Contribution to formal foundations of argumentation and case-based reasoning

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Abstract

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by Vivien BEUSELINCK

Argumentation, a sub-field of artificial intelligence, is a reasoning approach that justifies claims by interacting arguments. One of its key steps is evaluation of argument strength using formal methods, called semantics.

This thesis contributes to the understanding and development of the theoretical foundations of semantics. It's contribution are fourfold.

First , we investigated the critical notion of self-attacking arguments, proposed rationality postulates that describe how to deal wit them and introduces novel semantics that satisfy them.

Second, we characterized various semantics in terms of series, providing the mathematical counterparts of those semantics.

The third part of the thesis tackled the issue of comparing the plethora of existing semantics. We have shown that rationality postulates are not sufficient for a fair comparison of semantics, then we introduced the novel notion of equivalence which is based on the ranking induced by semantics. We compared most of existing semantics, and characterized a whole class of equivalence.

The last part of the thesis illustrates semantics in the context of the case-based reasoning (CBR). We started by laying the foundations of CBR by providing a set of principles that a model would satisfy, and proposed an argumentation model that satisfies most of them.

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This thesis is dedicated to René Poix, a retired Usinor employee and my grandfather.

Chapter 1

Introduction

As a sub-field of Artificial Intelligence, argumentation is a reasoning approach based on the justification of claims by arguments. It has been used for solving different problems including inconsistency handling (Besnard and Hunter, 2001; Amgoud and Cayrol, 2002), decision making (Amgoud and Prade, 2009; Zhong et al., 2019, case-based reasoning (Cyras, Satoh, and Toni, 2016; Zheng, Grossi, and Verheij, 2020), and negotiation (Dimopoulos, Mailly, and Moraitis, 2019). See (Rahwan and Simari(eds.), 2009) for more applications.

An argumentation-based system consists of a (flat or weighted) graph and an evaluation method, called a semantics. The nodes of the graph are *arguments* and its edges represent *attacks* between them. The graph is weighted when arguments are assigned basic weights and flat otherwise. The semantics is a formal method for evaluating the *strength* of every argument in the graph.

A great number of semantics have been proposed in the literature. They can roughly be classified into three families: *extension-based*, *gradual* and *ranking*. Initiated by Dung, 1995, the former look for sets of arguments that can be jointly accepted. Then, a dialectical status or strength is assigned to each argument according to its membership in the extensions. Introduced by Cayrol and Lagasquie-Schiex, 2005a, gradual semantics focus directly on individual arguments, and ascribe to each of them a value taken from an ordered scale representing its strength.
Ranking semantics have been introduced in (Amgoud and Ben-Naim, 2013), they rank arguments of a flat or weighted graph according to their strength. They thus return a preordering on the set of arguments.

1.1 Handling self-attacking arguments

Generally, a conflict occurs between two distinct arguments. However, it may also happen that an argument conflicts with itself, and such an argument is called self-attacking. Self-attacking arguments seem anecdotal at first sight;[1] however, the discussion on how to deal with them is subject of debate amongst argumentation scholars. There exist examples in the literature attempting to formally represent certain aspects with these arguments, such as the representation of the lottery paradox (Pollock, 1991). However, one quickly understands that the problem of representing the self-attacking arguments is mainly linked to the different choices made to formally represent an argument and the attacks between the arguments. This distinction can

[1] Bodanza and Tohmé (Bodanza and Tohmé, 2009) claim that there is a lack of "indisputably sound examples" concerning this type of arguments

be seen, for example, when comparing the approaches used in deductive argumentation and in abstract argumentation. As mentioned by Baumann and Woltran (Baumann and Woltran, 2016), in classical logic-based frameworks, self-attacking arguments do not occur at all (Besnard and Hunter, 2001), while other argumentation systems like ASPIC (Modgil and Prakken, 2014) allow such arguments. Within the abstract setting, several methods have been defined by proposing to deal with them directly (Bodanza and Tohmé, 2009; Baumann, Brewka, and Ulbricht, 2020b; Baumann, Brewka, and Ulbricht, 2020a; Dauphin, Rienstra, and Torre, 2020) or indirectly (e.g. when dealing with odd-length cycles because a self-attack is the smallest odd-length cycle) (Baroni and Giacomin, 2003). These methods essentially concern extension-based semantics.

In the context of ranking-based and gradual semantics, little research was conducted to find out how self-attacking arguments should be dealt with and what is the impact they have on the acceptability of other arguments. Existing studies are essentially done through the principle-based studies of these semantics. Indeed, defining and studying principles drew attention of many scholars in this area. Consider Equivalence, which is one of the well-known principles, stating that the acceptability degree of an argument should only depend on acceptability degrees of its direct attackers and observe the argumentation graph from Figure 3.1. Equivalence implies that a and b should be equally acceptable because a and b are both directly attacked by the same argument. However, this is debatable, since the intuition behind a self-attacking argument is that it is inconsistent in one way or another so we would tend to accept b being attacked by a (which is self-attacking) rather than accepting a.

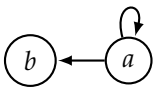


FIGURE 1.1: An argumentation graph with two arguments (a attacks itself and b) showing that Equivalence and Self-Contradiction are incompatible.

Note that, under all semantics returning conflict-free extensions, a self-attacking argument is always rejected, i.e. it does not belong to any extension. Also, regarding the ranking-based and gradual semantics, it was pointed out that it would be natural to attach the worst possible rank to self-attacking arguments (Matt and Toni, 2008). Furthermore, two principles were defined to formalise this intuition. The first one is called Strong Self-Contradiction, and was introduced by Matt and Toni (Matt and Toni, 2008). It says that the acceptability degree of an argument must be minimal if and only if that argument is self-attacking. The second principle, called Self-Contradiction, was introduced by Bonzon et al. (Bonzon et al., 2016) and states that each self-attacking argument is strictly less acceptable than each non self-attacking argument. Consider the argumentation graph illustrated in Figure 3.1 again and note that, under every semantics that satisfies Self-Contradiction, b is strictly more acceptable than a. This example shows that Equivalence and Self-Contradiction are not compatible, i.e. there exists no semantics that satisfies both of them.

To the best of our knowledge, there exists only one semantics proposed in (Matt and Toni, 2008) that satisfies Self-Contradiction and Strong Self-Contradiction. However, this semantics has a limitation that makes it inapplicable in practice. Namely, as noted by Matt and Toni themselves, as the space used to calculate the scores grows

exponentially with the number of arguments, even with the optimisation techniques they used it did not scale to more than a dozen of arguments.

The research objective of the first chapter of the thesis is to study the under-explored family of semantics that satisfy Strong Self-Contradiction. Our goals are thus to identify which principles are (in)compatible with Strong Self-Contradiction and to define a new argumentation semantics, called `nsa` (no self-attacks), that satisfies Strong Self-Contradiction as well as a maximal number of compatible principles.

After introducing the formal setting and recalling the existing principles from the literature:

- We prove the incompatibilities between some of the principles, and identify a maximal set of principles that contains (Strong) Self-Contradiction;
- We introduce an iterative algorithm in order to define a new semantics and prove that it always converges. The acceptability degree of each argument with respect to `nsa` is then defined as the limit of the corresponding sequence;
- We provide a characterisation of `nsa`, i.e. a declarative (non-iterative) definition and show that the two are equivalent: each semantics satisfying the declarative definition coincides with `nsa`;
- We check which principles are satisfied by `nsa` and compare it with the M&T semantics (Matt and Toni, 2008) and the h-categorizer semantics (Besnard and Hunter, 2001) in terms of principle satisfaction;
- We formally prove that no semantics can satisfy a strict super-set of the set of principles satisfied by `nsa`;
- We experimentally show that `nsa` is computationally efficient and compare it with the M&T semantics and the h-categorizer semantics. The results confirm the hypothesis that the M&T semantics does not scale.

1.2 Mathematical Foundations of semantics

There exists a plethora of semantics in the literature. Most of them have been motivated by example, or by adapting existing semantics from others domains like logic programming. In the last decade, several works have been done for understanding the foundations of semantics. They introduced formal properties, called principles, that a reasonable semantics would satisfy. While these principles shed light on the underpinnings of existing semantics, they do not tell much on the values assigned by semantics to argument. The second chapter of the thesis tackled this issue, and characterizes some existing numerical series like Fibonacci and the exponential series.

1.3 Equivalence of semantics

As said before, comparing existing semantics has been a hot topic in recent years. A dominant approach consists in identifying properties (called *principles*) that semantics may satisfy, then analysing every semantics against them. (Baroni and Giacomin, 2007) proposed several principles that (Torre and Vesic, 2017) used for comparing all the existing extension semantics. (Amgoud and Ben-Naim, 2016) introduced another

list of principles and used it for comparing some existing gradual/extension semantics. While these studies revealed conceptual differences between the analysed semantics, the messages they convey are not clear. Namely, the following questions remain unanswered:

- Are semantics satisfying the same principles similar, i.e. they provide the same evaluation of arguments?
- Are semantics following different principles incompatible, i.e. they may provide contradictory evaluations?

More generally, when are two semantics similar? We argue that to be similar, two semantics should agree not only on the evaluation of every individual argument but also on the ranking of arguments wrt their strengths. The first agreement depends on the principles followed by the two semantics. For instance, if both semantics satisfy the *maximality* principle from ((Amgoud and Ben-Naim, 2016)), then they would assign the highest strength to a non-attacked argument. However, satisfying the same principles does not guarantee agreement on the ranking of arguments. Consider for instance the two gradual semantics `Mbs` ((Amgoud et al., 2017)) and `EMbs` ((Amgoud and Doder, 2019)). They satisfy exactly the same set of principles. However, as we will see in the chapter, they may provide contradictory rankings of pairs of arguments when graphs are weighted. They are thus not similar and decision systems using them may make contradictory recommendations. Consider for instance the case of a committee which recruits a new researcher, and assume that the two candidates are supported respectively by the arguments a and b. Assume also that `Mbs` declares a as stronger than b and `EMbs` proposes the opposite ranking. Note that according to the semantics that is considered, the committee may make different recruitment's.

To sum up, the existing comparisons of semantics are incomplete as they focused only on the first requirement for similarity. This part of the thesis completes them by investigating the second requirement on rankings. Its contributions are threefold: First, it introduces three novel relations between semantics based on their rankings of arguments: *weak equivalence*, *strong equivalence* and *refinement*. They state respectively that two semantics do not disagree on their strict rankings, the rankings of the semantics coincide, and one semantics agrees with the strict comparisons of the second, furthermore it breaks some of its ties. Second, it investigates the properties of the three relations and their links with existing principles of semantics. Third, it studies the nature of relations between most of the existing semantics when dealing with flat or weighted graphs.

1.4 Case-base reasoning

Case-based reasoning (CBR) is an experience-based approach to solving problems. It uses stored cases describing similar prior problem-solving episodes and adapts their solutions to fit new needs (or new cases). For example, a car dealer would guess the price of a given car by comparing its characteristics with those of cars that have been sold. This form of reasoning has been used in the literature for solving various practical problems including some in the medical (eg. (Perez et al., 2021; Smiti and Nssibi, 2020; Schnell, 2020)) and legal (eg. (Ashley, 2011; Atkinson and Bench-Capon, 2005)) domains.

Several works have been devoted to modeling CBR, and various approaches can be distinguished including logic-based (Zheng, Grossi, and Verheij, 2020; Dubois et al., 1997) and argumentation-based (Paulino-Passos and Toni, 2021; Prakken et al., 2013) approaches (see (Richter and Weber, 2013; Hüllermeier, 2007; Aamodt and Plaza, 1994) for surveys). However, despite its popularity, there are few works on foundations, or properties, that may underlie CBR models. Foundations are important not only for a better understanding of case-based reasoning in general, but also for clarifying the basic assumptions underlying models, comparing different models, and also for comparing case-based reasoning with other kinds of reasoning like defeasible reasoning.

The last chapter of the thesis bridges this gap. It starts by analysing the basic assumption behind case-based reasoning, namely "*the more similar the cases, the more similar their outcomes*". It discusses three independent notions that capture (in different ways) the assumption.

Chapter 2

Argumentation Frameworks

2.1 Introduction

Humans engage in argumentation in almost all communications. They advance arguments and counter-arguments to support or refute claims. Let us consider the following simple dialogue between a parent and her child :

Parent (a_1) : Your grades are too low, your games distract your attention, and they make you lose time.

Child (a_2) : My grades have increased this semester, I am more efficient after playing.

Parent (a_3) : They are not increasing fast enough.

Here, the parents justifies the low grades by the fact that the child is distracted by games. The child disagrees and claims that his grades are better when playing games.

Argumentation is a reasoning process in which arguments are built and evaluated in order to increase or decrease the acceptability of a standpoint (Perelman and Olbrechts-Tyteca, 1958).

The most abstract and general argumentation framework in the literature is the one proposed by Dung (Dung, 1995). It takes as input a set of arguments and a binary relation encoding attacks between arguments. The framework is abstract since it does not specify neither the structure nor the origin of the two components. Thus, it can be instantiated in different ways.

It is general since no particular constraints are imposed on arguments and attacks.In the literature, arguments may have a basic weight which may represent the certainty degree of its premises (Benferhat et al., 1993) or the degree of reliability of its source (Villata et al., 2011) or votes (Leite and Martins, 2011).

Arguments may also be supported by other arguments. In this thesis, we will not consider this neither, and focus on weighted argumentation frameworks, which we call also weighted argumentation graphs.

Definition 1 (AG). *An* argumentation graph *(AG) is a tuple* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$*, with* $\mathcal{A}$ *is a finite set of arguments,* $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *is a binary relation representing attacks between*

arguments, $\sigma : \mathcal{A} \implies [0,1]$ *is a weighting on* $\mathcal{A}$. $\mathbf{G}$ *is* flat *if* $\sigma \equiv 1$[1] *and* weighted *otherwise. Let* AG *denote the set of all argumentation graphs.*

for $a, b \in \mathcal{A}$, $(a,b) \in \mathcal{R}$ (or aRb) means that a attacks b, we can abuse notation and say that a set E of arguments attacks an argument a if and only if it contains an argument b which attacks a. The argumentation graph corresponding to the dialogue between the parent and the child is depicted in the figure below where we assume that $\sigma(a_1) = \sigma(a_2) = \sigma(a_3)$ Attacks express conflicts between arguments. In

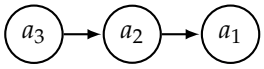


FIGURE 2.1: Argumentation framework representing the child example

some cases, an argument may attack itself.

Definition 2. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG. *An attack* $(a,b) \in \mathcal{R}$ *is called self-attack if and only if* $a = b$.

2.2 Semantics

Whatever the problem to solve, an argumentation process follows generally from steps: to justify claims by arguments, identify relations between arguments, evaluate the arguments and define outputs.

The last step depends on the results of the evaluation. For example, an inference system draws formulas that are justified by what is qualified at the evaluation step as "strong" arguments. Evaluation of arguments is thus crucial as it impacts the outcomes of argument-based systems. Consequently, a plethora of methods, called semantics, have been proposed in the literature. They can be organized into three families according to the type of their outcome : Extension semantics initiated by (Dung, 1995), Gradual semantics initiated by (Cayrol and Lagasquie-Schiex, 2005b) and Ranking semantics introduced in (Amgoud and Ben-Naim, 2013). They differ in the type of outcomes they produce. Extension semantics return sets of arguments, gradual semantics ascribe a (numerical or qualitative) value to every argument, and ranking semantics return a total preorder on the set of arguments.

Definition 3 (Semantics)**.** *A semantics is a function* π *that assigns to every* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$,

- *a set* $Ext^{\pi}_{\mathbf{G}} \subseteq P(\mathcal{A})$ *(Extension semantics)*
- *a weighting* $\mathtt{Deg}^{\pi}_{\mathbf{G}} : \mathcal{A} \to \mathcal{D}$ *(Gradual semantics)*
- *a preorder* $\succeq^{\pi}_{\mathbf{G}} \subseteq \mathcal{A} \times \mathcal{A}$ *(Ranking semantics)*

where $P(\mathcal{A})$ *stands for the powerset of* $\mathcal{A}$ *and* $\mathcal{D}$ *is a totally ordered scale.*

In what follows, we recall the main semantics of each family.

[1] $\sigma \equiv 1$ means $\forall a \in \mathcal{A}, \sigma(a) = 1$.

2.2.1 Extension-based semantics

Initiated in (Dung, 1995), extension semantics identify arguments that are acceptable for (thus can be accepted by) a rational agent. The following particular definition was used:

> An argument is acceptable for a rational agent if it can be defended against all attacks on it.

Dung proposed different ways of defining formally this notion of acceptability. They are all based on the same idea: identifying sets of arguments, called extensions, that defend their elements against all attacks. Each extension represents an alternative set of acceptable arguments. Extension semantics are grounded on three crucial notions: conflict-freeness, defence and extensions. In his paper, Dung considered flat graphes $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$ ie where $\sigma \equiv 1$.

Definition 4. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$ *and a subset* $\mathcal{S} \subseteq \mathcal{A}$.

- *We say that* $\mathcal{S}$ *is conflict-free if* $\nexists a, b \in \mathcal{S}$ *such that* $(a, b) \in \mathcal{R}$.
- *We say that* $\mathcal{S}$ *defends an argument* a *if* $\forall b \in \mathcal{A}$ *such that* $(b, a) \in \mathcal{R}$, $\exists c \in \mathcal{S}$ *such that* $(c, b) \in \mathcal{R}$, *and define* $\mathcal{F}(\mathcal{S}) = \{a \in \mathcal{A} \mid \mathcal{S} \text{ defends } a\}$.

We recall below the extension semantics proposed in (Dung, 1995).Interested readers can refer to (Torre and Vesic, 2017) for a complete presentation of all three extensions.

Definition 5. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$ *and* $\mathcal{S} \subseteq \mathcal{A}$ *is conflict free.*

- $\mathcal{S}$ *is* complete *iff* $\mathcal{S}$ *is conflict-free and and* $\mathcal{S} = \{a \in \mathcal{A} | \mathcal{S} \text{ defends } a\}$. $\mathcal{S}$ *defends all its elements and contains any argument it defends.*
- $\mathcal{S}$ *is a* preferred extension *if and only if* $\mathcal{S}$ *is a maximal for set inclusion complete extension.*
- $\mathcal{S}$ *is a* stable extension *if and only if* $\forall a \in \mathcal{A} | \mathcal{S}$, $\exists b \in \mathcal{S}$ *such that* $(b, a) \in \mathcal{R}$ *ie if and only if it attacks any arguments in* $\mathcal{A} | \mathcal{S}$
- $\mathcal{S}$ *is a* grounded extension *if it is a minimal for set inclusion complete extension.*
- $\mathcal{S}$ *is an* ideal extension *if and only if it is a maximal (wrt set inclusion) admissible set that is subset of every preferred extension.*

In (Dung, 1995), the grounded extension if defined using a characteristic function as follows.

Definition 6. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $\mathcal{S} \subseteq \mathcal{A}$ *is conflict free.* $\mathcal{S}$ *is the* grounded extension *if and only if* $\mathcal{S}$ *is the least (with respect to set inclusion) complete extension. It is the least fixpoint of the characteristic function* $\mathcal{F} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ *where for* $S \subseteq \mathcal{A}$,

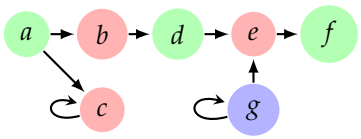
$$\mathcal{F}(S) = \{a \in \mathcal{A} \mid S \text{ defends } a\}.$$

Let $\texttt{Gr}$ *denote the grounded extension of* $\mathbf{G}$

Property 1. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *such that* $\mathcal{A}$ *is finite. The grounded extension of* $\mathbf{G}$ *is :*

$$\mathcal{S} = \cup \mathcal{F}^{i \geq 0}(\emptyset)$$

Example 1.



- $\mathcal{F}^1(\emptyset) = \{a\}$
- $\mathcal{F}^2(\emptyset) = \mathcal{F}(\mathcal{F}^1(\emptyset)) = \{a, d\}$
- $\mathcal{F}^3(\emptyset) = \mathcal{F}(\mathcal{F}^2(\emptyset)) = \{a, d, f\}$

Notation 1. *Throughout the thesis, for any $\mathcal{G} \in$ AG, $\mathcal{GR}(\mathcal{G})$ denotes the grounded extension of $\mathcal{G}$ and $Ext^x_{\mathbf{G}}$ the set of extensions of $\mathbf{G}$ under semantics x where $x \in \{c, p, s, g\}$ and c, p, s, g stand for complete, preferred, stable and grounded respectively.*

Once extension are identified, an acceptability status is assigned to every argument as follows (Baroni, Giacomin, and Guida, 2005; Cayrol and Lagasquie-Schiex, 2005a; Grossi and Modgil, 2015)) :

Definition 7. *Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in$ AG and $a \in \mathcal{A}$.*

- *a is sceptically accepted if it belongs to all extensions,ie a is sceptically accepted if $a \in \bigcup_{E_i \in Ext^x_{\mathbf{G}}} E_i$*
- *a is credulously accepted it belongs to some but not all extensions. Ie a is credulously accepted if $\exists E_i, E_j, i \neq j$, such that $a \in E_i$, $a \notin E_j$.*
- *a is undecided iff $a \notin \bigcup_{E_i \in Ext^x_{\mathbf{G}}} E_i$ and $\forall E_i \in Ext^x_{\mathbf{G}}$,$E_i$ does not attack a.*
- *a is rejected if it does not belong to any extension ie if only if $a \notin \bigcup_{E_i \in Ext^x_{\mathbf{G}}} E_i$ and $\exists E_i \in Ext^x_{\mathbf{G}}$ such that E_i attack a.*

with $x \in \{c, p, s, g\}$.

Example 2. *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework with $\mathcal{A} = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{R} = \{(b, a), (c, b), (a, a), (d, c), (e, f), (f, g), (g, e), (e, a), (h, a)\}$. The graphical representation of the framework is shown below.*

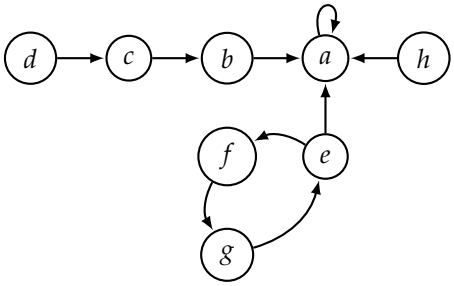


FIGURE 2.2: Argumentation framework representation

This graph contains one self-attack $(a, a) \in \mathcal{R}$. Indeed, the three arguments f, e, g form a cycle. In figure 2.2, $\{b, d, h\}$ is an example of a complete extension, as it defends all the arguments in the set $\{b, d, h\}$, $\{b, d, h\}$ is also an example of a

$f_{comp}(x_1, x_2) = x_1(1 - x_2)$	$g_{sum}(x_1, ..., x_n) = \sum_{i=1}^{n} x_i$
$f_{exp}(x_1, x_2) = x_1 e^{-x_2}$	$g_{sum,\alpha}(x_1, ..., x_n) = (\sum_{i=1}^{n}(x_i)^{\alpha})^{\frac{1}{\alpha}}$
$f_{frac}(x_1, x_2) = \frac{x_1}{1+x_2}$	$g_{max}(x_1, ..., x_n) = \max\{x_1, ..., x_n\}$

TABLE 2.1: Examples of functions f and g

preferred extension, an ideal extension and the grounded. There is no example of a stable extension in Figure 2.2. Indeed, arguments e, f, and g are not attacked by $\{b, d, h\}$. In order for $\{b, d, h\}$ to be stable, it would be necessary to add, for example, $\{(b, e), (b, f), (b, g)\}$ to $\mathcal{R}$.

2.2.2 Gradual semantics

Introduced for the first time in (Cayrol and Lagasquie-Schiex, 2005a), gradual semantics focus on individual arguments. They assign a value from a ordered scale to every argument of an argumentation graph. The value represents the strength of the argument ie, how robust is the argument against attacks on it. For any argumentation graph, $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \mathtt{AG}$, any argument $a \in \mathcal{A}$, the value (or strength) of a under a gradual semantics π is defined as follows:

$$\mathtt{Deg}^{\pi}_{\mathbf{G}}(a) = f(g(\mathtt{Deg}^{\pi}_{\mathbf{G}}(b_1), ..., \mathtt{Deg}^{\pi}_{\mathbf{G}}(b_n)))$$

where $b_1 \rightarrow b_n$ are the attackers of a, ie $\{x \in \mathcal{A} | (x, a) \in \mathcal{R}\} = \{b_1, .., b_n\}$. g is an aggregation function that evaluates how strongly a is attacked and f is an influence function that takes into account the strength of attacks. Examples of functions that are studied in the literature are given in the table below : It is worth mentioning that gradual semantics do not inform about the acceptability of arguments, ie, which argument to accept. It identify such arguments a further step is needed. For instance, one may accept any argument whose strength is greater than a given threshold. Extension semantics provide as output accepted arguments. However, their extension of argument strength is quite weak as they use a four-valued scale (sceptically accepted, credulously accepted, undecided, rejected). Hence, gradual semantics provide a finer-grained evaluation of individual arguments.

Several gradual semantics have been proposed in the literature. They differ in the way they consider attackers. Some semantics focus only on the strongest attacker and get rid of the others, others take into account all attackers, and finally some semantics focus on the number and quality of attackers.

In what follows, we recall the semantics that have been proposed for weighted argumentation graphs. The thirst two semantics... The first two semantics, Max-based and Euler-based, consider that the strongest attack is the sole decision-maker.

Definition 8 (Max-based). *((Amgoud et al., 2017)) Let $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ be an argumentation graph. The Max-based semantics (*Mbs*) is a gradual semantics such that $\forall a \in \mathcal{A}$:*

$$\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1 + \max\limits_{b \mathcal{R} a} \mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(b)}$$

Note that if $\mathbf{G}$ *is flat ie* $\sigma \equiv 1$*:*

$$\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \frac{1}{1 + \max_{b\mathcal{R}a} \mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(b)}$$

Note that Max-based uses f_{frac} as influence function and g_{max} as aggregation function. Euler-based (Amgoud and Doder, 2019) uses g_{max}, however its influence function is f_{exp}.

Example 3. *On the example 1,* $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(b) = 0.5$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(c) = 0.5$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(d) = \frac{2}{3}$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(e) = 0.6$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(f) = 0.625$ *and* $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(g) \approx 0.618$

Definition 9 (Euler-based)**.** *((Amgoud and Doder, 2019)) Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ *be an argumentation graph. The Euler-based (*EMbs*) semantics is a gradual semantics such that* $\forall a \in \mathcal{A}$*:*

$$\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = \sigma(a) \cdot e^{-\max_{b\mathcal{R}a} \mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b)}$$

Note that if $\mathbf{G}$ *is flat, ie* $\sigma \equiv 1$*, the strength of a is defined as follows:*

$$\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = e^{-\max_{b\mathcal{R}a} \mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b)}$$

Example 4. *On the example 1,* $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b) \approx 0.37$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(c) \approx 0.37$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(d) \approx 0.69$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(e) \approx 0.5$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(f) \approx 0.60$ *and* $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(g) \approx 0.57$

The third semantics, called *h*-categorizer (Hbs)(Besnard and Hunter, 2001), considers that all attacks have an impact on the strength of an argument. It uses for that purpose the g_{sum} aggregation function.

Definition 10 (*h*-categorizer semantics)**.** *((Besnard and Hunter, 2001)) Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ *be an argumentation graph. The h-categorizer semantics is a gradual semantics such that* $\forall a \in \mathcal{A}$*:*

$$\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1 + \sum_{b\mathcal{R}a} \mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(b)}$$

Note that if $\mathbf{G}$ *is flat, ie* $\sigma \equiv 1$*, the strength of a is:*

$$\mathtt{Deg}^{\mathtt{h}}_{\mathbf{G}}(x) = \frac{1}{1 + \sum_{y \in \mathtt{Att}_{\mathbf{G}}(x)} \mathtt{Deg}^{\mathtt{h}}_{\mathbf{G}}(y)}$$

Example 5. *On the example 1,* $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(b) = 0.5$, $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(c) \approx 0.41$, $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(d) = \frac{2}{3}$, $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(e) \approx 0.44$, $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(f) \approx 0.695$ *and* $\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(g) \approx 0.618$

The fourth semantics, Card-based (CBS), considers that the number of attackers plays a crucial role on argument strength indeed, the more an argument is attacked, the weaker it is.

Definition 11 (Card-based)**.** *((Amgoud et al., 2017)) Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ *be an argumentation graph. The Card-based semantics is a gradual semantics such that*

$$\mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1 + |\mathtt{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{b \in \mathtt{AttF}_{\mathbf{G}}(a)} \mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(b)}{|\mathtt{AttF}_{\mathbf{G}}(a)|}} \text{ where } \mathtt{AttF}_{\mathbf{G}}(a) = \{b \in \mathtt{Att}_{\mathbf{G}}(a) \mid \sigma(b) > 0\}$$

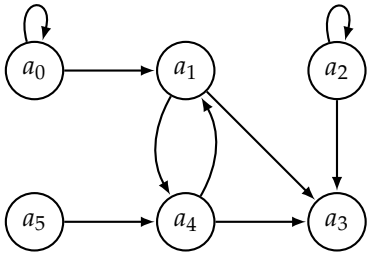


FIGURE 2.3: An argumentation graph example for M&T semantics

Note that if $\mathbf{G}$ *is flat, the strength of a :*

$$\mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(a) = \frac{1}{1 + |\mathtt{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{b \in \mathtt{AttF}_{\mathbf{G}}(a)} \mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(b)}{|\mathtt{AttF}_{\mathbf{G}}(a)|}} \text{ where } \mathtt{AttF}_{\mathbf{G}}(a) = \{b \in \mathtt{Att}_{\mathbf{G}}(a)\}$$

Example 6. *On the example 1,* $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b) \approx 0.33$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(c) \approx 0.27$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(d) \approx 0.429$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(e) \approx 0.29$, $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(f) \approx 0.436$ *and* $\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(g) \approx 0.33$

We can finally cite the gradual semantics introduced by Matt and Toni (Matt and Toni, 2008). It computes the acceptability degree of an argument using a two-person zero-sum strategic game. For an AG $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and an argument $x \in \mathcal{A}$, the set of strategies for the proponent is the set of all subsets of arguments that contain x: $S_P(x) = \{P \mid P \subseteq \mathcal{A}, x \in P\}$ and for the opponent it is the set of all subsets of arguments: $S_O = \{O \mid O \subseteq \mathcal{A}\}$. Given two strategies $X, Y \subseteq \mathcal{A}$, the set of attacks from X to Y is defined by $Y_{\mathcal{F}}^{\leftarrow X} = \{(x,y) \in X \times Y \mid (x,y) \in \mathcal{R}\}$. Then, the notion of degree of acceptability of a set of arguments w.r.t. another one used to compute the reward of a proponent's strategy is defined.

