Some colouring problems in edge/vertex-coloured graphs : Structural and extremal studies

Qiancheng Ouyang

université PARIS-SACLAY

Some colouring problems in edge/vertex-coloured graphs: Structural and extremal studies

Problèmes structuraux et extrémaux liés à la coloration des graphes sommet- ou arête-coloriés

Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 580 : Sciences et technologies de l'information et de la communication (STIC)
Spécialité de doctorat : Informatique
Graduate school: Informatique et sciences du numérique
Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'unité de recherche **Laboratoire interdisciplinaire des sciences du numérique (Université Paris-Saclay, CNRS)**, sous la direction de **Hao LI**, Directeur de recherche, CNRS, et le co-encadrement de **François PIROT**, Maître de conférences

Thèse soutenue à Paris-Saclay, le 16 novembre 2023, par

Qiancheng OUYANG

Composition du jury

Membres du jury avec voix délibérative

Xujin CHEN Professeure, Chinese Academy of Sciences	Présidente
Guantao CHEN Professeur, Georgia State University	Rapporteur & Examinateur
Louis ESPERET Directeur de recherche, CRNS, Université Grenoble Alpes	Rapporteur & Examinateur
Baogang XU Professeur, Nanjing Normal University	Examinateur

THESE DE DOCTORAT
NNT : 2023UPASG060

université PARIS-SACLAY

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Problèmes structuraux et extrémaux liés à la coloration des graphs sommet- ou arête- coloriés

Mots clés : Coloration propre; Coloration impaire; Hypergraphe; Graphe à arêtes colorées; Extensibilité des cycles; Nombre anti-Ramsey; Sous-graphe proprement coloré; Arbre couvrant faiblement propre.

Résumé : La coloration de graphes est l'un des sujets les plus connus, populaires et largement étudiés dans le domaine de la théorie des graphes, avec une vaste littérature comprenant des approches provenant de nombreux domaines ainsi que de nombreux problèmes qui sont encore ouverts et étudiés par divers mathématiciens et informaticiens à travers le monde. Tout d'abord, dans cette thèse, nous visons à optimiser la stratégie de coloration des sommets de graphes et d'hypergraphes avec certaines contraintes données, en combinant le concept de coloration propre et d'élément représentatif de certains sous-ensembles de sommets. D'autre part, en fonction du sujet à colorer, une grande quantité de recherches et de problèmes de graphes à arêtes colorées ont émergé, avec des applications importantes en biologie et en technologies web. Nous fournissons quelques résultats analogues pour certaines questions de connectivité, afin de décrire des graphes dont les arêtes sont attribuées suffisamment de couleurs, garantissant ainsi des arbres couvrants ou des cycles ayant une structure chromatique spécifique.

Title: Some colouring problems in edge/vertex-coloured graphs : Structural and extremal studies

Keywords: Proper colouring; Odd colouring; Hypergraph; Edge-coloured graph; Cycle extendable; Anti-Ramsey number; Properly coloured subgraph; Weakly proper spanning tree

Abstract: Graph colouring is one of the best known, popular and extensively researched subject in the field of graph theory, having a wide literature with approaches from many domains and a lot of problems, which are still open and studied by various mathematicians and computer scientists along the world. Firstly, in this thesis, we aim to optimize the strategy to colour the vertex of graphs and hypergraphs with some given constraints, which combines the concept of proper colouring and representative element of some vertex subsets. On the other hand, according to the subject to be coloured, a large amount of research and problems of edge-coloured graphs have emerged, which have important applications to biology and web technologies. We provide some analogous results for some connectivity issues—to describe graphs whose edges are assigned enough colours, that guarantee spanning trees or cycles of a specific chromatic structure.

Acknowledgement

First of all, I would like to give my heartfelt thanks to all the people who have ever helped me in this paper.

My sincere and hearty thanks and appreciations go firstly to my previous supervisor, Prof. Yannis Manoussakis, whose suggestions and encouragement have given me much insight into these theoretical studies. Manoussakis is a wise and trusted guide, and meanwhile a decisive leader. I could not able to integrate into the new academic environment smoothly without his experienced guidance and support when my doctoral career just began 4 years ago. Manoussakis always work methodically, to organize reading groups and seminars, to proceed our researches in an orderly way and advance step by step. I will be ever grateful for his supervision, and am sorry that he has not lived to witness my PhD defense.

I equally express my sincere gratitude to my previous supervisor, Prof. Hao Li, for his guidance, support, and encouragement throughout my doctoral research. I have been fortunate to have Li as my academic advisor, who not only imparted invaluable knowledge on research methodologies but also extended generous support in my personal life. Li instructed me on various aspects of research, including how to find problems, how to consider problems, how to give a presentation and more. Li, and his wife, also offered moral support to me when I plumbed the depths during the unprecedented COVID-19 pandemic and when I encountered unexpected events. I am extremely appreciated to Li for his unwavering guidance and support throughout my doctoral journey.

I am also immensely grateful to Prof. François Pirot, for successfully sharing his passion with me, and for generously guiding and supporting me during my studies. Pirot is a perfect researcher to work with. Whenever I had any questions, he never hesitate to pause the work at hand and to discuss with me. I would like to learn his conscientious and meticulous attitude towards academic research, which have taught me to keep enthusiasm to the bottom of a problem.

I express my sincere gratitude to my reading committee for dedicating their time to thoroughly review the entire content of my thesis and providing me with valuable feedback. I also appreciate the presence of each member of my jury at my defence, which is an honour to me.

I am grateful for my fruitful collaborations with Tianjiao Dai, Jie Hu, Shun-ichi Maezawa, Zengxian Tian, Yue Ma and of course Manoussakis, Li and Pirot, who work with me give me substantial support and urge me to do better. Without them, I would have walked a rather lonely path in my field.

I wish to thank Prof. Guanghui Wang, my master's supervisor, for introducing me to the field of graph colouring, for successfully sharing his passion with me, and for being very supportive all those years. Thanks to Prof. I would also like to thank Prof. Yue Ma, for introducing me to the applications of graph theory and exploring the mysteries of semantics together. I especially thank the guys from GaLac for the fun discussions during seminars. Thank LISN for providing me with everything that I needed in order to perform my research in excellent conditions.

Finally, I would like to thank my friends and my family, who, although not necessarily directly involved in my thesis, they have always been there to support me and believe in me. I therefore thank my parents who have nurtured my curious mind from my youngest age, and never doubted my abilities. I thank all my family for always supporting me to remain myself and to follow my passions.

Contents

List of Figures

Résumé

La coloration de graphes est l'un des sujets les plus connus, populaires et largement étudiés dans le domaine de la théorie des graphes, avec une vaste littérature comprenant des approches provenant de nombreux domaines ainsi que de nombreux problèmes qui sont encore ouverts et étudiés par divers mathématiciens et informaticiens à travers le monde. Le Problème des Quatre Couleurs, à l'origine de l'étude de la coloration des graphes, a été l'un des problèmes centraux en théorie des graphes au siècle dernier. Il demande s'il est possible de colorer proprement chaque graphe planaire avec quatre couleurs. Malgré son origine théorique, la coloration des graphes a trouvé de nombreuses applications pratiques telles que la planification, les problèmes d'assignation de fréquences, la segmentation, etc.

Le Problème des Quatre Couleurs est l'un des problèmes importants parmi de nombreux problèmes de la théorie des graphes chromatiques, à partir duquel de nombreuses variantes et généralisations ont été proposées. Tout d'abord, dans cette thèse, nous visons à optimiser la stratégie de coloration des sommets de graphes et d'hypergraphes avec certaines contraintes données, en combinant le concept de coloration propre et d'élément représentatif de certains sous-ensembles de sommets. D'autre part, en fonction du sujet à colorer, une grande quantité de recherches et de problèmes de graphes à arêtes colorées ont émergé, avec des applications importantes en biologie et en technologies web. Nous fournissons quelques résultats analogues pour certaines questions de connectivité, afin de décrire des graphes dont les arêtes sont attribuées suffisamment de couleurs, garantissant ainsi des arbres couvrants ou des cycles ayant une structure chromatique spécifique.

Le deuxième chapitre est dédié à la coloration des sommets impairs. étant donné un graphe G, une coloration des sommets σ de G et un sous-ensemble $X \subseteq V(G)$, une couleur $x \in \sigma(X)$ est dite *impair* pour X dans σ si elle apparaît un nombre impair de fois dans X. Nous disons que σ est une *coloration impaire* de G si elle est propre et que chaque voisinage (ouvert) a une couleur impaire dans σ. Le nombre chromatique impair d'un graphe G, noté $\chi_o(G)$, est le plus petit $k \in \mathbb{N}$

tel qu'il existe une coloration impaire $\sigma\colon V(G) \to [k]$. Dans cette thèse, nous utilisons la méthode probabiliste pour démontrer que pour tout graphe G avec un degré maximal Δ, $\chi_o(G) \leq \chi(G) + O(\Delta \ln \Delta/\delta)$ lorsque $\Delta \to \infty$. Nous prouvons également que $\chi_o(G) \leq \lfloor 3\Delta/2 \rfloor + 2$ pour tout Δ, avec une preuve inductive et constructive simple. Si de plus le degré minimal δ de G est suffisamment grand, alors $\chi_o(G) \leq \chi(G)+O(\Delta \ln \Delta/\delta)$ et $\chi_o(G) = O(\chi(G) \ln \Delta)$. Enfin, étant donné un entier $h \geq 1$, nous étudions la généralisation de ces résultats aux colorations h-impaires, où pour chaque sommet v, il doit y avoir au moins $\min\{\deg(v), h\}$ couleurs impaires dans $N(v)$. Beaucoup de nos résultats sont optimaux jusqu'à une constante multiplicatrice près.

Dans le troisième chapitre, nous nous intéressons à un nombre de Ramsey généralisé lié aux cycles colorés de longueurs données dans un graphe complet à arêtes colorées. Soit $\mathrm{pr}(K_n, G)$ le nombre maximum de couleurs dans une coloration des arêtes de K_n sans copie proprement colorée d'un graphe donné G. Dans cet article, nous déterminons le seuil exact pour les cycles de longueur ℓ, $\mathrm{pr}(K_n, C_\ell)$, ce qui confirme une conjecture proposée par Fang, Győri et Xiao, selon laquelle le nombre maximum de couleurs dans une coloration des arêtes de K_n sans copie proprement colorée de C_ℓ est $\max\left\{\binom{\ell-1}{2} + n - l + 1, \left\lfloor \frac{\ell-1}{3} \right\rfloor n - \binom{\lfloor \frac{\ell-1}{3} \rfloor + 1}{2} + 1 + r_{\ell-1}\right\}$, où $\ell - 1 \equiv r_{\ell-1} \mod 3$, et $0 \leq r_{\ell-1} \leq 2$. Il s'agit d'une légère modification d'une conjecture précédemment émise par Manoussakis, Spyratos, Tuza et Voigt. De plus, nous analysons la structure des colorations de K_n qui sont extrémales en ce qui concerne l'extensibilité des cycles, c'est-à-dire que chaque court cycle proprement coloré peut être étendu en ajoutant exactement un sommet pour former un cycle proprement coloré plus long.

Dans le quatrième chapitre, nous examinons les arbres couvrants faiblement propres dans les graphes à arêtes colorées. Un graphe à arêtes colorées G est un graphe avec une coloration des arêtes. Nous disons que G est proprement coloré si toutes les arêtes adjacentes de G ont des couleurs distinctes. Un arbre à arêtes colorées T avec une racine fixe r est faiblement propre s'il est proprement coloré pour tout chemin dans T, de la racine r à n'importe quelle feuille. Borozan et al. ont montré que pour un sommet donné r dans un graphe à arêtes colorées G, le problème de déterminer si G a un arbre couvrant faiblement propre avec la racine

r est NP-complet. Dans cette thèse, nous donnons une condition de type Dirac pour un graphe à arêtes colorées G telle qu'il existe un arbre couvrant faiblement propre avec la racine r pour un certain sommet $r \in V(G)$, ainsi que certaines conditions suffisantes pour qu'un graphe à arêtes colorées G ait un arbre couvrant faiblement propre avec la racine r pour n'importe quel sommet $r \in V(G)$.

Mots clés: Coloration propre; Coloration impaire; Hypergraphe; Graphe à arêtes colorées; Extensibilité des cycles; Nombre anti-Ramsey; Sous-graphe proprement coloré; Arbre couvrant faiblement propre.

Abstract

Graph colouring is one of the best known, popular and extensively researched subject in the field of graph theory, having a wide literature with approaches from many domains and a lot of problems, which are still open and studied by various mathematicians and computer scientists along the world. The Four Colour Problem, originating the study of graph colouring, was one of the central problem in graph theory in the last century, which asks if it is possible to colour every planar graph properly by four colours. Despite the theoretical origin, the graph colouring has found many applications in practice like scheduling, frequency assignment problems, segmentation, etc.

The Four Colour Problem is a significant one among many problems in chromatic graph theory, from which many variants and generalizations have been proposed. Firstly, in this thesis, we aim to optimize the strategy to colour the vertex of graphs and hypergraphs with some given constraints, which combines the concept of proper colouring and representative element of some vertex subsets. On the other hand, according to the subject to be coloured, a large amount of research and problems of edge-coloured graphs have emerged, which have important applications to biology and web technologies. We provide some analogous results for some connectivity issues—to describe graphs whose edges are assigned enough colours, that guarantee spanning trees or cycles of a specific chromatic structure.

The second chapter is dedicated to odd vertex colouring. Given a graph G, a vertex-colouring σ of G, and a subset $X \subseteq V(G)$, a colour $x \in \sigma(X)$ is said to be *odd* for X in σ if it has an odd number of occurrences in X. We say that σ is an *odd colouring* of G if it is proper and every (open) neighbourhood has an odd colour in σ. The odd chromatic number of a graph G, denoted by $\chi_o(G)$, is the minimum $k \in \mathbb{N}$ such that there exists an odd colouring $\sigma\colon V(G) \to [k]$. In this thesis, we use the probabilistic method to prove that for every graph G with maximum degree Δ, $\chi_o(G) \leq \Delta + O(\ln \Delta)$ as $\Delta \to \infty$. We also prove that $\chi_o(G) \leq \lfloor 3\Delta/2 \rfloor + 2$ for every Δ, with a simple inductive and constructive proof. If moreover the minimum degree δ of G is sufficiently large, we have

$\chi_o(G) \leq \chi(G) + O(\Delta \ln \Delta/\delta)$ and $\chi_o(G) = O(\chi(G) \ln \Delta)$. Finally, given an integer $h \geq 1$, we study the generalisation of these results to h-odd colourings, where for every vertex v there must be at least $\min\{\deg(v), h\}$ odd colours in $N(v)$. Many of our results are tight up to some multiplicative constant.

In the third chapter, we are interested in a generalised Ramsey number related to coloured cycles of given length in an edge-coloured complete graph. Let $\mathrm{pr}(K_n, G)$ be the maximum number of colours in an edge-coloring of K_n with no properly coloured copy of a given graph G. In this paper, we determine the exact threshold for cycles of length ℓ, $\mathrm{pr}(K_n, C_\ell)$, which proves a conjecture proposed by Fang, Győri, and Xiao, that the maximum number of colours in an edge-coloring of K_n with no properly coloured copy of C_ℓ is $\max\left\{\binom{\ell-1}{2} + n - l + 1, \left\lfloor \frac{\ell-1}{3} \right\rfloor n - \binom{\lfloor \frac{\ell-1}{3} \rfloor + 1}{2} + 1 + r_{\ell-1}\right\}$, where $\ell - 1 \equiv r_{\ell-1} \mod 3$, and $0 \leq r_{\ell-1} \leq 2$. It is a slight modification of a previous conjecture posed by Manoussakis, Spyratos, Tuza and Voigt. Also, we analyse the structure of colourings of K_n that are extremal with regard to cycle extendability, namely, each short properly coloured cycle can be extended to a longer properly coloured one by exact one more vertex.

In the fourth chapter, we consider weakly proper spanning trees in edge-coloured graphs. An edge-coloured graph G is a graph with an edge colouring. We say G is properly coloured if any two adjacent edges of G have distinct colours. An edge-coloured tree T with fixed root r is weakly proper if every path in T, from the root r to any leaf, is properly coloured. Borozan et al. showed that for a given vertex r in an edge-coloured graph G, the problem of determining whether G has a weakly proper spanning tree with root r is NP-complete. In this thesis, we give a Dirac type condition for an edge-coloured graph G such that there exists a weakly proper spanning tree with root r for some vertex $r \in V(G)$, and some sufficient conditions for an edge-coloured graph G to have a weakly proper spanning tree with root r for any vertex $r \in V(G)$.

Keywords: Proper colouring; Odd colouring; Hypergraph; Edge-coloured graph; Cycle extendable; Anti-Ramsey number; Properly coloured subgraph; Weakly proper spanning tree.

1 - Introduction

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations within some sets of elements. The paper written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736 is regarded as the first paper in the history of graph theory. In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams.

As it continues to develop, many research topics appeal in graph theory, including graph colouring, connectivity, decomposition and so on. While some of these problems are highly relevant to other fields of mathematical study such as algebra, topology, and probability.

Graphs are often represented using a visual diagram consisting of a set of points together with lines joining certain pairs of these points. On the other hand, these concepts can be used to conveniently describe many real-world situations. In a non-exhaustive list of domains in which we deal with networks, we can quote: anatomy (neural circuit), biology (protein interaction network), chemistry (crystal structures), computer sciences (web, peer-to-peer networks), artificial intelligence (artificial neural network), statistics (Bayesian network), electricity (electrical grid), telecom (telecommunication network), transportation (road network, rail network), urbanism (gas network, water distribution network).

In this thesis, we mainly study some colouring problems in graphs. Throughout this chapter, for the convenience of the description of our researching topic, let us first introduce the basics of graph theory in a more formal mathematical representation. Then we will mention the background in detail and related research of our work. In the last section, we will show the contributions and outline of this thesis.

1.1 . Basic notations

We first define the main objects of this work, which are graphs. Especially for graph theory, we follow the terminology of Bollobás [6].

1.1.1 . Graphs and digraphs

A *graph* G is a mathematical structure (V, E) consisting of a nonempty set $V(G)$ of *vertices*, and some binary relation $E(G) \subseteq \binom{V}{2}$ between them, which is described by a set of unordered pairs of vertices; these pairs are called the *edges* of the graph. The subscript might be omitted when there is no ambiguity on the relevant graph. In a drawing of G, the points represent the set V, and the lines represent the set E. An edge joining a vertex with itself is called *loop*. Several edges joining the same pair of vertices are called *multiple edges*. A graph is *finite*, if both its vertex set and edge set are finite. If a graph has no loops or multiple edges, then we say it is *simple*. This thesis focuses only on finite simple graphs.

A *digraph* or *directed graph* D is an ordered pair $(V(D), A(D))$ consisting of a nonempty set $V(D)$ of *vertices* and a set $A(D)$ of *arcs*, where $A(D)$ is made up of some ordered pairs of (not necessarily distinct) vertices. With each digraph D we can associate a graph G on the same vertex set; corresponding to each arc of D there is an edge of G with the same ends. This graph is the *underlying graph* of D. Conversely, given any graph G, we can obtain a digraph from D by specifying, for each edge, an order on its ends. Such a digraph is called an *orientation* of G.

The definition of graphs is quite simple and clear, yet it is sufficient to have a high expressivity. In order to understand how different types of graphs behave, we now introduce many basic tools which can help us analyse them.

Neighbourhood. Most of the definitions and concepts in graph theory are from the graphical representation intuitively. When a given edge e contains a vertex v, we say that v and e are *incident*. Two vertices incident with the same edge are *adjacent*, as are two edges which are incident with a common vertex.

Definition 1. Let G be a graph.

1. When there is an edge in G between two vertices u and v, we say that u is a neighbour of v, and vise versa. For any vertex $v \in V(G)$, we use

$N_G(v)$ to denote the set of all neighbours of v and call $N_G(v)$ the *(open) neighbourhood* of v. The degree $\deg_G(v)$ of a vertex v is the number of edges of G incident with v, i.e. the size of $N_G(v)$ when G is a simple graph;

$$N_G(v) \coloneqq \{u \in V(G) \mid uv \in E(G)\},$$

$$\deg_G(v) \coloneqq |N_G(v)| = |\{e \in E(G) \mid v \in e\}|,$$

2. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of G;

$$\delta(G) \coloneqq \min_{v \in V(G)} \deg_G(v),$$

$$\Delta(G) \coloneqq \max_{v \in V(G)} \deg_G(v).$$

3. The *closed neighbourhood* of a vertex v in G, denoted $N_G[v]$, is obtained by adding v to its open neighbourhood in G;

$$N_G[v] \coloneqq N_G(v) \cup \{v\}.$$

4. The notion of neighbourhood can be naturally extended to subset of vertices. For a vertex subset $S \subset V(G)$, namely we have

$$N_G(S) \coloneqq \bigcup_{v \in S} N_G(v) \setminus S,$$

$$N_G[S] \coloneqq N_G(S) \cup S.$$

5. A d*-regular* graph is a graph where every vertex has degree exactly d.

6. The *average degree* of G, denoted $\mathrm{ad}(G)$, is the average of all the degrees of the vertices in $V(G)$. The *maximum average degree* of G, denoted $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over all non-empty subgraphs H of G;

$$\mathrm{ad}(G) \coloneqq \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg_G(v) = \frac{2|E(G)|}{|V(G)|}.$$

$$\mathrm{mad}(G) \coloneqq \max_{H \subseteq G} \mathrm{ad}(H).$$

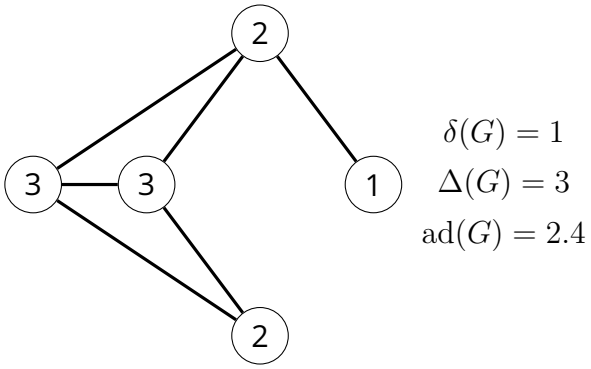


Figure 1.1: Each vertex is labelled with its degree

Subgraphs. Let G, H be two graphs. We say H is a *subgraph* of G if, up to some relabelling of vertices, we have $E(H) \subseteq E(G)$. Moreover, if H is a subgraph of G and H contains all the edges $uv \in E(G)$ with $u, v \in V(H)$, then H is an *induced* subgraph of G. We denote $G[S]$ the subgraph of G induced by a vertex subset $S \subseteq V(G)$. If H is a subgraph of G and $V(H) = V(G)$, then H is a *spanning* subgraph of G. We say G is *F-free* if G does not contain F as a subgraph, i.e. no subgraph of G is isomorphic to F. Let H_1, H_2 be two subgraphs of G. If $V(H_1) \cap V(H_2) = \emptyset$, then we say H_1 and H_2 are *vertex-disjoint*. If $E(H_1) \cap E(H_2) = \emptyset$, then we say H_1 and H_2 are *edge-disjoint*.

Subgraph is one of the most essential notions of graphs. A constitutive method to analyse the structure of a graph or its class is to consider whether a specific subgraph exists, appears frequently, or never at all. We list some of them which are the research object of this thesis.

- **Cliques and independent sets.**

  Definition 2. Let G be a graph.

  1. An *independent set* of G is a vertex subset $I \subset V(G)$ inducing no edge;

     $$E(G[I]) = \emptyset.$$

     It is *maximal* if there exists no independent set I' of G such that $I \subsetneq I'$. An independent set I is said to be *maximum* if there

exists no independent set I' of G such that $|I| < |I'|$. The number of vertices in a maximum independent set of G is called the *independence number* and is denoted $\alpha(G)$.

2. A *clique* of G is a vertex subset $S \subset V(G)$ inducing all possible edges;
$$E(G[S]) = \binom{S}{2}.$$
It is *maximal* if there exists no clique S' of G such that $S \subsetneq S'$, and *maximum* if there exists no clique with a larger cardinality. The number of vertices in a maximum clique of G is called the *clique number* and is denoted $\omega(G)$.

- **Paths and cycles.**

Definition 3. Let G be a graph.

1. A path in G is a finite non-null sequence of vertices $P_k = v_1 \dots v_k$, where two successively vertices are adjacent and all vertices are pairwise distinct. The vertices v_1 and v_k are called the *origin* and *terminus* of P_k, respectively (*extremity* for both), and $v_2, \dots, v_{k-1}$ its *internal vertices*. The number of edges $k-1$ is the *length* of P_k. We say P_k is a $v_1 - v_k$ path.
2. A cycle in G is a closed path, that is, a finite non-null sequence of vertices $C_k = v_0(= v_k)v_1 \dots v_k$, where two successively vertices are adjacent and all vertices are pairwise distinct. The length of C_k is the number of its edges k.

The following concepts can be naturally derived by paths.

Definition 4. Let G be a graph.

1. G is said to be *connected* if for every pair $\{u, v\}$ of distinct vertices there is a path from u to v.
2. Given vertices u and v, the *distance* $d_G(u, v)$ is the minimal length of a $u-v$ path. If there are no $u-v$ path, then we define $d_G(u, v) = \infty$.

3. A *vertex cut* of G is a subset V' of $V(G)$ such that $G - V'$ is disconnected. A k-*vertex cut* is a vertex cut of k elements.
4. If G has at least one pair of distinct nonadjacent vertices, the *connectivity* $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $\kappa(G)$ to be $|V(G)| - 1$. G is said to be k-*connected* if $\kappa(G) \geq k$.

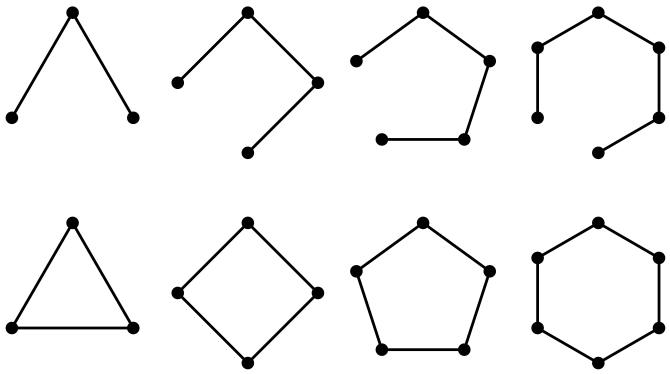


Figure 1.2: Paths and cycles of 3 to 6 vertices

- **Spanning trees.** Given a connected graph G, it is meaningful to determine a minimal connected spanning subgraph of G, which leads to the study of spanning trees.

  Definition 5. Let G be a graph.

  1. A graph without any cycles is a *forest*, or an *acyclic* graph.
  2. A *tree* is a connected forest.
  3. A *spanning tree* of G is a tree subgraph of G spanning all its vertices.

  We are often looking for a *minimum spanning tree (MST)* in a given weighted graph G, that is a spanning tree which minimises the sum of the weights of its edges. There are numerous efficient algorithms to find a MST of G, which works in $O(e(G) \log v(G))$ time, for instance.

In the main content of this thesis, we will use another kind of spanning trees, the *rooted tree*, that is a tree in which one vertex has been designated the root. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root, in which case the structure becomes a directed rooted tree. Given a connected graph G and a root $r \in V(G)$, many different kinds of spanning trees rooted at r have been defined. One of the most important, the *Breadth First Search Tree*, is a spanning tree of G rooted in some vertex r, with the property y that the distance from the root r to any vertex v in it is equal to $d_G(r, v)$. These trees are powerful tools in the context of structural analysis, especially when one is interested in the traversals of G.

Constructions. We are now going to see some convenient operations which let us construct a graph from another.

1. **Complement graph.** Given a graph G on n vertices, the complement of G, denoted $\overline{G}$, is obtained by removing all the edges of G from K_n;

$$V(\overline{G}) = V(G), \text{ and } E(\overline{G}) = \overline{E(G)} = \binom{V(G)}{2} \setminus E(G).$$

2. **Graph powers.** Given a graph G, the *t-th power* G^t of G is obtained from G by adding edges between all pairs of vertices at distance at most $t$in G;

$$V(G^t) = v(G), \text{ and } uv \in E(G) \iff d_G(u, v) \leq t.$$

3. Edge subdivision. Given a graph G and an edge $e = uv \in E(G)$, an edge subdivision of e is the insertion of a new vertex w in the middle of e accompanied by the joining of the original edge endpoints with the new vertex to form new edges $e' = uw$ and $e'' = wv$. Particularly, a 1-subdivision of G is a graph obtained by subdividing every edge of G exactly once.

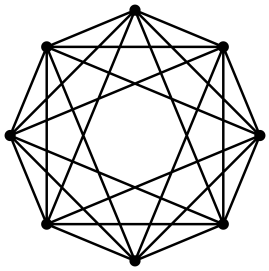


Figure 1.3: The distance-3 graph of C_8

Remark 1. Given a graph G on n vertices, the complement of G, denoted $\overline{G}$, is obtained by removing all the edges of G from K_n;

$$V(\overline{G}) = V(G), \text{ and } E(\overline{G}) = \overline{E(G)} = \binom{V(G)}{2} \setminus E(G).$$

Clearly, given a graph G, S is a clique of G if and only if S is an independent set of $\overline{G}$, and so the two concepts are complementary.

1.1.2 . Hypergraphs

For normal graphs, the edge set represents a (symmetric) binary relation on the vertex set. Sometimes we need to display information about relationships that feature more than one object. Here goes the concept of hypergraphs.

Many basic definitions are easy to illustrate by analogy with graphs. A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ consisting of a nonempty set $V(\mathcal{H})$ of *vertices* and a set $E(\mathcal{H})$ of *hyperedges*, where each hyperedge $e \in E(\mathcal{H})$ is a subset of the vertex set $e \subseteq V(\mathcal{H})$. Given a hypergraph $\mathcal{H}$, the *degree* of a vertex v, denoted by $\deg_{\mathcal{H}}(v)$, is the number of edges of $\mathcal{H}$ containing v. We denote by $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ the minimum and maximum degrees of $\mathcal{H}$, respectively. We denote $\epsilon(\mathcal{H})$ the minimum size of a hyperedge in $\mathcal{H}$. Particularly, we say $\mathcal{H}$ is k-*uniform* if every hyperedge of $\mathcal{H}$ has size exactly k.

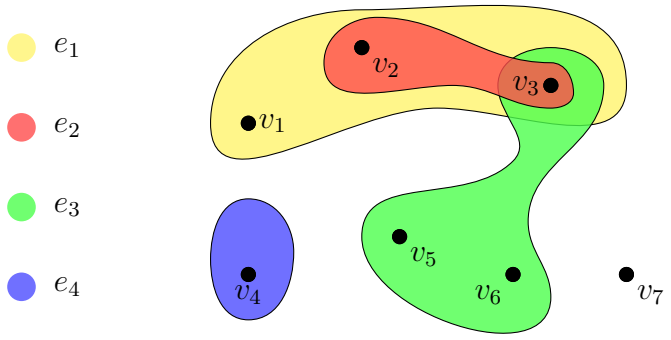


Figure 1.4: A hypergraph of order 7 and size 4.

Given a graph G, one can immediately construct a 2-uniform hypergraph $\mathcal{H}$ with the same vertex set and edge set isomorphic to G. In fact, there are many optional ways to associate a graph with a hypergraph, which provides possible approaches to solving or deriving many classic problems from Graph theory.

1. **Triangular graphs.** A graph G is said to be *triangular* if any edge of G is contained in a triangle of G. Triangular graphs (sometimes known as ternary graphs) offer an opportunity to display data based on three variables simultaneously. Let $\mathcal{H}$ be the 3-uniform hypergraph with $V(\mathcal{H}) = V(G)$ and $E(\mathcal{H})$ be the set of all 3-sets of $V(G)$ that induces a triangle. Hence for each triangular graph there lies an associating 3-uniform hypergraph, but the converse is not true.

2. **Neighbourhood hypergraphs.** In graph colouring theory, there are various problems asking a constraint of occurrence in the neighbourhood of each vertex or edge. Given a graph G, we can establish a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) \subseteq V(G) \cup E(G)$ where each hyperedge consists of the neighbourhood of an element in G.

1.1.3 . Graph colourings

In graph theory, graph colouring is a special case of graph labeling; it is an assignment of labels traditionally called "colours" to elements of a graph subject

to certain constraints. There are many different types of colouring, in terms of the research object, where vertex colouring and edge colouring are the most concerned categories.

Definition 6. Let G be a graph.

1. A k-*colouring* of G is a function $c\colon V(G) \to [k]$. This can be seen as a function which associates to every vertex $v \in V(G)$ a colour among a palette of k possible ones. A *partial colouring* of G is a colouring of an induced subgraph of G.

2. A colouring c of G is said to be *proper* whenever it follows the rule that no edge in G has two extremities of the same colour;

$$\forall uv \in E(G),\ c(u) \neq c(v).$$

   When there exists a proper k-colouring of G, we say that G is k-colourable.

3. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k for which G has a proper k-colouring.

4. For every colour $i \in [k]$ used in some k-colouring c of G, the subset of vertices $c^{-1}(\{i\})$ coloured with colour i is called a *colour class*, or *monochromatic class* of c. The colour classes of c yield a partition of $V(G)$, and if c is proper, every colour class is an independent set of G.

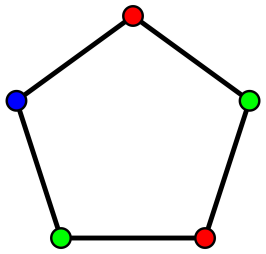


Figure 1.5: A proper 3-colouring of C_5

Remark 2. There are several equivalent definitions for a proper k-colouring of a given graph G on n vertices. A proper k-colouring of G equals to

1. a partition of $V(G)$ into k independent sets,

2. a graph homomorphism from G to the complete graph K_k on k vertices.

One can think of many variants of the colouring problem, depending on the context needed for its application. Let us present the most essential one here. In a k-colouring, every vertex receives a colour from a common set of colours $[k]$. In order to be more expressive, we consider a generalization of k-colourings where every vertex receives its colour from a private list of allowed ones. These are list colourings.

Definition 7. Let G be a graph.

1. A *list assignment* L with a graph G such that each $v \in V(G)$ is assigned a list of colours $L(v)$. A list assignment L is called a *t-list assignment* if $|L(u)| = t$ for every $u \in V(G)$. A list assignment L is called a *degree-list assignment* if $|L(v)| = \deg_G(v)$ for every $v \in V(G)$.

2. A graph G is *L-colourable* if there exists a proper colouring c of G such that
$$\forall v \in V(G),\ c(v) \in L(v).$$

3. A graph G is *k-choosable* if G is L-colourable for every k-list assignment L satisfying
$$\forall v \in V(G),\ |L(v)| \geq k.$$

4. The *choice number*, or *list chromatic number*, is the minimum k such that G is k-choosable. It is denoted $\chi_\ell(G)$.

5. A graph G is said to be *degree-choosable* if G admits an L-colouring for every degree-list assignment L.

As we mentioned, there are many equivalent definitions of the proper colouring of graphs, and hence there are many different ways to extend the notion of proper colouring to hypergraphs. As some edges may contain more than 2 vertices, to ask such an edge not to be monochromatic (all vertices of this edge have the same colour) or to be rainbow (all vertices of this edge have the distinct colour),

can lead to a radically different kind of behaviour, while either seems reasonable to be defined as "proper". The following conflict-free colouring with a moderate constraint attracts some attention to hypergraph study.

