submanifold. If a point does not degenerate (it corresponds to the walls that we add in the decomposition), then things are also going well.  $\square$

### Locality of the degree:

**Lemma 7.9.** *The degree  $\deg_{\overline{M}^\circ}(R^\circ)$  depends only on the boundary  $\overline{\mathcal{D}}^\circ(R^\circ)$  and  $\overline{M}^\circ$  and is denoted  $\deg_{\overline{M}^\circ}(\overline{\mathcal{D}}^\circ)$ .*

*Proof.* It's consequence of proposition 7.4 indeed an apparent degeneration concerns isolated vertices and are local. If  $R^\circ$  is an apparent degeneration, the degree corresponds to how many ways we can glue the marked vertex based on a given component  $c$  of the decorated subsurface  $\overline{\mathcal{D}}_{R^\circ}^\circ$  with  $\varsigma(c) = 0$ . Then the degree only depends on the parameter  $(\kappa_\gamma)_\gamma$ , which represents the degree of the "nodes". Moreover, degenerations that occur at different vertices are independent, and we can write

$$\deg_{\overline{M}^\circ}(R^\circ) = \deg_{\overline{M}^\circ}(R^\circ) = \prod_c H(\kappa(c))$$

For a function  $H(\kappa(c))$  defined on multi-indices.  $\square$

**Conjecture on the degree:** We give the following conjecture for the degree, which is true in many cases:

**Conjecture 7.1.** *The function  $H(\kappa)$  is given by*

$$H(\kappa) = \prod_{\gamma} \kappa_{\gamma} \frac{(d(\kappa) + n(\kappa) - 2)!}{d(\kappa)!}.$$

Using combinatorial bijection, we are allowed to prove this conjecture in a few cases but not in general.

## 7.3 Continuous extension

### 7.3.1 Behavior of the function $V_{R^\circ}$

**Image of the boundary lengths:** Let  $R^\circ$  be an oriented ribbon graph embedded in  $M^\circ$ , and

$$L_\partial : \text{Met}(R^\circ) \rightarrow \Lambda_{M^\circ}.$$

The image of this map is a convex cone, and we denote it  $\Lambda_{R^\circ}$ . If  $\overline{\Lambda}_{R^\circ}$  is the closure of  $\Lambda_{R^\circ}$ . A general consideration on image of a convex set gives:

$$L_\partial(\overline{\text{Met}}(R^\circ)) = \overline{\Lambda}_{R^\circ}.$$

We denote the boundary

$$\partial\Lambda_{R^\circ} = \overline{\Lambda}_{R^\circ} \setminus \Lambda_{R^\circ}.$$

Let  $W = W(R^\circ)$  be the wall associated with  $R^\circ$ , then the convex set  $\Lambda_{R^\circ}$  is supported by  $\Lambda_W$ .

**Proposition 7.7.** *There is a set  $\mathbf{Wall}(R^\circ) \subset \mathbf{Wall}(\overline{\mathcal{D}}^\circ)$  and a disjoint family of convex sets  $\Lambda_{R^\circ}(W)$  for  $W \in \mathbf{Wall}(R^\circ)$  such that:*

- $\overline{\Lambda}_{R^\circ} = \bigsqcup_W \Lambda_{R^\circ}(W)$ .
- For each  $W$ ,  $\Lambda_{R^\circ}(W)$  is an open convex set in  $\Lambda_W$ ; moreover, we have

$$\overline{\Lambda}_{R^\circ}(W) = T\Lambda_W \cap \overline{\Lambda}_{R^\circ}.$$

*Proof.* Let  $L \in \partial\Lambda_{R^\circ}$  and  $x \in \overline{\text{Met}}(R^\circ)$  with  $L_\partial(x) = L$ . Let  $R_1^\circ$  be the support of  $x$ , which is the ribbon graph such that  $x \in \text{Met}(R_1^\circ)$ . The subset  $\Lambda_{R_1^\circ}$  is contained in  $\Lambda_{W(R_1^\circ)}$  with  $\text{rk}(W(R_1^\circ)) > 0$ . Then we can set

$$\mathbf{Wall}(R^\circ) = \{W(R_1^\circ) | R_1^\circ \leq R^\circ\}.$$

We can consider the open set

$$\Lambda_{R^\circ}(W) = \bigcup_{R_1^\circ \leq R^\circ, W(R_1^\circ) = W} \Lambda_{R_1^\circ}.$$

Then if  $L \in \Lambda_W^* \cap \overline{\Lambda}_{R^\circ}$  there is  $R_1^\circ$  with  $W(R_1^\circ) = W$  such that  $L \in \Lambda_{R_1^\circ}$  then  $\Lambda_{R^\circ}(W)$  contains  $\Lambda_W^* \cap \overline{\Lambda}_{R^\circ}$ . The closure of  $\Lambda_{R^\circ}(W)$  then contains the set  $\overline{\Lambda}_W \cap \overline{\Lambda}_{R^\circ}$ . By definition of  $\Lambda_{R^\circ}(W)$ , it's contained in  $\overline{\Lambda}_W \cap \overline{\Lambda}_{R^\circ}$ , so we have equality. By definition, the subsets  $\Lambda_{R^\circ}(W)$  form a partition of  $\overline{\Lambda}_{R^\circ}$ .  $\square$

In other words,  $\{\Lambda_W(R^\circ) | W \in \mathbf{Wall}(R^\circ)\}$  corresponds to the faces of the polytope  $\Lambda_{R^\circ}$ .

