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Quelques résultats combinatoires autour de la décomposition de Littlewood

David Wahiche

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L'UNIVERSITE CLAUDE BERNARD LYON 1**

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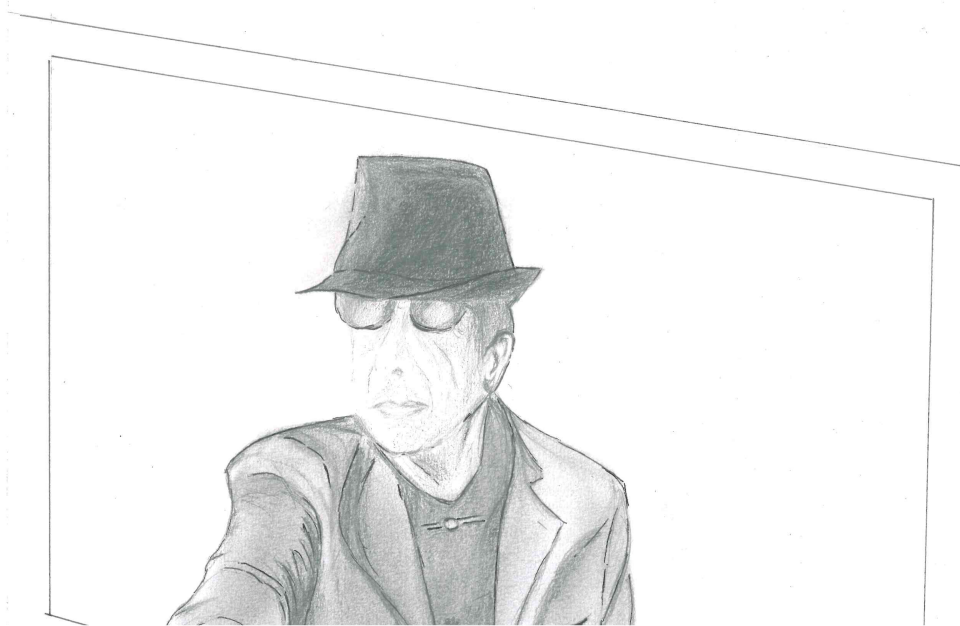
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**Quelques résultats combinatoires
autour de la décomposition de
Littlewood**

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Quelques résultats combinatoires autour de la décomposition de Littlewood



David WAHICHE
Thèse de doctorat

Université Claude Bernard Lyon 1
École doctorale InfoMaths (ED 512)
Spécialité : Mathématiques
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Quelques résultats combinatoires autour de la décomposition de Littlewood

Thèse présentée en vue d'obtenir le diplôme de
Doctorat de l'Université de Lyon

soutenue publiquement le 12/06/2023 par
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Jiang Zeng	(Université Lyon 1)	Examineur

L'enfer des vivants n'est pas chose à venir ; s'il y en a un, c'est celui qui est déjà là, l'enfer que nous habitons tous les jours, que nous formons d'être ensemble. Il y a deux façons de ne pas en souffrir. La première réussit aisément à la plupart : accepter l'enfer, en devenir une part au point de ne plus le voir ; la seconde est risquée et elle demande une attention, un apprentissage, continuel : chercher et savoir reconnaître qui et quoi, au milieu de l'enfer, n'est pas l'enfer, et le faire durer, et lui faire de la place.

– Italo Calvino, *Les villes invisibles*

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– Saint-Exupéry, *Le Petit Prince*

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À larguer les amarres

Pour aller n'importe où, pour aller jusqu'au bout des chemins de fortune

Pour cueillir, en rêvant, une rose des vents sur un rayon de lune.

– Serge Reggiani, Georges Moustaki

Avant-propos

Cette thèse porte essentiellement sur de la combinatoire énumérative, c'est-à-dire l'étude et le comptage de structures finies.

Le but premier de la combinatoire énumérative est le dénombrement, c'est-à-dire la détermination du nombre d'éléments d'un ensemble. Plus précisément on dit qu'un ensemble E fini ou discret si l'on dispose d'une application taille $h : E \rightarrow \mathbb{N}$, où \mathbb{N} est l'ensemble des entiers naturels, telle que les sections $E_n := \{x \in E \mid h(x) = n\}$ soient finies pour tout entier n . Dans le cas où E est fini, un exemple de taille peut correspondre à une indexation des éléments. C'est dans le cas où E est discret et infini que la notion de "section" prend tout son sens : on cherche effectivement à découper E par "tranches" d'ensembles sur lesquelles l'application taille est constante. Le problème initial potentiellement infini est alors discrétisé si on ne s'intéresse qu'aux éléments de E_n . Il n'est cependant pas toujours possible de trouver une formule explicite pour le cardinal de E_n . Il peut alors être utile d'étudier sa série génératrice. En effet, si cette dernière est égale à des séries génératrices correspondant à d'autres ensembles, cela implique alors qu'il existe une bijection entre les deux ensembles. Cette étude permet donc non seulement d'étudier le cardinal de l'ensemble E_n , mais donne aussi des informations sur certaines caractéristiques de cet ensemble comme certaines propriétés d'invariance.

Les problèmes liés à la combinatoire, de par leur aspect très appliqué, existent depuis au moins l'Antiquité, que cela soit en Europe comme en Grèce ou en Asie comme en Inde ou en Chine. D'ailleurs certains de ces objets asiatiques servent à ce jour d'inspiration pour des objets plus abstraits comme les abaqués. En Europe, Fibonacci au XIII^{ème} siècle, est un des pionniers de la discipline. La discipline a ensuite connu un essor particulier avec le développement des probabilités à partir du XVII^{ème} siècle avec, entre autres, les travaux de Pascal et Fermat. En effet, le dénombrement intervient lorsqu'on cherche à calculer la probabilité d'un événement : c'est le nombre d'événements divisé par le nombre total d'événements. On peut aussi citer Euler qui introduit et généralise l'usage des séries génératrices. Plus récemment, au XX^{ème} siècle, le développement et l'essor de l'informatique ont apporté à la combinatoire des questions nouvelles. En effet, l'algorithmique s'intéresse à la systématisation de résolution de problèmes sur des données finies. Il a par conséquent été nécessaire d'étudier la structure de ces problèmes afin d'améliorer l'efficacité des algorithmes, comme pour des algorithmes de tri. Une approche combinatoire est alors un des outils permettant cela. Le développement récent de l'apprentissage statistique a aussi contribué à la nécessité pour les personnes étudiant la combinatoire de se pencher sur de nouveaux modèles.

Les partitions d'entiers sont un des principaux thèmes de la combinatoire énumérative et algébrique ainsi que de cette thèse. Ces dernières ont été introduites en 1748 par Euler qui a démontré bon nombre d'identités comme le théorème des nombres pentagonaux. Leur énumération est déterminée par un algorithme récursif et leur série génératrice a été largement étudiée. Citons ici les travaux de Hardy et Ramanujan qui fournissent l'asymptotique du nombre de

partitions d'un entier. Cependant on ne connaît toujours à ce jour de formule close pour le nombre de partitions d'un entier n . Un des intérêts de la théorie des partitions d'entiers est le lien qu'elle entretient avec la théorie des représentations du groupe symétrique : il est en effet possible d'indexer les fonctions de Schur, à savoir les caractères des représentations irréductibles du groupe symétrique par des partitions.

Un sous-ensemble des partitions, le sous-ensemble des partitions t -cœurs, fut introduit en 1940 par Nakayama [Nak41] pour l'étude des représentations du groupe symétrique. C'est notamment pour l'étude de ces objets que Littlewood introduisit dans [Lit40] une décomposition entre partitions d'entiers et une paire constituée d'une partition t -cœur et de t partitions d'entiers qu'on appelle le t -quotient. James et Kerber [JK81] ont remis au goût du jour à partir du début des années 1980 cette décomposition sous la dénomination "décomposition de Littlewood". Cette décomposition est un des objets centraux de cette thèse. En 1990, Garvan, Kim et Stanton [GKS90] se servirent de cette décomposition afin d'obtenir une preuve bijective des congruences de Ramanujan. On peut ici citer les travaux d'Ono [Ono95] sur l'existence ou non de partitions t -cœurs pour un entier n donné. Des généralisations comme les partitions qui sont à la fois des t -cœurs et des s -cœurs ont été aussi étudiées plus récemment, citons ici les travaux d'Anderson [And02], Fayers [Fay19], Olsson et Stanton [OS07] ou encore Cho, Huh et Sohn [CHS20].

L'étude des t -cœurs apparaît aussi naturellement en théorie des nombres, notamment dans les développements en séries entières de la fonction η de Dedekind [Ser85]. Par exemple, la conjecture de Lehmer porte sur la non-nullité des coefficients de la fonction η . En 2006, Nekrasov et Okounkov, dans leur études de la théorie de Seiberg–Witten sur les jauges supersymétriques, ont obtenu une formule portant leur nom liant puissance de la fonction η d'une part et partitions d'entiers d'autre part. Ce résultat avait été déjà découvert simultanément par Westbury [Wes06] dans son travail sur les caractères universels des identités de Macdonald. Cette formule a été redécouverte par Han [Han10] en 2010 en utilisant une description bijective des t -cœurs ainsi qu'une spécialisation d'une identité de Macdonald pour le système de racines affines de type \tilde{A}_{t-1} . Cette démonstration crée un pont entre les identités de Macdonald pour les systèmes de racines affines d'une part, et les partitions t -cœurs d'autre part. Explorant les liens entre certains sous-ensembles de partitions t -cœurs et certaines spécialisations des identités de Macdonald, Pétréolle [Pé15b] a découvert deux nouvelles formules de Nekrasov–Okounkov pour d'autres systèmes de racines affines. En 2018, Carlsson et Rodriguez Villegas d'une part [CRV18], Rains et Warnaar [RW18] d'autre part, ont découvert un (q, t) -analogue de la formule de Nekrasov–Okounkov. Ce faisant, ils remarquent qu'un q -analogue était déjà obtenu grâce à des travaux combinatoires de Dehaye et Han [DH11] d'une part mais aussi grâce à des méthodes issues de la physique mathématique avec les travaux d'Iqbal, Nazir, Raza et Saleem [INRS12]. Ces derniers obtiennent très élégamment des q déformations des formules de Nekrasov–Okounkov grâce à la théorie des opérateurs de vertex.

Contenu de la thèse

Chapitre 1 (en anglais)

Ce chapitre a pour but d'introduire les définitions nécessaires à la compréhension de cette thèse. Il sert en particulier à rappeler les définitions des partitions d'entiers ainsi qu'à introduire les statistiques liées à ces dernières qui vont jouer un rôle particulier dans cette thèse. Un des ponts entre la combinatoire des partitions et la théorie des représentations est celui effectué par la notion introduite après, les tableaux de Young. Ces derniers peuvent servir à introduire les fonctions de Schur, qui peuvent être vues comme des caractères irréductibles de groupe symétrique ou de permutations. Ensuite sont introduites dans un premier temps les identités de Macdonald pour les 7 systèmes infinis de racines affines, ou plus exactement elles sont présentées en adoptant le point de vue de Rosengren–Schlosser ou Stanton, à savoir comme des

séries entières multivariées. Ces identités portent sur des vecteurs d’entiers et sont naturellement associés à des formes quadratiques. Cette connexion permet de motiver les interprétations de ces vecteurs d’entiers dans le monde de la combinatoire des partitions d’entiers présentes dans les chapitres suivants. L’interprétation combinatoire de ces identités peut, comme mentionné dans le paragraphe ci-avant, servir à obtenir des formules de type Nekrasov–Okounkov, qui sont introduites dans la dernière section. Ces dernières sont des identités de type “somme=produit” dont la partie somme porte sur les partitions d’entiers et dont les coefficients dépendent de statistiques données par le diagramme de Ferrers des partitions, à savoir les longueurs d’équerre. L’étude des connexions entre les identités de Macdonald d’une part et les formules de type Nekrasov–Okounkov d’autre part sont une des motivations principales de ce manuscrit.

Chapitre 2 (en anglais)

Comme mentionné ci-avant, des formes quadratiques apparaissent naturellement dans les identités de Macdonald. Or Garvan–Kim–Stanton [GKS90] ont montré que des vecteurs d’entiers associés à certaines formes quadratiques étaient en correspondance bijective avec des familles de partitions t -cœurs. Comme mentionné dans l’avant-propos, un moyen d’étude des t -cœurs est une bijection, appelée décomposition de Littlewood. Nous nous intéressons donc à une manière de décrire de façon explicite cette bijection. L’outil central pour ce faire est la correspondance bijective entre partitions d’entiers et mots binaires bi-infinis sur l’alphabet $\{0, 1\}$. Nous expliquons ensuite comment cette correspondance nous permet de décrire explicitement la décomposition de Littlewood sur les partitions ainsi que sur certaines sous-familles de partitions qui ont un intérêt algébrique : des partitions dont le diagramme de Ferrers a des propriétés de symétrie par rapport à la première bissectrice du plan (en notation à la française). Ensuite nous explicitons une correspondance bijective entre les vecteurs d’entiers associés aux formes quadratiques présentes dans les identités de Macdonald et certaines sous-familles de partitions qui ont pour propriété de pouvoir être définies explicitement par la décomposition de Littlewood. Enfin nous utilisons la correspondance entre partitions et mots binaires bi-infinis pour introduire une notion centrale qui est uniforme pour tous les types de systèmes infinis de racines affines à savoir le vecteur d’entier qui est un $V_{g,t}$ -codage d’une partition, où g est autre entier positif qui dépend de t . Ces vecteurs d’entiers peuvent être vus comme un t -uplet d’indices de lettres “0” dans le mot binaire bi-infini correspondant à une partition. Ces vecteurs d’entiers constituent l’ingrédient principal de la preuve de théorèmes d’énumération sur les produits d’équerres des partitions définies dans la section précédente pour chaque système infini de racines affines.

Chapitre 3 (en anglais)

Les preuves de ces théorèmes d’énumération d’équerres nous permettent de retranscrire, avec des statistiques de partitions d’entiers comme certains sous-ensembles de longueurs d’équerres ou encore le carré de Durfee, la signature de permutations permettant de réordonner des vecteurs d’entiers. Ce sont ces mêmes vecteurs d’entiers qui apparaissent dans les formes quadratiques des identités de Macdonald. Cette correspondance bijective entre vecteurs d’entiers et familles de partitions permet alors de réécrire chacune des 7 identités de Macdonald comme une somme portant sur des sous-familles de partitions et de fonctions de Schur. Enfin nous explicitons comment passer de la réécriture de ces identités à des formules de type Nekrasov–Okounkov.

Chapitre 4 (en anglais)

Si le chapitre précédent porte plutôt sur l’étude des partitions d’entiers t -cœurs ou presque t -cœurs, ici nous étudions l’autre pan de la décomposition de Littlewood, à savoir le t -quotient. Nous étudions plus particulièrement comment des propriétés fines de la décomposition de Littlewood permettent des analogues “modulaires” de formules d’équerre. Nous nous cantonnerons dans cette partie à l’étude des partitions auto-conjuguées.

Chapitre 5 (en français)

Ce chapitre constitue une brève conclusion à cette thèse et liste les perspectives potentielles de recherche que soulève cette thèse.

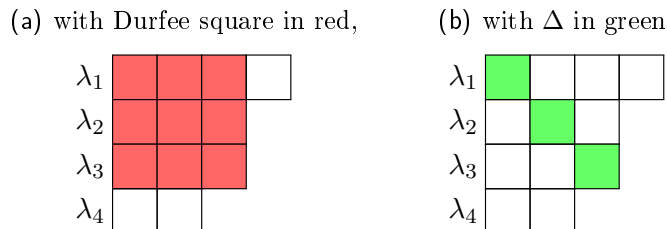
1.1 Partitions and compositions

A sequence of nonnegative integers $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ is called a *weak composition* of an integer n into k entries if $|\mu| := \mu_1 + \mu_2 + \dots + \mu_k = n$. By definition, negative integers do not have any weak composition. Moreover any nonnegative integer has an infinite number of weak compositions if the number of entries is unbounded. The set of all compositions of n into k parts is denoted by $\mathcal{C}_{n,k}$.

A *partition* λ is a composition of a positive integer n on an infinite number of entries, where the entries are in weakly decreasing order. Therefore there exists $\ell \in \mathbb{N}$ such that $\lambda = 0$ for any $i \geq \ell$. The positive entries of λ are called the *parts* and the number of parts is called the *length*, denoted by $\ell(\lambda)$. Partitions whose entries only differ by their number of zeros are held to be equal. The set of all partitions of n is denoted by $\mathcal{P}(n)$. Hence the set of all partitions is $\mathcal{P} := \cup_{n \geq 0} \mathcal{P}(n)$. For example, there are 5 partitions in $\mathcal{P}(4)$: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$. In this manuscript, λ and μ mostly denote partitions. The *symmetric group* S_n is the group of permutations on n indices, which acts on a sequence $\mathbf{x} = (x_1, \dots, x_n)$ by permutation of the entries x_i .

Each partition can be represented by its Ferrers diagram, which consists in a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. The *Durfee square* of λ is the maximal square fitting in the Ferrers diagram. Its diagonal, denoted by Δ , will be called the main diagonal of λ . Its size will be denoted $D_\lambda := \max\{s \mid \lambda_s \geq s\}$. As an example, in Figure 1.1a, the Durfee square of $\lambda = (4, 3, 3, 2)$, which is a partition of 12 of length 4, is colored in red. The main diagonal Δ is shaded in green in Figure 1.1b. The partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda'_1})$ is the *conjugate* of λ , where λ'_j denotes the number of boxes in the column j . For example, the conjugate of $\lambda = (4, 3, 3, 2)$ is $\lambda' = (4, 4, 3, 1)$. Note that the sum of parts of λ is invariant by conjugation, as well as its Durfee square and its main diagonal Δ . Indeed another way of describing the conjugation is to see it as the reflection of axis Δ .

Figure 1.1: A partition $\lambda = (4, 3, 3, 2) \in \mathcal{P}(12)$



When looking at this reflection with respect to Δ , one would want to consider the subset of partitions that remain invariant by this transformation. A partition λ is *self-conjugate* if its Ferrers diagram is symmetric along the main diagonal, that is equivalent to $\lambda = \lambda'$. Let \mathcal{SC} be the set of self-conjugate partitions. The set of doubled distinct partitions, denoted by \mathcal{DD} , is

that of all partitions λ of Durfee square size d_λ such that $\lambda_i = \lambda'_i + 1$ for all $i \in \{1, \dots, d_\lambda\}$. We also define the set of conjugates of doubled distinct partitions $\mathcal{DD}' := \{\lambda \mid \lambda' \in \mathcal{DD}\}$. For instance, in Figure 1.2a $\lambda = (5, 3, 3, 1, 1) \in \mathcal{SC}$ has its main diagonal Δ shaded in green while in Figure 1.2b $\lambda = (6, 4, 4, 1, 1) \in \mathcal{DD}$ has its main diagonal shaded in green. As for the strip shaded in yellow, it corresponds to the boxes added to a self-conjugate partition to obtain a doubled distinct partition. The conjugate of a doubled distinct partition is also illustrated in Figure 1.2c. These subsets of partitions already appear algebraically in particular within the works of Koike–Terada on Young diagrammatic representation of classical groups [KT90] and Macdonald [Mac95, p. 79] when he expresses the Weyl denominator formula for types B, C and D in terms of Schur functions. Another statistic that will play a particular role in this manuscript is ε_s , defined as -1 if the box s is strictly below Δ in the Ferrers diagram of λ and as 1 otherwise, as depicted in Figure 1.2. This signed statistic already appears algebraically within the work of King [Kin90] and combinatorially within the work of Pétréolle [Pé15b].

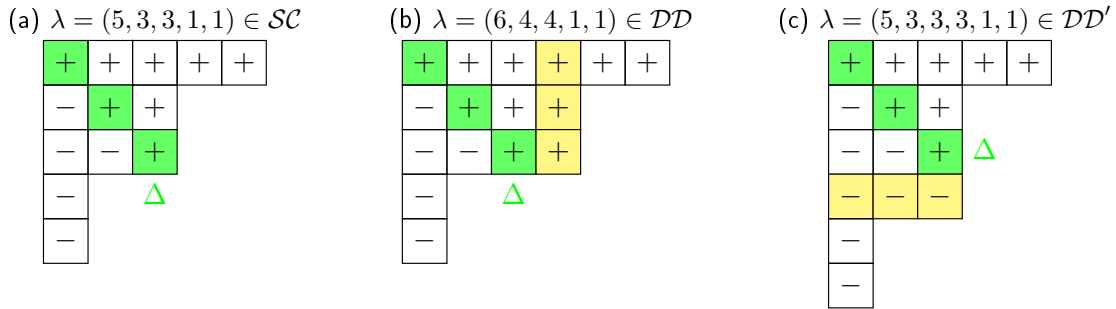


Figure 1.2: Self-conjugate, doubled distinct partitions and their conjugate filled with ε

For each box s in the Ferrers diagram of a partition λ (for short we will say for each box s in λ), one defines the *arm-length* (respectively *leg-length*) as the number of boxes in the same row (respectively in the same column) as v strictly to the right of (respectively strictly below) the box v . The *hook* associated to a box s is the collection of boxes u such that either $u = s$, or u lies strictly below (respectively to the right) of s in the same column (respectively row). The hook of the box of coordinates $(1, 2)$ is shaded in blue in Figure 1.3a. The cardinal of the hook is called the *hook length* of s , denoted by $h_s(\lambda)$ or h_s . The *hook length multiset* of λ , denoted by $\mathcal{H}(\lambda)$, is the multiset of all hook lengths of λ . For any positive integer t , the multiset of all hook lengths that are congruent to $0 \pmod{t}$ is denoted by $\mathcal{H}_t(\lambda)$. Notice that $\mathcal{H}(\lambda) = \mathcal{H}_1(\lambda)$. A partition ω is a t -core if it has no hook that is a multiple of t , i.e. $\mathcal{H}_t(\omega) = \emptyset$. For any $\mathcal{A} \subset \mathcal{P}$, we denote by $\mathcal{A}_{(t)}$ the subset of elements of \mathcal{A} that are t -cores. For example, the only 2-cores are the “staircase” partitions $(k, k - 1, \dots, 1)$, where k is any positive integer, which are also the only \mathcal{SC} 2-cores. In Figure 1.3b, the hook lengths of all boxes for the partition $\lambda = (4, 3, 3, 2)$ are written in their corresponding boxes, the boxes associated with $\mathcal{H}_3(\lambda)$ are shaded in red. The hook length multiset $\mathcal{H}(\lambda)$ is here given by $\{2, 1, 4, 3, 1, 5, 4, 2, 7, 6, 4, 1\}$ and $\mathcal{H}_3(\lambda) = \{3, 6\}$.



Figure 1.3: A hook and the hook lengths

A *rim hook* (or border strip, or ribbon) is a connected skew shape containing no 2×2 square. The length of a rim hook is the number of boxes in it, and its height is one less than its number of rows. By convention, the height of an empty rim hook is zero. Another possible definition of the t -core of a partition λ is to define it as the remaining partition after removing iteratively all the rim hooks of length t of λ until this operation is no longer possible. For example, if $t = 3$ and $\lambda = (4, 3, 3, 2)$, the 3-core of λ is the partition $\omega = (2, 2, 1, 1)$.

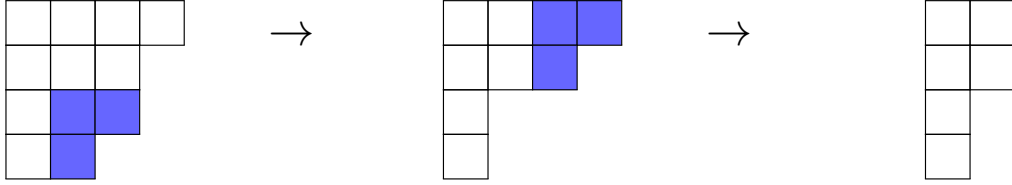


Figure 1.4: Successive removal of rim hooks of length 3 from $\lambda = (4, 3, 3, 2)$

Another statistic of integer partitions intervening in formulas bridging integer partitions and representation theory is the *content* c of a box s of coordinates (i, j) and is defined by $c(s) := j - i$. One can remark that the sign of $c(s)$ corresponds to ε_s .

1.2 Young tableaux

Our later discussions of the Schur functions might use the notion of Young tableaux. Here we briefly introduce these simple combinatorial objects. Fix a positive integer n . A *semi-standard Young tableau* of shape λ is a Young diagram of shape λ filled with the integers 1 to n such that the entries are weakly increasing from left to right along rows and strictly increasing down columns. A *standard Young tableau* is a semi-standard Young tableau of shape λ with $|\lambda| = n$, filled with integers from 1 to n so that each integer occurs exactly once. Formulas involving hook lengths abound in combinatorics and representation theory. One illustrative example is the hook length formula discovered in 1954 by Frame, Robinson and Thrall [FRT54]. It states the equality between the number f^λ of standard Young tableaux of shape λ and size n , and the number of permutations of $\{1, \dots, n\}$ divided by the product of the elements of the hook lengths multiset $\mathcal{H}(\lambda)$ of λ , namely:

$$f^\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}. \quad (1.1)$$

The *type* μ of a semi-standard Young tableau is a composition (μ_1, \dots, μ_n) where μ_i corresponds to the number of occurrences of the integer i as an entry of the semi-standard Young tableaux. We denote by $SSYT(\lambda, \mu)$ the finite set of semi-standard Young tableaux of shape λ and type μ such that $|\lambda| = |\mu|$ and by $SSYT(\lambda)$ the set of semi-standard Young tableaux such that $n = |\lambda| = |\mu|$ and $\mu = (1, \dots, 1)$: an element of $SSYT(\lambda)$ is a Young tableau filled with integers from 1 to n such that each number occurs exactly once. For example, the following semi-standard Young tableaux both have shape $(4, 4, 1, 1)$ and type $(2, 2, 3, 0, 2, 1)$:



Figure 1.5: Two semi-standard Young tableaux

1.3 Schur functions

Let n be a positive integer and let $\mathbf{x} = (x_1, \dots, x_n)$ be a set of any independent variables. Let $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, \dots, x_n]$ be the ring of polynomials in x_1, \dots, x_n with integer coefficients. A polynomial $P(\mathbf{x}) := P(x_1, \dots, x_n)$ in $\mathbb{Z}[\mathbf{x}]$ is *symmetric* if it is invariant under permutations of its variables, i.e. for every $\sigma \in S_n$:

$$P(x_1, \dots, x_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The set of all symmetric polynomials in n variables, denoted by Λ_n , is a vector subspace of $\mathbb{Z}[\mathbf{x}]$. The Schur polynomials have been an object of extensive studies as they can be thought of an orthonormal basis of Λ_n for the Hall inner product that we will not discuss in this manuscript. We will try to follow here Macdonald's path in [Mac95] and as presented in Bartlett's thesis [Bar14]. For $\alpha = (\alpha_1, \dots, \alpha_n)$, let $a_\alpha(\mathbf{x}) := a_\alpha$ be the anti-symmetrization of the monomial $\mathbf{x}^\alpha := (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$:

$$a_\alpha = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \mathbf{x}^{\sigma(\alpha)},$$

where $\text{sgn}(\sigma)$ is the *sign* (or *signature*) of the permutation σ . Moreover by construction, the polynomial is skew-symmetric:

$$\sigma(a_\alpha) = \text{sgn}(\sigma) a_\alpha.$$

An interesting consequence of the above property is that if there exists $(i, j) \in \{1, \dots, n\}^2$ such that $i \neq j$ and $\alpha_i = \alpha_j$, then the permutation (ij) merely introduces a sign, so we have $a_\alpha = -a_\alpha$. Therefore $a_\alpha = 0$ unless $\alpha_1, \dots, \alpha_n$ are all distinct. Therefore one can reorder α up to a sign so that $\alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$. Hence we can set $\alpha = \lambda + \delta$ where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition and $\delta := (n-1, \dots, 1, 0)$ is the staircase partition. With this rewriting, a_α naturally admits a determinant form

$$a_{\lambda+\delta} = \det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j} \right).$$

By the skew-symmetry property, if $x_i = x_j$ for $i \neq j$, $a_{\lambda+\delta} = 0$ and the polynomial $(x_i - x_j)$ is a divisor of $a_{\lambda+\delta}$ in $\mathbb{Z}[\mathbf{x}]$. Therefore the *Vandermonde determinant* $\Delta_A(\mathbf{x})$ (see (1.2)) is a divisor of $a_{\lambda+\delta}$. Recall

$$\Delta_A(\mathbf{x}) := \det_{1 \leq i, j \leq n} \left(x_i^{n-j} \right) = \prod_{1 \leq i < j \leq n} (x_j - x_i). \quad (1.2)$$

The *Schur polynomial* is given by

$$s_\lambda(\mathbf{x}) := \frac{a_{\lambda+\delta}}{a_\delta}. \quad (1.3)$$

First observe that $a_\delta = \Delta_A(\mathbf{x})$. By the alternating property of the determinant, a Schur polynomial is invariant by any $\sigma \in S_n$ as a quotient of determinants. One can think of a Schur polynomial as a *general linear (type A) character* of GL_n . We mention here an interesting specialization of the Schur polynomials called the hook-content formula (see [Sta71]):

$$s_\lambda(1, \dots, 1) = \prod_{s \in \lambda} \frac{n + c_s}{h_s}.$$

Schur polynomials have been extensively studied and there exist multiple approaches, such as (1.3), or as a sum over semi-standard Young tableaux (see [FH91]).

If one thinks of the Schur polynomials as invariant by the action of any element in S_n , one can introduce the analogues for other groups of signed permutations (see [AK22, FH91, Kra99, Kra05] for more details and background).

The *odd orthogonal (type B)* character of the group $\mathrm{SO}(2n+1)$ is given by

$$\begin{aligned} \mathrm{so}_\lambda(\mathbf{x}) &:= \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j + 1/2} - x_i^{-(\lambda_j + n - j + 1/2)} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j + 1/2} - x_i^{-(n - j + 1/2)} \right)} \\ &= \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j + 1} - x_i^{-(\lambda_j + n - j)} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j + 1} - x_i^{-(n - j)} \right)}, \end{aligned} \quad (1.4)$$

and the Weyl denominator formula here is:

$$\begin{aligned} \Delta_B(\mathbf{x}) &:= \det_{1 \leq i, j \leq n} \left(x_i^{j-n} - x_i^{-(j-n)+1} \right) \\ &= \prod_{1 \leq i \leq n} x_i^{1-n} (1 - x_i) \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - x_i x_j). \end{aligned} \quad (1.5)$$

The *symplectic (type C)* character of the group $\mathrm{Sp}(2n)$ is given by

$$\mathrm{sp}_\lambda(\mathbf{x}) := \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - i + 1} - x_i^{-(\lambda_j + n - i + 1)} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j + 1} - x_i^{-(n - j + 1)} \right)}, \quad (1.6)$$

and the corresponding Weyl denominator formula here is:

$$\begin{aligned} \Delta_C(\mathbf{x}) &:= \det_{1 \leq i, j \leq n} \left(x_i^{j-n-1} - x_i^{-(j-n-1)} \right) \\ &= \prod_{1 \leq i \leq n} x_i^{-n} (1 - x_i^2) \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - x_i x_j). \end{aligned} \quad (1.7)$$

Lastly, the *even orthogonal (type D)* character of the group $\mathrm{O}(2n)$ is given by

$$\mathrm{o}^{\mathrm{even}}_\lambda(\mathbf{x}) := \frac{2 \det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - i} + x_i^{-(\lambda_j + n - i)} \right)}{(1 + \delta_{\lambda_n, 0}) \det_{1 \leq i, j \leq n} \left(x_i^{n - j} + x_i^{-(n - j)} \right)}, \quad (1.8)$$

where δ is the Kronecker delta. There is an extra factor in the denominator because the last column becomes 2 if $\lambda_n = 0$. Note that the Weyl denominator formula here reads as follows:

$$\Delta_D(\mathbf{x}) := \det_{1 \leq i, j \leq n} \left(x_i^{j-n} + x_i^{-(j-n)} \right) = 2 \prod_{1 \leq i \leq n} x_i^{1-n} \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - x_i x_j). \quad (1.9)$$

1.4 Macdonald identities for affine root systems

As we will not need it here, we do not give the definitions of (affine) root systems. Nevertheless we refer an interested reader, who wants to read a proper introduction and the definitions of finite and affine root systems, to [Bar14, Part I Chapter 2]. Using the notations of Macdonald [Mac72], there are seven infinite reduced affine root systems: $\tilde{A}_t, \tilde{B}_t, \tilde{B}_t^\vee, \tilde{C}_t, \tilde{C}_t^\vee, \tilde{D}_t$ and $\tilde{B}C_t$. Note that the notations of Kac in his book [Kac90, p.80] slightly differ, \tilde{X}_t becomes $X_t^{(1)}$ for X in A, B, C, D , and $\tilde{B}_t^\vee, \tilde{C}_t^\vee$ and $\tilde{B}C_t$ are respectively denoted by $A_{2t-1}^{(2)}, D_{t+1}^{(2)}$ and $A_{2t}^{(2)}$.

The Macdonald identity for affine root systems can be thought of as a generalization of the Weyl denominator formulas, specialized before for the finite root systems A_{t-1}, B_t, C_t and D_t . The latter can be stated as follows for a reduced root system R

$$\sum_{\sigma \in W} \mathrm{sgn}(\sigma) e^{\sigma(\rho) - \rho} = \prod_{\alpha \in R_+} (1 - e^{-\alpha}), \quad (1.10)$$

where W is the Weyl group associated with R , R_+ is the set of positive roots, e^α is a formal exponential and $\rho = 1/2 \sum_{\alpha \in R_+} \alpha$. The Macdonald identity for affine root systems connects a sum depending on the associated Weyl group W , which is a group of (signed) permutation acting on the affine root system, and a parameter G , which can be seen as the dual Coxeter number h^\vee , except for type \tilde{C}_t , where $g = 2h^\vee$, to a product over the positive roots.

Even though none of the techniques used by Macdonald in [Mac72] are required in this manuscript, it is still interesting to see what the right-hand side of (1.10) becomes for infinite affine root systems. So we recall the roots and positive roots for each infinite type. Let $(e_i)_{1 \leq i \leq t}$ be a basis for \mathbb{R}^t , and let $k + \varepsilon_i$ be the affine function $e_j \mapsto k + \delta_{ij}$, with δ_{ij} the Kronecker symbol. Then in type

1. \tilde{A}_t for $t \geq 1$, the set of affine roots is $\{k \pm (\varepsilon_i - \varepsilon_j), k \in \mathbb{Z}, 1 \leq i < j \leq t\}$ and the one of positive roots is $\{k + (\varepsilon_i - \varepsilon_j), k \geq 0, 1 \leq i < j \leq t\} \cup \{k + (\varepsilon_j - \varepsilon_i), k \leq -1, 1 \leq i < j \leq t\}$.
2. \tilde{B}_t for $t \geq 3$, the affine roots are

$$\begin{aligned} k \pm \varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t. \end{aligned}$$

The positive roots are

$$\begin{aligned} k + \varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ k - \varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

3. \tilde{B}_t^\vee for $t \geq 3$, the affine roots are

$$\begin{aligned} 2k \pm 2\varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t. \end{aligned}$$

The positive roots are

$$\begin{aligned} 2k + 2\varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ 2k - 2\varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

4. \tilde{C}_t for $t \geq 2$, the affine roots are

$$\begin{aligned} k \pm 2\varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t. \end{aligned}$$

The positive roots are

$$\begin{aligned} k + 2\varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ k - 2\varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

5. \tilde{C}_t^\vee for $t \geq 3$, the affine roots are

$$\begin{aligned} \frac{1}{2}k \pm 2\varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t. \end{aligned}$$

The positive roots are

$$\begin{aligned} \frac{1}{2}k + \varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ \frac{1}{2}k - \varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

6. \tilde{BC}_t for $t \geq 1$, the affine roots are

$$\begin{aligned} k \pm \varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ 2k + 1 \pm 2\varepsilon_i, k \in \mathbb{Z}, 1 \leq i \leq t, \\ k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t. \end{aligned}$$

The positive roots are

$$\begin{aligned} k + \varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ k - \varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ 2k + 1 + 2\varepsilon_i, k \geq 0, 1 \leq i \leq t, \\ 2k + 1 - 2\varepsilon_i, k \leq -1, 1 \leq i \leq t, \\ k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

7. \tilde{D}_t for $t \geq 4$, the affine roots are

$$k \pm \varepsilon_i \pm \varepsilon_j, k \in \mathbb{Z}, 1 \leq i < j \leq t.$$

The positive roots are

$$\begin{aligned} k + \varepsilon_i + \varepsilon_j, k + \varepsilon_i - \varepsilon_j, k \geq 0, 1 \leq i < j \leq t, \\ k - \varepsilon_i + \varepsilon_j, k - \varepsilon_i - \varepsilon_j, k \leq -1, 1 \leq i < j \leq t. \end{aligned}$$

Stanton in [Sta89] and Rosengren–Schlosser in [RS06] prove the Macdonald identities in all these seven types by looking at (1.10) as multivariate power series. To illustrate this process, we give here the examples of the right-hand sides of (1.10) in types \tilde{A}_t and \tilde{C}_t . Using the set of positive roots for \tilde{A}_t , the right-hand side of (1.10) becomes

$$\prod_{1 \leq i < j \leq t} \prod_{k \geq 0} \left(1 - e^{-(k+\varepsilon_i-\varepsilon_j)}\right) \left(1 - e^{-(k+1+\varepsilon_j-\varepsilon_i)}\right), \quad (1.11)$$

and for \tilde{C}_t , it is:

$$\begin{aligned} \prod_{1 \leq i \leq t} \prod_{k \geq 0} \left(1 - e^{-(k+2\varepsilon_i)}\right) \left(1 - e^{-(k+1-2\varepsilon_i)}\right) \prod_{1 \leq i < j \leq t} \prod_{k \geq 0} \left(1 - e^{-(k+\varepsilon_i+\varepsilon_j)}\right) \left(1 - e^{-(k+\varepsilon_i-\varepsilon_j)}\right) \\ \times \left(1 - e^{-(k+1+\varepsilon_j+\varepsilon_i)}\right) \left(1 - e^{-(k+1-\varepsilon_j-\varepsilon_i)}\right). \quad (1.12) \end{aligned}$$

To rewrite the above products in a more compact way, we recall some classical notations. For formal variables, the T -Pochhammer symbol is defined as $(a; T)_0 = 1$ and for any integer $n \geq 1$:

$$(a; T)_n = (1-a)(1-aT) \dots (1-aT^{n-1}),$$

$$(a; T)_\infty = \prod_{j \geq 0} (1-aT^j),$$

and $(a_1, \dots, a_n; T)_\infty = (a_1; T)_\infty \dots (a_n; T)_\infty$.