Definition 8. Let $\mathcal{H}$ be a hypergraph.

1. A colouring c of the vertices of a hypergraph $\mathcal{H}$ is called *conflict-free* if each hyperedge e of $\mathcal{H}$ contains a vertex whose colour is not repeated in e.
$$\forall e \in E(\mathcal{H}),\ \exists v \in e,\ \forall\, u \in e \setminus \{v\},\ c(v) \neq c(u).$$
2. The smallest number of colours required for such a colouring is called the *conflict-free chromatic number* of $\mathcal{H}$, and is denoted by $\chi_{\mathrm{cf}}(\mathcal{H})$.

Remark 3. One can observe that a conflict-free colouring of a hypergraph $\mathcal{H}$ is in particular a proper colouring of the graph formed by the hyperedges of size 2 in $\mathcal{H}$.

A weakening of that notion, the odd colouring, is as follows.

Definition 9. Let $\mathcal{H}$ be a hypergraph. A colouring c of the vertices of a hypergraph $\mathcal{H}$ is called *odd* if in every hyperedge e of $\mathcal{H}$, there is a colour x with an odd number of occurrences in c; we say that x is an odd colour of e in c.
$$\forall e \in E(\mathcal{H}),\ \exists x \in c(e),\ |c^{-1}(x) \cap e| \text{ is odd.}$$

Then we introduce edge colouring and some relative definitions.

Definition 10. Let G be a graph.

1. A *k-edge colouring* of a graph G is an assignment of k colours $c\colon E(G) \to [k]$. A *partial edge colouring* of G is a colouring of an induced subgraph of G.
2. An edge colouring c of G is said to be *proper* if it satisfies that no 2 adjacent edges in G share the same colour;
$$\forall e_1, e_2 \in E(G) \text{ and } e_1 \cap e_2 \neq \emptyset,\ c(e_1) \neq c(e_2).$$
Whenever there exists a proper k-edge colouring of G, we say that G is k-edge colourable.

3. The *edge chromatic number* of G, denoted $\chi'(G)$ is the minimum k for which G has a proper k-edge colouring.

4. For every colour $i \in [k]$ used in some k-edge colouring c of G, the subset of edges $c^{-1}(\{i\})$ coloured with colour i is called a *colour class*, or *monochromatic class* of c. The colour classes of c yield a partition of $E(G)$, and if c is proper, every colour class is a matching of G.

Definition 11. Let G be an edge-coloured graph with colouring c.

1. G is said to be *properly coloured* if c is proper;

2. G is said to be *rainbow* if all edges have distinct colours;

$$\forall e_1, e_2 \in E(G),\ c(e_1) \neq c(e_2).$$

3. G is said to be *monochromatic* if all edges share the same colour;

$$\forall e_1, e_2 \in E(G),\ c(e_1) = c(e_2).$$

With a more precise consideration, we introduce the concept of degree in terms of colours, which describes how the colour of edges behaves in the neighbourhood of any vertex. Let G be an edge-coloured graph with colouring c and $v \in V(G)$ a vertex in G. The *colour degree* $d^c_G(v)$ of a vertex v is the number of distinct colours appearing on the incident edges of v. We denote by $\delta^c(G)$ and $\Delta^c(G)$ the minimum and maximum colour degrees, respectively, of vertices of G;

$$d^c_G(v) = |c(N_G(v))| = |\{c(e) \mid e \in E(G), v \in e\}|,$$

$$\delta^c(G) = \min_{v \in V(G)} d^c_G(v),$$
$$\Delta^c(G) = \max_{v \in V(G)} d^c_G(v).$$

1.2 . Motivation and relative works

1.2.1 . Classical chromatic number

The first and almost the most famous result about graph colouring, the Four Colour Problem, deals almost exclusively with planar graphs in the form of the colouring of maps. While trying to colour a map of the counties of England, Francis Guthrie postulated the Four Colour Conjecture, noting that four colours were sufficient to colour the map so that no regions sharing a common border received the same colour. It took more than a century before a valid proof was established by Kenneth Appel and Wolfgang Haken in 1976. They used the discharging method, which had been developed in the previous decade. However, this proof still relied on a computer program to check thousands of small statements and is too complex to be completed by hand. Although the proof has undergone several simplifications, a human-checkable proof has yet to be discovered.

Graph colouring theory has a central position in Discrete Mathematics. It appears in many places with seemingly no or little connection to colouring. A good example is the Erdős-Stone-Simonovits theorem in Extremal Graph Theory, showing that for a fixed graph G the behaviour of the maximum number $\mathrm{ex}(n, G)$ of edges in a graph on n vertices not containing G as a subgraph depends on the chromatic number $\chi(G)$ of G:

$$\mathrm{ex}(n, G) = \frac{1}{2}\left(1 - \frac{1}{\chi(G) - 1} + o(1)\right) n^2.$$

Even if many deep and interesting results have been obtained during the 100 years of graph colouring, there are very many easily formulated, interesting problems left.

To begin with, we introduce a naive upper bound of $\chi(G)$ by a greedy algorithm. Let G be a graph with maximum degree $\Delta(G)$. For the first vertex to be coloured, it could be assigned with any colour. Any time we colour a vertex $v \in V(G)$, let $H \subseteq G$ be the subgraph of G induced by all coloured vertices. Since v has at most $\Delta(G)$ neighbours in $V(H)$, there are at most $\Delta(G)$ colours that are forbidden for v. Hence we can greedily colour G with $\Delta(G) + 1$ colours, and so $\chi(G) \leq \Delta(G) + 1$.

A well-known objective in graph colouring is to establish adequate conditions

that can yield much-improved upper bounds on the chromatic number compared to the straightforward ones. Over the past few decades, considerable research has been conducted in this field, and presented below are some of the key findings in this domain.

Theorem 1.2.1 (4-colour theorem [72]) *For every planar graph G,*

$$\chi(G) \leq 4.$$

Theorem 1.2.2 (Grötzsch, 1959 [39]) *For every planar triangle-free graph G,*

$$\chi(G) \leq 3.$$

Theorem 1.2.3 (Brooks, 1941 [9]) *For every connected graph G, either G is a complete graph or an odd cycle, or*

$$\chi(G) \leq \Delta(G).$$

Theorem 1.2.4 (Johansson-Molloy theorem [46, 61]) *For every triangle-free graph G,*

$$\chi(G) \leq (1 + o(1))\frac{\Delta(G)}{\ln \Delta(G)}.$$

All these theorems are sharp, since for each of them there exist infinite families of graphs satisfying their constraints.

To handle more graph colouring problems, Vizing [79], and independently Erdős, Rubin, and Taylor [28] introduced list colourings. A graph G has a proper k-colouring if and only if G has an L-colouring with $L(v) = [k]$. Therefore, for any graph G we have $\chi_\ell(G) \geq \chi(G)$. Some well-known upper bounds on $\chi(G)$ in terms of vertex degrees hold for $\chi_\ell(G)$ as well. By the same greedy algorithm, one can get that $\chi_\ell(G) \leq \Delta(G) + 1$. Thomassen proved the following results of the choosability of planar graphs.

Theorem 1.2.5 (Thomassen, 1994 [76]) *For every planar graph G,*

$$\chi_\ell(G) \leq 5.$$

Theorem 1.2.6 (Thomassen, 1995 [77]) *For every planar graph G of girth at least 5,*

$$\chi_\ell(G) \leq 3.$$

Clearly, every k-choosable graph is k-colourable, but the converse is known not to be true. For instance, every planar graph is 4-colourable, but not all of them are 4-choosable [80].

Theorem 1.2.7 (Voigt, 1993 [80]) *There exist planar graphs which are not 4-choosable.*

In fact, the gap between them $\chi_\ell(G) - \chi(G)$ can be arbitrarily large. Even worse, one can have $\chi_\ell(G)$ unbounded even when $\chi(G) = 2$. Consider the complete bipartite graph K_{q,q^q}. On the small side of the partition, let the q vertices be given sets of colours that are pairwise disjoint, where there are q^q possible q-tuples if we draw exactly one colour from each vertex. Then we assign each tuple to a vertex on the large side. Then, K_{q,q^q} does not have a list colouring for L: no matter what set of colours is chosen for the vertices on the small side of the bipartition, this choice will conflict with all of the colours for one of the vertices on the other side of the bipartition.

Even though $\chi_\ell(G)$ can be arbitrarily larger than $\chi(G)$, the difference is bounded by some graph parameters.

Theorem 1.2.8 *Let G be a graph with n vertices. It holds that*

$$\chi_\ell(G) \leq \chi(G)(\ln n + 1).$$

Especially for bipartite graphs, in [28], Erdős et al. presented the following: What is the minimum number $N(2, k)$ of vertices in a graph G which is 2-colourable but not k-choosable? It was shown [28] that

$$2^{k-1} \leq N(2, k) \leq k^2 2^{k+2}.$$

For small values, they showed $N(2, 2) = 6$ and conjectured that $N(2, 3) = 14$, which was confirmed by Hanson, MacGillivray and Toft [42].

1.2.2 . Conflict-free colourings

The concept of conflict-free colourings was first introduced by Even, Lotker, Ron, and Smorodinsky [30] in a geometric setting, in connection with frequency assignment problems for cellular networks. Pach and Tardos [67] studied this notion and proved that every hypergraph with fewer than $\binom{s}{2}$ edges (for some integer s) has a conflict-free colouring with fewer than s colours.

Theorem 1.2.9 (Pach and Tardos, 2009 [67]) *Let $\mathcal{H}$ be a hypergraph with $|E(\mathcal{H})| < \binom{s}{2}$ edges for some integer s, and let Δ be the maximum degree of a vertex in $\mathcal{H}$. Then the conflict-free chromatic number of $\mathcal{H}$ satisfies:*

(a) $\chi_{\mathrm{cf}}(\mathcal{H}) < s$;

(b) $\chi_{\mathrm{cf}}(\mathcal{H}) \leq \Delta + 1$.

Both bounds are optimal and the corresponding colourings can be found in linear deterministic time.

Theorem 1.2.10 (Pach and Tardos, 2009 [67]) *For any positive integers t and Γ, the conflict-free chromatic number of any hypergraph in which each edge is of size at least $2t-1$ and each edge intersects at most Γ others, is $O(t\Gamma^{1/t}\log\Gamma)$.*

Kostochka, Kumbhat, and Łuczak [47] further studied conflict-free colouring for uniform hypergraphs.

Theorem 1.2.11 (Kostochka, Kumbhat, and Łuczak, 2012 [47]) *Let $\mathcal{H}$ be a r-uniform hypergraph with m edges and maximum degree Δ.*

(i) If $\Delta \leq 2^{r/2}$, and Δ (and thus r) is large enough, then

$$\chi_{\mathrm{cf}}(\mathcal{H}) \leq 120 \ln \Delta \leq 120 \ln m.$$

(ii) If $m \geq 2^{r/2}$, then

$$\chi_{\mathrm{cf}}(\mathcal{H}) \leq 4r(16m)^{2/(r+2)}.$$

In particular, let us consider two graphs or hypergraphs, given distinct constraints, sharing a common vertex set, as follows.

Definition 12. Let G be a graph, and $\mathcal{H}$ be a hypergraph.

1. We say that $(G, \mathcal{H})$ is a *graph-hypergraph pair* if they have the same vertex set $V(G) = V(\mathcal{H}) = V$.

2. A *proper conflict-free k-colouring of $(G, \mathcal{H})$* (*pcf k-colouring* for short) is a mapping $c\colon V \to [k]$ that is both a proper colouring of G and a conflict-free colouring of $\mathcal{H}$.

By the previous observation, this can be seen as a conflict-free colouring of the hypergraph $(V, E(\mathcal{H}) \cup E(G))$. There are many classical constraints for a proper colouring of a given graph G that are in particular satisfied by a pcf-colouring of $(G, \mathcal{H})$, for some hypergraph $\mathcal{H}$ carefully chosen. For instance, if $E(\mathcal{H})$ contains a maximum independent set of every even cycle in G, then a pcf-colouring of $(G, \mathcal{H})$ is in particular an acyclic colouring of G. As another example, if $E(\mathcal{H})$ contains all $(\beta+1)$-subsets of every neighbourhood in G, then a pcf-colouring of $(G, \mathcal{H})$ is in particular a β-frugal colouring of G.

There has been a specific focus on the special case of pcf-colourings σ of $(G, \mathcal{H})$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G. That is, $E(\mathcal{H}) = \{N(v) : v \in V(G), \deg(v) > 0\}$. In that case, we say that σ is a pcf-colouring of G (so when we omit $\mathcal{H}$, it is implicitly the neighbourhood-hypergraph of G). In other words, a *pcf-colouring* of G is a proper colouring of G such that for every non-isolated vertex v, there is a colour appearing exactly once among the neighbours of v. We let $\chi_{\mathrm{pcf}}(G)$ be the smallest integer k such that there exists a pcf k-colouring of G. This notion is the combination of proper colouring and the pointed conflict-free chromatic parameter introduced by Cheilaris [15].

The notion of pcf-colourings of graphs was formally introduced by Fabrici, Lužar, Rindošová, and Soták [31], where they investigated the pcf-colourings of planar and outerplanar graphs, among many other related variants of a proper conflict-free colouring. They proved that $\chi_{\mathrm{pcf}}(G) \leq 8$ for all planar graphs and $\chi_{\mathrm{pcf}}(G) \leq 5$ for all outerplanar graphs. Plenty of further studies in pcf-colourings of sparse graphs can be found in [12, 18, 31, 44, 54].

Theorem 1.2.12 (Caro, Petruševski, Škrekovski [12]) *Let G be a graph of girth $g \geq 6$, and let $K(G) = \{v \in V(G) \mid \deg_G(v) \geq 3\}$. If the distance $d_G(u, v) \geq 6$ for every two vertices $u, v \in K(G)$, then $\chi_{\mathrm{pcf}}(G) \leq 4$.*

Theorem 1.2.13 (Cho, Choi, Kwon, Park [18]) *If G is a planar graph with girth at least 5, then $\chi_{\mathrm{pcf}}(G) \leq 7$.*

Theorem 1.2.14 (Caro, Petruševski, Škrekovski [12]) *Let G be a graph.*

(i) If $\mathrm{mad}(G) \leq \frac{8}{3}$, then $\chi_{\mathrm{pcf}}(G) \leq 6$.

(ii) If $\mathrm{mad}(G) \leq \frac{5}{2}$, then $\chi_{\mathrm{pcf}}(G) \leq 5$.

(iii) If $\mathrm{mad}(G) \leq \frac{24}{11}$, *then* $\chi_{\mathrm{pcf}}(G) \leq 4$, *unless every maximal* 2*-connected subgraph of* G *is a* 5*-cycle.*

Theorem 1.2.15 (Cho, Choi, Kwon, Park [18]) *For* $c \geq 5$, *if* G *is a graph with* $\mathrm{mad}(G) \leq \mathrm{mad}(K^*_{c+1}) = \frac{4c}{c+2}$, *then* $\chi_{\mathrm{pcf}}(G) \leq c$, *unless* G *contains* K^*_{c+1} *as a subgraph. Here* K^*_{c+1} *is the* 1*-subdivision of* K_{c+1}.

Theorem 1.2.16 (Cho, Choi, Kwon, Park [18]) *If* G *is a graph with* $\mathrm{mad}(G) \leq \frac{12}{5}$ *and no induced* 5*-cycle, then* $\chi_{\mathrm{pcf}}(G) \leq 4$.

Theorem 1.2.17 (Fabrici, Lužar, Rindošová, Soták [31]) *For the class of planar graphs* $\mathcal{P}$, *it holds that*

$$6 \leq \chi_{\mathrm{pcf}}(\mathcal{P}) \leq 8.$$

For the class of outerplanar graphs $\mathcal{O}$, *it holds that*

$$\chi_{\mathrm{pcf}}(\mathcal{O}) = 5.$$

Theorem 1.2.18 (Hickingbotham [44]) *For every* K_t*-minor free graph* G, $\chi_{\mathrm{pcf}}(G) \leq 5(t-1)(t-2) - 1$.

Theorem 1.2.19 (Liu [54]) *For every graph* H, *there exists a real number* c_H *such that every graph that does not contain a subdivision of* H *(as a subgraph) is conflict-free* c_H*-choosable.*

Theorem 1.2.20 (Liu [54]) *Let* d *be a nonnegative integer. If* $\mathcal{F}$ *is a* d*-degenerate minor-closed family, then every graph* G *in* $\mathcal{F}$ *is conflict-free* $(2d+1)$*-choosable.*

Given a graph G, we let $\delta^*(G)$ denote the *degeneracy* of G, that is $\delta^*(G) = \max_{H \subseteq G} \delta(H)$. Caro, Petruševski, and Škrekovski [12] proposed the following conjecture about pcf-colourings.

Conjecture 1.2.1 (Caro, Petruševski, Škrekovski [12, Conjecture 6.4]) *If* G *is a connected graph of maximum degree* $\Delta \geq 3$, *then* $\chi_{\mathrm{pcf}}(G) \leq \Delta + 1$.

As a first step toward their conjecture, Caro, Petruševski, and Škrekovski [12] proved that for such a graph G, $\chi_{\mathrm{pcf}}(G) \leq \lfloor 2.5\Delta \rfloor$. Recently, it has been observed

by Cranston and Liu [22] that $\chi_{\text{pcf}}(G) \leq \Delta(G) + \delta^*(G) + 1$ (they actually more generally proved that there always exists a pcf $(\Delta(\mathcal{H})+\delta^*(G)+1)$-colouring of any given pair $(G, \mathcal{H})$). They further reduced the gap to Conjecture 1.2.1 by proving that $\chi_{\text{pcf}}(G) \leq \left\lceil 1.6550826\Delta + \sqrt{\Delta} \right\rceil$, given that Δ is large enough.

Theorem 1.2.21 (Cranston, Liu [22]) *Let G be a graph and $\mathcal{H}$ be a hypergraph with $V(G) = V(\mathcal{H})$. If G is d-degenerate, then $(G, \mathcal{H})$ has a proper conflict-free colouring with at most $d + \Delta(\mathcal{H}) + 1$ colours.*

Theorem 1.2.22 (Cranston, Liu [22]) *Fix a positive integer $\Delta \geq 1.24811 \cdot 10^8$, fix a real number β with $\Delta \geq \beta \geq 0.6550826\Delta$, and let $a := \left\lceil \Delta + \beta + \sqrt{\Delta} \right\rceil$. If G is a graph with maximum degree at most Δ and L is an a-assignment for G, then there are at least $\beta^{|V(G)|}$ proper conflict-free L-colourings of G.*

1.2.3 . Odd colourings

In [16], Cheilaris, Keszegh, and Pálvölgyi introduced odd colourings, a weakening of conflict-free colourings. It is straightforward that a conflict-free colouring of $\mathcal{H}$ is in particular an odd colouring of $\mathcal{H}$. Petruševski and Škrekovski [69] later considered that notion applied to the neighbourhood-hypergraph of a graph G. They defined an *odd colouring* of graph G as a proper colouring of G with the additional constraint that each non-isolated vertex has a colour appearing an odd number of times in its neighbourhood. The *odd chromatic number* of G, denoted by $\chi_o(G)$, is the minimum integer k such that there exists an odd k-colouring of G. Since odd colourings are a weakening of pcf-colourings, it always holds that $\chi_o(G) \leq \chi_{\text{pcf}}(G)$. In the last couple of years, there has been some interest in determining the extremal value of χ_o in various classes of graphs.

Even though all results about pcf-colouring can be directly used to bound $\chi_o(G)$, they could behave differetly on many classes of graphs. For the case of sparse graphs, Petruševski and Škrekovski [69] showed that $\chi_o(G) \leq 9$ for every planar graph G with a proof that relies on the discharging method. Furthermore, they conjectured that this bound may be reduced to 5. If true, this would be tight, since $\chi_o(C_5) = 5$. Recently there has been considerable attention in odd colourings of planar graphs [11, 19, 20, 68].

Conjecture 1.2.2 (Petruševski, Škrekovski [69]) *For every planar graph G it holds that $\chi_o(G) \leq 5$.*

Theorem 1.2.23 (Cho, Choi, Kwon, Park [19]) *Every planar graph with girth at least 5 is odd 6-colourable.*

A graph G is said to be 1-*planar* if it can be drawn in the plane so that each edge is crossed by at most one other edge. Cranston, Lafferty, and Song [21] proved that every 1-planar graph admits an odd 23-colouring. Later Liu, Wang, and Yu [55] improved this bound to 13.

Theorem 1.2.24 (Liu, Wang, Yu [55]) *Every 1-planar graph admits an odd 13-colouring.*

For graphs with a small maximum average degree, Cho, Choi, Kwon, and Park [19] proved the following result, which is a little bit stronger than the version of pcf.

Theorem 1.2.25 (Cho, Choi, Kwon, Park [19]) *If G is a graph with $\mathrm{mad}(G) \leq \frac{22}{9}$ and no induced 5-cycle, then $\chi_o(G) \leq 4$.*

Caro, Petruševski, and Škrekovski [11] also studied various properties of the odd chromatic number of general graphs; in particular, they proved the following facts: every graph of maximum degree three has an odd 4-colouring; every graph, except for C_5, of maximum degree Δ has an odd 2Δ-colouring. Moreover, they proposed a conjecture for general graphs, which is a weaker form of Conjecture 1.2.1.

Conjecture 1.2.3 (Caro, Petruševski, Škrekovski [11, Conjecture 5.5]) *If G is a connected graph of maximum degree $\Delta \geq 3$, then $\chi_o(G) \leq \Delta + 1$.*

1.2.4 . Edge-coloured subgraphs

While approaching the chromatic number $\chi(G)$ of a graph G can be highly complicated, the edge chromatic number $\chi'(G)$ is much more clear. In any proper edge colouring of G, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi'(G) \geq \Delta(G).$$

In fact, there are infinitely many cases when the equality $\chi'(G) = \Delta(G)$ holds for a graph G. An important theorem proved by Vizing (1964) and, independently, Gupta (1966), asserts that

Theorem 1.2.26 (Vizing's theorem) *For any graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

Since for any graph G, the edge chromatic number is either $\Delta(G)$ or $\Delta(G)+1$, it is meaningful to determine the exact value of $\chi'(G)$. When $\chi'(G) = \Delta(G)$, G is said to be of class 1; otherwise, it is said to be of class 2. For instance, every bipartite graph is of class 1, and almost all random graphs are of class 1. However, it is NP-complete to determine whether an arbitrary graph is of class 1.

Meanwhile, there are many efficient algorithms to assign proper edge-colouring of graphs, where most of them run in polynomial time. As an example, Gabow et al. [36] showed that every simple graph of maximum degree Δ can be properly edge-coloured by $\Delta + 1$ colours in $O(m\sqrt{n \log n})$ run-time.

Hence the majority of the study of edge colourings becomes analysing the colouring structure of them, in which an edge-coloured graph can be regarded as a special weighted graph. To begin with, we would like to mention the Hamiltonian problem, one of the most famous problems in Graph Theory and Algorithms, dealing with the connectivity of uncoloured graphs. A *Hamiltonian cycle* of a graph G is a cycle that spans the whole graph. If G has a Hamiltonian cycle, then G is *Hamiltonian*.

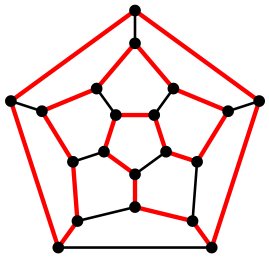


Figure 1.6: A Hamiltonian cycle of the dodecahedron

It is well-known that the Hamiltonian problem is NP-complete. Finding a Hamiltonian path or cycle in a general graph is computationally tricky and may

require significant computational resources. There was no elegant characterization of Hamiltonian graphs for a long time. A well-constructed necessary and sufficient condition for a graph to be Hamiltonian turns out to be impossible under the hypnothesis that $P! = NP$. Naturally, there has been much literature on graph theory studying sufficient conditions for Hamiltonicity, where the edge density is the most involved one. The earliest known result based on a degree condition was given by Dirac [25] in 1952.

Theorem 1.2.27 (Dirac, 1952 [25]) *Let G be a graph on n vertices. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

Over time, numerous researchers have established many results about finding a specific structure in various kinds of graphs, which include but never be limited to, normal graphs, directed graphs, coloured graphs and hypergraphs.

The study on the existence of properly coloured cycles in edge-coloured graphs has a long history. Grossman and Häggkvist [38] provided a sufficient condition for the existence of properly coloured cycles in edge-coloured graphs with two colours. Yeo [2] extended this result to any number of colours.

Theorem 1.2.28 (Grossman and Häggkvist, 1983 [38]) *Let G be a graph whose edges are coloured red and blue so that every vertex is incident with at least one edge of each colour. Then either G has a cut vertex separating colours, or G has an alternating cycle.*

Theorem 1.2.29 (Yeo, 1996 [2]) *If G is an edge-coloured graph with no alternating cycle, then there is a vertex $z \in V(G)$ such that every connected component of $G - z$ is joined to z of edges with monochromatic edges.*

During the past decades, establishing sufficient conditions forcing rainbow or properly coloured cycles of certain lengths has received considerable attention [3, 7, 24, 33, 49, 50, 52, 56]. For short properly coloured cycles, the case of triangles is the most studied one and obtains many significant results. The well-known Gallai colouring theory gives a structural characterisation of edge-coloured complete graphs containing no rainbow triangles [37, 40]. Gyárfás and Simonyi [40] proved that each edge-coloured complete graph K_n with $\Delta^{\text{mon}}(K_n) < 2n/5$ contains a properly coloured triangle and this bound is tight. Conditions for the existence of

rainbow triangles in edge-coloured graphs (not necessarily complete) are given in [49, 50].

Theorem 1.2.30 (Li, 2013 [50]) *Let G be an edge-coloured graph of order $n \geq 3$. If $\delta^c(G) \geq n/2$, then G has a rainbow triangle.*

Theorem 1.2.31 (Li, Ning, Xu and Zhang, 2014 [49]) *Let G be an edge-coloured graph on n vertices.*

1. *If $e(G) + c(G) \geq n(n+1)/2$, then G contains a rainbow triangle.*
2. *If $\sum_{v \in V(G)} d^c_G(v) \geq n(n+1)/2$, then G contains a rainbow triangle.*

Later, Fujita et al. [33] gave the colour degree condition for properly coloured triangles in edge-coloured complete graphs.

Theorem 1.2.32 *If $\delta^c(K_n) > \log_2 n$ with $n \geq 3$, then K_n contains a properly coloured C_3.*

In the case of properly coloured Hamiltonian cycles, Bollobás and Erdős [7] conjectured that every edge-coloured K_n with $\Delta^{\mathrm{mon}}(K_n) \leq \lfloor n/2 \rfloor - 1$ contains a properly coloured Hamiltonian cycle and this conjecture was asymptotically resolved by Lo [58].

Theorem 1.2.33 (Lo, 2016 [58]) *For any $\varepsilon > 0$, there exists an integer $N_0 = N_0(\varepsilon)$ such that every K_n with $N \geq N_0$ and $\Delta^{\mathrm{mon}}(K_n) \leq (1/2 - \varepsilon)n$ contains a properly coloured Hamiltonian cycle.*

Lo [57] also considered the existence of properly coloured Hamiltonian cycles under colour degree conditions.

Theorem 1.2.34 (Lo, 2014 [57]) *For every edge-coloured G with $\delta^c(G) \geq 2|G|/3$ contains a properly coloured 2-factor.*

Theorem 1.2.35 (Lo, 2014 [57]) *For every $\varepsilon > 0$, there exists an integer n_0 such that every edge-coloured graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| \geq n_0$ contains a properly coloured cycle of length ℓ for all $3 \leq \ell \leq |G|$.*

Using the absorbing technique and stability method, Lo [59] improved above result by the ε term.

Theorem 1.2.36 (Lo, 2019 [59]) *For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every edge-coloured graph G on $n \geq n_0$ vertices with $\delta^c(G) \geq (1/2+\varepsilon)n$ contains a properly coloured cycle of length at least $\min\{\lfloor 3\delta^c(G)/2 \rfloor, n\}$.*

This result actually partially answered another conjecture raised by Lo [56], which asks for sufficient conditions for an edge-coloured graph to contain properly coloured cycles or paths of any given length.

Conjecture 1.2.4 (Lo, 2013 [56]) *Every edge-coloured connected graph G with $\delta^(G) \geq d$ contains a properly coloured Hamiltonian cycle or a properly coloured path of length $\lfloor 2d/3 \rfloor$.*

Li and Wang [51] showed that every edge-coloured graph G with $\delta^c(G) \geq d$ contains a properly coloured path of length $2d$ or a properly coloured cycle of length at least $2d/3$. In [56], Lo improved $2d/3$ to $d+1$, which is best possible. Note that Theorem 1.2.36 implies Conjecture 1.2.4 when $d \geq (1/2+\varepsilon)n$ and n large. It would be interesting to know whether Conjecture 1.2.4 holds for $d < 1/2$.

1.2.5 . Generalised Ramsey number

If G has no large cliques, then one might ask whether G has a large independent set. That this is indeed the case was first proved by Ramsey.

Theorem 1.2.37 (Ramsey, 1930) *For every two integers r and s, there exists an integer $R(r,s)$ such that any graph on at least $R(r,s)$ vertices either contains a clique of size r or an independent set of size s.*

The number $R(r,s)$ are known as *Ramsey number* with parameters r and s. Computing the exact value or $R(r,s)$ for various values of r and s is a hard theoretical problem. The first exponential lower bound was obtained by Erdős using the probabilistic method. However, there is a vast gap between the tightest lower bounds and the tightest upper bounds. There are also very few numbers r and s for which we know the exact value of $R(r,s)$.

The Ramsey theory can be equivalently defined on edge-coloured graphs, which is isomorphic to that any edge-coloured graph on at least $R(r,s)$ with 2 colours (namely red and blue) either contains a red monochromatic clique of size r or a blue monochromatic clique of size s. More generally, we may allow arbitrarily

many colours to be used, and ask what types of subgraphs are forced if we replace monochromatic by some other condition on the subgraph's colouring. Erdős and Rado [27] were among the first to study problems of this type. In 1950 they proved a counterpart of Ramsey's theorem on colourings of finite sets using arbitrarily many colours. Their theorem is often known as the *Canonical Ramsey Theorem*. We paraphrase that theorem, in terms of the following notation. When the end vertices u and v of an edge e are integers, we call $\max(u, v)$ the higher endpoint and call $\min(u, v)$ the lower endpoint.

Theorem 1.2.38 (Erdős-Rado [27]) *Let p be a positive integer. Then there exists a least positive integer $N = er(p)$ such that if the edges of the complete graph K_N with vertex set $\{1, \ldots, N\}$ are coloured using an arbitrary number of colours, then there exists a complete subgraph with p vertices on which the colouring is of one of four canonical types:*

1. *monochromatic—all edges have the same colour;*
2. *rainbow—no two edges have the same colour;*
3. *upper lexical—two edges have the same colour if and only if they have the same higher endpoint;*
4. *lower lexical—two edges have the same colour if and only if they have the same lower endpoint.*

Both Ramsey theorem and the Canonical version focus on minimizing the order of graphs such that there exists an edge-coloured subgraph of any given type. Conversely, for a fixed complete graph K_n on n vertices, the number of colours used on the edges will determine its colouring structure—more multicoloured subgraphs may appear with the increment of the number of colours, for instance. Let us formalise this problem, which is called the anti-Ramsey problem.

Given a positive integer n and a graph G, the *anti-Ramsey number* $\mathrm{ar}(K_n, G)$ is the maximum number of colours in an edge-colouring of K_n that has no rainbow copy of G. This notion was introduced by Erdős, Simonovits and Sós [29]. They showed that $\mathrm{ar}(K_n, G) - \mathrm{ex}(n, \mathcal{G}) = o(n^2)$, where $\mathcal{G} = \{G - e : e \in E(G)\}$. Here $\mathrm{ex}(n, \mathcal{F})$ is the Turán number of $\mathcal{F}$, which is the maximum number of

edges of a simple graph of order n having no member of $\mathcal{F}$ as a subgraph. This determined $\mathrm{ar}(K_n, G)$ asymptotically when $\chi(H) \geq 3$ for any $H \in \mathcal{G}$. The remaining case is more complex. For paths, Simonovits and Sós [75] proved that $\mathrm{ar}(K_n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon$ for large n, where $\epsilon = 0$ or 1. Here P_ℓ denotes the path of ℓ vertices. For cycles, Erdős, Simonovits, and Sós [29] conjectured that for every fixed $\ell \geq 3$, $\mathrm{ar}(K_n, C_\ell) = (\frac{\ell-2}{2} + \frac{1}{\ell-1})n + O(1)$, and proved that $\mathrm{ar}(K_n, C_3) = n - 1$ for $\ell = 3$. Alon [1] proved this conjecture for $\ell = 4$ and gave some upper bounds for $k \geq 5$. Jiang, Schiermeyer, and West [45] proved this for $\ell \leq 7$. Finally, in 2005, Montellano-Ballesteros and Neumann-Lara [65] completely proved this conjecture. For cliques, Erdös, Simonovits, and Sós [29] showed that $\mathrm{ar}(K_n, K_{p+1}) = ex(n, K_p) + 1$ for $p \geq 3$ and sufficiently large n. Montellano-Ballesteros [64] and Neumann-Lara showed that this holds for every $3 \leq p \leq n$. We would mention a survey by Fujita, Magnant and Ozeki [35] for more conclusions towards these problems.

On the analogy of $\mathrm{ar}(K_n, G)$, we denote by $\mathrm{pr}(K_n, G)$ the maximum number of colours in an edge-colouring of K_n that has no properly coloured copy of G. The problem on the minimum number of colours, sufficient for the existence of given types of properly edge-coloured subgraphs in an edge-colouring of K_n, was first studied by Manoussakis, Spyratos, Tuza, and Voigt [60]. For cliques, they [60] obtained the approximate value of $\mathrm{pr}(K_n, K_t)$. Fang, Győri, and Xiao [32] generalized this result to an arbitrary graph G.

Theorem 1.2.39 (Fang, Győri, and Xiao, 2021 [32]) *Let G be a graph and $\mathcal{G}' = \{G - M : M$ is a matching of $G\}$, then $\mathrm{pr}(K_n, G) \geq \mathrm{ex}(n, \mathcal{G}') + 1$ and $\mathrm{pr}(K_n, G) = (\frac{d-1}{2d} + o(1))n^2$, where $d = \min\{\chi(F) : F \in \mathcal{G}'\} - 1$.*

For paths, Manoussakis, Spyratos, Tuza, and Voigt [60] proved that $\mathrm{pr}(K_n, P_n) = \binom{n-3}{2} + 1$ for large n. Fang, Győri, and Xiao [32] proved that $\mathrm{pr}(K_n, P_3) = 1$, $\mathrm{pr}(K_n, P_4) = 2$, and $\mathrm{pr}(K_n, P_5) = 3$. They [32] more generally proved the following.

