# Limites locales et profils de grands arbres Markov branchants

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À mon papa,
*How I wish you were here.*

# Résumé

Le but de cette thèse est d'étudier le comportement asymptotique de grands arbres aléatoires discrets. Par "arbre discret", nous entendrons un graphe acyclique connexe localement fini (tel que chaque nœud n'ait qu'un nombre fini de voisins) et enraciné, c'est-à-dire qu'un nœud de l'arbre que l'on appellera la racine jouera un rôle particulier. Il y a principalement deux points de vue quant à l'étude de ces comportements asymptotiques : les limites locales et d'échelle. Dans le premier cas, on s'intéresse à la structure de graphe autour de la racine tandis que dans le second, on étudie l'arbre dans sa globalité en tant qu'espace métrique normalisé.

Les arbres étudiés auront la propriété dite de Markov branchante : on dit qu'une suite $(T_n)_n$ d'arbres aléatoires respectivement à $n$ nœuds satisfait la propriété de Markov branchante si pour tout $n$, conditionnellement à l'évènement "$T_n$ a $p$ sous-arbres au dessus de sa racine avec respectivement $n_1 \geq \cdots \geq n_p$ nœuds", lesdits sous-arbres sont indépendants et le $i^{\text{ème}}$ plus gros sous-arbre a la même distribution que $T_{n_i}$. Les lois des éléments d'une telle suite sont caractérisées par une suite de mesures sur ce type d'évènements. Nous étudierons aussi des arbres Markov branchants avec un nombre donné de feuilles plutôt que de nœuds. Haas et Miermont [66] ont étudié les limites d'échelle de ces arbres et ont montré que les "arbres continus" correspondants appartiennent à la famille des *arbres de fragmentation*, voir Haas et Miermont [64], une famille à laquelle appartient notamment l'*arbre brownien* introduit par Aldous [7].

Après une introduction des modèles et méthodes utilisées dans cette thèse, ce manuscrit sera divisé en deux parties principales :

— Dans le Chapitre 1, nous introduirons la notion d'arbres Markov branchants infinis, arbres qui apparaissent naturellement comme limites locales d'arbres Markov branchants finis. Ensuite, nous étudierons les limites d'échelle de ces arbres infinis : asymptotiquement, nous verrons apparaître des arbres de fragmentation avec immigration. De ce résultat, on déduira à quelle vitesse le nombre de nœuds dans la boule de rayon $R$ autour de la racine dans un tel arbre croît quand $R$ tend vers l'infini. Le Chapitre 2 quant à lui donnera une extension naturelle de ces résultats à un modèle plus général d'arbres Markov branchants introduit par Rizzolo [111].

— Le Chapitre 3 sera un survol des résultats de la littérature sur le *profil* de modèles d'arbres aléatoires, c'est-à-dire sur la suite des nombres de nœuds à chaque génération desdits arbres. Ce chapitre sera divisé en deux sections : la première traitera de modèles d'arbres dont la hauteur est de l'ordre d'une puissance du nombre de nœuds alors que dans la seconde, nous considérerons des arbres dont la hauteur est d'ordre logarithmique.

Enfin, dans le Chapitre 4, nous étudierons le comportement asymptotique du profil d'arbres Markov branchants. Notre approche sera basée sur des résultats d'inversion de transformées de Fourier et nous serons alors amenés à étudier l'intégrabilité des fonctions caractéristiques du profil des arbres de fragmentation apparaissant à la limite, ce qui se traduira par des résultats sur la régularité de ces derniers.

De plus, nous trouverons dans l'Annexe A des programmes pour simuler certains modèles et objets étudiés dans cette thèse.

# Abstract

This thesis aims to study the asymptotic behaviour of large random discrete trees. By "discrete tree" we will mean a connected graph with no cycles which is locally finite (meaning that all vertices have only finitely many neighbors) and rooted, i.e. a vertex called root is chosen and plays a distinct role. There are mainly two points of view with which we can study the asymptotics of these objects: local or scaling limits. In the first case, we will be interested in the graph structure of the trees around their roots whereas in the latter, trees will be thought of as normalised metric spaces and will be studied as a whole.

The trees we will study will satisfy the Markov branching property: if $(T_n)_n$ is a sequence of random trees where $T_n$ has $n$ vertices for all $n$, the sequence will be said to have the Markov branching property if conditionally on the event "$T_n$ has $p$ sub-trees above its root with $n_1 \geq \cdots \geq n_p$ vertices respectively", these sub-trees are independent and the $i^{\text{th}}$ biggest sub-tree is distributed like $T_{n_i}$. The distributions of the elements of such sequences are characterised by a sequence of probability measures on the set of such events. We will also study Markov branching trees with a fixed number of leaves instead of vertices. Haas and Miermont [66] studied the scaling limits of Markov branching trees and proved that the limit "continuum trees" are *fragmentation trees*, see Haas and Miermont [64]. A prime example of a fragmentation tree is the Brownian tree, see Aldous [7].

In the introduction of this thesis, we will present the models we study as well as the methods we use. The rest of this manuscript will be split into two main parts:

— In Chapter 1, we will introduce infinite Markov branching trees. These trees appear as the natural local limits of finite Markov branching trees. Then, we will turn our focus to the study if the scaling limits of these infinite trees and will see that they converge to fragmentation trees with immigration. This result will allow us to determine how the number of vertices under some height $R$ in such a tree behaves when $R$ goes to infinity. Chapter 2 will give a natural extension of these results to a more general model of Markov branching trees introduced by Rizzolo [111].

— Chapter 3 is a survey on known results on the *profiles*, i.e. the sequence of the number of vertices in each successive layer of a given tree, of a few models of random trees. This chapter will contain two sections: the first of which will focus on models of trees whose height is roughly proportional to the number of vertices in the said trees while in the second section, we will consider trees whose height is roughly the logarithm of the number of vertices.
Finally, in Chapter 4, we will study the asymptotic behaviour of the profile of Markov branching trees. Our approach will rely on Fourier inversion results. That will lead us to study the integrability of the random characteristic functions of the profiles of corresponding fragmentation trees, which will translate to results on the regularity of the said profiles.

Furthermore, Appendix A will present some programs to simulate some models and objects studied in this thesis.

# Table des matières

# Introduction

## *i*.1 Arbres aléatoires

Cette section détaillera les différentes topologies utilisées dans cette thèse ainsi que quelques propriétés asymptotiques des arbres auxquelles nous nous intéresserons. Pour finir, nous illustrerons ces notions via l'exemple des arbres de Galton-Watson.

### *i*.1.1 Convergences d'arbres

*Topologie locale.* La topologie locale est utilisée pour définir une notion de convergence de graphes enracinés. Comme son nom l'indique, cette topologie compare la structure de graphe "locale" desdits graphes, c'est-à-dire autour de leurs racines.

Si $\mathtt{t}$ est un arbre, rappelons qu'il peut être naturellement équipé de la distance de graphe notée $\mathrm{d}_{\mathrm{gr}}$. Cette métrique est telle que pour tous nœuds $u$ et $v$ de $\mathtt{t}$, $\mathrm{d}_{\mathrm{gr}}(u,v)$ est le nombre d'arêtes dans le plus court chemin reliant $u$ et $v$. Pour tout entier positif $r$, notons $\mathtt{t}|_r$ les $r$ premières générations de $\mathtt{t}$, c'est-à-dire l'ensemble des nœuds de $\mathtt{t}$ à distance inférieure ou égale à $r$ de la racine. On dira qu'une suite $(\mathtt{t}_n)_{n\geq 1}$ d'arbres converge localement vers un arbre $\mathtt{t}$ si pour tout entier $r$, on a $\mathtt{t}_n|_r = \mathtt{t}|_r$ pour tout $n$ assez grand.

Si l'arbre $\mathtt{t}$ est fini, on a $\mathtt{t}|_r = \mathtt{t}$ pour $r$ assez grand. De ce fait, $\mathtt{t}_n \to \mathtt{t}$ localement ssi $\mathtt{t}_n = \mathtt{t}$ pour tout $n$ assez grand. Le cas de figure le plus intéressant se présente alors quand l'arbre limite $\mathtt{t}$ est infini (et vérifie donc $\mathtt{t} \neq \mathtt{t}|_r$ pour tout $r$ puisque l'on ne considèrera que des arbres localement finis).

Pour cette topologie, on aura le critère de convergence suivant : si $T_n$, $n \geq 1$ et $T$ sont des arbres aléatoires, alors $T_n$ converge en loi vers $T$ pour la topologie locale ssi pour tout arbre déterministe $\mathtt{t}$ et tout entier positif $r$ on a $\mathbb{P}[T_n|_r = \mathtt{t}|_r] \to \mathbb{P}[T|_r = \mathtt{t}|_r]$.

Cette topologie a été utilisée dans de nombreux travaux en rapport avec l'étude d'arbres aléatoires ou bien dans des cadres qui traitent d'autres types de graphes, voir [1, 2, 4, 6, 12, 13, 14, 15, 31, 34, 38, 76, 77, 87, 113, 114, 117, 118] par exemple.

*Topologie de Gromov-Hausdorff-Prokhorov.* Rappelons qu'un arbre discret est naturellement muni de la distance de graphe. On peut alors considérer un arbre non plus comme un graphe mais plutôt comme un espace métrique. Ce point de vue nous amène à utiliser la notion de convergence d'espaces métriques ce qui donnera des résultats asymptotiques radicalement différents de ceux obtenus pour la topologie locale.

Cette notion a été introduite par Gromov [58] qui eut l'idée de généraliser la distance de Hausdorff à des ensembles compacts n'étant pas inclus dans un même espace métrique. Rappelons que si $(E, d_E)$ est un espace métrique, alors la distance de Hausdorff $d_E^{\mathrm{H}}$ sur $E$ est définie pour tous ensembles compacts $A$ et $B$ inclus dans $E$ par

$$d_E^{\mathrm{H}}(A,B) := \inf\left\{r > 0 : x \in A \subset B^r, B \subset A^r\right\}$$

où pour tous $C \subset E$ et $r > 0$, $C^r := \{x \in E : \inf_{y\in C} d_E(x,y) \leq r\}$. L'idée de Gromov était de comparer différents ensembles en les plongeant dans un espace métrique commun. Ainsi, si $(X, d_X)$ et $(Y, d_Y)$ sont

deux espaces métriques compacts, la distance de Gromov-Hausdorff (non pointée) est définie par

$$\mathrm{d}^*_{\mathrm{GH}}\Big((X,d_X),(Y,d_Y)\Big) := \inf\, d^{\mathrm{H}}_M\Big(\varphi(X),\psi(Y)\Big)$$

où l'infimum est pris sur tous les espaces métriques $(M,d_M)$ et isométries $\varphi : X \to M$, $\psi : Y \to M$.

En plus d'être dotés d'un point particulier, la racine, les arbres que l'on considèrera seront munis d'une mesure Borélienne, typiquement la mesure de comptage sur leurs nœuds ou leurs feuilles.

Nous seront donc amenés à comparer des quadruplets $\mathbf{X} = (X, d_X, \rho_X, \mu_X)$ où $(X, d_X)$ est un espace métrique compact, $\rho_X$ est un point de $X$ que l'on appellera la racine et $\mu_X$ est une mesure Borélienne finie sur $X$. Pour cela nous utiliserons une généralisation de la distance de Gromov-Hausdorff : la distance de Gromov-Hausdorff-Prokhorov pointée, voir Evans [50], Villani [121], Abraham *et al* [3] ou bien Addario-Berry *et al* [5]. Pour deux tels quadruplets $\mathbf{X} = (X, d_X, \rho_X, \mu_X)$ et $\mathbf{Y} = (Y, d_Y, \rho_Y, \mu_Y)$, on définit alors

$$\mathrm{d}_{\mathrm{GHP}}(\mathbf{X},\mathbf{Y}) := \inf\, d_M\Big(\varphi(\rho_X),\psi(\rho_Y)\Big) \vee d^{\mathrm{H}}_M\Big(\varphi(X),\psi(Y)\Big) \vee d^{\mathrm{P}}_M\Big(\mu_X\circ\varphi^{-1},\mu_Y\circ\psi^{-1}\Big)$$

où l'infimum est de nouveau pris sur tous les espaces métriques $(M,d_M)$ et isométries $\varphi : X \to M$, $\psi : Y \to M$ et où pour tout espace métrique $(E,d_E)$, la distance de Prokhrov sur $E$ est donnée par

$$d^{\mathrm{P}}_E(\mu,\nu) := \inf\Big\{r > 0 : \forall A \subset E \text{ Borélien, } \mu(A) \le \nu(A^r) + r \text{ et } \nu(A) \le \mu(A^r) + r\Big\}$$

pour tout couple $(\mu,\nu)$ de mesures Boréliennes sur $(E,d_E)$.

Dans la topologie induite par cette métrique, les fonctions *hauteur* et *masse* qui à un espace métrique compact pointé et mesuré $\mathbf{X} = (X, d_X, \rho_X, \mu_X)$ associent respectivement $|\mathbf{X}| := \sup_{x\in X} d_X(\rho_X, x)$ et $\mu_X(X)$ sont continues. Ainsi, la convergence dans la topologie GHP assure la convergence de la masse et de la hauteur.

*Arbres réels.* Lorsque l'on étudie les limites d'échelle de la marche aléatoire simple, un objet discret, l'objet limite qui apparaît est le mouvement brownien, une fonction réelle et donc un objet continu. De la même manière, dans de nombreux cas, les limites d'échelle d'arbres finis et discrets appartiennent à une classe d'objets "continus" : les arbres réels ou $\mathbb{R}$-arbres, cf Le Gall [92] par exemple.

Un arbre réel est un espace métrique $(T,d_T)$ tel que si $x$ et $y$ sont deux points de $T$, alors

— Il existe une unique isométrie $\varphi : [0, d_T(x,y)] \to T$ telle que $\varphi(0) = x$ et $\varphi[d_T(x,y)] = y$.
— Si $\psi : [0,1] \to T$ est injective, continue et vérifie $\psi(0) = x$ et $\psi(1) = y$, alors $\varphi$ et $\psi$ ont la même image.

Plus informellement, tout couple de points de $T$ ne peut être relié que par un unique chemin injectif et continu, à sa paramétrisation près. Cette propriété est en quelque sorte l'équivalent continu de l'absence de cycles dans un arbre discret.

Soient $\mathbf{T}_n$, $n \ge 1$ des arbres réels compacts, enracinés et munis d'une mesure Borélienne finie. Si $\mathbf{T}_n$ converge vers un espace métrique compact enraciné et mesuré $\mathbf{T} = (T, d_T, \rho_T, \mu_T)$ pour la topologie GHP, alors $(T,d_T)$ est lui-même un arbre réel. En d'autres termes, l'ensemble des arbres réels enracinés et mesurés est fermé pour la topologie GHP.

Pour cette raison, lorsque l'on étudiera les limites d'échelle d'arbres discrets, on considèrera ces derniers comme des arbres réels en remplaçant chaque arête par une copie du segment $[0,1]$.

*Topologie GHP locale.* Les arbres infinis mais localement finis apparaissent naturellement comme limites locales de suites d'arbres finis mais ces arbres ne sont par contre pas compacts lorqu'on les munit de la distance de graphe. La métrique de Gromov-Hausdorff-Prokhorov ne peut donc pas être utilisée pour comparer deux tels arbres. Nous utiliserons alors une version "locale" de cette métrique qui a notamment été étudiée par Abraham *et al* [3].

Si $\mathbf{X} = (X, d_X, \rho_X, \mu_X)$ est un espace métrique localement compact, enraciné et muni d'une mesure Borélienne localement finie, pour tout $r$ positif, on notera $\mathbf{X}|_r = (X|_r, d_X, \rho_X, \mu_X|_r)$ où $X|_r := \{x \in X :$

$d_X(\rho_X, x) \leq r\}$ et $\mu_X|_r := \mathbb{1}_{X|_r}\mu_X$. Soient $\mathbf{T}$ et $\mathbf{T}_n$, $n \geq 1$ des arbres réels localement compacts, enracinés et munis d'une mesure localement finie ; on dira que $\mathbf{T}_n$ converge vers $\mathbf{T}$ pour la topologie GHP locale si pour tout $r$ positif qui est un point de continuité de la fonction $t \mapsto \mu_T(T|_t)$, on a $\mathbf{T}_n|_r \to \mathbf{T}|_r$ pour la topologie GHP usuelle.

Cette topologie est métrisable et coïncide avec la topologie GHP usuelle sur l'ensemble des arbres réels compacts enracinés et mesurés, cf Abraham *et al* [3].

*Profils.* Soit $\mathtt{t}$ un arbre discret fini et soit $\mu$ une mesure finie sur $\mathtt{t}$. Le profil de $\mathtt{t}$ associé à $\mu$ est la suite $\ell_\mu$ définie par $\ell_\mu(j) := \mu\big(u \in \mathtt{t} : \mathrm{d}_{\mathrm{gr}}(\rho, u) = j\big)$ où $\rho$ est la racine de $\mathtt{t}$. En d'autres termes, $\ell_\mu$ est la suite des mesures par rapport à $\mu$, de chacun des niveaux de $\mathtt{t}$.

On peut aussi définir une notion de profil pour certains arbres réels mesurés. Soit $\mathbf{T} = (T, d_T, \rho_T, \mu_T)$ un arbre réel localement compact muni d'une racine et d'une mesure localement finie. Si la fonction $t \mapsto \mu_T(T|_t)$ est absolument continue par rapport à la mesure de Lebesgue, alors on dit que $\mathbf{T}$ admet un profil qui est alors la densité $l_T$ de la mesure dont $\mu_T(T|.)$ est la fonction de répartition.

Notons que comme nous l'avons déjà remarqué, la fonction qui à un espace métrique compact enraciné et mesuré lui associe sa masse totale est continue pour la topologie GHP. Supposons que pour tout $n \geq 1$, $\mathtt{t}_n$ est un arbre discret, $\rho_n$ sa racine et $\mu_n$ une mesure sur $\mathtt{t}_n$ et supposons qu'il existe deux suites de réels positifs $(a_n)_n$ et $(b_n)_n$ telles que $(a_n\mathtt{t}_n, b_n\mu_n) := (\mathtt{t}_n, a_n\mathrm{d}_{\mathrm{gr}}, \rho_n, b_n\mu_n)$ converge vers $\mathbf{T}$ pour la topologie GHP locale. Dans ce cas, si $\mathbf{T}$ admet un profil, $b_n\mu_n(\mathtt{t}_n|_{a_n t})$ converge vers $\mu_T(T|_t)$ pour tout $t$ positif car la fonction $\mu_T(T|.)$ est continue. Il est alors naturel de se demander si le profil adéquatement renormalisé de $\mathtt{t}_n$ converge plus ou moins fortement vers celui de $\mathbf{T}$.

Si les mesures $b_n\mu_n$, $n \geq 1$ et $\mu_T$ sont des probabilités, cette convergence et ce problème admettent une interprétation probabiliste. Pour tout $n \geq 1$, définissons $I_n := d_n(\rho_n, U_n)$ avec $U_n$ une variable aléatoire de loi $b_n\mu_n$ à valeurs dans $\mathtt{t}_n$ ainsi que $I := d_T(\rho_T, U)$ où $U$ est de loi $\mu_T$. Si $(a_n\mathtt{t}_n, b_n\mu_n)$ converge vers $\mathbf{T}$ pour la topologie GHP, alors pour tout $t$ en lequel $\mu_T(T|.)$ est continue, $\mu(T_n|_t)$ converge vers $\mu_T(T|_t)$ ce qui implique que $a_nI_n$ converge en loi vers $I$. La convergence des profils correspondants est alors une version "locale" de cette convergence en loi, cf le théorème central limite local [56, 57].

### *i*.1.2 Arbres de Galton-Watson

Nous allons maintenant présenter un modèle d'arbres aléatoires qui a été étudié par de nombreux auteurs dont les résultats illustrent les notions de la Section *i*.1.1.

Les processus dits de Galton-Watson ont été introduits pour résoudre le problème, posé par Galton [54] et résolu par Watson [122], du calcul de la probabilité de survie des noms de grandes familles anglaises. Ces processus sont définis comme suit. Soit $\xi$ une loi de probabilité sur l'ensemble $\mathbb{N}$ des entiers naturels ; $\xi$ sera la loi de reproduction du processus. Soient $X_{n,k}$, $n \geq 0$, $k \geq 1$ des variables i.i.d. de loi $\xi$. Notons $Z_0 := 1$ et pour tout $n \geq 0$,

$$Z_{n+1} := \sum_{k=1}^{Z_n} X_{n,k}.$$

La variable $Z_n$ modélise alors le nombre d'individus constituant la $n^{\text{ème}}$ génération d'une population où chaque individu a un nombre aléatoire d'enfant de loi $\xi$ et se reproduit indépendamment des autres.

Watson [122] prouva que la probabilité d'extinction de la population, c'est-à-dire la probabilité que $Z_n = 0$ pour $n$ assez grand, est le plus petit point fixe de la fonction génératrice de $\xi$. En particulier, cette probabilité est égale à 1 si la moyenne de $\xi$ est inférieure ou égale à 1 et si $\xi_1 < 1$.

*Arbres de Galton-Watson et limites locales.* La généalogie de ces processus admet une représentation naturelle en tant qu'arbres localement finis : la racine représente l'individu originel et chaque nœud modélise un individu qui est relié aux nœuds représentant ses enfants.

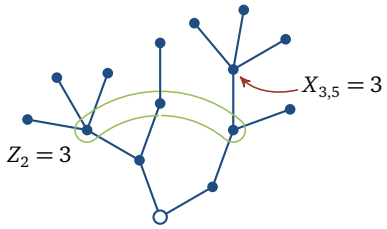



FIGURE 1 – Un arbre de Galton-Watson et le processus sous-jacent.

Lorsque la loi de reproduction $\xi$ est *critique*, i.e. de moyenne égale à 1 et telle que $\xi_1 < 1$, les processus de Galton-Watson s'éteignent presque sûrement et l'arbre correspondant est alors fini. Kennedy [81] et Kesten [83] se sont intéressés au comportement du processus $(Z_n)_{n\geq 1}$ sous différents conditionnements.

Leurs études respectives peuvent en fait être décrites en termes de limites locales d'arbres de Galton-Watson conditionnés. Soit $T$ un arbre de Galton-Watson de loi de reproduction $\xi$ :

— Le résultat de Kennedy décrit la limite en loi pour la topologie locale de $T$ conditionné à avoir $n$ nœuds quand $n$ tend vers l'infini, voir aussi Aldous et Pitman [12].

— Quant à celui de Kesten, il décrit la limite en loi de $T$ conditionnellement à $|T| \geq n$ quand $n$ tend vers l'infini.

Dans les deux cas, le même arbre apparaît à la limite. Cet arbre est appelé *arbre de Kesten*, il est infini mais localement fini et peut en quelque sorte être vu comme l'arbre qui encode un processus de Galton-Watson conditionné à ne jamais s'éteindre, ou encore conditionné à avoir une population totale infinie.

L'arbre de Kesten encode en fait un processus de Galton-Watson avec *immigration*, qui est défini comme suit : soient $(Y_n)_{n\geq 0}$ une suite de variables i.i.d. et telles que $\mathbb{P}[Y_n = j+1] = j\xi_j$ pour tout $j \in \mathbb{N}$ et $(X_{n,k})_{n\geq 0,k\geq 1}$ une famille de variables i.i.d. de loi $\xi$ et indépendante de $(Y_n)_{n\geq 0}$. On pose alors $Z_0 := 0$ et pour tout $n \geq 1$,

$$Z_{n+1} := Y_n + \sum_{k=1}^{Z_n} X_{n,k}.$$

La variable $Y_n$ peut être vue comme le nombre d'individus qui immigrent dans une population à la génération $n$. Ainsi, l'arbre de Kesten est en fait obtenu en greffant pour tout $n$ positif un nombre $Y_n$ d'arbres de Galton-Watson i.i.d. à hauteur $n$ de la *branche infinie*.

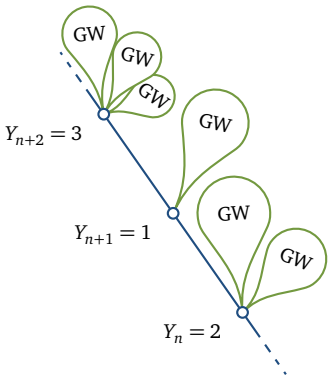



FIGURE 2 – L'arbre de Kesten.

Cet arbre apparaît de plus comme la limite locale d'arbres de Galton-Watson sous de nombreux autres conditionnements, cf Janson [76], Curien et Kortchemski [38] ou encore Abraham et Delmas [2].

*Limites d'échelle.* Dans une série de trois articles, Aldous [7, 8, 9] a étudié les limites d'échelles d'arbres de Galton-Watson conditionnés à avoir un grand nombre de nœuds ainsi que l'arbre réel limite quand leur

loi de reproduction a une variance finie : l'arbre brownien. Voir aussi Le Gall [91].

L'arbre brownien est un arbre réel compact enraciné et mesuré introduit dans [7] sous le nom de *Continuum Random Tree* (arbre continu aléatoire) et construit comme suit. Soit $(\mathbf{e}_t ; 0 \leq t \leq 1)$ une excursion brownienne de longueur 1 ; on définit la pseudo-métrique $\mathrm{d}_{\mathbf{e}}$ sur $[0,1]$ par

$$\mathrm{d}_{\mathbf{e}}(x,y) = \mathrm{d}_{\mathbf{e}}(y,x) := \mathbf{e}_x + \mathbf{e}_y - 2\inf_{t\in[x,y]} \mathbf{e}_t$$

pour $x \leq y$ dans $[0,1]$. On munit alors le segment $[0,1]$ d'une relation d'équivalence $\sim_{\mathbf{e}}$ telle que $x \sim_{\mathbf{e}} y$ ssi $\mathrm{d}_{\mathbf{e}}(x,y) = 0$. Pour tout $x \in [0,1]$, notons $\bar{x}$ la classe d'équivalence de $x$ dans $[0,1]/\sim_{\mathbf{e}}$ et $\mu$ la mesure image de la mesure de Lebesgue par l'application $x \mapsto \bar{x}$. L'arbre réel compact enraciné et mesuré

$$\mathcal{T}_{\mathbf{e}} := \left([0,1]/\sim_{\mathbf{e}}, \mathrm{d}_{\mathbf{e}}, \bar{0}, \mu\right)$$

est alors appelé l'arbre brownien et sa distribution sera notée $\mathcal{T}_B$.

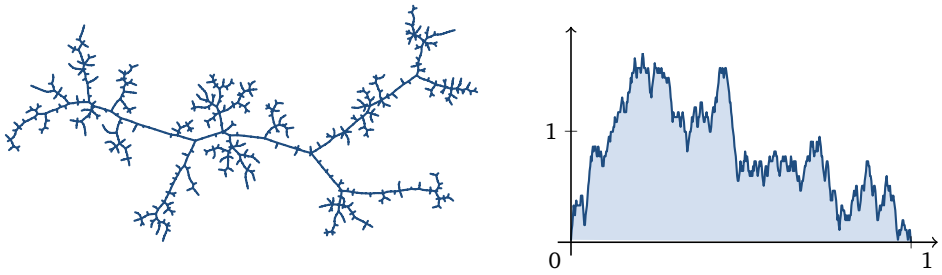



FIGURE 3 – Un arbre brownien et l'excursion sous-jacente.

Si $T$ est un arbre de Galton-Watson de loi de reproduction critique $\xi$, pour tout $n$ tel que $\mathbb{P}[\#T = n] > 0$, notons $T_n$ un arbre aléatoire de même loi que $T$ conditionné sur l'évènement $\{\#T = n\}$ et notons $\mu_n$ la mesure de comptage sur $T_n$. Aldous a prouvé le résultat suivant.

**Théorème 23 dans [9].** *Si $\xi$ est critique et à variance $\sigma^2 := \sum_{j\geq 0} j^2\xi_j - 1$ finie, alors*

$$\left(\frac{\sigma}{2\sqrt{n}} T_n, \frac{1}{n}\mu_n\right) \underset{n\to\infty}{\overset{GHP}{\Longrightarrow}} \mathcal{T}_B.$$

Pour ce faire, il a utilisé le fait que tout arbre ordonné est "encodé" par sa *fonction de contour* qui décrit la hauteur à chaque temps d'une particule parcourant les arêtes de l'arbre en question, voir la Figure 4. Aldous

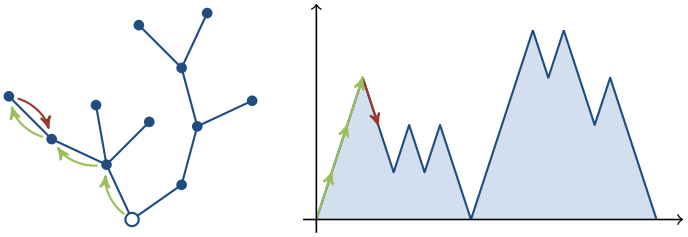


FIGURE 4 – Un arbre `t` (ordonné) et sa fonction de contour.

a montré que la fonction de contour de l'arbre $T_n$, après une remise à l'échelle adéquate, va converger en loi dans $\mathcal{C}[0,1]$ vers une excursion brownienne ce qui implique la convergence des arbres redimensionnés pour la topologie GHP.

Duquesne [47] étudia les limites d'échelle de $T_n$ quand $\xi$ est dans le domaine d'attraction d'une loi $\alpha$-stable avec $\alpha \in ]1,2[$. Dans ce cas, le coefficient de remise à l'échelle est de l'ordre de $n^{1/\alpha - 1}$ et l'arbre limite

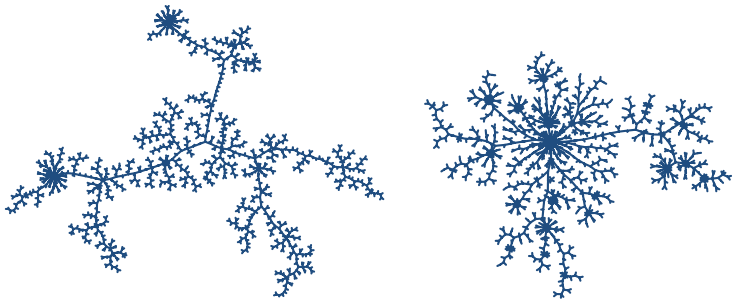


FIGURE 5 – Des arbres $\alpha$-stables avec $\alpha = 7/4$ et $\alpha = 3/2$.

n'est plus l'arbre brownien mais un arbre de Lévy $\alpha$-stable comme décrit par Duquesne et Le Gall [49]. Voir aussi Haas et Miermont [66], Korchemsky [85, 86] ou encore Rizzolo [111] pour des résultats connexes.

Dans un autre article, Duquesne [48] a aussi établi un résultat de convergence de l'arbre de Kesten au sens de la topologie GHP locale. Soit $T^*$ un arbre de Kesten et $\mu$ la mesure de comptage sur $T^*$. Si $\xi$ est à variance finie, le résultat de Duquesne assure que

$$\left(\frac{1}{R}T^*, \frac{1}{R^2}\mu\right) \underset{R\to\infty}{\Longrightarrow} \mathcal{T}_B^{\text{imm}}$$

pour la topologie GHP locale, où $\mathcal{T}_B^{\text{imm}}$ est l'arbre brownien avec immigration introduit dans [7] par Aldous sous le nom d'arbre continu aléatoire auto-similaire. Quand $\xi$ est dans le le domaine d'attraction d'une loi $\alpha$-stable avec $\alpha \in ]1,2[$, l'arbre réel infini obtenu à la limite par Duquesne est un arbre de Lévy avec immigration.

*Profil des arbres de Galton-Watson conditionnés.* De par sa construction, il est clair que l'arbre brownien $\mathcal{T}_{\mathbf{e}}$ admet un profil qui n'est autre que le temps local $L$ de l'excursion brownienne $\mathbf{e}$ défini pour tout $t$ positif par

$$L(t) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^1 \mathbb{1}_{[t,t+\varepsilon]}(u)\,\mathrm{d}u.$$

Voir aussi Delmas [39] pour une étude du profil des arbres de Lévy.

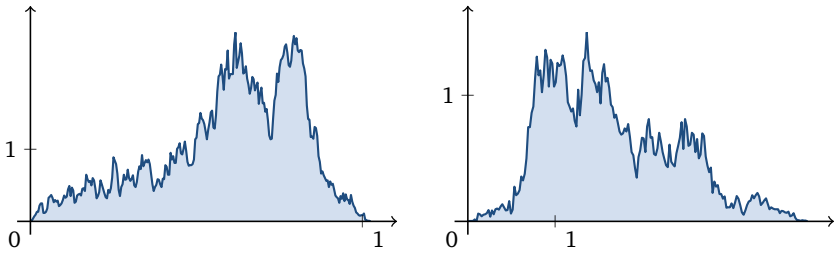



FIGURE 6 – Le profil d'un arbre brownien et d'un arbre $\alpha$-stable avec $\alpha = 7/4$.

Ayant prouvé que si $\xi$ est critique et à variance finie, alors la suite $(T_n)_n$ d'arbres de Galton-Watson conditionnés par leurs tailles converge, sous une renormalisation adéquate, vers l'arbre brownien, Aldous [8] conjectura que le profil renormalisé de $T_n$ devrait lui aussi converger vers celui de l'arbre brownien. Cette conjecture a été prouvée par Drmota et Gittenberger [43].

**Théorème 1.1 dans [43].** *Supposons que $\xi$ admet des moments exponentiels et notons $\sigma^2$ sa variance. Pour tout entier $n$, soit $\ell_n$ le profil de l'arbre $T_n$. Alors au sens de la topologie usuelle sur $\mathcal{D}[0,\infty[$, on a*

$$\left(\frac{2}{\sigma\sqrt{n}}\ell_n\Big(\lfloor 2\sqrt{n}t/\sigma\rfloor\Big)\,;\, t\geq 0\right)\underset{n\to\infty}{\Longrightarrow} L.$$

Kersting [82] élargit ce résultat aux cas où $\xi$ est dans le domaine d'attraction d'une loi $\alpha$-stable avec $\alpha\in]1,2[$. Voir la Section 3.1.1 du Chapitre 3 pour une esquisse des preuves de ces résultats.

## *i*.2 Modèles étudiés

Dans cette section, nous allons introduire les différents objets et modèles que nous étudions dans cette thèse, notamment les arbres de fragmentation auto-similaire et les arbres Markov branchants.

### *i*.2.1 Arbres de fragmentation

*Processus de fragmentation auto-similaire.* Ces processus ont été introduits par Bertoin [19] pour décrire un modèle aléatoire dans lequel un objet d'une certaine masse se désagrège en sous-objets qui vont alors eux-aussi se dégrader indépendamment les uns des autres, de la même manière que l'objet initial mais à des vitesses qui dépendent de leurs masses respectives.

Soit $\mathcal{S}^\downarrow$ l'ensemble des suites positives décroissantes sommables, i.e.

$$\mathcal{S}^\downarrow := \left\{\mathbf{s}=(s_n)_{n\geq 1}\in\ell_1\,:\, s_1\geq s_2\geq\cdots\geq 0\right\}$$

et notons $\mathcal{S}^\downarrow_{\leq 1} := \left\{\mathbf{s}\in\mathcal{S}^\downarrow\,:\,\|\mathbf{s}\|\leq 1\right\}$ où $\|\cdot\|$ dénote la norme $\ell_1$ usuelle. Un processus de fragmentation auto-similaire est un processus de Markov $\mathbf{F}$ à valeurs dans $\mathcal{S}^\downarrow_{\leq 1}$, continu en probabilité, tel que $\mathbf{F}(0)=(1,0,0,\dots)$ et tel qu'il existe une constante $\alpha$ pour lequel, pour tout $t_0$ positif, conditionnellement à $\mathbf{F}(t_0)=\mathbf{s}$,

$$\Big(\mathbf{F}(t_0+t), t\geq 0\Big)\overset{\text{loi}}{=}\Big(\big(s_i\,\mathbf{F}^{(i)}(s_i^\alpha t),\, i\geq 1\big)^\downarrow\,;\, t\geq 0\Big)$$

où $(\mathbf{F}^{(i)})_{i\geq 1}$ sont des copies i.i.d. de $\mathbf{F}$ et où pour toute suite $(x_1,x_2,\dots)$ de réels, $(x_1,x_2,\dots)^\downarrow$ est son réarrangement décroissant. Le réel $\alpha$ est appelé *l'indice d'auto-similarité* du processus $\mathbf{F}$.

Bertoin [19] et Berestycki [16] ont prouvé que la loi d'un tel processus est entièrement caractérisée par un triplet $(\alpha,c,\nu)$ où $\alpha$ est ledit indice d'auto-similarité, $c\geq 0$ est un coefficient dit d'érosion et où $\nu$ est une mesure $\sigma$-finie sur $\mathcal{S}^\downarrow_{\leq 1}$ qui ne charge pas $(1,0,0,\dots)$ et pour laquelle la fonction $\mathbf{s}\mapsto 1-s_1$ est intégrable. La mesure $\nu$ est appelée une *mesure de dislocation*.

Dans la suite, nous considèrerons uniquement des modèles de fragmentation dans lesquels l'indice d'auto-similarité $\alpha$ est strictement négatif, le coefficient d'érosion $c$ est nul et la mesure de dislocation est *conservatrice*, i.e. telle que $\nu(\|\mathbf{s}\|<1)=0$. La loi des processus de fragmentation sera alors caractérisée par un couple $(\gamma,\nu)$ où $\gamma>0$ est l'opposé de l'indice d'auto-similarité et $\nu$ est une mesure de dislocation conservatrice.

Bertoin [20] a montré qu'un processus de fragmentation $\mathbf{F}$ ainsi caractérisé par un couple $(\gamma,\nu)$ s'éteint presque sûrement en temps fini, c'est-à-dire que la variable $\inf\{t\geq 0\,:\,\mathbf{F}(t)=(0,0,\dots)\}$ est presque sûrement finie. Haas [59] montra de plus que cette variable admet des moments exponentiels finis.

Un exemple important de processus de fragmentation auto-similaire est donné par Bertoin [19] : la *fragmentation brownienne*. Soit $\mathbf{e}$ une excursion brownienne de durée 1. Pour tout $t$ positif, notons $\mathbf{F}_\mathbf{e}(t)$ le réarrangement décroissant de la suite des mesures de Lebesgue des composantes connexes de l'ensemble $\{u\in[0,1]:\mathbf{e}_u\geq t\}$. Le processus $\mathbf{F}_\mathbf{e}$ est alors un processus de fragmentation associé au couple $(1/2,\nu_B)$ où la mesure de dislocation $\nu_B$ est donnée par

$$\int_{\mathcal{S}^\downarrow} f(\mathbf{s})\,\nu_B(\mathrm{d}\mathbf{s}) := \frac{2}{\sqrt{2\pi}}\int_{1/2}^1 \frac{f(x,1-x,0,0,\dots)}{\sqrt{x^3(1-x)^3}}\,\mathrm{d}x$$

pour toute fonction mesurable $f : \mathbb{S}^{\downarrow} \to \mathbb{R}_+$.

*Arbres de fragmentation.* Rappelons que l'arbre brownien peut-être construit à partir d'une excursion brownienne. On peut alors se demander s'il est possible de construire un arbre brownien à partir de la fragmentation brownienne en oubliant l'excursion sous-jacente et si une telle construction est possible dans d'autres cas.

Cette question a été traitée par Haas et Miermont [64] qui ont montré que pour tout couple $(\gamma, \nu)$, on peut construire un arbre réel compact et mesuré qui retrace la généalogie d'un processus de fragmentation auto-similaire associé audit couple $(\gamma, \nu)$.

Plus précisément, il existe un arbre réel compact enraciné et mesuré $(\mathcal{T}, \mathrm{d}, \rho, \mu)$ tel que si pour tout $t$ positif, $\{\mathcal{T}_i(t) : i \geq 1\}$ est l'ensemble (possiblement fini voire même vide) des adhérences des composantes connexes de $\mathcal{T} \setminus \mathcal{T}|_t$, alors le processus

$$\Big( \big(\mu[\mathcal{T}_i(t)]\,;\, i \geq 1\big)^{\downarrow}\,;\, t \geq 0 \Big)$$

est un processus de fragmentation auto-similaire dont la loi est caractérisée par $(\gamma, \nu)$. La distribution de cet arbre sera notée $\mathcal{T}_{\gamma,\nu}$. Voir aussi Stephenson [116] dans un cadre plus général ainsi que Rembart et Winkel [109].

Notons que la notion d'arbre de fragmentation englobe, en plus de l'arbre brownien, les arbres $\alpha$-stables introduits par Duquesne et Le Gall [49] : Miermont [98] prouva que pour tout $\alpha \in ]1, 2[$, l'arbre de Lévy $\alpha$-stable peut lui aussi être décrit comme un arbre de fragmentation dont la loi est caractérisée par le couple $(1 - 1/\alpha, \nu_\alpha)$ pour une certaine mesure de dislocation $\nu_\alpha$.

*Profil des arbres de fragmentation.* Dans [60], Haas étudia l'existence ou non de ces profils. Elle prouva notamment le résultat suivant.

**Théorème 4 dans [60].** *Soient $\gamma > 0$ et $\nu$ une mesure de dislocation telle que $\int_{\mathbb{S}^{\downarrow}} \sum_{i\geq 1} s_i\,|\log s_i|\;\nu(\mathrm{d}\mathbf{s}) < \infty$. De plus, soit $(\mathcal{T}, \mu_{\mathcal{T}})$ un arbre de fragmentation de loi $\mathcal{T}_{\gamma,\nu}$.*

*Si $\gamma < 1$ et si $\nu$ vérifie*

$$\int_{\mathbb{S}^{\downarrow}} (1 - s_1)^{1-\varepsilon}\;\nu(\mathrm{d}\mathbf{s}) < \infty \qquad \text{et} \qquad \int_{\mathbb{S}^{\downarrow}} \sum_{i<j} s_i^{1-\gamma} s_j\;\nu(\mathrm{d}\mathbf{s}) < \infty$$

*pour un $\varepsilon \in ]0, 1[$, alors $\mathcal{T}$ admet p.s. un profil.*

*Si $\gamma \geq 1$, alors avec probabilité 1, $\mathcal{T}$ n'admet pas de profil.*

En d'autres termes, sous des hypothèses peu restrictives d'intégrabilité de la mesure de dislocation $\nu$, presque sûrement, un arbre de loi $\mathcal{T}_{\gamma,\nu}$ admet un profil ssi $\gamma < 1$.

*Arbres de fragmentation avec immigration.* Haas [61, 62] introduit les notions de processus et d'arbres de fragmentation avec immigration. Les processus de fragmentation avec immigration modélisent l'évolution des masses d'un système d'objets qui se désagrègent indépendamment avec le temps et où, en parallèle, de nouveaux objets viennent s'ajouter au système. Les arbres réels appartenant à cette classe ont alors une épine infinie sur laquelle s'attachent des arbres de fragmentation redimensionnés. Ces arbres sont localement compacts et munis d'une mesure localement finie.

Ils sont caractérisés par un triplet $(\gamma, \nu, I)$ où $\gamma > 0$, $\nu$ est une mesure de dislocation conservatrice et $I$ une mesure dite d'*immigration*, c'est-à-dire une mesure $\sigma$-finie sur $\mathbb{S}^{\downarrow}$ qui intègre la fonction $\mathbf{s} \to 1 \wedge |\mathbf{s}|$. On notera alors $\mathcal{T}^I_{\gamma,\nu}$ leur loi.

Pour construire un tel arbre, on procède de la manière suivante. Soit $\Sigma$ un processus ponctuel de Poisson sur $\mathbb{R}_+ \times \mathbb{S}^{\downarrow}$ d'intensité $\mathrm{d}t \otimes I(\mathrm{d}\mathbf{s})$ et notons $(u_n, \mathbf{s}^{(n)})_{n\geq 1}$ ses atomes. De plus, soit $[(\mathcal{T}_{n,k}, \mu_{n,k}); n \geq 1, k \geq 1]$ une famille indépendante de $\Sigma$ d'arbres de fragmentation i.i.d. de loi associée au couple $(\gamma, \nu)$ munis de

leurs mesures respectives. Enfin, considérons la demi-droite $[0,\infty[$ comme une branche (continue) infinie et pour tout $n \geq 1$, $k \geq 1$, à hauteur $u_n$ de cette branche, on attache l'arbre mesuré $\left((s_k^{(n)})^\gamma \mathcal{T}_{n,k}, s_k^{(n)} \mu_{n,k}\right)$.

Haas [62] a prouvé que sous une hypothèse mettant en relation $(\gamma, \nu)$ et $I$, si $(\mathcal{T}, \mu)$ est un arbre de fragmentation de loi caractérisée par $(\gamma, \nu)$, alors

$$\left(m^\gamma \mathcal{T}, m\mu\right) \underset{m\to\infty}{\Longrightarrow} \mathcal{T}^I_{\gamma,\nu}$$

au sens de la topologie GHP locale.

Notons que cette notion d'arbres englobe notamment l'arbre brownien avec immigration et les arbres de Lévy avec immigration, cf Haas [62].

### *i*.2.2 Arbres Markov branchants

Les modèles d'arbres discrets que nous étudierons auront un point commun : ils auront la propriété dite de Markov branchante. L'étude de ces arbres a été initiée par Aldous [10], qui a défini cette notion dans le cadre des arbres binaires. Voir aussi Haas *et al.* [66, 67, 63], Broutin *et al.* [30], Rizzolo [111] ou Lambert [89]. Nous suivrons Haas et Miermont [66].

Soit $(T_n)_n$ une suite d'arbres aléatoires indexés par leur nombre de nœuds, i.e. pour tout $n$, $T_n$ a $n$ nœuds. Informellement, la suite $(T_n)_n$ satisfait la propriété de Markov branchante si pour tout $n$, conditionnellement à l'évènement

"$T_n$ se scinde à la racine en $p$ sous-arbres $T_n^{(1)}, \dots, T_n^{(p)}$ avec respectivement $n_1 \geq \cdots \geq n_p$ nœuds"

lesdits sous-arbres sont indépendants et $T_n^{(i)}$ est distribué comme $T_{n_i}$.

La famille des lois de la suite $(T_n)_n$ est alors entièrement caractérisée par les lois des *partitions à la racine* des arbres $T_n$, c'est-à-dire les variables

$$\Lambda(T_n) := \left(\#T_n^{(1)}, \dots, \#T_n^{(p)}\right)^\downarrow$$

où $p$ est le nombre (aléatoire) de sous-arbres de $T_n$ au dessus de sa racine et $T_n^{(1)}, \dots, T_n^{(p)}$ sont ces derniers. Puisque $T_n$ a $n$ nœuds dont sa racine, $\Lambda(T_n)$ est une *partition entière* de $n-1$ c'est-à-dire une variable à valeurs dans l'ensemble

$$\mathcal{P}_{n-1} := \left\{(\lambda_1, \dots, \lambda_p) \in \mathbb{N}^p \ : \ p \geq 0, \lambda_1 \geq \cdots \geq \lambda_p > 0, \lambda_1 + \cdots + \lambda_p = n-1\right\},$$

avec la convention $\mathcal{P}_0 = \{\varnothing\}$.

*Construction.* Soit $(q_n)_n$ une suite de mesures de probabilité respectivement sur $\mathcal{P}_n$. Nous allons décrire la construction d'une suite d'arbres aléatoires $(T_n)_n$ ayant la propriété de Markov branchante et telle que pour tout $n \geq 1$, la partition à la racine $\Lambda(T_n)$ de $T_n$ soit de loi $q_{n-1}$. On notera alors $\mathrm{MB}_n^q$ la loi de $T_n$ pour tout $n$.

Pour construire l'arbre $T_n$ à $n$ nœuds, nous allons procéder récursivement sur $n$. Tout d'abord, quand $n = 1$, $T_1$ ne peut être que l'arbre ne contenant que sa racine. Pour $n \geq 2$, soit $\Lambda_n$ une partition de $n-1$ de loi $q_{n-1}$ ; conditionnellement à $\Lambda_n = (\lambda_1, \dots, \lambda_p)$, soient $T^{(1)}, \dots, T^{(p)}$ des arbres indépendants de lois respectives $\mathrm{MB}_{\lambda_i}^q$. Enfin relions les racines de ces arbres à un nouveau nœud qui sera la racine de l'arbre $T_n$ ainsi obtenu et appelons sa loi $\mathrm{MB}_n^q$.

Notons qu'une construction similaire, mais nécessitant des conditions techniques supplémentaires sur la suite $(q_n)_n$, permet de définir des suites Markov branchantes indexées par le nombre de feuilles et non plus le nombre de nœuds, cf Haas et Miermont [66]. Dans ce cas, la partition à la racine par rapport aux feuilles d'un arbre à $n \geq 2$ feuilles sera une partition de $n$. On notera $\mathrm{MB}_n^{\mathcal{L},q}$ la distribution d'un arbre associé à une famille $q$ de mesures de probabilité sur les partitions entières. Voir aussi Rizzolo [111] pour une construction plus générale.

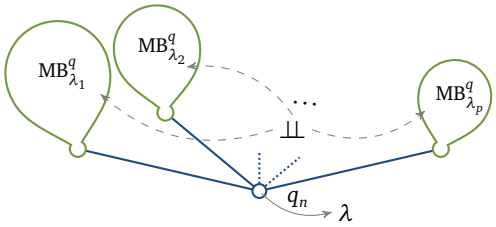



FIGURE 7 – Construction récursive d'un arbre Markov branchant à $n$ nœuds.

*Une autre heuristique.* Les arbres Markov branchants peuvent aussi être décrits comme les arbres retraçant la généalogie du processus suivant, ce qui n'est pas sans rappeler la description heuristique des processus et arbres de fragmentation.

Soit $(q_n)_n$ une suite de mesures de probabilité telle que $q_1$ est supportée par $\{(1), \varnothing\}$ et si $n \geq 2$, $q_n$ est supportée par l'ensemble $\mathcal{P}_n$ des partitions de $n$. Considérons une urne contenant $n$ boules indistinguables et avec probabilité $q_n(\lambda_1, \dots, \lambda_p)$, scindons ladite urne en $p$ sous-urnes contenant $\lambda_1, \dots, \lambda_p$ boules respectivement. Pour $i = 1, \dots, p$, on scinde alors la $i^{\text{ème}}$ sous-urne indépendamment des autres à partir de la mesure $q_{\lambda_i}$. Quand une sous-urne ne contient plus qu'une boule, avec probabilité $q_1(1)$ cette urne donne naissance à une nouvelle urne contenant elle-même une boule et avec probabilité $1 - q_1(1)$, la boule ne passe pas à l'étape suivante. Si pour tout $n$, $q_n(n) < 1$, ce processus s'arrête presque sûrement en temps fini, on peut alors représenter la généalogie de ces urnes par un arbre fini à $n$ feuilles et cet arbre aura pour loi $\mathrm{MB}_n^{\mathcal{L},q}$.

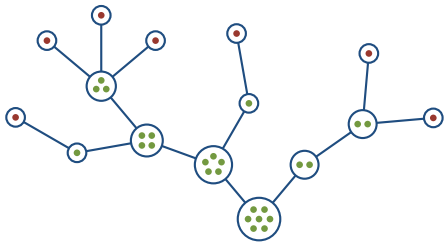


FIGURE 8 – L'évolution d'une urne contenant 7 boules et l'arbre à 7 feuilles correspondant.

*Limites d'échelle.* Haas et Miermont [66] ont montré que sous des hypothèses naturelles sur les lois des partitions à la racine, les arbres Markov branchants convergent, sous une bonne renormalisation, vers les arbres de fragmentation pour la topologie GHP.

Soit $(T_n)_n$ une suite d'arbres Markov branchants de lois respectives $\mathrm{MB}_n^q$ et pour tout $n$, soit $\mu_n$ la mesure de comptage sur $T_n$. Pour tout $n \geq 1$, soit $\bar{q}_n$ la mesure image de $q_n$ par l'application $\mathcal{P}_{n-1} \to \mathcal{S}^{\downarrow}$, $(\lambda_1, \dots, \lambda_p) \mapsto (\lambda_1/n, \dots, \lambda_p/n, 0, 0, \dots)$. Haas et Miermont ont prouvé le résultat suivant.

**Théorème 6 dans [66].** *S'il existe $\gamma \in ]0,1[$ et une mesure de dislocation $\nu$ tels que*

$$n^{\gamma}\,(1 - s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \xrightarrow[n\to\infty]{} (1 - s_1)\,\nu(\mathrm{d}\mathbf{s}) \tag{S}$$

*faiblement en tant que mesures finies sur $\mathcal{S}^{\downarrow}$, alors au sens de la topologie GHP,*

$$\left(\frac{T_n}{n^{\gamma}}, \frac{\mu_n}{n}\right) \xRightarrow[n\to\infty]{} \mathcal{T}_{\gamma,\nu}.$$

Si $\gamma > 0$ et si la suite $(q_n)_n$ satisfait (S), le Théorème 5 dans [66] donne la même convergence pour les arbres Markov branchants indexés par leur nombre de feuilles.

*Arbres de Galton-Watson conditionnés.* Un exemple important de suites d'arbres Markov branchants est encore une fois celui des arbres de Galton-Watson critiques conditionnés par leurs nombres de nœuds.

Soit $\xi$ une distribution critique et soit $T$ un arbre de Galton-Watson de loi de reproduction $\xi$. Pour tout $n$ tel que $\mathbb{P}[\#T = n] > 0$, on notera à nouveau $T_n$ l'arbre $T$ conditionné à avoir $n$ nœuds. Alors la suite $(T_n)_n$ est Markov branchante et pour tous $n$, $p$ et $n_1 \geq \cdots \geq n_p \geq 1$ entiers, on a

$$\mathbb{P}\Big[\text{“}T_n \text{ se scinde en } p \text{ sous-arbres à } n_1, \ldots, n_p \text{ nœuds”}\Big] = \mathrm{Arr}(n_1, \ldots, n_p) \cdot \frac{\prod_{i=1}^{p} \mathbb{P}[\#T = n_i]}{\mathbb{P}[\#T = n]}$$

où $\mathrm{Arr}(n_1, \ldots, n_p)$ est le nombre de manières différentes d'ordonner le $p$-uplet $(n_1, \ldots, n_p)$, voir Haas et Miermont [66].

Haas et Miermont ont alors utilisé leurs résultats pour redémontrer les résultats de limites d'échelle d'Aldous (quand $\xi$ est à variance finie) et de Duquesne (quand $\xi$ est dans le domaine d'attraction d'une loi $\alpha$-stable, $\alpha \in ]1, 2[$).

*Limites locales.* Stefánsson [113] a étudié les limites locales du modèle des arbres de Ford [52] en utilisant la nature Markov branchante du modèle. Sa preuve repose sur le fait que la loi de la partition à la racine converge en un certain sens, ce qui implique que les arbres eux-même convergent localement.

Le but de cette thèse sera d'obtenir des résultats sur les modèles Markov branchants similaires à ceux sur les arbres de Galton-Watson conditionnés exposés dans la Section *i*.1.2. À l'instar des résultats de Haas et Miermont sur les limites d'échelle ainsi que celui de Stefánsson pour les limites locales, nous verrons que le comportement asymptotique des modèles Markov branchants est dicté par celui des partitions à la racine associées.

Dans la suite de cette introduction, nous présenterons les résultats obtenus durant cette thèse. Notons que nous ferons alors référence aux résultats des Chapitres 1 et 4.

## *i*.3 Limites locales et croissance volumique des arbres Markov branchants

Dans le Chapitre 1, nous étudierons les limites locales des arbres Markov branchants vers une nouvelle famille d'arbres infinis satisfaisant une version de la propriété de Markov branchante. Ensuite nous nous consacrerons à l'étude de la convergence au sens de la topologie GHP locale de ces arbres infinis remis à l'échelle.

### *i*.3.1 Limites locales des arbres Markov branchants

*Arbres Markov branchants infinis.* Comme nous l'avons vu dans la Section *i*.2.2, les lois d'une suite d'arbres Markov branchants sont caractérisées par la donnée des lois des partitions à la racine desdits arbres. Nous avons de plus décrit un procédé récursif permettant de construire des arbres Markov branchants à partir d'une suite de telles lois. Notre but est de décrire une construction similaire pour des arbres infinis.

Soit $(q_n)_n$ une suite de mesures de probabilité telle que pour tout $n$, $q_n$ est supportée par $\mathcal{P}_n$, l'ensemble des partitions de l'entier $n$. Soit de plus $q_\infty$ une mesure de probabilité sur l'ensemble

$$\mathcal{P}_\infty := \Big\{(\lambda_1, \ldots, \lambda_p) \in (\mathbb{N} \cup \{\infty\})^p \,:\, p \geq 0, \lambda_1 \geq \cdots \geq \lambda_p > 0, \lambda_1 + \cdots + \lambda_p = \infty\Big\}$$

qui peut être vu comme l'ensemble des partitions entières finies de l'infini. Nous allons maintenant décrire la construction d'un arbre infini (mais localement fini) à partir de $(q_n)_n$ et $q_\infty$ de sorte que cet arbre satisfasse la propriété de Markov branchante.

Notons $m_\infty$ la fonction qui à un élément $\lambda$ de $\mathcal{P}_\infty$ associe son nombre de composantes infinies, i.e. $m_\infty(\lambda) := \sum_{i\geq 1} \mathbb{1}_{\lambda_i=\infty}$. Soit $\zeta$ la loi du nombre de composantes infinies d'une variable de loi $q_\infty$, c'est-à-dire que pour tout entier positif $j$, on pose $\zeta_j := q_\infty(m_\infty = j)$. Remarquons que les éléments de $\mathcal{P}_\infty$ ont nécessairement au moins une composante infinie donc $\zeta$ est une mesure de probabilité sur $\mathbb{N}^*$.

Soit $T^\circ$ un arbre de Galton-Watson (infini) de loi de reproduction $\zeta$. Conditionnellement à $T^\circ$, pour tout nœud $u$ de $T^\circ$, indépendamment des autres nœuds, soit le couple $(\Lambda_u, T_u)$ tel que

— $\Lambda_u$ est une variable de loi $q_\infty$ conditionnée sur l'évènement $m_\infty(\Lambda_u) = c_{T^\circ}(u)$, où $c_{T^\circ}(u)$ est le nombre d'enfants de $u$ dans $T^\circ$,

— Conditionnellement à $\Lambda_u = (\infty, \dots, \infty, \lambda_1, \dots, \lambda_r)$ avec $\lambda_1 < \infty$, soient $T_u^i$, $i = 1, \dots, r$ des arbres indépendants de lois respectives $\mathrm{MB}^q_{\lambda_i}$ ; on définit alors $T_u$ comme l'arbre obtenu en reliant par une arête les racines respectives de $T_u^1, \dots, T_u^r$ à une nouvelle racine. Si $r = 0$, on convient que $T_u$ sera l'arbre à 1 nœud.

Enfin, pour tout $u$ dans $T^\circ$, on attache $T_u$ sur le nœud $u$ de $T^\circ$ (sans arête intermédiaire) et on appelle $T_\infty$ l'arbre ainsi obtenu ainsi que $\mathrm{MB}^{q,q_\infty}_\infty$ sa distribution.

Dans la plupart de nos applications, la mesure $q_\infty$ sera portée par l'ensemble $\{\lambda \in \mathcal{P}_\infty : m_\infty(\lambda) = 1\}$. Dans ce cas, le "squelette infini" $T^\circ$ de $T_\infty$ sera simplement la branche infinie, qu'on appellera l'*épine infinie* de $T_\infty$. De plus, les variables $(\Lambda_u, T_u)$, $u \in T^\circ$ seront alors identiquement distribuées en plus d'être indépendantes.

*Limites locales.* Après avoir introduit ces arbres Markov branchants infinis, nous montrerons que ces arbres apparaissent naturellement comme la limite locale de suites d'arbres Markov branchants finis.

Fixons une suite $(q_n)_n$ de lois de partitions à la racine et pour tout $n$, soit $T_n$ un arbre de loi $\mathrm{MB}^q_n$. De même soit $q_\infty$ une mesure de probabilité sur $\mathcal{P}_\infty$ ainsi que $T_\infty$, un arbre Markov branchant infini de loi $\mathrm{MB}^{q,q_\infty}_\infty$. Rappelons que $T_n$ converge localement en loi vers $T_\infty$ ssi pour tout entier $r$ positif et pour tout arbre fini $\mathtt{t}$ on a

$$\mathbb{P}\big[T_n|_r = \mathtt{t}|_r\big] \xrightarrow[n\to\infty]{} \mathbb{P}\big[T_\infty|_r = \mathtt{t}|_r\big].$$

Soient $\mathtt{t}$ un arbre et $r$ un entier positif. Supposons que $\mathtt{t}|_{r+1}$ ait $d$ sous-arbres $\mathtt{t}_1, \dots, \mathtt{t}_d$ au desssus de sa racine. Pour tout $n$, la nature Markov branchante de $T_n$ assure que pour toute partition $\lambda$ de $n-1$ ayant $d$ composantes,

$$\mathbb{P}\big[T_n|_{r+1} = \mathtt{t}|_{r+1} \,|\, \Lambda(T_n) = \lambda\big] = \sum_{\sigma\in S(\mathtt{t}_1,\dots,\mathtt{t}_d)} \prod_{i=1}^d \mathbb{P}\big[T_{\lambda_i}|_r = \mathtt{t}_{\sigma\cdot i}\big]$$

où $S(\mathtt{t}_1, \dots, \mathtt{t}_d)$ est un sous-ensemble des permutations de $\{1, \dots, d\}$ qui encode les différentes manières d'arranger les arbres $\mathtt{t}_1, \dots, \mathtt{t}_d$. La même expression est valable pour $T_\infty$ et tout élément de $\mathcal{P}_\infty$ à $d$ composantes.

De ce fait, pour montrer que $\mathbb{P}[T_n|_{r+1} = \mathtt{t}|_{r+1}] \to \mathbb{P}[T_\infty|_{r+1} = \mathtt{t}|_{r+1}]$, il suffit de montrer que pour toute permutation $\sigma$ de $\{1, \dots, d\}$ fixée, la fonction

$$\lambda \longmapsto \prod_{i=1}^d \mathbb{P}[T_{\lambda_i}|_r = \mathtt{t}_{\sigma\cdot i}]\, \mathbb{1}_{\lambda \text{ a } d \text{ composantes}}$$

définie pour tout $\lambda$ dans l'ensemble $\mathcal{P} := \mathcal{P}_\infty \cup \bigcup_n \mathcal{P}_n$ est continue pour une topologie bien choisie sous laquelle on a de plus $q_n \Rightarrow q_\infty$. Nous introduirons alors la topologie suivante sur $\mathcal{P}$ qui n'est pas sans rappeler la topologie locale pour les arbres : on dira que $\lambda^n$ converge vers $\lambda$ dans $\mathcal{P}$ si pour tout $R$ positif, on a $\lambda^n \wedge R = \lambda \wedge R$ pour $n$ assez grand, où $\kappa \wedge R = (\kappa_1 \wedge R, \dots, \kappa_d \wedge R)$ pour tout $\kappa = (\kappa_1, \dots, \kappa_d)$ dans $\mathcal{P}$.

**Théorème 1.2.5.** *Si $q_n \Rightarrow q_\infty$ dans $\mathcal{P}$, alors $T_n \Rightarrow T_\infty$ localement.*

Nous verrons dans la Proposition 1.2.7 que la condition "$q_n \Rightarrow q_\infty$ dans $\mathcal{P}$" est optimale et donc que la topologie sur $\mathcal{P}$ est assez naturelle. Nous donnerons de plus dans le Lemme 1.2.4 un critère pour vérifier que $q_n \Rightarrow q_\infty$ dans le cas où $q_\infty(m_\infty = 1) = 1$. Le Théorème 1.2.5 aura donc le corollaire suivant, qui sera très utile dans nos applications.

**Corollaire 1.2.6.** *Soit $q_\infty$ une mesure de probabilité sur $\mathcal{P}_\infty$ portée par l'ensemble $\{m_\infty = 1\}$. Supposons que pour tout entier naturel $p$, et pour tous $\lambda_1 \geq \cdots \geq \lambda_p > 0$ entiers avec $L = \lambda_1 + \cdots + \lambda_p$ on a*

$$q_n\big(n - L, \lambda_1, \ldots, \lambda_p\big) \xrightarrow[n\to\infty]{} q_\infty\big(\infty, \lambda_1, \ldots, \lambda_p\big)$$

*alors $\mathrm{MB}_n^q \Rightarrow \mathrm{MB}_\infty^{q,q_\infty}$ pour la topologie locale.*

Notons que Broutin et Mailler [31] ont obtenu des résultats similaires.

## *i*.3.2 Limites d'échelle et croissance volumique

*Croissance volumique.* Si $\tau$ est un arbre aléatoire infini ayant une unique épine infinie, il est intéressant de savoir comment $\#\tau|_R$ se comporte quand $R$ est grand, voir Croydon et Kumagai [37] ainsi que Stefánsson et Zohren [115] pour des études en ce sens. Nous étudierons alors ce comportement asymptotique pour les arbres Markov branchants infinis ayant une unique épine infinie, c'est-à-dire les arbres de loi $\mathrm{MB}_\infty^{q,q_\infty}$ avec $q_\infty(m_\infty = 1) = 1$.

Comme nous l'avons évoqué dans la Section *i*.1.1, si $(\mathbf{T}_n, \mu_n)$, $n \geq 1$ et $\mathbf{T}$ sont des arbres réels localement compacts et mesurés, et si $\mathbf{T}_n \to \mathbf{T}$ pour la topologie GHP locale, alors pour tout $t$ positif qui est un point de continuité de la fonction $x \mapsto \mu_T(T|_x)$, $\mu_{T_n}(T_n|_t)$ converge vers $\mu_T(T|_t)$.

Soit $T$ un arbre de loi $\mathrm{MB}_\infty^{q,q_\infty}$ et soit $\mu$ la mesure de comptage sur $T$. Remarquons que pour tout entier $R$ positif, $\#T|_R$ peut alors s'écrire comme $\mu_T\big((T/R)|_1\big)$, c'est-à-dire comme la mesure de la boule de rayon 1 autour de la racine de l'arbre $T/R$. Ainsi, plutôt que d'étudier la suite aléatoire $(\#T|_R; R \geq 0)$, nous allons considérer les limites d'échelle de la suite $(R^{-1}T, \phi(R)\mu)$ quand $R$ tend vers l'infini pour une fonction $\phi$ bien choisie. Nous montrerons alors le résultat suivant.

**Théorème 1.4.1.** *Supposons qu'il existe un triplet $(\gamma, \nu, I)$ tel que :*

- *La suite $(q_n)_n$ satisfait l'hypothèse (S).*
- *$I$ est une mesure d'immigration.*
- *Si $\Lambda$ est telle que $(\infty, \Lambda)$ suit la loi $q_\infty$, alors pour toute fonction $g : \mathbb{S}^\downarrow \to \mathbb{R}$ continue et telle que $|g(\mathbf{s})| \leq 1 \wedge \|\mathbf{s}\|$,*

$$R\,\mathbb{E}\Big[g\big(\Lambda/R^{1/\gamma}\big)\Big] \xrightarrow[n\to\infty]{} \int_{\mathbb{S}^\downarrow} g(\mathbf{s})\, I(\mathrm{d}\mathbf{s}). \tag{I}$$

*Alors si $T$ est un arbre de loi $\mathrm{MB}_\infty^{q,q_\infty}$ et $\mu$ sa mesure de comptage*

$$\left(\frac{T}{R}, \frac{\mu}{R^{1/\gamma}}\right) \Longrightarrow_{R\to\infty} \mathcal{T}^I_{\gamma,\nu}$$

*pour la topologie GHP locale.*

De ce fait, et pour les raisons précédemment évoquées, ce théorème implique le corollaire suivant :

**Proposition 1.4.2.** *Supposons que les hypothèses du Théorème 1.4.1 sont vérifiées. Soit $(\mathcal{T}, \mu_\mathcal{T})$ un arbre de fragmentation avec immigration de loi associée au triplet $(\gamma, \nu, I)$. Alors*

$$\Big(t \longmapsto R^{-1/\gamma}\, \#T|_{Rt}\Big) \Longrightarrow_{R\to\infty} \Big(t \longmapsto \mu_\mathcal{T}(\mathcal{T}|_t)\Big)$$

*pour la topologie uniforme sur les compacts.*

Voir aussi la Remarque 1.4.2 pour la description d'un comportement différent sous des hypothèses différentes.

*Idées de la preuve du Théorème 1.4.1.* Rappelons que puisque $q_\infty$ est portée par les éléments de $\mathcal{P}_\infty$ ayant une unique composante infinie, on peut construire un arbre de loi $\mathrm{MB}_\infty^{q,q_\infty}$ comme suit. Soit $(\Lambda_n, \tau_n)_{n\geq 0}$ une suite de variables i.i.d. telle que

— Pour tout $n \geq 0$, $(\infty, \Lambda_n)$ est de loi $q_\infty$

— Conditionnellement à $\Lambda_n = (\lambda_1, \dots, \lambda_p)$, $\tau_n$ a $p$ sous-arbres indépendants et de lois respectives $\mathrm{MB}_{\lambda_i}^q$ au dessus de sa racine.

Enfin on colle la racine de $\tau_n$ à hauteur $n$ d'une branche infinie et l'arbre $T$ ainsi obtenu a bien comme loi $\mathrm{MB}_\infty^{q,q_\infty}$.

L'arbre $T$ est donc associé au processus ponctuel $\sum_{n\geq 0} \delta_{(n,\Lambda_n)}$ sur $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ de même qu'un arbre de fragmentation avec immigration est associé à un processus de Poisson, cf la Section *i*.2.1. Ainsi, la première étape dans l'étude des limites d'échelle de $(R^{-1}T, R^{-1/\gamma}\mu)$ quand $R$ tend vers l'infini est de trouver une hypothèse sur $q_\infty$ sous laquelle le processus ponctuel

$$\Sigma_R := \sum_{n\geq 0} \delta_{(n/R, \Lambda_n/R^{1/\gamma})}$$

converge en loi dans une certaine topologie sur les mesures localement finies sur $\mathbb{R}_+ \times \mathcal{S}^\downarrow$. Si $f : \mathbb{R}_+ \to \mathbb{R}$ est continue à support compact et si $g : \mathcal{S}^\downarrow \to \mathbb{R}$ est continue et bornée, observons que

$$\mathbb{E}\left[\int_{\mathbb{R}_+\times\mathcal{S}^\downarrow} f(u)\, g(\mathbf{s})\, \Sigma_R(\mathrm{d}u, \mathrm{d}\mathbf{s})\right] = \underbrace{\frac{1}{R}\sum_{n\geq 0} f(n/R)}_{\sim \int_0^\infty f(u)\,\mathrm{d}u}\; R\,\mathbb{E}\Big[g\big(\Lambda_0/R^{1/\gamma}\big)\Big].$$

Ainsi, pour que $\Sigma_R$ converge en loi, il semble naturel d'exiger que $R\,\mathbb{E}[g(\Lambda_0/R^{1/\gamma})]$ converge pour une classe assez grande de fonctions $g$ : c'est pour cette raison que l'on fait l'hypothèse (I).

Dans la Section 1.4.1, nous montrerons en particulier que si l'on suppose (I), $\Sigma_R$ va converger en loi vers un processus ponctuel de Poisson sur $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ d'intensité $\mathrm{d}u \otimes I(\mathrm{d}\mathbf{s})$. Si de plus on suppose (S), alors le processus ponctuel

$$\Pi_R := \sum_{n\geq 0} \delta_{\left(n/R, \Lambda_n/R^{1/\gamma}, (\tau_n/R, \mu_{\tau_n}/R^{1/\gamma})\right)}$$

convergera en loi pour la topologie définie dans la Section 1.3.3 vers un processus ponctuel de Poisson associé à un arbre de fragmentation avec immigration de loi $\mathcal{T}^I_{\gamma,\nu}$. Cette convergence nous permettra, dans la Section 1.4.2 de prouver le Théorème 1.4.1.

Remarquons que les Théorèmes 1.2.5 et 1.4.1 ainsi que la Proposition 1.4.2 s'appliquent également aux suites d'arbres Markov branchants indexés par leur nombre de feuilles. Ces théorèmes permettent en particulier de retrouver certains des résultats sur le modèle des arbres de Galton-Watson conditionnés présenté dans la Section *i*.1.2 de cette introduction. Voir aussi la Section 1.5 pour des applications des résultats que l'on vient d'énoncer à divers modèles Markov branchants comme certains arbres de coupe ou des arbres construits via des algorithmes récursifs. Enfin, une généralisation de ces résultats au cadre introduit par Rizzolo [111] est donnée dans le Chapitre 2.

## *i*.4 Comportement asymptotique du profil des arbres Markov branchants

Le but principal du Chapitre 4 est d'établir des résultats de convergences ponctuelle et fonctionnelle pour le profil d'arbres Markov branchants dont on sait qu'ils convergent vers des arbres de fragmentation. Pour des raisons techniques que l'on détaillera dans la Section *i*.4.2, nous nous concentrerons sur les arbres Markov branchants indexés par leur nombre de feuilles.

Soit $T$ un arbre aléatoire avec $n$ feuilles et soit $\mu_T$ la mesure de comptage sur l'ensemble des feuilles de $T$. On définit de plus $\ell_T$ le profil de $T$ par rapport à $\mu_T$, par $\ell_T(j) := \mu_T(u \in T : |u| = j)$ pour tout $j$ positif. Rappelons que pour tout $j$ positif, $\ell_T(j)/n$ est la probabilité conditionnellement à $T$ qu'une feuille uniformément choisie dans $T$ soit à la hauteur $j$.

Cette interprétation probabiliste du profil d'un arbre discret a un équivalent continu : si $(\mathcal{T}, \mu_\mathcal{T})$ est un arbre réel qui admet presque sûrement un profil, alors ledit profil est la densité, conditionnellement à $(\mathcal{T}, \mu_\mathcal{T})$ de la hauteur d'une variable de loi $\mu_\mathcal{T}$ à valeurs dans $\mathcal{T}$.

Notre approche pour étudier le comportement asymptotique des profils d'arbres Markov branchants sera basée sur cette idée que le profil est, à une renormalisation près, une loi de probabilité sur les entiers positifs. Nous utiliserons alors des résultats d'inversion de Fourier, de manière analogue à la preuve du Théorème Central Limite local de Gnedenko [56]. En ce sens, nous devrons tout d'abord établir des résultats d'intégrabilité de la transformée de Fourier (aléatoire) du profil d'un arbre de fragmentation, ce qui se traduira par des résultats de régularité pour ledit profil. Nous donnerons de plus dans le Chapitre 4 des conditions garantissant que les profils d'arbres Markov branchants ne peuvent converger dans $\mathcal{D}[0, \infty[$.

Dans cette section, nous présenterons tout d'abord nos résultats de régularité pour les profils d'arbres de fragmentation puis nos résultats sur la convergence fonctionnelle pour les profils d'arbres Markov branchants.

### *i*.4.1 Régularité du profil des arbres de fragmentation

Soit $(\mathcal{T}, \mu_\mathcal{T})$ un arbre de fragmentation de loi associée au couple $(\gamma, \nu)$ avec $\gamma > 0$ et $\nu$ une mesure de dislocation conservatrice. On définit la fonction $M : \mathbb{R}_+ \to [0,1]$, $t \mapsto \mu_\mathcal{T}(\mathcal{T}|_t)$, i.e. $M(t)$ est la masse de l'ensemble des points de l'arbre $\mathcal{T}$ à distance $t$ ou moins de sa racine.

Rappelons qu'on dit que $\mathcal{T}$ admet un profil si la fonction $M$ est absolument continue par rapport à la mesure de Lebesgue, c'est-à-dire s'il existe une fonction positive $L$ telle que $M(t) = \int_0^t L(u)\,\mathrm{d}u$. Dans ce cas, on dit que $L$ est le profil de l'arbre $\mathcal{T}$.

*Fonctionnelle exponentielle d'un subordinateur.* Conditionnellement à $(\mathcal{T}, \mu_\mathcal{T})$, soit $U$ une variable de loi $\mu_\mathcal{T}$ et posons $I := |U|$, la distance séparant $U$ de la racine de $\mathcal{T}$. Remarquons que pour tout $t \geq 0$,

$$\mathbb{P}[I \leq t | \mathcal{T}] = M(t),$$

c'est-à-dire que conditionnellement à $(\mathcal{T}, \mu_\mathcal{T})$, $I$ est de loi $\mathrm{d}M$. Pour montrer que sous certaines hypothèses sur le couple $(\gamma, \nu)$, l'arbre $\mathcal{T}$ admet p.s. un profil, Haas [60] a utilisé des résultats d'inversion de Fourier sur la mesure aléatoire $\mathrm{d}M$. Suivant une idée similaire, la première étape de notre étude sera d'obtenir des résultats de régularité sur la fonction caractéristique (déterministe) de $I$.

Bertoin [18, 19] a donné une autre description de la variable $I$ : c'est la fonctionnelle exponentielle d'un subordinateur, cf Yor *et al* [28, 32]. Plus précisément, soit $\xi$ un subordinateur dont l'exposant de Laplace $\psi$ est donné par

$$\psi(q) = -\log \mathbb{E}[\mathrm{e}^{-q\xi_1}] = \int_{\mathcal{S}^\downarrow} \left(1 - \textstyle\sum_{i \geq 1} s_i^{1+q}\right) \nu(\mathrm{d}\mathbf{s})$$

pour tout $q$ positif. Remarquons que $\psi$ se prolonge naturellement en une fonction holomorphe sur le demi plan $\{\mathrm{Re} > -\rho\}$ où $\rho := \sup\{q \geq 0 : \psi(-q) > -\infty\}$. Avec ces notations, on a

$$I \overset{(\mathrm{d})}{=} \int_0^\infty \mathrm{e}^{-\gamma \xi_t} \mathrm{d}t.$$

De plus, la variable $I$ admet une densité $k$ infiniment différentiable sur $]0, \infty[$, voir [32, 103].

Le résultat principal de la Section 4.3.1 porte sur la décroissance à l'infini de la fonction caractéristique de la variable $I$. En voici un énoncé simplifié :

**Proposition 4.3.1.** *Supposons que $\nu$ soit finie ou que $\psi$ soit de la forme $\psi(q) = \int_0^1 (1-x^q) f(x) \mathrm{d}x$ pour tout $q$ positif où la fonction $f : ]0,1[ \to \mathbb{R}_+$ est telle qu'il existe $\beta \in ]1,2[$ de sorte que pour tout $c > 0$, $f(1-cx)/f(1-x) \to c^{-\beta}$ quand $x \to 0$. Alors la fonction caractéristique $\varphi_I$ de $I$ est telle que pour tout $\theta \in [0, \rho/\gamma[$,*

$$\sup_{t\in\mathbb{R}} |t|^{1+\theta} |\varphi_I(t)| < \infty.$$

Nous verrons de plus que la Proposition 4.3.1 est, en un sens, optimale. En effet, la Proposition 4.3.2 montre que si $\theta \geq 0$ est tel que $t \mapsto |t|^{1+\theta}|\varphi_I(t)|$ est bornée, alors $\theta \leq \rho/\gamma$.

Pour prouver la Proposition 4.3.1, nous montrerons tout d'abord que si $\rho > 0$, la fonction $k$ se prolonge en une fonction holomorphe sur un ensemble de la forme $\{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$ pour $\theta < \pi/2$ puis que sur cet ensemble, $k(z) = \mathcal{O}(|z|^\lambda)$ pour tout $\lambda$ dans $]-\infty, \rho/\gamma[$, cf le Lemme 4.3.5.

*Régularité du profil.* Dans la Section 4.3.2, nous déduirons de la Proposition 4.3.1 des résultats sur la régularité du profil de l'arbre $\mathcal{T}$. Nous montrerons en particulier le théorème suivant :

**Théorème 4.3.6.** *Si $\gamma < 1/2$ et s'il existe $\alpha, \beta < 1 + \rho/\gamma$, tel que $2 < \alpha + \beta < 1/\gamma$ et*

$$\int_{\mathbb{S}^\downarrow} \sum_{i<j} s_i^{1-\alpha\gamma} s_j^{1-\beta\gamma} \mathbb{1}_{s_j>0} \, \nu(\mathrm{d}\mathbf{s}) < \infty$$

*alors l'arbre de fragmentation $(\mathcal{T}, \mu_{\mathcal{T}})$ admet presque sûrement un profil continu $L$. De plus, pour tout entier positif $d$ et $\theta \in ]0,1]$ tels que $d + \theta < (\alpha+\beta)/2 - 1$, $L$ est $d$ fois continument dérivable et $L^{(d)}$ est $\theta$-Hölderienne.*

Nous verrons que si l'on suppose que $\gamma < (1 \wedge \rho)/2$, alors les conditions du Théorème 4.3.6 seront remplies. En particulier, plus le paramètre $\gamma$ est proche de 0, plus le profil de $\mathcal{T}$ va être régulier.

Notons qu'à cause de la condition $\gamma < 1/2$, ce résultat ne peut s'appliquer à l'arbre brownien dont on sait par ailleurs qu'il admet un profil continu. Le Théorème 4.3.6 n'est donc probablement pas optimal. On pourra trouver dans l'Annexe A des simulations de profils d'arbres aléatoires.

Pour prouver le Théorème 4.3.6, nous procèderons comme suit. Conditionnellement à $(\mathcal{T}, \mu_{\mathcal{T}})$, soient $U$ et $V$ deux variables i.i.d. de loi $\mu_{\mathcal{T}}$ et notons $I := |U|$, $J := |V|$. Remarquons que la nature auto-similaire de l'arbre $\mathcal{T}$ assure que

$$I - J \overset{(\mathrm{d})}{=} \lambda_1^\gamma \tilde{I} - \lambda_2^\gamma \tilde{J}$$

où $\tilde{I}$, $\tilde{J}$ sont des copies i.i.d. de $I$ indépendantes des variables $\lambda_1$ et $\lambda_2$ qui sont les poids, par rapport à la mesure $\mu_{\mathcal{T}}$, des sous-arbres de $\mathcal{T}$ au dessus du plus grand ancêtre commun à $U$ et $V$ et qui contiennent $U$ et $V$ respectivement.