Definition 12 (Reward)**.** *Let* $\mathcal{F} = (\mathcal{A}, \sigma, \mathcal{R})$ *be an argumentation graph,* $x \in \mathcal{A}$ *be an argument,* $P \in S_P(x)$ *be a strategy chosen by the proponent and* $O \in S_O$ *be a strategy chosen by the opponent. The degree of acceptability of* P *w.r.t.* O *is :*
$\phi(P,O) = \frac{1}{2}\left[1 + f(|O_{\mathcal{F}}^{\leftarrow P}|) - f(|P_{\mathcal{F}}^{\leftarrow O}|)\right]$ with $f(n) = \frac{n}{n+1}$. *The reward of* P *over* O*, denoted by* $r_{\mathcal{F}}(P,O)$*, is defined by:*

$$r_{\mathcal{F}}(P,O) = \begin{cases} 0 & \text{iff } P \text{ is not conflict-free} \\ 1 & \text{iff } P \text{ is conflict-free and} \\ & \quad |P_{\mathcal{F}}^{\leftarrow O}| = 0 \\ \phi(P,O) & \text{otherwise} \end{cases}$$

Proponent and opponent have the possibility of using a strategy according to some probability distributions, respectively $p = (p_1, p_2, \ldots, p_m)$ and $q = (q_1, q_2, \ldots, q_n)$, with $m = |S_P(x)|$ and $n = |S_O|$. For each argument $x \in \mathcal{A}$, the proponent's expected payoff $E(x,p,q)$ is then given by $E(x,p,q) = \sum_{j=1}^{n}\sum_{i=1}^{m} p_i q_j r_{i,j}$ with $r_{i,j} = r_{\mathcal{F}}(P_i, O_j)$ where P_i (respectively O_j) represents the i^{th} (respectively j^{th}) strategy of $S_P(x)$ (respectively S_O). The proponent can expect to get at least $\min_q E(x,p,q)$, where the minimum is taken over all the probability distributions q available to the opponent. Hence the proponent can choose a strategy which will guarantee her a reward of $\max_p \min_q E(x,p,q)$. The opposite is also true with $\min_q \max_p E(x,p,q)$.

Definition 13 (M&T semantics). *The semantics M&T is a gradual semantics that assigns a score to each argument $x \in \mathcal{A}$ in $\mathcal{F}$ as follows:*

$$\mathtt{Deg}^{\mathtt{MT}}_{\mathtt{G}}(x) = \max_{p} \min_{q} E(x,p,q) = \min_{q} \max_{p} E(x,p,q)$$

Example 7. *Let us apply the semantics M&T on the argumentation graph 2.3, we obtain the following acceptability degrees :* $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_0) = 0.618$, $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_1) = 0.495$, $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_2) = 0.618$, $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_3) = 0.398$, $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_4) = 0.401$ *and* $\mathtt{Deg}^{\mathtt{h}}_{\mathtt{G}}(a_5) = 1$.

Definition 14 (Trust-based). *Let $\mathcal{F} = (\mathcal{A}, \sigma, \mathcal{R})$ be an argumentation graph. The Trust-based semantics is a gradual semantics such that $\forall a \in \mathcal{A}$:*
$\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(a) = \lim_{i \to +\infty} f_i(a)$, *where* $f_i(a) = \frac{1}{2} f_{i-1}(a) + \frac{1}{2} \min[w(a), 1 - \max_{b \mathcal{R} a} f_{i-1}(b)]$

Example 8. *On the example 1,* $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(b) = 0$, $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(c) = 0$, $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(d) = 1$, $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(e) = 0$, $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(f) = 1$ *and* $\mathtt{Deg}^{\mathtt{TB}}_{\mathtt{G}}(g) = 0.5$

Definition 15 (Iterative Schema). *Let $\mathcal{F} = (\mathcal{A}, \sigma, \mathcal{R})$ be an argumentation graph. The Iterative Schema semantics is a gradual semantics such that $\forall a \in \mathcal{A}$:*
$\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(a) = \lim_{i \to +\infty} f_i(a)$, *where* $f_i(a) = (1 - f_{i-1}(a)) \min(\frac{1}{2}, 1 - \max_{b \mathcal{R} a} \mathtt{f_{i-1}(b)}) + f_{i-1}(a) \max(\frac{1}{2}, 1 - \max_{b \mathcal{R} a} \mathtt{f_{i-1}(b)})$

Example 9. *On the example 1,* $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(a) = 1$, $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(b) = 0$, $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(c) = 0$, $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(d) = 1$, $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(e) = 0$, $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(f) = 1$ *and* $\mathtt{Deg}^{\mathtt{IS}}_{\mathtt{G}}(g) = 0.5$

Gradual semantics in a flat graph do not take into account the initial weight of an argument. Certain arguments may be given more importance initially compared to others. Therefore, a weighting is introduced, resulting in a weighted argumentation graph.

Let's note that it is also possible to enrich the argumentative graph by introducing weights on attacks or support relationship.

Exactly, one clear corollary of the notion of degree and value is the ability to compare the values assigned to arguments. This allows us to rank the arguments from the most relevant to the least relevant, which is the idea behind the next section. By comparing the degrees or values associated with each argument, we can establish a hierarchy of argument relevance, enabling a more nuanced and informative representation of the argumentation structure.

2.2.3 Ranking semantics

Ranking semantics were introduced in (Amgoud and Ben-Naim, 2013) with the aim of introducing graduality in acceptability, and thus to rank order arguments from the most to the least acceptable ones. The authors started by providing a list of properties (called principles) that a ranking semantics should satisfy, among which Void Precedence (VP) and Counter-Transitivity (CT). (VP) states that an argument that has no attackers is more acceptable than any attacked argument. (CT) states that an argument a should be at least as acceptable as an argument b if the attackers of b are at least as numerous and as acceptable as those of a. The authors proposed then two ranking semantics: Burden and Discussion based. Propagation semantics proposed in (Bonzon et al., 2016) and the one from (Dondio, 2018) are other examples of ranking semantics. We recall below one semantics proposed in (Bonzon et al., 2016).

Its basic idea is to give some power to non-attacked arguments by ascribing initial values to arguments. Before introducing the semantics, let us recall the definition of lexicographical order.

Let $V = \langle V_1, ..., V_n \rangle$ and $V' = \langle V'_1, ..., V'_n \rangle$ be two vectors of reals numbers. $V >_{lex} V'$ iff $\exists i \leq n$ such that $V_i > V'_i$ and $\forall j < i, V_j = V'_j$. $V \geq_{lex} V'$ means it is not the case that $V' >_{lex} V$.

Definition 16. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle$, $v : \mathcal{A} \to \{\epsilon, 1\}$ *where* $\epsilon \in [0, 1]$ *and* $\forall a \in \mathcal{A}, v(a) = 1$ *if* a *is not attacked and* $v(a) = \epsilon$ *else. The value of* $a \in \mathcal{A}$ *at step* $i \in \mathbb{N}$ *is* $P_i(a)$ *such that :*

$$P_i(a) = \begin{cases} v(a) & \text{iff } i = 0 \\ P_{i-1}(a) + (-1)^i \sum_{(b,a) \in \mathbb{R}} v(b) & \text{else} \end{cases}$$

The propagation vector of a *is* $P(a) = \langle P_0(a), P_1(a), ... \rangle$. *For* $a, b \in \mathcal{A}$, a *is at least as acceptable as* b *denoted by* $a \succeq^P_{\mathbf{G}} b$ *iff* $P(a) \geq_{lex} P(b)$.

Example 10. *Consider the graph* $\mathbf{G}_1$ *from figure 2.1 and let* $\epsilon = 0.75$. *Hence,* $v(c) = 1$ *and* $v(a) = v(b) = 0.75$. *It is easy to check that* $P_0(c) = P_1(c) = 1$, $P_0(a) = 0.75$,$P_1(a) = 0$ *and* $P_0(b) = 0.75$,$P_1(b) = -0.25$. *Thus,* $c \succeq^P_{\mathbf{G}_1} a \succeq^P_{\mathbf{G}_1} b$

Ranking semantics provide several levels of acceptability. While this allows fine-grained comparisons of arguments, it may lead to move away from the essence of acceptability, which is predicting whether an argument can be accepted or not.

We will not delve into ranking semantics in this thesis; therefore, we stop here for their presentation and proceed directly to the section on principles.

2.3 Principles

Gradual semantics should adhere to a sound behavior. They should satisfy a list of necessary axioms in order to be considered valid. Similarly, characterization principles have been introduced to divide gradual semantics into different families. For instance, the axioms of quality, compensation, and cardinality separate the families of Max-based, h-categorizer, and Card-based semantics. These principles play a crucial role in distinguishing and categorizing the various types of gradual semantics based on their behavior and characteristics.

We recall below the list of principles proposed in (Amgoud et al., 2017) for semantics $\mathbf{S} \in \texttt{Sem}$.

The first one, a mandatory principle called Anonymity, states that the name of an argument should not impact its acceptability degree. We also introduce the notion of Isomorphism. This notion will helps us to define Anonymity.

Definition 17 (Isomorphism)**.** *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}', \sigma, \mathcal{R}' \rangle$ *be two argumentation framework. An isomorphism from* $\mathbf{G}$ *to* $\mathbf{G}'$ *is is a bijective function* f *from* $\mathcal{A}$ *to* $\mathcal{A}'$ *such that* $\forall a, b \in \mathcal{A}$, aRb *iff* $f(a)R'f(b)$

Principle 1 (Anonymity)**.** *A semantics* $\mathcal{S}$ *satisfies Anonymity iff, for any* $\texttt{AG}$, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}', \sigma', \mathcal{R}' \rangle \in \texttt{AG}$, *for any isomorphism* f *from* $\mathbf{G}$ *to* $\mathbf{G}'$, $\forall a \in \mathcal{A}, \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}'}(f(a))$.

The second one from (Amgoud and Ben-Naim, 2013) is for ranking semantics.

Example 11. *Consider the two argumentation frameworks depicted in the figure below. Anonymity ensures that the degree (or ranking relation for abstraction) between a and b is*

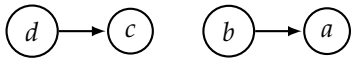


FIGURE 2.4: Example for Anonymity

the same as the one between c and d.

Independence says that the acceptability degree of an argument should be independent of unconnected arguments.

Principle 2 (Independance). *A semantics $\mathcal{S}$ satisfies independance iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}', \sigma', \mathcal{R}' \rangle \in$ AG *such that* $\mathcal{A} \cap \mathcal{A}' = \emptyset$, $\forall a \in \mathcal{A}$, $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G} \otimes \mathbf{G}'}(a)$.

Example 12. *Assume that the two graphs of Example 11 constitute a single argumentation framework. Then, independance ensures that the degree between a and b is still the same after the fusion of the two frameworks.*

Directionality states that the acceptability of argument x can depend on y only if there is a path from y to x.

Principle 3 (Directionality). *A semantics $\mathcal{S}$ satisfies Directionality iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}, \sigma, \mathcal{R}' \rangle \in$ AG *such that* $a, b \in \mathcal{A}$, $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$ *it holds that:* $\forall x \in \mathcal{A}$, *if there is no path from b to x, then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}'}(x)$.

Since an attack always weakens its target, the next postulate states that having attacked attackers is better than having non-attacked attackers (assuming the number of attackers is the same). In other words, being defended is better than not being defended. (Amgoud and Ben-Naim, 2013)

Notation 2. *Let $A = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ be an argumentation framework and $a \in \mathcal{A}$. We denote by $Def_A(a)$ the set of all defenders of a in A, that is, $Def_A(a) = \{b \in A | \exists c \in A, cRa$ and $bRc\}$.*

Neutrality states that an argument with an acceptability degree of 0 should have no impact on the arguments it attacks. One should not take into account an argument without any interest.

Principle 4 (Neutrality). *A semantics $\mathcal{S}$ satisfies Neutrality iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG, $\forall a, b \in \mathcal{A}$, *if* $\sigma(a) = \sigma(b)$, $\mathtt{Att}_{\mathbf{G}}(b) = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ *with* $x \in \mathcal{A} \setminus \mathtt{Att}_{\mathbf{G}}(a)$ *and* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) = 0$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Equivalence says that if two arguments have the same attackers, or more generally attackers of the same strength, they should have the same acceptability degree.

Principle 5 (Equivalence). *A semantics $\mathcal{S}$ satisfies Equivalence iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG, $\forall a, b \in \mathcal{A}$, *if* $\sigma(a) = \sigma(b)$ *and there exists a bijective function f from* $\mathtt{Att}_{\mathbf{G}}(a)$ *to* $\mathtt{Att}_{\mathbf{G}}(b)$ *s.t.* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(f(x))$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Maximality states that a non-attacked argument should have the highest acceptability degree.

Principle 6 (Maximality). *A semantics $\mathcal{S}$ satisfies Maximality iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a \in \mathcal{A}$, *if* $\texttt{Att}_{\mathbf{G}}(a) = \emptyset$, *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \sigma(a)$

Counting states that a non-zero degree attacker should impact the acceptability of the attacked argument.

Principle 7 (Counting). *A semantics $\mathcal{S}$ satisfies Counting iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a, b \in \mathcal{A}$, *if* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$ *and* $\texttt{Att}_{\mathbf{G}}(b) = \texttt{Att}_{\mathbf{G}}(a) \cup \{x\}$ *with* $x \in \mathcal{A} \backslash \texttt{Att}_{\mathbf{G}}(a)$ *and* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0$, *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Weakening says that the acceptability of an argument should be strictly lower than 1 if it has at least one attacker with a non-zero acceptability degree.

Principle 8 (Weakening). *A semantics $\mathcal{S}$ satisfies Weakening iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a \in \mathcal{A}$, *if* $\exists b \in \texttt{Att}_{\mathbf{G}}(a)$ *s.t.* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b) > 0$ *and* $\sigma(a) > 0$, *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) < \sigma(a)$.

Weakening Soundness states that if the acceptability degree of an argument is not maximal, it must be that it is attacked by at least one non-zero degree attacker.

Principle 9 (Weakening Soundness). *A semantics $\mathcal{S}$ satisfies Weakening Soundness iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, *if* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) < \sigma(a)$, *then* $\exists b \in \texttt{Att}_{\mathbf{G}}(a)$ *s.t* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b) > 0$.

Reinforcement states that the acceptability degree increases if the acceptability degrees of attackers decrease. If an attacker becomes weaker, it is reasonable to expect that its target gains in strength.

Principle 10 (Reinforcement). *A semantics $\mathcal{S}$ satisfies Reinforcement iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a, b \in \mathcal{A}$, *if i)* $\sigma(a) = \sigma(b)$, *ii)* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$ *or* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b) > 0$, *iii)* $\texttt{Att}_{\mathbf{G}}(a) \backslash \texttt{Att}_{\mathbf{G}}(b) = \{x\}$, *iv)* $\texttt{Att}_{\mathbf{G}}(b) \backslash \texttt{Att}_{\mathbf{G}}(a) = \{y\}$, *v)* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0$, *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Resilience states that no argument in an argumentation graph can have an acceptability degree of 0. It is certainly not a mandatory principle. The concept of resilience suggests that every argument, even a very weak one, should be given some importance; one can never completely eliminate an argument.

Principle 11 (Resilience). *A semantics $\mathcal{S}$ satisfies Resilience iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a \in \mathcal{A}$, *if* $\sigma(a) > 0$ *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$.

The concept of proportionality is relevant in the case of graphs with weights in argumentation. When two arguments receive the same attackers, but one argument had a higher initial weight, its degree of final acceptability should be higher. Proportionality ensures that the weights assigned to arguments play a significant role in determining their overall acceptability in the argumentation system. Arguments with higher initial weights carry more weight throughout the argumentation process, which reflects the idea that stronger or more significant arguments should have a greater impact on the final conclusions drawn from the debate.

Principle 12 (Proportionality). *A semantics $\mathcal{S}$ satisfies Proportionality iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \texttt{AG}$, $\forall a, b \in \mathcal{A}$, *if* $\sigma(a) > \sigma(b)$ *and* $\texttt{Att}_{\mathbf{G}}(a) = \texttt{Att}_{\mathbf{G}}(b)$, *then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

When two arguments have the same initial weight, but the set of attacks on one argument is a subset of the other, it implies that this second argument has a weaker weight. The additional attackers on the second argument must play a role in determining its final acceptability. The Monotony principle reflects the idea that arguments with more attacks against them should have a reduced acceptability in the argumentation system, as these additional attackers pose more challenges to their credibility and strength.

Principle 13 (Monotony). *A semantics $\mathcal{S}$ satisfies Monotony iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG, $\forall a, b \in \mathcal{A}$, *if* $\sigma(a) = \sigma(b)$ *and* $\mathtt{Att}(a) \subset \mathtt{Att}(b)$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) \geq \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

The last three principles are incompatible with each other. They exist in both gradual semantics and Extension-based semantics cases. Quality Precedence states that the greater the acceptability degree of the strongest attacker of an argument, the lower its acceptability degree.

Principle 14 (Quality Precedence). *A semantics $\mathcal{S}$ satisfies Quality Precedence iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG, $\forall a, b \in \mathcal{A}$, *if i)* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$ *and ii)* $\exists y \in \mathtt{Att}_{\mathbf{G}}(b)$ *s.t.* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x)$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

The second principle, called Cardinality Precedence states that the greater the number of direct attackers of an argument, the lower its acceptability degree.

Principle 15 (Cardinality Precedence). *A semantics $\mathcal{S}$ Cardinality Precedence iff, for any* AG, $\forall \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG, $\forall a, b \in \mathcal{A}$, *if i)* $\sigma(a) = \sigma(b)$ *ii)* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b) > 0$, *and iii)* $|\{x \in \mathtt{Att}_{\mathbf{G}}(a) \text{ s.t. } \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0\}| > |\{y \in \mathtt{Att}_{\mathbf{G}}(b) \text{ s.t. } \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > 0\}|$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) < \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Compensation states that several attacks from arguments with a low acceptability degree may compensate one attack from an argument with high acceptability degree. [2]

Principle 16 (Compensation). *A semantics $\mathcal{S}$ satisfies Compensation iff, for any* AG, $\exists \mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in$ AG *s.t for two arguments* $a, b \in \mathcal{A}$, *i)* $\sigma(a) = \sigma(b)$ *ii)* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$, *iii)* $|\{x \in \mathtt{Att}_{\mathbf{G}}(a) | \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0\}| > |\{y \in \mathtt{Att}_{\mathbf{G}}(b) | \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > 0\}|$, *iv)* $\exists y \in \mathtt{Att}_{\mathbf{G}}(b)$ *s.t.* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x)$ *and* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

One can note the existence of equivalent principles for ranking semantics, but they are not the subject of study in this thesis. This thesis is limited to weighted argumentation frameworks.

Proposition 1 ((Amgoud et al., 2017)). *The three following properties hold.*

- *Cardinality Precedence, Quality Precedence and Compensation are pairwise incompatible.*
- *Independence, Directionality, Equivalence, Resilience, Reinforcement, Maximality and Quality Precedence are incompatible.*
- *Cardinality Precedence (respectively Compensation) is compatible with all principles 1–11.*

2.4 Comparison of semantics

We can summarize the principles satisfied by the gradual and extension semantics in a table in order to have an overview of the principles satisfied by the different semantics.

[2] There are several version of this principle. We use the version that allows to clearly distinguish between the three cases (CP, QP, Compensation). Namely, each semantics satisfies *exactly* one of the three principles.

Principles	M&T	h-cat	Mbs	EMbs	Cbs	$\mathcal{GR}$	Stable	Preferred	Complete	IS	TB
Anonymity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Independence	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Directionality	✓	✓	✓	✓	✓	✓	×	✓	✓	✓	✓
Neutrality	×	✓	✓	✓	✓	✓	✓	×	×	✓	✓
Equivalence	×	✓	✓	✓	✓	×	×	×	×	✓	✓
Maximality	✓	✓	✓	✓	✓	×	×	×	×	×	✓
Weakening	✓	✓	✓	✓	✓	×	×	×	×	×	×
Counting	×	✓	×	×	×	×	×	×	×	×	×
Weakening Soundness	×	✓	✓	✓	✓	✓	×	×	×	✓	✓
Reinforcement	×	✓	×	×	✓	×	×	×	×	×	×
Resilience	×	✓	✓	✓	×	×	×	×	×	×	×
Cardinality Precedence	×	×	×	×	✓	×	×	×	×	×	×
Quality Precedence	×	×	✓	✓	×	×	×	×	×	×	×
Compensation	✓	✓	×	×	×	✓	✓	✓	✓	×	✓

TABLE 2.2: Principles satisfied by semantics (Amgoud, Doder, and Vesic, 2022).

2.5 Conclusion

Argumentation is a thriving field with numerous extensions, ranking semantics, and gradual semantics being introduced, along with principles that allow for their characterization and classification. However, there has been a lack of analysis regarding the ranking relations between these semantics. One possible research topic is to analyze the ranking behavior of extensions and gradual semantics. Which ones produce the same or similar rankings? Which ones complement the ranking of another? And which ones produce entirely different rankings? Will two semantics that satisfy the same principles be equivalent in terms of ranking ?

However, to successfully conduct this study, it is essential to ensure that we have the most comprehensive principles in place. Specifically, the handling of self-attacks is a potential research area that we will address in the next chapter.

Chapter 3

Dealing with Self-attacking Arguments

3.1 Introduction

We have seen previously that arguments may be conflicting, and thus may attack each other. The conflicts are represented by attacks between the arguments. Although in most cases a conflict occurs between two distinct arguments, sometimes an argument may conflict with itself. Such an argument is called a self-attacking argument. Self-attacking arguments seem anecdotal at first sight;[1] however, the discussion on how to deal with them is subject of debate amongst argumentation scholars. There exist examples in the literature attempting to formally represent certain aspects with these arguments, such as the representation of the lottery paradox (Pollock, 1991). However, one quickly understands that the problem of representing Self-attacking arguments is mainly linked to the different choices made to formally represent an argument and attacks between arguments. This distinction can be seen, for example, when comparing the approaches used in deductive argumentation and in abstract argumentation. As mentioned by Baumann and Woltran (Baumann and Woltran, 2016), in classical logic-based frameworks, self-attacking arguments do not occur at all (Besnard and Hunter, 2001), while other argumentation systems like ASPIC (Modgil and Prakken, 2014) allow such arguments. Within the abstract setting, several methods have been defined by proposing to deal with them directly (Bodanza and Tohmé, 2009; Baumann, Brewka, and Ulbricht, 2020b; Baumann, Brewka, and Ulbricht, 2020a; Dauphin, Rienstra, and Torre, 2020) or indirectly (e.g. when dealing with odd-length cycles because a self-attack is the smallest odd-length cycle) (Baroni and Giacomin, 2003). These methods essentially concern extension-based semantics.

In the context of ranking-based and gradual argumentation semantics (Amgoud and Ben-Naim, 2013; Amgoud and Doder, 2019), little research was conducted to find out how self-attacking arguments should be dealt with and what is the impact they have on the strength of other arguments. Existing approaches are essentially principle-based studies of these semantics. Indeed, defining and studying principles drew attention of many scholars in this area. Consider Equivalence, which is one of the well-known principles, stating that the degree of an argument should only depend on degrees of its direct attackers and observe the argumentation graph from Figure 3.1. Equivalence implies that a and b should be equally strong because a and b are both directly attacked by the same argument. However, this is debatable, since

[1] Bodanza and Tohmé (Bodanza and Tohmé, 2009) claim that there is a lack of "indisputably sound examples" concerning this type of arguments

the intuition behind a self-attacking argument is that it is inconsistent in one way or another so we would tend to accept b as stronger than a.

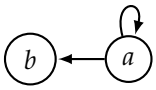


FIGURE 3.1: An argumentation graph with two arguments (a attacks itself and b) showing that Equivalence and Self-Contradiction are incompatible.

The research objective of the present thesis is to study the under-explored family of semantics that treat differently self-attacking arguments and others. Our goals to investigate principles and novel semantics that satisfy them.

3.2 Principles

Note that, under all semantics returning conflict-free extensions, a self-attacking argument is always rejected as it does not belong to any extension. Regarding ranking-based and gradual semantics, it was pointed out that it would be natural to attach the worst possible rank to self-attacking arguments. Two principles were defined to formalise this intuition. The first one is called Strong Self-Contradiction, and was introduced by (Matt and Toni, 2008). It says that the strength of an argument must be minimal if and only if that argument is self-attacking.

Principle 17 (Strong Self-Contradiction)**.** *A semantics S satisfies Strong Self-Contradiction iff, for any AG $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ with $a \in \mathcal{A}$, $\mathtt{Deg}^S_{\mathbf{G}}(a) = 0$ iff $(a, a) \in \mathcal{R}$.*

The second principle, called Self-Contradiction, was introduced by Bonzon et al. (Bonzon et al., 2016) and states that each self-attacking argument is strictly less acceptable than each non self-attacking argument.

Principle 18 (Self-Contradiction)**.** *A semantics S satisfies Self-Contradiction iff, for any AG $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$ with two arguments $a, b \in \mathcal{A}$, if $(a, a) \in \mathcal{R}$ and $(b, b) \notin \mathcal{R}$ then $\mathtt{Deg}^S_{\mathbf{G}}(b) > \mathtt{Deg}^S_{\mathbf{G}}(a)$.*

Consider the argumentation graph illustrated in Figure 3.1 again and note that, under every semantics that satisfies Self-Contradiction, b is strictly more acceptable than a. This example shows that Equivalence and Self-Contradiction are not compatible, i.e. there exists no semantics that satisfies both of them.

To the best of our knowledge, there exists only one semantics (known as M&T) that satisfies Self-Contradiction and Strong Self-Contradiction. That semantics was introduced by Matt and Toni (Matt and Toni, 2008). However, this semantics has a limitation that makes it inapplicable in practice. Namely, as noted by Matt and Toni themselves, as the space used to calculate the scores grows exponentially with the number of arguments, even with the optimisation techniques they used it did not scale to more than a dozen of arguments. Note that Strong Self-contradiction principle is more demanding than Self-contradiction; it completely destroys the weight of an argument that attacks itself and nullifies it.

Let us focus on the relationship between the two principles dealing with self-attacking arguments (both with each other and with the other principles). The first observation is that Strong Self-Contradiction implies Self-Contradiction. The next proposition follows directly from the definitions of the respective principles.

Proposition 2. *If a gradual semantics $\mathcal{S}$ satisfies Strong Self-Contradiction, it satisfies Self-Contradiction.*

Proof. Let us suppose that Strong Self-Contradiction is satisfied by $\mathcal{S}$. This means that those and only those arguments that have the minimum score are the self-attacking arguments ($\forall a \in \mathcal{A}$, $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = 0$ iff $(a,a) \in \mathcal{R}$). This implies that all arguments that do not attack themselves have an acceptability degree greater than 0. Formally, $\forall b \in \mathcal{A}$, $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b) > 0$ iff $(b,b) \notin \mathcal{R}$. Consequently, for two arguments $a, b \in \mathcal{A}$, if $(a,a) \in \mathcal{R}$ and $(b,b) \notin \mathcal{R}$ then $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b) > \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = 0$. □ □

As discussed in the introduction, the next result shows that Equivalence and Self-Contradiction are incompatible.

Proposition 3. *There exists no gradual semantics $\mathcal{S}$ that satisfies both Equivalence and Self-Contradiction.*

Proof. We provide a proof by contradiction. Let us suppose that a gradual semantics S satisfies both Equivalence and Self-Contradiction and consider the argumentation graph from Figure 3.1 on page 22. From Self-Contradiction, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) < \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b)$ because $(a,a) \in \mathcal{R}$ and $(b,b) \notin \mathcal{R}$. From Equivalence, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b)$ because $\texttt{Att}_{\mathbf{G}}(a) = \{a\}$ and $\texttt{Att}_{\mathbf{G}}(b) = \{a\}$ (and by using the identity function as the bijection from Definition 5).
Contradiction. Hence, S does not satisfy both Equivalence and Self-Contradiction.
Since S was arbitrary, we conclude that there exists no semantics that satisfies both Equivalence and Self-Contradiction. □ □

We show next that the Equivalence principle is not the only one that is incompatible with Strong Self-Contradiction. Some other incompatibilities exist mainly because self-attacking arguments are treated differently from other arguments. Indeed, according to Strong Self-Contradiction, self-attacking arguments are directly classified as the worst arguments, whereas the other principles like resilience just consider a self-attack as an attack like any other (i.e. an attack between two distinct arguments).

Proposition 4. *There exists no gradual semantics $\mathcal{S}$ that satisfies both Strong Self-Contradiction and Resilience.*

Proof. We provide a proof by contradiction. Let us suppose that a gradual semantics S satisfies both Strong Self-Contradiction and Resilience, and consider the argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ where $\mathcal{A} = \{a\}$ and $R = \{(a,a)\}$.
From Strong Self-Contradiction, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = 0$, while from Resilience, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) > 0$.
Contradiction. Hence, S does not satisfy both Strong Self-Contradiction and Resilience. Since S was arbitrary, there exists no semantics that satisfies both Resilience and Strong Self-Contradiction. □ □

Strong Self-Contradiction is also incompatible with Weakening Soundness.

Proposition 5. *There exists no gradual semantics $\mathcal{S}$ that satisfies both Strong Self-Contradiction and Weakening Soundness.*

Proof. We provide a proof by contradiction. Let us suppose that a gradual semantics S satisfies both Strong Self-Contradiction and Weakening Soundness, and consider the argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ where $\mathcal{A} = \{a\}$ and $R = \{(a,a)\}$.

From Strong Self-Contradiction, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = 0$, while from Weakening Soundness, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) > 0$ because a is the only attacker of a and $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = 0$. Contradiction. Hence, S does not satisfy both Strong Self-Contradiction and Weakening Soundness. Since S was arbitrary, there exists no semantics that satisfies both Strong Self-Contradiction and Weakening Soundness. □ □

Proposition 6. *There exists no gradual semantics $\mathcal{S}$ that satisfies both Strong Self-Contradiction and Reinforcement.*

Proof. We provide a proof by contradiction. Let us suppose that a gradual semantics S satisfies both Strong Self-Contradiction and Reinforcement, and consider the argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ represented in Figure 3.2.

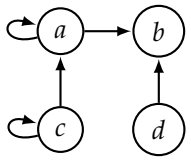


FIGURE 3.2: AG showing that Reinforcement and Strong Self-Contradiction are incompatible.

From Strong Self-Contradiction, we have $0 = \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) < \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b)$. From Reinforcement, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) > \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b)$ because i) $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b) > 0$, ii) $\texttt{Att}_{\mathbf{G}}(a) \backslash \texttt{Att}_{\mathbf{G}}(b) = \{c\}$, iii) $\texttt{Att}_{\mathbf{G}}(b) \backslash \texttt{Att}_{\mathbf{G}}(a) = \{d\}$, and iv) $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(d) > \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(c)$.
Contradiction. Hence, S does not satisfy both Strong Self-Contradiction and Reinforcement. Since S was arbitrary, there exists no semantics that satisfies both Strong Self-Contradiction and Reinforcement. □ □

Proposition 7. *There exists no gradual semantics $\mathcal{S}$ that satisfies both Strong Self-Contradiction and Neutrality.*

Proof. We provide a proof by contradiction. Let us suppose that a gradual semantics S satisfies both Strong Self-Contradiction and Neutrality, and consider the argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ represented in Figure 3.3.