**Proposition 7.8.** *Let  $R^\circ$  be an oriented ribbon graph and  $W \in \mathbf{Wall}(R^\circ)$ . There is a unique  $R_W^\circ \leq R^\circ$  such that:*

$$\Lambda_{R_W^\circ} = \Lambda_{R^\circ}(W).$$

*Moreover,  $R_W^\circ$  is maximal in the sense that if  $R_1^\circ \leq R^\circ$  satisfies  $\Lambda_{R_1^\circ} \subset \Lambda_{R^\circ}(W)$ , then  $R_1^\circ \leq R_W^\circ$*

*Proof.* If we consider the set  $\{R_1^\circ | W(R^\circ) = W\}$ , we can see that it has a partial order relation; moreover, if  $R_1^\circ, R_2^\circ$  are in this set and  $x_i \in \text{Met}(R_i^\circ)$ , then if  $x_3 = \frac{x_1 + x_2}{2}$  we still have  $L_\partial(x_3) \in \Lambda_W$ . Let  $R_3^\circ$  be the support of  $x_3$ , then we have  $\Lambda_{R_3^\circ} \cap \Lambda_W \neq \emptyset$ , and then  $\Lambda_{R_3^\circ} \subset \Lambda_W$ . This implies that  $W(R_3^\circ) = W$  and  $R_3^\circ \geq R_i^\circ$ . Using this, we see that the set  $\{R_1^\circ | W(R^\circ) = W\}$  admits a unique maximal element denoted  $R^\circ(W)$ .  $\square$

**Behavior near the boundaries of the polytopes:** It's possible to prove by induction the following lemma:

**Lemma 7.10.** *Let  $R^\circ$  the function  $V_{R^\circ}(L)$  is continuous and piece-wise polynomial on  $\Lambda_{R^\circ}$ .*

But to obtain continuity of the total volumes, we need to look at the boundary of the cone  $\Lambda_{R^\circ}$ .

**Proposition 7.9.** *Let  $W \in \mathbf{Wall}(R^\circ)$ , then if  $L_n \in \Lambda_{R^\circ}$  tends to  $L_* \in \Lambda_{R^\circ}(W)$ , we have*

$$V_{R^\circ}(L_n) = O(\|N_W(L_n)\|_1^{rk_W(R^\circ)}),$$

where  $rk_W(R^\circ) = \dim K_{R^\circ} - K_{R^\circ(W)}$  is the rank of the nodal graph.

In the case of apparent boundaries, we have the following:

**Proposition 7.10.** *If in the last proposition  $rk_W(R^\circ) = 0$  then*

$$\lim_n V_{R^\circ}(L_n) = V_{R^\circ(W)}(L)$$

To prove these two propositions, we need the following lemma:

**Lemma 7.11.** *Let  $R^\circ, W$  be an  $e \in X_1 R^\circ \setminus X_1 R^\circ(W)$ , then there is  $C > 0$  a constant such as*

$$m_e(x) \leq C |N_W(x)|_1.$$

Moreover, if  $rk_W(R^\circ) = 0$ , then there is  $\alpha_e : N_W \rightarrow \mathbb{R}$  a linear form such as

$$m_e(x) = \alpha_e(N_W(x))$$

### 7.3.2 Continuity

**Proof of the continuity theorem:** The aim of this section is to give a proof of the following proposition:

**Proposition 7.11.** *For each  $\overline{M}^\circ$ , the function  $V_{\overline{M}^\circ}(L)$  on  $\Lambda_{\overline{M}^\circ}^*$  admits an extension  $\overline{V}_{\overline{M}^\circ}(L)$  on  $\Lambda_{\overline{M}^\circ}$ . Moreover, the limit at a point  $L$ , which is in  $\Lambda_{\overline{M}^\circ}^*$ , is given by*

$$\lim_n V_{\overline{M}^\circ}(L_n) = V_{\overline{M}^\circ}(L) + \sum_{R^\circ \in \overline{rib}^{ap}(\overline{M}^\circ), W \leq W(R^\circ)} \deg_{R^\circ}(\overline{M}^\circ) V_{R^\circ}(L)$$

We then define the function:

$$\bar{V}_{\bar{M}^\circ}(L) = V_{\bar{M}^\circ}(L) + \sum_{R^\circ \in \overline{\text{rib}}^{ap}(\bar{M}^\circ)} \text{deg}_{R^\circ}(\bar{M}^\circ) V_{R^\circ}(L).$$

The quantity  $\text{deg}_{R^\circ}(\bar{M}^\circ)$  is the degree of the normal map:

$$TL_\partial : N_{R^\circ}(\bar{M}^\circ) \longrightarrow N_W(M^\circ).$$

Then, using the results of the last part, we can obtain the following:

**Theorem 7.1.** *The function  $\bar{V}_{\bar{M}^\circ}$  is continuous and given by*

$$\bar{V}_{\bar{M}^\circ} = \sum_{\bar{D}^\circ \in \overline{\text{sub}}^{max}(\bar{M}^\circ)} \text{deg}_{\bar{D}^\circ}(\bar{M}^\circ) \bar{V}_{\bar{D}^\circ}.$$

*and is continuous.*

*Proof.* We start by proving that the limit of  $V_{\bar{M}^\circ}$  is well defined. Let  $L_n \in \Lambda_{M^\circ}^*$ , which converges to  $L \in \Lambda_{M^\circ}$ . If  $L$  is in  $\Lambda_{M^\circ}^*$ , Then there is a finite family  $R^\circ$  of ribbon graphs such that  $L \in \Lambda_{R^\circ}$ ,  $\Lambda_{R^\circ}$  is open, and then for  $n$  sufficiently large we also have  $L_n \in \Lambda_{R^\circ}$ . Then,

$$V_{\bar{M}^\circ}(L_n) = \sum_{R^\circ} V_{R^\circ}(L_n).$$

Using lemma 7.10, we see that we have continuity in this case. Otherwise, if there is a wall  $W$  with  $L \in \Lambda_W^*$ . There is a family of non-nodal ribbon graphs that lie in the preimage of  $L$ ; for these graphs, we can apply the same argument as before, and then the contribution of these graphs is  $V_{\bar{M}^\circ}(L)$ . Among these ribbon graphs, some of them are apparent boundaries. If  $R^\circ$  is an apparent nodal ribbon graph with  $L \in \Lambda_{R^\circ}^*$ , then the sequence  $L_n$  induces a sequence on the normal bundle  $N_W^*(\bar{M}^\circ)$ ; this space is a union of disjoint cells; moreover, for all apparent ribbon graphs, the map

$$TL_\partial : N_{R^\circ}^*(\bar{M}^\circ) \longrightarrow N_W^*(M^\circ),$$

has a constant degree given by  $\text{deg}_{R^\circ}(\bar{M}^\circ)$ . If we restrict to the indices  $n'$  such that  $L_{n'}$  is contained in a given cell of  $N_W^*(M^\circ)$ . The contribution of  $R^\circ$  to the volume is given by

$$\sum_{R_1^\circ} V_{R_1^\circ}(L_{n'}),$$

where we sum over all the  $R_1^\circ$  that degenerate to  $R^\circ$  and which are in the preimage of the cell. Using lemma 7.9 when  $n' \rightarrow \infty$  we have

$$V_{R_1^\circ}(L_{n'}) \longrightarrow V_{R^\circ}(L).$$

Now from proposition 7.6, we can see that the limit of the last sum is equal to  $\text{deg}_{R^\circ}(\bar{M}^\circ) V_{R^\circ}(L)$  and does not depend on the choice of the cell in  $N_W^*(M^\circ)$ . Then the limit is still valid for the sequence  $(L_{n'})$ . By summing the contributions of all ribbon graphs and all apparent ribbon graphs in the preimage of  $L$ , we obtain the desired formula for the limit.  $\square$