Ainsi, si $\Phi$ dénote la fonction caractéristique de $I$ conditionnellement à $\mathcal{T}$, la Proposition 4.3.1 assure que pour tous $\alpha$ et $\beta$ positifs et strictement inférieurs à $1 + \rho/\gamma$, on obtient

$$\mathbb{E}[|\Phi(t)|^2] \leq \mathbb{E}\big[|\varphi_I(\lambda_1^\gamma t)|\,|\varphi_I(\lambda_2^\gamma t)|\big] \leq \frac{C}{t^{\alpha+\beta}} \mathbb{E}[\lambda_1^{-\alpha\gamma} \lambda_2^{-\beta\gamma} \mathbb{1}_{\lambda_1 \geq \lambda_2}]$$

pour tout réel $t$, avec $C$ une constante finie. Sous les hypothèses du Théorème 4.3.6, le Lemme 2 dans [60] assure que cette dernière quantité est finie. Ainsi, on aura en particulier prouvé que $t \mapsto |t|^\theta \Phi(t)$ est presque sûrement intégrable pour tout $\theta$ positif tel que $\theta < (\alpha+\beta)/2 - 1$, ce qui nous permettra d'utiliser le théorème d'inversion de Fourier et de conclure la preuve du Théorème 4.3.6.

Le reste de la Section 4.3 sera organisé comme suit : la Section 4.3.3 sera dédiée à prouver un critère pour garantir que le profil de l'arbre $\mathcal{T}$, s'il existe, ne peut être càdlàg (cf la Proposition 4.3.8) et la Section 4.3.4 appliquera ces résultats à divers modèles de fragmentation.

### *i*.4.2 Convergence des profils d'arbres Markov branchants

Comme nous l'avons évoqué au début de la Section *i*.4, pour des raisons techniques que l'on détaillera ci-après nous nous concentrerons sur l'étude du comportement asymptotique du profil des arbres Markov branchants indexés par leur nombre de feuilles.

Ainsi, soit $(T_n)_n$ une telle suite d'arbres Markov branchants et soit $(q_n)_n$ la suite correspondante des lois des partitions à la racine. Pour tout $n$, notons $\mu_n$ la mesure de comptage sur les $n$ feuilles de $T_n$. Nous supposerons dans tout cette section qu'il existe $\gamma > 0$ ainsi qu'une mesure de dislocation conservatrice $\nu$ tels que

$$n^\gamma (1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \underset{n\to\infty}{\Longrightarrow} (1-s_1)\,\nu(\mathrm{d}\mathbf{s}).$$

En particulier, l'hypothèse (S) est vérifiée donc le Théorème 5 de Haas et Miermont [66] assure qu'au sens de la topologie GHP, on a

$$\left(\frac{T_n}{n^\gamma}, \frac{\mu_n}{n}\right) \underset{n\to\infty}{\overset{\mathrm{GHP}}{\Longrightarrow}} \mathcal{T}_{\gamma,\nu}.$$

Pour tout $n$, notons de plus $\ell_n$ le profil de $T_n$ par rapport à $\mu_n$, i.e. pour tout $j \geq 0$,

$$\ell_n(j) := \mu_n(u \in T_n : |u| = j).$$

*Convergence fonctionnelle du profil renormalisé.* La Section 4.4 sera consacrée à l'étude du comportement asymptotique de la suite $(\ell_n)_n$. Plus particulièrement, nous nous intéresserons au profil renormalisé $L_n$ de $T_n$, c'est-à-dire la fonction définie pour tout $t$ positif par

$$L_n(t) := n^{\gamma-1}\ell_n(\lfloor n^\gamma t \rfloor).$$

Pour tout $n$, notons de plus $U_n$ une feuille uniformément choisie dans $T_n$ et $I_n$ sa hauteur. Nous montrerons alors le résultat suivant.

**Théorème 4.4.5.** *Supposons qu'il existe $\theta > 0$ tel que*

$$\sup_{n\geq 0} \sup_{t\in\mathbb{R}} |\varphi_{I_n/n^\gamma}(t)|\,\mathbb{1}_{|t|\leq n^\gamma \pi}\,|t|^{1+\theta} < \infty \qquad (\Phi)$$

*où $\varphi_{I_n/n^\gamma}$ est la fonction caractéristique de $I_n/n^\gamma$. Supposons de plus que $\gamma < 1/2$ et qu'il existe $\alpha, \beta < 1+\theta$ tels que $2 < \alpha+\beta < 1/\gamma$ et*

$$\limsup_{n\to\infty} \int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-\alpha\gamma} s_j^{1-\beta\gamma} \mathbb{1}_{s_j>0}\,\bar{q}_n(\mathrm{d}\mathbf{s}) < \infty.$$

*Alors l'arbre de fragmentation $\mathcal{T}$ admet p.s. un profil continu $L$ et*

$$L_n \underset{n\to\infty}{\overset{\mathcal{D}[0,\infty[}{\Longrightarrow}} L.$$

Notons que l'hypothèse (Φ) de ce théorème est en quelque sorte une version discrète de la conclusion de la Proposition 4.3.1. De plus, le reste des hypothèses est similaire à celles du Théorème 4.3.6.

La première étape pour prouver le Théorème 4.4.5, consiste à montrer que la suite $(L_n)_n$ ne peut converger dans $\mathcal{D}[0,\infty[$ que vers le profil de $\mathcal{T}$. Comme $(n^{-\gamma}T_n, n^{-1}\mu_n)$ converge en loi vers $(\mathcal{T}, \mu_\mathcal{T})$ au sens de la topologie GHP, et puisque $\int_0^t L_n(u)\,\mathrm{d}u \approx n^{-1}\mu_n(T_n|_{n^\gamma t})$ pour tout $t$, on obtient

$$\int_0^\cdot L_n(u)\,\mathrm{d}u \underset{n\to\infty}{\Longrightarrow} \mu_\mathcal{T}(\mathcal{T}|_\cdot) = \int_0^\cdot L(u)\,\mathrm{d}u$$

dans $\mathcal{C}[0,\infty[$. Ainsi, si $L_n$ converge en loi dans $\mathcal{D}[0,\infty[$ vers une variable $\tilde{L}$, on peut montrer que $L$ et $\tilde{L}$ ont les mêmes lois fini-dimensionnelles, voir le Lemme 4.2.5.

Pour tout $n$, notons $\Phi_n$ la fonction caractéristique de $I_n/n^\gamma$ conditionnellement à $T_n$. Sous l'hypothèse (Φ) nous montrerons dans la Proposition 4.4.6 que

$$\sup_n \int_{-n^\gamma\pi}^{n^\gamma\pi} \mathbb{E}[|\Phi_n(t)|]\,\mathrm{d}t < \infty.$$

Ainsi, nous déduirons que la suite $(L_n)_n$ est tendue dans $\mathcal{D}[0,\infty[$ ce qui conclura la preuve du Théorème 4.4.5.

*Contrôle des fonctions caractéristiques.* Rappelons que la variable $I_n$ est la hauteur d'une feuille $U_n$ choisie dans $T_n$ uniformément. Sous l'hypothèse de convergence de $n^{-\gamma}T_n$ vers l'arbre de fragmentation $\mathcal{T}$, on sait que $I_n/n^\gamma$ converge en loi vers la variable $I$ présentée précédemment. Pour tout $n$, notons $\psi_n : \mathbb{R} \to \mathbb{R}$ la fonction définie pour tout réel $u$ par

$$\psi_n(u) := n^\gamma \int_{\mathcal{S}^\downarrow} \left(1 - \sum_{i\geq 1} s_i^{1+u}\right) \bar{q}_n(\mathrm{d}\mathbf{s}).$$

Notons que cette définition est semblable à celle de $\psi$.

La Section 4.4.1 sera consacrée à prouver que la suite $(I_n)_n$ vérifie bien (Φ), qui est, comme nous l'avons déjà mentionné, une sorte d'équivalent discret à la Proposition 4.3.1. Bien que nous pensons que cette propriété est vérifiée par de nombreux modèles, nous ne savons le montrer pour l'instant que dans un cas particulier :

**Proposition 4.4.2.** *Supposons que la mesure $\nu$ est finie et que $n^\gamma[1-q_n(n)] \to \nu(\mathcal{S}^\downarrow)$ quand $n \to \infty$. Alors pour tout $\theta > 0$ tel que $\psi_n(-\gamma\theta) \to \psi(-\gamma\theta) > -\infty$, on a*

$$\sup_{n\geq 0} \sup_{t\in\mathbb{R}} |\varphi_{I_n/n^\gamma}(t)|\, \mathbb{1}_{|t|\leq n^\gamma \pi} |t|^{1+\theta} < \infty.$$

Pour prouver cette proposition, nous utiliserons la nature Markov branchante de l'arbre $T_n$. En effet, par construction, ce dernier est constitué d'une branche de longueur $G_n$, une variable géométrique de paramètre $1-q_n(n)$, au bout de laquelle sont attachés des sous-arbres indépendants de tailles strictement inférieures à $n$. Notons alors $Z_n$ le nombre de feuilles dans ledit sous-arbre qui contient $U_n$. La variable $I_n$ a alors la même loi que $G_n + I_{Z_n}$. Pour tout réel $t$, on pourra alors écrire

$$\varphi_{I_n/n^\gamma}(t) = \varphi_{G_n/n^\gamma}(t)\, \mathbb{E}\Big[\varphi_{I_{Z_n}/(Z_n)^\gamma}\big((Z_n/n)^{-\gamma} t\big)\Big].$$

Les hypothèses $n^\gamma[1-q_n(n)] \to \nu(\mathcal{S}^\downarrow)$ et $\nu(\mathcal{S}^\downarrow) < \infty$ nous permettrons alors d'affirmer que $G_n/n^\gamma$ converge en loi vers une variable exponentielle de paramètre $\nu(\mathcal{S}^\downarrow)$ et qu'il existe une constante finie $C$ telle que $|\varphi_{G_n/n^\gamma}(t)| \leq C$ pour tout $n$ et pour tout $|t| \leq n^\gamma \pi$. On pourra alors conclure la preuve de la Proposition 4.4.2 en injectant ce contrôle dans la décomposition de $\varphi_{I_n/n^\gamma}$ et en réitérant.

Notons que pour appliquer nos résultats à des modèles moins restrictifs, y compris les arbres Markov branchants indexés par leur nombre de nœuds, il suffit de prouver que (Φ) est satisfaite.

*Absence de convergence fonctionnelle.* La Section 4.4.4 donnera à l'inverse des critères pour garantir que $L_n$ ne puisse converger dans $\mathcal{D}[0,\infty[$. Ce sera en particulier le cas si $\gamma > 1$ ou si le couple $(\gamma, \nu)$ satisfait le critère établi à la Proposition 4.3.8. Notons que les hypothèses de ces critères n'excluent pas celles de la Proposition 4.4.2. Ainsi, il pourra être impossible pour $(L_n)_n$ de converger dans $\mathcal{D}[0,\infty[$ alors que $\mathbb{E}[L_n(\cdot)] \to \mathbb{E}[L(\cdot)]$ presque partout sur $[0,\infty[$.

Les hypothèses de la Proposition 4.4.2 sont, comme nous l'avons déjà dit, assez restrictives. Ces conditions ne semblent pas être nécessaires pour prouver cette proposition, mais il ne nous a pas été possible à cette date de les supprimer. Si tel avait été le cas, puisque la Proposition 4.4.2 est l'ingrédient principal de la preuve du Théorème 4.4.5, le champ d'application dudit théorème aurait été bien plus grand. En particulier, nous aurions pu étudier avec cette même approche le profil des arbres Markov branchants indexés par leur nombre de nœuds.

# Chapter 1

# Local limits of Markov branching trees and their volume growth

In this chapter, which was published as [101], we will introduce infinite Markov branching trees which appear naturally as the local limits of Markov branching trees with fixed number of either vertices or nodes. We will then study the scaling limits of a subset of this family of infinite trees. This will allow us to deduce how the "volume" of the ball with radius $R$ around the root of these trees grows when $R$ goes to infinity.

## 1.1 Introduction

The focus of this work is to study the asymptotic behaviour of sequences of random trees which satisfy the Markov branching property first introduced by Aldous in [10, Section 4] and later extended for example in [30, 66, 67]. See Haas [63] for an overview of this general model and Lambert [89] for applications to models used in evolutionary biology. Our study will therefore encompass various models, like Galton-Watson trees conditioned on their total progeny or their number of leaves, certain models of cut-trees (see Bertoin [22, 23, 24]) or recursively built trees (see Rémy [110], Chen, Ford and Winkel [36], Haas and Stephenson [69]) as well as models of phylogenetic trees (Ford's $\alpha$-model [52] and Aldous' $\beta$-splitting model [10]).

Informally, a sequence $(T_n)_n$ of random trees satisfies the Markov branching property if for all $n$, $T_n$ has "size" $n$, and conditionally on the event "$T_n$ has $p$ sub-trees above its root with respective sizes $n_1 \geq \cdots \geq n_p$", these sub-trees are independent and for each $i = 1, \ldots, p$, the $i^{\text{th}}$ largest sub-tree is distributed like $T_{n_i}$. The sequence of distributions of $(T_n)_n$ is characterised by a family $q = (q_n)_n$ of probability distributions, referred to as "first-split distributions" (see next paragraph), where $q_n$ is supported by the set of partitions of the integer $n$. We will detail two different constructions of Markov branching trees corresponding to a given sequence $q$ for two different notions of size: the number of leaves or the number of vertices.

Let $(q_n)_n$ be a sequence of first-split distributions. A tree with $n$ leaves with distribution in the associated Markov branching family is built with the following process. Consider a cluster of $n$ identical particles and with probability $q_n(\lambda_1, \ldots, \lambda_p)$, split it into $p$ smaller clusters containing $\lambda_1, \ldots, \lambda_p$ particles respectively. For each $i = 1, \ldots, p$, independently of the other sub-clusters, split the $i^{\text{th}}$ cluster according to $q_{\lambda_i}$. When a sub-cluster contains only 1 particle, with probability $q_1(1) < 1$, let it either give birth to a new sub-cluster which only contains 1 particle as well, or, with probability $1 - q_1(1)$, let the particle "die". Repeat this procedure until each of the particles are dead. The genealogy of these splits may be encoded as a tree with $n$ leaves (which correspond to the death of each particle). We'll denote by $\mathrm{MB}_n^{\mathcal{L},q}$ the distribution of such a tree.

A Markov branching tree with a given number of vertices, say $n$, is built with a slightly different procedure and we will call $\mathrm{MB}_n^q$ its distribution. Section 1.2.2 will rigorously detail the constructions of both

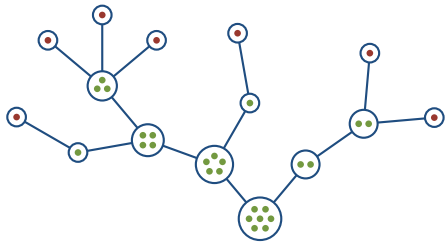


Figure 1.1 – Example of a tree with 7 leaves (in red) and first-split equal to $(5, 2)$.

$\mathrm{MB}_n^q$ and $\mathrm{MB}_n^{\mathcal{L},q}$. Rizzolo [111] considered a more general notion of size and described the construction of corresponding Markov branching trees.

One way of looking at the behaviour of large trees is through the local limit topology. For a given tree $\mathtt{t}$ and $R \geq 0$, we denote by $\mathtt{t}|_R$ the subset of vertices of $\mathtt{t}$ at graph distance less than $R$ from its root. We will say that a sequence $\mathtt{t}_n$ converges locally to a limit tree $\mathtt{t}_\infty$ if for any radius $R$, $\mathtt{t}_n|_R = \mathtt{t}_\infty|_R$ for sufficiently large $n$. There is considerable literature on the study of the local limits of certain classes of random trees or, more generally, of graphs. For instance, see Abraham and Delmas [1, 2], Stephenson [117], Stefánsson [113, 114] or a recent paper by Broutin and Mailler [31], as well as references therein, for studies related to our work.

Let us present in this Introduction the simplest, and most common, case in which Markov branching trees have local limits. Let $(T_n)_n$ be a sequence of Markov branching trees indexed by their size with corresponding family of first-split distributions $(q_n)_n$. Let $p$ be a non-negative integer and $\lambda_1 \geq \cdots \geq \lambda_p > 0$ be a non increasing family of integers with sum $L$. For $n$ large enough, consider $q_n(n-L, \lambda_1, \ldots, \lambda_p)$, that is the probability that $T_n$ gives birth to $p+1$ sub-trees among which the $p$ smallest have respective sizes $\lambda_1, \ldots, \lambda_p$. Assume that for any such $p$ and $\lambda$, $q_n(n-L, \lambda_1, \ldots, \lambda_p)$ converges to $q_*(\lambda_1, \ldots, \lambda_p)$ for some probability measure $q_*$ on the set of non-increasing finite sequences of positive integers. Under this natural assumption, we will prove in a rather straightforward way that $T_n$ locally converges to some "infinite Markov branching tree" $T_\infty$ with a single path from the root to infinity, called its *infinite spine*. The distribution of $T_\infty$ is characterised by the family $(q_n)_n$ and the measure $q_*$ which describes the distribution of the sizes of the finite sub-trees grafted on the spine of $T_\infty$. See Theorem 1.2.5 for a more precise and general statement.

A drastically different approach to understand the behaviour of large random trees is that of scaling limits. Aldous was the first to study scaling limits of random trees as a whole, see [7], and notably introduced the celebrated Brownian tree as the limit of rescaled critical Galton-Watson trees conditioned on their size with any offspring law that has finite variance. See also Le Gall [92] for a survey on random "continuous" trees.

In this context, we will consider $T_n$ as a metric space rescaled by some factor $a_n$, i.e. the edges of $T_n$ will be viewed as real segments of length $a_n$, and denote by $a_n T_n$ this rescaled metric space. Scaling limits for Markov branching trees were studied in [66, 67] by Haas *et al*. Their main result (see Theorems 5 and 6 in [66]) is that under simple conditions on the sequence $(q_n)_n$ of first-split distributions, $T_n$ converges in distribution, under appropriate rescaling, to a self-similar fragmentation tree. These objects were introduced by Haas and Miermont [64] and notably encompass Aldous' Brownian tree as well as Duquesne and Le Gall's stable trees [49].

Haas and Miermont's result from [66] in particular gives an asymptotic relation between the size and height of a finite Markov branching tree. When considering an infinite Markov branching tree $T$, we may wonder if a similar relation exists, namely how many vertices or leaves are typically found at height less

than some large integer $R$. This seemingly simple question, the study of the integer sequence $(\#T|_R)_R$, leads us to consider the scaling limits of the weighted tree $(T, \mu_T)$, where $\mu_T$ is the counting measure on either the vertices of $T$ or on its leaves.

In Theorem 1.4.1, we consider the case in which $T$ is an infinite Markov branching tree with a unique infinite spine with distribution characterised by a family $(q_n)_n$ of first-split distributions and a probability measure $q_*$ associated to the sizes of the finite sub-trees grafted on the spine. We prove that under the assumptions of Haas and Miermont's theorem on the family $(q_n)_n$ and an additional condition on the measure $q_*$, when $R$ goes to infinity, the tree $T/R$ endowed with the adequately rescaled measure $\mu_T$ converges in distribution to a self-similar fragmentation tree with immigration. These continuum random trees (CRTs) with infinite height were introduced by Haas [62]. They include Aldous' self-similar CRT [7] (which will appear as the limit in many of our applications) and Duquesne's Lévy trees with immigration [48].

As a result, under appropriate rescaling, the "volume" of the ball of radius $R$ centred at the root of $T$ converges in distribution to the measure of the ball with radius 1 centred at the root of a self-similar fragmentation tree with immigration. Proposition 1.4.2 actually gives the stronger convergence of the whole "volume growth" process.

The unified framework used here will yield multiple applications. As a first example, Theorem 1.2.5 will allow us to recover known results on the local limits of conditioned Galton-Watson trees towards Kesten's tree (see Abraham and Delmas [2] for instance) and Theorem 1.4.1 will give an alternative proof to Duquesne's results (see [48]) on the convergence of rescaled infinite critical Galton-Watson trees to Lévy trees with immigration. We will give similar results for some models of cut-trees, which encode the genealogy of the random dismantling of trees, studied by Bertoin [22, 23, 24]. We will also study some models of sequentially growing trees described in [36, 69, 95, 110] and models of phylogenetic trees [10, 52].

This article will be organised as follows. In Section 1.2, we will define finite and infinite Markov branching trees and give a natural criterion for their convergence under the local limit topology in Theorem 1.2.5. In Section 1.3 we will detail the background needed for our main result, Theorem 1.4.1, i.e. the study of the scaling limits of infinite Markov branching trees. Section 1.4 will focus on the proof of this result. Finally, Section 1.5 will give applications of our unified approach to various Markov branching models.

## 1.2 Markov branching trees and their local limits

### 1.2.1 Trees and partitions

**Background on trees.** Let $\mathcal{U} := \bigcup_{n\geq 0} \mathbb{N}^n$ be the set of finite words on $\mathbb{N}$ with the conventions $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$. We then call a *plane tree* or *ordered rooted tree* any non-empty subset $\mathtt{t} \subset \mathcal{U}$ such that:

- The empty word $\varnothing$ belongs to $\mathtt{t}$, it will be thought of as its "root",
- If $u = (u_1, \ldots, u_n)$, $u \neq \varnothing$ is in $\mathtt{t}$, then its parent $\mathrm{pr}(u) := (u_1, \ldots, u_{n-1})$ is also in $\mathtt{t}$,
- For all $u$ in $\mathtt{t}$, there exists a finite integer $c_u(\mathtt{t}) \geq 0$ such that $u\,i := (u_1, \ldots, u_n, i)$ is in $\mathtt{t}$ iff $1 \leq i \leq c_u(\mathtt{t})$. We will say that $c_u(\mathtt{t})$ is the number of children of $u$ in $\mathtt{t}$.

Let $\mathbb{T}^{\mathrm{ord}}$ be the set of plane trees. Observe that if $\mathtt{t}$ is an infinite plane tree, this definition requires the number of children of each of its vertices to be finite.

Plane trees are endowed with a total order which is of limited interest to us. Because of this, we define an equivalence relation on $\mathbb{T}^{\mathrm{ord}}$ to allow us to consider as identical two trees which have the same "shape" but different vertex orderings.

Say that two plane trees $\mathtt{t}$ and $\mathtt{t}'$ are equivalent (written $\mathtt{t} \sim \mathtt{t}'$) iff there exists a bijection $\sigma : \mathtt{t} \to \mathtt{t}'$ such that $\sigma(\varnothing) = \varnothing$ and for all $u \in \mathtt{t} \setminus \{\varnothing\}$, $\mathrm{pr}[\sigma(u)] = \sigma[\mathrm{pr}(u)]$. Finally, set $\mathtt{T} := \mathtt{T}^{\mathrm{ord}}/\sim$. From now on, unless otherwise stated, we will only consider unordered trees, i.e. by "tree" we will mean an element of $\mathtt{T}$.

Let $\mathtt{t}$ be a tree. We say that a vertex $u$ on $\mathtt{t}$ is a leaf if it has no children, i.e. if $c_u(\mathtt{t}) = 0$. Define $\#\mathtt{t}$ as the total number of vertices of $\mathtt{t}$ and $\#_{\mathcal{L}}\mathtt{t}$ as its number of leaves. For any positive integer $n$, let $\mathtt{T}_n$ and $\mathtt{T}_n^{\mathcal{L}}$ be the sets of finite trees with $n$ vertices and $n$ leaves respectively. Moreover, write $\mathtt{T}_\infty$ for the set of infinite trees.

We will use the following operations on trees:

— Let $\mathtt{t}_1, \dots, \mathtt{t}_d$ be trees; their *concatenation* is the tree $[\![\mathtt{t}_1, \dots, \mathtt{t}_d]\!]$ obtained by attaching each of their respective roots to a new common root, see Figure 1.2,

— Let $\mathtt{t}$ and $\mathtt{s}$ be two trees and $u$ be a vertex of $\mathtt{t}$; set $\mathtt{t} \otimes (u, \mathtt{s})$ the grafting of $\mathtt{s}$ on $\mathtt{t}$ at $u$, i.e. the tree obtained by glueing the root of $\mathtt{s}$ on $u$, see Figure 1.3,

— Fix $\mathtt{t}$ a tree, a non-repeating family $(u_i)_{i\in\mathcal{I}}$ of vertices of $\mathtt{t}$, and a family of trees $(\mathtt{s}_i)_{i\in\mathcal{I}}$; let $\mathtt{t}\bigotimes_{i\in\mathcal{I}}(u_i, \mathtt{s}_i)$ be the tree obtained by grafting $\mathtt{s}_i$ on $\mathtt{t}$ at $u_i$ for each $i$ in $\mathcal{I}$.

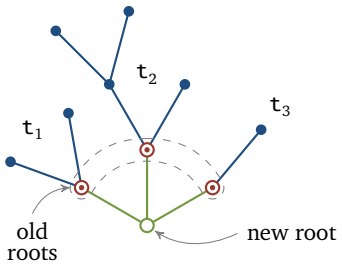



Figure 1.2 – The tree $[\![\mathtt{t}_1, \mathtt{t}_2, \mathtt{t}_3]\!]$.

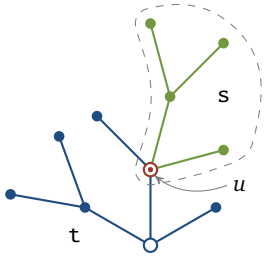



Figure 1.3 – The tree $\mathtt{t} \otimes (u, \mathtt{s})$.

For all $n \geq 0$, let $\mathtt{b}_n$ be the *branch* of length $n$, i.e. the tree with $n+1$ vertices among which is a single leaf. Similarly, define the infinite branch $\mathtt{b}_\infty$ and let $(\mathtt{v}_n)_{n\geq 0}$ be its vertices where $\mathtt{v}_0$ is its root and for all $n \geq 0$, $\mathtt{v}_n = \mathrm{pr}(\mathtt{v}_{n+1})$.

*The local limit topology.* If $\mathtt{t}$ is a tree, we may endow it with the *graph distance* $\mathrm{d}_{\mathrm{gr}}$ where for all $u$ and $v$ in $\mathtt{t}$, $\mathrm{d}_{\mathrm{gr}}(u, v)$ is defined as the number of edges in the shortest path between $u$ and $v$. For any non-negative integer $R$, we will write $\mathtt{t}|_R$ for the closed ball of radius $R$ centred at the root of $\mathtt{t}$, that is the tree $\mathtt{t}|_R := \{u \in \mathtt{t} : \mathrm{d}_{\mathrm{gr}}(\varnothing, u) \leq R\}$.

The local distance between two given trees $\mathtt{t}$ and $\mathtt{s}$ is defined as

$$\mathrm{d}_{\mathrm{loc}}(\mathtt{t}, \mathtt{s}) := \exp\Big[-\inf\{R \geq 0 : \mathtt{t}|_R \neq \mathtt{s}|_R\}\Big].$$

The function $\mathrm{d}_{\mathrm{loc}}$ is an ultra-metric on $\mathtt{T}$ and the resulting metric space $(\mathtt{T}, \mathrm{d}_{\mathrm{loc}})$ is Polish. The following well-known criterion for convergence in distribution with respect to the local limit topology will be useful. See for instance [2, Section 2.2] for a proof (which relies on [29, Theorem 2.3] and the fact that $\mathrm{d}_{\mathrm{loc}}$ is an ultra-metric).

**Lemma 1.2.1.** *Let $T_n$, $n \geq 1$ and $T$ be $\mathtt{T}$-valued random variables. Then, $T_n \to T$ in distribution with respect to $\mathrm{d}_{\mathrm{loc}}$ iff for all $\mathtt{t} \in \mathtt{T}$ and $R \geq 0$, $\mathbb{P}[T_n|_R = \mathtt{t}|_R] \to \mathbb{P}[T|_R = \mathtt{t}|_R]$ as $n$ tends to infinity.*

**Partitions of integers.** As discussed in the Introduction, Markov branching trees are closely related to "partitions of integers". This section thus aims to introduce a few notions on these objects which will be useful for our forthcoming purposes.

Set $\mathcal{P}_0 := \{\varnothing\}$, $\mathcal{P}_1 := \{\varnothing, (1)\}$ and for $n \geq 2$, let $\mathcal{P}_n$ be the set of *partitions* of $n$, i.e. of finite non-increasing integer sequences with sum $n$. More precisely, set

$$\mathcal{P}_n := \Big\{\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{N}^p \,:\, p \geq 1,\, \lambda_1 \geq \cdots \geq \lambda_p > 0 \text{ and } \lambda_1 + \cdots + \lambda_p = n\Big\}.$$

Similarly, let $\mathcal{P}_\infty$ be the set of finite non-increasing $\mathbb{N} \cup \{\infty\}$-valued sequences with infinite sum (and therefore at least one infinite part). In other words, define

$$\mathcal{P}_\infty := \Big\{\lambda = (\lambda_1, \dots, \lambda_p) \in \big(\mathbb{N} \cup \{\infty\}\big)^p \,:\, p \geq 1 \text{ and } \infty = \lambda_1 \geq \cdots \geq \lambda_p > 0\Big\}.$$

Set $\mathcal{P}_{<\infty} := \bigcup_{n\geq 0} \mathcal{P}_n$ and $\mathcal{P} := \mathcal{P}_{<\infty} \cup \mathcal{P}_\infty$.

Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be in $\mathcal{P}$. We will use the following notations:

— Let $p(\lambda) := p$ be its *length* and $\|\lambda\| = \lambda_1 + \cdots + \lambda_p$ its sum (with the conventions $p(\varnothing) = \|\varnothing\| = 0$).
— For $k \in \mathbb{N} \cup \{\infty\}$, let $m_k(\lambda) := \sum_i \mathbb{1}_{\lambda_i = k}$ be the number of occurrences of $k$ in the partition $\lambda$.
— For a non-negative integer $K$, set $\lambda \wedge K := (\lambda_1 \wedge K, \dots, \lambda_p \wedge K)$. This finite partition will be called the *truncation* of $\lambda$ at level $K$.

We endow $\mathcal{P}$ with an ultra-metric distance defined similarly to $\mathrm{d}_{\mathrm{loc}}$. For all $\lambda$ and $\mu$ in $\mathcal{P}$, let

$$\mathrm{d}_{\mathcal{P}}(\lambda, \mu) := \exp\Big[-\inf\{K \geq 0 \,:\, \lambda \wedge K \neq \mu \wedge K\}\Big].$$

**Lemma 1.2.2.** $(i)$ *The function* $\mathrm{d}_{\mathcal{P}}$ *is an ultra-metric distance,*
$(ii)$ *The metric space* $(\mathcal{P}, \mathrm{d}_{\mathcal{P}})$ *is Polish.*

*Remark 1.2.1.* For all $\lambda$ and $\mu$ in $\mathcal{P}$ and $K \geq 0$, $\lambda \wedge K = \mu \wedge K$ iff $\mathrm{d}_{\mathcal{P}}(\lambda, \mu) < \mathrm{e}^{-K}$. In particular, $\mathrm{d}_{\mathcal{P}}(\lambda, \mu) = 1$ iff $\lambda \wedge 0 \neq \mu \wedge 0$ in which case $p(\lambda) \neq p(\mu)$.

*Proof.* $(i)$ Clearly, $\mathrm{d}_{\mathcal{P}}$ is symmetric and $\mathrm{d}_{\mathcal{P}}(\lambda, \mu) = 0$ iff $\lambda = \mu$. Hence, we only need to prove that $\mathrm{d}_{\mathcal{P}}$ satisfies the ultra-metric triangular inequality. Let $\lambda$, $\mu$ and $\nu$ be in $\mathcal{P}$ and assume that $\mathrm{d}_{\mathcal{P}}(\lambda, \nu) > \mathrm{d}_{\mathcal{P}}(\lambda, \mu) \vee \mathrm{d}_{\mathcal{P}}(\mu, \nu)$. Then, there exists $K \geq 0$ such that $\lambda \wedge K = \mu \wedge K = \nu \wedge K$ and $\lambda \wedge K \neq \nu \wedge K$, which is absurd. Consequently, $\mathrm{d}_{\mathcal{P}}(\lambda, \nu) \leq \mathrm{d}_{\mathcal{P}}(\lambda, \mu) \vee \mathrm{d}_{\mathcal{P}}(\mu, \nu)$.

$(ii)$ Observe that $\mathcal{P} \subset \bigcup_{n\geq 0}(\mathbb{N} \cup \{\infty\})^n$ and is as a result both countable and separable. Therefore, it only remains to show that it is complete.

Let $(\lambda_n)_n$ be a Cauchy sequence with respect to $\mathrm{d}_{\mathcal{P}}$. By assumption, there exists an increasing sequence $(n_K)_K$ such that for all $K \geq 0$, $\lambda_n \wedge K = \lambda_m \wedge K$ when $n, m \geq n_K$. In particular, there exists a constant $p \geq 0$ such that $p(\lambda_{n_K}) = p$ for all $K$. Furthermore, notice that for all $i = 1, \dots, p$, the sequence $[\lambda_{n_K}(i) \wedge K]_K$ is non-decreasing. For each $i = 1, \dots, p$, set $\lambda(i) := \sup_K \lambda_{n_K}(i) \wedge K \leq \infty$. Clearly, $\lambda := [\lambda(1), \dots, \lambda(p)]$ is in $\mathcal{P}$ and is such that $\mathrm{d}_{\mathcal{P}}(\lambda_n, \lambda) \to 0$ when $n \to \infty$. This proves that $(\mathcal{P}, \mathrm{d}_{\mathcal{P}})$ is indeed complete. □

**Lemma 1.2.3.** *Let* $(\Lambda_n)_{n\geq 1}$ *and* $\Lambda$ *be* $\mathcal{P}$*-valued random variables. Then,* $\Lambda_n$ *converges to* $\Lambda$ *in distribution with respect to* $\mathrm{d}_{\mathcal{P}}$ *iff for all* $\lambda$ *in* $\mathcal{P}_{<\infty}$ *and all* $K \geq 0$*, we have* $\mathbb{P}[\Lambda_n \wedge K = \lambda \wedge K] \to \mathbb{P}[\Lambda \wedge K = \lambda \wedge K]$ *as* $n \to \infty$.

*Proof.* Uses the same arguments as in the proof of Lemma 1.2.1 (recall that $\mathrm{d}_{\mathcal{P}}$ is an ultra-metric and use [29, Theorem 2.3]). □

*Remark 1.2.2.* Elements of $\mathcal{P}_{<\infty}$ are closely related to elements of $\mathtt{T}$. Indeed, if $\mathtt{t}$ is a finite tree which can be written as the concatenation of $p$ trees $\mathtt{t}_1, \dots, \mathtt{t}_p$, i.e. $\mathtt{t} = [\![\mathtt{t}_1, \dots, \mathtt{t}_p]\!]$, then the decreasing rearrangement of $\#\mathtt{t}_1, \dots, \#\mathtt{t}_p$ is a partition of $n$ when $\mathtt{t}$ has $n+1$ vertices (the root plus $n$ descendants). We will write $\Lambda(\mathtt{t}) := (\#\mathtt{t}_1, \dots, \#\mathtt{t}_p)^{\downarrow}$, where $(x_1, \dots, x_k)^{\downarrow}$ stands for the decreasing rearrangement of $(x_1, \dots, x_p)$, and call $\Lambda(\mathtt{t})$ the *partition at the root* or *first split* of $\mathtt{t}$.

Similarly, if we consider leaves instead of vertices, then $\Lambda^{\mathcal{L}}(\mathtt{t}) := (\#_{\mathcal{L}}\mathtt{t}_1, \dots, \#_{\mathcal{L}}\mathtt{t}_p)^{\downarrow}$ is a partition of $n$ when $\mathtt{t}$ has $n$ leaves.

In this article, we will often have to consider sequences of random partitions $\Lambda_n \in \mathcal{P}_n$ that will weakly converge to a limit partition $\Lambda_\infty \in \mathcal{P}_\infty$ such that, $m_\infty(\Lambda_\infty) = 1$ a.s.. In this particular setting, the weak convergence can be characterised as follows.

**Lemma 1.2.4.** *For all $1 \leq n \leq \infty$, let $q_n$ be a probability measure on $\mathcal{P}_n$ and assume that $q_\infty(m_\infty = 1) = 1$. Then, $q_n \Rightarrow q_\infty$ with respect to $\mathrm{d}_\mathcal{P}$ iff for all $\lambda$ in $\mathcal{P}_{<\infty}$ we have $q_n(n - \|\lambda\|, \lambda) \to q_\infty(\infty, \lambda)$ as $n \to \infty$.*

*Proof.* $\Rightarrow$ Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be in $\mathcal{P}_{<\infty}$ and $K > \lambda_1$. In light of Lemma 1.2.3,

$$q_n(n - \|\lambda\|, \lambda) = q_n\big(\mu \in \mathcal{P}_n : \mu \wedge K = (K, \lambda) \wedge K\big) \xrightarrow[n\to\infty]{} q_\infty\big(\mu \in \mathcal{P}_\infty : \mu \wedge K = (K, \lambda) \wedge K\big) = q_\infty(\infty, \lambda).$$

$\Leftarrow$ For fixed $K \geq 0$ and $\lambda$ in $\mathcal{P}_{<\infty}$, Fatou's lemma ensures that

$$\begin{aligned}\liminf_{n\to\infty} q_n\big(\mu \in \mathcal{P}_n : \mu \wedge K = \lambda \wedge K\big) &= \liminf_{n\to\infty} \textstyle\sum_{\nu \in \mathcal{P}_{<\infty}} \mathbb{1}_{(\infty,\nu)\wedge K = \lambda\wedge K}\, q_n(n - \|\nu\|, \nu) \\ &\geq \textstyle\sum_{\nu \in \mathcal{P}_{<\infty}} \mathbb{1}_{(\infty,\nu)\wedge K = \lambda\wedge K}\, q_\infty(\infty, \nu) = q_\infty\big(\mu \in \mathcal{P}_\infty : \mu \wedge K = \lambda \wedge K\big).\end{aligned}$$

Similarly,

$$\liminf_{n\to\infty} q_n\big(\mu \in \mathcal{P}_n : \mu \wedge K \neq \lambda \wedge K\big) \geq q_\infty\big(\mu \in \mathcal{P}_\infty : \mu \wedge K \neq \lambda \wedge K\big).$$

As a result and thanks to Lemma 1.2.3, we get that $q_n \Rightarrow q_\infty$. □

## 1.2.2 The Markov-branching property

**Finite Markov branching trees.** We will now follow [66, Section 1.2] and define two types of family of probability measures on the set of finite unordered rooted trees, satisfying the Markov branching property discussed in the Introduction.

Informally, for a given sequence $q = (q_n)$ of probability measures respectively supported by $\mathcal{P}_n$ (referred to as "first-split distributions" in the Introduction), we want to define a sequence $\mathrm{MB}^q = (\mathrm{MB}^q_n)_n$ of probability measures on the set of finite trees where

- For all $n$, $\mathrm{MB}^q_n$ is supported by the set of trees with $n$ vertices,
- A tree $T$ with distribution $\mathrm{MB}^q_n$ is such that
  - The decreasing rearrangement $\Lambda(T)$ of the sizes of the sub-trees above its root is distributed according to $q_{n-1}$,
  - Conditionally on $\Lambda(T) = (\lambda_1, \dots, \lambda_p)$, the $p$ sub-trees of $T$ above its root are independent with respective distributions $\mathrm{MB}^q_{\lambda_i}$.

Similarly, if $q = (q_n)_n$ is a sequence of probability measures respectively on $\mathcal{P}_n$, we will define a sequence $\mathrm{MB}^{\mathcal{L},q}$ satisfying the same Markov branching property where we count leaves instead of vertices to measure the size of a tree.

*Markov branching tree with n vertices.* First of all, set $\mathcal{N}$ an infinite subset of $\mathbb{N}$ with $1 \in \mathcal{N}$. This set will index the possible number of vertices of the trees we want to generate. Let $q = (q_{n-1})_{n\in\mathcal{N}}$ be a sequence of probability measures such that $q_0(\varnothing) = 1$, $q_1[(1)] = 1$ (if $2 \in \mathcal{N}$), and for all $n$ in $\mathcal{N}$, $n \geq 2$, $q_{n-1}$ is supported by the set $\{\lambda \in \mathcal{P}_{n-1} : \lambda_i \in \mathcal{N}, i = 1, \dots, p(\lambda)\}$.

*Remark 1.2.3.* This last condition comes from the fact that if $T$ is distributed according to $\mathrm{MB}^q_n$, the blocks of $\Lambda(T)$ need to be in $\mathcal{N}$ because the distributions of the corresponding sub-trees belong to the family $(\mathrm{MB}^q_k)_{k\in\mathcal{N}}$.

We now detail a recursive construction for $\mathrm{MB}^q$. Let $\mathrm{MB}^q_1(\{\varnothing\}) = 1$ and for $n \geq 2$, proceed by a decreasing induction as follows:

— Let $\Lambda$ have distribution $q_{n-1}$,
— Conditionally on $\Lambda = (\lambda_1, \dots, \lambda_p) \in \mathcal{P}_{n-1}$, let $(T_1, \dots, T_p)$ be independent random trees such that $T_i$ is distributed according to $\mathrm{MB}^q_{\lambda_i}$ for each $1 \le i \le p$,
— Define $\mathrm{MB}^q_n$ as the law of the concatenation of these trees, i.e. that of $[\![T_1, \dots, T_{p(\Lambda)}]\!]$.

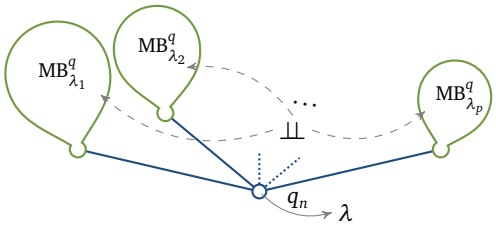



Figure 1.4 – The construction of a tree with distribution $\mathrm{MB}^q_n$.

*Markov branching tree with n leaves.* Similarly, fix an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $1 \in \mathcal{N}$ (corresponding to the possible number of leaves of the trees we will generate) and let $q = (q_n)_{n\in\mathcal{N}}$ be such that:

— $q_1$ is a probability measure on $\{\varnothing, (1)\}$ with $q_1(1) < 1$,
— For all $n > 1$ in $\mathcal{N}$, $q_n$ is a probability measure supported by the set $\{\lambda \in \mathcal{P}_n : \lambda_i \in \mathcal{N},\ i = 1, \dots, p(\lambda)\}$.

To define $\mathrm{MB}^{\mathcal{L},q}$, we will proceed by the same recursive method used for $\mathrm{MB}^q$: first choose how the size is split between the children sub-trees of the root, and then generate the said sub-trees adequately. However, if for some $n$ in $\mathcal{N}$ we have $q_n(n) = 1$, the recursion will be endless. For this reason, we also require that for all $n$ in $\mathcal{N}$, $q_n(n) < 1$ (i.e. with positive probability, a tree "splits" into smaller trees).

Let $\mathrm{MB}^{\mathcal{L},q}_1$ be the distribution of a branch of geometric length with parameter $1-q_1(1)$, i.e. $\mathrm{MB}^{\mathcal{L},q}_1(\mathsf{b}_k) = q_1(1)^k[1-q_1(1)]$ for all $k \ge 0$. For $n > 1$, we do as follows:

— Let $T_0$ be a branch with geometric length with parameter $1-q_n(n)$ and call $U$ its leaf,
— Let $\Lambda$ have distribution $q_n$ conditioned on the event $\{m_n = 0\}$,
— Conditionally on $\Lambda = (\lambda_1, \dots, \lambda_p)$, let $(T_1, \dots, T_p)$ be independent random trees respectively distributed according to $\mathrm{MB}^q_{\lambda_i}$ for $1 \le i \le p$,
— Graft the concatenation of these trees on the leaf $U$ of $T_0$, i.e. set $T := T_0 \otimes \big(U, [\![T_1, \dots, T_{p(\Lambda)}]\!]\big)$ and let $\mathrm{MB}^{\mathcal{L},q}_n$ be the distribution of $T$.

**Infinite Markov branching trees.** Using the same principle as before (split the number of vertices above the root and generate independent sub-trees with corresponding sizes) we will define a probability measure supported by the set of infinite trees which satisfies a version of the Markov branching property. Let $\mathcal{N}$ and $q = (q_{n-1})_{n\in\mathcal{N}}$ satisfy the conditions exposed in the construction of the sequence $\mathrm{MB}^q$.

In order to lighten notations, for any finite decreasing sequence of integers $\lambda = (\lambda_1, \dots, \lambda_p)$, we define $\mathrm{MB}^q_\lambda$ as the distribution of the concatenation of independent $\mathrm{MB}^q_{\lambda_i}$-distributed trees. More precisely:

— Let $\mathrm{MB}^q_\varnothing$ be the Dirac measure on the tree with a single vertex (its root), namely $\mathrm{MB}^q_\varnothing = \delta_{\{\varnothing\}}$,
— For any $\lambda \in \mathcal{P}_{<\infty}$ with $p = p(\lambda) > 0$ and $\lambda_i \in \mathcal{N}$ for $i = 1, \dots, p$, let $(T_1, \dots, T_p)$ be independent trees with respective distributions $\mathrm{MB}^q_{\lambda_i}$ for all $i = 1, \dots, p$. Set $\mathrm{MB}^q_\lambda$ as the distribution of the concatenation of these trees.

Observe that when $p(\lambda) = 1$, a tree with distribution $\mathrm{MB}^q_\lambda$ is obtained by attaching an edge "under" the root of a $\mathrm{MB}^q_{\lambda_1}$-distributed tree.

Consider $q_\infty$, a probability measure on $\mathcal{P}_\infty$ supported by the set

$$\big\{\lambda \in \mathcal{P}_\infty : \lambda_i \in \mathcal{N} \cup \{\infty\}, i = 1, \dots, p(\lambda)\big\}$$

and let $\Lambda$ follow $q_\infty$. Let $T^\circ$ be a Galton-Watson tree with offspring distribution the law of $m_\infty(\Lambda)$. Conditionally on $T^\circ$, let $(\Lambda_u, T_u)_{u\in T^\circ}$ be independent pairs and such that:

— $\Lambda_u$ has the same distribution as $\Lambda$ conditioned on the event $m_\infty(\Lambda) = c_u(T^\circ)$,

— Conditionally on $\Lambda_u = (\infty, \dots, \infty, \lambda)$ with $\lambda$ in $\mathcal{P}_{<\infty}$, $T_u$ follows $\mathrm{MB}_\lambda^q$.

Then, for every vertex $u$ in $T^\circ$, graft the corresponding tree $T_u$ on $T^\circ$ at $u$. Let $T$ be the tree hence obtained, i.e. set $T := T^\circ \bigotimes_{u\in T^\circ}(u, T_u)$. Finally, call $\mathrm{MB}_\infty^{q,q_\infty}$ the distribution of $T$.

*Remark 1.2.4.*

— Suppose that $q_\infty(m_\infty = 1) = 1$. In this case, the construction of $\mathrm{MB}_\infty^{q,q_\infty}$ is much simpler: the tree $T^\circ$ is simply the infinite branch and the family $(\Lambda_{\mathrm{v}_n}, T_{\mathrm{v}_n})_{n\geq 0}$ is i.i.d.. In particular, $T$ a.s. has a unique *infinite spine*, i.e. a unique infinite non-backtracking path originating from the root.

— A tree $T$ with distribution $\mathrm{MB}_\infty^{q,q_\infty}$ satisfies the Markov branching property: conditionally on $\Lambda(T)$, the sub-trees of $T$ above its root are independent and their respective distributions are either $\mathrm{MB}_\infty^{q,q_\infty}$ or in the family $(\mathrm{MB}_n^q)_{n\in\mathcal{N}}$, depending on their sizes.

— The same exact construction can be used to define a measure $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$.

### 1.2.3 Local limits of Markov-branching trees

Let $q$ be the sequence of first-split distributions associated to a Markov-branching family $\mathrm{MB}^q$ (respectively $\mathrm{MB}^{\mathcal{L},q}$). Suppose $q_\infty$ is a probability measure on $\mathcal{P}_\infty$ supported by the set of sequences $\lambda$ such that for all $i = 1, \dots, p(\lambda)$, $\lambda_i$ is either infinite or in $\mathcal{N}$. The aim of this section is to expose suitable conditions on $q$ and $q_\infty$ such that $\mathrm{MB}_n^q$ converges weakly to $\mathrm{MB}_\infty^{q,q_\infty}$ (or $\mathrm{MB}_n^{\mathcal{L},q} \Rightarrow \mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$) for the local limit topology.

**Theorem 1.2.5.** *Suppose that when $n$ goes to infinity, $q_n$ converges weakly to $q_\infty$ with respect to the topology induced by $\mathrm{d}_\mathcal{P}$. Then, with respect to $\mathrm{d}_{\mathrm{loc}}$, $\mathrm{MB}_n^q \Rightarrow \mathrm{MB}_\infty^{q,q_\infty}$ (respectively $\mathrm{MB}_n^{\mathcal{L},q} \Rightarrow \mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$).*

In many cases, the infinite trees we will consider will have a unique infinite spine, which corresponds to $q_\infty(m_\infty = 1) = 1$ and the particular construction mentioned in Remark 1.2.4. In this situation, we may use Theorem 1.2.5 alongside Lemma 1.2.4 to get the following corollary.

**Corollary 1.2.6.** *Assume that $q_\infty$ is such that $q_\infty(m_\infty = 1) = 1$ and suppose that for any finite partition $\lambda$ in $\mathcal{P}_\infty$ we have $q_n(n - \|\lambda\|, \lambda) \to q_\infty(\infty, \lambda)$. Then, $\mathrm{MB}_n^q \Rightarrow \mathrm{MB}_\infty^{q,q_\infty}$ (or $\mathrm{MB}_n^{\mathcal{L},q} \Rightarrow \mathrm{MB}_\infty^{\mathcal{L},q,,q_\infty}$) with respect to the local limit topology.*

*Proof of Theorem 1.2.5.* For all $n$ in $\mathcal{N}\cup\{\infty\}$, let $T_n$ follow $\mathrm{MB}_n^q$ and $\Lambda_{n-1}$ follow $q_{n-1}$. To prove this theorem, we will use Lemma 1.2.1 and proceed by induction on $R$. First, it clearly holds that for every tree $\mathtt{t}$, $\mathtt{t}|_0 = \{\varnothing\} = T_n|_0 = T_\infty|_0$ a.s..

Let $R$ be a non-negative integer and suppose that for any $\mathtt{s} \in \mathtt{T}$, $\mathbb{P}[T_n|_R = \mathtt{s}|_R] \to \mathbb{P}[T_\infty|_R = \mathtt{s}|_R]$ as $R \to \infty$. Fix $\mathtt{t} \in \mathtt{T}$ and set $d := c_\varnothing(\mathtt{t})$, the number of children of its root. We may write $\mathtt{t}|_{R+1} = [\![\mathtt{t}_1, \dots, \mathtt{t}_d]\!]$ for some $\mathtt{t}_1, \dots, \mathtt{t}_d$ in $\mathtt{T}$ with height $R$ or less. When $n > 1$, we can similarly write $T_n$ as the concatenation of its sub-trees: let $T_n = [\![T_n^{(1)}, \dots, T_n^{(p)}]\!]$ where $p = c_\varnothing(T_n)$. With these notations, for all $n > 1$ in $\mathcal{N}\cup\{\infty\}$, we have

$$\mathbb{P}\big[T_n|_{R+1} = \mathtt{t}|_{R+1}\big] = \mathbb{P}\Big[\big(c_\varnothing(T_n) = d\big) \cap \big(\exists \sigma \in \mathfrak{S}_d : T_n^{(i)}|_R = \mathtt{t}_{\sigma\cdot i}, i = 1, \dots, d\big)\Big],$$

where $\mathfrak{S}_d$ denotes the set of permutations of $\{1, \dots, d\}$. There exists a subset $S$ of $\mathfrak{S}_d$ such that for any $\sigma \in \mathfrak{S}_d$ there is a unique $\tau \in S$ satisfying $\mathtt{t}_{\sigma\cdot i} = \mathtt{t}_{\tau\cdot i}$ as elements of $\mathtt{T}$ for all $i = 1, \dots, d$. Observe that $S$ only depends on $\mathtt{t}$ and the (arbitrary) labelling of its sub-trees. Then,

$$\begin{aligned}
\mathbb{P}\big[T_n|_{R+1} = \mathtt{t}|_{R+1}\big] &= \sum_{\sigma\in S} \mathbb{P}\Big[\big(c_\varnothing(T_n) = d\big) \cap \big(T_n^{(i)}|_R = \mathtt{t}_{\sigma\cdot i}, i = 1, \dots, d\big)\Big] \\
&= \sum_{\sigma\in S} \mathbb{E}\Big[\prod\nolimits_{i=1}^d \mathbb{P}\big[T_{\Lambda_n(i)}|_R = \mathtt{t}_{\sigma\cdot i} \,\big|\, \Lambda_{n-1}\big] \mathbb{1}_{p(\Lambda_{n-1})=d}\Big] \\
&= \sum_{\sigma\in S} \int_\mathcal{P} \prod\nolimits_{i=1}^d \mathbb{P}\big[T_{\lambda_i}|_R = \mathtt{t}_{\sigma\cdot i}\big] \mathbb{1}_{p(\lambda)=d}\, q_{n-1}(\mathrm{d}\lambda),
\end{aligned}$$

where we have used the Markov branching property. Our induction assumption implies in particular that for all $i = 1, \dots, d$ and $\mathtt{s}$ in $\mathtt{T}$ with height $R$ or less, the function $\mathcal{P} \to [0,1]$, $\lambda \mapsto \mathbb{P}[T_{\lambda_i}|_R = \mathtt{s}]\mathbb{1}_{p(\lambda)=d}$ is continuous. As a result, $\mathbb{P}[T_n|_{R+1} = \mathtt{t}|_{R+1}]$ may be expressed as the integral against $q_{n-1}$ of a finite sum of continuous functions. Therefore, since $q_n \Rightarrow q_\infty$,

$$\mathbb{P}\Big[T_n|_{R+1} = \mathtt{t}|_{R+1}\Big] \xrightarrow[n\to\infty]{} \mathbb{P}\Big[T_\infty|_{R+1} = \mathtt{t}|_{R+1}\Big].$$

We proceed in the same way to prove the claim on $\mathrm{MB}^{\mathcal{L},q}$ trees. □

In the next proposition, we prove that the condition "$q_n \Rightarrow q_\infty$" in Theorem 1.2.5 is optimal for $\mathrm{MB}^q$ trees.

**Proposition 1.2.7.** *Let $q = (q_{n-1})_{n\in\mathcal{N}}$ be the sequence of first split distributions associated to a family $\mathrm{MB}^q$ of Markov branching trees with given number of vertices. If there exists a probability measure $q_\infty$ on $\mathcal{P}_\infty$ such that $\mathrm{MB}_n^q$ converges weakly to $\mathrm{MB}_\infty^{q,q_\infty}$ for the local limit topology, then $q_{n-1} \Rightarrow q_\infty$ in the sense of the $\mathrm{d}_\mathcal{P}$ topology.*

*Proof.* Observe that for all $K \geq 0$ and $\mathtt{t}, \mathtt{s} \in \mathtt{T}$, if $\mathtt{t}|_K = \mathtt{s}|_K$ then $\Lambda(\mathtt{t}) \wedge K = \Lambda(\mathtt{s}) \wedge K$. As a result, $\mathrm{d}_\mathcal{P}[\Lambda(\mathtt{t}), \Lambda(\mathtt{s})] \leq \mathrm{d}_{\mathrm{loc}}(\mathtt{t}, \mathtt{s})$ which proves in particular that $\Lambda : \mathtt{T} \to \mathcal{P}$ is a continuous function. Consequently, since for all possibly infinite $n$, $\Lambda(T_n)$ has distribution $q_{n-1}$, in the sense of the $\mathrm{d}_\mathcal{P}$ topology we have $q_{n-1} \Rightarrow q_\infty$ when $n \to \infty$. □

## 1.3 Background on scaling limits

In this section, we will introduce the framework needed to consider the scaling limits of both finite and infinite Markov branching trees as well as the corresponding limiting objects: self-similar fragmentation trees with or without immigration. Afterwards, we will also give a few useful results on point processes related to our models of trees.

### 1.3.1 $\mathbb{R}$-trees and the GHP topology

To talk about scaling limits of discrete trees, we need to introduce a continuous analogue. We use the framework of $\mathbb{R}$-trees. An $\mathbb{R}$-*tree* (or *real tree*) is a metric space $(T, d)$ such that for all $x$ and $y$ in $T$:

— There exists a unique isometry $\varphi : [0, d(x,y)] \to T$ such that $\varphi(0) = x$ and $\varphi[(d(x,y)] = y$,

— If $\gamma : [0,1] \to T$ is a continuous injection with $\gamma(0) = x$ and $\gamma(1) = y$, then the image of $\gamma$ is the same as that of $\varphi$, i.e. $\mathrm{Im}\,\gamma = \mathrm{Im}\,\varphi =: [\![x, y]\!]$.

This roughly means that any two points in an $\mathbb{R}$-tree can be continuously joined by a single path, up to its reparametrisation, which is akin to the acyclic nature of discrete trees.

To compare two such objects, we will use the Gromov-Hausdorff-Prokhorov distance. More precisely, we will follow the definition from [5] and extend it in a way similar to that of [3].

For any metric space $(X, \mathrm{d})$ let $\mathcal{M}_f(X)$ be the set of all finite non-negative Borel measures on $X$ and $\mathcal{M}(X)$ be the set of all non-negative and *boundedly finite* Borel measures on $X$, i.e. non-negative Borel measures $\mu$ on $X$ such that $\mu(A) < \infty$ for all measurable bounded $A \subset X$.

A *pointed* metric space is a 3-tuple $(X, \mathrm{d}, \rho)$ where $(X, \mathrm{d})$ is a metric space and $\rho \in X$ is a fixed point, which we will call its *root*. For any $x \in X$, set $|x| := \mathrm{d}(\rho, x)$ the *height* of $x$ in $(X, \mathrm{d}, \rho)$, and let $|X| := \sup_{x\in X} |x|$ be the height of $X$.

We will call *pointed weighted metric space* any 4-tuple $\mathbf{X} = (X, \mathrm{d}, \rho, \mu)$ where $(X, \mathrm{d})$ is a metric space, $\rho \in X$ is its root and $\mu$ is a boundedly finite Borel measure on $X$.

*Remark 1.3.1.* If $\mathbf{X}$ is a pointed weighted metric space, we will implicitly write $\mathbf{X} = (X, \mathrm{d}_X, \rho_X, \mu_X)$ unless otherwise stated.

Two pointed weighted metric spaces $\mathbf{X}$ and $\mathbf{Y}$ will be called *GHP-isometric* if there exists a bijective isometry $\Phi : X \to Y$ such that $\Phi(\rho_X) = \rho_Y$ and $\mu_X \circ \Phi^{-1} = \mu_Y$. Let $\mathbb{K}$ be the set of GHP-isometry classes of *compact* pointed weighted metric spaces.

**Comparing compact metric spaces.** Let $\mathbf{X}$ and $\mathbf{Y}$ be two pointed weighted compact metric spaces. A *correspondence* between $\mathbf{X}$ and $\mathbf{Y}$ is a measurable subset $C$ of $X \times Y$ which contains $(\rho_X, \rho_Y)$ such that for any $x \in X$ there exists $y \in Y$ with $(x, y) \in C$ and conversely, for any $y \in Y$ there is $x \in X$ such that $(x, y) \in C$. We will denote by $\mathrm{C}(\mathbf{X}, \mathbf{Y})$ (or $\mathrm{C}(X, Y)$ with a slight abuse of notation) the set of all pointed correspondences between $\mathbf{X}$ and $\mathbf{Y}$. For any $C \in \mathrm{C}(\mathbf{X}, \mathbf{Y})$, let its *distortion* be defined as follows:

$$\mathrm{dis}_{\mathbf{X},\mathbf{Y}}\, C := \sup\big\{|\mathrm{d}_X(x, x') - \mathrm{d}_Y(y, y')| \,:\, (x, y), (x', y') \in C\big\}.$$

When the setting is clear, we will simply write $\mathrm{dis}\, C := \mathrm{dis}_{\mathbf{X},\mathbf{Y}}\, C$. Observe that $\mathrm{dis}\, C \le 2\big(|X| \vee |Y|\big) < \infty$ and that $\mathrm{dis}\, C \ge \big||X| - |Y|\big|$.

For any finite Borel measure $\pi$ on $X \times Y$, we define its *discrepancy* with respect to $\mu_X$ and $\mu_Y$ as:

$$\mathrm{D}(\pi; \mu_X, \mu_Y) := \|\mu_X - \pi \circ p_X^{-1}\|_{\mathrm{TV}} + \|\mu_Y - \pi \circ p_Y^{-1}\|_{\mathrm{TV}}$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm, and $p_X : (x, y) \in X \times Y \mapsto x$, $p_Y : (x, y) \in X \times Y \mapsto y$ are the canonical projections from $X \times Y$ to $X$ and $Y$ respectively. The definition of the total variation norm and the triangular inequality give $\mathrm{D}(\pi; \mu_X, \mu_Y) \ge |\mu_X(X) - \mu_Y(Y)|$.

Following [5, Section 2.1], we define the *Gromov-Hausdorff-Prokhorov* distance (or GHP distance for short) between two pointed weighted compact metric spaces $\mathbf{X}$ and $\mathbf{Y}$ as:

$$\mathrm{d}_{\mathrm{GHP}}(\mathbf{X}, \mathbf{Y}) := \inf\left\{\frac{1}{2}\,\mathrm{dis}\, C \vee \mathrm{D}(\pi; \mu_X, \mu_Y) \vee \pi(C^c) \,:\, C \in \mathrm{C}(X, Y), \pi \in \mathcal{M}(X \times Y)\right\}$$

where $C^c = X \times Y \setminus C$.

*Remark 1.3.2.* Observe that $\mathrm{d}_{\mathrm{GHP}}(\mathbf{X}, \mathbf{Y}) \le \big(|X| \vee |Y|\big) \vee \big(\mu_X(X) + \mu_Y(Y)\big)$ and is consequently finite. Moreover, $\mathrm{d}_{\mathrm{GHP}}(\mathbf{X}, \mathbf{Y}) \ge \big(1/2 \cdot \big||X| - |Y|\big|\big) \vee \big|\mu_X(X) - \mu_Y(Y)\big|$. Therefore, the functions $\mathbb{K} \to \mathbb{R}_+$, $\mathbf{X} \mapsto |\mathbf{X}|$ and $\mathbf{X} \mapsto \mu_X(X)$ are both continuous with respect to $\mathrm{d}_{\mathrm{GHP}}$.

As was mentioned in [5, Section 2.1], $\mathrm{d}_{\mathrm{GHP}}$ is a well-defined distance on $\mathbb{K}$ and $(\mathbb{K}, \mathrm{d}_{\mathrm{GHP}})$ is both complete and separable and thus, Polish. Furthermore, it was also noted that $\mathrm{d}_{\mathrm{GHP}}$ gives rise to the same topology as the GHP distance defined in [3].

*Rescaling compact metric spaces.* For all $m \ge 0$, let $\mathbf{0}^{(m)} := \big(\{\varnothing\}, d, \varnothing, m\delta_\varnothing\big) \in \mathbb{K}$ be the degenerate metric space only made out of its root on which a mass $m$ is put. For a pointed weighted metric space $\mathbf{X}$ and any non-negative real numbers $a$ and $b$, we will write $(aX, b\mu_X) := (X, a\mathrm{d}_X, \rho_X, b\mu_X)$. When $\mathbf{X}$ is in $\mathbb{K}$ and $\mu_X(X) = m$, we will use the convention $(0X, \mu_X) = \mathbf{0}^{(m)}$ (which makes sense since $(\varepsilon X, \mu_X)$ converges to $\mathbf{0}^{(m)}$ as $\varepsilon$ goes to 0 with respect to $\mathrm{d}_{\mathrm{GHP}}$).

**Lemma 1.3.1.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two elements of $\mathbb{K}$. For any non-negative real numbers $a$, $b$, $c$ and $d$:*

$$(i) \quad \mathrm{d}_{\mathrm{GHP}}\big((aX, b\mu_X), (cX, d\mu_X)\big) \le \big(|a - c|\,|X|\big) \vee \big(|b - d|\,\mu_X(X)\big),$$

*and* $$(ii) \quad \mathrm{d}_{\mathrm{GHP}}\big((aX, b\mu_X), (aY, b\mu_Y)\big) \le (a \vee b)\,\mathrm{d}_{\mathrm{GHP}}(\mathbf{X}, \mathbf{Y}).$$

*Proof.* $(i)$ Let $C = \{(x, x) : x \in X\} \in \mathrm{C}(X, X)$. We have

$$\mathrm{dis}_{(aX, b\mu_X), (cX, d\mu_X)}\, C = \sup\big\{|a\,\mathrm{d}_X(x, y) - c\,\mathrm{d}_X(x, y)| : x, y \in X\big\} \le 2|a - c|\,|X|.$$

Let $\pi \in \mathcal{M}(X \times X)$ be defined for all measurable $A \subset X \times X$ by $\pi(A) = \int_X \mathbb{1}_A\big((x, x)\big)\, b\mu_X(\mathrm{d}x)$. Then $\mathrm{D}(\pi; b\mu_X, d\mu_X) = |b - d|\mu_X(X)$ and $\pi(C^c) = 0$.

$(ii)$ For every correspondence $C \in \mathrm{C}(X, Y)$, we clearly have $\mathrm{dis}_{(aX, b\mu_X), (aY, b\mu_Y)}\, C = a\,\mathrm{dis}_{\mathbf{X},\mathbf{Y}}\, C$. No less clearly, for any finite measure $\pi$ on $X \times Y$, $\mathrm{D}(b\pi; b\mu_X, b\mu_Y) = b\mathrm{D}(\pi; \mu_X, \mu_Y)$. □

**Corollary 1.3.2.** *The function* $\mathbb{K} \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{K}$ *defined by* $(\mathbf{X}, a, b) \longmapsto (aX, b\mu_X)$ *is continuous for the product topology.*

*Concatenated compact metric spaces.* Let $(\mathbf{X}_i)_{i\in\mathcal{I}}$ be a countable family of pointed weighted metric spaces with $\mathbf{X}_i = (X_i, \mathrm{d}_i, \rho_i, \mu_i)$. Let $(X, \mathrm{d}, \rho, \mu)$ where:

— $X = \{\rho\} \sqcup \bigsqcup_{i\in\mathcal{I}} X_i$,

— d is defined by:

  — For all $i, j \in \mathcal{I}$, $\mathrm{d}(\rho, \rho_i) := \mathrm{d}(\rho_i, \rho_j) = 0$,

  — For all $i \in \mathcal{I}$, and $x, y \in X_i$, $\mathrm{d}(x, y) := \mathrm{d}_i(x, y)$,

  — For all $i \neq j$ and $x \in X_i$, $y \in X_j$, $\mathrm{d}(x, y) := \mathrm{d}_i(x, \rho_i) + \mathrm{d}_j(y, \rho_j)$,

— For any Borel subset $A$ of $X$, $\mu(A) = \sum_{i\in\mathcal{I}} \mu_i(A \cap X_i)$.

With a slight abuse of notation, we will consider $(X, \mathrm{d})$ to be the quotient metric space $X/\sim_{\mathrm{d}}$ where $x \sim_{\mathrm{d}} y$ iff $\mathrm{d}(x, y) = 0$. For each $i$ in $\mathcal{I}$, we will also identify $X_i$ with its image in $X$ by the quotient map. Write $\mathbf{X} =: \langle \mathbf{X}_i \, ; \, i \in \mathcal{I} \rangle$.

*Remark 1.3.3.* If $(\mathbf{T}_i)_{i\in\mathcal{I}}$ is a countable family of weighted $\mathbb{R}$-trees, then $\langle \mathbf{T}_i \, ; \, i \in \mathcal{I} \rangle$ is clearly an $\mathbb{R}$-tree itself.

**Lemma 1.3.3.** *For all* $i \geq 1$*, let* $\mathbf{X}_i = (X_i, \mathrm{d}_i, \rho_i, \mu_i)$ *be in* $\mathbb{K}$*. Their concatenation* $\langle \mathbf{X}_i \, ; \, i \geq 1 \rangle$ *is an element of* $\mathbb{K}$ *iff the height* $|X_i|$ *of* $X_i$ *goes to* 0 *as* $i$ *goes to infinity and* $\sum_{i\geq 1} \mu_i(X_i)$ *is finite.*

*Proof.* Set $\mathbf{X} := \langle \mathbf{X}_i \, ; \, i \geq 1 \rangle$ and for all $x$ in $X$ and positive $r$, denote the open ball of $X$ centred at $x$ with radius $r$ by $\mathrm{B}_X(x, r) := \{y \in X : \mathrm{d}_X(x, y) < r\}$. Similarly, for all $i \geq 1$ and $x \in X_i$, write $\mathrm{B}_i(x, r) := \{y \in X_i : \mathrm{d}_i(x, y) < r\}$. Clearly, the measure $\mu_X$ is finite iff the sum $\sum_{i\geq 1} \mu_i(X_i)$ is.

If $|X_i| \to 0$, then in particular, for all positive $\varepsilon$, there exists an integer $n$ such that $\bigcup_{i>n} X_i \subset \mathrm{B}_X(\rho_X, \varepsilon)$. Moreover, since $X_i$ is compact for all $i = 1, \ldots, n$, we can find a finite $\varepsilon$-cover of $X_i$, i.e. a finite subset $A_i$ of $X_i$ such that $X_i \subset \bigcup_{x\in A_i} \mathrm{B}_i(x, \varepsilon)$. Set $A := \{\rho_X\} \cup A_1 \cup \cdots \cup A_n$. Observe that it is finite and that $X \subset \bigcup_{x\in A} \mathrm{B}_X(x, \varepsilon)$. Since this holds for all positive $\varepsilon$, it follows that $X$ is compact.

If $\limsup |X_i| > 0$, then there exists a positive $\varepsilon$ such that $|X_i| > \varepsilon$ for infinitely many indices $i$. As a result, $X$ cannot have a finite $\varepsilon$-cover, which implies that it is not compact. □

**Lemma 1.3.4.** *Let* $\mathbf{X}_i$*,* $\mathbf{Y}_i$*,* $i \geq 1$ *be in* $\mathbb{K}$ *and such that* $\mathbf{X} := \langle \mathbf{X}_i \, ; \, i \geq 1 \rangle$ *and* $\mathbf{Y} := \langle \mathbf{Y}_i \, ; \, i \geq 1 \rangle$ *both belong to* $\mathbb{K}$*. We have*

$$\mathrm{d}_{\mathrm{GHP}}\Big(\langle \mathbf{X}_i \, ; \, i \geq 1 \rangle, \langle \mathbf{Y}_i \, ; \, i \geq 1 \rangle\Big) \leq \sum_{i\geq 1} \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i, \mathbf{Y}_i).$$

*Proof.* Set $\mathbf{X} := \langle \mathbf{X}_i \, ; \, i \geq 1 \rangle$ and $\mathbf{Y} := \langle \mathbf{Y}_i \, ; \, i \geq 1 \rangle$. For all positive $\varepsilon$ and $i \geq 1$, there exists a correspondence $C_i$ in $\mathrm{C}(\mathbf{X}_i, \mathbf{Y}_i)$ and a finite Borel measure $\pi_i$ on $X_i \times Y_i$ such that

$$\frac{1}{2} \operatorname{dis} C_i \vee \mathrm{D}(\pi_i; \mu_{X_i}, \mu_{Y_i}) \vee \pi_i(C_i^c) < \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i, \mathbf{Y}_i) + 2^{-i}\varepsilon.$$

Set $C := \bigcup_{i\geq 1} C_i$, which is a correspondence between $\mathbf{X}$ and $\mathbf{Y}$. Let $(x, y)$ and $(x', y')$ be in $C$. If both $(x, y)$ and $(x', y')$ are in $C_i$ for some $i$, then clearly, $|\mathrm{d}_X(x, x') - \mathrm{d}_Y(y, y')| \leq \operatorname{dis} C_i$. Otherwise, if $(x, y) \in C_i$ and $(x', y') \in C_j$ with $i \neq j$, then using the definition of $\mathrm{d}_X$ and $\mathrm{d}_Y$ as well as the triangular inequality, we get $|\mathrm{d}_X(x, x') - \mathrm{d}_Y(y, y')| \leq \operatorname{dis} C_i + \operatorname{dis} C_j$. Therefore, $1/2 \cdot \operatorname{dis} C \leq \sum_{i\geq 1} \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i, \mathbf{Y}_i) + \varepsilon$.

For all $n \geq 0$, define the finite Borel measure $\pi^{(n)}$ on $X \times Y$ by $\pi^{(n)}(A) := \sum_{i=1}^n \pi_i\big[A \cap (X_i \times Y_i)\big]$ for any Borel set $A$. By definition,

$$\pi^{(n)}(C^c) = \sum_{i=1}^n \pi_i\big[C^c \cap (X_i \times Y_i)\big] = \sum_{i=1}^n \pi_i[C_i^c] \leq \sum_{i\geq 1} \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i, \mathbf{Y}_i) + \varepsilon.$$

Moreover, the discrepancy of $\pi^{(n)}$ with respect to $\mu_X$ and $\mu_Y$ satisfies

$$\begin{aligned}\mathrm{D}(\pi^{(n)};\mu_X,\mu_Y) &\leq \sum_{i=1}^n \|\mu_{X_i}-\pi_i\circ p_{X_i}^{-1}\|_{\mathrm{TV}}+\|\mu_{Y_i}-\pi_i\circ p_{Y_i}^{-1}\|_{\mathrm{TV}}+\sum_{j>n}\big(\|\mu_{X_j}\|_{\mathrm{TV}}+\|\mu_{Y_j}\|_{\mathrm{TV}}\big)\\ &\leq \sum_{i=1}^n \mathrm{D}(\pi_i;\mu_{X_i},\mu_{Y_i})+\sum_{j>n}\big(\mu_{X_j}(X_j)+\mu_{Y_j}(Y_j)\big)\\ &\leq \sum_{i\geq 1}^n \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i,\mathbf{Y}_i)+\varepsilon+\sum_{j>n}\big(\mu_{X_j}(X_j)+\mu_{Y_j}(Y_j)\big).\end{aligned}$$

In light of Lemma 1.3.3, there exists $n$ such that $\sum_{i>n}\mu_{X_i}(X_i)+\mu_{Y_i}(Y_i)<\varepsilon$. As a result, $\mathrm{d}_{\mathrm{GHP}}(\mathbf{X},\mathbf{Y})\leq \sum_{i\geq 1}\mathrm{d}_{\mathrm{GHP}}(\mathbf{X}_i,\mathbf{Y}_i)+2\varepsilon$ which holds for all positive $\varepsilon$. □

**Extension to locally compact $\mathbb{R}$-trees.** Let $\mathbf{X}=(X,\mathrm{d}_X,\rho_X,\mu_X)$ be a locally compact pointed weighted metric space such that $\mu_X$ is a boundedly finite measure. For all $r>0$, let $\mathbf{X}|_r:=\big(X|_r,\mathrm{d}_X,\rho_X,\mu_X|_r\big)$ where $X|_r:=\{x\in X:|x|\leq r\}$ is the closed ball with radius $r$ centred at $\rho_X$ and $\mu_X|_r:=\mathbb{1}_{X|_r}\mu_X$ is the restriction of $\mu_X$ to $X|_r$. Observe that if $r\leq R$, clearly $(\mathbf{X}|_R)|_r=(\mathbf{X}|_r)|_R=\mathbf{X}|_r$. We also define $\partial_r X:=\{x\in X:|x|=r\}$.

For any two locally compact pointed weighted metric spaces $\mathbf{X}$ and $\mathbf{Y}$, we define the extended Gromov-Hausdorff-Prokhorov distance between them as:

$$\mathrm{D}_{\mathrm{GHP}}(\mathbf{X},\mathbf{Y}):=\int_0^\infty \mathrm{e}^{-r}\Big[1\wedge \mathrm{d}_{\mathrm{GHP}}\big(\mathbf{X}|_r,\mathbf{Y}|_r\big)\Big]\mathrm{d}r.$$

This definition closely resembles that of the GHP distance on locally compact metric spaces defined and studied in [3].

*Remark 1.3.4.* Let $\mathbf{X}$ and $\mathbf{Y}$ be two weighted locally compact pointed metric spaces. For all $R\geq 0$,

$$\big|\mathrm{D}_{\mathrm{GHP}}(\mathbf{X},\mathbf{Y})-\mathrm{D}_{\mathrm{GHP}}(\mathbf{X}|_R,\mathbf{Y}|_R)\big|\leq \int_R^\infty \mathrm{e}^{-r}\underbrace{\big|1\wedge \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}|_r,\mathbf{Y}|_r)-1\wedge \mathrm{d}_{\mathrm{GHP}}(\mathbf{X}|_R,\mathbf{Y}|_R)\big|}_{\leq 1}\mathrm{d}r\leq \mathrm{e}^{-R}.$$

Let $\mathbb{T}$ be the set of GHP-isometry classes of locally compact rooted $\mathbb{R}$-trees endowed with a boundedly finite Borel measure and $\mathbb{T}_c$, be that of compact weighted and rooted $\mathbb{R}$-trees (i.e. $\mathbb{T}_c=\mathbb{K}\cap\mathbb{T}$).

**Proposition 1.3.5.** $(i)$ $\mathrm{D}_{\mathrm{GHP}}$ *is a metric on* $\mathbb{T}$,

$(ii)$ *If* $\mathbf{T}_n$, $n\geq 1$ *and* $\mathbf{T}$ *belong to* $\mathbb{T}$, *then* $\mathrm{D}_{\mathrm{GHP}}(\mathbf{T}_n,\mathbf{T})\to 0$ *iff* $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_n|_r,\mathbf{T}|_r)\to 0$ *for all* $r\geq 0$ *with* $\mu_T(\partial_r T)=0$,

$(iii)$ $(\mathbb{T},\mathrm{D}_{\mathrm{GHP}})$ *is a Polish metric space,*

$(iv)$ $\mathrm{d}_{\mathrm{GHP}}$ *and* $\mathrm{D}_{\mathrm{GHP}}$ *induce the same topology on* $\mathbb{T}_c$.

*Proof.* $(i)$ Since $\mathrm{d}_{\mathrm{GHP}}$ is a metric, $\mathrm{D}_{\mathrm{GHP}}$ is symmetric and clearly satisfies the triangular inequality. Moreover, if $\mathbf{T}$ and $\mathbf{T}'$ are two elements of $\mathbb{T}$ such that $\mathrm{D}_{\mathrm{GHP}}(\mathbf{T},\mathbf{T}')=0$, then for almost every $r\geq 0$, $\mathbf{T}|_r=\mathbf{T}'|_r$. In this case, $\mathbf{T}$ and $\mathbf{T}'$ are GHP-isometric (see [3, Proposition 5.3] for a similar proof).

$(ii)$ Suppose $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_n|_r,\mathbf{T}|_r)\to 0$ for all $r\geq 0$ with $\mu_T(\partial_r T)=0$. Since $\mu_T$ is a locally finite measure, the set $\{r>0:\mu_T(\partial_r T)>0\}$ is at most countable. As a result, the sequence $\big(r\mapsto 1\wedge \mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_n|_r,\mathbf{T}|_r)\big)_{n\geq 1}$ converges to $r\mapsto 0$ almost everywhere in $[0,\infty)$. Lebesgue's dominated convergence theorem then ensures that $\mathrm{D}_{\mathrm{GHP}}(\mathbf{T}_n,\mathbf{T})\to 0$.

Assume $\mathrm{D}_{\mathrm{GHP}}(\mathbf{T}_n,\mathbf{T})\to 0$ and let $r>0$ be such that $\mu_T(\partial_r T)=0$. For every subsequence $(n_k)_k$, there exists a sub-subsequence $(k_\ell)_\ell$ such that $1\wedge \mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_{n_{k_\ell}}|_t,\mathbf{T}|_t)\to 0$ for almost every $t\geq 0$ as $\ell\to\infty$. In particular, there exists $R>r$ such that $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_{n_{k_\ell}}|_R,\mathbf{T}|_R)\to 0$.

Recall that $\mathrm{d}_{\mathrm{GHP}}$ is topologically equivalent to the metric on $\mathbb{K}$ studied in [3]. Therefore, in light of the proof of [3, Proposition 2.10], if $\tau_n$, $n\geq 1$ and $\tau$ are compact $\mathbb{R}$-trees such that $\mathrm{d}_{\mathrm{GHP}}(\tau_n,\tau)\to 0$, then for all $r>0$ such that $\mu_\tau(\partial_r\tau)=0$, $\mathrm{d}_{\mathrm{GHP}}(\tau_n|_r,\tau|_r)\to 0$.

As a result, $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_{n_{k_\ell}}|_r,\mathbf{T}|_r)\to 0$. From every subsequence $(n_k)_k$ we can thus extract a sub-subsequence $(k_\ell)_\ell$ such that $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_{n_{k_\ell}}|_r,\mathbf{T}|_r)\to 0$, which is equivalent to saying that $\mathrm{d}_{\mathrm{GHP}}(\mathbf{T}_n|_r,\mathbf{T}|_r)\to 0$ as $n\to\infty$.

$(iii)$ Since a criterion similar to $(ii)$ holds for the metric studied in [3], this metric is topologically equivalent to $\mathrm{D}_{\mathrm{GHP}}$. As a result and thanks to Theorem 2.9 and Corollary 3.2 in [3], it follows that $(\mathbb{T}, \mathrm{D}_{\mathrm{GHP}})$ is completely metrisable and separable, i.e. it is Polish.