Moreover we introduce the three following shorthand notations:

$$(Ta_1^\pm; T)_\infty := (Ta_1, Ta_1^{-1}; T)_\infty,$$

$$(Ta_1^{\pm 2}; T)_\infty := (Ta_1^2, Ta_1^{-2}; T)_\infty,$$

$$(Ta_1^\pm a_2^\pm; T)_\infty := (Ta_1^{-1} a_2, Ta_1 a_2^{-1}, Ta_1 a_2, Ta_1^{-1} a_2^{-1}; T)_\infty.$$

Then setting $T = e^{-1}$ and $x_i = T^{-\varepsilon_i}$, formula (1.11) becomes

$$\prod_{1 \leq i < j \leq t} \left(x_i x_j^{-1}, x_i^{-1} x_j T; T \right)_\infty,$$

and this rewriting yields in (1.12)

$$\prod_{i=1}^t (x_i^2, T x_i^{-2}; T)_\infty \prod_{1 \leq i < j \leq t} \left(x_i x_j^{-1}, x_i x_j, T x_i^{-1} x_j, T x_i^{-1} x_j^{-1}; T \right)_\infty.$$

For a positive integer t , let us introduce the compact notation:

$$K_T(t, \mathbf{x}) := \prod_{1 \leq i < j \leq t} x_i^{-1} \left(T x_i^\pm x_j^\pm; T \right)_\infty. \quad (1.13)$$

In [RS06, Corollary 6.2], Macdonald identities are written with θ functions. We state here this result in a slightly different way writing the θ functions as infinite products in which we isolate the terms without any power of T . Moreover for types $\tilde{B}_t, \tilde{B}_t^\vee$ and \tilde{D}_t , Macdonald identities are stated as a sum of the sublattice $\{\mathbf{m} \in \mathbb{Z}^t \mid \sum_{i=1}^t m_i \equiv 0 \pmod{2}\}$. As remarked in the proof of [RS06, Corollary 6.2], the sum can be replaced as half the sum over \mathbb{Z}^t .

Proposition 1.4.1 ([RS06], Corollary 6.2). *The Macdonald identities are*

1. For type \tilde{A}_{t-1} with $t \geq 2$

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0 \\ \sigma \in \mathcal{S}_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{t \binom{m_i}{2} + (\sigma(i)-1)m_i} x_i^{tm_i + \sigma(i) - 1}$$

$$= (T; T)_\infty^{t-1} \Delta_A(\mathbf{x}) \prod_{1 \leq i < j \leq n} \left(T x_i x_j^{-1}, T x_i^{-1} x_j; T \right)_\infty, \quad (1.14)$$

where $\Delta_A(\mathbf{x})$ is defined in (1.2).

2. For type \tilde{B}_t with $t \geq 3$

$$\frac{1}{2} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in \mathcal{S}_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{(2t-1) \binom{m_i}{2} + (t-1)m_i} x_i^{(2t-1)m_i} \left((T^{m_i} x_i)^{\sigma(i)-t} - (T^{m_i} x_i)^{t+1-\sigma(i)} \right)$$

$$= (T; T)_\infty^t \Delta_B(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t (T x_i^\pm; T)_\infty,$$

where $\Delta_B(\mathbf{x})$ is defined in (1.5).

3. For type \tilde{B}_t^\vee with $t \geq 3$

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in S_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{2t \binom{m_i}{2} + t m_i} x_i^{2t m_i} \left((T^{m_i} x_i)^{\sigma(i)-t-1} - (T^{m_i} x_i)^{t+1-\sigma(i)} \right) \\ & = (T^2; T^2)_\infty (T; T)_\infty^{t-1} \Delta_C(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t x_i^{-1} (T^2 x_i^{\pm 2}; T^2)_\infty, \end{aligned}$$

where $\Delta_C(\mathbf{x})$ is defined in (1.7).

4. For type \tilde{C}_t with $t \geq 2$

$$\begin{aligned} & \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in S_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{(2t+2) \binom{m_i}{2} + (t+1)m_i} x_i^{(2t+2)m_i} \left((T^{m_i} x_i)^{\sigma(i)-t-1} - (T^{m_i} x_i)^{t+1-\sigma(i)} \right) \\ & = (T; T)_\infty^t \Delta_C(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t x_i^{-1} (T x_i^{\pm 2}; T)_\infty, \quad (1.15) \end{aligned}$$

where $\Delta_C(\mathbf{x})$ is defined in (1.7).

5. For type \tilde{C}_t^\vee with $t \geq 2$

$$\begin{aligned} & \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in S_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{2t \binom{m_i}{2} + (t-1/2)m_i} x_i^{2t m_i} \left((T^{m_i} x_i)^{\sigma(i)-t} - (T^{m_i} x_i)^{t+1-\sigma(i)} \right) \\ & = (T^{1/2}; T^{1/2})_\infty (T; T)_\infty^{t-1} \Delta_B(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t (T^{1/2} x_i^\pm; T^{1/2})_\infty, \end{aligned}$$

where $\Delta_B(\mathbf{x})$ is defined in (1.5).

6. For type $\tilde{B}C_t$ with $t \geq 1$

$$\begin{aligned} & \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in S_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{(2t+1) \binom{m_i}{2} + t m_i} x_i^{(2t+1)m_i} \left((T^{m_i} x_i)^{\sigma(i)-t} - (T^{m_i} x_i)^{t+1-\sigma(i)} \right) \\ & = (T; T)_\infty^t \Delta_B(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t (T x_i^\pm; T)_\infty (T x_i^{\pm 2}; T^2)_\infty, \end{aligned}$$

where $\Delta_B(\mathbf{x})$ is defined in (1.5).

7. For type \tilde{D}_t with $t \geq 4$

$$\begin{aligned} & \frac{1}{4} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ \sigma \in S_t}} \text{sgn}(\sigma) \prod_{i=1}^t T^{(2t-2) \binom{m_i}{2} + (t-1)m_i} x_i^{(2t-2)m_i} \left((T^{m_i} x_i)^{\sigma(i)-t} + (T^{m_i} x_i)^{t-\sigma(i)} \right), \\ & = (T; T)_\infty^t \Delta_D(\mathbf{x}) K_T(t, \mathbf{x}) \end{aligned}$$

where $\Delta_D(\mathbf{x})$ is defined in (1.9).

Hence the Macdonald identities can all be stated as sums over sublattices of \mathbb{Z}^t . To motivate the combinatorics in Chapter 2, we write (1.14) corresponding to type \tilde{A}_{t-1} and (1.15) corresponding to type \tilde{C}_t isolating the power of T . When we complete this step, a quadratic form will appear at the exponent of T , which can be mapped to some subsets of integer partitions defined explicitly through the Littlewood decomposition.

First we begin the rewriting of the Macdonald identity (1.14) for type \tilde{A}_{t-1} . Extracting the powers of T on the left-hand side of (1.14) yields

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0 \\ \sigma \in S_t}} \text{sgn}(\sigma) T^{t \sum_{i=1}^t m_i^2 + \sum_{i=1}^t (\sigma(i) - 1 - t/2) m_i} \prod_{i=1}^t x_i^{t m_i + \sigma(i) - 1}.$$

Substituting $m_i \mapsto m_{\sigma(i)}$ and setting $\|\mathbf{m}\|^2 := \sum_{i=1}^t m_i^2$, this becomes

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0 \\ \sigma \in S_t}} \text{sgn}(\sigma) T^{t \|\mathbf{m}\|^2 / 2 + \sum_{i=1}^t (i-1 - t/2) m_i} \prod_{i=1}^t x_i^{t m_{\sigma(i)} + \sigma(i) - 1}.$$

Hence, using the fact the sum of all m_i is equal to 0, one can rewrite (1.14) as follows

$$\begin{aligned} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0}} T^{t \|\mathbf{m}\|^2 / 2 + \sum_{i=1}^t (i-1) m_i} \frac{\det_{1 \leq i, j \leq t} \left(x_i^{t m_j + j - 1} \right)}{\det_{1 \leq i, j \leq t} \left(x_i^{t - j} \right)} \\ = (T; T)_{\infty}^{t-1} \prod_{1 \leq i < j \leq t} \left(T x_i x_j^{-1}, T x_i^{-1} x_j; T \right)_{\infty}. \end{aligned} \quad (1.16)$$

We proceed similarly for the Macdonald identity for type \tilde{C}_t . Expand the product on the left-hand side of (1.15) and denote by $\xi \in \{\pm 1\}^t$ the sign coming from the expansion of the terms

$$\prod_{i=1}^t T^{(2t+2) \binom{m_i}{2} + (t+1) m_i} x_i^{(2t+2) m_i} \left((T^{m_i} x_i)^{\sigma(i) - t - 1} - (x_i T^{m_i})^{t+1 - \sigma(i)} \right).$$

Then setting $m_i \rightarrow \xi_i m_i$ and finally noting that m_i^2 is invariant by this transformation, the left-hand side of (1.15) can be rewritten as follows:

$$\begin{aligned} \sum_{\sigma \in S_t} \text{sgn}(\sigma) \sum_{\xi \in \{\pm 1\}^t} \prod_{i=1}^t \xi_i \sum_{\mathbf{m} \in \mathbb{Z}^t} \prod_{i=1}^t T^{(t+1) m_i^2 + m_i \xi_i (\sigma(i) - t - 1)} x_i^{(2t+2) m_i + \xi_i (\sigma(i) - t - 1)} \\ = \sum_{\sigma \in S_t} \text{sgn}(\sigma) \sum_{\xi \in \{\pm 1\}^t} \prod_{i=1}^t \xi_i \sum_{\mathbf{m} \in \mathbb{Z}^t} \prod_{i=1}^t T^{(t+1) m_i^2 + m_i (\sigma(i) - t - 1)} x_i^{(2t+2) \xi_i m_i + \xi_i (\sigma(i) - t - 1)}. \end{aligned}$$

By extracting the powers of the variable T , replacing m_i with $m_{\sigma(i)}$ and inverting the sums, we get:

$$\begin{aligned} \sum_{\sigma \in S_t} \text{sgn}(\sigma) \sum_{\xi \in \{\pm 1\}^t} \prod_{i=1}^t \xi_i \sum_{\mathbf{m} \in \mathbb{Z}^t} T^{(t+1) \|\mathbf{m}\|^2 + \sum_{i=1}^t m_i (i-t-1)} \prod_{i=1}^t x_i^{(2t+2) \xi_i m_{\sigma(i)} + \xi_i (\sigma(i) - t - 1)} \\ = \sum_{\mathbf{m} \in \mathbb{Z}^t} T^{(t+1) \|\mathbf{m}\|^2 + \sum_{i=1}^t m_i (i-t-1)} \det_{1 \leq i, j \leq t} \left(x_i^{(2t+2) m_j + j - t - 1} - x_i^{-((2t+2) m_j + j - t - 1)} \right). \end{aligned}$$

The right-hand side of (1.15) is

$$(T; T)_\infty^t \Delta_C(\mathbf{x}) K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^{\pm 2}; T)_\infty$$

with $\Delta_C(\mathbf{x})$ as defined in (1.7). Hence (1.15) becomes

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^t} T^{(t+1)\|\mathbf{m}\|^2 + \sum_{i=1}^t m_i(i-t-1)} \frac{\det_{1 \leq i, j \leq t} \left(x_i^{(2t+2)m_j + j - t - 1} - x_i^{-((2t+2)m_j + j - t - 1)} \right)}{\det_{1 \leq i, j \leq t} \left(x_i^{j-t-1} - x_i^{-(j-t-1)} \right)} \\ = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^{\pm 2}; T)_\infty. \end{aligned} \quad (1.17)$$

1.5 Nekrasov–Okounkov formulas

As mentioned in Section 1.2, hook length formulas such as (1.1) are useful enumerative tools bridging combinatorics with other fields. A much more recent identity is the Nekrasov–Okounkov formula. It was discovered independently by Nekrasov and Okounkov in their work on random partitions and Seiberg–Witten theory [NO06], and by Westbury [Wes06] in his work on universal characters for \mathfrak{sl}_n . This formula is commonly stated as follows:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2} \right) = \prod_{k \geq 1} \left(1 - T^k \right)^{z-1}, \quad (1.18)$$

where z is a fixed complex number. This identity was later obtained independently by Han [Han10], using combinatorial tools and the Macdonald identities for affine type \tilde{A}_t [Mac72, Appendix 1, eq. (6.a) p.143] when all of the $x_i \mapsto 1$. If t is a positive integer and $z = t^2$, then the product over hook lengths on the left-hand side of (1.18) is zero unless λ has no hook of length t . The latter can be rephrased as λ is a t -core though it is not immediate to see that $\mathcal{H}_t(\lambda) = \emptyset$ is equivalent to λ not having any hook length equal to t . This will be explained in Section 2.3. The well-known generating series for \mathcal{P} can also be derived from (1.18) with $z = 0$:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = (T; T)_\infty^{-1}. \quad (1.19)$$

Formula (1.18) was first generalized to a two parameter identity in [IKS10, Equation (3.7)]. Later, simultaneously both Rains–Warnaar [RW18], by using refined skew Cauchy-type identities for Macdonald polynomials, and Carlsson–Rodriguez Villegas [CRV18], by means of vertex operators and the plethystic exponential, discovered a (q, t) -extension of the Nekrasov–Okounkov formula to settle a conjecture from Hausel–Rodriguez Villegas [HRV08, Conjecture 4.3.2] in their study of mixed Hodge polynomials of character varieties with genus 1.

Theorem 1.5.1. [CRV18, Theorem 1.0.2], [RW18, Theorem 1.3] *We have*

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{s \in \lambda} \frac{(1 - uq^{a(s)+1}t^{l(s)})(1 - u^{-1}q^{a(s)}t^{l(s)+1})}{(1 - q^{a(s)+1}t^{l(s)})(1 - q^{a(s)}t^{l(s)+1})} \\ = \prod_{i, j, k \geq 1} \frac{(1 - uq^i t^{j-1} T^k)^r (1 - u^{-1} q^{i-1} t^j T^k)^r}{(1 - q^{i-1} t^{j-1} T^k)^r (1 - q^i t^j T^k)^r}. \end{aligned}$$

As mentioned in [CRV18, RW18], the special case $q = t$ is a reformulation of a result due to Dehaye–Han [DH11] and Iqbal–Nazir–Raza–Saleem [INRS12], using the refined topological vertex. It can be written as follows:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1 - uq^h)(1 - u^{-1}q^h)}{(1 - q^h)^2} = \prod_{k, r \geq 1} \frac{(1 - uq^r T^k)^r (1 - u^{-1}q^r T^k)^r}{(1 - q^{r-1} T^k)^r (1 - q^{r+1} T^k)^r}. \quad (1.20)$$

Note that taking $u = q^z$ and letting $q \rightarrow 1$ in (1.20) yields (1.18), although it is not immediate for the product side.

The methods used by Han to prove the Nekrasov–Okounkov formula (1.18) and Dehaye–Han for its q -analogue (1.20), both start from the type \tilde{A} specialization of Macdonald’s formulas for affine root systems. Nevertheless one needs to keep track of the variables to derive (1.20). Therefore it is necessary to use the more general formulation given by Rosengren–Schlosser [RS06] (see also Stanton’s reformulation in [Sta89]). The next step in [DH11] for proving (1.20) uses new combinatorial notions such as exploded tableaux and a V_t -coding. In the next chapter, we extend this last notion to $V_{g,t}$ -codings (see Definition 2.5.1) adapted from Garvan–Kim–Stanton [GKS90], which comes naturally from the study of the Littlewood decomposition described in Section 2.3.

2.1 Introduction

This manuscript in particular aims to provide a transcription of Macdonald identities introduced in Proposition 1.4.1 in terms of integer partitions. From the discussion at the end of Section 1.4, this requires to establish a connection between vectors of integers and integer partitions. Such a connection can be obtained by considering integer partitions through their bijective correspondence with binary bi-infinite words. This bijection has many different names in the literature such as boson-fermion correspondence, Maya diagrams and is connected to many tools used within different theories such as beta sets (see for instance [Alb22, BN19]) or vertex operators. In this chapter, we first introduce this correspondence. Then we discuss the Littlewood decomposition, which is the main tool of this manuscript. As recalled with some history and context in [Alb22, Section 2.2], this decomposition has many equivalent descriptions, see for instance [GKS90, HJ11, WW20]. As mentioned in Section 2.4, Garvan–Kim–Stanton [GKS90] already gave a bijection between vectors of integers and t -cores. An important remark here is that the weight of the corresponding partition corresponds to a quadratic form on the corresponding vector of integers. We recall some of their results and introduce the subsets of partitions whose weights correspond to the quadratic forms obtained when extracting the power of T in Proposition 1.4.1. In the last section, we derive enumerative results on product of hook lengths of these subsets of partitions that will be necessary first to rewrite Macdonald identities for the seven infinite affine root systems, and then to derive q -Nekrasov–Okounkov type formulas in Section 2.5.7. Indeed, to derive the q -Nekrasov–Okounkov formula (1.20) from a specialization of the Macdonald identity (1.14) for type \tilde{A}_t , Dehaye–Han prove the following theorem, which we rephrase using the definition of $V_{t,t}$ -coding (see Definition 2.5.1):

Theorem 2.1.1. *Set t a positive integer. Let $\omega \in \mathcal{P}_{(t)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{t,t}$ -coding, and set $r_i = v_i$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\omega| = \frac{1}{2t} \sum_{i=1}^t r_i^2 - \frac{t(2t-1)}{12}. \quad (2.1)$$

Moreover, setting $\alpha_i(\omega) := \#\{s \in \omega, h_s = t - i\}$, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \omega} \frac{\tau(h_s - t)\tau(h_s + t)}{\tau(h_s)^2} = \prod_{i=1}^{t-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)}. \quad (2.2)$$

In order to mimic the proof of Dehaye–Han, we need to prove theorems that are similar to Theorem 2.1.1 for partitions arising in Section 2.4. The following theorem deals with the partitions in $\mathcal{DD}_{(2t+2)}$ arising in type \tilde{C}_t .

Theorem 2.1.2. *Set t a positive integer and $g = 2t + 2$. Let $\omega \in \mathcal{DD}_{(g)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding, and set $r_i = v_i - g/2$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\omega| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g/2 - 1)(g - 1)}{12}. \quad (2.3)$$

Moreover, setting $\alpha_i(\omega) := \#\{s \in \omega, h_s = g - i, \varepsilon_s = 1\}$, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{i=1}^t \frac{\tau(r_i)}{\tau(i)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g - i - j)}, \quad (2.4)$$

and

$$\begin{aligned} \prod_{s \in \omega} \frac{\tau(h_s + \varepsilon_s g)}{\tau(h_s)} &= \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{i=1}^t \frac{\tau(2r_i)}{\tau(2i)} \frac{\tau(r_i)}{\tau(i)} \frac{\tau(r_i + t + 1)}{\tau(i + t + 1)} \frac{\tau(r_i - t - 1)}{\tau(i - t - 1)} \\ &\quad \times \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g - i - j)}. \end{aligned} \quad (2.5)$$

2.2 Integer partitions and bi-infinite binary words

In this section, we use the formalism of Han and Ji of [HJ11], as already used by the author in [Wah22a, Wah22b].

Let $\partial\lambda$ be the border of the Ferrers diagram of λ . Each step on $\partial\lambda$ is either horizontal or vertical. Encode the walk along the border from the South-West to the North-East as depicted in Figure 2.6: take “0” for a vertical step and “1” for a horizontal step. This yields a 0/1 sequence and the resulting word over the $\{0, 1\}$ alphabet:

- contains infinitely many “0”’s at the beginning (respectively “1”’s at the end),
- is indexed by \mathbb{Z} ,
- and is written $(c_k)_{k \in \mathbb{Z}}$.

This writing as a sequence is not unique (since for any k sequences $(c_{k+i})_{i \in \mathbb{Z}}$ encode the same partition, note that this is related to the notion of charge in fermionic Fock space), hence the necessity to set the index 0 uniquely for that encoding to be bijective. To tackle that issue, we set the index 0 when the number of “0”’s to the right of that index is equal to the number of “1”’s to the left. In other words, the number of horizontal steps along $\partial\lambda$ corresponding to a “1” of negative index in $(c_k)_{k \in \mathbb{Z}}$ must be equal to the number of vertical steps corresponding to “0”’s of nonnegative index in $(c_k)_{k \in \mathbb{Z}}$ along $\partial\lambda$. The delimitation between the letter of index -1 and that of index 0 is called the *median* of the word, marked by a $|$ symbol. The size of the Durfee square is then equal to the number of “1”’s of negative index. Hence the application ψ bijectively associates a partition to the word:

$$\psi(\lambda) = (c_k)_{k \in \mathbb{Z}} = (\dots c_{-2} c_{-1} | c_0 c_1 c_2 \dots),$$

where $c_k \in \{0, 1\}$ for any $k \in \mathbb{Z}$, and such that

$$\#\{k \leq -1, c_k = 1\} = \#\{k \geq 0, c_k = 0\}.$$

In Figure 2.6, one can remark that the hook corresponds to the collection within $\partial\lambda$ of a ray coming from the south in parallel to the ordinate axis and exiting in parallel to the x-axis. This remark illustrates the following lemma.

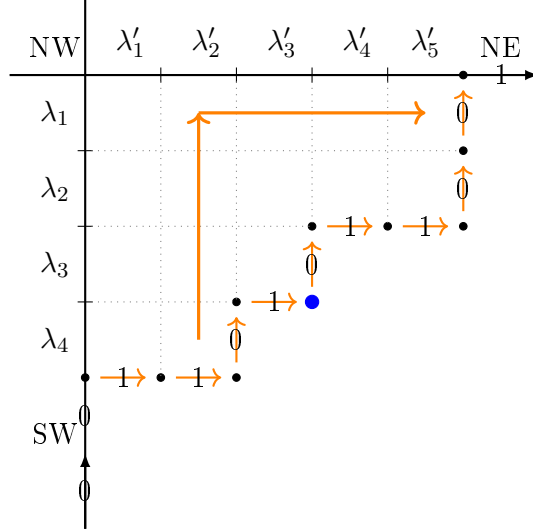


Figure 2.6: $\partial\lambda$ and its binary correspondence for $\lambda = (5, 5, 3, 2)$ with a hook.

Lemma 2.2.1. *The application ψ maps bijectively a box s of hook length h_s of the Ferrers diagram of λ to a pair of indices $(i_s, j_s) \in \mathbb{Z}^2$ of the word $\psi(\lambda)$ such that*

1. $i_s < j_s$,
2. $c_{i_s} = 1, c_{j_s} = 0$,
3. $j_s - i_s = h_s$,
4. s is a box above the main diagonal in the Ferrers diagram of λ if and only if the number of letters “1” with negative index greater than i_s is less than the number of letters “0” with nonnegative index less than j_s .

Proof. The first three points are immediate. We now prove the last point. Let s be a box and $(i_s, j_s) \in \mathbb{Z}^2$ the corresponding indices in $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ such that $c_{i_s} = 1$ and $c_{j_s} = 0$. If i_s and j_s share the same sign, this is equivalent to the fact that the hook defined by the sequence $c_{i_s} \dots c_{j_s}$ begins and ends on the same side of the median of $\psi(\lambda)$. Then the box s associated with this hook is either below the Durfee square or to its right. Hence s is below when i_s and j_s are negative as we also know that $i_s < j_s$, which concludes the proof of this case. Now we assume that i_s is negative and j_s nonnegative. The number of letters “1” with negative index greater than i_s corresponds to the number of horizontal steps before the Durfee square and the number of letters “0” with nonnegative index less than j_s corresponds to the number of vertical steps after the Durfee square. By definition of the Durfee square, this concludes the proof. \square

Let λ be a partition and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ be its corresponding word. Let λ' be the conjugate of λ and $\psi(\lambda') = (c'_i)_{i \in \mathbb{Z}}$. We have

$$\forall k \in \mathbb{Z}, c'_k = 1 - c_{-k-1}.$$

Given the properties of symmetries of self-conjugate and doubled distinct partitions, they admit a similar characterization:

$$\lambda \in \mathcal{DD} \iff \psi(\lambda) = (c_k)_{k \in \mathbb{Z}} \mid c_0 = 1 \text{ and } \forall k \in \mathbb{N}^*, c_{-k} = 1 - c_k, \quad (2.6)$$

and

$$\lambda \in \mathcal{SC} \iff \psi(\lambda) = (c_k)_{k \in \mathbb{Z}} \mid \forall k \in \mathbb{N}, c_{-k-1} = 1 - c_k. \quad (2.7)$$

From Remark 2.2 and (2.6), the set \mathcal{DD}' also admits a nice word interpretation characterization:

$$\lambda \in \mathcal{DD}' \iff \psi(\lambda) = (c_k)_{k \in \mathbb{Z}} \mid c_{-1} = 0 \text{ and } \forall k \in \mathbb{N}, c_{-k-2} = 1 - c_k. \quad (2.8)$$

From (2.6)–(2.8) and Lemma 2.2.1(4), we derive the following result:

Corollary 2.2.2. *Set $\lambda \in \{\mathcal{SC}, \mathcal{DD}, \mathcal{DD}'\}$ and $\psi(\lambda)$ its corresponding word. Let s be a box of the Ferrers diagram of λ . Let $(i_s, j_s) \in \mathbb{Z}^2$ be the indices in $\psi(\lambda)$ associated with s . Then s is a box on or above the main diagonal in the Ferrers diagram of λ if and only if*

$$\begin{cases} |i_s| \leq |j_s| \text{ if } \lambda \in \mathcal{DD}, \\ |i_s + 1| \leq |j_s| \text{ if } \lambda \in \mathcal{SC}, \\ |i_s + 2| \leq |j_s| \text{ if } \lambda \in \mathcal{DD}'. \end{cases}$$

2.3 The Littlewood decomposition

Now we recall the following classical map, often called the Littlewood decomposition (see for instance [GKS90, HJ11]).

Definition 2.3.1. Let $t \geq 2$ be an integer and consider:

$$\Phi_T : \begin{cases} \mathcal{P} & \rightarrow \mathcal{P}_{(t)} \times \mathcal{P}^t \\ \lambda & \mapsto (\omega, \nu^{(0)}, \dots, \nu^{(t-1)}), \end{cases}$$

where if we set $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}}$, then for all $k \in \{0, \dots, t-1\}$, one has $\nu^{(k)} := s^{-1}((c_{ti+k})_{i \in \mathbb{Z}})$. The tuple $\underline{\nu} = (\nu^{(0)}, \dots, \nu^{(t-1)})$ is called the t -quotient of λ and is denoted by $\text{quot}_t(\lambda)$, while ω is the t -core of λ denoted by $\text{core}_t(\lambda)$.

Obtaining the t -quotient is straightforward from $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}}$: we just look at subwords with indices congruent to the same values modulo t . The sequence 10 within these subwords are replaced iteratively by 01 until the subwords are all the infinite sequence of “0”’s before the infinite sequence of “1”’s (in fact it consists in removing all rim hooks in λ of length congruent to 0 (mod t)). Then ω is the partition corresponding to the word which has the subwords (mod t) obtained after the removal of the 10 sequences. For example, if we take $\lambda = (4, 4, 3, 2)$ and $t = 3$, then $\psi(\lambda) = \dots 001101|010011 \dots$

$$\begin{array}{l} \psi(\nu^{(0)}) = \dots 001|001 \dots \\ \psi(\nu^{(1)}) = \dots 000|111 \dots \\ \psi(\nu^{(2)}) = \dots 011|011 \dots \end{array} \quad \mapsto \quad \begin{array}{l} \psi(w_0) = \dots 000|011 \dots, \\ \psi(w_1) = \dots 000|111 \dots, \\ \psi(w_2) = \dots 001|111 \dots \end{array}$$

Thus

$$\psi(\omega) = \dots 000001|011111 \dots$$

and

$$\text{quot}_3(\lambda) = (\nu^{(0)}, \nu^{(1)}, \nu^{(2)}) = ((1, 1), \emptyset, (2)), \quad \text{core}_3(\lambda) = \omega = (1).$$

The following properties of the Littlewood decomposition are given in [HJ11].

Proposition 2.3.2. [HJ11, Theorem 2.1] *Let t be a positive integer. The Littlewood decomposition Φ_t maps bijectively a partition λ to $(\omega, \nu^{(0)}, \dots, \nu^{(t-1)})$ such that:*

(P1) ω is a t -core and $\nu^{(0)}, \dots, \nu^{(t-1)}$ are partitions,

(P2) $|\lambda| = |\omega| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$,

(P3) $\mathcal{H}_t(\lambda) = t\mathcal{H}(\underline{\nu})$,

where, for a multiset S ,

$$tS := \{ts, s \in S\} \quad \text{and} \quad \mathcal{H}(\underline{\nu}) := \bigcup_{i=0}^{t-1} \mathcal{H}(\nu^{(i)}).$$

When $t = 2$, Han and Ji also obtain additional properties on BG-rank and some other statistics. The first part of their Theorem 2.2 with respect to the BG-rank reads as follows.

Proposition 2.3.3. [[HJ11](#), Theorem 2.2] *When $t = 2$, the Littlewood decomposition Φ_2 has the further two properties:*

$$(P4) \quad \text{BG}(\lambda) = \begin{cases} \frac{\ell(\omega)+1}{2} & \text{if } \text{BG}(\lambda) > 0, \\ -\frac{\ell(\omega)}{2} & \text{if } \text{BG}(\lambda) \leq 0. \end{cases}$$

Proposition 2.3.2 (P3) in particular implies that λ is t -core is equivalent to the fact that λ has no hook of length t . Indeed if there exists $i \in \{1, \dots, t\}$ such that $|\nu^{(i)}| \neq \emptyset$, then 1 always belongs to the multiset $\mathcal{H}(\nu^{(i)})$ and then there always exists a hook of λ whose length is equal to t .

Now we discuss the Littlewood decomposition for \mathcal{SC} partitions. Let t be a positive integer, take $\lambda \in \mathcal{SC}$, and set $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ and $(\omega, \underline{\nu}) = (\text{core}_t(\lambda), \text{quot}_t(\lambda))$. Then one has the equivalence (see for instance [[GKS90](#), [Pé15a](#)]):

$$\begin{aligned} \lambda \in \mathcal{SC} & \iff \forall i_0 \in \{0, \dots, t-1\}, \forall j \in \mathbb{N}, c_{i_0+jt} = 1 - c_{-i_0-jt-1} \\ & \iff \forall i_0 \in \{0, \dots, t-1\}, \forall j \in \mathbb{N}, c_{i_0+jt} = 1 - c_{t-(i_0+1)-t(j-1)} \\ & \iff \forall i_0 \in \{0, \dots, t-1\}, \nu^{(i_0)} = \left(\nu^{(t-i_0-1)}\right)' \quad \text{and} \quad \omega \in \mathcal{SC}_{(t)}. \end{aligned} \quad (2.9)$$

Therefore λ is uniquely defined if its t -core is known as well as the $\lfloor t/2 \rfloor$ first elements of its quotient, which are partitions without any constraints. It implies that if t is even, there is a one-to-one correspondence between a self-conjugate partition and a pair made of one \mathcal{SC} t -core and $t/2$ generic partitions. If t is odd, the Littlewood decomposition is a one to one correspondence between a self-conjugate partition and a triple made of one \mathcal{SC} t -core, $(t-1)/2$ generic partitions and a self-conjugate partition $\mu = \nu^{((t-1)/2)}$. Hence the analogues of the above theorems when applied to self-conjugate partitions are as follows, which we will need in Chapter 3.

Proposition 2.3.4. [[Pé15b](#), Lemma 4.7] *Let t be a positive integer. The Littlewood decomposition Φ_t maps a self-conjugate partition λ to $(\omega, \nu^{(0)}, \dots, \nu^{(t-1)}) = (\omega, \underline{\nu})$ such that:*

(SC1) *the first component ω is a \mathcal{SC} t -core and $\nu^{(0)}, \dots, \nu^{(t-1)}$ are partitions,*

(SC2) $\forall j \in \{0, \dots, \lfloor t/2 \rfloor - 1\}, \nu^{(j)} = \left(\nu^{(t-1-j)}\right)'$,

(SC'2) *if t is odd, $\nu^{((t-1)/2)} = \left(\nu^{((t-1)/2)}\right)'$,*

(SC3) $|\lambda| = \begin{cases} |\omega| + 2t \sum_{i=0}^{(t-3)/2} |\nu^{(i)}| + t|\mu| & \text{if } t \text{ is odd,} \\ |\omega| + 2t \sum_{i=0}^{t/2-1} |\nu^{(i)}| & \text{if } t \text{ is even,} \end{cases}$

(SC4) $\mathcal{H}_t(\lambda) = t\mathcal{H}(\underline{\nu})$.

This way of computing the Littlewood decomposition enables to prove the following lemma which will be necessary to interpret Macdonald identities for types \tilde{B} , \tilde{B}^\vee and \tilde{D} in Section 2.5.7.

Lemma 2.3.5. [P615b, Lemma 4.3] Let t be a positive integer. The Littlewood decomposition Φ_t maps a doubled distinct partition λ to $(\omega, \nu^{(0)}, \dots, \nu^{(t-1)}) = (\omega, \underline{\nu})$ such that:

$$\begin{aligned}
(DD1) \quad & \text{the first component } \omega \text{ is a } \mathcal{DD} \text{ } t\text{-core and } \nu^{(0)}, \dots, \nu^{(t-1)} \text{ are partitions,} \\
(DD2) \quad & \forall j \in \{1, \dots, \lfloor t/2 \rfloor\}, \nu^{(j)} = \left(\nu^{(t-j)}\right)', \nu^{(0)} \in \mathcal{DD}, \\
& \text{and if } t \text{ is even, } \nu^{(t/2)} = \left(\nu^{(t/2)}\right)' \in \mathcal{SC}, \\
(DD3) \quad & |\lambda| = \begin{cases} |\omega| + 2t \sum_{i=1}^{(t-1)/2} |\nu^{(i)}| + t|\nu^{(0)}| & \text{if } t \text{ is odd,} \\ |\omega| + 2t \sum_{i=1}^{t/2-1} |\nu^{(i)}| + t|\nu^{(0)}| + t|\nu^{(t/2)}| & \text{if } t \text{ is even,} \end{cases} \\
(DD4) \quad & \mathcal{H}_t(\lambda) = t\mathcal{H}(\underline{\nu}).
\end{aligned}$$

By (2.8) and Lemma 2.3.5, one gets the following result for partitions in \mathcal{DD}' .

Lemma 2.3.6. Let t be a positive integer. The Littlewood decomposition Φ_t maps a conjugate doubled distinct partition λ to $(\omega, \nu^{(0)}, \dots, \nu^{(t-1)}) = (\omega, \underline{\nu})$ such that:

$$\begin{aligned}
(DD'1) \quad & \text{the first component } \omega \text{ is a } \mathcal{DD}' \text{ } t\text{-core and } \nu^{(0)}, \dots, \nu^{(t-1)} \text{ are partitions,} \\
(DD'2) \quad & \forall j \in \{0, \dots, \lfloor t/2 \rfloor - 1\}, \nu^{(j)} = \left(\nu^{(t-2-j)}\right)', \nu^{(t-1)} \in \mathcal{DD}', \\
& \text{and if } t \text{ even, } \nu^{(t/2-1)} = \left(\nu^{(t/2-1)}\right)' \in \mathcal{SC}, \\
(DD'3) \quad & |\lambda| = \begin{cases} |\omega| + 2t \sum_{i=0}^{(t-3)/2} |\nu^{(i)}| + t|\nu^{(t-1)}| & \text{if } t \text{ is odd,} \\ |\omega| + 2t \sum_{i=0}^{t/2-2} |\nu^{(i)}| + t|\nu^{(t-1)}| + t|\nu^{(t/2-1)}| & \text{if } t \text{ is even,} \end{cases} \\
(DD'4) \quad & \mathcal{H}_t(\lambda) = t\mathcal{H}(\underline{\nu}).
\end{aligned}$$

2.4 Integer partitions and some quadratic forms

Thanks to the properties of the Littlewood decomposition, one might see t -cores ω as partitions whose t -quotient in the Littlewood decomposition is empty: $\text{quot}_t(\omega) = (\emptyset, \dots, \emptyset)$. This is equivalent to say that all subwords $(\bmod t)$ in $\psi(\omega)$ are of the form $\dots 0011\dots$, which is an infinite sequence of “0”’s and then followed by an infinite sequence of “1”’s. For any $i \in \{0, \dots, t-1\}$ let us define $n_i := \min\{k \in \mathbb{Z} \mid c_{i+kt} = 1\}$. Each n_i corresponds to the index of the first “1” in the subword of $\psi(\omega)$ whose index is congruent to $i \pmod t$. Recall that the word $\psi(\omega)$ has as many “1”’s of negative index as “0”’s of positive index. That condition is equivalent to require $\sum_{i=0}^{t-1} n_i = 0$. Hence there is a natural bijective map

$$\phi : \mathcal{P}_{(t)} \rightarrow \mathbb{Z}^t$$

such that if we set $\phi(\omega) := \mathbf{n}$ then $\sum_{i=0}^{t-1} n_i = 0$.