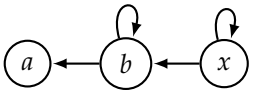


FIGURE 3.3: AG showing that Neutrality and Strong Self-Contradiction are incompatible.

From Strong Self-Contradiction, we have $0 = \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b) < \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a)$. From Neutrality, we have $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(a) = \texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(b)$ because $\texttt{Att}_{\mathbf{G}}(b) = \texttt{Att}_{\mathbf{G}}(a) \cup \{x\}$ with $\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}(x) = 0$.
Contradiction. Hence, S does not satisfy both Strong Self-Contradiction and Neutrality. Since S was arbitrary, there exists no semantics that satisfies both Strong Self-Contradiction and Neutrality. □ □

Taking these incompatibilities into account, our goal is now to study two maximal compatible sets of principles we are interested in. A compatible set of principles

is a set of principles such that two principles belonging to this set are not incompatible. In other words, a compatible set of principles is a set of principles that can be jointly satisfied by a semantics. In order to capture the idea of a maximal compatible sets of principles, let us define the notion of dominance.

Definition 18. *A semantics S dominates a semantics S' on the set of principles P if the subset of principles from P satisfied by S is a strict superset of the subset of principles from P satisfied by S'.*

In the rest of the discussion, we suppose that P is the set of all principles studied in Section 2.3. Note that if a semantics S satisfies a maximal (for set inclusion) set of principles, it is not dominated by any semantics.

A first maximal (for set inclusion) compatible set of principles has been identified by (Amgoud et al., 2017) and is a direct consequence of their Proposition 1. We define this set of principles as $P_{CREW} = \{$Anonymity, Independence, Directionality, Neutrality, Equivalence, Maximality, Weakening, Counting, Weakening Soundness, Reinforcement, Resilience and Compensation$\}$.

Theorem 1 ((Amgoud et al., 2017))**.** *P_{CREW} is a maximal compatible for set inclusion set of principles.*

We can formally show that there is a unique maximal compatible set of principles that includes Compensation, Resilience, Equivalence and Weakening Soundness.

Theorem 2. *Let P be the set of all principles defined in Section 2.3 (Principles 1-17). Let $\mathcal{S}$ be a gradual semantics that satisfies Compensation, Resilience, Equivalence and Weakening Soundness. If $\mathcal{S}$ is not dominated w.r.t. P, then $\mathcal{S}$ satisfies exactly the principles from P_{CREW}.*

Proof. On one hand, we know from the work by (Amgoud et al., 2017) that h-categorizer satisfies all the principles from P_{CREW}. On the other hand, it is clear from the incompatibility results between the principles that $\mathcal{S}$ cannot satisfy Strong Self-Contradiction which is incompatible with Resilience (see Proposition 4), Self-Contradiction which is incompatible with Equivalence (see Proposition 3), Cardinality/Quality Precedence which are both incompatible with Compensation (see (Amgoud et al., 2017)). Thus, in order not to be dominated by h-categorizer, $\mathcal{S}$ must satisfy all the principles from P_{CREW}; due to the incompatibilities, $\mathcal{S}$ cannot satisfy any more principles. □ □

In this thesis we choose to explore the space of principles compatible with Strong Self-Contradiction (which is not in P_{CREW}). One naturally wants to maximise the set of satisfied principles. Can we satisfy Strong Self-Contradiction and all the other principles? The answer is negative (see Propositions 3-7). First, one has to choose between Cardinality Precedence, Quality Precedence and Compensation. In this thesis, we explore the possibility of satisfying Compensation. This choice is based on the fact that this principle is satisfied by virtually all semantics, as showed by Amgoud et al. (Amgoud et al., 2017). Indeed, Cardinality Precedence and Quality Precedence represent, roughly speaking, *drastic* or *extreme* cases and are satisfied only by the semantics specifically designed to satisfy them, like max-based semantics and card-based semantics (Amgoud et al., 2017) or by semantics having other specificities. For instance, iterative schema (Gabbay and Rodrigues, 2015), which satisfies Quality Precedence, is a discrete semantics (it takes only three possible values). This yields another maximal compatible set of principles which includes those two principles. We define this set of principles as P_{2S2C} = {Anonymity, Independence, Directionality, Maximality, Weakening, Counting, Compensation, Self-Contradiction, Strong Self-Contradiction}.

Theorem 3. *P_{2S2C} is a maximal compatible for set inclusion set of principles.*

Proof. Note first that in this proof, we mention the `nsa` semantics, which is formally introduced in Definition 19 (see below). Firstly, all the principles in P_{2S2C} are compatible because `nsa` satisfies all of them (see Proposition 8 below). Secondly, P_{2S2C} is maximal because for each remaining principle $p \in \{$Equivalence, Weakening Soundness, Neutrality, Reinforcement, Cardinality Precedence, Quality Precedence and Resilience$\}$, there exists (at least) one principle in P_{2S2C} which is incompatible with p:

- Equivalence and Self-Contradiction are incompatible (see Proposition 3);
- Neutrality and Strong Self-Contradiction are incompatible (see Proposition 7);
- Reinforcement and Strong Self-Contradiction are incompatible (see Proposition 6);
- Weakening Soundness and Strong Self-Contradiction are incompatible (see Proposition 5);
- Cardinality Precedence and Compensation are incompatible (see (Amgoud et al., 2017));
- Quality Precedence and Compensation are incompatible (see (Amgoud et al., 2017));
- Resilience and Strong Self-Contradiction are incompatible (see Proposition 4).

□ □

We now show that there is a unique maximal compatible set of principles that includes Strong Self-Contradiction and Compensation. This follows from the fact that if a semantics satisfies Strong Self-Contradiction, this semantics cannot satisfy some existing principles (see the incompatibilities identified in Propositions 3-7).

Theorem 4. *Let P be the set of all principles defined in Section 2.3 (Principles 1-17). Let $\mathcal{S}$ be a gradual semantics that satisfies Strong Self-Contradiction and Compensation. If $\mathcal{S}$ is not dominated w.r.t. P, then $\mathcal{S}$ satisfies exactly the principles from P_{2S2C}.*

Proof. It is clear that from the incompatibility results between different principles, $\mathcal{S}$ cannot satisfy (i) Resilience, Equivalence and Weakening Soundness which are incompatible with Strong Self-Contradiction (or Self-Contradiction), and (ii) Cardinality Precedence and Quality Precedence which are both incompatible with Compensation.
The set of remaining principles corresponds exactly to P_{2S2C} which is a maximal for set inclusion set of principles. However, $\mathcal{S}$ cannot satisfy exactly a subset of P_{2S2C} because, in this case, $\mathcal{S}$ will be dominated by a semantics that satisfies the principles of P_{2S2C}. Consequently, when $\mathcal{S}$ satisfies Strong Self-Contradiction and Compensation, the only way to ensure that $\mathcal{S}$ is not dominated is when $\mathcal{S}$ satisfies exactly the principles from P_{2S2C}. □ □

To the best of our knowledge, no semantics that satisfies all the principles from P_{2S2C} has been presented in the literature. In the next section, we define a semantics that satisfies this set of principles.

Before doing that, let us comment on the non-satisfaction of some principles. It is tempting to change the principles in order to treat the self-attacks in another

way, and consequently make the principles fit some definitions or theorems. We argue that it is better to start by having a full picture of what happens with *existing* principles. Indeed, the principles should be the most stable part of a theory. We are not against the introduction of new principles (or changing the existing ones). This might be part of future work.

3.3 Novel gradual semantics (nsa)

In this section, we define a new gradual semantics, called no self-attack h-categorizer (nsa) semantics, inspired by the h-categorizer semantics. The main difference is that we assign degree 0 to the self-attacking arguments while the acceptability degrees of the other arguments, i.e. those that are not self-attacking, are calculated using the formula from h-categorizer semantics.

Definition 19. *Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ be an AG. We define $f^{\mathbf{G},i}_{\mathtt{nsa}} : \mathcal{A} \to [0, +\infty]$ as follows : for every argument $a \in \mathcal{A}$ for $i \in \{0, 1, 2, ..\}$,*

$$f^{\mathbf{G},i}_{\mathtt{nsa}}(a) = \begin{cases} 0 & \text{if } (a,a) \in \mathcal{R} \\ 1 & \text{if } (a,a) \notin \mathcal{R} \text{ and } i = 0 \\ \dfrac{1}{1 + \sum_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f^{\mathbf{G},i-1}_{\mathtt{nsa}}(b)} & \text{if } (a,a) \notin \mathcal{R} \text{ and } i > 0 \end{cases} \tag{3.1}$$

By convention, if $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$, $\sum_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f^{\mathbf{G},i-1}_{\mathtt{nsa}}(b) = 0$.

Although nsa is inspired by the h-categorizer semantics, the modifications made change the result obtained requiring the verification that nsa also converges to a unique result. Thus, in the next result, we show that for every argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$, for every argument $a \in \mathcal{A}$, $f^{\mathbf{G},i}_{\mathtt{nsa}}(a)$ converges as i approaches infinity. Roughly speaking, the goal of the next theorem is to formally check that assigning zero values to self-attacking arguments does not impact the convergence of the scores. Thus, applying nsa to the original argumentation graph $\mathbf{G}$ provides the same result as when the h-categorizer semantics is applied on a restricted version of $\mathbf{G}$ where the self-attacking arguments are deleted.

Theorem 5. *For every argumentation graph $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$, for every $a \in \mathcal{A}$, if $(a,a) \notin \mathcal{R}$, we have $\lim_{i \to \infty} f^{\mathbf{G},i}_{\mathtt{nsa}}(a) = \mathtt{Deg}^{\mathtt{h}}_{\mathbf{G}'}(a)$ where $\mathbf{G}' = \langle \mathcal{A}', \sigma \equiv 1, \mathcal{R}' \rangle \in \mathtt{AG}$ with the set of arguments $\mathcal{A}' = \{x \in \mathcal{A} \mid (x,x) \notin \mathcal{R}\}$ and $\mathcal{R}' = \{(x,y) \in \mathcal{R} \mid x \in A' \text{ and } y \in A'\}$.*

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ be an AG and $\mathbf{G}' = \langle \mathcal{A}', \sigma \equiv 1, \mathcal{R}' \rangle \in \mathtt{AG}$ be an AG such that $\mathcal{A}' = \{x \in \mathcal{A} | (x,x) \notin \mathcal{R}\}$ and $\mathcal{R}' = \{(x,y) \in \mathcal{R} \mid x \in A' \text{ and } y \in A'\}$. Without loss of generality, let us denote $\mathcal{A} = \{a_0, a_1, \ldots, a_n\}$.

Let us recall the iterative version of h-categorizer, that can be used to calculate the scores of arguments (Pu et al., 2014): for every a, for $i \in \mathbb{N}$

$$f^{\mathbf{G},i}_{h}(a) = \begin{cases} 1 & \text{if } i = 0 \\ \dfrac{1}{1 + \sum_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f^{\mathbf{G},i-1}_{h}(b)} & \text{if } i > 0 \end{cases} \tag{3.2}$$

We prove by induction on i that for each $a \in \mathcal{A}'$:

$$f^{\mathbf{G},i}_{\mathtt{nsa}}(a) = f^{\mathbf{G}',i}_{h}(a)$$

Base: Let $i = 0$. From the formal definition of nsa (Definition 19) and equation (3.2), we have $f_{\mathtt{nsa}}^{\mathbf{G},0}(a) = f_h^{\mathbf{G}',0}(a) = 1$. Thus, the inductive base holds.

Step: Let us suppose that the inductive hypothesis is true for every $k \in \{0, 1, \dots i\}$ and let us show that it is true for $i+1$. We need to prove :

$$f_{\mathtt{nsa}}^{\mathbf{G},i+1}(a) = f_h^{\mathbf{G}',i+1}(a)$$

From the inductive hypothesis, we know that for each argument $a \in \mathcal{A}'$, $f_{\mathtt{nsa}}^{\mathbf{G},i}(a) = f_h^{\mathbf{G}',i}(a)$. Thus, from equation (3.1), we have:

$$f_{\mathtt{nsa}}^{\mathbf{G},i+1}(a) = \frac{1}{1 + \sum_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f_{\mathtt{nsa}}^{\mathbf{G},i}(b)}$$

From equation (3.2), we have

$$f_h^{\mathbf{G}',i+1}(a) = \frac{1}{1 + \sum_{b \in \mathtt{Att}_{\mathbf{G}'}(a)} f_h^{\mathbf{G}',i}(b)}$$

Let us note $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}'}(a) \cup \{b_0, \dots, b_m\}$ with $m \geq 0$ and remark that $\forall b \in \{b_0, \dots, b_m\}$, we have $(b, b) \in \mathcal{R}$. According to equation (3.1), $\forall b \in \{b_0, \dots, b_m\}$, $f_{\mathtt{nsa}}^{\mathbf{G},i}(b) = 0$. Consequently, as 0 is the neutral element of the addition, we have $\forall a \in \mathcal{A}'$, $f_{\mathtt{nsa}}^{\mathbf{G},i+1}(a) = f_h^{\mathbf{G}',i+1}(a)$.

By induction, we conclude that for every $i \in \mathbb{N}$ and for every $a \in \mathcal{A}'$

$$f_{\mathtt{nsa}}^{\mathbf{G},i}(a) = f_h^{\mathbf{G}',i}(a)$$

Since f_h converges when $i \to \infty$ and $f_{\mathtt{nsa}}$ coincides with f_h for every argument of $\mathcal{A}'$, we conclude that $f_{\mathtt{nsa}}$ converges too. Formally, $\forall a \in cA'$,

$$\lim_{i \to \infty} f_{\mathtt{nsa}}^{\mathbf{G},i}(a) = \lim_{i \to \infty} f_{\mathtt{h}}^{\mathbf{G},i}(a) = \mathtt{Deg}_{\mathbf{G}'}^{\mathtt{h}}(a)$$

□
□

We can now introduce the formal definition of nsa.

Definition 20 (nsa semantics)**.** *The no self-attack h-categorizer semantics is a function* nsa *which associates to any argumentation framework* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ *a function* $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{nsa}}(a) : A \to [0, 1]$ *as follows:* $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{nsa}}(a) = \lim_{i \to \infty} f_{nsa}^{\mathbf{G},i}(a)$.

We can now show that the acceptability degrees attributed to arguments by nsa satisfy the equation from Definition 19 (naturally, not taking into account the second line of the equation, since it considers the case $i = 0$).

Theorem 6. *For any* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ *, for any* $a \in \mathcal{A}$*,*

$$\mathtt{Deg}_{\mathbf{G}}^{\mathtt{nsa}}(a) = \begin{cases} 0 & \text{if } (a, a) \in \mathcal{R} \\ \dfrac{1}{1 + \sum_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathtt{Deg}_{\mathbf{G}}^{\mathtt{nsa}}(b)} & \text{otherwise} \end{cases}$$

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ be an argumentation graph and $a \in \mathcal{A}$.
The case where a is a self-attacking argument is trivial.

In the rest of the proof we consider the case where a is not a self-attacking argument. Letting $\lim_{i\to\infty}$ in the following equality

$$f^{i+1}_{\texttt{nsa}}(a) = \frac{1}{1+\sum_{b\in \texttt{Att}_\texttt{G}(a)} f^{i}_{\texttt{nsa}}(b)}$$

and using the fact that arithmetical operations and sum are continuous functions, we obtain :

$$\lim_{i\to\infty} f^{i+1}_{\texttt{nsa}}(a) = \frac{1}{1+\sum_{b\in \texttt{Att}_\texttt{G}(a)} \lim_{i\to\infty} f^{i}_{\texttt{nsa}}(b)}$$

then

$$\texttt{Deg}^{\texttt{nsa}}_{\texttt{G}}(a) = \frac{1}{1+\sum_{b\in \texttt{Att}_\texttt{G}(a)} \texttt{Deg}^{\texttt{nsa}}_{\texttt{G}}(a)}$$

□ □

We now show that the equation from Theorem 6 is not only satisfied by nsa, but is also its characterization. More precisely, the next result proves that if an arbitrary semantics D satisfies that equation, it must be that D coincides with nsa.

Theorem 7. *Let* $\mathbf{G} = (\mathcal{A}, \sigma \equiv 1, \mathcal{R})$ *be an AG with* $a \in \mathcal{A}$ *and* $D : \mathcal{A} \to [0,1]$ *be a function with the following formula:*

$$D(a) = \begin{cases} 0 & \text{if } (a,a) \in \mathcal{R} \\ \dfrac{1}{1+\sum_{b\in Att_\mathbf{G}(a)} D(b)} & \text{otherwise} \end{cases} \tag{3.3}$$

then $D \equiv \texttt{Deg}^{\texttt{nsa}}_{\texttt{G}}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R}\rangle \in \texttt{AG}$ and suppose that $D : \mathcal{A} \to [0,1]$ is the function from equation (3.3).

Let $A = \{a_1, .., a_n\}$ and let $F : [0,1]^n \to [0,1]^n$ be the function such that $F(x_1, .., x_n) = (F_1(x_1, .., x_n), ..., F_n(x_1, ..., x_n))$ where the functions F_i are defined by the following equality:

$$F_i(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } (a_i, a_i) \in \mathcal{R} \\ \dfrac{1}{1+\sum\limits_{j:a_j\in Att_\mathbf{G}(a_i)} x_j} & \text{otherwise} \end{cases} \tag{3.4}$$

We also define the partial order $\leq$ on $\mathbb{R}^n$ in the following way: if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then $x \leq y$ iff for every i it holds that $x_i \leq y_i$.

Thus, from Equation (3.3), it follows that

$$F(D(a_1), ..., D(a_n)) = (D(a_1), ..., D(a_n)).$$

Observe that F is a non-increasing function and that $G = F \circ F$ is a non-decreasing function, and that :

$$(f^{i+1}_{\texttt{nsa}}(a_1), ..., f^{i+1}_{\texttt{nsa}}(a_n)) = F((f^{i}_{\texttt{nsa}}(a_1), ..., f^{i}_{\texttt{nsa}}(a_n)))$$

for every $i \in \mathbb{N}$. Since $(f^0_{\texttt{nsa}}(a_1), ..., f^0_{\texttt{nsa}}(a_n)) \in [0,1]^n$ with $f^0_{\texttt{nsa}}(a_i) = 0$ iff $(a_i, a_i) \in \mathcal{R}$ and $f^0_{\texttt{nsa}}(a_i) = 1$ otherwise, by the inequalities, we obtain

$$(f^0_{\texttt{nsa}}(a_1), ..., f^0_{\texttt{nsa}}(a_n)) \geq (D(a_1), ..., D(a_n)) \tag{3.5}$$

From (3.5), and since F is non-increasing, we have:

$$(f^1_{\texttt{nsa}}(a_1), ..., f^1_{\texttt{nsa}}(a_n)) \leq (D(a_1), ..., D(a_n)) \tag{3.6}$$

From (3.6), and since $G = F \circ F$ is non-decreasing, we have:

$$(f^{2i}_{\texttt{nsa}}(a_1), ..., f^{2i}_{\texttt{nsa}}(a_n)) \geq (D(a_1), ..., D(a_n)) \tag{3.7}$$

and

$$(f^{2i+1}_{\texttt{nsa}}(a_1), ..., f^{2i+1}_{\texttt{nsa}}(a_n)) \leq (D(a_1), ..., D(a_n)) \tag{3.8}$$

for every $i \in \mathbb{N}$.
Since all f^i converge, from (3.7) and (3.8) we obtain

$$(\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_1), \ldots, \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_n)) \geq (D(a_1), ..., D(a_n))$$

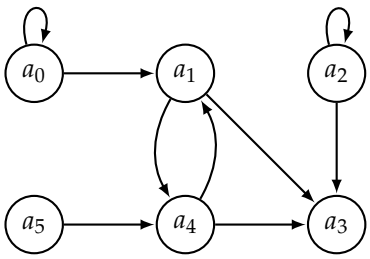
and

$$(\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_1), \ldots, \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_n)) \leq (D(a_1), ..., D(a_n))$$

and thus $\forall a \in \mathcal{A}, \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = D(a)$. □ □

Below is an example of the nsa semantics applied on an argumentation graph.

Example 13. *Let us apply the no self-attack h-categorizer semantics (*nsa*) on the argumentation graph illustrated in Figure 3.4.*



$\texttt{Deg}^{\mathcal{S}}_{\mathbf{G}}$	nsa	h	MT
a_0	0	0.618	0
a_1	0.732	0.495	0.25
a_2	0	0.618	0
a_3	0.477	0.398	0.167
a_4	0.399	0.401	0.25
a_5	1	1	1

FIGURE 3.4: On the left, an argumentation graph **G** and, on the right, the table containing the degrees of acceptability of each argument of **G** w.r.t. the no self-attack h-categorizer semantics (nsa), the h-categorizer semantics (h) and the semantics M&T (MT).

By definition, the self-attacking arguments have an acceptability degree of 0 : $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_0) = \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_2) = 0$. The non-attacked arguments or the arguments only attacked by self-attacking arguments have, by definition, the maximum score: $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_5) = 1$. Applying the formula from Theorem 6, we obtain the following strength degrees for a_1 and a_4 : $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_1) = 0.732$ and $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_4) = 0.399$. Finally,

following the same method, here are the details concerning a_3 :

$$\begin{aligned}\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_3) &= \frac{1}{1+\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_1)+\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_2)+\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a_4)}\\ &= \frac{1}{1+0.732+0+0.399}\\ &= 0.477\end{aligned}$$

In order to have an overview of the difference between nsa and the gradual semantics introduced in Section 2.2, the degrees of acceptability of arguments w.r.t. the h-categorizer semantics and the M&T semantics have also been added in the table of Figure 3.4. This comparison clearly shows that nullifying the impact of self-attacking arguments (i.e. a_0 and a_2) more or less significantly changes the degree of acceptability of other arguments (e.g. a_1, a_3 and a_4).

Proposition 8. *The gradual semantics* nsa *satisfies all the principles from* P_{2S2C}*. The other principles are not satisfied.*

In order to axiomatically compare nsa with the two other gradual semantics, let us check for the principles studied in this thesis those that are satisfied by M&T and recall those satisfied by the *h*-categorizer semantics.

Proposition 9. *The gradual semantics M&T satisfies Anonymity, Independence, Directionality, Maximality, Weakening, Compensation, Self-Contradiction and Strong Self-Contradiction. The other principles are not satisfied.*

We recall below the principles that are satisfied by *h*-categorizer.

Proposition 10 ((Amgoud and Ben-Naim, 2016))**.** *The gradual semantics h-categorizer satisfies all the principles from* P_{CREW}*. The other principles are not satisfied.*

3.4 Conclusion

We studied the question of handling self-attacks by gradual semantics following a principle-based approach. We first showed links and incompatibilities between existing principles before identifying two maximal compatible sets of principles (P_{CREW} which includes Equivalence and P_{2S2C} which includes Strong Self-Contradiction). Then, we defined a new semantics called no self-attack *h*-categorizer semantics and proved that it dominates the only existing semantics satisfying the Self-Contradiction principle. Moreover, we showed that our semantics satisfies a maximal possible amount of principles (i.e. no semantics satisfying Self-Contradiction can satisfy more principles) and is usable in practice as it returns results very quickly (on average less than 1 second) even on large and dense argumentation graphs.

We conclude by noting several considerations for future work on this topic.

Extend the methodology to other gradual semantics. It would be interesting to extend (if possible) the approach we used for the h-categorizer semantics (i.e. force self-attacking arguments to have the minimal strength value) to other existing gradual semantics.

Principles	M&T	h-cat	nsa
Anonymity	✓	✓	✓
Independence	✓	✓	✓
Directionality	✓	✓	✓
Neutrality	×	✓	×
Equivalence	×	✓	×
Maximality	✓	✓	✓
Weakening	✓	✓	✓
Counting	×	✓	✓
Weakening Soundness	×	✓	×
Reinforcement	×	✓	×
Resilience	×	✓	×
Cardinality Precedence	×	×	×
Quality Precedence	×	×	×
Compensation	✓	✓	✓
Self-Contradiction	✓	×	✓
Strong Self-Contradiction	✓	×	✓

TABLE 3.1: Principles satisfied by the M&T, h-categorizer and nsa semantics. The shaded cells contain the results already proved in the literature.

Identify all maximal sets of compatible principles. A second line of research would be to identify all maximal sets of consistent principles from the set of principles defined in Section 2.3. Indeed, we have chosen to include Compensation in P_{CREW} and P_{2S2C} but it would be interesting to look at and study the maximum sets which include Cardinality Precedence or Quality Precedence.

This set of principles is yet to be augmented. Another research direction concerns the principles dealing with self-attacking arguments. Indeed, Strong Self-Contradiction can be seen as a rather strong principle in that it expresses both necessary and sufficient conditions for an argument to have minimal degree (i.e. 0 in our case). It would be interesting to investigate weakened versions like for instance a principle that only expresses that self-contradiction is a sufficient condition for minimal degree.

Principle 19 (Weak Self-Contradiction)**.** *A gradual semantics S satisfies Weak Self-Contradiction iff, for any AG* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *with* $a \in \mathcal{A}$*, if* $(a, a) \in \mathcal{R}$ *then* $\texttt{Deg}^S_{\mathbf{G}}(a) = 0$.

In this case, some incompatibilities remain unchanged (e.g. with Resilience or with Weakening Soundness), whereas whether the same is still the case for all of them remains to be investigated. We could imagine that this way a new set of coherent principles appears (where Weak Self-Contradiction replaces Strong Self-Contradiction). However, it should be checked whether there is at least one gradual semantics that satisfies all these principles.

Towards an application-oriented axiomatic analysis. Concerning the principles, let us recall that we do not claim that all of the principles presented in Section 2.3 are mandatory. However, at this level of abstraction, they allow us to compare and better understand the gradual semantics. In line with the work initiated in (Vesic, Yun, and Teovanovic, 2022), it would be interesting to target the mandatory principles for some practical aspects of argumentation (persuasion, negotiation, online debate,

etc.).

Self-attacking arguments and gradual semantics in practical applications. There are several discussions about applications where gradual (or ranking-based) semantics can be used (Leite and Martins, 2011; Delobelle, 2017; Amgoud, 2019). One such application is online debates, for example, where participants propose, in the most basic form, arguments for or against a given topic or other arguments. As the arguments are given in textual format and the relationships between them are, in the vast majority of cases, given by the participants themselves, the arguments may not be correct and/or the set of attacks may not be complete. For example, some fallacious arguments (e.g. informal fallacies) may be put forward (this is sometimes the case in social networks or in fake news). These fallacious arguments could for example be spotted via argument mining methods (Goffredo et al., 2022) and considered, for some of them, as self-attacking arguments because of the false reasoning (e.g. sophism[2]). It is therefore necessary to be able to have reasoning tools that can deal with them in order to correctly analyse a given debate.

[2]A sophism is a confusing or slightly incorrect argument used for deceiving someone. For example, the following argument is a sophism : "Everything that is rare is expensive. A cheap horse is rare. So a cheap horse is expensive."

Chapter 4

Series-based Semantics

4.1 Introduction

We have seen previously that several gradual semantics have been proposed in the literature. They assign a numerical value to each argument in an argumentation framework. In this chapter, we investigate the mathematical counterparts of some of them. We show that they can be redefined using numerical series that satisfy some conditions.

4.2 Family of numerical series

Let us start by defining a family a series whose elements belong to the interval $[0, 1]$. Every series of the family is made of two sub-series: an increasing series and a decreasing one. In addition, every subseries admits a limit.

Definition 21 ($\mathbf{S}^*$). *We define $\mathbf{S}^*$ to be the set containing any numerical series $\mathcal{S} = (\mathcal{S}^n)^{n\geq 1}$ which satisfies the following conditions:*

- *for any $n \geq 1$, $\mathcal{S}^n \in [0, 1]$*
- *$\mathcal{S}$ contains two sub-series $\mathcal{S}_1 = (\mathcal{S}_1^n)^{n\geq 1}$ and $\mathcal{S}_2 = (\mathcal{S}_2^n)^{n\geq 1}$ s.t. for any $n \geq 1$:*
  - *$\mathcal{S}_1^n$ is strictly decreasing and $\mathcal{S}_2^n$ is strictly increasing*
  - $\lim_{n\to\infty} \mathcal{S}_1^n \geq \lim_{n\to\infty} \mathcal{S}_2^n$

Lemma 1. *Let $\mathcal{S} \in \mathbf{S}^*$. For all $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2$, the following hold:*

$$x > y \qquad x > \lim_{n\to\infty} \mathcal{S}_1^n \qquad y < \lim_{n\to\infty} \mathcal{S}_2^n$$

Proof Let $\mathcal{S} \in \mathbf{S}^*$. Let $\lim_{n\to\infty} \mathcal{S}_1^n = x$ and $\lim_{n\to\infty} \mathcal{S}_2^n = y$. Since $\mathcal{S}_1$ is strictly decreasing, then $x < \mathcal{S}_1^n$, $\forall n$. Since $\mathcal{S}_2$ is strictly increasing, then $y > \mathcal{S}_2^n$, $\forall n$. Since $\lim_{n\to\infty} \mathcal{S}_1^n \geq \lim_{n\to\infty} \mathcal{S}_2^n$, then we get for any $n > 1$, $\mathcal{S}_1^n > x \geq y > \mathcal{S}_2^n$, so $\mathcal{S}_1^n > \mathcal{S}_2^n$. ■

The following result shows that the set $\mathbf{S}^*$ is not empty.

Notation 3. *Let $S \in \mathbf{S}^*$. We denote by S_x^i the i^{th} element of the sub-sequence S_x where $x \in \{1, 2\}$.*

Proposition 11. *It holds that $\mathbf{S}^* \neq \emptyset$.*

To prove this result, we need to introduce a few examples of numerical series that belong to $\mathbf{S}^*$. We will present three of them initially. We show next that the well-known series are instances of the family $\mathbf{S}^*$. Let us start with the Fibonacci series defined as follows:

Definition 22. *From the well known Fibonacci sequence Philippou, 2015* $\{F^n\}_{n\geq 0}$ *:*

$$F^n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F^{n-1} + F^{n-2} & \text{if } n \geq 2 \end{cases} \tag{4.1}$$

Example of numbers of this series are 0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,... It is clear that this series tends to infinity and does not contain two sub-series. Hence, it is not a member of $\mathbf{S}^*$. However, we define another series from $\{F^n\}^{n\geq 0}$ which is part of $\mathbf{S}^*$.