$(iv)$ See Proposition 2.10 in [3]. □

*Continuum grafting.* Let $\{(u_i, \tau_i) : i \in \mathcal{I}\}$ be a family of elements of $\mathbb{R}_+ \times \mathbb{T}_c$ such that $\mathcal{I}$ is at most countable. We define the $\mathbb{R}$-tree $\mathbf{G}\big(\{(u_i, \tau_i) : i \in \mathcal{I}\}\big)$ as

$$\mathbf{G}\Big(\big\{(u_i, \tau_i) : i \in \mathcal{I}\big\}\Big) := \Big(\mathbb{R}_+ \sqcup \bigsqcup_{i\in\mathcal{I}} \tau_i,\, \mathrm{d},\, 0,\, \mu\Big)$$

where the metric $\mathrm{d}$ is defined by:

— $\mathrm{d}[u, v] = |u - v|$ for all $u$ and $v$ in $\mathbb{R}_+$,
— $\mathrm{d}[x, y] = \mathrm{d}_{\tau_i}(x, y)$ for all $i \in \mathcal{I}$, $x$ and $y$ in $\tau_i$,
— $\mathrm{d}[x, v] = \mathrm{d}_{\tau_i}(x, \rho_{\tau_i}) + |u_i - v|$ for all $i \in \mathcal{I}$, $x \in \tau_i$ and $v$ in $\mathbb{R}_+$,
— $\mathrm{d}[x, y] = \mathrm{d}_{\tau_i}(x, \rho_{\tau_i}) + \mathrm{d}_{\tau_j}(y, \rho_{\tau_j}) + |u_i - u_j|$ for all $i \neq j \in \mathcal{I}$, $x \in \tau_i$ and $y \in \tau_j$,

and $\mu$ is the measure defined for all Borel sets $A$ by $\mu(A) := \sum_{i\in\mathcal{I}} \mu_{\tau_i}(A \cap \tau_i)$. The function $\mathbf{G}$ grafts the trees $\tau_i$ at height $u_i$ for each $i \in \mathcal{I}$ on $\mathbb{R}_+$ which can be thought of as an infinite (continuous) branch. It is quite obvious that the weighted pointed metric space $\mathbf{G}\big(\{(u_i, \tau_i) : i \in \mathcal{I}\}\big)$ is an $\mathbb{R}$-tree.

**Lemma 1.3.6.** *Let $(u_i)_{i\geq 1}$ be a sequence of non-negative real numbers and $(\tau_i, \mathrm{d}_i, \rho_i, \mu_i)_{i\geq 1}$ be a sequence of compact weighted $\mathbb{R}$-trees. The weighted $\mathbb{R}$-tree $\mathbf{T} := \mathbf{G}\big(\{(u_i, \tau_i) : i \geq 1\}\big)$ is an element of $\mathbb{T}$ iff for all $K \geq 0$ and $\varepsilon > 0$ the set $\{i \geq 1 : u_i \leq K \text{ and } |\tau_i| \geq \varepsilon\}$ is finite and $\sum_{i\geq 1} \mathbb{1}_{u_i\leq K} \mu_{\tau_i}(\tau_i) < \infty$.*

*Proof.* For all $x$ in $T$ and positive $r$, denote by $\mathrm{B}_T(x, r) := \{y \in T : \mathrm{d}_T(x, y) < r\}$ the open ball of $T$ centred at $x$ with radius $r$ and similarly for all $i \geq 1$ and $x \in \tau_i$, write $\mathrm{B}_i(x, r) := \{y \in \tau_i : \mathrm{d}_i(x, y) < r\}$.

$\Leftarrow$ Assume that for all $K \geq 0$, $\sum_{i\geq 1} \mathbb{1}_{u_i\leq K} \mu_{\tau_i}(\tau_i) < \infty$ and for all positive $\varepsilon$, that the set $\{i \geq 1 : u_i \leq K, |\tau_i| \geq \varepsilon\}$ is finite. Observe that for all non-negative $K$, $\mu_T(T|_K) \leq \sum_{i\geq 1} \mu_{\tau_i}(\tau_i)\mathbb{1}_{u_i\leq K}$. Therefore, the measure $\mu_T$ is boundedly finite and we only need to prove that $T$ is locally compact.

Fix $K \geq 0$ and let $\varepsilon$ be positive. For all $i \geq 1$, because $\tau_i$ is compact, there exists a finite subset $A_i$ of $\tau_i$ such that $\tau_i \subset \bigcup_{x\in A_i} B_i(x, \varepsilon)$. To build an $\varepsilon$-cover of $T|_K$, first observe that if $i$ is such that $u_i \leq K$ and $|\tau_i| < \varepsilon/2$, then $\tau_i$ is contained in some open ball with radius $\varepsilon$ centred at some $n\varepsilon$ for $0 \leq n \leq K/\varepsilon$. Moreover, by assumption, there are only finitely many indices $i$ with $u_i \leq K$ and $|\tau_i| \geq \varepsilon/2$. Therefore, if we let $A := \{n\varepsilon; 0 \leq n \leq K/\varepsilon\} \cup \{x \in A_i; i \geq 1, u_i \leq K, |\tau_i| \geq \varepsilon/2\}$, then $A$ is finite and $T|_K$ is contained in $\bigcup_{x\in A} B_T(x, \varepsilon)$. As a result, $T|_K$ has a finite $\varepsilon$-cover for all positive $\varepsilon$ which means that it is compact.

$\Rightarrow$ Suppose the set $\{i \geq 1 : u_i \leq K, |\tau_i| \geq \varepsilon\}$ is infinite for some $K \geq 0$ and positive $\varepsilon$. In particular, we can find an increasing sequence $(i_n)_n$ with $u_{i_n} \leq K$ and $|\tau_{i_n}| \geq \varepsilon$ for all $n$. For each $n \geq 1$, let $x_n$ be in $\tau_{i_n}$ and such that $\varepsilon/2 < \mathrm{d}_{i_n}(\rho_{i_n}, x_n) \leq \varepsilon$. If $n \neq m$, the definition of the metric on $T$ gives $\mathrm{d}_T(x_n, x_m) > \varepsilon$. Therefore, $(x_n)_n$ has no Cauchy subsequence which implies that $T|_{K+\varepsilon}$ isn't compact and that $\mathbf{T} \notin \mathbb{T}$.

Assume that $\{i \geq 1 : u_i \leq K, |\tau_i| \geq \varepsilon\}$ is finite for all $K \geq 0$ and $\varepsilon > 0$, and that $\sum_{i\geq 1} \mathbb{1}_{u_i\leq K_0} \mu_{\tau_i}(\tau_i)$ is infinite for some finite $K_0$. By assumption, $\{|\tau_i| : u_i \leq K_0\}$ is bounded by a finite constant $R$. Therefore, $\mu_T(T|_{K_0+R}) \geq \sum_{i\geq 1} \mathbb{1}_{u_i\leq K_0} \mu_{\tau_i}(\tau_i) = \infty$. Consequently, $\mu_T$ isn't boundedly finite and $\mathbf{T} \notin \mathbb{T}$. □

*Remark 1.3.5.* In the following, when we consider discrete trees, we will see them as $\mathbb{R}$-trees by replacing their edges by segments of length 1.

### 1.3.2 Fragmentation trees

In this section, we will present a few results on certain classes of $\mathbb{T}_c$- and $\mathbb{T}$-valued random variables: self-similar fragmentation trees (introduced in [64]) and self-similar fragmentation trees with immigration (see [62]).

**Self-similar fragmentation trees.** Let $\mathcal{S}^{\downarrow} := \{\mathbf{s} = (s_n)_{n\geq 1} \in \ell_1 \,:\, s_1 \geq s_2 \geq \cdots \geq 0\}$ and endow it with the $\ell_1$ norm, i.e. for all $\mathbf{s}$ and $\mathbf{r}$ in $\mathcal{S}^{\downarrow}$, say that the distance between $\mathbf{s}$ and $\mathbf{r}$ is $\|\mathbf{s}-\mathbf{r}\| = \sum_{i\geq 1} |s_i - r_i|$. Moreover, set $\mathbf{0} := (0,0,\dots)$, $\mathbf{1} := (1,0,0,\dots)$ and $\mathcal{S}^{\downarrow}_{\leq 1} := \{\mathbf{s} \in \mathcal{S}^{\downarrow} \,:\, \|\mathbf{s}\| \leq 1\}$.

A *self-similar fragmentation process* is an $\mathcal{S}^{\downarrow}_{\leq 1}$-valued Markovian process $(\mathbf{X}(t); t \geq 0)$ which is continuous in probability, and satisfies $\mathbf{X}(0) = \mathbf{1}$ as well as the following so-called *fragmentation property*. There exists $\alpha \in \mathbb{R}$ such that for all $t_0 \geq 0$, conditionally to $\mathbf{X}(t_0) = \mathbf{s}$, $\big(\mathbf{X}(t_0 + t), t \geq 0\big)$ has the same distribution as

$$\Big(\big(s_i \mathbf{X}^{(i)}(s_i^{\alpha} t), \, i \geq 1\big)^{\downarrow} \,;\, t \geq 0\Big)$$

where $(\mathbf{X}^{(i)})_{i\geq 1}$ are i.i.d. copies of $\mathbf{X}$. The constant $\alpha$ is called the *self-similarity index* of the process $\mathbf{X}$.

These processes can be seen as the evolution of the fragmentation of an object of mass 1 into smaller objects which will each, in turn, split themselves apart independently from one another, at a rate proportional to their mass to the power $\alpha$.

It was shown in [16, 19] that the distribution of a self-similar fragmentation process is characterised by a 3-tuple $(\alpha, c, \nu)$ where $\alpha$ is the aforementioned self-similarity index, $c \geq 0$ is a so-called erosion coefficient which accounts for a continuous decay in the mass of each particle and $\nu$ is a *dislocation measure* on $\mathcal{S}^{\downarrow}_{\leq 1}$, i.e. a $\sigma$-finite measure such that $\int (1 - s_1)\, \nu(\mathrm{d}\mathbf{s}) < \infty$ and $\nu(\{\mathbf{1}\}) = 0$. Informally, at any given time, each particle with mass say $x$ will, independently from the other particles, split into smaller fragments of respective masses $xs_1, xs_2, \dots$ at rate $x^{\alpha}\nu(\mathrm{d}\mathbf{s})$.

We will be interested in fragmentation processes with negative self-similarity index $-\gamma < 0$ with no erosion, i.e. with $c = 0$. Furthermore, we will require the dislocation measure $\nu$ to be non-trivial, i.e. $\nu(\mathcal{S}^{\downarrow}_{\leq 1}) > 0$, and *conservative*, that is to satisfy $\nu(\|\mathbf{s}\| < 1) = 0$. Therefore, the fragmentation processes we will consider will be characterised by a *fragmentation pair* $(\gamma, \nu)$ and we will refer to them as $(\gamma, \nu)$-fragmentation processes.

Under these assumptions, each particle will split into smaller ones which will in turn break down faster, thus speeding up the global fragmentation rate. Let $\mathbf{X}$ be a $(\gamma, \nu)$-fragmentation process and set $\tau_{\mathbf{0}} := \inf\{t \geq 0 : \mathbf{X}(t) = \mathbf{0}\}$ the first time at which all the mass has been turned to dust. It was shown in [20, Proposition 2] that $\tau_{\mathbf{0}}$ is a.s. finite and in [59, Section 5.3] that it has exponential moments, i.e. that there exists $a > 0$ such that $\mathbb{E}\big[\exp(a\tau_{\mathbf{0}})\big] < \infty$.

Furthermore, a $\mathbb{T}_c$-valued random variable that encodes the genealogy of the fragmentation of the initial object was defined in [64]. This random $\mathbb{R}$-tree $(\mathcal{T}, \mathrm{d}, \rho, \mu)$ is such that $\mu(\mathcal{T}) = 1$ and if for all $t \geq 0$, $\{\mathcal{T}_i(t) : i \geq 1\}$ is the (possibly empty) set of the closures of the connected components of $\mathcal{T} \setminus (\mathcal{T}|_t)$, then

$$\Big(\big(\mu[\mathcal{T}_i(t)]\,;\, i \geq 1\big)^{\downarrow} \,;\, t \geq 0\Big)$$

is a $(\gamma, \nu)$-fragmentation process. We will denote the distribution of $(\mathcal{T}, \mathrm{d}, \rho, \mu)$ by $\mathcal{T}_{\gamma,\nu}$.

*Remark 1.3.6.*
- More general self-similar fragmentation trees, where both the assumptions "$c = 0$" and "$\nu$ is conservative" are dropped, were defined and studied in [116].
- Let $\mathcal{T}$ be a $(\gamma, \nu)$-self-similar fragmentation tree and $m > 0$. The tree $(m^{\gamma}\mathcal{T}, m\,\mu_{\mathcal{T}})$ encodes the genealogy of a $(\gamma, \nu)$-self-similar fragmentation process started from a single object with mass $m$.

*Classical examples.* It was observed in [19] that the Brownian tree, which was introduced in [7], may be described as a self-similar fragmentation tree with parameters $(1/2, \nu_B)$ where $\nu_B$ is called the *Brownian dislocation measure* and is defined for all measurable $f : \mathcal{S}^{\downarrow}_{\leq 1} \to \mathbb{R}_+$ by

$$\int f \,\mathrm{d}\nu_B = \int_{1/2}^{1} \left(\frac{2}{\pi x^3 (1-x)^3}\right)^{1/2} f(x, 1-x, 0, 0, \dots)\,\mathrm{d}x.$$

Another important example of fragmentation trees is the family of $\alpha$-stable trees from [49], where $\alpha$ belongs to $(1,2)$. Indeed, a result from [98] states that the $\alpha$-stable tree is a $(1-1/\alpha, \nu_\alpha)$-self-similar

fragmentation tree with $\nu_\alpha$ defined as follows: let $(\Sigma_t; t \geq 0)$ be a $1/\alpha$-stable subordinator with Laplace exponent $\lambda \mapsto -\log \mathbb{E}[\exp(-\lambda\Sigma_1)] = \lambda^{1/\alpha}$ and Lévy measure $\Pi_{1/\alpha}(\mathrm{d}t) := [\alpha\,\Gamma(1-1/\alpha)]^{-1}\, t^{-1-1/\alpha}\,\mathbb{1}_{t>0}\,\mathrm{d}t$, denote the decreasing rearrangement of its jumps on $[0,1]$ by $\Delta$ and for all measurable $f : \mathbb{S}^\downarrow \to \mathbb{R}_+$, let

$$\int_{\mathbb{S}^\downarrow} f \,\mathrm{d}\nu_\alpha = \frac{\Gamma(1-1/\alpha)}{k_\alpha}\mathbb{E}\Big[\Sigma_1 f\big(\Delta/\Sigma_1\big)\Big]$$

where $k_\alpha := \Gamma(2-\alpha)/[\alpha(\alpha-1)]$. Observe that the random point measure $\sum_{i\geq 1}\delta_{\Delta_i}$ on $(0,\infty)$ with atoms $(\Delta_i, i \geq 1)$ is a Poisson Point Process with intensity measure $\Pi_{1/\alpha}$.

*Scaling limits of Markov branching trees.* Self-similar fragmentation trees bear a close relationship with Markov branching trees. Let $\iota : \mathcal{P}_{<\infty} \to \mathbb{S}_1^\downarrow$ be such that if $\lambda = (\lambda_1, \ldots, \lambda_p)$ is in $\mathcal{P}_n$, then $\iota(\lambda) := (\lambda_1/n, \ldots, \lambda_p/n, 0, 0, \ldots)$.

**Theorem 1.3.7 ([66], Theorems 5 and 6).** — *Let $(q_n)_{n\in\mathbb{N}}$ be the sequence of first-split distributions of a Markov branching family $\mathrm{MB}^{\mathcal{L},q}$ and for all adequate $n \geq 1$, set $\bar{q}_n := q_n \circ \iota^{-1}$. Suppose there exists a fragmentation pair $(\gamma, \nu)$ and a slowly varying function $\ell$ such that, for the weak convergence of finite measures on $\mathbb{S}^\downarrow$,*

$$n^\gamma \ell(n)\,(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \xrightarrow[n\to\infty]{} (1-s_1)\,\nu(\mathrm{d}\mathbf{s}).$$

*For all $n \in \mathbb{N}$, let $T_n$ have distribution $\mathrm{MB}_n^{\mathcal{L},q}$ and set $\mu_n := \sum_{u\in\mathcal{L}(T_n)}\delta_u$ the counting measure on the leaves of $T_n$.*

— *Let $(q_{n-1})_{n\in\mathbb{N}}$ be the sequence associated to a Markov branching family $\mathrm{MB}^q$. Assume that there exists a fragmentation pair $(\gamma, \nu)$ and a slowly varying function $\ell$ with either $\gamma < 1$ or $\gamma = 1$ and $\ell(n) \to 0$ such that $n^\gamma \ell(n)\,(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu(\mathrm{d}\mathbf{s})$. For each $n \in \mathbb{N}$, let $T_n$ be a $\mathrm{MB}_n^q$ tree and endow it with its counting measure $\mu_n$.*

*Under either set of assumptions, with respect to the GHP topology on $\mathbb{T}_c$,*

$$\left(\frac{1}{n^\gamma\ell(n)}T_n, \frac{1}{n}\mu_n\right) \xrightarrow[n\to\infty]{} \mathcal{T}_{\gamma,\nu} \quad \textit{in distribution.}$$

The following useful result on the heights of Markov branching also holds.

**Lemma 1.3.8.** *Suppose that $(q_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Theorem 1.3.7 with respect to a given fragmentation pair $(\gamma, \nu)$ and a slowly varying function $\ell$. Then for any $p > 0$, there is a finite constant $h_p$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\frac{|T_n|}{n^\gamma\ell(n)}\right)^p\right] \leq h_p \quad \textit{and} \quad \mathbb{E}\big[|\mathcal{T}|^p\big] \leq h_p$$

*where $\mathcal{T}$ is a $(\gamma, \nu)$-fragmentation tree and, as in Theorem 1.3.7, $T_n$ has distribution either $\mathrm{MB}_n^q$ or $\mathrm{MB}_n^{\mathcal{L},q}$.*

*Proof.* See [59, Section 5.3] for the continuous setting and [66, Lemma 33] plus [66, Section 4.5] for the discrete one. □

*Concatenation of fragmentation trees.* Fix a fragmentation pair $(\gamma, \nu)$ and let $(\mathcal{T}_i)_{i\geq 1}$ be a sequence of i.i.d. $(\gamma, \nu)$-fragmentation trees. For all $i \geq 1$, call $\mu_i$ the measure of $\mathcal{T}_i$. Fix $\mathbf{s}$ in $\mathbb{S}^\downarrow$ and set $(\mathcal{T}_{\langle\mathbf{s}\rangle}, \mu_{\langle\mathbf{s}\rangle}) := \big\langle (s_i^\gamma \mathcal{T}_i, s_i\mu_i)\,;\, i \geq 1\big\rangle$.

**Lemma 1.3.9.** *With these notations, $(\mathcal{T}_{\langle\mathbf{s}\rangle}, \mu_{\langle\mathbf{s}\rangle})$ a.s. belongs to $\mathbb{T}_c$.*

*Proof.* Clearly $\mathcal{T}_{\langle\mathbf{s}\rangle}$ is an $\mathbb{R}$-tree and its total mass is $\mu_{\langle\mathbf{s}\rangle}(\mathcal{T}_{\langle\mathbf{s}\rangle}) = \sum_{i\geq 1} s_i\mu_i(\mathcal{T}_i) = \|\mathbf{s}\|$ which is finite. It only remains to show that it is compact or, in light of Lemma 1.3.3, that $s_i^\gamma\,|\mathcal{T}_i|$ a.s. converges to 0 as $i$ grows to infinity. Since $\mathbf{s}$ is summable, for any positive $\varepsilon$,

$$\sum_{i\geq 1}\mathbb{P}\big[s_i^\gamma|\mathcal{T}_i| > \varepsilon\big] \leq \sum_{i\geq 1}\frac{s_i}{\varepsilon^{1/\gamma}}\mathbb{E}\big[|\mathcal{T}_1|^{1/\gamma}\big] \leq \frac{1}{\varepsilon^{1/\gamma}}\mathbb{E}\big[|\mathcal{T}_1|^{1/\gamma}\big]\|\mathbf{s}\| < \infty$$

where we have used Markov's inequality and the fact that $|\mathcal{T}_i|^{1/\gamma} \in L^1$ (see Lemma 1.3.8). Borel-Cantelli's lemma then allows us to deduce that $s_i^\gamma |\mathcal{T}_i| \to 0$ a.s. as $i \to \infty$. □

**Lemma 1.3.10.** *For all fixed* $\mathbf{s}$ *in* $\mathcal{S}^\downarrow$, $\mathbb{E}\big[\mathrm{d}_{\mathrm{GHP}}(\mathcal{T}_{\langle \mathbf{s}\rangle}, \mathcal{T}_{\langle \mathbf{r}\rangle})\big]$ *converges to* 0 *as* $\mathbf{r} \to \mathbf{s}$ *in* $\mathcal{S}^\downarrow$.

*Proof.* For all $n \geq 0$, in light of Lemmas 1.3.1 and 1.3.4,

$$\mathrm{d}_{\mathrm{GHP}}\big(\mathcal{T}_{\langle \mathbf{s}\rangle}, \mathcal{T}_{\langle \mathbf{r}\rangle}\big) \leq \sum_{i=1}^{n} \Big[\big(|s_i^\gamma - r_i^\gamma| \,|\mathcal{T}_i|\big) \vee |s_i - r_i|\Big] + \sum_{i>n} \big(s_i + r_i\big) + \sup_{i>n}\Big(s_i^\gamma |\mathcal{T}_i|\Big) + \sup_{i>n}\Big(r_i^\gamma |\mathcal{T}_i|\Big).$$

If $\gamma \leq 1$, $t \mapsto t^\gamma$ is concave, hence Jensen's inequality gives

$$\mathbb{E}\Big[\sup_{i>n}\Big(s_i^\gamma |\mathcal{T}_i|\Big)\Big] = \mathbb{E}\Big[\Big(\sup_{i>n} s_i |\mathcal{T}_i|^{1/\gamma}\Big)^\gamma\Big] \leq \Big(\mathbb{E}\Big[\sup_{i>n} s_i |\mathcal{T}_i|^{1/\gamma}\Big]\Big)^\gamma \leq \mathbb{E}\big[|\mathcal{T}_1|^{1/\gamma}\big]^\gamma \Big(\textstyle\sum_{i>n} s_i\Big)^\gamma,$$

otherwise, if $\gamma > 1$, since $(s_i)$ is non-increasing, for all $i > n$, $s_i^\gamma \leq s_{n+1}^{\gamma-1} s_i$ which implies

$$\mathbb{E}\Big[\sup_{i>n}\Big(s_i^\gamma |\mathcal{T}_i|\Big)\Big] \leq s_{n+1}^{\gamma-1}\, \mathbb{E}\Big[\sup_{i>n}\Big(s_i |\mathcal{T}_i|\Big)\Big] \leq \mathbb{E}\big[|\mathcal{T}_1|\big] s_{n+1}^{\gamma-1} \sum_{i>n} s_i \leq \mathbb{E}\big[|\mathcal{T}_1|\big]\Big(\textstyle\sum_{i>n} s_i\Big)^\gamma.$$

Consequently, there is a constant $C \geq 0$ such that for all integer $n$ and $\mathbf{s}$ in $\mathcal{S}^\downarrow$, $\mathbb{E}\big[\sup_{i>n} s_i^\gamma |\mathcal{T}_i|\big] \leq C\big[\sum_{i>n} s_i\big]^\gamma$. Hence, for all $\mathbf{s}$ and $\mathbf{r}$ in $\mathcal{S}^\downarrow$ and any $n \geq 1$

$$\mathbb{E}\Big[\mathrm{d}_{\mathrm{GHP}}(\mathcal{T}_{\langle \mathbf{s}\rangle}, \mathcal{T}_{\langle \mathbf{r}\rangle})\Big] \leq \|\mathbf{s} - \mathbf{r}\| + \mathbb{E}\big[|\mathcal{T}_1|\big] \sum_{i=1}^{n} |s_i^\gamma - r_i^\gamma| + \sum_{i>n}\big(s_i + r_i\big) + C\Big[\Big(\sum_{i>n} s_i\Big)^\gamma + \Big(\sum_{i>n} r_i\Big)^\gamma\Big].$$

As a result,

$$\limsup_{\mathbf{r}\to\mathbf{s}} \mathbb{E}\Big[\mathrm{d}_{\mathrm{GHP}}(\mathcal{T}_{\langle \mathbf{s}\rangle}, \mathcal{T}_{\langle \mathbf{r}\rangle})\Big] \leq \inf_{n\geq 1} 2\textstyle\sum_{i>n} s_i + 2C\Big(\sum_{i>n} s_i\Big)^\gamma = 0.$$

□

**Fragmentation trees with immigration.** We say that a non-negative Borel measure $I$ on $\mathcal{S}^\downarrow$ is an *immigration measure* if it satisfies $\int_{\mathcal{S}^\downarrow} (1 \wedge \|\mathbf{s}\|) I(\mathrm{d}\mathbf{s}) < \infty$ and $I(\{\mathbf{0}\}) = 0$.

Fix an immigration measure $I$ such that $I(\mathcal{S}^\downarrow) > 0$ and let $(\gamma, \nu)$ be a fragmentation pair. Let $\Sigma = \sum_{n\geq 1} \delta_{(u_n, \mathbf{s}_n)}$ be a Poisson point process on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ with intensity $\mathrm{d}u \otimes I(\mathrm{d}\mathbf{s})$ independent of a family $(\mathbf{X}^{(n,k)}, n \geq 1, k \geq 1)$ of i.i.d. $(\gamma, \nu)$-fragmentation processes. Define the $\mathcal{S}^\downarrow$-valued process $\mathbf{X}$ as follows:

$$\mathbf{X} = \big(\mathbf{X}(t),\, t \geq 0\big) := \left(\left(s_{n,k} \mathbf{X}^{(n,k)}\big[s_{n,k}^{-\gamma}(t - u_n)\big];\, n \geq 1 : u_n \leq t,\, k \geq 1\right)^\downarrow;\; t \geq 0\right).$$

We call $\mathbf{X}$ a fragmentation process with immigration with parameters $(\gamma, \nu, I)$. It describes the evolution of the masses of a cluster of independently fragmenting objects, where new objects of sizes $\mathbf{s}_n$ appear, or immigrate, at time $u_n$. These processes were introduced in [61].

Similarly to pure fragmentation processes, the genealogy of these immigrations and fragmentations can be encoded as an infinite weighted $\mathbb{R}$-tree (see [62]), say $(\mathcal{T}^{(I)}, \mathrm{d}, \rho, \mu)$, such that if for all $t \geq 0$, we denote the set of the closures of the bounded connected components of $\mathcal{T}^{(I)} \setminus (\mathcal{T}^{(I)}|_t)$ by $\{\mathcal{T}_i(t) : i \geq 1\}$, then

$$\Big(\big(\mu[\mathcal{T}_i(t)];\, i \geq 1\big)^\downarrow;\; t \geq 0\Big)$$

is a $(\gamma, \nu, I)$-fragmentation process with immigration. Let $\mathcal{T}^I_{\gamma,\nu}$ be the distribution of $(\mathcal{T}^{(I)}, \mathrm{d}, \rho, \mu)$.

*Point process construction.* The construction of $(\gamma, \nu)$-fragmentation trees with immigration $I$ described in [62] can be expressed using Poisson point processes, concatenated $(\gamma, \nu)$-fragmentation trees and the continuum grafting function $\mathbf{G}$ from the end of Section 1.3.1. Let $\Sigma = \sum_{i\geq 1} \delta_{(u_i, \mathbf{s}_i)}$ be a Poisson point process on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ with intensity $\mathrm{d}u \otimes I(\mathrm{d}\mathbf{s})$ and $(\mathcal{T}_{i,j}, \mu_{i,j})_{i,j\geq 1}$ be i.i.d. $(\gamma, \nu)$-fragmentation trees independent of $\Sigma$. For all $i \geq 1$, set

$$\mathcal{T}_i := \Big\langle (s_{i,j}^\gamma \mathcal{T}_{i,j}, s_{i,j}\mu_{i,j}); j \geq 1 \Big\rangle,$$

the concatenation of $(\mathcal{T}_{i,j}; j \geq 1)$ with respective masses $s_{i,j}$. Define $\mathcal{T}^{(I)}$ as the tree obtained by grafting $\mathcal{T}_i$ at height $u_i$ on an infinite branch for each $i \geq 1$, i.e.

$$\mathcal{T}^{(I)} := \mathbf{G}\Big(\big\{(u_i, \mathcal{T}_i) \,:\, i \geq 1\big\}\Big).$$

The random tree $\mathcal{T}^{(I)}$ has distribution $\mathcal{T}^I_{\gamma,\nu}$.

Observe that for all $K \geq 0$, we can write the total mass grafted on the infinite branch at height less than $K$ as an integral against the point-process $\Sigma$:

$$\sum_{i\geq 1} \mathbb{1}_{u_i \leq K} \mu_{\mathcal{T}_i}(\mathcal{T}_i) = \sum_{i\geq 1} \mathbb{1}_{u_i \leq K} \|\mathbf{s}_i\| = \int \mathbb{1}_{u\leq K} \|\mathbf{s}\| \, \Sigma(\mathrm{d}u, \mathrm{d}\mathbf{s}).$$

Since $\int 1 \wedge \big(\mathbb{1}_{u\leq K} \|\mathbf{s}\|\big) \mathrm{d}u \, I(\mathrm{d}\mathbf{s}) = K \int (1 \wedge \|\mathbf{s}\|) I(\mathrm{d}\mathbf{s}) < \infty$, we may use Campbell's theorem (see [84, Section 3.2]) and claim that $\int \mathbb{1}_{u\leq K} \|\mathbf{s}\| \, \Sigma(\mathrm{d}u, \mathrm{d}\mathbf{s}) < \infty$ a.s.. The second condition of Lemma 1.3.6 is thus met. Moreover, for all $i \geq 1$,

$$\mathbb{E}\Big[|\mathcal{T}_i|^{1/\gamma}\Big|\Sigma\Big] = \mathbb{E}\Big[\sup_{j\geq 1} s_{i,j}|\mathcal{T}_{i,j}|^{1/\gamma}\Big|\Sigma\Big] \leq \sum_{j\geq 1} s_{i,j} \mathbb{E}\Big[|\mathcal{T}_{1,1}|^{1/\gamma}\Big] = \mathbb{E}\Big[|\mathcal{T}_{1,1}|^{1/\gamma}\Big] \|\mathbf{s}_i\|$$

where we have used the fact that $(\mathcal{T}_{i,j})_{i,j}$ is an i.i.d. family independent of $\Sigma$. Markov's inequality therefore implies that

$$\sum_{i\geq 1} \mathbb{1}_{u_i\leq K} \mathbb{P}\big[|\mathcal{T}_i| \geq \varepsilon|\Sigma\big] \leq \sum_{i\geq 1} \mathbb{1}_{u_i\leq K} \varepsilon^{-1/\gamma} \, \mathbb{E}\big[|\mathcal{T}_i|^{1/\gamma}|\Sigma\big] \leq \frac{\mathbb{E}\big[|\mathcal{T}_{1,1}|^{1/\gamma}\big]}{\varepsilon^{1/\gamma}} \sum_{i\geq 1} \mathbb{1}_{u_i\leq K} \|\mathbf{s}_i\|$$

which is, according to Campbell's formula, a.s. finite. Consequently, using Borel-Cantelli's lemma, we deduce that conditionally on $\Sigma$, with probability one, there are finitely many indices $i \geq 1$ such that $u_i \leq K$ and $\mathcal{T}_i$ is higher than $\varepsilon$. It follows from Lemma 1.3.6 that $\mathcal{T}^{(I)}$ is a.s. $\mathbb{T}$-valued.

*Self-similar immigration measures.* We will say that an immigration measure $I$ with $I(\mathcal{S}^\downarrow) > 0$ is *self-similar* with positive index $\gamma$ (or simply, $\gamma$-self-similar) if for all $c > 0$ and measurable $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$, $c \int F(\mathbf{s}) I(\mathrm{d}\mathbf{s}) = \int F(c^{1/\gamma}\mathbf{s}) I(\mathrm{d}\mathbf{s})$.

**Proposition 1.3.11.** *An immigration measure $I$ is $\gamma$-self-similar iff $\gamma \in (0,1)$ and there exists a positive constant $K$ as well as an $\mathcal{S}^\downarrow$-valued random variable $X$ with $\|X\| = 1$ a.s. and such that for all measurable $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$*

$$\int F \, \mathrm{d}I = \int_0^\infty \frac{K}{t^{1+\gamma}} \mathbb{E}\big[F(tX)\big] \mathrm{d}t.$$

*Proof.* Clearly, if $X$ is an $\mathcal{S}^\downarrow_1$-valued random variable, $K > 0$ and $\gamma \in (0,1)$, the measure $I$ on $\mathcal{S}^\downarrow$ defined for all measurable $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ by

$$\int F \, \mathrm{d}I = K \int_0^\infty t^{-1-\gamma} \mathbb{E}\big[F(tX)\big] \mathrm{d}t$$

is an immigration measure. Moreover, for all $c > 0$, a simple change of variable gives $\int F(c^{1/\gamma}\mathbf{s}) I(\mathrm{d}\mathbf{s}) = c \int F \, \mathrm{d}I$ which means that $I$ is indeed $\gamma$-self-similar.

Conversely, suppose $I$ is a $\gamma$-self-similar immigration. Define $\sigma$, the probability measure on $\mathcal{S}^\downarrow_1$ such that for all measurable $f : \mathcal{S}^\downarrow_1 \to \mathbb{R}_+$

$$\int_{\mathcal{S}^\downarrow_1} f(\mathbf{s}) \, \sigma(\mathrm{d}\mathbf{s}) := Z^{-1} \int_{\mathcal{S}^\downarrow} f(\mathbf{s}/\|\mathbf{s}\|) \mathbb{1}_{\|\mathbf{s}\|\geq 1} I(\mathrm{d}\mathbf{s}),$$

where $Z := I\big(\|\cdot\| \geq 1\big)$, and let $X$ be a $\sigma$-distributed random variable. Now, for any measurable $g : \mathcal{S}_1^{\downarrow} \to \mathbb{R}_+$ and $t > 0$, because $I$ is self-similar, we get that

$$\int_{\mathcal{S}^{\downarrow}} g(\mathbf{s}/\|\mathbf{s}\|)\mathbb{1}_{\|\mathbf{s}\|\geq t}\, I(\mathrm{d}\mathbf{s}) = t^{-\gamma}\, Z\, \mathbb{E}\big[g(X)\big] = \gamma\, Z \int_t^{\infty} u^{-1-\gamma}\, \mathbb{E}\big[g(X)\big]\mathrm{d}u.$$

Since this identity holds for any $t > 0$ and measurable $g : \mathcal{S}_1^{\downarrow} \to \mathbb{R}_+$ and because $I(\{\mathbf{0}\}) = 0$, it follows that $I$ may be written in the desired way. Finally, because $I$ is an immigration measure, it must integrate $\mathbf{s} \mapsto 1 \wedge \|\mathbf{s}\|$, which implies that $\gamma$ belongs to $(0, 1)$. □

The point process construction of fragmentation trees with immigration may be used to prove this next proposition.

**Proposition 1.3.12.** *Suppose $I$ is a $\gamma$-self-similar immigration measure and let $\nu$ be a dislocation measure. If $(\mathcal{T}, \mu)$ denotes a $(\gamma, \nu, I)$-fragmentation tree with immigration, then for any positive $m$,*

- *$(m^{\gamma}\mathcal{T}, m\,\mu)$ has the same distribution as $(\mathcal{T}, \mu)$,*
- *$(\mathcal{T}, c\mu)$ and $(c^{-\gamma}\mathcal{T}, \mu)$ are $(\gamma, c^{\gamma}\nu, c^{\gamma}I)$-fragmentation trees with immigration.*

*Relationship to compact fragmentation trees.* Let $(\gamma, \nu)$ be a fragmentation pair and $I$ an immigration measure with $I(\mathcal{S}^{\downarrow}) > 0$. Theorem 17 in [62] states that under suitable conditions, if $(\mathcal{T}, \mu_{\mathcal{T}})$ denotes a $(\gamma, \nu)$-self-similar fragmentation tree, then $(m^{\gamma}\mathcal{T}, m\mu_{\mathcal{T}})$ converges to $\mathcal{T}^{I}_{\gamma,\nu}$ in distribution as $m \to \infty$ with respect to the extended GHP topology.

For instance, Theorem 11 $(iii)$ in [7], states that if $(\mathcal{T}, \mu_{\mathcal{T}})$ is a standard Brownian tree then when $m \to \infty$, $(m^{1/2}\mathcal{T}, m\,\mu_{\mathcal{T}})$ converges in distribution to Aldous' "self-similar CRT". This result was reformulated in terms of fragmentation trees in [62, Section 1.2]: $(m^{1/2}\mathcal{T}, m\,\mu_{\mathcal{T}})$ converges in distribution as $m \to \infty$ to a $(1/2, \nu_B, I_B)$-fragmentation tree with immigration, where $\nu_B$ is the Brownian dislocation measure (see Section 1.3.2) and the *Brownian immigration measure* $I_B$ is defined for all measurable $f : \mathcal{S}^{\downarrow} \to \mathbb{R}_+$ by

$$\int f\, \mathrm{d}I_B := \left(\frac{2}{\pi}\right)^{1/2} \int_{[0,\infty)} \frac{f(x, 0, 0, \ldots)}{x^{3/2}}\, \mathrm{d}x.$$

We will call a $(1/2, \nu_B, I_B)$-fragmentation tree with immigration a *Brownian tree with immigration*. As mentioned in the Introduction, this tree will appear in many of our applications.

Set $\alpha \in (1, 2)$ and recall the notations used to define $\nu_{\alpha}$ in Section 1.3.2, in particular, that $\Delta$ denotes the decreasing rearrangement of the jumps on $[0, 1]$ of a $1/\alpha$-stable subordinator with Laplace exponent $\lambda \mapsto -\log \mathbb{E}[\exp(-\lambda\Sigma_1)] = \lambda^{1/\alpha}$ and that $k_{\alpha} = \Gamma(2-\alpha)/[\alpha(\alpha-1)]$. Let $I^{(\alpha)}$ be the immigration measure defined for all measurable $F : \mathcal{S}^{\downarrow} \to \mathbb{R}_+$ by

$$\int_{\mathcal{S}^{\downarrow}} F\, \mathrm{d}I^{(\alpha)} = \frac{1}{k_{\alpha}} \int_0^{\infty} \frac{\mathbb{E}\big[F(t^{\alpha}\Delta)\big]}{t^{\alpha}} \mathrm{d}t.$$

In [62, Section 5.1], it was observed that if $(\mathcal{T}, \mu_{\mathcal{T}})$ is an $\alpha$-stable tree, then $(m^{1-1/\alpha}\mathcal{T}, m\,\mu_{\mathcal{T}})$ converges in distribution to a $(1-1/\alpha, \nu_{\alpha}, I^{(\alpha)})$-fragmentation tree with immigration as $m \to \infty$. These trees coincide with the $\alpha$-stable Lévy trees with immigration introduced in [48, Section 1.2].

### 1.3.3 Convergence of point processes

With the notations used in Section 1.3.2, let $\Pi := \sum_{i\geq 1} \delta_{(u_i, \mathbf{s}_i, \mathcal{T}_i)}$. It is a Poisson point process on $\mathbb{R}_+ \times \mathcal{S}^{\downarrow} \times \mathbb{T}_c$ with intensity $\mathrm{d}u \otimes \mathcal{J}(\mathrm{d}\mathbf{s}, \mathrm{d}\tau)$ where the measure $\mathcal{J}$ on $\mathcal{S}^{\downarrow} \times \mathbb{T}_c$ is defined as follows: let $(\tau_i, \mu_i)_{i\geq 1}$ be a sequence of i.i.d. $(\gamma, \nu)$-fragmentation trees and for any $\mathbf{s}$ in $\mathcal{S}^{\downarrow}$, similarly to Section 1.3.2, set $\tau_{\langle \mathbf{s} \rangle} := \big\langle (s_i^{\gamma}\tau_i, s_i\mu_i)\,;\, i \geq 1 \big\rangle$ and for all $G : \mathcal{S}^{\downarrow} \times \mathbb{T}_c \to \mathbb{R}_+$, let $\int G\, \mathrm{d}\mathcal{J} := \int \mathbb{E}[G(\mathbf{s}, \tau_{\langle \mathbf{s} \rangle})]\, I(\mathrm{d}\mathbf{s})$.

Moreover, recall the construction of Markov branching trees with a unique infinite spine (see Remark 1.2.4). If $q_{\infty}$ is such that $q_{\infty}(m_{\infty} = 1) = 1$, then a tree $T$ with distribution $\mathrm{MB}_{\infty}^{q, q_{\infty}}$ can be built in

the following way: consider the infinite branch and for all $n \geq 0$, graft a tree $T_n$ at height $n$ (where the sequence $(T_n)_{n\geq 0}$ is i.i.d.), such that $\Lambda_n := \Lambda(T_n)$ has distribution $q_* = q_\infty(\infty, \cdot)$ and conditionally on $\Lambda_n = \lambda$ in $\mathcal{P}_{<\infty}$, $T_n$ has distribution $\mathrm{MB}^q_\lambda$. As a result, $T$ is characterised by the point process $\sum_{n\geq 0} \delta_{(n,\Lambda_n,T_n)}$ (or simply by $\sum_{n\geq 0} \delta_{(n,T_n)}$).

Therefore, when considering scaling limits of such trees, it seems natural to take a step back and instead consider the convergence of the underlying point processes on $\mathbb{R}_+ \times \mathcal{S}^\downarrow \times \mathbb{T}_c$. We will follow the spirit of [62, Section 2.1.2] and introduce a topology on the set of such point measures adequate for our forthcoming purposes.

Let $\mathcal{R}$ be the set of integer-valued Radon measures on $\mathbb{R}_+ \times \mathcal{S}^\downarrow \times \mathbb{T}_c$ which integrate the function $(u, \mathbf{s}, \tau) \longmapsto \mathbb{1}_{u\leq K}\|\mathbf{s}\|$ for all $K \geq 0$ and are such that $\mu(\mathbb{R}_+ \times \{\mathbf{0}\} \times \mathbb{T}_c) = 0$.

*Remark 1.3.7.* Recall that as an immigration measure, $I$ integrates the function $\mathbf{s} \in \mathcal{S}^\downarrow \to 1 \wedge \|\mathbf{s}\|$. Campbell's theorem (see [84, Section 3.2]) therefore ensures that $\Pi$, the Poisson point process associated to a $\mathcal{T}^I_{\gamma,\nu}$ tree, a.s. belong to $\mathcal{R}$.

Let $\mathcal{F}$ be the set of continuous functions $F : \mathbb{R}_+ \times \mathcal{S}^\downarrow \times \mathbb{T}_c \longrightarrow \mathbb{R}_+$ such that there is $K \geq 0$ satisfying $F(u, \mathbf{s}, \tau) \leq \mathbb{1}_{u\leq K}\|\mathbf{s}\|$ for all $(u, \mathbf{s}, \tau)$. If $\zeta$ is a random element of $\mathcal{R}$, we define its *Laplace transform* as the function $L_\zeta : \mathcal{F} \to \mathbb{R}_+$, defined by $L_\zeta(F) := \mathbb{E}\big[\exp\big(-\int F \,\mathrm{d}\zeta\big)\big]$ for all $F$ in $\mathcal{F}$.

If $\mu_n$, $n \geq 1$ and $\mu$ are elements of $\mathcal{R}$, we will say that $\mu_n \to \mu$ iff for all $F \in \mathcal{F}$, $\int F \,\mathrm{d}\mu_n \to \int F \,\mathrm{d}\mu$. Appendix A7 of [79] ensures that when endowed with the topology induced by this convergence, $\mathcal{R}$ is a Polish space. Moreover, Theorems 4.2 and 4.9 of [79] give the following criterion for convergence in distribution of elements of $\mathcal{R}$.

**Proposition 1.3.13 ([79]).** *Let $\xi_n$, $n \geq 1$ and $\xi$ be $\mathcal{R}$-valued random variables. Then $\xi_n$ converges to $\xi$ in distribution with respect to the topology on $\mathcal{R}$ iff for all $F \in \mathcal{F}$, $L_{\xi_n}(F) \to L_\xi(F)$.*

The following extension of the Portmanteau theorem to finite measures with any mass will be useful.

**Lemma 1.3.14.** *Set $(M, \mathrm{d})$ a metric space and let $\mu_n$, $n \geq 1$ and $\mu$ be finite Borel measures on $M$. Then $\mu_n$ converges weakly to $\mu$ iff for any bounded Lipschitz-continuous function $f : M \to \mathbb{R}$, $\int f \,\mathrm{d}\mu_n$ converges to $\int f \,\mathrm{d}\mu$ as $n$ goes to infinity.*

*Proof.* Suppose $\int f \,\mathrm{d}\mu_n \to \int f \,\mathrm{d}\mu$ for all Lipschitz-continuous functions $f : M \to \mathbb{R}$. Observe that since constants are Lipschitz-continuous functions, our assumption implies that $\mu_n(M) \to \mu(M)$. Therefore, if $\mu(M) = 0$, we directly get $\mu_n \Rightarrow \mu$.

Otherwise, there exists $n_0$ such that $\mu_n(M) > 0$ for all $n \geq n_0$. For all such $n$, let $\tilde{\mu}_n := [\mu_n(M)]^{-1}\mu_n$ and $\tilde{\mu} := [\mu(M)]^{-1}\mu$ which are probability measures. It ensues from the usual Portmanteau theorem and our assumption that $\tilde{\mu}_n \Rightarrow \tilde{\mu}$. As a result, for any bounded continuous function $f$, as $n$ goes to $\infty$, $\int f \,\mathrm{d}\mu_n = \mu_n(M) \int f \,\mathrm{d}\tilde{\mu}_n \to \mu(M) \int f \,\mathrm{d}\tilde{\mu} = \int f \,\mathrm{d}\mu$ which is to say that $\mu_n \Rightarrow \mu$. □

## 1.4 Scaling limits of infinite Markov-branching trees

In this section, we will state and prove our main result on scaling limits of infinite Markov branching trees as well as its corollary on their volume growth.

Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ containing 1 and let $q = (q_{n-1})_{n\in\mathcal{N}}$ be a sequence of first-split distributions where for each $n$, $q_{n-1}$ is supported by $\{\lambda \in \mathcal{P}_{n-1} : \lambda_i \in \mathcal{N}, i = 1, \ldots, p(\lambda)\}$. Recall from Section 1.2.2 that the associated Markov branching family $\mathrm{MB}^q$ is well defined. Furthermore, let $q_\infty$ be a probability measure on $\mathcal{P}_\infty$ supported by the set $\{(\infty, \lambda) : \lambda \in \mathcal{P}_{<\infty}, \lambda_i \in \mathcal{N}, i = 1, \ldots, p(\lambda)\}$. In this way,

the probability measure $\mathrm{MB}_\infty^{q,q_\infty}$ on $\mathtt{T}_\infty$ is also well defined and a.s. yields trees with a unique infinite spine. To lighten notations, let $q_* := q_\infty(\infty, \cdot)$ which is a probability measure on $\mathcal{P}_{<\infty}$.

In the remainder of this section, we will assume that:

(S) There exist some $\gamma > 0$ and a dislocation measure $\nu$ on $\mathcal{S}^\downarrow$, such that $n^\gamma(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu(\mathrm{d}\mathbf{s})$. In particular, Theorem 1.3.7 and Lemma 1.3.8 hold.

(I) There exists an immigration measure $I$ on $\mathcal{S}^\downarrow$ such that if $\Lambda$ has distribution $q_*$, for any continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ with $F(\mathbf{s}) \le 1 \wedge \|\mathbf{s}\|$, $R\,\mathbb{E}\big[F(\Lambda/R^{1/\gamma})\big] \to \int F\,\mathrm{d}I$ as $R \to \infty$.

*Remark 1.4.1.* Under Assumption (I), for any continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ such that $F \le 1 \wedge \|\cdot\|$ and positive $c$,

$$c\int F(\mathbf{s})\,I(\mathrm{d}\mathbf{s}) = \lim_{R\to\infty} cR\,\mathbb{E}\big[F(\Lambda/R^{1/\gamma})\big] = \lim_{S\to\infty} S\,\mathbb{E}\big[F(c^{1/\gamma}\,\Lambda/S^{1/\gamma})\big] = \int F(c^{1/\gamma}\mathbf{s})\,I(\mathrm{d}\mathbf{s})$$

where we have taken $S = cR$. As a result, the immigration measure $I$ is $\gamma$-self-similar, as defined in Section 1.3.2, and Proposition 1.3.12 therefore holds for $(\gamma, \nu, I)$-fragmentation trees with immigration.

**Theorem 1.4.1.** *Let $T$ be an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{q,q_\infty}$ endowed with its counting measure $\mu_T$. Under Assumptions (S) and (I), if $\gamma < 1$, with respect to the extended GHP topology,*

$$\left(\frac{T}{R}, \frac{\mu_T}{R^{1/\gamma}}\right) \xrightarrow[R\to\infty]{} \mathcal{T}^I_{\gamma,\nu}$$

*in distribution, where $\mathcal{T}^I_{\gamma,\nu}$ denotes the distribution of a $(\gamma, \nu, I)$-fragmentation tree with immigration.*

Let $\mathbf{T}$ be a fixed element of $\mathbb{T}$. We define its *volume growth function* as $V_{\mathbf{T}} : \mathbb{R}_+ \to \mathbb{R}_+$, $R \mapsto \mu_T(T|_R)$. In other words, $V_{\mathbf{T}}(R)$ is the mass or volume of the closed ball $T|_R$. Once Theorem 1.4.1 is proved, we will be interested in the volume growth processes associated to these trees.

**Proposition 1.4.2.** *Suppose the assumptions of Theorem 1.4.1 are met. Let $T$ be an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{q,q_\infty}$ and $(\mathcal{T}, \mu_\mathcal{T})$ be a $(\gamma, \nu, I)$-fragmentation tree with immigration. Then, the volume growth function of $(T/R, \mu_T/R^{1/\gamma})$ converges in distribution to that of $(\mathcal{T}, \mu_\mathcal{T})$ with respect to the topology of uniform convergence on compacts of $\mathbb{R}_+$. In particular*

$$\frac{\mu_T(T|_R)}{R^{1/\gamma}} \xrightarrow[R\to\infty]{\text{(d)}} \mu_\mathcal{T}(\mathcal{T}|_1).$$

We may adapt the proofs of Theorem 1.4.1 and Proposition 1.4.2 to get the following theorem.

**Theorem 1.4.3.** *Let $T$ be an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ and endow it with the counting measure $\mu_T$ on the set of its leaves. If $\gamma < 1$ and if Assumptions (S) and (I) hold for $(q_n)_n$ and $q_\infty$ respectively, then the conclusions of both Theorem 1.4.1 and Proposition 1.4.2 hold.*

*Remark 1.4.2.* Instead of Assumption (I), we may assume that

(I′) There exists $\alpha < 1/\gamma$ and an immigration measure $I$ on $\mathcal{S}^\downarrow$ such that if $\Lambda$ is distributed according to $q_*$, $R\,\mathbb{E}\big[F(\Lambda/R^\alpha)\big] \to \int F\,\mathrm{d}I$ for any continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ with $F(\mathbf{s}) \le 1 \wedge \|\mathbf{s}\|$.

If $T$ has distribution $\mathrm{MB}_\infty^{q,q_\infty}$ and is endowed with its counting measure $\mu_T$ under (S) and (I′), we get that $(T/R, \mu_T/R^\alpha)$ converges in distribution to the infinite branch $\mathbb{R}_+$ endowed with the random measure $\mu = \sum_{i\ge1} \|\mathbf{s}_i\|\,\delta_{u_i}$, where $\{(u_i, \mathbf{s}_i); i \ge 1\}$ are the atoms of a Poisson point process $\Sigma$ on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ with intensity $\mathrm{d}u \otimes I(\mathrm{d}\mathbf{s})$. The tree $(\mathbb{R}_+, \mu)$ encodes the genealogy of a pure immigration process. Furthermore, $\mu_T(T|_R)/R^\alpha$ converges in distribution to $\mu([0,1]) = \int_{[0,1]\times\mathcal{S}^\downarrow} \|\mathbf{s}\|\,\Sigma(\mathrm{d}u, \mathrm{d}\mathbf{s})$.

Similarly, if $T$ is distributed according to $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ and is endowed with the counting measure on its leaves, the same results hold under (S) and (I′).

To prove Theorem 1.4.1, we will first study the convergence of the underlying point processes in Section 1.4.1 which will give us more leeway to manipulate the corresponding trees and end the proof in Section 1.4.2. Section 1.4.3 will then focus on proving Proposition 1.4.2.

### 1.4.1 Convergence of the associated point processes

Since $(\mathbb{T}_c, \mathrm{d}_{\mathrm{GHP}})$ is Polish, in light of Assumption (S), Theorem 1.3.7 and Skorokhod's representation theorem, we can find an i.i.d. sequence $[(T_{i,n})_{n\in\mathbb{N}}, \mathcal{T}_i]_{i\geq 1}$, where for each $i \geq 1$, the family $(T_{i,n})_{n\in\mathbb{N}}, \mathcal{T}_i$ of random trees is such that:

— $T_{i,n}$ has distribution $\mathrm{MB}_n^q$,
— $\mathcal{T}_i$ is a $(\gamma, \nu)$ self-similar fragmentation tree,
— $(T_{i,n}/n^\gamma, \mu_{T_{i,n}}/n) =: \overline{T}_{i,n}$ a.s. converges to $\mathcal{T}_i$ as $n \to \infty$.

For $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathcal{P}_{<\infty}$, let $T_{[\lambda]} := [\![ T_{i,\lambda_i} \,;\, 1 \leq i \leq p ]\!]$. For any $\mathbf{s} \in \mathcal{S}^\downarrow$, let $\mathcal{T}_{\langle \mathbf{s}\rangle} := \langle (s_i^\gamma \mathcal{T}_i, s_i \mu_{\mathcal{T}_i}); i \geq 1\rangle$ which is a compact $\mathbb{R}$-tree (see Lemma 1.3.9).

Finally, let $\Lambda$ be a random finite partition with distribution $q_*$ independent of $[(T_{i,n})_{n\in\mathbb{N}}, \mathcal{T}_i]_{i\geq 1}$, and for any $R \geq 1$, set $q^{(R)}$ as the distribution of $\Lambda/R^{1/\gamma}$. With these notations, Assumption (I) becomes: $R(1 \wedge \|\mathbf{s}\|) q^{(R)}(\mathrm{d}\mathbf{s}) \Rightarrow (1 \wedge \|\mathbf{s}\|) I(\mathrm{d}\mathbf{s})$ as finite measures on $\mathcal{S}^\downarrow$.

**Lemma 1.4.4.** *Let $K \subset \mathcal{S}^\downarrow$ be compact. Then $\sup_{\mathbf{s}\in K} \sum_{i>n} s_i \to 0$ as $n$ goes to infinity.*

*Proof.* Assume the contrary, i.e. that there exists a sequence $(\mathbf{s}^{(n)})_{n\geq 1}$ in $K$ and a positive constant $c$ such that $\sum_{i>n} s_i^{(n)} > c$ for all $n \geq 1$. Since $K$ is compact, we can find a subsequence $(\mathbf{s}^{(n_k)})_k$ and $\mathbf{s} \in K$ such that $\|\mathbf{s}^{(n_k)} - \mathbf{s}\| \to 0$ as $k \to \infty$. Consequently, $0 < c \leq \sum_{i>n_k} s_i^{(n_k)} \leq \sum_{i>n_k} s_i + \|\mathbf{s}^{(n_k)} - \mathbf{s}\| \to 0$ as $k \to \infty$, which is a contradiction. □

Fix $G : \mathcal{S}^\downarrow \times \mathbb{T}_c \to \mathbb{R}_+$ a 1-Lipschitz function, i.e. such that for all $\mathbf{s}, \mathbf{s}'$ in $\mathcal{S}^\downarrow$ and $\tau, \tau'$ in $\mathbb{T}_c$, $\big|G(\mathbf{s},\tau) - G(\mathbf{s}',\tau')\big| \leq \|\mathbf{s} - \mathbf{s}'\| + \mathrm{d}_{\mathrm{GHP}}(\tau, \tau')$. Further assume that $G(\mathbf{s}, \cdot) \leq 1 \wedge \|\mathbf{s}\|$ for any $\mathbf{s} \in \mathcal{S}^\downarrow$. Finally, set $g : \mathcal{S}^\downarrow \to \mathbb{R}_+$ the function defined by $g(\mathbf{s}) := \mathbb{E}[G(\mathbf{s}, \mathcal{T}_{\langle \mathbf{s}\rangle})]$.

**Lemma 1.4.5.** *We have*

$$R\,\mathbb{E}\Big[ G\Big(R^{-1/\gamma}\Lambda, (R^{-1} T_{[\Lambda]}, R^{-1/\gamma}\mu_{T_{[\Lambda]}})\Big)\Big] \xrightarrow[R\to\infty]{} \int_{\mathcal{S}^\downarrow} \mathbb{E}\big[G(\mathbf{s}, \mathcal{T}_{\langle \mathbf{s}\rangle})\big] I(\mathrm{d}\mathbf{s}).$$

*Proof.* Clearly, $g(\mathbf{s}) \leq 1 \wedge \|\mathbf{s}\|$. Moreover, for any $\mathbf{s}$ and $\mathbf{r}$ in $\mathcal{S}^\downarrow$,

$$\big|g(\mathbf{s}) - g(\mathbf{r})\big| \leq \mathbb{E}\Big[\big|G(\mathbf{s}, \mathcal{T}_{\langle \mathbf{s}\rangle}) - G(\mathbf{r}, \mathcal{T}_{\langle \mathbf{r}\rangle})\big|\Big] \leq \|\mathbf{s} - \mathbf{r}\| + \mathbb{E}\big[\mathrm{d}_{\mathrm{GHP}}\big(\mathcal{T}_{\langle \mathbf{s}\rangle}, \mathcal{T}_{\langle \mathbf{r}\rangle}\big)\big] \xrightarrow[\mathbf{r}\to\mathbf{s}]{} 0$$

where we have used Lemma 1.3.10. Therefore, $g$ is continuous and Assumption (I) ensures that

$$R\,\mathbb{E}\Big[ G\Big(R^{-1/\gamma}\Lambda, \big(R^{-1}\mathcal{T}_{\langle\Lambda\rangle}, R^{-1/\gamma}\mu_{\mathcal{T}_{\langle\Lambda\rangle}}\big)\Big)\Big] = R\,\mathbb{E}\big[g(R^{-1/\gamma}\Lambda)\big] \xrightarrow[R\to\infty]{} \int_{\mathcal{S}^\downarrow} g(\mathbf{s}) I(\mathrm{d}\mathbf{s}).$$

Consequently, it will be sufficient to prove that as $R \to \infty$,

$$\begin{aligned} &R\,\mathbb{E}\Big[\Big|G\Big(R^{-1/\gamma}\Lambda, (R^{-1}T_{[\Lambda]}, R^{-1/\gamma}\mu_{T_{[\Lambda]}})\Big) - G\Big(R^{-1/\gamma}\Lambda, (R^{-1}\mathcal{T}_{\langle\Lambda\rangle}, R^{-1/\gamma}\mu_{\mathcal{T}_{\langle\Lambda\rangle}})\Big)\Big|\Big] \\ &\quad\leq R\,\mathbb{E}\Big[\big(1 \wedge R^{-1/\gamma}\|\Lambda\|\big) \wedge \mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T_{[\Lambda]}, R^{-1/\gamma}\mu_{T_{[\Lambda]}}), (R^{-1}\mathcal{T}_{\langle\Lambda\rangle}, R^{-1/\gamma}\mu_{\mathcal{T}_{\langle\Lambda\rangle}})\Big)\Big] =: \Delta_R \longrightarrow 0. \end{aligned}$$

For all $n \geq 1$, thanks to Lemma 1.3.4 and Remark 1.3.2 we get

$$\begin{aligned} &\mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T_{[\Lambda]}, R^{-1/\gamma}\mu_{T_{[\Lambda]}}), (R^{-1}\mathcal{T}_{\langle\Lambda\rangle}, R^{-1/\gamma}\mu_{\mathcal{T}_{\langle\Lambda\rangle}})\Big) \\ &\qquad\leq \sum_{i=1}^{n} \mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T_{i,\Lambda_i}, R^{-1/\gamma}\mu_{T_{i,\Lambda_i}}), (R^{-1}\Lambda_i^\gamma \mathcal{T}_i, R^{-1/\gamma}\Lambda_i \mu_{\mathcal{T}_i})\Big) \\ &\qquad\qquad + \sup_{i>n}\Big(\frac{\Lambda_i^\gamma}{R}|\overline{T}_{i,\Lambda_i}|\Big) + \sup_{i>n}\Big(\frac{\Lambda_i^\gamma}{R}|\mathcal{T}_i|\Big) + 2\sum_{i>n}\frac{\Lambda_i}{R^{1/\gamma}}, \end{aligned}$$

and for each $i \geq 1$, Lemma 1.3.1 gives

$$\mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T_{i,\Lambda_i}, R^{-1/\gamma}\mu_{T_{i,\Lambda_i}}), (R^{-1}\Lambda_i^\gamma\mathcal{T}_i, R^{-1/\gamma}\Lambda_i\mu_{\mathcal{T}_i})\Big) \leq \Big(\frac{\Lambda_i^\gamma}{R} \vee \frac{\Lambda_i}{R^{1/\gamma}}\Big)\mathrm{d}_{\mathrm{GHP}}\big(\overline{T}_{i,\Lambda_i}, \mathcal{T}_i\big).$$

Let $\varepsilon > 0$ be fixed. As a result of Assumption (I), the sequence $R(1 \wedge \|\mathbf{s}\|)q^{(R)}(\mathrm{d}\mathbf{s})$, $R \geq 1$ is tight and so there exists a compact subset $K$ of $\mathcal{S}^{\downarrow}$ such that $\sup_{R\geq 1} R\int (1\wedge\|\mathbf{s}\|)\big(1-\mathbb{1}_K(\mathbf{s})\big)q^{(R)}(\mathrm{d}\mathbf{s}) < \varepsilon$. Moreover, as a compact subset, $K$ is bounded, i.e. $\sup_{\mathbf{s}\in K}\|\mathbf{s}\| = C < \infty$.

For all $n \geq 1$, recall that $\overline{T}_{1,n}$ and $\mathcal{T}_1$ are endowed with probability measures. Remark 1.3.2 therefore ensures that $\mathrm{d}_{\mathrm{GHP}}(\overline{T}_{1,n}, \mathcal{T}_1) \leq 2 \vee |\overline{T}_{1,n}| \vee |\mathcal{T}_1|$. As a result, thanks to Lemma 1.3.8,

$$\sup_n \mathbb{E}\Big[\big(\mathrm{d}_{\mathrm{GHP}}(\overline{T}_{1,n}, \mathcal{T}_1)\big)^2\Big] \leq 3\Big(2^2 + \sup_n \mathbb{E}\big[|\overline{T}_{1,n}|^2\big] + \mathbb{E}\big[|\mathcal{T}_1|^2\big]\Big) \leq 12 + 6h_2 < \infty,$$

so the sequence $\big[\mathrm{d}_{\mathrm{GHP}}(\overline{T}_{1,n}, \mathcal{T}_1)\big]_n$ is bounded in $L^2$. Since by assumption, it converges to 0 a.s., it also does so in $L^1$. Furthermore, $\sup_n \mathbb{E}[\mathrm{d}_{\mathrm{GHP}}(\overline{T}_{1,n}, \mathcal{T}_1)] =: D$ is finite. Consequently, and because the sequence of families $\big\{(T_{i,n})_n, \mathcal{T}_i\big\}_{i\geq 1}$ is i.i.d., for any $\eta > 0$, there exists $N$ such that for all $i \geq 1$ and $n \geq N$, $\mathbb{E}\big[\mathrm{d}_{\mathrm{GHP}}\big(\overline{T}_{i,n}, \mathcal{T}_i\big)\big] < \eta$. This gives the rather crude following bound

$$\mathbb{E}\big[\mathrm{d}_{\mathrm{GHP}}(\overline{T}_{i,n}, \mathcal{T}_i)\big] \leq D\,\mathbb{1}_{n<N} + \eta.$$

For all $\delta > 0$, in light of Lemma 1.4.4, there exists an integer $m_{K,\delta}$ which depends only on $K$ and $\delta$ such that $\sup_{\mathbf{s}\in K}\sum_{i>m_{K,\delta}} s_i < \delta$. Then for all $R \geq 1$ and $\lambda \in \mathcal{P}_{<\infty}$ with $\lambda/R^{1/\gamma} \in K$, if $\gamma \leq 1$, Jensen's inequality gives

$$\mathbb{E}\left[\sup_{i>m_{K,\delta}}\left(\frac{\lambda_i^\gamma}{R}|\overline{T}_{i,\lambda_i}|\right)\right] \leq \left(\mathbb{E}\left[\sup_{i>m_{K,\delta}}\frac{\lambda_i}{R^{1/\gamma}}|\overline{T}_{i,\lambda_i}|^{1/\gamma}\right]\right)^\gamma \leq \left(\sum_{i>m_{K,\delta}}\frac{\lambda_i}{R^{1/\gamma}}\mathbb{E}\big[|\overline{T}_{i,\lambda_i}|^{1/\gamma}\big]\right)^\gamma \leq (h_{1/\gamma})^\gamma\,\delta^\gamma$$

where $h_{1/\gamma}$ is the constant from Lemma 1.3.8. Otherwise, if $\gamma > 1$, since $(\lambda_i)_{i\geq 1}$ is a non-increasing sequence,

$$\mathbb{E}\left[\sup_{i>m_{K,\delta}}\left(\frac{\lambda_i^\gamma}{R}|\overline{T}_{i,\lambda_i}|\right)\right] \leq \left(\frac{\lambda_{m_{K,\delta}+1}}{R^{1/\gamma}}\right)^{\gamma-1}\mathbb{E}\left[\sup_{i>m_{K,\delta}}\frac{\lambda_i}{R^{1/\gamma}}|\overline{T}_{i,\lambda_i}|\right] \leq \delta^{\gamma-1}\sum_{i>m_{K,\delta}}\frac{\lambda_i}{R^{1/\gamma}}\mathbb{E}\big[|\overline{T}_{i,\lambda_i}|\big] \leq h_1\,\delta^\gamma$$

where $h_1$ is defined as in Lemma 1.3.8. Similarly,

$$\mathbb{E}\left[\sup_{i>m_{K,\delta}}\left(\frac{\lambda_i^\gamma}{R}|\mathcal{T}_i|\right)\right] \leq \begin{cases}(h_{1/\gamma})^\gamma\,\delta^\gamma & \text{if } \gamma \leq 1,\\ h_1\,\delta^\gamma & \text{if } \gamma > 1.\end{cases}$$

In summary, for all $\lambda$ in $\mathcal{P}_{<\infty}$ such that $\lambda/R^{1/\gamma}$ belongs to $K$, we get that

$$\sum_{i>m_{K,\delta}}\frac{\lambda_i}{R^{1/\gamma}} \leq \delta \qquad \text{and} \qquad \mathbb{E}\left[\sup_{i>m_{K,\delta}}\left(\frac{\lambda_i^\gamma}{R}|\overline{T}_{i,\lambda_i}|\right) + \sup_{i>m_{K,\delta}}\left(\frac{\lambda_i^\gamma}{R}|\mathcal{T}_i|\right)\right] \leq B\,\delta^\gamma$$

for some finite constant $B$ independent of $\varepsilon$, $\eta$, $\delta$ and $K$.

Therefore, for all positive $\varepsilon$, $\delta$, $\eta$, and any $R \geq 1$,

$$\begin{aligned}\Delta_R \leq \varepsilon + R\,\mathbb{E}\Bigg[\mathbb{1}_K\Big(\frac{\Lambda}{R^{1/\gamma}}\Big)\Big(1\wedge\frac{\|\Lambda\|}{R^{1/\gamma}}\Big) \wedge \Bigg(\sum_{i=1}^{m_{K,\delta}}\Big(\frac{\Lambda_i^\gamma}{R} \vee \frac{\Lambda_i}{R^{1/\gamma}}\Big)\mathbb{E}\Big[\mathrm{d}_{\mathrm{GHP}}\big(\overline{T}_{i,\Lambda_i}, \mathcal{T}_i\big)\Big|\Lambda\Big] \\ + \mathbb{E}\Big[\sup_{i>m_{K,\delta}}\frac{\Lambda_i^\gamma}{R}|\overline{T}_{i,\Lambda_i}| + \sup_{i>m_{K,\delta}}\frac{\Lambda_i^\gamma}{R}|\mathcal{T}_i| + 2\sum_{i>m_{K,\delta}}\frac{\Lambda_i}{R^{1/\gamma}}\Big|\Lambda\Big]\Bigg)\Bigg] \\ \leq \varepsilon + R\,\mathbb{E}\left[\left(1\wedge\frac{\|\Lambda\|}{R^{1/\gamma}}\right) \wedge \left((C+C^\gamma)\,m_{K,\delta}\eta + \Big(\frac{N^\gamma}{R} + \frac{N}{R^{1/\gamma}}\Big)m_{K,\delta}D + 2\delta + B\delta^\gamma\right)\right].\end{aligned}$$

Let $\delta$ be such that $(2\delta + \delta^\gamma)B < \varepsilon$ and set $\eta < \varepsilon/[(C + C^\gamma)m_{K,\delta}]$. Because of Assumption (I), we therefore get that

$$\limsup_{R\to\infty}\Delta_R \leq \varepsilon + \int_{\mathcal{S}^{\downarrow}}(2\varepsilon)\wedge\|\mathbf{s}\|\,I(\mathrm{d}\mathbf{s}).$$

The monotone convergence theorem implies that the right hand side of this last inequality vanishes when $\varepsilon$ decreases to 0. This proves that $\Delta_R \to 0$, which concludes this proof. □

Since the conclusion of Lemma 1.4.5 is met for any Lipschitz continuous function $G : \mathcal{S}^{\downarrow} \times \mathbb{T}_c \to \mathbb{R}_+$ with $G(\mathbf{s}, \cdot) \leq 1 \wedge \|\mathbf{s}\|$, Lemma 1.3.14 gives the following corollary:

**Corollary 1.4.6.** *The convergence of Lemma 1.4.5 holds for any continuous $G$ with $G(\mathbf{s}, \cdot) \leq 1 \wedge \|\mathbf{s}\|$.*

We will now prove that the point processes associated to adequately rescaled Markov branching trees with a unique infinite spine converge in distribution to the point process associated to fragmentation trees with immigration. Let $\Pi$ be a Poisson point process on $\mathbb{R}_+ \times \mathcal{S}^{\downarrow} \times \mathbb{T}_c$ with intensity $\mathrm{d}u \otimes \mathcal{J}(\mathrm{d}\mathbf{s}, \mathrm{d}\tau)$, where $\mathcal{J}$ is the measure defined at the beginning of Section 1.3.3. Observe that for all $K \geq 0$,

$$\int \mathbb{1}_{u \leq K}\big(1 \wedge \|\mathbf{s}\|\big)\, \mathrm{d}u \otimes \mathcal{J}(\mathrm{d}\mathbf{s}, \mathrm{d}\tau) = K \int_{\mathcal{S}^{\downarrow}} \big(1 \wedge \|\mathbf{s}\|\big) I(\mathrm{d}\mathbf{s}) < \infty.$$

Campbell's theorem (see [84, Section 3.2]) therefore ensures that $\Pi$ a.s. satisfies the integrability conditions necessary to belong to the set $\mathcal{R}$ of point measures on $\mathbb{R}_+ \times \mathcal{S}^{\downarrow} \times \mathbb{T}_c$ defined in Section 1.3.3.

Let $T$ have distribution $\mathrm{MB}_{\infty}^{q,q_{\infty}}$. By construction of Markov branching trees with a unique infinite spine (see Remark 1.2.4), there exists a sequence $(\Lambda_n, T_n)_{n\geq 0}$ of i.i.d. random variables such that $T = \mathfrak{b}_{\infty} \bigotimes_{n\geq 0}(\mathfrak{v}_n, T_n)$, where $\Lambda_n$ is distributed according to $q_*$ and conditionally on $\Lambda_n = \lambda$, $T_n$ has distribution $\mathrm{MB}_{\lambda}^{q}$. For all $R \geq 1$, let $\Pi_R$ be the point process associated to $(T/R, \mu_T/R^{1/\gamma})$, i.e. the $\mathcal{R}$-valued random variable defined for all measurable $f : \mathbb{R}_+ \times \mathcal{S}^{\downarrow} \times \mathbb{T}_c \longrightarrow \mathbb{R}_+$ by

$$\int f \, \mathrm{d}\Pi_R := \sum_{n\geq 0} f\big[n/R,\, \Lambda_n/R^{1/\gamma},\, (T_n/R, \mu_{T_n}/R^{1/\gamma})\big].$$

**Lemma 1.4.7.** *With respect to the topology on $\mathcal{R}$ introduced in Section 1.3.3, $\Pi_R$ converges to $\Pi$ in distribution as $R$ goes to infinity.*

*Proof.* In light of Proposition 1.3.13, it will be enough to prove that for any function $F$ in the set $\mathcal{F}$, the Laplace transform of $\Pi_R$ evaluated in $F$ converges to that of $\Pi$. Fix such $F$ in $\mathcal{F}$ and recall that it is continuous and that there exists $K \geq 0$ such that $0 \leq F(u, \mathbf{s}, \tau) \leq \|\mathbf{s}\| \mathbb{1}_{u \leq K}$ for all $(u, \mathbf{s}, \tau)$. Campbell's theorem for Poisson point processes gives

$$L_{\Pi}(F) = \exp\Big(-\int \big[1 - \mathrm{e}^{-F(u,\mathbf{s},\tau)}\big]\, \mathrm{d}u \otimes \mathcal{J}(\mathrm{d}\mathbf{s}, \mathrm{d}\tau)\Big).$$

For all $R \geq 1$ and $u \geq 0$, set

$$\varphi_R(u) := R\,\mathbb{E}\Big[1 - \exp\Big(-F\big[u, \Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\big]\Big)\Big],$$
$$\text{and} \qquad \varphi(u) := \int \mathbb{E}\big[1 - \exp\big(-F[u, \mathbf{s}, \mathcal{T}_{\langle \mathbf{s} \rangle}]\big)\big] I(\mathrm{d}\mathbf{s}).$$

Using these notations, we may write $\log L_{\Pi}(F) = -\int_0^K \varphi(u)\, \mathrm{d}u$ and thanks to the i.i.d. nature of the sequence $(\Lambda_n, T_n)_{n\geq 0}$, for all $R \geq 1$,

$$\log L_{\Pi_R}(F) = -\sum_{n=0}^{\lfloor KR \rfloor} \log \mathbb{E}\Big[\exp\Big(-F\big[n/R, \Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\big]\Big)\Big]$$
$$= -\sum_{n=0}^{\lfloor KR \rfloor} \log\Big(1 - 1/R \cdot \varphi_R(n/R)\Big).$$

The functions $\varphi_R$, $R \geq 1$ and $\varphi$ all have support in $[0, K]$ and are continuous (in light of the dominated convergence theorem). Observe that $0 \leq 1 - \mathrm{e}^{-F(u,\mathbf{s},\tau)} \leq 1 \wedge \|\mathbf{s}\|$. From Corollary 1.4.6, we know that for all fixed $u \geq 0$, $\varphi_R(u) \to \varphi(u)$ as $R \to \infty$ and that furthermore

$$\sup_{R\geq 1} \sup_{u \geq 0} \varphi_R(u) \leq \sup_{R \geq 1} R\,\mathbb{E}\big[1 \wedge (\|\Lambda_0\|/R^{1/\gamma})\big] < \infty,$$

i.e. that the sequence $(\varphi_R)_{R\geq 1}$ is uniformly bounded by a finite constant, say $C$. Let $\varepsilon$ be positive. It also follows from Corollary 1.4.6 that there exists a compact subset $A$ of $\mathcal{S}^{\downarrow} \times \mathbb{T}_c$ with

$$\sup_{R\geq 1} R\,\mathbb{E}\Big[\big(1 \wedge (\|\Lambda_0\|/R^{1/\gamma})\big) \cdot \mathbb{1}_{A^c}\big(\Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\big)\Big] < \varepsilon.$$

Recall that $F$ is continuous, hence there exists $\delta > 0$ such that for any $(u, \mathbf{s}, \tau)$ and $(u', \mathbf{s}', \tau')$ in the compact set $[0, K] \times A$, if $|u - u'| + \|\mathbf{s} - \mathbf{s}'\| + \mathrm{d}_{\mathrm{GHP}}(\tau, \tau') < \delta$, then $|F(u, \mathbf{s}, \tau) - F(u', \mathbf{s}', \tau')| < \varepsilon$. As a result, and because $x \mapsto \mathrm{e}^{-x}$ is 1-Lipschitz continuous on $\mathbb{R}_+$, for all $R \geq 1$ and $u, v$ in $[0, K]$ with $|u - v| < \delta$,

$$\begin{aligned}\left|\varphi_R(u) - \varphi_R(v)\right| &\leq R\,\mathbb{E}\left[1 \wedge \left|F\left[u, \Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\right] - F\left[v, \Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\right]\right|\right] \\ &\leq \varepsilon + R\,\mathbb{E}\left[\left(\varepsilon \wedge (\|\Lambda_0\|/R^{1/\gamma})\right) \cdot \mathbb{1}_A\left(\Lambda_0/R^{1/\gamma}, (T_0/R, \mu_{T_0}/R^{1/\gamma})\right)\right],\end{aligned}$$

and in light of Corollary 1.4.6 and the monotone convergence theorem, we get

$$\limsup_{R\to\infty} \left|\varphi_R(u) - \varphi_R(v)\right| \leq \varepsilon + \int \varepsilon \wedge \|\mathbf{s}\|\, I(\mathrm{d}\mathbf{s}) \xrightarrow[\varepsilon\to 0]{} 0.$$

This ensures that the sequence $(\varphi_R)_{R\geq 1}$ is equicontinuous on $[0, K]$. It follows from the Arzelà-Ascoli theorem that $\varphi_R$ converges uniformly to $\varphi$. In turn, we deduce that

$$\left|\frac{1}{R}\sum_{n=0}^{\lfloor KR\rfloor} \varphi_R(n/R) - \frac{1}{R}\sum_{n=0}^{\lfloor KR\rfloor} \varphi(n/R)\right| \leq \frac{KR+1}{R} \sup_{0\leq u\leq K} \left|\varphi_R(u) - \varphi(u)\right| \xrightarrow[R\to\infty]{} 0.$$

Observe that for all $R \geq 1$, we may write

$$\log L_{\Pi_R}(F) - 1/R \cdot \sum_{n=0}^{\lfloor KR\rfloor} \varphi_R(n/R) = \sum_{n=0}^{\lfloor KR\rfloor} \Big( -\log\left[1 - 1/R \cdot \varphi_R(n/R)\right] - 1/R \cdot \varphi_R(n/R)\Big).$$

Recall that $\sup_{R\geq 1, u\geq 0} \varphi_R(u) \leq C$. Therefore, because the function $[0, 1) \to \mathbb{R}_+$, $x \mapsto -\log(1-x) - x$ increases with $x$, for any $R \geq C$ and $n \geq 0$, we get

$$\left|-\log\left[1 - 1/R \cdot \varphi_R(n/R)\right] - 1/R \cdot \varphi_R(n/R)\right| \leq \left|-C/R - \log(1 - C/R)\right| = o(1/R).$$

Consequently,

$$\left|\log L_{\Pi_R}(F) - 1/R \cdot \sum_{n=0}^{\lfloor KR\rfloor} \varphi_R(n/R)\right| \leq (KR+1)\left|-C/R - \log(1-C/R)\right| \xrightarrow[R\to\infty]{} 0.$$

Finally, as Riemann sums of the continuous function $\varphi$,

$$\frac{1}{R}\sum_{n=0}^{\lfloor KR\rfloor} \varphi(n/R) \xrightarrow[R\to\infty]{} \int_0^K \varphi(u)\,\mathrm{d}u = \log L_\Pi(F).$$

In summary, $\log L_{\Pi_R}(F) \to \log L_\Pi(F)$ when $R \to \infty$. □

### 1.4.2 Proof of Theorem 1.4.1

Now that we know that the underlying point processes converge, we can prove convergence of the trees themselves.

Recall that the topology we defined on $\mathscr{R}$ in Section 1.3.3 makes it a Polish topological space. As such, Skorokhod's representation theorem holds for $\mathscr{R}$-valued random variables. In particular, because of Lemma 1.4.7, there exist:

— A Poisson point process $\Pi$ with intensity $\mathrm{d}u \otimes \mathcal{I}(\mathrm{d}\mathbf{s}, \mathrm{d}\tau)$,
— A family $\left\{(\Lambda_n^{(R)}, \tau_n^{(R)})_{n\geq 0}; R \in \mathbb{N}\right\}$ such that for all fixed $R \geq 1$, $(\Lambda_n^{(R)}, \tau_n^{(R)})_{n\geq 0}$ is an i.i.d. sequence, $\Lambda_n^{(R)}$ follows $q_*$ and conditionally on $\Lambda_n^{(R)} = \lambda$, $\tau_n^{(R)}$ has distribution $\mathrm{MB}_\lambda^q$ and is endowed with the measure $\mu_{\tau_n^{(R)}} := \sum_{u\in\tau_n^{(R)}} \delta_u$,

such that if for any $R$ we let $\Pi_R$ be the random element of $\mathscr{R}$ defined for all measurable $f : \mathbb{R}_+ \times \mathcal{S}^\downarrow \times \mathbb{T}_c \longrightarrow \mathbb{R}_+$ by $\int f\,\mathrm{d}\Pi_R := \sum_{n\geq 0} f\left[n/R, \Lambda_n^{(R)}/R^{1/\gamma}, (\tau_n^{(R)}/R, \mu_{\tau_n^{(R)}}/R^{1/\gamma})\right]$, then $\Pi_R$ a.s. converges to $\Pi$ when $R \to \infty$.