**Easy application: zero value of the volumes:** There are two special cases in this theorem.

**Corollary 7.2.** *The volumes  $V_{M^\circ}$  of the principal stratum of combinatorial surfaces are continuous on  $\Lambda_{M^\circ} \cap \mathbb{R}_{>0}^{\partial M}$ .*

*Proof.* In this case, there is no apparent degeneration except when a boundary tends to zero.  $\square$

And this confirms the results obtained by recursion 4.6. The second special case is the one of surfaces with vertex of degree at most six, which is unramified. We can consider the volume  $V_{\overline{M}^\circ}^{comp}$  of the compactification of the moduli space. In general, there is a nontrivial degree  $\deg_{R^\circ}(\overline{M}^\circ)$ , and then the volume  $V_{\overline{M}^\circ}^{comp}$  is not equal to  $\overline{V}_{\overline{M}^\circ}$  and is discontinuous. Nevertheless, we have:

**Proposition 7.12.** *If there are only vertices of degree at most six, then the volume  $V_{\overline{M}^\circ}^{comp}(L)$  of the compactification  $\overline{\mathcal{M}}^{comb}(\overline{M}^\circ, L)$  is continuous and equal to  $\overline{V}_{\overline{M}^\circ}$ .*

*Proof.* Indeed, in the case of vertices of degree 6, the only possible apparent degeneration different from a ringlet are the ones in figure 7.2. We can see that in this case there is only one way to glue the two ribbon graphs over each cell in  $N_w(M^\circ)$ .  $\square$

The limit at zeros of the volume  $V_{g,n^+,n^-}^\nu$  is related to the degeneration of the ribbon graph when some boundaries tend to zeros.

**Proposition 7.13.** *The zero value of the volume is given by*

$$\overline{V}_{g,n^++1,n^-}^\nu(0, L^+|L^-) = \sum_{i \geq 1} (\nu(i) + 1)(i + 1) V_{g,n^+,n^-}^{\nu-(i+1)+(i)}(L^+|L^-).$$

*Proof.* We need to compute the apparent degeneration in this case. They are related to ribbon graphs such that the first boundary is a ringlet. In this case, the degree is  $(i + 1)(\nu(i) + 1)$ .  $\square$

We recover the string equation:

$$\overline{V}_{g,n^++1,n^-}(0, L^+|L^-) = \overline{V}_{g,n^+,n^-,1}(L^+|L^-).$$

## 7.4 Application to computation of Masur-Veech Volumes

### 7.4.1 Expression Masur-Veech volumes by using ribbon graph

**Period coordinates and Masur Veech volumes:** Fix  $\nu$  a partition; there are natural coordinates on the space  $\mathcal{H}(\nu)$  called period coordinates. To construct these coordinates rigorously, we fix  $M, X = \{x_1, \dots, x_n\}$  a subset of  $M$ , and  $(\kappa_1, \dots, \kappa_n)$  a multi-index with  $\nu_{\kappa_i} = \nu$ . We consider the Teichmüller space  $\mathcal{TH}^\bullet(M, X, \kappa)$  of abelian differentials with a zero of order  $\kappa_i$  on  $x_i$  for all  $i$  and no other zeros. The space  $\mathcal{TH}^\bullet(M, X, \kappa)$  is a finite cover over  $\mathcal{TH}(\nu)$ . For each homology class  $[\gamma] \in H_1(M, X, \mathbb{Z})$  and each  $\alpha \in \mathcal{TH}^\bullet(M, X, \kappa)$ , we can compute the integral

$$l_{[\gamma]}(\alpha) = \int_{[\gamma]} \alpha \in \mathbb{C}.$$

This defines a map

$$\mathcal{P} : \mathcal{TH}^\bullet(M, X, \kappa) \longrightarrow \text{Hom}(H_1(M, X, \mathbb{Z}), \mathbb{C}),$$

by using duality

$$H^1(M, X, \mathbb{C}) \simeq \text{Hom}(H_1(M, X, \mathbb{Z}), \mathbb{C}).$$

We obtain the period map

$$\mathcal{P} : \mathcal{TH}^\bullet(M, X, \kappa) \rightarrow H^1(M, X, \mathbb{C}).$$

The following theorem is well known in the theory and is the analog of proposition 4.12.

**Theorem 7.2.** *The period map is a local homeomorphism.*

We can wonder what happens if we look at the action of the mapping class group. Let  $\text{Mod}'(M, X, \kappa)$  be the subgroup of  $\text{Mod}'(M, X)$  that preserves  $\kappa$ , then we have

$$\mathcal{H}(\nu) = \mathcal{TH}^\bullet(M, X, \kappa) / \text{Mod}'(M, X, \kappa).$$

A point that is important is the fact that the period map is equivariant under this action. Moreover, the mapping class group  $\text{Mod}'(M, X, \kappa)$  acts on  $H^1(M, X, \mathbb{C})$  by linear transformations. This fact endows the space  $\mathcal{H}(\nu)$  with the structure of a linear orbifold of complex dimension  $2g - 2 + n(\nu)$ , which is the dimension of the cohomology.