Definition 23. *The series* $\{S^n\}^{n\geq 1}$ *is defined as follows :*

$$\mathcal{S}^n = \frac{F^n}{F^{n+1}} \tag{4.2}$$

Let $S = \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, ...\}$ the numbers of the series $\{S^n\}^{n\geq 1}$. It is worth noticing that this sequence contains two sub-sequences: The *decreasing* sub-sequence

$$\mathcal{S}_1 = \langle 1, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \frac{89}{144}, \frac{233}{377}, \ldots \rangle \tag{4.3}$$

made of the numbers that are at odd positions in $\mathcal{S}$, and the *increasing* sub-sequence

$$\mathcal{S}_2 = \langle \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{55}{89}, \frac{144}{233}, \ldots \rangle \tag{4.4}$$

which contains the numbers that are at even positions in $\mathcal{S}$.
For instance, $\mathcal{S}_1^1 = 1$, $\mathcal{S}_2^1 = \frac{1}{2}$ and $\mathcal{S}_2^3 = \frac{8}{13}$.
From (Philippou, 2015), it follows that the two sub-sequences converge, furthermore they have the same limit. Indeed,

$$\lim_{n\to\infty} \mathcal{S}_1^n = \lim_{n\to\infty} \mathcal{S}_2^n = \frac{1}{\varphi}.$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the so-called gold ratio. Consequently, $\{S^n\}^{n\geq 1}$ is a member of the family $\mathbf{S}^*$.

Proposition 12. $\{S^n\}^{n\geq 1} \in \mathbf{S}^*$.

The second well-known numerical series in the literature is the exponential one defined as follows:

Definition 24. *The exponential sequence is* $\{\mathcal{U}^n\}_{n\geq 0}$ *such that*

$$U^n = \begin{cases} 1 & \text{if } n = 1 \\ e^{-\mathcal{U}^{n-1}} & \text{if } n > 1 \end{cases} \tag{4.5}$$

Let $\mathcal{U} = \{1, 0.3675, 0.6922, 0.5004, 0.6062, 0.5453, 0.5796, ...\}$. Like the Fibonacci sequence, $\{\mathcal{U}^n\}_{n\geq 0}$ contains two sub-sequences: the decreasing sub sequence :

$$\mathcal{U}_1 = \langle 1, 0.6922, ... \rangle$$

made of the numbers that are at odd positions in $\mathcal{U}$, and the increasing sub-sequence:

$$\mathcal{U}_2 = \langle 0.3678, 0.5004, ... \rangle$$

which contains the numbers that are at even positions in $\mathcal{U}$. The two sub-sequences converge to the same limit

$$\lim_{n\to\infty} \mathcal{U}_1^n = \lim_{n\to\infty} \mathcal{U}_1^n = \omega$$

where $\omega \approx 0.5671$ is the so called Omega constant and $\mathcal{U}_2^n < \omega < \mathcal{U}_1^n$. Hence, the series $\{\mathcal{U}^n\}^{n\geq 0}$ is a member of the family $\mathbf{S}^*$

Proposition 13. $\{\mathcal{U}^n\}^{n\geq 0} \in \mathbf{S}^*$

The third sequence from the literature is the Jacobsthal (A.F., 1988) defined as follows:

Definition 25. $\mathcal{J} = (\mathcal{J}^n)^{n\geq 1}$ *where:*

$$\mathcal{J}_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2} & \text{if } n \geq 2 \end{cases} \tag{4.6}$$

We show next that we can define another series from $(\mathcal{J}_n)^{n\geq 1}$ which belongs to $\mathbf{S}^*$. From $\mathcal{J}$, we define a novel series which belongs to the set $\mathbf{S}^*$.

Proposition 14. *It holds that* $\mathtt{J} = (\frac{\mathcal{J}^{n+1}}{2^n})^{n\geq 0} \in \mathbf{S}^*$.

Lemma 2. *For any* $i \in \mathbb{N}$, $\mathtt{J}^i = 1 - \frac{\mathtt{J}^{i-1}}{2}$.

Proof . From (A.F., 1996), $\forall n \geq 0$, $\mathcal{J}^n = \frac{2^n - (-1)^n}{3}$. Then, $\mathcal{J}^{n+1} = 2^n - \mathcal{J}^n$.
Since $\mathtt{J}^n = \frac{\mathcal{J}^{n+1}}{2^n}$, we get $\mathtt{J}^n = \frac{2^n - \mathcal{J}_n}{2^n} = 1 - \frac{\mathcal{J}_n}{2^n} = 1 - \frac{\frac{\mathcal{J}_n}{2^{n-1}}}{2} = 1 - \frac{\mathtt{J}^{n-1}}{2}$. ∎

4.3 Mathematical counterparts of gradual semantics

This section investigates the mathematical counterparts of the two gradual semantics: Max-based (Mbs) proposed in (Amgoud et al., 2017) and Euler-based (EMbs) proposed in (Amgoud, 2019). We show that in case of flat graphs, (ie, $\sigma \equiv 1$), the two semantics can be expressed by numerical series. Let us first recall their definition in case of flat graphs. $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in$ AG . For any, $a \in$ ‘$\mathcal{A}$,

$$\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \frac{1}{1+\max\limits_{b\mathcal{R}a} \mathtt{Deg}_{\mathbf{G}}(b)} \qquad \mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = e^{-\max\limits_{b\mathcal{R}a} \mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b)}$$

In what follows, we show that the values assigned by Mbs to arguments are not arbitrary, they are rather the Fibonacci numbers, ie, elements of the series $\{\mathbf{S}^n\}^{n\geq 0}$. Indeed, in case of a flat graph $\mathbf{G}$, the semantics Mbs takes its values from the sequence $\mathcal{S}$, i.e. $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(.) \in \mathcal{S}$. It uses the three parts of the sequence $\mathcal{S}$, namely $\mathcal{S}_1, \mathcal{S}_2, \frac{1}{\varphi}$, for distinguishing between three groups of arguments in $\mathbf{G}$:

- The first group is composed of all elements of the set $\bigcup\limits_{i\geq 1} \mathcal{F}^i(\emptyset)$, with $\mathcal{F}$ being the characteristic function defined by (Dung, 1995) and that returns all the arguments defended by a given set of arguments. Hence, this group contains all arguments that are defended (directly or indirectly) by non attacked arguments in $\mathbf{G}$. Its members are stronger than any argument outside the group, and their values are taken from the sub-sequence $\mathcal{S}_1$.
- The second group contains the arguments that are attacked by at least one argument of the set $\bigcup_{i\geq 1} \mathcal{F}^i(\emptyset)$. These will be weaker than any argument of the two other groups, and their values are numbers of the sub-sequence $\mathcal{S}_2$.
- The third group contains all the remaining arguments, i.e. those that are neither in nor attacked by the set $\bigcup\limits_{i\geq 1} \mathcal{F}^i(\emptyset)$. The semantics Mbs ascribes the value $\frac{1}{\varphi}$ to every argument of this group.

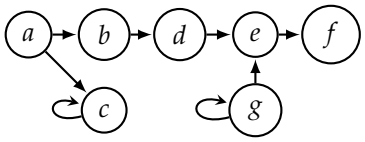
Unlike the third group, the semantics may assign different values to arguments of the first group. It is interesting to note that the value of an argument depends on the iteration at which it appears for the first time in the set $\bigcup\limits_{i\geq 1} \mathcal{F}^i(\emptyset)$. Recall that $\bigcup\limits_{i\geq 1} \mathcal{F}^i(\emptyset) = \mathcal{F}^1(\emptyset) \cup \mathcal{F}^2(\emptyset) \cup \ldots \mathcal{F}^n(\emptyset)$, with $\mathcal{F}^i(\emptyset) = \mathcal{F}(\mathcal{F}^{i-1}(\emptyset))$ is the set of arguments defended by $\mathcal{F}^{i-1}(\emptyset)$. If an argument a appears for the first time in $\mathcal{F}^i(\emptyset)$, then it gets the value of the i^{th} element of $\mathcal{S}_1$, namely $\mathcal{S}_1^i$. Furthermore, every argument it attacks receive the value $\mathcal{S}_2^i$ meaning that arguments of the second group do not necessarily have the same strength as well.

Theorem 8. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in$ AG *and* $i \in \mathbb{N}$. *For any* $a \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) \in \mathcal{S}$. *Furthermore,*

- $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \mathcal{S}_1^i$ *iff* $a \in \mathcal{F}^i(\emptyset)$ *and* $a \notin \bigcup\limits_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$.
- $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \mathcal{S}_2^i$ *iff* $\mathcal{F}^i(\emptyset)$ *attacks* a *and* $\bigcup\limits_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ *does not attack* a.
- $\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \frac{1}{\varphi}$ *iff* $a \notin \bigcup \mathcal{F}^{i\geq 1}(\emptyset)$ *and* $\bigcup \mathcal{F}^{i\geq 1}(\emptyset)$ *does not attack* a.

Note that according to Mbs, the longer the defence path of an argument, the weaker the argument.

Example 14. *Consider the flat graph depicted below:*



Note that $\mathcal{F}^1(\emptyset) = \{a\}$, $\mathcal{F}^2(\emptyset) = \{a, d\}$ *and* $\mathcal{F}^3(\emptyset) = \{a, d, f\}$. *It can be checked that:* $\texttt{Deg}^{\texttt{Mbs}}(a) = \mathcal{S}_1^1 = 1$, $\texttt{Deg}^{\texttt{Mbs}}(d) = \mathcal{S}_1^2 = \frac{2}{3}$, *and* $\texttt{Deg}^{\texttt{Mbs}}(f) = \mathcal{S}_1^3 = \frac{5}{8}$. *Since* $\mathcal{F}^1(\emptyset)$ *attacks both* b, c, *then* $\texttt{Deg}^{\texttt{Mbs}}(b) = \texttt{Deg}^{\texttt{Mbs}}(c) = \mathcal{S}_2^1 = \frac{1}{2}$, *and since* $\mathcal{F}^2(\emptyset)$ *attacks* e, *then* $\texttt{Deg}^{\texttt{Mbs}}(e) = \mathcal{S}_2^2 = \frac{3}{5}$. *Finally,* $\texttt{Deg}^{\texttt{Mbs}}(g) = \frac{1}{\varphi}$.

The second semantics EMbs takes its values from the expoential series. Like Mbs, the semantics EMbs divides the set of arguments into three groups: i) those that belong to the set $\bigcup_{i\geq 1} \mathcal{F}^i(\emptyset)$, ii) those attacked by $\bigcup_{i\geq 1} \mathcal{F}^i(\emptyset)$, and iii) the remaining ones.

Theorem 9. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $i \in \mathbb{N}$. *For any* $a \in \mathcal{A}$, $\texttt{Deg}^{\texttt{EMbs}}(a) \in \mathcal{U}$. *Furthermore,*

- $\texttt{Deg}_{\mathbf{G}}^{\texttt{EMbs}}(a) = \mathcal{U}_1^i$ *iff* $a \in \mathcal{F}^i(\emptyset)$ *and* $a \notin \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$.
- $\texttt{Deg}_{\mathbf{G}}^{\texttt{EMbs}}(a) = \mathcal{U}_2^i$ *iff* $\mathcal{F}^i(\emptyset)$ *attacks* a *and* $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ *does not attack* a.
- $\texttt{Deg}_{\mathbf{G}}^{\texttt{EMbs}}(a) = \Omega$ *iff* $a \notin \bigcup \mathcal{F}^{i\geq 1}(\emptyset)$ *and* $\bigcup \mathcal{F}^{i\geq 1}(\emptyset)$ *does not attack* a.

Example 14 (Cont) It can be checked that $\texttt{Deg}^{\texttt{EMbs}}(a) = \mathcal{U}_1^1 = 1$, $\texttt{Deg}^{\texttt{EMbs}}(d) = \mathcal{U}_1^2 = 0.6922$, and $\texttt{Deg}^{\texttt{EMbs}}(f) = \mathcal{U}_1^3 = 0.6062$. Since $\mathcal{F}^1(\emptyset)$ attacks both b, c, then $\texttt{Deg}^{\texttt{EMbs}}(b) = \texttt{Deg}^{\texttt{EMbs}}(c) = \mathcal{U}_2^1 = 0.3678$, and since $\mathcal{F}^2(\emptyset)$ attacks e, then $\texttt{Deg}^{\texttt{EMbs}}(e) = \mathcal{U}_2^2 = 0.5004$. Finally, $\texttt{Deg}^{\texttt{EMbs}}(g) = \Omega$.

4.4 Conclusion

We provided the mathematical counterparts of the two gradual semantics (Mbs,EMbs) in the context of flat graphs. We plan to investigate their counterparts in the general case (ie weighted graph). We also plain to explore the other gradual semantics.

Chapter 5

Equivalence of Semantics

5.1 Introduction

Comparing semantics has been a hot topic in recent years. A dominant approach consists in identifying *principles* that semantics may satisfy, then analysing every semantics against them. Baroni and Giacomin, 2007 proposed several principles that Torre and Vesic, 2017 used for comparing all the existing extension semantics. Amgoud and Ben-Naim, 2016 introduced another list of principles and used it for comparing some existing gradual/extension semantics. While these studies revealed conceptual differences between the analysed semantics, the messages they convey are not clear. Namely, the following questions remain unanswered:

- Are semantics satisfying the same principles similar, i.e. they provide the same evaluation of arguments?
- Are semantics following different principles incompatible, i.e. they may provide contradictory evaluations?

More generally, when are two semantics similar? In this thesis, we argue that to be similar, two semantics should agree not only on the evaluation of every individual argument but also on the ranking of arguments wrt their strengths. The first agreement depends on the principles followed by the two semantics. For instance, if both semantics satisfy the *maximality* principle from (Amgoud and Ben-Naim, 2016), then they would assign the highest strength to a non-attacked argument. However, satisfying the same principles does not guarantee agreement on the ranking of arguments. Consider for instance the two gradual semantics `Mbs` (Amgoud et al., 2017) and `EMbs` (Amgoud and Doder, 2019). They satisfy exactly the same set of principles. However, as we will see in the chapter, they may provide contradictory rankings of pairs of arguments when graphs are weighted. They are thus not similar and decision systems using them may make contradictory recommendations. Consider for instance the case of a committee which recruits a new researcher, and assume that the two candidates are supported respectively by the arguments a and b. Assume also that `Mbs` declares a as stronger than b and `EMbs` proposes the opposite ranking. Note that according to the semantics that is considered, the committee may make different recruitment's.

To sum up, the existing comparisons of semantics are incomplete as they focused only on the first requirement for similarity. This chapter completes them by investigating the second requirement on rankings. Its contributions are threefold: First, it introduces three novel relations between semantics based on their rankings of arguments: *weak equivalence*, *strong equivalence* and *refinement*. They state respectively that two semantics do not disagree on their strict rankings, the rankings of the semantics

coincide, and one semantics agrees with the strict comparisons of the second, furthermore it breaks some of its ties. Second, it investigates the properties of the three relations and their links with existing principles of semantics. Third, it studies the nature of relations between most of the existing semantics when dealing with flat or weighted graphs.

5.2 Relations between semantics

Before introducing the three novel relations between semantics, let us first recall the notion of preordering. It is a binary relation that is reflexive and transitive.

Definition 26 (Preordering)**.** *A* preordering *on a set X is a binary relation $\succeq$ on X such that:*

- *For any $x \in X$, $x \succeq x$* *(Reflexivity)*
- *For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$* *(Transitivity)*

A preordering $\succeq$ is total *iff for all $x, y \in X$, $x \succeq y$ or $y \succeq x$.*

Notation 4. *The notation $x \succ y$ is a shortcut for $x \succeq y$ and $y \not\succeq x$ and $x \approx y$ is a shortcut for $x \succeq y$ and $y \succeq x$. The relation $\succ$ is the strict version of $\succeq$, i.e. $\succ = \{(x, y) \in X^2 \mid x \succ y\}$, and $\approx$ is the equivalence relation, i.e. $\approx = \{(x, y) \in X^2 \mid x \succeq y \text{ and } y \succeq x\}$.*

Property 2. *If a binary relation $\succeq$ is a total preordering, then $\succeq = \succ \cup \approx$.*

We have seen previously that a semantics assigns a numerical value to each argument in argumentation framework. Those values are used for defining a preference relation between arguments as follows:

Definition 27 (Ranking)**.** *Let $\mathbf{S} \in \texttt{Sem}$ and $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$. A* ranking induced *from $\mathbf{S}$ is a binary relation $\succeq_{\mathbf{S}}^{\mathbf{G}}$ on $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$,*

$$a \succeq_{\mathbf{S}}^{\mathbf{G}} b \quad \textit{iff} \quad \texttt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \texttt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b).$$

The notation $a \succeq_{\mathbf{S}}^{\mathbf{G}} b$ expresses that the argument a is at least as strong as the argument b in the graph $\mathbf{G}$ under the semantics $\mathbf{S}$. Hence, $a \succ_{\mathbf{S}}^{\mathbf{G}}$ and $a \approx_{\mathbf{S}}^{\mathbf{G}} b$ state respectively that a is stronger than b and a is as strong as b under $\mathbf{S}$.
The relation $\succeq_{\mathbf{S}}^{\mathbf{G}}$ is obviously a total preordering, ie, it is reflexive, transitive and compares every pair or arguments.

Property 3. *For any $\mathbf{S} \in \texttt{Sem}$, for any $\mathbf{G} \in \texttt{AG}$, the ranking $\succeq_{\mathbf{S}}^{\mathbf{G}}$ is a total preordering.*

In what follows, we study when two semantics $\mathbf{S}_1$ and $\mathbf{S}_2$ are equivalent by comparing the rankings they provide. We introduce three relations between two rankings. The first relation is that of *refinement*. A semantics refines (or is more *discriminating* than) another semantics when it agrees with its strict comparisons and may break some of its ties.

Definition 28 (Refinement)**.** *Let $\mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$. We say that $\mathbf{S}_1$* refines *$\mathbf{S}_2$ iff $\forall \mathbf{G} \in \texttt{AG}$, $\succ_{\mathbf{S}_2}^{\mathbf{G}} \subseteq \succ_{\mathbf{S}_1}^{\mathbf{G}}$.*

Property 4. *The following properties hold:*

- *Let $\mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$. If $\mathbf{S}_1$ refines $\mathbf{S}_2$, then $\forall \mathbf{G} \in \texttt{AG}$, the inclusion $\approx_{\mathbf{S}_1}^{\mathbf{G}} \subseteq \approx_{\mathbf{S}_2}^{\mathbf{G}}$ holds.*

- *The refinement relation is transitive.*

The second notion, *weak equivalence*, ensures the compatibility of two rankings. More precisely, two semantics are weakly equivalent if they never provide opposite rankings. This notion is more general than refinement because neither of the two semantics should refine the other.

Definition 29 (Weak Equivalence)**.** *Let* $\mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$*. We say that* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are* weakly equivalent *iff* $\forall \mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$, $\nexists\, a, b \in \mathcal{A}$ *s.t.* $a \succ^{\mathbf{G}}_{\mathbf{S}_1} b$ *and* $b \succ^{\mathbf{G}}_{\mathbf{S}_2} a$*. The two semantics are said to be* incompatible *otherwise.*

Weak equivalence is a binary relation over the set of all possible semantics. It is reflexive but not transitive. Indeed, for three semantics $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$, it is possible that $\mathbf{S}_1, \mathbf{S}_2$ (resp. $\mathbf{S}_2, \mathbf{S}_3$) are weakly equivalent but $\mathbf{S}_1, \mathbf{S}_3$ are not.

Property 5. *The weak equivalence relation is not transitive.*

It is worthy to notice that two weakly equivalent semantics may not provide exactly the same ranking of arguments. Consider for instance four arguments a, b, c, d and two semantics $\mathbf{S}_1$ and $\mathbf{S}_2$. One may have: $a \succ_{\mathbf{S}_1} b$ and $a \approx_{\mathbf{S}_2} b$, while $c \approx_{\mathbf{S}_1} d$ and $c \succ_{\mathbf{S}_2} d$.

The third notion of *strong equivalence* ensures total agreement of two semantics regarding their rankings.

Definition 30 (Strong Equivalence)**.** *Let* $\mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$*. We say that* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are* strongly equivalent *iff for any* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$*, the equality* $\succeq^{\mathbf{G}}_{\mathbf{S}_1} = \succeq^{\mathbf{G}}_{\mathbf{S}_2}$ *holds.*

Since rankings induced by semantics are total preorderings, then two strongly equivalent semantics agree both on strict rankings and on ties.

Property 6. *The following properties hold:*

- *Two semantics* $\mathbf{S}_1, \mathbf{S}_2$ *are strongly equivalent iff* $\succ_{\mathbf{S}_1} = \succ_{\mathbf{S}_2}$ *and* $\approx_{\mathbf{S}_1} = \approx_{\mathbf{S}_2}$*.*
- *Strong equivalence is transitive.*

The following property summarizes the different links between the above three relations.

Proposition 15. *Let* $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3 \in \texttt{Sem}$*.*

- $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are strongly equivalent iff they refine each other.*
- *If* $\mathbf{S}_1$ *refines* $\mathbf{S}_2$*, then* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are weakly equivalent.*
- *If* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are strongly equivalent, then they are weakly equivalent. The converse does not hold.*
- *If* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are strongly equivalent and* $\mathbf{S}_2$ *and* $\mathbf{S}_3$ *are weakly equivalent, then* $\mathbf{S}_1$ *and* $\mathbf{S}_3$ *are weakly equivalent.*
- *If* $\mathbf{S}_1$ *refines* $\mathbf{S}_2$ *and* $\mathbf{S}_1$ *is strongly equivalent to* $\mathbf{S}_3$*, then* $\mathbf{S}_3$ *refines* $\mathbf{S}_2$*.*
- *If* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are strongly equivalent and* $\mathbf{S}_2$ *and* $\mathbf{S}_3$ *are incompatible, then* $\mathbf{S}_1$ *and* $\mathbf{S}_3$ *are incompatible.*
- *If* $\mathbf{S}_1$ *refines* $\mathbf{S}_2$ *and* $\mathbf{S}_2$ *and* $\mathbf{S}_3$ *are incompatible, then* $\mathbf{S}_1$ *and* $\mathbf{S}_3$ *are incompatible.*

We show next that the principles recalled in the previous section are necessary but not sufficient for the equivalence of two semantics. Indeed, two semantics may satisfy the same set of principles without being strongly or weakly equivalent. Conversely, two semantics may be strongly equivalent while they do not follow the same principles. Before introducing the formal result, let us denote by $\texttt{Princ}(\mathbf{S})$ the subset of principles satisfied by a given semantics $\mathbf{S}$, and by I the set containing the following principles: Neutrality, Equivalence, Counting, Reinforcement, Proportionality, and Monotony.

Proposition 16. *The following properties hold:*

- *Let* $\mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$ *and* $x \in I$. *If* $\mathbf{S}_1$ *satisfies* x *and* $\mathbf{S}_2$ *violates* x, *then* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are not strongly equivalent.*
- *If semantics* $\mathbf{S}_1$ *satisfies Cardinality Precedence and semantics* $\mathbf{S}_2$ *satisfies Quality Precedence, then* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are incompatible.*
- $\exists \mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$ *such that* $\mathbf{S}_1, \mathbf{S}_2$ *are strongly equivalent and* $\texttt{Princ}(\mathbf{S}_1) \neq \texttt{Princ}(\mathbf{S}_2)$.
- $\exists \mathbf{S}_1, \mathbf{S}_2 \in \texttt{Sem}$ *such that* $\texttt{Princ}(\mathbf{S}_1) = \texttt{Princ}(\mathbf{S}_2)$ *and* $\mathbf{S}_1, \mathbf{S}_2$ *are incompatible.*

The above results show that for being similar, it is not sufficient that two semantics evaluate in the same way every individual arguments. They should also provide the same ranking of arguments. Existing comparisons of semantics focused exclusively on the first criterion by relying on the principles followed by semantics. In the next section, we will compare semantics wrt their rankings.

5.3 Comparison of semantics

In what follows, we compare twelve semantics using the three novel notions (refinement, weak equivalence, strong equivalence). We consider the four extension semantics (grounded, ideal, stable and preferred) given in Definition 5 (Chapter 1), and the eight gradual semantics recalled in Table 1.

5.3.1 Comparison of extension semantics

The comparison of a pair of semantics amounts to comparing their rankings. The latter are obtained by comparing the strength degrees ascribed to arguments. In Definition 7 (chapter 1), the scale that has been used in the literature for extension semantics is qualitative and contains four values: sceptically accepted, credulously accepted, undecided and rejected. For the sake of consistency with gradual semantics, we replace this qualitative scale by a numerical one. The idea is to use the scale $\mathcal{T} = \{1, \alpha, \beta, 0\}$ with $1 > \alpha > \beta > 0$, and to assign the value 1 to sceptically accepted arguments, the value α to credulously accepted arguments, β to undecided arguments, and 0 to rejected ones.

Definition 31. *Let* $\mathbf{S} \in \{i, s, p, g\}$, $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$. *If* $\mathbf{G}$ *has no extensions, then* $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \beta$. *Otherwise,*

- $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = 1$ *iff* a *belongs to all extensions.*
- $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \alpha$ *iff* a *is in some but not all extensions.*
- $\texttt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \beta$ *iff* a *does not belong to any extension and is not attacked by any extension.*

Semantics	Definition	Graphs
h-Categorizer ($\mathtt{hCat}$) (Besnard and Hunter, 2001)	$\mathtt{Deg}^{h}_{\mathbf{G}}(a) = \frac{1}{1+\sum_{b\mathcal{R}a}\mathtt{Deg}^{h}_{\mathbf{G}}(b)}$	Flat
Compensation-based ($\alpha-\mathtt{BBS}$) (Amgoud et al., 2016)	$\mathtt{Deg}^{\alpha-\mathtt{BBS}}_{\mathbf{G}}(a) = 1+\left(\sum_{b\mathcal{R}a}\frac{1}{(s(b))^{\alpha}}\right)^{1/\alpha}$, $\alpha \in (0,+\infty)$	Flat
Weighted h-Categorizer ($\mathtt{Hbs}$) (Amgoud et al., 2017)	$\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1+\sum_{b\mathcal{R}a}\mathtt{Deg}^{\mathtt{Hbs}}_{\mathbf{G}}(b)}$	Weighted
Weighted Card-based ($\mathtt{Cbs}$) (Amgoud et al., 2017)	$\mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1+\lvert\mathtt{Att}^{\mathtt{F}}_{\mathbf{G}}(a)\rvert+\frac{\sum_{b\in\mathtt{Att}^{\mathtt{F}}_{\mathbf{G}}(a)}\mathtt{Deg}^{\mathtt{Cbs}}_{\mathbf{G}}(b)}{\lvert\mathtt{Att}^{\mathtt{F}}_{\mathbf{G}}(a)\rvert}}$ where $\mathtt{Att}^{\mathtt{F}}_{\mathbf{G}}(a) = \{b \in \mathtt{Att}_{\mathbf{G}}(a) \mid \sigma(b) > 0\}$	Weighted
Weighted Max-based ($\mathtt{Mbs}$) (Amgoud et al., 2017)	$\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(a) = \frac{\sigma(a)}{1+\max_{b\mathcal{R}a}\mathtt{Deg}^{\mathtt{Mbs}}_{\mathbf{G}}(b)}$	Weighted
Euler-Max-based ($\mathtt{EMbs}$) (Amgoud and Doder, 2019)	$\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(a) = \sigma(a)\cdot e^{-\max_{b\mathcal{R}a}\mathtt{Deg}^{\mathtt{EMbs}}_{\mathbf{G}}(b)}$	Weighted
Trust-based ($\mathtt{TB}$) (Costa Pereira, Tettamanzi, and Villata, 2011)	$\mathtt{Deg}^{\mathtt{TB}}_{\mathbf{G}}(a) = \lim_{i\to+\infty} f_i(a)$, where $f_i(a) = \frac{1}{2}f_{i-1}(a) + \frac{1}{2}\min[w(a), 1-\max_{b\mathcal{R}a} f_{i-1}(b)]$	Weighted
Iterative Schema ($\mathtt{IS}$) (Gabbay and Rodrigues, 2015)	$\mathtt{Deg}^{\mathtt{IS}}_{\mathbf{G}}(a) = \lim_{i\to+\infty} f_i(a)$, where $f_i(a) = (1-f_{i-1}(a))\min(\frac{1}{2}, 1-\max_{b\mathcal{R}a}\mathtt{f_{i-1}(b)}) +$ $f_{i-1}(a)\max(\frac{1}{2}, 1-\max_{b\mathcal{R}a}\mathtt{f_{i-1}(b)})$	Weighted

TABLE 5.1: Gradual semantics dealing with cycles.

- $\mathtt{Deg}^{\mathtt{S}}_{\mathbf{G}}(a) = 0$ *iff* a *does not belong to any extension and is attacked by at least one extension.*

Depending on the semantics some values of the scale T may not be used. Indeed, under grounded and ideal, an argument can never get the value α. When stable extensions exist, an argument cannot receive the value β.

Property 7. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R}\rangle \in \mathtt{AG}$ *and* $a \in \mathcal{A}$.

- $\mathtt{Deg}^{g}_{\mathbf{G}}(a) \in T \setminus \{\alpha\}$
- $\mathtt{Deg}^{i}_{\mathbf{G}}(a) \in T \setminus \{\alpha\}$
- *If stable extensions exist,* $\mathtt{Deg}^{s}_{\mathbf{G}}(a) \in T \setminus \{\beta\}$.

From their definitions, the four semantics seem compatible. For instance, grounded is more sceptical than ideal, which in turn is more sceptical than preferred and stable. The following result confirms this observation, however it shows that the four semantics are only weakly equivalent. None of them refines or is strongly equivalent to the others.

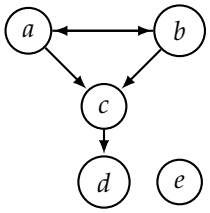
Theorem 10. *The four semantics (stable, preferred, grounded, ideal) are pairwise weakly equivalent.*

Consider the following example.

Example 15. *Consider the flat argumentation graph depicted below. Its grounded extension is* $\{e\}$ *and its stable extensions are* $\{a,d,e\}$ *and* $\{b,d,e\}$. *Note that* $\mathtt{Deg}^{g}_{\mathbf{G}}(e) = 1$, $\mathtt{Deg}^{g}_{\mathbf{G}}(d) = \beta$, *and* $\mathtt{Deg}^{s}_{\mathbf{G}}(e) = \mathtt{Deg}^{s}_{\mathbf{G}}(d) = 1$. *Hence,* $e \succ^{g} d$ *while* $e \approx^{s} d$. *We have also* $\mathtt{Deg}^{g}_{\mathbf{G}}(a) = \mathtt{Deg}^{g}_{\mathbf{G}}(d) = \beta$, $\mathtt{Deg}^{s}_{\mathbf{G}}(a) = \alpha$. *So,* $a \approx^{g} d$ *while* $d \succ^{s} a$.

5.3.2 Comparison of gradual semantics

As said above, we focus here on the eight gradual semantics recalled in Table 5.1. Note that the two first semantics ($\mathtt{hCat}$ and $\alpha-\mathtt{BBS}$) deal only with flat graphs.