Let $\{(u_i, \mathbf{s}_i, \mathcal{T}_i); i \geq 1\}$ be the atoms of $\Pi$ and set $\Sigma := \sum_{i\geq 1} \delta_{(u_i, \mathbf{s}_i)}$. By definition of the intensity measure of $\Pi$, there exists a family $\{\mathcal{T}_{i,j}\,;\, i, j \geq 1\}$ of i.i.d. $(\gamma, \nu)$-fragmentation trees independent of $\Sigma$ such that for all $i \geq 1$, $\mathcal{T}_i = \langle (s_{i,j}^\gamma \mathcal{T}_{i,j}, s_{i,j}\mu_{\mathcal{T}_{i,j}}); j \geq 1\rangle$. Set $\mathcal{T}^{(I)} := \mathbf{G}(\{(u_i, \mathcal{T}_i); i \geq 1\})$ where $\mathbf{G}$ is the continuum grafting

function defined in Section 1.3.1 and recall that it is a $(\gamma, \nu)$-fragmentation tree with immigration $I$ (see Section 1.3.2). For all $\varepsilon > 0$, let

$$\mathcal{T}_\varepsilon^{(I)} := \mathbf{G}\big(\{(u_i, \mathcal{T}_i); i \geq 1, \|\mathbf{s}_i\| \geq \varepsilon\}\big).$$

This tree can be thought of as $\mathcal{T}^{(I)}$ on which all sub-trees grafted on the spine with mass less than $\varepsilon$ have been cut away. Observe that because of the definition of the function $\mathbf{G}$, the measure on $\mathcal{T}_\varepsilon^{(I)}$ is simply the restriction of $\mu_{\mathcal{T}^{(I)}}$ to $\mathcal{T}_\varepsilon^{(I)}$.

For all $R$, set $\tau^{(R)} := \mathfrak{b}_\infty \bigotimes_{n\geq 0}(\mathbf{v}_n, \tau_n^{(R)})$ and denote its counting measure by $\mu_{\tau^{(R)}}$. Observe that $\tau^{(R)}$ is distributed according to $\mathrm{MB}_\infty^{q,q_\infty}$. Let $T^{(R)} := (R^{-1}\tau^{(R)}, R^{-1/\gamma}\mu_{\tau^{(R)}})$ be the rescaled infinite Markov branching tree associated to $\Pi_R$. Moreover, for all positive $\varepsilon$, let $T_\varepsilon^{(R)}$ be the tree obtained by removing from $T^{(R)}$ all the sub-trees grafted on its spine with mass less than $\varepsilon$, i.e. set

$$T_\varepsilon^{(R)} := \mathbf{G}\Big(\Big\{\big[n/R, (R^{-1}\tau_n^{(R)}, R^{-1/\gamma}\mu_{\tau_n^{(R)}})\big] \;\Big|\; n \geq 0 : \|\Lambda_n^{(R)}\| \geq R^{1/\gamma}\varepsilon\Big\}\Big).$$

The tree $T_\varepsilon^{(R)}$ is clearly a subset of $T^{(R)}$ and it is endowed with the restriction of $\mu_{T^{(R)}}$.

In this section we will endeavour to prove Theorem 1.4.1. In order to do so, we will use the following criterion for convergence in distribution.

**Theorem 1.4.8 ([29], Theorem 3.2).** *Let $(M, \mathrm{d})$ be a metric space. If $X_n$, $X_n^{(k)}$, $X^{(k)}$, $n \geq 1$, $k \geq 1$ and $X$ are $M$-valued random variables satisfying:*

*$(i)$ For all $k \geq 1$, $X_n^{(k)} \Rightarrow X^{(k)}$ as $n \to \infty$,*

*$(ii)$ $X^{(k)} \Rightarrow X$ as $k \to \infty$,*

*$(iii)$ For any positive $\eta$, $\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\big[\mathrm{d}(X_n^{(k)}, X_n) > \eta\big] = 0$,*

*Then $X_n$ converges to $X$ in distribution.*

*Remark 1.4.3.* Condition $(i)$ is akin to finite-dimensional convergence of $X_n$ to $X$ and Conditions $(ii)$ and $(iii)$ to tightness of $(X_n)_n$.

In our setting, the sequence $(T^{(R)}; R \in \mathbb{N})$ of rescaled $\mathrm{MB}_\infty^{q,q_\infty}$ trees will play the role of $(X_n)_n$ and the limit variable $X$ will be $\mathcal{T}^{(I)}$, a $(\gamma, \nu)$-fragmentation tree with immigration $I$. The intermediate family $(X_n^{(k)})_{n,k}$ will be replaced by $(T_\varepsilon^{(R)}; R \geq 1)$ with $\varepsilon \to 0$ along some countable subset of $(0, \infty)$. Similarly, we'll consider $\mathcal{T}_\varepsilon^{(I)}$ trees instead of $(X^{(k)})_k$.

**Lemma 1.4.9.** *With these notations, $\mathcal{T}_\varepsilon^{(I)}$ a.s. converges to $\mathcal{T}^{(I)}$ as $\varepsilon \to 0$ with respect to $\mathrm{D}_{\mathrm{GHP}}$.*

*Proof.* For all $\varepsilon > 0$, let $C_\varepsilon$ be the correspondence between $\mathcal{T}^{(I)}$ and $\mathcal{T}_\varepsilon^{(I)}$ defined by $C_\varepsilon := \big\{(x, x) : x \in \mathcal{T}_\varepsilon^{(I)}\big\} \cup \bigcup_{i\geq 1:\|\mathbf{s}_i\|<\varepsilon} \mathcal{T}_i \times \{u_i\}$ and set $\pi_\varepsilon$, the boundedly finite Borel measure on $\mathcal{T}^{(I)} \times \mathcal{T}_\varepsilon^{(I)}$, such that for all Borel $A$, $\pi_\varepsilon(A) := \int_{\mathcal{T}_\varepsilon^{(I)}} \mathbb{1}_A(x, x)\, \mu_{\mathcal{T}_\varepsilon^{(I)}}(\mathrm{d}x)$. Let $K \geq 0$ be fixed. Call $\pi_\varepsilon|_K$ the restriction of $\pi_\varepsilon$ to $\mathcal{T}^{(I)}|_K \times \mathcal{T}_\varepsilon^{(I)}|_K$. The monotone convergence theorem yields

$$\mathrm{D}\big(\pi_\varepsilon|_K; \mu_{\mathcal{T}^{(I)}}|_K, \mu_{\mathcal{T}_\varepsilon^{(I)}}|_K\big) = \pi_\varepsilon|_K(C_\varepsilon^c) \leq \int \|\mathbf{s}\| \mathbb{1}_{\|\mathbf{s}\|<\varepsilon} \mathbb{1}_{u\leq K} \Sigma(\mathrm{d}u, \mathrm{d}\mathbf{s}) \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0.$$

Let $C_\varepsilon|_K := C_\varepsilon \cap \big(\mathcal{T}^{(I)}|_K \times \mathcal{T}_\varepsilon^{(I)}|_K\big)$ and observe that it is a correspondence between $\mathcal{T}^{(I)}|_K$ and $\mathcal{T}_\varepsilon^{(I)}|_K$. Its distortion satisfies

$$\mathrm{dis}\, C_\varepsilon|_K \leq 2 \sup\big\{|\mathcal{T}_i| : i \geq 1, u_i \leq K, \|\mathbf{s}_i\| < \varepsilon\big\} \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0.$$

As a result, $\mathrm{d}_{\mathrm{GHP}}\big(\mathcal{T}^{(I)}|_K, \mathcal{T}_\varepsilon^{(I)}|_K\big) \to 0$ a.s. as $\varepsilon \to 0$. Since this holds for all $K \geq 0$, Proposition 1.3.5 $(ii)$ ensures that $\mathrm{D}_{\mathrm{GHP}}\big(\mathcal{T}^{(I)}, \mathcal{T}_\varepsilon^{(I)}\big)$ a.s. converges to 0 when $\varepsilon \to 0$. □

**Lemma 1.4.10.** *For all positive $\eta$,*

$$\lim_{\varepsilon\to 0} \limsup_{R\to\infty} \mathbb{P}\Big[\mathrm{D}_{\mathrm{GHP}}\big(T^{(R)}, T_\varepsilon^{(R)}\big) > \eta\Big] = 0.$$

*Proof.* We will proceed in a way similar to the proof of Lemma 1.4.9. For all $R \geq 1$ and $\varepsilon > 0$, define the correspondence $C_\varepsilon^{(R)}$ between $T^{(R)}$ and $T_\varepsilon^{(R)}$ as $C_\varepsilon^{(R)} := \{(u,u) : u \in T_\varepsilon^{(R)}\} \cup \{(u, n/R) : n \geq 1, \|\Lambda_n\| < R^{1/\gamma}\varepsilon, u \in \tau_n^{(R)}\}$ and let $\pi_\varepsilon^{(R)}$ be the boundedly finite measure $T^{(R)} \times T_\varepsilon^{(R)}$ defined for all Borel sets $A$ by $\pi_\varepsilon^{(R)}(A) := \int_{T_\varepsilon^{(R)}} \mathbb{1}_A(x,x)\,\mu_{T_\varepsilon^{(R)}}(\mathrm{d}x)$.

For all $K \geq 0$, set $C_\varepsilon^{(R)}\big|_K := C_\varepsilon^{(R)} \cap \big(T^{(R)}|_K \times T_\varepsilon^{(R)}|_K\big)$, which is a correspondence between $T^{(R)}|_K$ and $T_\varepsilon^{(R)}|_K$, and let $\pi_\varepsilon^{(R)}\big|_K$ be the restriction of $\pi_\varepsilon^{(R)}$ to $T^{(R)}|_K \times T_\varepsilon^{(R)}|_K$. Then, for any non-negative $K$,

$$\mathrm{dis}_{T^{(R)}|_K, T_\varepsilon^{(R)}|_K} C_\varepsilon^{(R)}\big|_K \leq \frac{2}{R} \sup\Big\{|\tau_n^{(R)}| : 0 \leq n \leq RK,\, \|\Lambda_n^{(R)}\| < R^{1/\gamma}\varepsilon\Big\}.$$

For all $n \geq 0$ and $R \geq 1$, $|\tau_n^{(R)}| = 1 + \sup\{|\tau_{n,i}^{(R)}| : 1 \leq i \leq p(\Lambda_n^{(R)})\}$. Further observe that thanks to Lemma 1.3.8, we can find a finite constant $h$ such that for all $n \geq 0$, $R \geq 1$ and $i = 1, \ldots, p(\Lambda_n^{(R)})$, $\mathbb{E}[(1+|\tau_{n,i}^{(R)}|)^{1/\gamma} | \Lambda_n^{(R)}] \leq h\,\Lambda_n^{(R)}(i)$. Therefore, since the sequence $(\Lambda_n^{(R)}, \tau_n^{(R)})_{n\geq 1}$ is i.i.d.,

$$\begin{aligned}\mathbb{E}\Big[\Big(\mathrm{dis}_{T^{(R)}|_K, T_\varepsilon^{(R)}|_K} C_\varepsilon^{(R)}\big|_K\Big)^{1/\gamma}\Big] &\leq (KR+1)\,\frac{2^{1/\gamma}}{R^{1/\gamma}}\,\mathbb{E}\Big[\textstyle\sum_{i=1}^{p(\Lambda_0^{(R)})}(1+|\tau_{0,i}^{(R)}|)^{1/\gamma}\mathbb{1}_{\|\Lambda_0^{(R)}\|<R^{1/\gamma}\varepsilon}\Big] \\ &\leq (KR+1)\,\frac{2^{1/\gamma}h}{R^{1/\gamma}}\,\mathbb{E}\Big[\|\Lambda_0^{(R)}\|\,\mathbb{1}_{\|\Lambda_0^{(R)}\|<R^{1/\gamma}\varepsilon}\Big].\end{aligned}$$

Similarly,

$$\mathbb{E}\Big[\mathrm{D}\Big(\pi_\varepsilon^{(R)}\big|_K; \mu_{T^{(R)}}\big|_K, \mu_{T_\varepsilon^{(R)}}\big|_K\Big)\Big] = \mathbb{E}\Big[\pi_\varepsilon^{(R)}\big|_K\big[(C_\varepsilon^{(R)})^c\big]\Big] = (KR+1)\,\frac{1}{R^{1/\gamma}}\,\mathbb{E}\Big[\|\Lambda_0^{(R)}\|\,\mathbb{1}_{\|\Lambda_0^{(R)}\|<R^{1/\gamma}\varepsilon}\Big].$$

In light of Assumption (I),

$$(KR+1)\,\frac{1}{R^{1/\gamma}}\,\mathbb{E}\Big[\|\Lambda_0^{(R)}\|\,\mathbb{1}_{\|\Lambda_0^{(R)}\|<R^{1/\gamma}\varepsilon}\Big] \leq (KR+1)\,\mathbb{E}\Big[\varepsilon \wedge \frac{\|\Lambda_0^{(R)}\|}{R^{1/\gamma}}\Big] \xrightarrow[R\to\infty]{} K\int(\varepsilon \wedge \|\mathbf{s}\|)\,I(\mathrm{d}\mathbf{s}).$$

Finally, for any positive $\eta$, if $K > -\log(\eta/2)$, using Markov's inequality and the monotone convergence theorem,

$$\begin{aligned}\limsup_{R\to\infty} \mathbb{P}\Big[\mathrm{D}_{\mathrm{GHP}}\big(T^{(R)}, T_\varepsilon^{(R)}\big) > \eta\Big] &\leq \limsup_{R\to\infty} \mathbb{P}\Big[\mathrm{D}_{\mathrm{GHP}}\big(T^{(R)}|_K, T_\varepsilon^{(R)}|_K\big) > \eta - 2\mathrm{e}^{-K}\Big] \\ &\leq \limsup_{R\to\infty}\left(\frac{\mathbb{E}\Big[\Big(\mathrm{dis}_{T^{(R)}|_K, T_\varepsilon^{(R)}|_K} C_\varepsilon^{(R)}\big|_K\Big)^{1/\gamma}\Big]}{(\eta - 2\mathrm{e}^{-K})^{1/\gamma}} + \frac{\mathbb{E}\Big[\mathrm{D}\Big(\pi_\varepsilon^{(R)}\big|_K; \mu_{T^{(R)}}\big|_K, \mu_{T_\varepsilon^{(R)}}\big|_K\Big)\Big]}{\eta - 2\mathrm{e}^{-K}}\right) \\ &\leq \left(\frac{2^{1/\gamma}Kh}{(\eta - 2\mathrm{e}^{-K})^{1/\gamma}} + \frac{K}{\eta - 2\mathrm{e}^{-K}}\right)\int(\varepsilon \wedge \|\mathbf{s}\|)\,I(\mathrm{d}\mathbf{s}) \xrightarrow[\varepsilon\to 0]{} 0.\end{aligned}$$

□

The next result is both intuitive and easy to prove. Its proof will therefore be left to the reader.

**Lemma 1.4.11.** *Fix $n$ a positive integer and let $\mathbf{G}_n$ be the restriction of $\mathbf{G}$ to $(\mathbb{R}_+ \times \mathbb{T}_c)^n$; $\mathbf{G}_n$ is a continuous function for the product topology.*

**Lemma 1.4.12.** *Let $K \geq 0$ and $\varepsilon > 0$ be fixed. Almost surely, for any continuous $F : \mathbb{R}_+ \times \mathbb{S}^\downarrow \times \mathbb{T}_c \to \mathbb{R}_+$ bounded by 1,*

$$\limsup_{R\to\infty} \int F(u,\mathbf{s},\tau)\,\mathbb{1}_{u\leq K, \|\mathbf{s}\|\geq\varepsilon}\,\mathrm{d}\Pi_R(u,\mathbf{s},\tau) \leq \int F(u,\mathbf{s},\tau)\,\mathbb{1}_{u\leq K, \|\mathbf{s}\|\geq\varepsilon}\,\mathrm{d}\Pi(u,\mathbf{s},\tau),$$

*and*

$$\liminf_{R\to\infty} \int F(u,\mathbf{s},\tau)\,\mathbb{1}_{u<K, \|\mathbf{s}\|>\varepsilon}\,\mathrm{d}\Pi_R(u,\mathbf{s},\tau) \geq \int F(u,\mathbf{s},\tau)\,\mathbb{1}_{u<K, \|\mathbf{s}\|>\varepsilon}\,\mathrm{d}\Pi(u,\mathbf{s},\tau).$$

*Proof.* Let $\varphi$ and $\varphi_n$, $n \geq 1$ be the functions from $\mathbb{R}_+ \times \mathbb{S}^\downarrow \times \mathbb{T}_c$ to $\mathbb{R}_+$ defined for all $(u,\mathbf{s},\tau)$ by $\varphi(u,\mathbf{s},\tau) := \mathbb{1}_{u\leq K}\mathbb{1}_{\|\mathbf{s}\|\geq\varepsilon}$ and $\varphi_n(u,\mathbf{s},\tau) := [1-n(u-K)_+]_+ \times [1-n(\varepsilon - \|\mathbf{s}\|)_+]_+$ respectively (where $x_+ = x \vee 0$ for any real number $x$). Observe that for all $n \geq 1$, $\varphi_n$ is continuous and that for $n$ large enough, $\varepsilon\varphi_n F$ is an element of $\mathscr{F}$. Therefore, everywhere on the event $\{\Pi_R \to \Pi\}$, $\int \varphi_n F\,\mathrm{d}\Pi_R \to \int \varphi_n F\,\mathrm{d}\Pi$ for any fixed $n \geq 1$.

Furthermore, $\varphi_n \downarrow_n \varphi$ so the monotone convergence theorem yields $\inf_{n\geq 1}\int \varphi_n F\,\mathrm{d}\Pi = \int \varphi F\,\mathrm{d}\Pi$ and for all $R \geq 1$, $\inf_{n\geq 1}\int \varphi_n F\,\mathrm{d}\Pi_R = \int \varphi F\,\mathrm{d}\Pi_R$. As a result, on $\{\Pi_R \to \Pi\}$,

$$\limsup_{R\to\infty} \int \varphi F\,\mathrm{d}\Pi_R \leq \inf_{n\geq 1}\left[\limsup_{R\to\infty} \int \varphi_n F\,\mathrm{d}\Pi_R\right] = \int \varphi F\,\mathrm{d}\Pi.$$

Similarly, if we let $\psi(u,\mathbf{s},\tau) := \mathbb{1}_{u<K}\mathbb{1}_{\|\mathbf{s}\|>\varepsilon}$, there exists a sequence $(\psi_n)_n$ of continuous functions such that $\psi_n \uparrow_n \psi$ and for $n$ large enough, $\varepsilon\psi_n F$ is in $\mathscr{F}$. The same kind of arguments lead to

$$\liminf_{R\to\infty} \int \psi F\,\mathrm{d}\Pi_R \geq \sup_{n\geq 1}\left[\liminf_{R\to\infty} \int \psi_n F\,\mathrm{d}\Pi_R\right] = \int \psi F\,\mathrm{d}\Pi$$

everywhere on $\{\Pi_R \to \Pi\}$. □

**Lemma 1.4.13.** *Let $\varepsilon$ be positive and such that $\Pi\big((u,\mathbf{s},\tau) : \|\mathbf{s}\| = \varepsilon\big) = 0$ a.s.. Then $T_\varepsilon^{(R)}$ a.s. converges to $\mathcal{T}_\varepsilon^{(I)}$ as $R \to \infty$.*

*Proof.* Observe that for any $K \geq 0$, $\Pi\big((u,\mathbf{s},\tau) : u = K\big) = 0$ a.s. which implies that with probability 1, for any continuous bounded $F : \mathbb{R}_+ \times \mathcal{S}^\downarrow \times \mathbb{T}_c \to \mathbb{R}_+$,

$$\int F(u,\mathbf{s},\tau)\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\mathrm{d}\Pi(u,\mathbf{s},\tau) = \int F(u,\mathbf{s},\tau)\mathbb{1}_{u<K,\,\|\mathbf{s}\|>\varepsilon}\,\mathrm{d}\Pi(u,\mathbf{s},\tau).$$

Consequently, in light of Lemma 1.4.12,

$$\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi_R(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau) \xrightarrow[R\to\infty]{\text{a.s.}} \mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau).$$

Furthermore, the measures $\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi_R(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau)$, $R \geq 1$ and $\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau)$ may be written as finite sums of Dirac measures. As a result, almost surely, the atoms of $\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi_R(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau)$ converge to those of $\mathbb{1}_{u\leq K,\,\|\mathbf{s}\|\geq\varepsilon}\,\Pi(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}\tau)$ when $R \to \infty$. Lemma 1.4.11 then ensures that $T_\varepsilon^{(R)}|_K$ a.s. converges to $\mathcal{T}_\varepsilon^{(I)}|_K$. Since this holds for any $K \geq 0$, Proposition 1.3.5 allows us to conclude. □

*Proof of Theorem 1.4.1.* Observe that the set of positive $\varepsilon$ such that $\mathbb{P}\big[\Pi\big((u,\mathbf{s},\tau) : \|\mathbf{s}\| = \varepsilon\big) = 0\big] < 1$ is at most countable. As a result, we may consider a sequence $(\varepsilon_k)_{k\geq 1}$ of positive real numbers which converges to 0 and such that for all $k$, $\Pi\big((u,\mathbf{s},\tau) : \|\mathbf{s}\| = \varepsilon_k\big) = 0$ a.s.. Lemmas 1.4.9, 1.4.10 and 1.4.13 then respectively prove that conditions $(ii)$, $(iii)$ and $(i)$ of Theorem 1.4.8 are met for $T^{(R)}$, $T_{\varepsilon_k}^{(R)}$, $\mathcal{T}_{\varepsilon_k}^{(I)}$, $R \geq 1$, $k \geq 1$ and $\mathcal{T}^{(I)}$. Therefore, $T^{(R)} \Rightarrow \mathcal{T}^{(I)}$ with respect to $\mathrm{D}_{\mathrm{GHP}}$. □

### 1.4.3 Volume growth of infinite Markov branching trees

We now turn to the proof of Proposition 1.4.2. Recall that if $\mathbf{T} \in \mathbb{T}$ is fixed, then $V_{\mathbf{T}}$, the volume growth function of $\mathbf{T}$, is given by

$$V_{\mathbf{T}} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+,\ R \longmapsto \mu_T(T|_R).$$

Notice that $V_{\mathbf{T}}$ is a non-negative, non-decreasing càdlàg function.

*Proof of Proposition 1.4.2.* Proposition 1.3.5 ensures that $(\mathbb{T}, \mathrm{D}_{\mathrm{GHP}})$ is a Polish metric space. In light of Skorokhod's representation theorem and since the assumptions of Theorem 1.4.1 are met, there exist a sequence $(\tau_R)_{R\geq 1}$ of $\mathrm{MB}_\infty^{q,q_\infty}$ trees as well as a $(\gamma,\nu,I)$-fragmentation tree with immigration $\mathcal{T}^{(I)}$ such that $(R^{-1}\tau_R, R^{-1/\gamma}\mu_{\tau_R}) =: T^{(R)}$ a.s. converges to $\mathcal{T}^{(I)}$.

Proposition 1.3.5 and Remark 1.3.2 ensure that a.s., for all $t \geq 0$ such that $\mu_{\mathcal{T}^{(I)}}[\partial_t\mathcal{T}^{(I)}] = 0$, $V_{T^{(R)}}(t)$ converges to $V_{\mathcal{T}^{(I)}}(t)$. Now observe that $\mu_{\mathcal{T}^{(I)}}[\partial_t\mathcal{T}^{(I)}] = 0$ iff $V_{\mathcal{T}^{(I)}}$ is continuous at $t$. Therefore, if we prove that $V_{\mathcal{T}^{(I)}}$ is a.s. continuous on $\mathbb{R}_+$, since volume growth functions are monotone, we may use the following classical result to conclude this proof:

*If $(f_n)_n$ is a sequence of monotone functions from a compact interval $I$ to $\mathbb{R}$ such that $f_n \to f$ point-wise for some continuous function $f$, then $f_n \to f$ uniformly on $I$.*

Following the construction of fragmentation trees with immigration detailed in Section 1.3.2, there exist a Poisson point process $\Sigma = \sum_{i\geq 1} \delta_{(u_i,\mathbf{s}_i)}$ on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ with intensity $\mathrm{d}u \otimes I(\mathrm{d}\mathbf{s})$ and a family $\big[\mathcal{T}_{i,j}; i,j \geq 1\big]$ of i.i.d. $(\gamma, \nu)$-fragmentation trees independent of $\Sigma$ such that

$$\mathcal{T}^{(I)} = \mathbf{G}\left(\left\{\left(u_i, \left\langle \left(s_{i,j}^{\gamma}\mathcal{T}_{i,j}, s_{i,j}\mu_{\mathcal{T}_{i,j}}\right); j \geq 1\right\rangle\right) \,:\, i \geq 1\right\}\right).$$

With these notations, we may write $V_{\mathcal{T}^{(I)}} = \sum_{i\geq 1}\sum_{j\geq 1} s_{i,j} V_{\mathcal{T}_{i,j}}\big[(\cdot - u_i)_+ / s_{i,j}^{\gamma}\big]$. Furthermore, for any non-negative $K$, since $V_{\mathcal{T}_{i,j}} \leq 1$ for all $i,j \geq 1$,

$$\sum_{i\geq 1}\sum_{j\geq 1} s_{i,j}\mathbb{1}_{u_i\leq K} = \int \mathbb{1}_{u\leq K}\,\|\mathbf{s}\|\,\Sigma(\mathrm{d}u, \mathrm{d}\mathbf{s})$$

which is a.s. finite, as already noticed. As a result and in light of the Weierstrass $M$-test, the restriction of $V_{\mathcal{T}^{(I)}}$ to the compact interval $[0,K]$ is a series which a.s. converges uniformly on $[0,K]$.

Proposition 1.9 in [21] implies that the volume growth function of $(\gamma, \nu)$-fragmentation trees is a.s. continuous. In particular, with probability one, $V_{\mathcal{T}_{i,j}}$ is continuous for all $i$ and $j$. As a uniformly converging series of continuous functions, $V_{\mathcal{T}^{(I)}}|_{[0,K]}$ is a.s. continuous on $[0,K]$. Since this holds for any $K \geq 0$, $V_{\mathcal{T}^{(I)}}$ is a.s. continuous on $\mathbb{R}_+$, which concludes this proof. □

### 1.4.4 Unary immigration measures

Before concluding this section, we will state a useful criterion to prove Assumption (I) when the limit immigration measure is unary, i.e. supported by the set $\{(s,0,0,\dots) : s > 0\}$. In light of Remark 1.4.1, we will only study *self-similar* unary immigration measures.

Let $\gamma \in (0,1)$. Proposition 1.3.11 ensures that any unary $\gamma$-self-similar immigration measure may be written as $c\, I_\gamma^{\mathrm{un}}$ where $c$ is a positive constant and $I_\gamma^{\mathrm{un}}$ is the measure defined by

$$\int_{\mathcal{S}^\downarrow} f \,\mathrm{d}I_\gamma^{\mathrm{un}} = \int_0^\infty f(x,0,0,\dots)\, x^{-1-\gamma}\,\mathrm{d}x$$

for any measurable $f : \mathcal{S}^\downarrow \to \mathbb{R}_+$.

*Remark 1.4.4.* Recall the immigration measures defined in Section 1.3.2. The Brownian immigration measure $I_B$ is unary and may be written as $I_B = (2/\pi)^{1/2} I_{1/2}^{\mathrm{un}}$. On the other hand, for any $\alpha \in (1,2)$, $I^{(\alpha)}$ isn't unary.

**Lemma 1.4.14.** *Let $X$ be an integer valued random variable such that there exist $\gamma \in (0,1)$ and a positive constant $c$ satisfying $n^{1+\gamma}\mathbb{P}[X = n] \to c$. In this case, for all continuous $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) \leq 1 \wedge x$, $R\,\mathbb{E}[f(X/R^{1/\gamma})] \to \int_0^\infty c\, f(x)\, x^{-1-\gamma}\,\mathrm{d}x$ as $R$ goes to infinity.*

*Proof.* By assumption, for all $\varepsilon > 0$, there exists an integer $N$ such that for all $n \geq N$, $\left|n^{1+\gamma}\mathbb{P}[X=n] - c\right| < \varepsilon$. As a result

$$R\sum_{n>N}(c-\varepsilon)\frac{1}{n^{1+\gamma}} f\left(\frac{n}{R^{1/\gamma}}\right) \leq R\,\mathbb{E}\left[f\left(\frac{X}{R^{1/\gamma}}\right)\right] \leq R\sum_{n=1}^{N}\frac{n}{R^{1/\gamma}} + R\sum_{n>N}(c+\varepsilon)\frac{1}{n^{1+\gamma}} f\left(\frac{n}{R^{1/\gamma}}\right).$$

As a Riemann sum, $R\sum_{n>N} n^{-1-\gamma} f(n/R^{1/\gamma})$ converges toward $\int_0^\infty f(x)\,x^{-1-\gamma}\,\mathrm{d}x$ as $R$ goes to infinity. The desired result then follows. □

**Proposition 1.4.15.** *Let $\Lambda$ be a random finite partition such that as $n \to \infty$, $n^{1+\gamma}\mathbb{P}[\|\Lambda\| = n] \to c$ for some $\gamma \in (0,1)$, $c > 0$ and $n^\gamma \mathbb{P}[\Lambda_1 \geq n]$ converges to $c/\gamma$. For all $R \geq 1$, let $q^{(R)}$ be the distribution of $\Lambda/R^{1/\gamma}$. Then, $R(1 \wedge \|\mathbf{s}\|)q^{(R)}(\mathrm{d}\mathbf{s})$ converges weakly to $(1\wedge\|\mathbf{s}\|)\,c\,I_\gamma^{\mathrm{un}}(\mathrm{d}\mathbf{s})$ as $R \to \infty$ in the sense of finite measures on $\mathcal{S}^\downarrow$.*

*Proof.* The main idea for this proof is to show that the tail of $\Lambda$ is asymptotically negligible when its first component is large, or more precisely, that $R\,\mathbb{E}\big[1 \wedge \big([\|\Lambda\| - \Lambda_1]/R^{1/\gamma}\big)\big]$ converges to 0 when $R$ goes to infinity. Since $\|\Lambda\|$ fulfils the assumptions of Lemma 1.4.14,

$$R\,\mathbb{E}\big[1 \wedge (\|\Lambda\|/R^{1/\gamma})\big] \xrightarrow[R\to\infty]{} c\int 1\wedge\|\mathbf{s}\|\, I_\gamma^{\mathrm{un}}(\mathrm{d}\mathbf{s}) = c/[\gamma(1-\gamma)] =: C_\gamma$$

Furthermore, $\Lambda_1 \le \|\Lambda\|$, so we get that $\limsup_{R\to\infty} R\,\mathbb{E}\big[1\wedge(\Lambda_1/R^{1/\gamma})\big] \le C_\gamma$. In light of Fatou's lemma and the assumption on the probability tail of $\Lambda_1$,

$$\liminf_{R\to\infty} R\,\mathbb{E}\left[1\wedge\frac{\Lambda_1}{R^{1/\gamma}}\right] = \liminf_{R\to\infty}\int_0^1 R\,\mathbb{P}\big[\Lambda_1 \ge R^{1/\gamma}t\big]\,\mathrm{d}t \ge \int_0^1 c\gamma^{-1}t^{-\gamma}\,\mathrm{d}t = C_\gamma.$$

In summary, when $R\to\infty$, $R\,\mathbb{E}\big[1\wedge(\Lambda_1/R^{1/\gamma})\big]\to C_\gamma$.

Now observe that if $a$, $b$, $x$ and $y$ are four real numbers, then $a\wedge x + b\wedge y \le (a+b)\wedge(x+y)$. In particular, for all $\varepsilon\in(0,1)$, $1\wedge(\|\Lambda\|/R^{1/\gamma}) \ge (1-\varepsilon)\wedge(\Lambda_1/R^{1/\gamma}) + \varepsilon\wedge\big([\|\Lambda\|-\Lambda_1]/R^{1/\gamma}\big)$. Moreover,

$$\begin{aligned}\lim_{R\to\infty} R\,\mathbb{E}\left[(1-\varepsilon)\wedge\frac{\Lambda_1}{R^{1/\gamma}}\right] &= \lim_{R\to\infty}(1-\varepsilon)R\,\mathbb{E}\left[1\wedge\frac{\Lambda_1}{[(1-\varepsilon)^\gamma R]^{1/\gamma}}\right]\\ &= (1-\varepsilon)^{1-\gamma}\left(\lim_{S\to\infty} S\,\mathbb{E}\left[1\wedge\frac{\Lambda_1}{S^{1/\gamma}}\right]\right) = (1-\varepsilon)^{1-\gamma}\,C_\gamma\end{aligned}$$

where we have taken $S=(1-\varepsilon)^\gamma R$. Similarly, for $S=\varepsilon^\gamma R$,

$$\limsup_{R\to\infty} R\,\mathbb{E}\left[\varepsilon\wedge\frac{\|\Lambda\|-\Lambda_1}{R^{1/\gamma}}\right] = \varepsilon^{1-\gamma}\left(\limsup_{S\to\infty} S\,\mathbb{E}\left[1\wedge\frac{\|\Lambda\|-\Lambda_1}{S^{1/\gamma}}\right]\right).$$

Therefore,

$$\limsup_{R\to\infty} R\,\mathbb{E}\left[1\wedge\frac{\|\Lambda\|-\Lambda_1}{R^{1/\gamma}}\right] \le \inf_{\varepsilon\in(0,1)}\frac{C_\gamma-(1-\varepsilon)^{1-\gamma}C_\gamma}{\varepsilon^{1-\gamma}} = 0.$$

Let $f:\mathbb{S}^\downarrow\to\mathbb{R}_+$ be a Lipschitz-continuous function bounded by 1 and set $g(x):=f(x,0,0,\dots)$ for all $x\ge 0$. There exists a constant $K\ge 0$ such that for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{S}^\downarrow$, $|f(\mathbf{x})-f(\mathbf{y})|\le 1\wedge(K\,\|\mathbf{x}-\mathbf{y}\|)$. Therefore

$$\left|R\,\mathbb{E}\left[\left(1\wedge\frac{\|\Lambda\|}{R^{1/\gamma}}\right)f\left(\frac{\Lambda}{R^{1/\gamma}}\right)-\left(1\wedge\frac{\|\Lambda\|}{R^{1/\gamma}}\right)g\left(\frac{\|\Lambda\|}{R^{1/\gamma}}\right)\right]\right| \le R\,\mathbb{E}\left[1\wedge\frac{2K\,(\|\Lambda\|-\Lambda_1)}{R^{1/\gamma}}\right]\xrightarrow[R\to\infty]{} 0.$$

Used jointly with our assumption on $\|\Lambda\|$ and Lemma 1.4.14, this ensures that $R\,\mathbb{E}\big[(1\wedge\|\Lambda\|/R^{1/\gamma})f(\Lambda/R^{1/\gamma})\big]$ converges to $\int(1\wedge\|\mathbf{s}\|)f(\mathbf{s})\,c\,I_\gamma^{\mathrm{un}}(\mathrm{d}\mathbf{s})$ as $R\to\infty$. Lemma 1.3.14 concludes this proof. □

## 1.5 Applications

In this section, we will develop applications of our three main results (Theorems 1.2.5, 1.4.1 and Proposition 1.4.2) to various models of random trees which satisfy the Markov branching property. With our unified approach, we will recover known results and get new ones.

### 1.5.1 Galton-Watson trees

Let $\xi$ be a probability measure on $\mathbb{Z}_+$ with mean 1 and $\xi(1)<1$ (*critical regime*). We will be interested in *unordered* Galton-Watson trees with offspring ditribution $\xi$, the law of which we will write $\mathrm{GW}_\xi$. For any finite tree $\mathtt{t}$,

$$\mathrm{GW}_\xi(\mathtt{t}) := \sum_{\mathtt{t}'\in\mathbb{T}^{\mathrm{ord}}:\,\mathtt{t}'\sim\mathtt{t}}\ \prod_{u\in\mathtt{t}'}\xi\big[c_u(\mathtt{t}')\big].$$

For each positive integer $n$ such that $\mathrm{GW}_\xi(\mathbb{T}_n)>0$, let $\mathrm{GW}_\xi^n$ be the measure $\mathrm{GW}_\xi$ conditioned on the set $\mathbb{T}_n$ of trees with $n$ vertices. Similarly, if $n$ satisfies $\mathrm{GW}_\xi(\mathbb{T}_{\mathcal{L},n})>0$, define $\mathrm{GW}_\xi^{\mathcal{L},n}$ as $\mathrm{GW}_\xi$ conditioned on the set $\mathbb{T}_{\mathcal{L},n}$ of trees with $n$ leaves. Moreover, let $d:=\gcd\{n-1;\mathrm{GW}_\xi(\mathbb{T}_n)>0\}$ and $d_{\mathcal{L}}:=\gcd\{n-1;\mathrm{GW}_\xi(\mathbb{T}_{\mathcal{L},n})>0\}$.

*Kesten's tree.* Let $\hat{\xi}$ be the *size-biased distribution* of $\xi$, that is $\hat{\xi}(k) = k\xi(k)$ for all $k \geq 0$. By assumption, the mean of $\xi$ is 1, so $\hat{\xi}$ is a probability measure. We define $\mathrm{GW}_\xi^\infty$ as the distribution of *Kesten's tree* which is obtained as follows:

— Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. random variables such that $X_n + 1$ follows $\hat{\xi}$,
— Independently of this sequence, let $(T_{n,k}; n \geq 0, k \geq 1)$ be i.i.d. $\mathrm{GW}_\xi$ trees,
— For each $n \geq 0$, let $T_n := [\![ T_{n,1}, \dots, T_{n,X_n} ]\!]$,
— For all $n \geq 0$, graft $T_n$ on an infinite branch at height $n$ respectively, i.e. set $T := \mathfrak{b}_\infty \bigotimes_{n\geq 0}(\mathfrak{v}_n, T_n)$ and denote its distribution by $\mathrm{GW}_\xi^\infty$.

*Remark 1.5.1.* These infinite trees were first indirectly introduced in [83] by Kesten who studied the genealogy of Galton-Watson processes conditioned to hit 0 after a large time. This result entails that if $T$ is a $\mathrm{GW}_\xi$ tree, the conditional distribution of $T$ on $|T| \geq n$ converges to $\mathrm{GW}_\xi^\infty$ as $n \to \infty$. Kesten's tree can thus be, in a way, considered as a $\mathrm{GW}_\xi$ tree conditioned to have infinite height.

This tree also appears as the local limit of conditioned critical Galton-Watson trees under various types of conditioning, see [2]. In particular, it was first proved in [81] (in terms of Galton-Watson processes) and in [12] (in terms of trees) that if $\xi$ is critical and has finite variance, then $\mathrm{GW}_\xi^n \Rightarrow \mathrm{GW}_\xi^\infty$. In [38], it was shown that under the same assumptions, $\mathrm{GW}_\xi^{\mathcal{L},n} \Rightarrow \mathrm{GW}_\xi^\infty$. In both cases, the finite variance assumption may be dropped, see [76] and [2].

The local limits of Galton-Watson trees conditioned on their size with offspring distribution with means less than 1 were studied in [77], [76] and [1]. See also [117] for the study of the local limits of multi-type critical Galton-Watson trees.

Using Theorem 1.2.5, we will recover the following proposition in Section 1.5.1.

**Proposition 1.5.1.** *In the sense of the* $\mathrm{d_{loc}}$ *topology,* $\mathrm{GW}_\xi^n$ *and* $\mathrm{GW}_\xi^{\mathcal{L},n}$ *both converge weakly towards* $\mathrm{GW}_\xi^\infty$.

Afterwards, we will study scaling limits of Kesten's tree in the spirit of Theorem 1.4.1. Recall the descriptions of the Brownian tree with immigration and $\alpha$-stable Lévy trees with immigration from Section 1.3.2.

**Proposition 1.5.2.** *Let $T$ be a tree with distribution $\mathrm{GW}_\xi^\infty$ and define $\mu_T := \sum_{u\in T}\delta_u$ and $\mu_T^{\mathcal{L}} := \sum_{u\in\mathcal{L}(T)}\delta_u$ the counting measures on the set of its vertices and leaves respectively.*

$(i)$ Finite variance: *Suppose $\xi$ has finite variance $\sigma^2$ and that $d = 1$. Then, with respect to the $\mathrm{D_{GHP}}$ topology,*

$$\left(\frac{T}{R}, \frac{\mu_T}{R^2}\right) \xrightarrow[R\to\infty]{(\mathrm{d})} \left(\mathcal{T}_B, \frac{\sigma^2}{4}\mu_B\right)$$

*where $(\mathcal{T}_B, \mu_B)$ is the Brownian tree with immigration.*

$(i')$ *If $\xi$ has finite variance $\sigma^2$ and if $d_{\mathcal{L}} = 1$, then*

$$\left(\frac{T}{R}, \frac{\mu_T^{\mathcal{L}}}{R^2}\right) \xrightarrow[R\to\infty]{(\mathrm{d})} \left(\mathcal{T}_B, \frac{\sigma^2\,\xi(0)}{4}\mu_B\right).$$

$(ii)$ Stable case: *Suppose that $\xi(n) \sim c\, n^{-1-\alpha}$ as $n \to \infty$ for some positive constant $c$ and $\alpha \in (1,2)$. Then,*

$$\left(\frac{T}{R}, \frac{\mu_T}{R^{\alpha/(\alpha-1)}}\right) \xrightarrow[R\to\infty]{(\mathrm{d})} \left(\mathcal{T}_\alpha, (ck_\alpha)^{1/(\alpha-1)}\mu_\alpha\right)$$

*where $(\mathcal{T}_\alpha, \mu_\alpha)$ is the $\alpha$-stable immigration Lévy tree and $k_\alpha = \Gamma(2-\alpha)/[\alpha(\alpha-1)]$.*

*Remark 1.5.2.* Both $(i)$ and $(ii)$ were proved in [48] and $(i')$ seems to be a new, if predictable, result.

We also mention that under the assumptions of $(ii)$, $(T/R, \mu_T^{\mathcal{L}}/R^{\alpha/(\alpha-1)})$ should converge in distribution to $\left(\mathcal{T}_\alpha, (ck_\alpha)^{1/(\alpha-1)}\,\xi(0)\mu_\alpha\right)$. We won't prove this statement as Assumption (S) hasn't been proved in this case and to do so would require quite a bit of computation. The scaling limits of Galton-Watson trees with such an offspring distribution conditioned on their number of leaves were however studied in [85].

Section 1.5.1 will focus on the finite variance case, first on $(i)$ and then on $(i')$. We will prove Proposition 1.5.2 in the stable case $(ii)$ in Section 1.5.1.

**Markov branching property and local limits.** Let $\mathcal{N} := \{n \geq 1 : \mathrm{GW}_\xi(\mathrm{T}_n) > 0\}$. Proposition 37 in [66] states that the sequence of probability measures $(\mathrm{GW}_\xi^n)_{n\in\mathcal{N}}$ satisfies the Markov branching property, i.e. we have $\mathrm{GW}_\xi^n = \mathrm{MB}_n^q$ for all adequate $n$ with $q_{n-1}$ defined for all $\lambda = (\lambda_1, \dots, \lambda_p)$ in $\mathcal{P}_{n-1}$ by

$$q_{n-1}(\lambda) = \frac{p!\xi(p)}{\prod_{j\geq 1} m_j(\lambda)!} \frac{\prod_{i=1}^p \mathbb{P}[\#T = \lambda_i]}{\mathbb{P}[\#T = n]}$$

where $T$ is a $\mathrm{GW}_\xi$ tree.

Similarly, if we let $\mathcal{N}_\mathcal{L} := \{n \geq 1 : \mathrm{GW}_\xi(\mathrm{T}_{\mathcal{L},n}) > 0\}$, then in light of [111, Lemma 8], the family $(\mathrm{GW}_\xi^{\mathcal{L},n})_{n\in\mathcal{N}_\mathcal{L}}$ of probability measures satisfies the Markov branching property and the associated sequence $q^\mathcal{L}$ of first-split distributions such that $\mathrm{GW}_\xi^{\mathcal{L},n} = \mathrm{MB}_n^{\mathcal{L},q^\mathcal{L}}$ is given for all $n$ in $\mathcal{N}_\mathcal{L}$ and $\lambda = (\lambda_1, \dots, \lambda_p)$ in $\mathcal{P}_n$ by

$$q_n^\mathcal{L}(\lambda) = \frac{p!\xi(p)}{\prod_{j\geq 1} m_j(\lambda)!} \frac{\prod_{i=1}^p \mathbb{P}[\#_\mathcal{L} T = \lambda_i]}{\mathbb{P}[\#T_\mathcal{L} = n]}$$

where $T$ still denotes a $\mathrm{GW}_\xi$ tree.

A Kesten tree with distribution $\mathrm{GW}_\xi^\infty$ can be seen as an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{q,q_\infty}$ where $q_\infty$ is defined for any $\lambda = (\lambda_2, \dots, \lambda_p)$ in $\mathcal{P}_{<\infty}$ by

$$q_\infty(\infty, \lambda) := \hat{\xi}(p) \frac{(p-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \prod_{i=2}^p \mathbb{P}[\#T = \lambda_i].$$

The distribution of Kesten's tree may also be rewritten as $\mathrm{GW}_\xi^\infty = \mathrm{MB}_\infty^{\mathcal{L},q^\mathcal{L},q_\infty^\mathcal{L}}$ where $q_\infty^\mathcal{L}$ is given for all $\lambda \in \mathcal{P}_{<\infty}$ by

$$q_\infty^\mathcal{L}(\infty, \lambda) = \hat{\xi}(p) \frac{(p-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \prod_{i=2}^p \mathbb{P}[\#_\mathcal{L} T = \lambda_i].$$

Proposition 1.5.1 is a direct consequence of the following results from Sections 4.3 and 4.4 in [2] used alongside Theorem 1.2.5.

**Lemma 1.5.3.** *If $T$ is a $\mathrm{GW}_\xi$ tree, then*

$$\frac{\mathbb{P}[\#T = (n+1)d + 1]}{\mathbb{P}[\#T = nd + 1]} \xrightarrow[n\to\infty]{} 1 \quad \text{and} \quad \frac{\mathbb{P}[\#_\mathcal{L} T = (n+1)d_\mathcal{L} + 1]}{\mathbb{P}[\#_\mathcal{L} T = nd_\mathcal{L} + 1]} \xrightarrow[n\to\infty]{} 1.$$

*Proof of Proposition 1.5.1.* Let $\lambda = (\lambda_2, \dots, \lambda_p)$ be an element of $\mathcal{P}_{<\infty}$. If there exists $2 \leq i \leq p$ such that $\lambda_i - 1$ isn't divisible by $d$, then for all $n \in \mathcal{N}$, $q_{n-1}(n-1-\|\lambda\|, \lambda) = 0 = q_\infty(\infty, \lambda)$. Otherwise, for $n \in \mathcal{N}$ large enough, in light of Lemma 1.5.3

$$\begin{aligned} q_{n-1}\big(n-1-\|\lambda\|, \lambda\big) &= \frac{p!\xi(p)}{\prod_{j\geq 1} m_j(\lambda)!} \frac{\mathbb{P}[\#T = n - \|\lambda\|]}{\mathbb{P}[\#T = n]} \prod_{i=1}^p \mathbb{P}[\#T = \lambda_i] \\ &\xrightarrow[n\to\infty]{} \hat{\xi}(p) \frac{(p-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \prod_{i=2}^p \mathbb{P}[\#T = \lambda_i] = q_\infty(\infty, \lambda). \end{aligned}$$

Similarly, as $n$ goes to infinity, $q_n^\mathcal{L}(n - \|\lambda\|, \lambda) \to q_\infty^\mathcal{L}(\infty, \lambda)$. Since these hold for any $\lambda$ in $\mathcal{P}_{<\infty}$, we end this proof by using Corollary 1.2.6. □

**Scaling limits, finite variance.** In the remainder of this section, $(T_i)_{i\geq 1}$ will denote i.i.d. Galton-Watson trees with offspring distribution $\xi$, $(Y_n)_{n\geq 1}$, i.i.d. $\xi$ distributed random variables and for all $n \geq 1$, $S_n := Y_1 + \cdots + Y_n - n$. We will also consider $N$, a random variable independent of both $(T_i)_i$ and $(Y_n)_n$ and such that $N + 1$ follows $\hat{\xi}$.

The following so called *Otter-Dwass' formula* or *cyclic lemma* (see [106, Chapter 6] for instance) will be the cornerstone of many forthcoming computations.

**Lemma 1.5.4 (Otter-Dwass' formula).** *With these notations, for all $k \geq 1$ and $n \geq 1$,*

$$\mathbb{P}\big[\#T_1 + \cdots + \#T_k = n\big] = \frac{k}{n}\mathbb{P}\big[S_n = -k\big].$$

Let $q_*$ be the probability distribution on $\mathcal{P}_{<\infty}$ defined by $q_* = q_\infty(\infty, \cdot)$. Let $\Lambda$ follow $q_*$ and recall that it has the same distribution as $(\#T_1, \ldots, \#T_N)^\downarrow$.

In this paragraph, we'll assume that the variance $\sigma^2$ of $\xi$ is finite and that $d = 1$. Recall that the Brownian tree with immigration is a $(1/2, \nu_B, I_B)$-fragmentation tree with immigration. It was proved in [66, Section 5.1] that Assumption (S) of Theorem 1.4.1 is fulfilled for $\gamma = 1/2$ and $\nu = \sigma/2 \cdot \nu_B$. To prove Proposition 1.5.2, it will therefore be sufficient to show that Assumption (I) is satisfied for $\gamma = 1/2$ and $I = \sigma/2 \cdot I_B$. For all $R \geq 1$, let $q^{(R)}$ be the distribution of $\Lambda/R^2$.

**Proposition 1.5.5.** *In the sense of weak convergence of finite measures on $\mathbb{S}^\downarrow$, $R(1 \wedge \|\mathbf{s}\|)q^{(R)}(\mathrm{d}\mathbf{s})$ converges as $R$ goes to infinity toward $(1 \wedge \|\mathbf{s}\|)\,\sigma/2 \cdot I_B(\mathrm{d}\mathbf{s})$.*

Since $I_B$ is unary, in order to prove Proposition 1.5.5, it will be enough to show that $\Lambda$ satisfies the assumptions of Proposition 1.4.15. The next two lemmas will prove that both are met.

**Lemma 1.5.6.** *When $n$ goes to infinity, $n^{3/2}\mathbb{P}[\|\Lambda\| = n] \to (\sigma^2/2\pi)^{1/2}$.*

*Proof.* In light of Otter-Dwass' formula, for all $n \geq 1$,

$$\begin{aligned} n^{3/2}\mathbb{P}\big[\|\Lambda\| = n\big] &= n^{3/2}\textstyle\sum_{k\geq 1}\mathbb{P}\big[\#T_1 + \cdots + \#T_k = n \,|\, N = k\big]\mathbb{P}\big[N = k\big] \\ &= \textstyle\sum_{k\geq 1} k\hat{\xi}(k+1)\, n^{1/2}\,\mathbb{P}\big[S_n = -k\big]. \end{aligned}$$

Recall the local central limit theorem in the finite variance case:

$$\sup_{k\in\mathbb{Z}}\Big|n^{1/2}\,\mathbb{P}[S_n = k] - (2\pi\sigma^2)^{-1/2}\,\mathrm{e}^{-k^2/2n\sigma^2}\Big| \xrightarrow[n\to\infty]{} 0.$$

As a result, there exists a finite constant $C$ such that $n^{1/2}\mathbb{P}[S_n = -k] \leq C$ for all $n \geq 1$ and $k \geq 1$ and if $k \geq 1$ is fixed, $n^{1/2}\mathbb{P}[S_n = -k] \to (2\pi\sigma^2)^{-1/2}$. Furthermore, $\sum_{k\geq 1} k\hat{\xi}(k+1) = \sigma^2$ so Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} n^{3/2}\mathbb{P}[\|\Lambda\| = n] = \sum_{k\geq 1} k\hat{\xi}(k+1)\left(\lim_{n\to\infty} n^{1/2}\mathbb{P}[S_n = -k]\right) = \left(\sigma^2/2\pi\right)^{1/2}.$$

□

**Lemma 1.5.7.** *When $n \to \infty$, $n^{1/2}\mathbb{P}[\Lambda_1 \geq n]$ converges to $(2\sigma^2/\pi)^{1/2}$.*

*Proof.* Observe that for all $n \geq 0$, the event $\{\Lambda_1 \geq n\}$ has the same probability as $\{N \geq 1, \exists i \leq N : \#T_i \geq n\}$. Therefore $\mathbb{P}[\Lambda_1 \geq n] = \sum_{k\geq 1}\hat{\xi}(k+1)\big(1 - \mathbb{P}[\#T_1 < n]^k\big)$. Let $G$ be the generating function of $\xi$, i.e. $G(s) = \sum_{k\geq 0}\xi(k)s^k$ for all $s \in [0,1]$. This function is twice-differentiable on $[0,1]$ and we may write $\mathbb{P}[\Lambda_1 \geq n] = G'(1) - G'\big(1 - \mathbb{P}[\#T_1 \geq n]\big)$.

For all $n \geq 1$, Otter-Dwass' formula gives $n^{1/2}\mathbb{P}[\#T_1 \geq n] = n^{1/2}\sum_{m\geq n} m^{-1}\,\mathbb{P}[S_m = -1]$. The local central limit theorem ensures that $m^{1/2}\mathbb{P}[S_m = -1] \to (2\pi\sigma^2)^{-1/2}$ as $m \to \infty$. Therefore, for all positive $\varepsilon$ and $n$ large enough,

$$n^{1/2}\left|\mathbb{P}[\#T_1 \geq n] - \textstyle\sum_{m\geq n} m^{-3/2}(2\pi\sigma^2)^{-1/2}\right| \leq n^{1/2}\sum_{m\geq n} m^{-3/2}\,\varepsilon \xrightarrow[n\to\infty]{} 2\varepsilon.$$

Incidentally, $n^{1/2}\mathbb{P}[\#T_1 \geq n]$ and $n^{1/2}\sum_{m\geq n} m^{-3/2}(2\pi\sigma^2)^{-1/2}$ have the same limit when $n \to \infty$ which is to say that $n^{1/2}\mathbb{P}[\#T_1 \geq n] \to (2/\pi\sigma^2)^{1/2}$ as $n \to \infty$. As a result,

$$n^{1/2}\mathbb{P}[\Lambda_1 \geq n] = n^{1/2}\Big[G'(1) - G'\big(1 - \mathbb{P}[\#T_1 \geq n]\big)\Big] \xrightarrow[n\to\infty]{} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} G''(1) = \left(\frac{2\sigma^2}{\pi}\right)^{1/2}.$$

□

Lemmas 1.5.6 and 1.5.7 and Proposition 1.4.15 prove Proposition 1.5.5. Theorem 1.4.1 therefore implies that $(T/R, \mu_T/R^2)$ converges in distribution to a $(1/2, \sigma/2\cdot\nu_B, \sigma/2\cdot I_B)$ fragmentation tree with immigration. Using Proposition 1.3.12, we may restate this last result as Proposition 1.5.2 $(i)$. Furthermore, as a result of Proposition 1.4.2, we get that in particular, $\mu_T(T|_R)/R^2$ converges in distribution to $(\sigma^2/4)\,\mu_{\mathcal{T}_B}(\mathcal{T}_B|_1)$ or equivalently to $\mu_{\mathcal{T}_B}(\mathcal{T}_B|_{\sigma/2})$.

We will now prove Proposition 1.5.2 $(i')$. Assume that $d_{\mathcal{L}} = 1$. Theorem 7 in [111] proves that the family $(q_n^{\mathcal{L}})_n$ of first split distributions associated to Galton-Watson trees conditioned on their number of leaves satisfies Assumption (S): $n^{1/2}(1-s_1)\,\bar{q}_n^{\mathcal{L}} \Rightarrow \sigma\,\xi(0)^{1/2}/2\cdot(1-s_1)\,\nu_B(\mathrm{d}\mathbf{s})$. As a result, we only need to prove Assumption (I) for $\gamma = 1/2$ and $I = \sigma\,\xi(0)^{1/2}/2\cdot I_B$.

*Proof of Proposition 1.5.2* $(i')$. Theorem 6 in [111] states that there exists a critical probability distribution $\zeta$ on $\mathbb{Z}_+$ such that $\#_{\mathcal{L}}T_1$, the number of leaves of $T_1$, has the same distribution as $\#\tau$, where $\tau$ follows $\mathrm{GW}_\zeta$. Lemma 6 further states that if $\xi$ has finite variance $\sigma^2$, then $\zeta$ has variance $\sigma^2/\xi(0)$.

Let $\Lambda^{\mathcal{L}}$ be such that $(\infty, \Lambda^{\mathcal{L}})$ is distributed according to $q_\infty^{\mathcal{L}}$. The random partition $\Lambda^{\mathcal{L}}$ is distributed like $(\#_{\mathcal{L}}T_1, \dots, \#_{\mathcal{L}}T_N)^{\downarrow}$, or equivalently, like $(\#\tau_1, \dots, \#\tau_N)^{\downarrow}$, where $(\tau_n)_{n\geq 1}$ are i.i.d. $\mathrm{GW}_\zeta$ trees independent of $N$. Therefore, if $(V_n)_{n\geq 1}$ is a sequence of i.i.d. $\zeta$-distributed random variables and if $Z_n := V_1 + \cdots + V_n - n$, proceeding as in the proof of Lemma 1.5.6 gives:

$$n^{3/2}\mathbb{P}[\|\Lambda^{\mathcal{L}}\| = n] = \textstyle\sum_{k\geq 0} k\,\hat{\xi}(k+1)n^{1/2}\mathbb{P}[Z_n = -k] \xrightarrow[n\to\infty]{} \left[\sigma^2\xi(0)/(2\pi)\right]^{1/2}.$$

Similarly, the same kind of computations as in Lemma 1.5.7 yields

$$n^{1/2}\mathbb{P}[\Lambda_1^{\mathcal{L}} \geq n] = n^{1/2}\Big[G'(1) - G'\big(1-\mathbb{P}[\#\tau_1 \geq n]\big)\Big] \xrightarrow[n\to\infty]{} \left[2\sigma^2\xi(0)/\pi\right]^{1/2}$$

where $G$ still denotes the generating function of $\xi$. As a result, because of Theorem 1.4.1 and Proposition 1.4.15, when $R \to \infty$, $(T/R, \mu_T^{\mathcal{L}}/R^2)$ converges in distribution to a $(1/2, \sigma\xi(0)^{1/2}/2\cdot\nu_B, \sigma\xi(0)^{1/2}/2\cdot I_B)$ fragmentaion tree with immigration. Proposition 1.3.12 then allows us to conlude. □

**Scaling limits, stable case.** In this paragraph, we'll suppose that there exist $\alpha \in (1,2)$ and a positive constant $c$ such that $n^{1+\alpha}\xi(n) \to c$ when $n \to \infty$.

Recall that $\Lambda$ denotes a $q_*$-distributed variable and has the same distribution as $(\#T_1, \dots, \#T_N)^{\downarrow}$ where $N+1$ is distributed according to $\hat{\xi}$ and is independent of the sequence $(T_n)_{n\geq 1}$ of i.i.d. $\mathrm{GW}_\xi$ trees. Moreover, we will use the notations introduced to define $\nu_\alpha$ and $I^{(\alpha)}$ in Section 1.3.2: $(\Sigma_t; t \geq 0)$ will denote a $1/\alpha$-stable subordinator with Laplace exponent $\lambda \mapsto -\log\mathbb{E}[\exp(-\lambda\Sigma_1)] = \lambda^{1/\alpha}$ and $\Delta$ will be the decreasing rearrangement of its jumps on $[0,1]$.

It was proved in [66, Section 5.2] that the family $q = (q_n)_{n\in\mathcal{N}}$ of first-split distributions associated to $(\mathrm{GW}_\xi^n)_{n\in\mathcal{N}}$ satisfies Assumption (S) of Theorem 1.4.1 for $\gamma = 1 - 1/\alpha$ and $\nu = (c\,k_\alpha)^{1/\alpha}\cdot\nu_\alpha$. Proposition 1.5.2 $(ii)$ will therefore be a consequence of the next proposition. For all $R \geq 1$, write $q^{(R)}$ for the distribution of $R^{-\alpha/(\alpha-1)}\Lambda$.

**Proposition 1.5.8.** *When $R \to \infty$, $R(1 \wedge \|\mathbf{s}\|)\,q^{(R)}(\mathrm{d}\mathbf{s})$ converges weakly to $(c\,k_\alpha)^{1/\alpha}(1 \wedge \|\mathbf{s}\|)\,I^{(\alpha)}(\mathrm{d}\mathbf{s})$.*

*Proof.* As shown in [66, Section 5.2], $n^{1+1/\alpha}\,\mathbb{P}[\#T_1 = n]$ converges to $[(c\,k_\alpha)^{1/\alpha}\alpha\,\Gamma(1-1/\alpha)]^{-1}$. Therefore, $(\#T_n)_{n\geq 1}$ lies in the domain of attraction of a $1/\alpha$-stable distribution. More accurately, in the Skorokhod topology,

$$\left(\frac{\#T_1 + \cdots + \#T_{\lfloor nt\rfloor}}{n^\alpha}\,;\, t \geq 0\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \frac{1}{c\,k_\alpha}\big(\Sigma_t\,;\, t \geq 0\big).$$

This, in conjunction with Skorokhod's representation theorem, implies that there exists a sequence $(X_n)_{n\geq 0}$, where for all $n \geq 1$,

$$X_n \overset{(\mathrm{d})}{=} \frac{c\,k_\alpha}{n^\alpha}\big(\#T_1, \dots, \#T_n, 0, 0, \dots\big)^{\downarrow},$$

which a.s. converges to (a version of) $\Delta$.

Let $F : \mathbb{S}^{\downarrow} \to \mathbb{R}_+$ be a Lipschitz continuous function such that $F(\mathbf{s}) \leq 1 \wedge \|\mathbf{s}\|$ and set $f : \mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto \mathbb{E}[F(t^\alpha/(c\,k_\alpha)\cdot\Delta)]$. The dominated convergence theorem ensures that the function $f$ is continuous. It is clearly bounded by 1 and

$$f(t) \leq \mathbb{E}\Big[1 \wedge \big(t^\alpha/(c\,k_\alpha)\cdot\|\Delta\|\big)\Big] = \mathbb{E}\Big[1 \wedge \Sigma_{(c\,k_\alpha)^{-1/\alpha}t}\Big] \leq \frac{t}{(c\,k_\alpha)^{1/\alpha}} \int_{\mathbb{R}_+} (1\wedge x)\,\Pi_{1/\alpha}(\mathrm{d}x).$$

Since $n^\alpha \mathbb{P}[N=n] \to c$, Lemma 1.4.14 ensures that when $R$ goes to infinity, $R\,\mathbb{E}\Big[f\big(N/R^{1/(\alpha-1)}\big)\Big]$ converges to $c\int_0^\infty t^{-\alpha} f(t)\,\mathrm{d}t = (c\,k_\alpha)^{1/\alpha}\int F\,\mathrm{d}I^{(\alpha)}$. Furthermore, because $\Lambda$ is distributed like $(c\,k_\alpha)^{-1}N^\alpha X_N$,

$$\left|R\,\mathbb{E}\left[F\left(\frac{\Lambda}{R^{\alpha/(\alpha-1)}}\right) - f\left(\frac{N}{R^{1/(\alpha-1)}}\right)\right]\right| \leq R\,\mathbb{E}\left[1 \wedge \left(K\left(\frac{N}{R^{1/(\alpha-1)}}\right)^\alpha \big\|X_N - \Delta\big\|\right)\right]$$

where $K\cdot(c\,k_\alpha)$ is bigger than the Lipschitz constant of $F$. We will now endeavour to prove that this last quantity goes to 0 when $R\to\infty$. For all $\mathbf{s}$ in $\mathbb{S}^{\downarrow}$, let $\mathbf{s}\wedge 1$ be the sequence $(s_i\wedge 1)_{i\geq 1}$. Then for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{S}^{\downarrow}$, we may write $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}\wedge 1 - \mathbf{y}\wedge 1\| + \|(\mathbf{x}-\mathbf{x}\wedge 1)-(\mathbf{y}-\mathbf{y}\wedge 1)\|$.

In light of Lemma 1.4.14, $n\,\mathbb{E}\Big[1\wedge(\#T_1/n^\alpha)\Big]$ converges to $[(c\,k_\alpha)^{1/\alpha}\,\Gamma(2-1/\alpha)]^{-1}$. It ensues from the i.i.d. nature of the sequence $(\#T_i)_{i\geq 1}$ that

$$\sup_{n\geq 1}\mathbb{E}\big[\|X_n\wedge 1\|^2\big] = \sup_{n\geq 1}\left(n\,\mathbb{E}\left[\left(\frac{\#T_1}{n^\alpha}\wedge 1\right)^2\right] + n(n-1)\,\mathbb{E}\left[\frac{\#T_1}{n^\alpha}\wedge 1\right]^2\right) < \infty.$$

Fatou's lemma (or classical results on Poisson Point Process, see [84, Section 3.2]) ensures that $\mathbb{E}[\|\Delta\wedge 1\|^2]$ is also finite. As a result, the sequence $(\|X_n\wedge 1 - \Delta\wedge 1\|)_{n\geq 1}$ is bounded in $L^2$. Since $\|X_n\wedge 1-\Delta\wedge 1\|\to 0$ a.s., we also have $\mathbb{E}\big[\|X_n\wedge 1-\Delta\wedge 1\|\big]\to 0$.

If $\beta < 1/\alpha$, then $\mathbb{E}[\|\Delta-\Delta\wedge 1\|^\beta] \leq \mathbb{E}[\|\Delta\|^\beta] = \mathbb{E}[\Sigma_1^\beta] < \infty$. Moreover, since it converges, the sequence $\big(m^{1+1/\alpha}\mathbb{P}[\#T_1 = m]\big)_m$ is bounded by a finite constant, say $Q$. Consequently,

$$\mathbb{E}\big[\|X_n - X_n\wedge 1\|^\beta\big] = n\,\mathbb{E}\left[\left(\frac{\#T_1}{n^\alpha}-1\right)_+^\beta\right] \leq Q\,n\sum_{k>n^\alpha}\frac{k^\beta}{n^{\alpha\beta}}\,\frac{1}{k^{1+1/\alpha}} \xrightarrow[n\to\infty]{} Q\int_1^\infty \frac{\mathrm{d}t}{t^{1+1/\alpha-\beta}} = \frac{\alpha Q}{1-\alpha\beta}$$

which proves that the sequence $\big(\mathbb{E}[\|X_n - X_n\wedge 1\|^\beta]\big)_{n\geq 1}$ is bounded. Since this holds for all $\beta<1/\alpha$, if $\varepsilon$ is positive and such that $(1+\varepsilon)\beta =: \beta' < 1/\alpha$, then

$$\sup_{n\geq 1}\mathbb{E}\Big[\big(\|(X_n - X_n\wedge 1)-(\Delta-\Delta\wedge 1)\|^\beta\big)^{1+\varepsilon}\Big] \leq \sup_{n\geq 1}\mathbb{E}\Big[\|X_n - X_n\wedge 1\|^{\beta'} + \|\Delta-\Delta\wedge 1\|^{\beta'}\Big] < \infty.$$

Hence, the sequence $\big(\|(X_n - X_n\wedge 1)-(\Delta-\Delta\wedge 1)\|^\beta\big)_{n\geq 1}$ is bounded in $L^{1+\varepsilon}$. Because it converges to 0 almost surely, its mean also goes to 0 as $n$ tends to infinity.

For all $\beta<1/\alpha$ and $\varepsilon>0$, there exist a finite constant $C$ and a finite integer $n_\varepsilon$ such that for all $n\geq 1$

$$\mathbb{E}\big[\|X_n\wedge 1-\Delta\wedge 1\|\big] \vee \mathbb{E}\big[\|(X_n - X_n\wedge 1)-(\Delta-\Delta\wedge 1)\|^\beta\big] \leq \varepsilon + C\mathbb{1}_{n<n_\varepsilon}.$$

Using the same arguments as in the proof of Lemma 1.4.14 it is easy to prove that for any $\kappa > \alpha-1$,

$$R\,\mathbb{E}\big[1\wedge(N/R^{1/(\alpha-1)})^\kappa\big] \xrightarrow[R\to\infty]{} c\int_0^\infty \frac{1\wedge t^\kappa}{t^\alpha}\,\mathrm{d}t = \frac{c}{\kappa-(\alpha-1)} + \frac{c}{\alpha-1}.$$

Consequently, if $\beta\in(1-1/\alpha, 1/\alpha)$, we get

$$\begin{aligned}
&\limsup_{R\to\infty} R\,\mathbb{E}\left[1\wedge\left(K\left(\frac{N}{R^{1/(\alpha-1)}}\right)^\alpha\big\|X_N-\Delta\big\|\right)\right]\\
&\quad\leq \limsup_{R\to\infty} R\,\mathbb{E}\left[1\wedge\left(K\left(\frac{N}{R^{1/(\alpha-1)}}\right)^\alpha \mathbb{E}\Big[\big\|X_N\wedge 1-\Delta\wedge 1\big\|\;\Big|\; N\Big]\right)\right]\\
&\qquad + R\,\mathbb{E}\left[1\wedge\left(K^\beta\left(\frac{N}{R^{1/(\alpha-1)}}\right)^{\alpha\beta}\mathbb{E}\Big[\big\|(X_N - X_N\wedge 1)-(\Delta-\Delta\wedge 1)\big\|^\beta\;\Big|\; N\Big]\right)\right]
\end{aligned}$$

$$\leq \limsup_{R\to\infty} \; R\mathbb{E}\left[1\wedge\left(K\,\frac{N^\alpha}{R^{\alpha/(\alpha-1)}}\left(\varepsilon + C\mathbb{1}_{N<n_\varepsilon}\right)\right)\right] + R\mathbb{E}\left[1\wedge\left(K^\beta\,\frac{N^{\alpha\beta}}{R^{\alpha\beta/(\alpha-1)}}\left(\varepsilon + C\mathbb{1}_{N<n_\varepsilon}\right)\right)\right]$$

$$\leq \limsup_{R\to\infty} \; \frac{K\,C\,n_\varepsilon^\alpha}{R^{\alpha/(\alpha-1)-1}} + \frac{K^\beta C\,n_\varepsilon^{\alpha\beta}}{R^{\alpha\beta/(\alpha-1)-1}} + K^{\alpha/(\alpha-1)}\varepsilon^{\alpha/(\alpha-1)}\,R\mathbb{E}\left[1\wedge\frac{N^\alpha}{R^{\alpha/(\alpha-1)}}\right]$$
$$+ K^{\alpha/(\alpha-1)}\varepsilon^{[\alpha/(\alpha-1)]/\beta}\,R\mathbb{E}\left[1\wedge\frac{N^{\alpha\beta}}{R^{\alpha\beta/(\alpha-1)}}\right]$$

$$= O(\varepsilon^{\alpha/(\alpha-1)}).$$

Since this holds for any positive $\varepsilon$, it follows that

$$R\mathbb{E}\left[1\wedge\left(K\left(\frac{N}{R^{1/(\alpha-1)}}\right)^\alpha \left\|X_N - \Delta\right\|\right)\right] \xrightarrow[R\to\infty]{} 0,$$

which in turn proves that $R\mathbb{E}\left[F(\Lambda/R^{\alpha/(\alpha-1)})\right]$ indeed converges to $(c\,k_\alpha)^{1/\alpha}\int_{S^\downarrow} F\,\mathrm{d}I^{(\alpha)}$. We conclude with Lemma 1.3.14. □

### 1.5.2 Cut-trees

Let $\tau$ be a finite labelled tree. If $\tau$ is made out of a single vertex, let its cut-tree $\mathrm{Cut}(\tau)$ be the tree with a single vertex. Otherwise, define the cut-tree of $\tau$ as the (unordered) binary tree $\mathrm{Cut}(\tau)$ obtained by the following recursive process:

— Pick $a \to b$ uniformly at random among the edges of $\tau$ and remove that edge,
— Let $\tau_1$ and $\tau_2$ be the two sub-trees of $\tau$ formerly connected by $a \to b$,
— Define the cut-tree of $\tau$ as the concatenation of the cut-trees of $\tau_1$ and $\tau_2$, i.e. set $\mathrm{Cut}(\tau) := [\![\mathrm{Cut}(\tau_1), \mathrm{Cut}(\tau_2)]\!]$.

With this definition, if $\tau$ has $n$ vertices, then $\mathrm{Cut}(\tau)$ has $n$ leaves. The cut-tree of $\tau$ represents the genealogy of its dismantling when we remove edge after edge, until all have been deleted.

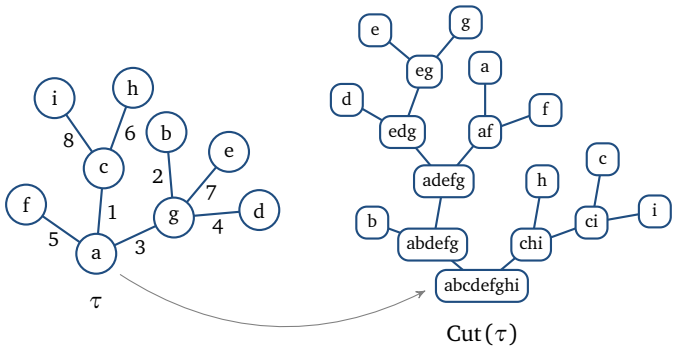



Figure 1.5 – A labelled tree $\tau$ and its cut-tree
(the edges of $\tau$ are labelled in the order they are removed).

Cut-trees were introduced in [22] as a means of generalising the study of the number of cuts necessary to isolate a marked vertex or a finite number of marked vertices. In this section, we will study the local and scaling limits of two models of cut-trees, studied in [22] and [24], which both satisfy the Markov branching property. Also see [26] and [41] for the study of the cut-trees of conditioned Galton-Watson trees.

**Cut-trees of Cayley trees.** A *Cayley tree* of size $n \geq 1$ is a labelled tree $\tau_n$ chosen uniformly at random in the set of trees with $n$ labelled vertices (for convenience, with labels 1 through $n$). It is well-known

that, viewed as an unlabelled tree, $\tau_n$ has the same distribution as an unordered Galton-Watson tree with offspring law Poisson$(1)$ conditioned to have $n$ vertices. For all $n \geq 1$, let $T_n := \mathrm{Cut}(\tau_n)$ be the cut-tree of a Cayley tree with size $n$.

Let $(\vartheta_n)_{n\geq 0}$ be a sequence of i.i.d. unconditioned $\mathrm{GW}_{\mathrm{Poisson}(1)}$ trees. Let $T_\infty$ be the tree obtained by attaching for each $n \geq 0$ the cut-tree of $\vartheta_n$ to the vertex of an infinite branch at height $n$ by an edge. In other words, set $T_\infty := \mathtt{b}_\infty \bigotimes_{n\geq 0} \big(\mathtt{v}_n, [\![\mathrm{Cut}(\vartheta_n)]\!]\big)$.

The aim of this section will be to prove the next two results.

**Proposition 1.5.9.** *When $n \to \infty$, $T_n$ converges to $T_\infty$ in distribution with respect to the local limit topology.*

**Proposition 1.5.10.** *Endow $T_\infty$ with counting measure on its leaves $\mu_\infty$. Then $(T_\infty/R, \mu_\infty/R^2)$ converges as $R$ goes to infinity to $(\mathcal{T}_B, 1/2\cdot\mu_B)$ in distribution with respect to the $\mathrm{D}_{\mathrm{GHP}}$ topology, where $(\mathcal{T}_B, \mu_B)$ denotes the Brownian tree with immigration.*

*Markov branching property.* It was stated in [22] that $(T_n)$ satisfies the Markov branching property and more specifically, that the distribution of $T_n$ is $\mathrm{MB}_n^{\mathcal{L},q}$ where the associated first-split distributions are given by $q_1(1) = 1$, for all $n \geq 2$, $q_n(p \neq 2) = 0$ and if $1 \leq k < n/2$,

$$q_n(n-k,k) = \frac{(n-k)^{n-k-1}}{(n-k)!}\,\frac{k^{k-1}}{k!}\,\frac{(n-2)!}{n^{n-3}}.$$

The tree $T_\infty$ can be described as an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ where the probability measure $q_\infty$ is defined by $q_\infty(p \neq 2) = q_\infty(m_\infty \neq 1) = 0$ and for all positive $k$, $q_\infty(\infty, k) = \mathbb{P}[\#\vartheta = k]$ where $\vartheta$ is a $\mathrm{GW}_{\mathrm{Poisson}(1)}$ tree. Recall that the size of $\vartheta$ has Borel distribution with parameter 1, therefore, for any positive $k$, $q_\infty(\infty, k) = k^{k-1}\,\mathrm{e}^{-k}/k!$.

*Local limits.* For any $k \geq 1$, when $n \to \infty$, Stirling's approximation gives

$$q_n(n-k,k) \sim \frac{k^{k-1}\,\mathrm{e}^{2-k}}{k!}(1-2/n)^n \xrightarrow[n\to\infty]{} \frac{k^{k-1}\,\mathrm{e}^{-k}}{k!} = q_\infty(\infty,k).$$

We may then use Corollary 1.2.6 and thus prove Proposition 1.5.9.

*Scaling limits.* Section 2.1 in [22] proves that $n^{1/2}(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s})$ converges weakly to $(1-s_1)\,1/2\cdot\nu_B(\mathrm{d}\mathbf{s})$ in the sense of finite measures on $\mathcal{S}^{\downarrow}_{\leq 1}$.

Moreover, $q_\infty$ is a.s. binary, and Stirling's approximation ensures that $n^{3/2}q_\infty(\infty, n) \to (2\pi)^{-1/2}$. Therefore, if $\Lambda$ is such that $(\infty, \Lambda)$ follows $q_\infty$ and if $q^{(R)}$ is the distribution of $\Lambda/R^2$, then Proposition 1.4.15 implies that $R(1 \wedge \|\mathbf{s}\|)\,q^{(R)}(\mathrm{d}\mathbf{s})$ weakly converges to $(1 \wedge \|\mathbf{s}\|)\,1/2\cdot I_B(\mathrm{d}\mathbf{s})$ as $R \to \infty$. In other words, Assumption (I) is also satisfied.

Consequently, Theorem 1.4.1 ensures that when $R \to \infty$, $(T_\infty/R, \mu_\infty/R^2)$ converges in distribution to a $(1/2, 1/2\cdot\nu_B, 1/2\cdot I_B)$ fragmentation tree with immigration with respect to the topology induced by $\mathrm{D}_{\mathrm{GHP}}$. Proposition 1.3.12 then concludes the proof of Proposition 1.5.10.

**Cut-trees of uniform recursive trees.** A *recursive tree* with $n$ vertices is a labelled tree (with labels 1 through $n$) such that the labels on the shortest path from 1 to any given leaf are increasing. For all $n \geq 1$, let $\tau_n$ denote a labelled tree chosen uniformly at random among the set of recursive trees with $n$ vertices and call $T_n$ its cut-tree.

Define a probability measure $\pi$ on $\mathbb{N}$ by $\pi(n) = 1/[n(n+1)]$ and let $(X_n, \vartheta_n)_{n\geq 0}$ be a sequence of i.i.d. variables, where for each $n$, $X_n$ follows $\pi$ and conditionally on $X_n = \ell$, $\vartheta_n$ is a recursive tree with $\ell$ vertices. Define $T_\infty$ as the tree obtained by attaching the cut-tree of $\vartheta_n$ by an edge to an infinite branch at height $n$, i.e. set $T_\infty := \mathtt{b}_\infty \bigotimes_{n\geq 0} \big(\mathtt{v}_n, [\![\mathrm{Cut}(\vartheta_n)]\!]\big)$.

**Proposition 1.5.11.** *In the sense of the local limit topology, $T_n$ converges in distribution to $T_\infty$ when $n \to \infty$.*

It was observed in [23] and [24] that the sequence $(T_n)_{n\geq 1}$ is Markov branching. Moreover, we may deduce from [23, Section 2] the expression of the respective distributions $q_n$ of $\Lambda^{\mathcal{L}}(T_n)$. Clearly, $q_1(1) = 1$, and for $n \geq 2$, if $X$ denotes a random variable with distribution $\pi$, then for all $k \leq n/2$, $q_n(n-k,k) = \mathbb{P}[X = k | X < n] + \mathbb{P}[X = n-k | X < n]\mathbb{1}_{k\neq n/2}$. In particular,

$$q_n(n-k,k) = \begin{cases} \dfrac{n}{n-1}\left(\dfrac{1}{k(k+1)} + \dfrac{1}{(n-k)(n-k+1)}\right) & \text{if } k < n/2, \\ \dfrac{4}{(n-1)(n+2)} & \text{if } k = n/2. \end{cases}$$

The tree $T_\infty$ may also be described as an infinite Markov branching tree with distribution $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ where the measure $q_\infty$ is given by $q_\infty(p \neq 2) = q_\infty(m_\infty \neq 1) = 0$ and for all $k \geq 1$, $q_\infty(\infty, k) = \pi(k)$.

If $k$ is a fixed integer, then $q_n(n-k,k)$ clearly converges to $q_\infty(\infty,k)$. We conclude the proof of Proposition 1.5.11 with Corollary 1.2.6.

*Remark 1.5.3.* It was shown in [24] that $(n/\log n)^{-1}T_n$ converges to the real interval $[0,1]$ rooted at 0 and endowed with the Lebesgue measure. However, Assumption (S) doesn't hold.

### 1.5.3 The $\alpha$-$\gamma$ model

In this section, we will study trees generated according to the algorithm of the $\alpha$-$\gamma$ model described in [36]. This algorithm was introduced as an interpolation between various models of sequentially growing trees such as Rémy's algorithm [110], used to generate uniform binary trees with any number of leaves, Marchal's [95], which gives the $n$-dimensional marginal of Duquesne-Le Gall's stable trees (the discrete tree spanned by $n$ leaves chosen uniformly at random in a stable tree), and Ford's $\alpha$-model [52], used for instance in phylogeny.