Theorem 1.2.40 (Fang, Győri, and Xiao, 2021 [32]) *Let P_ℓ be a path with $\ell \geq 27$ and $\ell \equiv r_\ell \mod 3$, where $0 \leq r_\ell \leq 2$. For $n \geq 2\ell^3$, we have*

$$\mathrm{pr}(K_n, P_\ell) = (\lfloor \frac{\ell}{3} \rfloor - 1)n - \binom{\lfloor \frac{\ell}{3} \rfloor}{2} + 1 + r_\ell.$$

For cycles, we can observe that $\mathrm{pr}(K_n, C_3) = ar(K_n, C_3) = n - 1$. Li, Broersma, and Zhang [53] and Xu, Magnant, and Zhang [83] showed that $\mathrm{pr}(K_n, C_4) = n$ for $n \geq 4$. Fang, Győri, and Xiao [32] showed that $\mathrm{pr}(K_n, C_5) = n + 2$ for $n \geq 5$ and $\mathrm{pr}(K_n, C_6) = n + 5$ for $n \geq 6$. For classes of cycles, Han, Zhang, Bai, and Li [41] completely characterized edge-coloured complete graphs containing no properly coloured odd cycles. Manoussakis, Spyratos, Tuza, and Voigt [60] proved that $\mathrm{pr}(K_n, C_n) = \binom{n-1}{2} + 1$, and proposed an interesting conjecture for cycles as follows.

Conjecture 1.2.5 (Manoussakis et al. 1996 [60]) *Let $n > \ell \geq 4$. Assume that K_n is coloured with at least k colours, where*

$$k = \begin{cases} \frac{1}{2}\ell(\ell+1) + n - \ell + 1, & \text{if } n < \frac{10\ell^2 - 6\ell - 18}{6(\ell-3)}; \\ \frac{1}{3}ln - \frac{1}{18}l(\ell+3) + 2, & \text{if } n \geq \frac{10\ell^2 - 6\ell - 18}{6(\ell-3)}, \end{cases}$$

then K_n admits a properly coloured cycle of length $\ell + 1$.

In the same paper they [60] proved that if K_n is edge-coloured with at least $\frac{\ell n}{2}$ colours, then it contains a properly coloured cycle of length at least $\ell + 1$. Fang, Győri, and Xiao [32] slightly improved the lower bound of Conjecture 1.2.5 and modified it as follows.

Conjecture 1.2.6 (Fang, Győri, and Xiao, 2021 [32, Conjecture 2]) *Let C_ℓ be a cycle on ℓ vertices and $\ell - 1 \equiv r_{\ell-1} \mod 3$, where $0 \leq r_{\ell-1} \leq 2$. For $n \geq \ell$, $\mathrm{pr}(K_n, C_\ell)$ equals to*

$$\max\left\{\binom{\ell-1}{2} + n - \ell + 1, \left\lfloor \frac{\ell-1}{3} \right\rfloor n - \binom{\lfloor \frac{\ell-1}{3} \rfloor + 1}{2} + 1 + r_{\ell-1}\right\}$$

.

1.2.6 . Weakly proper spanning trees

The connectivity of graphs play an important role in the study of Structural and Algorithmic Graph Theory. Similarly, some researchers have considered rainbow or properly coloured subgraphs. Aside from the its applications, properly colored

paths and cycles appear in a variety of other fields including Genetics and Social Sciences. Chartrand, Johns, Mckeon, and Zhang [14] first introduced the rainbow connection in graphs.

Definition 13. Let G be an edge-coloured graph. G is *rainbow connected* if G contains a rainbow $u-v$ path for every two vertices u and v of G.

This notion have been further studied in [10, 13, 23, 48, 74]. In this thesis, we mainly focus a weakening properly coloured version. In [81], Wang and Li considered what they called "alternating cycles" in edge-coloured graphs, cycles in which no two adjacent edges share a colour. Since this terminology can be a bit misleading when the number of colours is greater than 2, Fujita and Magnant [34] introduced the following.

Definition 14. Let G be an edge-coloured graph. G is *properly connected* if G contains a properly coloured $u-v$ path for every two vertices u and v of G.

In the same paper, they proved a Dirac type condition for an edge-coloured graph to be properly connected.

Theorem 1.2.41 ([34]) *Let G be an edge-coloured graph of order n. If $n \geq 3$ and $\delta^c(G) \geq n/2$, then G is properly connected.*

The lower bound in Theorem 1.2.41 is sharp, which means that there exists an edge-coloured graph G with $\delta^c(G) = (n-1)/2$ having two vertices $u, v \in V(G)$ such that there is no properly coloured $u-v$ path.

Another significant issue of connectivity is spanning trees. The problem of searching for properly coloured trees in edge-coloured graphs is a generalization of the well-known bounded degree spanning tree problem for uncolored graphs, as the number of colours bounds the degree here. In [17], Cheng, Kano, and Wang gave a minimum colour degree condition forcing properly coloured spanning trees, and the lower bound is sharp.

Theorem 1.2.42 ([17]) *Let G be an edge-coloured graph of order n. If $\delta^c(G) \geq n/2$, then G has a properly coloured spanning tree.*

In 2019, Borozan, Fernandez de La Vega, Manoussakis, Martinhon, Muthu, Pham, and Saad [8] introduced the concept of weakly proper tree and proved the following theorem.

Definition 15. An edge-coloured tree T with fixed root r is *weakly proper* if every path in T, from the root r to any leaf, is properly coloured.

Theorem 1.2.43 ([8]) *For a given vertex r in an edge-coloured graph G, the problem of determining whether G has a weakly proper spanning tree with root r is NP-complete.*

1.3 . Contribution and outline of the thesis

In this section, we summarize the main works and the organization of this thesis.

(1) We study the general upper bound for the odd chromatic number of graphs and graph-hypergraph pairs in terms of maximum degree conditions. The concept of odd colouring was first introduced by Cheilaris, Keszegh, and Pálvölgyi. They conjectured that $\chi_o(G) \leq \Delta(G)+1$ for any G with $\Delta(G) \geq 3$. We asymptotically solve this conjecture. On the other hand, it is showed that the gap between $\chi(G)$ and $\chi_o(G)$ can be arbitrarily large for infinitely many graphs. We give many optimal bounds up to a multiplicative constant. Moreover, we generalise the above two kinds of results to that guarantee multiple odd colours. This work is illustrated precisely in Chapter 2.

[Corresponding paper: *New bounds for odd colourings of graphs*, Submited to Electron. J. Combin. in Jun. 2023, with Tianjiao Dai and François Pirot.]

(2) Given a graph G, let $\mathrm{pr}(n, G)$ be the maximum number of colours in an edge-colouring of K_n with no properly coloured copy of G. We focus on the case when G is a cycle of length ℓ.

This problem was raised by Manoussakis, Spyratos, Tuza, and Voigt. Later Fang, Győri, and Xiao gave a precise conjecture and showed the extremal edge-coloured graphs K_n with exact $\mathrm{pr}(n, G)$ colours without properly coloured C_ℓ. We prove this conjecture by an accurate characterization. Moreover, we prove a sufficient condition for extending a properly coloured

cycle by exactly one more vertex in an edge-colouring of K_n. This work is shown in Chapter 3.

[Corresponding paper: *Properly colored cycles in edge-colored complete graphs*, Submitted to Discrete Math. in Feb. 2023, with Tianjiao Dai, Hao Li, and Yannis Manoussakis.]

(3) An edge-coloured tree T with fixed root r is weakly proper if every path in T, from the root r to any leaf, is properly coloured. Fujita and Magnant proved the sharp bound of degree condition for a graph to be properly connected, while the case of properly coloured spanning trees was showed by Cheng, Kano, and Wang, both of them equals to $|V(G)|/2$.

We gives some sufficient conditions for an edge-coloured graph G to have a weakly proper spanning tree with root r for any vertex $r \in V(G)$. We also analyse the difference of our results from a graph to be properly connected, or to have a properly coloured spanning tree. This work is discussed in Chapter 4.

[Corresponding paper: *On sufficient conditions for weakly proper spanning trees in edge-colored graphs*, in preprint, with Jie Hu, Shun-ichi Maezawa, and Yannis Manoussakis.]

2 - New bounds for odd colourings of graphs

2.1 . Introduction

In this chapter we study the odd chromatic number of graphs and graph-hypergraph pairs. Caro, Petruševski, and Škrekovski [11] conjectured that any connected graph G of maximum degree $\Delta \geq 3$ has odd chromatic number at most $\Delta + 1$ (see Conjecture 1.2.3).

Our main result states that Conjecture 1.2.3 holds asymptotically as $\Delta \to \infty$.

Theorem 2.1.1 *For every graph G of maximum degree Δ,*

$$\chi_o(G) \leq \Delta + \lceil 4(\ln \Delta + \ln \ln \Delta + 3) \rceil .$$

For small values of Δ, we provide another bound on $\chi_o(G)$ that is derived from a relatively simple (deterministic) colouring procedure.

Theorem 2.1.2 *For every graph G of maximum degree Δ,*

$$\chi_o(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2.$$

The proofs of our results stated in Theorem 2.1.1 and Theorem 2.1.2 rely highly on the structure of neighbourhood hypergraphs. In a more general setting, we could wonder how the odd colouring problem behaves on any graph-hypergraph pair $(G, \mathcal{H})$. We were able to extend Theorem 2.1.1 to that more general setting, at the cost of requiring a lower bound on the minimum hyper-edge size $\epsilon(\mathcal{H})$ in $\mathcal{H}$.

Theorem 2.1.3 *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, if $\epsilon(\mathcal{H}) \geq C \log \Delta(\mathcal{H})$ then there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leq \Delta(G) + C \log \Delta(\mathcal{H}).$$

With that extra condition on $\epsilon(\mathcal{H})$, we can actually derive an upper bound on $\chi_o(G, \mathcal{H})$ that mainly depends on $\chi(G)$ rather than $\Delta(G)$. Moreover, we show with a construction that this bound is tight up to the precise value of the constant C.

Theorem 2.1.4 *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, if $\epsilon(\mathcal{H}) \geq C \log \Delta(\mathcal{H})$ then there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leq \chi(G) \cdot C \log \Delta(\mathcal{H}).$$

When $\epsilon(\mathcal{H})$ gets closer to $\Delta(G)$, we show that the difference $\chi_o(G, \mathcal{H}) - \chi(G)$ gets relatively small.

Theorem 2.1.5 *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leq \chi(G) + C \frac{\Delta(G) \log \Delta(\mathcal{H})}{\epsilon(\mathcal{H})}.$$

A direct consequence of Theorem 2.1.5 is that for quasi-regular graphs G (that is, the ratio $\Delta(G)/\delta(G)$ is bounded by a uniform constant), the difference $\chi_o(G) - \chi(G)$ is small, namely $O(\log \Delta(G))$. This contrasts with the general case where that difference can be much larger: if G is the 1-subdivision of the complete graph on $\Delta + 1 \geq 5$ vertices, then $\chi(G) = 2$ while $\chi_o(G) = \Delta + 1$.

2.2 . Probabilistic tools.

The first probabilistic result that we need is the following lopsided version of the Symmetric Lovász Local Lemma (LLL for short) (see e.g. [4]).

Lemma 2.2.1 (Lopsided Lovász Local Lemma) *Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a finite set of random (bad) events, and let d be a fixed integer. Suppose that, for every $i \in [n]$, there is a set $\Gamma(i) \subseteq [n]$ of size at most d such that, for every $Z \subseteq [n] \setminus \Gamma(i)$,*

$$\mathbb{P}\left[B_i \,\middle|\, \bigcap_{j \in Z} \overline{B_j}\right] \leq p.$$

If $epd \leq 1$, then $\mathbb{P}\left[\bigcap_{i \in [n]} \overline{B_i}\right] > 0$.

Many random variables we analyse in this paper are highly concentrated around their mean. This is a consequence of Chernoff's bounds as stated hereafter.

Lemma 2.2.2 (Chernoff's bounds) *Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. $(0,1)$-valued random variables, and let $\mathbf{S}_n \coloneqq \sum_{i=1}^n \mathbf{X}_n$. Let us write $\mu \coloneqq \mathbb{E}[\mathbf{S}_n]$. Then*

(i) *for every $0 \leq \eta < \mu$,*

$$\mathbb{P}[\mathbf{S}_n \leq \mu - \eta] \leq e^{-\frac{\eta^2}{2\mu}}, \text{ and}$$

(ii) *for every $\eta > 0$,*

$$\mathbb{P}[\mathbf{S}_n \geq \mu + \eta] \leq e^{-\frac{\eta^2}{2(\mu+\eta)}}.$$

Finally, we will need to analyse a specific Markovian process as described in the following lemma. First we observe that, using the well-known Stirling bounds

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

for every integer $n \geq 1$, it is straightforward to derive that

$$\frac{n!}{(\frac{n}{2})!} \leq \sqrt{2}\left(\frac{2n}{e}\right)^{n/2} \tag{2.1}$$

for every even integer $n \geq 2$. We will also use the classical upper bound $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

Lemma 2.2.3 *Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be a sequence of binary random variables with values in $\{-1, 1\}$, let $S_0 \in \mathbb{N}$ be a non-negative integer, and let $\mathbf{S}_i \coloneqq S_0 + \sum_{j=1}^i \mathbf{X}_j$ for every integer $i \geq 0$. If there exists a real number $\tau > 0$ such that, for every $i \geq 0$, it holds that $\mathbb{P}[\mathbf{X}_i = -1 \mid \mathbf{S}_{i-1} = s] \leq s/\tau$, then*

$$\mathbb{P}[\mathbf{S}_n \leq k] \leq \sqrt{2}\binom{n}{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}},$$

for every integers $0 \leq k \leq n$.

Proof of Lemma 2.2.3. Given a possible outcome $(X_1, \dots X_n)$ of $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ that yields that $\mathbf{S}_n \leq k$, let $I \coloneqq \{i : X_i = -1\}$. Noting that $\mathbf{S}_n = S_0 + n - 2|I|$ conditioned on $\mathbf{X}_1 = X_1, \dots, \mathbf{X}_n = X_n$, we infer that $|I| \geq \frac{S_0+n-k}{2} \geq \frac{n-k}{2}$. Moreover, if i_j is the j-th last element in I, then it deterministically holds that $\mathbf{S}_{i_j-1} \leq k + j$, and hence

$$\begin{aligned}&\mathbb{P}\left[\mathbf{X}_{i_j} = -1 \;\middle|\; \mathbf{X}_1 = X_1, \dots, \mathbf{X}_{i_j-1} = X_{i_j-1}\right]\\ \leq &\mathbb{P}\left[\mathbf{X}_{i_j} = -1 \;\middle|\; \mathbf{S}_{i_j-1} \leq k + j\right] \leq \frac{k+j}{\tau}.\end{aligned}$$

We therefore have the following (crude) upper-bound:

$$\begin{aligned}\mathbb{P}\left[\mathbf{X}_1 = X_1, \dots, \mathbf{X}_n = X_n\right] &\leq \prod_{i \in I} \mathbb{P}\left[\mathbf{X}_i = -1 \mid \mathbf{X}_1 = X_1, \dots, \mathbf{X}_{i-1} = X_{i-1}\right] \\ &\leq \prod_{j=1}^{\frac{n-k}{2}} \frac{k+j}{\tau} = \frac{\left(\frac{n+k}{2}\right)!}{k!\,\tau^{\frac{n-k}{2}}}.\end{aligned}$$

Since there are at most $\binom{n}{\frac{n-k}{2}}$ possibles choices for the $\frac{n-k}{2}$ last elements of I, we have

$$\begin{aligned}\mathbb{P}\left[\mathbf{S}_n \leq k\right] &\leq \binom{n}{\frac{n-k}{2}} \cdot \frac{\left(\frac{n+k}{2}\right)!}{k!\,\tau^{\frac{n-k}{2}}} \\ &= \frac{n!}{k!\left(\frac{n-k}{2}\right)!\,\tau^{\frac{n-k}{2}}} = \binom{n}{k} \frac{(n-k)!}{\left(\frac{n-k}{2}\right)!\,\tau^{\frac{n-k}{2}}} \\ &\leq \sqrt{2}\binom{n}{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}} \qquad \text{by (2.1)} \\ &\leq \sqrt{2}\left(\frac{ne}{k}\right)^{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}}.\end{aligned}$$

□

Let G be a graph, and $\mathscr{C}(G) \subseteq [k]^{V(G)}$ a prescribed set of k-colourings of G. Given a colouring $\sigma \in \mathscr{C}(G)$ and a vertex $v \in V(G)$, we let $L_\sigma(v)$ be the set of colours x such that, if we redefine $\sigma(v) \leftarrow x$, we still have $\sigma \in \mathscr{C}(G)$. For instance, if $\mathscr{C}(G)$ is the set of proper k-colourings of G, then $L_\sigma(v) = [k] \setminus \sigma(N(v))$.

Lemma 2.2.4 *Let G be a graph, and $\mathscr{C}(G)$ a set of colourings of G. Let $\boldsymbol{\sigma}$ be drawn uniformly at random from $\mathscr{C}(G)$. For a given subset of vertices $X \subseteq V(G)$, let $B_{\boldsymbol{\sigma}}$ be the bad event that X has no colour in $\boldsymbol{\sigma}$, and let $M \subseteq X$ be a subset of size $m \leq |X|$. If there exists an integer τ such that we deterministically have $|L_{\boldsymbol{\sigma}}(v)| \geq \tau$ for every vertex $v \in M$, then for every possible realisation σ_0 of $\boldsymbol{\sigma}_{|V(G)\setminus M}$, we have*

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}} \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M} = \sigma_0\right] \leq \sqrt{2}\left(\frac{2m}{e\tau}\right)^{m/2}.$$

Proof. Let $M = \{u_1, \dots, u_m\}$ be a fixed subset of X. Let σ_0 be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M}$. Let $\boldsymbol{\sigma}_0$ be drawn uniformly at random from the

extensions of σ_0 to $\mathscr{C}(G)$. For every $1 \leq i \leq m$, we let $\boldsymbol{\sigma}_i \in \mathscr{C}(G)$ be obtained from $\boldsymbol{\sigma}_{i-1}$ by resampling the colour of u_i uniformly at random from $L_{\boldsymbol{\sigma}_{i-1}}(u_i)$. For every $i \leq m$, let $\mathbf{S}_i$ be the number of odd colours of X in $\boldsymbol{\sigma}_i$. For every $i \geq 1$, we have $\mathbf{S}_i = \mathbf{S}_{i-1} - 1$ if $\boldsymbol{\sigma}_i(u_i)$ is one of the $\mathbf{S}_{i-1}$ odd colours of X in $\boldsymbol{\sigma}_{i-1}$; since there are at least τ choices for $\boldsymbol{\sigma}_i(u_i)$ this happens with probability at most k/τ if $\mathbf{S}_{i-1} = k$. Otherwise, we have $\mathbf{S}_i = \mathbf{S}_{i-1}+1$. So the sequence $(\mathbf{S}_i)_{i\leq m}$ satisfies the hypotheses of Lemma 2.2.3, hence by setting $k \coloneqq 0$ we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}_m}\right] = \mathbb{P}\left[\mathbf{S}_m = 0\right] \leq \sqrt{2}\left(\frac{2m}{e\tau}\right)^{m/2}.$$

Since we resample the colours uniformly at random, the random colourings $(\boldsymbol{\sigma}_i)_{i\leq m}$ are identically distributed. Therefore, if $\boldsymbol{\sigma}$ is drawn uniformly at random from $\mathscr{C}(G)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}} \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M} = \sigma_0\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_0}\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_m}\right],$$

and the conclusion follows. □

2.3 . A greedy bound

Given a proper k-colourings $\sigma\colon V(G) \to [k]$, and a vertex $u \in V(G)$, we denote $\mathcal{U}_\sigma(u)$ the set of odd colours of $N_G(u)$ in σ. So σ is an odd k-colouring if $|\mathcal{U}_\sigma(u)| \geq 1$ for every vertex $u \in V(G)$. If $\mathcal{U}_\sigma(u) = \{x\}$, we say that u is *σ-critical*, and that x is its *witness colour*; we denote it $w_\sigma(u) \coloneqq x$.

Proof of Theorem 2.1.2. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G. We let $H_i \coloneqq G[\{v_1, \ldots, v_i\}]$ for every $i \in [n]$. Let $k \coloneqq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2$, and let $\mathscr{C}(H)$ denote the set of odd k-colourings of each induced subgraph $H \subseteq G$. We construct an odd k-colouring of G greedily by constructing a sequence of partial colourings $(\sigma_i)_{i\in[n]}$ that satisfies the following induction hypothesis.

$$\sigma_i \in \mathscr{C}(H_i) \text{ and } |\mathcal{U}_{\sigma_i}(u)| \geq 1 \text{ for every vertex } u \in N_G(V(H_i)). \qquad \text{(IH)}$$

For the base case, we may begin with the empty colouring σ_0. Let us now assume that we have constructed σ_i that satisfies

(IH). In order to maintain (IH), we forbid that $\sigma_{i+1}(v_{i+1})$ is one of $\{\sigma_i(u) : u \in N_{H_i}(v_{i+1})\} \cup \{w_{\sigma_i}(u) : u \in N_G(v_{i+1})$ and u is σ_i-critical$\}$. If at most $k-1$ colours are forbidden for v_{i+1}, then there remains at least one possible choice for $\sigma_{i+1}(v_{i+1})$, and letting $\sigma_{i+1}(u) = \sigma_i(u)$ for every $u \in V(H_i)$ we have that σ_{i+1} satisfies (IH).

We may now assume that all k colours are forbidden for v_{i+1}. Let $X \subseteq N_G(v_{i+1})$ be the set of neighbours of v_{i+1} that forbid exactly one colour for v_{i+1}, and $Y \subseteq N_G(v_{i+1}) \setminus X$ be the set of neighbours of v_{i+1} that forbid exactly two colours for v_{i+1} (so $Y \subseteq V(H_i)$ and every vertex $y \in Y$ is σ_i-critical). We claim that there is a vertex $y \in Y$ such that $\sigma_i(y)$ is forbidden only by y for v_{i+1}. Indeed, otherwise the number of forbidden colours for v_{i+1} would be at most $|X| + \frac{3}{2}|Y| \leq \frac{3}{2}\Delta < k$, a contradiction. We also claim that $\sigma_i(y)$ is not a witness colour of v_{i+1} in σ_i. Indeed, otherwise every colour of $N(v_{i+1}) \setminus y$ appears at least twice, hence v_{i+1} has at most $\left\lfloor \frac{\Delta-1}{2} \right\rfloor + 1$ adjacent colours in σ_i. Since there are at most Δ witness colours in $N_G(v_{i+1})$, there are at most $\left\lfloor \frac{3\Delta+1}{2} \right\rfloor \leq k-1$ forbidden colours for v_{i+1}, a contradiction. Let us set $\sigma_{i+1}(v_{i+1}) \coloneqq \sigma_i(y)$, and $\sigma_{i+1}(u) \coloneqq \sigma_i(u)$ for every vertex $u \in V(H_i) \setminus y$. This colouring satisfies the conditions of (IH) on the graph $H_{i+1} \setminus y$.

There remains to define $\sigma_{i+1}(y)$. Since $|\mathcal{U}_{\sigma_i}(y)| = 1$, it means that every colour in $N_G(y) \cap V(H_{i+1})$ appears at least twice in σ_{i+1} except $w_{\sigma_i}(y)$ and $\sigma_{i+1}(v_{i+1}) = \sigma_i(y)$. So y has at most $\lfloor \frac{\Delta-2}{2} \rfloor + 2 = \lfloor \frac{\Delta}{2} \rfloor + 1$ adjacent colours in σ_{i+1}. Since there are at most Δ witness colours in $N_G(y)$, there are at most $\lfloor \frac{3\Delta}{2} \rfloor + 1 = k-1$ forbidden colours for y in σ_{i+1}, and so there remains at least one possible choice for $\sigma_{i+1}(y)$. We claim that there is no vertex in $N_{H_i}(y)$ such that $\sigma_i(y)$ is its witness colour in σ_i. Otherwise, we assume that the vertex $x \in N_{H_i}(y)$ and $\sigma_i(y)$ is the witness colour of x in σ_i, then the degree of x must be even. It means that $|\mathcal{U}_{\sigma_i}(x)| \leq 2$. This ends the proof of the induction.

We conclude that σ_n is an odd k-colouring of G, which proves that $\chi_o(G) \leq k$, as desired. □

2.4 . Proof of Theorem 2.1.1

2.4.1 . Set-up

Given a vertex colouring $\sigma\colon V(H) \to [k]$ of some induced subgraph H of G, for each $v \in V(H)$ a colour x is said to be an *odd colour* of v if x is an odd colour of $N_H(v)$ in σ. Let $w_\sigma(v)$ denote the unique odd colour of v in σ if such a colour exists; otherwise $w_\sigma(v)$ is undefined. When it exists, we say that $w_\sigma(v)$ is the *witness colour* of v in σ.

Let G be a graph of maximum degree Δ, and let $k > \Delta$ be some integer. Let V^- be the subset of $V(G)$ consisting of all vertices of degree less than $k/2$, and $V^+ \coloneqq V(G) \setminus V^+$ be the set of vertices of degree at least $k/2$. We denote $G^+ \coloneqq G[V^+]$. For every $X \subseteq V^+$, we say that a proper partial k-colouring $\sigma\colon X \to [k]$ of G^+ is *admissible* if every vertex $v \in V^-$ having $N_G(v) \subseteq X$ has an odd colour in σ. Finally, we let V^{++} be the set of vertices $v \in V^+$ having $N_G(v) \subseteq V^+$.

Since the bound on $\chi_o(G)$ that we want to prove is weaker than that of Corollary 2.1.2 if $\Delta \leq 65$, we may assume that $\Delta \geq 66$.

2.4.2 . Colouring vertices of large degree.

Let $\boldsymbol{\sigma}\colon V^+ \to [k]$ be a uniformly random admissible colouring of G^+. For every $v \in V^{++}$, we let $B_{\boldsymbol{\sigma}}(v)$ be the random event that $N_G(v)$ has no odd colour in $\boldsymbol{\sigma}$. The goal of this section will be to show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(v)$ occurs.

Lemma 2.4.1 *If $\Delta \geq 49$ and $k \geq \Delta + 4(\ln \Delta + \ln\ln \Delta + 3)$, then there exists an admissible colouring $\sigma\colon V^+ \to [k]$ of G^+ such that every vertex $v \in V^{++}$ has an odd colour in σ.*

Proof of Lemma 2.4.1. Fix $k \coloneqq \Delta + \eta$, for some integer $\eta \geq 1$ whose precise value will be determined later in the proof, and let $\boldsymbol{\sigma}$ be a uniformly random admissible k-colouring of G^+. Such a colouring exists, since each vertex v has at most Δ constraints (at most $\deg_{V^+}(v)$ constraints because of the adjacent colours, and at most $\deg_{V^-}(v)$ constraints because of the adjacent witness colours). In particular, we

have $|L_{\boldsymbol{\sigma}}(v)| \geq \eta$ for every $v \in V^+$. We want to show that, with non-zero probability, no bad event $B_{\boldsymbol{\sigma}}(v)$ occurs for $v \in V^{++}$.

Let $m \leq k/2$ be some integer whose explicit value will be determined later in the proof. For every $v \in V^{++}$, we pick an arbitrary subset $M(v) \subseteq N(v)$ of size m. Then we let $\Gamma(v) \coloneqq N[M(v)]$. For a vertex $u \in V^{++}$, the outcome of $B_{\boldsymbol{\sigma}}(u)$ is entirely determined by the colours assigned to vertices in $N(u)$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(v)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(u)$ such that $M(v) \cap N[u] = \emptyset$. This holds for every $u \notin \Gamma(v)$. We wish to apply Lemma 2.2.1 to those bad events, with that definition of $\Gamma(v)$. To that end, let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V^+ \setminus M(v)}$ such that no event $B_{\boldsymbol{\sigma}}(u)$ occurs for $u \notin \Gamma(v)$. For every $Z \subseteq V^{++} \setminus \Gamma(v)$, we have

$$\begin{aligned}\mathbb{P}\left[B_{\boldsymbol{\sigma}}(v) \;\middle|\; \bigcap_{u \in Z} \overline{B_{\boldsymbol{\sigma}}(u)}\right] &\leq \sup_{\sigma_0 \in \Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(v) \;\middle|\; \boldsymbol{\sigma}_{|V^+ \setminus M(v)} = \sigma_0\right] \\ &\leq \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2},\end{aligned}$$

by Lemma 2.2.4 applied to the graph G^+ with $\mathscr{C}(G)$ being the set of admissible k-colourings of G^+.

Let us fix $\eta \coloneqq 2m$, so that this is at most $\sqrt{2}e^{-m/2}$. Since $v \in V^{++}$, we know that $\deg_{G^+}(v) \geq k/2 \geq \eta/2 \geq m$, so this lets us pick any value for m. To apply Lemma 2.2.1, we need an upper bound of $\frac{1}{em\Delta}$ for that probability, which holds precisely when $m \geq -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta}\right)$. We may therefore pick $m \coloneqq \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta}\right)\right\rceil$; a careful analysis of that value yields that $2m \leq \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil$ when $\Delta \geq 49$. □

2.4.3 . Colouring vertices of small degree.

By Lemma 2.4.1, G^+ has an admissible k-colouring $\sigma\colon V^+ \to [k]$ with $k = \Delta + \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil$, such that every $v \in V^{++}$ has an odd colour. Then we colour the vertices of V^- greedily. Each time we colour $v \in V^-$, each neighbour u of v yields at most 2 forbidden colours (its colour $\sigma(u)$, and its witness colour if it exists), so there are less than k forbidden colours for v. This ensures that the greedy colouring terminates, and ends the proof of Theorem 2.1.1.

2.5 . Odd colourings of hypergraphs with constrained edge sizes

We recall that, given a graph-hypergraph pair $(G, \mathcal{H})$, an *odd k-colouring of* $(G, \mathcal{H})$ is a mapping $\sigma \colon V \to [k]$ that is both a proper colouring of G and an odd colouring of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, we let $\mathcal{H}[S] = (V(\mathcal{H}), \{e \cap S : e \in E(\mathcal{H})\})$ be the sub-hypergraph of $\mathcal{H}$ induced by S.

2.5.1 . A bound in terms of the chromatic number for quasi-regular graphs

The proof of Theorem 2.1.1 relies on the probabilistic method by analysing the behaviour of a uniformly random admissible colouring of a given graph G. It turns out that we have exploited the randomness of only a subset of the vertices of G: a subset of m neighbours of each vertex $v \in V^{++}$, where $m = \Theta(\ln \Delta(G))$. If G has a large minimum degree, we may restrict the random choices to a small subset of vertices that should suffice to have an odd colour in every neighbourhood, and colour the other vertices with an optimal proper colouring.

Theorem 2.5.1 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3) \rceil$, and assume that $\Delta(\mathcal{H}) \geq 49$. For every subset of vertices $S \subseteq V(\mathcal{H})$, if $\epsilon(\mathcal{H}[S]) \geq \eta/2$, then $(G, \mathcal{H})$ has an odd k-colouring, where*

$$k \leq \chi(G \setminus S) + \Delta(G[S]) + \eta.$$

Proof. Let $k_0 \coloneqq \chi(G \setminus S)$ and let σ_0 be a proper k_0-colouring of $G \setminus S$. Let $k \coloneqq k_0 + \Delta(G[S]) + \eta$, and let $\boldsymbol{\sigma}$ be a uniformly random proper k-colouring of G that satisfies $\boldsymbol{\sigma}_{|G\setminus S} = \sigma_0$. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random event that e has no colour appearing at odd times in $\boldsymbol{\sigma}$. Let us show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. Let $m \leq \eta/2$ be an integer, and for every edge $e \in E(\mathcal{H})$ let $M(e) = \{u_1, u_2, \ldots, u_m\} \subseteq e \cap S$ be a subset of m vertices in e. Let us recolour the vertices in $M(e)$ in turn with a uniformly random available colour. Each time we recolour u_i, the neighbours of u_i in S forbid at most $\deg_S(u_i) \leq \Delta(G[S])$ colours, and the neighbours of u_i not in S forbid at most $k_0 = \chi(G \setminus S)$ colours (these colours are fixed by σ_0).

So there are at least η available colours for u_i. In particular, we have $|L_{\boldsymbol{\sigma}}(v)| \geq \eta$ for each $v \in V(G)$.

We apply Lemma 2.2.1 with $\Gamma(e) \coloneqq \{e' \in E(\mathcal{H}) : e' \cap M(e) \neq \emptyset\}$ for every edge $e \in E(\mathcal{H})$, and obtain that, with non-zero probability, none of the events $B_{\boldsymbol{\sigma}}(e)$ occurs. The size of $\Gamma(e)$ is at most $m\Delta(\mathcal{H})$. Let Σ_1 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$ occurs for $e' \notin \Gamma(e)$. For every $Z \subseteq E(\mathcal{H}) \setminus \Gamma(e)$, we have

$$\begin{aligned}\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \bigcap_{e' \in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] &\leq \sup_{\sigma_1 \in \Sigma_1} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_1\right] \\ &\leq \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2} = \sqrt{2}\, e^{-m/2},\end{aligned}$$

by Lemma 2.2.4 applied to the graph G with $\mathscr{C}(G)$ being the set of proper k-colourings of G. As in the proof of Lemma 2.4.1, by fixing $m \coloneqq \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta}\right)\right\rceil \leq \eta/2$, the above probability is at most $\frac{1}{e \cdot m\Delta(\mathcal{H})}$. This proves the existence of a proper k-colouring σ of G such that every vertex has an odd colour in σ, as desired.

□

By taking $S = V(G)$, Theorem 2.5.1 has the following result as a corollary.

Corollary 2.5.1 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair, and fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)\rceil$. If $\Delta(\mathcal{H}) \geq 49$ and $\epsilon(\mathcal{H}) \geq \eta/2$, then there exists an odd $(\Delta(G) + \eta)$-colouring of $(G, \mathcal{H})$.*

We next show how to find a set S satisfying the hypothesis of Theorem 2.5.1 such that $\Delta(G[S])$ is as small as possible.

Lemma 2.5.1 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Let $\Delta \coloneqq \Delta(G) + \Delta(\mathcal{H})$, and let us assume that the minimum edge size in $\mathcal{H}$ is $\epsilon(\mathcal{H}) \geq 12 \ln \Delta$. Let $r \coloneqq \min\{\epsilon(\mathcal{H}), \Delta(G)\}$. Then for every m satisfying $4 \ln \Delta \leq m \leq r/3$, if Δ is large enough, there is a subset $S \subseteq V(G)$ such that for every edge $e \in E(\mathcal{H})$, $|e \cap S| \geq m$, and $\Delta(G[S]) \leq \frac{\Delta(G)}{r}(m + \sqrt{60\, m \ln \Delta})$.*

Proof. Let us for short denote $D \coloneqq (m + \sqrt{60\, m \ln \Delta})\, \frac{\Delta(G)}{r}$. Let $\mathbf{S}$ be a random subset of $V(G)$ obtained by taking each vertex independently

uniformly at random with probability $p = \frac{m+\sqrt{11\,m\ln\Delta}}{r}$. We note that, since $4\ln\Delta \leq m \leq r/3$ by assumption, we have $p < 1$ and so this probability is well-defined. The result follows if there exists a realization S such that every $e \in E(\mathcal{H})$ has $|e \cap S| \geq m$ and every $v \in V(G)$ satisfies $\deg_S(v) \leq D$. Let $V(G)$ be ordered arbitrarily. For every edge $e \in E(\mathcal{H})$, we let $\widetilde{e}$ consist of the first r vertices (with respect to that order) of e. For every edge $e \in E(\mathcal{H})$, let $\mathbf{X}_e \coloneqq |\widetilde{e} \cap \mathbf{S}|$ be the random variable that counts the number of vertices of $\widetilde{e}$ in $\mathbf{S}$, and we write $\mu_e \coloneqq \mathbb{E}[\mathbf{X}_e] = m + \sqrt{11m\ln\Delta}$. For every vertex $v \in V(G)$, we let $\mathbf{X}_v \coloneqq |N_G(v) \cap \mathbf{S}|$ be the random variable that counts the number of neighbours of v in $\mathbf{S}$, and we write $\mu_v \coloneqq \mathbb{E}[\mathbf{X}_v] = \frac{m+\sqrt{11m\ln\Delta}}{r} \cdot \deg_G(v) \leq \frac{\Delta(G)}{r}(m + \sqrt{11m\ln\Delta})$. For every edge $e \in E(\mathcal{H})$, let $B_{\boldsymbol{\sigma}}(e)$ be the random event that $\mathbf{X}_e < m$, and for every vertex $x \in V(G)$, let B_v be the random event that $\mathbf{X}_v > D$.