The tangent space of the moduli space is then locally identified to  $H^1(M, X, \mathbb{C})$ , the cohomology contains the natural lattice  $H^1(M, X, \mathbb{Z}[i])$  of integral points. The mapping class group acts on  $H^1(M, X, \mathbb{Z})$ , and then it also preserves  $H^1(M, X, \mathbb{Z}) \otimes \mathbb{Z}[i]$ . Let  $d\lambda$  be the volume form that represents the Lebesgue measure on  $H^1(M, X, \mathbb{C})$  normalized by  $H^1(M, X, \mathbb{Z}[i])$  (the orientation comes from the direct orientation of  $\mathbb{C}$ ). If  $([\gamma_i])_i$  is a basis of  $H^1(M, X, \mathbb{Z})$ , then

$$d\lambda = \bigwedge_i d\text{Re}(l_{[\gamma_i]}) \wedge d\text{Im}(l_{[\gamma_i]}).$$

The mapping class group preserves the volume form  $\mathcal{P}^*d\lambda$ , which is the pullback of  $d\lambda$  under the period map.

**Definition 7.3.** *The measure  $\mathcal{P}^*d\lambda$  on  $\mathcal{H}(\nu)$  is called the Masur-Veech measure, we denote it  $d\mu^{MV}(\nu)$ .*

But this measure is infinite, and there are several ways to reduce the space and obtain a measure with a finite volume. Let  $A$  be the function:

$$A : \mathcal{TH}(\nu) \longrightarrow \mathbb{R}_{>0},$$

that associates to a flat surface the area of the volume form associated to the flat metric. Then we can decompose the measure as

$$d\mu^{MV} = d\mu^{MV}(A)dA.$$

For all  $a$ , the measure  $d\mu^{MV}(a)$  is supported on  $\mathcal{H}^{(a)}(\overline{M})$ , the subset of surfaces with an area equal to  $a$ . It's has been proved the following theorem:

**Theorem 7.3.** *The volumes  $\int_{\mathcal{H}^{(1)}(\nu)} d\mu^{MV}(1)$  are finite, and we denote them  $\vartheta^\circ(\nu)$  and they are called Masur-Veech volumes.*

There are several normalizations that make sense for Masur-Veech volumes, and we chose one of them. We have

$$\vartheta^\circ(\nu) = \int_{A \leq 1} d\mu^{MV}$$

**Square tiled surfaces:** A way to compute the Masur Veech volumes is to count the integral points in the moduli space. This strategy has been used in [Zoro2] and [DGZZ21]. On the moduli space  $\mathcal{H}(\nu)$ , there is a local coordinate given by the period coordinates. The Masur-Veech measure is given by the Lebesgue measure in these coordinates normalized by the lattice  $H^1(M, X, \mathbb{Z}[i])$  of surfaces with integral period coordinates. Such surfaces are square, tiled surfaces; the developing map

$$x \in M \longrightarrow \int_{x_1}^x \alpha,$$

is well defined up to  $\mathbb{Z}[i]$ . Then a square tiled surface defines covering over the torus  $\mathbb{C}/\mathbb{Z}[i]$  ramified at the zeros  $x_i$  and can be seen as squares of paper glued together along their boundaries. A zippered rectangle construction also allows us to prove that the horizontal foliation of a square tiled surface is periodic. These cylinders define an oriented multi-curve, and this multi-curve must be non-degenerate.

**Masur-Veech volumes of stratum of Abelian differentials:** We give proof of the following result:

**Theorem 7.4** ([DGZZ21]). *The volumes are given by the formula*

$$\vartheta^\circ(\nu) = \sum_{\overline{\mathcal{G}^\circ}} \frac{1}{\#\text{Aut}(\overline{\mathcal{G}^\circ})} \int_{L \in \Lambda_{\mathcal{G}^\circ}} \prod_{\gamma} L_{\gamma} \frac{e^{-L_{\gamma}}}{1 - e^{-L_{\gamma}}} \prod_c V_{\overline{\mathcal{G}^\circ}(c)}(L(c)) d\sigma_{\overline{\mathcal{G}^\circ}}$$

Where the sum runs over  $\overline{\text{stab}^\circ}(\nu)$ .

The set  $\overline{\text{stab}^\circ}(\nu)$  correspond to all decorated directed connected stable graphs of the form  $\overline{\mathcal{G}^\circ} = (\mathcal{G}^\circ, \nu_{\mathcal{G}^\circ})$  with  $\sum_c \nu_{\mathcal{G}^\circ}(c) = \nu$ .

*Proof.*

**Lemma 7.12.** *The Masur-Veech volume of  $\mathcal{H}(\nu)$  is given by*

$$\vartheta^\circ(\nu) = \frac{1}{(2g - 1 + n(\nu))!} \int_{\mathcal{H}(\nu)} e^{-A} d\mu^{MV}$$

*Proof.* The space  $\mathcal{H}(\nu)$  admits an action of  $\mathbb{R}_{>0}$  by multiplying the one forms, and the function  $A$  is homogeneous of degree two. The measure  $d\mu^{MV}(a)$  is then of degree  $4g - 2 + 2n(\nu)$  under the action. According to this, the function  $a \rightarrow \int_{\mathcal{H}(a)(\nu)} d\mu^{MV}(a)$  is homogeneous of degree  $2g - 1 + n(\nu)$ .

$$\int_{\mathcal{H}(a)(\overline{M})} d\mu^{MV}(a) = a^{2g-2+n(\nu)} \vartheta^\circ(\nu)$$

On the other hand, by decomposing the measure, we have

$$\int_{\mathcal{H}(\nu)} e^{-A} d\mu^{MV} = \int_0^\infty e^{-A} \int_{\mathcal{H}(A)(\nu)} d\mu_{\nu}^{MV}(A) dA = \int_0^\infty e^{-A} A^{2g-1+n(\nu)} \vartheta^\circ(\nu) dA.$$

And then we have

$$\vartheta^\circ(\nu) = \frac{1}{(2g - 1 + n(\nu))!} \int_{\mathcal{H}(\nu)} e^{-A} d\mu^{MV}.$$

□

We reproduce more or less the proof developed in [DGZZ21] for the principal stratum. We compute the integral:

$$\int_{\mathcal{H}(\nu)} e^{-A} d\mu^{MV}.$$

The function is integrable over the moduli space, and we can use the Riemann sum. The Masur-Veech measure is the Lebesgue measure normalized by the set  $\mathcal{H}_{\mathbb{Z}}(\nu)$  of abelian differentials with integral periods. Then we can apply the formula:

$$\int_{\mathcal{H}(\nu)} e^{-A} d\mu^{MV} = \lim_{N \rightarrow \infty} \frac{1}{N^{2g-2+n(\nu)}} \sum_{S \in \mathcal{H}_{\mathbb{Z}}(\nu)} e^{\frac{A(S)}{N}}.$$