Let $0 \leq \gamma \leq \alpha \leq 1$. Start with $T_1 := \{\varnothing\}$, the trivial tree, and $T_2 := \{\varnothing, (1), (2)\}$, a tree with two leaves attached to its root. Then for $n \geq 3$, conditionally on the tree $T_{n-1}$:

- Assign to each edge of $T_{n-1}$ (considered as a planted tree, i.e. a tree in which a phantom edge has been attached under the root) the weight $1-\alpha$ if the edge ends with a leaf or $\gamma$ otherwise,
- Also assign to each non-leaf vertex $u$ the weight $[c_u(T_{n-1}) - 1]\alpha - \gamma$,
- Pick an edge or a vertex in $T_{n-1}$ with probability proportional to these weights,
  - If an edge was picked, place a new vertex at its middle and attach a new leaf to it,
  - If a vertex was selected, attach a new leaf to it,

and let $T_n$ be the tree thus obtained. We will also call $\mathrm{AG}^n_{\alpha,\gamma}$ its distribution for all $n \geq 1$ and $0 \leq \gamma \leq \alpha \leq 1$.

*Remark 1.5.4.* As mentioned at the beginning of this section, some particular choices of parameters give previously studied algorithms:

- When $\alpha = \gamma = 1/2$, we get Rémy's algorithm [110],
- If $\beta \in (1,2)$, taking $\alpha = 1/\beta$ and $\gamma = 1-\alpha$ gives Marchal's algorithm [95],
- When $\alpha = \gamma$, this algorithm coincides with that of Ford's $\alpha$-model [52].

*The Beta geometric distribution.* Fix $\theta$ in $(0,1)$. Let $\Pi$ be a Beta random variable with parameters $(1-\theta, \theta)$, and conditionally on $\Pi$, let $X$ have geometric distribution with parameter $1-\Pi$, meaning that $\mathbb{P}[X = n \,|\, \Pi] = \Pi^n(1-\Pi)$ for every integer $n \geq 0$. We say that $X$ is a *beta geometric* variable of parameters $(\theta, 1-\theta)$. For all integers $n \geq 0$,

$$\mathbb{P}[X = n] = \mathbb{E}\big[\Pi^n(1-\Pi)\big] = \frac{1}{\mathrm{B}(1-\theta,\theta)}\int_0^1 x^{n-\theta}(1-x)^\theta \mathrm{d}x = \frac{\theta\,\Gamma(n+1-\theta)}{\Gamma(1-\theta)(n+1)!}.$$

We will also use the convention $X = 0$ a.s. if $\theta = 1$ and $X = \infty$ a.s. if $\theta = 0$.

*Infinite $\alpha$-$\gamma$ tree.* Assume that $0<\gamma\le\alpha\le 1$. Let $(X_n)_{n\ge 0}$ be a sequence of i.i.d. beta geometric random variables with parameters $(\gamma/\alpha, 1-\gamma/\alpha)$. Let $(Y_{n,k},\tau_{n,k})$ be a sequence of i.i.d. variables independent of $(X_n)_n$ such that $Y_{n,k}$ is a $(\alpha,1-\alpha)$ beta geometric variable and conditionally on $Y_{n,k}=\ell$, $\tau_{n,k}$ is an $\alpha$-$\gamma$ tree with $\ell+1$ leaves, i.e. $\tau_{n,k}$ follows $\mathrm{AG}^{\ell+1}_{\alpha,\gamma}$.

Finally, conditionally on $(X_n, Y_{n,k},\tau_{n,k}; n\ge 0, k\ge 0)$, define $T_\infty$ as the tree obtained by grafting for each $n\ge 0$ the concatenation of $\tau_{n,i}$, $0\le i\le X_n$ at height $n$ on an infinite branch. In other words,

$$T_\infty := \mathrm{b}_\infty \otimes_{n\ge 0}\left(\mathrm{v}_n, [\![\tau_{n,0},\dots\tau_{n,X_n}]\!]\right)$$

and denote by $\mathrm{AG}^{\infty}_{\alpha,\gamma}$ its distribution.

*Remark 1.5.5.* In Ford's $\alpha$-model, i.e. when $\alpha=\gamma>0$, $X_n=0$ a.s. for all $n$, so a single tree is grafted at each height. Similarly, when $\alpha=1$ and $0<\gamma\le\alpha$, $Y_{n,k}=0$ a.s..

We will start our study of the $\alpha$-$\gamma$ model by proving this next proposition with the help of Theorem 1.2.5. Similar results for $\alpha=\gamma$ were already proved in [113] and in [31, Lemma 3.8] for any $0<\gamma\le\alpha\le 1$.

**Proposition 1.5.12.** *For any $0<\gamma\le\alpha\le 1$, the probability measure $\mathrm{AG}^n_{\alpha,\gamma}$ converges weakly to $\mathrm{AG}^\infty_{\alpha,\gamma}$ as $n$ grows to $\infty$ in the sense of the local limit topology.*

We will then study the scaling limits of these infinite trees: Section 1.5.3 will focus on the case $0<\gamma<\alpha<1$ and Section 1.5.3, on $\alpha=\gamma$.

**Markov branching property and local limits.** Proposition 1 in [36] states that the sequence $(\mathrm{AG}^n_{\alpha,\gamma})_n$ satisfies the Markov branching property. Moreover, the sequence $q=(q_n)_n$ associated to the first split distributions of $T_n$, i.e. such that $q_n$ is the law of $\Lambda^{\mathcal{L}}(T_n)$ for all $n\ge 1$, is given by $q_1(\varnothing)=1$, and for any $n\ge 2$, for all $\lambda=(\lambda_1,\dots,\lambda_p)\in\mathcal{P}_n$,

$$q_n(\lambda)=\frac{1}{\prod_{j\ge 1} m_j(\lambda)}\left(\gamma+\frac{1-\alpha-\gamma}{n(n-1)}\sum_{i\ne j}\lambda_i\lambda_j\right)\frac{\Gamma(1-\alpha)\,n!}{\Gamma(n-\alpha)}\,\frac{\alpha^{p-2}\,\Gamma(p-1-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)}\prod_{i=1}^{p}\frac{\Gamma(\lambda_i-\alpha)}{\Gamma(1-\alpha)\,\lambda_i!},$$

with the conventions $\Gamma(0)=\infty$ and $\Gamma(0)/\Gamma(0)=1$ (which will be used throughout this section).

We can also write $\mathrm{AG}^\infty_{\alpha,\gamma}=\mathrm{MB}^{\mathcal{L},q,q_\infty}_\infty$ where $q_\infty$ is the measure on $\mathcal{P}_\infty$ given by

$$q_\infty(\infty,\lambda)=\frac{\gamma/\alpha\,\Gamma(p-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)\,p!}\,\frac{p!}{\prod_{j\ge 1} m_j(\lambda)!}\prod_{i=1}^{p}\frac{\alpha\,\Gamma(\lambda_i-\alpha)}{\Gamma(1-\alpha)\,\lambda_i!}$$

for all $\lambda=(\lambda_1,\dots,\lambda_p)$ in $\mathcal{P}_{<\infty}$ and $q_\infty(\mu)=0$ for all $\mu$ in $\mathcal{P}_\infty$ with either $p(\mu)=1$ or $m_\infty(\mu)>1$.

If $X$ has beta geometric distribution with parameters $(\gamma/\alpha,1-\gamma/\alpha)$ and is independent of the i.i.d. sequence $(Y_i)_{i\ge 0}$ of beta geometric variables with parameters $(\alpha,1-\alpha)$, for any $\lambda=(\lambda_1,\dots,\lambda_p)$ in $\mathcal{P}_{<\infty}$, we get that

$$q_\infty(\infty,\lambda)=\mathbb{P}\big[X=p-1,\,(Y_1+1,\dots,Y_p+1)^{\downarrow}=\lambda\big]$$

which ensures that $q_\infty$ is a probability measure on $\mathcal{P}_\infty$.

*Proof of Proposition 1.5.12.* Let $\lambda=(\lambda_1,\dots,\lambda_p)$ be in $\mathcal{P}_{<\infty}$. Then, for $n$ large enough, in light of Stirling's

approximation,

$$q_n(n-\|\lambda\|,\lambda) = \frac{1}{\prod_{j\geq 1} m_j(\lambda)!}\Bigg(\gamma + \overbrace{\frac{1-\alpha-\gamma}{n(n-1)}\sum_{i\neq j}\lambda_i\lambda_j}^{\uparrow 0 \ n\to\infty}\Bigg)\overbrace{\frac{\Gamma(n-\|\lambda\|-\alpha)\,n!}{\Gamma(n-\alpha)(n-\|\lambda\|)!}}^{\uparrow 1 \ n\to\infty}$$
$$\times \frac{\alpha^{p-1}\,\Gamma(p-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)}\prod_{i=1}^{p}\frac{\Gamma(\lambda_i-\alpha)}{\Gamma(1-\alpha)\,\lambda_i!}$$
$$\xrightarrow[n\to\infty]{} \frac{\gamma/\alpha\,\Gamma(p-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)\,p!}\,\frac{p!}{\prod_{j\geq 1} m_j(\lambda)!}\prod_{i12}^{p}\frac{\alpha\,\Gamma(\lambda_i-\alpha)}{\Gamma(1-\alpha)\,\lambda_i!} = q_\infty(\infty,\lambda).$$

We conclude with Corollary 1.2.6. □

**Scaling limits.** In this paragraph, we will assume that $0<\gamma<\alpha<1$. Let $\Sigma$ be an $\alpha$-stable subordinator with Laplace exponent $\lambda\mapsto\lambda^\alpha$ and Lévy measure $\Pi_\alpha(\mathrm{d}t) = \alpha/\Gamma(1-\alpha)\,t^{-1-\alpha}\mathbb{1}_{t>0}\,\mathrm{d}t$. Define $\Delta$ as the decreasing rearrangement of its jumps on $[0,1]$. We define the dislocation measure $\nu_{\alpha,\gamma}$ for all measurable functions $f:\mathcal{S}^\downarrow_{\leq 1}\to\mathbb{R}_+$ by

$$\int_{\mathcal{S}^\downarrow_{\leq 1}} f\,\mathrm{d}\nu_{\alpha,\gamma} := \frac{\Gamma(1-\alpha)}{\alpha\,\Gamma(1-\gamma/\alpha)}\mathbb{E}\Big[\Sigma_1^{\alpha+\gamma}\big(\gamma+(1-\alpha-\gamma)\textstyle\sum_{i\neq j}\Delta_i\Delta_j\big)f\big(\Delta/\Sigma_1\big)\Big].$$

Results from [36] and [67] ensure that the family $q$ satisfies Assumption (S): when $n\to\infty$, $n^\gamma(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s})$ converges weakly towards $(1-s_1)\,\nu_{\alpha,\gamma}(\mathrm{d}\mathbf{s})$.

We also define the immigration measure $I_{\alpha,\gamma}$ for all measurable functions $F:\mathcal{S}^\downarrow\to\mathbb{R}_+$ by

$$\int_{\mathcal{S}^\downarrow} F\,\mathrm{d}I_{\alpha,\gamma} := \frac{\gamma/\alpha}{\Gamma(1-\gamma/\alpha)}\int_0^\infty \frac{\mathbb{E}\big[F(t^{1/\alpha}\Delta)\big]}{t^{1+\gamma/\alpha}}\,\mathrm{d}t.$$

**Proposition 1.5.13.** *Let $T$ be distributed according to $\mathrm{AG}^\infty_{\alpha,\gamma}$ and endow it with $\mu_T$, the counting measure on the set of its leaves. With respect to the $\mathrm{D}_{\mathrm{GHP}}$ topology, $(T/R,\mu_T/R^{1/\gamma})$ converges in distribution to a $(\gamma,\nu_{\alpha,\gamma},I_{\alpha,\gamma})$ fragmentation tree with immigration.*

*Proof.* Let $\Lambda$ be such that $(\infty,\Lambda)$ follows $q_\infty$. For all $R\geq 1$, set $q^{(R)}$ as the distribution of $R^{-1/\gamma}\Lambda$. In light of Theorem 1.4.1, it is sufficient to prove that $R(1\wedge\|\mathbf{s}\|)\,q^{(R)}(\mathrm{d}\mathbf{s})\Rightarrow(1\wedge\|\mathbf{s}\|)\,I_{\alpha,\gamma}(\mathrm{d}\mathbf{s})$ when $R\to\infty$.

To prove this claim, we may proceed as in the proof of Proposition 1.5.8. The only significant difference is that the constant $\beta$ used near the end of that proof must now belong to the open interval $(\gamma,\alpha)$. □

*Remark 1.5.6.* Let $\beta$ be in $(1,2)$ and set $\alpha=1/\beta$, $\gamma=1-\alpha$. It was proved in [95] that the distribution $\mathrm{AG}^n_{1/\beta,1-1/\beta}$ coincides with $\mathrm{GW}^{\mathcal{L},n}_\xi$, where the generating function of $\xi$ is given by $s\mapsto s+\beta^{-1}(1-s)^\beta$. The results of Propositions 1.5.12 and 1.5.13 are then consistent with those of Proposition 1.5.1 and Remark 1.5.2.

**Ford's $\alpha$-model.** When $\alpha=\gamma$, no weight is ever assigned to vertices. Consequently, the trees generated by this algorithm are a.s. binary (i.e. each vertex has either two children or none). Furthermore, the sequence $(q_n)_n$ of associated first split distributions is much simpler: $q_1(\varnothing)$ still equals 1, and for $n\geq 2$, if $\alpha<1$, for all $1\leq k\leq n/2$,

$$q_n(n-k,k) = (2-\mathbb{1}_{2k=n})\binom{n}{k}\frac{\Gamma(n-k-\alpha)\,\Gamma(k-\alpha)}{\Gamma(1-\alpha)\,\Gamma(n-\alpha)}\left(\frac{\alpha}{2}+\frac{(1-2\alpha)(n-k)\,k}{n(n-1)}\right),$$

finally if $\alpha=1$, $q_n(n-1,1)=1$.

Moreover, if $\alpha$ is positive, for all $n\geq 1$, $q_\infty(\infty,n) = \alpha\,\Gamma(n-\alpha)/[\Gamma(1-\alpha)\,n!]$ and $q_\infty(\lambda)=0$ if $p(\lambda)\neq 2$ or $m_\infty(\lambda)\neq 1$. As a result, a tree with distribution $\mathrm{AG}^\infty_{\alpha,\alpha}$ is obtained by grafting at each height of an infinite

spine a single tree with distribution $\mathrm{AG}_{\alpha,\alpha}^{N+1}$ where $N$, its number of leaves minus 1, has beta geometric distribution of parameters $(\alpha, 1-\alpha)$.

*Scaling limits of Ford's α model.* Let $\alpha \in (0,1)$. Results from [67, Section 5.2] ensure that $(T_n)_n$ satisfies Assumption (S): when $n \to \infty$, $n^\alpha (1-s_1)\, \tilde{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\, \nu_\alpha^{(\mathrm{F})}(\mathrm{d}\mathbf{s})$ where $\nu_\alpha^{(\mathrm{F})}$ is the binary dislocation measure defined for all measurable $f : \mathcal{S}^\downarrow_{\leq 1} \to \mathbb{R}_+$ by

$$\int f \,\mathrm{d}\nu_\alpha^{(\mathrm{F})} = \frac{1}{\Gamma(1-\alpha)} \int_{1/2}^1 \left( \frac{\alpha}{[x(1-x)]^{1+\alpha}} + \frac{2-4\alpha}{[x(1-x)]^\alpha} \right) f(x, 1-x, 0, 0, \dots)\, \mathrm{d}x.$$

Furthermore, $q_\infty$ is a.s. binary and Stirling's approximation ensures that $q_\infty(\infty, n)$ is equivalent to $[\alpha/\Gamma(1-\alpha)]\, n^{-1-\alpha}$ when $n \to \infty$. Consequently, if $\Lambda$ is such that $(\infty, \Lambda)$ follows $q_\infty$ and $q^{(R)}$ denotes the distribution of $\Lambda/R^{1/\alpha}$, Proposition 1.4.15 proves that $R(1 \wedge \|\mathbf{s}\|)\, q^{(R)}(\mathrm{d}\mathbf{s}) \Rightarrow (1 \wedge \|\mathbf{s}\|)\, [\alpha/\Gamma(1-\alpha)]\, I_\alpha^{\mathrm{un}}(\mathrm{d}\mathbf{s})$ as $R \to \infty$. Therefore, if we set $I_\alpha^{(\mathrm{F})} := \alpha/\Gamma(1-\alpha) \cdot I_\alpha^{\mathrm{un}}$, we may use Theorem 1.4.1 and Proposition 1.4.2 to get the following result:

**Proposition 1.5.14.** *Let $T$ be an $\mathrm{AG}_{\alpha,\alpha}^\infty$ tree with $\alpha$ in $(0,1)$ and endow it with the counting measure on the set of its leaves. Then, $(T/R, \mu_T^{\mathcal{L}}/R^{1/\alpha})$ converges in distribution to a $(\alpha, \nu_\alpha^{(\mathrm{F})}, I_\alpha^{(\mathrm{F})})$-fragmentation tree with immigration with respect to the topology induced by $\mathrm{D}_{\mathrm{GHP}}$.*

*Remark 1.5.7.* When $\alpha = 1/2$, i.e. in Rémy's algorithm, these results coincide with Proposition 1.5.1 and Proposition 1.5.2 ($i'$) for $\xi(0) = \xi(2) = 1/2$.

*When $\alpha = 1$.* In this case, the algorithm's output is deterministic: for each $n \geq 2$, a tree $T_n$ with distribution $\mathrm{AG}_{1,1}^n$ is simply equal to a branch of length $n-1$ upon which a single leaf has been grafted at each non-leaf vertex (a "comb" of length $n$). Similarly, an infinite tree with distribution $\mathrm{AG}_{1,1}^\infty$ is the "infinite comb", obtained by attaching a single leaf to all the vertices of the infinite branch.

As a result, if $T$ has distribution $\mathrm{AG}_{1,1}^\infty$ and $\mu_T$ denotes the counting measure on the set of its leaves, then clearly, $(T/R, \mu_T/R)$ converges as $R \to \infty$ to the metric space $\mathbb{R}_+$ rooted at 0 and endowed with the usual Lebesgue measure.

*When $\alpha = 0$.* Observe that $q_n(n-k, k) = (2 - \mathbb{1}_{k=n/2})/(n-1)$. Then for all $K \geq 1$ and $n$ large enough,

$$\mathbb{P}[\Lambda^{\mathcal{L}}(T_n) \wedge K = \infty_2 \wedge K] = 1 - \frac{K-1}{n-1} \xrightarrow[n\to\infty]{} 1,$$

which implies $\Lambda^{\mathcal{L}}(T_n) \to (\infty, \infty)$ a.s. when $n \to \infty$. Theorem 1.2.5 then ensures that $T_n$ converges in distribution to the complete infinite binary tree (in which every vertex has 2 children). Moreover, since $T_n \subset T_{n+1}$ a.s., this convergence happens almost surely.

*With Assumptions (S) and (I′).* Let us give an application of the result from Remark 1.4.2.

For any $\alpha$ in $(0,1)$ and $n \in \mathbb{N} \cup \{\infty\}$, denote by $q_n^{(\alpha)}$ the first-split distribution (with respect to the number of leaves) associated to a tree with distribution $\mathrm{AG}_{\alpha,\alpha}^n$. Now, fix $0 < \alpha < \beta < 1$, and consider a tree $T$ with distribution $\mathrm{MB}_\infty^{\mathcal{L}, q^{(\alpha)}, q_\infty^{(\beta)}}$ endowed with $\mu_T$, the counting measure on the set of its leaves. We may deduce from previous results that $(q_n^{(\alpha)})_{n\geq 1}$ and $q_\infty^{(\beta)}$ satisfy Assumptions (S) and (I′).

Consequently, $(T/R, \mu_T/R^{1/\beta})$ converges in distribution to the metric space $\mathbb{R}_+$ rooted at 0 and endowed with a random measure $\mu = \sum_{i\geq 1} \|\mathbf{s}_i\|\, \delta_{u_i}$, where $\sum_{i\geq 1} \delta_{(u_i, \mathbf{s}_i)}$ is a Poisson point process on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ with intensity measure $\mathrm{d}u \otimes I_\beta^{(\mathrm{F})}(\mathrm{d}\mathbf{s})$.

### 1.5.4 Aldous' $\beta$-splitting model

This section will focus on the study a model of binary random trees introduced in [10, Section 4] as a Markov branching model. Let $\beta > -2$ be fixed. Set $q_1(\varnothing) := 1$ and for all $n \geq 2$ and $1 \leq k \leq n/2$,

$$q_n(n-k,k) := \frac{2-\mathbb{1}_{2k=n}}{Z_n} \frac{\Gamma(n-k+1+\beta)}{(n-k)!} \frac{\Gamma(k+1+\beta)}{k!}$$

where $Z_n$ is a normalising constant. For all $n \geq 1$, let $T_n$ be a random tree with distribution $\mathrm{MB}_n^{\mathcal{L},q}$.

*Remark 1.5.8.* — The constant $Z_n$ is given by

$$Z_n := \sum_{k=1}^{n-1} \frac{\Gamma(n-k+1+\beta)}{(n-k)!} \frac{\Gamma(k+1+\beta)}{k!}.$$

When $\beta > -1$, it simplifies to $Z_n = [\mathrm{B}(1+\beta,1+\beta) - 2\,\mathrm{B}(n+1+\beta,1+\beta)] \cdot \Gamma(n+2+2\beta)/n!$ (where B denotes the usual Beta function) and when $\beta = -1$, it becomes $Z_n = 2/n \cdot \sum_{k=1}^{n-1} k^{-1}$.

— When $\beta = -3/2$, observe that the sequence $(q_n)_n$ is the same as that of the $\alpha$-model with $\alpha = 1/2$ (see Section 1.5.3). Therefore, like Rémy's algorithm, this model generates uniform binary trees with any given number of leaves.

There are three regimes in this model, respectively $\beta > -1$, $\beta = -1$ and $\beta \in (-2,-1)$. The asymptotic behaviour of $q_n$ were studied in [10, Section 5] in these three regimes.

**Local limits.** In this paragraph, we will focus on the study of the local limits of $T_n$. We will once again rely on the Markov branching nature of the model and on Theorem 1.2.5.

**Proposition 1.5.15.** $\beta \geq -1$ : *In the sense of the local limit topology, $T_n$ converges in distribution to the infinite binary tree.*

$\beta \in (-2,-1)$ : *Let $X$ follow the beta geometric distribution with parameters $(2+\beta,-1-\beta)$ (see Section 1.5.3). Define $q_\infty$, a probability measure on $\mathcal{P}_\infty$, by $q_\infty(\infty,k) = \mathbb{P}[X = k-1]$ for any $k \geq 1$ and $q_\infty(\lambda) = 0$ if $p(\lambda) \neq 2$ or $m_\infty(\lambda) \neq 1$. With these notations, $T_n$ converges in distribution to $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ with respect to the local limit topology.*

*Remark 1.5.9.* Suppose $\beta \in (-2,-1)$ and let $(X_n,\tau_n)_{n\geq 0}$ be an i.i.d. sequence such that for each $n$, $X_n$ has beta geometric distribution with parameters $(2+\beta,-1-\beta)$ and conditionally on $X_n = k-1$, $\tau_n$ is distributed like $T_k$. Finally, denote by $T_\infty$ the tree obtained by attaching by a single edge the tree $\tau_n$ respectively at each height $n$ of an infinite branch, i.e. $T_\infty := \mathrm{b}_\infty \bigotimes_{n\geq 0} \big(\mathrm{v}_n, [\![\tau_n]\!]\big)$. The tree $T_\infty$ hence obtained has distribution $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$.

*Proof.* Observe that in light of Stirling's approximation, $\Gamma(n+1+\beta)/n! \sim n^\beta$ when $n \to \infty$.

$\beta \geq -1$ : When $\beta > -1$, using Stirling's approximation once again, we get that $Z_n \sim \mathrm{B}(1+\beta,1+\beta)\, n^{-1-2\beta}$ so if $k \geq 1$ is a fixed integer, $q_n(n-k,k) = O(n^{1+\beta})$ when $n \to \infty$.

When $\beta = -1$, $Z_n \sim 2/n \cdot \log n$ hence, for any fixed $k \geq 1$, $q_n(n-k,k) \sim 1/(k \log n)$ as $n \to \infty$.

Therefore, for any $\beta \geq -1$, if $K \geq 1$,

$$q_n\big[\mu \in \mathcal{P}_n : \mu \wedge K = (K,K)\big] = 1 - \textstyle\sum_{k=1}^K q_n(n-k,k) \xrightarrow[n\to\infty]{} 1.$$

Lemma 1.2.3 then ensures that $q_n \Rightarrow \delta_{(\infty,\infty)}$. It follows from Theorem 1.2.5 that $T_n$ converges in distribution to the (deterministic) infinite binary tree.

$\beta \in (-2,-1)$ : Let $\beta \in (-2,-1)$. Stirling's formula ensures that the sequence $\left(i^{-\beta}\,\Gamma(i+1+\beta)/i!\right)_{i\geq 1}$ is bounded by a finite constant. As a result, the dominated convergence theorem ensures that

$$\frac{Z_n}{n^\beta} = \sum_{k\geq 1} \frac{\Gamma(k+1+\beta)}{k!}\,\frac{\Gamma(n-k+1+\beta)}{(n-k)^\beta\,(n-k)!}\,\frac{(n-k)^\beta}{n^\beta}\,(2-\mathbb{1}_{2k=n})\,\mathbb{1}_{2k\leq n}$$
$$\xrightarrow[n\to\infty]{} 2\sum_{k\geq 1}\frac{\Gamma(k+1+\beta)}{k!} = 2\frac{\Gamma(2+\beta+)}{-1-\beta}$$

where we have used the definition of the beta geometric distribution with parameters $(2+\beta,-1-\beta)$ as introduced in Section 1.5.3.

Consequently, for any fixed positive integer $k$,

$$\lim_{n\to\infty} q_n(n-k,k) = \lim_{n\to\infty} 2\,\frac{\Gamma(k+1+\beta)}{k!}\,\frac{\Gamma(n-k+1+\beta)}{Z_n\,(n-k)!} = \frac{(-1-\beta)\,\Gamma(k+1+\beta)}{\Gamma(2+\beta)\,k!} = q_\infty(\infty,k).$$

We may then conclude with Corollary 1.2.6. □

**Scaling limits.** We will now study the scaling limits of the $\beta$-splitting model when $\beta \in (-2,-1)$ with the help of Theorem 1.4.1.

Let $\nu_\beta^{(\mathrm{B})}$ be the dislocation measure such that for all measurable $f : \mathcal{S}^{\downarrow}_{\leq 1} \to \mathbb{R}_+$,

$$\int f\,\mathrm{d}\nu_\beta^{(\mathrm{B})} := \frac{-1-\beta}{\Gamma(2+\beta)}\int_0^{1/2} t^\beta\,(1-t)^\beta\,f(1-t,t,0,0,\dots)\,\mathrm{d}t.$$

It follows from Section 5.1 in [67] that $(q_n)_{n\geq 1}$ satisfies Assumption (S) for $\gamma = -1-\beta$ and $\nu = \nu_\beta^{(\mathrm{B})}$. More precisely, $n^{-1-\beta}\,(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s})$ converges weakly to $(1-s_1)\,\nu_\beta^{(\mathrm{B})}(\mathrm{d}\mathbf{s})$ as finite measures on $\mathcal{S}^{\downarrow}_{\leq 1}$.

Let $\Lambda$ denote a random integer such that $(\infty,\Lambda)$ has distribution $q_\infty$ and for all $R \geq 1$, set $q^{(R)}$ as the distribution of $\Lambda/R^{1/(-1-\beta)}$. Just like in Section 1.5.3, Stirling's approximation and Proposition 1.4.15 ensure that Assumption (I) is met for $\gamma = -1-\beta$ and the immigration measure $I_\beta^{(\mathrm{B})} := (-1-\beta)/\Gamma(2+\beta)\cdot I^{\mathrm{un}}_{-1-\beta}$. As a result,

**Proposition 1.5.16.** *Fix $\beta \in (-2,-1)$. Let $T$ be a $\mathrm{MB}_\infty^{\mathcal{L},q,q_\infty}$ tree and endow it with $\mu_T$, the counting measure on the set of its leaves. In the topology induced by $\mathrm{D}_{\mathrm{GHP}}$, $(T/R, \mu_T^{\mathcal{L}}/R^{1/(-1-\beta)})$ converges in distribution to a $(-1-\beta, \nu_\beta^{(\mathrm{B})}, I_\beta^{(\mathrm{B})})$-fragmentation tree with immigration.*

## 1.5.5 $k$-ary growing trees

Let $k \geq 2$ be an integer. In this section, we will study a model of $k$-ary trees, i.e. trees in which vertices have either 0 or $k$ children, described in [69]. This model is yet another generalisation of Rémy's algorithm [110] (which corresponds to $k=2$).

The following algorithm allows us to get a sequence $(T_n)_{n\geq 0}$ of $k$-ary trees such that for all $n$, $T_n$ has $n$ internal vertices (vertices that aren't leaves) or, equivalently, $kn+1$ vertices or $(k-1)n+1$ leaves. First, let $T_0$ be the trivial tree $\{\varnothing\}$ and for $n \geq 1$, conditionally on $T_{n-1}$:

— Pick an edge of $T_{n-1}$ (considered as a planted tree) uniformly at random,

— Place a new vertex on that edge and attach $k-1$ new leaves to it,

and call $T_n$ the resulting tree. We will denote the distribution of $T_n$ by $\mathrm{GT}_k^n$.

*The negative Dirichlet multinomial distribution.* Let $\Pi$ be a $(k-1)$-dimensional Dirichlet variable with $k$ parameters $(1/k,\dots,1/k)$, i.e. $\Pi$ takes its values in the $(k-1)$-dimensional simplex $\{\boldsymbol{x} \in (0,\infty)^k : x_1+\cdots+x_k = 1\}$. Conditionally on $\Pi$, let $X = (X_1,\dots,X_{k-1})$ have negative multinomial distribution of parameters $(1;\Pi)$, i.e. for each $i \in \{1,\dots,k-1\}$, $X_i$ counts the number of type $i$ results before the first type

$k$ result (failure) in a sequence of i.i.d. trials with $k$ possible results with respective probabilities $\Pi_1, \dots, \Pi_k$. For any non-negative integers $n_1, \dots, n_{k-1}$ and with $N = n_1 + \cdots + n_{k-1}$, we have

$$\mathbb{P}\big[X = (n_1, \dots, n_{k-1})\big] = \mathbb{E}\left[\frac{N!}{n_1! \dots n_{k-1}!} \prod_{i=1}^{k-1} \Pi_i^{n_i}\, \Pi_k\right] = \frac{1}{k}\,\frac{1}{1+N} \prod_{i=2}^{k} \frac{\Gamma(n_i + 1/k)}{\Gamma(1/k)\, n_i!}.$$

The random variable $X$ is said to follow a $(k-1)$-dimensional *negative Dirichlet multinomial* distribution with parameters $(1; 1/k, \dots, 1/k)$ which is a multidimensional generalisation of the beta geometric distribution. Further observe that the sum $\|X\| = X_1 + \cdots + X_{k-1}$ has beta geometric distribution with parameters $(1/k, 1-1/k)$ and that conditionally on $\|X\| = n$, $X$ follows a $(k-1)$-dimensional Dirichlet multinomial distribution with parameters $(n\,; 1/k, \dots, 1/k)$.

*Corresponding infinite tree.* Let $(X_n, \tau_{n,1}, \dots \tau_{n,k-1})_{n\geq 0}$ be a sequence of i.i.d. variables such that for all $n \geq 0$, $X_n$ is distributed according to a $(k-1)$-dimensional $(1; 1/k, \dots, 1/k)$ negative Dirichlet multinomial distribution and conditionally on $X_n = (m_1, \dots, m_{k-1})$, $\tau_{n,1}, \dots, \tau_{n,k-1}$ are independent and have respective distributions $\mathrm{GT}_k^{m_1}, \dots, \mathrm{GT}_k^{m_{k-1}}$.

Conditionally on $(X_n, \tau_{n,1}, \dots \tau_{n,k-1})_{n\geq 0}$, let $T_\infty$ be the tree obtained after grafting at each height $n \geq 0$ of an infinite branch the concatenation of $\tau_{n,i}$, $1 \leq i \leq k-1$, i.e. set

$$T_\infty := \mathtt{b}_\infty \bigotimes_{n\geq 0} \big(\mathtt{v}_n, [\![\tau_{n,1}, \dots \tau_{n,k-1}]\!]\big),$$

and let $\mathrm{GT}_k^\infty$ be the distribution of $T_\infty$.

Section 1.5.5 will prove the following proposition.

**Proposition 1.5.17.** *In the sense of the local limit topology, $\mathrm{GT}_k^n$ converges weakly to $\mathrm{GT}_k^\infty$ when $n$ goes to $\infty$.*

Let $\Pi$ be a $(k-1)$-dimensional Dirichlet variable with parameters $(1/k, \dots, 1/k)$. Following [69, Section 3.1], we define the dislocation measure $\nu_k^{\mathrm{GT}}$ such that for all measurable $f : \mathcal{S}^\downarrow_{\leq 1} \to \mathbb{R}_+$

$$\int_{\mathcal{S}^\downarrow_{\leq 1}} f \,\mathrm{d}\nu_k^{\mathrm{GT}} = \frac{\Gamma(1/k)}{k}\,\mathbb{E}\left[\frac{f\big[(\Pi, 0, 0, \dots)^\downarrow\big]}{1-\Pi_1}\right].$$

Let $\Delta$ be a $(k-2)$-dimensional Dirichlet variable with parameters $(1/k, \dots, 1/k)$. We also define the immigration measure $I_k^{\mathrm{GT}}$ for all measurable functions $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ by

$$\int_{\mathcal{S}^\downarrow} F(\mathbf{s})\, I_k^{\mathrm{GT}}(\mathrm{d}\mathbf{s}) := \frac{1/k}{\Gamma(1-1/k)} \int_0^\infty t^{-1-1/k}\mathbb{E}\Big[F\big(t\,(\Delta, 0, 0, \dots)^\downarrow\big)\Big]\mathrm{d}t.$$

The aim of Section 1.5.5 will be to prove the next proposition.

**Proposition 1.5.18.** *Let $T$ be a $\mathrm{GT}_k^\infty$-distributed tree and endow it with $\mu_T^\circ$, the counting measure on the set of its internal vertices. With respect to the topology induced by $\mathrm{D_{GHP}}$, when $R$ grows to infinity, $(T/R, \mu_T^\circ/R^k)$ converges in distribution to a $(1/k, \nu_k^{\mathrm{GT}}, I_k^{\mathrm{GT}})$-fragmentation tree with immigration.*

**Markov branching property and local limits.** For any $\mathtt{t}$ in $\mathtt{T}$, we define $\Lambda^\circ(\mathtt{t})$ as the decreasing rearrangement of the number of internal vertices of the sub-trees of $\mathtt{t}$ attached to its root, i.e. we let $\Lambda^\circ(\mathtt{t}) := \Lambda(\mathtt{t}) - \Lambda^{\mathcal{L}}(\mathtt{t})$. In the setting of $k$-ary growing trees, $\Lambda^\circ(T_0) = \varnothing$ a.s. and if $n \geq 1$, $\Lambda^\circ(T_n)$ takes its values in the set of decreasing families of $(\mathbb{Z}_+)^k$ with sum $n-1$. Because of the deterministic relationship between $n$, $\#T_n$ and $\#_{\mathcal{L}} T_n$, we have $\Lambda(T_0) = \Lambda^{\mathcal{L}}(T_0) = \varnothing$ and for $n \geq 1$, $\Lambda(T_n) = k\Lambda^\circ(T_n) + (1, \dots, 1)$ in $\mathcal{P}_{kn}$ and $\Lambda^{\mathcal{L}}(T_n) = (k-1)\Lambda^\circ(T_n) + (1, \dots, 1)$ in $\mathcal{P}_{(k-1)n+1}$. For all $n \geq 1$, call $q^\circ_{n-1}$ the distribution of $\Lambda^\circ(T_n)$, that is the first-split distribution of $T_n$ with respect to internal vertices.

Proposition 3.3 from [69] states that $(T_n)_{n\geq 0}$ satisfies the Markov branching property and the distribution of $T_n$ may be expressed as either $\mathrm{MB}^q_{kn+1}$ or $\mathrm{MB}^{\mathcal{L}, q^{\mathcal{L}}}_{(k-1)n+1}$ where $q$ and $q^{\mathcal{L}}$ are both easily obtained from

$(q_n^\circ)_{n\geq 0}$. Rewriting the formula from this last proposition for our purposes (where partition blocks are arranged in decreasing order), for all $n \geq 1$ and $\lambda = (\lambda_1, \dots, \lambda_k)$ decreasing with sum $n$, we get that

$$q_{n-1}^\circ(\lambda) = \frac{(k-1)!}{\prod_{j\geq 1} m_j(\lambda)!}\frac{1}{k}\frac{\Gamma(1/k)}{\Gamma(n+1+1/k)}\prod_{i=1}^{k}\frac{\Gamma(\lambda_i+1/k)}{\Gamma(1/k)\,\lambda_i!}\sum_{i=1}^{k}\left(m_{\lambda_i}(\lambda)\lambda_i!\sum_{j=0}^{\lambda_i}\frac{(j+n-\lambda_i)!}{j!}\right).$$

We can rewrite $\mathrm{GT}_k^\infty$ as the distribution $\mathrm{MB}_\infty^{q,q_\infty}$ or $\mathrm{MB}_\infty^{\mathcal{L},q^{\mathcal{L}},q_\infty^{\mathcal{L}}}$ of an infinite Markov branching tree. The corresponding measures $q_\infty$ and $q_\infty^{\mathcal{L}}$ on $\mathcal{P}_\infty$ can also be easily deduced from the measure $q_\infty^\circ$ on the set of decreasing $k$-tuples of $\mathbb{Z}_+ \cup \{\infty\}$ with infinite sum such that $q_\infty^\circ(\lambda) = 0$ if $\lambda_2$ is infinite and

$$q_\infty^\circ(\infty, \lambda_2, \dots, \lambda_k) = \frac{(k-1)!}{\prod_{j\geq 1} m_j(\lambda)!}\frac{1}{k}\frac{1}{\|\lambda\|+1}\prod_{i=2}^{k}\frac{\Gamma(\lambda_i+1/k)}{\Gamma(1/k)\,\lambda_i!}$$

for any integers $\infty > \lambda_2 \geq \cdots \geq \lambda_p \geq 0$. Observe that $q_\infty^\circ(\infty, \lambda_2, \dots, \lambda_k) = \mathbb{P}\big[X^\downarrow = (\lambda_2, \dots, \lambda_k)\big]$ where $X$ is a $(k-1)$-dimensional negative Dirichlet multinomial variable with parameters $(1; 1/k, \dots, 1/k)$. As a result, $q_\infty^\circ$ is a probability measure.

*Proof of Proposition 1.5.17.* Let $\lambda = (\lambda_2, \dots, \lambda_k)$ be a decreasing sequence of $(\mathbb{Z}_+)^{k-1}$ and set $L = \lambda_2 + \cdots + \lambda_k$. For $n$ large enough, we have

$$\begin{aligned} q_n^\circ(n-L, \lambda_2, \dots, \lambda_k) &= \frac{(k-1)!}{\prod_{j\geq 1} m_j(\lambda)!}\frac{1}{k}\prod_{i=2}^{k}\frac{\Gamma(\lambda_i+1/k)}{\Gamma(1/k)\,\lambda_i!}\overbrace{\frac{\Gamma(n-L+1/k)}{\Gamma(n+1+1/k)}}^{\asymp\, n^{-L-1}}\left[\overbrace{\sum_{j=0}^{n-L}\frac{(j+L)!}{j!}}^{\asymp\, n^{L+1}\int_0^1 x^L\mathrm{d}x} + \overbrace{\sum_{i=2}^{k}\sum_{j=0}^{\lambda_i}\frac{\lambda_i!\,(j+n-\lambda_i)!}{(n-L)!\,j!}}^{=\,O(n^L)}\right] \\ &\xrightarrow[n\to\infty]{} \frac{(k-1)!}{\prod_{j\geq 1} m_j(\lambda)!}\frac{1}{k}\frac{1}{L+1}\prod_{i=2}^{k}\frac{\Gamma(\lambda_i+1/k)}{\Gamma(1/k)\,\lambda_i!} = q_\infty^\circ(\infty, \lambda). \end{aligned}$$

Corollary 1.2.6 concludes this proof. □

**Scaling limits.** Proposition 3.1 in [69] states that $n^{1/k}(1-s_1)\,\bar{q}_n^\circ(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu_k^{\mathrm{GT}}(\mathrm{d}\mathbf{s})$ as $n \to \infty$ in the sense of finite measures on $\mathcal{S}_{\leq 1}^\downarrow$. Assumption (S) of Theorem 1.4.1 is thus met for the sequence $q^\circ$. To prove Proposition 1.5.18, we will need the following lemma. Let $X = (X_1, \dots, X_{k-1})$ denote a negative Dirichlet multinomial variable with parameters $(1; 1/k, \dots, 1/k)$.

**Lemma 1.5.19.** *Let $\Delta$ be a $(k-2)$-dimensional Dirichlet $(1/k, \dots, 1/k)$ variable. For all Lipschitz-continuous functions $G : [0,\infty)^{k-1} \longrightarrow \mathbb{R}_+$ such that $G(\boldsymbol{x}) \leq 1 \wedge \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$ in $[0,\infty)^{k-1}$,*

$$R\,\mathbb{E}\left[G\left(\frac{X}{R^k}\right)\right] \xrightarrow[R\to\infty]{} \frac{1/k}{\Gamma(1-1/k)}\int_0^\infty t^{-1-1/k}\mathbb{E}[G(t\,\Delta)]\,\mathrm{d}t.$$

*Proof.* Let $(Y_n)_{n\geq 1}$ be i.i.d. and such that conditionally on $\Delta$, $Y_n$ is multinomial with parameters $(1; \Delta)$. Moreover, set $Z_n := Y_1 + \cdots + Y_n$. The law of large numbers ensures that $Z_n/n$ converges almost surely to $\Delta$. Let $N$ be independent of $\Delta$ and $(Z_n)_n$ and have beta geometric distribution with parameters $(1/k, 1-1/k)$. Observe that $X$ has the same distribution as $Z_N$.

Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by $g(t) := \mathbb{E}[G(t\,\Delta)]$. The dominated convergence theorem implies that it is continuous and it clearly satisfies $g(t) \leq 1 \wedge t$. Lemma 1.4.14 then ensures that $R\,\mathbb{E}[g(N/R^k)] \to \big[k\,\Gamma(1-1/k)\big]^{-1}\int_0^\infty t^{-1-1/k}g(t)\,\mathrm{d}t$.

Since $Z_n/n$ a.s. converges to $\Delta$ and because $\|(Z_n/n) - \Delta\| \leq 2$, we can use the dominated convergence theorem to state that for all positive $\varepsilon$, there exists $n_\varepsilon$ such that $\mathbb{E}[\|(Z_n/n) - \Delta\|] < \varepsilon$ as soon as $n \geq n_\varepsilon$.

Therefore, if $K$ is the Lipschitz constant of $G$,

$$\begin{aligned}\left|R\,\mathbb{E}\left[G\left(\frac{X}{R^k}\right)\right]-R\,\mathbb{E}\left[g\left(\frac{N}{R^k}\right)\right]\right| &\leq R\,\mathbb{E}\left[\left|G\left[\frac{N}{R^k}\frac{Z_N}{N}\right]-G\left[\frac{N}{R^k}\Delta\right]\right|\right]\\ &\leq R\,\mathbb{E}\left[1\wedge\left(K\varepsilon\frac{N}{R^k}\right)\right]+\frac{2Kn_\varepsilon}{R^{k-1}}\xrightarrow[R\to\infty]{}\frac{1/k}{\Gamma(1-1/k)}\int_0^\infty\frac{1\wedge(K\varepsilon\, t)}{t^{1+1/k}}\mathrm{d}t\end{aligned}$$

where we have used Lemma 1.4.14. This last quantity in turn converges to 0 when $\varepsilon\to 0$ which proves the desired result. □

*Proof of Proposition 1.5.18.* Recall that if $\Lambda$ is such that $(\infty,\Lambda)$ follows $q^\circ_\infty$, then $\Lambda$ is distributed like $X^\downarrow$. We may then deduce from Lemma 1.5.19 and Lemma 1.3.14 that Assumption (I) holds for $q^\circ_\infty$, $I=I_k^{\mathrm{GT}}$ and $\gamma=1/k$. As a result, Theorem 1.4.1 concludes this proof. □

# Chapter 2

# Local limits of generalised Markov branching trees and their volume growth

In Chapter 1, we considered two notions of "size" for trees: namely their number of vertices and their number of leaves. Furthermore, we described the two constructions that Haas and Miermont [66] gave of Markov branching trees corresponding to these two notions of size.

Rizzolo [111] considered a more general notion of size and introduced a unified and more general construction of Markov branching trees with a prescribed number of vertices whose degree belongs to a given subset of $\mathbb{Z}_+$. He also extended Haas and Miermont's Theorems 5 and 6 from [66] to this new family of generalised Markov branching trees and gave an application of his result to Galton-Watson trees under some conditioning.

In this chapter, after recalling Rizzolo's construction and extending it to infinite Markov branching trees, we will broaden the scope of the results of Chapter 1. We will freely use the notations introduced in this last chapter.

## 2.1 Generalised Markov branching trees and their local limits

Let $A$ be a fixed non-empty subset of $\mathbb{Z}_+$. For any finite tree $\mathtt{t}$ in $\mathtt{T}_{<\infty}$, let $\mathcal{V}_A(\mathtt{t}) := \{u \in \mathtt{t} : c_u(\mathtt{t}) \in A\}$ be the set of the vertices of $\mathtt{t}$ whose degree lies in $A$, and let $\#_A\mathtt{t} := \#\mathcal{V}_A(\mathtt{t})$ be the number of such vertices. For all $n \in \mathbb{Z}_+$, let $\mathtt{T}_n^A$ be the set of all finite trees with $n$ vertices with degree in $A$, that is set $\mathtt{T}_n^A := \{\mathtt{t} \in \mathtt{T}_{<\infty} : \#_A\mathtt{t} = n\}$.

*Remark 2.1.1.*
— If $A = \mathbb{Z}_+$, $\#_A\mathtt{t} = \#\mathtt{t}$ is simply the number of vertices of $\mathtt{t}$. Similarly, if $A = \{0\}$, $\#_A\mathtt{t} = \#_{\mathcal{L}}\mathtt{t}$ is the number of leaves of $\mathtt{t}$.
— When $0 \notin A$, observe that $\mathtt{T}_0^A \neq \varnothing$.

*Partitions of integers.* Let $\mathtt{t}$ be in $\mathtt{T}_n^A$ and be such that $\mathtt{t} = [\![\mathtt{t}_1, \dots, \mathtt{t}_p]\!]$ for some integer $p$ and finite trees $\mathtt{t}_1, \dots, \mathtt{t}_p$. Set $\Lambda^A(\mathtt{t}) := (\#_A\mathtt{t}_1, \dots, \#_A\mathtt{t}_p)^{\downarrow}$ and observe that $\|\Lambda^A(\mathtt{t})\|$ is either $n$ or $n-1$ depending on whether $p \in A$ or not and that some blocks of $\Lambda^A(\mathtt{t})$ might be 0. As a result, to construct Markov branching trees with given sizes with respect to $\#_A$, we will need to tweak the notion of partitions of integers.

For all $n$ in $\mathbb{Z}_+ \cup \{\infty\}$, set

$$\mathcal{P}_n^* := \Big\{\lambda = (\lambda_1, \dots, \lambda_p) \in (\mathbb{Z}_+)^p : p \geq 0, \lambda_1 \geq \cdots \geq \lambda_p \geq 0, \lambda_1 + \cdots + \lambda_p = n\Big\},$$

and $\mathcal{P}^*_{<\infty} := \bigcup_{n\geq 0} \mathcal{P}^*_n$ as well as $\mathcal{P}^* := \mathcal{P}^*_{<\infty} \cup \mathcal{P}^*_{\infty}$. Observe the only difference between $\mathcal{P}^*$ and the set $\mathcal{P}$ from Section 1.2 is that elements of $\mathcal{P}^*$ are allowed to have blocks equal to 0.

We will use similar notations as in Section 1.2. Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be in $\mathcal{P}^*$ and set

— $p(\lambda) := p$ its length.
— $\|\lambda\| = \lambda_1 + \cdots + \lambda_p$ its sum.
— $\lambda \wedge K := (\lambda_1 \wedge K, \dots, \lambda_p \wedge K)$ its truncation at level $K$, for all non-negative integers $K$.
— $m_k(\lambda) := \sum_i \mathbb{1}_{\lambda_i = k}$, the number of blocks of $\lambda$ equal to $k$ where $k \in \mathbb{Z}_+ \cup \{\infty\}$.
— For all $p \geq 0$, we will also write $0_p$ for the element of $\mathcal{P}^*$ such that $p(0_p) = p$ and $\|0_p\| = 0$.

Now for all $n \in \mathbb{Z}_+$, set

$$\mathcal{P}^A_n := \big\{\lambda \in \mathcal{P}^*_n : p(\lambda) \notin A\big\} \cup \big\{\lambda \in \mathcal{P}^*_{n-1} : p(\lambda) \in A\big\}.$$

We also endow $\mathcal{P}^*$ with the ultra-metric $\mathrm{d}_{\mathcal{P}^*}$ defined for all $\lambda$ and $\mu$ in $\mathcal{P}$, let

$$\mathrm{d}_{\mathcal{P}^*}(\lambda, \mu) := \exp\big[-\inf\{K \geq 0 \,:\, \lambda \wedge K \neq \mu \wedge K\}\big].$$

Compare with the definition of $\mathrm{d}_{\mathcal{P}}$. This metric satisfies results similar to Lemmas 1.2.2 to 1.2.4. In particular,

**Lemma 2.1.1.** *(i)* $(\mathcal{P}^*, \mathrm{d}_{\mathcal{P}^*})$ *is Polish.*

*(ii)* *If $\Lambda_n$, $n \geq 1$ and $\Lambda$ are $\mathcal{P}^*$-valued random variables, then $\Lambda_n \Rightarrow \Lambda$ with respect to $\mathrm{d}_{\mathcal{P}^*}$ iff for all $K \geq 0$ and $\lambda \in \mathcal{P}^*_{<\infty}$,*

$$\mathbb{P}[\Lambda_n \wedge K = \lambda \wedge K] \xrightarrow[n\to\infty]{} \mathbb{P}[\Lambda \wedge K = \lambda \wedge K].$$

*(iii)* *If furthermore, $\Lambda_n$ is $\mathcal{P}^A_n$-valued for all $n$ and $m_\infty(\Lambda) = 1$ a.s., then $\Lambda_n \Rightarrow \Lambda$ with respect to $\mathrm{d}_{\mathcal{P}^*}$ iff for all $\lambda \in \mathcal{P}^*_{<\infty}$,*

$$\mathbb{P}\big[\Lambda_n = (n - L, \lambda)\big] \xrightarrow[n\to\infty]{} \mathbb{P}\big[\Lambda = (\infty, \lambda)\big]$$

*with $L = \|\lambda\| + \mathbb{1}_A[p(\lambda) + 1]$.*

*Finite Markov branching trees.* Let $\mathcal{N}$ be an infinite subset of $\mathbb{Z}_+$ with $1 \in \mathcal{N}$ and such that $0 \in \mathcal{N}$ iff $0 \notin A$. This set will once again index all the sizes, with respect to $\#_A$, of the random trees we will want to generate.

Now let $q = (q_n)_{n\in\mathcal{N}}$ be a sequence of probability measures such that

— If $0 \in \mathcal{N}$, $q_0$ is supported by $\mathbb{T}^A_0$.
— For all $n \neq 0$, $q_n$ is supported by

$$\big\{\lambda \in \mathcal{P}^A_n : \lambda_i \in \mathcal{N}, 1 \leq i \leq p(\lambda)\big\}$$

and $q_n(m_n = 1) < 1$.

We will now detail how to build a sequence $\mathrm{MB}^{A,q} = (\mathrm{MB}^{A,q}_n)_{n\in\mathcal{N}}$ of probability measures respectively supported by $\mathbb{T}^A_n$, which satisfies a version of the Markov branching property. More precisely $\mathrm{MB}^{A,q}$ will be such that for all $n$, if $T$ has distribution $\mathrm{MB}^{A,q}_n$, then the law of $\Lambda^A(T)$ is $q_n$ and conditionally on $\Lambda^A(T) = (\lambda_1, \dots, \lambda_p)$ and $T = [\![T^{(1)}, \dots, T^{(p)}]\!]$ with $\#_A T^{(1)} \geq \cdots \geq \#_A T^{(p)}$, the trees $T^{(1)}, \dots, T^{(p)}$ are independent and such that the law of $T^{(i)}$ is $\mathrm{MB}^{A,q}_{\lambda_i}$.

To build $\mathrm{MB}^{A,q} = (\mathrm{MB}^{A,q}_n)_{n\in\mathcal{N}}$, we will use a similar construction as for $\mathrm{MB}^{\mathcal{L},q}$ in Section 1.2.2. In particular, we will proceed inductively on $n$. If $0 \in A$ (and thus $0 \notin \mathcal{N}$), let $\mathrm{MB}^{A,q}_1$ be the law of a branch with geometric length with parameter $1 - q_1(1)$, i.e. $\mathrm{MB}^{A,q}_1(\mathrm{b}_k) = q_1(1)^k[1 - q_1(1)]$, where for all $k \geq 0$, $\mathrm{b}_k$ denotes the branch with length $k$. Otherwise, when $0 \notin A$, set $\mathrm{MB}^{A,q}_0 := q_0$. Now, for $n > \mathbb{1}_A(0)$,

— Let $T_0$ be a branch of geometric length with parameter $1 - q_n(n)$ and denote its leaf $U$,
— Let $\Lambda$ have distribution $q_n$ conditioned on the event $\{m_n = 0\}$,
— Conditionally on $\Lambda = (\lambda_1, \dots, \lambda_p)$, let $(T_1, \dots, T_p)$ be independent random trees respectively distributed according to $\mathrm{MB}^q_{\lambda_i}$ for $1 \leq i \leq p$,

— Graft the concatenation of these trees on the leaf $U$ of $T_0$, i.e. set $T := T_0 \otimes \left(U, [\![T_1, \dots, T_{p(\Lambda)}]\!]\right)$ and let $\mathrm{MB}_n^{\mathcal{L},q}$ be the distribution of $T$.

Additionally to $q = (q_n)_{n\in\mathcal{N}}$, consider $q_\infty$, a probability measure on $\mathcal{P}^*_\infty$ and supported by

$$\left\{\lambda \in \mathcal{P}^*_\infty : \lambda_i \in \mathcal{N} \cup \{\infty\}, 1 \le i \le p(\lambda)\right\}.$$

To define an infinite Markov branching tree with probability measure $\mathrm{MB}_\infty^{A,q,q_\infty}$ supported by $\mathtt{T}_\infty$, we can proceed in the exact same manner as in Section 1.2.2. When $q_\infty(m_\infty \neq 1) = 0$, we can use the construction detailed in Remark 1.2.4.

The proof of Theorem 1.2.5 can be adapted to get the following result:

**Proposition 2.1.2.** *If $q_n \Rightarrow q_\infty$ with respect to $\mathrm{d}_{\mathcal{P}^*}$, then $\mathrm{MB}_n^{A,q} \Rightarrow \mathrm{MB}_\infty^{A,q,q_\infty}$ with respect to $\mathrm{d}_{\mathrm{loc}}$.*

## 2.2 Scaling limits and volume growth

As mentioned at the beginning of Chapter 2, Rizzolo [111] extended Haas and Miermont's result from [66] to the framework of generalised Markov branching trees.

Let $\mathtt{t}$ be a tree and let $\mu_\mathtt{t}$ be the counting measure on $\mathcal{V}_A(\mathtt{t})$. Recall that if $u$ is in $\mathtt{t}$, then $\mathtt{t}_u$ denotes the sub-tree of $\mathtt{t}$ above $u$. We will need the following notations:

— Let $\Pi_A(\mathtt{t})$ be the tree obtained by cutting away all the finite sub-trees of $\mathtt{t}$ which contain no vertices in $\mathcal{V}_A(\mathtt{t})$, i.e. set

$$\Pi_A(\mathtt{t}) := \{u \in \mathtt{t} : \#\mathtt{t}_u = \infty \text{ or } \#_A\mathtt{t}_u > 0\}.$$

These cut-away sub-trees are often called *fringe sub-trees*.

— Observe that the leaves of $\Pi_A(\mathtt{t})$ are attached to the main body of $\Pi_A(\mathtt{t})$ by branches with varying lengths. Define $\Psi_A(\mathtt{t})$ as the tree obtained by shortening all those branches to have length 1. More precisely,

$$\Psi_A(\mathtt{t}) := \{u \in \mathtt{t} : \#\mathtt{t}_u = \infty \text{ or } \#_A\mathtt{t}_{pr(u)} > 1\}.$$

— Finally, let $\Phi_A(\mathtt{t})$ be the tree obtained by attaching a new leaf to each non-leaf vertex in $\Psi_A(\mathtt{t})$ which belongs to $\mathcal{V}_A[\Psi_A(\mathtt{t})]$, in other words, set

$$\Phi_A(\mathtt{t}) := \Psi_A(\mathtt{t}) \bigotimes_{u\in\mathcal{V}_{A\setminus\{0\}}\Psi_A(T)} (u, \mathtt{b}_1)$$

where $\mathtt{b}_1$ denotes the branch with length 1.

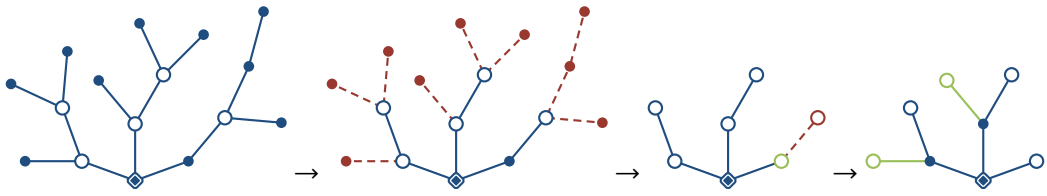


Figure 2.1 – A tree $\mathtt{t}$, $\Pi_A(\mathtt{t})$, $\Psi_A(\mathtt{t})$ and $\Phi_A(\mathtt{t})$ with $A = \{2\}$.
(The red edges and vertices are deleted and the green ones are added)

Let $q = (q_n)_{n\in\mathcal{N}}$ be a sequence of first-split distributions. For all $n \neq 0$ in $\mathcal{N}$, let $T_n$ have distribution $\mathrm{MB}_n^{A,q}$ and endow it with the counting measure on $\mathcal{V}_A(T_n)$ denoted by $\mu_n$. Let $\iota : \mathcal{P}^*_{<\infty} \to \mathcal{S}^\downarrow$ be defined by $\iota(\lambda) := (\lambda_1/\|\lambda\|, \dots, \lambda_{p(\lambda)}/\|\lambda\|, 0, \dots)$ if $\|\lambda\| > 0$ and $\iota(\lambda) := (1, 0, 0, \dots)$ otherwise. Finally, set $\bar{q}_n := q_n \circ \iota^{-1}$ the pushforward of $q_n$ on $\mathcal{S}^\downarrow$ by the function $\iota$.

Also fix a probability measure $q_\infty$ on $\mathcal{P}^*_\infty$ and suppose that $q_\infty(m_\infty \neq 1) = 0$. Let $T$ be an infinite Markov branching tree associated to $q$ and $q_\infty$, i.e. a $\mathrm{MB}_\infty^{A,q,q_\infty}$ distributed random tree. Endow $T$ with

the counting measure $\mu_T$ on $\mathcal{V}_A(T)$. Let $\Lambda$ be a $\mathcal{P}^*_{<\infty}$ valued random variable such that the law of $(\infty, \Lambda)$ is $q_\infty$.

In the remainder of this section, we will make assumptions similar to those from Section 1.4:

(S) There exist $\gamma \in (0,1)$ and a conservative dislocation measure $\nu$ such that

$$n^\gamma (1-s_1)\, \bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\, \nu(\mathrm{d}\mathbf{s}).$$

(I) There exists an immigration measure $I$ such that

$$R\,\mathbb{E}\big[F(\Lambda/R^{1/\gamma})\big] \xrightarrow[R\to\infty]{} \int_{\mathcal{S}^\downarrow} F(\mathbf{s})\, I(\mathrm{d}\mathbf{s})$$

for any continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$ with $F(\mathbf{s}) \le 1 \wedge \|\mathbf{s}\|$.

Under Assumption (S), Rizzolo proved the following result:

**Theorem 2.2.1 ([111], Theorem 5).** *If $0 \notin A$, suppose that*

$$\mathrm{d}_{\mathrm{GHP}}\Big(\big(n^{-\gamma} T_n, n^{-1}\mu_n\big), \big(n^{-\gamma}\Pi_A(T_n), n^{-1}\mu_n\big)\Big) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

*Then*

$$\left(\frac{T_n}{n^\gamma} T_n, \frac{\mu_n}{n}\right) \xrightarrow[n\to\infty]{\mathrm{d}_{\mathrm{GHP}}} \mathcal{T}_{\gamma,\nu}.$$

To prove Theorem 2.2.1, Rizzolo first proved that for all positive $n$ in $\mathcal{N}$, the law of $T_n^\circ := \Phi_A(T_n)$ is $\mathrm{MB}_n^{\mathcal{L},q^\circ}$ for a well-chosen family $q^\circ = (q_n^\circ)_{n\in\mathcal{N}\setminus\{0\}}$ defined as follows:

— Consider $\chi : \mathcal{P}^* \to \mathcal{P}$ such that for all $\lambda$ in $\mathcal{P}$ and any integer $k$,

$$\chi(\lambda, 0_k) := \begin{cases} \lambda & \text{if } p+k \notin A, \\ (\lambda, 1) & \text{if } p+k+1 \in A, \end{cases}$$

— For all $n \neq 0$ in $\mathcal{N}$, set $q_n^\circ := q_n \circ \chi^{-1}$ and observe that it is supported by $\mathcal{P}_n$.

See Lemma 2 in [111]. He then went on to prove that $q^\circ$ satisfies Assumption (S) and that as a result, Haas and Miermont's Theorem 1.3.7 holds: if $\mu_n^\circ$ denotes the counting measure on the set of leaves of $T_n^\circ$, $(n^{-\gamma} T_n^\circ, n^{-1}\mu_n^\circ)$ converges in law to a $(\gamma,\nu)$ fragmentation tree, see Lemma 3 in [111]. Finally, he proved that $T_n$ and $T_n^\circ$ are "asymptotically close". We will use a similar approach to extend Theorem 1.4.1:

**Theorem 2.2.2.** *Under Assumption (S) and (I), if we further assume that*

$$\mathrm{D}_{\mathrm{GHP}}\Big(\big(R^{-1}T, R^{-1/\gamma}\mu_T\big), \big(R^{-1}\Pi_A(T), R^{-1/\gamma}\mu_T\big)\Big) \xrightarrow[R\to\infty]{\mathbb{P}} 0, \tag{$\Delta$}$$

*then*

$$\left(\frac{T}{R}, \frac{\mu_T}{R^{1/\gamma}}\right) \xrightarrow[R\to\infty]{\mathrm{d}_{\mathrm{GHP}}} \mathcal{T}^I_{\gamma,\nu}.$$

*Remark 2.2.1.*

— When $0 \in A$, $\Pi_A(T) = T$ so Assumption ($\Delta$) holds.

— When $0 \notin A$, if we assume that $q_0$ is supported by a set of trees with bounded heights, then Assumption ($\Delta$) is fulfilled.

— If Theorem 2.2.2 holds, its also entails functional convergence of the associated volume growth processes, compare with Proposition 1.4.2.

The proof of Theorem 2.2.2 will be split into the next few lemmas.

**Lemma 2.2.3.** *Let $T^\circ := \Phi_A(T)$. The law of the infinite tree $T^\circ$ is $\mathrm{MB}_\infty^{\mathcal{L},q^\circ,q_\infty^\circ}$ where $q^\circ$ is the previously defined sequence and $q_\infty^\circ := q_\infty \circ \chi^{-1}$.*

*Proof.* Write $T = [\![T_0, \dots, T_N]\!]$ with $\#T_0 = \infty$, $\#_A T_1 \geq \cdots \geq \#_A T_N$. By construction of $T$, $T_0$ is distributed like $T$ and $\Lambda := (\#_A T_1, \dots \#_A T_N)$ is such that the law of $(\infty, \Lambda)$ is $q_\infty$ and is independent of $T_0$.

Similarly, let $T^\circ$ be written as $T^\circ = [\![T_0^\circ, \dots, T_{N'}^\circ]\!]$ for some integer $N'$, with $\#T_0^\circ = \infty$ and $\#_{\mathcal{L}} T_1^\circ \geq \cdots \geq \#_{\mathcal{L}} T_{N'}^\circ$. Let $\Lambda^\circ := (\#_{\mathcal{L}} T_1^\circ, \dots, \#_{\mathcal{L}} T_{N'}^\circ)$.

By definition of the function $\Phi_A$, we get that conditionally on $\Lambda = (\lambda_1, \dots, \lambda_p, 0_j)$ with $\lambda_p > 0$,

$$(\infty, \Lambda^\circ) = \begin{cases} (\infty, \lambda_1, \dots, \lambda_p) & \text{if } p + j + 1 \notin A, \\ (\infty, \lambda_1, \dots, \lambda_p, 1) & \text{if } p + j + 1 \in A. \end{cases}$$

In other words, $(\infty, \Lambda^\circ) = \chi(\infty, \Lambda)$ which means that its distribution is $q_\infty^\circ$.

Conditionally on $\Lambda$, the sub-trees $T_0, \dots, T_N$ are independent so conditionally on $\Lambda^\circ$, $T_0^\circ, \dots, T_{N'}^\circ$ are independent as well. Moreover, conditionally on $\Lambda^\circ = \lambda$, if $\lambda_i \geq 2$, $T_i^\circ = \Phi_A(T_i)$ and thus has distribution $\mathrm{MB}_{\lambda_i}^{\mathcal{L}, q^\circ}$ in light of [111, Lemma 2], and if $\lambda_i = 1$, then $T_i^\circ = \mathfrak{b}_0$ so is $\mathrm{MB}_1^{\mathcal{L}, q^\circ}$-distributed.

Finally, we have $T_0^\circ = \Phi_A(T_0)$, so $T_0^\circ$ and $T^\circ$ are identically distributed. Consequently, $T^\circ$ is indeed Markov branching and its distribution is $\mathrm{MB}_\infty^{\mathcal{L}, q^\circ, q_\infty^\circ}$. □

**Lemma 2.2.4.** *If $q_\infty$ satisfies Assumption (I) then so does $q_\infty^\circ$. In particular, if $\Lambda^\circ$ is such that $(\infty, \Lambda^\circ)$ is $q_\infty^\circ$-distributed, then for all continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}$ with $|F(\mathbf{s})| \leq 1 \wedge \|\mathbf{s}\|$,*

$$R\,\mathbb{E}\Big[F(\Lambda^\circ / R^{1/\gamma})\Big] \xrightarrow[R\to\infty]{} \int_{\mathcal{S}^\downarrow} F(\mathbf{s})\, I(\mathrm{d}\mathbf{s}).$$

*Proof.* Let $\Lambda$ be such that the distribution of $(\infty, \Lambda)$ is $q_\infty$ and let $\Lambda^\circ$ satisfy $(\infty, \Lambda^\circ) = \chi(\infty, \Lambda)$. As such, the law of $(\infty, \Lambda^\circ)$ is $q_\infty^\circ$. Moreover, observe that a.s., $\|\Lambda^\circ - \Lambda\| \leq 1$.

Let $G : \mathcal{S}^\downarrow \to \mathbb{R}$ be bounded and Lipschitz continuous, i.e. there exists a finite $K$ such that for any two $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{S}^\downarrow$, $|G(\mathbf{x}) - G(\mathbf{y})| \leq K\|\mathbf{x} - \mathbf{y}\|$. Then observe that for all positive $R$

$$R\left|\mathbb{E}\Big[G(\Lambda^\circ/R^{1/\gamma}) - G(\Lambda/R^{1/\gamma})\Big]\right| \leq R\,\mathbb{E}\left[\frac{K}{R^{1/\gamma}}\big\|\Lambda^\circ - \Lambda\big\|\right] \leq K R^{1-1/\gamma} \xrightarrow[R\to\infty]{} 0$$

because $\gamma < 1$. Consequently, for all bounded and Lipschitz continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}$ with $|F(\mathbf{s})| \leq 1 \wedge \|\mathbf{s}\|$,

$$\lim_{R\to\infty} R\,\mathbb{E}\Big[F(\Lambda^\circ/R^{1/\gamma})\Big] = \lim_{R\to\infty} R\,\mathbb{E}\Big[F(\Lambda/R^{1/\gamma})\Big] = \int_{\mathcal{S}^\downarrow} F(\mathbf{s})\, I(\mathrm{d}\mathbf{s}).$$

Classical results on weak convergence of finite measures entail that this convergence holds for all bounded and continuous $F : \mathcal{S}^\downarrow \to \mathbb{R}$ with $|F(\mathbf{s})| \leq 1 \wedge \|\mathbf{s}\|$, which concludes this proof. □

**Lemma 2.2.5.** *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. non-negative random variables and assume that $\mathbb{E}[\exp(aX_1)]$ is finite for some positive $a$. Then*

$$\limsup_{n\to\infty} \frac{X_1 \vee \cdots \vee X_n}{\log n}$$

*is a.s. bounded by a finite constant.*

*Proof.* Observe that for all positive $t$,

$$\mathbb{P}[X_1 \vee \cdots \vee X_n / \log n > t] \leq n\,\mathbb{P}[X_1 > t \log n] = n\,\mathbb{P}[\mathrm{e}^{aX_1} > n^{at}] \leq n^{1-at}\mathbb{E}[\mathrm{e}^{aX_1}]$$

where we used Markov's inequality. Taking $t > 2/a$, we can use Borel-Cantelli's lemma to ensure that

$$\limsup_{n\to\infty} (X_1 \vee \cdots \vee X_n)/\log n \leq t$$

almost surely, which concludes this proof. □

*Proof of Theorem 2.2.2.* Let $T^\circ := \Phi_A(T)$ and denote by $\mu_{T^\circ}$ the counting measure on the set of its leaves. Lemmas 2.2.3 and 2.2.4 and Lemma 2 from [111] allow us to use Theorem 1.4.3 and deduce that

$$\big(R^{-1}T^\circ, R^{-1/\gamma}\mu_{T^\circ}\big) \xrightarrow[R\to\infty]{\mathrm{D}_{\mathrm{GHP}}} \mathcal{T}_{\gamma,\nu}^I.$$

We will now prove that $\mathrm{D}_{\mathrm{GHP}}[(R^{-1}T, R^{-1/\gamma}\mu_T), (R^{-1}T^\circ, R^{-1/\gamma}\mu_{T^\circ})] \to 0$ in probability, which will suffice to conclude. We will proceed in two steps: first we will prove that $T^\circ = \Phi_A(T)$ and $\Psi_A(T)$ are asymptotically close and we will then prove the same for $\Psi_A(T)$ and $\Pi_A(T)$.

$(i)$ Endow $\Psi_A(T)$ with the counting measure on the set $\mathcal{V}_A[\Psi_A(T)] \cup \mathcal{L}[\Psi_A(T)]$ denoted by $\mu'_T$. Let $C$ be the correspondence between $\Psi_A(T)$ and $\Phi_A(T)$ given by

$$C := \Big\{(u,u) : u \in \Psi_A(T)\Big\} \cup \Big\{(\mathrm{pr}(v), v) : v \in \Phi_A(T), v \notin \Psi_A(T)\Big\}$$

and for all $k \geq 0$, set $C_k := \{(u,v) \in C : |u| \leq k, |v| \leq k\}$, which is a correspondence between $\Psi_A(T)|_k$ and $\Phi_A(T)|_k$. Similarly, let $\pi$ be the measure on $\Psi_A(T) \times \Phi_A(T)$ defined by

$$\pi := \sum_{u \in \mathcal{L}[\Psi_A(T)]} \delta_{(u,u)} + \sum_{u \in \mathcal{L}[\Phi_A(T)] \setminus \Psi_A(T)} \delta_{(\mathrm{pr}(u),u)},$$

i.e. the atoms of $\pi$ are the pairs $(u,v)$ of $\Psi_A(T) \times \Phi_A(T)$ where $u$ is an atom of $\mu_{T^\circ}$ and $v$ of $\mu'_T$ and either $u = v$ or $u = \mathrm{pr}(v)$. For all $k \geq 0$, set $\pi_k$, the restriction of $\pi$ to $\Psi_A(T)|_k \times \Phi_A(T)|_k$.

Observe that $T^\circ = \Phi_A(T)$ differs from $\Psi_A(T)$ by the addition of some leaves to some vertices and their measures differ from the corresponding displacement of unit masses from the said vertices to these new leaves. If $(u,v)$ is an atom of $\pi$, the graph distance between $u$ and $v$ is therefore at most one. As a result, for all $k \geq 0$, $\mathrm{dis}\, C_k \leq 2$, $\pi_k(C_k^c) = 0$ and

$$\mathrm{D}\Big(\pi_k; \mu'_T|_k, \mu_{T^\circ}|_k\Big) \leq 2\mu_{T^\circ}\Big(\Phi_A(T)|_{k+1}\Big) - 2\mu_{T^\circ}\Big(\Phi_A(T)|_k\Big)$$

Lemma 1.3.1 and Proposition 1.4.2 then ensure that for all $t \geq 0$,

$$\mathrm{d}_{\mathrm{GHP}}\Big(\Big(R^{-1}\,\Psi_A(T)|_{tR}, R^{-1/\gamma}\mu'_T|_{tR}\Big), \Big(R^{-1}\,T^\circ|_{tR}, R^{-1/\gamma}\mu_{T^\circ}|_{tR}\Big)\Big) \leq \frac{1}{R} \vee \frac{2\mu_{T^\circ}(T^\circ|_{tR+1}) - 2\mu_{T^\circ}(T^\circ|_{tR})}{R^{1/\gamma}} \xrightarrow[R\to\infty]{\mathbb{P}} 0.$$

$(ii)$ Let $\Pi_A(T)$ be endowed with $\mu_T$, the counting measure on $\mathcal{V}_A(T)$. The tree $\Psi_A(T)$ is obtained by cutting some branches with i.i.d. geometric lengths from $\Pi_A(T)$ and suitably moving the atoms of the corresponding measures. In other words, we may write

$$\Pi_A(\mathtt{t}) = \Psi_A(\mathtt{t}) \bigotimes_{u \in \mathcal{L}[\Psi_A(T)]} (u, \mathtt{b}_{G_u})$$

where $(G_u)_u$ are i.i.d. geometric random variables with parameter $q_1(m_1 = 1) > 0$ and where $\mathtt{b}_k$ denotes the branch with length $k$ for any non-negative integer $k$.

Thus there is a natural correspondence $C$ between $\Pi_A(T)$ and $\Psi_A(T)$ given by

$$C := \Big\{(u,u) : u \in \Psi_A(T)\Big\} \cup \bigcup_{u \in \mathcal{L}[\Psi_A(T)]} \Big\{(v,u) : v \in \Pi_A(T), v \text{ is above } u\Big\}.$$

If for all $k \geq 0$ we set $C_k := \{(u,v) \in C : |u| \leq k, |v| \leq k\}$, then

$$\mathrm{dis}\, C_k \leq \max_{u \in \mathcal{L}[\Psi_A(T)] : |u| \leq k} G_u =: M_k.$$

Observe that $\#\{u \in \mathcal{L}[\Psi_A(T)] : |u| \leq k\} \leq \mu_{T^\circ}(T^\circ|_{k+1})$. In light of Lemma 2.2.5, there is a finite deterministic constant say $K$ such that for all $t \geq 0$

$$\limsup_{R\to\infty} \frac{\mathrm{dis}\, C_{\lfloor tR \rfloor}}{\log \mu_{T^\circ}(T^\circ|_{tR+1})} \leq K.$$

Moreover, it follows from Theorem 2.2.1 and lemma 2.2.4 that we may use Proposition 1.4.2 and get that $\log \mu_{T^\circ}(T^\circ|_{tR+1}) = o(R^{-1})$ in probability.

Now, let $\pi$ be the measure on $\Pi_A(T) \times \Psi_A(T)$ such that its atoms are the pairs $(u,v)$ of $\Pi_A(T) \times \Psi_A(T)$ where $u$ is an atom of $\mu_T$, $v$ of $\mu'_T$ and where $u$ is the highest ancestor of $v$ in $\Psi_A(T)$. More precisely, set

$$\pi(\mathrm{d}u, \mathrm{d}v) := \mathbb{1}_{v \notin \mathcal{L}[\Psi_A(T)]}\, \delta_v(\mathrm{d}u)\, \mu'_T(\mathrm{d}v) + \mathbb{1}_{v \in \mathcal{L}[\Psi_A(T)]}\, \mathbb{1}_{u \text{ is above } v}\, \mu_T(\mathrm{d}u)\, \mu'_T(\mathrm{d}v)$$

and denote by $\pi_k$, the restriction of $\pi$ to $\Pi_A(T)|_k \times \Psi_A(T)|_k$ for all integers $k \geq 0$. Observe that for all $k$, $\pi_k(C_k^c) = 0$ and that if $(u, v)$ is an atom of $\pi$ with $|v| \leq k$, then $|u| \leq k + M_k$. As a result,

$$\begin{aligned} \mathrm{D}\Big(\pi_k; \mu_T|_k, \mu'_T|_k\Big) &\leq 2\mu'_T\Big(\Pi_A(T)|_{k+M_k}\Big) - 2\mu'_T\Big(\Psi_A(T)|_k\Big) \\ &\leq 2\mu_{T^\circ}\Big(T^\circ|_{k+M_k+1}\Big) - 2\mu_{T^\circ}\Big(T^\circ|_{k-1}\Big). \end{aligned}$$

Proceeding as we did before, we can use Lemma 2.2.5 and Proposition 1.4.2 to get that for all $t \geq 0$, $\mathrm{D}\big(\pi_{tR}; \mu_T|_{tR}, \mu'_T|_{tR}\big) = o(R^{-1/\gamma})$ in probability. Consequently, for all $t \geq 0$,

$$\mathrm{d}_{\mathrm{GHP}}\Big(\Big(R^{-1}\,\Pi_A(T)|_{tR}, R^{-1/\gamma}\mu_T|_{tR}\Big), \Big(R^{-1}\,\Psi_A(T)|_{tR}, R^{-1/\gamma}\mu'_T|_{tR}\Big)\Big) \xrightarrow[R\to\infty]{\mathbb{P}} 0.$$

Using the results proved in steps $(i)$ and $(ii)$ along with Assumption $(\Delta)$, we get that for all $t \geq 0$,

$$\mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T, R^{-1/\gamma}\mu_T)|_t, (R^{-1}T^\circ, R^{-1/\gamma}\mu_{T^\circ})|_t\Big) \xrightarrow[R\to\infty]{\mathbb{P}} 0.$$

Because this convergence holds for all $t \geq 0$, we deduce from Theorem 1.4.8 that

$$\mathrm{D}_{\mathrm{GHP}}\Big((R^{-1}T, R^{-1/\gamma}\mu_T), (R^{-1}T^\circ, R^{-1/\gamma}\mu_{T^\circ})\Big) \xrightarrow[R\to\infty]{\mathbb{P}} 0.$$

Now recall that $(T^\circ/R, \mu_{T^\circ}/R^{1/\gamma})$ converges to $\mathcal{T}^I_{\gamma,\nu}$ in distribution for $\mathrm{D}_{\mathrm{GHP}}$. Slutsky's theorem concludes this proof. □

## 2.3 Application to conditioned Galton-Watson trees

In this section, we will generalise results from Section 1.5.1. Let $\xi$ denote a critical probability distribution on $\mathbb{Z}_+$, i.e. with mean 1 and such that $\xi(1) < 1$. Denote by $\hat{\xi}$ the size-biased distribution of $\xi$, that is set $\hat{\xi}(k) := k\,\xi(k)$ for all $k \in \mathbb{Z}_+$. Recall that $\mathrm{GW}_\xi$ is the distribution of a Galton-Watson tree with offspring distribution $\xi$ and that $\mathrm{GW}^\infty_\xi$ denotes the distribution of the corresponding Kesten's tree.

Let $A \subset \mathbb{Z}_+$ be such that $0 < \xi(A) < 1$ as well as $\mathcal{N}_A := \{n \geq 0 : \mathrm{GW}_\xi(\mathsf{T}^A_n) > 0\}$, the set of possible $\#_A$-sizes for a Galton-Watson tree with offspring distribution $\xi$. Moreover, set $d_A := \gcd\{n-1 : n \in \mathcal{N}_A\}$.

Let $T$ be a Galton-Watson tree with offspring distribution $\xi$. For all $n \in \mathcal{N}_A$, let $\mathrm{GW}^{A,n}_\xi$ be the law of a $T$ conditioned on the event "$\#_A T = n$".

**Proposition 2.3.1.** $(i)$ *In the local topology on* $\mathsf{T}$, $\mathrm{GW}^{A,n}_\xi \Rightarrow \mathrm{GW}^\infty_\xi$.

$(ii)$ *Let* $T_\infty$ *be distributed following* $\mathrm{GW}^\infty_\xi$ *and let* $\mu_A$ *be the counting measure on* $\mathcal{V}_A(T_\infty)$. *Suppose that* $\xi$ *has finite variance* $\sigma^2$ *and that* $d_A = 1$. *Then*

$$\left(\frac{T_\infty}{R}, \frac{\mu_A}{R^2}\right) \xRightarrow[R\to\infty]{\mathrm{D}_{\mathrm{GHP}}} \left(\mathcal{T}_B, \frac{\sigma^2\,\xi(A)}{4}\mu_B\right)$$

*where* $(\mathcal{T}_B, \mu_B)$ *denotes the Brownian tree with immigration.*

*Remark 2.3.1.* Proposition 2.3.1 $(i)$ was proved in [2], Theorem 5.1.