The first result states that hCat and $\alpha-$BBS are strongly equivalent. Furthermore, in case of flat graphs (i.e. graphs with $\sigma \equiv 1$), hCat and Weighted h-categorizer (Hbs) are also strongly equivalent since they coincide. These three semantics are fully similar since they provide both the same evaluations of individual arguments and the same rankings.

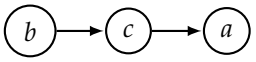
Theorem 11. *In case of flat graphs,* hCat, $\alpha-$BBS *(with* $\alpha = 1$*) and* Hbs *are pairwise strongly equivalent.*

The six semantics that deal with weighted graphs are pairwise incompatible, that is, they may return contradictory rankings for pairs of arguments. They are thus all different, and the choice of the one to use depends on the application and the nature of arguments (deductive, analogical, ...).

Theorem 12. *The six semantics* Hbs, Mbs, Cbs, EMbs, TB *and* IS *are pairwise incompatible.*

It is worthy to recall that Mbs and EMbs satisfy exactly the same subset of principles (see Amgoud and Doder, 2019). However, they may disagree on their ranking of arguments as shown in the following example.

Example 16. *Consider the weighted graph below where* $\sigma(a) = 0.6252$, $\sigma(b) = 0.3939$ *and* $\sigma(c) = 0.8107$.



It can be checked that $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = 0.3953$, $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(b) = 0.3939$, $\text{Deg}_{\mathbf{G}}^{\text{EMbs}}(a) = 0.362$, *and* $\text{Deg}_{\mathbf{G}}^{\text{EMbs}}(b) = 0.3939$. *Note that* $a \succ_{\text{Mbs}} b$ *and* $b \succ_{\text{EMbs}} a$. *This shows that the two semantics may lead to different results.*

Let us now consider only flat argumentation graphs (i.e. graphs of the form $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle$), and let us focus on the four gradual semantics (Mbs, EMbs, TB, IS). These all use the same function, namely max, for aggregating the strengths of an argument's attackers. In other words, they consider only the strongest attacker when computing the strength of any argument.

We show that the two semantics (TB, IS) coincide, i.e. they assign exactly the same value to an argument. Recall that this property is not true in the general case of weighted graphs. We show also that the value of an argument depends on whether the argument is in or is attacked by the grounded extension Gr of the graph. This means that (TB, IS) are two alternative characterizations of the grounded semantics.

Theorem 13. *For any* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in$ AG, *it holds that*

$$\text{Deg}_{\mathbf{G}}^{\text{TB}} \equiv \text{Deg}_{\mathbf{G}}^{\text{IS}}.$$

Moreover, for any $x \in \{\text{TB}, \text{IS}\}$, *for any* $a \in \mathcal{A}$,

- $\texttt{Deg}^x_{\mathbf{G}}(a) = 1$ *iff* $a \in \texttt{Gr}$.
- $\texttt{Deg}^x_{\mathbf{G}}(a) = 0$ *iff* $\texttt{Gr}$ *attacks* a.
- $\texttt{Deg}^x_{\mathbf{G}}(a) = \frac{1}{2}$ *iff* $a \notin \texttt{Gr}$ *and* $\texttt{Gr}$ *does not attack* a.

The two semantics (TB, IS) are thus strongly equivalent when dealing with flat graphs.

Theorem 14. *In case of flat argumentation graphs,* TB *and* IS *are strongly equivalent.*

The two semantics Mbs and EMbs also are strongly equivalent when applied to flat graphs. **Notations:** Let $\mathbf{G}$ be a flat graph, A, B be two sets of arguments and $x \in \texttt{Sem}$. The notation $A \succ_x B$ stands for $\forall a \in A, \forall b \in B$, it holds $\texttt{Deg}^x(a) > \texttt{Deg}^x(b)$. Let A^+ denote the set of arguments that are attacked by at least one argument of A, and A^o be the set of arguments that are neither in nor attacked by the set A. Let $\mathcal{X}_i = \mathcal{F}^i(\emptyset) \setminus \big(\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)\big)$ with $i > 1$ and by convention, $\mathcal{X}_1 = \mathcal{F}^1(\emptyset)$. This means that $\mathcal{X}_1 = \mathcal{F}(\emptyset)$, $\mathcal{X}_2 = \mathcal{F}^2(\emptyset) \setminus \mathcal{F}^1(\emptyset)$, and so on.

The following result follows from the characterizations of Mbs, EMbs by numerical series as seen in the previous chapter. It shows how the two semantics refine the grounded extension of a graph, and how they refine the set of arguments attacked by the grounded extension.

Theorem 15. *Let* $\mathbf{G}$ *be a flat graph and* $x \in \{\texttt{Mbs}, \texttt{EMbs}\}$.

- $\mathcal{X}_1 \succ_x \mathcal{X}_2 \succ_x \ldots \succ_x \mathcal{X}_n$.
- $\mathcal{X}_n^+ \succ_x \mathcal{X}_{n-1}^+ \succ_x \ldots \succ_x \mathcal{X}_1^+$.
- $\texttt{Gr} \succ_x \texttt{Gr}^o \succ_x \texttt{Gr}^+$.

From the characterizations of Mbs and EMbs, it follows that the two semantics are strongly equivalent.

Theorem 16. *In case of flat argumentation graphs,* Mbs *and* EMbs *are strongly equivalent.*

The previous results show that TB and IS assign the same value to any argument that belongs to (or is attacked by) the grounded extension. Mbs and EMbs are more discriminating and provide finer-grained evaluations and comparisons of arguments. Thus, they both refine TB and IS.

Theorem 17. *In case of flat argumentation graphs,* Mbs *(resp.* EMbs*) refines* TB *(resp.* IS*).*

The novel characterizations of the four semantics Mbs, EMbs, TB and IS show that they are based on the grounded extension of a graph $\mathbf{G}$ as $\texttt{Gr} = \bigcup_{i \geq 1} \mathcal{F}^i(\emptyset)$. They can also be seen as different characterizations of the grounded semantics in terms of various gradual semantics.

5.3.3 Extension Semantics vs Gradual Semantics

Throughout this section, we focus on flat argumentation graphs. We show that hCat, $\alpha-$BBS, Hbs and Cbs are incompatible with any of the recalled extension semantics. This is mainly due to the fact that the first group takes into account all the attackers of an argument while the second focuses only on the strongest one.

Semantics	hCat	α−BBS	Hbs	Cbs	Mbs	EMbs	TB	IS	Grounded	Ideal	Stable	Preferred
hCat	–	s	s	i	i	i	i	i	i	i	i	i
α−BBS	s	–	s	i	i	i	i	i	i	i	i	i
Hbs	s	s	–	i	i	i	i	i	i	i	i	i
Cbs	i	i	i	–	i	i	i	i	i	i	i	i
Mbs	i	i	i	i	–	s	r	r	r	w	w	w
EMbs	i	i	i	i	s	–	r	r	r	w	w	w
TB	i	i	i	i	w	w	–	s	s	w	w	w
IS	i	i	i	i	w	w	s	–	s	w	w	w
Grounded	i	i	i	i	w	w	s	s	–	w	w	w
Ideal	i	i	i	i	w	w	w	w	w	–	w	w
Stable	i	i	i	i	w	w	w	w	w	w	–	w
Preferred	i	i	i	i	w	w	w	w	w	w	w	–

TABLE 5.2: Case of Flat Argumentation Graphs. s, w, r, i stand resp. for strong equivalence, weak equivalence, refinement, incompatibility.

Theorem 18. hCat *(resp.* α−BBS, Hbs, Cbs*) is incompatible with grounded, ideal, stable and preferred.*

Unsurprisingly, the two semantics TB and IS are strongly equivalent with the grounded semantics. Furthermore, they are weakly equivalent to the three other extension semantics.

Theorem 19. *In case of flat argumentation graphs, the following properties hold:*

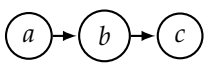
- TB, IS *and the grounded are pairwise strongly equivalent.*
- TB *(resp.* IS*) is weakly equivalent with ideal (resp. stable and preferred).*

We show also that Mbs (resp. EMbs) refines the grounded semantics when applied to flat argumentation graphs. However, the converse does not hold, thus the two semantics are not strongly equivalent (see Example 14).

Theorem 20. *In case of flat argumentation graphs,* Mbs *(resp.* EMbs*) refines the grounded semantics.*

Mbs (resp. EMbs) does not refine ideal, stable and preferred semantics. It is thus not strongly equivalent with any of these three semantics as shown below.

Example 17. *Consider the flat graph below. It has a single preferred (respectively stable) extension which coincides with the ideal and grounded extension* $\{a, c\}$*. Thus,* $a \equiv_x c$*, with* $x \in \{i, s, p\}$*, while* $a \succ_{\texttt{Mbs}} c$ *and* $a \succ_{\texttt{EMbs}} c$*.*



The last result shows that the Mbs (resp. EMbs) semantics is weakly equivalent with ideal, stable and Preferred semantics. They provide thus compatible rankings of arguments.

Theorem 21. *In case of flat argumentation graphs,* `Mbs` *(resp.* `EMbs`*) is weakly equivalent with ideal, stable and preferred semantics.*

Table 5.2 summarizes the relations between the reviewed semantics in case of flat argumentation graphs. There are 5 groups of semantics. The first group contains (`hCat`, `Hbs`, $\alpha-$`BBS`), which are strongly equivalent, provide the same evaluations for individual arguments, and are incompatible with all the other semantics. The second group contains only `Cbs`, which is incompatible with any other semantics. Like the semantics of the first group, with `Cbs` every attacker is harmful to its target. However, `Cbs` favors the quantity of attackers while the former promote compensation. The third group is made of `Mbs`, `EMbs`, which are strongly equivalent, and refine some semantics that consider only one attacker, namely `TB`, `IS` and grounded. The fourth group is made of these three semantics, which are strongly equivalent. The last group contains ideal, stable and preferred, which are incompatible with semantics of the first and second groups, and are weakly equivalent with any of the remaining semantics.

5.4 An equivalence class of semantics

we have seen previously that in case of flat graph argumentation frameworks, the two gradual semantics `Mbs` and `EMbs` are strongly equivalent, ie, they provide the same rankings. In what follows, we go further by characterizing the whole equivalence class to which they belong. Recall that the notion of strong equivalence structures the universe few of semantics into equivalence classes.
Before introducing the class, let us first recall the family $\mathbf{S}^*$ of numerical series.

Definition. *We define* $\mathbf{S}^*$ *to be the set containing any numerical series* $\mathcal{S} = (\mathcal{S}^n)^{n\geq 1}$ *which satisfies the following conditions:*

- *for any* $n \geq 1$, $\mathcal{S}^n \in [0, 1]$
- $\mathcal{S}$ *contains two sub-series* $\mathcal{S}_1 = (\mathcal{S}_1^n)^{n\geq 1}$ *and* $\mathcal{S}_2 = (\mathcal{S}_2^n)^{n\geq 1}$ *s.t. for any* $n \geq 1$*:*
  - $\mathcal{S}_1^n$ *is strictly decreasing and* $\mathcal{S}_2^n$ *is strictly increasing*
  - $\lim\limits_{n\to\infty} \mathcal{S}_1^n \geq \lim\limits_{n\to\infty} \mathcal{S}_2^n$

In what follows, we define a family of gradual semantics that are based on series of the set $\mathbf{S}^*$. Every instance (i.e., semantics) of the family partitions the set of arguments into three groups. It assigns to arguments of the first group values taken from the decreasing sub-sequence of the series on which it is based. The exact value of an argument depends on the iteration at which it appears for the first time in the set $\mathtt{Gr} = \bigcup\limits_{n\geq 1} \mathcal{F}^n(\emptyset)$. The arguments of the second group are assigned values from the increasing sub-sequence and the value of an argument depends on the first appearance of its strongest attacker in `Gr`. Finally, the semantics ascribes the same value, which is between the limits of the two sub-sequences, to all arguments of the third group.

Definition 32 (`Sem`*)**.** *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in$ `AG`*. A gradual semantics* $\sigma \in$ `Sem` *based on a series* $\mathcal{S} \in \mathbf{S}^*$ *is a mapping from* $\mathcal{A}$ *to* $\mathcal{S} \cup \{\delta\}$*, with* $\lim\limits_{n\to\infty} \mathcal{S}_1^n \geq \delta \geq \lim\limits_{n\to\infty} \mathcal{S}_2^n$*, such that for any* $a \in \mathcal{A}$*,*

- $\mathtt{Deg}^{\sigma}_{\mathbf{G}}(a) = \mathcal{S}_1^i$ *if* $a \in \mathcal{F}^i(\emptyset)$ *and* $a \notin \bigcup\limits_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$*.*

- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(a) = \mathcal{S}^i_2$ *if* $\mathcal{F}^i(\emptyset)$ *attacks* a *and* $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ *does not attack* a.
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(a) = \delta$ *otherwise.*

We denote by $\texttt{Sem}^*$ *the set of all semantics that are based on a series from* $\mathbf{S}^*$.

Example 18. *Assume* $\sigma \in \texttt{Sem}^*$ *and* σ *is based on* $\mathcal{S} \in \mathbf{S}^*$. *It is easy to check that* $\texttt{Deg}^{\sigma}_{\mathbf{G}}(g) = \delta$ *with* $\lim_{n\to\infty} \mathcal{S}^n_1 \geq \delta \geq \lim_{n\to\infty} \mathcal{S}^n_2$.

- $a \in \mathcal{F}^1(\emptyset)$
- $d \in \mathcal{F}^2(\emptyset) \setminus \mathcal{F}^1(\emptyset)$
- $f \in \mathcal{F}^3(\emptyset) \setminus \bigcup_{i=1}^{2} \mathcal{F}^i(\emptyset)$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(a) = \mathcal{S}^1_1$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(d) = \mathcal{S}^2_1$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(f) = \mathcal{S}^3_1$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(b) = \mathcal{S}^1_2$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(c) = \mathcal{S}^1_2$
- $\texttt{Deg}^{\sigma}_{\mathbf{G}}(e) = \mathcal{S}^2_2$

We show next that the set $\texttt{Sem}^*$ is non-empty as it contains $\texttt{Mbs}$ and $\texttt{EMbs}$.

Proposition 17. *It holds that* $\{\texttt{Mbs}, \texttt{EMbs}\} \subseteq \texttt{Sem}^*$.

We show below that all semantics in the set $\texttt{Sem}^*$ are pairwise strongly equivalent.

Theorem 22. *For all* $\sigma, \sigma' \in \texttt{Sem}^*$, σ *and* σ' *are strongly equivalent.*

We go one step further by showing that $\texttt{Sem}^*$ is the whole equivalence class as it contains all semantics that are strongly equivalent to $\texttt{Mbs}$.

Theorem 23. *For any* $\sigma \in \texttt{Sem}$, *if* σ *and* $\texttt{Mbs}$ *are strongly equivalent, then* $\sigma \in \texttt{Sem}^*$.

We show next that every semantics of the class $\texttt{Sem}^*$ refines the *grounded semantics* from Dung, 1995. Indeed, it keeps all its strict rankings and breaks some ties. Before presenting the formal result, let us first recall the grounded extension and the ranking it induces. The latter considers arguments of the grounded extension as equally strong and strictly stronger than all remaining arguments. Arguments that are attacked by the extension are equally strong and strictly weaker than all remaining arguments.

Definition 33 ($\succeq^{\texttt{g}}$). *Let* $\mathcal{A} \in \texttt{AG}$. *The* grounded extension *of* $\mathbf{G}$ *is the set* $\texttt{Gr} = \bigcup_{n=1}^{\infty} \mathcal{F}^n(\emptyset)$. *It induces a total preordering* $\succeq^{\texttt{g}}$ *such that* $\texttt{Gr} \succ^{\texttt{g}} \texttt{Gr}^o \succ^{\texttt{g}} \texttt{Gr}^+$ *and for any* $x \in \{\texttt{Gr}, \texttt{Gr}^+, \texttt{Gr}^o\}$, *for all* $a, b \in x$, $a \approx^{\texttt{g}} b$.

Example 19. *running The grounded extension of the flat graph is* $\texttt{Gr} = \{a, d, f\}$. *Hence,* $a \approx^{\texttt{g}} d \approx^{\texttt{g}} f \succ^{\texttt{g}} g \succ^{\texttt{g}} b \approx^{\texttt{g}} c \approx^{\texttt{g}} e$. *However, for any* $\sigma \in \texttt{Sem}^*$, *we have the following ranking:* $a \succ^{\sigma} d \succ^{\sigma} f \succ^{\sigma} g \succ^{\sigma} e \succ^{\sigma} b \approx^{\sigma} c$.

Theorem 24. *For any* $\sigma \in \texttt{Sem}^*$, σ *refines the grounded semantics.*

From the definition of the ranking $\succeq^{\texttt{g}}$ and the previous result, it follows that any semantics $\sigma \in \texttt{Sem}^*$ preserves the strict ordering of $\succeq^{\texttt{g}}$.

Theorem 25. *For any* $\sigma \in \texttt{Sem}^*$, *the strict relations* $\texttt{Gr} \succ^{\sigma} \texttt{Gr}^o \succ^{\sigma} \texttt{Gr}^+$ *hold.*

5.5 Another Instance of the Equivalence Class

We have seen in the previous section that the equivalence class $\texttt{Sem}^*$ contains at least two semantics: Mbs and EMbs. In what follows, we discuss another instance of the class. It is based on the series $\mathcal{J}$ defined from Jacobsthal sequence (see Definition 25 in chapter 4)

We now characterize the gradual semantics which is based on the above series J. Like Mbs and EMbs, its only considers the strongest attacker of an argument when computing its strength. Furthermore, it takes half of that attacker's strength.

Theorem 26. *Let* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$, $a \in \mathcal{A}$ *and* Jac *a mapping from* $\mathcal{A}$ *to* $[0,1]$ *such that for any* $a \in \mathcal{A}$, $\texttt{Deg}^{\texttt{Jac}}_{\mathbf{G}}(a) = 1 - \frac{\max_{b \mathcal{R} a} \texttt{Deg}^{\texttt{Jac}}_{\mathbf{G}}(b)}{2}$. *The following hold:*

- Jac *is based on the series* J
- $\texttt{Jac} \in \texttt{Sem}^*$
- *for any* $a \in \mathcal{A}$, $\texttt{Deg}^{\texttt{Jac}}_{\mathbf{G}}(a) \in [\frac{1}{2}, 1]$

Example 20. *running The values of the arguments are summarized in the table below.*

	a	b	c	d	e	f	g
Mbs	1	0.50	0.50	0.66	0.60	0.62	0.61
EMbs	1	0.36	0.36	0.69	0.50	0.60	0.56
Jac	1	0.5	0.5	0.75	0.625	0.69	0.66

In the example, Jac assigns higher values than Mbs, EMbs. The following result confirms this property which is valid in general. It also shows that Mbs assigns higher values than EMbs but only to arguments that are attacked by the grounded extension.

Theorem 27. *Let* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$. *The following properties hold:*

- $\texttt{Deg}^{\texttt{Jac}}_{\mathbf{G}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}_{\mathbf{G}}(a)$
- $\texttt{Deg}^{\texttt{Jac}}_{\mathbf{G}}(a) \geq \texttt{Deg}^{\texttt{EMbs}}_{\mathbf{G}}(a)$
- *If* $a \in \texttt{Gr}^+$, *then* $\texttt{Deg}^{\texttt{Mbs}}_{\mathbf{G}}(a) > \texttt{Deg}^{\texttt{EMbs}}_{\mathbf{G}}(a)$

5.6 Conclusion

An important question is which semantics to choose in a given application ? (Amgoud and Ben-Naim, 2013), the authors have argued that the choice of the suitable semantics depends on the nature of arguments (deductive, analogical, casual,...) and the specificities of the application. It is worth mentioning that existing principles provide a good basis for assessing the suitability of a semantics. However, they are insufficient since two semantics may satisfy the same principles and still return opposite rankings. This means the two semantics may lead to different outcomes in decision making, for instance.

We tackled the question of comparing semantics, be them extension-based or gradual. We have shown that existing comparisons in the literature are not conclusive, since they were only based on the evaluations of individual arguments. We argued that comparing rankings is also crucial for the similarity of semantics. Thus investigated a notion of equivalence which is based on the comparison of rankings, and we studied several existing semantics. The results have shown compatibilities

between the classical extension semantics and the gradual semantics that consider only the stronger attacker of an argument when computing its strength. Some gradual semantics like TB, IS go further by providing characterizations for the grounded semantics, while Mbs, EMbs refine the latter in a reasoned way. The gradual semantics that consider all attackers are incompatible with extension semantics. Furthermore, they are themselves pairwise incompatible in case of weighted graphs. The chapter has tackled the question of comparing semantics, be them extension-based or gradual. It has shown that existing comparisons in the literature are not conclusive, since they were only based on the evaluations of individual arguments. It has argued that comparing rankings is also crucial for the similarity of semantics. The chapter has thus investigated a notion of equivalence which is based on the comparison of rankings, and has studied several existing semantics. The results have shown compatibilities between the classical extension semantics and the gradual semantics that consider only the stronger attacker of an argument when computing its strength. Some gradual semantics like TB, IS go further by providing characterizations for the grounded semantics, while Mbs, EMbs refine the latter in a reasoned way. The gradual semantics that consider all attackers are incompatible with extension semantics. Furthermore, they are themselves pairwise incompatible in case of weighted graphs.

Finally, we contributed to setting up the mathematical foundations of computational argumentation and precisely gradual semantics. We defined a mathematical counterpart of a large class of semantics. We showed that a number of semantics including Mbs, EMbs and Jac can be defined with numerical series having specific characteristics. Furthermore, those semantics provide all the same rankings of arguments of a flat graph while they may assign different values to the same argument. This show that a ranking is more expressive than the number assigned to arguments. Finally, the semantics of the class refine the grounded extension and the value of an argument depends on the iteration at which it appears for the first time in the grounded extension or is attacked by the grounded extension. This characterization allowed us to define a very efficient algorithm which computes strength of arguments.

Chapter 6

Case-based Reasoning

6.1 Introduction

Case-based reasoning (CBR) is an experience-based approach to solving problems. It uses stored cases describing similar prior problem-solving episodes and adapts their solutions to fit new needs (or new cases). For example, a car dealer would guess the price of a given car by comparing its characteristics with those of cars that have been sold. This form of reasoning has been used in the literature for solving various practical problems. Indeed, several CBR models have been proposed in the medical domain including CASSEY (Tsatsoulis and Kashyap, 1993), BOLERO (López and Plaza, 1993), SHRINK (Kolodner and Kolodner, 1987), MNAOMIA (Bichindaritz, 1996), AUGUST (Marling and Whitehouse, 2001), SOMNUS (Kwiatkowska and Atkins, 2004), CARE PARTNER (Bichindaritz, Kansu, and Sullivan, 1998), KASIMIR (Lieber and Bresson, 2000), SIDSTOU (Sharaf-El-Deen, Moawad, and Khalifa, 2014) and MARLING (Marling, Shubrook, and Schwartz, 2008)

Another practical domain where CBR is commonly used is law. Several works have been done, some of which are based on argumentation eg., a model of legal reasoning with cases incorporating theories and values (Bench-Capon and Sartor, 2003), A formalisation of argumentation schemes for legal case-based reasoning in ASPIC+ (Prakken et al., 2013).

Several works have been devoted to modeling CBR, and various approaches can be distinguished including logic-based (Zheng, Grossi, and Verheij, 2020; Dubois et al., 1997) and argumentation-based (Paulino-Passos and Toni, 2021; Prakken et al., 2013) approaches (see (Richter and Weber, 2013; Hüllermeier, 2007; Aamodt and Plaza, 1994) for surveys). However, despite its popularity, there are few works on foundations, or properties, that may underlie CBR models. Foundations are important not only for a better understanding of case-based reasoning in general, but also for clarifying the basic assumptions underlying models, comparing different models, and also for comparing case-based reasoning with other kinds of reasoning like defeasible reasoning.

This chapter bridges this gap. It starts by analysing the basic assumption behind case-based reasoning, namely "*the more similar the cases, the more similar their outcomes*". It discusses three independent notions that capture (in different ways) the assumption. Then, the chapter proposes principles that a case-based reasoning model would satisfy and analyses their properties. Some principles ensure the three forms of the CBR assumption, and we show that they are incompatible with some axioms underlying non-monotonic reasoning (NMR) (Kraus, Lehmann, and Magidor, 1990),

namely cautious monotonicity and cut. This shows that CBR and NMR are different forms of reasoning, and sheds light on the reasons behind their differences.

6.2 Case-based reasoning models

Before introducing formally the basic notions of a CBR problem, let us introduce the notion of similarity measure, which plays a key note in CBR. (Dubois et al., 1997).

Definition 34 (Similarity). *A similarity measure* $\mathbf{S}$ *on a set* X *is a function* $\mathbf{S} : X \times X \to [0,1]$ *where:*

- $\forall x \in X, \mathbf{S}(x,x) = 1$
- $\forall x,y \in X, \mathbf{S}(x,y) = \mathbf{S}(y,x)$

Let us now illustrate a CBR problem whith a simple example borrowed from (Dubois et al., 1997)

Example 21. *Consider the problem of identifying the price of second-hand cars. A car is described with five attributes, namely years old, power, mileage, the state of equipment, and shape. Knowing the characteristics and the prices of four cars* (C_1, C_2, C_3, C_4) *(summarized in the table below), the problem is to identify the price of the new car* (C_n) *whose characteristics are also known.*

Cases	*Years old*	*Power*	*Mileage*	*Equipment*	*Shape*	*Price*
C_1	*1*	*1300*	*20 000*	*poor*	*good*	*8000*
C_2	*2*	*1600*	*30 000*	*excellent*	*poor*	*7000*
C_3	*2*	*1600*	*40 000*	*good*	*good*	*5000*
C_4	*3*	*1500*	*60 000*	*excellent*	*poor*	*5000*
C_n	*2*	*1600*	*50 000*	*poor*	*good*	*?*

To identify the price of C_n*, any CBR model would compare the characteristics of cars as well as their prices. Hence, it would use two similarity measures: one for comparing prices* $(\mathbf{S}^o)$ *and another for comparing attributes-values* $(\mathbf{S}^i)$*. In (Dubois et al., 1997),* $\mathbf{S}^o$ *is for instance defined as follows:*

$$\mathbf{S}^o(u,v) = \begin{cases} 1 & \text{if } |u-v| \leq 500 \\ 0 & \text{if } |u-v| \geq 2000 \\ 1 - \frac{1}{1500} * (|u-v| - 500) & \text{if } 500 < |u-v| < 2000 \end{cases}$$

It is easy to check that $\mathbf{S}^o(x,x) = 1$, $\mathbf{S}^o(5000,7000) = \mathbf{S}^o(5000,8000) = 0$ *and* $\mathbf{S}^o(7000,8000) = \frac{2}{3}$.

Regarding $\mathbf{S}^i$*, it combines five measures, each of which compares the values of an attribute.* $\mathbf{S}^1$ *compares years old (respectively mileage) as follows:*

$$\mathbf{S}^1(u,v) = \frac{\min(u,v)}{\max(u,v)}$$

. For instance, $\mathbf{S}^1(1,2) = \frac{1}{2}$ *and* $\mathbf{S}^1(20000,30000) = \frac{2}{3}$*. The measure that compares the powers of two cars is defined as follows:*

$$\mathbf{S}^2(u,v) = 1 - (\frac{|u-v|}{1000})$$

. *For instance,* $\mathbf{S}^2(1300, 1600) = \frac{7}{10}$. *Finally, equipment and shape are compared using the measure* $\mathbf{S}^3$*, which assumes the ordering bad < poor < good < excellent.*

$$\mathbf{S}^3(v, v') = \begin{cases} 1 & \text{if} \quad v = v' \\ \frac{2}{3} & \text{if} \quad v \text{ and } v' \text{ are consecutive} \\ \frac{1}{3} & \text{if} \quad \text{there is exactly one element between } v \text{ and } v' \\ 0 & \text{otherwise.} \end{cases}$$

The similarity between (the characteristics of) two cars is the minimal similarity of the characteristics. For instance, $\mathbf{S}^i(C_1, C_2) = \min(\mathbf{S}^1(1,2), \mathbf{S}^2(1300,1600), \mathbf{S}^1(20000,30000), \mathbf{S}^3(poor, excellent), \mathbf{S}^3(good, poor)) = \min(\frac{1}{2}, \frac{7}{10}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$. *The table below summarises the values returned by* $\mathbf{S}^i$ *for each pair of cars.*

Cases	C_1	C_2	C_3	C_4	C_n
C_1	*1*	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$
C_2	$\frac{1}{3}$	*1*	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
C_3	$\frac{1}{2}$	$\frac{2}{3}$	*1*	$\frac{2}{3}$	$\frac{2}{3}$
C_4	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	*1*	$\frac{1}{3}$
C_n	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	*1*

Throughout the thesis, we assume a finite and non-empty set $\mathcal{F} = \{f_1, \ldots, f_n, \mathtt{f}\}$ of *features*, where $f_1, \ldots, f_n$ describe the cases (eg. Power, Mileage, Shape) and `f` is the feature being solved (price in the example). Let `dom` be a function on $\mathcal{F}$ which returns the domain of every $f \in \mathcal{F}$. Hence, $\mathtt{dom}(\mathtt{f})$ is the set of possible outcomes of a CBR problem, which is finite in classification tasks. In addition to this set, we assume the special symbols ? and `Und`, which denote respectively that the value of `f` is *pending* and *undecided* by a CBR model.

Definition 35 (Input space)**.** *We call literal every pair* (f, v) *such that* $f \in \mathcal{F} \setminus \{\mathtt{f}\}$ *and* $v \in \mathtt{dom}(f)$*, and instance every set of literals, where each feature* $f_1, \ldots, f_n$ *appears exactly once. We denote by* `Inst` *the set of all possible instances, and call it input space.*

The feature space is endowed with a similarity measure $\mathbf{S}^i$, which assesses how close are instances and the set $\mathtt{dom}(\mathtt{f})$ is endowed with a similarity measure $\mathbf{S}^o$, which compares outcomes.
We consider two additional parameters $0 < \delta^i \leq 1$ and $0 < \delta^o \leq 1$, which represent the *thresholds* for considering respectively two instances and two outcomes as somewhat similar. More precisely, for $x, y \in \mathtt{Inst}$, x is dissimilar to y iff $\mathbf{S}^i(x, y) < \delta^i$ and for $v, v' \in \mathtt{dom}(\mathtt{f})$, v is dissimilar to v' iff $\mathbf{S}^o(v, v') < \delta^o$.

Let us now introduce the backbone of a CBR problem, the notion of *case*. It is an instance labelled with an outcome.