For example, if we take $\omega = (4, 2)$ and $t = 3$, then

$$\psi(\omega) = \dots 000011|01101111\dots \mapsto \begin{cases} \psi(w_0) = \dots 000 & | \underbrace{001}_2 1 \dots, \\ \psi(w_1) = \dots 00 \underbrace{1}_{n_1=-1} & | \underbrace{111}_{n_0=2} \dots, \\ \psi(w_2) = \dots 00 \underbrace{1}_{n_2=-1} & | \underbrace{111}_{n_0=2} \dots \end{cases}$$

Therefore in the above example, $\phi(\omega) = (2, -1, -1) \in \mathbb{Z}^3$.

In [Joh18], Johnson uses the fermionic description of partitions (which is equivalent to the definition of ψ) to prove that this bijection is the one defined by Garvan–Kim–Stanton in [GKS90, Bijection 2]. We reformulate what Johnson wrote in [Joh18, Section 2] in terms of indices of words. Let λ be a partition and t be a positive integer. The abaci correspond exactly to the t -subwords of $\psi(\lambda)$ with fixed residue $(\bmod t)$ and the n_i 's, as defined in Theorem 2.4.1, correspond to the charge of the i -th runner on the abaci.

Theorem 2.4.1. [Joh18, Theorem 2.10][GKS90, Bijection 2] *Let ω be a t -core and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$ be its corresponding word. The bijection ϕ , where if we set $\phi(\omega) = (n_0, \dots, n_{t-1})$, then satisfies $n_i = \min\{k \in \mathbb{Z} \mid c_{kt+i} = 1\}$ and $\sum_{i=0}^{t-1} n_i = 0$. Moreover, we have:*

$$|\omega| = \frac{t}{2} \sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} i n_i. \quad (2.10)$$

When studying Macdonald identities for the seven infinite types of affine root systems, some quadratic forms arise. For instance, as pointed out by Han [Han10] and Dehaye–Han [DH11], the quadratic form (2.10) appears in the Macdonald identity for type \tilde{A}_{t-1} . When it comes to the study of other types, different quadratic forms arise. The goal of this section is to introduce the subsets of partitions linked to these quadratic forms. We will see that all of them can be described through the Littlewood decomposition.

First, when studying the Macdonald identity for type \tilde{C}_t , the subset $\mathcal{DD}_{(2t+2)}$ appears. A restriction of the Littlewood decomposition to $\mathcal{DD}_{(2t+2)}$ yields some additional conditions on the associated vector of integers already studied by Garvan–Kim–Stanton in [GKS90] but stated here in a slightly different way. Set $\omega \in \mathcal{DD}_{(2t+2)}$ and $\phi(\omega) = \mathbf{n} \in \mathbb{Z}^{2t+2}$, where ϕ is the bijection from Theorem 2.4.1. Equivalence (2.6) ensures that the following additional conditions are satisfied:

$$n_0 = 0, \text{ and } \forall i \in \{1, \dots, 2t+1\}, n_i = -n_{2t+2-i}. \quad (2.11)$$

So in particular, the last condition implies $n_{t+1} = 0$ and ω is bijectively associated with a vector of t integers. Applying Theorem 2.4.1, we then have:

$$|\omega| = (t+1) \sum_{i=0}^{2t+1} n_i^2 + \sum_{i=0}^{2t+1} i n_i = 2 \left((t+1) \sum_{i=1}^t n_i^2 + \sum_{i=1}^t (i-t-1) n_i \right). \quad (2.12)$$

As underlined in [Joh18], Figure 2.7 below illustrates how the word interpretation together with the computation of the Littlewood decomposition for a partition in $\mathcal{DD}_{(6)}$ when $t = 2$, and $2t + 2 = 6$ provide the vector of integers given in [GKS90]. The arrows are sorted in six different colors, each of them corresponding to a fixed residue $(\bmod 6)$ of the index of the corresponding word of ω . The word corresponding to ω writes as follows:

$$\psi(\omega) = \dots \mathbf{00000100001001} | \mathbf{101101111101111111} \dots$$

By extracting the subwords of fixed residue $(\bmod 6)$, we obtain:

$$\begin{aligned} \psi(w_0) &= \dots \mathbf{000} | \mathbf{111} \dots \\ \psi(w_1) &= \dots \mathbf{000} | \mathbf{011} \dots \\ \psi(w_2) &= \dots \mathbf{011} | \mathbf{111} \dots \\ \psi(w_3) &= \dots \mathbf{000} | \mathbf{111} \dots \\ \psi(w_4) &= \dots \mathbf{000} | \mathbf{001} \dots \\ \psi(w_5) &= \dots \mathbf{001} | \mathbf{111} \dots \end{aligned}$$

so that

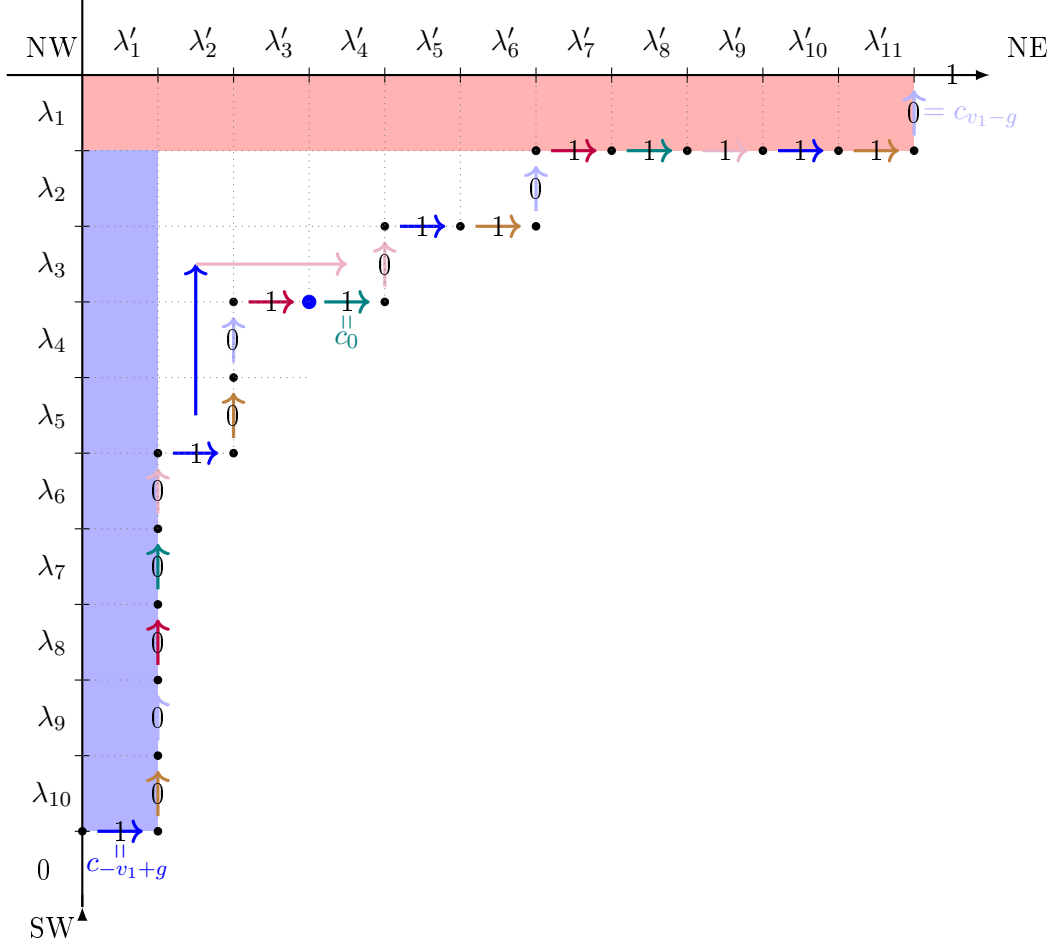


Figure 2.7: $\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1) \in \mathcal{DD}_{(6)}$ and its binary correspondence. Here $g = 2t + 2 = 6$ and v_1 is defined in Section 2.5.

$$(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6.$$

Equivalence (2.6) enables us to keep two components of \mathbf{n} :

$$\omega \in \mathcal{DD}_{(6)} \longleftrightarrow (1, -2) \in \mathbb{Z}^2.$$

Similarly the subset $\mathcal{DD}_{(2t+1)}$ arise in the Macdonald identity for type \tilde{BC}_t . Set $\omega \in \mathcal{DD}_{(2t+1)}$ and $\phi(\omega) = \mathbf{n} \in \mathbb{Z}^{2t+1}$ its image by the bijection of Theorem 2.4.1. Equivalence (2.6) yields in this case $n_0 = 0$ and for all $i \in \{1, \dots, 2t\}$, $n_i = -n_{2t+1-i}$. Once again, ω is bijectively associated with a vector of t integers. Then (2.12) becomes:

$$|\omega| = \frac{2t+1}{2} \sum_{i=0}^{2t} n_i^2 + \sum_{i=0}^{2t} i n_i = (2t+1) \sum_{i=1}^t n_i^2 + \sum_{i=1}^t (2i-2t-1)n_i. \quad (2.13)$$

The subset of self-conjugate $2t$ -cores arises in the Macdonald identity for \tilde{C}_t^\vee . A restriction of this previous results to $\mathcal{SC}_{(2t)}$ yields some additional conditions on the associated vector of integers. Let $\omega \in \mathcal{SC}_{(2t)}$ and $\phi(\omega) = \mathbf{n} \in \mathbb{Z}^{2t}$ by the bijection of Theorem 2.4.1. Equivalence (2.7) ensures that for all $i \in \{0, \dots, 2t-1\}$, $n_i = -n_{2t-1-i}$. Hence ω is also bijectively associated with a vector of t integers. By application of Theorem 2.4.1, we then have:

$$|\omega| = \frac{2t}{2} \sum_{i=0}^{2t-1} n_i^2 + \sum_{i=0}^{2t-1} i n_i = 2t \sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} (2i-2t+1)n_i. \quad (2.14)$$

Regarding types \tilde{B}_t , \tilde{B}_t^\vee , and \tilde{D}_t , the leading coefficients of the quadratic forms on vectors of \mathbb{Z}^t appearing in the Macdonald identities are respectively $2t - 1$, $2t$ and $2t - 2$. We start by examining the properties of the restriction of the bijection ϕ to the set $\mathcal{DD}'_{(2t)}$ of partitions whose conjugate is a doubled distinct partition and that are $(2t)$ -cores. Set $\omega \in \mathcal{DD}'_{(2t)}$ and $\phi(\omega) = \mathbf{n} \in \mathbb{Z}^{2t}$. From (2.8), one has

- $n_{2t-1} = 0$,
- $\forall i \in \{0, \dots, 2t - 2\}, n_i = -n_{2t-2-i}$.

So in particular, the last condition implies $n_{t-1} = 0$ and ω is bijectively associated with a vector of $t - 1$ integers. Applying Theorem 2.4.1, we then have

$$|\omega| = \frac{2t}{2} \sum_{i=0}^{2t-1} n_i^2 + \sum_{i=0}^{2t-1} i n_i = 2 \left(t \sum_{i=0}^{t-2} n_i^2 + \sum_{i=0}^{t-2} (i - t + 1) n_i \right) \quad (2.15)$$

Following the exact same path, if we set $\omega \in \mathcal{DD}'_{(2t-1)}$ and $\phi(\omega) = \mathbf{n} \in \mathbb{Z}^{2t-1}$, $n_{2t-2} = 0$ and ω is bijectively associated with a vector of $t - 1$ integers such that

$$|\omega| = \frac{2t-1}{2} \sum_{i=0}^{2t-3} n_i^2 + \sum_{i=0}^{2t-3} i n_i = 2 \left(\left(t - \frac{1}{2} \right) \sum_{i=0}^{t-2} n_i^2 + \sum_{i=0}^{t-2} \left(i - t + 1 + \frac{1}{2} \right) n_i \right) \quad (2.16)$$

As remarked above, ϕ maps $\mathcal{DD}'_{(2t)}$ and $\mathcal{DD}'_{(2t-1)}$ to \mathbb{Z}^{t-1} whereas it maps $\mathcal{DD}'_{(2t-2)}$ to \mathbb{Z}^{t-2} . Therefore these quadratic forms on vectors of t integers cannot be interpreted as the weight of (conjugate or not) doubled distinct core partitions. Nevertheless these quadratic forms can be interpreted as weights of elements of subsets of partitions with an explicit Littlewood decomposition whose quotient is almost empty but for one or two components.

For instance, in the case of type \tilde{B}_t , let m_1 be a positive integer and let us introduce the set $\mathcal{DD}'_{2t-1} := \{\lambda \in \mathcal{DD}' \mid \Phi_{2t-1}(\lambda) = (\omega, \emptyset, \dots, \emptyset, \nu^{(2t-2)})\}$ where $\nu^{(2t-2)}$ is the rectangular conjugate doubled distinct partition whose parts are all equal to $m_1 - 1$ and are repeated m_1 times. Hence $|\nu^{(2t-2)}| = m_1(m_1 - 1)$. Note that its corresponding word is:

$$\psi(\nu^{(2t-2)}) = \dots 0 \underbrace{1 \dots 1}_m 0 \mid 0 \dots 0 \underbrace{1 \dots 1}_m \dots \quad (2.17)$$

Now if we set $\phi(\omega) = (n_0, \dots, n_{t-2})$, $\lambda \in \mathcal{DD}'_{2t-1}$ is mapped bijectively to $(n_0, \dots, n_{t-2}, m_1) \in \mathbb{Z}^{t-1} \times \mathbb{N}^*$. Moreover using (2.13) and Lemma 2.3.6 ($DD'3$), we derive:

$$\begin{aligned} |\lambda| &= (2t-1) \sum_{i=0}^{t-2} n_i^2 + \sum_{i=0}^{t-2} (2(i-t)+1) n_i + (2t-1)(m_1^2 - m_1) \\ &= (2t-1)(m_1^2 + \sum_{i=1}^{t-1} n_i^2) + \left(-(2t-1)m_1 + \sum_{i=0}^{t-2} (2i-2t+1)n_i \right). \end{aligned} \quad (2.18)$$

For type \tilde{B}_t^\vee , let m_1 be a positive integer and let us introduce the set $\mathcal{DD}'_{2t} := \{\lambda \in \mathcal{DD}' \mid \Phi_{2t}(\lambda) = (\omega, \emptyset, \dots, \emptyset, \nu^{(2t-1)})\}$ where $\nu^{(2t-1)}$ is the same rectangular partition as in type \tilde{B}_t whose corresponding word is (2.17). Now if we set $\phi(\omega) = (n_0, \dots, n_{t-2})$, $\lambda \in \mathcal{DD}'_{2t}$ is mapped bijectively to $(n_0, \dots, n_{t-2}, m_1) \in \mathbb{Z}^{t-1} \times \mathbb{N}^*$. Moreover using (2.13) and Lemma 2.3.6 ($DD'3$), we get this time:

$$\begin{aligned} |\lambda| &= 2t \sum_{i=0}^{t-2} n_i^2 + \sum_{i=0}^{t-2} 2(i-t+1)n_i + 2t(m_1^2 - m_1) \\ &= 2t(m_1^2 - m_1 + \sum_{i=0}^{t-2} n_i^2) + \left(-2tm_1 + 2 \sum_{i=0}^{t-2} (i-t+1)n_i \right). \end{aligned} \quad (2.19)$$

For type \tilde{D}_t , let m_1 be a positive integer and m_t be a nonnegative integer and let us introduce the set $\mathcal{DD}'_{2t-2} := \{\lambda \in \mathcal{DD}' \mid \Phi_{2t-2}(\lambda) = (\omega, \emptyset, \dots, \emptyset, \nu^{(t-2)}, \emptyset, \dots, \emptyset, \nu^{(2t-3)})\}$ where $\nu^{(2t-3)}$ is the same rectangular partition as in types \tilde{B}_t and \tilde{B}_t^\vee , and $\nu^{(t-2)}$ is the $m_t \times m_t$ square self-conjugate partition. Hence $|\nu^{(2t-3)}| = m_1(m_1 - 1)$ and $|\nu^{(t-2)}| = m_t^2$. Note that the corresponding word is:

$$\psi(\nu^{(t-2)}) = \dots 0 \underbrace{1 \dots 1}_{m_t} \mid \underbrace{0 \dots 0}_{m_t} 1 \dots \quad (2.20)$$

Now if we set $\phi(\omega) = (n_0, \dots, n_{t-3})$, $\lambda \in \mathcal{DD}'_{2t-2}$ is mapped bijectively to $(n_0, \dots, n_{t-3}, m_1, m_t) \in \mathbb{Z}^{t-2} \times \mathbb{N}^* \times \mathbb{N}$. Moreover using (2.13) and Lemma 2.3.6 ($DD'3$), we derive:

$$\begin{aligned} |\lambda| &= (2t-2) \sum_{i=0}^{t-3} n_i^2 + \sum_{i=0}^{t-3} 2(i-t+1)n_i + (2t-2)(m_1^2 - m_1 + m_t^2) \\ &= (2t-2) \left(\sum_{i=1}^{t-1} n_i^2 + m_1^2 + m_t^2 \right) + \left(-(2t-2)m_1 + \sum_{i=1}^{t-1} 2(i-t+1)n_i \right). \end{aligned} \quad (2.21)$$

2.5 The enumeration of hook lengths products

A common property of all of the subsets of partitions introduced above is that their elements can be associated with a vector of t integers. Given the properties of symmetry of these partitions, we will bijectively associate them with t positive integers through their word interpretations: it consists in the greatest t indices of letters “0” in the corresponding word such that all of those t indices have a different congruence class modulo an integer g to which we add g . To be more precise, first we introduce the notion of $V_{g,t}$ -coding:

Definition 2.5.1. *Let t and g be two positive integers such that $t \leq g$. Set $\lambda \in \mathcal{P}$ and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ its corresponding binary word. For $i \in \{0, \dots, g-1\}$, define $\beta_i := \max\{(k+1)g+i \mid c_{kg+i} = 0\}$. Let $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ be the unique bijection such that $\beta_{\sigma(1)} > \dots > \beta_{\sigma(g)}$. The vector $\mathbf{v} := (\beta_{\sigma(1)}, \dots, \beta_{\sigma(t)})$ is called the $V_{g,t}$ -coding corresponding to λ .*

Remark 2.5.2. *When $g = t$, the map between $\omega \in \mathcal{P}_{(t)}$ and its $V_{t,t}$ -coding is bijective. Indeed one can deduce σ from $v_i \equiv \sigma(i) \pmod{t}$ and $0 \in \{\sigma(1), \dots, \sigma(t)\}$. Moreover we have $n_{\overline{\sigma(i)}} = \lfloor v_i/t \rfloor$, where $\overline{\sigma(i)}$ is the remainder of $\sigma(i)$ by t . Therefore*

$$\omega = \phi^{-1} \left(\left\lfloor \frac{v_{\sigma^{-1}(1)}}{t} \right\rfloor, \dots, \left\lfloor \frac{v_{\sigma^{-1}(t)}}{t} \right\rfloor \right),$$

where ϕ is the bijection from Theorem 2.4.1.

For any condition C , we will use the boolean notation

$$\mathbf{1}_C := \begin{cases} 1 & \text{if } C \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition bridges the gap between Definition 2.5.1 and all of the subsets of partitions introduced in Section 2.4. Indeed it shows that the $V_{g,t}$ -coding of a partition is equivalent to the vector of integers used to characterize the subsets of partitions defined in the previous subsection.

Proposition 2.5.3. *Let t be a positive integer. Any λ in one of the sets $\mathcal{P}_{(t)}$, $\mathcal{DD}_{(2t+2)}$, $\mathcal{DD}_{(2t+1)}$, $\mathcal{SC}_{(2t)}$, \mathcal{DD}'_{2t} , \mathcal{DD}'_{2t-1} , and \mathcal{DD}'_{2t-2} is in bijective correspondence with its $V_{g,t}$ -coding, where g is the index of the corresponding set.*

Proof. We will show that the correspondence between the $V_{g,t}$ -coding of λ and the vector of integers $\phi(\lambda)$ is bijective, and therefore so is the correspondence between λ and its $V_{g,t}$ -coding. Let λ be a partition in one of the sets of Proposition 2.5.3. Set $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ the word corresponding to λ . Let \mathbf{v} be its $V_{g,t}$ -coding. The case $\lambda \in \mathcal{P}_{(t)}$ is already handled in Remark 2.5.2.

If $\text{quot}_g(\lambda)$ is empty, then λ is a g -core. By definition $\beta_i - g$ is the last letter “0” in the subword of $\psi(\lambda)$ whose indices are congruent to $i \pmod{g}$. This implies that for all i , $\beta_i = \min_{k \in \mathbb{Z}} \{kg + i \mid c_{kg+i} = 1\}$. Set $\phi(\lambda) = \mathbf{n} \in \mathbb{Z}^g$. Let σ' be the unique permutation of $\{0, \dots, g-1\}$ such that the sequence $(n_{\sigma'(i)}, \sigma'(i))_i$ is strictly decreasing with respect to the lexicographic order. By definition of σ' , note that $\beta_{\sigma'(0)} > \dots > \beta_{\sigma'(g-1)}$. By definition of the $V_{g,t}$ -coding, we define σ from σ' as $\sigma(i) = \sigma'(i-1)$ for all $i \in \{1, \dots, g\}$ and $v_i = \beta_{\sigma(i)}$ for all $i \in \{1, \dots, t\}$. Moreover setting

$$g' := \begin{cases} g-1 & \text{if } \lambda \in \mathcal{SC}, \\ g & \text{otherwise,} \end{cases}$$

recall that we have $n_{i-1} \lambda \in \mathcal{SC} = -n_{\overline{-i}}$ for all $i \in \{1, \dots, t\}$ where $\overline{-i}$ is the remainder of $-i$ by g . Hence for all $i \in \{0, \dots, t-1\}$, $\beta_{\sigma'(i)} = -\beta_{\sigma'(g-1-i)} + g'$. Hence knowing the $V_{g,t}$ -coding of λ is equivalent to knowing $\beta_{\sigma'(i)}$ for $i \in \{0, \dots, g-1\}$. Whence the $V_{g,t}$ -coding of λ is in bijective correspondence with λ .

Moreover we have that for all $i \in \{1, \dots, t\}$:

$$-v_i + g \leq \frac{g}{2} \leq v_i. \quad (2.22)$$

If $\text{quot}_g(\lambda)$ is not empty, it implies that λ belongs to one of the subsets $\mathcal{DD}'_{2t}, \mathcal{DD}'_{2t-1}$ or \mathcal{DD}'_{2t-2} . We now have that for all $i \in \{0, \dots, g-2\} \setminus \{g/2\}$, $\beta_i = \min_{k \in \mathbb{Z}} \{kg + i \mid c_{kg+i} = 1\}$. By Section 2.4, note that the index congruent to $g-1 \pmod{g}$ of the last letter “0” in $\psi(\lambda)$ is $(m_1 - 1)g - 1$. Hence $\beta_{g-1} \geq g-1$. Similarly if g is even, $\beta_{g/2} \geq g/2$. If g is equal to $2t$ or $2t-1$ and if we set $\omega = \text{core}_g(\lambda)$ and $\phi(\omega) = \mathbf{n}' \in \mathbb{Z}^g$, let σ' be once again the unique permutation of $\{0, \dots, g-1\}$ such that the sequence $(n'_{\sigma'(i)}, \sigma'(i))_i$ is strictly decreasing with respect to the lexicographic order. By the word interpretation of elements in \mathcal{DD}'_g given in Section 2.4 and following the exact reasoning as for elements of $\mathcal{DD}_{(g)}$, we derive the same inequalities as (2.22). Hence there are exactly $t-1$ elements k in $\{0, \dots, g-2\}$ such that $\beta_k \geq g/2$ and exactly $t-1$ elements such that $\beta_k \leq g/2$. Thus there exists $i_0 \in \{0, \dots, t-1\}$ such that $\beta_{\sigma'(i_0)} > \beta_{g-1} > \beta_{\sigma'(i_0-1)}$. By setting $\sigma(i) = \sigma'(i-1)$ for any $1 \leq i \leq i_0$, $\sigma(i_0) = g-1$ and $\sigma(i) = \sigma'(i-2)$ for $i_0+1 \leq i \leq t-1$, we obtain $v_i = \beta_{\sigma(i)}$ for all $i \in \{1, \dots, t\}$. By (2.8), the $V_{g,t}$ -coding associated with λ is in bijective correspondence with $(n'_0, \dots, n'_{t-2}, m_1)$, hence as remarked in Section 2.4, this is in bijective correspondence with λ .

The case $g = 2t-2$ is exactly the same except that there exists $(k, l) \in \{1, \dots, t\}^2$ such that $v_k = m_t g + t - 2$ and $v_l = m_1 g - 1$. □

For example, the $V_{6,2}$ -coding of the partition $\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1) \in \mathcal{DD}_{(6)}$ is $(6 \times 2 + 4, 6 \times 1 + 1) = (16, 7)$ and $\sigma : \{1, \dots, 6\} \rightarrow \{0, \dots, 5\}$ such that $\sigma(1) = 4$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 0$, $\sigma(5) = 5$ and $\sigma(6) = 2$.

The aim of this section is to show Theorem 2.1.2 stated in Section 2.1, which is a general formula for hook lengths. This is done by induction on the length of the Durfee square of a partition.

We need to introduce the g -intervals as $I_{M,M}^{g,+} := \{k \in \mathbb{Z} \mid m \leq k < M, k \equiv m \pmod{g}\}$ and $I_{M,M}^{g,-} := \{l \in \mathbb{Z} \mid m < l \leq M, l \equiv M \pmod{g}\}$ for m, M two integers. This notion is of

particular interest in our case ever since we are trying to enumerate hook lengths with a fixed residue \pmod{g} . We will need the following lemma all along this section.

Lemma 2.5.4. *Let g be a positive integer, let τ be a function defined over \mathbb{Z} , let m and M be two integers such that $m < M$. We have*

$$\prod_{k \in I_{m,M}^{g,+}} \frac{\tau(M - k - g)}{\tau(M - k)} = \frac{\tau(M - \max(I_{m,M}^{g,+}) - g)}{\tau(M - m)}, \quad (2.23)$$

and

$$\prod_{l \in I_{m,M}^{g,-}} \frac{\tau(l - m + g)}{\tau(l - m)} = \frac{\tau(M - m + g)}{\tau(\min(I_{m,M}^{g,-}) - m)}.$$

The proof is straightforward since all products are telescopic.

The proofs of Theorems 2.1.1 and 2.1.2, as well as those of all theorems present in the next subsections, are done by induction on the maximal element of the $V_{g,t}$ -coding of the partitions: it means that, given \mathbf{v} a $V_{g,t}$ -coding and λ the corresponding partition, v_1 is the maximal element. By definition of \mathbf{v} , $v_1 - g$ corresponds to the maximal index of “0” in $\psi(\lambda)$. To use the induction property, one needs to consider the partition associated with \mathbf{v}' a $V_{g,t}$ -coding such that there exists $i \in \{1, \dots, t\}$ such that $v'_i = v_1 - g$. For generic t -core partitions (i.e. when $g = t$), in order to use the induction property, one needs to consider the $V_{t,t}$ -coding corresponding to the partition stripped from its largest part (i.e. stripped from the colored part in Figure 2.8). Whereas for all the other subsets of partitions discussed in Section 2.4, given their property of symmetry along Δ , one needs to consider the partition stripped from its largest hook (i.e. stripped from colored hook in Figure 2.7) to use the induction property on the maximal element of their associated $V_{g,t}$ -coding.

In the next subsections, we first state the lemma corresponding to the indices of the boxes in either the largest part or the largest hook. Then we use these lemmas combined with the induction property to derive the theorems of hook lengths product. These hook lengths enumerations and more specifically their proofs are to be applied in the next chapter, both to rewrite Macdonald identities for all types and also to derive q -Nekrasov–Okounkov formulas. As seen in Section 2.4, quadratic forms appear in Macdonald identities and are related to some subsets of partitions. Each of the following subsections hence give the product of hook lengths on the subset of partitions defined in Section 2.4. The case of type \tilde{C}_t is fully detailed, whereas the other subsections contain only the main technical results.

2.5.1 Type \tilde{A}_{t-1} and $\mathcal{P}_{(t)}$

As mentioned above, the case of t -core partitions is a bit particular: it is the only subset of partitions studied in this chapter with no properties of symmetry along Δ . For instance, when $t = 6$, to prove Theorem 2.1.1 in the example of $\omega = (7, 3, 3, 2, 2, 1, 1) \in \mathcal{P}_{(6)}$ (see Figure 2.8), the induction property is used on the partition $(3, 3, 2, 2, 1)$, which is still a 6-core partition.

ω_1	13	10	7	5	4	3	2	1	$\leftarrow c_{v_1-6} = 0$
ω_2	8	5	2						
ω_3	7	4	1						
ω_4	5	2							
ω_5	4	1							
ω_6	2								
ω_7	1								

Figure 2.8: $\omega \in \mathcal{P}_{(6)}$

Lemma 2.5.5. *Let t be a positive integer. Set $\omega \in \mathcal{P}_{(t)}$ and $(v_1, \dots, v_t) \in \mathbb{Z}^t$ its associated $V_{t,t}$ -coding. Then the largest part of ω , corresponds to the collection of boxes of indices in $\mathbb{I}_{v_i, v_1-t}^{t,+}$.*

The proof is straightforward from the fact that the boxes in ω_1 are those whose index of the letter “0” is $v_1 - g$ and the definition of g -intervals.

Lemma 2.5.4 is necessary to prove Theorem 2.1.1, but given the product over hook lengths slightly differ, one needs also the following result, which is proved as (2.23). Let g be a positive integer, let τ be a function defined over \mathbb{Z} , let m and M be two integers such that $m < M$. We have

$$\prod_{k \in \mathbb{I}_{m,M}^{g,+}} \frac{\tau(M - k + g)}{\tau(M - k)} = \frac{\tau(M - m + g)}{\tau(M - \max(\mathbb{I}_{m,M}^{g,+}))}.$$

The proof of Theorem 2.1.1 follows after Lemmas 2.5.4 and 2.5.5 and manipulations of products as (but simpler than) in the proof of Theorem 2.1.2 detailed below.

2.5.2 Type \tilde{C}_t and $\mathcal{DD}_{(2t+2)}$

The following lemma is useful in order to decompose the product over hook lengths of doubled distinct $(2t + 2)$ -core as a product over g -intervals.

Lemma 2.5.6. *Let t be a positive integer and set $g = 2t + 2$. Set $\lambda \in \mathcal{DD}_{(g)}$ and $(v_1, \dots, v_t) \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Then the largest hook of λ , denoted by H_1 corresponds to the collection of boxes of indices in*

$$H_1 := \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\begin{aligned} \mathcal{H}_{1,+} &= \mathbb{I}_{-v_1+g, v_1-g}^{g,+} \cup \mathbb{I}_{0, v_1-g}^{g,+} \cup \mathbb{I}_{g/2, v_1-g}^{g,+} \bigcup_{i=2}^t \left(\mathbb{I}_{v_i, v_1-g}^{g,+} \cup \mathbb{I}_{-v_i+g, v_1-g}^{g,+} \right), \\ \mathcal{H}_{1,-} &= \mathbb{I}_{-v_1+g, v_1-2g}^{g,-} \cup \mathbb{I}_{-v_1+g, -g}^{g,-} \cup \mathbb{I}_{-v_1+g, -g/2}^{g,-} \bigcup_{i=2}^t \left(\mathbb{I}_{-v_1+g, v_i-g}^{g,-} \cup \mathbb{I}_{v_1-g, -v_i}^{g,-} \right). \end{aligned}$$

Proof. Set $\lambda \in \mathcal{DD}_{(g)}$, $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$, and set $(v_i)_{i \in \{1, \dots, t\}}$ its associated $V_{g,t}$ -coding and σ as defined in Definition 2.5.1. By Lemma 2.2.1, the box with the largest hook length has the corresponding pair of indices (i_{\min}, j_{\max}) where $i_{\min} := \min\{i \in \mathbb{Z} \mid c_i = 1\}$ and $j_{\max} := \max\{j \in \mathbb{Z} \mid c_j = 0\}$. Since λ is a g -core, $v_1 = \max\{j \in \mathbb{Z} \mid c_{j-g} = 0\}$ by Proposition 2.5.3. Therefore $j_{\max} = v_1 - g$. Moreover by (2.6), the maximal index of a letter “0” corresponds to the minimal index of a letter “1”, whence $i_{\min} = -v_1 + g$. Therefore the box of coordinates $(1, 1)$ in the Ferrers diagram of λ corresponds to word indices $(-v_1 + g, v_1 - g)$ as shown in the boxes shaded in Figure 2.7.

Moreover, as illustrated in Figure 2.7, each box $s \in \lambda$ is by Lemma 2.2.1 in bijection with a pair of indices (i_s, j_s) such that $i_s \not\equiv j_s \pmod{g}$, $c_{i_s} = 1$, $c_{j_s} = 0$, and $i_s < j_s$. By Proposition 2.5.3, it implies that

$$\begin{aligned} \exists! k \in \{1, \dots, t\} \mid i_s &\equiv \pm \sigma(k) \pmod{g}, \\ \exists! l \in \{1, \dots, t\} \setminus \{k\} \mid j_s &\equiv \pm \sigma(l) \pmod{g}. \end{aligned}$$

By Theorem 2.4.1, we have

$$\begin{aligned} i_s &\geq \begin{cases} v_k & \text{if } i_s \equiv \sigma(k) \pmod{g}, \\ -v_k + g & \text{if } i_s \equiv -\sigma(k) \pmod{g}, \end{cases} \\ \text{and} & \\ j_s &\leq \begin{cases} v_l - g & \text{if } i_s \equiv \sigma(l) \pmod{g}, \\ -v_l & \text{if } i_s \equiv -\sigma(l) \pmod{g}. \end{cases} \end{aligned} \quad (2.24)$$

Ultimately, and as illustrated in Figure 2.7 in the red and blue shaded areas, let s be a box of λ in the first hook. Then $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) implies $j_s = v_1 - g$ (respectively $i_s = -v_1 + g$) and $j_s < v_1 - g$. \square

From the proof above, note that the largest hook length in the boxes of the largest hook of λ is $2(v_1 - g) = 2(n_{\sigma(1)} - 1)g + 2\sigma(1)$ and that it corresponds to the hook length of a box on the main diagonal Δ . From this remark, we derive by induction on the length of Δ

$$\{h_s \mid s \in \Delta\} = \bigcup_{i=1}^t \{2kg + 2\sigma(i) \mid 0 \leq k \leq n_{\sigma(i)} - 1\}. \quad (2.25)$$

We now prove Theorem 2.1.2.

Proof of Theorem 2.1.2. First we show (2.3). By Theorem 2.4.1, the weight of ω is:

$$|\omega| = \frac{g}{2} \sum_{i=0}^{2t+1} n_i^2 + \sum_{i=0}^{2t+1} i n_i = \frac{1}{2g} \left(\sum_{i=0}^{2t+1} (g n_i)^2 + \sum_{i=0}^{2t+1} g n_i (i - t - 1) \right).$$

From (2.11), we derive (2.3).

To prove (2.4), we proceed by induction on the size of the Durfee square by removing the hook with largest length (which is the largest hook on the main diagonal). When $\omega = \emptyset$, its corresponding $V_{g,t}$ -coding is $\mathbf{v} \in \mathbb{Z}^t$ whose components are $v_i = g - i$. Hence the vector \mathbf{r} as defined in Theorem 2.1.2 has components $r_i = g - i - t - 1 = t + 1 - i$. The left-hand side of (2.4) is empty, thus equals 1 by convention. So is the right-hand side. Hence the initialization.