Lemma 8 in [111] states that $(\mathrm{GW}^{A,n}_\xi; n \in \mathcal{N}_A) = \mathrm{MB}^{A,q^A}$ where, if $0 \notin A$, $q^A_0 := \mathrm{GW}^{A,0}_\xi$, and for all $n \neq 0$ in $\mathcal{N}_A$, $q^A_n$ is given by

$$q^A_n(\lambda) = \frac{p!\,\xi(p)}{\prod_{j\geq 1} m_j(\lambda)!} \frac{\prod_{i=1}^p \mathbb{P}[\#_A T = \lambda_i]}{\mathbb{P}[\#_A T = n]}$$

for all $\lambda = (\lambda_1, \ldots, \lambda_p)$ in $\mathcal{P}^A_n$.

Further observe that Kesten's tree may be described as an infinite Galton-Watson tree. Indeed, $\mathrm{GW}_\xi^\infty = \mathrm{MB}_\infty^{A,q^A,q^A_\infty}$ where the probability measure $q^A_\infty$ is defined for all $\lambda=(\lambda_2,\dots,\lambda_p)$ in $\mathcal{P}^*_{<\infty}$ by

$$q^A_\infty(\infty,\lambda) := \hat{\xi}(p)\,\frac{(p-1)!}{\prod_{j\geq 1} m_j(\lambda)!}\prod_{i=2}^p \mathbb{P}[\#_A T=\lambda_i].$$

As a result, we will use Proposition 2.1.2 and Theorem 2.2.2 to prove Proposition 2.3.1. We will also need the following results from [111]:

**Proposition 2.3.2.** $(i)$ *There exists a critical probability measure $\zeta$ on $\mathbb{Z}_+$ such that if $\tau$ is a Galton-Watson tree with offspring distribution $\zeta$, then conditionally on $\{\#_A T\neq 0\}$,*

$$\#_A T \overset{(\mathrm{d})}{=} \#\tau.$$

*Moreover, $d_A = \gcd\{n-1;\mathrm{GW}_\zeta(\mathrm{T}_n)>0\}$.*

$(ii)$ *If $\xi$ has finite variance $\sigma^2$, then $\zeta$ has finite variance $\mathbb{P}[\#_A T>0]^2\,\sigma^2/\xi(A)$.*

*Remark 2.3.2.* See also Theorem 5.1 in [2].

*Proof of Proposition 2.3.1 $(i)$.* We will proceed like in Proposition 1.5.1. Using Lemma 2.1.1 $(iii)$ and Proposition 2.1.2, it will be enough to prove that for all $\lambda=(\lambda_2,\dots,\lambda_p)$ in $\mathcal{P}^*_{<\infty}$ and $L:=\|\lambda\|+\mathbb{1}_A(p)$,

$$q^A_n(n-L,\lambda)\xrightarrow[n\to\infty]{} q^A_\infty(\infty,\lambda).$$

To get this convergence, we therefore need to prove that

$$\frac{\mathbb{P}[\#_A T=n-L]}{\mathbb{P}[\#_A T=n]}\xrightarrow[n\to\infty]{} 1,$$

which is a consequence of Lemma 1.5.3 and Proposition 2.3.2. □

We now turn to the proof of Proposition 2.3.1 $(ii)$.

**Lemma 2.3.3.** *The measure $q^A_\infty$ satisfies Assumption (I) of Theorem 2.2.2 for $\gamma=1/2$ and $[\sigma^2\xi(A)/4]^{1/2}$ times the immigration Brownian measure $I_B$.*

*Proof.* We will proceed as in Proposition 1.5.5 with the help of Proposition 1.4.15. Let us first introduce the necessary notations.

Let $(T_n)_{n\geq 1}$ be i.i.d. $\mathrm{GW}_\xi$ trees and set $w_A:=\mathbb{P}[\#_A T_1>0]$. Let $\zeta$ be as in Proposition 2.3.2 and $\rho^2:=w_A^2\sigma^2/\xi(A)$, which is the variance of $\zeta$. If $(\tau_n)_{n\geq 1}$ is a sequence of i.i.d. $\mathrm{GW}_\xi$ trees, Corollary 2 in [111] gives the following Otter-Dwass type formula: for all positive integers $n$ and $p$

$$\begin{aligned}\mathbb{P}\big[\#_A T_1+\cdots+\#_A T_p=n\big] &= \sum_{k=1}^p \binom{p}{k} w_A^k(1-w_A)^{p-k}\mathbb{P}\big[\#\tau_1+\cdots+\#\tau_k=n\big]\\ &= \sum_{k=1}^p \binom{p}{k} w_A^k(1-w_A)^{p-k}\frac{k}{n}\mathbb{P}\big[S_n=-k\big]\end{aligned}$$

where $(Y_n)_{n\geq 1}$ are i.i.d. $\zeta$-distributed random variables and $S_n:=Y_1+\cdots+Y_n-n$ for all $n\geq 0$.

Recall that if the law of $(\infty,\Lambda)$ is $q^A\infty$, then $\Lambda$ is distributed like $(\#_A T_1,\dots,\#_A T_N)^\downarrow$ where the random variable $N$ is independent of $(T_n)_n$ and is such that the law of $1+N$ is $\hat{\xi}$. As a result, proceeding as Lemma 1.5.6 yields

$$\begin{aligned}n^{3/2}\,\mathbb{P}[\|\Lambda\|=n] &= n^{3/2}\sum_{p\geq 0}\hat{\xi}(p+1)\sum_{k=0}^p\binom{p}{k}w_A^k(1-w_A)^{p-k}\mathbb{P}[\#\tau_1+\cdots+\#\tau_k=n]\\ &= n^{3/2}\sum_{p\geq 0}\hat{\xi}(p+1)\sum_{k=0}^p\binom{p}{k}w_A^k(1-w_A)^{p-k}\frac{k}{n}\mathbb{P}[S_n=-k]\\ &\xrightarrow[n\to\infty]{} \frac{w_A\sigma^2}{(2\pi\rho^2)^{1/2}} = \left(\frac{\xi(A)\,\sigma^2}{2\pi}\right)^{1/2}\end{aligned}$$

where we have used the fact that $\sum_{k=0}^{p}\binom{k}{p}w_A^k(1-w_A)^{p-k}\,k = p\,w_A$. Similarly, if $G$ denotes the generating function of $\xi$, we may proceed as is Lemma 1.5.7 to get

$$n^{1/2}\,\mathbb{P}\big[\Lambda_1 \geq n\big] = n^{1/2}\Big[G'(1) - G'\big(1-w_A\mathbb{P}[\#\tau_1 \geq n]\big)\Big] \xrightarrow[n\to\infty]{} \left(\frac{2w_A}{(2\pi\,\rho^2)^{1/2}}\right)^{1/2} G''(1) = \left(\frac{2\xi(A)\,\sigma^2}{\pi}\right)^{1/2}.$$

Proposition 1.4.15 now concludes this proof. □

**Lemma 2.3.4.** *The pair $(T,\mu_A)$ satisfies Assumption $(\Delta)$ of Theorem 2.2.2 with $\gamma = 1/2$.*

*Proof.* As we pointed out in Remark 2.2.1, when $0 \in A$, the assumption is automatically satisfied. Thus, let us assume that $0 \notin A$.

Let $\tau$ be a $\mathrm{GW}_\xi$ tree and let $\tau_0$ be distributed like $\tau$ conditioned on the event $\#_A T = 0$; in other words, the law of $\tau_0$ is $\mathrm{GW}_\xi^{A,0} = \mathrm{MB}_0^{A,q}$. Let $\mathtt{t}$ be a finite tree such that $\mathtt{t} = [\![\mathtt{t}_1,\dots,\mathtt{t}_p]\!]$, and denote by $C_\mathtt{t}$ the number of permutations of the $p$-tuple $(\mathtt{t}_1,\dots,\mathtt{t}_p)$. If we write $w_A := \mathbb{P}[\#_A\tau > 0]$, then

$$\begin{aligned}\mathbb{P}[\tau_0 = \mathtt{t}] &= (1-w_A)^{-1}\,\mathbb{P}[\tau = \mathtt{t}]\mathbb{1}_{\#_A\mathtt{t}=0}\\ &= (1-w_A)^{-1}\,\xi(p)\mathbb{1}_{A^c}(p)\,C_\mathtt{t}\textstyle\prod_{i=1}^p\mathbb{P}[\tau=\mathtt{t}_i]\mathbb{1}_{\#_A\mathtt{t}_i=0}\\ &= (1-w_A)^{p-1}\,\xi(p)\mathbb{1}_{A^c}(p)\,C_\mathtt{t}\textstyle\prod_{i=1}^p\mathbb{P}[\tau_0=\mathtt{t}_i].\end{aligned}$$

For all $p \geq 0$, set $\eta_p := (1-w_A)^{p-1}\,\xi(p)\mathbb{1}_{A^c}(p)$ and observe that

$$\textstyle\sum_{p\geq 0}\eta_p = \dfrac{1}{\mathbb{P}[\#_A\tau=0]}\sum_{p\notin A}\xi(p)\,\mathbb{P}[\#_A T=0]^p = \dfrac{1}{\mathbb{P}[\#_A\tau=0]}\,\mathbb{P}[\#_A\tau=0] = 1$$

so $\eta$ is a probability distribution. As a result, we get that $\tau_0$ is a $\mathrm{GW}_\eta$ tree.

Moreover, for all positive $p$, $(1-w_A)^{p-1}\mathbb{1}_{A^c}(p) < 1$ so the mean $m_\eta$ of $\eta$ is clearly less than that of $\xi$, i.e. $m_\eta < 1$. Consequently, if $(Z_k)_{k\geq 0}$ denotes a Galton-Watson process with offspring distribution $\eta$ and started from 1, for all integers $j \geq 0$, we get that

$$\mathbb{P}\big[|\tau_0| > j\big] = \mathbb{P}\big[Z_j \geq 1\big] \leq \mathbb{E}[Z_j] = m_\eta^j.$$

This bound entails that $\mathbb{E}[\mathrm{e}^{a|\tau_0|}]$ is finite for some positive $a$.

Now recall that $\Pi_A(T)$ is the tree obtained by deleting all the sub-trees of $T$ which contain no vertices of $\mathcal{V}_A(T)$ and is endowed with $\mu_A$. Note that the deleted sub-trees are i.i.d. $\mathrm{GW}_\xi^{A,0}$ trees and are attached to $\Pi_A(T)$ by a single edge. As a result, we may write

$$T = \Pi_A(T)\textstyle\bigotimes_{u\in\Upsilon}\big(\mathrm{pr}(u),[\![\tau_u]\!]\big)$$

where $\Upsilon = \{u \in T : \#_A T_u = 0, \#_A T_{\mathrm{pr}(u)} > 0\}$ is the set of the roots of the deleted sub-trees and $(\tau_u)_{u\in T}$ are i.i.d. $\mathrm{GW}_\xi^{A,0}$ trees. A natural correspondence between $T$ and $\Pi_A(T)$ is therefore given by

$$C := \Big\{(u,u) : u \in \Pi_A(T)\Big\} \cup \textstyle\bigcup_{u\in\Upsilon}\Big\{(\mathrm{pr}(u),v) : v \in \tau_u\Big\}$$

For all $k \geq 0$, set $C_k := \{(u,v) \in C : |u| \leq k\}$ and observe that

$$\mathrm{dis}\,C_k \leq 2\,\max_{u\in\Upsilon:|u|\leq k} 1 + |\tau_u|.$$

Clearly, $\#\{u \in \Upsilon : u \leq k\} \leq \#T|_k$ so Lemma 2.2.5 and the fact that $|\tau_u|$ has finite exponential moments ensure that we can find a finite deterministic constant $K$ such that for all $t \geq 0$

$$\limsup_{R\to\infty}\frac{\mathrm{dis}\,C_{\lfloor tR\rfloor}}{\log \#T|_{tR}} \leq K.$$

Moreover, Propositions 1.4.2 and 1.5.2 ensure that $\log \#T|_{tR} = o(R^{-1})$ in probability. Now recall that the measures on $T$ and $\Pi_A(T)$ are equal so for all $t \geq 0$,

$$\mathrm{d}_{\mathrm{GHP}}\Big((R^{-1}T, R^{-2}\mu_A)|_t, (R^{-1}\Pi_A(T), R^{-2}\mu_{T^\circ})|_t\Big) \xrightarrow[R\to\infty]{\mathbb{P}} 0$$

which, along with Theorem 1.4.8, concludes this proof. □

*Proof of Proposition 2.3.1* $(ii)$. Theorem 7 in [111] ensures that

$$n^{1/2}(1-s_1)\,\bar{q}_n^A(\mathrm{d}\mathbf{s}) \Longrightarrow_{n\to\infty} \frac{\sigma\,\xi(A)^{1/2}}{2}(1-s_1)\,\nu_B(\mathrm{d}\mathbf{s}),$$

i.e. Assumption (S) of Theorem 2.2.2 holds for $q^A$, $\gamma = 1/2$ and $\nu = [\sigma^2\xi(A)/4]^{1/2}\,\nu_B$. This fact along with Lemmas 2.3.3 and 2.3.4 allows us to use Theorem 2.2.2 and prove that $(T/R, \mu_A/R^2)$ converges in distribution to a $\big(1/2, [\sigma^2\xi(A)/4]^{1/2}\nu_B, [\sigma^2\xi(A)/4]^{1/2}I_B\big)$ fragmentation tree with immigration. Proposition 1.3.12 then yields the desired result. □

# Chapter 3

# On the profile of random trees: A survey of existing results

In this chapter, we will survey some known results on the asymptotic behaviour of the *profile* of various models of random trees. Here, if $\mathtt{t}$ is a finite rooted tree, we will call its *profile* the sequence $\ell_{\mathtt{t}}$ of the number of vertices in each layer of $\mathtt{t}$, i.e. for any non-negative integer $j$,

$$\ell_{\mathtt{t}}(j) := \#\big\{u \in \mathtt{t} : |u| = j\big\}$$

where $|u|$ denotes the *height* of $u$, that is the graph distance between $u$ and the root of $\mathtt{t}$. A similar notion is that of the *external profile* of $\mathtt{t}$, i.e. the sequence $\ell^\circ_{\mathtt{t}}$ such that

$$\ell^\circ_{\mathtt{t}}(j) := \#\big\{u \in \mathcal{L}(\mathtt{t}) : |u| = j\big\}$$

where $\mathcal{L}(\mathtt{t})$ is the set of *leaves* of $\mathtt{t}$. In other words, $\ell^\circ_{\mathtt{t}}(j)$ is the number of leaves of $\mathtt{t}$ at height $j$. Some authors refer to leaves as *external vertices*, hence the name.

We will be interested in two families of random trees: those whose height is roughly proportional to a power of their size and those whose height is proportional to the logarithm of their size. The asymptotic profiles of these two types of trees are very unalike and the methods used in their study thus differ widely. We point out that this chapter will not cover all known models but only some illustrative examples of both families. Namely, in Section 3.1, we will present results on the profiles of conditioned Galton-Watson trees and Pólya trees, which belong to the former family, and in Section 3.2, we will turn to recursive trees and binary search trees, which are in the latter.

## 3.1 Conditioned Galton-Watson trees and Pólya trees

It is known that conditioned Galton-Watson trees and Pólya trees exhibit scaling limits under appropriate rescaling, see e.g. [9, 47, 66]. Moreover, the corresponding limits belong to the class of $\mathbb{R}$-trees. Informally, $\mathbb{R}$-trees are metric spaces in which any two points can be continuously joined by a single path (up to its parametrisation). See Le Gall [92] for instance for some background on the topic of $\mathbb{R}$-trees.

It would then be natural to expect that the adequately rescaled profiles of these discrete trees converge to some "continuum profile" associated to the corresponding $\mathbb{R}$-trees. We will first present this notion of continuum profile and then turn to the study of the asymptotic properties of the (discrete) profiles of conditioned Galton-Watson trees and Pólya trees.

Let $\mathbf{T}$ be a compact rooted $\mathbb{R}$-tree endowed with a probability measure, i.e. a 4-tuple $\mathbf{T} = (T, d_T, \rho_T, \mu_T)$ where $(T, d_T)$ is a compact $\mathbb{R}$-tree, $\rho_T$ is an element of $T$ referred to as its root, and $\mu_T$ is a Borel probability measure on $(T, d_T)$. Consider the function $m_{\mathbf{T}} : \mathbb{R}_+ \to [0, 1]$ such that $m_{\mathbf{T}}(r) := \mu_T(T|_r)$ where

$T|_r := \{x \in T : d_T(\rho_T, x) \leq r\}$. Observe that $m_{\mathbf{T}}$ is the Cumulative Distribution Function of a probability measure $\mathrm{d}m_{\mathbf{T}}$. When $\mathrm{d}m_{\mathbf{T}}$ is absolutely continuous with respect to the Lebesgue measure, we will say that $\mathbf{T}$ has a profile which is the density $l_{\mathbf{T}}$ of $\mathrm{d}m_{\mathbf{T}}$.

In general, the existence of a profile for random real trees is not a trivial question. Let us give an important example of a random real tree which a.s. has a profile: Aldous' [7] Brownian tree.

*The Brownian tree.* Recall that the contour path of a Brownian tree is distributed like a standard Brownian excursion with length 1. As a result, if $(\mathcal{T}, \mu)$ is a Brownian tree and $\mathbf{e}$ is a Brownian excursion, then the process $\left[\mu(\mathcal{T}|_t); t \geq 0\right]$ is distributed like

$$\big(M(t); t \geq 0\big) := \Big( \textstyle\int_0^1 \mathbb{1}_{\mathbf{e}_s \leq t} \mathrm{d}s \,;\, t \geq 0 \Big) \tag{3.1}$$

which is almost surely a continuous Cumulative Distribution Function. The random measure $\mathrm{d}M$ is called the *occupation measure* of the excursion $\mathbf{e}$. It a.s. admits a density $L$ called the *local time* of the underlying Brownian excursion and given by

$$L(t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^1 \mathbb{1}_{]t-\varepsilon, t+\varepsilon[}(\mathbf{e}_u)\, \mathrm{d}u. \tag{3.2}$$

Informally, $L(t)$ records the time spent by the excursion at height $t$. Consequently, the Brownian tree a.s. has a profile which is distributed like $L$.

*Remark 3.1.1.* Duquesne and Le Gall's [49] $\alpha$-stable trees also a.s. admit a profile, see Kersting [82] for instance. In [60], Haas gave a simple criterion to establish existence or non-existence of the profile of fragmentation trees, a large class of random $\mathbb{R}$-trees to which both the Brownian tree and the stable trees belong. See also Ged [55] for a similar study on self-similar growth-fragmentation models.

*Convergence of rescaled profiles.* Let $(\mathtt{t}_n)_n$ be a sequence of unordered trees such that for all $n$, $\#\mathtt{t}_n = n$. For all $n$, set $\mu_n$ as the counting measure on $\mathtt{t}_n$. Suppose that there exist some sequence $(a_n)_n$ of positive real numbers and a compact rooted $\mathbb{R}$-tree $\mathbf{T}$ endowed with a probability measure such that

$$\big(a_n^{-1}\mathtt{t}_n, n^{-1}\mu_n\big) \xrightarrow[n\to\infty]{\mathrm{d_{GHP}}} \mathbf{T}. \tag{3.3}$$

When $m_{\mathbf{T}}$ is continuous, we have

$$\frac{1}{n}\Big(\mu_n\big(\mathtt{t}_n|_{a_n r}\big)\,;\, r \geq 0\Big) \xrightarrow[n\to\infty]{} m_{\mathbf{T}} \tag{3.4}$$

uniformly on $[0, \infty[$. This is a consequence of Proposition 2.10 in [3], which gives point-wise convergence, and the well-known fact that a sequence of CDFs that converges point-wise to a continuous CDF converges uniformly on $\mathbb{R}$.

When $\mathbf{T}$ has a profile, it is then natural to wonder whether once suitably rescaled, the profile of $\mathtt{t}_n$ converges to that of $\mathbf{T}$ in some sense. More precisely, we want to know if

$$\frac{a_n}{n}\Big(\ell_{\mathtt{t}_n}\big(\lfloor a_n r \rfloor\big); r \geq 0\Big) \xrightarrow[n\to\infty]{} l_{\mathbf{T}} \tag{3.5}$$

either point-wise or in the space $\mathcal{D}[0, \infty[$. In the latter case, we naturally have to assume that $l_{\mathbf{T}}$ is a càdlàg function.

*Remark 3.1.2.* Suppose that $\mathbf{T}$ admits a càdlàg profile $l_{\mathbf{T}}$. Further assume that the convergence (3.3) holds and that $(a_n/n) \cdot \ell_{\mathtt{t}_n}\big(\lfloor a_n \cdot \rfloor\big) \to l_{\mathbf{T}}$ in $\mathcal{D}[0, \infty[$. Then, in particular, since there exists some finite $K$ such that $\ell_{\mathtt{t}_n}(\lfloor a_n\, t \rfloor) = 0$ for all $t \geq K$ and $n \geq 1$, we get that

$$\sup_{j \geq 0} \frac{a_n}{n} \ell_{\mathtt{t}_n}(j) \xrightarrow[n\to\infty]{} \sup_{r \geq 0} l_{\mathbf{T}}(r).$$

This means that the "width" of $\mathtt{t}_n$ times $a_n/n$ converges to that of $\mathbf{T}$ when $n \to \infty$.

Also observe that if $(a_n/n) \cdot \ell_{\mathtt{t}_n}(\lfloor a_n \cdot \rfloor)$ converges to $l_{\mathbf{T}}$ almost everywhere on $[0, \infty[$, then by Scheffé's lemma, this convergence also happens in $L^1(\mathrm{d}r)$.

We point out that (3.5) has a natural probabilistic interpretation. For all $n \geq 1$, let $U_n$ be drawn uniformly in $\mathtt{t}_n$ and observe that for all $r$, $\mathbb{P}[|U_n|/a_n \leq r] = \mu_n(\mathtt{t}_n|_{a_n r})$. Similarly, if $U$ denotes a $\mu_T$ distributed random variable in $T$, we have $\mathbb{P}[|U| \leq r] = m_{\mathbf{T}}(r)$ for all $r$. The convergence (3.4) therefore translates to

$$\frac{1}{a_n}|U_n| \underset{n\to\infty}{\Longrightarrow} |U|. \tag{3.6}$$

If $\mathbf{T}$ has a profile, $l_{\mathbf{T}}$ is the density of the random variable $|U|$. Similarly, for all $j \geq 0$, $\mathbb{P}[|U_n| = j] = \ell_{\mathtt{t}_n}(j)/n$. As a result, the convergence (3.5) is akin to a "local" version of (3.6) (compare with the local Central Limit Theorem [56] or with [57]).

### 3.1.1 Critical Galton-Watson trees

We will now describe one of the most important models of trees for which a convergence like (3.5) holds: conditioned critical Galton-Watson trees.

Let $\xi$ be a probability distribution on the set $\mathbb{Z}_+$ of non-negative integers such that $\sum_{k\geq 0} k\xi_k = 1$. For technical reasons, assume that the g.c.d. of the support $\{k \geq 0 : \xi(k) > 0\}$ equals one; we will say that $\xi$ is aperiodic. Recall that a (plane) Galton-Watson tree $T$ with offspring distribution $\xi$ is a random ordered tree such that for any finite tree $\mathtt{t}$,

$$\mathbb{P}[T = \mathtt{t}] = \prod_{u\in\mathtt{t}} \xi_{c_{\mathtt{t}}(u)}$$

where $c_{\mathtt{t}}(u)$ denotes the number of children of the vertex $u$ in the tree $\mathtt{t}$. Observe that the profile of $T$ is simply a Galton-Watson process started from 1 and with offspring distribution $\xi$.

For any positive integer $n$ such that $\mathbb{P}[\#T = n] > 0$, let $\mathrm{GW}_\xi^n$ be the distribution of the tree $T$ conditioned to have $n$ vertices. The assumption $\gcd\{k : \xi(k) > 0\} = 1$ ensures that $\mathbb{P}[\#T = n] > 0$ for all $n$ large enough. Moreover, for all suitable $n$, let $T_n$ be distributed according to $\mathrm{GW}_\xi^n$. The scaling limits of the sequence $(T_n)_n$ have been studied quite extensively, see Aldous [8, 9], Duquesne [47], Duquesne and Le Gall [49], Haas and Miermont [66], Kortchemski [86] or Rizzolo [111] for instance.

Most notably, Aldous proved:

**Theorem 3.1.1 (Theorem 23 [9]).** *If $\xi$ has a finite and positive variance $\sigma^2$, then*

$$\left(\frac{\sigma}{2\sqrt{n}} T_n, \frac{1}{n}\mu_n\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{T}_B \tag{3.7}$$

*with respect to the GHP topology and where $\mathcal{T}_B$ denotes a* Brownian tree.

For all $n$, let $M_n$ be the rescaled cumulative profile of $T_n$, defined for all non-negative $t$ by

$$M_n(t) := \frac{1}{n}\#T_n|_{2\sqrt{n}t/\sigma}.$$

Corollary 3 in [8] states that as a result of (3.7), $M_n$ converges in law to $M$ for the uniform topology on $[0, \infty[$, compare with (3.4). Aldous then conjectured in [8] that the rescaled profiles of the trees $T_n$ should also converge once adequately rescaled toward the local time $L$. More precisely, Conjecture 4 in [8] inferred that if $\xi$ has finite and positive variance $\sigma^2$,

$$L_n \underset{n\to\infty}{\Longrightarrow} L \tag{3.8}$$

for the Skorokhod topology on $\mathcal{D}[0,\infty[$ where for all $n \geq 1$, $L_n$ denotes the rescaled profile of $T_n$, i.e. is given for all $t \geq 0$ by

$$L_n(t) := \frac{2}{\sigma\sqrt{n}} \cdot \#\Big\{u \in T_n : |u| = \lfloor 2\sqrt{n}t/\sigma \rfloor\Big\}. \tag{3.9}$$

Observe that for all $n \geq 1$, $\int_0^\infty L_n(t)\,\mathrm{d}t = 1$. Furthermore, if $h_n := \sup\{t \geq 0 : L_n(t) > 0\}$, we get that $h_n := (|T_n|+1)\cdot\sigma/(2\sqrt{n})$ and as such, in light of (3.7), $h_n \Rightarrow |\mathcal{T}_B| := \sup_{0\leq u\leq 1} \mathbf{e}_u$.

This conjecture was proved by Drmota and Gittenberger [43] under the additional assumption that $\xi$ has finite exponential moments. Their methods also allowed them to prove a similar statement for the *external* profile of $T_n$. See also Kersting [82], who used a very different approach which also applies when $\xi$ belongs to the domain of attraction of a stable distribution. Also see Pitman [105] who, in order to study an SDE satisfied by the process $L$, proved that (3.8) holds when $\xi$ is the Poisson distribution with parameter 1. We will now outline these three different methods in the next few paragraphs.

**Generating functions.** In [43], Drmota and Gittenberger endeavoured to prove (3.8) with a precise study of generating functions. See also Section 4.2 in [42].

Recall that $\xi$ is an aperiodic probability measure on $\mathbb{Z}_+$ with mean 1 and finite variance $\sigma^2 > 0$. Recall that $T_n$ is a Galton-Watson tree conditioned to have $n$ vertices and $\ell_n$ is its profile. Let $X$ be a $\xi$-distributed random variable and denote by $\phi$ its generating function, i.e. for all complex $z$ with $|z| \leq 1$, set $\phi(z) := \mathbb{E}[z^X] = \sum_{k\geq 0} \xi_k z^k$. We will also suppose that the radius of convergence of $\phi$ is strictly bigger than 1; in particular, $X$ has exponential moments.

In this paragraph, we will consider a continuous version of the rescaled profile. More precisely, for all $t \geq 0$, set

$$L_n(t) := \frac{1}{\sqrt{n}}\Big(\big(\lfloor\sqrt{n}t+1\rfloor/\sqrt{n} - t\big)\,\ell_n(\lfloor\sqrt{n}t\rfloor) + \big(t - \lfloor\sqrt{n}t\rfloor/\sqrt{n}\big)\ell_n(\lfloor\sqrt{n}t\rfloor+1)\Big). \tag{3.10}$$

Drmota and Gittenberger's main result from [43] is the following.

**Theorem 3.1.2 (Theorem 1.1 [43]).** *Suppose that $\xi$ has finite exponential moments, i.e. that there exists some $r > 1$ such that $\sum_{n\geq 0} \xi_n r^n < \infty$. Then*

$$\Big(L_n(t); t \geq 0\Big) \underset{n\to\infty}{\Longrightarrow} \left(\frac{\sigma}{2} L\Big(\frac{\sigma\, t}{2}\Big); t \geq 0\right) \tag{3.11}$$

*in the space $\mathcal{C}[0,\infty[$ of continuous processes.*

*Remark 3.1.3.* For all $n$ let $\ell_n^\circ$ be the external profile of $T_n$, i.e. $\ell_n^\circ(j)$ is the number of leaves of $T_n$ at height $j$. Let $L_n^\circ(j/\sqrt{n}) := \ell_n^\circ(j)/\sqrt{n}$ for all integers $j$ and define $L_n^\circ(t)$ by linear interpolation when $\sqrt{n}\,t$ isn't an integer. Using the same techniques, Drmota and Gittenberger also proved that

$$\Big(L_n^\circ(t); t \geq 0\Big) \underset{n\to\infty}{\Longrightarrow} \left(\frac{\xi_0\,\sigma}{2} L\Big(\frac{\sigma\, t}{2}\Big); t \geq 0\right)$$

in $\mathcal{C}[0,\infty[$, see Theorem 1.2 [43]. This is closely linked to the fact that in a large Galton-Watson tree, the number of its leaves is roughly $\xi_0$ times its total size, see Section 2 in [85].

To prove the functional convergence (3.11), Drmota and Gittenberger first proved convergence of the finite-dimensional distributions, i.e. that for all $p \geq 1$, $0 \leq t_1 \leq \ldots t_p$,

$$\big(L_n(t_1), \ldots, L_n(t_p)\big) \underset{n\to\infty}{\Longrightarrow} \frac{\sigma}{2}\big(L(\sigma t_1/2), \ldots, L(\sigma t_p/2)\big), \tag{3.12}$$

see Theorem 2.1 [43]. Then they proved that the sequence $(L_n)_n$ is tight in $\mathcal{C}[0,\infty[$ with the help of a tightness criterion [29, Theorem 7.3] and the following result (see Theorem 6.1 in [43]): there exists a finite constant $C$ such that

$$\mathbb{E}\Big[\big(\ell_n(k+j) - \ell_n(k)\big)^4\Big] \leq C\, j^2\, n \tag{3.13}$$

holds for all integers $n$, $k$, and $j$. The rather involved and computation heavy proof of (3.13) is the main topic of Section 6 in [43]. Let us now introduce the appropriate objects and outline the proof of (3.12).

Let $T$, $T^{(i)}$, $i \geq 1$ be i.i.d. Galton-Watson trees with offspring distribution $\xi$ independent of $X$. We will denote by $\ell_T$, $\ell_{T^{(i)}}$, $i \geq 1$ their respective profiles. Let $a$ be the generating function of $\#T$, the total size of $T$. In other words, for all $z$ such that $|z| \leq 1$, set $a(z) := \mathbb{E}[z^{\#T}]$. Observe that

$$\#T \overset{(\mathrm{d})}{=} 1 + \sum_{i=1}^{X} \#T^{(i)}, \tag{3.14}$$

which ensures that the generating function $a$ satisfies $a(z) = z\,\phi[a(z)]$ for all $z$ in the unit disk.

Thus, the function $(z,w) \mapsto z\,\phi(w) - w$ vanishes for $w = a(z)$ for all $z$ in the unit disk. Moreover, its derivative with respect to the second variable $w$ is given by $(z,w) \mapsto z\,\phi'(w) - 1$. Observe that for all $z \neq 1$ with $|z| = 1$, $\sum_{n\geq 1} n\xi_n a(z)^{n-1}$ is a convex combination of the non-constant family $[a(z)^n; n \geq 1]$ which belongs to the unit disk (because $\sum_{n\geq 1} n\xi_n = 1$ and $\xi$ is aperiodic) so its modulus is less than 1. In particular, $z\,\phi'[a(z)] - 1 \neq 0$ for all $z \neq 1$ on the unit disk. As a result, the implicit function theorem ensures that the function $a$ can be extended analytically on an open set $U_a := \{z : |z| < 1 + \delta, \arg z \neq 0\} \cup [0,1[$ for some positive $\delta$.

Moreover, the Otter-Dwass formula [106, Chapter 6] and the local central limit theorem [56] ensure that $\mathbb{P}[\#T = n] \sim (2\pi\sigma^2 n^3)^{-1/2}$ as $n \to \infty$. From this fact, we deduce that as $z \to 0$ with $\mathrm{Re}\, z > 0$,

$$a(1-z) = 1 - \sqrt{2z}/\sigma + \mathcal{O}(|z|). \tag{3.15}$$

Now, for all non-negative $k$, let $a_k$ be the generating function of the pair $[\#T, \ell_T(k)]$, i.e. the function given by

$$a_k(z,u) := \mathbb{E}\Big[z^{\#T}\, u^{\ell_T(k)}\Big]$$

for all complex $z$ and $u$ with $|z| \leq 1$ and $|u| \leq 1$. Similarly to (3.14), we may write

$$a_k(z,u) = \mathbb{E}\Big[z^{\#T|_{k-1}} \prod_{i=1}^{\ell_T(k)} u\, z^{\#T^{(i)}}\Big].$$

If we set $y_k(z,u) := \mathbb{E}[z^{\#T|_{k-1}} u^{\ell_T(k)}]$ for all $z$ and $u$ in the unit circle (with the convention $y_0(z,u) = u$), we get $a_k(z,u) = y_k[z, u\,a(z)]$. Furthermore, for all $k \geq 0$, we also get

$$y_{k+1}(z,u) = z\,\mathbb{E}\Big[\prod_{i=1}^{X} u^{\ell_{T^{(i)}}(k)}\, z^{\#T^{(i)}}\Big] = z\,\phi[y_k(z,u)]. \tag{3.16}$$

Let $0 \leq k \leq j$ be integers and let $a_{k,j}$ denote the generating function of the 3-tuple $[\#T, \ell_T(k), \ell_T(j)]$. For all $z$, $u$ and $v$ in the unit circle, the same properties of Galton-Watson trees yield

$$\begin{aligned} a_{k,j}(z,u,v) &:= \mathbb{E}\Big[z^{\#T}\, u^{\ell_T(k)}\, v^{\ell_T(j)}\Big] = \mathbb{E}\Big[z^{\#T|_{k-1}} \prod_{i=1}^{\ell_T(k)} \big(u\, z^{\#T^{(i)}}\, v^{\ell_{T^{(i)}}(j-k)}\big)\Big] \\ &= y_k\Big[z, u\, a_{j-k}(z,v)\Big] = y_k\Big[z, u\, y_{j-k}[z, v\, a(z)]\Big]. \end{aligned}$$

By iterating these computations, we get that for any integers $p \geq 1$ and $0 \leq k_1 \leq \cdots \leq k_p$, for all $z, u_1, \ldots, u_p$ in the unit circle,

$$\begin{aligned} a_{k_1,\ldots,k_p}(z,u_1,\ldots,u_p) &:= \mathbb{E}\Big[z^{\#T} \prod_{j=1}^{p} u_j^{\ell_T(k_j)}\Big] \\ &= y_{k_1}\Big(z, u_1\, y_{k_2-k_1}\Big(z, u_2\, y_{k_3-k_2}\Big(z, \ldots y_{k_p-k_{p-1}}\big(z, u_p\, a(z)\big)\ldots\Big)\Big)\Big). \end{aligned} \tag{3.17}$$

For any integers $p \geq 1$, $0 \leq k_1 \leq \cdots \leq k_p$, we can express the characteristic function $\varphi_{n,k_1,\ldots,k_n}$ of the $p$-tuple $[\ell_n(k_1), \ldots, \ell_n(k_p)]$ in terms of the generating function $a_{k_1,\ldots,k_p}$. Indeed, for all $t_1, \ldots, t_p$, the Cauchy integral formula ensures that if $C$ is a closed contour around 0, then

$$\begin{aligned} \varphi^{(n)}_{k_1,\ldots,k_n}(x_1,\ldots,x_p) &:= \mathbb{E}\Big[\prod_{j=1}^{p} \mathrm{e}^{ix_j \ell_n(k_j)}\Big] \\ &= \frac{1}{\mathbb{P}[\#T = n]} \frac{1}{2i\pi} \oint_C a_{k_1,\ldots,k_n}\big(z, \mathrm{e}^{ix_1}, \ldots, \mathrm{e}^{ix_p}\big)\, z^{-n-1}\, \mathrm{d}z. \end{aligned} \tag{3.18}$$

Section 5 in [43] focuses on giving an appropriate integral representation of the characteristic function of the finite-dimensional distributions of the local time $L$. For all $p \geq 1$, $0 \leq t_1 \leq \cdots \leq t_p$ and real numbers $x_1, \ldots, x_p$,

$$\begin{aligned}\varphi_{t_1,\ldots,t_p}(x_1,\ldots,x_p) &:= \mathbb{E}\Big[\prod_{j=1}^{p} \mathrm{e}^{ix_j L(t_j)}\Big] \\ &= 1 + \frac{1}{i\sqrt{\pi/2}} \int_\gamma f^{(2)}_{t_1,\ldots,t_p}(z; x_1, \ldots, x_p)\,\mathrm{d}z \end{aligned} \tag{3.19}$$

where $\gamma := \{z \in \mathbb{C} : \inf_{x\geq 0} |z - x| = 1\}$, see Figure 3.1, and the function $f^{(\sigma)}_{x_1,\ldots,x_p}$ is given by

$$f^{(\sigma)}_{t_1,\ldots,t_p}(z; x_1, \ldots, x_p) := \Psi^{(\sigma)}_{t_1}\Big(z, ix_1 + \Psi^{(\sigma)}_{t_2-t_1}\Big(z, ix_2 + \Psi^{(\sigma)}_{t_3-t_2}\big(z, \ldots \Psi^{(\sigma)}_{t_p-t_{p-1}}\big(z, ix_p\big)\ldots\big)\Big)\Big) \tag{3.20}$$

for all positive $\sigma$ and the function $\Psi^{(\sigma)}_x$ is defined by

$$\Psi^{(\sigma)}_t(z,u) := \frac{u\sqrt{-z}\exp(-\sigma\, t\,\sqrt{z/2})}{\sqrt{-z}\exp(\sigma\, t\,\sqrt{z/2}) - u\sqrt{\sigma^2/2}\sinh(\sigma\, t\,\sqrt{z/2})}. \tag{3.21}$$

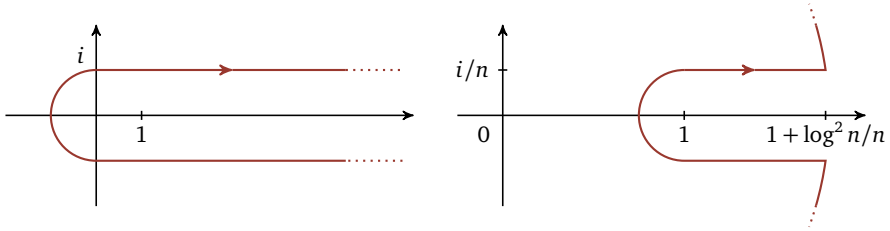



Figure 3.1 – The contours $\gamma$ and $\gamma_n$.

Lemma 2.1 in [43] then establishes in particular that if $\arg z \neq 0$, $x$ is real and $t \geq 0$, then as $n \to \infty$

$$\frac{y_{\lfloor\sqrt{n}t\rfloor}\big[1 + z/n, \big(1 + w/\sqrt{n}\big)\cdot a\big(1 + z/n\big)\big]}{a\big(1 + z/n\big)} - 1 \sim \frac{1}{\sqrt{n}}\Psi^{(\sigma)}_t(z,w). \tag{3.22}$$

Now, plugging (3.22) into (3.17) we get that for all $0 \leq t_1 \leq \cdots \leq t_p$, for all $z$ with $\arg z \neq 0$ and $x_1, \ldots, x_p$ in $\mathbb{R}$,

$$\sqrt{n}\Big[a_{\lfloor\sqrt{n}t_1\rfloor,\ldots,\lfloor\sqrt{n}t_p\rfloor}\big(1 + z/n, \mathrm{e}^{ix_1/\sqrt{n}}, \ldots, \mathrm{e}^{ix_p/\sqrt{n}}\big) - a\big(1 + z/n\big)\Big] \xrightarrow[n\to\infty]{} f^{(\sigma)}_{x_1,\ldots,x_p}(z; t_1, \ldots, t_p).$$

Finally, we will take $C = \gamma_n$ in (3.18) with

$$\begin{aligned}\gamma_n := \{&z \in \mathbb{C} : 1 \leq \mathrm{Re}\, z \leq \log^2 n/n, |\mathrm{Im}\, z| = 1/n\} \\ &\cup \{z \in \mathbb{C} : |1 - z| = 1/n, \mathrm{Re}\, z < 1\} \\ &\cup \{z \in \mathbb{C} : |z| = |1 + (\log^2 n + i)/n|, |\arg z| > \arg[1 + (\log^2 n + i)/n]\}\end{aligned}$$

see Figure 3.1. Recall that $a$ is analytic on an open subset $U_a$ which contains $\gamma_n$ for $n$ large enough. As a result,

$$\begin{aligned}&\varphi^{(n)}_{\lfloor\sqrt{n}t_1\rfloor,\ldots,\lfloor\sqrt{n}t_p\rfloor}\big(x_1/\sqrt{n}, \ldots, x_p/\sqrt{n}\big) \\ &\quad\sim \frac{1}{\mathbb{P}[\#T = n]}\frac{1}{2i\pi}\oint_{\gamma_n}\frac{a(z)\,,\mathrm{d}z}{z^{n+1}} + \frac{1}{i\sqrt{2\pi/\sigma^2}}\int_\gamma f^{(\sigma)}_{t_1,\ldots,t_p}(z; x_1, \ldots, x_p)\,\mathrm{d}z \\ &\quad= \varphi_{\sigma t_1/,\ldots,\sigma t_p/2}(\sigma x_1/2, \ldots, \sigma x_p/2)\end{aligned}$$

which proves (3.19).

**The profile as a functional of a Random Walk.** Kersting's strategy in [82] relies on the correspondence between Galton-Watson trees and random walks. The main idea here is to write the profile of a tree as a functional of some random walk which, in a way, encodes the said tree.

In this paragraph, we will make the same assumption as in [82], that is we'll assume that $\xi$ is an aperiodic probability measure on $\mathbb{Z}_+$ with mean 1 and that $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution with $\alpha \in ]1,2]$. In particular, if $(X_n)_{n\geq 1}$ denotes a sequence of i.i.d. $\xi$ distributed random variables, then there exists $(a_n)_{n\geq 1}$ such that

$$\frac{X_1+\cdots+X_n-n}{a_n} \xRightarrow[n\to\infty]{} \mu_\alpha$$

where $\mu_\alpha$ is the law of a non-degenerate $\alpha$-stable random variable, see [57] and Chapter XVII in [51]. Under this assumption, Duquesne proved that

$$\left(\frac{a_n}{n}T_n, \frac{1}{n}\mu_n\right)$$

converges in distribution to the $\alpha$-stable tree. Let $L$ denote its profile.

For all $n$, denote by $L_n$ the rescaled profile of $T_n$ defined for all non-negative $t$ by

$$L_n(t) := \frac{1}{a_n}\ell_n\Big(\lfloor nt/a_n\rfloor\Big), \tag{3.23}$$

compare with (3.5). Kersting proved the following result.

**Theorem 3.1.3 (Theorem 1 [82]).** *Under these assumptions on $\xi$, $L_n \Rightarrow L$ in $\mathcal{D}[0,\infty[$.*

Let us start by describing the aforementioned correspondence between trees and random walk excursions. The set $\mathcal{U}$ of finite words on $\mathbb{N}$ can be endowed with the two followings orders.

— The *depth first* order (or lexicographical order) $\preceq_{\mathrm{d}}$: let $u$ and $v$ be in $\mathcal{U}$, set $w := u\wedge v$, $u'$ and $v'$ such that $u = w\cdot u'$ and $v = w\cdot v'$. Then

$$u \preceq_{\mathrm{d}} v \quad\Longleftrightarrow\quad u' = \varnothing \text{ or } u'_1 < v'_1.$$

— The *breadth first* order $\preceq_{\mathrm{b}}$: for any two elements $u$ and $v$ in $\mathcal{U}$,

$$u \preceq_{\mathrm{b}} v \quad\Longleftrightarrow\quad |u| < |v| \text{ or } |u| = |v| \text{ and } u \preceq_{\mathrm{d}} v.$$

Let $\mathtt{t}$ be a plane tree with $\#\mathtt{t} = n$ and label its vertices $x_1,\dots,x_n$ in increasing breadth first order. Let the sequence $[s_{\mathtt{t}}(i); i = 0,\dots,n]$ be defined by $s_{\mathtt{t}}(0) = 1$ and $s_{\mathtt{t}}(i) = s_{\mathtt{t}}(i-1) + c_{\mathtt{t}}(x_i) - 1$ for $i = 1,\dots,n$. Observe that $s_{\mathtt{t}}(i) > 0$ if $i \neq n$ and that $s_{\mathtt{t}}(n) = 0$. Moreover, for any $k \leq |\mathtt{t}|$,

$$s_{\mathtt{t}}\big[\ell_{\mathtt{t}}(1)+\cdots+\ell_{\mathtt{t}}(k)\big] = 1 + \sum_{i=0}^{k} \underbrace{\sum_{u\in\mathtt{t}:|u|=i} \big[c_{\mathtt{t}}(u)-1\big]}_{=\ell_{\mathtt{t}}(i+1)-\ell_{\mathtt{t}}(i)} = \ell_{\mathtt{t}}(k+1). \tag{3.24}$$

Conversely, consider $s : \{0,\dots,n\} \to \mathbb{N}$ such that $s(0) = 1$, $s(n) = 0$, $s(i) > 0$ for $0 < i < n$ and $s(i) - s(i-1) \geq -1$ for all $1 \leq i \leq n$. We can build a tree $\mathtt{t}$ with $\#\mathtt{t} = n$ and such that if $x_1 \preceq_{\mathrm{b}} \cdots \preceq_{\mathrm{b}} x_n$ are its $n$ vertices, then $c_{\mathtt{t}}(x_i) = s(i) - s(i-1) + 1$, i.e. $\mathtt{t}$ satisfies $s_{\mathtt{t}} = s$. In other words, there is a one-to-one correspondence between finite trees and such finite sequences.

If $T_n$ has distribution $\mathrm{GW}^n_\xi$, i.e. is a Galton-Watson tree with offspring distribution $\xi$ conditioned to have size $n$, observe that $s_{T_n}$ is a random walk excursion conditioned to have length $n$ such that its increments are $\xi$-distributed random variables minus one. Recall that $\xi$ belongs to the domain of attraction of a stable distribution, so intuitively, under appropriate rescaling, the random walk excursion $s_{T_n}$ should converge to some Lévy excursion. This intuition driven result indeed holds and is one of the key steps in Kersting's study.

More precisely, for all $n$, set

$$S_n : [0,1] \longrightarrow \mathbb{R}_+,\ t \longmapsto \frac{1}{a_n} s_{T_n}(\lfloor nt \rfloor).$$

Following Proposition 15 in Chapter VIII of [17], we define a Lévy excursion with unit length. Let $X$ be a Lévy process such that the law of $X(1)$ is $\mu_\alpha$. Define

$$\tau^- := \sup\left\{t \le 1 : X(t) = \inf_{0\le s\le 1} X(s)\right\}$$

the last instant at which $X$ is at its minimum before time 1 as well as

$$\tau^+ := \inf\left\{t \ge 1 : X(t) = \inf_{0\le s\le 1} X(s)\right\}$$

the first time after 1 at which $X$ goes back to said minimum. Both $\tau^-$ and $\tau^+$ are a.s. well-defined because $X$ has no negative jumps. Then, define the process $Y$ by

$$Y(t) := \frac{X\left[\tau^- \cdot (1-t) + \tau^+ \cdot t\right] - X[\tau^-]}{(\tau^+ - \tau^-)^{1/\alpha}}$$

for all $t \in [0,1]$. The process $Y$ hence defined is a Lévy excursion. With these notations, Theorem 11 in [82] states that

$$S_n \underset{n\to\infty}{\overset{\mathcal{D}}{\Longrightarrow}} Y. \tag{3.25}$$

The next step in the proof is to rewrite (3.24) for $T_n$ in terms of $S_n$ and $L_n$. Define $C_n(t) := \int_0^t L_n(u)\mathrm{d}u$ for all $t \ge 0$ and observe that $h_n := \inf\{t \ge 0 : C_n(t) = 1\} = (1 + |T_n|) \cdot a_n/n$. Moreover, $\operatorname{supp} L_n = [0, h_n]$. For all $k \in \mathbb{Z}_+$,

$$C_n(k \cdot a_n/n) = \frac{1}{n} \# T_n|_{k-1}$$

so that, in light of (3.24),

$$S_n\left[C_n(k \cdot a_n/n)\right] = L_n(k \cdot a_n/n).$$

For all $t \in [0,1[$, set $Y_n(t) := S_n[C_n(k \cdot a_n/n)]$ where $k \in \mathbb{Z}_+$ is such that $C_n[a_n k/n] \le t < C_n[a_n(k+1)/n]$ and $Y_n(1) := 0$. Then, for all $t \in [0,1[$, this definition yields

$$Y_n[C_n(t)] = L_n(t) = \frac{\mathrm{d}_+}{\mathrm{d}t} C_n(t).$$

Therefore, for all non-negative $x \le h_n$,

$$C_n^{-1}(x) = \int_0^x \frac{\mathrm{d}u}{Y_n(u)} \tag{3.26}$$

where $C_n^{-1} : [0, h_n] \to [0,1]$ is the inverse of $C_n$. Let $\Psi$ and $\Phi$ be the functionals defined on the set of non-negative càdlàg functions by

$$\Psi(f)(t) := \sup\left\{u \in [0,1] : \int_0^u \frac{\mathrm{d}s}{f(s)} \le t\right\} \quad \text{and} \quad \Phi(f)(t) := \frac{\mathrm{d}_+}{\mathrm{d}t}\Psi(f)(t)$$

for all such functions $f$ and $t \ge 0$. With this notation, because $C_n$ is also the inverse of $C_n^{-1}$ and because $L_n$ is the right derivative of $C_n$, we get that $C_n = \Psi(Y_n)$ and $L_n = \Phi(Y_n)$.

The remainder of Kersting's proof is then dedicated to proving that $\Phi(Y_n) \Rightarrow \Phi(Y)$ in $\mathcal{D}[0,\infty[$. In order to do this, Theorem 4 proves that if

$(i)$ For all $\delta \in ]0, 1/2[$,

$$\inf_{\delta\le t\le 1-\delta} Y(t) > 0 \quad \text{a.s.},$$

$(ii)$ $Y_n \Rightarrow Y$ in $\mathcal{D}$,

$(iii)$ For all positive $t$,

$$\lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty} \mathbb{P}[C_n(t) \le \varepsilon]\right) = 0,$$

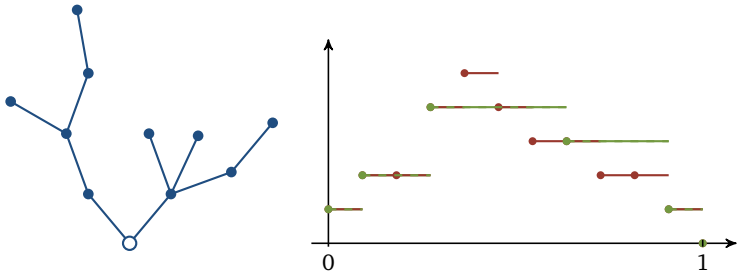



Figure 3.2 – A drawing of $T_n$, $S_n$ in red and $\Psi(S_n)$ in green with $n = 11$.

then $L_n = \Phi(Y_n)$ converges in distribution to $\Phi(Y)$ in $\mathcal{D}[0,\infty[$. Condition $(i)$ is satisfied by construction of the Lévy excursion $Y$. The convergence in law of $Y_n$ to $Y$ is a consequence of (3.24) and (3.25) and is proved in Section 5. Finally, Condition $(iii)$ is proved in Section 4. As a result, Theorem 1 in [82] states that $L_n \Rightarrow \Phi(Y)$ in $\mathcal{D}[0,\infty[$.

If $\xi$ is a distribution with finite variance $\sigma^2$, we may choose $a_n = \sigma\sqrt{n}/2$. In this case, the process $Y$ is a standard Brownian excursion and the process $\Phi(Y)$ is then its local time. Theorem 1 in [82] therefore provides a proof of Aldous' conjecture (3.8).

*Remark 3.1.4.* Let $\mathtt{t}$ be a fixed plane tree with $\#\mathtt{t} = n$ and label its vertices $y_1, \dots, y_n$ in increasing depth first order. Set $\tilde{s}_{\mathtt{t}}(0) = 1$, $\tilde{s}_{\mathtt{t}}(i) = \tilde{s}_{\mathtt{t}}(i-1) + c_{\mathtt{t}}(y_i) - 1$ for $1 \le i \le n$. The sequence $\tilde{s}_{\mathtt{t}}$ is called the *Lukasiewicz path* of $\mathtt{t}$. There is also a one-to-one relationship between finite plane trees and their Lukasiewicz paths.

Furthermore, if $T$ is a Galton-Watson tree with offspring distribution $\xi$, its Lukasiewicz path is also a Random walk excursion started from 1, with length $\#T$ and such its increments plus one have distribution $\xi$. Because of this, the study of the asymptotic behaviour of these Lukasiewicz paths has been an important tool in the study of the scaling limits of Galton-Watson trees, see [47, 49, 85, 86, 96] for instance.

**Stochastic Differential Equation.** In [105], Pitman was interested in proving that the local time of a *reflected Brownian bridge* is the unique solution to a certain Stochastic Differential Equation (SDE for short).

In order to do so, he studied the asymptotics of Galton-Watson processes with Poisson offspring distribution started with $k \ge 1$ individuals and conditioned to have total progeny $n$. These processes can be encoded as a forest, that is a family of $k$ independent Galton-Watson trees with total size $n$. Aldous and Pitman studied these random forests in [11] and proved that they are encoded by a random walk which, under appropriate rescaling, converges to a reflected Brownian bridge.

Pitman proved that, in a sense, the SDE in which he was interested asymptotically drives the behaviour of the profile of the said forest when its size goes to infinity. Results from [11] allowed Pitman to deduce the desired result. Let us first introduce the necessary notation before describing the main ideas that Pitman used in this study.

Let $[W(u); u \ge 0]$ be a standard Brownian motion. The continuous process $[W(u) - uW(1); 0 \le u \le 1]$ is called a *Brownian bridge* (with unit length) and its absolute value is called a reflected Brownian bridge. Let $B^{|\mathrm{br}|}$ be such a reflected Brownian bridge, and define its local time $L^{|\mathrm{br}|}$ as

$$L^{|\mathrm{br}|}(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathbb{1}_{[t,t+\varepsilon[}[B^{|\mathrm{br}|}(u)]\mathrm{d}u.$$

Let also $\mathbf{e}$ be a standard Brownian excursion and denote its local time by $L^{\mathrm{ex}}$.

Pitman was interested in the following SDE: if $\beta$ is a standard Brownian motion and $\ell$ is non-negative,

$$X(0) = \ell \quad \text{and} \quad \mathrm{d}X(t) = \delta_t(X)\,\mathrm{d}t + 2\sqrt{X(t)}\mathrm{d}\beta_t \tag{3.27}$$

where

$$\delta_t(X) := 4 - X(t)^2 \big/ \Big(1 - \int_0^t X(s)\mathrm{d}s\Big) \tag{3.28}$$

with the convention that (3.27) is to be solved for $t$ in $[0, V(X)[$ and $X(t) = 0$ for all $t \geq V(X)$ with $V(X) := \inf\big\{s \geq 0 : \int_0^s X(u)\mathrm{d}u = 1\big\}$. Lemma 1 in [105] establishes that (3.27) has a unique strong solution which we will denote by $\big(X_\ell(t); t \geq 0\big)$. Proposition 3 gives some basic properties of the family $(X_\ell; \ell \geq 0)$, notably that for all $\ell \geq 0$, almost surely, the process $X_\ell$ is continuous on $[0, \infty[$ and $\int_0^\infty X_\ell(t)\,\mathrm{d}t = 1$. Moreover, Pitman proved the following result:

**Theorem 3.1.4 (Theorem 4 [105]).** *For all fixed $\ell \geq 0$, $X_\ell$ is distributed like the local time of $B^{|\mathrm{br}|}$ conditioned on the event $L^{|\mathrm{br}|}(0) = \ell$, that is*

$$X_\ell \overset{(\mathrm{d})}{=} \Big(L^{|\mathrm{br}|}(t); t \geq 0 \,\Big|\, L^{|\mathrm{br}|}(0) = \ell\Big).$$

*If $\ell = 0$, $X_0$ is distributed like $L^{\mathrm{ex}}$. In particular, the profile of a Brownian tree is distributed like $X_0$.*

The second claim of this result is a consequence of the first coupled with the intuitively clear fact that conditionally on the event $L^{|\mathrm{br}|}(0) = 0$, the process $B^{|\mathrm{br}|}$ is distributed like a standard Brownian excursion; See Lemma 12 [105].

As was previously mentioned, in order to prove Theorem 4 [105], Pitman studied the asymptotics of Galton-Watson processes started with a certain number of individuals and conditioned to have large total population. Let us now outline the main ideas of Pitman's approach.

Let $k \geq 1$ be an integer and consider $[Z_k(j); j \geq 0]$ a Galton-Watson process with offspring distribution $\xi$ started from $k$. For all $n \geq k$, let $Z_{k,n}$ denote the process $Z_k$ conditioned on the event $\sum_{j\geq 0} Z_k(j) = n$. For all $n \geq k$ and $h \geq 0$, set $A_{k,n}(h) := n - \sum_{j=0}^h Z_{k,n}(j)$. Using Otter-Dwass' formula (see Chapter 6 in [106] for instance), Lemma 8 [105] states that the process $[Z_{k,n}(h), A_{k,n}(h)]_{h\geq 0}$ is a Markov chain.

When $\xi$ is the Poisson distribution with parameter 1, this last lemma allows to precisely describe the dynamics of the process $[Z_{k,n}(j); j \geq 0]$. As was stated in Lemma 9 [105], for all $n \geq k \geq 1$ and $h \geq 0$, we get that conditionally on $Z_{k,n}(0), \ldots, Z_{k,n}(h)$, $Z_{k,n}(h+1) - 1$ is distributed according to a binomial distribution with parameters

$$A_{k,n}(h) - 1 \quad \text{and} \quad \frac{Z_{k,n}(h)}{A_{k,n}(h) + Z_{k,n}(h)}$$

with the convention that almost surely, a Binomial$(0, p)$ variable equals 0 and a Binomial$(-1, p)$ variable equals $-1$. As a result, for all $h \geq 0$, we get that

$$\mathbb{E}\Big[Z_{k,n}(h+1) - Z_{k,n}(h) \,\Big|\, Z_{k,n}(0), \ldots, Z_{k,n}(h)\Big] = 1 - \frac{Z_{k,n}(h)^2 + Z_{k,n}(h)}{A_{k,n}(h) + Z_{k,n}(h)} \tag{3.29}$$

and

$$\mathrm{Var}\Big(Z_{k,n}(h+1) - Z_{k,n}(h) \,\Big|\, Z_{k,n}(0), \ldots, Z_{k,n}(h)\Big) = \frac{[A_{k,n}(h) - 1]A_{k,n}(h)\, Z_{k,n}(h)}{[A_{k,n}(h) + Z_{k,n}(h)]^2}. \tag{3.30}$$

Now, let $\ell \geq 0$ and $(k_n)_n$ be such that $2k_n/\sqrt{n} \to \ell$ as $n \to \infty$. For all $n \geq 1$ and $t \geq 0$, set

$$Y_n(t) := \frac{2}{\sqrt{n}} Z_{k_n,n}\big(\lfloor 2\sqrt{n}\, t \rfloor\big),$$

compare with (3.9) for the choice of scaling. Further set $[\mathcal{F}^{Y_n}(t); t \geq 0]$ as the natural filtration of $Y_n$. Observe that

$$\frac{1}{n} A_{k_n,n}\big(\lfloor 2\sqrt{n}\, t \rfloor\big) \underset{n\to\infty}{\sim} 1 - \int_0^t Y_n(s)\,\mathrm{d}s.$$

Rewriting (3.29) and (3.30) in terms of $Y_n$, we get that for all $t \geq 0$, as $n \to \infty$,

$$\mathbb{E}\Big[Y_n(t + 1/2\sqrt{n}) - Y_n(t) \,\Big|\, \mathcal{F}^{Y_n}(t)\Big] \sim \frac{1}{2\sqrt{n}}\left(4 - \frac{Y_n(t)^2}{1 - \int_0^t Y_n(s)\,\mathrm{d}s}\right) = \frac{1}{2\sqrt{n}}\,\delta_t(Y_n), \tag{3.31}$$

where $\delta_t$ is the function defined in (3.28), and

$$\mathrm{Var}\Big(Y_n(t+1/2\sqrt{n}) - Y_n(t) \,\Big|\, \mathcal{F}^{Y_n}(t)\Big) \sim \frac{1}{2\sqrt{n}}\, 4\, Y_n(t). \tag{3.32}$$

Furthermore, by assumption, $Y_n(0) \to \ell$. Equations (3.31) and (3.32) indicate that asymptotically, $Y_n$ is driven by the SDE (3.27). Using some results from [88] on the functional convergence of Markov chains to the solution of an SDE and the uniqueness of strong solutions of (3.27), Pitman then deduces that $Y_n \Rightarrow X_\ell$ in the space $\mathcal{D}[0,\infty[$, see Theorem 7 in [105].

Finally, results from [11] imply in particular that

$$\left(\int_0^t Y_n(s)\,\mathrm{d}s\,;\, t\geq 0\right) \underset{n\to\infty}{\Longrightarrow} \left(\int_0^1 \mathbb{1}_{[0,t]}\big(B^{|\mathrm{br}|}(u)\big)\,\mathrm{d}u\,;\, t\geq 0 \,\middle|\, L^{|\mathrm{br}|}(0)=\ell\right)$$

in $\mathcal{C}[0,\infty[$. Moreover, for all $t \geq 0$, we have

$$\int_0^1 \mathbb{1}_{[0,t]}\big(B^{|\mathrm{br}|}(u)\big)\,\mathrm{d}u = \int_0^t L^{|\mathrm{br}|}(s)\,\mathrm{d}s.$$

As a result, we get that conditionally on $L^{|\mathrm{br}|}(0)=\ell$, $L^{|\mathrm{br}|}$ is distributed like $X_\ell$, which concludes the proof of Theorem 4 [105].

*Remark 3.1.5.* Theorems 4 and 7 from [105] combine to prove that (3.8) holds when $\xi$ is the Poisson distribution.

### 3.1.2 Pólya trees

For all $n \geq 1$, denote by $\mathrm{T}_n$ the set of unordered rooted trees with $n$ vertices and let $T_n$ be drawn uniformly in $\mathrm{T}_n$. The random variable $T_n$ is then called a Pólya tree with size $n$. The first investigation into these trees was done by Pólya, hence their name, in [108]. He studied the generating function of these trees, that is the power series $a$ given by

$$a(z) := \textstyle\sum_{n\geq 1} a_n z^n$$

with $a_n := \#\mathrm{T}_n$. The radius of convergence $\rho$ of $a$ was proved to belong to $]0,1[$. Furthermore, $\rho$ is the only singularity of $a$ on the circle $\{|z|=\rho\}$. Pólya also proved that $a$ satisfies

$$a(z) = z\,\exp\Big(\textstyle\sum_{n\geq 1} a(z^n)\Big). \tag{3.33}$$

Otter [100] then proved that $a(\rho)=1$ and that $a(\rho-x) = 1-\kappa\sqrt{x}+\mathcal{O}(x)$ for some positive constant $\kappa$; he also gave numerical approximations of $\rho$ and $\kappa$: $\rho \approx 0.3383219$ and $\kappa \approx 2.6811266$. Finally, he proved that as $n\to\infty$,

$$a_n \sim \frac{\kappa \rho^{1/2-n}}{2\sqrt{\pi n^3}}.$$

Haas and Miermont [66] studied the scaling limits of $(T_n)_n$. Theorem 9 [66] states that

$$\left(\frac{\kappa\sqrt{\rho}}{2\sqrt{2n}} T_n, \frac{1}{n}\mu_n\right) \underset{n\to\infty}{\overset{\mathrm{d_{GHP}}}{\Longrightarrow}} \mathcal{T}_B$$

where $\mathcal{T}_B$ denotes the Brownian tree and $\mu_n$ is the counting measure on $T_n$. See also Panagiotou and Stufler [102].

For all $n$, let $\ell_n$ be the profile of $T_n$ and define $L_n$ as in (3.10). Drmota and Gittenberger's main result in [44] is about the asymptotic behaviour of $L_n$.

**Theorem 3.1.5 (Theorem 3 [44]).** *With these notations,*

$$L_n \underset{n\to\infty}{\Longrightarrow} \left(\frac{\kappa\sqrt{\rho}}{2\sqrt{2}} \cdot L\left(\frac{\kappa\sqrt{\rho}\,t}{2\sqrt{2}}\right);\, t\geq 0\right) \tag{3.34}$$

*in* $\mathcal{C}[0,\infty[$ *where* $L$ *denotes the local time of a Brownian excursion.*

They used an approach similar to the one they used in [43]: they proved convergence of the finite-dimensional distributions using similar methods as those used for (3.12) and then proved tightness in a way similar to their proof of (3.13). Notably, this result was proved before the convergence of the corresponding trees in [66].

## 3.2 Recursive trees and Binary search trees

In this section, we will present some up-to-date results found in the literature on the profile of recursive trees with $n$ vertices and binary search trees with $n$ leaves. These two important families of random trees do not fit the framework of Section 3.1 and arise naturally as combinatorial structures. We will see that functional convergence of the rescaled profiles cannot happen as in Section 3.1. The object of interest will then be the *normalised profile* of the trees, that is the profile divided by its expectation.

The very nature of the results, and thus the methods used to establish them, are widely different from those presented in the previous section. The methods rely heavily on the recursive structure of both recursive and binary search trees.

Drmota [42] presented an overview of many results on the profiles of these two models and other related models.

### 3.2.1 Recursive trees

A *recursive tree* with size $n$ is a tree with $n$ vertices labelled 1 through $n$ such that for any $k = 2, \dots, n$, the labels on the shortest path from vertex 1 to vertex $k$ are increasing. For such trees, we will consider them to be rooted at vertex 1. Let $\mathrm{T}_n^{\mathrm{rec}}$ be the set of recursive trees with size $n$.

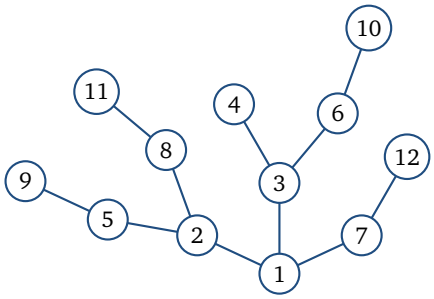



Figure 3.3 – A recursive tree with 12 vertices.

It is easy to compute $\#\mathrm{T}_n^{\mathrm{rec}}$. Indeed, an element of $\mathrm{T}_{n+1}^{\mathrm{rec}}$ can be seen as a recursive tree with $n$ vertices to which a vertex with label $n+1$ was added. There are $n$ locations at which to attach the said vertex so $\#\mathrm{T}_{n+1}^{\mathrm{rec}} = n\#\mathrm{T}_n^{\mathrm{rec}}$. Moreover, as there is only one recursive tree with with a single vertex, clearly $\#\mathrm{T}_1^{\mathrm{rec}} = 1$ so that $\#\mathrm{T}_n^{\mathrm{rec}} = (n-1)!$ for all $n \geq 1$.

The same line of reasoning ensures that the following algorithm generates a sequence $(T_n)_n$ such that for all $n \geq 1$, $T_n$ is uniformly drawn in $\mathrm{T}_n^{\mathrm{rec}}$:

— Let $(U_n)_{n\geq 2}$ be independent and such that for all $n \geq 2$, $U_n$ has the uniform distribution on the set $\{1, \dots, n-1\}$,
— Set $T_1$ as the tree with a single vertex, which is labelled 1, i.e. the only element of $\mathrm{T}_1^{\mathrm{rec}}$,
— For $n \geq 1$, conditionally on $T_n$, attach a new vertex with label $n+1$ to the vertex $U_{n+1}$ in $T_n$ and call $T_{n+1}$ the tree thus obtained.

Pittel [107] proved that with this algorithm,

$$\frac{|T_n|}{\log n} \xrightarrow[n\to\infty]{\text{a.s.}} \exp(1), \tag{3.35}$$

a fact alluded to in the introduction of this chapter.

Let $\hat{T}_n$ be the sub-tree of $T_n$ made out of 2 and its descendants and let $\check{T}_n$ be the complement sub-tree of $T_n$. Elementary computations involving conditional probabilities ensure that $\#\hat{T}_n$ is uniformly distributed on the set $\{1,\dots,n-1\}$ and that conditionally on $\#\hat{T}_n = k$, $\hat{T}_n$ and $\check{T}_n$ are independent and respectively distributed like $T_k$ and $T_{n-k}$. This fact was observed by van der Hofstad *et al.* [120].

*The expected profile.* For all $n \geq 1$, denote by $\ell_n$ the profile of $T_n$, that is

$$\ell_n(k) := \#\big\{j \leq n : \mathrm{d}_{\mathrm{gr}}(1,j) = k\big\}$$

for every non-negative integer $k$. Using the recursive partitioning of the sequence $(T_n)_n$ mentioned before, we get that if $n \geq 2$ and $k \geq 1$,

$$\ell_n(k) \overset{(\mathrm{d})}{=} \ell_{X_n}(k-1) + \tilde{\ell}_{n-X_n}(k) \tag{3.36}$$

where $(\tilde{T}_k, \tilde{\ell}_k)_{k\geq 1}$ is an independent copy of $(T_k, \ell_k)_{k\geq 1}$ and both families are independent of the random variable $X_n$, which is uniform on $\{1,\dots,n-1\}$. Taking the expected values of both sides of (3.36), we get for all integers $n \geq 2$ and $k \geq 1$

$$\mathbb{E}\big[\ell_n(k)\big] = \frac{1}{n-1}\sum_{j=1}^{n-1} \mathbb{E}\big[\ell_j(k)\big] + \mathbb{E}\big[\ell_j(k-1)\big]$$

and thus $n\,\mathbb{E}[\ell_{n+1}(k)] - (n-1)\,\mathbb{E}[\ell_n(k)] = \mathbb{E}[\ell_n(k)] + \mathbb{E}[\ell_n(k-1)]$. As a result, for all $n \geq k \geq 0$,

$$n!\,\mathbb{E}\big[\ell_{n+1}(k)\big] = n(n-1)!\,\mathbb{E}\big[\ell_n(k)\big] + (n-1)!\,\mathbb{E}\big[\ell_n(k-1)\big]. \tag{3.37}$$

Moreover, $\mathbb{E}[\ell_n(0)] = 0$ and $\mathbb{E}\big[\ell_1(k)\big] = \mathbb{1}_{k=0}$. The family $\big(n!\,\mathbb{E}[\ell_{n+1}(k)]; n \geq k \geq 0\big)$ therefore satisfies the same recurrence relation with the same initial conditions as the family $\big(\left[\begin{smallmatrix} n+1 \\ k+1 \end{smallmatrix}\right]; n \geq k \geq 0\big)$ of *unsigned Stirling numbers of the first kind*: for all $n \geq 0$, $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$ are defined such that for all $x$,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{j=0}^{n-1} (x+j). \tag{3.38}$$

As a result, we get the following formula, first established by Meir and Moon in [97], for the expected profile of $T_n$:

**Theorem 3.2.1 (Corollary 7.1 [97]).** *For all $n \geq k \geq 0$, we get that*

$$\mathbb{E}[\ell_n(k)] = \frac{1}{(n-1)!} \begin{bmatrix} n \\ k+1 \end{bmatrix}. \tag{3.39}$$

To get an asymptotic equivalent of this expected profile, some results on Stirling numbers are needed. In particular, Theorem 2 in [71] or Lemma 6.1 in [42] ensure that if $k \sim \alpha \log n$ for some positive $\alpha$, then as $n \to \infty$,

$$\begin{bmatrix} n \\ k \end{bmatrix} \sim \frac{(n-1)!\,(\log n)^k}{k!\,\Gamma(\alpha)}. \tag{3.40}$$

As a result, if $\alpha > 0$, when $n \to \infty$, we can use (3.40) and Stirling's approximation in (3.39) to get the following asymptotic expression, see [53],

$$\mathbb{E}\Big[\ell_n\big(\lfloor \alpha \log n \rfloor\big)\Big] \sim \frac{n^{\alpha(1-\log\alpha)}}{\sqrt{2\pi\alpha\log n}\,\Gamma(1+\alpha)}. \tag{3.41}$$

Now observe that $\alpha(1-\log\alpha) < 1$ for all $\alpha \neq 1$. Therefore, for all $\alpha \neq 1$,

$$\frac{\log n}{n}\,\mathbb{E}\Big[\ell_n\big(\lfloor \alpha \log n \rfloor\big)\Big] \xrightarrow[n\to\infty]{} 0, \tag{3.42}$$

yet (3.35) would seem to suggest that this is the suitable rescaling in order to obtain results on the functional convergence of the profile similar to those from Section 3.1. For this reason, (3.41) and (3.42) indicate that such functional convergence results cannot happen for the profile of recursive trees.

Nevertheless, the asymptotic behaviour of $\ell_n(k)$ has been studied. Results essentially fall into one of two categories: when $k \sim \alpha \log n$ for some positive $\alpha$ and when $k$ is fixed.

**Profile asymptotics at logarithmic height.** In this paragraph, the object of interest will be the so-called normalised profile of $T_n$, that is the random variables

$$\Lambda_n(k) := \frac{\ell_n(k)}{\mathbb{E}[\ell_n(k)]}$$

for all integers $n \geq 1$ and $0 \leq k \leq n$. We will be interested in the asymptotic behaviour of $\Lambda_n(k)$ when $k \sim \alpha \log n$ with $\alpha \in ]0, \mathrm{e}[$, in light of (3.35).

There are mainly two methods used in the literature to study this problem: a "contraction method" used on a fixed-point equation in distribution and a study of the sequence of the normalised profile polynomials of $T_n$, which is a martingale. Let us outline both methods.

*The fixed-point equation.* Rewriting (3.36) in terms of $(\Lambda_n)_{n\geq 1}$, we get that if $(\tilde{\Lambda}_n)_n$ is an i.i.d. copy of $(\Lambda_n)_{n\geq 1}$ and if both families are independent of $X_n$, a uniformly distributed random variable on $\{1, \dots, n-1\}$, then, for all $k \geq 1$

$$\Lambda_n(k) \overset{(\mathrm{d})}{=} \frac{\mathbb{E}[\ell_{X_n}(k-1) \,|\, X_n]}{\mathbb{E}[\ell_n(k)]} \Lambda_{X_n}(k-1) + \frac{\mathbb{E}[\ell_{n-X_n}(k) \,|\, X_n]}{\mathbb{E}[\ell_n(k)]} \tilde{\Lambda}_{n-X_n}(k).$$

Then, using (3.39) and (3.40), if $k \sim \alpha \log n$ for some positive $\alpha$, as $n \to \infty$ we get

$$\frac{\mathbb{E}[\ell_{X_n}(k-1) \,|\, X_n]}{\mathbb{E}[\ell_n(k)]} \sim \frac{(\log X_n)^{k-1}}{(k-1)!} \frac{k!}{(\log n)^k} \sim \alpha \left( \frac{\log X_n}{\log n} \right)^{\alpha \log n} \sim \alpha \left( \frac{X_n}{n} \right)^{\alpha}$$

and similarly,

$$\frac{\mathbb{E}[\ell_{n-X_n}(k) \,|\, X_n]}{\mathbb{E}[\ell_n(k)]} \sim \left( 1 - \frac{X_n}{n} \right)^{\alpha}.$$

As a result, if $\Lambda_n(\lfloor \alpha \log n \rfloor)$ were to converge weakly to some random variable $X(\alpha)$, then intuition would seem to imply that

$$X(\alpha) \overset{(\mathrm{d})}{=} \alpha\, U^{\alpha} X(\alpha) + (1-U)^{\alpha} \tilde{X}(\alpha) \tag{3.43}$$

where $\tilde{X}(\alpha)$ denotes an independent copy of $X(\alpha)$ and $U$ is a uniform random variable on $]0, 1[$ independent of both $X(\alpha)$ and $\tilde{X}(\alpha)$.

This heuristically derived identity was one of the main ideas used by Fuchs *et al.* [53] along with the contraction method, see [99], in the study of the asymptotic behaviour of $\Lambda_n(k)$ when $k \sim \alpha \log n$. In Proposition 1 [53], they proved that for any $\alpha \in ]0, \mathrm{e}[$, there is a unique probability measure $\pi_\alpha$ on $[0, \infty[$ such that if $X(\alpha)$ is $\pi_\alpha$-distributed, then $X(\alpha)$ satisfies (3.43) and

$$\mathbb{E}[X(\alpha)] = 1, \qquad \mathbb{E}[X(\alpha)^s] < \infty,$$

where, if $\alpha < 2$, $s = 2$ and if $\alpha \in [2, \mathrm{e}[$, $s \in ]1, 2[$ is such that $s > \alpha^{s-1}$. This allowed them to prove the next result:

**Theorem 3.2.2 (Theorem 1 [53]).** *For all $\alpha \in [0, \mathrm{e}[$, there exists a positive random variable $X(\alpha)$ with $\mathbb{E}[X(\alpha)] = 1$ which satisfies the identity (3.43) and such that*

$$\Lambda_n\big(\lfloor \alpha \log n \rfloor\big) \underset{n\to\infty}{\Longrightarrow} X(\alpha). \tag{3.44}$$

*Furthermore, if $p \geq 2$ is an integer and $\alpha < p^{1/(p-1)}$, then $X(\alpha) \in L^p$, the $p$ first moments of $\Lambda_n(\lfloor \alpha \log n \rfloor)$ converge to those of $X(\alpha)$ and convergence fails for higher moments.*

*With martingales.* Another method to prove (3.44) on a sub-interval of $[0, \mathrm{e}[$ was presented in [42] and was used in [45] for search trees. This method is based on the study of the so-called *profile polynomials*, that is, for all $n$, the random polynomial $W_n$ (whose degree is at most $n-1$) defined for all complex $z$ by

$$W_n(z) := \sum_{k\geq 0} \ell_n(k) z^k.$$

Its expectation is given by

$$\mathbb{E}[W_n(z)] = \frac{1}{(n-1)!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k+1 \end{bmatrix} z^k = \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (z+j) \tag{3.45}$$

where we have used (3.39), (3.38) and the fact that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ if $n > 0$. Now observe that

$$\mathbb{E}\big[\ell_{n+1}(k) \,|\, T_n\big] = \ell_n(k) + \frac{1}{n}\ell_n(k-1)$$

so that

$$\mathbb{E}\big[W_{n+1}(z) \,|\, T_n\big] = \frac{z+n}{n} \cdot W_n(z).$$

As a result, for all complex $z$ with $-z \notin \mathbb{N}$, the sequence $[M_n(z)]_{n\geq 1}$ defined by

$$M_n(z) := \frac{W_n(z)}{\mathbb{E}[W_n(z)]}$$

is a martingale. In particular, for any $x \geq 0$, $M_n(x)$ is a non-negative martingale and thus converges a.s. to a non-negative random variable $M(x)$. The next step is then to prove that this convergence happens uniformly on an open complex domain, which ensures that $M$ is a random analytic function, and to observe that for any positive $\alpha$, the Cauchy integral formula gives

$$\ell_n(k) = \frac{1}{2\pi\, \alpha^k} \int_{-\pi}^{\pi} \mathbb{E}[W_n(\alpha \mathrm{e}^{i\theta})] M_n(\alpha \mathrm{e}^{i\theta}) \mathrm{e}^{-ik\theta}\, \mathrm{d}\theta.$$

Precise asymptotic expansions of this relation give the expected result for $0 \leq \alpha < 2$. Moreover, $M(\alpha)$ is distributed like $X(\alpha)$.

*When $\alpha = 1$.* For $\alpha = 1$, with the help of (3.43) we get

$$\mathbb{E}\big[X(1)^2\big] = \frac{2}{3}\mathbb{E}\big[X(1)^2\big] + \frac{1}{3}\mathbb{E}\big[X(1)\big]^2$$

so that $\mathbb{E}[X(1)^2] = 1$ and thus $\mathrm{Var}(X(1)) = 0$. As a result, $X(1) = 1$ almost surely. Fuchs *et al.* [53] refined the convergence (3.44) for $\alpha = 1$:

**Theorem 3.2.3 (Theorem 2 and Corollary 3 in [53]).** *If $k = \log n + a_n$ with $|a_n| \to \infty$ and $|a_n| = o(\log n)$, then*

$$k! \, \frac{\ell_n(k) - \mathbb{E}[\ell_n(k)]}{a_n (\log n)^{k-1}} \Longrightarrow_{n\to\infty} X'(1)$$

*where the random variable $X'(1)$ satisfies*

$$X'(1) \overset{(\mathrm{d})}{=} U X'(1) + (1-U)\tilde{X}(1) + U + U \log U + (1-U)\log(1-U)$$

*with $\tilde{X}'(1)$ an independent copy of $X'(1)$ and where $U$ is uniformly distributed on $]0,1[$ and independent of both $X'(1)$ and $\tilde{X}'(1)$.*

**Profile asymptotics at fixed height.** Here, we will focus on the limit distribution of $\ell_n(k)$ as $n \to \infty$ and $k$ is a fixed positive integer.