We now compute the RHS. If  $S \in \mathcal{H}_{\mathbb{Z}}(\nu)$ , its horizontal foliation is periodic, and the cylinders define an integral-oriented multi-curve  $\bar{\Gamma}^{\circ} = \bar{\Gamma}_S^{\circ}$  and then a decorated stable graph. Let  $\bar{\mathcal{G}}^{\circ} \in \overline{\text{stab}}^{\circ}(\nu)$  be a decorated directed stable graph. We can consider  $\mathcal{H}_{\mathbb{Z}}(\bar{\mathcal{G}}^{\circ})$  the set of square tilled surfaces  $S$  such that  $\bar{\mathcal{G}}_S^{\circ} = \bar{\mathcal{G}}^{\circ}$  where  $\bar{\mathcal{G}}_S^{\circ}$  is the stable graph of  $S$ . From the results of this section, we have a map.

$$\mathcal{H}_{\mathbb{Z}}(\bar{\mathcal{G}}^{\circ}) \longrightarrow \mathcal{BM}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})$$

The fiber of this map corresponds to all the possible weights on the curves. These weights correspond to the height of the cylinders. And then the push forward of the function  $e^{-\frac{A}{N}}$  is given by

$$\sum_{m_{\gamma} \geq 1} \prod_{\gamma} e^{-\frac{l_{\gamma}(S)}{N}} = \prod_{\gamma} \frac{e^{-\frac{l_{\gamma}(S)}{N}}}{1 - e^{-\frac{l_{\gamma}(S)}{N}}}.$$

for  $S \in \mathcal{BM}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})$ . We also have a map

$$\mathcal{BM}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ}) \longrightarrow \mathcal{M}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ}),$$

and the cardinal of the pre-image of a point is  $\prod_{\gamma} l_{\gamma}(S)$  due to all the possible ways to perform gluing. The subsets  $\mathcal{H}_{\mathbb{Z}}(\bar{\mathcal{G}}^{\circ})$  form a partition of  $\mathcal{H}_{\mathbb{Z}}(\bar{M})$ , and then we have

$$\sum_{S \in \mathcal{H}_{\mathbb{Z}}(\nu)} e^{\frac{A(S)}{N}} = \sum_{\bar{\mathcal{G}}^{\circ}} \sum_{\mathcal{BM}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})} \prod_{\gamma} l_{\gamma} \frac{e^{-\frac{l_{\gamma}(S)}{N}}}{1 - e^{-\frac{l_{\gamma}(S)}{N}}}.$$

The function  $S \longrightarrow \prod_{\gamma} l_{\gamma} \frac{e^{-l_{\gamma}(S)}}{1 - e^{-l_{\gamma}(S)}}$  is integrable on  $\mathcal{M}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})$ , and we have

$$\dim(\mathcal{M}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})) = d(\nu) + n(\nu) - \#X_1\mathcal{G}^{\circ}.$$

Then, as the measure on  $\mathcal{M}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})$  is normalized by integral points, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^{d(\nu)+n(\nu)-\#X_1\mathcal{G}^{\circ}}} \sum_{\mathcal{M}_{\mathbb{Z}}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})} \prod_{\gamma} \frac{l_{\gamma}}{N} \frac{e^{-\frac{l_{\gamma}(S)}{N}}}{1 - e^{-\frac{l_{\gamma}(S)}{N}}} = \int_{\mathcal{M}^{\text{comb}}(\bar{\mathcal{G}}^{\circ})} \prod_{\gamma} l_{\gamma}(S) \frac{e^{-l_{\gamma}(S)}}{1 - e^{-l_{\gamma}(S)}} d\mu_{\bar{\mathcal{G}}^{\circ}}^{\text{comb}}.$$

The RHS is equal to

$$\vartheta_{\bar{\mathcal{G}}^{\circ}}^{\circ} = \frac{1}{\#\text{Aut}(\bar{\mathcal{G}}^{\circ})} \int_{\Lambda_{\mathcal{G}^{\circ}}} \prod_{\gamma} l_{\gamma} \frac{e^{-l_{\gamma}}}{1 - e^{-l_{\gamma}}} \prod_c V_{\bar{\mathcal{G}}^{\circ}(c)}(L(c)) d\sigma_{\mathcal{G}^{\circ}}.$$

Then, putting all of this together, we obtain the desired formula:

$$\vartheta^\circ(\nu) = \sum_{\bar{\mathcal{G}}^\circ} \vartheta_{\bar{\mathcal{G}}^\circ}^\circ.$$

□

## 7.4.2 Degeneration of the formula

In this part, we give a formula that relies on the Masur–Veech volumes  $\vartheta^\circ(\nu)$  and the integrals of the augmented volumes  $\bar{V}_{\bar{M}^\circ}(L)$  of the moduli spaces of metric ribbon graphs. A similar formula for quadratic differential was conjectured by E. Goujard and A. Sauvaget.

**Domain of the integrals:** Let  $\mathcal{G}^\circ$  be a directed stable graph on a surface  $M \in \mathbf{top}$ . For a family  $F = (f_c)_{c \in X_0 \mathcal{G}^\circ}$  of integrable continuous functions, we can consider

$$V_{\mathcal{G}^\circ}(F) = \int_{\Lambda_{\mathcal{G}^\circ}} \prod_{c \in X_0 \mathcal{G}^\circ} f_c d\sigma_{\mathcal{G}^\circ}.$$

It happens that in general, this formula is not well defined if the functions  $F = (f_c)_{c \in X_0 \mathcal{G}^\circ}$  are defined almost surely on  $\Lambda_{\mathcal{G}^\circ(c)}$ . Indeed, it happens that the image of the projection

$$\mathrm{pr}_c : \Lambda_{\mathcal{G}^\circ} \longrightarrow \Lambda_{\mathcal{G}^\circ(c)}$$

is contained in a linear subspace. We have the following lemma:

**Lemma 7.13.** *There is a wall  $W_{\mathcal{G}^\circ(c)} \in \mathbf{Wall}(\mathcal{G}^\circ(c))$  such that the image of the projection  $\mathrm{pr}_c$  is an open set in  $\Lambda_{W_{\mathcal{G}^\circ(c)}}$ .*

Then the integral  $V_{\mathcal{G}^\circ}(F)$  is well defined for a family of functions such that  $f_c$  is measurable for each  $c \in X_0 \mathcal{G}^\circ$  on  $\Lambda_{\mathcal{G}^\circ(c)}$ .