Observe that, for every pair of vertices (u, v) such that $N(u) \cap N(v) = \emptyset$, the random events B_u and B_v are independent. So the bad event B_v is dependent with at most $\Delta(G)^2$ bad events B_u, and at most $\Delta(G)\Delta(\mathcal{H})$ bad events $B_{\boldsymbol{\sigma}}(e)$. So the dependency-degree of B_v is at most $\Delta(G)\Delta \leq \Delta^2$. Moreover, for every pair of edges (e, e'), such that $\widetilde{e} \cap \widetilde{e}' = \emptyset$, the random events $B_{\boldsymbol{\sigma}}(e)$ and $B_{\boldsymbol{\sigma}}(e')$ are independent. So the bad event $B_{\boldsymbol{\sigma}}(e)$ is dependent with at most $r\Delta(\mathcal{H})$ bad events $B_{\boldsymbol{\sigma}}(e')$, and at most $r\Delta(G)$ bad events B_v. So the dependency-degree of $B_{\boldsymbol{\sigma}}(e)$ is at most $r\Delta \leq \Delta^2$. Hence we may apply the LLL in order to show that, with non-zero probability, no event B_v or $B_{\boldsymbol{\sigma}}(e)$ occurs. Regarding that the maximum degree of the dependency-graph of those random events is at most Δ^2, it suffices to prove that

$$\mathbb{P}[\mathbf{X}_e < m] \leq \frac{1}{e\Delta^2} \quad \text{and} \quad \mathbb{P}[\mathbf{X}_v > D] \leq \frac{1}{e\Delta^2}.$$

We do so by applying Chernoff bounds on the random variable $\mathbf{X}_e$ of

expectancy μ_e:

$$\begin{aligned}\mathbb{P}\left[\mathbf{X}_e < m\right] &\leq \exp\left(-\frac{(\mu_e - m)^2}{2\mu_e}\right)\\ &= \exp\left(-\frac{11m\ln\Delta}{2m + 2\sqrt{11m\ln\Delta}}\right) = \exp\left(-\frac{11\ln\Delta}{2 + 2\sqrt{\frac{11\ln\Delta}{m}}}\right)\\ &\leq \Delta^{-\frac{11}{2+\sqrt{11}}} \leq \frac{1}{e\Delta^2} \qquad \text{if } \Delta \text{ is large enough.}\end{aligned}$$

We do the same with the random variable $\mathbf{X}_v$ of expectancy μ_v:

$$\begin{aligned}\mathbb{P}\left[\mathbf{X}_v > D\right] &= \exp\left(-\frac{(D - \mu_v)^2}{2D}\right)\\ &\leq \exp\left(-\frac{(\sqrt{60} - \sqrt{11})^2 m\ln\Delta}{2m + 2\sqrt{60m\ln\Delta}} \cdot \frac{\Delta(G)}{r}\right)\\ &= \exp\left(-\frac{(\sqrt{60} - \sqrt{11})^2\ln\Delta}{2 + 2\sqrt{\frac{60\ln\Delta}{m}}} \cdot \frac{\Delta(G)}{r}\right)\\ &\leq \Delta^{-\frac{(\sqrt{60}-\sqrt{11})^2}{2+2\sqrt{15}}} \leq \frac{1}{e\Delta^2} \qquad \text{if } \Delta \text{ is large enough.}\end{aligned}$$

By Lemma 2.2.1, with non-zero probability, $\mathbf{S}$ satisfies the conclusion of Lemma 2.5.1. □

We note that it is possible to drop the condition that Δ is large enough in the statement of Lemma 2.5.1 if we set $D \coloneqq \frac{\Delta(G)}{r}(m + \sqrt{30m(1 + 2\ln\Delta)})$ and $p \coloneqq \frac{1}{r}(m + \sqrt{5.5m(1 + 2\ln\Delta)})$ instead, and assume that $m \geq 2 + 4\ln\Delta$ and $\epsilon(\mathcal{H}) \geq 6 + 12\ln\Delta$.

We may combine Theorem 2.5.1 and Lemma 2.5.1 in order to obtain that, for a quasi-regular graph G (that is, a graph G where the ratio $\Delta(G)/\delta(G)$ is bounded), the odd chromatic number of G is not too far from its chromatic number.

Corollary 2.5.2 *Let G be a graph of maximum degree Δ large enough, and minimum degree $\delta \geq 12\ln(2\Delta)$. Then*

$$\begin{aligned}\chi_o(G) &\leq \chi(G) + \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil + \frac{20\Delta\ln(2\Delta)}{\delta}\\ &= \chi(G) + O\left(\frac{\Delta\ln\Delta}{\delta}\right) \quad \textit{as } \Delta \to \infty.\end{aligned}$$

Note that the minimum degree condition in Corollary 2.5.2 can be dropped, since if it is not fulfilled, then the upper bound on $\chi_o(G)$ is larger than that given by Theorem 2.1.2.

2.5.2 . Graphs of small chromatic number

Given a graph-hypergraph pair $(G, \mathcal{H})$, if we can find a set S that satisfies the hypothesis of Theorem 2.5.1 such that $\chi(G[S])$ is much smaller than $\Delta(G[S])$, we can use another approach to obtain a better bound.

Theorem 2.5.2 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)\rceil$, and assume that $\Delta(\mathcal{H}) \geq 49$. For every subset of vertices $S \subseteq V(\mathcal{H})$, if $\epsilon(\mathcal{H}[S]) \geq \eta/2$, then $(G, \mathcal{H})$ has an odd k-colouring, where*

$$k \leq \chi(G \setminus S) + \eta\, \chi(G[S]).$$

Proof. Let $G_0 \coloneqq G \setminus S$ and $G_1 \coloneqq G[S]$. For each $i \in \{0, 1\}$, we write $k_i \coloneqq \chi(G_i)$, and we let σ_i be a proper k_i-colouring of G_i.

We define a random proper k-colouring $\boldsymbol{\sigma}$ of G as follows, where $k = k_0 + \eta k_1$. For every $v \in S$, draw some random value $\mathbf{x}_v$ uniformly at random from $[\eta]$, and let $\boldsymbol{\sigma}(v) \coloneqq (\sigma_1(v), \mathbf{x}_v)$. For every $v \notin S$, let $\boldsymbol{\sigma}(v) \coloneqq \sigma_0(v)$. Let us order the vertices in $V(G)$ arbitrarily. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random (bad) event that e has no odd colour. We let $M(e)$ contain the smallest m vertices of $e \cap S$. Let σ be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. By construction, for every $v \in S$, there are η choices in $L_{\boldsymbol{\sigma}}(v)$. Hence we may apply Lemma 2.2.4 and obtain that

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma\right] \leq \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2} = \sqrt{2}\, e^{-m/2}.$$

As explained in the proof of Theorem 2.1.1, this is at most $\frac{1}{em\Delta(\mathcal{H})}$. For an edge $e' \in E(\mathcal{H})$, the outcome of $B_{\boldsymbol{\sigma}}(e')$ is entirely determined by the realisation of $\boldsymbol{\sigma}_{|e'}$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(e)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(e')$ such that $M(e) \cap e' = \emptyset$. So we set $\Gamma(e) \coloneqq \{e' : e' \cap M(e) \neq \emptyset\}$, and observe that these

sets have size at most $m\Delta(\mathcal{H})$. We may now apply Lemma 2.2.1 to the bad events $(B_{\boldsymbol{\sigma}}(e))$, with that definition of $\Gamma(e)$, and conclude that with positive probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. So there is a realisation of $\boldsymbol{\sigma}$ that is an odd k-colouring of $(G, \mathcal{H})$. This concludes the proof. □

Corollary 2.5.3 *Let G be a graph of maximum degree $\Delta \geq 49$, and fix $\eta \coloneqq \lceil 4(\ln \Delta + \ln\ln \Delta + 3)\rceil$. If the minimum degree of G is at least $\eta/2$, then*

$$\chi_o(G) \leq \eta\, \chi(G),$$

and this is tight up to a multiplicative constant for a family of graphs of increasing chromatic numbers.

Proof. The upper bound on k is a direct consequence of Theorem 2.5.2 where $\mathcal{H}$ is the neighbourhood-hypergraph of G, and $S = V(G)$. Let us prove the tightness of the bound.

Fix an integer $k_0 \geq 2$, and let $n_0 \geq k_0$ be an even integer. Let G_0 be a complete k_0-partite graph, with parts $X_1, \dots, X_{k_0}$ all of size $2n_0$. For every $i \in [k_0]$, and for every $S \in \binom{X_i}{n_0}$, we add a vertex with neighbourhood S in G_0. Let G be the obtained graph; let us show that $\chi_o(G) \geq k_0(n_0+1)$. We write $k \coloneqq \chi_o(G)$, and let σ be an odd k-colouring of G. First observe that we must have $\sigma(X_i) \cap \sigma(X_j) = \emptyset$ for every $i \neq j$, otherwise we would find a monochromatic edge in σ. So it suffices to show that $|\sigma(X_i)| \geq n_0 + 1$ for every $i \in [k_0]$. Let us assume for the sake of contradiction that $|\sigma(X_i)| \leq n_0$. For every odd colour of X_i, we remove one vertex with that colour from X_i. We are left with at least n_0 vertices. We now remove monochromatic pairs of vertices from X_i until exactly n_0 vertices remain. We obtain a set S with no odd colour, and by construction there is a vertex in $V(G)$ such that $N(v) = S$. So v has no odd colour in σ, a contradiction.

The maximum degree of G is $\frac{1}{2}\binom{2n_0}{n_0} + (k_0 - 1)\cdot 2n_0 < 4^{n_0}$ when n_0 is large enough, and $\chi(G) = \chi(G_0) = k_0$. So we have

$$\chi_o(G) > \chi(G) \log_4 \Delta(G),$$

while the minimum degree of G is $n_0 \geq \log_4 \Delta(G)$.

Note that the minimum degree of G is smaller than what is required by a factor $4 \ln 2$. If we want to meet the required lower bound for the minimum degree of G, we may let the size of each X_i be $\frac{7}{6}n_0$ instead of $2n_0$. We can now prove that more than $\frac{n_0}{6}$ colours must appear on each part X_i. We still have $\chi(G) = k_0$ and $\delta(G) = n_0$, while

$$\begin{aligned}\Delta(G) &= (k_0 - 1) \cdot \frac{7}{6}n_0 + \binom{7n_0/6}{n_0} \\ &= (k_0 - 1) \cdot \frac{7}{6}n_0 + \binom{7n_0/6}{n_0/6} \\ &\leq (k_0 - 1) \cdot \frac{7}{6}n_0 + (7e)^{n_0/6}.\end{aligned}$$

Since $6/\ln(7e) > 2.03$, when n_0 is large enough we have $\delta(G) = n_0 \geq 2(\ln \Delta(G) + \ln \ln \Delta(G) + 3)$, as required. On the other hand, we have $\chi_o(G) > \frac{1}{6}k_0 n_0$, so we are only a factor 12 away from the upper bound guaranteed in that regime[1]. □

2.6 . The effect of multiplying the constraints

We may now wonder what happens when we seek for an odd colouring of a given graph-hypergraph pair $(G, \mathcal{H})$, such that every edge $e \in E(\mathcal{H})$ has *many* odd colours. For a positive integer h, an *h-odd colouring* of a hypergraph $\mathcal{H}$ satisfies the constraint that there are at least $\min\{h, |e|\}$ odd colours in every edge $e \in E(\mathcal{H})$. For a graph-hypergraph pair $(G, \mathcal{H})$, an *h-odd k-colouring* of $(G, \mathcal{H})$ is a mapping $\sigma \colon V \to [k]$ that is both a proper colouring of G and an h-odd colouring of $\mathcal{H}$. The least k for which G is h-odd k-colourable is in turn called the *h-odd chromatic number* of $(G, \mathcal{H})$ and we denote it by $\chi_o^h(G, \mathcal{H})$. We say that σ is an *h-odd k-colouring* of G if σ is an h-odd k-colouring of $(G, \mathcal{H})$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G. We denote $\chi_o^h(G)$ the *h-odd chromatic number* of G. By a greedy algorithm, we can immediately get $\chi_o^h(G) \leq (h+1)\Delta(G)+1$ for any graph G and integer h, or more generally that $\chi_o^h(G, \mathcal{H}) \leq h\Delta(\mathcal{H})+\Delta(G)+1$ given a graph-hypergraph pair $(G, \mathcal{H})$. We will show that we can ensure much better upper bounds with an additional reasonable minimum edge size condition

[1] With a more refined estimate of the binomial coefficient, one can replace $7/6$ with $53/45$, and conclude that we are only a factor $45/4$ away from best possible.

in $\mathcal{H}$.

We will rely on an extension of Lemma 2.2.4 to h-odd colourings, that we present hereafter.

Lemma 2.6.1 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair, and $\mathscr{C}(G)$ a set of colourings of G. Let $\boldsymbol{\sigma}$ be drawn uniformly at random from $\mathscr{C}(G)$, and assume that there exists an integer τ such that we deterministically have $|L_{\boldsymbol{\sigma}}(v)| \geq \tau$ for every integer $v \in V(G)$. Let $h \geq 1$ and $t \geq 0$ be integers that satisfy $m \coloneqq h - 1 + t \leq \epsilon(\mathcal{H})$, and $B_{\boldsymbol{\sigma}}(e)$ be the bad event that e has less than h odd colours in $\boldsymbol{\sigma}$. Then, for every subset $M(e) \subseteq e$ of size m, and for every possible realisation σ_0 of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$, we have*

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] \leq \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\tau}\right)^{t/2},$$

for every $m \leq \epsilon(\mathcal{H})$.

Proof. Let $e \in E(\mathcal{H})$, and let $M(e) = \{u_1, \dots, u_m\}$ be a fixed subset of m vertices in e. Let σ_0 be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. Let $\boldsymbol{\sigma}_0$ be drawn uniformly at random from the extensions of σ_0 to $\mathscr{C}(G)$. For every $1 \leq i \leq m$, we let $\boldsymbol{\sigma}_i \in \mathscr{C}(G)$ be obtained from $\boldsymbol{\sigma}_{i-1}$ by resampling the colour of u_i uniformly at random from $L_{\boldsymbol{\sigma}_{i-1}}(u_i)$. For every $i \leq m$, let $\mathbf{S}_i$ be the number of odd colours of e in $\boldsymbol{\sigma}_i$. For every $i \geq 1$, we have $\mathbf{S}_i = \mathbf{S}_{i-1} - 1$ if $\boldsymbol{\sigma}_i(u_i)$ is one of the $\mathbf{S}_{i-1}$ odd colours of e in $\boldsymbol{\sigma}_{i-1}$; since there are at least τ choices for $\boldsymbol{\sigma}_i(u_i)$ this happens with probability at most $\frac{h-1}{\tau}$ if $\mathbf{S}_{i-1} = h - 1$. Otherwise, we have $\mathbf{S}_i = \mathbf{S}_{i-1} + 1$. So the sequence $(\mathbf{S}_i)_{i\leq m}$ satisfies the hypotheses of Lemma 2.2.3, which yields

$$\begin{aligned}\mathbb{P}\left[B_{\boldsymbol{\sigma}_m}(e)\right] &= \mathbb{P}\left[\mathbf{S}_m \leq h-1\right] \\ &\leq \sqrt{2}\binom{m}{h-1}\left(\frac{2(m-h+1)}{e\tau}\right)^{\frac{m-h+1}{2}} \\ &= \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\tau}\right)^{t/2}.\end{aligned}$$

Since we resample the colours uniformly at random, the random colourings $(\boldsymbol{\sigma}_i)_{i\leq m}$ are identically distributed. Therefore, if $\boldsymbol{\sigma}$ is drawn

uniformly at random from $\mathscr{C}(G)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_0}(e)\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_m}(e)\right],$$

and the conclusion follows. □

Theorem 2.6.1 *Let $(G, \mathcal{H})$ be a graph-hypergraph pair with $\Delta(\mathcal{H}) \geq 49$, and let h be a given integer. If there exists a subset of vertices $S \subseteq V(\mathcal{H})$ such that $\min\{h-1, \epsilon(\mathcal{H}[S]) - h + 1\} \geq 2(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)$, then $(G, \mathcal{H})$ has an h-odd k-colouring, where*

$$k \leq \begin{cases} \chi(G \setminus S) + \Delta(G[S]) + 32(h-1) & \text{if } \epsilon(\mathcal{H}[S]) \geq 2(h-1); \\ \chi(G \setminus S) + \Delta(G[S]) + 2e^2 \frac{\epsilon(\mathcal{H}[S])^{2+1/\ln \Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1} & \text{otherwise.} \end{cases}$$

Proof. Let $k_0 \coloneqq \chi(G \setminus S)$ and let σ_0 be a proper k_0-colouring of $G \setminus S$. Fix $k \coloneqq k_0 + \Delta(G[S]) + \eta$, for some integer $\eta \geq 1$ whose precise value will be determined later in the proof, and let $\boldsymbol{\sigma}$ be a uniformly random proper k-colouring of G that satisfies $\boldsymbol{\sigma}_{|G\setminus S} = \sigma_0$. For every edge $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the (bad) random event that e contains less than h odd colours in $\boldsymbol{\sigma}$. Let us show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs.

Let us write $t \coloneqq \min\{\epsilon(\mathcal{H}[S]) - h + 1, h-1\}$ and $m \coloneqq h - 1 + t$. Let $e \in E(\mathcal{H})$ and $M(e) = \{v_1, v_2, \ldots, v_m\} \subseteq e \cap S$ be a subset of m vertices in e. Let us recolour the vertices in $M(e)$ in turn with a uniformly random available colour. Each time we recolour v_i, the neighbours of v_i in S forbid at most $\deg_S(v_i) \leq \Delta(G[S])$ colours, and the neighbours of v_i not in S forbid at most $k_0 = \chi(G \setminus S)$ colours (these colours are fixed by σ_0). So there are at least η available colours for v_i. In particular, we have $|L_{\boldsymbol{\sigma}}(v)| \geq \eta$ for each $v \in M(e)$.

We apply Lemma 2.2.1 with $\Gamma(e) \coloneqq \{e' \in E(\mathcal{H}) : e' \cap M(e) \neq \emptyset\}$ for every edge $e \in E(\mathcal{H})$, and obtain that, with non-zero probability, none of the events $B_{\boldsymbol{\sigma}}(e)$ occurs. The size of $\Gamma(e)$ is at most $m\Delta(\mathcal{H})$. Let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$

occurs for $e' \notin \Gamma(e)$. For every $Z \subseteq E(\mathcal{H}) \setminus \Gamma(e)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \bigcap_{e' \in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] \leq \sup_{\sigma_0 \in \Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] \leq \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\eta}\right)^{\frac{t}{2}},$$

by Lemma 2.6.1 applied to the graph G with $\mathscr{C}(G)$ being the set of proper k-colourings of G.

Let us fix $\eta \coloneqq 2t\,(\frac{m}{t} \times \binom{m}{t})^{2/t}$, so that the above probability is at most $\sqrt{2}\frac{t}{m}e^{-t/2}$. We can apply Lemma 2.2.1 if this is at most $\frac{1}{em\Delta(\mathcal{H})}$, which is equivalent to

$$\sqrt{2}e^{-t/2} \leq \frac{1}{et\Delta(\mathcal{H})}.$$

As in the proof of Lemma 2.4.1, this holds by fixing $t \coloneqq \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta(\mathcal{H})}\right)\right\rceil \leq \lceil 2(\ln\Delta(\mathcal{H}) + \ln\ln\Delta(\mathcal{H}) + 3\rceil$ when $\Delta(\mathcal{H}) \geq 49$.

If $t = h-1$ then

$$\eta = 2(h-1)\left(2\binom{2h-2}{h-1}\right)^{\frac{2}{h-1}} \leq 2(h-1)\left(2^{2h-2}\right)^{\frac{2}{h-1}} = 32(h-1),$$

Otherwise, we have $m = \epsilon(\mathcal{H}[S]) > 2t$, and

$$\eta = 2t\left(\frac{m}{t}\binom{m}{t}\right)^{\frac{2}{t}} \leq 2t\left(\frac{m}{t}\right)^{\frac{2}{t}}\left(\frac{me}{t}\right)^{2} \leq 2e^2\,\frac{m^{2+\frac{1}{\ln\Delta(\mathcal{H})}}}{m-h+1}.$$

This proves the existence of a proper k-colouring σ of $(G,\mathcal{H})$ such that every edge has at least h odd colours in σ, as desired.

□

Remark 4. The second bound of Theorem 2.6.1 is at most $\chi(G\setminus S)+\Delta(G[S])+2e^3\,\frac{\delta(G[S])^2}{\delta(G[S])-h+1}$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G.

The following corollary could be derived from Theorem 2.6.1 by setting $S \coloneqq V(G)$.

Corollary 2.6.1 *Let $(G,\mathcal{H})$ be a graph-hypergraph pair with $\Delta(\mathcal{H}) \geq 49$, and let h be a given integer. Let us assume that* $\min\{h-1, \epsilon(\mathcal{H}[S]) -$

$h+1\} \geq 2(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)$. *If* $\epsilon(\mathcal{H}) \geq 2(h-1)$, *then there exists an* h*-odd* $(\Delta(G)+32(h-1))$*-colouring of* $(G,\mathcal{H})$*; Otherwise,* $(G,\mathcal{H})$ *admits an* h*-odd* $(\Delta(G) + 2e^2 \frac{\epsilon(\mathcal{H})^{2+1/\ln\Delta(\mathcal{H})}}{\epsilon(\mathcal{H})-h+1})$*-colouring.*

We also extend Theorem 2.5.2 to h-odd colourings.

Theorem 2.6.2 *Let* $(G,\mathcal{H})$ *be a graph-hypergraph pair with* $\Delta(\mathcal{H}) \geq 49$*, and let* h *be a given integer. If there exists a subset of vertices* $S \subseteq V(\mathcal{H})$ *such that* $\min\{h-1, \epsilon(\mathcal{H}[S]) - h + 1\} \geq 2(\ln\Delta(\mathcal{H}) + \ln\ln\Delta(\mathcal{H}) + 3)$*, then* $(G,\mathcal{H})$ *has an* h*-odd* k*-colouring, where*

$$k \leq \begin{cases} \chi(G\setminus S) + 32(h-1)\,\chi(G[S]) & \text{if } \epsilon(\mathcal{H}[S]) \geq 2(h-1); \\ \chi(G\setminus S) + 2e^2 \frac{\epsilon(\mathcal{H}[S])^{2+1/\ln\Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1}\,\chi(G[S]) & \text{otherwise.} \end{cases}$$

Proof. Let $G_0 \coloneqq G \setminus S$ and $G_1 \coloneqq G[S]$. For each $i \in \{0,1\}$, we write $k_i \coloneqq \chi(G_i)$, and we let σ_i be a proper k_i-colouring of G_i.

We define a random proper k-colouring $\boldsymbol{\sigma}$ of G as follows, where $k = k_0 + \eta k_1$ and η is some positive integer whose precise value will be determined later in the proof. For every $v \in S$, draw some random value $\mathbf{x}_v$ uniformly at random from $[\eta]$, and let $\boldsymbol{\sigma}(v) \coloneqq (\sigma_1(v), \mathbf{x}_v)$. For every $v \notin S$, let $\boldsymbol{\sigma}(v) \coloneqq \sigma_0(v)$. Let us order the vertices in $V(G)$ arbitrarily. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random (bad) event that e contains less than h odd colours in $\boldsymbol{\sigma}$. Let us write $t \coloneqq \min\{\epsilon(\mathcal{H}[S]) - h + 1, h-1\}$, and let us fix $m \coloneqq h-1+t$. We let $M(e)$ contain the smallest m vertices of $e \cap S$. Let σ be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. By construction, for every $v \in S$, there are η choices in $L_{\boldsymbol{\sigma}}(v)$.

For an edge $e' \in E(\mathcal{H})$, the outcome of $B_{\boldsymbol{\sigma}}(e')$ is entirely determined by the realisation of $\boldsymbol{\sigma}_{|e'}$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(e)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(e')$ such that $M(e) \cap e' = \emptyset$. So we set $\Gamma(e) \coloneqq \{e' : e' \cap M(e) \neq \emptyset\}$, and observe that these sets have size at most $m\Delta(\mathcal{H})$. Let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$ occurs for $e' \notin \Gamma(e)$. Hence we may apply Lemma 2.6.1 and obtain that for every $Z \subseteq E(\mathcal{H})\setminus$

$\Gamma(e)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \bigcap_{e'\in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] \leq \sup_{\sigma_0\in\Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right]$$
$$\leq \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\eta}\right)^{\frac{t}{2}}.$$

As explained in the proof of Theorem 2.6.1, this is at most $\frac{1}{em\Delta(\mathcal{H})}$ by fixing $\eta \coloneqq 2t\,(\frac{m}{t} \times \binom{m}{t})^{2/t}$ when $\Delta(\mathcal{H}) \geq 49$. We also pick that $\eta = 32(h-1)$ if $\epsilon(\mathcal{H}[S]) \geq 2(h-1)$; otherwise, $\eta = 2e^2\,\frac{\epsilon(\mathcal{H}[S])^{2+1/\ln\Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1}$. We may now apply Lemma 2.2.1 to the bad events $(B_{\boldsymbol{\sigma}}(e))$, with that definition of $\Gamma(e)$, and conclude that with positive probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. So there is a realisation of $\boldsymbol{\sigma}$ that is an h-odd k-colouring of $(G,\mathcal{H})$. This concludes the proof. □

2.6.1 . Constructions

In the current section, we have derived upper bound for $\chi_o^h(G,\mathcal{H})$ given an integer h and a graph-hypergraph pair $(G,\mathcal{H})$ that satisfies that $\epsilon(\mathcal{H}) - h$ is sufficiently large. We now show that, if no such restriction holds, then the bound obtained by a greedy colouring may be close to best possible.

We propose a construction that relies on the existence of Steiner 2-designs, which was proven in [82]. Given an integer q, a Steiner 2-design on $[q]$ is a collection of sets of uniform size k (that we call *blocks*), such that every pair of vertices from $[q]$ is contained in exactly one set. We denote it $S(2,k;q)$. Among its many properties, it must contain exactly $\binom{q}{2}/\binom{k}{2}$ sets, and each vertex is contained in exactly $\frac{q-1}{k-1}$ sets.

Proposition 1 *For every integer $h \geq 1$, there exists a family of graphs G of increasing maximum degree Δ and of minimum degree $h+1$, such that*

$$\chi_o^h(G) \geq h(\Delta+1).$$

Proof. By [82, Theorem 2.1], we may find an arbitrarily large integer q such that a Steiner 2-design $S(2,h+1;q)$ exists. We let $H=(X,Y,E)$ be its (bipartite) incidence graph. We have $|X|=q$, $|Y| = \binom{q}{2}/\binom{h+1}{2}$, every vertex $y\in Y$ has degree $h+1$, every vertex $x\in X$ has degree $\frac{q-1}{h}$, and

every pair of vertices $x, x' \in X$ is contained in the neighbourhood of some vertex $y \in Y$. In particular, $\delta(H) = h + 1$ and $\Delta(H) = \frac{q-1}{h}$.

Let σ be an h-odd k-colouring of H, where $k = \chi_o^h(H)$. So every neighbourhood of size h must be rainbow in σ (i.e. contains no pair of vertices with the same colour). Since every pair of vertices from X is contained in a neighbourhood of size h, we infer that X is rainbow, and so $k \geq |X| = q = h(\Delta(H) + 1)$.

□

One interesting special case of Theorem 2.6.1 is the following.

Corollary 2.6.2 *Let G be a Δ-regular graph, let $2(\ln \Delta + \ln \ln \Delta + 4) \leq t \leq \Delta$ be a given integer, and let $h \coloneqq \Delta + 1 - t$. Then*

$$\chi_o^h(G) = O\left(\frac{\Delta^2}{t}\right).$$

We now show that Corollary 2.6.2 is tight up to a multiplicative constant.

Proposition 2 *For every even integer $\Delta \geq 2$ and $1 \leq t \leq \Delta$, there is a Δ-regular graph G such that, letting $h \coloneqq \Delta + 1 - t$, one has*

$$\chi_o^h(G) > \frac{1}{2}\frac{\Delta^2}{t+1}.$$

Proof. Let $n \coloneqq \frac{\Delta}{2} + 1$, and let $G \coloneqq L(K_{n,n})$ be the line-graph of a complete bipartite graph. Let $k \coloneqq \chi_o^h(G)$, and let σ be a h-odd k-colouring of G.

Let $v \in V(G)$. The neighbourhood of v can be covered with two cliques, so each colour in $N(v)$ has at most 2 occurrences. Let $s_\sigma(v)$ denote the number of colours with 2 occurrences in $N(v)$; we must have $s_\sigma(v) \leq (t-1)/2$. So

$$S \coloneqq \sum_{v \in V(G)} s_\sigma(v) \leq \frac{t-1}{2} n^2. \tag{2.2}$$

Let $(M_1, \ldots, M_k)$ be the colour classes of σ. If M_i has size m_i, then it has a contribution of $m_i(m_i - 1)$ to S. Indeed, M_i is a matching of size m_i in $K_{n,n}$, and there are $m_i(m_i - 1)$ edges incident to 2 edges from M_i in

$K_{n,n}$; each of them corresponds to a vertex in G with a monochromatic pair of colour i in its neighbourhood. So by convexity we have

$$S = \sum_{i=1}^{k} m_i(m_i - 1) \geq k \cdot \frac{n^2}{k}\left(\frac{n^2}{k} - 1\right) = n^2\left(\frac{n^2}{k} - 1\right).$$

Combining this with (2.2), we obtain

$$k \geq \frac{2n^2}{t+1} > \frac{1}{2}\frac{\Delta^2}{t+1}.$$

□

We finish this section with the following consequence of Theorem 2.6.2.

Corollary 2.6.3 *Let G be a graph of maximum degree Δ, and let $h \geq 2(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)$ be a given integer. If the minimum degree of G is at least $2h$, then*

$$\chi_o^h(G) \leq 32(h-1)\,\chi(G),$$

and this is tight up to a multiplicative constant for a family of graphs of increasing chromatic numbers.

Proof. The upper bound can be derived from Theorem 2.6.2 by letting $\mathcal{H}$ be the neighbourhood-hypergraph of G, and setting $S \coloneqq V(G)$.

To prove the tightness of the bound, we show that given any $\varepsilon > 0$, there is a graph G such that

$$\chi_o^h(G) \geq (2-\varepsilon)h\,\chi(G).$$

Fix an integer $k_0 \geq 2$, and let $n_0 \geq k_0$ be an even integer. Let G_0 be a complete k_0-partite graph, with parts $X_1, \ldots, X_{k_0}$. For every $i \in [k_0]$, X_i contains n_0 h-sets and the size of X_i is hn_0. For every $i \in [k_0]$ and for every pair of h-sets S in X_i, we add a vertex with neighbourhood S in G_0. Note that for each X_i, we add $\binom{n_0}{2}$ vertices and put these vertices in a vertex set Y_i. Let G be the obtained graph (see Fig. 2.1); let us show that

$$\chi_o^h(G) \geq \left(2 - \frac{2}{n_0}\right)hk_0.$$

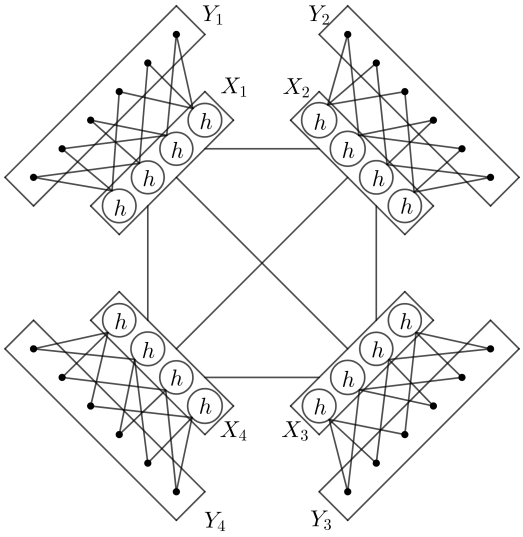


Figure 2.1: The construction in Corollary 2.6.3 when $k_0 = n_0 = 4$. A line between two sets stands for a complete bipartite graph.

We write $k \coloneqq \chi_o^h(G)$, and let σ be an h-odd k-colouring of G. First observe that we must have $\sigma(X_i) \cap \sigma(X_j) = \emptyset$ for every $i \neq j$, otherwise we would find a monochromatic edge in σ. So it suffices to show that $|\sigma(X_i)| \geq \left(2 - \frac{2}{n_0}\right) h$ for every $i \in [k_0]$. On the one hand, each vertex $v \in Y_i$ has at least h odd colours. On the other hand, for each X_i and each colour c in σ, let $0 \leq p \leq n_0$ be the number of h-sets in X_i where c is an odd colour. Clearly, c is an odd colour of $p(n_0 - p) \leq \frac{n_0^2}{4}$ vertices in Y_i. So we need at least $\frac{4\binom{n_0}{2}}{n_0^2} h = \left(2 - \frac{2}{n_0}\right) h$ colours to make every vertex have h odd colours in $\sigma(X_i)$. We also have $\chi(G) = \chi(G_0) = k_0$, so this concludes the proof.

□

2.7 . A slight improvement of Theorem 2.1.1

In Section 2.4 we compute the probability of the bad event of a vertex v whose degree is at least $\Delta(G)/2$ by observing only m neighbours of v. Clearly some factors was lost during the calculation. Then we will show that even if we take

all neighbours of v into consideration, the logarithmic term of Theorem 2.1.1 can be only improved up to a small multiplicative constant.

We first introduce a convergence theorem for finite Markov chains.

Definition 16. A *finite Markov chain* with finite *state space* Ω and $|\Omega| \times |\Omega|$ transition matrix P is a sequence of random variables $X_0, X_1, \ldots$ where

$$\mathbb{P}[X_{t+1} = y \mid X_t = x] = P(x, y).$$

Then $P(x, \cdot)$, the x-th row of P, gives the distribution of X_{t+1} given $X_t = x$. Thus, given $X_0 = x \in \Omega$, we have $\mathbb{P}[X_t = y \mid X_0 = x] = P^t(x, y)$, as multiplying a distribution by the transition matrix P advances the distribution one step along the Markov chain, so multiplying by P^t advances it by t steps from $X_0 = x$.

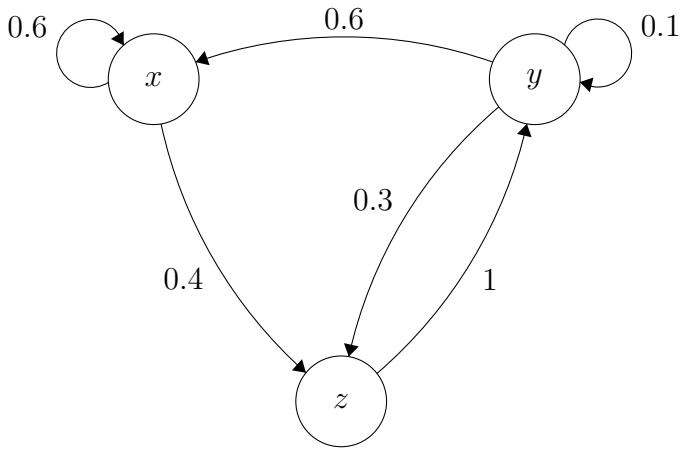


Figure 2.2: An example Markov chain with three states x, y, and z.