Let us first observe the following. Let $(\varepsilon_i)_{i\geq 1}$ be independent Bernoulli random variables with respective parameters $1/i$. For all $n \geq 2$, we get that

$$\ell_n(1) \overset{(\mathrm{d})}{=} \sum_{i=1}^{n-1} \varepsilon_i.$$

As a result, when $n \to \infty$, clearly $\mathbb{E}[\ell_n(1)] \sim \mathrm{Var}(\ell_n(1)) \sim \log n$. It is then easy to deduce a central limit theorem:

$$\frac{\ell_n(1) - \log n}{\sqrt{\log n}} \underset{n\to\infty}{\Longrightarrow} \mathcal{N}(0,1) \tag{3.46}$$

where $\mathcal{N}(0,1)$ denotes the standard Gaussian distribution.

The recursive structure of $(T_n)_n$ and (3.46) suggest that a similar CLT should hold at each fixed height $k > 0$ for well chosen parameters. Indeed, Fuchs *et al.* [53] proved in particular the following result:

**Theorem 3.2.4 (Theorem 3 [53]).** *If $k > 0$ is fixed then*

$$\frac{\ell_n(k) - (\log n)^k/k!}{(k-1)!\,(\log n)^{k-1/2}\sqrt{2k-1}} \underset{n\to\infty}{\Longrightarrow} \mathcal{N}(0,1).$$

Fuchs *et al.*'s result was recently extended by Iksanov and Kabluchko [74]. Their result treats all levels jointly and also reflects the dynamic construction of the sequence $(T_n)_n$:

**Theorem 3.2.5 (Theorem 1.1 [74]).** *Let $(B_s; s \geq 0)$ denote a standard Brownian motion, then*

$$\left( \frac{\ell_{\lfloor n^t \rfloor}(k) - (t\,\log n)^k/k!}{(k-1)!\,(\log n)^{k-1/2}\sqrt{2k-1}}\,;\, t \geq 0 \right)_{k\geq 1} \underset{n\to\infty}{\Longrightarrow} \left( \int_{[0,t]} (t-s)^{k-1}\,\mathrm{d}B_s\,;\, t \geq 0 \right)_{k\geq 1}$$

*for the product topology on $\mathcal{D}[0,\infty[^{\mathbb{N}}$.*

## 3.2.2 Search trees

Another important model of random trees whose profiles have been studied is that of *search trees*. In this section, we will focus on binary search trees.

Let $n$ be a non-negative integer and $x_1, \ldots, x_n$ be distinct real numbers. A binary search tree with $n$ internal vertices (and thus $n+1$ leaves) with *input key* $(x_1, \ldots, x_n)$ is a binary tree $\mathtt{t}_{(x_1,\ldots,x_n)}$ in which internal nodes have labels $x_1$ through $x_n$ and leaves are unlabelled. Search trees are constructed as follows:

— For $n = 0$, $\mathtt{t}_\varnothing$ is the tree with a single unlabelled vertex.

— For $n > 0$, let $I := \{i : 1 \leq i \leq n, x_i < x_1\}$ and $J := \{j : 1 \leq j \leq n, x_j < x_1\}$. Now $\mathtt{t}_{(x_1,\ldots,x_n)}$ is the tree with a root labelled $x_1$ and such that its two sub-trees attached at the said root are $\mathtt{t}_{(x_i; i\in I)}$ on the left and $\mathtt{t}_{(x_j; j\in J)}$ on the right.

Input keys for a binary search tree with $n$ internal vertices are usually simply permutations of $\{1, \ldots, n\}$, see Figure 3.4. A natural distribution on the set of binary search trees with $n$ internal vertices is then to pick such a permutation $\Sigma_n$ uniformly at random and to consider the associated binary search tree $\mathtt{t}_{\Sigma_n}$. These trees are sometimes referred to as the *Random Permutation Model*.

There is a recursive algorithm to build a sequence $(T_n)_{n\geq 0}$ of random binary trees such that for all $n \geq 0$, $T_n$ is distributed like $\mathtt{t}_{\Sigma_n}$:

— Let $T_0$ be the tree with a single vertex.

— For all $n \geq 0$, conditionally on $T_n$, chose a leaf uniformly in $T_n$, attach two new leaves to it (making it an internal vertex) and call $T_{n+1}$ the resulting tree.

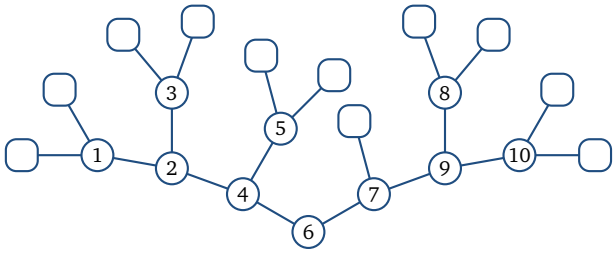



Figure 3.4 – The binary search tree associated to $(6,4,2,5,7,3,9,1,8,10)$.

It is easy to compute the distributions of the sub-trees of $T_n$ originating at its root. Indeed, if for $n \geq 1$, $\hat{T}_n$ and $\check{T}_n$ denote the sub-trees on the left and right of the root in $T_n$ respectively, then the sequence $[\#\mathcal{L}(\hat{T}_n)]_{n\geq 1}$ is a Pólya urn and therefore, for each $n \geq 1$, $\#\mathcal{L}(\hat{T}_n)$ is uniformly distributed on $\{1,\dots,n\}$. Moreover, conditionally on $\#\mathcal{L}(\hat{T}_n) = k+1$ for $0 \leq k < n$, $\hat{T}_n$ and $\check{T}_n$ are independent and distributed like $T_k$ and $T_{n-1-k}$ respectively.

*Remark 3.2.1.* This algorithm is the same as Ford's [52] when its parameter $\alpha$ equals 0. As a result, the sequence $(T_n)_{n\geq 0}$ indeed has the Markov branching property, see Lemma 27 in [52].

There are two types of vertices in binary search trees, namely internal vertices and leaves (sometimes called *external* vertices). Thus for all $n$, we can either study the *internal* or the *external* profile of $T_n$, that is the sequences $[\ell_n^\bullet(k)]_{k\geq 0}$ and $[\ell_n^\circ(k)]_{k\geq 0}$ given by

$$\ell_n^\bullet(k) := \#\big\{u \in T_n : u \text{ is an internal vertex}, |u| = k\big\}$$

and

$$\ell_n^\circ(k) := \#\big\{u \in T_n : u \text{ is a leaf}, |u| = k\big\}.$$

Recall that in a binary tree, internal vertices have 2 children. Consequently, for all $n, k \geq 0$, we have

$$2\ell_n^\bullet(k) = \ell_n^\bullet(k+1) + \ell_n^\circ(k+1).$$

Then, by induction,

$$\ell_n^\bullet(k) = \textstyle\sum_{j\geq 1} 2^{-j}\, \ell_n^\circ(k+j).$$

As a result, we can recover $\ell_n^\bullet$ from $\ell_n^\circ$ and the other way around. We will now focus on the study of the asymptotic behaviour of the external profile $\ell_n^\circ$ of $T_n$

Observe that the recursive structure of $T_n$ allows us to get the following identity in law for $\ell_n^\circ$: for all non-negative integers $k$,

$$\ell_n^\circ(k+1) \overset{(\mathrm{d})}{=} \ell_{U_n}^\circ(k) + \tilde{\ell}_{n-1-U_n}^\circ(k) \tag{3.47}$$

where $(\tilde{\ell}_m^\circ)_{m\geq 0}$ is an independent copy of $(\ell_m^\circ)_{m\geq 0}$ and $U_n$ is uniformly distributed on $\{0,\dots,n-1\}$ and independent of $\ell_m$, $\tilde{\ell}_m$, $m \geq 0$.

*Expected external profile.* For all non-negative $n$ and $k$, because of the recursive construction of $(T_n)_n$,

$$\mathbb{E}\big[\ell_{n+1}^\circ(k) - \ell_n^\circ(k) \,|\, T_n\big] = 2\,\frac{\ell_n^\circ(k-1)}{n+1} - \frac{\ell_n^\circ(k)}{n+1}.$$

Now let $W_n$ be the (external) profile polynomial of $T_n$ by

$$W_n(z) = \textstyle\sum_{k\geq 0} \ell_n^\circ(k) z^k$$

for all complex $z$ and observe that consequently,

$$\mathbb{E}\big[W_{n+1}(z) \,|\, T_n\big] = W_n(z) + \frac{2}{n+1}\, z\, W_n(z) - \frac{1}{n+1}\, W_n(z) = \frac{n+2z}{n+1}\, W_n(z).$$

Because $W_0(z) = 1$ a.s., this last identity entails that

$$\mathbb{E}\big[W_n(z)\big] = \prod_{j=0}^{n-1} \frac{j+2z}{j+1} = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (2z)^k$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the unsigned Stirling numbers of the first kind defined at (3.38). As a result, we get:

**Theorem 3.2.6 ([93]).** *For all non-negative $k$,*

$$\mathbb{E}\big[\ell_n^\circ(k)\big] = \frac{2^k}{n!} \begin{bmatrix} n \\ k \end{bmatrix} \tag{3.48}$$

When $n \to \infty$, if $\alpha > 0$, (3.40) and Stirling's approximation give the following asymptotic expansion:

$$\mathbb{E}\big[\ell_n^\circ(\lfloor \alpha \log n \rfloor)\big] \sim \frac{n^{\alpha[1-\log(\alpha/2)]-1}}{\sqrt{2\pi\,\alpha \log n}}. \tag{3.49}$$

Observe that $\alpha[1-\log(\alpha/2)]-1 < 1$ iff $\alpha \neq 2$ so that the leaves of $T_n$ seem to be concentrated around height $2\log n$. Moreover, let $\alpha_- < \alpha_+$ be such that

$$\alpha[1-\log(\alpha/2)]-1 \geq 0 \quad \Longleftrightarrow \quad \alpha \in [\alpha_-, \alpha_+].$$

Then, if $\alpha \notin [\alpha_-, \alpha_+]$, $\mathbb{E}\big[\ell_n^\circ(\lfloor \alpha \log n \rfloor)\big] \to 0$ as $n \to \infty$. Compare with Devroye [40] who studied the asymptotic height of $T_n$ and proved that

$$\frac{|T_n|}{\log n} \xrightarrow[n\to\infty]{\mathbb{P}} \alpha_+.$$

*Asymptotic behaviour of the normalised external profile.* Just like it was the case for recursive trees, we will study the normalised (external) profile of $T_n$: for all $n \geq 0$, $0 \leq k \leq n$, let

$$\Lambda_n(k) := \frac{\ell_n^\circ(k)}{\mathbb{E}[\ell_n^\circ(k)]}.$$

We will study the asymptotic behaviour of $\Lambda_n(k)$ when $k \sim \alpha \log n$ and $\alpha \in [\alpha_-, \alpha_+]$.

Let $U_n$ be uniformly distributed on $\{0, \dots, n-1\}$. Equations (3.40) and (3.48) ensure that if $k \sim \alpha \log n$, then

$$\frac{\mathbb{E}[\ell_{U_n}^\circ(k-1)|U_n]}{\mathbb{E}[\ell_n^\circ(k)]} \sim \frac{k}{2}\frac{n}{U_n}\left(\frac{\log U_n}{\log n}\right)^{k-1} \frac{\Gamma(k/\log n)}{\Gamma((k-1)/\log U_n} \sim \frac{\alpha}{2}\left(\frac{U_n}{n}\right)^{\alpha-1}.$$

As a result, if $\Lambda_n(\lfloor \alpha \log n \rfloor)$ converges in law to some random variable $X(\alpha)$, then heuristically, in light of (3.47),

$$X(\alpha) \overset{(\mathrm{d})}{=} \frac{\alpha}{2} U^{\alpha-1} X(\alpha) + \frac{\alpha}{2}(1-U)^{1-\alpha} \tilde{X}(\alpha) \tag{3.50}$$

where $\tilde{X}(\alpha)$ is an independent copy of $X(\alpha)$ and $U$ is uniformly distributed on $]0,1[$ and independent of both $X(\alpha)$ and $\tilde{X}(\alpha)$.

This fixed-point identity was used by Fuchs *et al.* [53] with the same techniques as in their study of recursive trees to obtain the next theorem.

**Theorem 3.2.7 (Theorem 5 [53]).** *If $\alpha \in ]\alpha_-, \alpha_+[$, then*

$$\Lambda_n\big(\lfloor \alpha \log n \rfloor\big) \Longrightarrow_{n\to\infty} X(\alpha) \tag{3.51}$$

*where $X(\alpha)$ satisfies (3.50).*

In [35], Chauvin *et al* proved that the convergence (3.51) happens a.s. by a careful study of the normalised external profile polynomial $M_n$ given for all complex $z$ with $-2z \notin \mathbb{Z}_+$ by

$$M_n(z) := \frac{W_n(z)}{\mathbb{E}[W_n(z)]}.$$

Just like for recursive trees, the sequence $[M_n(z)]_{n\geq 0}$ is a martingale [75]. Drmota *et al* [45] proved a functional version of (3.51).

The profiles of some models related to recursive trees and binary search trees were also studied. See for instance Drmota *et al* [45] for $m$-ary search trees, and Hwang [72] as well as Sulzbach [119] for plane-oriented recursive trees (PORTs). Kabluchko *et al* [78] presented a unified approach to give Edgeworth expansions for the profiles of recursive trees, search trees and PORTs. See also Hwang *et al* [73], Drmota and Szpankowski [46], Magner and Szpankowski [94] as well as references therein for similar studies on digital search trees and *tries*.

# Chapter 4

# Asymptotic behaviour of the profile of Markov branching trees

This chapter is comprised of results obtained in collaboration with Bénédicte Haas which are intended to be eventually submitted for publication. Before that, we want to improve upon some of these results. We mainly want to broaden the scope of Proposition 4.4.2 and to make clear whether the assumptions of Theorems 4.3.6 and 4.4.5 could be lightened significantly.

## 4.1 Introduction

The focus of this work will be models of trees which satisfy the *Markov branching* property. Such trees were introduced by Aldous [10] and later studied by quite a few authors, see for instance Haas *et al.* [63, 66, 67], Broutin *et al.* [30], Rizzolo [111] and references therein.

A sequence $(T_n)_n$ of random trees is said to be Markov branching if for all $n$, $T_n$ has $n$ leaves and conditionally on the event "$T_n$ gives birth to $p$ sub trees with $n_1 \geq \cdots \geq n_p$ leaves", the said sub-trees are independent and the $i^{\text{th}}$ largest is distributed like $T_{n_i}$. The distributions of the elements of the sequence $(T_n)_n$ are then characterised by the sequence $(q_n)_n$ of probability measures such that $q_n$ is supported by the partitions of the integer $n$ and is given by $q_n(n_1, \ldots, n_p) := \mathbb{P}\big[\text{"}T_n \text{ splits into } p \text{ sub-trees with } n_1, \ldots, n_p \text{ leaves"}\big]$ for all $p \geq 1$ and $n_1, \geq \cdots \geq n_p > 0$ with sum $n$.

Haas and Miermont [66] proved that under mild assumptions on this sequence $(q_n)_n$, the adequately normalised tree $T_n$ will converge in distribution to a random measured "continuum tree" in the Gromov-Hausdorff-Prokhorov topology. The limit trees are *self-similar fragmentation trees*, a family of continuum random trees introduced in [64] which in particular encompasses Aldous' Brownian tree introduced in [7]. Section 4.2.2 will give some background on both fragmentation trees and the GHP topology.

Here, we will be interested in the asymptotic study of the *profile* of large Markov branching trees. The profile of a given discrete tree $\mathtt{t}$ endowed with some measure $\mu_{\mathtt{t}}$ is the sequence of the sizes, with respect to $\mu_{\mathtt{t}}$, of each successive layer of $\mathtt{t}$. In our setting, we will consider Markov branching trees endowed with the counting measure on the set of their leaves so this notion of profile will coincide with the *external profile* we considered in Chapter 3. The asymptotic behaviour of the profile of some models of random trees has been studied, see Drmota and Gittenberger [43, 44], Kersting [82], Pitman [105] as well as [35, 45, 53] for instance.

There is a notion of profile for continuum trees as well. Consider $\mathcal{T}$, such a tree endowed with a measure and let $m_{\mathcal{T}}$ be the function such that $m_{\mathcal{T}}(t)$ is the mass of all points in $\mathcal{T}$ at distance less than or equal to $t$ from the root of $\mathcal{T}$. When this function is absolutely continuous, we say that $\mathcal{T}$ has a profile which is the density of the measure $\mathrm{d}m_{\mathcal{T}}$. As was the case for discrete trees, the profile of a continuum tree records how

much mass is concentrated at each height of the said tree. In [60], Haas gave a simple condition which, when satisfied, guarantees that a fragmentation tree admits such a continuum profile, see Section 4.2.3.

Thanks to Haas and Miermont's result from [66], we know that adequately rescaled Markov branching trees converge to fragmentation trees. Our main goal in this chapter is to study the convergence of the associated profiles. The starting point of our investigation is the close link between the profile of a discrete tree and the distribution of a particular random variable. If $\mathtt{t}$ is a tree with $n$ leaves and whose profile we denote by $\ell_{\mathtt{t}}$ and if $U$ is a leaf chosen uniformly in $\mathtt{t}$, then the probability that $U$ is at some height $j$ is $\ell_{\mathtt{t}}(j)/n$. In a similar way, the profile of a continuum tree $\mathcal{T}$ endowed with a probability measure $\mu_{\mathcal{T}}$ is the density of the height of a leaf drawn in $\mathcal{T}$ following $\mu_{\mathcal{T}}$.

In their study, Haas and Miermont [66] proved that under appropriate rescaling, the height $I_n$ of a leaf picked uniformly in the Markov branching tree $T_n$ converges in distribution to the height $I$ of a randomly chosen leaf in the corresponding fragmentation tree. We will endeavour to strengthen this last convergence in order to deduce convergence of the profiles. For that purpose, we will use Fourier inversion results on the characteristic function of $I_n$ conditionally on $T_n$ in a way akin to how Gnedenko proved the local Central Limit Theorem, see [56]. This will lead us to study the integrability of the Fourier transform of the profile of fragmentation trees, which will translate to results on the regularity of the said profile.

This chapter will be organised as follows. Section 4.2 will introduce Markov branching trees, fragmentation trees and their respective profiles. Section 4.3 will then focus on fragmentation trees; in particular, Section 4.3.2 will give results on the regularity of their profile. In contrast, Section 4.3.3 will give a criterion to guarantee that the said profile is not càdlàg. Applications of these results to various fragmentation models are given in Section 4.3.4. In Section 4.4, we will turn to the study of the profile of Markov branching trees. Section 4.4.1 focuses on the expectation of the said profiles, Section 4.4.3 establishes functional convergence under appropriate assumptions while in Section 4.4.4, we will give conditions under which such functional convergence cannot happen.

Throughout this chapter, if $X$ is a given real valued random variable, $\varphi_X$ will denote its characteristic function, i.e. for any real $t$, $\varphi_X(t) := \mathbb{E}[\mathrm{e}^{itX}]$.

## 4.2 Background on Markov branching trees and their scaling limits

### 4.2.1 Markov branching trees

*Basic notations.* We will denote by $\mathtt{T}$ the set of finite rooted unordered tree. Let $\mathtt{t}$ be in $\mathtt{T}$; we will use the following notations:

- A vertex of $\mathtt{t}$ will be called a *leaf* if it has no offspring. The set of leaves of $\mathtt{t}$ will be denoted by $\mathcal{L}(\mathtt{t})$.
- If $u$ is a vertex of $\mathtt{t}$, we will write $|u|$ for its *height*, that is the number of edges in the shortest path from the root of $\mathtt{t}$ to $u$.
- Moreover, $|\mathtt{t}| := \sup_{u\in\mathtt{t}} |u|$ will be the height of $\mathtt{t}$.
- For any non-negative integer $k$, $\mathtt{t}|_k$ will denote the sub-tree of all the vertices of $\mathtt{t}$ with height $k$ or less. In other words, $\mathtt{t}|_k$ is made out of the layers 0 through $k$ of $\mathtt{t}$.
- For any two vertices $u$ and $v$ of $\mathtt{t}$, let $u \wedge v$ be their *most recent common ancestor*, that is the lowest vertex in the shortest path from $u$ to $v$.
- We will denote by $\#$ the counting measure on the set of vertices $\mathtt{t}$. Similarly, $\#_{\mathcal{L}}$ will be the counting measure on the set of leaves of $\mathtt{t}$.
- For any positive integer $n$, let $\mathtt{T}_n$ and $\mathtt{T}_n^{\mathcal{L}}$ be the sets of finite trees with $n$ vertices and $n$ leaves respectively.

If $\mu$ is a measure on $\mathtt{t}$ and $k$ is a non-negative integer. Define $\ell_{\mathtt{t}}^{\mu}(k)$ as the measure (with respect to $\mu$) of the $k^{\text{th}}$ layer of $\mathtt{t}$, i.e. set

$$\ell_{\mathtt{t}}^{\mu}(k) := \mu\big(u \in \mathtt{t} : |u| = k\big).$$

The sequence $[\ell_{\mathtt{t}}^{\mu}(k)]_{k\geq 0}$ will be referred to as the *profile* of $\mathtt{t}$ with respect to $\mu$. When the setting is clear, $\ell_{\mathtt{t}}^{\mu}$ will simply be called the profile of $\mathtt{t}$.

*Remark 4.2.1.* The profile with respect to $\#$ (resp. $\#_{\mathcal{L}}$) is the usual profile of $\mathtt{t}$ (resp. *external profile* of $\mathtt{t}$); see Chapter 3.

*Partitions of integers.* Recall that Markov branching trees are closely related to "partitions of integers".

Set $\mathcal{P}_0 := \{\varnothing\}$, $\mathcal{P}_1 := \{\varnothing, (1)\}$ and for $n \geq 2$, let $\mathcal{P}_n$ be the set of *partitions* of $n$, i.e. the set of finite non-increasing sequences of positive integers with sum $n$. More precisely, set

$$\mathcal{P}_n := \Big\{\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{N}^p \,:\, p \geq 1,\, \lambda_1 \geq \cdots \geq \lambda_p \text{ and } \lambda_1 + \cdots + \lambda_p = n\Big\}.$$

Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be in $\mathcal{P}_n$ for some integer $n$. We will use the following notations:

— Its *length* will be the integer $p(\lambda) := p$, with the convention $p(\varnothing) = 0$.

— For a positive integer $k$, $m_k(\lambda) := \sum_i \mathbb{1}_{\lambda_i = k}$ will be the number of occurrences of $k$ in the partition $\lambda$.

*Markov branching trees.* Informally, a sequence $(T_n)_n$ of random trees is said to satisfy the Markov branching property if for all $n$, $\#_{\mathcal{L}} T_n = n$, and conditionally on the event

"$T_n$ has $p$ sub-trees $T^{(1)}, \dots, T^{(p)}$ attached to its root and $\#_{\mathcal{L}} T^{(1)} = \lambda_1 \geq \cdots \geq \#_{\mathcal{L}} T^{(p)} = \lambda_p$"

these sub-trees are independent and $T^{(i)}$ is distributed like $T_{\lambda_i}$. The sequence of distributions of $(T_n)_n$ is characterised by the probabilities of such events.

Fix an infinite subset $\mathcal{N}$ of $\mathbb{N}$ with $1 \in \mathcal{N}$; this set will index the possible numbers of leaves for our random trees. Let $q_1$ be a probability measure on $\{\varnothing, (1)\}$ with $q_1(1) < 1$ and for all $n > 1$, let $q_n$ be a probability measure on $\mathcal{P}_n$ such that

— $q_n[(n)] < 1$,

— $q_n$ is supported by $\{\lambda \in \mathcal{P}_n \,:\, \lambda_i \in \mathcal{N},\, i = 1, \dots, p(\lambda)\}$.

Set $q := (q_n)_{n\in\mathcal{N}}$. We will now define a sequence $\mathrm{MB}^{\mathcal{L},q} = (\mathrm{MB}_n^{\mathcal{L},q})_{n\in\mathcal{N}}$ with the Markov branching property and first-split distribution $q$. Let $\mathrm{MB}_1^{\mathcal{L},q}$ be the distribution of a branch of geometric length with parameter $1 - q_1(1)$. For $n > 1$, proceed as follows:

— Let $T_0$ be a branch with geometric length with parameter $1 - q_n(n)$ and call $U$ its leaf,

— Let $\Lambda$ be independent of $T_0$ and have distribution $q_n$ conditioned on the event $\{m_n = 0\}$,

— Conditionally on $\Lambda = (\lambda_1, \dots, \lambda_p)$, let $(T_1, \dots, T_p)$ be independent random trees respectively distributed according to $\mathrm{MB}_{\lambda_i}^{q}$ for $1 \leq i \leq p$,

— Attach each of these trees to $U$ by a single edge,

finally, let $\mathrm{MB}_n^{\mathcal{L},q}$ be the distribution of the tree given by this procedure.

### 4.2.2 Self-similar fragmentation trees

*The GHP topology.* In this paragraph, we describe a distance used to compare compact and weighted metric spaces, namely the Gromov-Hausdorff-Prokhorov distance. More details about this metric can be found in Chapter 4 of [50], Chapter 27 of [121] or in [3] for instance. Let us first recall the necessary notations.

Let $(M, \mathrm{d}_M)$ be a separable metric space. For any two compact subsets $A$ and $B$ of $M$, the Hausdorff distance between the two is defined as:

$$\mathrm{d}_M^{\mathrm{H}}(A, B) := \inf\Big\{\varepsilon > 0 : A \subset B^{\varepsilon} \text{ and } B \subset A^{\varepsilon}\Big\}$$

where $C^r := \{x \in M : \inf_{y\in C} \mathrm{d}_M(x,y) \le r\}$ for any $C \subset M$ and $r \ge 0$. We also define the Prokhorov metric on the set of finite Borel measures on $M$ by

$$\mathrm{d}^{\mathrm{P}}(\mu,\nu) := \inf\Big\{\varepsilon > 0 \,:\, \forall F \subset X \text{ closed}, \mu(F) \le \nu(F^\varepsilon) + \varepsilon \text{ and } \nu(F) \le \mu(F^\varepsilon) + \varepsilon\Big\}$$

for any two such measures $\mu$ and $\nu$.

Let $(X,\mathrm{d})$ be a compact metric space, let $\rho_X$ be in $X$, we will call it its *root*, and let $\mu_X$ be a Borel probability measure on $X$. The 4-tuple $\mathbf{X} = (X, \mathrm{d}_X, \rho_X, \mu_X)$ will be called a *compact pointed weighted metric space*. If $x$ is in $X$, let $|x| := \mathrm{d}_X(\rho_X, x)$ be the height of $x$ and set $|\mathbf{X}| := \sup_{x\in X} |x|$. For all $r \ge 0$, let $X|_r := \{x \in X : |x| \le r\}$ and set $\mathbf{X}|_r := (X|_r, \mathrm{d}_X, \rho_X, \mathbb{1}_{|\cdot|\le t}\mu_X)$.

Two compact pointed weighted metric spaces $\mathbf{X}$ and $\mathbf{Y}$ will be called *GHP-isometric* if there exists a bijective isometry $\Phi : X \to Y$ such that $\Phi(\rho_X) = \rho_Y$ and $\mu_X \circ \Phi^{-1} = \mu_Y$. Let $\mathbb{K}$ be the set of GHP-isometry classes of *compact* pointed weighted metric spaces.

Following [5, Section 2.1], we define the *Gromov-Hausdorff-Prokhorov* distance (or GHP distance for short) between two elements $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbb{K}$ as

$$\mathrm{d}_{\mathrm{GHP}}(\mathbf{X},\mathbf{Y}) := \inf\Big\{\mathrm{d}_M\big[\varphi(\rho_X), \psi(\rho_Y)\big] \vee \mathrm{d}_M^{\mathrm{H}}\big[\varphi(X), \psi(Y)\big] \vee \mathrm{d}_M^{\mathrm{P}}\big[\mu_X \circ \varphi^{-1}, \mu_Y \circ \psi^{-1}\big]\Big\}$$

where the infimum runs over all metric spaces $(M, \mathrm{d}_M)$ and isometries $\varphi : X \to M$, $\psi : Y \to M$. Theorem 2.5 in [3] states that $\mathrm{d}_{\mathrm{GHP}}$ is a well-defined distance on $\mathbb{K}$ and that $(\mathbb{K}, \mathrm{d}_{\mathrm{GHP}})$ is both complete and separable and thus, Polish.

$\mathbb{R}$-*trees.* To talk about scaling limits of discrete trees, we need to introduce a continuous analogue. We use the framework of $\mathbb{R}$-trees, see [92] for instance. An $\mathbb{R}$-*tree* (or *real tree*) is a metric space $(T,d)$ such that for all $x$ and $y$ in $T$,

— There exists a unique isometry $\varphi : [0, d(x,y)] \to T$ such that $\varphi(0) = x$ and $\varphi[(d(x,y)] = y$,
— If $\gamma : [0,1] \to T$ is a continuous injection with $\gamma(0) = x$ and $\gamma(1) = y$, then $\gamma$ and $\varphi$ have the same images, which we will denote by $[\![x,y]\!]$.

This roughly means that any two points in an $\mathbb{R}$-tree can be continuously joined by a single path, up to its reparametrisation, which is akin to the acyclic nature of discrete trees. We will denote by $\mathbb{T}$ the set of GHP isometry classes of compact rooted and weighted $\mathbb{R}$-trees.

**Proposition 4.2.1.** $(i)$ *The set $\mathbb{T}$ is closed in the Polish metric space $(\mathbb{K}, \mathrm{d}_{\mathrm{GHP}})$.*
$(ii)$ *If $\mathbf{T}_n$, $n \ge 1$ and $\mathbf{T}$ are elements of $\mathbb{T}$ and if $\mathbf{T}_n \to \mathbf{T}$ under $\mathrm{d}_{\mathrm{GHP}}$, then for all $r \ge 0$ such that $\mu_T(x \in T : |x| = r) = 0$, we have*

$$\mu_{T_n}(T_n|_r) \xrightarrow[n\to\infty]{} \mu_T(T|_r).$$

Proposition 4.2.1 $(i)$ was proved in [3, Corollary 3.2] while $(ii)$ was a step in the proof of [3, Proposition 2.10].

Let $\mathbf{T} = (T, \mathrm{d}_T, \rho_T, \mu_T)$ be an element of $\mathbb{T}$. If $x$ and $y$ belong to $T$, let $x \wedge y$ be their *most recent common ancestor*, i.e. $x \wedge y$ is the unique element of $T$ such that $[\![\rho_T, x]\!] \cup [\![\rho_T, x]\!] = [\![\rho_T, x \wedge y]\!]$.

*Remark 4.2.2.* If $\mathtt{t}$ is in $\mathtt{T}$, i.e. a finite discrete tree, it is naturally endowed with the *graph distance* $\mathrm{d}_{\mathrm{gr}}$ where $\mathrm{d}_{\mathrm{gr}}(u,v)$ is the number of edges in the shortest path between $u$ and $v$ in $\mathtt{t}$. Let $\mu$ be the uniform measure on the set of leaves of $\mathtt{t}$ and let $\rho$ denote its root. We can then consider $(\mathtt{t}, \mathrm{d}_{\mathrm{gr}}, \rho, \mu)$ as an element of $\mathbb{K}$.

Furthermore, if we let $C_{\mathtt{t}}$ be the tree $\mathtt{t}$ in which all the edges have been replaced with a copy of the unit interval $[0,1]$, $C_{\mathtt{t}}$ belongs to $\mathbb{T}$ and for all positive $a$, it is easy to check that $\mathrm{d}_{\mathrm{GHP}}(a\mathtt{t}, aC_{\mathtt{t}}) \le a$. Consequently, in order to use this last proposition, when we study the scaling limits of discrete trees, we will implicitly consider them to be $\mathbb{R}$-trees.

*Profile of an $\mathbb{R}$-tree.* If $\mathbf{T}$ belongs to $\mathbb{T}$, then the function $m_{\mathbf{T}} : t \mapsto \mu_T(T|_t)$ is clearly non-decreasing. Moreover, $\mu_T(T|_t) = 0$ for all $t < 0$ and $\mu_T(T|_t) = 1$ if $t \geq |T|$. Finally, for all $t$, the monotone convergence theorem ensures that $\mu_T(T|_{t+\varepsilon}) \downarrow \mu_T(T|_t)$ as $\varepsilon \downarrow 0$. The function $m_T$ is thus càdlàg and is consequently the cumulative distribution function (CDF for short) of a probability measure $\mathrm{d}m_{\mathbf{T}}$ with support included in the compact interval $[0, |T|]$.

When this measure is absolutely continuous with respect to the Lebesgue measure, we call its density the *profile* of $\mathbf{T}$. Informally, the profile records how much the mass of $\mu$ is concentrated at each height, compare with the profiles of discrete trees introduced in Section 4.2.1.

*Remark 4.2.3.* Observe that if $U$ is a $T$ valued random variable with law $\mu_T$, the distribution of its height is $\mathrm{d}m_{\mathbf{T}}$.

*Self-similar fragmentation processes.* Let $\mathcal{S}^{\downarrow} := \left\{\mathbf{s} = (s_n)_{n\geq 1} \in \ell_1 \,:\, s_1 \geq s_2 \geq \cdots \geq 0\right\}$ and endow it with the usual $\ell_1$ norm $\|\mathbf{s}\| = \sum_{i\geq 1} |s_i|$. Moreover, set $\mathbf{0} := (0, 0, \dots)$, $\mathbf{1} := (1, 0, 0, \dots)$ and $\mathcal{S}^{\downarrow}_{\leq 1} := \left\{\mathbf{s} \in \mathcal{S}^{\downarrow} \,:\, \|\mathbf{s}\| \leq 1\right\}$.

A *self-similar fragmentation process* is an $\mathcal{S}^{\downarrow}_{\leq 1}$-valued Markovian process $(\mathbf{F}(t); t \geq 0)$ which is continuous in probability, and satisfies $\mathbf{F}(0) = \mathbf{1}$ as well as the following so-called *fragmentation property*. There exists an $\alpha \in \mathbb{R}$ such that for all $t_0 \geq 0$, conditionally on $\mathbf{F}(t_0) = \mathbf{s}$, $\big(\mathbf{F}(t_0 + t), t \geq 0\big)$ has the same distribution as

$$\Big(\big(s_i F^{(i)}_j(s_i^{\alpha} t),\, i \geq 1, j \geq 1\big)^{\downarrow};\, t \geq 0\Big)$$

where $(\mathbf{F}^{(i)})_{i\geq 1}$ are i.i.d. copies of $\mathbf{F}$. The constant $\alpha$ is called the *self-similarity index* of the process $\mathbf{F}$.

These processes can be seen as the evolution of the fragmentation of an object of mass 1 into smaller objects which will each, in turn, split themselves apart independently from one another, at a rate proportional to their mass to the power $\alpha$.

Bertoin and Berestycki proved in [16, 19] that the distribution of a self-similar fragmentation process is characterised by a 3-tuple $(\alpha, c, \nu)$ where $\alpha$ is the aforementioned self-similarity index, $c \geq 0$ is a so-called erosion coefficient which accounts for a continuous decay in the mass of each particle and $\nu$ is a *dislocation measure* on $\mathcal{S}^{\downarrow}_{\leq 1}$, i.e. a $\sigma$-finite measure such that $\int (1 - s_1)\, \nu(\mathrm{d}\mathbf{s}) < \infty$ and $\nu(\{\mathbf{1}\}) = 0$. Informally, at any given time, each particle with mass say $x$ will, independently from the other particles, split into smaller fragments of respective masses $xs_1, xs_2, \dots$ at rate $x^{\alpha}\nu(\mathrm{d}\mathbf{s})$.

We will be interested in fragmentation processes with negative self-similarity index $-\gamma < 0$ with no erosion, i.e. with $c = 0$. Furthermore, we will require the dislocation measure $\nu$ to be non-trivial, i.e. $\nu(\mathcal{S}^{\downarrow}_{\leq 1}) > 0$, and *conservative*, that is to satisfy $\nu(\|\mathbf{s}\| < 1) = 0$. Therefore, the fragmentation processes we will consider will be characterised by a *fragmentation pair* $(\gamma, \nu)$ and we will refer to them as $(\gamma, \nu)$-fragmentation processes.

Under these assumptions, each particle will split into smaller ones which will in turn break down faster, thus speeding up the global fragmentation rate. Let $\mathbf{F}$ be a $(\gamma, \nu)$-fragmentation process and set $\tau_{\mathbf{0}} := \inf\{t \geq 0 : \mathbf{F}(t) = \mathbf{0}\}$ the first time at which all the mass has been turned to dust. It was shown in [20, Proposition 2] that $\tau_{\mathbf{0}}$ is a.s. finite and in [59, Section 5.3] that it has exponential moments, i.e. that there exists $a > 0$ such that $\mathbb{E}\big[\exp(a\tau_{\mathbf{0}})\big] < \infty$.

*Self-similar fragmentation trees.* Let $\mathbf{F}$ be a $(\gamma, \nu)$-fragmentation process. In [64], Haas and Miermont exhibited a random rooted and weighted $\mathbb{R}$-tree which encodes the genealogy of the fragmentation of the initial object.

More precisely, this $\mathbb{T}$-valued random variable say $(\mathcal{T}, \mathrm{d}, \rho, \mu)$ is such that $\mu(\mathcal{T}) = 1$ and if for all $t \geq 0$, $\{\mathcal{T}_i(t) : i \geq 1\}$ is the (possibly empty) set of the closures of the connected components of $\mathcal{T} \setminus (\mathcal{T}|_t)$, then

$$\Big(\big(\mu[\mathcal{T}_i(t)];\, i \geq 1\big)^{\downarrow};\, t \geq 0\Big)$$

is a $(\gamma, \nu)$-fragmentation process. We will denote the distribution of $(\mathcal{T}, \mathrm{d}, \rho, \mu)$ by $\mathcal{T}_{\gamma,\nu}$.

Self-similar fragmentation trees bear a close resemblance to Markov branching trees. Let $\iota : \bigcup_{n\geq 1} \mathcal{P}_n \to \mathcal{S}_1^{\downarrow}$ be defined by $\iota(\lambda) := (\lambda_1/n, \dots, \lambda_p/n, 0, 0, \dots)$ if $\lambda = (\lambda_1, \dots, \lambda_p)$ is in $\mathcal{P}_n$.

**Theorem 4.2.2 ([66], Theorem 5).** *Let $(q_n)_{n\in\mathbb{N}}$ be the sequence of first-split distributions of a Markov branching family $\mathrm{MB}^{\mathcal{L},q}$. For all $n$ in $\mathbb{N}$, set $\bar{q}_n := q_n \circ \iota^{-1}$. Suppose there exists a fragmentation pair $(\gamma, \nu)$ and a slowly varying function $\ell$ such that, for the weak convergence of finite measures on $\mathcal{S}^{\downarrow}$,*

$$n^{\gamma}\ell(n)(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \xrightarrow[n\to\infty]{} (1-s_1)\,\nu(\mathrm{d}\mathbf{s}).$$

*For all $n$ in $\mathbb{N}$, let $T_n$ have distribution $\mathrm{MB}_n^{\mathcal{L},q}$ and let $\mu_n$ be the counting measure on the leaves of $T_n$. Then, with respect to the GHP topology on $\mathbb{T}$,*

$$\left(\frac{1}{n^{\gamma}\ell(n)} T_n, \frac{\mu_n}{n}\right) \xrightarrow[n\to\infty]{} \mathcal{T}_{\gamma,\nu} \quad \textit{in distribution.}$$

### 4.2.3 Profiles of fragmentation and Markov branching trees

**Fragmentation trees.** Let $\mathcal{T}$ be a $(\gamma, \nu)$-fragmentation tree. For all $t \geq 0$, set

$$M(t) := \mu_{\mathcal{T}}\big(\mathcal{T}|_t\big)$$

the mass of the ball of radius $t$ centred around the root of $\mathcal{T}$. Recall that $M$ is the CDF of a (random) probability measure $\mathrm{d}M$ on $\mathbb{R}_+$ the support of which is a subset of $[0, |\mathcal{T}|]$.

Proposition 1.9 in [21] further states that $M$ is a.s. continuous. A natural question is then whether $\mathrm{d}M$ has a density with respect to the Lebesgue measure or in other words if $\mathcal{T}$ admits a profile. This question was part of the focus of [60].

**Theorem 4.2.3 (Theorem 4 in [60]).** *Suppose that $\nu$ is such that $\int_{\mathcal{S}^{\downarrow}} \sum_{i\geq 1} s_i\,|\log s_i|\;\nu(\mathrm{d}\mathbf{s}) < \infty$.*

*(i) If $\gamma < 1$ and if*

$$\int_{\mathcal{S}^{\downarrow}} (1-s_1)^{\beta}\;\nu(\mathrm{d}\mathbf{s}) < \infty, \qquad \int_{\mathcal{S}^{\downarrow}} \sum_{i<j} s_i^{1-\gamma} s_j\;\nu(\mathrm{d}\mathbf{s}) < \infty$$

*for some $\beta$ in $]0,1[$, then $\mathrm{d}M$ a.s. has a (random) density $L$ with respect to the Lebesgue measure and*

$$\mathbb{E}\Big[\int_0^{\infty} L^2(t)\,\mathrm{d}t\Big] < \infty.$$

*(ii) If $\gamma \geq 1$, then the measure $\mathrm{d}M$ is a.s. singular.*

When the measure $\mathrm{d}M$ has a density $L$, this density is called the *profile* of $\mathcal{T}$.

*Tagged fragment.* To prove Theorem 4.2.3 $(i)$, Haas first studied the height of a leaf drawn in a fragmentation tree.

Conditionally on $\mathcal{T}$, pick $V$ following $\mu_{\mathcal{T}}$ and set $I := |V|$. For any $t \geq 0$, let $\mathcal{T}_V(t) := \{u \in \mathcal{T} : |u \wedge V| \geq t\}$ be the closed connected component of $\mathcal{T} \setminus (\mathcal{T}|_t)$ containing $V$, with the convention $\mathcal{T}_V(t) = \varnothing$ if $t > I$. For all $t \geq 0$, set $X(t) := \mu_{\mathcal{T}}[\mathcal{T}_V(t)]$. Note that conditionally on $\mathcal{T}$, $I$ is distributed according to $\mathrm{d}M$.

In [18, 19], Bertoin studied the joint distributions of $X$ and $I$. He proved in particular that $X$ is a self-similar non-increasing Markov process started from 1 (see Lamperti [90]) and that $I$ is its absorption time.

Let us describe the distribution of $X$ and $I$. Let $\xi$ be a subordinator with Laplace exponent $\psi$ given by

$$\begin{aligned} \psi : [0,\infty[ &\longrightarrow [0,\infty[ \\ q &\longmapsto -\log \mathbb{E}[\mathrm{e}^{-q\xi_1}] = \int_{\mathcal{S}^{\downarrow}} \big(1 - \textstyle\sum_{i\geq 1} s_i^{1+q}\big)\,\nu(\mathrm{d}\mathbf{s}), \end{aligned} \tag{4.1}$$

and define the Lamperti time change $\tau$ such that for all $t \geq 0$,

$$\tau(t) := \inf\Big\{r \geq 0 : \textstyle\int_0^r \mathrm{e}^{-\gamma\xi_u}\mathrm{d}u \geq t\Big\}.$$

Then Bertoin proved that

$$(X, I) \overset{(\mathrm{d})}{=} \left( \left( \exp[-\xi_{\tau(t)}]\,; t \geq 0 \right), \int_0^\infty \mathrm{e}^{-\gamma \xi_t} \mathrm{d}t \right). \tag{4.2}$$

The process $X$ is called a self-similar Markov process and the random variable $I$ is called the *exponential functional* of the subordinator $\gamma\,\xi$, see [28, 32].

**Markov branching trees.** Suppose that $(q_n)_n$ is a sequence of first-split distributions which satisfies the assumptions of Theorem 4.2.2. For all $n$, let $T_n$ denote a random tree with distribution $\mathrm{MB}_n^{\mathcal{L},q}$ endowed with the counting measure $\mu_n$ on the set of its leaves. Also let $(\mathcal{T}, \mu_{\mathcal{T}})$ be a $\mathcal{T}_{\gamma,\nu}$ distributed random variable. With these notations and assumptions, Theorem 4.2.2 ensures that $(n^{-\gamma} T_n, n^{-1}\mu_n) \Rightarrow (\mathcal{T}, \mu_{\mathcal{T}})$ for the GHP topology.

For all $t \geq 0$, recall that $M(t) = \mu_{\mathcal{T}}(\mathcal{T}|_t)$ and let

$$M_n(t) := \frac{1}{n} \mu_n\big(x \in T_n : |x| \leq n^\gamma t\big),$$

i.e. $M_n(t)$ is the proportion of leaves of $T_n$ at height $n^\gamma t$ or less. Observe that $M_n$ is the CDF of a random discrete probability measure $\mathrm{d}M_n$. Moreover, if $V_n$ denotes a leaf drawn conditionally on $T_n$ uniformly among its leaves, $\mathrm{d}M_n$ is the distribution of $|V_n|/n^\gamma$ conditionally on $T_n$.

**Lemma 4.2.4.** *Under our assumptions, $M_n \Rightarrow M$ in the uniform topology on $\mathbb{R}_+$ jointly with $n^{-\gamma} T_n \Rightarrow \mathcal{T}$.*

*Proof.* The metric space $(\mathbb{T}, \mathrm{d}_{\mathrm{GHP}})$ is Polish so we can use Skorokhod's representation theorem and assume that (a version of) $n^{-\gamma} T_n$ a.s. converges to $\mathcal{T}$ in the GHP topology. Proposition 1.9 in [21] ensures that the process $M : t \mapsto \mu_{\mathcal{T}}(\mathcal{T}|_t)$ is a.s. continuous. By Proposition 4.2.1 $(ii)$, we then get that (a version of) $M_n$ is such that $M_n(t) \to M(t)$ a.s. for all $t \geq 0$. Finally, we use the well known fact that a sequence of cumulative distributions functions which converge point-wise to a continuous CDF also converges uniformly. Therefore, a.s., $\sup_{t \geq 0} |M_n(t) - M(t)| \to 0$ as $n \to \infty$. □

For all $n$, let $\ell_n := \ell_{T_n}^{\mu_n}$ be the profile of $T_n$ with respect to $\mu_n$. In other words, for all $j \geq 0$,

$$\ell_n(j) := \#\{u \in \mathcal{L}(T_n) : |u| = j\}$$

denotes the number of leaves at height $j$ in the tree $T_n$. For any real $t$, set

$$L_n(t) := n^{\gamma - 1} \ell_n(\lfloor n^\gamma t \rfloor).$$

Observe that $L_n$ is a càdlàg process and is such that $\int_0^\infty L_n(t)\,\mathrm{d}t = 1$. Moreover, with these notations

$$\mathrm{d}M_n = \frac{1}{n} \sum_{j \geq 0} \ell_n(j) \delta_{j/n^\gamma} = n^{-\gamma} \sum_{j \geq 0} L_n(j/n^\gamma) \delta_{j/n^\gamma}.$$

Finally, if $U$ is a uniform random variable on $]0,1[$ independent of $T_n$ and $V_n$, then for all real $t$,

$$\mathbb{P}\big[|V_n| + U \leq n^\gamma t \,|\, T_n\big] = \int_0^t L_n(s)\,\mathrm{d}s.$$

Slutsky's theorem and Lemma 4.2.4 therefore ensure that

$$\Big( \int_0^t L_n(s)\,\mathrm{d}s ; t \geq 0 \Big) \underset{n \to \infty}{\Longrightarrow} M \tag{4.3}$$

in the uniform topology on $\mathbb{R}_+$.

We want to study the asymptotic behaviour of $L_n$ when $n \to \infty$. In particular, we will endeavour to find suitable conditions under which $\mathbb{E}[L_n(t)] \to \mathbb{E}[L(t)]$ uniformly on $\mathbb{R}_+$ and $L_n \Rightarrow L$ in $\mathcal{D}[0, \infty[$. We will also give conditions under which $(L_n)_n$ cannot converge weakly in $\mathcal{D}[0, \infty[$.

This next lemma is a uniqueness result. It proves that if $(L_n)_n$ converges weakly in $\mathcal{D}[0, \infty[$, its limit is distributed like the profile of the $(\gamma, \nu)$ fragmentation tree $\mathcal{T}$.

**Lemma 4.2.5.** *Suppose that $L_n$ converges in distribution to some càdlàg process $\tilde{L}$ in the Skorokhod topology. Then $\mathcal{T}$ a.s. has a profile which is distributed like $\tilde{L}$.*

*Proof.* We may deduce from (4.3) that $L_n(t)\,\mathrm{d}t \Rightarrow \mathrm{d}M$ in the weak topology on finite measures. We will now endeavour to prove that $L_n(t)\,\mathrm{d}t \Rightarrow \tilde{L}(t)\,\mathrm{d}t$ in the same topology. This will entail that the two random probability measures $\mathrm{d}M$ and $\tilde{L}(t)\,\mathrm{d}t$ have the same distribution. Lemma 2.10 $(i)$ in [80] will then allow us to get the desired result.

Because the Skorokhod space $\mathcal{D}[0,\infty[$ is Polish, we may use the Skorokhod representation theorem and assume that $L_n \to \tilde{L}$ a.s. for the Skorokhod topology.

Recall that for all non-negative $t$, the function $\mathcal{D}[0,\infty[ \to \mathbb{R}$, $f \mapsto \int_0^t f(s)\,\mathrm{d}s$ is continuous. As a result, on the set $\{L_n \to \tilde{L}\}$, for all $t \geq 0$,

$$\int_0^t L_n(s)\,\mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t \tilde{L}(s)\,\mathrm{d}s. \tag{4.4}$$

Convergence in $\mathcal{D}[0,\infty[$ entails convergence a.e. so by Fatou's lemma, on the set $\{L_n \to \tilde{L}\}$, we get that $\int_0^\infty \tilde{L}(t)\,\mathrm{d}t \leq 1$. Furthermore, for all non-negative $R$,

$$\mathbb{E}\Big[\int_0^\infty \tilde{L}(t)\,\mathrm{d}t\Big] \geq \mathbb{E}\Big[\int_0^R \tilde{L}(t)\,\mathrm{d}t\Big] = \lim_{n\to\infty} \mathbb{E}\Big[\int_0^R L_n(t)\,\mathrm{d}t\Big] = \mathbb{E}\big[M(R)\big] \xrightarrow[R\to\infty]{} 1$$

where we have used (4.3) as well as the dominated and the monotone convergence theorems. Consequently, $\int_0^\infty \tilde{L}(t)\,\mathrm{d}t = 1$ a.s. or, in other words, $\tilde{L}(t)\,\mathrm{d}t$ is a.s. a probability measure. Equation (4.4) then ensures that on the event $\big\{L_n \to \tilde{L}\big\} \cap \big\{\int_0^\infty \tilde{L}(t)\,\mathrm{d}t = 1\big\}$, $L_n(t)\,\mathrm{d}t \to \tilde{L}(t)\,\mathrm{d}t$ in the weak topology on finite measures.

Theorem 2.2 in [80] yields that $\tilde{L}(t)\,\mathrm{d}t \overset{(\mathrm{d})}{=} \mathrm{d}M$ as finite measures. Lemma 2.10 $(i)$ in [80] then ensures that $\mathrm{d}M$ a.s. admits a density, say $L$, with respect to the Lebesgue measure and that $L$ and $\tilde{L}$ have the same finite dimensional distributions. □

*Tagged fragments.* Conditionally on $T_n$, let $V_n$ be drawn uniformly among the leaves of $T_n$. Conditionally on both $T_n$ and $V_n$, for every $j \geq 0$, let $T_n(j) := \{u \in T_n : |u \wedge V_n| \geq j\}$ be the (possibly empty) connected sub-tree of $T_n \setminus T_n|_{j-1}$ which contains $V_n$. Set $X_n(j) := \#_{\mathcal{L}} T_n(j)$. With these notations, $[X_n(j); j \geq 0]$ is a non-increasing Markov chain started from $n$ with transition matrix $P^q = \big(p_i(j)\big)_{i\geq 0, j\geq 0}$ where for any integer $i$, $p_i$ is the probability measure supported by $\{0,\ldots,i\}$ defined by

$$p_i(j) := \sum_{\lambda\in\mathcal{P}_i} \frac{j}{i} m_j(\lambda) q_i(\lambda). \tag{4.5}$$

Observe that $P^q(i,j) = 0$ if $j > i$. Let $I_n := 1 + |V_n|$; it is the absorption time at 0 of the Markov chain $X_n$. Observe that for all $t \geq 0$,

$$\mathbb{P}[I_n/n^\gamma \leq t | T_n] = M_n(t - n^{-\gamma}).$$

In other words, conditionally on $T_n$, the distribution of $(I_n - 1)/n^\gamma$ is $\mathrm{d}M_n$.

*Remark 4.2.4.* The study of the asymptotic behaviour of the Markov chains $X_n$ hence defined was one of the steps in Haas and Miermont's proof of Theorem 4.2.2, see Lemma 28 in [66].

The asymptotic behaviour of such Markov chains and their absorption times were studied by Haas and Miermont [65]. See also Bertoin and Kortchemski [25] as well as Haas and Stephenson [70].

## 4.3 Regularity of the profile of self-similar fragmentation trees

In this section, we will establish some results on the regularity of the profile of fragmentation trees characterized by some fragmentation pairs $(\gamma, \nu)$.

Let $(\mathcal{T}, \mu_{\mathcal{T}})$ be such a tree; conditionally on it, let $V$ have distribution $\mu_{\mathcal{T}}$ and set $I := |V|$. In Section 4.3.1 we will prove that the characteristic function $\varphi_I$ of $I$ decays faster at infinity than $|t|^{-1-\theta}$ for any $\theta$ smaller than some parameter depending on both $\gamma$ and $\nu$, see Proposition 4.3.1.

This result will then be used in Section 4.3.2 along with Fourier inversion to prove that under some assumptions, if $\gamma$ is small enough, $(\mathcal{T}, \mu_{\mathcal{T}})$ a.s. has a continuous profile; a precise statement is given in Theorem 4.3.6.

Conversely, in Section 4.3.3, we will give conditions under which the profile, if it exists, cannot be càdlàg.

Finally in Section 4.3.4, we will give applications of these results to various fragmentation models.

### 4.3.1 On the exponential functional of a subordinator

Recall (4.2); as a result, instead of considering the height of a leaf drawn in a fragmentation tree, in this section we will study exponential functionals of subordinators, which is slightly more general.

Let us first introduce similar notations as around (4.2). Let $\gamma$ be positive and let $\mu$ be a $\sigma$-finite measure on $]0,1[$. Define the function $\psi$ by

$$\psi(q) = \int_{]0,1[}(1-x^q)\,\mu(\mathrm{d}x) \tag{4.6}$$

for all real $q$. We will assume that $\psi(1) < \infty$, i.e. that $\mu$ integrates $x \mapsto 1-x$. This ensures that $\psi$ is finite on $[0,\infty[$. We will often extend $\psi$ on the set of complex numbers $z$ such that $x \mapsto |1-x^z|$ is integrable with respect to $\mu$. Let $\xi$ be a subordinator with Laplace exponent $\psi$ and let $I$ be the exponential functional of $\gamma\,\xi$, i.e. set

$$I := \int_0^\infty \mathrm{e}^{-\gamma\xi_t}\mathrm{d}t. \tag{4.7}$$

The random variable $I$ has been studied quite extensively, see [32, 33, 28, 103] and references therein for instance.

Let $\rho(\mu)$ be defined as

$$\rho(\mu) := \sup\{q \geq 0 : \psi(-q) > -\infty\}. \tag{4.8}$$

This parameter will be important in this section's main result.

*Remark 4.3.1.* When $I$ is the height of a leaf drawn in a $(\gamma, \nu)$ fragmentation tree, the measure $\mu$ will be such that

$$\int_{]0,1[} f\,\mathrm{d}\mu := \int_{\mathcal{S}^\downarrow} \sum_{i\geq 1} s_i f(s_i)\mathbb{1}_{s_i>0}\,\nu(\mathrm{d}\mathbf{s}), \tag{4.9}$$

for non-negative and measurable functions $f$, compare with (4.1). In this case, we will set $\rho(\nu) := \rho(\mu)$. For any real $q$,

$$\psi(-q) = \int_{\mathcal{S}^\downarrow}(1-s_1^{1-q})\,\nu(\mathrm{d}\mathbf{s}) - \int_{\mathcal{S}^\downarrow}\sum_{i\geq 2} s_i^{1-q}\,\nu(\mathrm{d}\mathbf{s}).$$

Recall that as a conservative dislocation measure, $\nu$ integrates $\mathbf{s} \mapsto 1-s_1$ and that $\sum_{i\geq 1} s_i = 1$ $\nu$-a.e.; further observe that provided $q \leq 1$, for all $\mathbf{s} \in \mathcal{S}^\downarrow$, $0 \leq 1-s_1^{1-q} \leq 1-s_1$. As a result, we have the following useful identity

$$1 \wedge \rho(\nu) = \sup\left\{q \in [0,1] : \int_{\mathcal{S}^\downarrow}\sum_{i\geq 2} s_i^{1-q}\,\nu(\mathrm{d}\mathbf{s}) < \infty\right\}. \tag{4.10}$$

We say that a function $f$ is regularly varying at 0 with exponent $\alpha \in \mathbb{R}$ if for any $c > 0$, $f(cx)/f(x) \to c^\alpha$ when $x \to 0$ or equivalently, if $f$ can be written as $f(x) = x^\alpha \ell(x)$ where $\ell$ is slowly varying around 0, i.e. $\ell$ is regularly varying with exponent 0. A function $g$ is said to be quasi-monotone if it is of bounded variation around 0 and for any positive $\beta$, when $x \downarrow 0$,

$$\int_0^x t^\beta |\mathrm{d}g(t)| = \mathcal{O}[x^\beta g(x)].$$

As the name suggests, if $g$ is monotone, it is quasi-monotone as well.

The main result of this section is the following proposition which gives a bound on the speed of the decay of the characteristic function $\varphi_I$ of $I$.

**Proposition 4.3.1.** *Suppose that the constant $\rho(\mu)$ defined in (4.8) is positive, that the function $x \mapsto \mu(]0,1-x])$ is regularly varying around 0 with exponent $-\alpha$, $0 \le \alpha < 1$, and that $x \mapsto x^{\alpha}\mu(]0,1-x])$ is quasi-monotone. Then, for any $\theta \in [0, \rho(\mu)/\gamma[$, there exists a finite constant $C_\theta$ such that $|\varphi_I(t)| \le C_\theta |t|^{-1-\theta}$ for any real $t$.*

*Remark 4.3.2.* It is sometimes easy to check the assumptions on $\mu$:

— If $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ and $x \mapsto f(1-x)$ is regularly varying at 0 with exponent $-\beta$, $\beta > 1$, then $x \mapsto \mu(]0,1-x])$ is regularly varying with index $1-\beta$ see Karamata's theorem (Theorem 1.15 in [112] for instance) and $x \mapsto x^{\beta-1}\mu(]0,1-x])$ is quasi-monotone (see Example 1.30 in [112]).

— Observe that $x \mapsto \mu(]0,1-x])$ is monotone, and thus quasi-monotone so if it further is slowly varying, $\mu$ fulfils the assumptions of Proposition 4.3.1. That is in particular the case when $\mu$ is finite.

We will also prove the following converse to Proposition 4.3.1:

**Proposition 4.3.2.** *If $\theta$ is positive and such that $|\varphi_I(t)| = \mathcal{O}(|t|^{-1-\theta})$ when $|t| \to \infty$, then $\theta \le \rho(\mu)/\gamma$.*

*Basic properties of $I$.* In order to prove Proposition 4.3.1 and Proposition 4.3.2, we will use results on the Mellin transform of $I$ which were proved in [104]. Mellin inversion results will then allow us to get fine results on the distribution of $I$ which we will in turn use to prove Proposition 4.3.1 and Proposition 4.3.2. Let us first recall some known properties of the distribution of $I$.

**Proposition 4.3.3.** *The random variable $I$ has a density $k$ with respect to the Lebesgue measure, $k$ is infinitely differentiable on $]0,\infty[$, and*

$$\forall x > 0, \qquad k(x) = \int_x^\infty \mu\big(]0,(x/u)^{1/\gamma}[\big)\, k(u)\,\mathrm{d}u, \tag{4.11}$$

Proposition 4.3.3 was first proved in [32], Proposition 2.1, under the assumption $\psi'(0_+) < \infty$ and then in [103], Theorem 2.3, with this last assumption dropped.

A consequence of (4.11) is that for all real $t \neq 0$,

$$\mathbb{E}[I^{t-1}] = \frac{\psi(\gamma t)}{t}\mathbb{E}[I^t] \tag{4.12}$$

where the two sides may be infinite. This identity for positive $t$ was proved in Proposition 3.1 of [32] which also states that

$$\mathbb{E}[I^{-1}] = \gamma\,\psi'(0_+) := \lim_{\varepsilon\downarrow 0} \gamma\,\psi(\varepsilon)/\varepsilon = \int_{]0,1[} |\log x|\,\mu(\mathrm{d}x) = \gamma\,\mathbb{E}[\xi_1] \in ]0,\infty]. \tag{4.13}$$

Proposition 2 in [27] deals with negative moments of $I$ under different assumptions. As a result, for any non-negative integer $n$, we get

$$\mathbb{E}[I^n] = n! \Big/ \prod_{i=1}^n \psi(\gamma i). \tag{4.14}$$

A rather straightforward consequence of (4.12) is the following result.

**Corollary 4.3.4.** *For all $a < 1+\rho(\mu)/\gamma$, $\mathbb{E}[I^{-a}] < \infty$ and if $b > 1+\rho(\mu)/\gamma$, then $\mathbb{E}[I^{-b}] = \infty$.*

*Proof.* Let $d$ be a non-negative integer and let $\theta \in ]0,1[$. By iterating (4.12), we get

$$\mathbb{E}[I^{-d-\theta}] = \prod_{i=0}^{d-1} \frac{-\psi[-\gamma(\theta+i)]}{\theta+i} \cdot \frac{\psi[\gamma(1-\theta)]}{1-\theta}\,\mathbb{E}[I^{1-\theta}]. \tag{4.15}$$

In light of (4.14), $I$ has positive moments of all orders and in particular, $\mathbb{E}[I^{1-\theta}] < \infty$. As a result,

$$\mathbb{E}[I^{-d-\theta}] < \infty \quad \Longleftrightarrow \quad \psi[-\gamma(\theta+d-1)] > -\infty$$

which gives us, along with the definition of $\rho(\mu)$ (see (4.8)), the desired result. □

*Remark 4.3.3.* If $q > 0$ and $x \in ]0,1[$,

$$|\log x| \leq (x^{-q} - 1)/q. \tag{4.16}$$

As a result, if $\rho(\mu) > 0$, $\psi'(0_+) = \int |\log x|\, \mu(\mathrm{d}x)$ is necessarily finite and (4.13) ensures that $\mathbb{E}[I^{-1}] < \infty$.

Define $\mathcal{M}_I$, the aforementioned *Mellin transform* of $I$, by $\mathcal{M}_I(z) := \mathbb{E}[I^{z-1}]$ whenever $\mathbb{E}[|I^{z-1}|] = \mathbb{E}[I^{\mathrm{Re}\, z-1}] < \infty$. Corollary 4.3.4 ensures that $\mathcal{M}_I$ is well defined and holomorphic on $\{z : \mathrm{Re}\, z > -\rho(\mu)/\gamma\}$. Moreover, just like for (4.12), as soon as $\mathrm{Re}\, z > -\rho(\mu)/\gamma$ and $z \neq 0$,

$$\mathcal{M}_I(z) = \frac{\psi(\gamma z)}{z} \mathcal{M}_I(z+1). \tag{4.17}$$

*Regularity of $I$.* From this point on, assume that the constant $\rho(\mu)$ from (4.8) is positive. In order to prove Proposition 4.3.1, we will need the following lemma.

**Lemma 4.3.5.** *Under the assumptions of Proposition 4.3.1, the density $k$ of $I$ coincides on the complex domain $\{z : |z| > 0, |\arg z| < (1-\alpha)\pi/2\}$ with a holomorphic function, which we will also denote by $k$.*

*Furthermore, for any $\lambda \in ]-\infty, \rho(\mu)/\gamma[$, there is a finite constant $C_\lambda$ such that for all $z$ in the aforementioned complex domain, $|k(z)| \leq C_\lambda |z|^\lambda$.*

*Remark 4.3.4.* See Lemma 1 in [60] and Theorem 5 in [103] for similar results on $k$ on the real line.

*Proof.* The cornerstone of this proof is Theorem 2.3 (2) in [104] which ensures that under our assumptions, for any $\varepsilon > 0$, fixed $s > 0$, and any $t \in \mathbb{R}$,

$$|\mathcal{M}_I(s+it)| \leq C \exp\big(-[(1-\alpha)\pi/2 - \varepsilon] \cdot |t|\big) \tag{4.18}$$

for some finite constant $C$ which depends on $s$ and $\varepsilon$. We will endeavour to prove that (4.18) also holds when $s \in ]-\rho(\mu)/\gamma, 0]$ with the help of (4.17). This will allow us to conclude this proof by using the Mellin inversion theorem.

If $a \in ]-\rho(\mu), \infty[$, and $b \in \mathbb{R}$,

$$|\psi(a+ib)| \leq \int_{(0,1)} |1 - x^a \mathrm{e}^{ib\log x}|\, \mu(\mathrm{d}x) \leq |\psi(a)| + \int_{(0,1)} x^a |1 - \mathrm{e}^{ib\log x}|\, \mu(\mathrm{d}x).$$

Furthermore, for any positive $\eta$ and $x \in ]0,1[$, $|1 - \mathrm{e}^{ib\log x}| \leq |b||\log x|$ so that, using (4.16), if $a < 0$, $x^a |1 - \mathrm{e}^{ib\log x}| \leq |b|(x^{a-\eta} - 1)/\eta$ and thus

$$|\psi(a+ib)| \leq |\psi(a)| + \frac{|b|}{\eta} |\psi(a-\eta)|.$$

Similarly, if $a \geq 0$, $x^a |1 - \mathrm{e}^{ib\log x}| \leq |b|(x^{-\eta} - 1)/\eta$ and

$$|\psi(a+ib)| \leq |\psi(a)| + \frac{|b|}{\eta} |\psi(-\eta)|.$$

Therefore, by choosing $\eta$ fixed and small enough, when $|b| \to \infty$, we get

$$|\psi(a+ib)| = \mathcal{O}(|b|). \tag{4.19}$$

In light of (4.17) and (4.19), for any $s > -\rho(\mu)/\gamma$ and positive $\varepsilon$, for all $t \in \mathbb{R}$

$$|\mathcal{M}_I(s+it)| \leq C' \exp\big(-[(1-\alpha)\pi/2 - \varepsilon] \cdot |t|\big), \tag{4.20}$$

for some constant $C'$ which depends on $s$ and $\varepsilon$, i.e. (4.18) also holds for $s \in ]-\rho(\mu)/\gamma, 0]$.

Let $z = r\mathrm{e}^{i\omega}$ with $r > 0$ and $|\omega| < (1-\alpha)\pi/2$. Fix $\varepsilon \in ]0, (1-\alpha)\pi/2 - |\omega|[$. Equation (4.20) ensures that for any $s > -\rho(\mu)/\gamma$, we have

$$\int_{-\infty}^{\infty} |\mathcal{M}_I(s+it) z^{-s-it}|\, \mathrm{d}t \leq \frac{C'}{r^s} \int_{-\infty}^{\infty} \exp\big(-[(1-\alpha)\pi/2 - \varepsilon] \cdot |t| + \omega t\big)\, \mathrm{d}t \leq \frac{C''}{r^s} \tag{4.21}$$

where $C''$ denotes yet another constant which only depends on $s$. In particular, the integral on the left hand side of (4.21) is absolutely convergent. We now deduce from the Mellin inversion theorem that the density $k$ of $I$ coincides on $\{z : |z| > 0, |\arg z| < (1-\alpha)\pi/2\}$ with a holomorphic function, which we will also denote by $k$, given by

$$k(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}_I(s+it) z^{-s-it} \, \mathrm{d}t \tag{4.22}$$

for any appropriate $z$ and where $s$ is any real number in $]-\rho(\mu)/\gamma, \infty[$. Observe that (4.21) and (4.22) also ensure that, provided $\lambda \in ]-\infty, \rho(\mu)/\gamma[$, there is a finite constant $C_\lambda$ such that $|k(z)| \leq C_\lambda |z|^\lambda$ for all complex $z$ with $\mathrm{Re}\, z > 0$ and $|\arg z| < (1-\alpha)\pi/2$. □

*Remark 4.3.5.* We point out that [104, Theorem 2.3 (2)] only deals with $\alpha > 0$. Yet the same proof can extend to $\alpha = 0$.

*Proof of Proposition 4.3.1.* Lemma 4.3.5 ensures that for any real $t$, the function $z \mapsto \mathrm{e}^{itz} k(z)$ is holomorphic on the domain $\{z : |z| > 0, |\arg z| < (1-\alpha)\pi/2\}$. Therefore, for any $\omega \in ]0, (1-\alpha)\pi/2[$ and $0 < r < R$, if

$$C := [r, R] \cup \{R\mathrm{e}^{i\sigma} : 0 < \sigma < \omega\} \cup \{u\mathrm{e}^{i\omega} : r < u < R\} \cup \{r\mathrm{e}^{i\sigma} : 0 < \sigma < \omega\},$$

see Figure 4.1, the Cauchy integral theorem ensures that $\oint_C \mathrm{e}^{itz} k(z) \, \mathrm{d}z = 0$ from which we deduce that

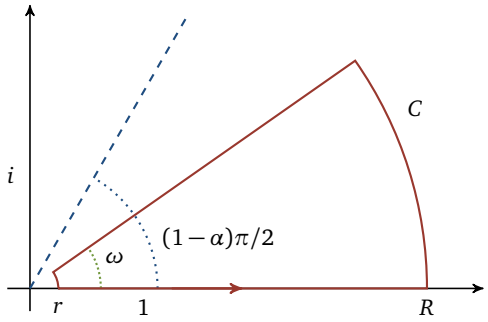



Figure 4.1 – The contour $C$.

$$\int_r^R \mathrm{e}^{itu} k(u) \mathrm{d}u - \mathrm{e}^{i\omega} \int_r^R \mathrm{e}^{itue^{i\omega}} k(u\mathrm{e}^{i\omega}) \, \mathrm{d}u = ir \int_0^\omega \mathrm{e}^{itre^{i\sigma}} k(r\mathrm{e}^{i\sigma}) \mathrm{d}\sigma - iR \int_0^\omega \mathrm{e}^{itRe^{i\sigma}} k(R\mathrm{e}^{i\sigma}) \mathrm{d}\sigma.$$

Recall from Lemma 4.3.5 that there is a finite constant $C$ such that $|k(z)| \leq C \cdot (1 \wedge |z|^{-2})$ for all $z$ with $|z| > 0$ and $|\arg z| < (1-\alpha)\pi/2$. As a result

$$\left| ir \int_0^\omega \mathrm{e}^{itre^{i\sigma}} k(r\mathrm{e}^{i\sigma}) \, \mathrm{d}\sigma \right| \leq r \int_0^\omega \mathrm{e}^{-tr\sin(\sigma)} |k(r\mathrm{e}^{i\sigma})| \, \mathrm{d}\sigma = \mathcal{O}(r)$$

when $r \to 0$. Similarly, when $R \to \infty$

$$\left| iR \int_0^\omega \mathrm{e}^{itRe^{i\sigma}} k(R\mathrm{e}^{i\sigma}) \, \mathrm{d}\sigma \right| \leq R \int_0^\omega \mathrm{e}^{-tR\sin(\sigma)} |k(R\mathrm{e}^{i\sigma})| \, \mathrm{d}\sigma = \mathcal{O}(R^{-1})$$

because $\sin\sigma > 0$ for all $\sigma \in ]0, \omega[$. Consequently,

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \int_r^R \mathrm{e}^{itu} k(u) \mathrm{d}u - \mathrm{e}^{i\omega} \int_r^R \mathrm{e}^{itue^{i\omega}} k(u\mathrm{e}^{i\omega}) \mathrm{d}u = 0.$$

Because of Lemma 4.3.5, it is easy to see that the two integrals are absolutely convergent, so that

$$\varphi_I(t) := \int_0^\infty \mathrm{e}^{itu} k(u) \mathrm{d}u = \mathrm{e}^{i\omega} \int_0^\infty \mathrm{e}^{itue^{i\omega}} k(u\mathrm{e}^{i\omega}) \mathrm{d}u.$$

As a result, for any $\theta \in [-1, \rho(\mu)/\gamma[$ and any $t \in \mathbb{R}$, Lemma 4.3.5 and a change of variables give

$$|\varphi_I(t)| \leq \int_0^\infty \mathrm{e}^{-tu\sin(\omega)} |k(u\mathrm{e}^{i\omega})| \mathrm{d}u \leq C_\theta \int_0^\infty \mathrm{e}^{-tu\sin(\omega)} u^\theta \mathrm{d}u \leq \frac{C'_\theta}{t^{1+\theta}}$$

for some finite constant $C'_\theta$ and all real $t$. □

We now turn to the proof of the converse to Proposition 4.3.1.

*Proof of Proposition 4.3.2.* Under these assumptions, for any $b$ in $]0, a[$

$$\int_{-\infty}^{\infty} |t|^b \, |\varphi_I(t)| \, \mathrm{d}t \leq \int_{-\infty}^{\infty} |t|^b \cdot 1 \wedge C|t|^{-1-a} \, \mathrm{d}t < \infty.$$

Classical Fourier theory results then ensure that for any non-negative integer $d$ and $\theta$ in $]0, 1]$ with $d+\theta < a$, the density $k$ of $I$ is $d$ times continuously differentiable on $\mathbb{R}$ and that for any two real numbers $x$ and $y$,

$$\begin{aligned} \left|k^{(d)}(x) - k^{(d)}(y)\right| &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \overbrace{|\mathrm{e}^{-itx} - \mathrm{e}^{-ity}|}^{\leq h_\theta |tx-ty|^\theta} |t|^d \, |\varphi_I(t)| \mathrm{d}t \\ &\leq \frac{h_\theta}{2\pi} |x-y|^\theta \int_{-\infty}^{\infty} |t|^{d+\theta} \, |\varphi_I(t)| \mathrm{d}t \leq C_{d+\theta} |x-y|^\theta. \end{aligned}$$

where $h_\theta$ and $C_{d+\theta}$ are two finite constants which only depend on $d$ and $\theta$. In other words, $k^{(d)}$ is Hölder continuous with exponent $\theta$.

Recall that $k, k', \ldots, k^{(d)}$ are continuous and supported by $\mathbb{R}_+$ so that $k^{(i)}(0) = 0$ for each $i \in \{0, \ldots, d\}$. Taylor's inequality then yields that $|k(x)| = \mathcal{O}(x^{d+\theta})$ as $x \to 0$. As a result, if $b < d + \theta$, $\mathbb{E}[I^{-1-b}] < \infty$ so that necessarily, using Corollary 4.3.4, $b \leq \rho(\mu)/\gamma$. Because this holds for any $d + \theta < a$ and $b < d + \theta$, it follows that $a \leq \rho(\mu)/\gamma$. □

## 4.3.2 Regularity of the profile of fragmentation trees

Let $\nu$ be a dislocation measure and recall that the parameter $\rho(\nu)$ is defined in Remark 4.3.1. Define $\sigma_\nu(\gamma)$ as follows:

$$\sigma_\nu(\gamma) := \sup \left\{ a + b \, : \, a, b < 1 + \rho(\nu)/\gamma, \, \int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-a\gamma} s_j^{1-b\gamma} \mathbb{1}_{s_j>0} \, \nu(\mathrm{d}\mathbf{s}) < \infty \right\}. \tag{4.23}$$

Observe that $\sigma_\nu(\gamma)$ decreases when $\gamma$ increases. For any positive $a$ we have

$$\int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-a\gamma} s_j \mathbb{1}_{s_j>0} \, \nu(\mathrm{d}\mathbf{s}) \leq \int_{\mathcal{S}^\downarrow} \sum_{i\geq 1} (1-s_i) s_i^{1-a\gamma} \mathbb{1}_{s_i>0} \, \nu(\mathrm{d}\mathbf{s})$$

which is finite iff

$$\int_{\mathcal{S}^\downarrow} \sum_{i\geq 1} s_i (1 - s_i^{-a\gamma}) \mathbb{1}_{s_i>0} \, \nu(\mathrm{d}\mathbf{s}) > -\infty,$$

and thus, if $a < \rho(\nu)/\gamma$. As a result, $\sigma_\nu(\gamma) \geq \rho(\nu)/\gamma$ so if $\rho(\nu)$ is positive, $\sigma_\nu(\gamma) \to \infty$ when $\gamma \to 0$. Now, define $\kappa(\nu)$ as

$$\kappa(\nu) := \sup \left\{ \gamma : \sigma_\nu(\gamma) > 2 \right\} \tag{4.24}$$

and observe that $\kappa(\nu) \geq \rho(\nu)/2$.

*Remark 4.3.6.* In Section 4.3.4, we will see that there are dislocation measures $\nu$ such that $\kappa(\nu) = \rho(\nu)/2$, i.e. the lower bound is sometimes reached. On the other hand, there are also dislocation measures $\nu$ such that $\kappa(\nu) > \rho(\nu)/2$. In particular, if $\nu$ is $N$-ary, i.e. supported by $\{\mathbf{s} \in \mathcal{S}^\downarrow : s_{N+1} = 0\}$, we will prove that $\kappa(\nu) \geq (1/2) \wedge [2\rho(\nu)]$, see Proposition 4.3.13.

With these notations, we can state the main result of this section. Let us now assume that $\nu$, or rather the measure associated to $\nu$ as in (4.9), satisfies the conditions of Proposition 4.3.1.

**Theorem 4.3.6.** *Let $\gamma$ be positive and assume that $\gamma < 1/2 \wedge \kappa(\nu)$. Then, if $\mathcal{T}$ is a $(\gamma, \nu)$-fragmentation tree, it a.s. has a profile $L$. Furthermore, for all non-negative integer $d$ and $\theta \in ]0,1]$ such that $d + \theta < [\sigma_\nu(\gamma) \wedge \gamma^{-1}]/2 - 1$, $L$ is $d$ times continuously differentiable and $L^{(d)}$ is $\theta$-Hölder continuous. In particular, $\mathcal{T}$ a.s. admits a continuous profile.*

*Proof.* In [60, Theorem 4 $(i)$] Fourier inversion was used to prove the existence of the profile of $\mathcal{T}$. We will proceed similarly to get some results on its regularity.

Conditionally on $\mathcal{T}$, let $V$ and $W$ be two leaves in $\mathcal{T}$ drawn independently following $\mu_\mathcal{T}$. Let $\lambda_1$ and $\lambda_2$ be the measures of the sub-trees of $\mathcal{T}$ above $U \wedge V$, the most recent common ancestor of $U$ and $V$, containing $U$ and $V$ respectively. In other words, let

$$\lambda_1 := \mu_\mathcal{T}\Big(x \in \mathcal{T} : |U \wedge x| \geq |U \wedge V|\Big)$$
$$\text{and} \quad \lambda_2 := \mu_\mathcal{T}\Big(x \in \mathcal{T} : |V \wedge x| \geq |U \wedge V|\Big).$$

Now set $I := |V|$ and $J := |W|$ and recall that conditionally on $\mathcal{T}$, the law of $I$ is $\mathrm{d}M$. Finally, let $\tilde{I}, \tilde{J}$ be i.i.d., independent of $\mathcal{T}$ and distributed like $I$. Observe that because of the self-similar fragmentation property,

$$I - J = \Big(I - |U \wedge V|\Big) - \Big(J - |U \wedge V|\Big) \overset{(\mathrm{d})}{=} \lambda_1^\gamma \tilde{I} - \lambda_2^\gamma \tilde{J}. \tag{4.25}$$

For any real $u$, set $\Phi(u) := \mathbb{E}[\mathrm{e}^{iuI}|\mathcal{T}]$. The function $\Phi$ is simply the (random) characteristic function of $I$ (and that of $J$) conditionally on $\mathcal{T}$, i.e. the Fourier transform of the probability measure $\mathrm{d}M$ conditionally on $\mathcal{T}$. For any $u \in \mathbb{R}$, in light of (4.25),

$$\begin{aligned}\mathbb{E}\Big[|\Phi(u)|^2\Big] &= \mathbb{E}\Big[\mathbb{E}[\mathrm{e}^{iuI}|\mathcal{T}] \cdot \mathbb{E}[\mathrm{e}^{-iuJ}|\mathcal{T}]\Big] = \mathbb{E}\Big[\exp\Big(iu(I-J)\Big)\Big] \\ &= \mathbb{E}\Big[\exp\Big(iu(\lambda_1^\gamma \tilde{I} - \lambda_2^\gamma \tilde{J})\Big)\Big] \leq 2\mathbb{E}\Big[|\varphi_I(\lambda_1^\gamma u)| \cdot |\varphi_I(\lambda_2^\gamma u)| \mathbb{1}_{\lambda_1 \geq \lambda_2}\Big].\end{aligned}$$

Using Proposition 4.3.1, we get that if $a$ and $b$ are in $[0, 1 + \rho(\nu)/\gamma[$, then for any real $u$,

$$\mathbb{E}\Big[|\Phi(u)|^2\Big] \leq C\mathbb{E}\Big[(\lambda_1^\gamma |u|)^{-a} \cdot (\lambda_2^\gamma |u|)^{-b} \mathbb{1}_{\lambda_1 \geq \lambda_2}\Big] = \frac{C}{u^{a+b}} \mathbb{E}\Big[\lambda_1^{-a\gamma} \lambda_2^{-b\gamma} \mathbb{1}_{\lambda_1 \geq \lambda_2}\Big]$$

where $C$ denotes a constant which only depends on $a$ and $b$. In light of [60, Lemma 2], this last quantity is finite iff $a + b < 1/\gamma$ and

$$\int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-a\gamma} s_j^{1-b\gamma} \, \nu(\mathrm{d}\mathbf{s}) < \infty. \tag{4.26}$$

By assumption $\gamma < 1/2 \wedge \kappa(\nu)$ so we can find $a, b < 1 + \rho(\nu)/\gamma$ with $2 < a + b < 1/\gamma$ and such that (4.26) holds. As a result,

$$\sup_{u \in \mathbb{R}} |u|^{(a+b)/2} \mathbb{E}\Big[|\Phi(u)|\Big] \leq \sup_{u \in \mathbb{R}} \Big(|u|^{a+b} \mathbb{E}\Big[|\Phi(u)|^2\Big]\Big)^{1/2} < \infty$$

which ensures that

$$\mathbb{E}\Big[\int_{-\infty}^{\infty} |u|^{d+\theta} |\Phi(u)| \mathrm{d}u\Big] < \infty$$

for any non-negative integer $d$ and $\theta \in ]0,1]$ with $d + \theta < (a+b)/2 - 1$. Consequently for all $t \in \mathbb{R}$,

$$L(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-iut} \Phi(u) \mathrm{d}u$$

is a.s. well defined, $d$ times continuously differentiable and $L^{(d)}$ is $\theta$-Hölder continuous. Finally, the Fourier inversion theorem ensures that $\Phi$ is the Fourier transform of $L$ and therefore, $\mathrm{d}M(t) = L(t)\,\mathrm{d}t$. In particular, $L$ is the profile of $\mathcal{T}$. □

*Remark 4.3.7.* It is known that the Brownian tree, which is a fragmentation tree with self-similarity index $1/2$, a.s. admits a continuous profile, see Section 3.1. In particular, the assumption $\gamma < 1/2$ of Theorem 4.3.6 is not optimal. Yet we do not known whether it could be lightened significantly with a different approach. See Appendix A for simulations intended to shed light on this matter.