*Proof.* Let  $c \in X_0 \mathcal{G}^\circ$ . We can consider  $\mathcal{G}_c^\circ$  the stable graph obtained by taking the quotient of  $\mathcal{G}^\circ$  by the set of curves that are not in the boundary of  $\mathcal{G}_c^\circ$ . We have a projection:

$$K_{\mathcal{G}^\circ} \longrightarrow K_{\mathcal{G}_c^\circ}.$$

Applying lemma 3.6, this map is surjective. Note that all the curves in  $\mathcal{G}_c^\circ$  are boundaries of  $\mathcal{G}^\circ(c)$ , and then we have an injective map

$$K_{\mathcal{G}_c^\circ} \longrightarrow T_{\mathcal{G}^\circ(c)}.$$

The graph  $\mathcal{G}_c^\circ$  is simple; it has a central vertex  $c$ , and all the other vertices are glued to this vertex only. For each of these vertex, the sum of the boundaries is equal to zero, and this gives a relation on  $T_{\mathcal{G}^\circ(c)}$ . These relations are disjoint and define a wall that we denote  $W_{\mathcal{G}^\circ(c)}$ . The map  $K_{\mathcal{G}_c^\circ} \longrightarrow K_{\mathcal{G}_c^\circ}$  and then the tangent map  $K_{\mathcal{G}_c^\circ} \longrightarrow T_{W_{\mathcal{G}^\circ(c)}}$  is also surjective, and then the image of  $\Lambda_{\mathcal{G}^\circ}$  is open in  $\Lambda_{M^\circ}$ . □

Then we can derive the following corollary:

**Corollary 7.3.** Let  $\mathcal{G}^\circ$  be as before, and  $F = (f_c)_{c \in X_0 \mathcal{G}^\circ}$  be a family of measurable functions. Let  $F_{\mathcal{G}^\circ} = (f_c 1_{W_{\mathcal{G}^\circ}(c)})_{c \in X_0 \mathcal{G}^\circ}$  (with  $1_W$  the characteristic function of  $\Lambda_W$ ). Then

$$V_{\mathcal{G}^\circ}(F) = V_{\mathcal{G}^\circ}(F_{\mathcal{G}^\circ}).$$

Moreover, if there is  $c$  such that  $f_c = 0$   $d\sigma_{W_{\mathcal{G}^\circ}(c)}$  almost surely then

$$V_{\mathcal{G}^\circ}(F) = 0.$$

We do not use it here, but these two results have a straightforward generalization for graphs on a directed surface with boundaries.

**Continuity issues and degeneration formula:** It happens that the volumes  $\bar{V}_{M^\circ}$  are easier to compute in practice because they are piecewise polynomials. The functions  $\bar{V}_{M^\circ}$  and  $V_{M^\circ}$  only differ on walls in  $\mathbf{Wall}(M^\circ)$ . But as we see in the last section, we need to restrict the functions when we compute the expression of the form:

$$\bar{\vartheta}_{\mathcal{G}^\circ}^\circ = \int_{\Lambda_{\mathcal{G}^\circ}} \prod_{\gamma} \frac{l_{\gamma} e^{-l_{\gamma}}}{1 - e^{-l_{\gamma}}} \prod_c \bar{V}_{\mathcal{G}^\circ(c)} d\sigma_{\mathcal{G}^\circ}.$$

Sometimes the domain of integration falls into a subspace where a function  $V_{\mathcal{G}^\circ(c)}$  jumps. Then, in general, the two quantities  $\bar{\vartheta}_{\mathcal{G}^\circ}^\circ$  and  $\vartheta_{\mathcal{G}^\circ}^\circ$  are not equal. In this section, we express the difference. Let  $\bar{\vartheta}^\circ(\nu)$  be the quantity defined by

$$\bar{\vartheta}^\circ(\nu) = \sum_{\mathcal{G}^\circ} \bar{\vartheta}_{\mathcal{G}^\circ}^\circ,$$

which is the augmented volume. Let  $\mathcal{T}_\nu$  be the subset of trees  $T$  with the following additional structures:

- $\mathcal{T}$  is bipartite, i.e., there is  $\varsigma : X_0 \mathcal{T} \rightarrow \{0, 1\}$  and two adjacent vertices have different labels.
- For each component  $c \in X_0^1 \mathcal{T}$ , there is  $\nu(c)$  a partition.
- For each  $\gamma \in X_1 \mathcal{T}$ , there is an integer  $\kappa_\gamma \geq 0$ .

As in the case of subsurfaces, they have the following additional structure: For each  $c \in X_0^0 \mathcal{T}$ , we denote

$$\nu_c = \left( \sum_{\gamma \simeq c} (\kappa_\gamma + 2) - 2 \right) \quad \nu(c) = \delta_{\nu_c}.$$

We impose the constraint

$$\nu = \sum_c \nu(c).$$

For each  $\bar{\mathcal{T}}$ , we denote

$$\vartheta_{\bar{\mathcal{T}}}^\circ = \prod_{c, \varsigma(c)=1} \vartheta_{\nu(c), \kappa(c)}^\circ$$

With  $\vartheta_{\nu(c), \kappa(c)}^\circ$  are the Masur-Veech volumes of surfaces with unmarked singularities prescribed by  $\nu(c)$  and marked singularities given by  $\kappa(c)$ . and

$$\deg(\bar{\mathcal{T}}) = \prod_{c, \varsigma(c)=0} H(\kappa(c))$$

Then we prove the following theorem.