We can illustrate a Markov chain with a state diagram, in which an arrow from one state to another indicates the probability of going to the second state given we were just in the first. For instance, in this diagram, given that the Markov chain is currently in x, we have probability 0.4 of staying in x, probability 0.6 of going to z, and probability 0 of going to y in the next time step (Figure 2.2). This Markov chain would be represented by the transition

matrix

$$P = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.6 & 0.1 & 0.3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 17. A distribution π is called a *stationary distribution* of a Markov chain P if

$$\pi P = \pi,$$

or equivalently,

$$\pi(y) = \sum_{x \in \Omega} \pi(x) P(x, y) \text{ for any } y \in \Omega.$$

For the graph in Figure 2.2, we have

$$\pi = \left(\frac{15}{34}, \frac{5}{17}, \frac{9}{34} \right).$$

We now make note of two simple properties possessed by the most interesting chains. Both will turn out to be necessary for the Convergence Theorem to be true.

Definition 18. A Markov chain is *irreducible* if for all states $x, y \in \Omega$, there exists a $t \geq 0$ such that $P^t(x, y) > 0$.

This means that it is possible to get from any state to any other state using only transitions of positive probability.

Definition 19. A Markov chain is *aperiodic* if $\gcd\{t \colon P^t(x, x) > 0\} = 1$ for all states $x \in \Omega$. Otherwise we call it *periodic*.

In order to test convergence, we would like to bound the following measurement of the distance between distributions.

Definition 20. The *total variation* between two distributions μ and ν is defined as

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_{A \subset \Omega} |\mu(A) - \nu(A)|,$$

where $\mu(A) = \sum_{x \in A} \mu(x)$.

This definition is explicitly probabilistic: the distance between μ and ν is the maximum difference between the probabilities assigned to a single event by the two measures.

Proposition 3 *Let μ and ν be two probability distributions on Ω. Then*

$$\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

Now we are ready to illustrate the convergence theorem, which tells us that an irreducible, aperiodic Markov chain will converge at an exponential rate to a stationary distribution over time.

Theorem 2.7.1 (Convergence Theorem) *Suppose that P is irreducible and aperiodic, with stationary distribution π. Then there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that*

$$\max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{\mathrm{TV}} \leq C\alpha^t.$$

Lemma 2.7.1 *Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ be a sequence of binary random variables with values in $\{-1, 1\}$, let $T_0 \in \mathbb{N}$ be a non-negative integer, and let $\mathbf{T}_i \coloneqq T_0 + \sum_{j=1}^{i} \mathbf{Y}_j$ for every integer $i \geq 0$. If there exists a even integer $\tau > 0$ such that, for any integers $i \geq 0$ and $0 \leq t \leq \tau$, it holds that $\mathbb{P}[\mathbf{Y}_i = -1 \mid \mathbf{T}_{i-1} = t] = t/\tau$, then there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that*

$$\mathbb{P}[\mathbf{T}_n = 0] \leq 2^{1-\tau} + C\alpha^n.$$

Proof. Let $\mathbf{Y}'_1, \mathbf{Y}'_2, \ldots$ be a sequence of random variables satisfying that $\mathbf{Y}'_i = \mathbf{Y}_{2i-1} + \mathbf{Y}_{2i}$ for any positive integer i, let $T'_0 = T_0$, and let $\mathbf{T}'_i \coloneqq T'_0 + \sum_{j=1}^{i} \mathbf{Y}'_j$ for every integer $i \geq 0$. Clearly $\{\mathbf{T}'_i\}_{i=0}^{\infty}$ is a Markov process with finite state space $\Omega = \{0, 2, 4, \ldots, \tau\}$ and transition matrix P where

$$P(x, y) = \begin{cases} \frac{x(x-1)}{\tau^2}, & \text{if } y = x - 2; \\ \frac{(\tau - x)(\tau - x - 1)}{\tau^2}, & \text{if } y = x + 2; \\ \frac{1}{\tau}, & \text{if } x = y = 0 \text{ or } x = y = \tau; \\ \frac{x(\tau - x + 1) + (\tau - x)(x + 1)}{\tau^2}, & \text{if } x = y \notin \{0, \tau\}; \\ 0, & \text{otherwise.} \end{cases}$$

Here we assume that x, y are all even integers by the property of P. Let $\pi = (p_0, p_2, \ldots, p_\tau)$ be the stationary distribution of P. We claim that π satisfies the following recurrence.

$$\frac{p_{x+2}}{p_x} = \frac{(\tau - x)(\tau - x - 1)}{(x + 1)(x + 2)} \quad \text{for any } x \in \{0, 2, \ldots, \tau - 2\}. \tag{2.3}$$

For the first step, since $\pi = \pi \cdot P$, we have

$$p_0 = p_0 \cdot P(0,0) + p_2 \cdot P(2,0) = \frac{1}{\tau} p_0 + \frac{2}{\tau^2} p_2.$$

and hence Equation (2.3) follows. The case of $p_\tau / p_{\tau-2}$ is the same. Now assume that Equation (2.3) holds for $x-2$ where $2 \leq x \leq \tau/2$, then we have

$$\begin{aligned} p_x = & \frac{x(\tau - x + 1) + (\tau - x)(x+1)}{\tau^2} p_x \\ & + \frac{(\tau - x + 2)(\tau - x + 1)}{\tau^2} p_{x-2} \\ & + \frac{(x+2)(x+1)}{2} p_{x+2}. \end{aligned}$$

By the induction hypothesis we have

$$\frac{(\tau - x + 2)(\tau - x + 1)}{\tau^2} p_{x-2} = \frac{x(x-1)}{\tau^2} p_x.$$

Hence we have

$$\frac{(\tau - x)(\tau - x - 1)}{\tau^2} p_x = \frac{(x+2)(x+1)}{\tau^2} p_{x+2},$$

the equality holds. By symmetry this proves Equation (2.3).

By Equation (2.3) and symmetry that $p_0 = p_\tau$, it holds that

$$1 = \sum_{0 \leq x \leq \tau,\ x \text{ even}} p_x = \left(1 + \binom{\tau}{2} + \binom{\tau}{4} + \ldots + 1 \right) p_0 = 2^{\tau - 1} p_0.$$

Therefore we have $p_0 = 2^{1-\tau}$.

Clearly there exists $i \geq 0$ such that $P^i(x,y) > 0$ for any states x and y, and we have $P(x,x) > 0$ for any state x. Hence P is irreducible and aperiodic, with stationary distribution π. Applying Theorem 2.7.1, there exists constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\max_{x \in \Omega} \| P^n(x, \cdot) - \pi \|_{\mathrm{TV}} \leq C \alpha^{n/2}.$$

Since we have $\mathbb{P}[\mathbf{T}_{2i} = 0] = \mathbb{P}[\mathbf{T}'_i = 0]$ and $\mathbb{P}[\mathbf{T}_{2i+1} = 0] = 0$ for any $i \in \mathbb{N}$, the result follows. □

Lemma 2.7.2 *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be a sequence of binary random variables with values in $\{-1, 1\}$, let $S_0 \in \mathbb{N}$ be a non-negative integer, and let $\mathbf{S}_i \coloneqq S_0 + \sum_{j=1}^{i} \mathbf{X}_j$ for every integer $i \geq 0$. If there exists a even integer $\tau > 0$ such that, for every $i \geq 0$, it holds that $\mathbb{P}[\mathbf{X}_i = -1 \mid \mathbf{S}_{i-1} = s] = p_{i,s} \leq s/\tau$, then*

$$\mathbb{P}[\mathbf{S}_n = 0] \leq 2^{2-\tau},$$

for any large enough n.

Proof. Consider the following coupling of two sequences of random variables. Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ be random variables with values in $\{-1, 1\}$, let $T_0 \coloneqq S_0$, and let $\mathbf{T}_i \coloneqq T_0 + \sum_{j=1}^{i} \mathbf{Y}_j$ for every integer $i \geq 0$. Furthermore for every $i \geq 0$, it holds that $\mathbb{P}[\mathbf{Y}_i = -1 \mid \mathbf{T}_{i-1} = t] = t/\tau$. We claim that

$$\mathbb{P}[\mathbf{S}_i = 0] \leq \mathbb{P}[\mathbf{T}_i = 0] \text{ for any } i \geq 0. \tag{2.4}$$

In order to prove Equation (2.4) we now construct a random process that has the same distribution as $\mathbf{S}_i$ and never goes behind $\mathbf{T}_i$ at any time step i. Let $\mathbf{X}'_1, \mathbf{X}'_2$ be random variables with values in $\{-1, 1\}$, let $S'_0 \coloneqq S_0$, and let $\mathbf{S}'_i \coloneqq S'_0 + \sum_{j=1}^{i} \mathbf{X}'_j$ for every integer $i \geq 0$. We describe each coupling $(\mathbf{X}'_i, \mathbf{Y}_i)$ as follows.

- When $\mathbf{S}'_{i-1} > \mathbf{T}_{i-1}$, we let $\mathbf{X}'_i$ and $\mathbf{Y}$ be mutually independent, where

$$\mathbb{P}\left[\mathbf{X}'_i = -1 \mid \mathbf{S}'_{i-1} = s\right] = \mathbb{P}[\mathbf{X} = -1 \mid \mathbf{S}_{i-1} = s].$$

- When $\mathbf{S}'_{i-1} = \mathbf{T}_{i-1} = s$, if $\mathbf{Y}_i$ takes value -1, then we let $\mathbf{X}'_i$ take value -1 with probability $p_{i,s}/(s/\tau)$ conditioning on that;

$$\mathbb{P}\left[\mathbf{X}'_i = -1 \mid \mathbf{S}'_{i-1} = \mathbf{T}_{i-1} = s,\ \mathbf{Y}_i = -1\right] = p_{i,s}/(s/\tau).$$

  Otherwise $\mathbf{X}'_i$ takes value 1. By the dependency of them, it holds that $\mathbf{X}'_i \geq \mathbf{Y}_i$ with probability 1.

Clearly $\mathbf{S}'_i - \mathbf{T}_i$ is an even number at any time step i, and so $\mathbf{S}'_i \geq \mathbf{T}_i$ always holds, which implies that Equation (2.4) is true.

Now we can apply Lemma 2.7.1 for random variables $\{\mathbf{Y}_i\}$ and $\{\mathbf{T}_i\}$, there exist constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\mathbb{P}\left[\mathbf{S}_n = 0\right] \leq \mathbb{P}\left[\mathbf{T}_n = 0\right] \leq 2^{1-\tau} + C\alpha^n \leq 2^{2-\tau}$$

for large n. □

Remark 5. In the proof of Theorem 2.1.1 we prove the probability of any bad event has an upper bound $\sqrt{2}e^{-m/2}$, which can be replaced by $4^{1-m/2}$ for sufficiently large Δ. Hence in order to apply LLL, we need an upper bound of $\frac{1}{e\Delta^2}$ for that probability, which holds precisely when $m \geq 4\log_4 \Delta + 2\log_4(4e)$. The inequality holds when then number of colours $k \geq \Delta + 4\log_4 \Delta + 2\log_4(4e)$ since we take $m \coloneqq \frac{k-\Delta}{2}$.

3 - Properly coloured cycles in edge-coloured complete graphs

3.1 . Introduction

In this chapter we prove that Conjecture 1.2.6 is true, which gives the maximum number of colours in an edge-colouring of K_n with no properly coloured C_ℓ as a subgraph.

Theorem 3.1.1 *Let C_ℓ be a cycle on ℓ vertices and $\ell - 1 \equiv r_{\ell-1} \mod 3$, where $0 \leq r_{\ell-1} \leq 2$. For $n \geq \ell$, $\mathrm{pr}(K_n, C_\ell)$ equals to*

$$\max\left\{\binom{\ell-1}{2} + n - \ell + 1, \lfloor\frac{\ell-1}{3}\rfloor n - \binom{\lfloor\frac{\ell-1}{3}\rfloor + 1}{2} + 1 + r_{\ell-1}\right\}.$$

3.2 . Preliminaries

Let us for short denote $f(n, \ell)$ to be

$$\max\left\{\binom{\ell-1}{2} + n - \ell + 1, \lfloor\frac{\ell-1}{3}\rfloor n - \binom{\lfloor\frac{\ell-1}{3}\rfloor + 1}{2} + 1 + r_{\ell-1}\right\}.$$

We write a cycle of length ℓ as $C_\ell = v_0v_1...v_{\ell-1}(v_\ell = v_0)$, with $V(C_\ell) = \{v_0, v_1, ..., v_{\ell-1}\}$ and $E(C_\ell) = \{v_0v_1, v_1v_2, ..., v_{\ell-1}v_0\}$. The lower bound of $\mathrm{pr}(K_n, C_\ell)$ was given roughly by Manoussakis, Spyratos, Tuza, and Voigt in [60]. Fang, Győri, and Xiao [32] gave the precise following one.

Lemma 3.2.1 *[32] Let C_ℓ be a cycle on ℓ vertices and $\ell - 1 \equiv r_{\ell-1} \mod 3$, where $0 \leq r_{\ell-1} \leq 2$. For $n \geq \ell$,*

$$\mathrm{pr}(K_n, C_\ell) \geq f(n, \ell).$$

The extremal graph when $\mathrm{pr}(K_n, C_\ell)$ reaches its lower bound will be helpful to our constructive proof in the next section. For the first lower bound, consider a complete graph K_n with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. For any edge v_iv_j where $i < j$ and $i \leq n - \ell + 1$, we colour this edge by colour i. All the other edges

form a complete graph of order $\ell - 1$, and we colour it rainbow distinct from the previous $n - \ell + 1$ colours.

For the second lower bound, we partition $V(K_n)$ into two parts A and B with $|A| = \lfloor \frac{\ell-1}{3} \rfloor$ and $|B| = n - \lfloor \frac{\ell-1}{3} \rfloor$. Let G_1 be the complete graph induced by B and $G_2 = K_n - G_1$. Clearly G_2 consists of the complete graph induced by A and the complete bipartite graph of A and B. First we colour G_2 rainbow. Then we colour G_1 by $1 + r_{\ell-1}$ new colours without producing a properly coloured path of $3 + r_{\ell-1}$ vertices (see [32, Proposition 1]).

A cycle C in a graph G is *extendable* if there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. If such a cycle C' exists, then C can be *extended* to C' or C' is an *extension* of C. This problem was first introduced by Hendry [43] as an extension of hamiltonian problems. Similarly, in edge-coloured graphs, a properly coloured (rainbow) cycle C in an edge-coloured graph G is *properly coloured* (*rainbow*) *extendable*, if there exists a properly coloured (rainbow) cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$, respectively.

For a vertex $u \notin V(C)$, let $E(u, C) = \{uv : v \in V(C)\}$ be the set of edges between u and $V(C)$. Let $\varphi\colon E(K_n) \to \Phi$ be the assignment of that edge-colouring of K_n, where Φ is the set of all colours of $E(K_n)$. Let $\varphi(E)$ be the set of colours of edges in E, where E is a subset of $E(G)$. For any properly coloured cycle C of K_n, we partition Φ into 4 parts as follows:

$$\begin{aligned}
\Phi_0(C) :=& \{c \in \Phi : c = \varphi(e), e \in E(C)\};\\
\Phi_1(C) :=& \{c \in \Phi : c \notin \Phi_0, c = \varphi(e), e \text{ is a chord of } C\};\\
\Phi_2(C) :=& \{c \in \Phi : c \notin (\Phi_0 \cup \Phi_1), c = \varphi(e),\\
&\qquad e \text{ has exact 1 endpoint on } V(C)\};\\
\Phi_3(C) :=& \Phi \setminus (\Phi_0 \cup \Phi_1 \cup \Phi_2).
\end{aligned}$$

Then we partition $V(K_n)$ into 3 parts as follows:

$$\begin{aligned}
V_0(C) :=& V(C);\\
V_1(C) :=& \{v \in V(K_n) : v \notin V_0(C),\\
&\quad \varphi(e) \notin \Phi_3(C) \text{ for any } e \text{ incident to } v\};\\
V_2(C) :=& \{v \in V(K_n) : v \notin V_0(C),\\
&\quad \varphi(e) \in \Phi_3(C) \text{ for some } e \text{ incident to } v\}.
\end{aligned}$$

Obviously we have that $\Phi_0(C), \Phi_1(C), \Phi_2(C), \Phi_3(C)$ are pairwise disjoint, and $V_0(C)$, $V_1(C)$, $V_2(C)$ are pairwise disjoint by the definition.

3.3 . Proof of Theorem 3.1.1

We prove the upper bound of $\mathrm{pr}(K_n, C_\ell)$ by induction on both the order of K_n and the length of C_ℓ. Note that $\mathrm{pr}(K_n, C_\ell) = f(n, \ell)$ for $3 \leq \ell \leq 6$. Assume that $\mathrm{pr}(K_{n'}, C_{\ell'})$ satisfies the theorem for any $n' \leq n$ and $\ell' \leq \ell$ with at most one equality holds. Let K_n be edge-coloured by $f(n, \ell) + 1$ colours where $\ell \geq 7$. Let us assume for the sake of contradiction that there is no properly coloured cycle of length ℓ in K_n. Our aim is to find a contradiction on the number of colours.

Clearly $f(n, \ell-1) < f(n, \ell)$. Then K_n must contain a properly coloured cycle of any length $\ell' < \ell$ by the induction hypothesis. For the convenience of the proof, the labeling of vertices on a cycle is taken as module its length. The following two lemmas give some properties of the properly coloured cycle of length less than ℓ, which would be used to prove the main theorem directly.

Lemma 3.3.1 *Let K_n be an edge-coloured graph with at least $f(n, \ell) + 1$ colours, where $3 \leq \ell \leq n$. If either*

- *ℓ is even and there is a properly coloured cycle C of length $\ell - 1$ with $\Phi_3(C) \neq \emptyset$, or*
- *ℓ is odd and there is a properly coloured cycle C of length $\ell - 2$ with $\Phi_3(C) \neq \emptyset$,*

then K_n contains a properly coloured cycle of length ℓ.

The second lemma gives a general result about properly coloured extendability.

Lemma 3.3.2 *Let K_n be an edge-coloured graph with at least $f(n,\ell)+1$ colours, where $3 \leq \ell \leq n$. If K_n has a properly coloured cycle C of length at most $\ell-1$ with $\Phi_3(C)=\emptyset$, then C is properly coloured extendable.*

Proof of Theorem 3.1.1 (assuming Lemma 3.3.1 and Lemma 3.3.2).
Assume for the sake of contradiction that K_n does not contain a properly coloured cycle of length ℓ. By the hypothesis induction there exist properly coloured cycles of length $\ell-1$ and $\ell-2$. When ℓ is even, the properly coloured cycle C of length $\ell-1$ satisfies that $\Phi_3(C)=\emptyset$ by Lemma 3.3.1, and is therefore properly coloured extendable by Lemma 3.3.2. When ℓ is odd, the properly coloured cycle C of length $\ell-2$ is properly coloured extendable, and the resulting cycle of length $\ell-1$ is still properly coloured extendable. In both cases we can find a properly coloured cycle of length ℓ, which contradicts to our assumption that there is no properly coloured cycle of length ℓ in K_n. □

3.3.1 . Proof of Lemma 3.3.1

First we prove the case when ℓ is even. Assume for the sake of contradiction there is no properly coloured cycle of length ℓ. Since $\Phi_3(C)$ is not empty, there exists an edge u_1u_2 such that $\varphi(u_1u_2) \in \Phi_3(C)$. We first consider the colours of $E(u_1,C)$ and $E(u_2,C)$. Since C is properly coloured, we have $\varphi(v_0v_1) \neq \varphi(v_0v_{\ell-2})$. Without loss of generality we assume that $\varphi(u_1v_0) \neq \varphi(v_0v_{\ell-2})$. We characterise the colours of $E(u_1,C)$ and $E(u_2,C)$ in the following claim:

Claim 3.3.1 $\varphi(u_iv_j)=\varphi(v_jv_{j+1})$ *for any* $1 \leq i \leq 2$ *and* $0 \leq j \leq \ell-2$.

Proof of claim. Let us begin with the edge u_2v_2. Since $\varphi(u_1v_0) \neq \varphi(v_0v_{\ell-2})$, we must have $\varphi(u_2v_2)=\varphi(v_2v_3)$, otherwise

$$u_1u_2, v_2v_3v_4\ldots v_{\ell-2}v_0, u_1$$

is a properly coloured cycle of length ℓ, which contradicts to our assumption. Here we use that $\varphi(u_1u_2) \notin \varphi(E(v,C))$ by the definition of $\Phi_3(C)$.

Now we have $\varphi(u_2v_2) \neq \varphi(v_2v_1)$. By the same argument it holds that $\varphi(u_1v_4)=\varphi(v_4v_5)$. And we can get the following property:

- For any $1 \leq i \leq 2$ and $0 \leq j \leq \ell - 2$, if $\varphi(u_i v_j) = \varphi(v_j v_{j+1})$, then we have $\varphi(u_i v_{j+4}) = \varphi(v_{j+4} v_{j+5})$.

Combined with the conditions that $\varphi(u_2 v_2) = \varphi(v_2 v_3)$ and $\varphi(u_1 v_4) = \varphi(v_4 v_5)$, and noting that 4 and $\ell - 1$ are relatively prime, the claim follows. □

Let $v_j v_k$ be a chord of C with $2 \leq k - j \leq \frac{\ell-1}{2}$. We must have $\varphi(v_j v_k) = \varphi(v_j v_{j+1})$ or $\varphi(v_j v_k) = \varphi(v_k v_{k+1})$, otherwise there exists a properly coloured cycle

$$u_1 u_2, v_{k-1} v_{k-2} \dots v_j, v_k v_{k+1} \dots v_{j-2}, u_1$$

of length ℓ, a contradiction.

Let $u \in V_1(C)$. The colour of $E(u, C)$ are from $\Phi_0(C) \cup \Phi_1(C) \cup \Phi_3(C)$ by definition. Assume uv_j is an edge with $\varphi(uv_j) \in \Phi_2(C)$. We must have $\varphi(uu_1) = \varphi(uv_j)$, otherwise there is a properly coloured cycle

$$uu_1 u_2, v_{j-3} v_{j-4} \dots v_{j+1} v_j, u$$

of length ℓ. Here we use that $\varphi(uu_1) \neq \varphi(u_1 u_2)$. Hence all edges of $E(u, V(C))$ can receive at most 1 colour from $\Phi_2(C)$.

Now we have that $|\Phi_0(C)| \leq \ell - 1$, $|\Phi_1(C)| = 0$, $|\Phi_2(C)| \leq |V_1(C)|$, and $|\Phi_3(C)| \leq f(|V_2(C)|, \ell)$. Therefore, the number of colours of $E(K_n)$ is

$$|\Phi| = \sum_{i=0}^{3} |\Phi_i(C)| \leq \ell - 1 + |V_1(C)| + f(|V_2(C)|, \ell) \leq f(n, \ell),$$

which is a contradiction.

When ℓ is odd. Suppose that C is a properly coloured cycle of length $\ell - 2$ such that $\Phi_3(C) \neq \emptyset$. And without loss of generality we assume that $\varphi(u_1 v_0) \neq \varphi(v_0 v_{\ell-3})$. Then we have the following claim:

Claim 3.3.2 *$\varphi(u_i v_j) = \varphi(v_j v_{j+1})$ for any $1 \leq i \leq 2$ and $0 \leq j \leq \ell - 3$.*

Proof of claim. Let us begin with the edge $u_2 v_1$. Since $\varphi(u_1 v_0) \neq \varphi(v_0 v_{\ell-3})$, we must have $\varphi(u_2 v_1) = \varphi(v_1 v_2)$, otherwise

$$u_1 u_2, v_1 v_2 v_3 \dots v_{\ell-3} v_0, u_1$$

is a properly coloured cycle of length ℓ, a contradiction.

Now we have $\varphi(u_2v_1) \neq \varphi(v_1v_0)$. By the same argument it holds that $\varphi(u_1v_2) = \varphi(v_2v_3)$. And we can get the following property:

- For any $1 \leq i \leq 2$ and $0 \leq j \leq \ell - 3$, if $\varphi(u_iv_j) = \varphi(v_jv_{j+1})$, then we have $\varphi(u_iv_{j+2}) = \varphi(v_{j+2}v_{j+3})$.

Combined with the conditions that $\varphi(u_2v_1) = \varphi(v_1v_2)$ and $\varphi(u_2v_1) \neq \varphi(v_1v_0)$, and noting that 2 and $\ell - 2$ are relatively prime, the claim follows. □

Let v_jv_k be a chord of C with $2 \leq k - j \leq \frac{\ell-2}{2}$. We must have $\varphi(v_jv_k) = \varphi(v_jv_{j+1})$ or $\varphi(v_jv_k) = \varphi(v_kv_{k+1})$, otherwise there exists a properly coloured cycle

$$u_1u_2, v_{k-1}v_{k-2} \dots v_j, v_kv_{k+1} \dots v_{j-1}, u_1$$

of length ℓ, a contradiction.

Let $u \in V_1(C)$. The colour of $E(u, C)$ are from $\Phi_0(C) \cup \Phi_1(C) \cup \Phi_3(C)$ by definition. Assume uv_j is an edge with $\varphi(uv_j) \in \Phi_2(C)$. We must have $\varphi(uu_1) = \varphi(uv_j)$, otherwise there is a properly coloured cycle

$$uu_1u_2, v_{j-2}v_{j-3} \dots v_{j+1}v_j, u$$

of length ℓ. Hence all edges of $E(u, V(C))$ can receive at most 1 colour from $\Phi_2(C)$.

By a same argument as above we can get a contradiction on the number of colours of $E(K_n)$. We finish the proof. □

3.3.2 . Proof of Lemma 3.3.2

The proof of Lemma 3.3.2 splits into several steps as follows. First, there is a cycle C of small length by induction. Consider a vertex where the number of "new" colours between this vertex and C reaches the maximum. Then we give the exact depiction of maximal colouring of K_n when the number of "new" colours is very large or small. In the last step, we show that K_n can not have more colours in other cases.

We prove the case when the length of the properly coloured cycle is $\ell - 1$. Suppose for the sake of contradiction that a properly coloured cycle $C =$

$v_0v_2\dots v_{\ell-2}$ with $\Phi_3(C)=\emptyset$ is not properly coloured extendable. Since we are considering the extendability of C, it is meaningful to choose one special vertex $w\notin V(C)$ and analyse the colour structure of $K_n[V(C)\cup\{w\}]$. We first show that which vertex $w\notin V(C)$ could be an appropriate object.

Choosing a special vertex outside the cycle

Clearly we have $\Phi_2(C)\neq\emptyset$, otherwise we have

$$|\Phi|\leq\binom{\ell-1}{2}<f(n,\ell).$$

Let $t\coloneqq\max_{u\notin V(C)}|\varphi(E(u,C))\cap\Phi_2(C)|$ be the maximum number of new colours (in terms of C and its chords) between $V(C)$ and a vertex outside. Write

$$g(t)\coloneqq\max\{|\Phi_0(C)|+|\Phi_1(C)|:t\}.$$

Our aim is to prove that $g(t)+t(n-\ell+1)\leq f(n,\ell)$ for any possible value of t.

When $t=1$, for any $u\notin V(C)$, if uv_j has the same colour for all j, then C can not be properly coloured extended by u. This implies that $g(1)\leq\binom{\ell-1}{2}$, trivially. When $t\geq 2$, choose a vertex $w\notin V(C)$ reaching the number, that is, $|\varphi(E(w,C))\cap\Phi_2(C)|=t$. Then pick

$$V_w=\{v_{a_1},v_{a_2},\dots,v_{a_t}\}$$

such that $\varphi(wv_{a_j})\in\Phi_2(C)$ for $0\leq a_1<a_2<\ldots<a_t\leq\ell-2$, and these colours are pairwise distinct. For every two consecutive v_{a_i} and $v_{a_{i+1}}$, we say the set of vertices $\{w,v_{a_i+1},\dots,v_{a_{i+1}-1}\}$ is a *segment* of w. Moreover, we can label the segment between v_{a_i} and $v_{a_{i+1}}$ the i-th segment.

Let us characterise some typical structures of $K_n[V(C)\cup\{w\}]$ under the assumption that C is not properly coloured extendable.

Some typical structures

Rule 3.3.1 *If $\varphi(wv_j)\neq\varphi(v_{j\mp1}v_j)$, then we must have $\varphi(wv_{j\pm1})=\varphi(wv_j)$ or $\varphi(wv_{j\pm1})=\varphi(v_{j\pm1}v_{j\pm2})$, respectively.*

Clearly any two vertices in V_w can not be consecutive on C, otherwise C can be properly coloured extended via some v_jwv_{j+1}. For any vertex $v_{a_j}\in V_w$, if

we look at the vertices clockwise for each, we could find a labeling r' such that $\varphi(wv_i) = \varphi(wv_{a_j})$ for any $a_j \leq i \leq r'$ and $\varphi(wv_{r'+1}) = \varphi(v_{r'+1}v_{r'+2})$. Then we have $\varphi(wv_{r'+2}) = \varphi(wv_{r'+1})$ or $\varphi(wv_{r'+2}) = \varphi(v_{r'+2}v_{r'+3})$ by Rule 3.3.1. If the latter case holds we can repeat this observation until we find a monochromatic triangle wv_rv_{r+1}. For the counterclockwise direction of $v_{a_{j+1}}$ we do the same observation. Then we can get the following rule.

Rule 3.3.2 *For two vertices $v_{a_j}, v_{a_{j+1}} \in V_w$, there exist r', r, s, s' with $a_j \leq r' < r < s < s' \leq a_{j+1}$ satisfying the following properties:*

1. *$\varphi(wv_i) = \varphi(wv_{a_j})$ for any $a_j \leq i \leq r'$, $\varphi(wv_i) = \varphi(wv_{a_{j+1}})$ for any $s' \leq i \leq a_{j+1}$;*
2. *$\varphi(wv_i) = \varphi(v_iv_{i+1})$ for any $r' < i \leq r$, $\varphi(wv_i) = \varphi(v_iv_{i-1})$ for any $s \leq i < s'$;*
3. *wv_rv_{r+1} and wv_sv_{s-1} are monochromatic triangles (identity if $r+1 = s$).*

We call those triangles noted in Rule 3.3.2 *typical*. In fact we can write r and s as functions of a_j with $1 \leq j \leq t$. For any segment of w, if two triangles $wv_{r(a_j)}v_{r(a_j)+1}$ and $wv_{s(a_{j+1})}v_{s(a_{j+1})-1}$ are distinct, we say $wv_{r(a_j)}v_{r(a_j)+1}$ is a *clockwise typical* (CT) triangle, $wv_{s(a_{j+1})}v_{s(a_{j+1})-1}$ is a *counterclockwise typical* (CCT) triangle. Otherwise we say the unique monochromatic triangle is a *doubly typical* (DT) triangle.

We use $\Delta(w)$ to denote the number of typical triangles. Note that there may be many other monochromatic triangles, but we focus on the typical triangles. Hence we have $t \leq \Delta(w) \leq 2t$. It immediately follows from Rule 3.3.2 that $t \leq \frac{\ell-1}{3}$.

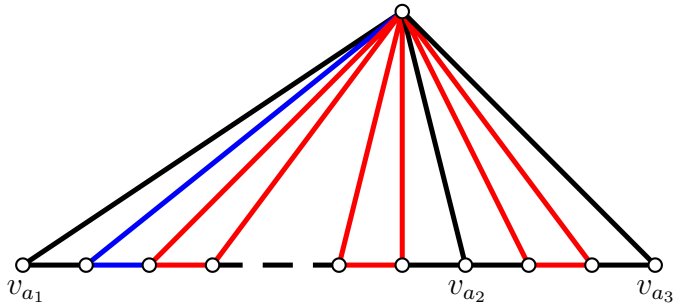


Figure 3.1: Three kinds of typical triangles

Let us denote V_T to be the set of vertices $v_j \in V(C)$ such that wv_jv_{j+1} is a typical triangle. Let V_{CT}, V_{CCT}, V_{DT} be that of CT, CCT, DT triangle, respectively. Next we observe the colours of chords of C.

Rule 3.3.3 *Let v_jv_k be a chord of C. If $\varphi(wv_{j+1}) \neq \varphi(v_{j+1}v_{j+2})$, $\varphi(wv_{k+1}) \neq \varphi(v_{k+1}v_{k+2})$ and $\varphi(wv_{j+1}) \neq \varphi(wv_{k+1})$, then we must have $\varphi(v_jv_k) = \varphi(v_jv_{j-1})$ or $\varphi(v_jv_k) = \varphi(v_kv_{k-1})$. If $\varphi(wv_{j-1}) \neq \varphi(v_{j-1}v_{j-2})$, $\varphi(wv_{k-1}) \neq \varphi(v_{k-1}v_{k-2})$ and $\varphi(wv_{j-1}) \neq \varphi(wv_{k-1})$, then we must have $\varphi(v_jv_k) = \varphi(v_jv_{j+1})$ or $\varphi(v_jv_k) = \varphi(v_kv_{k+1})$.*

Proof. Suppose on the contrary we have that $\varphi(v_jv_k) \neq \varphi(v_{j-1}v_j)$ and $\varphi(wv_k) \neq \varphi(v_{k-1}v_k)$,while $\varphi(wv_{j+1}) \neq \varphi(v_{j+1}v_{j+2})$, $\varphi(wv_{k+1}) \neq \varphi(v_{k+1}v_{k+2})$ and $\varphi(wv_{j+1}) \neq \varphi(wv_{k+1})$ hold, then there exists a properly coloured cycle $C' = wv_{j+1}v_{j+2}\dots v_kv_jv_{j-1}\dots v_{k+1}$ of length l, which is a contradiction. □

We say a pair of vertices $\{v_{j\pm1}, v_{k\pm1}\}$ *restrict* a chord v_jv_k if it satisfies the condition of Rule 3.3.3, respectively. Naturally, we need to consider whether a chord v_jv_k would be restricted by two pairs of vertices $\{v_{j+1}, v_{k+1}\}$ and $\{v_{j-1}, v_{k-1}\}$. We say such a chord *doubly restricted*.

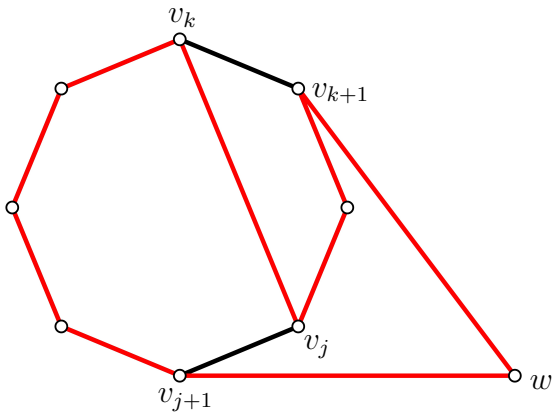


Figure 3.2: The way to extend C if a chord v_jv_k is not restricted

Rule 3.3.4 *Let v_jv_k be a doubly restricted chord restricted by $\{v_{j+1}, v_{k+1}\}$ and $\{v_{j-1}, v_{k-1}\}$. Then we must have $\varphi(v_jv_k) = \varphi(v_jv_{j-1}) = \varphi(v_kv_{k+1})$ or $\varphi(v_jv_k) = \varphi(v_jv_{j+1}) = \varphi(v_kv_{k-1})$.*

Rule 3.3.4 shows that each doubly restricted chord makes two edges on C share a common colour. We call such two edges a *restricted pair*. For instance, let v_jv_k be a doubly restricted chord and furthermore we have $\varphi(v_jv_k) = \varphi(v_jv_{j-1}) = \varphi(v_kv_{k+1})$. Then $\{v_jv_{j-1}, v_kv_{k+1}\}$ is a restricted pair.