We can rewrite the left-hand side of (2.4) as

$$\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{\substack{s \in \mathcal{H}_{1,+}(\omega) \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - g)}{\tau(h_s)} \prod_{\substack{s \in \mathcal{H}_{1,-}(\omega) \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s + g)}{\tau(h_s)} \prod_{\substack{s \in \omega \setminus H_1 \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)}. \quad (2.26)$$

By Lemma 2.5.6, the first product on the right-hand side of (2.26) writes

$$\begin{aligned} &\prod_{x \in \mathbb{I}_{-v_1+g, v_1-g}^{g,+}} \frac{\tau(v_1 - g - x - g)}{\tau(v_1 - g - x)} \prod_{x \in \mathbb{I}_{0, v_1-g}^{g,+}} \frac{\tau(v_1 - g - x - g)}{\tau(v_1 - g - x)} \\ \times &\prod_{x \in \mathbb{I}_{g/2, v_1-g}^{g,+}} \frac{\tau(v_1 - g - x - g)}{\tau(v_1 - g - x)} \prod_{i=2}^t \prod_{x \in \mathbb{I}_{v_i, v_1-g}^{g,+}} \frac{\tau(v_1 - g - x - g)}{\tau(v_1 - g - x)} \prod_{x \in \mathbb{I}_{-v_i+g, v_1-g}^{g,+}} \frac{\tau(v_1 - g - x - g)}{\tau(v_1 - g - x)}. \end{aligned} \quad (2.27)$$

By definition of g -intervals, if $\mathbb{I}_{m,M}^{g,+} \neq \emptyset$, then

$$x \in \mathbb{I}_{m,M}^{g,+} \setminus \{m\} \implies x - g \in \mathbb{I}_{m,M}^{g,+}.$$

Moreover by Lemma 2.5.6 there exists a unique box s in the Ferrers diagram of ω such that $h_s = v_1 - g - \max(\mathbb{I}_{m, v_1 - g}^{g, +}) < g$. Products in (2.27) are therefore telescopic and by Lemma 2.5.4 it can be rewritten as

$$\begin{aligned} & \left(\frac{\tau(2\sigma(1) - g(1 + \mathbb{1}_{\sigma(1) > g/2})}{\tau(2(v_1 - g))} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g)}{\tau(v_1 - g)} \right)^{\mathbb{1}_{g < v_1}} \\ & \quad \times \left(\frac{\tau(\sigma(1) - g/2 - \mathbb{1}_{\sigma(1) > g/2} g)}{\tau(v_1 - \frac{3g}{2})} \right)^{\mathbb{1}_{3g/2 < v_1}} \\ & \times \prod_{i=2}^t \left(\frac{\tau(\sigma(1) - \sigma(i) - \mathbb{1}_{\sigma(1) > \sigma(i)g})}{\tau(v_1 - v_i - g)} \right)^{\mathbb{1}_{g < v_1 - v_i}} \left(\frac{\tau(\sigma(1) + \sigma(i) - g(1 + \mathbb{1}_{\sigma(1) + \sigma(i) > g}))}{\tau(v_1 + v_i - 2g)} \right)^{\mathbb{1}_{2g < v_1 + v_i}}. \end{aligned} \quad (2.28)$$

To complete the rewriting of the first hook product H_1 , it remains to proceed the same way for the second product on the right-hand side of (2.26). Note that the latter can be written as follows

$$\begin{aligned} & \prod_{y \in \mathbb{I}_{-v_1 + g, v_1 - 2g}^{g, -}} \frac{\tau(v_1 + y)}{\tau(v_1 - g + y)} \prod_{y \in \mathbb{I}_{-v_1 + g, -g}^{g, -}} \frac{\tau(v_1 + y)}{\tau(v_1 - g + y)} \prod_{y \in \mathbb{I}_{-v_1 + g, -g/2}^{g, -}} \frac{\tau(v_1 + y)}{\tau(v_1 - g + y)} \\ & \quad \times \prod_{i=2}^t \prod_{y \in \mathbb{I}_{-v_1 + g, v_i - g}^{g, -}} \frac{\tau(v_1 + y)}{\tau(v_1 - g + y)} \prod_{y \in \mathbb{I}_{-v_1 + g, -v_i}^{g, -}} \frac{\tau(v_1 + y)}{\tau(v_1 - g + y)}. \end{aligned} \quad (2.29)$$

Again the products in (2.29) are telescopic and the same computations as before yield

$$\begin{aligned} & \left(\frac{\tau(2(v_1 - g))}{\tau(2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2})} \right)^{\mathbb{1}_{3g/2 < v_1}} \left(\frac{\tau(v_1 - g)}{\tau(\sigma(1))} \right)^{\mathbb{1}_{2g < v_1}} \\ & \quad \times \left(\frac{\tau(v_1 - g/2)}{\tau(\sigma(1) - g/2 + \mathbb{1}_{\sigma(1) < g/2} g)} \right)^{\mathbb{1}_{3g/2 < v_1}} \\ & \quad \times \prod_{i=2}^t \left(\frac{\tau(v_1 + v_i - g)}{\tau(\sigma(1) + \sigma(i) - g\mathbb{1}_{\sigma(1) + \sigma(i) > g})} \right)^{\mathbb{1}_{2g < v_1 + v_i}} \left(\frac{\tau(v_1 - v_i)}{\tau(\sigma(1) - \sigma(i) + \mathbb{1}_{\sigma(1) < \sigma(i)g})} \right)^{\mathbb{1}_{g < v_1 - v_i}}. \end{aligned} \quad (2.30)$$

First note that $g < v_1 < 3g/2$ is equivalent to $g < g\mathbb{1}_{\sigma(1)} + \sigma(1) < 3g/2$. This implies $n_{\sigma(1)} = 1$ and $\sigma(1) < g/2$. Hence $2(v_1 - g) = 2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2}$ and $v_1 - g/2 = \sigma(1) - g/2 + g\mathbb{1}_{\sigma(1) < g/2}$. Similarly if $g < v_1 < 2g$, then $v_1 - g = \sigma(1)$. We can then rewrite (2.30) as

$$\begin{aligned} & \left(\frac{\tau(2(v_1 - g))}{\tau(2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2})} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(v_1 - g)}{\tau(\sigma(1))} \right)^{\mathbb{1}_{g < v_1}} \\ & \quad \times \left(\frac{\tau(v_1 - g/2)}{\tau(\sigma(1) - g/2 + \mathbb{1}_{\sigma(1) < g/2} g)} \right)^{\mathbb{1}_{g < v_1}} \\ & \quad \times \prod_{i=2}^t \left(\frac{\tau(v_1 + v_i - g)}{\tau(\sigma(1) + \sigma(i) - g\mathbb{1}_{\sigma(1) + \sigma(i) > g})} \right)^{\mathbb{1}_{2g < v_1 + v_i}} \left(\frac{\tau(v_1 - v_i)}{\tau(\sigma(1) - \sigma(i) + \mathbb{1}_{\sigma(1) < \sigma(i)g})} \right)^{\mathbb{1}_{g < v_1 - v_i}}. \end{aligned} \quad (2.31)$$

Hence using (2.28) and (2.31), the red terms $\tau(2(v_1 - g))$ cancel and (2.26) can be rewritten

as

$$\begin{aligned}
\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} &= \left(\frac{\tau(2\sigma(1) - g(1 + \mathbf{1}_{\sigma(1) > g/2}))}{\tau(2\sigma(1) - g\mathbf{1}_{\sigma(1) > g/2})} \right)^{\mathbf{1}_{g < v_1}} \\
&\quad \times \left(\frac{\tau(\sigma(1) - g)}{\tau(\sigma(1))} \right)^{\mathbf{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g/2 - \mathbf{1}_{\sigma(1) > g/2} g)}{\tau(\sigma(1) - g/2 + \mathbf{1}_{\sigma(1) < g/2} g)} \right)^{\mathbf{1}_{g < v_1}} \\
&\times \prod_{i=2}^t \left(\frac{\tau(\sigma(1) + \sigma(i) - g(1 + \mathbf{1}_{\sigma(1) + \sigma(i) > g}))}{\tau(\sigma(1) + \sigma(i) - g\mathbf{1}_{\sigma(1) + \sigma(i) > g})} \right)^{\mathbf{1}_{2g < v_1 + v_i}} \left(\frac{\tau(\sigma(1) - \sigma(i) - \mathbf{1}_{\sigma(1) > \sigma(i)} g)}{\tau(\sigma(1) - \sigma(i) + \mathbf{1}_{\sigma(1) < \sigma(i)} g)} \right)^{\mathbf{1}_{g < v_1 - v_i}} \\
&\quad \times \left(\frac{\tau(v_1 - \frac{g}{2})}{\tau(v_1 - \frac{g}{2} - g)} \right)^{\mathbf{1}_{g < v_1}} \prod_{i=2}^t \left(\frac{\tau(v_1 - v_i)}{\tau(v_1 - v_i - g)} \right)^{\mathbf{1}_{g < v_1 - v_i}} \left(\frac{\tau(v_1 + v_i - g)}{\tau(v_1 + v_i - 2g)} \right)^{\mathbf{1}_{2g < v_1 + v_i}} \\
&\quad \times \prod_{\substack{s \in \omega \setminus H_1 \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)}. \quad (2.32)
\end{aligned}$$

To complete the induction step, we need to investigate the second largest hook of ω , denoted H_2 . There are two cases according to the coordinates of the box on the main diagonal of H_2 (that of largest hook length), which has coordinates

1. (first case) either $(-v_1 + 2g, v_1 - 2g)$, then for any $i \in \{2, \dots, t\}$, $v_1 - v_i > g$,
2. (second case) or $(-v_2 + g, v_2 - g)$, then in particular one has $v_1 - v_2 < g$. To study this case, let l be the integer such that $l := \max\{i \in \{2, \dots, t\} \mid v_1 - v_i < g\}$.

We handle the first case. Let $\omega \setminus H_1$ be the partition ω deprived from its largest hook. As $\mathcal{H}(\omega \setminus H_1) \subset \mathcal{H}(\omega)$, $\omega \setminus H_1$ still belongs to $\mathcal{DD}_{(g)}$. In this case, $r'_1 = v_1 - g - t - 1$ and $r'_i = v_i - t - 1$ for $2 \leq i \leq t$. It also implies by (2.22) that $v_1 > 3g/2$. Moreover for all $2 \leq i \leq t$, $v_1 - v_i > g$ and $2g < v_1 + v_i$. Hence all of the boolean conditions in (2.32) are satisfied.

Now we examine the residual terms depending on σ in the product (2.32). Set $(i, j) \in \{1, \dots, t\}$, and note that

$$\sigma(1) - \sigma(i) + \mathbf{1}_{\sigma(1) < \sigma(i)} g = \sigma(1) - \sigma(j) + \mathbf{1}_{\sigma(1) < \sigma(j)} g \implies \sigma(i) \equiv \sigma(j) \pmod{g}.$$

By definition of σ this cannot be true, so we have $\sigma(1) - \sigma(i) + \mathbf{1}_{\sigma(1) < \sigma(i)} g \neq \sigma(1) - \sigma(j) + \mathbf{1}_{\sigma(1) < \sigma(j)} g$, hence:

$$\begin{aligned}
\{1, \dots, g-1\} &= \{\sigma(1), 2\sigma(1) - g\mathbf{1}_{\sigma(1) > g/2}, \sigma(1) - g/2 + \mathbf{1}_{\sigma(1) < g/2} g\} \\
&\quad \bigcup_{i=2}^t \{\sigma(1) - \sigma(i) + \mathbf{1}_{\sigma(1) < \sigma(i)} g, \sigma(1) + \sigma(i) - g\mathbf{1}_{\sigma(1) + \sigma(i) > g}\}, \quad (2.33)
\end{aligned}$$

and

$$\begin{aligned}
\{-g+1, \dots, -1\} &= \{\sigma(1) - g, 2\sigma(1) - g(1 + \mathbf{1}_{\sigma(1) > g/2}), \sigma(1) - g/2 - \mathbf{1}_{\sigma(1) > g/2} g\} \\
&\quad \bigcup_{i=2}^t \{\sigma(1) - \sigma(i) - \mathbf{1}_{\sigma(1) < \sigma(i)} g, \sigma(1) + \sigma(i) - g(1 + \mathbf{1}_{\sigma(1) + \sigma(i) > g})\}. \quad (2.34)
\end{aligned}$$

Hence by (2.33) and (2.34), the products depending on σ in (2.32) can be rewritten as

$$\prod_{i=1}^{g-1} \frac{\tau(-i)}{\tau(i)}. \quad (2.35)$$

By the induction property, we have:

$$\begin{aligned}
\prod_{\substack{s \in \omega \setminus H_1 \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} &= \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega \setminus H_1)} \\
&\times \frac{\tau(v_1 - 3g/2)}{\tau(1)} \prod_{j=1}^t \frac{\tau(v_1 - g - v_j)}{\tau(j-1)} \prod_{2 \leq j \leq t} \frac{\tau(v_1 + v_j - 2g)}{\tau(1+j)} \\
&\times \prod_{i=2}^t \frac{\tau(v_i - g/2)}{\tau(i)} \prod_{2 \leq i < j \leq t} \frac{\tau(v_i + v_j - g)}{\tau(i+j)} \prod_{2 \leq i < j \leq t} \frac{\tau(v_i - v_j)}{\tau(j-i)}. \quad (2.36)
\end{aligned}$$

Plugging (2.35) and (2.36) into (2.32) concludes the proof of this first case.

The second case is yet more subtle. To handle the second point, let us first remark that $\omega \setminus H_1 \in \mathcal{DD}_{(g)}$ and let \mathbf{v}' be its associated $V_{g,t}$ -coding. Let l be the integer such that, if $g < v_1 < 3g/2$, then $v'_l = g - \sigma(1) = -v_1 + 2g$ and $\mathbf{v}' = (v_2, \dots, v_l, -v_1 + 2g, v_{l+1}, \dots, v_t)$ and $\mathbf{v}' = (v_2, \dots, v_l, v_1 - g, v_{l+1}, \dots, v_t)$ otherwise. We start with the case $v_1 > 3g/2$.

By the induction property, we have:

$$\begin{aligned}
\prod_{\substack{s \in \omega \setminus H_1 \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} &= \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega \setminus H_1)} \\
&\times \frac{\tau(v_1 - 3g/2)}{\tau(1)} \prod_{j=2}^l \frac{\tau(v_j + g - v_1)}{\tau(j-1)} \prod_{j=l+1}^t \frac{\tau(v_1 - g - v_j)}{\tau(j-1)} \prod_{2 \leq j \leq t} \frac{\tau(v_1 + v_j - 2g)}{\tau(1+j)} \\
&\times \prod_{i=2}^t \frac{\tau(v_i - g/2)}{\tau(i)} \prod_{2 \leq i < j \leq t} \frac{\tau(v_i + v_j - g)}{\tau(g-i-j)} \prod_{2 \leq i < j \leq t} \frac{\tau(v_i - v_j)}{\tau(j-i)}. \quad (2.37)
\end{aligned}$$

Hence (2.32) and (2.37) lead to:

$$\begin{aligned}
\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} &= \left(\frac{\tau(2\sigma(1) - g(1 + \mathbb{1}_{\sigma(1) > g/2}))}{\tau(2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2})} \right)^{\mathbb{1}_{g < v_1}} \\
&\times \left(\frac{\tau(\sigma(1) - g)}{\tau(\sigma(1))} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g/2 - \mathbb{1}_{\sigma(1) > g/2} g)}{\tau(\sigma(1) - g/2 + \mathbb{1}_{\sigma(1) < g/2} g)} \right)^{\mathbb{1}_{g < v_1}} \\
&\times \prod_{i=2}^t \left(\frac{\tau(\sigma(1) + \sigma(i) - g(1 + \mathbb{1}_{\sigma(1) + \sigma(i) > g}))}{\tau(\sigma(1) + \sigma(i) - g\mathbb{1}_{\sigma(1) + \sigma(i) > g})} \right)^{\mathbb{1}_{2g < v_1 + v_i}} \left(\frac{\tau(\sigma(1) - \sigma(i) - \mathbb{1}_{\sigma(1) > \sigma(i)} g)}{\tau(\sigma(1) - \sigma(i) + \mathbb{1}_{\sigma(1) < \sigma(i)} g)} \right)^{\mathbb{1}_{g < v_1 - v_i}} \\
&\times \prod_{j=2}^l \frac{\tau(v_j + g - v_1)}{\tau(v_1 - v_j)} \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega \setminus H_1)} \\
&\times \prod_{i=1}^t \frac{\tau(v_i - g/2)}{\tau(i)} \prod_{1 \leq i < j \leq t} \frac{\tau(v_i - v_j)}{\tau(j-i)} \frac{\tau(v_i + v_j - g)}{\tau(i+j)}. \quad (2.38)
\end{aligned}$$

Moreover we have:

$$\begin{aligned}
\prod_{j=2}^l \frac{\tau(v_j + g - v_1)}{\tau(v_1 - v_j)} &= \prod_{j=2}^l \frac{\tau(\sigma(j) + g(1 - \mathbb{1}_{\sigma(1) < \sigma(j)}) - \sigma(1))}{\tau(\sigma(1) - \sigma(j) + \mathbb{1}_{\sigma(1) < \sigma(j)} g)} \\
&= \prod_{j=2}^l \frac{\tau(\sigma(j) + \mathbb{1}_{\sigma(1) > \sigma(j)} g - \sigma(1))}{\tau(\sigma(1) - \sigma(j) + \mathbb{1}_{\sigma(1) < \sigma(j)} g)}.
\end{aligned}$$

Let us introduce

$$\begin{aligned}
A := & \left(\frac{\tau(2\sigma(1) - g(1 + \mathbb{1}_{\sigma(1) > g/2}))}{\tau(2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2})} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g)}{\tau(\sigma(1))} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g/2 - \mathbb{1}_{\sigma(1) > g/2}g)}{\tau(\sigma(1) - g/2 + \mathbb{1}_{\sigma(1) < g/2}g)} \right)^{\mathbb{1}_{g < v_1}} \\
& \times \prod_{i=2}^t \left(\frac{\tau(\sigma(1) + \sigma(i) - g(1 + \mathbb{1}_{\sigma(1) + \sigma(i) > g}))}{\tau(\sigma(1) + \sigma(i) - g\mathbb{1}_{\sigma(1) + \sigma(i) > g})} \right)^{\mathbb{1}_{2g < v_1 + v_i}} \left(\frac{\tau(\sigma(1) - \sigma(i) - \mathbb{1}_{\sigma(1) > \sigma(i)}g)}{\tau(\sigma(1) - \sigma(i) + \mathbb{1}_{\sigma(1) < \sigma(i)}g)} \right)^{\mathbb{1}_{g < v_1 - v_i}} \\
& \times \prod_{j=2}^l \tau(v_j + g - v_1).
\end{aligned}$$

By (2.33), we can rewrite A as

$$\begin{aligned}
\frac{\prod_{i=1}^{g-1} \tau(-i)}{\prod_{i=1}^{g-1} \tau(i)} \prod_{j=2}^l \frac{\tau(v_1 - v_j)}{\tau(v_1 - v_j - g)} \prod_{j=2}^l \tau(v_j + g - v_1) &= \prod_{i=1}^{g-1} \frac{\tau(-i)}{\tau(i)} \prod_{j=2}^l \frac{\tau(-(v_1 - v_j - g))}{\tau(v_1 - v_j - g)} \prod_{j=2}^l \tau(v_1 - v_j) \\
&= \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(H_1)} \prod_{j=2}^l \tau(v_1 - v_j), \quad (2.39)
\end{aligned}$$

where $\alpha_i(H_1) = \begin{cases} 1 & \text{if } g - i \in H_{1,+}, \\ 0 & \text{otherwise.} \end{cases}$

Plugging (2.38) in (2.39) completes the induction.

The case $g < v_1 < 3g/2$ is settled by the induction property and the remarks used to obtain (2.32).

The proof of (2.5) follows the exact same steps except for the first terms of each product of the right-hand sides of (2.30) and (2.28) where the products in red and in green are now:

$$\begin{aligned}
& \left(\frac{\tau(2\sigma(1) - g(1 + \mathbb{1}_{\sigma(1) > g/2}))}{\tau(2(v_1 - g) - g)} \right)^{\mathbb{1}_{3/2g < v_1}} \left(\frac{\tau(2(v_1 - g) + g)}{\tau(2\sigma(1) - g\mathbb{1}_{\sigma(1) > g/2})} \right)^{\mathbb{1}_{g < v_1}} \\
& \times \left(\frac{\tau(v_1)\tau(v_1 - g)}{\tau(\sigma(1))\tau(v_1 - g)} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(\sigma(1) - g)\tau(v_1 - g)}{\tau(v_1 - 2g)\tau(v_1 - g)} \right)^{\mathbb{1}_{2g < v_1}}. \quad (2.40)
\end{aligned}$$

To handle the boolean conditions, we use once again that if $g < v_1 < 3g/2$ then $2\sigma(1) - g(1 + \mathbb{1}_{\sigma(1) > g/2}) = 2v_1 - 3g$ and if $g < v_1 < 2g$ then $\sigma(1) - g = v_1 - 2g$ which concludes the proof. \square

2.5.3 Type \tilde{B}_t and \mathcal{DD}'_{2t-1}

As mentioned in Section 2.4, interpreting the geometric conditions on lattices in Macdonald identities for types \tilde{B} , \tilde{B}^\vee and \tilde{D} as g -cores of \mathcal{DD} or \mathcal{SC} is not possible but these can be interpreted as a subset of doubled distinct partitions with a specific g -quotient. In this subsection, we set $g = 2t - 1$.

If we take $\lambda \in \mathcal{DD}'_g$ such that $\nu^{(2t-2)}$ has its corresponding word equal to (2.17), and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$, remark that the set of indices congruent to $2t - 2 \pmod{g}$ corresponding to a letter "0" is $\{kg - 1 \mid k \in \mathbb{Z}, k \leq -m_1 \text{ or } 0 \leq k \leq m_1 - 1\}$. Using (2.8) and the same considerations as for the proof of Lemma 2.5.6, we have the following similar result.

Lemma 2.5.7. *Let t be a positive integer and set $g = 2t - 1$. Set $\lambda \in \mathcal{DD}'_g$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Let $i_0 \in \{1, \dots, t\}$ be such that $v_{i_0} = m_1g - 1$. Then the largest hook of λ , denoted by H_1 , corresponds to the collection of boxes of indices*

$$H_1 = \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\mathcal{H}_{1,+} = \mathbb{I}_{-v_1+g-2, v_1-g}^{g,+} \cup \mathbb{I}_{v_{i_0}, v_1-g}^{g,+} \cup \left(\mathbb{I}_{-v_{i_0}+g-2, v_1-g}^{g,+} \setminus \mathbb{I}_{-1, v_1-g}^{g,+} \right) \bigcup_{\substack{i=2 \\ i \neq i_0}}^t \left(\mathbb{I}_{v_i, v_1-g}^{g,+} \cup \mathbb{I}_{-v_i+g-2, v_1-g}^{g,+} \right),$$

$$\mathcal{H}_{1,-} = \mathbb{I}_{-v_1+g-2, v_1-2g}^{g,-} \cup \mathbb{I}_{-v_1+g-2, -v_{i_0}-2}^{g,-} \cup \left(\mathbb{I}_{-v_1+g-2, v_{i_0}-g}^{g,-} \setminus \mathbb{I}_{-v_1+g-2, -g-1}^{g,-} \right) \bigcup_{\substack{i=2 \\ i \neq i_0}}^t \left(\mathbb{I}_{-v_1+g-2, v_i-g}^{g,-} \cup \mathbb{I}_{v_1-g, -v_i-2}^{g,-} \right).$$

As for (2.25), we derive similarly that for any partition in $\mathcal{DD}'_{2t}, \mathcal{DD}'_{2t-1}$ or \mathcal{DD}'_{2t-2} and g is the index of the corresponding set:

$$\{h_s \mid s \in \Delta\} = \bigcup_{i=1}^t \{2kg + 2(\sigma(i) + 1) \mid 0 \leq k \leq n_{\sigma(i)} - 1\}. \quad (2.41)$$

The issues arising when using (2.4) is that if the g -quotient of a partition is not empty, we have at least one box s such that $h_s = g$. As when we derive q -Nekrasov–Okounkov type formulas, we specialize $\tau(x) = 1 - q^x$ or $\tau = \text{Id}$, the product can be equal to 0. Therefore one needs the following result for type B_t .

Theorem 2.5.8. *Set t a positive integer and $g = 2t - 1$. Let $\lambda \in \mathcal{DD}_g^1$ and $\mathbf{v} \in \mathbb{Z}^t$ the $V_{g,t}$ -coding associated to λ , and set $r_i = v_i - g/2 + 1$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\lambda| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g+2)(\lfloor g/2 \rfloor + 1)}{12}.$$

Moreover setting $\alpha_i(\lambda)$ as in Theorem 2.1.2, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \lambda} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\lambda)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g + 2 - i - j)}. \quad (2.42)$$

Proof. The proof is exactly the same as the one of Theorem 2.1.2 but without the products in brown in (2.27) and (2.29). These products would have been over the sets $\mathbb{I}_{g/2-1, v_1-g}^{g,+}$ and $\mathbb{I}_{-v_1+g-2, -v_{i_0}-2}^{g,-}$ in this case. Remark that these products in brown are the one which yields the terms $\tau(v_i - g/2)/\tau(i)$ in (2.4). Hence there are no terms $\tau(r_i)/\tau(i)$ in (2.42). The other difference with (2.4) is that $\Phi_{2t-1}(\lambda) = (\omega, \emptyset, \dots, \emptyset, \nu^{(2t-2)})$ where $\nu^{(2t-2)}$ has a corresponding word verifying (2.17). Then there exists $i_0 \in \{1, \dots, t\}$ such that $\sigma(i_0) = g-1$ and $v_{i_0} = m_1 g - 1$. The product of hook lengths belonging to $\mathcal{H}_{2t-1}(\lambda)$ corresponding to pairs of indices (i, j) such that $i < j$ and $i \equiv j \pmod{g} \equiv g-1 \pmod{g}$ are telescopic as well as the other products over hook lengths corresponding to pairs of indices (i, j) with $i < j$ such that i or j is congruent to $g-1$ modulo g . Using Lemma 2.5.7, the products give

$$\frac{\tau(2v_{i_0} - 2g + 2)}{\tau(g)} \frac{\tau(g)}{\tau(2v_{i_0} - 2g + 2)} = 1.$$

If $i_0 > 1$, the products in green in (2.40) are now

$$\begin{aligned} & \left(\frac{\tau(v_1 + v_{i_0} - g + 2)}{\tau(v_1 + 1 - g)} \right)^{\mathbb{1}_{g < v_1}} \left(\frac{\tau(v_1 - v_{i_0})}{\tau(\sigma(1) + 1)} \right)^{\mathbb{1}_{g < v_1 - v_{i_0}}} \\ & \quad \times \left(\frac{\tau(\sigma(1) + 1 - g)}{\tau(v_1 - v_{i_0} - g)} \right)^{\mathbb{1}_{g < v_1 - v_{i_0}}} \left(\frac{\tau(v_i + 1 - g)}{\tau(v_1 + v_{i_0} - 2g + 2)} \right)^{\mathbb{1}_{0 < v_1}}. \end{aligned}$$

The case $i_0 = 1$ is handled in the same way and the proof is completed with the same computations as in the previous theorems. \square

2.5.4 Type \tilde{B}_t^\vee and \mathcal{DD}'_g

If we take $\lambda \in \mathcal{DD}'_g$ such that $\nu^{(2t-1)}$ has its corresponding word equal to (2.17), and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$, remark that the set of indices congruent to $2t - 1 \pmod{g}$ corresponding to a letter “0” is $\{kg - 1 \mid k \in \mathbb{Z}, k \leq -m_1 \text{ or } 0 \leq k \leq m_1 - 1\}$. Using (2.8) and the same considerations as for the proof of Lemma 2.5.6, we have the following similar result.

Lemma 2.5.9. *Let t be a positive integer and set $g = 2t$. Set $\lambda \in \mathcal{DD}'_g$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Let $i_0 \in \{1, \dots, t\}$ be such that $v_{i_0} = m_1 g - 1$. Then the largest hook of λ , denoted by H_1 , corresponds to the collection of boxes of indices*

$$H_1 = \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\begin{aligned} \mathcal{H}_{1,+} &= \mathbf{I}_{-v_1+g-2, v_1-g}^{g,+} \cup \mathbf{I}_{v_{i_0}, v_1-g}^{g,+} \cup \left(\mathbf{I}_{-v_{i_0}+g-2, v_1-g}^{g,+} \setminus \mathbf{I}_{-1, v_1-g}^{g,+} \right) \\ & \quad \cup \mathbf{I}_{g/2-1, v_1-g}^{g,+} \bigcup_{\substack{i=2 \\ i \neq i_0}}^t \left(\mathbf{I}_{v_i, v_1-g}^{g,+} \cup \mathbf{I}_{-v_i+g-2, v_1-g}^{g,+} \right), \\ \mathcal{H}_{1,-} &= \mathbf{I}_{-v_1+g-2, v_1-2g}^{g,-} \cup \mathbf{I}_{-v_1+g-2, -v_{i_0}-2}^{g,-} \cup \left(\mathbf{I}_{-v_1+g-2, v_{i_0}-g}^{g,-} \setminus \mathbf{I}_{-v_1+g-2, -g-1}^{g,-} \right) \\ & \quad \cup \mathbf{I}_{-v_1+g, -g/2-1}^{g,-} \bigcup_{\substack{i=2 \\ i \neq i_0}}^t \left(\mathbf{I}_{-v_1+g-2, v_i-g}^{g,-} \cup \mathbf{I}_{v_1-g, -v_i-2}^{g,-} \right). \end{aligned}$$

Following the steps of the proof of Theorem 2.1.2, using Lemma 2.5.9, we derive the following result which will be necessary to derive a Nekrasov–Okounkov formula for type \tilde{B}_t^\vee .

Theorem 2.5.10. *Set t a positive integer and $g = 2t$. Set $\lambda \in \mathcal{DD}'_g$ and $\mathbf{v} \in \mathbb{Z}^t$ the $V_{g,t}$ -coding associated to λ , and set $r_i = v_i - g/2 + 1$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\lambda| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g+1)(g/2+1)}{12}.$$

Moreover setting $\alpha_i(\lambda)$ as in Theorem 2.1.2, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \lambda} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\lambda)} \prod_{i=1}^t \frac{\tau(r_i)}{\tau(i)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g + 2 - i - j)}, \quad (2.43)$$

2.5.5 Type \tilde{C}_t^V and $\mathcal{SC}_{(2t)}$

In this subsection, set $g = 2t$. Recall that the quadratic forms appearing to the exponent of T in the Macdonald identity for type \tilde{C}_t^V can be interpreted as half the weight of an element in $\mathcal{SC}_{(g)}$. Using (2.7) and the same considerations as for the proof of Lemma 2.5.6, we have the following similar result.

Lemma 2.5.11. *Let t be a positive integer and set $g = 2t$. Set $\lambda \in \mathcal{SC}_{(g)}$ and $(v_1, \dots, v_t) \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Then the largest hook of λ , denoted by H_1 corresponds to the collection of boxes of indices in*

$$H_1 := \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\begin{aligned} \mathcal{H}_{1,+} &= \mathbb{I}_{-v_1+g-1, v_1-g}^{g,+} \bigcup_{i=2}^t \left(\mathbb{I}_{v_i, v_1-g}^{g,+} \cup \mathbb{I}_{-v_i+g-1, v_1-g}^{g,+} \right), \\ \mathcal{H}_{1,-} &= \mathbb{I}_{-v_1+g, v_1-2g}^{g,-} \bigcup_{i=2}^t \left(\mathbb{I}_{-v_1+g-1, v_i-g}^{g,-} \cup \mathbb{I}_{v_1-g, -v_i}^{g,-} \right). \end{aligned}$$

As for (2.25), we derive similarly that for any self-conjugate g -core partition

$$\{h_s \mid s \in \Delta\} = \bigcup_{i=1}^t \{2kg + 2\sigma(i) + 1 \mid 0 \leq k \leq n_{\sigma(i)} - 1\}. \quad (2.44)$$

Using Lemmas 2.5.11 and 2.5.4, we derive the following analogue of Theorem 2.1.2

Theorem 2.5.12. *Set t a positive integer and $g = 2t$. Let $\omega \in \mathcal{SC}_{(g)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding, and set $r_i = v_i - g/2 + 1/2$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\omega| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{g^2 - 1}{24}.$$

Moreover setting $\alpha_i(\omega)$ as in Theorem 2.1.2, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g + 1 - i - j)}, \quad (2.45)$$

2.5.6 Type $\tilde{B}C_t$ and $\mathcal{DD}_{(2t+1)}$

In this subsection, we set $g = 2t + 1$. Recall that the quadratic forms appearing to the exponent of T in the Macdonald identity for type $\tilde{B}C_t$ can be interpreted as half the weight of an element in $\mathcal{DD}_{(g)}$. Similarly to Lemma 2.5.6, one derives the following lemma.

Lemma 2.5.13. *Let t be a positive integer and set $g = 2t + 1$. Set $\lambda \in \mathcal{DD}_{(g)}$ and $(v_1, \dots, v_t) \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Then the largest hook of λ , denoted by H_1 corresponds to the collection of boxes of indices in*

$$H_1 := \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\begin{aligned}\mathcal{H}_{1,+} &= \mathbf{I}_{-v_1+g, v_1-g}^{g,+} \cup \mathbf{I}_{0, v_1-g}^{g,+} \bigcup_{i=2}^t \left(\mathbf{I}_{v_i, v_1-g}^{g,+} \cup \mathbf{I}_{-v_i+g, v_1-g}^{g,+} \right), \\ \mathcal{H}_{1,-} &= \mathbf{I}_{-v_1+g, v_1-2g}^{g,-} \cup \mathbf{I}_{-v_1+g, -g}^{g,-} \bigcup_{i=2}^t \left(\mathbf{I}_{-v_1+g, v_i-g}^{g,-} \cup \mathbf{I}_{v_1-g, -v_i}^{g,-} \right).\end{aligned}$$

The proof of Theorem 2.5.14 stated below is exactly the same as the one of (2.4) in Theorem 2.1.2 but without the products in brown in (2.27) and (2.29), as underlined in the proof of Theorem 2.5.8. Using Lemmas 2.5.13 and 2.5.4, the proof follows the same step as the one of Theorem 2.1.2.

Theorem 2.5.14. *Set t a positive integer and $g = 2t + 1$. Set $\omega \in \mathcal{DD}(g)$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding, and set $r_i = v_i - g/2$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\omega| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g-2)\lfloor g/2 \rfloor}{12},$$

Moreover, setting $\alpha_i(\omega)$ as in Theorem 2.1.2, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g - i - j)}. \quad (2.46)$$

2.5.7 Type \tilde{D}_t and $\mathcal{DD}_{2t-2}^{\prime 2}$

Set $g = 2t + 2$. If we take $\lambda \in \mathcal{DD}_g^{\prime 2}$ such that $\nu^{(2t-3)}$ and $\nu^{(t-2)}$ has their corresponding word equal to (2.17) and (2.20), and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$, remark that the set of indices congruent to $t - 2 \pmod{g}$ corresponding to a letter "0" is $\{kg - t \mid k \in \mathbb{Z}, k \leq -m_t \text{ or } 1 \leq k \leq m_t\}$. Using (2.8) and the same considerations as for the proof of Lemma 2.5.6, we have the following similar result.

Lemma 2.5.15. *Let t be a positive integer and set $g = 2t - 2$. Set $\lambda \in \mathcal{DD}_g^{\prime 1}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding. Let $(i_0, i_1) \in \{1, \dots, t\}^2$ be such that $v_{i_0} = m_1 g - 1$ and $v_{i_1} = m_t g + t - 2$. Then the largest hook of λ , denoted by H_1 , corresponds to the collection of boxes of indices*

$$H_1 = \mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$$

where $\mathcal{H}_{1,+}$ (respectively $\mathcal{H}_{1,-}$) denotes the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) and

$$\begin{aligned}\mathcal{H}_{1,+} &= \mathbf{I}_{-v_1+g-2, v_1-g}^{g,+} \cup \mathbf{I}_{v_{i_0}, v_1-g}^{g,+} \cup \left(\mathbf{I}_{-v_{i_0}+g-2, v_1-g}^{g,+} \setminus \mathbf{I}_{-1, v_1-g}^{g,+} \right) \\ &\quad \cup \mathbf{I}_{v_{i_1}, v_1-g}^{g,+} \cup \left(\mathbf{I}_{-v_{i_1}+g-2, v_1-g}^{g,+} \setminus \mathbf{I}_{g/2-1, v_1-g}^{g,+} \right) \bigcup_{\substack{i=2 \\ i \neq i_0 \\ i \neq i_1}}^t \left(\mathbf{I}_{v_i, v_1-g}^{g,+} \cup \mathbf{I}_{-v_i+g-2, v_1-g}^{g,+} \right), \\ \mathcal{H}_{1,-} &= \mathbf{I}_{-v_1+g-2, v_1-2g}^{g,-} \cup \mathbf{I}_{-v_1+g-2, -v_{i_0}-2}^{g,-} \cup \left(\mathbf{I}_{-v_1+g-2, v_{i_0}-g}^{g,-} \setminus \mathbf{I}_{-v_1+g-2, -g-1}^{g,-} \right) \\ &\quad \cup \mathbf{I}_{-v_1+g-2, -v_{i_1}-2}^{g,-} \cup \left(\mathbf{I}_{-v_1+g-2, v_{i_1}-g}^{g,-} \setminus \mathbf{I}_{-v_1+g-2, -g/2-1}^{g,-} \right) \bigcup_{\substack{i=2 \\ i \neq i_0 \\ i \neq i_1}}^t \left(\mathbf{I}_{-v_1+g-2, v_i-g}^{g,-} \cup \mathbf{I}_{v_1-g, -v_i-2}^{g,-} \right).\end{aligned}$$

Following the steps of the proof of Theorem 2.5.8, using Lemmas 2.5.15 and 2.5.4, we derive the following result on product of hook lengths for elements of \mathcal{DD}'_{2t-2} related to type \tilde{D}_t as underlined in Section 2.4.

Theorem 2.5.16. *Set t a positive integer and $g = 2t - 2$. Set $\lambda \in \mathcal{DD}'_g$ and $\mathbf{v} \in \mathbb{Z}^t$ the $V_{g,t}$ -coding associated to ω , and set $r_i = v_i - g/2 + 1$ for any $i \in \{1, \dots, t\}$. Then we have*

$$|\lambda| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g/2 + 1)(g + 1)}{12}.$$

Moreover setting $\alpha_i(\lambda)$ as in Theorem 2.1.2, for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \lambda} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\lambda)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g + 2 - i - j)}. \quad (2.47)$$

3.1 Introduction

Motivated by the computations from Section 1.4, the last chapter detailed some enumerative results about subsets of partitions linked to the quadratic forms that appear in the Macdonald identities of each infinite affine root system. In this chapter, the goal is to show how these enumerations enable one to transcribe the Macdonald identities within the language of integer partitions. The $V_{g,t}$ -codings introduced in Definition 2.5.1 (that are some specific (decreasingly) ordered vectors of integers) are going to be the key tool for this purpose. These are heavily linked to the bijection $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ involved in their definition and which can be thought of as a shifted permutation $\sigma' \in S_g$, i.e. $\sigma' := g - \sigma(i)$. In Section 3.2, we detail how the sign of these σ , denoted by $\text{sgn}(\sigma) := -\text{sgn}(\sigma')$ and which can be thought of as a signature of a permutation sorting in decreasing order, is characterized by both the cardinal of some subsets of hook lengths of the partitions introduced in Section 2.4 and some statistics attached to the partitions (such as Δ or the size of the Durfee square).