**Theorem 7.5.** *We have the relation*

$$\bar{\vartheta}^\circ(\nu) = \sum_{\bar{\mathcal{T}} \in \mathcal{T}_\nu} \deg(\bar{\mathcal{T}}) \vartheta_{\bar{\mathcal{T}}}^\circ$$

*Proof.* Let  $\bar{\mathcal{G}}^\circ$  by applying the formula of theorem 7.1 for each  $c$  we have on  $\Lambda_{W_{\mathcal{G}}^\circ(c)}$

$$\bar{\vartheta}^\circ_{\bar{\mathcal{G}}^\circ(c)} = \sum_{\bar{\mathcal{D}}^\circ(c), W_{\bar{\mathcal{D}}^\circ(c)} \geq W_{\mathcal{G}^\circ(c)}} \deg_{\bar{\mathcal{D}}^\circ(c)}(\bar{\mathcal{G}}^\circ(c)) V_{\bar{\mathcal{D}}^\circ(c)}$$

Now, given  $\bar{\mathcal{G}}^\circ$  and  $(\bar{\mathcal{D}}^\circ(c))_c$ , we obtain a nodal surface  $\bar{\mathcal{N}}^\circ$  by gluing the component of  $(\bar{\mathcal{N}}^\circ(c))_c$  along the curves in  $X_1 \mathcal{G}^\circ$ . Moreover, we also obtain a directed, stable graph on each component at level 1. According to proposition 7.5 and lemma 7.13, we can see that these nodal surfaces are exactly the trees given before. Then we see that we recover all the possible directed, decorated stable graphs for each tree.  $\square$

# Bibliography

- [ABC<sup>+</sup>20] Jørgen E. Andersen, Gaëtan Borot, Severin Charbonnier, Alessandro Giacchetto, Danilo Lewanski, and Campbell Wheeler. On the kontsevich geometry of the combinatorial teichmuller space. *arXiv preprint arXiv:2010.11806*, 2020.
- [ABC<sup>+</sup>23] Jørgen E. Andersen, Gaëtan Borot, Séverin Charbonnier, Vincent Delecroix, Alessandro Giacchetto, Danilo Lewanski, and Campbell Wheeler. Topological recursion for masur–veech volumes. *Journal of the London Mathematical Society*, 2023.
- [ABO17] Jørgen E. Andersen, Gaëtan Borot, and Nicolas Orantin. Geometric recursion. *arXiv preprint arXiv:1711.04729*, 2017.
- [AC94] Enrico Arbarello and Maurizio Cornalba. Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves. *arXiv preprint/9406008*, 1994.
- [ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. Geometry of algebraic curves: volume ii with a contribution by joseph daniel harris. 2011.
- [ADG<sup>+</sup>20] Amol Aggarwal, Vincent Delecroix, Elise Goujard, Peter Zograf, and Anton Zorich. Conjectural large genus asymptotics of masur-veech volumes and of area siegel-veech constants of strata of quadratic differentials. *Arnold Mathematical Journal*, 2020.
- [AEZ15] Jayadev S. Athreya, Alex Eskin, and Anton Zorich. Right-angled billiards and volumes of moduli spaces of quadratic differentials on  $cp^1$ . *Annales scientifiques de l'école normale supérieure*, 2015.
- [Agg20] Amol Aggarwal. Large genus asymptotics for volumes of strata of abelian differentials. *Journal of the American Mathematical Society*, 2020.
- [Agg21] Amol Aggarwal. Large genus asymptotics for intersection numbers and principal strata volumes of quadratic differentials. 2021.
- [Ale11] Alexander Alexandrov. Cut-and-join operator representation for kontsevich–witten tau-function. *Modern Physics Letters A*, 2011.
- [Ale22] Alexandrov Alexander. Cut-and-join operators in cohomological field theory and topological recursion. *arXiv preprint arXiv:2202.09090*, 2022.
- [AO06] Eskin Alex and Andrei Okounkov. *Pillowcases and quasimodular forms*. 2006.
- [Baro8] Alexander Barvinok. Integer points in polyhedra. 2008.

- [BBC<sup>+</sup>21] Gaëtan Borot, Vincent Bouchard, Nitin K. Chidambaram, Thomas Creutzig, and Dmitry Noshchenko. Higher airy structures, w algebras and topological recursion. 2021.
- [BCG<sup>+</sup>22] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller. The moduli space of multi-scale differentials. 2022.
- [BKS23] Gaëtan Borot, Reinier Kramer Kramer, and Yannik Schüler. Higher airy structures and topological recursion for singular spectral curves, 2023.
- [BM08] Vincent Bouchard and Marcos Marino. Hurwitz numbers, matrix models and enumerative geometry, 2008.
- [Bol12] Bela Bollobas. Graph theory: an introductory course. 63, 2012.
- [CMSZ20] Dawei Chen, Martin Moller, Adrien Sauvaget, and Don Zagier. Masur–veech volumes and intersection theory on moduli spaces of abelian differentials. *Inventiones mathematicae*, 2020.
- [DBOSS14] Petr Dunin-Barkowski, Nicolas Orantin, Shadrin Sergey, and Loek Spitz. Identification of the givental formula with the spectral curve topological recursion procedure. *Communications in Mathematical Physics*, 2014.
- [DGZZ21] Vincent Delecroix, Elise Goujard, Peter Zograf, and Anton Zorich. Masur–veech volumes, frequencies of simple closed geodesics, and intersection numbers of moduli spaces of curves. *Duke Mathematical Journal*, 2021.
- [DGZZ22] Vincent Delecroix, Elise Goujard, Peter Zograf, and Anton Zorich. Large genus asymptotic geometry of random square-tiled surfaces and of random multicurves. *Inventiones mathematicae*, 2022.
- [DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l’IHES*, 1969.
- [DM14] Norman Do and David Manescu. Quantum curves for the enumeration of ribbon graphs and hypermaps, 2014.
- [DM18] Olivia Dumitrescu and Motohico Mulase. Edge contraction on dual ribbon graphs and 2d tqft. *Journal of Algebra*, 2018.
- [DMSS13] Olivia Dumitrescu, Motohico Mulase, Brad Safnuk, and Adam Sorkin. The spectral curve of the eynard-orantin recursion via the laplace transform. *Algebraic and geometric aspects of integrable systems and random matrices*, 2013.
- [EGFEN21] Bertrand Eynard, Olivier Garcia-Failde Elba, Marchal, and Orantin Nicolas. Quantization of classical spectral curves via topological recursion, 2021.
- [EKR15] Bertrand Eynard, Taro Kimura, and Sylvain Ribault. Random matrices. *arXiv preprint arXiv:1510.04430*, 2015.
- [EMS11] Bertrand Eynard, Motohico Mulase, and Brad Safnuk. The laplace transform of the cut-and-join equation and the bouchard-marino conjecture on hurwitz numbers, 2011.