We aim to count the number of edges whose colouring are restricted. For the purpose of proof, we partition $V(C)$ in terms of the vertex w as follows:

$$U_0(w) \coloneqq \{v_i \in V(C) : \varphi(wv_i) \in \Phi_2(C)\};$$
$$U_1(w) \coloneqq \{v_i \in V(C) : v_i \text{ belongs to some typical triangle}\};$$
$$U_2(w) \coloneqq \{v_i \in V(C) \setminus U_0(w) : a_j < i < r(a_j) \text{ for some } v_{a_j} \in V_w\};$$
$$U_3(w) \coloneqq \{v_i \in V(C) \setminus U_0(w) : s(a_j) < i < a_j \text{ for some } v_{a_j} \in V_w\};$$
$$U_4(w) \coloneqq V(C) \setminus (U_0(w) \cup U_1(w) \cup U_2(w) \cup U_3(w)).$$

For instance, in the following figure we have

$$v_{a_1}, v_{a_2}, v_{a_3} \in U_0, \quad v_2, v_3, v_5, v_6, v_8, v_9 \in U_1, \quad v_1 \in U_2, \quad v_4 \in U_4.$$

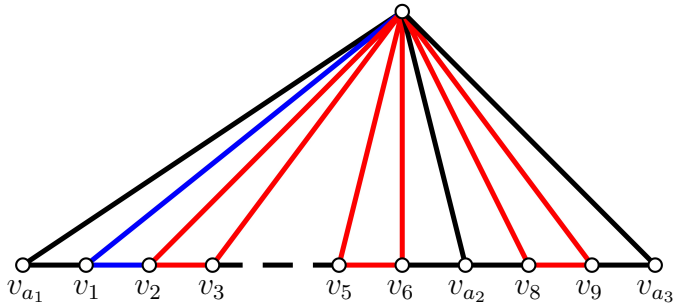


Figure 3.3: Partition of $V(C)$ with respect to typical triangles

We say a vertex $v_j \in V(C)$ to be *potentially restricted* if $\varphi(wv_{j+1}) \neq \varphi(v_{j+1}v_{j+2})$ and $\varphi(wv_{j-1}) \neq \varphi(v_{j-1}v_{j-2})$. Moreover, if a chord whose end-vertices are two potentially restricted vertices v_j, v_k satisfies that $\varphi(wv_{j+1}) \neq \varphi(wv_{k+1})$, $\varphi(wv_{j-1}) \neq \varphi(wv_{k-1})$, then this chord is restricted by two pairs of vertices. For any j with $1 \leq j \leq t$, $v_{r(a_j)}$ and $v_{s(a_j)}$ are both potentially restricted by Rule 3.3.2. For any $v_j \in U_0(w)$, if $v_{j-1}, v_{j+1} \in U_0(w)$, then v_j is potentially restricted; otherwise it is not potentially restrcited. Any vertex of $U_2(w)$ or $U_3(w)$ is not potentially restricted.

Colours restricted in the cycle

Let us denote

$$R'(w) \coloneqq \{\{e, u, v\} \mid e \text{ is restricted by } \{u, v\}\},$$
$$DR'(w) \coloneqq \{e \mid e \text{ is doubly restricted, } e \text{ belongs to 2 elements of } R'(w)\}.$$

Note that when we define $R'(w)$, the pair of vertices corresponding to a restricted chord should also be considered. Hence a doubly restricted chord and its corresponding pairs of vertices are counted twice in $R'(w)$. Furthermore, let $re(w)$ be the repetitive number of colours on $E(C)$, which means that, for instance, if m edges of $E(C)$ share a common colour, then they contribute $m-1$ to $re(w)$. It immediately follows that

$$g(t) \leq \binom{\ell - 1}{2} - |R'(w)| + |DR'(w)| - re(w).$$

Every pair of non-adjacent vertices on C can correspond to elements of $R'(w)$. Let us consider some special subsets of $R'(w)$ for the purpose of counting. Let $R(w)$ be the set of $\{e, u, v\} \in R'(w)$ with

1. $u, v \in V_w$;
2. $u \in V_w$, $v \in U_1(w) \cup U_2(w) \cup U_3(w)$;
3. $u, v \in U_1(w)$;
4. $u \in U_2(w) \cup U_3(w)$, $v \in U_1(w) \cup U_2(w) \cup U_3(w)$;
5. $u \in V_w$, $v \in U_0(w) \setminus V_w$;
6. $u \in V_w$, $v \in U_4(w)$.

We denote them $R_i(w)$ of size $r_i(w)$, where $1 \leq i \leq 6$, respectively.

Let $DR(w)$ be the set of doubly restricted chords that occur twice in $R(w)$. Then any chord $v_jv_k \in DR(w)$ satisfies one of the following:

1. $v_j, v_k \in U_1(w)$;
2. $v_j \in U_0(w)$, $v_k \in U_0(w) \cup U_1(w)$, either $v_{j+1}, v_{k-1} \in V_w$ or $v_{j-1}, v_{k+1} \in V_w$.

We denote them $DR_i(w)$ of size $dr_i(w)$, where $1 \leq i \leq 2$, respectively. We claim that $DR(w)$ does not contain any chord of other types. In fact, every potentially restricted vertex v_j satisfies one of the following:

1. $v_j \in U_1(w)$;
2. $v_{j-1}, v_j, v_{j+1} \in U_0(w)$;
3. $v_{j-1}, v_j, v_{j+1} \in U_4(w)$.

Let v_jv_k be a doubly restricted chord in $DR'(w)$, we prove that the following can not belong to $DR(w)$ and omit some symmetric cases.

- If $v_j \in U_1(w)$, $v_k \in U_4(w)$, and assuming that $v_{j+1} \in U_1(w)$, then $\{v_jv_k, v_{j+1}, v_{k+1}\} \notin R(w)$;

- If $v_j \in U_0(w)$, $v_k \in U_0(w) \cup U_1(w)$, and assuming that $v_{j+1} \notin V_w$, $v_{k+1} \notin V_w$, then $\{v_jv_k, v_{j+1}, v_{k+1}\} \notin R(w)$;
- If $v_j \in U_0(w)$, $v_k \in U_4(w)$, and assuming that $v_{j+1} \notin V_w$, then $\{v_jv_k, v_{j+1}, v_{k+1}\} \notin R(w)$;
- If $v_j, v_k \in U_4(w)$, then $\{v_jv_k, v_{j+1}, v_{k+1}\} \notin R(w)$.

Let us first consider $R_1(w)$ and $R_2(w)$. For any two vertices $v_{a_j}, v_{a_k} \in V_w$, there exist two chords $v_{a_j+1}v_{a_k+1}$ and $v_{a_j-1}v_{a_k-1}$ restricted by them. Hence we have

$$r_1(w) = 2\binom{t}{2}.$$

For any two vertices v_{a_j}, v_k with $v_{a_j} \in V_w$ and $v_k \in U_1(w) \cup U_2(w) \cup U_3(w)$, there exists one chord restricted by them, unless $v_{a_j}v_k \in E(C)$. Hence we have

$$r_2(w) \geq t(2\Delta(w) + |U_2(w)| + |U_3(w)| - 2).$$

For the case of typical triangles, we need to compute the constraints of $R_3(w)$, $DR_1(w)$ and the restricted pairs corresponding to $DR_1(w)$ together. We define the following.

Definition 21. For two typical triangles wv_jv_{j+1} and wv_kv_{k+1}, we let

$$\mathbf{x}(j,k) \coloneqq (x_1, x_2, x_3, x_4)$$

be the *constraint* of these triangles where

- they contribute x_1 to $r_3(w)$;
- they contribute x_2 to $dr_1(w)$;
- there exist x_3 edges among $v_{j-1}v_j, v_jv_{j+1}, v_{j+1}v_{j+2}$ that have the same colour as some edge among $v_{k-1}v_k, v_kv_{k+1}, v_{k+1}v_{k+2}$;
- there exist x_4 edges among $v_{k-1}v_k, v_kv_{k+1}, v_{k+1}v_{k+2}$ that have the same colour as some edge among $v_{j-1}v_j, v_jv_{j+1}, v_{j+1}v_{j+2}$;

Clearly we have $x_1(j,k) = 0$ if wv_jv_{j+1} and wv_kv_{k+1} have the same colour, otherwise $x_1(j,k) = 2$. For other coefficients we simply have $x_2 \leq 4$ and $x_3, x_4 \leq x_2$. We prove the following few claims to characterise the colours with respect to x_2.

Claim 3.3.3 *Let wv_jv_{j+1} and wv_kv_{k+1} be two DT triangles. If they have the same colour then we have $x_2(j,k) \leq 2$. Otherwise we have $x_2(j,k) \leq 3$.*

Proof. When wv_jv_{j+1} and wv_kv_{k+1} have the same colour, let us consider the chord v_jv_k. Clearly we have $\{v_jv_k, v_{j+1}, v_{k+1}\} \notin R_3(w)$ since $\varphi(wv_{j+1}) = \varphi(wv_{k+1})$. And so $v_jv_k \in DR(w)$. For the same reason $v_{j+1}v_{k+1}$ is not doubly restricted. So we have $x_2(j,k) \leq 2$ for this case.

When wv_jv_{j+1} and wv_kv_{k+1} have distinct colours and there are at least 2 doubly restricted chords, let us assume that 2 of them share a common end vertex. Without loss of generality let them be v_jv_k and v_jv_{k+1}. By the properness of C the restricted pairs corresponding to v_jv_k and v_jv_{k+1} must be $\{v_jv_{j+1}, v_kv_{k-1}\}$ and $\{v_jv_{j-1}, v_{k+1}v_{k+2}\}$, respectively. We claim that $v_{j+1}v_k$ can not be doubly restricted, since otherwise it corresponds to a restricted pair $\{v_{j+1}v_j, v_kv_{k+1}\}$ or $\{v_{j+1}v_{j+2}, v_kv_{k-1}\}$, each of which would lead to a contradiction to the properness of C. Hence another doubly restricted chord must be $v_{j+1}v_{k+1}$ and the corresponding restricted pair must be $\{v_{j+1}v_{j+2}, v_{k+1}v_k\}$. Therefore, we have $x_2(j,k) \leq 3$. □

When one of wv_jv_{j+1} and wv_kv_{k+1} is CT or CCT triangle, we have the following claim with some additional constraints.

Claim 3.3.4 *Assume that wv_jv_{j+1} is a CT triangle. If wv_kv_{k+1} is also a CT triangle, then there is at most one chord v_jv_k belonging to $DR(w)$. If wv_kv_{k+1} is a CCT or DT triangle, then the chord $v_{j+1}v_k$ can not belong to $DR(w)$.*

Proof. A chord belongs to $DR(w)$ if and only if it is contained in 2 distinct elements of $R(w)$. By definition, we know that $v_{j+2} \in U_4(w)$ or $v_{j+2} \in V_{CCT}$. If wv_kv_{k+1} is a CT triangle, then we have $\{v_{j+1}v_k, v_{j+2}, v_{k+1}\} \notin R(w)$ by our definition of $R(w)$. Hence $v_{j+1}v_k \notin DR(w)$. Similarly we have $v_jv_{k+1}, v_{j+1}v_{k+1} \notin DR(w)$. And we have $v_{j+1}v_k \notin DR(w)$ if wv_kv_{k+1} is a CCT or DT triangle. □

Now we can give the following two claims which show the relationship among the coordinates of $\mathbf{x}$.

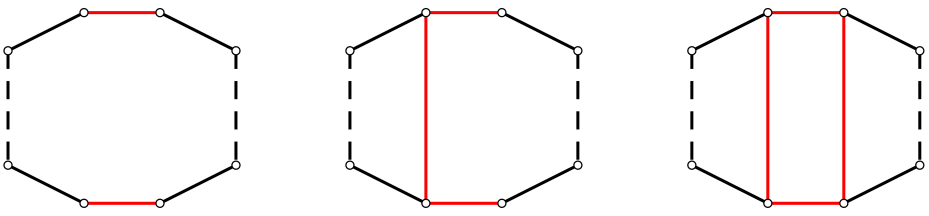
Claim 3.3.5 *Let wv_jv_{j+1} and wv_kv_{k+1} be 2 typical triangles of the same colour.*

- *If they are both CT or CCT triangles, then $x_2(j,k) = 0$;*
- *if they are both DT triangles, then $x_2(j,k) \leq 2$, while the restricted pair of these two chords can be $\{v_jv_{j+1}, v_kv_{k+1}\}$ in the meantime;*
- *otherwise, $x_2(j,k) \leq 1$.*

Claim 3.3.6 *Let wv_jv_{j+1} and wv_kv_{k+1} be 2 typical triangles of distinct colours.*

- *If wv_jv_{j+1} and wv_kv_{k+1} are both CT or CCT triangles, then $x_2(j,k) \leq 1$; otherwise,*
- *if $x_4(j,k) = 0$, then $x_2(j,k) = 0$;*
- *if $x_4(j,k) = 1$, then $x_2(j,k) \leq 2$, and we have $\varphi(v_{j-1}v_j) = \varphi(v_{j+1}v_{j+2})$ if the equality holds;*
- *if $x_4(j,k) = 2$, then $x_2(j,k) = 2$;*
- *if $x_4(j,k) = 3$, then $x_2(j,k) = 3$.*

There are many possible colouring structures for the cases above, where the number of restrictions are different from each other. Here we give a list of figures to show some "best" cases with respect to minimizing the number of restrictions.



$\mathbf{x} = (0,0,1,1)$ $\mathbf{x} = (0,1,1,1)$ $\mathbf{x} = (0,2,1,1)$

Figure 3.4: 2 triangles with the same colour

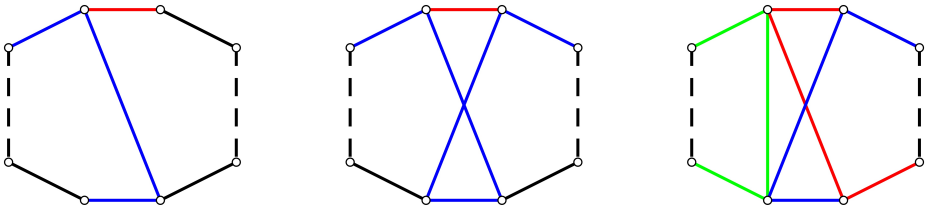


Figure 3.5: 2 triangles with distinct colours

Since the colours of typical triangles highly correspond to $R_3(w)$ and $DR_1(w)$, and noting that $re(w)$ is mainly due to $DR(w)$, we give a bound of $r_3(w) - dr_1(w) + re(w)$ as follows.

Claim 3.3.7 *If all typical triangles have the same colour, then we have*

$$r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + t - 1.$$

If all typical triangles have distinct colours and $\Delta(w) = t$, then we have

$$r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + (3t - 2).$$

Otherwise, we have

$$r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + (\Delta(w) + t - 1).$$

Proof. When all typical triangles are assigned the same colour, there are $(\Delta(w) - t)$ CT triangles, $(\Delta(w) - t)$ CCT triangles and $(2t - \Delta(w))$ DT triangles. Hence we have

$$r_3(w) = 0,$$
$$dr_1(w) \leq 2\binom{t}{2} + \Delta(w) - t,$$
$$re(w) \geq \Delta(w) - 1.$$

The bound follows and the equality holds by taking $\mathbf{x}$ as following:

- If v_j, v_k are both in V_{CT} or V_{CCT}, then $\mathbf{x}(j, k) = (0, 0, 1, 1)$;

- if $v_j, v_k \in V_{DT}$, then $\mathbf{x}(j,k) = (0,2,1,1)$;
- otherwise, $\mathbf{x}(j,k) = (0,1,1,1)$.

When all typical triangles are assigned the same colour, in order to compute the value of $r_3(w) - dr_1(w) + re(w)$ in this situation, let us first consider a single pair of typical triangles. Specifically, we consider the difference between the case when two typical triangles have distinct colours and that of the same colour.

1. wv_jv_{j+1} and wv_kv_{k+1} are both CT or CCT triangles. We take $\mathbf{x} = (2,1,1,1)$ here while $\mathbf{x} = (0,0,1,1)$ for the former case. The difference of $x_1 - x_2$ is 1.
2. wv_jv_{j+1} and wv_kv_{k+1} are both DT triangles. We take $\mathbf{x} = (2,2,2,1)$ or $\mathbf{x} = (2,3,3,3)$ here while $\mathbf{x} = (0,2,1,1)$ for the former case. The difference of $x_1 - x_2$ is 2 or 1, respectively.
3. wv_jv_{j+1} and wv_kv_{k+1} are other types of typical triangles. We take $\mathbf{x} = (2,2,2,1)$ or $\mathbf{x} = (2,3,3,3)$ here while $\mathbf{x} = (0,1,1,1)$ for the former case. The difference of $x_1 - x_2$ is 1 or 0, respectively.

It holds that for each pair of typical triangles, the case of the same colour always leads to less constraints (with respect to the value of $x_1 - x_2$, x_3 and x_4). Hence, $r_3(w) - dr_1(w) + re(w)$ reaches the minimum value only if all triangles share a common colour. Let wv_iv_{i+1} be the unique triangle with distinct colour. If $v_i \in V_{DT}$, then there exist at most $2\binom{t}{2} + \Delta(w) - t - 2(t-1)$ doubly restricted chords among other typical triangles. Hence we have

$$r_3(w) = 2(\Delta(w) - 1),$$

$$dr_1(w) \leq 2\binom{t}{2} + \Delta(w) - (3t-2) + \sum_{k \neq i} x_2(i,k),$$

$$re(w) \geq \sum_{k \neq i} x_4(i,k) + \tau,$$

where $\tau = 1$ if there exist v_k such that $\mathbf{x}(i,k) = (2,2,2,1)$, otherwise $\tau = 0$. So we have

$$\begin{aligned} r_3(w) - dr_1(w) + re(w) \geq & -2\binom{t}{2} + (3t-2) \\ & + \Delta(w) + \sum_{k \neq i}(x_4(i,k) - x_2(i,k)) - 2 + \tau \\ \geq & -2\binom{t}{2} + (3t-2), \end{aligned}$$

where the equality holds only if $\mathbf{x}(i,k) = (2,2,2,1)$ for every $k \neq i$.

If $v_i \in V_{CT}$, then there exist at most $2\binom{t}{2} + \Delta(w) - t - t$ doubly restricted chords among other typical triangles. Hence we have

$$r_3(w) = 2(\Delta(w) - 1),$$

$$dr_1(w) \leq 2\binom{t}{2} + \Delta(w) - 2t + \sum_{k \neq i} x_2(i,k),$$

$$re(w) \geq \sum_{k \neq i} x_4(i,k) + \tau,$$

where $\tau = 1$ if there exist v_k such that $\mathbf{x}(i,k) = (2,2,2,1)$, otherwise $\tau = 0$. So we have

$$\begin{aligned} r_3(w) - dr_1(w) + re(w) \geq & -2\binom{t}{2} + 2t \\ & + \Delta(w) + \sum_{k \neq i}(x_4(i,k) - x_2(i,k)) - 2 + \tau \\ \geq & -2\binom{t}{2} + (\Delta(w) + t - 1), \end{aligned}$$

where the equality holds only if $\mathbf{x}(i,k) = (2,2,2,1)$ for every $k \in V_{CCT} \cup V_{DT}$. □

We could give a rough bound of $g(t)$ so far. Specially when $\Delta(w) = t$ and all

typical triangles have the same colour, we have

$$\begin{aligned} g(t) \leq & \binom{\ell-1}{2} - r_1(w) - r_2(w) - (r_3(w) - dr_1(w) + re(w)) \\ \leq & \binom{\ell-1}{2} - 2\binom{t}{2} - t(2t + |U_2(w)| + |U_3(w)| - 2) \\ & - (r_3(w) - dr_1(w) + re(w)) \\ \leq & \binom{\ell-1}{2} - (2t^2 - t - 1). \end{aligned}$$

In the leftover part of counting, we mainly focus on those chords restricted by a vertex in $V_w \cup U_1(w)$ and a vertex out of them. We give a specific bound when $t = \lfloor \frac{\ell-1}{3} \rfloor$.

Claim 3.3.8 *If $t = \lfloor \frac{\ell-1}{3} \rfloor$, then we have $|\Phi| \leq \lfloor \frac{\ell-1}{3} \rfloor n - \binom{\lfloor \frac{\ell-1}{3} \rfloor + 1}{2} + 1 + r_{\ell-1}$.*

Proof. Since almost all vertices of C are in $U_0(w) \cup U_1(w)$, we first state that all typical triangles must be the same colour except for some special cases. If there exists a typical triangle wv_jv_{j+1} such that $v_{j-1}, v_{j+2} \in U_0(w)$, then for any typical triangle wv_kv_{k+1} of different colour, one of the following statements holds.

1. One of v_k or v_{k+1} is not potentially restricted.
2. Both v_k and v_{k+1} are potentially restricted. At least one of wv_{k-1} or wv_{k+2} has the same colour as wv_jv_{j+1}.

If neither hold, then there are 4 doubly restricted chords, which is a contradiction.

Case 1. $\ell - 1 = 3t$.

In this case, we have that $V(C) = U_0(w) \cup U_1(w)$, and all typical triangles have the same colour. Hence we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) = \frac{\ell-1}{3}n - \binom{\frac{\ell-1}{3} + 1}{2} + 1.$$

Case 2. $\ell - 1 = 3t + 1$.

The size of $U_0(w)$ could be two cases.

2.1. $|U_0(w)| = |V_w| + 1$.

We have that $V(C) = U_0(w) \cup U_1(w)$, and all typical triangles have the same colour. There exist two edges wv_j and wv_{j+1} coloured by the same colour of $\Phi_2(C)$. Assume that $v_{j+1} \notin V_w$, without loss of generality. Then we have $r_2(w) + r_5(w) = t(2t-2) + 2t$ and $dr_2(w) = 0$. And so

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - 2t.$$

2.2. $U_0(w) = V_w$.

There exists a vertex v_j such that $V(C)\backslash(U_0(w) \cup U_1(w)) = \{v_j\}$. Without loss of generality, let $v_j \in U_2(w)$, and $wv_{j+1}v_{j+2}$ is a DT triangle. The colouring of all typical triangles could be separated into two cases.

2.2.1 All typical triangles have the same colour.

We have $r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + t - 1$, $r_2(w) = t(2t-1)$ and $r_4(w) = t - 1$. And so

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (2t - 1).$$

2.2.2 All typical triangles have the same colour except $wv_{j+1}v_{j+2}$.

We have $r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + 3t - 2$ and $r_2(w) = t(2t-1)$. And so

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (3t - 1).$$

Therefore, we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (2t - 1) = \frac{\ell-2}{3}n - \binom{\frac{\ell-2}{3}+1}{2} + 2.$$

Case 3. $l - 1 = 3t + 2$.

The size of $U_0(w)$ could be 3 cases.

3.1. $|U_0(w)| = |V_w| + 2$.

We have that $V(C) = U_0(w) \cup U_1(w)$, and all typical triangles have the same colour. Now, we need to see the positions of the vertices of $U_0(w)\backslash V_w$.

3.1.1. There exist three vertices $v_j, v_{j+1}, v_{j+2} \in U_0(w)$.
Let us assume that $v_{j+1} \in V_w$, which is reasonable since we have that $r(w) - dr(w) \leq |R'(w)| - |DR'(w)|$ for any $R(w) \subset R'(w)$. Then we have $r_2(w) + r_5(w) = t(2t-2) + 4t$ and $dr_2(w) = 0$. And so

$$|\Phi| \leq \binom{\ell - 1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - 4t.$$

3.1.2. There exist two pairs of vertices $v_j, v_{j+1} \in U_0(w)$ and $v_k, v_{k+1} \in U_0(w)$ with $|j - k| \geq 2$.
We have $r_2(w) + r_5(w) = t(2t-2) + 4t$ and $dr_2(w) = 0$. And so

$$|\Phi| \leq \binom{\ell - 1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - 4t.$$

3.2. $|U_0(w)| = |V_w| + 1$
There exist 2 vertices v_j, v_k such that $v_j \in U_0(w) \setminus V_w$ and $v_k \in U_2(w) \cup U_3(w)$. Then we have $r_2(w) + r_5(w) = t(2t-2) + 4t$. And so

$$|\Phi| \leq \binom{\ell - 1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (2t - 1) - (4t - 1).$$

3.3. $U_0(w) = V_w$
Now we have $|V(C) \backslash U_0(w)| = 2t + 2$, which means that the number of typical triangles $\Delta(w) = t$ or $\Delta(w) = t + 1$. We first consider when $\Delta(w) = t$ and all typical triangles have the same colour. Then there exist two vertices v_j and v_k such that $\{v_j, v_k\} = V(C) \backslash (U_0(w) \cup U_1(w))$. The type of v_j and v_k could be varied into several subcases.
3.3.1. $v_j \in U_2(w)$, $v_k \in U_3(w)$.
We have $r_2(w) = t \cdot 2t$ and $r_4(w) = 2t - 2$. If $k = j + 3$, then consider chords $v_{j-1}v_{j+2}$ and v_jv_{k+2}, which are not restricted. If they are assigned distinct colours from $\Phi_0(C)$, then we have that

$$v_{j-1}, v_{j+2}v_{j+1}v_j, v_{k+2}v_{k+1}v_k, w, v_{k+3}v_{k+4} \dots v_{j-1}$$

is properly coloured, which shows that C can be properly coloured extended by w, a contradiction. If $k \neq j + 3$, then consider chords

$v_{j-1}v_{j+2}$, v_jv_k, and $v_{k-1}v_{k+1}$. These 3 chords are not restricted. If they are assigned distinct colours from $\Phi_0(C)$, then we have that

$$v_{j-1}, v_{j+2}v_{j+1}v_j, v_k, w, v_{j+3}v_{j+4}\dots v_{k-2}v_{k-1}, v_{k+1}v_{k+2}\dots v_{j-1}$$

is properly coloured, which shows that C can be properly coloured extended by w, a contradiction. Hence we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2-t-1) + t(n-\ell+1) - (2t+(2t-2)+1).$$

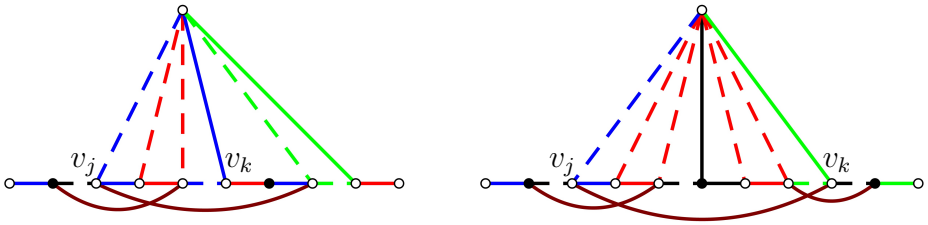


Figure 3.6: Case 3.3.1

3.3.2. $v_j, v_k \in U_2(w)$ or $v_j, v_k \in U_3(w)$, both wv_j and wv_k have different colours from those typical triangles.

We have $r_2(w) = t \cdot 2t$. If $\varphi(wv_j) \neq \varphi(wv_k)$, then we have $r_4(w) = 2t-1$ and $re(w) \geq t-1$. If $\varphi(wv_j) = \varphi(wv_k)$, then we have $r_4(w) = 2t-2$ and $re(w) \geq t$. Hence we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2-t-1) + t(n-\ell+1) - (4t-1).$$

3.3.3. $v_j, v_k \in U_2(w)$ or $v_j, v_k \in U_3(w)$, wv_j is assigned the same colour as those typical triangles.

Consider $v_j, v_k \in U_2(w)$. By Rule 3.3.1, it holds that $j = k-1$ and $wv_{j+2}v_{j+3}$ is a typical triangle. So we have $r_2(w) = t \cdot 2t$, $r_4(w) = t-1$ and $re(w) \geq t$. For any typical triangle wv_iv_{i+1} with $i \neq j+2$ and $i \neq j-3$, consider chords v_kv_{i+1}, v_jv_{i+2} and $v_{j-1}v_i$. These three chords are not in $R(w)$. If they are assigned distinct colours from $\Phi_0(C)$, then we have that

$$v_{j-1}, v_iv_{i+1}, v_kv_{k+1}\dots v_{i-1}, w, v_j, v_{i+2}v_{i+3}\dots v_{j-1}$$

is properly coloured, which shows that C can be properly extended by w, a contradiction. Specially when $i = j - 3$, consider two chords $v_k v_{i+1}$ and $v_{j-1}v_i$. If they are assigned distinct colours from $\Phi_0(C)$, then we have that

$$v_{i-1}, w, v_j v_{j-1}, v_i v_{i+1}, v_k v_{k+1} \dots v_{i-1}$$

is properly coloured, which shows that C can be properly extended by w, a contradiction. Moreover, these $t-1$ tuples of chords are pairwise disjoint for all $i \neq j+2$. Hence we have

$$|\Phi| \leq \binom{\ell - 1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (2t + (t-1) + 1 + (t-1)).$$

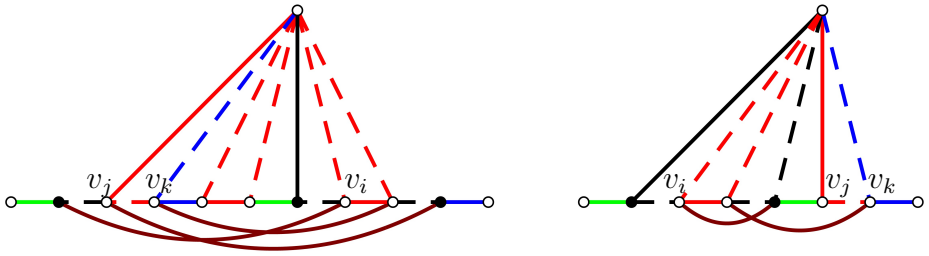


Figure 3.7: Case 3.3.3

3.3.4. $\Delta(w) = t$ and not all typical triangles have the same colour.
We have that $r_3(w) - dr_1(w) + re(w) \geq -2\binom{t}{2} + (3t-2)$ by Claim 3.3.7. Also, there exist two vertices v_j and v_k belong to $U_2(w) \cup U_3(w)$. We have $r_2(w) = t \cdot 2t$. And so

$$|\Phi| \leq \binom{\ell - 1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - ((2t-1) + 2t).$$

3.3.5. $\Delta(w) = t + 1$ and all typical triangles have the same colour.
There exists v_j such that wv_jv_{j+1} and $wv_{j+2}v_{j+3}$ are typical triangles, where v_{j+1} and v_{j+2} are not potentially restricted. We have $r_2(w) = t \cdot 2t$, $r_3(w) = 0$, $dr_1(w) = 2\binom{t}{2} + 1$, and $re(w) \geq t$. For those non-restricted chords, there are $2(t-1)$ triples of chords that can not be coloured without restriction, by a similar argument to case 3.3.3, where these

triples are pairwise disjoint. Moreover, consider two non-restricted chords v_jv_{j+2} and $v_{j+1}v_{j+3}$. If these two chords have distinct colours from C, then C can be properly extended to

$$v_{j-1}v_j, v_{j+2}v_{j+1}, v_{j+3}, w, v_{j+4}v_{j+5}\dots v_{j-1}.$$

Hence we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2-t-1) + t(n-\ell+1) - (2t-1+1+2(t-2)+1).$$

3.3.6. $\Delta(w) = t+1$ and not all typical triangles have the same colour. There exists v_j such that wv_jv_{j+1} and $wv_{j+2}v_{j+3}$ are typical triangles, where v_{j+1} and v_{j+2} are not potentially restricted. Note that all DT triangles must have the same colour. Let us denote the colour of these DT triangles by major colour, even though there may exist only one DT triangle.

Suppose that wv_jv_{j+1} has a distinct colour from the major colour. Then for any typical triangle wv_kv_{k+1} with $k \neq j$ and $k \neq j+2$, we have that v_jv_k and v_jv_{k+1} are doubly restricted, and so we have $\varphi(v_jv_k) = \varphi(v_jv_{j+1}) = \varphi(v_{k-1}v_k)$ and $\varphi(v_jv_{k+1}) = \varphi(v_{j-1}v_j) = \varphi(v_{k+1}v_{k+2})$. Now the colour of $wv_{j+2}v_{j+3}$ can not be same as wv_jv_{j+1}, otherwise we have that $\varphi(v_{j+2}v_{j+3}) = \varphi(v_{k+1}v_{k+2}) = \varphi(v_{j-1}v_j)$, which contradicts to that C is properly coloured. Since v_jv_{j+3} is doubly restricted, we have $\varphi(v_jv_{j+3}) = \varphi(v_{j-1}v_j) = \varphi(v_{j+3}v_{j+4})$.

If $wv_{j+2}v_{j+3}$ has the major colour, then we have $r_3(w) = 2t$, $dr_1(w) = 2\binom{t-1}{2} + 2(t-1) + (t-1) + 1$, and $re(w) \geq 2(t-1)+1$. Hence we have

$$g(t) \leq \binom{\ell-1}{2} - 2\binom{t}{2} - 2t((t+1)-1) - (-2\binom{t-1}{2} + t + 1).$$

If $wv_{j+2}v_{j+3}$ has the distinct colour from the major colour and wv_jv_{j+1}, then we have $r_3(w) = 4t-2$, $dr_1(w) = 2\binom{t-1}{2} + 4(t-1) + 1$, and $re(w) \geq 2(t-1)+2$. Hence we have

$$g(t) \leq \binom{\ell-1}{2} - 2\binom{t}{2} - 2t((t+1)-1) - (-2\binom{t-1}{2} + 2t + 1).$$

And so

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - 4t.$$

Summarizing all the cases of $l - 1 = 3t + 2$ above, we have

$$|\Phi| \leq \binom{\ell-1}{2} - (2t^2 - t - 1) + t(n - \ell + 1) - (4t - 1) = \frac{\ell - 3}{3}n - \binom{\frac{\ell-3}{3}+1}{2} + 3.$$

□

We give an upper bound when $2 \leq t \leq \lfloor \frac{\ell-1}{3} \rfloor - 1$.

Claim 3.3.9 *If $2 \leq t \leq \lfloor \frac{\ell-1}{3} \rfloor - 1$ and $\ell \geq 10$, then we have $g(t) \leq \binom{\ell-1}{2} - (2t^2 - t - 1) - t(\ell - 3t) + 1$.*

Proof. By the previous argument we have

$$\begin{aligned} & r_1(w) + r_2(w) + r_3(w) - dr_1(w) + re(w) \\ \geq & (2t\Delta(w) - t - 1) + t(|U_2(w)| + |U_3(w)|) \\ = & (2t^2 - t - 1) + t(2(\Delta(w) - t) + |U_2(w)| + |U_3(w)|) \end{aligned}$$

if all typical triangles are monochromatic. Otherwise we can find at least $\Delta(w)$ more restrictions. It is expected that every vertex of $U_0(w) \setminus V_w$ and $U_4(w)$ restricts t more chords with V_w. Hence it suffices to find $(t-1)$ more restrictions.