The purpose of Section 3.3 is then to provide the desired rewriting of Macdonald identities for all infinite affine types in the language of partition. These formulas are of type “sum = product”, and recall that in (1.16) and (1.17), the sum is computed over \mathbf{m} in some subsets of \mathbb{Z}^t without any order on its components. However, partitions come with an order, hence when sending the vector \mathbf{m} to the vector \mathbf{r} from Theorems 2.1.1-2.1.2 and the theorems from Section 2.5 we have chosen a specific order: that is the translation by a constant of a $V_{g,t}$ -coding. We apply this formalism to rewrite the sum part of Macdonald identities for all seven infinite root systems as a sum over symplectic, special orthogonal and even orthogonal Schur functions. For instance the rewriting of the Macdonald identity (1.16) for type \tilde{A}_{t-1} gives the following result.

Theorem 3.1.1. *Set $t \in \mathbb{N}^*$. The Macdonald identity for type \tilde{A}_{t-1} can be rewritten as follows:*

$$\sum_{\omega \in \mathcal{P}(t)} (-1)^{|H_{<t}|} T^{|\omega|} s_{\mu}(\mathbf{x}) \prod_{i=1}^t x_i^{-\ell(\omega)} = (T; T)_{\infty}^{t-1} \prod_{1 \leq i < j \leq t} \left(T x_i x_j^{-1}, T x_i^{-1} x_j; T \right)_{\infty}. \quad (3.1)$$

where \mathbf{v} is the $V_{t,t}$ -coding corresponding to ω (see Definition 2.5.1), $H_{<t} := \{s \in \omega \mid h_s < t\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i - v_t + i - t$ for all $1 \leq i \leq t$ and $\ell(\mu) = t - 1$.

Recall the notation K_T defined in (1.13)

$$K_T(t, \mathbf{x}) = \prod_{1 \leq i < j \leq t} x_i^{-1} \left(T x_i^{\pm} x_j^{\pm}; T \right)_{\infty}.$$

Formula (1.17) becomes with the same techniques:

Theorem 3.1.2. *Set $t \in \mathbb{N}^*$ such that $t \geq 2$ and $g = 2t + 2$. The Macdonald identity for type \tilde{C}_t can be rewritten as follows:*

$$\sum_{\omega \in \mathcal{DD}(g)} (-1)^{d_\omega + |H_{<g,+}|} T^{|\omega|/2} \text{sp}_\mu(\mathbf{x}) = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^2, Tx_i^{-2})_\infty, \quad (3.2)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to ω (see Definition 2.5.1), $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$, d_ω the length of the main diagonal of ω and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i + i - g$ for all $1 \leq i \leq t$.

Then in Section 3.4, we show how one can derive by specializing the rewriting of Macdonald identities obtained in Section 3.3 and combining them with the theorems from Section 2.5 q -Nekrasov–Okounkov type formulas such as (1.20) but also new results. For instance setting $x_i = q^i$ in Theorem 3.1.2, and $\tau(x) = 1 - q^x$ in Theorem 2.1.2, with a polynomiality argument, we derive a q -Nekrasov–Okounkov formula for type \tilde{C}_t .

Theorem 3.1.3. *[A q -Nekrasov–Okounkov formula for type \tilde{C}_t] For formal variables T, q and any complex number u , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{DD}} (-u)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{h_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + uq^{h_s/2}}{1 + u^{-1}q^{h_s/2}} \\ = \prod_{m,r \geq 1} \frac{1 + uq^{r-1}T^m}{1 + u^{-1}q^rT^m} \frac{(1 - u^{-2}q^{r+2}T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2q^{r-1}T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^rT^m)^{r - \lfloor r/2 \rfloor} (1 - q^{r+1}T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned} \quad (3.3)$$

We finally examine in Section 3.5 how we can specialize these techniques to derive for all the identities in [Mac72, Appendix 1] for the infinite affine root systems $\tilde{B}_t, \tilde{B}_t^\vee, \tilde{C}_t, \tilde{C}_t^\vee, \tilde{B}C_t$ and \tilde{D}_t as Nekrasov–Okounkov formulas.

3.2 Signs of permutations and integer partitions

In this section, we establish the lemmas that bridge the sign of $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in 2.5.1 together with some subsets of the corresponding partitions for each type. The order of the types in this Section slightly differs from the rest of the paper because these enumerations depend on the subset of partitions attached to the quadratic form as introduced in Section 2.4.

3.2.1 Type \tilde{A}_{t-1} and $\mathcal{P}_{(t)}$

The case of type \tilde{A}_t is once again a bit particular. Indeed one of the differences between Theorems 2.1.1 (stated for type \tilde{A}_t) and 2.1.2 (stated for type \tilde{C}_t) is that in Theorem 2.1.1 the left-hand side $\tau(h-t)\tau(h+t)/\tau(h)^2$ does not depend on a signed statistic ε , whereas in Theorem 2.1.2 this left-hand-side is a single quotient of functions τ depending on ε .

Lemma 3.2.1. *Let t be a positive integer and $g = t$. Let $\omega \in \mathcal{P}_{(t)}$ and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be its associated $V_{t,t}$ -coding and σ as defined in Definition 2.5.1. Then*

$$|H_{<t}| = \#\{s \in \omega, h_s < t\} \equiv \text{sgn}(\sigma) \pmod{2}. \quad (3.4)$$

Proof. Set $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$ the word corresponding to ω by Lemma 2.2.1 then $H_{<t}$ as defined in Theorem 3.1.1 is equivalently described by $H_{<t} = \{(k, l) \in \mathbb{Z} \mid k < l, c_k = 1, c_l = 0 \text{ and } l - k < t\}$. We decompose $H_{<t}$ with respect to the remainders \pmod{t} of the indices of the letters “0” and “1”. More precisely, let k, l be two integers such that $1 \leq k < l \leq t$. Note that there exists

a unique element that belongs to both $H_{<t}$ and the set $\{s \in \omega \mid i_s \in \mathbb{I}_{v_l+mt, v_k-t-t}^{t,+}\}$ (where $\mathbb{I}_{v_l+mt, v_k-t-t}^{t,+}$ is the t -interval defined in Section 2.5) if and only if $v_l + mt \leq v_k - t - mt$ and $m \geq 0$, whence $0 \leq m \leq n_{\sigma(k)} - n_{\sigma(l)} - \mathbb{1}_{\sigma(k) < \sigma(l)}$. Therefore this enumeration of $H_{<t}$ yields

$$|H_{<t}| = \sum_{\substack{(i,j) \in \{1, \dots, t\}^2 \\ i < j}} (n_{\sigma(i)} - n_{\sigma(j)} - \mathbb{1}_{\sigma(i) < \sigma(j)}). \quad (3.5)$$

Ever since we are interested here in having a relation on the parity of $\text{sgn}(\sigma)$, rewriting (3.5) (mod 2) yields

$$|H_{<t}| = \sum_{\substack{(i,j) \in \{1, \dots, t\}^2 \\ i < j}} (n_{\sigma(i)} + n_{\sigma(j)} + \mathbb{1}_{\sigma(i) < \sigma(j)}) \pmod{2}. \quad (3.6)$$

Gathering the common terms, one derives from (3.6)

$$|H_{<t}| = (t-1) \sum_{i=0}^{t-1} n_{\sigma(i)} + \sum_{\substack{(i,j) \in \{1, \dots, t\}^2 \\ i < j}} \mathbb{1}_{\sigma(i) < \sigma(j)} \pmod{2}.$$

We then conclude using Theorem 2.4.1 which states that for elements $\omega \in \mathcal{P}_{(t)}$ and $\phi(\omega) = (n_0, \dots, n_{t-1})$ one has the vanishing condition $\sum_{i=0}^{t-1} n_i = 0$. □

3.2.2 Type \tilde{C}_t and $\mathcal{DD}_{(2t+2)}$

We now prove the following lemma that will be quite useful when applying Theorem 2.1.2 to specializations in Macdonald identities. Its proof follows the same path as the proof of Lemma 3.2.1, but requires an extra discussion due to the presence of g -intervals of different kinds in this case (as explained in Section 2.5). This explains the presence of an extra term (which is the size of the Durfee square).

Lemma 3.2.2. *Set t be a positive integer and $g = 2t + 2$. Set $\omega \in \mathcal{DD}_{(g)}$ and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be the $V_{g,t}$ -coding associated to ω together with $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < g, \varepsilon_s = 1\} \equiv d_\omega + \text{sgn}(\sigma) \pmod{2}. \quad (3.7)$$

Proof. By (2.24) and Lemma 2.2.1 (4), we can split the boxes in $H_{<g,+}$ according to the corresponding pair of indices. By Lemma 2.2.1, we associate a box $s \in \omega$ to (i_s, j_s) . Then $H_{<g,+} = \{j_s - i_s \mid i_s < j_s, c_{j_s} = 0, c_{i_s} = 1\}$ and we can rewrite $H_{<g,+}$ as the disjoint union of the respective congruence classes (mod g) of (i_s, j_s) . Let k, l be two integers such that $1 \leq k < l \leq t$. Note that there exists a unique element that belongs to both $H_{<g,+}$ and the set $\{s \in \omega \mid i_s \in \mathbb{I}_{v_l+mg, v_k-g-mg}^{g,+}\}$ if and only if $v_l + mg \leq v_k - g - mg$ and $m \geq 0$. Hence $0 \leq m \leq n_{\sigma(k)} - n_{\sigma(l)} - \mathbb{1}_{\sigma(k) < \sigma(l)}$. For any $(k, l) \in \{1, \dots, t\}^2$, the case of the elements of $H_{<g,+} \cap \{s \in \omega \mid i_s \in \mathbb{I}_{-v_l+g+mg, v_k-g+mg}^{g,+}\}$ is trickier. If $\sigma(k) + \sigma(l) > g$, there are exactly $n_{\sigma(k)}$ elements. If $\sigma(k) + \sigma(l) < g$, then it remains to determine whether the box u associated to the pair of indices $(-\sigma(l), \sigma(k))$ is above the main diagonal or not. Hence by Corollary 2.2.2, the number of boxes in $H_{<g,+}$ corresponding to this enumeration corresponds

to $n_{\sigma(k)} - \mathbb{1}_{\sigma(k)+\sigma(l)<g} \times \mathbb{1}_{\sigma(k)>\sigma(l)}$. This enumeration yields:

$$|H_{<g,+}| = \sum_{\substack{(i,j) \in \{1,\dots,t\}^2 \\ i < j}} (n_{\sigma(i)} - n_{\sigma(j)} - \mathbb{1}_{\sigma(i)<\sigma(j)} + n_{\sigma(i)} + n_{\sigma(j)} - \mathbb{1}_{\sigma(i)+\sigma(j)<g}) \\ + \sum_{i=1}^t (n_{\sigma(i)} + n_{\sigma(i)} + n_{\sigma(i)} - \mathbb{1}_{\sigma(i)<g/2}). \quad (3.8)$$

By gathering the common terms, we derive the following from the above equality:

$$|H_{<g,+}| \equiv \sum_{i=1}^t n_{\sigma(i)} + \sum_{\substack{(i,j) \in \{1,\dots,t\}^2 \\ i < j}} (\mathbb{1}_{\sigma(i)<\sigma(j)} + \mathbb{1}_{\sigma(i)<g-\sigma(j)} + \mathbb{1}_{\sigma(i)<g/2}) \pmod{2}. \quad (3.9)$$

The last step of the proof consists in noticing that the sum over $n_{\sigma(i)}$ for $i \in \{1, \dots, t\}$ is the sum over the positive $(n_k)_{k \in \{0, \dots, 2t+1\}}$. It is therefore the size of the Durfee square of ω , whence the desired equality. \square

3.2.3 Type \tilde{BC}_t and $\mathcal{DD}_{(2t+1)}$

Similarly we get the following lemma needed in the rewriting of Macdonald identities for affine type \tilde{BC}_t as a sum of special orthogonal Schur functions.

Lemma 3.2.3. *Set t be a positive integer and $g = 2t + 1$. Set $\omega \in \mathcal{DD}_{(g)}$ and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be the $V_{g,t}$ -coding associated to ω together with $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < 2t + 1, \varepsilon_s = 1\} \equiv \text{sgn}(\sigma) + |H_{<g,+} \cap \Delta| \pmod{2}. \quad (3.10)$$

Proof. The proof follows the same exact steps as for Lemma 3.2.2 except that (3.8) becomes in this case

$$|H_{<g,+}| = \sum_{\substack{(i,j) \in \{1,\dots,t\}^2 \\ i < j}} (n_{\sigma(i)} - n_{\sigma(j)} - \mathbb{1}_{\sigma(i)<\sigma(j)} + n_{\sigma(i)} + n_{\sigma(j)} - \mathbb{1}_{\sigma(i)+\sigma(j)<g}) \\ + \sum_{i=1}^t (n_{\sigma(i)} + n_{\sigma(i)}).$$

Hence (3.9) becomes

$$|H_{<g,+}| \equiv \sum_{\substack{(i,j) \in \{1,\dots,t\}^2 \\ i < j}} (\mathbb{1}_{\sigma(i)<\sigma(j)} + \mathbb{1}_{\sigma(i)<g-\sigma(j)}) \pmod{2}. \quad (3.11)$$

To complete the proof, it remains to add $\mathbb{1}_{\sigma(i)<g/2} - \mathbb{1}_{\sigma(i)<g/2}$ and note that by (2.25)

$$|H_{<g,+} \cap \Delta| = \#\{i \in \{1, \dots, t\} \mid \sigma(i) < g/2\}.$$

\square

3.2.4 Type \tilde{C}_t^\vee and $\mathcal{SC}_{(2t)}$

Following the same kind of enumeration as for Lemma 3.2.2, one derives the following lemma which will be applied when rewriting the Macdonald identity for affine type \tilde{C}_t^\vee as a sum over orthogonal Schur functions.

Lemma 3.2.4. *Set t be a positive integer and $g = 2t$. Set $\omega \in \mathcal{SC}_{(g)}$ and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be its associated $V_{g,t}$ -coding and σ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < 2t, \varepsilon_s = 1\} \equiv \text{sgn}(\sigma) + |H_{<g,+} \cap \Delta| + d_\omega \pmod{2}. \quad (3.12)$$

3.2.5 Type \tilde{B}_t^\vee and \mathcal{DD}'_{2t}

Adapting the proof of Lemma 3.2.2, we derive the following lemma.

Lemma 3.2.5. *Set t be a positive integer and $g = 2t$. Set $\lambda \in \mathcal{DD}'_g$ and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be the $V_{g,t}$ -coding associated to λ together with $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < 2t, \varepsilon_s = 1\} \equiv d_\lambda + \text{sgn}(\sigma) \pmod{2}. \quad (3.13)$$

Proof. The only difference with the proof of Lemma 3.2.2 resides in the fact that λ has a $(2t)$ -quotient not necessarily empty. Let m_1 be a positive integer as in Section 2.4. Set $i_0 \in \{1, \dots, t\}$ such that $\sigma(i_0) = g-1$ and $v_{i_0} = m_1g-1$. Set $i \in \{1, \dots, t\}$. If $i < i_0$, then $\sigma(i) < \sigma(i_0)$ and there are exactly $n_{\sigma(i)} - m_1$ pairs of indices (k, l) such that $l - k \in H_{<g,+}$ and $l \equiv \sigma(i) \pmod{g}$ with $c_l = 0$ and $k \equiv g-1 \pmod{g}$ and $c_k = 1$. Similarly there are $m_1 - 1$ pair of indices (k, l) such that $l - k \in H_{<g,+}$ and $l \equiv g-1 \pmod{g}$ with $c_l = 0$ and $k \equiv \sigma(i) \pmod{g}$ and $c_k = 1$. Similarly if $i > i_0$, this number becomes $2(m_1 - 1) - n_{\sigma(i)}$. Remarking that $d_\lambda = m_1 - 1 + \sum_{\substack{i=1 \\ i \neq i_0}}^t n_{\sigma(i)}$ and following the same steps as for Lemma 3.2.2 where $\mathbb{1}_{\sigma(i) < g - \sigma(j)}$ is replaced by $\mathbb{1}_{\sigma(i) < g - 2 - \sigma(j)}$ concludes the proof. \square

3.2.6 Type \tilde{B}_t and \mathcal{DD}'_{2t-1}

Following the same path as for Lemma 3.2.5 yields the following lemma

Lemma 3.2.6. *Set t be a positive integer and $g = 2t - 1$. Set $\lambda \in \mathcal{DD}'_g$ and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be the $V_{g,t}$ -coding associated to λ together with $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < 2t - 1, \varepsilon_s = 1\} \equiv \text{sgn}(\sigma) + |H_{<g,+} \cap \Delta| \pmod{2}. \quad (3.14)$$

3.2.7 Type \tilde{D}_t and \mathcal{DD}'_{2t-2}

Following the same path as for Lemma 3.2.5 yields the following lemma

Lemma 3.2.7. *Set t be a positive integer and $g = 2t - 2$. Set $\lambda \in \mathcal{DD}'_g$ and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$. Let \mathbf{v} be the $V_{g,t}$ -coding associated to λ together with $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ as defined in Definition 2.5.1. Then*

$$|H_{<g,+}| = \#\{s \in \omega, h_s < 2t - 3, \varepsilon_s = 1\} \equiv d_\lambda + \text{sgn}(\sigma) \pmod{2}. \quad (3.15)$$

3.3 Rewriting of the Macdonald identities for affine root systems

The Macdonald identities of Proposition 1.4.1 are formulas of the form “sum=product” where the sum is a power series over sublattices of \mathbb{Z}^t of a formal variable T depending on a vector of t variables \mathbf{x} . The proofs in this section all follow the same steps:

1. expand the products in the sum part and interpret them as determinants,
2. interpret the exponent of the formal variable T as one of the quadratic forms from Section 2.4 and then use the correspondence between vectors of integers and partitions studied in Section 2.4 together with the $V_{g,t}$ -coding.

3.3.1 Type \tilde{A}_{t-1}

Recall the left-hand side from (1.16)

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0}} T^{t\|\mathbf{m}\|/2 + \sum_{i=1}^t (i-1)m_i} \frac{\det_{1 \leq i, j \leq t} \left(x_i^{tm_j + j - 1} \right)}{\det_{1 \leq i, j \leq t} \left(x_i^{t-j} \right)}. \quad (3.16)$$

Set $\omega \in \mathcal{P}_{(t)}$ such that $\phi(\omega) = (m_1, \dots, m_t)$, where ϕ is the bijection from Theorem 2.4.1 and \mathbf{v} its associated $V_{t,t}$ -coding. By (2.1), the exponent of T is the weight of ω . It remains to deal with the quotient of determinants.

Note that the exponents $tm_j + j - 1$ of the x_i 's in (3.16) can be negative, whereas in the definition of a Schur polynomial in (1.3) the exponents of the x_i 's in the determinant are positive. To prove Theorem 3.1.1, we will first reorder the columns of the matrix $\left(x_i^{tm_j + j - 1} \right)_{1 \leq i, j \leq t}$ so that the sequence $(tm_{\sigma(j)} + \sigma(j) - 1)$ is in decreasing order. Set $\sigma : \{1, \dots, t\} \mapsto \{0, \dots, t-1\}$ as defined in Definition 2.5.1. Take $\sigma' \in S_t$ such that $\sigma' := t - \sigma$ and σ'^{-1} the inverse of σ . Therefore for all $j \in \{1, \dots, t\}$, $tm_j + j - 1 = v_{(\sigma')^{-1}(j)}$.

Hence, using Lemma 3.2.1, (3.16) becomes

$$\sum_{\omega \in \mathcal{P}_{(t)}} (-1)^{|H_{<t}|T|\omega|} \frac{\det_{1 \leq i, j \leq t} \left(x_i^{v_j} \right)}{\det_{1 \leq i, j \leq t} \left(x_i^{t-j} \right)}, \quad (3.17)$$

where \mathbf{v} is the $V_{t,t}$ -coding associated bijectively with ω , as underlined in Remark 2.5.2.

The final step of the proof to rewrite the quotient of determinants as a Schur polynomial is to transform it so that all of the exponents of the x_i 's become nonnegative. By definition of \mathbf{v} , $v_t = \min(v_1, \dots, v_t)$. Therefore $v_i - v_t \geq 0$ for any $i \in \{1, \dots, t\}$ and factorizing by $\prod_{i=1}^t x_i^{v_t}$, one has

$$\det_{1 \leq i, j \leq t} \left(x_i^{v_j} \right) = \det_{1 \leq i, j \leq t} \left(x_i^{v_j - v_t} \right) \prod_{i=1}^t x_i^{v_t}.$$

To complete the proof of Theorem 3.1.1, it remains to prove that $v_t = -\ell(\omega)$ where $\ell(\omega)$ is the number of parts of ω . Let $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$ be the word corresponding to ω . Then by Definition 2.5.1, $v_t = \min\{k \in \mathbb{Z} \mid c_k = 1\}$. The number of parts of ω then corresponds to the number of letters "0" whose indices are greater than v_t . Therefore

$$\begin{aligned} \ell(\omega) &= \sum_{i=1}^{t-1} (m_{\sigma(i)} - m_{\sigma(t)} - \mathbb{1}_{\sigma(i) < \sigma(t)}) \\ &= \sum_{i=1}^{t-1} m_{\sigma(i)} - (t-1)m_{\sigma(t)} - \sigma(t). \end{aligned}$$

By using $\sum_{i=1}^t m_{\sigma(i)} = 0$, one derives $\ell(\omega) = -v_t$. By setting $\mu \in \mathcal{P}$ such that $\mu_i = v_i - v_t + i - t$, we conclude the proof.

3.3.2 Type \tilde{C}_t

Proof of Theorem 3.1.2. We recall the left-hand side of (1.17) here:

$$\sum_{\mathbf{m} \in \mathbb{Z}^t} T^{(t+1)\|\mathbf{m}\|^2 + \sum_{i=1}^t m_i(i-t-1)} \frac{\det_{1 \leq i, j \leq t} \left(x_i^{(2t+2)m_j + j - t - 1} - x_i^{-((2t+2)m_j + j - t - 1)} \right)}{\det_{1 \leq i, j \leq t} \left(x_i^{j-t-1} - x_i^{-(j-t-1)} \right)}. \quad (3.18)$$

It remains to order the m_j 's to derive a sum of symplectic Schur functions sp (defined in (1.6)). Set $\omega \in \mathcal{DD}_{(g)}$ such that $\phi(\omega) = (0, m_1, \dots, m_t, 0, -m_t, \dots, -m_1)$, where ϕ is the bijection from Theorem 2.4.1, \mathbf{v} its $V_{g,t}$ -coding with $g = 2t + 2$ as in Definition 2.5.1 and $\mathbf{r} = (v_1 - g/2, \dots, v_t - g/2)$ the vector defined in Theorem 2.1.2. Let $\sigma' \in S_g$ be such that the sequence $(\phi(\omega)_{\sigma'(i)-1}, \sigma'(i))_{i \in \{1, \dots, g\}}$ is increasing with respect to the lexicographic order. Note that by definition of σ in Definition 2.5.1, for any $i \in \{1, \dots, t\}$, $\sigma(i) = g - \sigma'(i)$ and $\sigma'(g - i) = g - \sigma(i)$. Hence for any $i \in \{1, \dots, t\}$

$$gm_i + i - g/2 = \begin{cases} r_{(\sigma')^{-1}(i)} & \text{if } gm_i + i - g/2 < -(gm_i + i - g/2), \\ -r_{(\sigma')^{-1}(i)} & \text{otherwise.} \end{cases}$$

Moreover the exponent of T in (1.17) corresponds to the quadratic form (2.12). Recall that in Theorem 2.1.2 the vector \mathbf{r} is in bijective correspondence with \mathbf{v} and so with $\omega \in \mathcal{DD}_{(g)}$. Hence the left-hand side of (1.17) becomes

$$\sum_{\mathbf{r} \in \mathbb{Z}^t} \text{sgn}(\sigma) T^{|\omega|/2} \frac{\det \left(x_i^{r_j} - x_i^{-r_j} \right)}{\det \left(x_i^{t+1-j} - x_i^{j-t-1} \right)}.$$

Set $\mu_j := r_j - g/2 + j$ for all $j \in \{1, \dots, t\}$. By definition of the $V_{g,t}$ -coding and (2.22), we have $\mu_1 \geq \dots \geq \mu_t \geq 0$ which concludes the proof by (1.6). \square

3.3.3 Type \tilde{B}_t

Following the same path as before, one rewrites the Macdonald identity for type \tilde{B}_t as follows.

Theorem 3.3.1. *Set $t \in \mathbb{N}^*$ such that $t \geq 3$ and $g = 2t - 1$. The Macdonald identity for type \tilde{B}_t can be rewritten as follows:*

$$\sum_{\lambda \in \mathcal{DD}'_g} (-1)^{|H_{<g,+}| + |H_{<g,+} \cap \Delta|} T^{|\lambda|/2} \text{so}_\mu(\mathbf{x}) = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^\pm; T)_\infty \quad (3.19)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to λ (see Definition 2.5.1), Δ is the main diagonal of λ , $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := (v_i + i - g)$ for all $1 \leq i \leq t$.

Proof. The change of variables and the swap of sums performed in the proof of (3.2) are the same and (1.17) is replaced by

$$\begin{aligned} \frac{1}{2\Delta_B(\mathbf{x})} \sum_{\mathbf{m} \in \mathbb{Z}^t} T^{(2t-1)\|\mathbf{m}\|^2/2 + \sum_{i=1}^t m_i(i-t-1/2)} \det_{1 \leq i, j \leq t} \left(x_i^{(2t-1)m_j + j - t} - x_i^{-((2t-1)m_j + j - t - 1)} \right) \\ = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^\pm; T)_\infty. \end{aligned} \quad (3.20)$$

We split the sum on the left-hand side of (3.20) in two sums. The first is over $\mathbf{m} \in \mathbb{Z}^t$ such that $m_1 \in \mathbb{N}^*$ and the other is over $\mathbf{m} \in \mathbb{Z}^t$ such that $m_1 \leq 0$. By (2.18), the exponent of T in the first sum is the quadratic form that corresponds to half the weight of $\lambda \in \mathcal{DD}'_{2t-1}$ such that

- setting $\text{core}_{2t-1}(\lambda) = \omega$ then $\phi(\omega) = (m_2, \dots, m_{t-1}, -m_{t-1}, \dots, m_2, 0)$,
- and $\text{quot}_{2t-1}(\lambda) = \nu^{(2t-2)}$ with $|\nu^{(2t-2)}| = m_1(m_1 - 1)$.

The second sum reads as follows

$$\frac{1}{\Delta_B(\mathbf{x})} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 \leq 0}} T^{(2t-1)\|\mathbf{m}\|^2 + \sum_{i=1}^t m_i(i-t-1/2)} \det_{1 \leq i, j \leq t} \left(x_i^{(2t-1)m_j + j - t} - x_i^{-((2t-1)m_j + j - t - 1)} \right).$$

Note that the exponent of T in the second sum is the quadratic form corresponding to (2.18). This quadratic form is invariant by the transformation $m_1 \rightarrow 1 - m_1$. Hence it corresponds to half the weight of $\lambda \in \mathcal{DD}'_{2t-1}$ such that

- setting $\text{core}_{2t-1}(\lambda) = \omega$, then $\phi(\omega) = (m_2, \dots, m_{t-1}, -m_{t-1}, \dots, m_2, 0)$,
- and $\text{quot}_{2t-1}(\lambda) = \nu^{(2t-2)}$ with $|\nu^{(2t-2)}| = m_1(m_1 - 1)$.

Hence λ corresponds to the vector $(m_2, \dots, m_t, 1 - m_1)$. Therefore by setting $m_1 \rightarrow 1 - m_1$, using then (2.42) and Lemma 3.2.6, the sum over nonpositive m_1 is the same as the sum over positive m_1 which concludes the proof. \square

3.3.4 Type \tilde{B}_t^\vee

Using Lemma 3.2.5 and the same computations, one derives the following result.

Theorem 3.3.2. *Set $t \in \mathbb{N}^*$ such that $t \geq 3$ and $g = 2t$. The Macdonald identity for type \tilde{B}_t^\vee can be rewritten as follows:*

$$\sum_{\lambda \in \mathcal{DD}'_g} (-1)^{d_\lambda + |H_{<g,+}|} T^{|\lambda|/2} \text{sp}_\mu(\mathbf{x}) = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (T^2 x_i^{\pm 2}; T^2)_\infty \quad (3.21)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to λ (see Definition 2.5.1), d_λ is the length of the main diagonal of λ , $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i + i - g$ for all $1 \leq i \leq t$.

3.3.5 Type \tilde{C}_t^\vee

Following the same steps and using Lemma 3.2.4 one can also rewrite the Macdonald identities for type \tilde{C}_t^\vee as follows.

Theorem 3.3.3. *Set $t \in \mathbb{N}^*$ such that $t \geq 2$ and $g = 2t$. The Macdonald identity for type \tilde{C}_t^\vee can be rewritten as follows:*

$$\begin{aligned} \sum_{\omega \in \mathcal{SC}_{(g)}} (-1)^{|H_{<g,+}| + |H_{<g,+} \cap \Delta| + d_\omega} T^{|\omega|/2} \text{so}_\mu(\mathbf{x}) \\ = \left(T^{1/2}; T^{1/2}\right)_\infty (T; T)_\infty^{t-1} K_T(t, \mathbf{x}) \prod_{i=1}^t \left(T^{1/2} x_i^\pm; T^{1/2}\right)_\infty \end{aligned} \quad (3.22)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to ω (see Definition 2.5.1), Δ is the main diagonal of ω , $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i + i - g$ for all $1 \leq i \leq t$.

Proof. The proof is the same as for Theorem 3.1.2 but the determinants appearing are now determinants of type B and $\Delta_C(\mathbf{x})$ is replaced by $\Delta_B(\mathbf{x})$ defined in (1.5). Therefore the intermediate (1.17) now becomes

$$\begin{aligned} \frac{1}{\Delta_B(\mathbf{x})} \sum_{\mathbf{m} \in \mathbb{Z}^t} T^{t\|\mathbf{m}\|^2 + \sum_{i=1}^t m_i(i-t-1/2)} \det_{1 \leq i, j \leq t} \left(x_i^{gm_j + j - t} - x_i^{-(gm_j + j - t - 1)} \right) \\ = (T; T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t (T x_i^\pm; T)_\infty. \end{aligned} \quad (3.23)$$

The exponent of the formal variable T on the left-hand side is the same quadratic form as (2.14) when taking $\omega \in \mathcal{SC}_{(2t)}$ such that $\phi(\omega) = (m_1, \dots, m_t, -m_t, \dots, -m_1)$. Using the correspondence between ω and its $V_{g,t}$ -coding and \mathbf{r} as defined in Theorem 2.5.12, note that for any $i \in \{1, \dots, t\}$ there exists $k \in \{1, \dots, t\}$ such that $r_k = \max(gm_i + i - t, -gm_i - i + 1 + t)$. By reordering the columns in a decreasing order, one derives (3.22). \square

3.3.6 Type \tilde{BC}_t

Following the same steps as before using Theorem 2.5.14 and Lemma 3.2.3, one derives the following results

Theorem 3.3.4. *Set $t \in \mathbb{N}^*$ and $g = 2t + 1$. The Macdonald identity for type \tilde{BC}_t can be rewritten as follows:*

$$\begin{aligned} \sum_{\omega \in \mathcal{DD}_{(g)}} (-1)^{|H_{<g,+}| + |\Delta|} T^{|\omega|/2} \text{so}_{\mu}(\mathbf{x}) \\ = (T; T)_{\infty}^t K_T(t, \mathbf{x}) \prod_{i=1}^t (Tx_i^{\pm}; T)_{\infty} (Tx_i^{\pm 2}; T^2)_{\infty} \end{aligned} \quad (3.24)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to ω (see Definition 2.5.1), Δ is the main diagonal of ω , $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i + i - g$ for all $1 \leq i \leq t$.

3.3.7 Type \tilde{D}_t

Similarly, using Lemma 3.2.7, Theorem 2.5.16 and the same computations, one derives the rewriting of Macdonald identity for type \tilde{D}_t .

Theorem 3.3.5. *Set $t \in \mathbb{N}^*$ such that $t \geq 4$ and $g = 2t - 2$. The Macdonald identity for type \tilde{D}_t can be rewritten as follows:*

$$\sum_{\lambda \in \mathcal{DD}'_g} (-1)^{d_{\lambda} + |H_{<g,+}|} T^{|\lambda|/2} \text{o}^{\text{even}}_{\mu}(\mathbf{x}) = \frac{1}{2} (T; T)_{\infty}^t K_T(t, \mathbf{x}) \quad (3.25)$$

where \mathbf{v} is the $V_{g,t}$ -coding corresponding to λ (see Definition 2.5.1), d_{λ} the length of the main diagonal of ω , $H_{<g,+} := \{s \in \omega, h_s < g, \varepsilon_s = 1\}$ and $\mu \in \mathcal{P}$ is such that $\mu_i := (v_i + i - g)$ for all $1 \leq i \leq t$.

The proof is done using (2.43) in Theorem 2.5.10 and Lemma 3.2.7. The only difference is that instead of splitting the sum in two parts as we did for type \tilde{B}_t , the sum is split with a condition on the positivity of m_1 or not and the sign of m_t . The case $m_t = 0$ is equivalent to $\mu_t = 0$ and hence by definition of $\text{o}^{\text{even}}_{\mu}$, this yields (3.25).

3.4 q -Nekrasov–Okounkov formulas

In this section, we combine the results obtained in Sections 2.5 and 3.3 to derive q -Nekrasov–Okounkov type formulas for all infinite types except for the type \tilde{A} already given in (1.20). The proofs of all theorems below follow the same steps:

1. Specialize all the Macdonald identities stated in Section 3.3 either taking $x_i = q^i$ or $x_i = q^{2i-1}$. Then

(a) reinterpret the specialized symplectic, special orthogonal Schur functions as a product over hook lengths using one of the theorems in Section 2.5 with an appropriate choice of the function τ on the sum part,

(b) and rewrite the products with some technical lemmas.

2. Then setting $u = q^g$ or $u = q^{g/2}$, verify that both sides of the equalities derived at the end of the previous step can be seen as Laurent polynomials in u .

For instance, to derive Theorem 3.1.3, we will use a polynomiality argument by remarking that for any positive integer n , the coefficients of T^n on both sides of (3.3) can be seen as Laurent polynomials with coefficient in $\mathbb{C}(q)$. To see that, note that for any $\lambda \in \mathcal{DD}$ we have:

$$\prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{h_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + uq^{h_s/2}}{1 + u^{-1}q^{h_s/2}} = \prod_{\substack{s \in \lambda \\ s \notin \Delta}} \frac{1 - u^{-2\varepsilon_s} q^{h_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{(1 + uq^{h_s/2})(1 + u^{-1}q^{h_s/2})}{1 - q^{h_s}}.$$

Hence the left-hand side is a sum of Laurent polynomials in u with coefficients in $\mathbb{C}(q)$. By expanding the quotient on the right-hand side of (3.3) and extracting the coefficients in T^n , we obtain the Laurent polynomiality in u of the coefficients. Hence it is enough to prove that both sides coincide at $u = q^t$ for infinitely many positive integers t .

The following Lemma is the key ingredient in step 1.(b).