- [EO01] Alex Eskin and Andrei Okounkov. Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials. *Inventiones Mathematicae*, 2001.
- [EO07] Bertrand Eynard and Nicolas Orantin. Invariants of algebraic curves and topological expansion, 2007.
- [Eps66] David Epstein. Curves on 2-manifolds and isotopies. 1966.
- [Eyn05] Bertrand Eynard. Topological expansion for the 1-hermitian matrix model correlation functions. *Journal of High Energy Physics*, 2005.
- [Eyn14a] Bertrand Eynard. Invariants of spectral curves and intersection theory of moduli spaces of complex curves, 2014.
- [Eyn14b] Bertrand Eynard. A short overview of the "topological recursion". *arXiv preprint arXiv:1412.3286*, 2014.
- [Eyn16] Bertrand Eynard. Counting surfaces. *Progress in Mathematical Physics*, 2016.
- [Eyn19] Bertrand Eynard. Topological recursion, airy structures in the space of cycles, 2019.
- [EZ15] Alex Eskin and Anton Zorich. Volumes of strata of abelian differentials and siegel-veech constants in large genera. *Arnold Mathematical Journal*, 2015.
- [FLP21] Albert Fathi, Francois Laudenbach, and Valentin Poenaru. *Thurston's Work on Surfaces (MN-48)*, volume 48. Princeton University Press, 2021.
- [Giv01] Alexander B. Givental. Gromov - witten invariants and quantization of quadratic hamiltonians, 2001.
- [GK21] Alexander Goncharov and Maxim Kontsevich. Spectral description of non-commutative local systems on surfaces and non-commutative cluster varieties. *arXiv preprint arXiv:2108.04168*, 2021.
- [Gou15] Elise Goujard. Siegel-veech constants for strata of moduli spaces of quadratic differentials. *Geometric and Functional Analysis*, 2015.
- [HKL18] Marvin Anas Hahn, Reinier Kramer, and Danilo Lewanski. Wall-crossing formulae and strong piecewise polynomiality for mixed grothendieck dessins d'enfant, monotone, and simple double hurwitz numbers, 2018.
- [HM79] John Hubbard and Howard Masur. Quadratic differentials and foliations. *Acta Mathematica*, 1979.
- [KLo7] Maxim Kazarian and Sergey K. Lando. An algebro-geometric proof of witten's conjecture. *Journal of the American Mathematical Society*, 2007.
- [KM94] Maxim Kontsevich and Yu Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 1994.
- [Kong92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. *Communications in Mathematical Physics*, 1992.

- [KS17] Maxim Kontsevich and Yan Soibelman. Airy structures and symplectic geometry of topological recursion, 2017.
- [KZo3] Maxim Kontsevich and Anton Zorich. Connected components of the moduli spaces of abelian differentials with prescribed singularities. *Inventiones mathematicae*, 2003.
- [LMo8] Elon Lindenstrauss and Maryam Mirzakhani. Ergodic theory of the space of measured laminations. *International mathematics research notices*, 2008.
- [LZZo4] Sergei K. Lando, Alexander K. Zvonkin, and Don Bernard Zagier. *Graphs on surfaces and their applications*. Springer, 2004.
- [MA23] Andrey Mironov and Morozov Andrey. Spectral curves and w-representations of matrix models. *Journal of High Energy Physics*, 2023.
- [Mas82] Howard Masur. Interval exchange transformations and measured foliations. *Annals of Mathematics*, 1982.
- [McS98] Greg McShane. Simple geodesics and a series constant over teichmuller space. *Inventiones mathematicae*, 1998.
- [Miro7] Maryam Mirzakhani. Simple geodesics and weil-petersson volumes of moduli spaces of bordered riemann surfaces. *Inventiones mathematicae*, 2007.
- [Miro8a] Maryam Mirzakhani. Ergodic theory of the earthquake flow. *International Mathematics Research Notices*, 2008.
- [Miro8b] Maryam Mirzakhani. Growth of the number of simple closed geodesies on hyperbolic surfaces. *Annals of Mathematics*, 2008.
- [Noro8] Paul Norbury. Counting lattice points in the moduli space of curves. *Mathematical Research Letters*, 2008.
- [Nor13] Paul Norbury. String and dilaton equations for counting lattice points in the moduli space of curves. *Transactions of the American Mathematical Society*, 2013.
- [Ok0] Andrei Okounkov. Random matrices and random permutations. *International Mathematics Research Notices*, 2000.
- [Str84] Kurt Strebel. Quadratic differentials. In *Quadratic Differentials*. Springer, 1984.
- [Tel11] Constantin Teleman. The structure of 2d semi-simple field theories. *Inventiones mathematicae*, 2011.
- [Tut63] William. T. Tutte. A census of planar maps. *Canadian Journal of Mathematics*, 1963.
- [Vee82] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Annals of Mathematics*, 1982.
- [Wit90] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys in differential geometry*, 1990.

- [WLZZ22] Rui Wang, Fan Liu, Chun-Hong Zhang, and Wei-Zhong Zhao. Superintegrability for beta-deformed partition function hierarchies with w-representations. *The European Physical Journal C*, 2022.
- [Yak22] Ivan Yakovlev. Contribution of  $n$ -cylinder square-tiled surfaces to masur-veech volume of  $\mathcal{H}(2g - 2)$ . *arXiv preprint arXiv:2209.12348*, 2022.
- [Zhu12] Shengmao Zhu. On the recursion formula for double hurwitz numbers. *Proceedings of the American Mathematical Society*, 2012.
- [Zor96] Anton Zorich. Finite gauss measure on the space of interval exchange transformations. lyapunov exponents. 1996.
- [Zor97] Anton Zorich. Deviation for interval exchange transformations. *Ergodic Theory and Dynamical Systems*, 1997.
- [Zor02] Anton Zorich. Square tiled surfaces and teichmuller volumes of the moduli spaces of abelian differentials. *Rigidity in Dynamics and Geometry: Contributions from the Programme Ergodic Theory*, 2002.
- [Zvo02] Dimitri Zvonkine. Strebel differentials on stable curves and kontsevich's proof of witten's conjecture. *arXiv preprint math/0209071*, 2002.
- [Zvo12] Dimitri Zvonkine. An introduction to moduli spaces of curves and their intersection theory. *Handbook of Teichmuller theory*, 3, 2012.