We first consider $r_5(w) - dr_2(w)$. In fact, for any chord $v_j v_k \in DR_2(w)$, it satisfies that $v_{j\pm1} \in V_w$, $v_{j\mp1} \in U_0(w) \setminus V_w$ and $v_{k\mp1} \in V_w$, respectively. Hence for any potentially restricted vertex $v \in U_0(w)$, there are at most t doubly restricted chords in $DR(w)$ incident to v. So we have

$$r_5(w) - dr_2(w) \geq (t+1)(|U_0(w)| - |V_w|) + (t-1).$$

Clearly we have $r_6(w) = t \cdot |U_4(w)|$. Hence, if the typical triangles have distinct colours or $U_0(w) \neq V_w$, we have

$$r(w) - dr(w) + re(w) \geq (2t^2 - t - 1) + t(\ell - 1 - 3t) + t.$$

Let us assume that all typical triangles have the same colour and $U_0(w) = V_w$. If $U_2(w)$ or $U_3(w)$ is not empty, then we have $r_4(w) \geq t$, which is sufficient to prove the bound. So let us further assume that $U_2(w) \cup U_3(w) = \emptyset$.

Now we have $U_4(w) \neq \emptyset$. There exists a segment that contains CT and CCT triangles $v_j \in V_{CT}$ and $v_k \in V_{CCT}$. Consider the vertices of $U_4(w)$ in this segment. If there is a vertex v_i such that $\varphi(wv_m) \neq \varphi(v_m v_{m+1})$ and $\varphi(wv_m) \neq \varphi(v_m v_{m-1})$, then v_m and V_w contribute $2t$ to $R_6(w)$, which is enough to prove the bound. Hence let us assume that such a vertex does not exist. In this case there exists a vertex v_m satisfying that:

- $j < m < k$,
- $\varphi(wv_m) = \varphi(v_m v_{m-1})$ and
- $\varphi(wv_{m+1}) = \varphi(v_{m+1} v_{m+2})$.

For any other segment with typical triangles $wv_i v_{i+1}$ and $wv_{i'} v_{i'+1}$ (can be the same DT one), where $i \leq i'$ and $i \neq j$. Consider two chords $v_m v_{i'+1}$ and $v_{m+1} v_i$. They are not in $R(w)$. If they are assigned distinct colours from $\Phi_0(C)$, then we have that

$$v_m, v_{i'+1} v_{i'} \ldots v_i, v_{m+1} v_{m+2} \ldots v_{i-1}, w, v_{i'+2} v_{i'+3} \ldots v_m$$

is a properly colored cycle of length ℓ, a contradiction. Hence we can find $t-1$ pairs of chords that are pairwise disjoint, in which there are $t-1$ more restrictions.

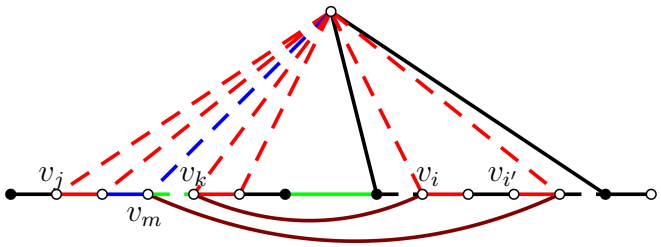


Figure 3.8: the case when $m = j + 2 = k - 1$

The bound follows. □

Now we can give the upper bound of $|\Phi|$ by Claim 3.3.8 and Claim 3.3.9. Define

$$h(t) = \begin{cases} (n-\ell+1)+\binom{\ell-1}{2} & t=1; \\ t(n-\ell+1)+\binom{\ell-1}{2}-(2t^2-t-1)-t(\ell-3t)+1 & 2\le t\le \lfloor\frac{\ell-1}{3}\rfloor-1,\ \ell\ge 10; \\ t(n-\ell+1)+\binom{\ell-1}{2}-(2t^2-t-1) & 3t=\ell-1; \\ t(n-\ell+1)+\binom{\ell-1}{2}-(2t^2-t-1)-(2t-1) & 3t=\ell-2; \\ t(n-\ell+1)+\binom{\ell-1}{2}-(2t^2-t-1)-(4t-1) & 3t=\ell-3. \end{cases}$$

If $2 \le t \le \lfloor\frac{\ell-1}{3}\rfloor - 1$, then $h(t)$ reaches the maximum value when $t = 2$ or $t = \lfloor\frac{\ell-1}{3}\rfloor - 1$. When $\ell \le \frac{3n}{5}$, we always have $h(\lfloor\frac{\ell-1}{3}\rfloor - 1) \ge h(2)$ by the property of quadratic function. In this case, we have

$$h\left(\left\lfloor\frac{\ell-1}{3}\right\rfloor\right) - h\left(\left\lfloor\frac{\ell-1}{3}\right\rfloor - 1\right) \ge h\left(\frac{\ell-3}{3}\right) - h\left(\frac{\ell-3}{3}-1\right) = n - \frac{5\ell}{3} \ge 0.$$

When $\ell > \frac{3n}{5}$, $2 \le t \le \lfloor\frac{\ell-1}{3}\rfloor - 1$, we have

$$h(1) - h(t) = -t^2 - (n-2\ell+2)t + (n-\ell-1) > 0$$

since $\ell \ge 10$. Therefore, we have

$$|\Phi| \le \max\left\{h(1), h\left(\left\lfloor\frac{\ell-1}{3}\right\rfloor\right)\right\} = f(n,\ell),$$

which is a contradiction. We finish the proof of Lemma 3.3.2.

Remark 6. We did not get the exact upper bound in Claim 3.3.9. However, it is enough to get a less number of colours to compare to the cases when t get largest or smallest. We conjecture that it reaches the maximum when the assignment of colours is one of the extremal structure Lemma 3.2.1.

4 - On sufficient conditions for weakly proper spanning trees in edge-coloured graphs

4.1 . Introduction

In 2019, Borozan et al. [8] first studied the notion of weakly proper tree. Inspired by it, we are interested in sufficient conditions for edge-coloured graphs to have weakly proper spanning trees. There are two types of problems:

(I) For an edge-coloured graph G, find sufficient conditions forcing a weakly proper spanning tree with root r for any vertex $r \in V(G)$.

(II) For an edge-coloured graph G, find sufficient conditions forcing a weakly proper spanning tree with root r for some vertex $r \in V(G)$.

In general, Dirac-type problems in edge-coloured graphs ask for the minimum colour degree threshold, namely, the smallest possible minimum colour degree condition that ensures certain properly coloured (spanning) subgraphs in an edge-coloured graph. To begin with, we study the Dirac-type problem for weakly proper spanning tree, i.e., find a minimum colour degree condition for an edge-coloured graph G to have a weakly proper spanning tree with root r for any vertex $r \in V(G)$. Our first main result is a sufficient condition for type **(II)**.

Theorem 4.1.1 *For $\varepsilon > 0$, there exist an integer n_0 such that every connected edge-coloured graph G on $n \geq n_0$ vertices with $\delta^c(G) \geq (1/3 + \varepsilon)n$, there is a vertex $r \in V(G)$, and G contains a weakly proper spanning tree rooted at r.*

For the problem of type **(I)**, by Theorem 1.2.41 and Theorem 1.2.42, there is no difference between the minimum colour degree condition forcing a properly coloured spanning tree and the one forcing a weakly proper spanning tree with root r for any $r \in V(G)$.

Note that if an edge-coloured graph G has a weakly proper spanning tree with root r for any $r \in V(G)$, then G is properly connected. But the converse is

not true, i.e., there is a properly connected graph G which has no weakly proper spanning tree with root r for some $r \in V(G)$. In fact, the edge-coloured graph H in Figure 4.1 is properly connected but has no weakly proper spanning tree with root r.

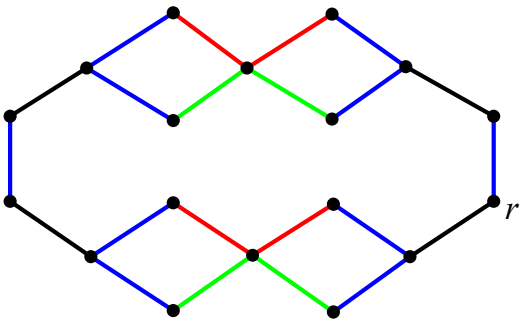


Figure 4.1: The edge-coloured graph H.

We further study some other sufficient conditions forcing weakly proper spanning trees, and obtain Theorems 4.1.2, 4.1.3 and 4.1.4, where the conditions in Theorems 4.1.2 and 4.1.4 cannot guarantee properly coloured spanning trees.

Theorem 4.1.2 *Let $k \geq 3$ be an integer and G be an edge-coloured connected graph containing a properly coloured cycle C_k. If every properly coloured P_k is contained in a properly coloured C_k, then G has a weakly proper spanning tree with root r for any $r \in V(G)$.*

If $k = 2$, then the condition in Theorem 4.1.2 is that every edge is contained in a properly coloured C_2, which means G is a multigraph, and it is easy to find a weakly proper spanning tree with root r for any vertex $r \in V(G)$. As mentioned above, we only consider simple graphs. For $k = 2$, we try to modify the condition as follows: Every edge is contained in a properly coloured C_3. However, we obtain the following result showing that this condition cannot guarantee a weakly proper spanning tree with root r for any $r \in V(G)$.

Theorem 4.1.3 *There are infinitely many edge-coloured graphs G such that $\delta^c(G) \geq 2$ and every edge is contained in a properly coloured C_3, but G has no weakly proper spanning tree with root r for some $r \in V(G)$.*

Inspired by the condition "every edge is contained in a properly coloured C_3", we further consider "every properly coloured P_3 is contained in a properly coloured C_4", and obtain the following theorem.

Theorem 4.1.4 *Let G be an edge-coloured connected graph with $\delta^c(G) \geq 2$. Suppose that G satisfies the following two conditions:*

1. *Every properly coloured P_3 is contained in a properly coloured C_4.*
2. *For every $K_{1,4}$ with four edges e_1, e_2, e_3, e_4 and $c(e_1) = c(e_2) \neq c(e_3) = c(e_4)$, either $c(x_1x_2) = c(e_1)$ or $c(x_3x_4) = c(e_3)$, where x_i is the vertex incident with e_i and not the center of $K_{1,4}$ for any $1 \leq i \leq 4$.*

Then G has a weakly proper spanning tree with root r for any vertex $r \in V(G)$.

There exist many edge-coloured graphs which satisfy conditions of Theorem 4.1.2 or 4.1.4 but do not have properly coloured spanning trees. We give two examples here.

Example 4.1.1 Let $k \geq 4$ be an integer and $C = v_1v_2 \dots v_kv_1$ be a properly coloured cycle. Let $l > 6$ be an integer and K_l be a monochromatic complete graph with $V(K_l) = \{u_1, u_2, \dots, u_l\}$ so that the colour appearing in K_l is $c(v_1v_2)$. Let G be an edge-coloured graph obtained from C and K_l by adding new edges v_1u_i, v_2u_i, v_3u_i and v_ku_i with $c(v_1u_i) = c(v_2u_i) = c(v_1v_2)$, $c(v_3u_i) = c(v_2v_3)$ and $c(v_ku_i) = c(v_1v_k)$ for any $1 \leq i \leq l$. Then G satisfies the conditions in Theorem 4.1.2 but has no properly coloured spanning tree. This is because there are at most 6 vertices in $V(K_l)$ which can be contained in a properly coloured tree of G.

Example 4.1.2 Let $l \geq 3$ and k be positive integers, K_l be a properly coloured complete graph and $V(K_l) = \{u_1, u_2, \dots, u_l\}$. Let $v_1, v_2, \dots, v_{l+k}$ be distinct vertices not belonging to $V(K_l)$. Let $c_1, c_2, \dots, c_l$ be distinct colours such that for each $1 \leq i \leq l$, c_i does not appear on edges incident with u_i in K_l. Basing on the properly coloured K_l and $\{v_1, v_2, \dots, v_{l+k}\}$, we construct an edge-coloured graph G by

adding a new edge u_iv_j and colouring it with c_i for every $1 \leq i \leq l$ and $1 \leq j \leq l+k$. Then G satisfies the conditions in Theorem 4.1.4. However, for any properly coloured tree T in G, the number of edges between K_l and $\{v_1, v_2, \ldots, v_{l+k}\}$ contained in T is at most l. Hence, $|T| \leq 2l$ and G does not have properly coloured spanning trees.

4.2 . Finding WPST for some root

Here we present the proof of Theorem 4.1.1, which naturally splits into two lemmas. First, we consider the case when G has a low connectivity and have the following lemma.

Lemma 4.2.1 *For $\varepsilon > 0$ and a graph G on n vertices with $\delta^c(G) \geq (1/3 + \varepsilon)n$, if G is not $\varepsilon n/2$-connected, then G has a properly coloured spanning tree.*

Proof. Since $\kappa(G) < \varepsilon n/2$, there exists a vertex cut $W \subset V(G)$ of order $k = \kappa(G)$. Let $W = \{v_1, \ldots, v_\kappa\}$. Noting that $\delta^c(G) \geq (1/3 + \varepsilon)n$, $G - W$ has exactly two connected components of vertex set W_1 and W_2, each of which has at least $(1/3 + \varepsilon/2)n$ vertices. By the minimality of W each vertex $v \in W$ has neighbours both in W_1 and W_2. Hence there exists a vertex subset $U = \{w, u_1, u_2, \ldots, u_\kappa\}$ of size $\kappa + 1$ in $G - W$ satisfies:

(i) w and u_1 are in the distinct connected components in $G - W$,

(ii) $v_1w \in E(G)$,

(iii) $v_iu_i \in E(G)$ for $1 \leq i \leq \kappa$.

Let Σ be the set of colours of edges v_1w and v_iu_i for $1 \leq i \leq \kappa$. Σ has size at most $\kappa + 1$. Then we remove all edges in $G - W$ whose colour is in Σ, and let the G' be the graph. Now each component of G' has at most $(2/3 - \varepsilon)n$ vertices and minimum colour degree at least $n/3$. By Theorem 1.2.42 each component has a properly coloured spanning tree. Combined with edges v_1w and v_iu_i for $1 \leq i \leq \kappa$, we can get a properly coloured spanning tree of G. □

Hence Lemma 4.2.1 implies that Theorem 4.1.1 holds if $\kappa(G) < \varepsilon n$. If G has a larger connectivity, then we proceed using the *absorption technique* introduced by Rödl, Ruciński and Szemerédi [73]. First we find a small "absorbing tree" T in G using the following lemma.

Lemma 4.2.2 *Let $0 < 1/n \ll \varepsilon < 1/2$. Suppose that G is a $\varepsilon n/2$-connected edge-coloured graph on n vertices with $\delta^c(G) \geq (1/3 + \varepsilon)n$. Then there exists a weakly proper tree T rooted at some vertex $r \in V(G)$ of order at most $\varepsilon n/2$ such that for any vertex subset $U \subset V(G)$, there exists a weakly proper tree rooted at r with vertex set $V(T) \cup U$.*

Theorem 4.1.1 follows from these two lemmas immediately.

Proof. Let G be the graph on n vertices with $\delta^c(G) \geq (1/3 + \varepsilon)n$ such that $1/n \ll \varepsilon < 1/2$. If G is not $\varepsilon n/2$-connected, then by Lemma 4.2.1 G has a properly coloured spanning tree, and hence has a weakly proper spanning tree for any root $r \in V(G)$. Otherwise, such a root r and the weakly proper spanning tree rooted at r can be found by Lemma 4.2.2. □

4.2.1 . Absorbing trees

In this section, we prove Lemma 4.2.2. We need the following definition. Given a vertex x, we say that a path P is an *absorbing path for* x if the following conditions hold:

(i) $P = z_1 z_2 z_3$ is a properly coloured path of length 2;

(ii) $x \notin V(P)$;

(iii) the colour of xz_2 is distinct from $z_1 z_2$ and $z_2 z_3$.

Given a vertex x, let $\mathcal{L}(x)$ be the set of absorbing paths for x. Since $\delta^c(G) \geq (1/3 + \varepsilon)n$, clearly we have $|\mathcal{L}(x)| \geq n^3/27$ for every x. Lemma 4.2.4 is proved by a simple probabilistic argument since each of $\mathcal{L}(x)$ is large. We will need the following Chernoff's bound for the binomial distribution.

Lemma 4.2.3 (Chernoff's bound) *Suppose that $\mathbf{X}$ has the binomial distribution and $0 < a < 3/2$. Then*

$$\mathbb{P}\left[|\mathbf{X} - \mathbb{E}\left[\mathbf{X}\right]| \geq a\mathbb{E}\left[\mathbf{X}\right]\right] \leq 2e^{-a^2\mathbb{E}[\mathbf{X}]/3}.$$

Lemma 4.2.4 *Let $0 < \gamma < 1$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Let G be an edge-coloured graph on n vertices. Suppose that $|\mathcal{L}(x)| \geq \gamma n^3$ for every $x \in V(G)$. Then there exists a family $\mathcal{F}$ of vertex-disjoint properly coloured paths each of length 2, which satisfies the following properties:*

$$|\mathcal{F}| \leq 2^{-5}\gamma n, \qquad |\mathcal{L}(x) \cap \mathcal{F}| \geq 2^{-8}\gamma^2 n$$

for all $x \in V(G)$.

Proof. Choose $n_0 \in \mathbb{N}$ large so that

$$\exp(-\gamma n_0/(3 \times 2^6)) + n_0 \exp(-\gamma^2 n_0/(3 \times 2^8)) \leq 1/6. \tag{4.1}$$

Recall that each path is assumed to be connected. So a path $z_1z_2z_3$ will be considered as a triple (z_1, z_2, z_3). Choose a family $\mathcal{F}'$ of triples in $V(G)$ by selecting each of the $n!/(n-3)!$ possible triples independently at random with probability

$$p = 2^{-6}\gamma \frac{(n-3)!}{(n-1)!} \geq 2^{-6}\gamma n^{-2}.$$

Notice that

$$\mathbb{E}\left[|\mathcal{F}'|\right] = p\frac{n!}{(n-3)!} = 2^{-6}\gamma n,$$
$$\mathbb{E}\left[|\mathcal{L}(x) \cap \mathcal{F}'|\right] = p|\mathcal{L}(x)| \geq 2^{-6}\gamma^2 n$$

for every $x \in V(G)$. Then by Lemma 4.2.3, the union bound and Equation (4.1) with probability $2/3$, the family $\mathcal{F}'$ satisfies the following properties:

$$|\mathcal{F}'| \leq 2\mathbb{E}\left[|\mathcal{F}'|\right] = 2^{-5}\gamma n, \tag{4.2}$$
$$|\mathcal{L}(x) \cap \mathcal{F}'| \geq 2^{-1}\mathbb{E}\left[|\mathcal{L}(x) \cap \mathcal{F}'|\right] \geq 2^{-7}\gamma^2 n \tag{4.3}$$

for every $x \in V(G)$.

We say two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) are *intersecting* if $z_i = w_j$ for some $1 \leq i, j \leq 3$. We can bound the expected number of pairs of triples in $\mathcal{F}'$ that are intersecting from above by

$$\frac{n!}{(n-3)!} \times 3^2 \times \frac{(n-1)!}{(n-3)!} \times p^2 = 9 \times 2^{-12}\gamma^2 n.$$

Thus, using Markov's inequality, it holds that with probability at least $7/16$,

$$\mathcal{F}' \text{ contains at most } 2^{-8}\gamma^2 n \text{ intersecting pairs of triples.} \tag{4.4}$$

Hence, with positive probability the family $\mathcal{F}'$ satisfies properties (4.1)-(4.4). Remove one triple in each intersecting pair in $\mathcal{F}'$. Further remove those triples that are not absorbing paths. We get a family $\mathcal{F}$ consisting of pairwise disjoint triples, which satisfies

$$|\mathcal{L}(x) \cap \mathcal{F}| \geq 2^{-7}\gamma^2 n - 2^{-8}\gamma^2 n = 2^{-8}\gamma^2 n$$

for every $x \in V(G)$. Since each triple in $\mathcal{F}$ is an absorbing path, this lemma follows. □

By Lemma 4.2.4 we get a family $\mathcal{F}$ of absorbing paths satisfying the properties in the lemma. Then we are going to package all these paths into a weakly proper tree. For an edge-coloured graph G and a vertex $v \in V(G)$, let $C_G(v) := \{c(uv) \colon u \in N_G(v)\}$. We need the following lemma from Lemma 4.4 of [59].

Lemma 4.2.5 (Lo, 2019 [59]) *Let $0 < 1/n \ll \varepsilon < 1/2$. Suppose that G is an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n + 1$. Let $x, y \in V(G)$ be distinct and let c_y be any colour. Then there exists a properly coloured path P from x to y of length at most ε^{-2} such that $C_P(y) \neq \{c_y\}$.*

We prove the following by Lemma 4.2.5.

Lemma 4.2.6 *Let $0 < 1/n \ll \varepsilon < 1/2$. Suppose that G is a 2-connected edge-coloured graph on n vertices with $\delta^c(G) \geq (1/3 + \varepsilon)n + 2$. Let $x \in V(G)$ and let c_x, c'_x be any two colours. Then there exists a vertex subset $S \subset V(G)$ satisfying:*

(i) $|S| \geq (1/2 + \varepsilon)n$,

(ii) for every $v \in S$ there exists a properly coloured path P from x to v of length at most $\varepsilon^{-2} + 2$ such that $C_P(x) \not\subset \{c_x, c'_x\}$.

Proof. Let $\ell \coloneqq \lfloor \varepsilon^{-2} \rfloor$. Remove all edges at x with colour c_x or c'_x and let H be the resulting graph. Clearly H is connected and we have $N_H(x) \subset S$. Let $S_0 = N_H(x)$ and $T_0 = N_H[x]$. If $|S_0| \geq (1/2 + \varepsilon)n$ then S is found. Hence we assume that $|S_0| < (1/2+\varepsilon)n$. If there exists $v \in S_0$ such that

$$d^c_{H-T_0}(v) \geq (1/2 + \varepsilon)n - |T_0| + 1,$$

then we can find at least $d^c_{H-T_0}(v) - 1$ vertices that can be reached by x by a properly coloured path of length 2. S is found.

We further assume that for every $v \in S_0$, $d^c_{H-T_0}(v) \leq (1/2+\varepsilon)n-|T_0|$. So

$$\begin{aligned} d^c_{T_0}(v) &\geq (1/3 + \varepsilon)n - [(1/2 + \varepsilon)n - |T_0|] \\ &= |T_0| - n/6 \\ &\geq (1/2 + \varepsilon)|T_0| + 1. \end{aligned}$$

Hence for any $vu \in E(H)$ with $v \in S_0$ and $u \in V(H) \setminus T_0$, by Lemma 4.2.5, there exists a properly coloured path P from x to v of length at most ε^{-2} such that $C_P(v) \neq \{c(vu)\}$. And so Pu is a properly coloured path from x to u of length at most $\varepsilon^{-2} + 1$, which implies that $N_{H^2}(x) \subset S$.

Let $S_1 = N_{H^2}(x)$ and $T_1 = N_{H^2}[x]$. Repeat the above argument until we get S_i and T_i such that

(i) either $|S_i| \geq (1/2 + \varepsilon)n$,

(ii) or there exists $v \in S_i$ such that $d^c_{H-T_i}(v) \geq (1/2 + \varepsilon)n - |T_i| + 1$.

In both case we can find a set S satisfying the properties in the lemma.

□

We now proof Lemma 4.2.2.

Proof of Lemma 4.2.2. Let ε_0 be such that $1/n \ll \varepsilon_0 \ll \varepsilon$. Since $\delta^c(G) \geq (1/3 + \varepsilon)n$, apply Lemma 4.2.4 and obtain a family $\mathcal{F}$ of vertex-disjoint properly coloured paths each of length 2 such that for all $x \in V(G)$,

$$|\mathcal{F}| \leq 2^{-5}\gamma n, \qquad |\mathcal{L}(x) \cap \mathcal{F}| \geq 2^{-8}\gamma^2 n.$$

Let $P_1, \ldots, P_{|\mathcal{F}|}$ be paths in $\mathcal{F}$. Let x_i be one of the end vertex of P_i for all $1 \leq i \leq |\mathcal{F}|$. Then P_i can be regarded as a weakly proper tree rooted at x_i. To join all the paths in $\mathcal{F}$ we have the following claim.

Claim 4.2.1 *Let T_1 and T_2 be two disjoint weakly proper tree of G rooted at r_1 and r_2 respectively, where $d_{T_i}(r_i) \leq 2$ for $1 \leq i \leq 2$. Each has order at most $\varepsilon n/6$. Then there exists a vertex $r \in V(G) \setminus (V(T_1) \cup V(T_2))$ and two internally disjoint paths P_1 and P_2 such that for $1 \leq i \leq 2$,*

(i) P_i is a properly coloured $r_i - r$ path of length at most $\varepsilon_0^{-2} + 2$,

(ii) $C_{P_i}(r_i) \not\subset C_{T_i}(r_i)$.

Proof. Let G' be the graph of order n' obtained from G by deleting all vertices of T_1 and T_2 except their roots. Let c_{r_i}, c'_{r_i} be the colours incident to r_i in T_i for $1 \leq i \leq 2$. Then G' is a 2-connected graph of minimum colour degree $\delta^c(G') \geq (1/3 + \varepsilon/2)n \geq (1/3 + \varepsilon_0)n' + 2$. By Lemma 4.2.6 there exist two vertex subset $S_1, S_2 \subset V(G')$ such that for $1 \leq i \leq 2$,

(i) $|S_i| \geq (1/2 + \varepsilon_0)n'$,

(ii) for every $v \in S$ there exists a properly coloured path P from r_i to v of length at most $\varepsilon_0^{-2} + 2$ such that $C_P(r_i) \not\subset \{c_{r_i}, c'_{r_i}\}$.

Since $S_1 \cap S_2 \neq \emptyset$, we can find a vertex $r \in S_1 \cap S_2$ satisfying the claim. □

The family $\mathcal{F}$ contains at most $2^{-5}\gamma n$ paths. By Claim 4.2.1 we can join $P_1, \ldots, P_{|\mathcal{F}|}$ into a weakly proper tree T of order at most $2^{-5}\gamma n(3 + 2\varepsilon_0^{-2} + 4) \leq \varepsilon n/2$. For each $P_i = z_{i_1} z_{i_2} z_{i_3}$, the middle vertex z_{i_2} still has degree 2 in T, where $1 \leq i \leq |\mathcal{F}|$. Hence for any vertex subset $U \subset V(G) \setminus V(T)$, U can be absorbed in T into a weakly proper tree. □

4.3 . Proof of Theorem 4.1.2

Let $C = v_0 v_1 \ldots v_{k-1} v_1$ be an arbitrary properly coloured cycle with order k in G. Note that the following indices of vertices are taken modulo k. We have the following claim.

Claim 4.3.1 *For any vertex $u \in V(G) \setminus V(C)$, u is adjacent to some vertex in $V(C)$.*

Proof. Suppose that there is a vertex $u \in V(G) \setminus V(C)$ not adjacent to any vertices in $V(C)$. Since G is connected, there exists a vertex $v \in V(G) \setminus V(C)$ adjacent to some vertex in $V(C)$. Let v_i ($0 \leq i \leq k-1$) be a vertex in $V(C)$ adjacent to v. Since C is properly coloured, we have $c(vv_i) \neq c(v_{i-1}v_i)$ or $c(vv_i) \neq c(v_iv_{i+1})$. Without loss of generality, we assume $c(vv_i) \neq c(v_iv_{i+1})$. Then the path $uvv_iv_{i+1}\dots v_{i-3}$ is a properly coloured P_k. By the assumption of Theorem 4.1.2, $uvv_iv_{i+1}\dots v_{i-3}u$ is a properly coloured C_k. Hence, u is adjacent to v_{i-3}, a contradiction. □

By Claim 4.3.1 and the assumption of Theorem 4.1.2, every vertex in $V(G)$ is contained in a properly coloured C_k. Let $C(r)$ be the properly coloured C_k containing the vertex $r \in V(G)$. We use v_0^r to denote the vertex r and let $C(r) = v_0^rv_1^r\dots v_{k-1}^rv_0^r$. For any vertex $v \in V(G) \setminus V(C_r)$, there are two vertices v_i^r, v_{i+2}^r adjacent to v for some $0 \leq i \leq k-1$, and $v_i^rvv_{i+2}^r\dots v_{i-1}^rv_i^r$ is a properly coloured C_k. If one of the two vertices v_i^r, v_{i+2}^r is v_0^r, then let $f(v) = v_0^r$. Otherwise, let $f(v) = v_i^r$. Let $P(r) = v_0^rv_1^r\dots v_{k-1}^r$. Then $P(r) + \sum_{v\in V(G)\setminus V(C_r)} vf(v)$ is a weakly proper spanning tree with root r.

4.4 . Proof of Theorem 4.1.3

Let T be an edge-coloured triangle with $u, v, w \in V(T)$ and $c(vw) = c_1, c(uv) = c_2, c(uw) = c_3$. Let m and ℓ be positive integers. Let T_i be an edge-coloured triangle with $x_i, y_i, z_i \in V(T_i)$ for $1 \leq i \leq 3m+2$ and

$$c(x_iy_i) = \begin{cases} c_2 & \text{if } i \equiv 1 \\ c_3 & \text{if } i \equiv 2 \\ c_1 & \text{if } i \equiv 0 \end{cases} \quad c(y_iz_i) = \begin{cases} c_1 & \text{if } i \equiv 1 \\ c_2 & \text{if } i \equiv 2 \\ c_3 & \text{if } i \equiv 0 \end{cases} \quad c(x_iz_i) = \begin{cases} c_3 & \text{if } i \equiv 1 \\ c_1 & \text{if } i \equiv 2 \\ c_2 & \text{if } i \equiv 0, \end{cases}$$

where "$\equiv$" means that i is taken modulo three. Let S_i be an edge-coloured triangle with $x_i', y_i', z_i' \in V(S_i)$ for $1 \leq i \leq 3\ell+2$ and

$$c(x'_iy'_i) = \begin{cases} c_3 & \text{if } i \equiv 1 \\ c_2 & \text{if } i \equiv 2 \\ c_1 & \text{if } i \equiv 0 \end{cases} \quad c(y'_iz'_i) = \begin{cases} c_1 & \text{if } i \equiv 1 \\ c_3 & \text{if } i \equiv 2 \\ c_2 & \text{if } i \equiv 0 \end{cases} \quad c(x'_iz'_i) = \begin{cases} c_2 & \text{if } i \equiv 1 \\ c_1 & \text{if } i \equiv 2 \\ c_3 & \text{if } i \equiv 0. \end{cases}$$

Let G be a graph obtained from $T, T_1, \ldots, T_{3m+2}, S_1, \ldots, S_{3\ell+2}$ by identifying $\{z_i, x_{i+1}\}$ for $1 \leq i \leq 3m+1$, $\{z'_j, x'_{j+1}\}$ for $1 \leq j \leq 3\ell+1$, $\{x_1, z_{3m+2}, v\}$ and $\{x'_1, z'_{3\ell+2}, w\}$ (see Figure 4.2 as an example).

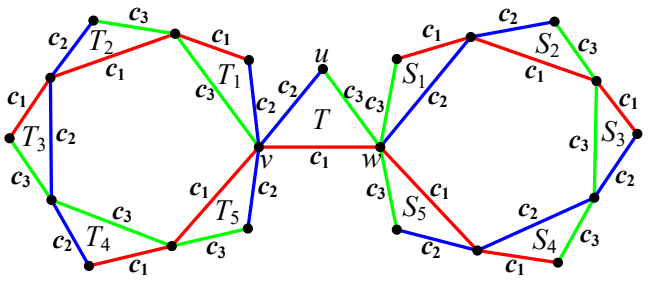


Figure 4.2: The graph G with $m = \ell = 1$.

We will show that G has no weakly proper spanning tree with root u. Suppose that there is a weakly proper spanning tree H with root u. It is easy to see that one of uv and uw must be contained in H. Without loss of generality, we assume that $uv \in E(H)$. Let a_1 be the vertex identified by $\{x_1, z_{3m+2}, v\}$ and a_{i+1} be the vertex identified by $\{z_i, x_{i+1}\}$ for each $1 \leq i \leq 3m+1$. Let C be the cycle consisting of $a_1, a_2, \ldots, a_{3m+2}$. Since H is a spanning tree with root u, for every a_i $(2 \leq i \leq 3m+2)$, there is a unique path in H connecting a_1 and a_i, and we denote it by P_i. We say P_i is *maximal* if there is no P_j $(2 \leq j \leq 3m+2$ and $j \neq i)$ taking P_i as a subpath. Obviously, there are at most two maximal paths in $\{P_2, P_3, \ldots, P_{3m+2}\}$. Let P be a maximal path in $\{P_2, P_3, \ldots, P_{3m+2}\}$. For convenience, write $P = b_1b_2 \ldots b_i$, where $b_1 = a_1$ and $b_i \in \{a_2, \ldots, a_{3m+2}\}$.

Claim 4.4.1 *For each* $1 \leq j \leq i-1$, $b_jb_{j+1} \in E(C)$.

Proof. We prove this claim by induction on j. Since $uv \in E(H)$ and $c(uv) = c(a_1y_1) = c(a_1y_{3m+2})$, we have $b_1b_2 \in E(C)$. Suppose that $b_jb_{j+1} \notin E(C)$, where $j \geq 2$. Since $b_{j-1}b_j \in E(C)$, b_j is in $V(C)$ and

so b_{j+1} is not in $V(C)$. Since $b_i \in \{a_2, \ldots, a_{3m+2}\}$, $b_{j+1} \neq b_i$ and b_{j+1} is not contained in a triangle containing $b_{j-1}b_j$. By the colouring of G, $c(b_{j-1}b_j) = c(b_jb_{j+1})$, which contradicts that H is a weakly proper tree. □

If there are two maximal paths in $\{P_2, P_3, \ldots, P_{3m+2}\}$, then let $Q = b'_1b'_2 \ldots b'_k$ be the maximal path different from P, where $b'_1 = a_1$ and $b'_k \in \{a_2, \ldots, a_{3m+2}\}$. Otherwise, let $Q = b'_1 = b'_k = a_1$. Since H is a spanning tree, b_i and b'_k are contained in a same triangle, and let b be the vertex in this triangle different from b_i, b'_k. By Claim 4.4.1 and the colouring of G, $c(b_{i-1}b_i) = c(b_ib)$ and $c(b'_{k-1}b'_k) = c(b'_kb)$, where $b'_0 = u$. Since H is a weakly proper tree, b is not contained in H, which contradicts that H is a spanning tree. Hence G has no weakly proper coloured spanning tree with root u.

4.5 . Proof of Theorem 4.1.4

Suppose that G has no weakly proper spanning tree with root r for some $r \in V(G)$. Let T be a maximum weakly proper tree with root r. For a vertex $v \in V(T)$, let $e(v)$ be the edge incident with v such that $e(v)$ is on the path from r to v in T. We use $L(T)$ to denote the leaves of T.

Claim 4.5.1 *For each vertex $v \in V(T)$ and $u \in V(G) \setminus V(T)$ adjacent to v, we have $c(uv) = c(e(v))$.*

Proof. Suppose that there exists a vertex $v \in V(T)$ such that v is adjacent to a vertex $u \in V(G) \setminus V(T)$ and $c(uv) \neq c(e(v))$. Then $T + uv$ is a weakly proper tree with order $|T| + 1$, which contradicts to the maximality of T. □

Claim 4.5.2 *Let $\ell \in L(T)$ adjacent to a vertex $u \in V(G) \setminus V(T)$. If ℓ is adjacent to a vertex $v \in V(T)$ such that $c(\ell u) \neq c(\ell v)$, then $c(\ell v) = c(e(v))$.*

Proof. Suppose that $c(\ell v) \neq c(e(v))$. Then $T + \ell u + \ell v - e(\ell)$ is a weakly proper tree with order $|T| + 1$, a contradiction. □

Since G is connected, there exists a vertex $u \in V(G) \setminus V(T)$ adjacent to a vertex in $V(T)$.