Definition 36 (Case)**.** *A* case *is a pair* $c = \langle I, v \rangle$ *such that* $I \in \mathtt{Inst}$ *and* $v \in \mathtt{dom}(\mathtt{f}) \cup \{?\}$*. We call* c *a* past case *when* $v \in \mathtt{dom}(\mathtt{f})$*, and a* new case *when* $v =?$*. A* case base *is a sample that consists of n past cases* $c_i = \langle I_i, v_i \rangle$ $(1 \leq i \leq n)$*.*

Throughout the thesis, we call CBR theory, or theory for short, a tuple containing a set of attributes, their domains, two similarity measures $\mathbf{S}^i$, $\mathbf{S}^o$ and their thresholds.

Definition 37 (Theory)**.** *A* theory *is a tuple* $\mathbf{T} = \langle \mathcal{F}, \mathtt{dom}, \mathbf{S}^i, \mathbf{S}^o, \delta^i, \delta^o \rangle$.

A CBR model is a function, which takes as input a theory and a new case, and returns possible outcomes of the latter. Since every instance is assigned exactly one

label, one expects that a model provides a single solution. However, this is not always possible since the new case may be close to several differently labelled cases, and the model cannot discriminate between those labels. So, each of label is considered as a *candidate* outcome. It is also possible that the new case is dissimilar to all past cases of a base. Hence, instead of returning an arbitrary outcome, we assume that a model may rerun the symbol Und (for undecided), meaning no solution is proposed.

Definition 38 (CBR Model)**.** *Let* $\mathbf{T} = \langle \mathcal{F}, \mathtt{dom}, \mathbf{S}^i, \mathbf{S}^o, \delta^i, \delta^o \rangle$ *be a theory. A* CBR model *is a function* R *mapping every case base* Σ *and new case* $\langle I, ? \rangle$ *into a set* $O \subseteq \mathtt{dom(f)} \cup \{\mathtt{Und}\}$ *such that* $O \neq \emptyset$ *and either* $O = \{\mathtt{Und}\}$ *or* $O \subseteq \mathtt{dom(f)}$*. We write* $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathtt{R}} O$*.*

Through this chapter, we consider a fixed but arbitrary theory $\mathbf{T} = \langle \mathcal{F}, \mathtt{dom}, \mathbf{S}^i, \mathbf{S}^o, \delta^i, \delta^o \rangle$, and its set Inst of instances.

6.3 CBR assumption

Case-based reasoning is heavily based on similarities between cases. It looks for the most similar past cases to the new case, then adapts their outcomes following the key rule below :

> *The more similar the cases (in the sense of* $\mathbf{S}^i$*), the more similar their outcomes (in the sense of* $\mathbf{S}^o$*).*

Formalizing this rule is important for developing reasonable CBR models, and also for checking whether existing models obey the rule. In what follows, we propose four alternative ways for formalizing it. The first one is the notion of consistency.

In (Dubois et al., 1997), a case base is said to be *consistent* if identical cases in the base have identical outcomes (i.e., for all cases $\langle I, v \rangle$ and $\langle I', v' \rangle$ in a base, if $I = I'$ then $v = v'$). In some problems like the one described in the above example, this constraint may be strong as the same instances may have different but similar outcomes. It is also possible that similar instances have the same or similar outcomes. Imagine a second-hand car $C*$ which has the same characteristics as C_1, but its price is 8400. Note that $\mathbf{S}^o(8000, 8400) = 1$, which means that the difference between the two prices is negligible. In what follows, we generalize this notion of consistency using similarity measures. The idea is that fully similar instances get fully similar outcomes.

Definition 39 (Consistency)**.** *A case base* Σ *is* consistent *iff* $\forall \langle I, v \rangle, \langle I', v' \rangle \in \Sigma$*, if* $\mathbf{S}^i(I, I') = 1$ *then* $\mathbf{S}^o(v, v') = 1$*. It is* inconsistent *otherwise.*

It is easy to see that in a consistent case base, identical instances may receive different but fully similar outcomes.

Property 8. *If a case base* Σ *is consistent, then* $\forall \langle I, v \rangle, \langle I', v' \rangle \in \Sigma$*, if* $I = I'$*, then* $\mathbf{S}^o(v, v') = 1$*.*

Proof. Let Σ be a consistent case base. Assume that $\langle I, v \rangle, \langle I', v' \rangle \in \Sigma$ such that $I = I'$. Since $\mathbf{S}^i$ is a similarity measure, then $\mathbf{S}^i(I, I') = 1$. From Consistency of Σ, $\mathbf{S}^o(v, v') = 1$. □

In (Dubois et al., 1997), the rule has been formalized as a *fuzzy gradual* rule, which states that the similarity of two instances should be lower or equal to the similarity of their outcomes. Throughout the thesis, we refer to this notion as *strong coherence*.

Definition 40 (Strong Coherence). *A case base* Σ *is* strongly coherent *iff* $\forall \langle I, v\rangle, \langle I', v'\rangle \in \Sigma$, $\mathbf{S}^i(I, I') \leq \mathbf{S}^o(v, v')$.

Example 22. *The case base* $\Sigma_1 = \{C_i = \langle I_i, v_i\rangle, i = 1, \ldots, 4\}$ *is not strongly coherent. For instance,* $\mathbf{S}^i(I_1, I_3) = \frac{1}{2}$ *while* $\mathbf{S}^o(v_1, v_3) = 0$.

Example 23. *Consider the case base* $\Sigma_2 = \{C = \langle I, v\rangle, C' = \langle I', v'\rangle\}$. *If* $\mathbf{S}^i(I, I') = 0.7$ *and* $\mathbf{S}^o(v, v') \geq 0.7$, *then* Σ *is strongly coherent. Assume now that* $\mathbf{S}^i(I, I') = 0.1$ *and* $\mathbf{S}^o(v, v') = 1$. *Again,* Σ_2 *is strongly coherent even if the two cases are dissimilar (let* $\delta^i = 0.5$*).*

It is easy to show that fully similar cases in a strongly coherent case base have fully similar outcomes.

Property 9. *Let* Σ *be a strongly coherent case base. For all* $\langle I, v\rangle, \langle I', v'\rangle \in \Sigma$, *if* $\mathbf{S}^i(I, I') = 1$, *then* $\mathbf{S}^o(v, v') = 1$.

Proof. Let Σ be a strongly coherent case base. Let $\langle I, v\rangle, \langle I', v'\rangle \in \Sigma$ such that $\mathbf{S}^i(I, I') = 1$. Strong coherence of Σ implies $\mathbf{S}^i(I, I') \leq \mathbf{S}^o(v, v')$. Since $\mathbf{S}^o(v, v') \in [0, 1]$, then $\mathbf{S}^o(v, v') = 1$. □

It is also easy to show that any strongly coherent case base is consistent. The converse is false as shown in Example 22 (the base Σ_1 is consistent but not strongly coherent).

Property 10. *If a case base is strongly coherent, then it is consistent. The converse does not hold.*

Proof. Let Σ be a case base and assume it is strongly coherent. Let $\langle I, v\rangle, \langle I', v'\rangle \in \Sigma$ such that $\mathbf{S}^i(I, I') = 1$. From Property 9, it follows that $\mathbf{S}^o(v, v') = 1$. □

By directly linking the similarity of outcomes with the similarity of instances, the property of strong coherence ensures that the former is proportional to the latter. However, the similarity measures $\mathbf{S}^i$ and $\mathbf{S}^o$ as well as their corresponding thresholds (δ^i and δ^o) may be different and not necessarily commensurate. This makes the satisfaction of the property difficult in case of such measures. Let us illustrate the issue with the following example.

Example 24. *Suppose we have a case base* Σ_3 *on student grades. There are 4 attributes corresponding to courses and which take values from the interval* $[0, 20]$*; the outcome is a global appreciation whose range consists of 4 qualitative levels: bad < poor < good < excellent. Let* $\mathbf{S}^o$ *be the similarity measure* $\mathbf{S}^3$ *in Example 26. Similarity between any pair of grades obtained in a course is defined by* $\mathbf{S}(u, v) = 1 - (\frac{|u-v|}{20})$. *The similarity measure* $\mathbf{S}^i$ *takes the minimal value returned by* $\mathbf{S}$ *on the four courses. Assume* Σ_3 *contains two students who got respectively* $I = \langle 20, 20, 20, 20\rangle$ *and* $v =$ *"excellent" as global appreciation, and* $I' = \langle 20, 20, 15, 15\rangle$ *with appreciation* $v' =$ *"good". Hence,* $\mathbf{S}^i(I, I') = 0.75$ *and* $\mathbf{S}^o(v, v') = \frac{2}{3}$. *Note that the base is not strongly coherent. In order to be coherent,* $\mathbf{S}^o(v, v')$ *should be equal to 1, which is not reasonable in the example as the two instances are different and deserve different appreciations. Furthermore, the scale of* $\mathbf{S}^o$ *does not have an intermediate value between* $\frac{2}{3}$ *and 1.*

In what follows, we introduce a novel notion of *weak coherence*, which makes use of the two thresholds for judging similar instances/outcomes. It states that similar cases should receive similar outcomes.

Definition 41 (Weak Coherence)**.** *A case-base* Σ *is* weakly coherent *iff* $\forall \langle I, v\rangle, \langle I', v'\rangle \in \Sigma$*, if* $\mathbf{S}^i(I, I') \geq \delta^i$*, then* $\mathbf{S}^o(v, v') \geq \delta^o$*.*

Example 25. *If* $\delta^i \geq 0.75$ *and* $\delta^o \geq \frac{2}{3}$*, then* Σ_2 *is weakly coherent.*

The above example shows that a case base may be weakly but not strongly coherent. However, weak coherence follows from the strong version when $\delta^i \geq \delta^o$.

Proposition 18. *Let* Σ *be a case base and* $\delta^i \geq \delta^o$*. If* Σ *is strongly coherent, then* Σ *is also weakly coherent.*

Proof. Assume $\delta^i \geq \delta^o$. Let Σ be a strongly coherent case base, and $\langle I, v\rangle, \langle I', v'\rangle \in \Sigma$. Assume $\mathbf{S}^i(I, I') \geq \delta^i$. From strong coherence, $\delta^i \leq \mathbf{S}^i(I, I') \leq \mathbf{S}^o(v, v')$. Hence, $\mathbf{S}^o(v, v') \geq \delta^o$. □

It is worth mentioning that consistency does not follow from weak coherence. Indeed, it is possible to find a weakly coherent case base which contains two cases such that $\mathbf{S}^i(I, I') = 1$, thus $\mathbf{S}^i(I, I') \geq \delta^i$, while $\delta^o \leq \mathbf{S}^o(v, v') < 1$.

The two versions of coherence compare pairs of cases of a case base. Our next notion, called *regularity*, is defined on the whole set of cases and ensures that the closest instances receive the closest outcomes. Indeed, if an instance I is closer to I' than to I'', then its outcome should be closer to that of I'.

Definition 42 (Regularity)**.** *A case-base* Σ *is* regular *iff* $\forall \langle I, v\rangle, \langle I', v'\rangle, \langle I'', v''\rangle \in \Sigma$*, if* $\mathbf{S}^i(I, I') \geq \mathbf{S}^i(I, I'')$ *then* $\mathbf{S}^o(v, v') \geq \mathbf{S}^o(v, v'')$*.*

Example 26. *The case base* $\Sigma_1 = \{C_i = \langle I_i, v_i\rangle, i = 1, \ldots, 4\}$ *is not regular. For instance,* $\mathbf{S}^i(I_1, I_3) > \mathbf{S}^i(I_1, I_2)$ *while* $\mathbf{S}^o(v_1, v_3) < \mathbf{S}^o(v_1, v_2)$*.*

Regularity is different from the two forms of coherence, and thus it does not imply or follow from them. It is also independent from consistency.

6.4 Principles for CBR

In what follows, we assume an arbitrary but fixed theory $\mathbf{T}$, a case base Σ, a new case $\langle I, ? \rangle$ and a CBR model $\mathbf{R}$. We introduce some principles (or properties) that a reasonable CBR model would satisfy. The first two principles concern the situation where the new case is dissimilar to all the past cases of the base. There are two possibilities. The first consists of proposing outcomes of the closest cases. This may be undesirable in applications like medical diagnosis, where a CBR model looks for a diagnosis of patients of the basis of their symptoms.

Principle 20 (Strong Completeness)**.** $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} O$ *with* $O \subseteq \mathtt{dom(f)}$.

The second possibility consists of abstaining from choosing an arbitrary outcome, and ensures that the model returns the symbol `Und`.

Principle 21 (Weak Completeness)**.** $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} \{\mathtt{Und}\}$ *iff* $\forall \langle I, v \rangle \in \Sigma$, $\mathbf{S}^i(I_n, I) < \delta^i$.

Note that any model which satisfies weak completeness returns `Und` when the case base is empty. This is reasonable as arbitrariness is avoided.

Proposition 19. *If a model* $\mathbf{R}$ *satisfies weak completeness, then* $\emptyset \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} \{\mathtt{Und}\}$.

The strong and weak versions of completeness are incompatible, i.e., there is no CBR model which can satisfies both at the same time. Indeed, they recommend different outcomes in the above mentioned particular case.

Proposition 20. *Strong completeness and weak completeness are incompatible*

The third principle ensures that the model preserves the consistency of the case base. Of course, this therefore assumes that the base is consistent.

Principle 22 (Consistency)**.** *Let* Σ *be consistent and* $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} O$ *such that* $O \neq \{\mathtt{Und}\}$. *For any* $v \in O$, $\Sigma \cup \{\langle I_n, v \rangle\}$ *is consistent.*

The three next principles are those that capture the CBR rule discussed previously. Strong coherence states that adding the new case labelled with any of its candidate outcomes to a strongly coherent base would preserve coherence.

Principle 23 (Strong Coherence)**.** *Let* Σ *be strongly coherent and* $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} O$ *such that* $O \neq \{\mathtt{Und}\}$. *For any* $v \in O$, $\Sigma \cup \{\langle I_n, v \rangle\}$ *is strongly coherent.*

In the same way, weak coherence ensures that a CBR model preserves the weak coherence of a case base.

Principle 24 (Weak Coherence)**.** *Let* Σ *be weakly coherent and* $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} O$ *such that* $O \neq \{\mathtt{Und}\}$. *For any* $v \in O$, $\Sigma \cup \{\langle I_n, v \rangle\}$ *is weakly coherent.*

Proposition 21. *Let* $\mathbf{T} = \langle \mathcal{F}, \mathtt{dom}, \mathbf{S}^i, \mathbf{S}^o, \delta^i, \delta^o \rangle$ *be a theory such that* $\delta^i \geq \delta^o$. *If a CBR model satisfies strong coherence, then it satisfies weak coherent.*

Proof. Let $\delta^i \geq \delta^o$, $\langle I, ? \rangle$ a new case, and a CBR model which satisfies strong coherence. Let $\Sigma \oplus \langle I, ? \rangle \mid\sim_{\mathbf{T},\mathbf{R}} O$. It holds that for any $v \in O$, $\Sigma \cup \{\langle I_n, v \rangle\}$ is strongly coherent. From Proposition 18, since $\delta^i \geq \delta^o$, then $\Sigma \cup \{\langle I_n, v \rangle\}$ is weakly coherent. □

Regularity principle ensures that a CBR model preserves the regularity of a case base.

Principle 25 (Regularity). *Let Σ be regular and $\Sigma \oplus \langle I, ?\rangle \ \mid\!\sim_{\mathbf{T},\mathbf{R}} O$ such that $O \neq \{\texttt{Und}\}$. For any $v \in O$, $\Sigma \cup \{\langle I_n, v\rangle\}$ is regular.*

In what follows, we show that case-based reasoning is non-monotonic as its conclusions can be revised when a base is extended with additional cases. Let us first define formally the principle of non-monotonicity.

Principle 26 (Monotonicity).

$$\begin{cases} \Sigma \oplus \langle I, ?\rangle \mid\!\sim_{\mathbf{T},\mathbf{R}} O \\ \\ \Sigma \subseteq \Sigma' \end{cases} \Longrightarrow \Sigma' \oplus \langle I, ?\rangle \mid\!\sim_{\mathbf{T},\mathbf{R}} O$$

The following result shows that non-monotonicity follows from weak completeness.

Proposition 22. *If a CBR model satisfies weak completeness, then it violates monotonicity.*

Proof. Assume a CBR model $\mathbf{R}$ which satisfies weak completeness. Let Σ be a case base and $\langle I, ?\rangle$ a new case. Assume $\Sigma = \emptyset$, then $\emptyset \oplus \langle I, ?\rangle \mid\!\sim_{\mathbf{T},\mathbf{R}} \{\texttt{Und}\}$. Let now $\Sigma' = \{\langle I, v\rangle\}$ such that $I = I_n$. Obviously, $\mathbf{S}^i(I, I_n) = 1$ and $\mathbf{S}^i(I, I_n) \geq \delta^i$ (since $0 < \delta^i \leq 1$). So, $\Sigma' \oplus \langle I, ?\rangle \ \not\mid\!\sim_{\mathbf{T},\mathbf{R}} \{\texttt{Und}\}$. □

6.5 Conclusion

This chapter presented a preliminary contribution on foundations of case-based reasoning. It started by formalizing the key rule behind this form of reasoning, then proposed a set of principles that any model would satisfy. We have shown that CBR is non-monotonic in that conclusion could be revised in light of additional information (cases).

Our plan is to analyze existing CBR models against those principles, and propose argumentation-based models that satisfy the principles.

Chapter 7

Conclusion and Future works

In this thesis, we investigated theoretical foundations of semantics in argumentation. I studied formal approaches for describing semantics and comparing the plethora of existing (extension, gradual) semantics. Indeed, i proposed various principles showing how a semantics would deal with self-attacking arguments, investigated their compatibility with other existing principles, and finally proposed the first semantics which satisfies a maxima set of principles. In another chapter, i proposed for the first time the mathematical counterpart of two gradual semantics. Indeed, i redefined them using numerical series.

Another contribution of the thesis consists of showing that principles are not sufficient for comparing pairs of semantics. Their rankings should also be checked. I thus introduced the novel notion of equivalence, compared a large set of semantics with this notion, and identified some strong equivalence classes.

The thesis tackled also the topic of case-based reasoning. I introduced some formal properties that a CBR model would satisfy. Some of them formalize the key rule (or assumption) behind CBR.

Our aim is to continue investigating this part of the thesis by analyzing existing CBR against the properties, and proposing novel models that satisfy all the properties. We plan to define argumentation based one that use our novel gradual semantics. The reason is that argumentation is powerful approach for modeling different types of reasoning, and is more akine with the way human reasons. Finally, the novel gradual semantics are richer than extension semantics, consequently, they would allow us to define efficient CBR models.

Another future work consists of pursuing the work on characterizing families of gradual semantics, especially in the general case of weighted graphs.

Chapter 8

Appendix of proofs

Proposition 9. *The gradual semantics* nsa *satisfies all the principles from* P_{2S2C}*. The other principles are not satisfied.*

Proof.

Anonymity. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AG. Let γ be an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$. Recall the iterative version of $f^{\mathcal{F},i}_{\texttt{nsa}}$ from Definition 19. Let us prove Anonymity by induction on i, where i is the step of the iterative algorithm. The inductive hypothesis is: for every $a \in \mathcal{A}$, $f^{\mathcal{F},i}_{\texttt{nsa}}(a) = f^{\mathcal{F},i}_{\texttt{nsa}}(\gamma(a))$.

Base: Let $i = 0$. From the formal definition of nsa we have that for each $a \in \mathcal{A}$, $f^{\mathcal{F},0}_{\texttt{nsa}}(a) = 0$ if and only if a is self-attacking in $\mathcal{F}$ and that $f^{\mathcal{F},0}_{\texttt{nsa}}(a) = 1$, otherwise. Likewise, $f^{\mathcal{F}',0}_{\texttt{nsa}}(a') = 0$ if and only if a' is self-attacking in $\mathcal{F}'$ and that $f^{\mathcal{F}',0}_{\texttt{nsa}}(a') = 1$, otherwise.

Step: Let us suppose that the inductive hypothesis is true for every $k \in \{0, 1, \dots i\}$ and let us show that it is true for $i+1$. Let $a \in \mathcal{A}$ and let $a' \in \mathcal{A}$ such that $a' = \gamma(a)$. Let $\texttt{Att}_{\mathcal{F}}(a) = \{b_1, \dots, b_n\}$. From the inductive hypothesis, for each $j \in \{1, \dots, n\}$, $f^{\mathcal{F},i}_{\texttt{nsa}}(b_j) = f^{\mathcal{F},i}_{\texttt{nsa}}(\gamma(b_j))$. Hence, $f^{\mathcal{F},i+1}_{\texttt{nsa}}(a) = f^{\mathcal{F},i+1}_{\texttt{nsa}}(\gamma(a))$.

By induction, we conclude that for every i, for every $a \in \mathcal{A}$, $f^{\mathcal{F},i}_{\texttt{nsa}}(a) = f^{\mathcal{F},i}_{\texttt{nsa}}(\gamma(a))$. Hence, for every $a \in \mathcal{A}$, $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = Deg^{\texttt{nsa}}_{\mathcal{F}'}(\gamma(a))$.

Independence. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$.
Let us recall that $\forall a \in \mathcal{A}$,

$$\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = \begin{cases} 0 & \text{if } (a,a) \in \mathcal{R} \\ \dfrac{1}{1 + \sum_{b \in \texttt{Att}_{\mathcal{F}}(a)} \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(b)} & \text{otherwise} \end{cases} \tag{8.1}$$

Let $\texttt{X} \subseteq \mathcal{A}$ be a set of arguments. Let us define $\texttt{Att}^0_{\mathcal{F}}(\texttt{X}) = \bigcup_{x \in \texttt{X}} \texttt{Att}_{\mathcal{F}}(x)$ as the union of the set of direct attackers of each $x \in \texttt{X}$ and $\texttt{Att}^{i+1}_{\mathcal{F}}(\texttt{X}) = \texttt{Att}_{\mathcal{F}}(\texttt{Att}^i_{\mathcal{F}}(\texttt{X}))$. Let $\texttt{Att}^\star_{\mathcal{F}}(\texttt{X}) = \bigcup_{i \geq 0} \texttt{Att}^i_{\mathcal{F}}(\texttt{X})$. Since $\mathcal{A} \cap \mathcal{A}' = \emptyset$ and $\texttt{Att}^\star_{\mathcal{F}}(\{a\}) \subseteq \mathcal{A}$, since the definition of $\texttt{Deg}^{\texttt{nsa}}_{\mathcal{F}}(a)$ depends only on attackers of a and, in view of the recursion, on $\texttt{Att}^\star_{\mathcal{F}}(\{a\})$, we have $\texttt{Deg}^{\texttt{nsa}}_{\mathcal{F}}(a) = \texttt{Deg}^{\texttt{nsa}}_{\mathcal{F} \oplus \mathcal{F}'}(a)$.

Directionality. Trivial

Maximality. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AG and $a \in \mathcal{A}$ such that $\texttt{Att}_{\mathcal{F}}(a) = \emptyset$. By definition, if $\texttt{Att}_{\mathcal{F}}(a) = \emptyset$ then we have $\sum_{b \in \texttt{Att}_{\mathcal{F}}(a)} \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(b) = 0$. Consequently, $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = 1$.

Weakening. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AG and let argument $a \in \mathcal{A}$ such that $\exists b \in \mathtt{Att}_{\mathcal{F}}(a), \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b) > 0$. Clearly, argument b cannot be a self-attacking argument because $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b) > 0$.
We have two possibilities for a:

- If a is a self-attacking argument then, by definition, we have $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) = 0 < 1$ which satisfies the principle.
- If a is not a self-attacking argument then we have

$$\begin{aligned} \sum_{b' \in \mathtt{Att}_{\mathcal{F}(a)}} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &> 0 \\ 1 + \sum_{b' \in \mathtt{Att}_{\mathcal{F}(a)}} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &> 1 \\ \frac{1}{1 + \sum_{b' \in \mathtt{Att}_{\mathcal{F}(a)}} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b')} &< 1 \\ \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) &< 1 \end{aligned}$$

  showing that the principle is satisfied.

Counting. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AG and $a, b \in \mathcal{A}$ such that i) $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) > 0$ and ii) $\mathtt{Att}_{\mathcal{F}}(b) = \mathtt{Att}_{\mathcal{F}}(a) \cup \{y\}$ with $y \in \mathcal{A} \backslash \mathtt{Att}_{\mathcal{F}}(a)$ and $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(y) > 0$.
Clearly, a cannot be a self-attacking argument because $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) > 0$.
In addition, if b is a self-attacking argument then $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b) = 0 < \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a)$ which satisfies the principle.
So, if a and b are not self-attacking arguments, by definition, we have:

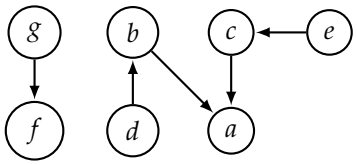
$$\begin{aligned} \sum_{b' \in \mathtt{Att}_{\mathcal{F}}(b) \backslash \{y\}} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &= \sum_{a' \in \mathtt{Att}_{\mathcal{F}}(a)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a') \\ \sum_{b' \in \mathtt{Att}_{\mathcal{F}}(b)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &= \sum_{a' \in \mathtt{Att}_{\mathcal{F}}(a)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a') + \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(y) \end{aligned}$$

Since $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(y) > 0$, we have:

$$\begin{aligned} \sum_{b' \in \mathtt{Att}_{\mathcal{F}}(b)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &> \sum_{a' \in \mathtt{Att}_{\mathcal{F}}(a)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a') \\ 1 + \sum_{b' \in \mathtt{Att}_{\mathcal{F}}(b)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b') &> 1 + \sum_{a' \in \mathtt{Att}_{\mathcal{F}}(a)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a') \\ \frac{1}{1 + \sum_{b' \in \mathtt{Att}_{\mathcal{F}}(b)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b')} &< \frac{1}{1 + \sum_{a' \in \mathtt{Att}_{\mathcal{F}}(a)} \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a')} \\ \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b) &< \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) \end{aligned}$$

Compensation. Figure 8.1 is an example showing that there exists an AG such that i) $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) > 0$; ii) $|\mathtt{Att}_{\mathcal{F}}(a)| = |\{b, c\}| = 2 > 1 = |\{g\}| = |\mathtt{Att}_{\mathcal{F}}(f)|$; iii) $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(g) > \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(b)$ and $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(g) > \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(c)$ and $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(f)$.

Strong Self-Contradiction. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AG and $a \in \mathcal{A}$.
($\Leftarrow$) Let us suppose that $(a, a) \in \mathcal{R}$. By definition of $\mathtt{nsa}$, $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) = 0$.
($\Rightarrow$) Let us suppose that $\mathtt{Deg}^{\mathtt{nsa}}_{\mathbf{G}}(a) = 0$. Again, by definition, for any AG and any



$\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(d) = \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(e) = \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(g) = 1$
$\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(b) = \texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(c) = 0.5$
$\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(f) = 0.5$
$\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = 0.5$

FIGURE 8.1: nsa satisfies Compensation

non-self-attacking argument a, we have $\frac{1}{1+\sum_{b\in\texttt{Att}_{\mathcal{F}}(a)}\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(b)} > 0$. Consequently, the only way to obtain $\texttt{Deg}^{\texttt{nsa}}_{\mathbf{G}}(a) = 0$ is when $(a,a) \in \mathcal{R}$.
Self-Contradiction. Implied by Strong Self-Contradiction which is satisfied by nsa.

The other principles are not satisfied because of incompatibilities :

- Equivalence and Self-Contradiction are incompatible.
- Neutrality and Strong Self-Contradiction are incompatible.
- Reinforcement and Strong Self-Contradiction are incompatible.
- Weakening Soundness and Strong Self-Contradiction are incompatible.
- Cardinality Precedence and Compensation are incompatible.
- Quality Precedence and Compensation are incompatible.
- Resilience and Strong Self-Contradiction are incompatible.

□

Proposition 10. *The gradual semantics M&T satisfies Anonymity, Independence, Directionality, Maximality, Weakening, Compensation, Self-Contradiction and Strong Self-Contradiction. The other principles are not satisfied.*

Proof.

Satisfied principles

Anonymity. See (Bonzon et al., 2016).

Independence. See proof of Proposition 9 in (Matt and Toni, 2008).

Directionality. The sets of strategies of the players are the same in the (G,x) and (G',x) games because the set of arguments remains unchanged. This implies that there is no impact on the payoff matrix. Therefore, the acceptability of a given argument only depends on arguments which have a path to this argument.

Maximality. See proof of Proposition 4 in (Matt and Toni, 2008).

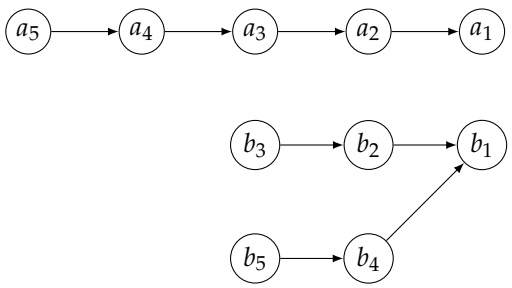
Weakening. This result is a direct consequence of the result from Proposition 5.b in (Matt and Toni, 2008) stating that if there exist n attacks against an argument x, then $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(x) < 1 - \frac{1}{2}f(n)$ where $f(n) = \frac{n}{n+1}$. Indeed, one can easily deduce from this formula that whatever the number of attackers of x and regardless of their degree of acceptability, the degree of x will always be strictly less than 1 because $\forall n > 0, \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(x) < 1 - \frac{1}{2}f(n) < 1$, in accordance with the Weakening principle.

Compensation. Figure 8.2 is an example showing that there exists an AG such that :

i) $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_1) > 0$;

ii) $|\texttt{Att}_{\mathcal{F}}(b_1)| = |\{b_2, b_4\}| = 2 > 1 = |\{a_2\}| = |\texttt{Att}_{\mathcal{F}}(a_1)|$;

iii) $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_2) > \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_2)$ and $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_2) > \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_4)$ and $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_1) = \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_1)$.



$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_5) = \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_3) = \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_5) = 1$
$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_4) = \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_2) = \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_4) = 0.25$
$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_3) = 0.5$
$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_2) \simeq 0.386$
$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_1) = 0.5$
$\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(b_1) = 0.5$

FIGURE 8.2: The gradual semantics M&T satisfies the Compensation principle

Self-Contradiction. See (Bonzon et al., 2016).

Strong Self-Contradiction. See proof of Proposition 3 in (Matt and Toni, 2008).

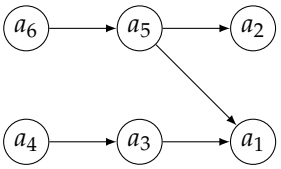
Unsatisfied principles

Neutrality. Incompatible with Self-Contradiction which is satisfied

Equivalence. Incompatible with Self-Contradiction which is satisfied

Counting. To show that the semantics M&T does not satisfy the Counting principle, consider the AG represented in Figure 8.3.