Lemma 3.4.1. *Let x be a formal variable and t a positive integer. Then we have:*

$$\prod_{i=1}^t (1 - q^i x) = \prod_{r \geq 1} \frac{1 - q^r x}{1 - q^{r+t} x}, \quad (3.26)$$

$$\prod_{i=1}^t (1 - q^{-i} x) = \prod_{r \geq 1} \frac{1 - q^{r-1-t} x}{1 - q^{r-1} x}, \quad (3.27)$$

and

$$\prod_{1 \leq i < j \leq t} (1 - q^{i+j} x) = \prod_{r \geq 1} \frac{(1 - q^{r+2} x)^{\lfloor (r+1)/2 \rfloor} (1 - q^{r+1+2t} x)^{r - \lfloor r/2 \rfloor}}{(1 - q^{r+1+t} x)^r}, \quad (3.28)$$

$$\prod_{1 \leq i < j \leq t} (1 - q^{-i-j} x) = \prod_{r \geq 1} \frac{(1 - q^{r-1} x)^{\lfloor (r+1)/2 \rfloor} (1 - q^{r-2t} x)^{r - \lfloor r/2 \rfloor}}{(1 - q^{r-1-t} x)^r}, \quad (3.29)$$

and

$$\prod_{1 \leq i < j \leq t} (1 - q^{j-i} x) (1 - q^{i-j} x) = \frac{1}{(1-x)^{t-1}} \prod_{r \geq 1} \frac{(1 - q^{r-t} x)^r (1 - q^{r+t} x)^r}{(1 - q^{r+1} x)^r (1 - q^{r-1} x)^r}. \quad (3.30)$$

Proof. Equalities (3.26) and (3.27) are just obtained immediately but noting that they are telescopic products. In order to prove (3.28), we will need to gather $i + j$ by values. Hence if we set $3 \leq k \leq 2t - 1$, we need to count how many $(i, j) \in \{1, \dots, t\}^2$ such that $i < j$ and $i + j = k$ there are. For $i + j = k$, we have:

$$\begin{cases} 1 \leq i \leq t-1 \\ i+1 \leq j \leq t \end{cases} \iff \begin{cases} 1 \leq i \leq t-1 \\ 2i+1 \leq k \leq i+t. \end{cases}$$

Hence it is equivalent to

$$\begin{cases} t+1 \leq k \leq 2t-1 \\ k-t \leq i \leq \lfloor \frac{k-1}{2} \rfloor \end{cases} \quad \text{or} \quad \begin{cases} 3 \leq k \leq t \\ 1 \leq i \leq \lfloor \frac{k-1}{2} \rfloor. \end{cases}$$

Therefore gathering by values of k in the left-hand side of (3.28) yields:

$$\begin{aligned} \prod_{k=3}^t (1-q^k x)^{\lfloor (k-1)/2 \rfloor} \prod_{k=t+1}^{2t-1} (1-q^k x)^{\lfloor (k-1)/2 \rfloor - k + t + 1} \\ = \prod_{k=1}^{2t-1} (1-q^k x)^{\lfloor (k-1)/2 \rfloor} \prod_{k=t+1}^{2t-1} (1-q^k x)^{t+1-k} \end{aligned}$$

The above equation is equivalent shifting $k \rightarrow k+t$ in the second product to

$$\begin{aligned} \prod_{k=1}^{2t-1} (1-q^k x)^{\lfloor (k-1)/2 \rfloor} \prod_{k=1}^{t-1} (1-q^{k+t} x)^{1-k} &= \frac{\prod_{r \geq 1} (1-q^r x)^{\lfloor (r-1)/2 \rfloor} \prod_{r \geq t} (1-q^{r+t} x)^{r-1}}{\prod_{r \geq 2t} (1-q^r x)^{\lfloor (r-1)/2 \rfloor} \prod_{r \geq 1} (1-q^{r+t} x)^{r-1}} \\ &= \prod_{r \geq 1} \frac{(1-q^r x)^{\lfloor (r-1)/2 \rfloor}}{(1-q^{r-1+2t} x)^{t-1+\lfloor r/2 \rfloor}} \prod_{r \geq 1} \frac{(1-q^{r-1+2t} x)^{t+r-2}}{(1-q^{r+t} x)^{r-1}} \\ &= \prod_{r \geq 1} \frac{(1-q^r x)^{\lfloor (r-1)/2 \rfloor} (1-q^{r-1+2t} x)^{r-1-\lfloor r/2 \rfloor}}{(1-q^{r+t} x)^{r-1}}, \end{aligned}$$

which is the desired result by shifting $r \rightarrow r+2$ at the numerator and $r \rightarrow r+1$ at the denominator. The other equalities (3.29) and (3.30) are proved similarly. \square

3.4.1 Type \tilde{A}_{t-1}

This corresponds to the result (1.20), obtained in parallel by Dehaye–Han [DH11] and Iqbal–Nazir–Raza–Saleem [INRS12].

3.4.2 Type \tilde{C}_t

The goal of this subsection is to specialize symplectic or orthogonal Schur functions as product of hook lengths using the theorems from Section 2.5. The left-hand side of (3.3) can be obtained by setting $\tau(x) = 1 - q^x$ in Theorem 2.1.2 and multiplying the resulting expression by the product of hook lengths on the main diagonal Δ . We prove the following result.

Lemma 3.4.2. *Let t be a positive integer, set $\lambda \in \mathcal{DD}_{(2t+2)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t+2,t}$ -coding. Set $\mu = (v_1 - 2t - 1, \dots, v_t - t - 2) \in \mathcal{P}$. Then*

$$\text{sp}_\mu(q, q^2, \dots, q^t) = (-1)^{|H+|} q^{(t+1)d_\lambda} \prod_{s \in \lambda} \frac{1 - q^{h_s - (2t+2)\varepsilon_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + q^{t+1+h_s/2}}{1 + q^{-t-1+h_s/2}}. \quad (3.31)$$

Proof. The proof of (3.31) is done by induction on the length of the main diagonal Δ of λ . We show that it follows the same induction as for Theorem 2.1.2 with $\tau(x) = 1 - q^x$. Set $g = 2t + 2$. When λ is the empty partition, $\mathbf{v} = (2t + 1, \dots, t + 2)$ then μ is the empty partition. This concludes the initialization step. If we set $\lambda \in \mathcal{DD}_{(2t+2)}$, \mathbf{v} its associated $V_{2t+2,t}$ -coding, then $\mu = (v_1 - 2t - 1, \dots, v_t - t - 2) \in \mathcal{P}$ and H_1 the largest hook of λ . Then define $\tilde{\lambda} = \lambda \setminus H_1 \in \mathcal{DD}_{(2t+2)}$ the partition whose Ferrers diagram is the one of λ where H_1 as

been removed. Let \mathbf{v}' be its associated $V_{2t+2,t}$ -coding and $\tilde{\mu} = (v'_1 - 2t - 1, \dots, v'_t - t - 2) \in \mathcal{P}$. Setting $r_i = v_i - t - 1$ and $r'_i = v'_i - t - 1$, we have

$$\frac{\text{sp}(\mu)}{\text{sp}(\tilde{\mu})} = \frac{\det_{1 \leq i, j \leq t} (q^{i(v_j - t - 1)} - q^{-i(v_j - t - 1)})}{\det_{1 \leq i, j \leq t} (q^{i(v'_j - t - 1)} - q^{-i(v'_j - t - 1)})} = \frac{\det_{1 \leq i, j \leq t} (q^{(i-t-1)r_j} - q^{-(i-t-1)r_j})}{\det_{1 \leq i, j \leq t} (q^{(i-t-1)r'_j} - q^{-(i-t-1)r'_j})}. \quad (3.32)$$

We first handle the case $g < v_1 < 3g/2$. In this case, $v_1 = g + \sigma(1)$ and $\sigma(1) < g/2$, with σ as defined in Definition 2.5.1. Then $\sigma(1) < g - \sigma(1)$, λ has for largest hook length $h_{(1,1)} = 2\sigma(1) \in \Delta$ and $h_{(1,1)}/2 - t - 1 = \sigma(1) - g/2 < 0$. There exists $2 \leq l \leq t$ such that $r'_l = g/2 - \sigma(1) = -r_1 + g$, and for any $i \in \{1, \dots, l-1\}$ we have $r'_i = r_{i+1}$, and $r'_i = r_i$ otherwise. Moreover if we take $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ the word corresponding to λ , then $c_{-\sigma(1)} = 1$. For $1 \leq i \leq l$, we have $g - \sigma(1) < v_i < g + \sigma(1)$. This implies that $-\sigma(1) < v_i$ and recall that $c_{v_i - g} = 0$. Similarly for $l+1 \leq i \leq t$, $v_i - g < -\sigma(1)$. Hence $\mathcal{H}_{1,+} = \{2\sigma(1), \sigma(1), r_1 + r_i - g \text{ for } 1 \leq i \leq l\}$ with $\mathcal{H}_{1,+}$ as defined in Lemma 2.5.6.

By (1.7), (3.32) becomes

$$\begin{aligned} & \prod_{i=1}^t \frac{q^{-r_i}(1 - q^{2r_i})}{q^{-r'_i}(1 - q^{2r'_i})} \prod_{i=1}^{t-1} \frac{q^{-(t-i)r_i}}{q^{-(t-i)r'_i}} \prod_{j=i+1}^t \frac{(1 - q^{r_i - r_j})(1 - q^{r_i + r_j})}{(1 - q^{r'_i - r'_j})(1 - q^{r'_i + r'_j})} \\ &= q^{-2r_1 + g + \sum_{i=2}^l r_i - r_1(2t-l-1) + g(t-l)} \frac{(1 - q^{2r_1})}{1 - q^{2(g-r_1)}} \\ & \times \prod_{j=2}^{l-1} \frac{(1 - q^{r_1 - r_j})(1 - q^{r_1 + r_j})}{(1 - q^{r_j + r_1 - g})(1 - q^{r_j - r_1 + g})} \prod_{j=l+1}^t \frac{(1 - q^{r_1 - r_j})(1 - q^{r_1 + r_j})}{(1 - q^{g - r_j - r_1})(1 - q^{r_j - r_1 + g})}. \quad (3.33) \end{aligned}$$

Remark that the first term in the right-hand side of (3.33) can be rewritten using $tg = -g/2 + \sum_{i=1}^{2t+1} i$ as follows

$$q^{-tg + 2(g-r_1) + \sum_{j=2}^t (r_j - r_1 + g) - \sum_{j=l+1}^t (r_1 + r_i - g)} = q^{-g/2 + \sum_{i=1}^{2t+1} i + 2(g-r_1) + \sum_{j=2}^t (r_j - r_1 + g) - \sum_{j=l+1}^t (r_1 + r_i - g)}.$$

Then using that for any $x \in \mathbb{C}^*$ $q^x = -(1 - q^x)/(1 - q^{-x})$, this term becomes

$$q^{t+1} \left(-\frac{(1 - q^{g-r_1})(1 + q^{g-r_1})}{(1 - q^{r_1-g})(1 + q^{r_1-g})} \right) \prod_{i=1}^{2t+1} \left(-\frac{1 - q^{-i}}{1 - q^i} \right) \prod_{j=2}^t \left(-\frac{1 - q^{r_j - r_1 + g}}{1 - q^{r_1 - g - r_j}} \right) \prod_{j=l+1}^t \left(-\frac{1 - q^{g-r_1-r_j}}{1 - q^{r_1+r_j-g}} \right).$$

This can be rewritten as

$$\frac{1 + q^{t+1+h_{(1,1)}/2}}{1 + q^{-t-1+h_{(1,1)}/2}} \prod_{i=1}^{2t+1} \left(-\frac{1 - q^{-i}}{1 - q^i} \right)^{\alpha_i(H_1)}.$$

Using the induction property, it concludes the proof of the case $g < v_1 < 3g/2$. As in the proof of Theorem 2.1.2, there are two cases left to deal with to perform the induction:

1. either $v'_1 = v_1 - 2t - 2$ and $v'_i = v_i$ for $2 \leq i \leq t$,
2. or there exists $1 \leq l \leq t$ such that for any $1 \leq i < l$ $v'_i = v_{i+1}$, $v'_l = v_1 - 2t - 2$ and for any $l+1 \leq i \leq t$, $v'_i = v_i$.

Moreover by (2.25), remark that in both cases $h \in H_1 \cap \Delta$ implies that $h = 2r_1 - 2t - 2$.

We handle these cases by using $1 - x^2 = (1 - x)(1 + x)$, $q^{-\sum_{i=1}^{2t+1} i} = \prod_{i=1}^{2t+1} \left(-\frac{1 - q^{-i}}{1 - q^i} \right)$ and the same reasoning as for the proof of Theorem 2.1.2. □

When $u = q^{t+1}$, the product over hook lengths on the left-hand side of (3.3) is zero unless λ has no hook length equal to $2t + 2$, i.e. $\lambda \in \mathcal{DD}_{(2t+2)}$. By Lemmas 3.4.2 and 3.4.1, (3.3) coincides with the Macdonald identity (3.2) for type \tilde{C}_t setting $x_i = q^i$. This completes the proof of Theorem 3.1.3.

3.4.3 Type \tilde{B}_t

Following the same steps as for Lemma 3.4.7, using that $\sum_{i=1}^{2t-2} i = (t-1)(2t-1)$, we derive the following from Theorem 2.5.8 with $\tau(x) = 1 - q^{2x}$.

Lemma 3.4.3. *Let t be a positive integer, set $\lambda \in \mathcal{DD}_{2t-1}^1$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t-1,t}$ -coding. Set $\mu = (v_1 - 2t + 2, \dots, v_t - t + 1) \in \mathcal{P}$. Then*

$$\text{so}_\mu(q, q^3, \dots, q^{2t-1}) = (-1)^{|H_{<g,+}| + |H_{<g,+ \cap \Delta}|} q^{-(2t-1)d_\lambda} \prod_{s \in \lambda} \frac{1 - q^{2h_s - 2(2t-1)\varepsilon_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - q^{2t-1+h_s}}{1 - q^{-2t+1+h_s}}.$$

Following the same steps, using Theorem 3.3.1, rewriting the products with Lemma 3.4.1 and Lemma 3.4.3, setting $x_i = q^{2i-1}$, $u = q^{2t-1}$, one derives

Theorem 3.4.4. *[A q -Nekrasov–Okounkov formula for type \tilde{B}_t] For formal variables T , q and any complex number u , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{DD}'} u^{-d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{2h_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - uq^{h_s}}{1 - u^{-1}q^{h_s}} \\ = \prod_{m,r \geq 1} \frac{1 - u^{-1}q^{2(r-1)}T^m}{1 - uq^{2r}T^m} \frac{(1 - u^{-2}q^{2r}T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2q^{2(r+1)}T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^{2r}T^m)^{r+1 - \lfloor (r+1)/2 \rfloor} (1 - q^{2(r+2)}T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned}$$

Like in the case of Theorem 3.1.3, if we set $u = q^{2t-1}$ and $x_i = q^{2i-1}$ in Theorem 3.3.1, (3.19) coincides with the formula of Theorem 3.4.4. We conclude by remarking that the two sides are Laurent polynomials in u .

3.4.4 Type \tilde{B}_t^\vee

Following the same steps of the proof of Lemma 3.4.2, showing that the induction steps are the same as the ones of Theorem 2.5.10 with $\tau(x) = 1 - q^{2x}$, (2.41) and using the fact that $4t(t-1) - \sum_{i=1}^{2t-1} i$, we derive the following result.

Lemma 3.4.5. *Let t be a positive integer, set $g = 2t$, $\lambda \in \mathcal{DD}_g^1$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t,t}$ -coding. Set $\mu = (v_1 - 2t + 1, \dots, v_t - t) \in \mathcal{P}$. Then*

$$\text{sp}_\mu(q, q^2, \dots, q^t) = (-1)^{|H+|} q^{-td_\lambda} \prod_{s \in \lambda} \frac{1 - q^{h_s - 2t\varepsilon_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + q^{t+h_s/2}}{1 + q^{-t+h_s/2}}.$$

Following the same steps, using Theorem 3.3.2, rewriting the products with Lemma 3.4.1 and Lemma 3.4.5, setting $x_i = q^i$, $u = q^t$, one derives the following q -Nekrasov–Okounkov formula.

Theorem 3.4.6. *[A q -Nekrasov–Okounkov formula for type \tilde{B}_t^\vee] For formal variables T , q and any complex number u , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{DD}'} (-u)^{-d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{h_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + uq^{h_s/2}}{1 + u^{-1}q^{h_s/2}} \\ = \prod_{m \geq 1} (1 + u^{-1}T^m) \prod_{r \geq 1} \frac{(1 + u^{-1}q^r T^m)}{(1 + uq^r T^m)} \frac{(1 - q^{r+1}u^2 T^m)^{r - \lfloor r/2 \rfloor} (1 - q^r u^{-2} T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^r T^m)^{r - \lfloor r/2 \rfloor} (1 - q^{r+1} T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned}$$

Like in the case of Theorem 3.1.3, if we set $u = q^t$ and $x_i = q^i$ in Theorem 3.3.2, (3.21) coincides with the formula of Theorem 3.4.6. We conclude by remarking that the two sides are Laurent polynomials in u .

3.4.5 Type \tilde{C}_t^\vee

The following lemma corresponds to the specialized odd orthogonal Schur functions specialized for $x_i = q^{2i-1}$.

Lemma 3.4.7. *Let t be a positive integer, set $\lambda \in \mathcal{SC}_{(2t)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t,t}$ -coding. Set $\mu = (v_1 - 2t + 1, \dots, v_t - t) \in \mathcal{P}$. Then*

$$\mathrm{so}_\mu(q, q^3, \dots, q^{2t-1}) = (-1)^{|H_{<g,+}| + |H_{<g,+} \cap \Delta|} \prod_{s \in \lambda} \frac{1 - q^{2h_s - 4t\varepsilon_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - q^{2t+h_s}}{1 - q^{-2t+h_s}}. \quad (3.34)$$

The proof of Lemma 3.4.7 follows the exact same path as the one of Lemma 3.4.2 with the difference that in Theorem 2.5.12 with $\tau(x) = 1 - q^{2x}$, the product $\prod_{i=1}^t (1 - q^{2r_i}) / (1 - q^{2i})$ is missing and instead of using (1.7), we use (1.5). As in the proof of Lemma 3.4.2, this product comes from the product over the terms belonging to the main diagonal of $\lambda \in \mathcal{SC}_{(2t)}$. Using $(1 - q^{-i}) / (1 - q^i) = -q^{-i}$, this product induces a term $(-1)^{|H_{<g,+} \cap \Delta|}$ in (3.34). Whereas in all other lemmas, there is a factor q to some power, this factor is missing in (3.34) because of the fact that, at each step of the induction, this power of q yielding the dependency on d_λ for the other types, is equal to $2t(2t-1) - 2 \sum_{i=1}^{2t-1} i = 0$.

Following the same steps, using Theorem 3.3.3 rewriting the products with Lemma 3.4.1 and Lemma 3.4.7, setting $x_i = q^{2i-1}$, $u = q^{2t}$ and a Laurent polynomiality argument, one derives the following Nekrasov–Okounkov type formulas.

Theorem 3.4.8. *[A q -Nekrasov–Okounkov formula for type \tilde{C}_t^\vee] For formal variables T , q and any complex number u , we have:*

$$\begin{aligned} & \sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{2h_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - uq^{h_s}}{1 - u^{-1}q^{h_s}} \\ &= \prod_{m \geq 1} \frac{1}{1 + T^{m/2}} \prod_{r \geq 1} \frac{1 - u^{-1}q^{2r-1}T^{m/2}}{1 - uq^{2r-1}T^{m/2}} \frac{(1 - u^{-2}q^{2(r+1)}T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2q^{2r}T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^{2r}T^m)^{r+1 - \lfloor (r+1)/2 \rfloor} (1 - q^{2(r+1)}T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned}$$

3.4.6 Type $\tilde{B}C_t$

Following the same steps as in the proof of Lemma 3.4.7, one derives the following from Theorem 2.5.14 with $\tau(x) = 1 - q^{2x}$.

Lemma 3.4.9. *Let t be a positive integer, set $\lambda \in \mathcal{DD}_{(2t+1)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t+1,t}$ -coding. Set $\mu = (v_1 - 2t, \dots, v_t - t - 1) \in \mathcal{P}$. Then*

$$\mathrm{so}_\mu(q, q^3, \dots, q^{2t-1}) = (-1)^{|H_{<g,+}| + |H_{<g,+} \cap \Delta|} q^{(2t+1)d_\lambda} \prod_{s \in \lambda} \frac{1 - q^{2h_s - 2(2t+1)\varepsilon_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - q^{2t+1+h_s}}{1 - q^{-2t-1+h_s}}.$$

Using Theorem 3.3.4, rewriting the products with Lemma 3.4.1 and Lemma 3.4.9, setting $x_i = q^{2i-1}$, $u = q^{2t+1}$, one derives

Theorem 3.4.10. *[A q -Nekrasov–Okounkov formula for type $\tilde{B}C_t$] For formal variables T , q and any complex number u , we have:*

$$\begin{aligned} & \sum_{\lambda \in \mathcal{DD}} u^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{2h_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - uq^{h_s}}{1 - u^{-1}q^{h_s}} \\ &= \prod_{m, r \geq 1} \frac{1 - u^{-1}q^{2r}T^m}{1 - uq^{2r}T^m} \frac{1 - u^{-2}q^{4r}T^{2m+1}}{1 - u^2q^{4r}T^{2m+1}} \frac{(1 - u^{-2}q^{2(r+1)}T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2q^{2r}T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^{2r}T^m)^{r+1 - \lfloor (r+1)/2 \rfloor} (1 - q^{2(r+1)}T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned}$$

Like in the case of Theorem 3.1.3, if we set $u = q^{2t+1}$ and $x_i = q^{2i-1}$ in Theorem 3.3.4, (3.24) coincides with the formula of Theorem 3.4.10. We conclude by remarking that the two sides are Laurent polynomials in u .

3.4.7 Type \tilde{D}_t

Similarly one derives from the induction step of Theorem 2.5.16 with $\tau(x) = 1 - q^{2x}$

Lemma 3.4.11. *Let t be a positive integer, set $\lambda \in \mathcal{DD}'_{2t-2}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{2t,t}$ -coding. Set $\mu = (v_1 - 2t + 3, \dots, v_t - t + 2) \in \mathcal{P}$. Then*

$$o^{even}_{\mu}(q, q^3, \dots, q^{2t-1}) = (-1)^{|H+|} q^{-4(t-1)d_{\lambda}} \prod_{s \in \lambda} \frac{1 - q^{2h_s - 2(2t-2)\varepsilon_s}}{1 - q^{2h_s}}.$$

Following the same steps, using Theorem 3.3.5, rewriting the products with Lemma 3.4.1 and Lemma 3.4.11, setting $x_i = q^{2i-1}$, $u = q^{2t-2}$, one derives

Theorem 3.4.12. *[A q -Nekrasov–Okounkov formula for type \tilde{D}_t] For formal variables T, q and any complex number u , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{DD}'} (-u^{-2})^{d_{\lambda}} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{2h_s}}{1 - q^{2h_s}} \\ = \prod_{m, r \geq 1} \frac{(1 - q^{2(r+2)} u^2 T^m)^{r - \lfloor r/2 \rfloor} (1 - q^{2(r-1)} u^{-2} T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^{2(r+1)} T^m)^{r - \lfloor r/2 \rfloor} (1 - q^{2(r+2)} T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned}$$

As for Theorem 3.1.3, if we set $u = q^{2t-2}$ and $x_i = q^{2i-1}$ in Theorem 3.3.5, then (3.25) coincides with the formula of Theorem 3.4.12. We conclude by remarking that the two sides are Laurent polynomials in u .

3.5 Confluences and connections with the Appendix 1 of Macdonald's paper

In this section, we consider all of the specializations for infinite root systems in [Mac72, Appendix 1] as specializations of q -Nekrasov–Okounkov formulas above. These specializations yields some Nekrasov–Okounkov type formulas that are already known and some others which are new. To explicit how Macdonald's specializations are to be transcribed in terms of the set of variables \mathbf{x} , recall from Chapter 2 the following. Let $(e_i)_{1 \leq i \leq t}$ be a basis for \mathbb{R}^t and write $k + \varepsilon_i$ for the affine function $e_j \rightarrow k + \delta_{i,j}$. Then Rosengren–Schlosser set $T = e^{-1}$ and $x_i = T^{-\varepsilon_i}$ in their rewritings of Macdonald identities for all infinite affine types.

As for technical aspects and in order to avoid repetitions, we remind the reader of the following limit, which will be used in all of the specializations introduced in [Mac72, Appendix 1] to get new Nekrasov–Okounkov formulas. For any $(x, y) \in \mathbb{R}^{*2}$, we have:

$$\lim_{q \rightarrow 1} \frac{1 - q^x}{1 - q^y} = \frac{x}{y}.$$

3.5.1 Type \tilde{A}_{t-1}

Both specializations in [Mac72, Appendix 1] correspond to the same Nekrasov–Okounkov formula (1.18).

3.5.2 Type \tilde{B}_t

Let t be an integer greater than 3. In this case, in [Mac72], $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = \varepsilon_t$. In all this section, we set $u = q^{2t-1}$.

(a) The first specialization in [Mac72] corresponds to $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.3.1. By letting $q \rightarrow 1$ in Theorem 3.4.4, using this time the fact that both sides are polynomials in t , one derives

$$\sum_{\lambda \in \mathcal{DD}'} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z-1)\varepsilon_s}{h_s} \right) \prod_{s \in \Delta} \frac{2z-1+h_s}{h_s-2z+1} = (T)_\infty^{2z^2+z}, \quad (3.35)$$

where z is any complex number. With this formulation, the issue is that it is not immediate that the product on the left-hand side of (3.35) is well-defined because of the term $h_s - 2z + 1$ at the denominator. Moreover this formula was stated in a different way in [P 15b, Theorem 3.20]: the sum part was over the set of doubled distinct partitions and the product over hook lengths slightly differs. Recall that the set $\varepsilon\mathcal{H}(\lambda) := \{\varepsilon_s h_s, s \in \lambda\}$ and $\varepsilon\mathcal{H}(\lambda')$ only differ for the boxes shaded in yellow in Figure 1.2b and Figure 1.2c: a horizontal (respectively vertical) strip of length d_λ with sign $\varepsilon_s = 1$ (respectively $\varepsilon_s = -1$) for $\lambda \in \mathcal{DD}'$ (respectively λ'). Therefore for $\lambda \in \mathcal{DD}'$, the left-hand side of (3.35) becomes

$$\prod_{s \in \lambda \setminus \Delta} \left(1 - \frac{(2z-1)\varepsilon_s}{h_s} \right) \prod_{s \in \Delta} (h_s + 2z - 1) = (-1)^{d_{\lambda'}} T^{|\lambda|/2} \prod_{s \in \lambda'} \left(1 + \frac{(2z-1)\varepsilon_s}{h_s} \right).$$

Hence one can write (3.35) as [P 15b, Theorem 3.20]

$$\sum_{\lambda \in \mathcal{DD}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 + \frac{(2z-1)\varepsilon_s}{h_s} \right) = (T; T)_\infty^{2z^2+z}, \quad (3.36)$$

where z is any complex number.

(b) The second specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t-1$ and $e^{a_t} \rightarrow -1$. This specialization is equivalent to taking $x_i = -1$ in Theorem 3.3.1. It is equivalent to letting $q \rightarrow -1$ in Theorem 3.4.4. This implies that $u = q^{2t-1} \rightarrow -1$. Moreover recall that if $\lambda \in \mathcal{DD}'$, all the hook lengths of the boxes of Δ are even. Hence for any complex number z , we get

$$\sum_{\lambda \in \mathcal{DD}'} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z-1)\varepsilon_s}{h_s} \right) = \left((T; T)_\infty^{2z-3} (T^2; T^2)_\infty^2 \right)^z, \quad (3.37)$$

which is new.

(c) The third specialization is $e^{a_i} \rightarrow 1$ for $0 \leq i \leq t-1$. This specialization is equivalent to taking $x_i \rightarrow 1/\sqrt{T}$ in Theorem 3.3.1. In this case, one cannot derive a formula with the q -analogues written in this manuscript. Nevertheless there is a q -Nekrasov–Okounkov formula that could be obtained using the same techniques as the ones developed in this manuscript. To do so, one should first replace $x_i \rightarrow x_i T^{-1/2}$ in the Macdonald identity for type \tilde{B}_t from Proposition 1.4.1. In this case, the quadratic form to the exponent of T changes. The subset of partitions linked to the latter is another subset of partitions. Namely it is the subset \mathcal{SC}_{2t-1}^1 of self-conjugate partitions whose $(2t-1)$ -quotient is empty except for the self-conjugate element of the quotient which is a square. Analogues of Theorem 2.5.12 and Lemma 3.2.4 can be obtained using the same methods but no details are given here. The Nekrasov–Okounkov formula corresponding to this specialization is therefore the following:

$$\sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 + \frac{(2z-1)\varepsilon_s}{h_s} \right) = \left((T^{1/2}; T^{1/2})_\infty^2 (T; T)_\infty^{2z+3} \right)^z, \quad (3.38)$$

where z is a complex number. Moreover (3.38) can actually be obtained from a Nekrasov–Okounkov formula (3.43) for type \tilde{C}_t^\vee , as already noted in [P 15b, Theorem 3.33]. The proof of (3.43) is detailed in the subsection dedicated to the type \tilde{C}_t^\vee : it suffices to substitute $T \rightarrow T^{1/2}$ and $z \rightarrow -z + 1/2$ in (3.43) to derive (3.38).

3.5.3 Type \tilde{B}_t^\vee

Let t be an integer greater than 3. In this case, in [Mac72], $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = 2\varepsilon_t$. In all of this section, we set $u = q^t$.

(a) The first specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.3.2. By letting $q \rightarrow 1$ in Theorem 3.4.6, using this time the fact that both sides are polynomials in t , one derives

$$\sum_{\lambda \in \mathcal{DD}'} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{2z\varepsilon_s}{h_s} \right) = \left((T; T)_\infty^{z-1} (T^2; T^2)_\infty \right)^{2z+1}, \quad (3.39)$$

where z is any complex number.

(b) The second specialization is $e^{a_i} \rightarrow 1$ for $0 \leq i \leq t-1$. Once again, this specialization is equivalent to taking $x_i \rightarrow 1/\sqrt{T}$ in Theorem 3.3.2 and one cannot derive a formula with the q -analogues written in this manuscript. Once again a q analogue could be obtained using the same techniques but once more one can derive a Nekrasov–Okounkov type formula here by shifting $z \rightarrow -z + 1/2$ in (3.37) and one derives the new Nekrasov–Okounkov type formula:

$$\sum_{\lambda \in \mathcal{DD}'} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 + \frac{2z\varepsilon_s}{h_s} \right) = \left((T; T)_\infty^{z+1} (T^2; T^2)_\infty^{-1} \right)^{2z-1}, \quad (3.40)$$

where z is a complex number.

3.5.4 Type \tilde{C}_t

Let t be an integer greater than 2. In this case, in [Mac72], $a_0 = 1 - 2\varepsilon_1$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = 2\varepsilon_t$. We set $u = q^{2t+2}$.

The specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.1.2. By letting $q \rightarrow 1$ in Theorem 3.4.4, using this time the fact both sides are polynomials in t , one derives as [Pé15b, Theorem 3.1]:

$$\sum_{\lambda \in \mathcal{DD}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z+2)\varepsilon_s}{h_s} \right) = (T; T)_\infty^{2z^2+z}, \quad (3.41)$$

where z is any complex number.

3.5.5 Type \tilde{C}_t^\vee

Let t be an integer greater than 2. In this case, in [Mac72], $a_0 = 1/2 - \varepsilon_1$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = \varepsilon_t$. In this section, we set $u = q^{2t}$.

(a) The first specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.3.3. Moreover in [Mac72, Appendix 1], T is replaced with T^2 . By letting $q \rightarrow 1$ in Theorem 3.4.8, using this time the fact that both sides are polynomials in t , one derives this new Nekrasov–Okounkov type formula

$$\sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|} \prod_{s \in \lambda \setminus \Delta} \left(1 - \frac{2z\varepsilon_s}{h_s} \right) \prod_{s \in \Delta} \left(1 + \frac{2z}{h_s} \right) = \left((T; T)_\infty (T^2; T^2)_\infty^{z-1} \right)^{2z+1}, \quad (3.42)$$

where z is any complex number.

(b) The second specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t-1$ and $e^{a_t} \rightarrow -1$. This is equivalent to taking $x_i = -1$ in Theorem 3.3.3. Moreover in [Mac72, Appendix 1], T is replaced with T^2 . It is equivalent to letting $q \rightarrow -1$ in Theorem 3.4.8. This implies that $u \rightarrow 1$. Moreover if s is

a box on the main diagonal Δ of a self-conjugate partition, then h_s is odd. Therefore we derive the following result already proved in [P 15b, Theorem 3.24]

$$\sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|} \prod_{s \in \lambda} \left(1 - \frac{2z\varepsilon_s}{h_s} \right) = \left((T; T)_\infty^{-1} (T^2; T^2)_\infty^{z+1} \right)^{2z-1}, \quad (3.43)$$

where z is any complex number.

3.5.6 Type \tilde{BC}_t

Let t be an integer greater than 1. In this case, in [Mac72], $a_0 = 1 - 2\varepsilon_1$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = \varepsilon_t$. In this section, we set $u = q^{2t+1}$. **(a)** The first specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.3.4. By letting $q \rightarrow 1$ in Theorem 3.4.10, using this time the fact that both sides are polynomials in t , one derives this new Nekrasov–Okounkov type formula

$$\sum_{\lambda \in \mathcal{DD}} T^{|\lambda|/2} \prod_{s \in \lambda \setminus \Delta} \left(1 - \frac{(2z+1)\varepsilon_s}{h_s} \right) \prod_{s \in \Delta} \left(1 + \frac{2z+1}{h_s} \right) = \left((T; T)_\infty^{2z+3} (T^2; T^2)_\infty^{-2} \right)^z, \quad (3.44)$$

where z is any complex number.

(b) The second specialization is $e^{a_i} \rightarrow 1$ for $0 \leq i \leq t-1$. This specialization is equivalent to taking $x_i \rightarrow 1/\sqrt{T}$ in Theorem 3.3.4. As for specializations **(c)** of type \tilde{B}_t and **(b)** of type \tilde{B}_t^\vee , one can derive a q -Nekrasov–Okounkov formula by substituting x_i by $x_i T^{-1/2}$ in the Macdonald identity for type \tilde{BC}_t of Proposition 1.4.1. The quadratic form to the exponent of T changes and the subset of partitions naturally arising is $\mathcal{SC}_{(2t+1)}$. Using the same techniques, one has analogues of Theorem 2.5.12 and Lemma 3.2.4 and one derives the following new Nekrasov–Okounkov type formula:

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z+1)\varepsilon_s}{h_s} \right) = \left((T^{1/2}; T^{1/2})_\infty^2 (T; T)_\infty^{2z-3} (T^2; T^2)_\infty^2 \right)^z, \quad (3.45)$$

where z is any complex number.

(c) The third specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t-1$ and $e^{a_t} \rightarrow -1$. This is equivalent to $q \rightarrow -1$ and $u \rightarrow -1$ in Theorem 3.4.10, we derive the following new Nekrasov–Okounkov type formula:

$$\sum_{\lambda \in \mathcal{DD}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z+1)\varepsilon_s}{h_s} \right) = (T; T)_\infty^{2z^2-z}, \quad (3.46)$$

where z is any complex number.

(d) The fourth specialization is $e^{a_0/2} \rightarrow -1$ and $e^{a_i} \rightarrow 1$ for $0 \leq i \leq t-1$. This specialization is equivalent to taking $x_i \rightarrow -1/\sqrt{T}$ in Theorem 3.3.4. One could obtain this specialization by letting q go to -1 in the q -Nekrasov–Okounkov derived from the specialization **(b)** mentioned above. Nevertheless this specialization can also be obtained as remarked by P tr olle [P 15b, Theorem 3.33] by the shift $z \rightarrow z + 1/2$ and $T \rightarrow T^{1/2}$ in (3.43) and one has the following Nekrasov–Okounkov type formula:

$$\sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z+1)\varepsilon_s}{h_s} \right) = \left((T^{1/2}; T^{1/2})_\infty^{-2} (T; T)_\infty^{2z+3} \right)^z, \quad (3.47)$$

where z is any complex number.

3.5.7 Type \tilde{D}_t

Let t be an integer greater than 4. In this case, in [Mac72], $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, and $a_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq t-1$ and $a_t = \varepsilon_{t-1} + \varepsilon_t$. We set $u = q^{2t-2}$.

The specialization is $e^{a_i} \rightarrow 1$ for $1 \leq i \leq t$. This is equivalent to taking $x_i = 1$ in Theorem 3.3.5. By letting $q \rightarrow 1$ in Theorem 3.4.12, using this time the fact that both sides are polynomials in t , we have the new Nekrasov–Okounkov type formula

$$\sum_{\lambda \in \mathcal{DD}'} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \left(1 - \frac{(2z-2)\varepsilon_s}{h_s} \right) = (T; T)_\infty^{2z^2-z}, \quad (3.48)$$

where z is any complex number.

4.1 Introduction

In the last two chapters, the main focus was to provide products of hook lengths over some subsets of t -core partitions or with a t -quotient very constrained. Motivated by the work of Han–Ji [HJ11], we explore here the properties of the Littlewood decomposition applied to self-conjugate partitions on the subset $\mathcal{H}_t(\lambda)$ of hook lengths of a partition λ that are divisible by t . The work of Han–Ji can be thought of as a way of systematically getting some generalizations of known hook length formulas, including more variables, via the Littlewood decomposition. One of the new variables keeps track of a statistic not introduced so far, the BG-rank.

We give in Figure 4.9 some illustrations of the statistics already introduced in Section 1.1 plus a new statistic, the aforementioned BG-rank. In Figure 4.9a, the hook lengths of all boxes for the partition $\lambda = (4, 3, 3, 2)$ have been written in their corresponding boxes and the boxes associated with $\mathcal{H}_3(\lambda)$ shaded in red. In the example, we have $\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 7, 6, 4, 1\}$ and $\mathcal{H}_3(\lambda) = \{3, 6\}$. Recall from the work of Berkovich–Garvan [BG06] that the BG-rank of the partition λ , denoted by $\text{BG}(\lambda)$, is defined as follows. First fill each box in the Ferrers diagram of λ with alternating ± 1 's along rows and columns beginning with a “+1” in the $(1, 1)$ position (see Figure 4.9b). Then sum their values over all the boxes. Note that all boxes belonging to the diagonal of a Ferrers diagram are filled with a “+1”. For instance, the BG-rank of $\lambda = (4, 3, 3, 2)$ is 0.



Figure 4.9: Ferrers diagram and some partition statistics

In [HJ11], Han–Ji underline some important properties of the Littlewood decomposition, which enable them to prove the following multiplication-addition theorem.

Theorem 4.1.1. [HJ11, Theorem 1.1] *Let t be a positive integer and set ρ_1, ρ_2 two functions defined on \mathbb{N} . Let f_t and g_t be the following formal power series:*

$$f_t(T) := \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th),$$

$$g_t(T) := \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th) \sum_{h \in \mathcal{H}(\lambda)} \rho_2(th).$$

Then we have

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) = t \frac{(T^t; T^t)_\infty^t}{(T; T)_\infty} (f_t(xT^t))^{t-1} g_t(xT^t).$$

Note that Walsh–Warnaar in [WW20] also prove multiplication theorems giving rise to hook length formulas. They also prove interesting extensions regarding leg lengths.

Theorem 4.1.1 gives modular analogues of many classical formulas. For instance, setting $\rho_1(h) = 1 - z/h^2$ for any complex number z and $\rho_2(h) = 1$, it provides the modular analogue of the Nekrasov–Okounkov formula (1.18) originally proved in [Han10, Theorem 1.2]:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{z}{h^2}\right) = \frac{(T^t; T^t)_\infty^t}{(xT^t; xT^t)_\infty^{t-z/t} (T; T)_\infty}. \quad (4.1)$$

In the present work, we extend Theorem 4.1.1 to an important subset of \mathcal{P} , namely the self-conjugate partitions, and derive several applications regarding these.