Claim 4.5.3 *The edges between u and $L(T)$ have the same colour.*

Proof. Suppose that there exist leaves ℓ_1 and ℓ_2 of T with $c(u\ell_1) \neq c(u\ell_2)$. Since $\ell_1 u \ell_2$ is a properly coloured P_3, it follows from the assumption (i) of Theorem 4.1.4 that there exists a vertex v such that $v\ell_1 u \ell_2 v$ is a properly coloured C_4. Since $c(v\ell_i) \neq c(u\ell_i)$ for $i \in \{1, 2\}$, we have $v \in V(T)$ by Claim 4.5.1. This contradicts to Claim 4.5.2 since $c(v\ell_1) \neq c(v\ell_2)$. Hence the edges between u and $L(T)$ have the same colour. □

Claim 4.5.4 *There exists a vertex in $V(T) \setminus L(T)$ adjacent to u.*

Proof. Assume it is false. Then there exists a leaf ℓ of T adjacent to u. Since $\delta^c(G) \geq 2$, there exists a vertex v adjacent to u with $c(uv) \neq c(u\ell)$. By Claim 4.5.3, v is not in $V(T)$. Since ℓuv is a properly coloured P_3, there exists a vertex w such that $w\ell uvw$ is a properly coloured C_4. Since $c(u\ell) \neq c(w\ell)$, it follows from Claim 4.5.1 that w is in $V(T)$. This contradicts to Claim 4.5.1 or Claim 4.5.2 since $c(w\ell) \neq c(vw)$. □

For two vertices a and b in $V(T)$, we use $P_T(a, b)$ to denote the unique path in T connecting a and b. Let w be a vertex in $V(T) \setminus L(T)$ adjacent to u such that

$$|P_T(r, w)| = \min\{|P_T(r, v)| : v \in N_G(u) \cap V(T)\}.$$

Let x be a child of w in T. Let $\mathcal{T}$ be a set of maximum weakly proper trees S with root r in G such that

(i) $V(S) = V(T)$,

(ii) $|P_S(r, w)| = \min\{|P_S(r, v)| : v \in N_G(u) \cap V(S)\}$, and

(iii) x is a child of w in S.

Note that Claims 4.5.3 and 4.5.4 hold for any tree in $\mathcal{T}$. Choose a maximum weakly proper tree $T' \in \mathcal{T}$ such that

(P1) $\deg_{T'}(x)$ is as large as possible.

Since T' is a weakly proper tree, it follows from Claim 4.5.1 that uwx is a properly coloured P_3. Hence there exists a vertex y such that $yuwxy$ is a properly coloured C_4. Moreover, by Claims 4.5.1 and 4.5.2, we have $y \in V(T') \setminus L(T')$ and $e(y) \neq xy$.

By the choice of w, the unique cycle of $T' + xy$ contains $e(y)$. By Claim 4.5.1, $c(uy) = c(e(y))$. We show that there exists a child z of y such that $c(xy) = c(yz)$. Suppose that $c(xy) \neq c(yz)$ for each child z of y. Then $T'' = T' + xy + uy - e(y)$ is order $|T'| + 1$, a contradiction. Hence there exists a child z of y such that $c(xy) = c(yz)$.

We define a 2-*edge-coloured* $K_{1,4}$ as an edge-coloured star $K_{1,4}$ with $c(e_1) = c(e_2) \neq c(e_3) = c(e_4)$, where e_1, e_2, e_3, e_4 are four edges of $K_{1,4}$. Let $e(y) = yy'$. Then the subgraph induced by yy', yz, xy and uy is a 2-edge-coloured $K_{1,4}$. By the assumption (ii) in Theorem 4.1.4, one of the following cases happens:

- $uy' \in E(G)$ and $c(uy') = c(e(y))$;
- $xz \in E(G)$ and $c(xz) = c(yz)$.

If the first case happen, then $T' + uy'$ is a weakly proper tree with order $|T| + 1$, a contradiction. Hence the latter case must happen. Let $T'' = T' + xz - yz$. Then T'' is in $\mathcal{T}$ and $\deg_{T''}(x) > \deg_{T'}(x)$, which contradicts (P1). Thus Theorem 4.1.4 holds.

In fact, if we replace the assumption (i) of Theorem 4.1.4 with "every edge is contained in a properly coloured C_3", then we can get the same conclusion as Theorem 4.1.4. Since the proof is similar, we omit it.

4.6 . Concluding remarks

We give some sufficient conditions for graphs G to have a weakly proper spanning tree with root r for any $r \in V(G)$. By Theorem 4.1.3, "every edge is contained in a properly coloured C_3" is not a sufficient condition for the existence of such weakly proper spanning trees. But we do not know whether "every properly coloured P_3 is contained in a properly coloured C_4" is a sufficient condition or not.

More generally, we can consider whether "every properly coloured P_k is contained in a properly coloured C_{k+1}" is a sufficient condition or not, where $k \geq 3$. Hence we propose the following conjecture.

Conjecture 4.6.1 *Let G be an edge-coloured connected graph with $\delta^c(G) \geq 2$. If every properly coloured P_3 is contained in a properly coloured C_4, then G has a weakly proper spanning tree with root r for any $r \in V(G)$.*

By Example 4.1.2, the condition in Conjecture 4.6.1 cannot guarantee properly coloured spanning trees, which makes this conjecture more interesting. Theorem 4.1.4 provides some evidence for this conjecture. In the following proposition, we give two other properties of an edge-coloured graph G which satisfies the assumption of Conjecture 4.6.1. The second property is that G is properly connected, which is a necessary condition for weakly proper spanning trees.

Proposition 4 *Let G be an edge-coloured graph with $\delta^c(G) \geq 2$. If every properly coloured P_3 is contained in a properly coloured C_4, then*

1. *G is 2-connected,*
2. *G is properly connected.*

Proof. (i) Suppose that G is not 2-connected. Then there is a vertex v such that $G - v$ has at least two components. Since $\delta^c(G) \geq 2$, there exist two different components C_1 and C_2 of $G - v$ such that there are two vertices $w_1 \in V(C_1)$ and $w_2 \in V(C_2)$ with $c(vw_1) \neq c(vw_2)$. By the assumption of this proposition, there is a properly coloured C_4 containing v, w_1, w_2 in G. But C_1 and C_2 are different components of $G - v$, a contradicition.

(ii) For an edge-coloured path P having distinct monochromatic paths $Q_1, Q_2, \ldots, Q_t$ with order at least three, let $\text{mon}(P) = \sum_{1 \leq i \leq t} |E(Q_i)|$. Suppose that there are two vertices u and v such that there is no properly coloured path connecting u and v. Let P be a path in G connecting u and v. We assign an orientation in P from u to v. For $x \in V(P)$, we denote the successor and the predecessor of x on P by x^+ and x^-, respectively. Since there is no properly coloured path connecting u and v, there are monochromatic paths with order at least three in P. Let $Q_1 = w_1 w_2 \ldots w_m$ be a monochromatic

path in P with order at least three such that uPw_1 has no monochromatic path, where uPw_1 is a subpath of P connecting u and w_1. We choose P so that

(P1) mon(P) is as small as possible,

(P2) $|P|$ is as small as possible, and

(P3) $|uPw_1|$ is as large as possible.

Suppose that $w_m = v$. Since $\delta^c(G) \geq 2$, v is adjacent to a vertex x such that $c(vx) \neq c(vw_{m-1})$. Note that $x \neq u$. Suppose that $x \in V(P)$. Then $P' = uPx + vx$ is a path in G connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-2}w_{m-1}, w_{m-1}v\}| + |\{x^-x, xv\}| =$ mon(P) and $|P'| < |P|$. This contradicts the choice of (P1) or (P2). Hence $x \notin V(P)$. Since $w_{m-1}vx$ is a properly coloured P_3 in G, it follows from the assumption of the proposition that there is a vertex y such that $w_{m-1}vxyw_{m-1}$ is a properly coloured C_4 in G. Suppose that $y \in V(P)$. Note that $y \neq u$ and $y \neq w_{m-2}$. Then $P' = uPy + xy + xv$ is a path in G connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-2}w_{m-1}, w_{m-1}v\}| + |\{y^-y, xy\}| =$ mon(P) and $|P'| < |P|$. This contradicts the choice of (P1) or (P2). Hence $y \notin V(P)$. Then $P' = P + w_{m-1}y + xy + vx - vw_{m-1}$ is a path connecting u and v with mon$(P') <$ mon(P), which contradicts (P1).

Hence we may assume that $w_m \neq v$. Then $w_{m-1}w_mw_m^+$ is a properly coloured P_3 and so by the assumption of the proposition, there is a vertex z such that $w_{m-1}w_mw_m^+zw_{m-1}$ is a properly coloured C_4. Suppose that $z \in V(w_mPv)$. Suppose further that either $|Q_1| = 3$ or $c(w_{m-1}z) \neq c(zz^+)$. Let $P' = uPw_{m-1} + w_{m-1}z + zPv$. Then

$$\text{mon}(P') \leq \begin{cases} \text{mon}(P) - |E(Q_1)| + |\{w_{m-1}z, zz^+\}| & \\ \quad = \text{mon}(P) & \text{if } |Q_1| = 3 \\ \text{mon}(P) - |\{w_{m-1}w_m\}| & \\ \quad < \text{mon}(P) & \text{if } c(w_{m-1}z) \neq c(zz^+), \end{cases}$$

and $|P'| < |P|$. This contradicts (P1) or (P2). Hence $|Q_1| \geq 4$ and $c(w_{m-1}z) = c(zz^+)$. Since $w_{m-2}w_{m-1}z$ is a properly coloured P_3, it follows

from the assumption of the proposition that there is a vertex z' such that $w_{m-2}w_{m-1}zz'w_{m-2}$ is a properly coloured C_4. If $z' \notin V(P)$, then $P' = uPw_{m-2} + w_{m-2}z' + zz' + z'Pv$ is a path connecting u and v with mon$(P') <$ mon(P) since $|Q_1| \geq 4$ and $c(w_{m-1}z) = c(zz^+)$. This contradicts the choice of (P1). Hence $z' \in V(P)$. Suppose that $z' \in V(w_{m-1}Pv)$. Then $P' = uPw_{m-2} + w_{m-2}z' + z'Pv$ is a path connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-2}w_{m-1}, w_{m-1}w_m\}| + |\{w_{m-2}z', z'z'^+\}| =$ mon(P) and $|P'| < |P|$. This contradicts the choice of (P1) or (P2). Hence $z' \in V(uPw_{m-3})$. Note that $z' \neq w_{m-3}$ Then $P' = uPz' + z'z + zPv$ is a path connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-3}w_{m-2}, w_{m-2}w_{m-1}, w_{m-1}w_m\}| + |\{z'^-z', z'z\}| <$ mon(P), which contradicts the choice of (P1). Hence $z \notin V(w_mPv)$.

Suppose that $z \in V(uPw_{m-1})$. Note that $z \neq w_{m-2}$. Suppose further that $c(w_{m-1}z) \neq c(zz^-)$. Then $P' = uPz + w_{m-1}z + w_{m-1}Pv$ is a path connecting u and v with mon$(P') <$ mon(P), which contradicts the choice of (P1). Hence $c(w_{m-1}z) = c(zz^-)$ and so $c(w_m^+z) \neq c(zz^-)$. Then $P' = uPz + zw_m^+ + w_m^+Pv$ is a path connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-2}w_{m-1}, w_{m-1}w_m\}| + |\{zw_m^+, w_m^+(w_m^+)^+\}| \leq$ mon(P) and $|P'| < |P|$. This contradicts the choice of (P1) or (P2).

Hence we may assume that $z \notin V(P)$. Let $P' = P + w_{m-1}z + w_m^+z - w_{m-1}w_m - w_mw_m^+$. Note that $|P'| = |P|$. If $|Q_1| = 3$, then mon$(P') \leq$ mon(P) and $|uPz| > |uPw_1|$, which contradicts the choice of (P1) or (P3). Hence $|Q_1| \geq 4$. Since $w_{m-2}w_{m-1}z$ is a properly coloured P_3, it follows from the assumption of the proposition that there is a vertex z' such that $w_{m-2}w_{m-1}zz'w_{m-2}$ is a properly coloured C_4. If $z' \notin V(P)$, then $P' = P + w_{m-2}z' + zz' + w_{m-1}z - w_{m-2}w_{m-1}$ is a path connecting u and v with mon$(P') <$ mon(P), which contradicts the choice of (P1). Hence $z' \in V(P)$. Suppose that $z' \in V(w_{m-1}Pv)$. Then $P' = uPw_{m-2} + w_{m-2}z' + z'Pv$ is a path connecting u and v with mon$(P') \leq$ mon$(P) - |\{w_{m-2}w_{m-1}, w_{m-1}w_m\}| + |\{w_{m-2}z', z'z'^+\}| \leq$ mon(P) and $|P'| < |P|$. This contradicts the choice of (P1) or (P2). Hence $z' \in V(uPw_{m-3})$. If $c(w_{m-2}z') \neq c(z'z'^-)$, then $P' = uPz' + w_{m-2}z' + w_{m-2}Pv$ is a path connecting u and v with mon$(P') <$ mon(P), which contradicts the choice of (P1). Hence we obtain $c(w_{m-2}z') = c(z'z'^-)$ and so $c(zz') \neq c(z'z'^-)$. Then $P' = uPz' + zz' + w_{m-1}z + w_{m-1}Pv$ is a path connecting u and v

with mon$(P') <$ mon(P), which contradicts the choice of (P1). Therefore the proposition holds. □

5 - Conclusion and Perspectives

There are still many open problems related to our results in the thesis.

1. **Odd colouring and pcf colouring.** In Chapter 2, we prove some asymptotic bounds for odd colouring. There remains a gap of $\log \Delta$ to the Conjecture 1.2.3. It is meaningful to mention a graceful result of Molloy and Reed [63], which proved that every graph with large maximum degree Δ can be properly $(\Delta + 1)$-coloured so that no colour appears more than $O(\log \Delta / \log \log \Delta)$ times in the neighbourhood of any vertex. We are quite convinced that Conjecture 1.2.3 can be reached for graphs with large maximum degree. Also, we are working to give an asymptotic result on $Conjecture$ 1.2.1 of the pcf colouring.

2. **Proper version of anti-Ramsey numbers.** There are 2 possible way to generalise. The first one is to consider this kind of structural problem in dense graphs, such as a graph G with minimum degree $\delta(G) \geq 3n/4$, rather than in complete graphs. One can find many results about the minimum (colour) degree conditions forcing some given subgraphs. We hope to apply these useful and strong tools combined with anti-Ramsey problems. On the other hand, we are working to extend our results on $\mathrm{pr}(n, C_l)$ to other cycle-based graphs, like theta graphs and cell graphs.

Then I would like to mention the prospects of my research in other topics.

1. **Fractional colouring.** The fractional chromatic number $\chi_f(G)$ of a graph G is a refinement of the chromatic number. It is the fractional solution to a linear program, the integer solution of which is the chromatic number. If G is a graph, then we define $\mathscr{I}(G)$ to be the set of all independent sets of G, and the *fractional chromatic number* $\chi_f(G)$ of G is the solution of

the following linear program.

$$\min \sum_{I \in \mathscr{I}(G)} w_I$$

$$\text{such that} \begin{cases} w_I \in [0,1] & \text{for each } I \in \mathscr{I}(G) \\ \sum_{\substack{I \in \mathscr{I}(G) \\ v \in I}} w_I \geq 1 & \text{for each } v \in V(G). \end{cases}$$

A *fractional colouring of weight* w *of* G is any instance within the domain of the above linear program such that $\sum w_I = w$.

Given a graph H, we let $\chi_f(d, H)$ be the supremum of the fractional chromatic numbers over all H-free graphs of maximum degree at most d.

When H is a complete graph, the study of $\chi_f(d, H)$ falls in the domain of Ramsey theory, a domain which emerged in the 1930s following seminal results by van der Waerden [78] and by Ramsey [71], and has attracted a lot of attention ever since. An important result in this case is due to Molloy and Reed [62, Theorem 21.7, p. 244]: known as "the fractional Reed bound", it states that $\chi_f(d, K_n) \leq \frac{d+n}{2}$ for all integers $d, n \geq 2$.

we focus on the case $H = K_3$, which is closely related to *off-diagonal Ramsey numbers*. It has been established [26] that $\chi_f(3, K_3) = 14/5$. The same question for larger values of the maximum degree is still open. At one end of the spectrum, we know that $\chi_f(4, K_3)$ lies between 3.25 (see Figure 5.1) and 3.5 (by the fractional Reed bound). At the other end of the spectrum, one has $\chi_f(d, K_3) \leq (1 + o(1))\, d/\ln d$ as $d \to \infty$, which is a consequence of a result by Molloy [61], and one can infer from a study of random d-regular graphs by Bollobás [5] that $\chi_f(d, K_3) \geq \frac{d}{2 \ln d}$.

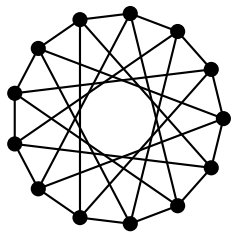


Figure 5.1: A graph certifying that $\chi_f(4, K_3) \geq 3.25$.

A first study of $\chi_f(d, K_3)$ has been made by the last two authors [70], with the help of a so-called *greedy fractional colouring algorithm* (GFCA). This algorithm takes as input a graph G and a probability distribution π on the independent sets of any induced subgraph of G, and returns a fractional colouring of G whose weight is bounded by a function of π and G. They proved the following.

Theorem 5.0.1 (Pirot & Sereni, 2021) *For every integer d,*

$$\chi_f(d, K_3) \leq 1 + \min_{k \in \mathbb{N}} \inf_{\lambda > 0} \frac{(1+\lambda)^k + \lambda(1+\lambda)}{\lambda(1+k\lambda)}.$$

2. **Caccetta-Häggvist conjecture on bipartite graphs.** In 1978, Caccetta and Häggvist proposed a famous conjecture about the girth of oriented graphs.

Conjecture 5.0.1 *For every integer $k \geq 1$, and all $n \geq 0$, every n-vertex digraph in which every vertex has out-degree at least n/k has girth at most k.*

One of its generalisation is to consider a similar condition on (balanced) bipartite graphs.

Conjecture 5.0.2 *For every integer $k \geq 1$, if G is a bipartite digraph, with $n > 0$ vertices in each part, and every vertex has out-degree more than $n/(k+1)$, then G has girth at most $2k$.*

In 2020, Seymour and Spirkl proved that Conjecture 5.0.1 implies Conjecture 5.0.2 if it is true. They also proved the following theorem.

Theorem 5.0.2 *For $k = 1, 2, 3, 4, 6$, and all $k \geq 224,539$, if G is a bipartite digraph, with $n > 0$ vertices in each part, and every vertex has out-degree more than $n/(k+1)$, then G has girth at most $2k$.*

Bibliography

[1] N. Alon. On a conjecture of erdös, simonovits, and sós concerning anti-ramsey theorems. *J. Graph Theory*, 7(1):91–94, 1983.

[2] Y. Anders. A note on alternating cycles in edge-coloured graphs. *J. Combin. Theory Ser. B*, 69:222–225, 1977.

[3] J. Balogh and T. Molla. Long rainbow cycles and hamiltonian cycles using many colors in properly edge-colored complete graphs. *European J. Combin.*, 79:140–151, 2019.

[4] A. Bernshteyn. The johansson-molloy theorem for dp-coloring. *Random Structures & Algorithms*, 54(4):653–664, 2019.

[5] B. Bollobás. The independence ratio of regular graphs. *Proceedings of the American Mathematical Society*, pages 433–436, 1981.

[6] B. Bollobás. *Modern Graph Theory*. Springer, 1998.

[7] B. Bollobás and P. Erdős. Alternating hamiltonian cycles. *Israel J. Math.*, 23:126–131, 1976.

[8] V. Borozan, W. Fernandez de La Vega, Y. Manoussakis, C. Martinhon, R. Muthu, H. P. Pham, and R. Saad. Maximum colored trees in edge-colored graphs. *European J. Combin.*, 80:296–310, 2019.

[9] R. L. Brooks. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 37, pages 194–197. Cambridge University Press, 1941.

[10] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster. On rainbow connection. *Electron. J. Combin.*, 15(1):Research paper 57, 13, 2008.

[11] Y. Caro, M. Petruševski, and R. Škrekovski. Remarks on odd colorings of graphs. *Discrete Applied Mathematics*, 321:392–401, 2022.

[12] Y. Caro, M. Petruševski, and R. Škrekovski. Remarks on proper conflict-free colorings of graphs. *arXiv preprint arXiv:2203.01088*, 2022.

[13] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. On the rainbow connectivity of cages, in: Proceedings of the thirty-eighth southeastern international conference on combinatorics. *Graph Theory and Comput.*, 184, 2007.

[14] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. Rainbow connection in graphs. *Math. Bohem.*, 133(1), 2008.

[15] P. Cheilaris. *Conflict-free coloring*. PhD thesis, City University of New York, 2009.

[16] P. Cheilaris, B. Keszegh, and D. Pálvölgyi. Unique-maximum and conflict-free colouring for hypergraphs and tree graphs. *SIAM J. Discrete Math*, 27, 2013.

[17] Y. Cheng, M. Kano, and G. Wang. Properly colored spanning trees in edge-colored graphs. *Discrete Math.*, 343:111629, 2020.

[18] E.-K. Cho, I. Choi, H. Kwon, and B. Park. Proper conflict-free coloring of sparse graphs. *arXiv preprint arXiv:2203.16390*, 2022.

[19] E.-K. Cho, I. Choi, H. Kwon, and B. Park. Odd coloring of sparse graphs and planar graphs. *Discrete Mathematics*, 346(5):113305, 2023.

[20] D. W. Cranston. Odd colorings of sparse graphs. *arXiv preprint arXiv:2201.01455*, 2022.

[21] D. W. Cranston, M. Lafferty, and S. Zixia. A note on odd colorings of 1-planar graphs. *Discrete Applied Math.*, 330:112–117, 2023.

[22] D. W. Cranston and C.-H. Liu. Proper conflict-free coloring of graphs with large maximum degree. *arXiv preprint arXiv:2211.02818*, 2022.

[23] D. Dellamonica Jr., C. Magnant, and D. Martin. Rainbow paths. *Discrete Math.*, 310(4):774–781, 2010.

[24] L. Ding, J. Hu, G. Wang, and D. Yang. Properly colored short cycles in edge-colored graphs. *European J. Combin.*, 100:103436, 2022.

[25] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.

[26] Z. Dvořák, J.-S. Sereni, and J. Volec. Subcubic triangle-free graphs have fractional chromatic number at most 14/5. *Journal of the London Mathematical Society*, 89(3):641–662, 2014.

[27] P. Erdő and R. Rado. A combinatorial theorem. *J. London Math. Soc.*, 25:249–255, 1950.

[28] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.

[29] P. Erdős, M. Simonovits, and V. T. Sós. Anti-ramsey theorems. infinite and finite sets, vol. ii, pp. 633-643. colloq. math. soc. jános bolyai, vol. 10. north-holland, amsterdam, 1975.

[30] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM Journal on Computing*, 33(1):94–136, 2003.

[31] I. Fabrici, B. Lužar, S. Rindošová, and R. Soták. Proper conflict-free and unique-maximum colorings of planar graphs with respect to neighborhoods. *arXiv preprint arXiv:2202.02570*, 2022.

[32] C. Fang, E. Győri, and J. Xiao. On the maximal colorings of complete graphs without some small properly colored subgraphs. *Graphs Combin.*, 37:2287–2304, 2021.

[33] S. Fujita, R. Li, and S. Zhang. Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs. *J. Graph Theory*, 87(3):362–373, 2018.

[34] S. Fujita and C. Magnant. Properly colored paths and cycles. *Discrete Applied Math.*, 159:1391–1397, 2011.

[35] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of ramsey theory - a dynamic survey. *Theory and Applications of Graphs*, 2014.

[36] H. N. Gabow, T. Nishizeki, O. Kariv, D. Leven, and O. Terada. Algorithms for edge-coloring graphs. *Technical report*, TRECIS-8501, 1985.

[37] T. Gallai. Transitiv orientierbare graphen. *Acta Math. Hungar.*, 18:25–66, 1967.

[38] J. W. Grossman and R. Häggkvist. Alternating cycles in edge-partitioned graphs. *J. Combin. Theory Ser. B*, 34:77–81, 1983.

[39] H. Grötzsch. Ein dreifarbensatz fur dreikreisfreie netze auf der kugel. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math. Nat. Reihe*, 8:109–120, 1959.

[40] A. Gyárfás and G. Simonyi. Edge colorings of complete graphs without tricolored triangles. *J. Graph Theory*, 46:211–216, 2004.

[41] T. Han, S. Zhang, Y. Bai, and R. Li. Colored complete graphs containing no properly colored odd cycles. *Graphs and Combin.*, 37:1129–1138, 2021.

[42] D. Hanson, G. MacGillivray, and B. Toft. Choosability of bipartite graphs. *Ars Combin.*, 44:183–192, 1996.

[43] G. R. T. Hendry. Extending cycles in graphs. *Discrete Math.*, 85:59–72, 1990.

[44] R. Hickingbotham. Odd colourings, conflict-free colourings and strong colouring numbers. *arXiv preprint arXiv:2203.10402*, 2022.

[45] T. Jiang, I. Schiermeyer, and D. B. West. The erdös-simonovits-sós conjecture for $k \leq 7$. unpublished manuscript.

[46] A. Johansson. Asymptotic choice number for triangle free graphs. Technical report, 91-95 DIMACS, 1996.

[47] A. Kostochka, M. Kumbhat, and T. Łuczak. Conflict-free colourings of uniform hypergraphs with few edges. *Combinatorics, Probability and Computing*, 21(4):611–622, 2012.

[48] M. Krivelevich and R. Yuster. The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory*, 63(3):185–191, 2010.

[49] B. Li, B. Ning, C. Xu, and S. Zhang. Rainbow triangles in edge-colored graphs. *European J. Combin.*, 36:453–459, 2014.

[50] H. Li. Rainbow c_3's and c_4's in edge-colored graphs. *Discrete Math.*, 313:1893–1896, 2013.

[51] H. Li and G. Wang. Color degree and alternating cycles in edge-colored graphs. *Discrete Math.*, 309:4349–4354, 2009.

[52] R. Li. Properly colored cycles in edge-colored complete graphs without monochromatic triangle: A vertex-pancyclic analogous result. *Discrete Math.*, 344:112753, 2021.

[53] R. Li, H. Broersma, and S. Zhang. Properly edge-colored theta graphs in edge-colored complete graphs. *Graphs Combin.*, 35:261–286, 2019.

[54] C.-H. Liu. Proper conflict-free list-coloring, subdivisions, and layered treewidth. *arXiv preprint arXiv:2203.12248*, 2022.

[55] R. Liu, W. Wang, and G. Yu. 1-planar graphs are odd 13-colorable. *Discrete Math.*, 346(8):113423, 2023.

[56] A. Lo. A dirac type condition for properly coloured paths and cycles. *J. Graph Theory*, 76:60–87, 2014.

[57] A. Lo. An edge-colored version of dirac's theorem. *SIAM J. Discrete Math.*, 28:18–36, 2014.

[58] A. Lo. Properly coloured hamiltonian cycles in edge-coloured complete graphs. *Combinatorica*, 36(4):471–492, 2016.

[59] A. Lo. Long properly coloured cycles in edge-coloured graphs. *J. Graph Theory*, 90:416–442, 2019.

[60] Y. Manoussakis, M. Spyratos, Z. Tuza, and M. Voigt. Minimal colorings for properly colored subgraphs. *Graphs Combin.*, 12:345–360, 1996.

[61] M. Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.

[62] M. Molloy, M. S. Molloy, and B. Reed. *Graph colouring and the probabilistic method*, volume 23. Springer Science & Business Media, 2002.

[63] M. Molloy and B. Reed. Asymptotically optimal frugal colouring. *Journal of Combinatorial Theory, Series B*, 100(2):226–246, 2010.

[64] J. J. Montellano-Ballesteros and V. Neumann-Lara. An anti-ramsey theorem. *Combinatorica*, 22:445–449, 2002.

[65] J. J. Montellano-Ballesteros and V. Neumann-Lara. An anti-ramsey theorem on cycles. *Graphs Combin.*, 21(3):343–354, 2005.

[66] Q. Ouyang, T. Dai, and Y. Ma. A hypergraph approach for logic-based abduction. *DBKDA 2023*, page 31, 2023.

[67] J. Pach and G. Tardos. Conflict-free colourings of graphs and hypergraphs. *Combinatorics, Probability and Computing*, 18(5):819–834, 2009.

[68] J. Petr and J. Portier. Odd chromatic number of planar graphs is at most 8. *arXiv preprint arXiv:2201.12381*, 2022.

[69] M. Petruševski and R. Škrekovski. Colorings with neighborhood parity condition. *Discrete Applied Mathematics*, 321:385–391, 2022.

[70] F. Pirot and J.-S. Sereni. Fractional chromatic number, maximum degree, and girth. *SIAM Journal on Discrete Mathematics*, 35(4):2815–2843, 2021.

[71] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc*, 30, 1930.

[72] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. A new proof of the four-colour theorem. *Electronic research announcements of the American Mathematical Society*, 2(1):17–25, 1996.

[73] V. Rödl, A. Ruciński, and E. Szemerédi. An approximate dirac-type theorem for k-uniform hypergraphs. *Combinatorica*, 28(2):229–260, 2008.

[74] I. Schiermeyer. On minimally rainbow k-connected graphs. *Discrete Applied Math.*, 161(4-5):702–705, 2013.

[75] M. Simonovits and V. T. Sós. On restricted colorings of k_n. *Combinatorica*, 4(1):101–110, 1984.

[76] C. Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory Series B*, 62(1):180–181, 1994.

[77] C. Thomassen. 3-list-coloring planar graphs of girth 5. *Journal of Combinatorial Theory, Series B*, 64(1):101–107, 1995.

[78] B. L. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wiskd., II. Ser.*, 15:212–216, 1927.

[79] V. G. Vizing. Vertex colorings with given colors. *Diskret. Analiz*, 29:3–10, 1976.

[80] M. Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1-3):215–219, 1993.

[81] G. Wang and H. Li. Color degree and alternating cycles in edge-colored graphs. *Discrete Math.*, 309:4349–4354, 2009.

[82] R. M. Wilson. Constructions and uses of pairwise balanced designs. In *Combinatorics: Proceedings of the NATO Advanced Study Institute held at Nijenrode Castle, Breukelen, The Netherlands 8–20 July 1974*, pages 19–42. Springer, 1975.

[83] C. Xu, C. Magnant, and S. Zhang. Properly colored c_4's in edge-colored graphs. *Discrete Math.*, 343(12):112116, 2020.

Publications

International journals and manuscripts

(1) Jie Hu, **Qiancheng Ouyang**, Guanghui wang, Local antimagic orientations of d-degenerate graphs, Discrete Applied Mathematics, 257: 206-215, 2019.

(2) Yulin Chang, **Qiancheng Ouyang**, Guanghui wang, Adjacent vertex distinguishing total choosability of planar graphs with maximum degree at least 10, Journal of Combinatorial Optimization, 38: 185–196, 2019.

(3) **Qiancheng Ouyang**, Guanghui Wang, Local antimagic labeling on directed graphs, submitted.

(4)* Jie Hu, Shun-ichi Maezawa, **Qiancheng Ouyang**, On sufficient conditions for weakly proper spanning trees in edge-colored graphs, manuscript.

(5) Tianjiao Dai, Hao Li, **Qiancheng Ouyang**, Zengxian Tian, On $(2, k)$-Hamilton-connected graphs, accepted to *Discrete Applied Mathematics*.

(6)* Tianjiao Dai, Hao Li, Yannis Manoussakis, **Qiancheng Ouyang**, Properly colored cycles in edge-colored complete graphs, submitted.

(7) Tianjiao Dai, **Qiancheng Ouyang**, François Pirot, Jean-Sébastien Sereni, Beyond the fractional Reed bound for triangle-free graphs, manuscript.

(8)* Tianjiao Dai, **Qiancheng Ouyang**, François Pirot, New bounds for odd colourings of graphs, submitted to *Electronic Journal of Combinatorics*.

(9) Tianjiao Dai, **Qiancheng Ouyang**, François Pirot, An asymptotic version of the pcf-colouring conjecture, in preparation.

International conferences

(1) Qiancheng Ouyang, **Tianjiao Dai**, Yue Ma, A Hypergraph Approach for Logic-based Abduction, In *15th International Conference on Advances in Databases, Knowledge, and Data Applications (DBKDA 2023)*, Barcelona, Spain, 2023.

Note that the papers marked with * form the main content of this thesis. Other papers are introduced briefly in Appendix.

Appendix

Some of my works are not included in this thesis, which have been (or will be) contained in theses of my collaborators. I introduce these works briefly here.

(1) A graph G is called (k_1, k_2)-Hamilton-connected, if for any two disjoint vertex subsets $X = \{x_1, x_2, \ldots, x_{k_1}\}$ and $U = \{u_1, u_2, \ldots, u_{k_2}\}$, there are k_1k_2 internally disjoint paths connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, which span the whole graph. Let $\sigma_2(G)$ be the minimum value of $deg(u) + deg(v)$ over all pairs $\{u, v\}$ of non-adjacent vertices in G. In this paper, we prove that an n-vertex graph G is $(2, k)$-Hamilton-connected if G is $(5k - 4)$-connected with $\sigma_2(G) \geq n + k - 2$ where $k \geq 2$. We also prove that if $\sigma_2(G) \geq n + k_1k_2 - 2$ with $k_1, k_2 \geq 2$, then G is (k_1, k_2)-Hamilton-connected. Moreover, these bounds of the two results are sharp.

(2) Given a graph H, we let $\chi_f(d, H)$ be the supremum of the fractional chromatic numbers over all H-free graphs of maximum degree at most d. We focus on the case $H = K_3$.

It has been settled in a work including the fourth author that $\chi_f(3, K_3) = 14/5$. Moreover, in a more recent work, the third and fourth authors have used hard-core distributions on the independent sets of triangle-free graphs in order to derive the best known upper bounds for $\chi_f(d, K_3)$ when $d \geq 17$, while for $d \in \{4, \ldots, 16\}$ the best known upper bound remained $\chi_f(d, K_3) \leq \frac{d+3}{2}$, which follows from the fractional Reed bound, established by Molloy and Reed.

Here, we use mixed probability distributions in order to further improve these bounds — which, in particular, yields that the fractional Reed bound is never tight for triangle-free graphs of maximum degree at least 3. We prove that $\chi_f(4, K_3) < 3.4663$, and that $\chi_f(d, K_3) < \min\{\frac{2d+10}{5}, \frac{d+8}{3}\}$ for every $d \geq 4$.

(3) We present a hypergraph approach for logic-based abduction in [66].

Abduction reasoning, which finds possible hypotheses from existing observations, has been studied in many different areas. We consider an abduction problem that takes into account a user's interest. We propose a new approach to solving such an abduction problem based on a hypergraph representation of an ontology and obtain a linear algorithm for a description logic.