The principle says that $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_2) > \texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_1)$ because i) $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_2) > 0$ and ii) $\texttt{Att}_{\mathcal{F}}(a_1) = \texttt{Att}_{\mathcal{F}}(a_2) \cup \{a_3\}$ where $a_3 \notin \texttt{Att}_{\mathcal{F}}(a_2)$ and $\texttt{Deg}^{\texttt{MT}}_{\mathbf{G}}(a_3) > 0$. However,



$\mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_6) = \mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_4) = 1$
$\mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_5) = \mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_3) = 0.25$
$\mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_2) = 0.5$
$\mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_1) = 0.5$

FIGURE 8.3: The gradual semantics M&T falsifies the Counting principle

when the semantics is applied on $\mathcal{F}$, we have $\mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_2) = \mathtt{Deg}^{\mathtt{MT}}_{\mathbf{G}}(a_1)$, contradicting the principle.

Weakening Soundness. Incompatible with Strong Self-Contradiction which is satisfied.

Cardinality Precedence. Incompatible with Compensation which is satisfied.

Quality Precedence. Incompatible with Compensation which is satisfied. □

Lemma 3. *Let* $\langle \mathcal{A}, \mathcal{R} \rangle \in \mathtt{AG}$, $a, b \in \mathcal{A}$ *and* $i \in \mathbb{N}$.

- *If* $a \in \bigcup_{j=1}^{i} \mathcal{F}^j(\emptyset)$ *and* $b \notin \bigcup_{j=1}^{i} \mathcal{F}^j(\emptyset)$, *then* $\mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(a) > \mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(b)$.
- *If the set* $\bigcup \mathcal{F}^{i \geq 1}(\emptyset)$ *attacks* a *and does not attack* b, *then* $\mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(a) < \mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(b)$.

Proof of Proposition 14. From (A.F., 1996), $\forall n \geq 0$, $\mathcal{J}^n = \frac{2^n - (-1)^n}{3}$ (called Binet formulae). Since $\mathtt{J} = (\frac{\mathcal{J}^{n+1}}{2^n})^{n \geq 0}$, it follows that $\mathtt{J}^n = \frac{1}{3}(2 + (-\frac{1}{2})^n)$. Let $\mathtt{J}_1 = (\mathtt{J}^n_1)^{n \geq 0}$ and $\mathtt{J}_2 = (\mathtt{J}^n_2)^{n \geq 0}$ be two sub-series of $\mathtt{J}$ such that for any $n \geq 0$: $\mathtt{J}^n_1 = \mathtt{J}^{2n}$ and $\mathtt{J}^n_2 = \mathcal{S}^{2n+1}$.

We show that $\mathtt{J}_1$ is strictly decreasing while $\mathtt{J}_2$ is strictly increasing. Obviously, $\mathtt{J}^{n+1}_1 - \mathtt{J}^n_1 = \frac{1}{3}(2 + (\frac{1}{2})^{n+1}) - \frac{1}{3}(2 + (\frac{1}{2})^n) = -\frac{1}{3} \times \frac{1}{2^{n+1}}$. Since $\frac{1}{2^{n+1}} > 0$, then $\mathtt{J}^{n+1}_1 - \mathtt{J}^n_1 < 0$ and so $\mathtt{J}^n_1 > \mathtt{J}^{n+1}_1$.

In a similar way we have $\mathtt{J}^{n+1}_2 - \mathtt{J}^n_2 = \frac{1}{3}(2 - (\frac{1}{2})^{n+1}) - \frac{1}{3}(2 - (\frac{1}{2})^n) = \frac{1}{3} \times \frac{1}{2^{n+1}}$. Since $\frac{1}{2^{n+1}} > 0$, then $\mathtt{J}^{n+1}_1 - \mathtt{J}^n_1 > 0$ and so $\mathtt{J}^n_1 < \mathtt{J}^{n+1}_1$.

Obviously, $\mathtt{J}^n_1 > \mathtt{J}^n_2$ because $\frac{\frac{2^{n+1}+1}{3}}{2^n} > \frac{\frac{2^{n+1}-1}{3}}{2^n}$. Since $\mathtt{J}^n_1$ is strictly decreasing with $\mathtt{J}^1_1 = 1$ then $\forall n, \mathtt{J}^n \leq 1$. From (A.F., 1988), $\forall n \geq 0$, $\mathcal{J}^n \geq 0$ and thus $\mathtt{J}^n \geq 0$.

Finally, it is easy to check that $\lim_{n \to \infty} \mathtt{J}^n_1 = \lim_{n \to \infty} \mathtt{J}^n_2 = \frac{2}{3}$. Then, $\mathtt{J} \in \mathbf{S}^*$. ■

Theorem 10. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle$, $a \in \mathcal{A}$ *and* $\mathtt{Jac}$ *a mapping from* $\mathcal{A}$ *to* $[0, 1]$ *such that for any* $a \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(a) = 1 - \frac{\max_{b \mathcal{R} a} \mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(b)}{2}$. *The following hold:*

- $\mathtt{Jac}$ *is based on the series* $\mathtt{J}$
- $\mathtt{Jac} \in \mathtt{Sem}^*$
- *for any* $a \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Jac}}_{\mathbf{G}}(a) \in [\frac{1}{2}, 1]$

Proof of Theorem 10. Let $\langle \mathcal{A}, \mathcal{R} \rangle \in \mathtt{AG}$, $a \in \mathcal{A}$, and $i \in \mathbb{N}$.

▶ Let: (P) $\mathtt{Deg}^{\mathtt{Jac}}_{\mathtt{G}}(a) = \mathtt{J}^i_1$ if $a \in \mathcal{F}^i(\emptyset) \setminus \bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$,

(Q) $\mathtt{Deg}^{\mathtt{Jac}}_{\mathtt{G}}(a) = \mathtt{J}^i_2$ if $\mathcal{F}^i(\emptyset)$ attacks a and $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ does not attack a.

We prove by induction that the property $P \wedge Q$ is true for any $i \in \mathbb{N}$.

Case $i = 1$. $\mathcal{F}^1(\emptyset) = \{x \in \mathcal{A} \mid \mathtt{Att}(a) = \emptyset\}$. Furthermore, $\mathtt{Deg}^{\mathtt{Jac}}(a) = 1 = \mathtt{J}^1_1$ if $\mathtt{Att}(a) = \emptyset$, and so (P) holds for $i = 1$.

By definition of Jac, for any $a \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Jac}}(a) = \frac{1}{2}$ iff $\exists b \in \mathtt{Att}(a)$ such that $\mathtt{Deg}^{\mathtt{Jac}}(b) = 1$. Hence, $\mathtt{Deg}^{\mathtt{Jac}}(a) = \frac{1}{2} = \mathtt{J}^1_2$. Thus, (Q) holds for $i = 1$.

Case $i > 1$. Assume that the property P&Q is true at step i and let us show that it holds at step $i + 1$.

Assume $a \in \mathcal{F}^{i+1}(\emptyset) \setminus \big(\bigcup_{j=1}^{i} \mathcal{F}^j(\emptyset)\big)$. Hence, $\mathtt{Att}(a) \neq \emptyset$ and a is defended by $\mathcal{F}^i(\emptyset)$, i.e., $\forall b \in \mathtt{Att}(a)$, $\exists c \in \mathcal{F}^i(\emptyset)$ such that $c \mathcal{R} b$. There are two sub-cases:

- $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ does not attack any $b \in \mathtt{Att}(a)$. Thus, $\forall b \in \mathtt{Att}(a)$, $\mathtt{Deg}^{\mathtt{Jac}}(b) = \mathtt{J}^i_2$ (by assumption). Then, $\mathtt{Deg}^{\mathtt{Jac}}(a) = 1 - \frac{\mathtt{J}^i_2}{2} = \mathtt{J}^{i+1}_1$. (from Lemma 2).
- $\bigcup_{j=1}^{i-1} \mathcal{F}^j(\emptyset)$ attacks some $b \in \mathtt{Att}(a)$. This means that $\exists j < i$ such that $\mathcal{F}^j(\emptyset)$ attacks b and $\bigcup_{k=1}^{j-1} \mathcal{F}^k(\emptyset)$ does not attack b. By assumption, $\mathtt{Deg}^{\mathtt{Jac}}(b) = \mathtt{J}^j_2$. But since $a \notin \bigcup_{k=1}^{i} \mathcal{F}^k(\emptyset)$, then $\exists b' \in \mathtt{Att}(a)$ such that $\mathcal{F}^i(\emptyset)$ attacks b' and $\bigcup_{k=1}^{i-1} \mathcal{F}^k(\emptyset)$ does not attack b'. Thus, by assumption, $\mathtt{Deg}^{\mathtt{Jac}}(b') = \mathtt{J}^i_2$. Since the subsequence $\{\mathtt{J}_2\}^n$ is strictly increasing, then $\mathtt{J}^i_2 > \mathtt{J}^j_2$ and so $\max_{b \mathcal{R} a} \mathtt{Deg}^{\mathtt{Jac}}(b) = \mathtt{J}^i_2$ and $\mathtt{Deg}^{\mathtt{Jac}}(a) = 1 - \frac{\mathtt{J}^i_2}{2} = \mathtt{J}^{i+1}_1$ (from Lemma 2).

Assume now that $\mathcal{F}^{i+1}(\emptyset)$ attacks a and $\bigcup_{j=1}^{i} \mathcal{F}^j(\emptyset)$ does not attack a. Thus, $\mathtt{Att}(a) = X_1 \cup X_2$ such that:

- $X_1 \subseteq \mathcal{F}^{i+1}(\emptyset) \setminus \big(\bigcup_{j=1}^{i} \mathcal{F}^j(\emptyset)\big)$
- $X_2 \cap \big(\bigcup_{j=1}^{i+1} \mathcal{F}^j(\emptyset)\big) = \emptyset$.

From Property (P) above, $\forall b \in X_1$, $\mathtt{Deg}^{\mathtt{Jac}}(b) = \mathtt{J}^{i+1}_1$. There are two possibilities:

- i) $X_2 = \emptyset$. Hence, $\mathtt{Deg}^{\mathtt{Jac}}(a) = 1 - \frac{\mathtt{J}^{i+1}_1}{2} = \mathtt{J}^{i+1}_2$.
- ii) $X_2 \neq \emptyset$. From Lemma 3, $X_1 >^m X_2$. So, $\max_{b \mathcal{R} a} \mathtt{Deg}^{\mathtt{Jac}}(b) = \max_{(b \mathcal{R} a) \text{ and } b \in X_1} \mathtt{Deg}^{\mathtt{Jac}}(b)$. So, $\mathtt{Deg}^{\mathtt{Jac}}(a) = 1 - \frac{\mathtt{J}^{i+1}_1}{2} = \mathtt{J}^{i+1}_2$.

▶ Let now $a \in \mathcal{A}$ such that $a \notin \bigcup \mathcal{F}^{i\geq 1}(\emptyset)$ and a is not attacked by $\bigcup \mathcal{F}^{i\geq 1}(\emptyset)$. From Lemma 3, for all $b \in \mathtt{Gr}$, $\mathtt{Deg}^{\mathtt{Jac}}(a) < \mathtt{Deg}^{\mathtt{Jac}}(b)$ and from the first property of Theorem 10, $\mathtt{Deg}^{\mathtt{Jac}}(a) < \mathtt{J}_1^i$ for all $i \in \mathbb{N}$. From Lemma 3, for all $c \in \mathtt{Gr}^+$, $\mathtt{Deg}^{\mathtt{Jac}}(a) > \mathtt{Deg}^{\mathtt{Jac}}(c)$ and from the second property of Theorem 10, $\mathtt{Deg}^{\mathtt{Jac}}(a) > \mathtt{J}_2^i$ for all $i \in \mathbb{N}$. Hence, for all $i \in \mathbb{N}$, $\mathtt{J}_1^i > \mathtt{Deg}^{\mathtt{Jac}}(a) > \mathtt{J}_2^i$. $\mathtt{Deg}^{\mathtt{Jac}}(a) = \lim_{n\to\infty} \mathtt{J}_1^n = \lim_{n\to\infty} \mathtt{J}_2^n = \frac{2}{3}$. ■

Property 6. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ *and* $a \in \mathcal{A}$.

- $\mathtt{Deg}^g_{\mathbf{G}}(a) \in T \setminus \{\alpha\}$
- $\mathtt{Deg}^i_{\mathbf{G}}(a) \in T \setminus \{\alpha\}$
- *If stable extensions exist,* $\mathtt{Deg}^s_{\mathbf{G}}(a) \in T \setminus \{\beta\}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ and $a \in \mathcal{A}$. Grounded (resp. ideal) extension is unique, thus α cannot be assigned to an argument. By definition, a stable extension attacks every argument it does not contain, hence for every argument a, either a is in all stable extensions or attacked by at least one. Hence, β can never be ascribed to a. □

The following lemma shows links between degrees assigned by stable semantics and those ascribed by preferred semantics to an argument.

Lemma 4. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\mathtt{Deg}^s(a) = 1$, *then* $\mathtt{Deg}^p(a) \in \{\alpha, 1\}$
- *If* $\mathtt{Deg}^s(a) = \alpha$, *then* $\mathtt{Deg}^p(a) = \alpha$
- *If* $\mathtt{Deg}^s(a) = \beta$, *then* $\mathtt{Deg}^p(a) \in \{0, \beta, \alpha, 1\}$
- *If* $\mathtt{Deg}^s(a) = 0$, *then* $\mathtt{Deg}^p(a) \in \{0, \alpha\}$

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $a \in \mathcal{A}$. Let $\mathtt{SE}(\mathbf{G})$ and $\mathtt{PE}(\mathbf{G})$ denote respectively the set of stable extensions and preferred extensions of $\mathbf{G}$.

$\mathtt{Deg}^s(a) = 1$ This means that $\mathtt{SE}(\mathbf{G}) \neq \emptyset$. Since $\mathtt{SE}(\mathbf{G}) \subseteq \mathtt{PE}(\mathbf{G})$, there exists at least one preferred extension that contains a. So, $\mathtt{Deg}^p(a) \geq \alpha$.

$\mathtt{Deg}^s(a) = \alpha$ Thus, $\mathtt{SE}(\mathbf{G}) \neq \emptyset$ and $\mathtt{SE}(\mathbf{G}) \subseteq \mathtt{PE}(\mathbf{G})$. By Def. 9, there exists at least one stable (thus one preferred) extension that contains a, and at least one stable (thus one preferred) extension that does not contain a. Then, $\mathtt{Deg}^p(a) = \alpha$.

$\mathtt{Deg}^s(a) = \beta$ This means $\mathtt{SE}(\mathbf{G}) = \emptyset$, hence $\mathtt{Deg}^p(a) \in \{0, \beta, \alpha, 1\}$.

$\mathtt{Deg}^s(a) = 0$ Thus, $\mathtt{SE}(\mathbf{G}) \neq \emptyset$. By Def. 9, a does not belong to any stable extension, then a does not belong to all preferred extensions (i.e. $\mathtt{Deg}^p(a) \neq 1$). Furthermore, a is attacked by at least one stable extension, and so a is attacked by at least one preferred extension (since $\mathtt{SE}(\mathbf{G}) \subseteq \mathtt{PE}(\mathbf{G})$). This means that $\mathtt{Deg}^p(a) \neq \beta$. There are two possibilities: i) a belongs to at least one preferred extension, then $\mathtt{Deg}^p(a) = \alpha$; ii) otherwise $\mathtt{Deg}^p(a) = 0$.

□

The following results show how strengths vary from preferred semantics to stable.

Lemma 5. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$. *If* $\texttt{SE}(\mathbf{G}) = \emptyset$, *then for any* $\texttt{Deg}^p(a) \in T$, *it holds* $\texttt{Deg}^s(a) = \beta$. *Else,*

- *If* $\texttt{Deg}^p(a) = 1$, *then* $\texttt{Deg}^s(a) = 1$.
- *If* $\texttt{Deg}^p(a) = \alpha$, *then* $\texttt{Deg}^s(a) \in \{0, \alpha, 1\}$.
- *If* $\texttt{Deg}^p(a) = \beta$, *then* $\texttt{Deg}^s(a) = 0$.
- *If* $\texttt{Deg}^p(a) = 0$, *then* $\texttt{Deg}^s(a) = 0$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$. Assume that $\texttt{SE}(\mathbf{G}) = \emptyset$, then by definition, $\texttt{Deg}^s(a) = \beta$. Assume now that $\texttt{SE}(\mathbf{G}) \neq \emptyset$.

$\texttt{Deg}^p(a) = 1$ Then, a belongs to every preferred extension. Since $\texttt{SE}(\mathbf{G}) \subseteq \texttt{PE}(\mathbf{G})$, then a belongs to every stable extension, so $\texttt{Deg}^s(a) = 1$.

$\texttt{Deg}^p(a) = \alpha$ Thus, a belongs to at least one preferred extension and does not belong to another extension. There are two cases: i) a belongs to at least one stable extension, then $\texttt{Deg}^s(a) \in \{\alpha, 1\}$, or it does not belong to any extension, in which case $\texttt{Deg}^s(a) = 0$ (since stable extensions exist, and they attack every argument they do not contain).

$\texttt{Deg}^p(a) = \beta$ By definition, a does not belong to any preferred extension, thus it does not belong to any stable extension. Since the latter attack every argument left outside, then $\texttt{Deg}^s(a) = 0$.

$\texttt{Deg}^p(a) = 0$ By definition, a does not belong to any preferred extension, thus it does not belong to any stable extension. Since the latter attack every argument left outside, then $\texttt{Deg}^s(a) = 0$.

□

The following two lemmas establish links between the degrees assigned to arguments by preferred and grounded semantics. Recall that $T = \{1, \alpha, \beta, 0\}$.

Lemma 6. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^g(a) = 1$, *then* $\texttt{Deg}^p(a) = 1$.
- *If* $\texttt{Deg}^g(a) = \beta$, *then* $\texttt{Deg}^p(a) \in T$.
- *If* $\texttt{Deg}^g(a) = 0$, *then* $\texttt{Deg}^p(a) = 0$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$.

$\texttt{Deg}^g(a) = 1$ This means that a belongs to the grounded extension. Since the latter is included in every preferred extension, then $\texttt{Deg}^g(p) = 1$.

$\texttt{Deg}^g(a) = \beta$ This means that a is not in the grounded extension and is not attacked by it. Hence, $\texttt{Deg}^p(a) \in \{0, \beta, \alpha, 1\}$.

$\texttt{Deg}^g(a) = 0$ This means that the grounded extension attacks a. Since the latter is included in every preferred extension, then each preferred extension attacks a, and so by definition $\texttt{Deg}^p(a) = 0$.

□

Lemma 7. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^p(a) = 1$, *then* $\texttt{Deg}^g(a) \in \{\beta, 1\}$.
- *If* $\texttt{Deg}^p(a) = \alpha$, *then* $\texttt{Deg}^g(a) = \beta$.
- *If* $\texttt{Deg}^p(a) = \beta$, *then* $\texttt{Deg}^g(a) = \beta$.
- *If* $\texttt{Deg}^p(a) = 0$, *then* $\texttt{Deg}^g(a) \in \{0, \beta\}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$.

$\texttt{Deg}^p(a) = 1$ There are two possibilities: i) a is in the grounded extension, then $\texttt{Deg}^g(a) = 1$, and ii) a is not in the grounded extension, then $\texttt{Deg}^g(a) = \beta$. Indeed, the grounded extension does not attack a since it is part of every preferred extension, which are conflict-free.

$\texttt{Deg}^p(a) = \alpha$ If the grounded extension is empty, then $\texttt{Deg}^g(a) = \beta$. Assume it is not empty. Since a does not belong to at least one preferred extension, then a is not in the grounded extension. Furthermore, the latter does not attack a. Indeed, if the grounded extension attacks a, then every preferred extension attacks a including the one that contains a. This contradicts the conflict-freeness property of extensions. Hence, $\texttt{Deg}^g(a) = \beta$.

$\texttt{Deg}^p(a) = \beta$ This means that a does not belong to any preferred extension, thus does not belong to the grounded extension. Furthermore, a is not attacked by any preferred extension, then so is for the grounded extension. Then, $\texttt{Deg}^g(a) = \beta$.

$\texttt{Deg}^p(a) = 0$ This means that a does not belong to any preferred extension, thus it does not belong to the grounded extension. Thus, $\texttt{Deg}^g(a) \leq \beta$.

□

Lemma 8. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^s(a) = 1$, *then* $\texttt{Deg}^g(a) \in \{\beta, 1\}$.
- *If* $\texttt{Deg}^s(a) = \alpha$, *then* $\texttt{Deg}^g(a) = \beta$.
- *If* $\texttt{Deg}^s(a) = 0$, *then* $\texttt{Deg}^g(a) \in \{0, \beta\}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$.

$\texttt{Deg}^s(a) = 1$ Thus, $\texttt{SE}(\mathbf{G}) \neq \emptyset$ and $\forall E \in \texttt{SE}(\mathbf{G})$, $a \in E$. Since $\texttt{Gr} \subseteq E_i$, $\forall E_i \in \texttt{SE}(\mathbf{G})$, then $\texttt{Gr}$ does not attack a. So, $\texttt{Deg}^g(a) \in \{\beta, 1\}$.

$\texttt{Deg}^s(a) = \alpha$ Hence, $\exists E, E' \in \texttt{SE}(\mathbf{G})$ s.t. $a \in E$ and $a \notin E'$. Since $\texttt{Gr} \subseteq E$ and $\texttt{Gr} \subseteq E'$, then $a \notin \texttt{Gr}$. Furthermore, $\texttt{Gr}$ does not attack a since E is conflict-free. Hence, $\texttt{Deg}^g(a) = \beta$.

$\texttt{Deg}^s(a) = 0$ Thus, $\texttt{SE}(\mathbf{G}) \neq \emptyset$ and $\forall E \in \texttt{SE}(\mathbf{G})$, $a \notin E$. Since $\forall E \in \texttt{SE}(\mathbf{G}), \texttt{Gr} \subseteq E$, then $a \notin \texttt{Gr}$. Thus, $\texttt{Deg}^g(a) \in \{0, \beta\}$.

□

Lemma 9. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^g(a) = 1$, *then* $\texttt{Deg}^s(a) \in \{\beta, 1\}$.
- *If* $\texttt{Deg}^g(a) = \beta$, *then* $\texttt{Deg}^s(a) \in T$.

- *If* $\texttt{Deg}^g(a) = 0$, *then* $\texttt{Deg}^s(a) \in \{0, \beta\}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$.

$\texttt{Deg}^g(a) = 1$ Thus, $a \in \texttt{Gr}$. There are two possibilities: i) $\texttt{SE}(\mathbf{G}) = \emptyset$, hence $\texttt{Deg}^s(a) = \beta$, ii) $\texttt{SE}(\mathbf{G}) \neq \emptyset$. Since $\forall E \in \texttt{SE}(\mathbf{G}), \texttt{Gr} \subseteq E$, then $\texttt{Deg}^s(a) = 1$.

$\texttt{Deg}^g(a) = \beta$ Then $\texttt{Deg}^s(a) \in T$, i.e., all three values are possible.

$\texttt{Deg}^g(a) = 0$ This means that $a \notin \texttt{Gr}$ and $\texttt{Gr}$ attacks a. There are two possibilities: i) $\texttt{SE}(\mathbf{G}) = \emptyset$, hence $\texttt{Deg}^s(a) = \beta$, ii) $\texttt{SE}(\mathbf{G}) \neq \emptyset$. Since $\forall E \in \texttt{SE}(\mathbf{G}), \texttt{Gr} \subseteq E$, then $\texttt{Deg}^s(a) = 0$ as every extension attacks a.

□

Lemma 10. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^g(a) = 1$, *then* $\texttt{Deg}^i(a) = 1$.
- *If* $\texttt{Deg}^g(a) = \beta$, *then* $\texttt{Deg}^i(a) \in \{0, \beta, 1\}$.
- *If* $\texttt{Deg}^g(a) = 0$, *then* $\texttt{Deg}^i(a) = 0$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$. Let $\texttt{Gr}$ and $\texttt{IE}(\mathbf{G})$ denote respectively the grounded extension and the ideal extension of $\mathbf{G}$.

$\texttt{Deg}^g(a) = 1$ This means that $a \in \texttt{Gr}$. Since $\texttt{Gr} \subseteq \texttt{IE}(\mathbf{G})$ then $a \in \texttt{IE}(\mathbf{G})$ and so $\texttt{Deg}^i(a) = 1$.

$\texttt{Deg}^g(a) = \beta$ This means that a does not belong to the grounded extension. Furthermore, a is not attacked by the grounded extension. Then, $\texttt{Deg}^g(a) \in \{0, \beta, 1\}$.

$\texttt{Deg}^g(a) = 0$ This means $a \notin \texttt{Gr}$ and $\texttt{Gr}$ attacks a. Since $\texttt{Gr} \subseteq \texttt{IE}(\mathbf{G})$ and $\texttt{IE}(\mathbf{G})$ is conflict-free, then $a \notin \texttt{IE}(\mathbf{G})$. Thus, $\texttt{Deg}^i(a) = 0$.

□

Lemma 11. *Let* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ *and* $a \in \mathcal{A}$.

- *If* $\texttt{Deg}^i(a) = 1$, *then* $\texttt{Deg}^g(a) \in \{\beta, 1\}$.
- *If* $\texttt{Deg}^i(a) = \beta$, *then* $\texttt{Deg}^g(a) = \beta$.
- *If* $\texttt{Deg}^i(a) = 0$, *then* $\texttt{Deg}^g(a) \in \{0, \beta\}$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \texttt{AG}$ and $a \in \mathcal{A}$. Let $\texttt{Gr}$ and $\texttt{IE}(\mathbf{G})$ denote respectively the grounded extension and the ideal extension of $\mathbf{G}$.

$\texttt{Deg}^i(a) = 1$ Then $a \in \texttt{IE}(\mathbf{G})$. There are two possibilities: i) $a \in \texttt{Gr}$ in which case $\texttt{Deg}^g(a) = 1$, and ii) $a \notin \texttt{Gr}$. Since $\texttt{Gr} \subseteq \texttt{IE}(\mathbf{G})$, $\texttt{Gr}$ does not attack a (due to the conflict-freeness of $\texttt{IE}(\mathbf{G})$). Hence, $\texttt{Deg}^g(a) = \beta$.

$\texttt{Deg}^i(a) = \beta$ Thus $a \notin \texttt{IE}(\mathbf{G})$ and $\texttt{IE}(\mathbf{G})$ does not attack a. Since $\texttt{Gr} \subseteq \texttt{IE}(\mathbf{G})$, then $a \notin \texttt{Gr}$ and $\texttt{Gr}$ does not attack a. Thus, $\texttt{Deg}^g(a) = \beta$.

$\texttt{Deg}^i(a) = 0$ Thus $a \notin \texttt{IE}(\mathbf{G})$ and so $a \notin \texttt{Gr}$. Consequently, $\texttt{Deg}^g(a) \leq \beta$.

□

Theorem 18. *The four semantics (stable, preferred, grounded, ideal) are pairwise weakly equivalent.*

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma, \mathcal{R} \rangle \in \mathtt{AG}$ and $a, b \in \mathcal{A}$.
Let us show that preferred and stable are weakly equivalent.

- Assume that $a \succ^s b$. Thus, $\mathtt{Deg}^s(a) > \mathtt{Deg}^s(b)$, by definition $\mathtt{SE}(\mathbf{G}) \neq \emptyset$, and from Lemma 6, $\mathtt{Deg}^s(a) \in \{\alpha, 1\}$ and $\mathtt{Deg}^s(b) \in \{0, \alpha, 1\}$.

  $\mathtt{Deg}^s(a) = 1$ From Lemma 4, $\mathtt{Deg}^p(a) \geq \alpha$. Furthermore, if $\mathtt{Deg}^s(b) = \alpha$, then $\mathtt{Deg}^p(b) = \alpha$; and if $\mathtt{Deg}^s(b) = 0$, $\mathtt{Deg}^p(b) \leq \alpha$. Hence, $\mathtt{Deg}^p(a) \geq \mathtt{Deg}^p(b)$ and $a \succeq^p b$.

  $\mathtt{Deg}^s_{\mathbf{G}}(a) = \alpha$ From Lemma 4, $\mathtt{Deg}^p(a) = \alpha$. Since $\mathtt{Deg}^s(a) > \mathtt{Deg}^s(b)$, then $\mathtt{Deg}^s(b) = 0$. From Lemma 4, $\mathtt{Deg}^p(b) \leq \alpha$. Thus, $\mathtt{Deg}^p(a) \geq \mathtt{Deg}^p(b)$ and $a \succeq^p b$.

- Assume that $a \succ^p b$, i.e., $\mathtt{Deg}^p(a) > \mathtt{Deg}^p(b)$. If $\mathtt{SE}(\mathbf{G}) = \emptyset$, then $\mathtt{Deg}^s(a) = \mathtt{Deg}^s(b) = \beta$ (from Lemma 5). Thus, $a \approx^s b$. Assume now that $\mathtt{SE}(\mathbf{G}) \neq \emptyset$.

  $\mathtt{Deg}^p(a) = 1$ From Lemma 5, $\mathtt{Deg}^s(a) = 1$. Hence, $\mathtt{Deg}^s(a) \geq \mathtt{Deg}^s(b)$ and $a \succeq^s b$.

  $\mathtt{Deg}^p(a) = \alpha$ From Lemma 5, $\mathtt{Deg}^s(a) \in \{0, \alpha, 1\}$. Since $\mathtt{Deg}^p(b) < \alpha$, then $\mathtt{Deg}^s(b) = 0$ (from Lemma 5). So, $\mathtt{Deg}^s(a) \geq \mathtt{Deg}^s(b)$ and $a \succeq^s b$.

  $\mathtt{Deg}^p(a) = \beta$ This means that $\mathtt{Deg}^p(b) = 0$. From Lemma 5, $\mathtt{Deg}^s(a) = \mathtt{Deg}^s(b) = 0$. Thus, $a \approx^s b$.

We show that preferred and grounded are weakly equivalent.

- Assume that $a \succ^g b$, then $\mathtt{Deg}^g(a) > \mathtt{Deg}^g(b)$.

  $\mathtt{Deg}^g(a) = 1$ From Lemma 6, $\mathtt{Deg}^p(a) = 1$, thus $\mathtt{Deg}^p(a) \geq \mathtt{Deg}^p(b)$, and so $a \succeq^p b$.

  $\mathtt{Deg}^g(a) = \beta$ Hence, $\mathtt{Deg}^g(b) = 0$. From Lemma 6, $\mathtt{Deg}^p(b) = 0$, thus $\mathtt{Deg}^p(a) \geq \mathtt{Deg}^p(b)$, and so $a \succeq^p b$.

- Assume that $a \succ^p b$, then $\mathtt{Deg}^p(a) > \mathtt{Deg}^p(b)$.

  $\mathtt{Deg}^p(a) = 1$ From Lemma 7, $\mathtt{Deg}^g(a) \geq \beta$. Besides, $\mathtt{Deg}^p(b) \in \{0, \beta, \alpha\}$, and from Lemma 7, $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

  $\mathtt{Deg}^p(a) = \alpha$ From Lemma 7, $\mathtt{Deg}^g(a) = \beta$. Since, $\mathtt{Deg}^p(b) \in \{0, \beta\}$, and from Lemma 7, $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

  $\mathtt{Deg}^p(a) = \beta$ and $\mathtt{Deg}^p(b) = 0$. From Lemma 7, $\mathtt{Deg}^g(a) = \beta$ and $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

Let us show that grounded and ideal are weakly equivalent.