Recall that we denote the set of self-conjugate partitions by \mathcal{SC} . As mentioned in Section 1.1, this subset of partitions has been of particular interest within the works of Pétréolle [Pé15b, Pé15a] where two Nekrasov–Okounkov type formulas for $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}^\vee$ are derived. See also the work of Han–Xiong [HX19] or Cho–Huh–Sohn [CHS20]. The already mentioned Littlewood decomposition, when restricted to \mathcal{SC} , also has interesting properties and can be stated as follows (see for instance [GKS90, Pé15b]):

$$\begin{aligned} \lambda \in \mathcal{SC} &\mapsto (\omega, \tilde{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2} && \text{if } t \text{ even,} \\ \lambda \in \mathcal{SC} &\mapsto (\omega, \tilde{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC} && \text{if } t \text{ odd.} \end{aligned}$$

Indeed, as detailed in Section 2.3, in the particular case of self-conjugate partitions, elements of the t -quotient $\underline{\nu} \in \mathcal{P}^t$ can be gathered two by two through conjugation (except $\nu^{((t-1)/2)}$ when t is odd), therefore yielding the above vectors $\tilde{\nu}$ and $(\tilde{\nu}, \mu)$.

As can be seen above, to provide an analogue of Theorem 4.1.1 for self-conjugate partitions, the t even case is simpler to handle, therefore we first restrict ourselves to this setting. Nevertheless, it yields a slightly more general result than Theorem 4.1.1, as the BG-rank can be incorporated.

Theorem 4.1.2. *Let t be a positive even integer and set ρ_1, ρ_2 two functions defined on \mathbb{N} . Let f_t and g_t be the formal power series defined as:*

$$\begin{aligned} f_t(T) &:= \sum_{\nu \in \mathcal{P}} T^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2, \\ g_t(T) &:= \sum_{\nu \in \mathcal{P}} T^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2 \sum_{h \in \mathcal{H}(\nu)} \rho_2(th). \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) \\ = t (f_t(x^2 T^{2t}))^{t/2-1} g_t(x^2 T^{2t}) (T^{2t}; T^{2t})_\infty^{t/2} (-bT; T^4)_\infty (-T^3/b; T^4)_\infty. \end{aligned}$$

Remark 4.1.3. *Note that the functions f_t and g_t in Theorem 4.1.2 are close to the ones in Theorem 4.1.1. The explanation is that when t is even, there is no additional self-conjugate partition μ in the Littlewood decomposition.*

We will derive several consequences of this result, including a new trivariate generating function for \mathcal{SC} , new hook length formulas, new modular versions of the Han–Carde–Loubert–Potechin–Sanborn, the Nekrasov–Okounkov, the Bessenrodt–Bacher–Manivel, the Okada–Panova, and the Stanley–Panova formulas. Among them, we highlight here the self-conjugate version of (4.1):

Corollary 4.1.4. *For any complex number z and t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{z}{h^2}\right)^{1/2} \\ = (x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(z/t-t)/2} (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}. \end{aligned}$$

As some combinatorial signs naturally appear in the work of Pétréolle regarding Nekrasov–Okounkov type formulas for self-conjugate partitions, we will also prove a signed refinement of Theorem 4.1.2 (see Theorem 4.4.2 in Section 4.4, which actually generalizes Theorem 4.1.2).

It is also possible to prove a result similar to Theorem 4.1.2 when t is odd; nevertheless more difficulties arise due to the additional $\mu \in \mathcal{SC}$ appearing in the Littlewood decomposition. However, as will be seen later, the subset of \mathcal{SC} for which μ is empty, can be handled almost similarly as for Theorem 4.1.2 (see Theorem 4.5.2 in Section 4.5). The interesting thing here is that this subset of \mathcal{SC} actually corresponds to partitions called BG_t in [Ber21], which are algebraically involved in representation theory of the symmetric group over a field of characteristic t when t is an odd prime number.

This chapter is organized as follows. Section 4.2 is devoted to the proof of Theorem 4.1.2, together with some useful special cases. Many interesting modular self-conjugate analogues of the above mentioned classical formulas are then listed and proved in Section 4.3. In Section 4.4, our signed generalization of Theorem 4.1.2 is proved, and finally in Section 4.5 we study the odd case.

The set $D(\lambda) = \{h_s \mid s \in \Delta\}$ is called the *set of main diagonal hook lengths of λ* . For short, we will denote $h_{(i,i)}$ by δ_i . It is clear that if $\lambda \in \mathcal{SC}$, then $D(\lambda)$ determines λ , and elements of $D(\lambda)$ are all distinct and odd. Hence, as observed in [CHS20], for a self-conjugate partition λ , the set $D(\lambda)$ can be divided into the following two disjoint subsets:

$$\begin{aligned} D_1(\lambda) &:= \{\delta_i \in D(\lambda) : \delta_i \equiv 1 \pmod{4}\}, \\ D_2(\lambda) &:= \{\delta_i \in D(\lambda) : \delta_i \equiv 3 \pmod{4}\}. \end{aligned}$$

We have the following result which is the analogue to Proposition 2.3.3.

Lemma 4.1.5. *For a self-conjugate partition λ , set $r := |D_1(\lambda)|$ and $s := |D_2(\lambda)|$. Then*

$$\text{BG}(\lambda) = r - s.$$

Proof. Set $a_1 > a_2 > \dots > a_r \geq 0$ and $b_1 > b_2 > \dots > b_s \geq 0$ integers such that:

$$\begin{aligned} D_1(\lambda) &= \{4a_1 + 1, \dots, 4a_r + 1\}, \\ D_2(\lambda) &= \{4b_1 + 3, \dots, 4b_s + 3\}. \end{aligned}$$

Let us consider a hook in the main diagonal of λ whose length is $4a + 1$ for a nonnegative integer a . Then its leg and arm are both of length $2a$. As the BG-rank alternates in sign, we have $\text{BG}(4a + 1) = 1$. In the same way, we can observe that $\text{BG}(4b + 3) = -1$ for any main diagonal hook length $4b + 3 \in D_2(\lambda)$. Hence

$$\text{BG}(\lambda) = \sum_{i=1}^r \text{BG}(4a_i + 1) + \sum_{j=1}^s \text{BG}(4b_j + 3) = r - s.$$

□

Remark 4.1.6. Note that as its diagonal is filled with “+1”, we can consider λ hook by hook. In the following example are depicted two hooks of length congruent to 1 (mod 4) and 3 (mod 4) respectively.

+	-	+	-
-			
+			
-			

(a) A hook of length $7 = 4 \times 1 + 3$.

+	-	+	-	+
-				
+				
-				
+				

(b) A hook of length $9 = 4 \times 2 + 1$.

In the case $t = 2$, we can combine Lemma 4.1.5 and Proposition 2.3.3 (P4) to derive the following additional result.

Proposition 4.1.7. The Littlewood decomposition Φ_2 has the further property:

$$(SC5) \quad \text{BG}(\lambda) = r - s = \begin{cases} \frac{\ell(\omega)+1}{2} & \text{if } \text{BG}(\lambda) > 0, \\ -\frac{\ell(\omega)}{2} & \text{if } \text{BG}(\lambda) \leq 0. \end{cases}$$

4.2 Multiplication-addition theorems for self-conjugate partitions

In this section, we prove Theorem 4.1.2 stated in the introduction and we exhibit some interesting special cases.

4.2.1 A preliminary result on BG-rank and $\mathcal{SC}_{(t)}$ -core partitions

In order to prove Theorem 4.1.2, we will need the following result, which can be of independent interest.

Proposition 4.2.1. For any positive even integer t , the generating series of $\mathcal{SC}_{(t)}$ with prescribed BG-rank is:

$$\sum_{\omega \in \mathcal{SC}_{(t)}} T^{|\omega|} b^{\text{BG}(\omega)} = (T^{2t}, T^{2t})_{\infty}^{t/2} (-bq; T^4)_{\infty} (-T^3/b; T^4)_{\infty}.$$

Proof. For an integer k , let $c_{t/2}(k)$ be the number of $t/2$ -core partitions of k . Following [CHS20], define for a nonnegative integer m :

$$\mathcal{SC}^{(m)}(n) := \{ \lambda \in \mathcal{SC}(n) : |D_1(\lambda)| - |D_3(\lambda)| = (-1)^{m+1} \lceil m/2 \rceil \}.$$

Setting $p = 1$ in [CHS20, proposition 4.7], we get that for any integer $m \geq 0$, the number of self-conjugate t -core partitions ω such that $|D_1(\omega)| - |D_3(\omega)| = (-1)^{m+1} \lceil m/2 \rceil$ is

$$sc_{(t)}^{(m)}(n) = \begin{cases} c_{t/2}(k) & \text{if } n = 4k + \frac{m(m+1)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, the authors define a bijection $\phi^{(m)}$ in [CHS20, Corollary 4.6] between $\omega \in \mathcal{SC}_{(t)}^{(m)}$ and $\kappa \in \mathcal{P}_{(t/2)}$ with $|\omega| = 4|\kappa| + m(m+1)/2$ and κ independent of m .

Recall from Lemma 4.1.5 that $\text{BG}(\lambda) = r - s = |D_1(\lambda)| - |D_3(\lambda)|$. Therefore

$$m = \begin{cases} 2 \text{BG}(\lambda) - 1 & \text{if } \text{BG}(\lambda) > 0, \\ -2 \text{BG}(\lambda) & \text{if } \text{BG}(\lambda) \leq 0. \end{cases}$$

Hence the bijection $\phi^{(m)}$ maps a t -core self-conjugate partition ω with BG-rank j to a $t/2$ -core partition independent of j . Then property (SC5) from Proposition 4.1.7 implies that $|\omega| = j(2j - 1) + 4|\kappa|$ with κ independent of j . Therefore we deduce

$$\sum_{\omega \in \mathcal{SC}(t)} T^{|\omega|} b^{\text{BG}(\omega)} = \sum_{j=-\infty}^{\infty} b^j T^{j(2j-1)} \times \sum_{\kappa \in \mathcal{P}(t/2)} T^{4|\kappa|}. \quad (4.2)$$

Now we compute the sum over j . Recall that the Jacobi triple product [HS99] can be stated as

$$\sum_{j=-\infty}^{+\infty} (-1)^j z^j T^{j(j-1)/2} = (z; T)_{\infty} (T/z; T)_{\infty} (T; T)_{\infty}.$$

Therefore, setting $z = -bT$ and then replacing T by T^4 in the above identity, yields

$$\sum_{j=-\infty}^{+\infty} b^j T^{j(2j-1)} = (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} (T^4; T^4)_{\infty}. \quad (4.3)$$

Finally, to complete the proof of Theorem 4.1.2, it remains to compute the generating function of $t/2$ -core partitions which is well-known (see [GKS90, Han10]). However we shortly recall its computation. By direct application of the Littlewood decomposition, using (SC3) and the generating series (1.19) for \mathcal{P} where T is replaced by $T^{t/2}$, we have for $\omega \in \mathcal{P}(t/2)$:

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \text{core}_{t/2}(\lambda) = \omega}} T^{|\lambda|} = T^{|\omega|} \sum_{i=0}^{t/2-1} \sum_{\nu^{(i)} \in \mathcal{P}} T^{t|\nu^{(i)}|/2} = \frac{T^{|\omega|}}{(T^{t/2}; T^{t/2})_{\infty}^{t/2}}.$$

As by (1.19)

$$\frac{1}{(T; T)_{\infty}} = \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = \sum_{\omega \in \mathcal{P}(t/2)} \sum_{\substack{\lambda \in \mathcal{P} \\ \text{core}_{t/2}(\lambda) = \omega}} T^{|\lambda|},$$

we derive

$$\sum_{\omega \in \mathcal{P}(t/2)} T^{|\omega|} = \frac{(T^{t/2}; T^{t/2})_{\infty}^{t/2}}{(T; T)_{\infty}}. \quad (4.4)$$

Replacing T by T^4 in (4.4), and using (4.2) and (4.3), this proves Proposition 4.2.1. \square

4.2.2 Proof of Theorem 4.1.2

Let t be a fixed positive even integer. Let ρ_1 and ρ_2 be two functions defined on \mathbb{N} . First we will compute the term

$$\sum_{\substack{\lambda \in \mathcal{SC} \\ \text{core}_t(\lambda) = \omega}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h), \quad (4.5)$$

where $\omega \in \mathcal{SC}(t)$ is fixed. Let us remark that for $\lambda \in \mathcal{SC}$ and $\omega = \text{core}_t(\lambda)$, one has $\text{BG}(\lambda) = \text{BG}(\omega)$. Indeed ω is obtained by removing from λ ribbons of even lengths t and these have BG-rank 0. Hence (4.5) can be rewritten as follows

$$b^{\text{BG}(\omega)} T^{|\omega|} \sum_{\substack{\lambda \in \mathcal{SC} \\ \text{core}_t(\lambda) = \omega}} T^{|\lambda| - |\omega|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h).$$

Hence using properties (SC3) and (SC4) from Proposition 2.3.4, this is equal to

$$b^{\text{BG}(\omega)} T^{|\omega|} \sum_{\underline{\nu} \in \mathcal{P}^t} T^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{h \in \mathcal{H}(\underline{\nu})} \rho_1(th) \sum_{h \in \mathcal{H}(\underline{\nu})} \rho_2(th), \quad (4.6)$$

where $|\underline{\nu}| := \sum_{i=0}^{t-1} |\nu^{(i)}|$.

The product part $T^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{h \in \mathcal{H}(\underline{\nu})} \rho_1(th)$ inside the sum over $\underline{\nu}$ can be rewritten as follows

$$\prod_{i=0}^{t/2-1} T^{t(|\nu^{(i)}| + |\nu^{(t-1-i)}|)} x^{|\nu^{(i)}| + |\nu^{(t-1-i)}|} \prod_{h \in \mathcal{H}(\nu^{(i)})} \rho_1(th) \prod_{h \in \mathcal{H}(\nu^{(t-1-i)})} \rho_1(th).$$

When t is even, as mentioned in the introduction, Proposition 2.3.4 (SC2) implies that the t -quotient $\underline{\nu}$ is uniquely determined by its first $t/2$ components, which are any partitions. It also implies that $|\nu^{(i)}| = |\nu^{(t-1-i)}|$ and $\mathcal{H}(\nu^{(i)}) = \mathcal{H}(\nu^{(t-1-i)})$ for any $i \in \{0, \dots, t/2 - 1\}$ because sizes and hook lengths multisets of partitions are invariant by conjugation. Therefore

$$T^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{h \in \mathcal{H}(\underline{\nu})} \rho_1(th) = \prod_{i=0}^{t/2-1} T^{2t|\nu^{(i)}|} x^{2|\nu^{(i)}|} \prod_{h \in \mathcal{H}(\nu^{(i)})} \rho_1^2(th).$$

Moreover by application of Proposition 2.3.4 (SC2) and (SC4), the sum part $\sum_{h \in \mathcal{H}(\underline{\nu})} \rho_2(th)$ in (4.6) is

$$\sum_{i=0}^{t/2-1} \left(\sum_{h \in \mathcal{H}(\nu^{(i)})} \rho_2(th) + \sum_{h \in \mathcal{H}(\nu^{(t-1-i)})} \rho_2(th) \right) = 2 \sum_{i=0}^{t/2-1} \sum_{h \in \mathcal{H}(\nu^{(i)})} \rho_2(th).$$

Therefore (4.6), and thus (4.5), become

$$2b^{\text{BG}(\omega)} T^{|\omega|} \sum_{i=0}^{t/2-1} \left(\sum_{\nu^{(i)} \in \mathcal{P}} T^{2t|\nu^{(i)}|} x^{2|\nu^{(i)}|} \prod_{h \in \mathcal{H}(\nu^{(i)})} \rho_1^2(th) \sum_{h \in \mathcal{H}(\nu^{(i)})} \rho_2(th) \right) \times \left(\sum_{\nu \in \mathcal{P}} T^{2t|\nu|} x^{2|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1^2(th) \right)^{t/2-1}.$$

Hence we get:

$$\sum_{\substack{\lambda \in \mathcal{SC} \\ \text{core}_t(\lambda) = \omega}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) = tb^{\text{BG}(\omega)} T^{|\omega|} (f_t(x^2 T^{2t}))^{t/2-1} g_t(x^2 T^{2t}).$$

To finish the proof, it remains to sum both sides over all $\mathcal{SC}_{(t)}$ -core partitions ω and apply Proposition 4.2.1.

4.2.3 Special cases

Here we list useful special cases of Theorem 4.1.2. First, by setting $\rho_2 = 1$, we have the following result.

Corollary 4.2.2. *Set ρ_1 a function defined on \mathbb{N} , and let t be a positive even integer and f_t be defined as in Theorem 4.1.2. Then we have*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \\ = (f_t(x^2 T^{2t}))^{t/2} (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}. \end{aligned}$$

Proof. Take $\rho_2 = 1$ in Theorem 4.1.2. This yields $g_t = \sum_{\nu \in \mathcal{P}} |\nu| T^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2$. Therefore we get

$$g_t(x^2 T^{2t}) = \frac{x}{2} \frac{d}{dx} f_t(x^2 T^{2t}).$$

The right-hand side of Theorem 4.1.2 is then

$$\frac{t}{2} (f_t(x^2 T^{2t}))^{t/2-1} x \frac{d}{dx} f_t(x^2 T^{2t}) \times (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty},$$

while its left-hand side becomes

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} |\mathcal{H}_t(\lambda)| x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h).$$

We complete the proof by dividing both sides by x and integration with respect to x . \square

Similarly, as when we take $\rho_1 = 1$ in Theorem 4.1.2, then f_t becomes the generating function (1.19) of \mathcal{P} (with T replaced by $x^2 T^{2t}$), we immediately derive the following special case.

Corollary 4.2.3. *Set ρ_2 a function defined on \mathbb{N} and let t be a positive even integer and g_t be defined as in Theorem 4.1.2. Then*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) = t g_t(x^2 T^{2t}) \\ \times \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(x^2 T^{2t}; x^2 T^{2t})_{\infty}^{t/2-1}} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}. \end{aligned}$$

4.3 Applications

In [HJ11], Han–Ji derive from Theorem 4.1.1 modular versions of many classical identities for partitions. In this section, we give self-conjugate modular analogues of most of them as consequences of Theorem 4.1.2 and its corollaries. The specificity for \mathcal{SC} is that we have to consider t even in all this section. Nevertheless, our results are slightly more general than in [HJ11], as the BG-rank can be included in our formulas, although this was only possible in [HJ11] for $t = 2$.

4.3.1 A generating function

Setting $\rho_1(h) = 1$ in Corollary 4.2.2, we derive the following trivariate generating function for \mathcal{SC} :

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} = \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(x^2 T^{2t}; x^2 T^{2t})_{\infty}^{t/2}} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}.$$

If we take $x = 1$, we obtain the generating function with respect to the BG-rank for \mathcal{SC} :

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} b^{\text{BG}(\lambda)} = (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}.$$

4.3.2 Two classical hook length formulas

Recall the following hook length formulas:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{h^2} = \exp(T), \quad (4.7)$$

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{h} = \exp\left(T + \frac{T^2}{2}\right). \quad (4.8)$$

These formulas are direct consequences of the Robinson–Schensted–Knuth correspondence (see for example [Sta99] p.324). Again, we can use Corollary 4.2.2 to find self-conjugate modular versions for them. The difference between the case of \mathcal{P} treated in [HJ11] and the case of self-conjugate partitions is that now ρ_1 is replaced by its square leading to applications with $1/h$ and $1/\sqrt{h}$ instead of $1/h^2$ and $1/h$.

The modular \mathcal{SC} version of (4.7) is as follows.

Corollary 4.3.1. *For t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \\ = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp\left(\frac{x^2 T^{2t}}{2t}\right). \end{aligned}$$

Proof. Taking $\rho_1(h) = 1/h$ in Corollary 4.2.2, we have by using (4.7):

$$f_t(T) = \exp\left(\frac{T}{t^2}\right).$$

□

Setting $x = 1$ and comparing coefficients b^0 on both sides of Corollary 4.3.1, we get:

$$\sum_{\substack{\lambda \in \mathcal{SC} \\ \text{BG}(\lambda)=0}} T^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(T^4; T^4)_{\infty}} \exp\left(\frac{T^{2t}}{2t}\right).$$

Note that in [HJ11], a similar formula was given for \mathcal{P} only when $t = 2$. By identification of the coefficients of $b^j x^{2n} T^{2tn+j(2j-1)}$ on both sides of Corollary 4.3.1 and using (4.3) on the right-hand side, we have for all integers j and all nonnegative integers n :

$$\sum_{\substack{\lambda \in \mathcal{SC}, \lambda \vdash 2tn+j(2j-1) \\ |\mathcal{H}_t(\lambda)|=2n \\ \text{BG}(\lambda)=j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{1}{n! 2^n t^n}.$$

The case $j = 0$ is the same result as Pétréolle ([Pé15b] Corollary 4.24):

$$\sum_{\substack{\lambda \in \mathcal{SC}, \lambda \vdash 2tn \\ |\mathcal{H}_t(\lambda)|=2n}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{1}{n! 2^n t^n},$$

as the conditions on λ in the summation necessarily imply by the Littlewood decomposition that $\omega = \emptyset$, which is equivalent to $\text{BG}(\lambda) = 0$.

Now we prove the following modular \mathcal{SC} version of (4.8).

Corollary 4.3.2. *For t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^{1/2}} \\ = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp\left(\frac{x^2 T^{2t}}{2} + \frac{x^4 T^{4t}}{4t}\right). \end{aligned}$$

Proof. Take $\rho_1(h) = 1/h^{1/2}$ in Corollary 4.2.2. Then by direct application of (4.8), we have:

$$f_t(T) = \exp\left(\frac{T}{t} + \frac{T^2}{2t^2}\right).$$

□

Setting $x = 1$ and comparing coefficients b^0 on both sides of Corollary 4.3.2, we derive:

$$\sum_{\substack{\lambda \in \mathcal{SC} \\ \text{BG}(\lambda)=0}} T^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^{1/2}} = \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(T^4; T^4)_{\infty}} \exp\left(\frac{T^{2t}}{2} + \frac{T^{4t}}{4t}\right).$$

On the other hand, by comparing coefficients of $T^{2tn+j(2j-1)} x^{2n} b^j$ on both sides of Corollary 4.3.2 and using (4.3) on the right-hand side, we have the following result, which is true for all integers j and all positive integers n :

$$\sum_{\substack{\lambda \in \mathcal{SC} \\ \lambda \vdash 2tn+j(2j-1) \\ |\mathcal{H}_t(\lambda)|=2n \\ \text{BG}(\lambda)=j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^{1/2}} = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k!(n-2k)! t^k}. \quad (4.9)$$

As underlined by one of the reviewers of my article [Wah22a], one can remark that the right-hand side of (4.9) can be actually rewritten as a Hermite polynomial. Indeed, for n a nonnegative integer, the *Hermite polynomial* H_n can be defined as follows (see for instance [Ism05, Sze75]):

$$\forall z \in \mathbb{C}, \quad H_n(z) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!} (-1)^k (2z)^{n-2k}.$$

Therefore (4.9) yields the following unusual expansion for this polynomial:

$$H_n\left(\frac{i\sqrt{t}}{2}\right) = (2i)^n t^{n/2} n! \sum_{\substack{\lambda \in \mathcal{SC} \\ \lambda \vdash 2tn+j(2j-1) \\ |\mathcal{H}_t(\lambda)|=2n \\ \text{BG}(\lambda)=j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^{1/2}}. \quad (4.10)$$

4.3.3 The Han–Carde–Loubert–Potechin–Sanborn formula

The following formula is an interpolation between (4.7) and (4.8) conjectured by Han in [Han09] and proved by Carde–Loubert–Potechin–Sanborn in [CLPS08] :

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{h} \frac{1+z^h}{1-z^h} = \exp\left(\frac{1+z}{1-z} T + \frac{T^2}{2}\right). \quad (4.11)$$

Here is a modular \mathcal{SC} version of (4.11).

Corollary 4.3.3. *For t an even positive integer, for any complex number z , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \left(\frac{1}{h} \frac{1+z^h}{1-z^h} \right)^{1/2} \\ = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp \left(\frac{1+z^t}{1-z^t} \frac{x^2 T^{2t}}{2} + \frac{x^4 T^{4t}}{4t} \right). \end{aligned}$$

Proof. Take $\rho_1(h) = \left(\frac{1}{h} \frac{1+z^h}{1-z^h} \right)^{1/2}$ in Corollary 4.2.2. By direct application of (4.11), we have:

$$f_t(T) = \exp \left(\frac{1+z^t}{1-z^t} \frac{T}{t} + \frac{T^2}{2t^2} \right).$$

□

4.3.4 The Nekrasov–Okounkov formula

In [Pé15a], Pétreolle discovered and proved analogues of the Nekrasov–Okounkov formula (1.18) for \mathcal{SC} and \mathcal{DD} , as this has been mentioned in Section 3.5. Recall that both in his work and in Sections 3.4, a sign appears combinatorially, which corresponds to the algebraic sign in the associated Littlewood formulas for Schur functions [Lit40, 11.9.5 p.238]. Here it is possible to avoid the sign and only use (1.18) with Theorem 4.1.2 to derive a modular \mathcal{SC} version of Nekrasov–Okounkov type when t is even. This is given in Corollary 4.1.4 that we prove below. In Section 4.4 we will prove refined versions of our results which take the signs into account.

Proof of Corollary 4.1.4. Take $\rho_1(h) = (1 - z/h^2)^{1/2}$ in Corollary 4.2.2, we have by application of (1.18):

$$f_t(T) = (T; T)_{\infty}^{z/t^2-1}.$$

The conclusion follows when this result is plugged in the right-hand side of Corollary 4.2.2. □

By setting $z = -c^2/x^2$ and letting $x \rightarrow 0$, the left-hand side of Corollary 4.1.4 becomes:

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{c}{h}.$$

On the right-hand side, the three first terms remain unchanged, while we can write for all $j \geq 1$:

$$(1 - x^{2j} T^{2tj})^{(z/t-t)/2} = \exp \left(\frac{1}{2} \left(\frac{c^2}{tx^2} + t \right) \sum_{k \geq 1} \frac{x^{2jk} T^{2tjk}}{k} \right),$$

therefore

$$\begin{aligned} (x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(z/t-t)/2} &= \exp \left(\frac{1}{2} \left(\frac{c^2}{tx^2} + t \right) \sum_{k \geq 1} \frac{x^{2k} T^{2tk}}{k(1 - x^{2k} T^{2tk})} \right) \\ &= \exp \left(\frac{c^2 T^{2t}}{2t} + O(x^2) \right) \\ &\xrightarrow{x \rightarrow 0} \exp \left(\frac{c^2 T^{2t}}{2t} \right). \end{aligned}$$

Therefore

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{c}{h} = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp \left(\frac{c^2 T^{2t}}{2t} \right),$$

which is equivalent to the identity in Corollary 4.3.1.

4.3.5 The Bessenrodt–Bacher–Manivel formula

The following formula deals with power sums of hook lengths. Its proof is based on a result due to Bessenrodt, and Bacher–Manivel [Bes98, BM02] which provides a mapping, for any couple of positive integers $j < k$, the total number of occurrences of the part k among all partitions of n to the number of boxes whose hook-type is $(j, k - j - 1)$. In [HJ11], Han–Ji explain that this result can be embedded in the following generalization, which is true for any complex number β :

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \sum_{h \in \mathcal{H}} h^\beta = \frac{1}{(q; T)_\infty} \sum_{k \geq 1} k^{\beta+1} \frac{T^k}{1 - T^k}. \quad (4.12)$$

The modular \mathcal{SC} version of (4.12) takes the following form.

Corollary 4.3.4. *For any complex number β and t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \sum_{h \in \mathcal{H}_t(\lambda)} h^\beta \\ = \frac{(T^{2t}; T^{2t})_\infty^{t/2}}{(x^2 T^{2t}; x^2 T^{2t})_\infty^{t/2}} (-bT; T^4)_\infty (-T^3/b; T^4)_\infty \sum_{k \geq 1} (tk)^{\beta+1} \frac{x^{2k} T^{2kt}}{1 - x^{2k} T^{2kt}}. \end{aligned}$$

Proof. Take $\rho_2(h) = h^\beta$ in Corollary 4.2.3 and then use (4.12) to compute:

$$g_t(T) = \frac{t^\beta}{(T; T)_\infty} \sum_{k \geq 1} k^{\beta+1} \frac{T^k}{1 - T^k}.$$

□

4.3.6 The Okada–Panova formula

The following formula is the generating function form of the Okada–Panova formula, which was conjectured by Okada and proved by Panova in [Pan12]:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} \prod_{i=1}^r (h^2 - i^2) = C(r) T^{r+1} \exp(T), \quad (4.13)$$

where

$$C(r) := \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1}.$$

To find a modular \mathcal{SC} version of (4.13), we want to use Theorem 4.1.2 with $\rho_1(h) = 1/h$ and $\rho_2(h) = \prod_{i=1}^r (h^2 - i^2)$. Recall from [HJ11] that:

$$f_\alpha(T) := \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{(\alpha h)^2} = \exp\left(\frac{T}{\alpha^2}\right). \quad (4.14)$$

We also define as in [HJ11]:

$$g_\alpha(T) := \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{(\alpha h)^2} \sum_{h \in \mathcal{H}(\lambda)} \prod_{i=1}^r ((\alpha h)^2 - i^2).$$

In order to evaluate $g_\alpha(T)$, Han and Ji introduce the polynomials defined by the following relations:

$$\begin{aligned} B_{r,0}(\alpha) &= \prod_{j=1}^r (\alpha^2 - j^2), \\ B_{r,k}(\alpha) &= \left[\alpha^2 (k+1)^2 - r^2 \right] B_{r-1,k}(\alpha) + \alpha^2 B_{r-1,k-1}(\alpha) \quad \text{for } k \in \{1, \dots, r-1\}, \\ B_{r,r}(\alpha) &= \alpha^{2r}. \end{aligned}$$

This enables them to rewrite $g_\alpha(T)$ in [HJ11, Proposition 8.2] as:

$$g_\alpha(T) = \exp\left(\frac{T}{\alpha^2}\right) \sum_{k=0}^r B_{r,k}(\alpha) C(k) \left(\frac{T}{\alpha^2}\right)^{k+1}. \quad (4.15)$$

We prove the following modular \mathcal{SC} version of (4.13).

Corollary 4.3.5. *For any positive integer r and t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} \prod_{i=1}^r (h^2 - i^2) \\ = t (T^{2t}, T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \\ \times \exp\left(\frac{x^2 T^{2t}}{2t}\right) \sum_{k=\lceil (r-t+1)/t \rceil}^r B_{r,k}(t) C(k) \left(\frac{x^2 T^{2t}}{t^2}\right)^{k+1}. \end{aligned}$$

Proof. Take $\rho_1(h) = 1/h$ and $\rho_2(h) = \prod_{i=1}^r (h^2 - i^2)$ in Theorem 4.1.2 and $\alpha = t$ in (4.14) and (4.15) to rewrite f_t and g_t , respectively. \square

4.3.7 The Stanley–Panova formula

Panova and Stanley proved the following formula [Pan12, Sta10]:

$$n! \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^{2k} = \sum_{i=0}^k T(k+1, i+1) C(i) \prod_{j=0}^i (n-j) \quad (4.16)$$

where $T(k, i)$ is a central factorial number [Sta99, ex.5.8] defined for $k \geq 1$ and $i \geq 1$ by:

$$\begin{aligned} T(k, 0) &= T(0, i) = 0, \quad T(1, 1) = 1, \\ T(k, i) &= i^2 T(k-1, i) + T(k-1, i-1) \quad \text{for } (k, i) \neq (1, 1). \end{aligned}$$

By setting $\rho_1(h) = 1/(\alpha h)$ and $\rho_2(h) = (\alpha h)^{2k}$, we have as in (4.14)

$$f_\alpha(T) = \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{(\alpha h)^2} = \exp\left(\frac{T}{\alpha^2}\right), \quad (4.17)$$

and by using (4.16)

$$\begin{aligned} g_\alpha(T) &= \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}} \frac{1}{(\alpha h)^2} \sum_{h \in \mathcal{H}(\lambda)} \alpha^{2k} h^{2k} \\ &= \alpha^{2k} \exp\left(\frac{T}{\alpha^2}\right) \sum_{i=0}^k T(k+1, i+1) C(i) \left(\frac{T}{\alpha^2}\right)^{i+1}. \end{aligned} \quad (4.18)$$

Now we prove the following modular \mathcal{SC} version of (4.16).

Corollary 4.3.6. *For any positive integer k and t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} h^{2k} \\ = t^{2k+1} (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \\ \exp\left(\frac{x^2 T^{2t}}{2t}\right) \sum_{i=0}^k T(k+1, i+1) C(i) \left(\frac{x^2 T^{2t}}{t^2}\right)^{i+1}. \end{aligned}$$

Proof. Take $\rho_1(h) = 1/h$ and $\rho_2(h) = h^{2k}$ in Theorem 4.1.2 and $\alpha = t$ in (4.17) and (4.18) to rewrite f_t and g_t , respectively. \square

4.4 Signed refinements

In [P 15b], P tr olle proved the following \mathcal{SC} Nekrasov–Okounkov type formula similar to (1.18), which stands for any complex number z : Recall from Section 3.5 the following Nekrasov–Okounkov formula (3.43), which we state here stressing the fact that the product over hook lengths is computed over the multiset of all hook lengths of the partitions/

$$\sum_{\lambda \in \mathcal{SC}} \delta_{\lambda} T^{|\lambda|} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}(\lambda)}} \left(1 - \frac{2z}{h_s \varepsilon_s}\right) = \left(\frac{(T^2; T^2)_{\infty}^{z+1}}{(T; T)_{\infty}}\right)^{2z-1},$$

for any complex number z .

We recall here some the definitions of some statistics widely discussed in the previous chapters. Here, δ_{λ} and ε_s are signs depending on the partition λ , and the position of any box s in its Ferrers diagram (written $s \in \lambda$ in the above formula), respectively. If the Durfee square of λ has size d_{λ} , then one simply defines $\delta_{\lambda} := (-1)^{d_{\lambda}}$. Recall that this sign also has an algebraic meaning regarding Littlewood summations for Schur functions indexed by partitions in \mathcal{SC} . Next, for any partition $\lambda \in \mathcal{SC}$ and a box $s = (i, j) \in \lambda$, ε_s is defined as -1 if s is a box strictly below the diagonal of the Ferrers diagram and as 1 otherwise.

Our goal in this section is to prove a multiplication-addition theorem similar to Theorem 4.1.2 including the above signs. Nevertheless one can notice that for $\lambda \in \mathcal{SC}$, we have actually $\delta_{\lambda} = (-1)^{|\lambda|}$. Indeed, by Lemma 4.1.5 in Section 2.3 and by definition of the BG-rank, one has $|\lambda| \equiv r - s \pmod{2}$; and moreover $d = r + s$ by definition of $D_1(\lambda)$ and $D_3(\lambda)$. This means that the sign δ_{λ} can readily be omitted, by replacing T by $-T$ in formulas like (3.43) and their modular analogues.

Recall that Corollary 2.2.2 allows to determine the position with respect to the main diagonal of the Ferrers diagram, thanks to the correspondence between a box of λ and a pair of indices of the corresponding word $\psi(\lambda)$. Next, to include the sign ε , we will need a refinement of Proposition 2.3.2 (P3), which is an immediate consequence of the Littlewood decomposition: for $\lambda \in \mathcal{P}$ and any box $s \in \lambda$ with hook length $h_s \in \mathcal{H}_t(\lambda)$ (here t is any positive integer), there exists a unique $k \in \{0, \dots, t-1\}$ and a unique box $u_k \in \nu^{(k)}$ such that $h_s = t h_{u_k}$, where h_{u_k} is the hook length of s_k in the partition $\nu^{(k)}$. We will say that the box s_k is *associated to* the box s . We have the following result for self-conjugate partitions.

Lemma 4.4.1. *Set $\lambda \in \mathcal{SC}$, let t be a positive even integer. Set $u \in \lambda$ such that $h_s \in \mathcal{H}_t(\lambda)$. Then the following properties hold true:*

1. *The box s does not belong to the main diagonal of λ .*
2. *The application $s \mapsto s'$, where us' is the symmetric of s with respect to the main diagonal of λ , is well-defined on λ , bijective and satisfies $h_{u'} = h_s \in \mathcal{H}_t(\lambda)$ and $\varepsilon_s = -\varepsilon_{u'}$.*

3. If s_k and s_l are the boxes associated to s and s' respectively, then $l = t - 1 - k$.

Proof. For any \mathcal{SC} partition, all hook lengths of boxes on the main diagonal are odd numbers. As t is even, the result (1) follows.

Next (2) is a direct consequence of (1) and the definitions of \mathcal{SC} and ε_s .

Finally, to prove (3) we need to explicit the bijection between the coordinates of a box of λ and a pair of indices of the corresponding word $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}}$ defined in Section 2.3. Let us introduce the two following sets:

$$\begin{aligned} I &:= \{i \in \mathbb{Z} \mid c_i = 1 \text{ and } \exists j \in \mathbb{Z} \text{ such that } i < j, c_j = 0\}, \\ J &:= \{j \in \mathbb{Z} \mid c_j = 0 \text{ and } \exists i \in \mathbb{Z} \text{ such that } i < j, c_i = 1\}. \end{aligned}$$

By definition of $\psi(\lambda)$, those sets are finite. Therefore one can write $I = \{i_1, \dots, i_{\lambda_1}\}$ and $J = \{j_1, \dots, j_{\lambda_1}\}$ such that the sequence $(i_a)_{a \in \{1, \dots, \lambda_1\}}$ (resp. $(j_b)_{b \in \{1, \dots, \lambda_1\}}$) is strictly increasing (resp. strictly decreasing).