- Assume that $a \succ^g b$, then $\mathtt{Deg}^g(a) > \mathtt{Deg}^g(b)$.

  $\mathtt{Deg}^g(a) = 1$ From Lemma 10, $\mathtt{Deg}^i(a) = 1$, thus $\mathtt{Deg}^i(a) \geq \mathtt{Deg}^i(b)$, and so $a \succeq^i b$.

  $\mathtt{Deg}^g(a) = \beta$ Hence, $\mathtt{Deg}^g(b) = 0$. From Lemma 10, $\mathtt{Deg}^i(b) = 0$, thus $\mathtt{Deg}^i(a) \geq \mathtt{Deg}^i(b)$, and so $a \succeq^i b$.

- Assume that $a \succ^i b$, then $\mathtt{Deg}^i(a) > \mathtt{Deg}^i(b)$.

$\mathtt{Deg}^i(a) = 1$ From Lemma 11, $\mathtt{Deg}^g(a) \geq \beta$. Besides, $\mathtt{Deg}^i(b) \leq \beta$, and from Lemma 11, $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

$\mathtt{Deg}^i(a) = \beta$ and thus $\mathtt{Deg}^i(b) = 0$. From Lemma 11, $\mathtt{Deg}^g(a) = \beta$ and $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

Let us show that grounded and stable are weakly equivalent.

- Assume that $a \succ^g b$, then $\mathtt{Deg}^g(a) > \mathtt{Deg}^g(b)$.

  $\mathtt{Deg}^g(a) = 1$ and $\mathtt{Deg}^g(b) = \beta$. There are two cases: i) $\mathtt{SE}(\mathbf{G}) = \emptyset$, so $\mathtt{Deg}^s(a) = \mathtt{Deg}^s(b) = \beta$ and $a \succeq^s b$. ii) $\mathtt{SE}(\mathbf{G}) \neq \emptyset$. Since $\forall E \in \mathtt{SE}(\mathbf{G}), \mathtt{Gr} \subseteq E$ and $a \in \mathtt{Gr}$, then $\mathtt{Deg}^s(a) = 1$ and consequently, $\mathtt{Deg}^s(a) \geq \mathtt{Deg}^s(b)$ and $a \succeq^s b$.

  $\mathtt{Deg}^g(a) = 1$ and $\mathtt{Deg}^g(b) = 0$. From Lemma 9, $\mathtt{Deg}^s(a) \geq \beta$ and $\mathtt{Deg}^s(b) \leq \beta$. Consequently, $\mathtt{Deg}^s(a) \geq \mathtt{Deg}^s(b)$ and $a \succeq^s b$.

  $\mathtt{Deg}^g(a) = \beta$ and so $\mathtt{Deg}^g(b) = 0$. There are two cases: i) $\mathtt{SE}(\mathbf{G}) = \emptyset$, so $\mathtt{Deg}^s(a) = \mathtt{Deg}^s(b) = \beta$ and $a \succeq^s b$. ii) $\mathtt{SE}(\mathbf{G}) \neq \emptyset$ and $\forall E \in \mathtt{SE}(\mathbf{G}), \mathtt{Gr} \subseteq E$. Since $\mathtt{Deg}^g(b) = 0$, then $\mathtt{Gr}$ attacks b. Hence, every stable extension attacks b. Consequently, $\mathtt{Deg}^s(b) = 0$, $\mathtt{Deg}^s(a) \geq \mathtt{Deg}^s(b)$ and $a \succeq^s b$.

- Assume that $a \succ^s b$, then $\mathtt{Deg}^s(a) > \mathtt{Deg}^s(b)$.

  $\mathtt{Deg}^s(a) = 1$ From Lemma 8, $\mathtt{Deg}^g(a) \geq \beta$. Besides, $\mathtt{Deg}^s(b) \leq \alpha$, and from Lemma 8, $\mathtt{Deg}^g(b) \leq \beta$. So, $\mathtt{Deg}^g(a) \geq \mathtt{Deg}^g(b)$ and $a \succeq^g b$.

  $\mathtt{Deg}^s(a) = \alpha$, thus $\mathtt{Deg}^g(a) = \beta$ from Lemma 8. Furthermore, $\mathtt{Deg}^s(b) = 0$ and from the same lemma $\mathtt{Deg}^g(b) \leq \beta$. So, $a \succeq^g b$.

The proofs of weak equivalence of ideal with (preferred, stable) are similar to those of grounded with the same semantics. □

Theorem 21. *For any* $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$, *it holds that*

$$\mathtt{Deg}^{\mathtt{TB}}_{\mathbf{G}} \equiv \mathtt{Deg}^{\mathtt{IS}}_{\mathbf{G}}.$$

Moreover, for any $x \in \{\mathtt{TB}, \mathtt{IS}\}$, *for any* $a \in \mathcal{A}$,

- $\mathtt{Deg}^x_{\mathbf{G}}(a) = 1$ *iff* $a \in \mathtt{Gr}$.
- $\mathtt{Deg}^x_{\mathbf{G}}(a) = 0$ *iff* $\mathtt{Gr}$ *attacks* a.
- $\mathtt{Deg}^x_{\mathbf{G}}(a) = \frac{1}{2}$ *iff* $a \notin \mathtt{Gr}$ *and* $\mathtt{Gr}$ *does not attack* a.

Proof. Let $\langle \mathcal{A}, \sigma \equiv 1, \mathcal{R} \rangle \in \mathtt{AG}$ and $a \in \mathcal{A}$.

▶ We show the result for Trust-based semantics.

Let (O_i) and (Q_i) be the following two properties:

(P_i) $\forall a \in \mathcal{F}^i(\emptyset) \setminus \cup_{j=1}^{i-1} \mathcal{F}^j(\emptyset), \mathtt{Deg}^{\mathtt{TB}}(a) = 1.$

(Q_i) $\forall a \in \mathcal{A}$ s.t. a is attacked by $\mathcal{F}^i(\emptyset) \setminus \cup^{i-1} \mathcal{F}^j(\emptyset), \mathtt{Deg}^{\mathtt{TB}}(a) = 0.$

Let us show that (P_1) holds. Let $a \in \mathcal{F}^1(\emptyset)$, i.e., $\mathtt{Att}(a) = \emptyset$. By definition $\max_{b \mathcal{R} a} \mathtt{Deg}(b) = 0$. So, $f_n(a) = \frac{1}{2} f_{n-1}(a) + \frac{1}{2}$. From ((Costa Pereira, Tettamanzi, and Villata, 2011)), $\lim_{n \to \infty} f_n(a) = l$ with $l \in [0, 1]$. Then, $l = \frac{1}{2} l + \frac{1}{2}$ and so $l = 1$, which shows (P_1).

Let's show that (Q_1) holds. Let $b \in \mathcal{A}$ such that $\texttt{Att}(b) = \{a_1, \ldots, a_k\}$ with $a_1 \in \mathcal{F}^1(\emptyset)$. $\texttt{Deg}^{\texttt{TB}}(b) = \lim_{n\to\infty} f_n(b) = \frac{1}{2} \lim_{n\to\infty} f_{n-1}(b) - \frac{1}{2} \max \big[\lim_{n\to\infty} f_{n-1}(a_1), \ldots, \lim_{n\to\infty} f_{n-1}(a_k) \big] + \frac{1}{2}$. From ((Costa Pereira, Tettamanzi, and Villata, 2011)), $\forall i = 1, k$, $\lim_{n\to\infty} f_{n-1}(a_i) = l_i$ with $l_i \in [0, 1]$. From (P_1), $\lim_{n\to\infty} f_{n-1}(a_1) = 1$. Hence, $\lim_{n\to\infty} f_n(b) = \frac{1}{2} \lim_{n\to\infty} f_{n-1}(b)$. Since $\lim_{n\to\infty} f_n(b) = \lim_{n\to\infty} f_{n-1}(b)$, then $\lim_{n\to\infty} f_n(b) = 0$. Thus, $\texttt{Deg}^{\texttt{TB}}(b) = 0$. This shows (Q_1).

Assume now that the property (P_i) holds and let us show that (P_{i+1}) holds as well. Let $a \in \mathcal{F}^{i+1}(\emptyset)$ and $a \notin \bigcup_1^i \mathcal{F}^j(\emptyset)$.

$$\texttt{Deg}^{\texttt{TB}}(a) = \lim_{n\to\infty} f_n(a) =$$

$$\lim_{n\to\infty} \frac{1}{2}[f_{n-1}(a) - \max_{b\mathcal{R}a} f_{n-1}(b) + 1]$$

$$= \frac{1}{2}[\lim_{n\to\infty} (f_{n-1}(a) - \max_{b\mathcal{R}a} f_{n-1}(b) + 1)]$$

$$= \frac{1}{2} \lim_{n\to\infty} f_{n-1}(a) - \frac{1}{2} \max_{b\mathcal{R}a} \lim_{n\to\infty} f_{n-1}(b) + \frac{1}{2}$$

Since $a \in \mathcal{F}^{i+1}(\emptyset) \setminus \bigcup_{j=1}^{i} \mathcal{F}^{j'}(\emptyset)$, then $\mathcal{F}^i(\emptyset)$ defends a. This means, $\forall x \mathcal{R} a, \exists y \in \mathcal{F}^i(\emptyset)$ such that $y\mathcal{R}x$ and . $\texttt{Deg}^{\texttt{TB}}(y) = 1$ (from (P_i) and $\texttt{Deg}^{\texttt{TB}}(x) = 0$ (from (Q_i). Hence,

$$\max \lim_{n\to\infty} f_{n-1}(b) = \max_{b\mathcal{R}a} \lim_{n\to\infty} \texttt{Deg}^{\texttt{TB}}(b) = 0.$$

Thus, $\texttt{Deg}^{\texttt{TB}}(a) = \frac{1}{2} \lim_{n\to\infty} f_{n-1}(a) + \frac{1}{2}$. Since the series f_n converges and $\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} f_{n-1}(a) = l$, then $l = \frac{1}{2}l + \frac{1}{2}$ and so $l = 1$ and $\texttt{Deg}^{\texttt{TB}}(a) = 1$.

Assume now that the property (Q_i) holds and let us show that (Q_{i+1}) holds as well. Let now $x \in \mathcal{A}$ such that $\mathcal{F}^{i+1}(\emptyset)$ attacks x.

$$\texttt{Deg}^{\texttt{TB}}(x) = \lim_{n\to\infty} f_n(x) =$$

$$\frac{1}{2}[\lim_{n\to\infty} (f_{n-1}(x) - \max_{y\mathcal{R}x} \lim_{n\to\infty} f_{n-1}(y) + 1)].$$

Note that $\max_{y\mathcal{R}x} \lim_{n\to\infty} f_{n-1}(y) = \max_{y\mathcal{R}x} \lim_{n\to\infty} f_n(y) = \max_{y\mathcal{R}x} \texttt{Deg}^{\texttt{TB}}(y)$. Since $\mathcal{F}^{i+1}$ attacks x, then $\exists y* \in \mathcal{F}^{i+1}(\emptyset) \cap \texttt{Att}(x)$. From (P_i), $\texttt{Deg}^{\texttt{TB}}(y*) = 1$, thus $\max_{y\mathcal{R}x} \texttt{Deg}^{\texttt{TB}}(y) = 1$ and consequently, $\lim_{n\to\infty} f_n(x) = \frac{1}{2} \lim_{n\to\infty} f_{n-1}(x)$. Since f_n converges to some $l \in [0, 1]$, thus $l = \frac{1}{2}l$ and so $l = 0$.

Let us now that $\texttt{Deg}^{\texttt{TB}}(a) = \frac{1}{2}$ iff $a \notin \texttt{Gr}$ and $\texttt{Gr}$ does not attack a. Let $a \notin \texttt{Gr}$ and a not attacked by $\texttt{Gr}$. From (P), $\texttt{Deg}^{\texttt{TB}}(a) < 1$ and from (Q), $\texttt{Deg}^{\texttt{TB}}(a) > 0$. Hence, $0 < \texttt{Deg}^{\texttt{TB}}(a) < 1$. Furthermore, $\texttt{Att}(a) = \{x_1, .., x_n\} \neq \emptyset$ and $\forall x_i \in \texttt{Att}(a)$, $x_i \notin \texttt{Gr}$, so $\texttt{Deg}^{\texttt{TB}}(x_i) < 1$. Since $a \notin \texttt{Gr}$, it is not defended against all its attacks by $\texttt{Gr}$. Let $\texttt{Att}(a) = X \cup Y$ with $X = \{x \text{ not attacked by } \texttt{Gr}\}$ and $Y = \{x \text{ attacked by } \texttt{Gr}\}$. Obviously, (i) $X \neq \emptyset$ and (ii) $\forall x \in Y$, $\texttt{Deg}^{\texttt{Mbs}}(x) = 0$ (from the second property of the

theorem). Hence, $\texttt{Deg}^{\texttt{Mbs}}(a) = \lim_{n\to\infty} f_{n+1}(a) = \frac{1}{2}\lim_{n\to\infty} f_n(a) - \frac{1}{2}\max_{x\in\texttt{Att}(a)}[\lim_{n\to\infty} f_n(x)] + \frac{1}{2}$. By definition, $\lim_{n\to\infty} f_n(x) = \texttt{Deg}^{\texttt{Mbs}}(x)$. From (ii) it follows that

$$\max_{x\in\texttt{Att}(a)}[\lim_{n\to\infty} f_n(x)] = \max_{y\in X}[\lim_{n\to\infty} f_n(y)]. \qquad (1)$$

It has been shown in ((Costa Pereira, Tettamanzi, and Villata, 2011)) that the series f_n converges to a value $l \in [0,1]$. Furthermore, $\lim_{n\to\infty} f_{n+1}(a) = \lim_{n\to\infty} f_n(a) = l$. Let $l' = \max_{y\in X} \lim_{n\to\infty} f_n(y)$. Hence $l + l' = 1$.

Let us show that $l = l' = \frac{1}{2}$. Equation (1) shows that $\texttt{Deg}^{\texttt{TB}}(a)$ depends only on its stronger attacker, and the latter is neither in $\texttt{Gr}$ nor attacked by $\texttt{Gr}$. Let us define the finite sequence $\langle x_1, x_2, \ldots, x_k\rangle$ such that:

- $x_1 = a$,
- x_i is the strongest attacker of x_{i-1}, for $i > 1$.
- x_i appears at most twice in the sequence, and there is no other argument x_j that appears twice and $j < i$.

Since the number of arguments is finite and $\forall x_i$, $\texttt{Att}(x_i) \neq \emptyset$, then $\exists j < k$ such that $x_k = x_j$. This means the sequence contains a cycle C. Let $l_i = \lim_{n\to\infty} f_n(x_i)$. From the above reasoning, $\forall i = 1, k$, $l_i + l_{i+1} = 1$. Thus, $l_i = l_{i+2}$. Indeed, $l_1 = l_3 = l_5 = \ldots$ and $l_2 = l_4 = l_6 = \ldots$. There are thus only two values at most: $l = l_1 = l_3 = l_5 = \ldots$ and $l' = l_2 = l_4 = l_6 = \ldots$. There are two cases:

- C is an odd-length cycle: Hence, $l_j = l$ and $l_k = l'$ or the converse. But $x_j = x_k$ and an argument has one value, thus $l = l'$. Since $l + l' = 1$, then $l = l' = \frac{1}{2}$.
- C is an even-length cycle: Recall that $l_j = \frac{1}{2}l_j + \frac{1}{2} - \frac{1}{2}l_{j+1}$. If $l_{j+1} \geq \frac{1}{2}$, then $l_j \leq \frac{1}{2}$ and if $l_{j+1} \leq \frac{1}{2}$, then $l_j \geq \frac{1}{2}$. Thus, $\frac{1}{2} \leq l_j \leq \frac{1}{2}$ and so $l_j = \frac{1}{2}$. Since $l_j + l_{j+1} = 1$, then $l_{j+1} = \frac{1}{2}$. Thus, $l = l' = \frac{1}{2}$.

▶ We show the result for Iterative Schema semantics. Let $a \in \mathcal{A}$. From Corollary 2.4 in ((Gabbay and Rodrigues, 2015)), $\texttt{Deg}^{\texttt{IS}}(a) \in \{0, \frac{1}{2}, 1\}$. The rest of the proof is similar to the above one for $\texttt{TB}$. □

Theorem 27. *In case of flat argumentation graphs,* $\texttt{Mbs}$ *(resp.* $\texttt{EMbs}$*) is weakly equivalent with ideal, stable and preferred semantics.*

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \sigma \equiv 1, \mathcal{R}\rangle \in \texttt{AG}$, and $a, b \in \mathcal{A}$.

▶ Let us show that Max-based and stable are weakly equivalent. If $\texttt{SE}(\mathbf{G}) = \emptyset$, then $a \approx_s b$ for all $a, b \in \mathcal{A}$. Assume now that $\texttt{SE}(\mathbf{G}) \neq \emptyset$ and $a \succ^s b$. So, $\texttt{Deg}^s(a) > \texttt{Deg}^s(b)$. There are three cases:

$\texttt{Deg}^s(a) = 1$. This means that $a \in \bigcap_{\mathcal{E}_i\in\texttt{SE}(\mathbf{G})} \mathcal{E}_i$. Since $\texttt{Gr} \subseteq \bigcap_{\mathcal{E}_i\in\texttt{SE}(\mathbf{G})} \mathcal{E}_i$, then there are two possibilities:

1) $a \in \texttt{Gr}$, in which case $\texttt{Deg}^{\texttt{Mbs}}(a) = \mathcal{S}_1^i$, for some $i \in \mathbb{N}$ from Theorem 8, and

2) $a \notin \texttt{Gr}$, and since $\texttt{Gr}$ does not attack a (as stable extensions are conflict-free), hence from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(a) = \frac{1}{\varphi}$.

Since $\forall i \in \mathbb{N}$, $\mathcal{S}_1^i > \frac{1}{\varphi}$, then $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \frac{1}{\varphi}$.

- $\texttt{Deg}^s(b) = \alpha$. Thus, $b \notin \texttt{Gr}$ and $\texttt{Gr}$ does not attack b (as stable extensions are conflict-free), hence from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$. This means that $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.
- $\texttt{Deg}^s(b) = 0$. Either b is attacked by $\texttt{Gr}$, in which case from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \mathcal{S}_2^i$ for some $i \in \mathbb{N}$, or b is not attacked by $\texttt{Gr}$, in which case from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$. Since $\forall i \in \mathbb{N}, \mathcal{S}_2^i < \frac{1}{\varphi}$, then $\texttt{Deg}^{\texttt{Mbs}}(b) \leq \frac{1}{\varphi}$. Consequently, $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.

$\texttt{Deg}^s(a) = \alpha$. As seen above, $\texttt{Deg}^{\texttt{Mbs}}(a) = \frac{1}{\varphi}$. Since $\texttt{Deg}^s(b) = 0$, then from above $\texttt{Deg}^{\texttt{Mbs}}(b) \leq \frac{1}{\varphi}$. Consequently, $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.

If $\texttt{Gr} = \emptyset$, then $\forall a, b \in \mathcal{A}, \texttt{Deg}^{\texttt{Mbs}}(a) = \texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$ and thus $a \approx^{\texttt{Mbs}} b$. Assume now that $\texttt{Gr} \neq \emptyset$ and let $a \succ^{\texttt{Mbs}} b$, i.e., $\texttt{Deg}^{\texttt{Mbs}}(a) > \texttt{Deg}^{\texttt{Mbs}}(b)$.

Case $a \in \texttt{Gr}$. Since $\texttt{Gr} \subseteq \bigcap_{\mathcal{E}_i \in \texttt{SE}(\mathbf{G})} \mathcal{E}_i$, then $\texttt{Deg}^s(a) = 1$ and so $a \succeq_s b$.

Case $a \in \texttt{Gr}^+$. This means that both a, b are attacked by the grounded extension. Indeed, Corollary 15 shows that arguments of $\texttt{Gr}^+$ are weaker than those of $\texttt{Gr}$ and $\texttt{Gr}^o$. Hence, $\texttt{Deg}^s(a) = \texttt{Deg}^s(b) = 0$ and so $a \approx_s b$.

Case $a \in \texttt{Gr}^o$. From Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(a) = \frac{1}{\varphi}$ and $b \in \texttt{Gr}^+$, which means that $\texttt{Deg}^s(b) = 0$. So, $\texttt{Deg}^s(a) \geq \texttt{Deg}^s(b)$ and so $a \succeq_s b$.

▶ Let us show that Max-based and preferred are weakly equivalent. Let $a, b \in \mathcal{A}$. If $\texttt{PE}(\mathbf{G}) = \{\emptyset\}$, then $a \approx_p b$ since $\texttt{Deg}^p(a) = \texttt{Deg}^p(b) = \beta$. Since $\texttt{Gr} \subseteq \bigcap_{\mathcal{E}_i \in \texttt{PE}(\mathbf{G})} \mathcal{E}_i$, then $\texttt{Gr} = \emptyset$. From Theorem 8, $\forall x \in \mathcal{A}, \texttt{Deg}^{\texttt{Mbs}}(x) = \frac{1}{\varphi}$. Consequently, $a \approx_{\texttt{Mbs}} b$.

Assume that $\texttt{PE}(\mathbf{G}) \neq \{\emptyset\}$ and $a \succ_p b$. There are three cases:

$\texttt{Deg}^p(a) = 1$. This means that $a \in \bigcap_{\mathcal{E}_i \in \texttt{PE}(\mathbf{G})} \mathcal{E}_i$. Since $\texttt{Gr} \subseteq \bigcap_{\mathcal{E}_i \in \texttt{PE}(\mathbf{G})} \mathcal{E}_i$, then there are two possibilities:

1) $a \in \texttt{Gr}$, in which case $\texttt{Deg}^{\texttt{Mbs}}(a) = \mathcal{S}_1^i$, for some $i \in \mathbb{N}$ (from Theorem 8), or 2) $a \notin \texttt{Gr}$, and since $\texttt{Gr}$ does not attack a (as preferred extensions are conflict-free), hence from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(a) = \frac{1}{\varphi}$. Since $\forall i \in \mathbb{N}, \mathcal{S}_1^i > \frac{1}{\varphi}$, then $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \frac{1}{\varphi}$.

- $\texttt{Deg}^p(b) = \alpha$. Thus, $b \notin \texttt{Gr}$ and $\texttt{Gr}$ does not attack b (as preferred extensions are conflict-free), hence from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$. This means that $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.
- $\texttt{Deg}^p(b) = \beta$. Thus, $b \notin \bigcup_{\mathcal{E}_i \in \texttt{PE}(\mathbf{G})} \mathcal{E}_i$ and is not attacked by $\bigcup_{\mathcal{E}_i \in \texttt{PE}(\mathbf{G})} \mathcal{E}_i$. Consequently, $b \notin \texttt{Gr}$ and b is not attacked by $\texttt{Gr}$. From Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$. Hence, $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.
- $\texttt{Deg}^p(b) = 0$. Either b is attacked by $\texttt{Gr}$, in which case from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \mathcal{S}_2^i$ for some $i \in \mathbb{N}$, or b is not attacked by $\texttt{Gr}$, in which case from Theorem 8, $\texttt{Deg}^{\texttt{Mbs}}(b) = \frac{1}{\varphi}$. Since $\forall i \in \mathbb{N}, \mathcal{S}_2^i < \frac{1}{\varphi}$, then $\texttt{Deg}^{\texttt{Mbs}}(b) \leq \frac{1}{\varphi}$. Consequently, $\texttt{Deg}^{\texttt{Mbs}}(a) \geq \texttt{Deg}^{\texttt{Mbs}}(b)$.

$\mathtt{Deg}^p(a) = \alpha$. Following the same reasoning as above, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \frac{1}{\varphi}$. From above also, we have $\mathtt{Deg}^{\mathtt{Mbs}}(b) \leq \frac{1}{\varphi}$. Consequently, $\mathtt{Deg}^{\mathtt{Mbs}}(a) \geq \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

$\mathtt{Deg}^p(a) = \beta$. Thus, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \frac{1}{\varphi}$. Since $a \succ_p b$, then $\mathtt{Deg}^p(b) = 0$. From above, $\mathtt{Deg}^{\mathtt{Mbs}}(b) \leq \frac{1}{\varphi}$. Thus, $\mathtt{Deg}^{\mathtt{Mbs}}(a) \geq \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

If $\mathtt{Gr} = \emptyset$, then $\forall a, b \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \mathtt{Deg}^{\mathtt{Mbs}}(b) = \frac{1}{\varphi}$ and thus $a \approx^{\mathtt{Mbs}} b$. Assume now that $\mathtt{Gr} \neq \emptyset$ and let $a \succ^{\mathtt{Mbs}} b$, i.e., $\mathtt{Deg}^{\mathtt{Mbs}}(a) > \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

Case $a \in \mathtt{Gr}$. Since $\mathtt{Gr} \subseteq \bigcap_{\mathcal{E}_i \in \mathtt{PE}(\mathbf{G})} \mathcal{E}_i$, then $\mathtt{Deg}^p(a) = 1$ and so $a \succeq_p b$.

Case $a \in \mathtt{Gr}^+$. This means that both a, b are attacked by the grounded extension. Indeed, Corollary 15 shows that arguments of $\mathtt{Gr}^+$ are weaker than those of $\mathtt{Gr}$ and $\mathtt{Gr}^o$. Hence, $\mathtt{Deg}^p(a) = \mathtt{Deg}^p(b) = 0$ and so $a \approx_p b$.

Case $a \in \mathtt{Gr}^o$. From Corollary 15, $b \in \mathtt{Gr}^+$, which means that $\mathtt{Deg}^p(b) = 0$. So, $\mathtt{Deg}^p(a) \geq \mathtt{Deg}^p(b)$ and so $a \succeq_p b$.

▶ Let us show that Max-based and ideal are weakly equivalent. Let $a, b \in \mathcal{A}$.
If $\mathtt{IE}(\mathbf{G}) = \emptyset$, then $a \approx_p b$ since $\mathtt{Deg}^i(a) = \mathtt{Deg}^i(b) = \beta$. Since $\mathtt{Gr} \subseteq \mathtt{IE}(\mathbf{G})$, then $\mathtt{Gr} = \emptyset$ and so from Theorem 8, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \mathtt{Deg}^{\mathtt{Mbs}}(b) = \frac{1}{\varphi}$.
Assume that $\mathtt{IE}(\mathbf{G}) \neq \emptyset$ and $a \succ_i b$. There are two cases:

$\mathtt{Deg}^i(a) = 1$. From Lemma 11, $\mathtt{Deg}^g(a) \geq \beta$. From Theorem 8, $\mathtt{Deg}^{\mathtt{Mbs}}(a) \geq \frac{1}{\varphi}$. Since $\mathtt{Deg}^i(a) > \mathtt{Deg}^i(b)$, then $\mathtt{Deg}^i(b) \leq \beta$. From Lemma 11, $\mathtt{Deg}^g(b) \leq \beta$. From Theorem 8, $\mathtt{Deg}^{\mathtt{Mbs}}(b) \leq \frac{1}{\varphi}$. Hence, $\mathtt{Deg}^{\mathtt{Mbs}}(a) \geq \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

$\mathtt{Deg}^i(a) = \beta$. From Lemma 11, $\mathtt{Deg}^g(a) = \beta$ and from Theorem 8, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \frac{1}{\varphi}$. Since $\mathtt{Deg}^i(a) > \mathtt{Deg}^i(b)$, then $\mathtt{Deg}^i(b) = 0$. From Lemma 11, $\mathtt{Deg}^g(b) \leq \beta$. From Theorem 8, $\mathtt{Deg}^{\mathtt{Mbs}}(b) \leq \frac{1}{\varphi}$. Hence, $\mathtt{Deg}^{\mathtt{Mbs}}(a) \geq \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

If $\mathtt{Gr} = \emptyset$, then $\forall a, b \in \mathcal{A}$, $\mathtt{Deg}^{\mathtt{Mbs}}(a) = \mathtt{Deg}^{\mathtt{Mbs}}(b) = \frac{1}{\varphi}$ and thus $a \approx^{\mathtt{Mbs}} b$. Assume now that $\mathtt{Gr} \neq \emptyset$ and let $a \succ^{\mathtt{Mbs}} b$, i.e., $\mathtt{Deg}^{\mathtt{Mbs}}(a) > \mathtt{Deg}^{\mathtt{Mbs}}(b)$.

Case $a \in \mathtt{Gr}$. Since $\mathtt{Gr} \subseteq \mathtt{IE}(\mathbf{G})$, then $\mathtt{Deg}^i(a) = 1$ and so $a \succeq_i b$.

Case $a \in \mathtt{Gr}^+$. This means that both a, b are attacked by the grounded extension. Indeed, Corollary 15 shows that arguments of $\mathtt{Gr}^+$ are weaker than those of $\mathtt{Gr}$ and $\mathtt{Gr}^o$. Hence, $\mathtt{Deg}^i(a) = \mathtt{Deg}^i(b) = 0$ and so $a \approx_i b$.

Case $a \in \mathtt{Gr}^o$. From Corollary 15, $b \in \mathtt{Gr}^+$, which means that $\mathtt{Deg}^i(b) = 0$. So, $\mathtt{Deg}^i(a) \geq \mathtt{Deg}^i(b)$ and so $a \succeq_i b$.

□

□

Chapter 9

Résumé

L'argumentation, un sous-domaine de l'intelligence artificielle, est une approche du raisonnement qui justifie les affirmations en interagissant avec des arguments. L'une de ses étapes clés est l'évaluation de la force des arguments à l'aide de méthodes formelles, appelées sémantiques.

Cette thèse contribue à la compréhension et au développement des fondements théoriques des sémantiques. Ses contributions sont quadruples.

Premièrement, nous avons étudié la notion cruciale d'arguments qui s'auto-attaque, proposé des postulats de rationalité décrivant comment les traiter et introduit de nouvelles sémantiques qui les satisfont.

Deuxièmement, nous avons caractérisé diverses sémantiques en termes de séries, fournissant les homologues mathématiques de ces sémantiques.

La troisième partie de la thèse a abordé le problème de la comparaison de la pléthore de sémantiques existantes. Nous avons montré que les postulats de rationalité ne sont pas suffisants pour une comparaison équitable des sémantiques, puis nous avons introduit la nouvelle notion d'équivalence basée sur le classement produit par les sémantiques. Nous avons comparé la plupart des sémantiques existantes et caractérisé toute une classe d'équivalence.

La dernière partie de la thèse illustre les sémantiques dans le contexte du raisonnement par cas (CBR). Nous avons commencé par poser les bases du CBR en fournissant un ensemble de principes qu'un modèle devrait satisfaire, et nous avons proposé un modèle d'argumentation qui en satisfait la plupart.

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