Let $F(\lambda)$ be the Ferrers diagram of λ and define the application

$$\Psi : \begin{array}{l} F(\lambda) \rightarrow I \times J \\ (x, y) \mapsto (i_y, j_x). \end{array}$$

Note that Ψ is injective by monotony of the sequences (i_a) and (j_b) .

Recall that $\lambda \in \mathcal{SC}$ translates in terms of the associated word to:

$$c_j = 1 - c_{-1-j} \quad \forall j \in \mathbb{N}. \quad (4.19)$$

This implies that $|I| = \lambda'_1 = |J| = \lambda_1$. Let $\psi : I \rightarrow \psi(I)$ be the application such that $\psi(i_m) := -1 - i_m$. The aforementioned property actually guarantees that $\psi(I) \subset J$. As $|I| = |J|$, we deduce that ψ is bijective. Moreover, as $(i_a)_{a \in \{1, \dots, \lambda_1\}}$ is strictly increasing, we derive that $(\psi(i_a))$ is strictly decreasing and for any $a \in \{1, \dots, \lambda_1\}$, we have $j_a = -1 - i_a$.

Suppose that $(i_y, j_x) \in \Psi(F(\lambda))$ is such that $i_y \equiv k \pmod{t}$ and $j_x \equiv k \pmod{t}$. By (2.9) and the bijectivity of ψ sending (i_a) to (j_b) , we have that $(i_x, j_x) \in \Psi(F(\lambda))$ and $i_x \equiv t - 1 - k \pmod{t}$ and $j_y \equiv t - 1 - k \pmod{t}$. As s' has coordinates (i_x, j_y) and is associated to the box s_l , we derive that $l = t - 1 - k$, which concludes the proof. \square

4.4.1 A signed addition-multiplication theorem

We will now prove a generalization of Theorem 4.1.2 which includes the sign mentioned above.

Theorem 4.4.2. *Set t an even integer and let $\tilde{\rho}_1, \tilde{\rho}_2$ be two functions defined on $\mathbb{Z} \times \{-1, 1\}$. Set also $f_t(T), g_t(T)$ the formal power series defined by:*

$$\begin{aligned} f_t(T) &:= \sum_{\nu \in \mathcal{P}} T^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \tilde{\rho}_1(th, 1) \tilde{\rho}_1(th, -1), \\ g_t(T) &:= \sum_{\nu \in \mathcal{P}} T^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \tilde{\rho}_1(th, 1) \tilde{\rho}_1(th, -1) \sum_{h \in \mathcal{H}(\nu)} (\tilde{\rho}_2(th, 1) + \tilde{\rho}_2(th, -1)). \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_s, \varepsilon_s) \sum_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_2(h_s, \varepsilon_s) \\ = \frac{t}{2} (f_t(x^2 T^{2t}))^{t/2-1} g_t(x^2 T^{2t}) (T^{2t}, T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}. \end{aligned}$$

Proof. The proof follows the same steps as the one of Theorem 4.1.2, but now (4.6) becomes

$$b^{\text{BG}(\omega)} T^{|\omega|} \sum_{\underline{\nu} \in \mathcal{P}^t} T^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{s \in \underline{\nu}} \tilde{\rho}_1(th_s, \varepsilon_s) \sum_{s \in \underline{\nu}} \tilde{\rho}_2(th_s, \varepsilon_s), \quad (4.20)$$

where ω is in $\mathcal{SC}(t)$. The product part $T^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{s \in \underline{\nu}} \tilde{\rho}_1(th_s, \varepsilon_s)$ inside the sum over $\underline{\nu}$ can be rewritten as follows

$$\prod_{i=0}^{t/2-1} T^{t(|\nu^{(i)}| + |\nu^{(t-1-i)}|)} x^{|\nu^{(i)}| + |\nu^{(t-1-i)}|} \prod_{h \in \mathcal{H}(\nu^{(i)})} \tilde{\rho}_1(th, 1) \tilde{\rho}_1(th, -1).$$

Indeed, by Lemma 4.4.1, each box $s \in \nu^{(i)}$, with $0 \leq i \leq t-1$, is bijectively paired with a box $s' \in \nu^{(t-1-i)}$ satisfying $\tilde{\rho}_1(th_{u'}, \varepsilon_{u'}) = \tilde{\rho}_1(th_u, -\varepsilon_s)$. The sum part $\sum_{s \in \underline{\nu}} \tilde{\rho}_2(th_s, \varepsilon_s)$ in (4.20) can be rewritten in a similar fashion. We leave the rest of the proof to the reader as the remaining computations are similar to the ones used to prove Theorem 4.1.2. \square

Note that Theorem 4.1.2 is a consequence of Theorem 4.4.2, by choosing $\tilde{\rho}_1(a, \varepsilon) = \rho_1(a)$ and $\tilde{\rho}_2(a, \varepsilon) = \rho_2(a)$. Moreover by choosing $\tilde{\rho}_1 = 1$ or $\tilde{\rho}_2 = 1$, we have special cases similar to Corollaries 4.2.2 and 4.2.3. However we will only highlight the case where $\tilde{\rho}_2 = 1$, as this one yields interesting consequences.

Corollary 4.4.3. *Set $\tilde{\rho}_1$ a function defined on $\mathbb{Z} \times \{-1, 1\}$, and let t be a positive even integer and f_t be defined as in Theorem 4.4.2. Then we have*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_s, \varepsilon_s) \\ = (f_t(x^2 T^{2t}))^{t/2} (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty}. \end{aligned}$$

4.4.2 Applications

We have applications similar to the ones obtained in Sections 4.3.1–4.3.7. However we only highlight the cases concerning Sections 4.3.1–4.3.4, which are the most interesting in our opinion and are all derived from Corollary 4.4.3.

First note that the generating series obtained with $\tilde{\rho}_1 = 1$ is the same as the one in Section 4.3.1.

Next, when t is an even positive integer and $\lambda \in \mathcal{SC}$, notice that

$$\prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \varepsilon_s = (-1)^{|\mathcal{H}_t(\lambda)|/2}.$$

Therefore the specialization $\tilde{\rho}_1(a, \varepsilon) = 1/(a\varepsilon)$ yields a hook length formula equivalent to the one in Corollary 4.3.1 when x is replaced by $x\sqrt{-1}$. Similarly, the specialization $\tilde{\rho}_1(a, \varepsilon) = 1/(a\varepsilon)^{1/2}$ yields a hook length formula equivalent to the one in Corollary 4.3.2 when x is replaced by $x\sqrt[4]{-1}$.

Now the signed modular analogue of (4.11) is as follows.

Corollary 4.4.4. *For t an even positive integer, for any complex number z , we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \frac{1}{h_s^{1/2}} \frac{1 + \varepsilon_s z^{h_s} \sqrt{-1}}{1 - \varepsilon_s z^{h_s}} \\ = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp\left(\frac{1 + z^t}{1 - z^t} \frac{x^2 T^{2t}}{2} + \frac{x^4 T^{4t}}{4t}\right). \end{aligned}$$

Proof. Take

$$\tilde{\rho}_1(a, \varepsilon) = \frac{1}{a^{1/2}} \frac{1 + \varepsilon z^a \sqrt{-1}}{1 - \varepsilon z^a}$$

in Corollary 4.4.3 and use the identity $\tilde{\rho}_1(a, 1)\tilde{\rho}_1(a, -1) = (1+z^a)/(a(1-z^a))$ and Formula (4.11) to conclude. \square

The signed modular SC analogue of the Nekrasov–Okounkov formula (1.18), which is actually a modular analogue of (3.43), is the following.

Corollary 4.4.5. *For any complex number z and t an even positive integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \left(1 - \frac{z}{h_s \varepsilon_s}\right) \\ = (T^{2t}; T^{2t})_{\infty}^{t/2} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} (x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(z^2/t-t)/2}. \end{aligned}$$

Proof. Take $\tilde{\rho}_1(a, \varepsilon) = 1 - z/(a\varepsilon)$ in Corollary 4.4.3, then use the identity $\tilde{\rho}_1(a, 1)\tilde{\rho}_1(a, -1) = 1 - z^2/a^2$ and (1.18) to conclude. \square

Note that taking $b = 1$ in the above formula, one gets Pétréolle’s result [Pé15b, Théorème 4.22], in which T, y, z have to be replaced by $-T, x, z/t$, respectively.

By identifying coefficients on both sides of the previous formula, we get the following consequence.

Corollary 4.4.6. *For all positive integers n and all integers j , we have*

$$\sum_{\substack{\lambda \in \mathcal{SC}, \lambda \vdash 2nt+j(2j-1) \\ \text{BG}(\lambda)=j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{h^2}{2} = \frac{1}{2^n t^{n-1} (n-1)!} (t + 3n - 3). \quad (4.21)$$

Proof. By Lemma 4.4.1, the left-hand side of Corollary 4.4.5 can be rewritten as follows

$$\sum_{\lambda \in \mathcal{SC}} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\text{BG}(\lambda)} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda) \\ \varepsilon_s = 1}} \left(1 - \frac{z^2}{h_s^2}\right). \quad (4.22)$$

The left-hand side of (4.21) is the coefficient of $T^{2tn+j(2j-1)} x^{2n} b^j (-z^2)^{n-1}$ in (4.22). Using the following identity

$$\prod_{m \geq 1} \frac{1}{1 - T^m} = \exp \left(\sum_{k \geq 1} \frac{T^k}{k(1 - T^k)} \right),$$

the right-hand side of Corollary 4.4.5 can be rewritten:

$$R = \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(x^2 T^{2t}; x^2 T^{2t})_{\infty}^{t/2}} (-bT; T^4)_{\infty} (-T^3/b; T^4)_{\infty} \exp \left(\frac{-z^2}{2t} \sum_{k \geq 1} \frac{(x^2 T^{2t})^k}{k(1 - (x^2 T^{2t})^k)} \right).$$

Thus, by also using (4.3), our desired coefficient is equal to

$$\begin{aligned}
& [T^{2tn+j(2j-1)}x^{2n}b^j(-z^2)^{n-1}] R \\
&= [T^{2tn}x^{2n}(-z^2)^{n-1}] \frac{(T^{2t}; T^{2t})_{\infty}^{t/2}}{(x^2T^{2t}; x^2T^{2t})_{\infty}^{t/2} (T^4; T^4)_{\infty}} \exp\left(\frac{-z^2}{2t} \sum_{k \geq 1} \frac{(x^2T^{2t})^k}{k(1 - (x^2T^{2t})^k)}\right) \\
&= [T^{2tn}x^{2n}] \frac{1}{2^{n-1}t^{n-1}(n-1)!} \frac{1}{(x^2T^{2t}; x^2T^{2t})_{\infty}^{t/2}} \left(\sum_{k \geq 1} \frac{(x^2T^{2t})^k}{k(1 - (x^2T^{2t})^k)}\right)^{n-1} \\
&= [T^{2t}x^2] \frac{1}{2^{n-1}t^{n-1}(n-1)!} \left(1 + \frac{t}{2}x^2T^{2t}\right) \left(\frac{1}{1 - x^2T^{2t}} + \frac{x^2T^{2t}}{2(1 - (x^2T^{2t})^2)}\right)^{n-1} \\
&= \frac{1}{2^{n-1}t^{n-1}(n-1)!} \left(\frac{t}{2} + \frac{3(n-1)}{2}\right) \\
&= \frac{1}{2^n t^{n-1} (n-1)!} (t + 3n - 3).
\end{aligned}$$

□

Corollary 4.4.6 could also be derived from Corollary 4.3.6 by setting $k = 1$ and comparing the coefficients of $T^{2tn+j(2j-1)}x^{2n}b^j$ on both sides.

4.5 The odd case

In this section, we analyse the case where t is a positive odd integer. Recall that in this case the Littlewood decomposition can be written as follows

$$\lambda \in \mathcal{SC} \mapsto (\omega, \tilde{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}. \quad (4.23)$$

When t is odd, Formula (3.4) in [GKS90] gives a connection between the BG-rank of a partition, and its t -quotient and its t -core. However the formula implies a dependence between t -core and t -quotient, which is not convenient for multiplication-addition type theorems. This is why we will formulate multiplication-addition type theorems without the BG-rank.

Moreover, because of the partition $\mu \in \mathcal{SC}$ appearing in (4.23), more difficulties arise which make a general result less elegant than in the even case. Even if it is possible to prove a general odd analogue to Theorem 4.1.2 (without the BG-rank), formulas on self-conjugate partitions would be required to derive interesting applications. These are, to our knowledge, missing in the literature. This is why we will focus here on a subset of self-conjugate partitions for which μ is empty, which, as will be explained, is algebraically interesting.

For a fixed positive odd integer t , let us define

$$\text{BG}^t := \{\lambda \in \mathcal{SC}, \Phi_t(\lambda) = (\omega, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^t \text{ with } \nu^{((t-1)/2)} = \emptyset\}.$$

Note that λ is in BG^t if and only if the partition μ is empty in (4.23). Following [Ber21], we also define for an odd prime number p , the set of self-conjugate partitions with no diagonal hook length divisible by p :

$$\text{BG}_p := \{\lambda \in \mathcal{SC} \mid \forall i \in \{1, \dots, d\}, p \nmid h_{(i,i)}\}.$$

Algebraically, this set yields interesting properties in representation theory of the symmetric group over a field of characteristic p , see for instance [BG10, Ber21]. Combinatorially, it is natural to extend this definition to a set BG_t for any positive odd number t .

The following result explains the connection between the two above sets and is proved in [BG10, Lemma 3.4] for any prime number p . Nevertheless, we give a proof here to generalize it to any positive odd integer t .

Lemma 4.5.1. *For any positive odd integer t , we have:*

$$\text{BG}^t = \text{BG}_t.$$

Proof. Take $\lambda \in \mathcal{SC} \setminus \text{BG}_t$. There exists $(x, x) \in \lambda$ such that $t \mid h_{(x,x)}$. Recall that $h_{(x,x)}$ is necessarily odd. Take m such that $h_{(x,x)} = t(2m + 1)$. Let $(i_x, j_x) \in \mathbb{Z}^2$ be the pair of indices in $\psi(\lambda)$ associated with the box (x, x) . Then $j_x \geq 0$ and $i_x < 0$. Moreover, by (4.19), one has $i_x = -j_x - 1$. As $h_{(x,x)} = j_x - i_x$, we get $h_{(x,x)} = 2j_x + 1$. This yields $2j_x + 1 = t(2m + 1)$. Therefore we have

$$j_x = tm + \frac{t-1}{2}.$$

This implies that there exists a sequence “10” in the subword $(c_{kt+(t-1)/2})_{k \in \mathbb{Z}} = \psi(\mu)$, where $\mu = \nu^{((t-1)/2)}$ is the partition uniquely defined by the Littlewood decomposition. Hence $\mu \neq \emptyset$ and therefore $\lambda \notin \text{BG}_t$.

Conversely, let $\lambda \in \mathcal{SC} \setminus \text{BG}_t$. So $\mu \neq \emptyset$. Set $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ the corresponding word. Remark that $\mu \neq \emptyset$ is equivalent to the existence of $i_1 \in \mathbb{N}$ such that $c_{ti_1+(t-1)/2} = 0$ and $c_{-ti_1+(t-1)/2} = 0$. This implies that there exists a hook of length $t(2i_1 + 1)$ which is on the main diagonal of λ . Therefore $\lambda \notin \text{BG}_t$. \square

We now prove the following result which is the analogue of Theorem 4.4.2 for t odd, restricted to the set $\text{BG}^t = \text{BG}_t$.

Theorem 4.5.2. *Let t be a positive odd integer and set $\tilde{\rho}_1, \tilde{\rho}_2$ two functions defined on $\mathbb{Z} \times \{-1, 1\}$. Let f_t and g_t be the formal power series defined in Theorem 4.4.2. Then we have*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_s, \varepsilon_s) \sum_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_2(h_s, \varepsilon_s) \\ = (t-1) (f_t(x^2 T^{2t}))^{(t-3)/2} g_t(x^2 T^{2t}) \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}}. \end{aligned}$$

Proof. The proof follows the same lines as the ones of Theorems 4.1.2 and 4.4.2 but with $b = 1$. Here t is odd and the summation on the left-hand side is over partitions in BG^t (therefore $\nu^{((t-1)/2)} = \mu = \emptyset$), so the Littlewood decomposition shows that, in our situation, (4.20) takes the form

$$T^{|\omega|} \sum_{\nu \in \mathcal{P}^{t-1}} T^{|\nu|} x^{|\nu|} \prod_{s \in \nu} \tilde{\rho}_1(th_s, \varepsilon_s) \sum_{s \in \nu} \tilde{\rho}_2(th_s, \varepsilon_s),$$

where ω is a fixed t -core in BG^t . Next we can transform the above expression by using Proposition 2.3.4 and Lemma 4.4.1: although the latter was proved in the t even case only, it is possible to extend it to t odd for partitions $\lambda \in \text{BG}^t$, by noticing that a box s is on the main diagonal of λ and satisfies $h_s \in \mathcal{H}_t(\lambda)$ only if s is associated by the Littlewood decomposition to a box in $\nu^{((t-1)/2)} = \mu$, which is empty in our situation. Therefore we can proceed as in the proof of Theorem 4.4.2, but the factor t in Theorem 4.4.2 now becomes $t - 1$.

The remaining part to finish the proof is the computation of the generating series of partitions ω in BG^t that are t -cores, that are partitions in the set $\text{BG}_{(t)}^t$. As remarked in [AO91], the generating series of $\text{BG}_{(t)}^t$ is given by

$$\sum_{\lambda \in \text{BG}_{(t)}^t} T^{|\lambda|} = \prod_{\substack{k \geq 1 \\ t \nmid 2k+1}} (1 + T^{2k+1}) = \frac{(-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}}. \quad (4.24)$$

By using Proposition 2.3.4 (SC3) of the Littlewood decomposition and the generating series (1.19) for partitions, the left-hand side of (4.24) can be rewritten as

$$\sum_{\omega \in \text{BG}_{(t)}^t} T^{|\omega|} \left(\sum_{\nu \in \mathcal{P}} T^{2t|\nu|} \right)^{(t-1)/2} = \frac{1}{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2}} \sum_{\omega \in \text{BG}_{(t)}^t} T^{|\omega|}.$$

Hence the generating series of $\text{BG}_{(t)}^t$ is

$$\sum_{\omega \in \text{BG}_{(t)}^t} T^{|\omega|} = \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}}.$$

The rest of the proof follows the exact same steps as for Theorem 4.1.2, without taking the BG-rank into account. \square

Note that by taking $\tilde{\rho}_1(a, \varepsilon) = \rho_1(a)$ and $\tilde{\rho}_2(a, \varepsilon) = \rho_2(a)$ in the above result, we get an analogue of Theorem 4.1.2 for t odd and $b = 1$, restricted to the set $\text{BG}^t = \text{BG}_t$.

We now derive applications of Theorem 4.5.2 in the same spirit as the ones proved in Sections 4.3 and 4.4, but for odd t . As the specializations are the same here, we do not give details for the proofs.

First, our bivariate generating function takes the form:

$$\sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} = \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(t-1)/2} (-T^t; T^{2t})_{\infty}}.$$

Next, the odd analogues of Corollaries 4.3.1 and 4.3.2 for BG^t are summarized in the following result.

Corollary 4.5.3. *For t a positive odd integer, we have:*

$$\sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} \exp \left((t-1) \frac{x^2 T^{2t}}{2t^2} \right),$$

and

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^{1/2}} \\ = \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} \exp \left((t-1) \left(\frac{x^2 T^{2t}}{2t} + \frac{x^4 T^{4t}}{4t^2} \right) \right). \end{aligned}$$

The odd version of Corollary 4.4.4 is as follows.

Corollary 4.5.4. *For t a positive odd integer, for any complex number z , we have:*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \frac{1}{h_s^{1/2}} \frac{1 + \varepsilon_s \sqrt{-1} z^{h_s}}{1 - \varepsilon_s z^{h_s}} \\ = \frac{(T^{2t}; T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} \exp \left((t-1) \left(\frac{1 + z^t x^2 T^{2t}}{1 - z^t} \frac{x^2 T^{2t}}{2t} + \frac{x^4 T^{4t}}{4t^2} \right) \right). \end{aligned}$$

Now the odd version of the modular signed Nekrasov–Okounkov type formula given in Corollary 4.4.5 is given below.

Corollary 4.5.5. *For t a positive odd integer, for any complex number z , we have:*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}_t(\lambda)}} \left(1 - \frac{z}{h_s \varepsilon_s} \right) \\ = \frac{(T^{2t}, T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} (x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(t-1)(z^2/t^2-1)/2}. \end{aligned}$$

Finally, the odd analogues of Corollaries 4.3.4, 4.3.5 and 4.3.6 are given in the three results below.

Corollary 4.5.6. *For any complex number β and t a positive odd integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \sum_{h \in \mathcal{H}_t(\lambda)} h^{\beta} = (t-1) \frac{(T^{2t}, T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(x^2 T^{2t}; x^2 T^{2t})_{\infty}^{(t-1)/2} (-T^t; T^{2t})_{\infty}} \\ \times \sum_{k \geq 1} (tk)^{\beta+1} k \frac{x^{2k} T^{2kt}}{1 - x^{2k} T^{2kt}}. \end{aligned}$$

Corollary 4.5.7. *For any positive integer r and t a positive odd integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} \prod_{i=1}^r (h^2 - i^2) = (t-1) \frac{(T^{2t}, T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} \\ \times \exp \left((t-1) \left(\frac{x^2 T^{2t}}{2t^2} \right) \right) \sum_{k=\lceil (r-t+1)/t \rceil}^r B_{r,k}(t) C(k) \left(\frac{x^2 T^{2t}}{t^2} \right)^{k+1}. \end{aligned}$$

Corollary 4.5.8. *For any positive integer k and t a positive odd integer, we have:*

$$\begin{aligned} \sum_{\lambda \in \text{BG}^t} T^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} h^{2k} = (t-1) t^{2k} \frac{(T^{2t}, T^{2t})_{\infty}^{(t-1)/2} (-T; T^2)_{\infty}}{(-T^t; T^{2t})_{\infty}} \\ \times \exp \left((t-1) \frac{x^2 T^{2t}}{2t^2} \right) \sum_{i=0}^k T(k+1, i+1) C(i) \left(\frac{x^2 T^{2t}}{t^2} \right)^{i+1}. \end{aligned}$$

5.1 Conclusion

Nous avons vu dans le chapitre 3 comment obtenir des q formules de type Nekrasov–Okounkov pour chacun des systèmes de racines affines infinis d’une part et d’autre part dans le chapitre 4 comment obtenir des analogues “modulaires” de formules d’équerres pour le seul cas des partitions auto-conjuguées. Notons toutefois que les techniques et preuves développées dans le chapitre 4 s’adaptent sans problème aux partitions \mathcal{DD} ou \mathcal{DD}' . En fait, on pourrait même obtenir des théorèmes de multiplication-addition sur les partitions z -*asymétriques* pour z un entier relatif, telles qu’introduites par Ayer–Kumari dans [AK22]. Ces dernières sont des partitions qui ont exactement z bandelettes (verticales ou horizontales selon le signe de z) ayant pour longueur la longueur de la diagonale Δ du carré de Durfee. En pratique ces bandelettes correspondent aux cases coloriées en jaune dans les figures 1.2b et 1.2c. Notons que le cas $z = 0$ correspond aux éléments \mathcal{SC} , le cas $z = -1$ aux éléments de \mathcal{DD} et le cas $z = 1$ aux éléments de \mathcal{DD}' . Toutefois la décomposition de Littlewood sur ces sous-ensembles de partitions donne des éléments du t -quotient qui ne sont plus seulement des partitions mais qui peuvent aussi être des partitions distinctes à parts doublées ou leurs partitions conjuguées, ainsi que des partitions auto-conjuguées. Dans ce cas, comme souligné à la Section 4.5, on peut obtenir des théorèmes de multiplication-addition sur ces sous-ensembles de partitions. Toutefois, obtenir des applications intéressantes nécessite de connaître à la fois les séries entières f_t et g_t dont les sommes portent sur les partitions, mais aussi leurs analogues lorsque la somme porte sur \mathcal{DD} , \mathcal{SC} ou \mathcal{DD}' . Enfin obtenir par des théorèmes de multiplication des analogues modulaires des formules de q -Nekrasov–Okounkov du Chapitre 3 est possible mais nécessiterait une version raffinée des théorèmes du Chapitre 4 en raison du rôle particulier joué par la diagonale principale Δ .

5.2 Perspectives

Cette section se divise en quatre sous-parties. La première porte sur les potentielles connexions algébriques des q -analogues des formules de Nekrasov–Okounkov. La seconde, quant à elle, liste les potentielles pistes à explorer dans l’obtention de (q, t) analogues.

5.2.1 q -analogues de la formule de Nekrasov–Okounkov

Théorèmes de multiplication

Mon premier article [Wah22a] utilise des propriétés fines de la décomposition de Littlewood pour établir des théorèmes de multiplication sur les partitions \mathcal{SC} , ce qui me permet d’obtenir des versions modulaires (pour les \mathcal{SC}) de formules classiques sur les longueurs d’équerre. Lorsque le paramètre fixé pour appliquer la décomposition de Littlewood est pair, j’obtiens des formules qui n’utilisent que des identités sur les partitions et font apparaître le BG-rang (une statistique

introduite par Berkovich et Garvan [BG06] pour obtenir des raffinements des congruences de Ramanujan). Le cas du paramètre impair est plus complexe. Une question intéressante serait d’obtenir des théorèmes de multiplication du même genre mais qui font apparaître la diagonale du carré de Durfee afin de pouvoir dériver des q -analogues de la formule de Nekrasov–Okounkov à partir de théorèmes de multiplication. Notons que les techniques développées dans ma thèse permettent l’obtention de q -analogues d’identités de type Nekrasov–Okounkov grâce à la spécialisation de la reformulation des identités de Macdonald obtenues.

Il est également a priori possible d’obtenir le même type de théorème de multiplication pour les partitions \mathcal{DD} . Il serait intéressant d’étudier quelles formules pourraient être obtenues à partir de ce cadre.

Par ailleurs, je souhaiterais explorer les connections entre la décomposition de Littlewood et les formules de caractères en caractéristique positive. Dans [Wah22a], une première connexion semble apparaître, puisque je démontre un théorème de multiplication pour les BG-partitions mais les aspects algébriques de ces formules d’équerres restent encore à explorer.

5.2.2 (q, t) -analogue de la formule de Nekrasov–Okounkov

Les paragraphes qui suivent s’intéressent à l’obtention de q, t -analogues de la formule de Nekrasov–Okounkov. Ces (q, t) -analogues permettraient par exemple d’obtenir des formules de caractères sur des variétés algébriques [CRV18, RW18]. Si la littérature laisse apparaître assez clairement une méthodologie à adopter pour obtenir un q -analogue, la situation est moins déterministe pour l’obtention du (q, t) -analogue. La seule approche combinatoire devrait se révéler insuffisante ici, bien qu’elle permette de dégager certains axes d’étude. Ainsi, le recours à des théorèmes de multiplication semble permettre de mieux appréhender la nature des questions qui s’y rattachent (en faisant par exemple apparaître fortement le lien entre l’obtention de cet analogue et l’étude des propriétés de certaines fonctions elliptiques).

Théorèmes de multiplication et conjecture de Walsh–Warnaar

Ce paragraphe est dédié à la description d’un travail commun en cours avec Isaac Konan autour de la conjecture de Walsh–Warnaar sur un analogue modulaire de la (q, t) -formule de Nekrasov–Okounkov ([WW20, Conjecture 8.1]).

Cette conjecture résulte d’un travail préalable sur des variations autour des théorèmes de multiplication de Han–Ji [HJ11] qui utilisent la décomposition de Littlewood pour obtenir des analogues modulaires de certaines formules portant sur les longueurs d’équerres. Ma thèse applique notamment la méthodologie de Han–Ji à l’étude d’autres sous-familles de partitions, à savoir les partitions auto-conjuguées, pour obtenir des théorèmes de multiplications et formuler des analogues modulaires, en particulier des q -formules de Nekrasov–Okounkov [Wah22a]. Il ressort de l’article de Walsh–Warnaar que les théorèmes de multiplications ne permettent a priori pas d’obtenir des (q, t) -analogues du fait de la forme des produits qui apparaissent dans ces égalités de type “somme = produit”. En particulier, les auteurs expliquent qu’une approche uniquement combinatoire ne devrait pas permettre la résolution de la conjecture car même les cas simples impliquent des propriétés fines de fonctions elliptiques.

Il semble qu’un raffinement de la décomposition de Littlewood qui garderait la trace des bras et des jambes des partitions et non seulement des longueurs de jambes permettrait une reformulation de la conjecture qui ferait apparaître plus explicitement les propriétés des fonctions elliptiques.

Par ailleurs, notons que les travaux de Walsh–Warnaar [WW20] s’intéressent au lien entre théorèmes de multiplication et la formule de Buryak–Feigin–Nakajima [BFN15] qui propose une décomposition de la série génératrice des polynômes de Hilbert–Poincaré des schémas de Hilbert des points du plans. Cette formule a récemment obtenu une preuve combinatoire à l’aide de modèles de mots bi-infinis dans une prépublication de Vidalis [Vid22].

Mots binaires et formule de caractères de Weyl-Kac

Dans sa thèse, Bartlett [Bar14, BW15], avec son directeur de thèse Warnaar, donne des formules combinatoires pour les caractères des algèbres affines de Kac–Moody pour les types $\tilde{C}_t, \tilde{BC}_t$ et \tilde{B}_t^\vee comme des sommes sur des polynômes modifiés de Hall–Littlewood. Les spécialisations de ces identités donnent des identités ressemblant aux identités de l’appendice de l’article de Macdonald [Mac72, Appendix 1], et j’aimerais regarder dans quelle mesure les résultats obtenus pendant ma thèse pourraient s’étendre à ce type d’identités.

Dans une prépublication récente, Guillot–Lecouvey–Tarrago expliquent en quoi les algèbres positivement multiplicatives et les graphes positivement multiplicatifs permettent d’uniformiser le cadre de nombreux problèmes à l’intersection entre algèbre, combinatoire et probabilités [GLT22], ce qui est le cas de la question formulée ici. Ces outils interviennent en particulier dans l’obtention d’une description combinatoire des algèbres de fusion, qui semblent être en lien avec l’ensemble $H_{<2t+2,+}$ de longueurs d’équerres inférieures à $2t + 2$ qui intervient dans ma recherche (voir par exemple [Gri07]).

Opérateurs de vertex, modèles de particules et (q, t) -analogues

Comme mentionné dans la Section 1.5, les formules de type Nekrasov–Okounkov ont des connections avec la physique et certains outils employés pour trouver ce type de formules sont appelés opérateurs de vertex. La méthodologie employée dans ma thèse est très similaire à ce type de techniques comme souligné par Johnson [Joh18]. Il serait intéressant d’uniformiser les notations entre les deux méthodes afin de pouvoir les comparer et d’établir un langage commun.

En effet, dans l’article de Rains–Warnaar, un (q, t) -analogue de la formule de Nekrasov–Okounkov pour le type \tilde{A} est obtenu en utilisant des identités déformées de type Cauchy sur des partitions dont le diagramme de Ferrers est contenue dans un rectangle. Les paramètres q et t désignent ici respectivement la taille du bras d’une équerre et de la jambe. Soit s une case d’une partition. Ce travail est en lien avec un article de Bouttier–Chapuy–Corteeel [BCC17], dans lequel un dictionnaire semble être déjà construit. Ce dernier article utilise un formalisme d’opérateur de vertex sur des modèles de particules et de partitions entrelacées. Il serait intéressant d’étudier les connexions avec les modèles de particules afin de déterminer en particulier à quel niveau ces travaux se situent dans le cadre des hiérarchies intégrables.

Dans [IKV09] Iqbal–Kozçaz–Vafa définissent un opérateur de vertex raffiné qui permet à Iqbal–Kozçaz–Shabbir d’obtenir dans [IKS10] non seulement les q -analogues des formules de Nekrasov–Okounkov, mais aussi des formules plus générales incluant les longueurs de bras et de jambes des partitions considérées, qui sont en fait des cas limites des (q, t) -analogues, et qui semblent être des spécialisations des polynômes de Jack. Là encore, une première étape serait d’effectuer un vrai travail d’uniformisation du cadre théorique, en établissant un dictionnaire entre l’article de Carlsson–Rodriguez Villegas [CRV18] et celui d’Iqbal–Kozçaz–Vafa. Si le dictionnaire s’avère efficace cela devrait permettre de retrouver des résultats déjà connus en type \tilde{A} et d’obtenir des formules similaires pour tous les autres types, en adoptant une méthodologie similaire à celle que j’ai développé dans ma thèse.

Par ailleurs, la combinatoire des partitions intervient dans l’étude par l’angle de la géométrie algébrique de problèmes issus de la physique théorique. C’est par exemple l’approche proposée par les travaux mentionnés précédemment mais également dans la recherche de Szendrői. Un autre objectif serait de mieux comprendre les liens entre mes résultats de thèse et les schémas de Hilbert. Un point de départ pourrait être un article de Gyenge–Némethi–Szendrői [GNS18] qui étudie la géométrie et la topologie de schémas de Hilbert de points sur certaines surfaces en lien avec des sous-groupes de types A ou D .

\mathcal{P} set of all integer partitions. 3

Δ set of all boxes on the main diagonal of the Ferrers diagram of a partition. 3

d_λ length of the side of the Durfee square, alternatively cardinal of Δ . 3

\mathcal{SC} set of all self-conjugate partitions. 3

\mathcal{DD} set of all doubled distinct partitions. 3

\mathcal{DD}' set of all partitions whose conjugate is a doubled distinct partition. 4

$\mathcal{H}(\lambda)$ multiset of all hook lengths of the partition λ . 4

$\mathcal{H}_t(\lambda)$ multiset of all hook lengths divisible by an integer t . 4

g parameter arising in Macdonald identities, can also be seen as equal to h^\vee for all types but for type \tilde{C}_t where $g = 2h^\vee$. 8

$\mathcal{DD}_{(g)}$ set of g -core doubled distinct partitions. 16, 36

ψ bijection between partitions and binary bi-infinite words. 16

Φ_t Littlewood decomposition. 18

ϕ Garvan–Kim–Stanton bijection between t -cores and vectors of integers. 20

\mathcal{DD}'_{2t-1} set of all conjugate doubled distinct partitions whose $(2t-1)$ -quotient is empty but the $2t-1$ -th component whose Ferrers diagram is a $(a-1) \times a$ rectangle, where a is an integer. 23

\mathcal{DD}'_{2t} set of all conjugate doubled distinct partitions whose $(2t)$ -quotient is empty but the $2t$ -th component whose Ferrers diagram is a $(a-1) \times a$ rectangle, where a is an integer. 23

\mathcal{DD}'_{2t-2} set of all conjugate doubled distinct partitions whose $(2t-2)$ -quotient is empty but the $2t-2$ -th component whose Ferrers diagram is a $(a-1) \times a$ rectangle, where a is an integer and the $t-1$ -th component's Ferrers diagram is a square. 24

$\mathbb{I}_{m,M}^{g,+}$ set of all integers lower than M and whose residue \pmod{g} is m . 25

$\mathbb{I}_{m,M}^{g,-}$ set of all integers greater than m and whose residue \pmod{g} is M . 25

\mathcal{DD}'_g set of conjugate doubled distinct partitions whose g -quotient is almost empty but for one component. [33](#)

$\mathcal{SC}_{(g)}$ set of g -core self-conjugate partitions. [35](#)

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Quelques résultats combinatoires autour de la décomposition de Littlewood

Résumé. Cette thèse s'intéresse à des objets de combinatoire énumérative et plus particulièrement aux partitions d'entiers.

Les partitions et les tableaux de Young sont des objets combinatoires qui jouent un rôle important en théorie des représentations et des fonctions symétriques. La correspondance de Robinson–Schensted–Knuth est une correspondance bijective entre des permutations et des tableaux de Young dont le nombre est donné par une formule d'équerres.

On s'intéresse dans ce manuscrit principalement à l'obtention de formules d'équerres par le biais de la décomposition de Littlewood, qui est une bijection entre partitions d'entiers et une paire constituée d'une partition bien spécifique, appelée t -core et d'un t -uplet de partitions.

Dans un premier temps, nous nous intéressons aux partitions t -cores et nous montrons comment la correspondance bijective entre partitions d'entiers et mots binaires bi-infinis permet non seulement de réécrire les identités de Macdonald, qui peuvent être vues comme une généralisation des déterminants de Weyl aux systèmes de racines affines infinis, mais aussi d'obtenir des q -déformations de formules de type Nekrasov–Okounkov. Les formules de Nekrasov–Okounkov sont des identités de type "somme=produit" qui interviennent dans divers autres domaines tels que la théorie des représentations, la théorie de jauge ou encore la combinatoire des partitions d'entiers. Ces formules, de part leur nature, font intervenir naturellement des partitions t -core.

Dans un second temps, nous nous intéressons à des généralisations de formules d'équerres grâce à la décomposition de Littlewood dans le cas particulier des partitions auto-conjuguées, à savoir des partitions dont le diagramme de Ferrers (en notation à la française) est symétrique par rapport à la première bissectrice du plan. Cela permet d'obtenir des théorèmes de multiplication-addition pour les partitions auto-conjuguées, dans la lignée des travaux de Han et Ji.

Mots-clés : Identités de Macdonald, partitions d'entiers, partitions t -cores, formules de Nekrasov–Okounkov

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