*On the expectation of $L$.* Since the process $L$ is the random density of $I$ conditionally on $\mathcal{T}$, it would seem intuitive for the expectation of $L$ to be the density $k$ of $I$. We will end this section by proving this fact.

**Lemma 4.3.7.** $(i)$ *For almost every $t \geq 0$, $\mathbb{E}[L(t)] = k(t)$.*
$(ii)$ *Under the assumptions of Theorem 4.3.6, $\mathbb{E}[L(t)] = k(t)$ for all $t \geq 0$.*

*Proof.* $(i)$ For all non-negative $t$ we have

$$\begin{aligned}\int_0^t k(s)\,\mathrm{d}s = \mathbb{P}[I \leq t] = \mathbb{E}\big[\mathbb{P}[I \leq t|\mathcal{T}]\big] = \mathbb{E}[M(t)]\\ = \mathbb{E}\Big[\int_0^t L(s)\,\mathrm{d}s\Big] = \int_0^t \mathbb{E}[L(s)]\,\mathrm{d}s\end{aligned}$$

where we used the Fubini-Tonelli theorem at the end. As a result, $k = \mathbb{E}[L(\cdot)]$ for almost every non-negative $t$.
$(ii)$ Under the assumptions of Theorem 4.3.6, as we saw in its proof, the characteristic function of $I$ conditionally on $\mathcal{T}$ is integrable with respect to the product measure $\mathrm{d}u \otimes \mathrm{d}\mathbb{P}$. As a result,

$$\mathbb{E}\Big[\sup_{t\geq 0} L(t)\Big] = \frac{1}{2\pi}\mathbb{E}\Big[\sup_{t\geq 0}\int_{-\infty}^{\infty} \mathrm{e}^{-iut}\,\mathbb{E}[\mathrm{e}^{iuI}|\mathcal{T}]\,\mathrm{d}u\Big] \leq \frac{1}{2\pi}\mathbb{E}\Big[\int_{-\infty}^{\infty}\big|\mathbb{E}[\mathrm{e}^{iuI}|\mathcal{T}]\big|\,\mathrm{d}u\Big] < \infty.$$

The continuity under the integral sign theorem then ensures that the function $t \mapsto \mathbb{E}[L(t)]$ is continuous and by $(i)$, we know that this function coincides with $k$, another continuous function, almost everywhere. Therefore, $\mathbb{E}[L(t)] = k(t)$ for all $t \geq 0$. □

### 4.3.3 Unboundedness of the profile

When a fragmentation tree has a profile, it is not necessarily càdlàg. This paragraph aims to provide a criterion to ensure that even if a fragmentation tree has a profile, it may be locally unbounded and therefore not càdlàg.

In this section, $(\gamma, \nu)$ will be a fixed fragmentation pair. Let $(\mathcal{T}, \mu_{\mathcal{T}})$ be a $(\gamma, \nu)$ fragmentation tree and denote by $M$ the function $t \mapsto \mu_{\mathcal{T}}(\mathcal{T}|_t)$. Also denote by $\mathbf{F}$ the underlying fragmentation process encoded by this tree and let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $\mathbf{F}$.

**Proposition 4.3.8.** *Let $\gamma < 1$ and suppose that there exists some positive $\alpha < \gamma$ such that*

$$\int_{\mathcal{S}^\downarrow}(s_1^{-1} - 1)\,\nu(\mathrm{d}\mathbf{s}) < \infty \quad \text{and} \quad \nu\Big(\sum_{i\geq 1} s_i^{1-\alpha} = \infty\Big) > 0.$$

*Then the varying rate of $M$ is a.s. not locally bounded.*

*Remark 4.3.8.* Under the assumptions of Proposition 4.3.8, the tree $\mathcal{T}$ cannot have a càdlàg profile.

On the event $\{\mathcal{T}$ admits a profile$\}$, let $L$ denote the said profile. If $0 \leq s \leq t$ and $\varepsilon \in ]0, 1]$, then

$$\frac{1}{\varepsilon}\big[M(s+\varepsilon) - M(s)\big] = \frac{1}{\varepsilon}\int_s^{s+\varepsilon} L(u)\,\mathrm{d}u \leq \sup_{0\leq u\leq t+1} L(u). \tag{4.27}$$

On the event $\{L \in \mathcal{D}[0, \infty[\}$, $L$ is bounded on compacts and the varying rate of $M$ is therefore locally bounded. Proposition 4.3.8 ensures that the probability for this to happen is 0. Consequently,

$$\mathbb{P}\big[\text{“}L \text{ exists and is càdlàg”}\big] = 0.$$

We will need the next few lemmas in the proof of Proposition 4.3.8.

**Lemma 4.3.9.** *Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that $x_n \to 0$ and $\sum x_n = \infty$. Then, there exists a subset $J$ of positive integers satisfying*

$$\sum_{j\in J} x_j = \infty \quad \textit{and} \quad \forall \varepsilon > 0,\ \sum_{j\in J} x_j^{1+\varepsilon} < \infty.$$

*Proof.* The assumptions on $(x_n)_n$ ensure that we may find a sequence $(J_k)_{k\geq 1}$ of pairwise disjoint sets of integers such that for all $k \geq 1$,

$$1/k \leq \sum_{j\in J_k} x_j \leq 2/k.$$

As a result, if we set $J = \bigcup_{k\geq 1} J_k$, then on the one hand

$$\sum_{j\in J} x_j = \sum_{k\geq 1} \sum_{j\in J_k} x_j \geq \sum_{k\geq 1} 1/k = \infty,$$

and on the other, for any positive $\varepsilon$,

$$\sum_{j\in J} x_j^{1+\varepsilon} = \sum_{k\geq 1} \sum_{j\in J_k} x_j^{1+\varepsilon} \leq \sum_{k\geq 1} \Big(\sum_{j\in J_k} x_j\Big)^{1+\varepsilon} \leq \sum_{k\geq 1} (2/k)^{1+\varepsilon} < \infty. \qquad \square$$

**Lemma 4.3.10.** *Let $\chi$ be a finite measure on $[0,\infty[$. Then,*

$$\chi\big([0,\varepsilon]\big) = \mathcal{O}(\varepsilon^\beta) \quad \Longrightarrow \quad \forall \alpha \in (0,\beta),\ \int_{[0,\infty[} u^{-\alpha}\chi(\mathrm{d}u) < \infty.$$

*Proof.* By assumption, there exists a non-negative constant $C$ such that $\chi\big([0,\varepsilon]\big) \leq C\varepsilon^\beta$ for all $\varepsilon > 0$. As a result, we may write

$$\begin{aligned}\int_{[0,\infty[} u^{-\alpha}\,\chi(\mathrm{d}u) &= \int_{[0,\infty[\times[0,\infty[} \mathbb{1}_{t\leq u^{-\alpha}}\,\mathrm{d}t\,\chi(\mathrm{d}u) = \int_0^\infty \chi\big([0,t^{-1/\alpha}]\big)\,\mathrm{d}t \\ &\leq \chi([0,\infty[) + C\int_1^\infty t^{-\beta/\alpha}\mathrm{d}t\end{aligned}$$

which is finite provided $0 < \alpha < \beta$. $\square$

**Lemma 4.3.11.** *Under the assumptions of Proposition 4.3.8, there exists an a.s. positive and finite stopping time $T$ such that with probability 1,*

$$\sum_{i\geq 1} F_i(T)^{1-\alpha} = \infty.$$

*Proof.* Recall the Poissonian construction of homogeneous fragmentation processes and the subsequent time change to make them self-similar with index $-\gamma$ (see Chapter 3 in [21] for instance).

Because of the assumptions on $\nu$, we can find a positive $\varepsilon$ such that

$$\nu\Big(1-s_1 > \varepsilon, \sum_{i\geq 1} s_i^{1-\alpha} = \infty\Big) > 0.$$

Set $A := \big\{1-s_1 > \varepsilon, \sum_{i\geq 1} s_i^{1-\alpha} = \infty\big\}$. Conditionally on $\mathbf{F}$ tag a fragment chosen uniformly at random and observe that the first time $T$ at which the said tagged fragment splits into sub-particles with relative sizes in $A$ is both finite and positive. Moreover, $T$ is clearly a stopping time and its definition entails that $\sum_{i\geq 1} F_i(T)^{1-\alpha} = \infty$ almost surely. $\square$

**Lemma 4.3.12.** *Let $\zeta := |\mathcal{T}|$ be the height of $\mathcal{T}$. For all $\theta < 1 + 1/\gamma$, we have $\mathbb{E}[\zeta^{-\theta}] < \infty$.*

*Proof.* Conditionally on $\mathbf{F}$, tag the particle which belongs to the biggest fragment at each splitting time along its lifeline. Set $I^\circ$ as the first time the said fragment vanishes. The random variable $I^\circ$ is the height of an element of $\mathcal{T}$ so $\zeta \geq I^\circ$.

Let $\mu^\circ$ be the measure on $]0,1[$ defined by

$$\int_{]0,1[} f\,\mathrm{d}\mu^\circ := \int_{\mathcal{S}^\downarrow} f(s_1)\,\nu(\mathrm{d}\mathbf{s})$$

for any non-negative and measurable $f$ and let $\xi^\circ$ be a subordinator with Laplace exponent

$$q \longmapsto \int_{]0,1[} (1-x^q)\,\mu^\circ(\mathrm{d}x).$$

The random variable $I^\circ$ is distributed like the exponential functional of $\gamma\,\xi^\circ$. Corollary 4.3.4 now ensures that $\mathbb{E}[\zeta^{-\theta}] \leq \mathbb{E}[(I^\circ)^{-\theta}] < \infty$ for all $\theta \in [0, 1+\rho(\mu^\circ)/\gamma[$. Finally, by assumption on $\nu$, $\rho(\mu^\circ) \geq 1$. $\square$

We can now turn to the proof of Proposition 4.3.8.

*Proof of Proposition 4.3.8.* Lemma 4.3.11 entails that we can find a finite and positive stopping time $T$ such that $\sum_{i\geq 1} F_i(T)^{1-\alpha} = +\infty$ a.s. We will now endeavour to prove that the varying rate of $M$ is a.s. unbounded around $T$.

Let $\eta$ be positive and such that $\gamma-\alpha-(1-\gamma)<\eta<\gamma-\alpha$. The strong Markov property ensures that there are i.i.d. $(\gamma,\nu)$ fragmentation processes $\mathbf{F}^{(i)}$, $i\geq 1$ independent of $\mathcal{F}_T$ such that a.s.,

$$\Big(\big[F_j(T+s)\,;\,j\geq 1\big],s\geq 0\Big)=\left(\Big[F_i(T)F_j^{(i)}\big(F_i(T)^{-\gamma}s\big)\,;\,i\geq 1,j\geq 1\Big]^{\downarrow},s\geq 0\right).$$

For all $i\geq 1$, let $M^{(i)}$ be the process defined by $M^{(i)}(t)=1-\sum_{j\geq 1}F_j^{(i)}(t)$ as well as $\zeta^{(i)}$ the extinction time of $\mathbf{F}^{(i)}$. With these notations, for any positive $\varepsilon$,

$$\begin{aligned}M(T+\varepsilon)-M(T)&=\textstyle\sum_{i\geq 1}F_i(T)\cdot M^{(i)}\big[F_i(T)^{-\gamma}\varepsilon\big]\\&\geq\textstyle\sum_{i\geq 1}F_i(T)\cdot\mathbb{1}_{\zeta^{(i)}\leq F_i(T)^{-\gamma}\varepsilon}\\&\geq\textstyle\sum_{i\in J}F_i(T)\cdot\mathbb{1}_{\zeta^{(i)}\leq F_i(T)^{-\gamma}\varepsilon}=:N_J(\varepsilon)\end{aligned}\tag{4.28}$$

where $J$ is chosen in such a way that $\sum_{i\in J}F_i(T)^{\beta}<\infty$ iff $\beta>1-\alpha-\eta$, see Lemma 4.3.9.

Observe that

$$\begin{aligned}\mathbb{E}\big[N_J(\varepsilon)\,|\,\mathcal{F}_T\big]&=\textstyle\sum_{i\in J}F_i(T)\,\mathbb{E}\big[\mathbb{1}_{F_i(T)^{\gamma}\zeta^{(i)}\leq\varepsilon}\,|\,\mathcal{F}_T\big]\\&=\Big(\textstyle\sum_{i\in J}F_i(T)\,\mathbb{E}\big[\delta_{F_i(T)^{\gamma}\zeta^{(1)}}|\mathcal{F}_T\big]\Big)\big([0,\varepsilon]\big)=:\Theta_J\big([0,\varepsilon]\big)\end{aligned}$$

and that the (random) measure $\Theta_J$ hence defined is finite. Moreover, for any positive $\beta$,

$$\int x^{-\beta}\Theta_J(\mathrm{d}x)=\mathbb{E}[(\zeta^{(1)})^{-\beta}]\textstyle\sum_{i\in J}F_i(T)^{1-\beta\gamma}.$$

Taking $\beta=(\alpha+\eta)/\gamma$, in light of Lemma 4.3.10 we get that $\limsup\varepsilon^{-\beta'}\Theta_J\big([0,\varepsilon]\big)=\infty$ for any $\beta'>\beta$ as $\varepsilon\to 0$. In particular, since $(\alpha+\eta)/\gamma<1$,

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\big[N_J(\varepsilon)\,|\,\mathcal{F}_T\big]=\infty\ \text{ a.s.}\tag{4.29}$$

For all positive $\beta$, the Markov inequality ensures that

$$\begin{aligned}\mathrm{Var}\big(N_J(\varepsilon)/\varepsilon\,|\,\mathcal{F}_T\big)&\leq\varepsilon^{-2}\textstyle\sum_{i\in J}F_i(T)^2\mathbb{E}\big[\mathbb{1}_{F_i(T)^{\gamma}\zeta^{(i)}>\varepsilon}\,|\,\mathcal{F}_T\big]\\&\leq\mathbb{E}[(\zeta^{(1)})^{-\beta}]\varepsilon^{\beta-2}\textstyle\sum_{i\in J}F_i(T)^{2-\gamma\beta}\end{aligned}$$

and because of the definition of $J$, $\sum_{i\in J}F_i(T)^{2-\gamma\beta}<\infty$ iff $2-\gamma\beta>1-\alpha-\eta$. Now observe that $\eta$ was chosen in such a way that $2(1-\gamma)>1-\alpha-\eta$. In particular and thanks to Lemma 4.3.12, we can find $\beta>2$ small enough such that $\sum_{i\in J}F_i(T)^{2-\gamma\beta}<\infty$ and $\mathbb{E}[(\zeta^{(1)})^{-\beta}]<\infty$. Consequently, we see that

$$\mathrm{Var}\big(N_J(\varepsilon)/\varepsilon\,|\,\mathcal{F}_T\big)\xrightarrow[\varepsilon\to 0]{}0.\tag{4.30}$$

As a result of (4.29) and (4.30), we get that almost surely

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}N_J(\varepsilon)=+\infty$$

which, thanks to (4.28), concludes this proof. □

### 4.3.4 Applications

This section provides applications of our main results, namely Proposition 4.3.1, Theorem 4.3.6 and Proposition 4.3.8, to various fragmentation models. To that effect, if $\nu$ is a given dislocation measure, recall the definitions of $\rho(\nu)$ and $\kappa(\nu)$ from Remark 4.3.1 and Equation (4.24) respectively.

**$N$-ary fragmentation.** Let $\nu$ be a dislocation measure and suppose that there exists an integer $N \geq 2$ such that $\nu(s_{N+1} > 0) = 0$. Let us assume that $\nu$ meets the assumptions of Proposition 4.3.1. In particular, $\rho(\nu)$ is positive.

**Proposition 4.3.13.** *Let $\gamma \in ]0,1[$ and suppose $\mathcal{T}$ is a $(\gamma, \nu)$-fragmentation tree. Then $\mathcal{T}$ a.s. has a profile and if $\gamma < (1/2) \wedge [2\rho(\nu)]$, then this profile is a.s. continuous.*

*Remark 4.3.9.* The first claim of Proposition 4.3.13 is Corollary 5 in [60].

*Proof.* For any two non-negative real numbers $a$ and $b$, if $a \leq 1$ we get

$$\int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-a} s_j^{1-b} \mathbb{1}_{s_j>0}\, \nu(\mathrm{d}\mathbf{s}) \leq (N-1) \int_{\mathcal{S}^\downarrow} \sum_{j\geq 2} s_j^{1-b} \mathbb{1}_{s_j>0}\, \nu(\mathrm{d}\mathbf{s}).$$

Observe that

$$\begin{aligned}\int_{\mathcal{S}^\downarrow} \sum_{j\geq 2} s_j^{1-b} \mathbb{1}_{s_j>0}\, \nu(\mathrm{d}\mathbf{s}) &= \int_{\mathcal{S}^\downarrow} \Big(1 - s_1 + \sum_{j\geq 2} s_j (s_j^{-b} - 1)\mathbb{1}_{s_j>0}\Big)\nu(\mathrm{d}\mathbf{s}) \\ &\leq \int_{\mathcal{S}^\downarrow} (1-s_1)\, \nu(\mathrm{d}\mathbf{s}) + \int_{\mathcal{S}^\downarrow} \sum_{j\geq 1} s_j |1 - s_j^{-b}| \mathbb{1}_{s_j>0}\, \nu(\mathrm{d}\mathbf{s})\end{aligned}$$

which by definition of $\rho(\nu)$ is finite for all $b < \rho(\nu)$. In particular and because we made the assumption that $\gamma < 1$, $\int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-\gamma} s_j\, \nu(\mathrm{d}\mathbf{s}) < \infty$. Moreover, we get that

$$\sigma_\nu(\gamma) \geq \frac{1}{\gamma} \sup\Big\{a + b : a < 1 \wedge [\gamma + \rho(\nu)], b \leq \rho(\nu)\Big\} = \frac{[\gamma + 2\rho(\nu)] \wedge [1 + \rho(\nu)]}{\gamma}.$$

If $\gamma < 1/2$, we get that $\sigma_\nu(\gamma) \geq 2$ provided $\gamma \leq 2\rho(\nu)$. As a result, $\kappa(\nu) \geq (1/2) \wedge [2\rho(\nu)]$. We may now conclude this proof with the help of Theorem 4.3.6. □

There are a few notable examples of $N$-ary dislocation measures.

*Ford's $\alpha$-model.* Let $\alpha \in (0,1)$. Ford's $\alpha$-model was introduced in [52]. The aim was to build random binary trees with a given number of leaves to model phylogenetic trees. The algorithm described was a generalisation of Rémy's algorithm [110]. The scaling limits of this model were studied in [67, Section 5.2] in the context of fragmentation trees.

The law of the corresponding fragmentation trees is characterised by the pair $(\alpha, \nu_\alpha^{(\mathrm{F})})$ where $\nu_\alpha^{(\mathrm{F})}$ is the binary dislocation measure defined for all measurable $f : \mathcal{S}^\downarrow \to \mathbb{R}_+$ by

$$\int_{\mathcal{S}^\downarrow} f\, \mathrm{d}\nu_\alpha^{(\mathrm{F})} = \frac{1}{\Gamma(1-\alpha)} \int_{1/2}^1 \left(\frac{\alpha}{[x(1-x)]^{1+\alpha}} + \frac{2-4\alpha}{[x(1-x)]^\alpha}\right) f(x, 1-x, 0, 0, \dots)\, \mathrm{d}x.$$

Clearly, $\rho(\nu_\alpha^{(\mathrm{F})}) = 1 - \alpha$. Furthermore, the measure $\mu$ associated to $\nu_\alpha^{(\mathrm{F})}$ as in (4.9) is given by

$$\mu(\mathrm{d}x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\alpha}{x^\alpha (1-x)^{1+\alpha}} + \frac{2-4\alpha}{x^{\alpha-1}(1-x)^\alpha}\right) \mathrm{d}x.$$

Therefore, as pointed out in Remark 4.3.2, the function $x \mapsto \mu(]0, 1-x])$ is regularly varying at 0 with exponent $-\alpha \in ]-1, 0[$ and is quasi-monotone. The assumptions of Proposition 4.3.1 are thus met and Proposition 4.3.13 holds.

In particular, if $\alpha < 1/2$, then an $(\alpha, \nu_\alpha^{(\mathrm{F})})$ fragmentation tree a.s. admits a continuous profile.

Note that when $\alpha = 1/2$, $(\alpha, \nu_\alpha^{(\mathrm{F})})$-fragmentation trees are simply (a multiple of) the Brownian tree, which a.s. admits a continuous profile.

*Aldous' $\beta$-model.* These binary random trees were introduced in [10, Section 4]. In [67, Section 5.1], it was proved that when the parameter $\beta$ is in $]-2,-1[$, the corresponding scaling limits are $(|\beta|-1, \nu_\beta^{(\mathrm{B})})$ fragmentation trees where $\nu_\beta^{(\mathrm{B})}$ is the dislocation measure such that for all measurable $f : \mathcal{S}^\downarrow \to \mathbb{R}_+$,

$$\int f \,\mathrm{d}\nu_\beta^{(\mathrm{B})} := \frac{-1-\beta}{\Gamma(2+\beta)} \int_0^{1/2} x^\beta (1-x)^\beta f(1-x, x, 0, 0, \dots)\,\mathrm{d}x.$$

A simple computation gives $\rho(\nu_\beta^{(\mathrm{B})}) = 2-|\beta|$. Moreover, the measure $\mu$ associated to $\nu_\beta^{(\mathrm{B})}$ like in (4.9) is given by

$$\mu(\mathrm{d}x) = \frac{1}{\Gamma(2+\beta)} x^{1+\beta}(1-x)^\beta \,\mathrm{d}x.$$

Remark 4.3.2 then ensures that the function $x \mapsto \mu(]0, 1-x])$ is regularly varying at 0 with exponent $1-|\beta| \in ]-1, 0[$ and quasi-monotone. The assumptions of Proposition 4.3.1 are met and Proposition 4.3.13 therefore holds.

In particular, if $\beta \in ]-3/2, -1[$, then a $(|\beta|-1, \nu_\beta^{(\mathrm{B})})$ fragmentation tree a.s. admits a continuous profile.

*Haas and Stephenson's $k$-ary growing trees.* Let $k \geq 2$ be an integer. In [69], a model of $k$-ary trees built with an algorithm similar to Rémy's [110] was introduced. Under appropriate rescaling, these trees converge in distribution to a $(1/k, \nu_k^{\mathrm{GT}})$ fragmentation tree for some dislocation measure $\nu_k^{\mathrm{GT}}$ which we will now describe.

Let $\Pi$ be a $(k-1)$-dimensional Dirichlet variable with parameters $(1/k, \dots, 1/k)$. We define the dislocation measure $\nu_k^{\mathrm{GT}}$ such that for all measurable $f : \mathcal{S}^\downarrow_{\leq 1} \to \mathbb{R}_+$

$$\int_{\mathcal{S}^\downarrow_{\leq 1}} f \,\mathrm{d}\nu_k^{\mathrm{GT}} = \frac{\Gamma(1/k)}{k} \mathbb{E}\left[\frac{f\left[(\Pi, 0, 0, \dots)^\downarrow\right]}{1-\Pi_1}\right].$$

The measure $\mu$ built from $\nu_k^{\mathrm{GT}}$ as in Equation (4.9) satisfies

$$\int_{]0,1[} f \,\mathrm{d}\mu := \int_{\mathcal{S}^\downarrow} \sum_{i\geq 1} s_i f(s_i)\, \nu_k^{\mathrm{GT}}(\mathrm{d}\mathbf{s}) = \frac{\Gamma(1/k)}{k}\left(\mathbb{E}\left[\frac{\Pi_1}{1-\Pi_1} f(\Pi_1)\right] + \sum_{i=2}^{k} \mathbb{E}\left[\frac{\Pi_i}{1-\Pi_1} f(\Pi_i)\right]\right).$$

Recall that $\Pi_1$ has the Beta distribution with parameters $(1/k, 1-1/k)$ so

$$\mathbb{E}\left[\frac{\Pi_1}{1-\Pi_1} f(\Pi_1)\right] = \frac{1}{\mathrm{B}(1/k, 1-1/k)} \int_0^1 x^{1/k}(1-x)^{-1-1/k} f(x)\,\mathrm{d}x.$$

For $i = 2, \dots, k$, recall that $(\Pi_1, \Pi_i, 1-\Pi_1-\Pi_i)$ is a Dirichlet random vector with parameters $(1/k, 1/k, 1-2/k)$. As a result,

$$\begin{aligned}\mathbb{E}\left[\frac{\Pi_i}{1-\Pi_1} f(\Pi_i)\right] &= \frac{1}{\Gamma(1/k)^2\Gamma(1-2/k)} \int_0^1 \int_0^1 \frac{x^{1/k-1}}{1-x} y^{1/k}(1-x-y)^{-2/k} \mathbb{1}_{x+y<1} f(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{\Gamma(1/k)^2\Gamma(1-2/k)} \int_0^1 y^{1/k} f(y) \left(\int_0^{1-y} x^{1/k-1}(1-x-y)^{-2/k}\,\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Consequently, the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure and its density, say $g$, is

$$g(x) := \frac{1/k}{\Gamma(1-1/k)} x^{1/k}(1-x)^{-1-1/k} + \frac{(k-1)/k}{\Gamma(1/k)\,\Gamma(1-2/k)} x^{1/k} \int_0^{1-x} t^{1/k-1}(1-x-t)^{-2/k}\,\mathrm{d}t.$$

Furthermore, when $x \to 1$,

$$\int_0^{1-x} t^{1/k-1}(1-x-t)^{-2/k}\,\mathrm{d}t \sim (1-x)^{-2/k}$$

so $g(x) \sim C(1-x)^{-1-1/k}$ for some positive $C$ when $x \to 1$. Thanks to Remark 4.3.2, we may now state that $x \mapsto \mu(]0, 1-x])$ is regularly varying at 0 with exponent $-1/k \in ]0,1[$ and quasi-monotone. Moreover, it follows that $\rho(\nu_k^{\mathrm{GT}}) = 1 - 1/k$

As a result, if $k \geq 3$, $\nu_k^{\mathrm{GT}}$ satisfies the assumptions of Proposition 4.3.13. Therefore, a $(1/k, \nu_k^{\mathrm{GT}})$ fragmentation tree a.s. admits a continuous profile.

Our results cannot be used when $k = 2$. Yet, the dislocation measure $\nu_2^{\mathrm{GT}}$ is a multiple of the Brownian dislocation measure. As a result, when $k = 2$, $(1/k, \nu_k^{\mathrm{GT}})$ fragmentation trees are simply rescaled Brownian trees and therefore a.s. admit a continuous profile.

**Poisson-Dirichlet fragmentation.** We refer to Section 3 in [68]. Fix two real numbers $\alpha \in ]0,1[$ and $\theta > -2\alpha$.

Let $\Pi$ be a Poisson Point Process on $]0,\infty[$ with intensity measure $\alpha/\Gamma(1-\alpha) \cdot x^{-1-\alpha}\mathrm{d}x$. Campbell's theorem (see Section 3.2 in [84]) ensures that $T := \int_{]0,\infty[} x\,\Pi(\mathrm{d}x)$ is a.s. finite. Let $\Delta_1 > \Delta_2 > \dots$ be the atoms of $\Pi$ in decreasing order. Then we define the measure $\nu_{\alpha,\theta}^{\mathrm{PD}}$ by

$$\int_{\mathcal{S}^\downarrow} f \,\mathrm{d}\nu_{\alpha,\theta}^{\mathrm{PD}} := \mathbb{E}[T^{-\theta} f(\Delta_1/T, \Delta_2/T, \dots)]$$

for any measurable $f : \mathcal{S}^\downarrow \to \mathbb{R}_+$. Lemma 7 in [68] ensures that $\nu_{\alpha,\theta}^{\mathrm{PD}}$ is a dislocation measure and is finite iff $\theta > -\alpha$.

**Proposition 4.3.14.** *Let $\gamma$ be positive and let $\mathcal{T}$ denote a $(\gamma, \nu_{\alpha,\theta}^{\mathrm{PD}})$ fragmentation tree.*

*(i) We have $\rho(\nu_{\alpha,\theta}^{\mathrm{PD}}) = 1-\alpha$,*

*(ii) If $\gamma < 2(1-\alpha) \wedge 1$ then $\mathcal{T}$ a.s. admits a profile,*

*(iii) If $\gamma < (1-\alpha) \wedge 1/2$, then the profile of $\mathcal{T}$ is continuous with probability 1,*

*(iv) If $\gamma > 1-\alpha$, then the profile of $\mathcal{T}$, if it exists, is a.s. not càdlàg.*

*Remark 4.3.10.* This proposition shows in particular that there is a phase transition for the regularity of the profile of fragmentation trees in this model.

*Proof.* For convenience, in this proof, we will now write $\nu$ instead of $\nu_{\alpha,\theta}^{\mathrm{PD}}$.

The following measure will make some computations much easier. Let $\nu^*$ be the measure defined for all measurable and non-negative functions $f$ on the set $\mathcal{S} := \{\mathbf{x} \in \ell_1 : \sum_i x_i = 1, x_j \geq 0, j \geq 1\}$ by

$$\int_{\mathcal{S}} f(\mathbf{x})\,\mathrm{d}\nu^*(\mathbf{x}) := \int_{\mathcal{S}^\downarrow} \mathbb{E}[f(\mathbf{s}^*)]\,\nu(\mathrm{d}\mathbf{s})$$

where $\mathbf{s}^*$ is a sized biased ordering of $\mathbf{s}$ (see Section 2.1.3 in [21]). If $\theta > -\alpha$, Equations (13) and (15) in [68] ensure that $\nu^*$ is given by

$$\int_{\mathcal{S}} f(\mathbf{x})\,\mathrm{d}\nu^*(\mathbf{x}) := \frac{\Gamma(1+\theta/\alpha)}{\Gamma(1+\theta)} \mathbb{E}\Big[f\big(Y_1, Y_2, \dots\big)\Big] \tag{4.31}$$

where $(X_n)_n$ are independent $\mathrm{Beta}(1-\alpha, n\alpha+\theta)$ random variables respectively and $Y_n := \prod_{i=1}^{n-1}(1-X_i)X_n$.

As a result, if $\theta > -\alpha$, the measure $\mu$ which characterises the distribution of the process of the size of a tagged fragment in a $(\gamma, \nu)$ fragmentation process is given by

$$\begin{aligned}
\int_{]0,1[} f\,\mathrm{d}\mu &:= \int_{\mathcal{S}^\downarrow} \textstyle\sum_{i\geq 1} s_i f(s_i)\,\nu(\mathrm{d}\mathbf{s}) = \mathbb{E}\Big[T^{-\theta}\textstyle\sum_{i\geq 1}\Delta_i/T\, f(\Delta_i/T)\Big] \\
&= \int_{\mathcal{S}} f(x_1)\,\nu^*(\mathrm{d}\mathbf{x}) = \frac{\Gamma(1+\theta/\alpha)}{\Gamma(1+\theta)}\mathbb{E}\big[f(X_1)\big] \\
&= \frac{\alpha\Gamma(2+\theta/\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha+\theta)} \int_0^1 x^{-\alpha}(1-x)^{\alpha+\theta-1} f(x)\,\mathrm{d}x.
\end{aligned}$$

for all measurable $f : ]0,1[ \to \mathbb{R}_+$. By analytic continuation, for any $\theta > -2\alpha$, we get

$$\mu(\mathrm{d}x) := \frac{\alpha\Gamma(2+\theta/\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha+\theta)} x^{-\alpha}(1-x)^{\alpha+\theta-1}\mathbb{1}_{]0,1[}(x)\,\mathrm{d}x.$$

Therefore, either $\theta > -\alpha$ and $\mu$ is finite, or $\theta \leq -\alpha$ and $x \mapsto \mu(]0,1-x])$ is $\alpha+\theta \in ]0,1[$ regularly varying at 0 with exponent $\alpha+\theta \in ]-1,0]$ (see Remark 4.3.2). Moreover, for any $q \geq 0$,

$$\int_0^1 (1-x^{-q})\,\mu(\mathrm{d}x) > -\infty \iff \int_0^1 x^{-q-\alpha}(1-x)^{\alpha+\theta}\,\mathrm{d}x < \infty.$$

As a result, $\rho(\mu) = \rho(\nu) = 1-\alpha > 0$ which proves claim $(i)$. The dislocation measure $\nu$ therefore meets the conditions of Proposition 4.3.1.

Using (4.31) once again, for any non-negative $u$ and $w$

$$\begin{aligned}\int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-u} s_j^{1-w}\,\nu(\mathrm{d}\mathbf{s}) &= \int_{\mathcal{S}} x_1^{-u}(1-x_1)\,x_2^{-w}\mathbb{1}_{x_1\geq x_2}\,\nu^*(\mathrm{d}\mathbf{x}) \\ &= \frac{\Gamma(1+\theta/\alpha)}{\Gamma(1+\theta)}\mathbb{E}\Big[X_1^{-u}(1-X_1)^{1-w}X_2^{-w}\mathbb{1}_{(1-X_1)X_2\leq X_1}\Big] \\ &= \frac{\alpha\,\Gamma(2+\theta/\alpha)}{\Gamma(1-\alpha)^2\Gamma(2\alpha+\theta)}\int_0^1\int_0^1 \frac{(1-x)^{\alpha+\theta-w}}{x^{\alpha+u}}\frac{(1-y)^{2\alpha-\theta-1}}{y^{\alpha+w}}\mathbb{1}_{y\leq x/(1-x)}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.32}$$

By analytic continuation, the last equality of (4.32) holds for all $\theta > -2\alpha$. Observe that

$$\int_0^1 y^{-\alpha-w}(1-y)^{2\alpha+\theta-1}\mathbb{1}_{y\leq x/(1-x)}\mathrm{d}y \sim \int_0^x y^{-\alpha-w}\mathrm{d}y$$

when $x \to 0$ and thus that (4.32) is infinite if $w \geq 1-\alpha$. Otherwise, if $w < 1-\alpha$, then

$$\int \textstyle\sum_{i<j} s_i^{1-u}s_j^{1-w}\,\nu(\mathrm{d}\mathbf{s}) < \infty \iff \int_0^1 x^{2\alpha+u+w-1}(1-x)^{\alpha+\theta-w}\,\mathrm{d}x < \infty \tag{4.33}$$

which is in turn equivalent to having $u+w < 2(1-\alpha)$. Taking $w=0$, we see that if $u < 2(1-\alpha) = 2\rho(\nu)$, then (4.32) is finite. Theorem 4.2.3 then proves Proposition 4.3.14 $(ii)$.

Equation (4.33) also gives $\sigma_\nu(\gamma) = 2(1-\alpha)/\gamma$ so that $\sigma_\nu(\gamma) > 2$ iff $\gamma < 1-\alpha$. Therefore, $\kappa(\nu) = 1-\alpha = \rho(\nu)$. Theorem 4.3.6 therefore entails Proposition 4.3.14 $(iii)$.

Finally, let us prove claim $(iv)$. If $\beta \geq 0$, we get that

$$\nu\Big(\textstyle\sum_{i\geq 1} s_i^{1-\beta} = +\infty\Big) = \mathbb{E}\Big[T^{-\theta}\mathbb{1}_{T^{\beta-1}\int x^{1-\beta}\,\Pi(\mathrm{d}x)=\infty}\Big] > 0 \iff \mathbb{P}\Big[\int_{]0,1[} x^{1-\beta}\,\Pi(\mathrm{d}x) = \infty\Big] > 0.$$

Campbell's theorem ensures that $\int x^{1-\beta}\,\Pi(\mathrm{d}x) < \infty$ a.s. iff $\int_0^\infty [1 \wedge x^{1-\beta}]\,x^{-1-\alpha}\,\mathrm{d}x < \infty$. This last integral is finite iff $\beta < 1-\alpha = \rho(\nu)$.

It only remains to prove that $\nu$ integrates $\mathbf{s} \mapsto s_1^{-1} - 1$. For all $q \geq 0$, we have

$$\int_{\mathcal{S}^\downarrow}(s_1^{-q}-1)\,\nu(\mathrm{d}\mathbf{s}) = \int_{\mathcal{S}}\left(\inf_{n\geq 1} x_n^{-q} - 1\right)\nu^*(\mathrm{d}\mathbf{x}) = \inf_{n\geq 1}\int_{\mathcal{S}}\left(x_1^{-q}\wedge\cdots\wedge x_n^{-q} - 1\right)\nu^*(\mathrm{d}\mathbf{x}). \tag{4.34}$$

If $\theta > -\alpha$, using (4.31), for any fixed $n \geq 1$,

$$\begin{aligned}&\frac{\Gamma(1-\alpha)^n\Gamma(n+\alpha+\theta)}{\alpha^{1-n}\Gamma(n+\theta/\alpha)}\int_{\mathcal{S}}\left(x_1^{-q}\wedge\cdots\wedge x_n^{-q}-1\right)\nu^*(\mathrm{d}\mathbf{x}) \\ &\quad= \int_{]0,1[^n}\prod_{k=1}^n x_k^{-\alpha}(1-x_k)^{k\alpha+\theta-1}\cdot\left[\min_{1\leq j\leq n}\left(\prod_{i=1}^{j-1}(1-x_i)x_j\right)^{-q} - 1\right]\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &\quad= \int_0^1 \frac{q}{t^{q+1}}\left(\int_{]0,1[^n}\prod_{k=1}^n x_k^{-\alpha}(1-x_k)^{k\alpha+\theta-1}\cdot\mathbb{1}_{\max_{1\leq j\leq n}\prod_{i=1}^{j-1}(1-x_i)x_j\leq t}\,\mathrm{d}x_1\ldots\mathrm{d}x_n\right)\mathrm{d}t. \end{aligned} \tag{4.35}$$

By analytic continuation, this equality holds for all $\theta > -2\alpha$. Now observe that

$$\int_{]0,1[^n}\prod_{k=1}^n x_k^{-\alpha}(1-x_k)^{k\alpha+\theta-1}\cdot\mathbb{1}_{\max_{1\leq j\leq n}\prod_{i=1}^{j-1}(1-x_i)x_j\leq t}\,\mathrm{d}x_1\ldots\mathrm{d}x_n \underset{t\to 0}{\sim} \prod_{i=1}^n\int_0^t x_k^{-\alpha}\,\mathrm{d}x_k = \left(\frac{t^{1-\alpha}}{1-\alpha}\right)^n \tag{4.36}$$

and that when $t \to 1$,

$$\begin{aligned}\int_{]0,1[^n} \prod_{k=1}^{n} x_k^{-\alpha}(1-x_k)^{k\alpha+\theta-1} \cdot \mathbb{1}_{\max_{1\le j\le n}\prod_{i=1}^{j-1}(1-x_i)x_j\le t}\, \mathrm{d}x_1 \dots \mathrm{d}x_n \\ = \prod_{i=2}^{n} \mathrm{B}(1-\alpha, i\alpha+\theta) \cdot \int_0^t x_1^{-\alpha}(1-x_1)^{\alpha+\theta-1}\, \mathrm{d}x_1 \\ \le \prod_{i=2}^{n} \mathrm{B}(1-\alpha, i\alpha+\theta) \cdot \int_0^1 x_1^{-\alpha}\frac{(1-x_1)^{2\alpha+\theta-1}}{(1-t)^{\alpha}}\, \mathrm{d}x_1 = \mathcal{O}[(1-t)^{-\alpha}]. \end{aligned} \tag{4.37}$$

Equations (4.36) and (4.37) ensure that (4.35) is finite iff $q < n(1-\alpha)$. As a result and thanks to (4.34), if we take $n$ large enough, it follows that for all $q \ge 0$, $\int (s_1^{-q}-1)\, \nu(\mathrm{d}\mathbf{s}) < \infty$. The conditions of Proposition 4.3.8 are then met which ensures that Proposition 4.3.14 $(iv)$ holds. □

*Remark 4.3.11.* Fix $\beta \in ]1,2[$. Miermont proved in [98] that Duquesne and Le Gall's $\beta$-stable tree from [49] can be described as a fragmentation tree. Its self-similarity index is $1-1/\beta$ and its dislocation measure is

$$\frac{\beta^2\Gamma(2-1/\beta)}{\Gamma(2-\beta)}\, \nu^{\mathrm{PD}}_{1/\beta,-1}.$$

Proposition 4.3.14 then recovers the known fact that a $\beta$-stable tree a.s. has a profile, see [39] for instance. However, our approach does not allow us to get any more information on the said profile. It is also known that a.s. the profile of stable trees is not continuous, see Delmas [39].

**Deterministic fragmentation with power decay.** Set $\alpha > 1$ and consider the finite dislocation measure $\nu := \delta_{\zeta(\alpha)^{-1}(n^{-\alpha})_{n\ge1}}$ where $\zeta$ is the Riemann zeta function, i.e. $\zeta(\alpha) = \sum_{n\ge1} n^{-\alpha}$.

**Proposition 4.3.15.** *Set $\gamma > 0$ and let $\mathcal{T}$ be a $(\gamma, \nu)$ fragmentation tree. If $\gamma < 2(1-1/\alpha) \wedge 1$ then $\mathcal{T}$ a.s. admits a profile. If $\gamma < (1-1/\alpha)\wedge 1/2$, then the profile of $\mathcal{T}$ is a.s. continuous. Otherwise, if $\gamma > 1-1/\alpha$, then the profile of $\mathcal{T}$, if it exists, is a.s. not càdlàg.*

For any $q \in [0,1]$, in light of (4.10), we get that

$$\int \sum_{i\ge2} s_i^{1-q}\, \nu(\mathrm{d}\mathbf{s}) < \infty \quad \Longleftrightarrow \quad q < 1-1/\alpha =: \rho(\nu)$$

and for non-negative $u$ and $w$,

$$\int \sum_{i<j} s_i^{1-u} s_j^{1-w}\, \nu(\mathrm{d}\mathbf{s}) < \infty \quad \Longleftrightarrow \quad w < \rho(\nu) \text{ and } u + w < 2\rho(\nu). \tag{4.38}$$

Because $\nu$ is finite, the assumptions of Theorem 4.2.3 are then met as soon as $\gamma < 2\rho(\nu)\wedge 1$. This proves the first claim of Proposition 4.3.15.

Equation (4.38) also ensures that

$$\sigma_\nu(\gamma) = \frac{1}{\gamma}\sup\{u+w : u < \gamma+\rho(\nu), w < \rho(\nu), u+w < 2\rho(\nu)\} = \frac{[2\rho(\nu)]\wedge[1+\rho(\nu)}{\gamma}$$

so that $\sigma_\nu(\gamma) > 2$ iff $\gamma < \rho(\nu)$. As a result, $\kappa(\nu) = \rho(\nu) = 1-1/\alpha$ and the second claim of Proposition 4.3.15 is then a consequence of Theorem 4.3.6.

Finally, observe that $\nu\big(\sum_i s_i^{1-\beta} = \infty\big) > 0$ iff $\beta \ge 1-1/\alpha = \rho(\nu)$ and that $\int (s_1^{-1}-1)\nu(\mathrm{d}\mathbf{s}) < \infty$. We can therefore use Proposition 4.3.8 to finish the proof of Proposition 4.3.15.

**Balanced fragmentation with random number of fragments.** Let $(a_n)_n$ be a sequence of non-negative real numbers and consider $\nu = \sum_{n\geq 1} a_n \delta_{(1/n,\dots,1/n)}$. It is a dislocation measure iff $A := \sum_n a_n < \infty$, in which case it is even finite. If $A < \infty$, let $X$ be an integer valued random variable with $\mathbb{P}[X = n] = a_n/A$. Then for any measurable $f : \mathcal{S}^{\downarrow} \to \mathbb{R}_+$,

$$\int_{\mathcal{S}^{\downarrow}} f \, \mathrm{d}\nu = A\mathbb{E}\big[f(1/X, \cdot, 1/X, 0, \dots)\big]. \tag{4.39}$$

For all $q \geq 0$, using (4.39) we get that

$$\left|\int_{\mathcal{S}^{\downarrow}} \sum_{i\geq 1} s_i (1 - s_i^{-q}) \, \nu(\mathrm{d}\mathbf{s})\right| = A\mathbb{E}[X^q - 1]$$

which is finite iff $\mathbb{E}[X^q]$ is, so that $\rho(\nu) = \sup\{q \geq 0 : \mathbb{E}[X^q] < \infty\}$ with the convention $\sup \varnothing = 0$. Note that $X$ is the random number of fragments produced when a particle splits.

**Proposition 4.3.16.** *Let $\gamma > 0$ and let $\mathcal{T}$ be a $(\gamma, \nu)$ fragmentation tree. Finally, suppose that $\rho(\nu) > 0$. If $\gamma < \rho(\nu) \wedge 1$, then $\mathcal{T}$ a.s. has a profile. If we further assume that $\gamma < 1/2 \wedge [\rho(\nu)/2]$, then, this profile is a.s. continuous.*

For non-negative $u$ and $w$, we get

$$\int \sum_{i<j} s_i^{1-u} s_j^{1-w} \, \nu(\mathrm{d}\mathbf{s}) = A\mathbb{E}[X^{u+w-2} \cdot X(X-1)/2]$$

which is finite iff $\mathbb{E}[X^{u+w}]$ is. Therefore, if $\gamma < 1 \wedge \rho(\nu)$, then, in light of Theorem 4.2.3, $\mathcal{T}$ indeed a.s. admits a profile.

Using the previous notations, we also get $\sigma_\nu(\gamma) = \rho(\nu)/\gamma$ and $\kappa(\nu) = \rho(\nu)/2$. If $\gamma < 1/2 \wedge \rho(\nu)/2$, because $\nu$ is finite, Theorem 4.3.6 holds and the profile, say $L$, of $\mathcal{T}$ is a.s. continuous. In particular, if $\mathbb{E}[X] < \infty$, $L$ is continuous for any choice of $\gamma < 1/2$.

## 4.4 Asymptotic behaviour of the profiles of Markov branching trees

In this section, we will study the asymptotic behaviour of the profiles of a sequence $(T_n)_n$ of Markov branching trees indexed by their number of leaves. Our approach will be similar to the one we used in Section 4.3.

First, in Section 4.4.1, we will establish Proposition 4.4.2, a discrete analogue to Proposition 4.3.1. Proposition 4.4.2 states that for all $n$ the characteristic function of the height $I_n$ of a leaf chosen uniformly at random in $T_n$ satisfies

$$\forall t \in \mathbb{R}, \qquad |\varphi_{I_n}(t/n^\gamma)| \mathbb{1}_{|t|\leq n^\gamma \pi} \leq C_\theta |t|^{-1-\theta}$$

for all non-negative $\theta$ smaller than some constant depending on the model and where $C_\theta$ denotes a constant. We point out that the assumptions of Proposition 4.4.2 are rather strong and restrictive but could most probably be lightened.

Then in Section 4.4.2, we will prove that Proposition 4.4.2 entails uniform convergence of the first moment of the profiles of $(T_n)_n$ which, in other words, is a local limit theorem for the sequence $(I_n)_n$, see Corollary 4.4.3.

In Section 4.4.3, we will use Proposition 4.4.2 and Fourier inversion results to prove functional convergence of the rescaled profile $L_n$ of $T_n$ to that of a fragmentation tree. Finally in Section 4.4.4, we will give sufficient conditions under which $(L_n)_n$ cannot converge in distribution in $\mathcal{D}[0, \infty[$.

### 4.4.1 Absorption times of non-increasing Markov chains

As we discussed at the end of Section 4.2.3, if $I_n$ is the height of a uniformly drawn leaf in $T_n$, $I_n + 1$ is the first hitting time of 0 by a non-decreasing Markov chain. In this section and in the next, we will therefore consider this slightly more general framework.

*Functional convergence of non-increasing Markov chains.* For all $n \geq 0$, let $p_n$ be a probability measure on $\{0, \dots, n\}$ and set $P : \mathbb{N} \times \mathbb{N} \to [0,1]$ such that $P(n,j) := p_n(j)$ (with the convention $p_n(j) = 0$ if $j > n$). Moreover, for all $n \geq 0$, let $[X_n(i); i \geq 0]$ be a Markov chain with transition matrix $P$ and such that $X_n(0) = n$.

Let $\mathcal{A} := \{n \geq 0 : p_n(n) = 1\}$ be the set of absorbing states of $P$. For all $n$, let $I_n := \inf\{j \geq 0 : X_n(j) \in \mathcal{A}\}$ be the absorption time of $X_n$. For all $n \geq 0$, $j \geq 1$, set $\tilde{X}_n(j) = X_n(j)$ if $X_n(j) \notin \mathcal{A}$ or $\tilde{X}_n(j) = 0$ otherwise. The process $\tilde{X}_n$ is a Markov chain started from $n$ with transition matrix $\tilde{P}$ given by $\tilde{P}(n,0) = \sum_{i \in \mathcal{A}} P(n,i)$, $\tilde{P}(n,j) = 0$ if $j \in \mathcal{A}$ and $j > 0$ and $\tilde{P}(n,j) = P(n,j)$ otherwise. Let $\tilde{I}_n$ be the absorption time of $\tilde{X}_n$ and observe that if $n \neq 0$, then $I_n = \tilde{I}_n$. As such, we do not lose generality in the study of $(I_n)_n$ if we assume that $\mathcal{A} = \{0\}$. Therefore, from now on, we will make this assumption and as a result, $I_n = \inf\{j \geq 0 : X_n(j) = 0\}$.

Fix $\mu$, a $\sigma$-finite measure on $]0,1[$ which integrates $x \mapsto 1-x$ and let $\gamma$ be positive. For all real $q$, define $\psi(q) := \int_{]0,1[}(1-x^q)\,\mu(\mathrm{d}x)$. Let the pair $(X, I)$ be as in (4.2). In particular, $[X(t); t \geq 0]$ is a self-similar Markov process and its extinction time $I$ is the exponential functional of a subordinator with Laplace exponent $\psi$.

Set $\bar{p}_0 := \delta_0$ and for all $n \geq 1$, let $\bar{p}_n$ be the distribution of $X_n(1)/n$, i.e. $\bar{p}_n = \sum_{i=0}^{n} p_n(i)\delta_{i/n}$. Assume that

$$n^\gamma (1-x)\bar{p}_n(\mathrm{d}x) \underset{n\to\infty}{\Longrightarrow} (1-x)\mu(\mathrm{d}x) \tag{4.40}$$

as finite measures on $[0,1]$.

**Theorem 4.4.1 (Theorems 1 and 2 in [65]).** *Under Assumption* (4.40),

$$\left(\frac{1}{n}X_n\big(\lfloor n^\gamma t \rfloor\big)\,;\, t \geq 0\right) \underset{n\to\infty}{\Longrightarrow} X \qquad \text{in } \mathcal{D}[0,\infty[$$

*jointly with* $I_n/n^\gamma \underset{n\to\infty}{\Longrightarrow} I$.

*Furthermore, for all $a \geq 0$,*

$$\mathbb{E}\Big[\big(I_n/n^\gamma\big)^a\Big] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big[I^a\big].$$

In what follows, we will always assume that Assumption (4.40) holds.

*Power decay of $\varphi_{I_n}$.* From now on, assume that $\mu$ is a finite measure and set $\mathsf{m} := \mu\big(]0,1[\big)$. Also assume that $\rho(\mu)$ is positive, see (4.8) for its definition. As in Remark 4.3.2, observe that $\mu$ satisfies the assumptions of Proposition 4.3.1 and that as a result, $|\varphi_I(t)| \leq C_\theta |t|^{-1-\theta}$ for all $0 \leq \theta < \rho(\mu)/\gamma$ where $C_\theta$ denotes a constant independent of $t$.

We will also assume that

$$n^\gamma\big(1-p_n(n)\big) \underset{n\to\infty}{\longrightarrow} \mu\big(]0,1[\big) = \mathsf{m}. \tag{4.41}$$

Moreover, for convenience, for all $n$, we will write $p_n = (1-a_n)\delta_{(n)} + a_n\pi_n$ where $a_n = 1-p_n(n)$ and $\pi_n$ is a probability measure with $\pi_n(n) = 0$.

*Remark 4.4.1.* We point out that we make this rather restrictive assumption to be able to write an identity like $I_n \overset{(\mathrm{d})}{=} G_n + I_{Z_n}$, where, on the right hand side, the variables are independent, $G_n$ is a geometric variable independent of $Z_n$ and $Z_n$ is distributed like $X_n$ after its first jump. This identity will allow us to prove Proposition 4.4.2 with our methods.

As mentioned at the beginning of Section 4.4, Proposition 4.4.2 will be the main tool in the proofs of some results in Sections 4.4.2 and 4.4.3. To extend their scope, it would therefore be enough to find a different method requiring lighter assumptions to prove Proposition 4.4.2.

For all $n$, let $Z_n$ be distributed according to $\pi_n$. Assumption (4.40) then gives

$$\mathbb{E}\big[(Z_n/n)^q\big] \xrightarrow[n\to\infty]{} \frac{1}{\mathrm{m}} \int_{]0,1[} x^q \,\mu(\mathrm{d}x)$$

for all $q \geq 0$. The support of the probability measure $\mathrm{m}^{-1}\,\mu$ is bounded, so $\mathrm{m}^{-1}\,\mu$ is characterised by its moments. Therefore, $Z_n/n \Rightarrow \mathrm{m}^{-1}\,\mu$.

**Proposition 4.4.2.** *Suppose that the sequence $(p_n)_n$ satisfies (4.41) and that there exists a positive $\theta < \rho(\mu)/\gamma$ such that*

$$n^\gamma \textstyle\int_{[0,1]} (1-x^{-\gamma\theta})\,\bar{p}_n(\mathrm{d}x) \xrightarrow[n\to\infty]{} \psi(-\gamma\theta).$$

*Then we have*

$$\sup_{n\geq 0} \sup_{t\in\mathbb{R}} |\varphi_{I_n/n^\gamma}(t)|\,\mathbb{1}_{|t|\leq n^\gamma\pi}\,|t|^{1+\theta} < \infty.$$

*Remark 4.4.2.* By assumption $n^\gamma a_n \to \mathrm{m} = \mu(]0,1[)$ so we can rewrite $n^\gamma \int_{[0,1]}(1-x^{-\gamma\theta})\,\bar{p}_n(\mathrm{d}x) \to \psi(-\gamma\theta)$ simply as

$$\mathbb{E}\Big[(Z_n/n)^{-\gamma\theta}\Big] \xrightarrow[n\to\infty]{} \frac{1}{\mathrm{m}} \textstyle\int_{]0,1[} x^{-\gamma\theta}\mu(\mathrm{d}x).$$

In particular, $\mathbb{P}[Z_n = 0] = 0$ for $n$ large enough which ensures that $I_n > 1$ almost surely.

Recall that $Z_n/n$ converges in distribution to $\mathrm{m}^{-1}\mu$. As a result, if $\mathbb{E}[1-(Z_n/n)^{-a}] \to \psi(-a)/\mathrm{m}$ for some positive $a$, then $\mathbb{E}[1-(Z_n/n)^{-b}] \to \psi(-b)/\mathrm{m}$ for all $b \in [0,a]$ which in turn translates to

$$n^\gamma \textstyle\int_{[0,1]} (1-x^{-b})\,\bar{p}_n(\mathrm{d}x) \xrightarrow[n\to\infty]{} \psi(-b).$$

*Proof.* Let $(J_n)_{n\geq 0}$ be an independent copy of the sequence $(I_n)_n$. Let $(G_n)_{n\geq 1}$ be a sequence of independent geometric random variables with respective parameters $a_n$. Further suppose that $(G_n)_n$ is independent of $(J_n, Z_n)_{n\geq 0}$. Observe that if $n \geq 1$,

$$I_n \overset{(\mathrm{d})}{=} G_n + J_{Z_n}. \tag{4.42}$$

For all $n \geq 1$ and $u \in \mathbb{R}$, elementary computation gives

$$|\varphi_{G_n}(u)|^2 = \left|\frac{a_n}{1-(1-a_n)\mathrm{e}^{iu}}\right|^2 = \frac{1}{1+2(1-a_n)/a_n^2\cdot(1-\cos u)}. \tag{4.43}$$

Now observe that if we use the convention $u^2/(1-\cos u) = 2$ for $u = 0$, then the function $u \mapsto u^2/(1-\cos u)$ is continuous and thus bounded on $[-\pi,\pi]$. Furthermore, recall that $n^\gamma a_n \to \mathrm{m}$ which ensures that $a_n < 1$ for all $n$ large enough. Therefore, there is a positive constant $C$ such that, for all $n$, and any $\alpha \in [0,1]$,

$$|\varphi_{G_n/n^\gamma}(t)|\,\mathbb{1}_{|t|\leq n^\gamma\pi} \leq 1 \wedge \frac{C}{|t|}. \tag{4.44}$$

Using (4.42) and (4.44), for any $t \in [-n^\gamma\pi, n^\gamma\pi]$ we get

$$|\varphi_{I_n/n^\gamma}(t)| \leq |\varphi_{G_n/n^\gamma}(t)| \leq C/|t|. \tag{4.45}$$

Now suppose that we have proved that for all $n$ and $t$, $|\varphi_{I_n/n^\gamma}(t)|\,\mathbb{1}_{|t|\leq n^\gamma\pi} \leq C_\beta\,|t|^{-\beta}$ for some $\beta \leq \theta$ and where $C_\beta$ is a constant independent of $t$ and $n$. Recall that by assumption, since $\beta \leq \theta$, $\mathbb{E}[(Z_n/n)^{-\gamma\beta}]$ converges to a finite limit. Using (4.42) and (4.44) once again, we get that for all $\alpha \in [0,1]$, for any real $t$ if $n$ is large enough

$$\begin{aligned}|\varphi_{I_n/n^\gamma}(t)|\,\mathbb{1}_{|t|\leq n^\gamma\pi} &\leq |\varphi_{G_n/n^\gamma}(t)|\,\mathbb{E}\Big[\big|\varphi_{J_{Z_n}/Z_n^\gamma}\big[(Z_n/n)^\gamma t\big]\big|\Big]\mathbb{1}_{|t|\leq n^\gamma\pi} \\ &\leq \frac{C^\alpha}{t^\alpha}\,\frac{C_\beta}{|t|^\beta}\,\mathbb{E}\Big[(Z_n/n)^{-\gamma\beta}\Big] \leq \frac{C'}{|t|^{\alpha+\beta}}\end{aligned} \tag{4.46}$$

for some finite constant $C'$ independent of $n$ and $t$. The combination of (4.45) and (4.46) concludes this proof. □

### 4.4.2 A local limit theorem

We will now give a local version of the convergence of $I_n/n^\gamma$ to $I$. The proof of this result will rely on the conclusion of Proposition 4.4.2 and Fourier inversion.

**Corollary 4.4.3.** *Under the assumptions of Proposition 4.4.2,*

$$\sup_{j\geq 0}\left|n^\gamma \mathbb{P}[I_n = j] - k\big(j/n^\gamma\big)\right| \xrightarrow[n\to\infty]{} 0. \tag{4.47}$$

*Proof.* Recall that $\mu$ is finite so Proposition 4.3.1 holds. In particular, the characteristic function $\varphi_I$ of $I$ is integrable and we may use the Fourier inversion theorem to get

$$\forall x \geq 0, \qquad k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-itx}\,\varphi_I(t)\mathrm{d}t.$$

On the other hand, for all $n \geq 1$ and $j \geq 0$, we have

$$n^\gamma \mathbb{P}[I_n = j] = \frac{n^\gamma}{2\pi}\mathbb{E}\Big[\int_{-\pi}^{\pi}\mathrm{e}^{-itj}\mathrm{e}^{itI_n}\mathrm{d}t\Big] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-itj/n^\gamma}\,\varphi_{I_n/n^\gamma}(t)\mathbb{1}_{|t|\leq n^\gamma\pi}\mathrm{d}t.$$

As a result,

$$\left|n^\gamma \mathbb{P}[I_n = j] - k(j/n^\gamma)\right| \leq \frac{1}{2\pi}\int \left|\varphi_I(t) - \varphi_{I_n/n^\gamma}(t)\mathbb{1}_{|t|\leq n^\gamma\pi}\right|\mathrm{d}t \tag{4.48}$$

which holds uniformly for all $j \in \mathbb{Z}_+$. Theorem 4.4.1 entails that $\varphi_{I_n/n^\gamma} \to \varphi_I$ point-wise. Propositions 4.3.1 and 4.4.2 then allow us to use the dominated convergence theorem to prove that the right hand side of (4.48) vanishes when $n \to \infty$. □

### 4.4.3 Convergence of the profiles of Markov branching trees

Let $\nu$ be a dislocation measure and $\gamma$ a positive real number. Furthermore, let $(\mathcal{T}, \mu_\mathcal{T})$ denote a $(\gamma, \nu)$-fragmentation tree, let $I$ be the height of a $\mu_\mathcal{T}$-distributed random variable and $k$ the density of the law of $I$. Furthermore, $\psi$ will be defined as in (4.9).

Set $(q_n)_n$ a sequence of first-split distributions. For all $n$ let $T_n$ be distributed according to $\mathrm{MB}_n^{\mathcal{L},q}$ and endow it with $\mu_n$, the counting measure on the set of its leaves. Also let $\ell_n$ be the profile of $T_n$ with respect to $\mu_n$. Recall that $\bar{q}_n$ is the measure on $\mathcal{S}^\downarrow$ such that

$$\int_{\mathcal{S}^\downarrow} f(\mathbf{s})\,\bar{q}_n(\mathrm{d}\mathbf{s}) = \sum_{\lambda\in\mathcal{P}_n} f(\lambda/n)\,q_n(\lambda)$$

for all measurable $f : \mathcal{S}^\downarrow \to \mathbb{R}_+$. We will suppose that $n^\gamma(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu(\mathrm{d}\mathbf{s})$. In particular, Theorem 4.2.2 holds and $(n^{-\gamma}T_n, n^{-1}\mu_n) \Rightarrow (\mathcal{T}, \mu_\mathcal{T})$ in the GHP topology.

This section will focus on the asymptotic behaviour of the rescaled profile of $T_n$ which we will denote by $L_n$ and define for all $t \geq 0$ by $L_n(t) := n^{\gamma-1}\,\ell_n\big(\lfloor n^\gamma t\rfloor\big)$. Let us first observe that by combining Lemma 4.3.7 $(i)$ and Corollary 4.4.3, we get the following result on the convergence of the first moment of $L_n$.

**Proposition 4.4.4.** *If the conclusion of Proposition 4.4.2 holds,*

$$\sup_{t\geq 0}\big|\mathbb{E}[L_n(t)] - k(t)\big| \xrightarrow[n\to\infty]{} 0.$$

*In particular, if $(\mathcal{T}, \mu_\mathcal{T})$ a.s. admits a profile say $L$, then for almost every non-negative $t$,*

$$\mathbb{E}[L_n(t)] \xrightarrow[n\to\infty]{} \mathbb{E}[L(t)].$$

Notably, this last result can hold for any $\gamma > 0$. We will now endeavour to prove that in some cases, the sequence $(L_n)_n$ will converge in distribution in $\mathcal{D}[0,\infty[$ when $\gamma$ is small enough.

*Convergence to a continuous profile.* Before stating this section's main result, we need some more notations. For all $n$, define the measure $p_n$ on $\{0,\dots,n\}$ by

$$p_n(j) := \sum_{\lambda\in\mathcal{P}_n} \frac{j}{i} m_j(\lambda) q_n(\lambda), \tag{4.49}$$

compare with (4.5). Set

$$\rho(q) := \sup\left\{\theta \geq 0 : \lim_{n\to\infty} n^\gamma \int_{[0,1]} (1-x^{-\theta})\,\bar{p}_n(\mathrm{d}x) = \psi(-\theta) > -\infty\right\} \tag{4.50}$$

and observe that $\rho(q) \leq \rho(\nu)$. Let $\sigma^q(\gamma)$ be defined as

$$\sigma^q(\gamma) := \sup\left\{\alpha+\beta : \alpha,\beta < 1+\rho(q)/\gamma, \limsup \int_{\mathcal{S}^\downarrow} \sum_{i<j} s_i^{1-\alpha\gamma} s_j^{1-\beta\gamma} \mathbb{1}_{s_j>0} \bar{q}_n(\mathrm{d}\mathbf{s}) < \infty\right\}$$

Observe that $\sigma^q(\gamma)$ decreases when $\gamma$ increases. Finally, set $\kappa(q) := \sup\{\gamma : \sigma^q(\gamma) > 2\}$.

**Theorem 4.4.5.** *Suppose that $n^\gamma(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu(\mathrm{d}\mathbf{s})$ where $\nu$ is finite and that $n^\gamma\,[1-q_n(n)] \to \nu(\mathcal{S}^\downarrow)$ when $n\to\infty$. Further assume that $\rho(q)$ is positive and that $\gamma < 1/2 \wedge \kappa(q)$. Then $\mathcal{T}$ a.s. admits a continuous profile $L$ and $L_n \Rightarrow L$ in $\mathcal{D}[0,\infty[$.*

*Remark 4.4.3.* Under these assumptions:

— The sequence $(p_n)_n$ defined in (4.49) satisfies the assumptions of Proposition 4.4.2.

— The fragmentation pair $(\nu,\gamma)$ meets the assumptions of Theorem 4.3.6 which guarantees the a.s. existence and continuity of the profile $L$ of $\mathcal{T}$.

We will need a few results before we are able to prove Theorem 4.4.5. Conditionally on $T_n$, let $I_n$ be the height of a uniformly drawn leaf in $T_n$. let $\Phi_n$ be the characteristic function of $I_n/n^\gamma$ conditionally on $T_n$, i.e. for all $u\in\mathbb{R}$, set $\Phi_n(u) := \mathbb{E}[\mathrm{e}^{iuI_n/n^\gamma}|T_n]$.

**Proposition 4.4.6.** *For all $\theta < \sigma^q(\gamma)\wedge 1/\gamma$, we have*

$$\sup_{n\geq 0} \sup_{|u|\leq n^\gamma\pi} |u|^\theta \mathbb{E}[|\Phi_n(u)|^2] < \infty.$$

*In particular, if $\gamma < 1/2\wedge\kappa(q)$, then*

$$\sup_{n\geq 1} \int_{-n^\gamma\pi}^{n^\gamma\pi} \mathbb{E}[|\Phi_n(u)|]\mathrm{d}u < \infty.$$

*Proof.* We will proceed as we did in the proof of Theorem 4.3.6.

For all $n$, conditionally on $T_n$, let $V_n$, $W_n$ be two leaves of $T_n$ drawn uniformly and independently. Set $I_n := |V_n|$ and $J_n := |W_n|$. Let $R_n$ be the number of leaves of $T_n$ above $V_n\wedge W_n$, the most recent common ancestor of $V_n$ and $W_n$, and set $\Delta_n(1)$ and $\Delta_n(2)$ as the numbers of leaves in the sub-trees above $V_n\wedge W_n$ containing $V_n$ and $W_n$ respectively. Observe that for any $f:\mathbb{N}^2\to\mathbb{R}_+$ and $r>0$,

$$\mathbb{E}\Big[f\big(\Delta_n(1),\Delta_n(2)\big)\,\Big|\,R_n = r\Big] = \int_{\mathcal{S}^\downarrow} \sum_{i\neq j} s_i s_j f(rs_i, rs_j)\,\bar{q}_r(\mathrm{d}\mathbf{s})$$

and that for any $r\leq n$, $\mathbb{P}[R_n\leq r] \leq r/n$ which means that $R_n/n$ is stochastically bigger than a uniform variable on $]0,1[$.

Then, for any $|u|\leq n^\gamma\pi$ and $\alpha,\beta < 1+\rho(q)/\gamma$, we get

$$\begin{aligned}\mathbb{E}\big[|\Phi_n(u)|^2\big] &= \mathbb{E}\big[\mathrm{e}^{iu(I_n-J_n)/n^\gamma}\big] = \mathbb{E}\big[\mathrm{e}^{iu(\tilde{I}_{\Delta_n(1)}-\tilde{J}_{\Delta_n(2)})/n^\gamma}\big)\big] \\ &\leq 2\,\mathbb{E}\left[\left|\varphi_{\tilde{I}_{\Delta_n(1)}/\Delta_n(1)^\gamma}\Big[\big(\Delta_n(1)/n\big)^\gamma u\Big]\right| \cdot \left|\varphi_{\tilde{J}_{\Delta_n(2)}/\Delta_n(2)^\gamma}\Big[\big(\Delta_n(2)/n\big)^\gamma u\Big]\right| \cdot \mathbb{1}_{\Delta_n(1)\geq\Delta_n(2)}\right]\end{aligned}$$

where for all $\ell \geq 0$, $\tilde{I}_\ell$, $\tilde{J}_\ell$ are i.i.d. copies of $I_\ell$ independent of $(T_n)_n$. Proposition 4.4.2 now ensures that there we can find a finite constant $C$ such that

$$\mathbb{E}\big[|\Phi_n(u)|^2\big] \leq C\,\mathbb{E}\left[\frac{n^{\gamma\alpha}}{|u|^\alpha \Delta_n(1)^{\gamma\alpha}}\frac{n^{\gamma\beta}}{|u|^\beta \Delta_n(2)^{\gamma\beta}}\mathbb{1}_{\Delta_n(1)\geq\Delta_n(2)}\right]$$
$$\leq \frac{2C}{|u|^{\alpha+\beta}}\,\mathbb{E}\left[\left(\frac{R_n}{n}\right)^{-\gamma(\alpha+\beta)}\int_{\mathbb{S}^\downarrow}\sum_{i<j}s_i^{1-\alpha\gamma}s_j^{1-\beta\gamma}\,\bar{q}_{R_n}(\mathrm{d}\mathbf{s})\right].$$

By assumption, we can choose $\alpha,\beta < 1+\rho(q)/\gamma$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{S}^\downarrow}\sum_{i<j}s_i^{1-\alpha\gamma}s_j^{1-\beta\gamma}\,\bar{q}_n(\mathrm{d}\mathbf{s}) < \infty.$$

Because $R_n/n$ is stochastically bigger than a uniform variable on $]0,1[$, $\mathbb{E}[(R_n/n)^{-(\alpha+\beta)\gamma}] \leq \int_0^1 x^{-(\alpha+\beta)\gamma}\mathrm{d}x$ which is finite provided $(\alpha+\beta)\gamma<1$. Consequently

$$\sup_n \sup_{|u|\leq n^\gamma\pi}|u|^{\alpha+\beta}\,\mathbb{E}\big[|\Phi_n(u)|^2\big] < \infty.$$

As a result, if $\gamma < 1/2\wedge\kappa(q)$, we may find $\theta>1$ such that

$$\mathbb{E}\big[|\Phi_n(u)|\mathbb{1}_{u\leq n^\gamma\pi}\big] \leq 1\wedge(C/|u|^\theta)$$

for some positive constant $C$ independent of $n$ and $u$. The second claim of Proposition 4.4.6 follows immediately. □

*Remark 4.4.4.* Like for Corollary 4.4.3, the cornerstone of the proof of Proposition 4.4.6 is Proposition 4.4.2 which is why we need these burdensome assumptions.

**Corollary 4.4.7.** *Under the assumptions of Proposition 4.4.6, the family $(L_n)_n$ is tight in $\mathcal{D}[0,\infty[$.*

*Proof.* Observe that for any $t\geq 0$ and $n\geq 1$,

$$L_n(t) = \frac{1}{2\pi}\int_{-n^\gamma\pi}^{n^\gamma\pi}\Phi_n(u)\,\mathrm{e}^{-iu\lfloor n^\gamma t\rfloor/n^\gamma}\,\mathrm{d}u,$$

which ensures, by Proposition 4.4.6 that

$$\mathbb{E}\big[\sup_{t\geq0}L_n(t)\big] \leq \frac{1}{2\pi}\int_{-n^\gamma\pi}^{n^\gamma\pi}\mathbb{E}\big[|\Phi_n(u)|\big]\mathrm{d}u \leq \frac{C_\theta}{2\pi}\int_{-\infty}^{\infty}1\wedge|u|^{-1-\theta}\mathrm{d}u \tag{4.51}$$

for some positive $\theta$ and finite constant $C_\theta$. Since exp is Hölder continuous of any order, if $\alpha\leq1$, there is a finite constant $h_\alpha$ such that for any $x$ and $y$ in $\mathbb{R}$, $|\mathrm{e}^{ix}-\mathrm{e}^{iy}|\leq h_\alpha|x-y|^\alpha$. As a result, for any $t,s\geq0$ with $|t-s|\leq\varepsilon$,

$$|L_n(t)-L_n(s)| \leq \frac{H_\alpha}{2\pi}\int_{-n^\gamma\pi}^{n^\gamma\pi}|\Phi_n(u)|\,|u|^\alpha\,\frac{|\lfloor n^\gamma t\rfloor-\lfloor n^\gamma s\rfloor|^\alpha}{n^{\alpha\gamma}}\,\mathrm{d}u \leq \frac{H_\alpha(\varepsilon^\alpha+n^{-\alpha\gamma})}{2\pi}\int_{-n^\gamma\pi}^{n^\gamma\pi}|u|^\alpha\,|\Phi_n(u)|\,\mathrm{d}u$$

so that for $\alpha<\theta$, we get

$$\limsup_{n\to\infty}\mathbb{E}\left[\sup_{|t-s|<\varepsilon}|L_n(t)-L_n(s)|\right] \leq \frac{C_\theta H_\alpha\,\varepsilon^\alpha}{2\pi}\int_{-\infty}^{\infty}1\wedge|u|^{\alpha-1-\theta}\mathrm{d}u \xrightarrow[\varepsilon\to0]{} 0. \tag{4.52}$$

We may therefore use Aldous' tightness criterion (see [29], Theorem 16.10) to conclude that $(L_n)_n$ is indeed tight in $\mathcal{D}[0,\infty[$. □

*Proof of Theorem 4.4.5.* Corollary 4.4.7 states that $(L_n)_n$ is tight in $\mathcal{D}[0,\infty[$. Lemma 4.2.5 ensures that $L$ is the only possible limit in distribution for a subsequence of $(L_n)_n$. As a result, $L_n\Rightarrow L$ in $\mathcal{D}[0,\infty[$. □

### 4.4.4 Non-convergence of the profiles

In this section, we will give some sufficient conditions under which functional convergence of the rescaled profiles cannot happen. We stress that these conditions might be satisfied along with the assumptions of Proposition 4.4.4. In particular, functional convergence of $L_n$ might be impossible while its expectation might converge almost everywhere.

*Unbounded limit profile.* Suppose that $(\gamma, \nu)$ satisfies the assumptions of Proposition 4.3.8 and that $n^{\gamma}(1-s_1)\,\bar{q}_n(\mathrm{d}\mathbf{s}) \Rightarrow (1-s_1)\,\nu(\mathrm{d}\mathbf{s})$. We may prove the following result.

**Proposition 4.4.8.** *The sequence $(L_n)_n$ cannot converge in distribution in $\mathcal{D}[0,\infty[$.*

*Remark 4.4.5.* Let us stress that the $(\gamma,\nu)$ fragmentation tree $\mathcal{T}$ may a.s. have a profile $L$. However Proposition 4.3.8 guarantees that $L$ is not càdlàg.

*Proof.* Using the Skorokhod representation theorem and Lemma 4.2.4, we may assume that $M_n \to M$ a.s. in uniform topology on $[0,\infty[$, i.e. that with probability one, for all $t \geq 0$,

$$\sup_{s\leq t} |M_n(s) - M(s)| \xrightarrow[n\to\infty]{} 0. \tag{4.53}$$

Proposition 4.3.8 ensures in particular that there exists a random finite time $T$ such that, a.s. for all positive $a$, we have

$$\frac{1}{\varepsilon_a}\big[M(T+\varepsilon_a) - M(T)\big] > a \tag{4.54}$$

for some random positive $\varepsilon_a$. Now because of (4.53), almost surely

$$\left|\frac{1}{\varepsilon_a}\big[M_n(T+\varepsilon_a) - M_n(T)\big] - \frac{1}{\varepsilon_a}\big[M(T+\varepsilon_a) - M(T)\big]\right| \xrightarrow[n\to\infty]{} 0.$$

As a result, (4.54) yields that

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_a}\big[M_n(T+\varepsilon_a) - M_n(T)\big] \geq a. \tag{4.55}$$

Now observe that if $0 \leq s \leq T$ and $0 < \varepsilon \leq 1$, for all $n$ we have

$$\frac{1}{\varepsilon}\big[M_n(s+\varepsilon) - M_n(s)\big] \leq \frac{1}{\varepsilon}\int_{s-n^{-\gamma}}^{s+\varepsilon+n^{-\gamma}} L_n(u)\,\mathrm{d}u \leq (1+2n^{-\gamma}/\varepsilon)\cdot \sup_{u\leq t+1} L_n(u). \tag{4.56}$$

Combining (4.55) and (4.56), we conclude that almost surely,

$$\sup_{n\geq 0}\sup_{t\leq T+1} L_n(t) = \infty.$$

In other words, with probability one, the sequence $(L_n)_n$ is not uniformly bounded on all compacts. As a result, almost surely, this sequence cannot converge in the $\mathcal{D}[0,\infty[$ topology. □

*When $\gamma > 1$.* In this paragraph, we will drop all previous assumptions and will only suppose that $\gamma > 1$. In this case, recall that a.s. $\mathcal{T}$ does not have a profile.

Let $\mathrm{d}_{\mathbb{P}}$ denote the metric defined by $\mathrm{d}_{\mathbb{P}}(X,Y) := \mathbb{E}[1\wedge|X-Y|]$ for any two real valued random variables. It is well-known that $\mathrm{d}_{\mathbb{P}}$ induces the topology of convergence in probability.

Let $f_n$, $n\geq 1$, $f$ be real valued measurable functions defined on $\mathbb{R}$. We'll say that $f_n$ converges to $f$ in measure (with respect to the Lebesgue measure) iff for any positive $\varepsilon$, the Lebesgue measure of the set $\{t\in\mathbb{R} : |f_n(t)-f(t)| > \varepsilon\}$ goes to 0 as $n$ goes to infinity.

**Proposition 4.4.9.** *Suppose that $\gamma > 1$. Then, almost surely, $L_n$ converges to 0 in measure and $t \mapsto \mathrm{d}_{\mathbb{P}}[L_n(t),0]$ converges to 0 in $L^1(\mathrm{d}t)$. In particular, from all subsequences $(n_k)_k$, we can extract a sub-subsequence $(k_\ell)_\ell$ such that for almost every $t\geq 0$, $L_{n_{k_\ell}}(t)$ converges to 0 in probability.*

*Remark 4.4.6.* Suppose that Proposition 4.3.1 holds. In this case, for all $t > 0$, $\mathbb{E}[L_n(t)] \to k(t) > 0$.

*Proof.* Recall that $L_n(t) := n^{\gamma-1}\#\{u \in \mathcal{L}(T_n) : |u| = \lfloor n^\gamma t \rfloor\}$ and therefore a.s. belongs to $n^{\gamma-1}\mathbb{Z}_+$. If $\varepsilon \in ]0,1[$, then $\mathbb{1}_{L_n(t)>\varepsilon} = \mathbb{1}_{L_n(t)>0} = 1 \wedge L_n(t) \leq n^{1-\gamma} L_n(t)$. As a result,

$$\int_0^\infty \mathbb{1}_{L_n(t)>\varepsilon} \mathrm{d}t \leq n^{1-\gamma} \int_0^\infty L_n(t) \mathrm{d}t = n^{1-\gamma} \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

and using Fubini's theorem,

$$\int_0^\infty \mathrm{d}_{\mathbb{P}}[L_n(t), 0]\, \mathrm{d}t = \mathbb{E}\Big[\int_0^\infty 1 \wedge L_n(t) \mathrm{d}t\Big] \leq n^{1-\gamma} \mathbb{E}\Big[\int_0^\infty L_n(t) \mathrm{d}t\Big] = n^{1-\gamma} \xrightarrow[n\to\infty]{} 0. \qquad \square$$

# Appendix

# Simulating random trees

## Random trees

Here are some programs written in *Mathematica* used to efficiently generate certain random trees with a given number of vertices or leaves. These trees will be encoded as follows: vertices are integers, the root is 0, and edges are written as $\mathtt{p}[i] \to i$ where $\mathtt{p}[i]$ denotes the parent of $i$. The trees can be drawn with the command

```
Graph[tree, DirectedEdges -> False,
  GraphLayout -> {"SpringElectricalEmbedding",
    "RepulsiveForcePower" -> -3/4,
    "InferentialDistance" -> 2,
    "Octree" -> False}]]
```

for instance, where `tree` denotes the result of any of the following programs. We point out that it is much more efficient in *Mathematica* to generate everything that is random at once instead of only generating what is needed when it is needed. Because of this, these programs might seem more complicated than they need to be.

*Conditioned Poisson Galton-Watson trees.* This algorithm was described by Aldous [7]. It generates a critical Poisson Galton-Watson tree conditioned to have $n$ vertices.

```
PoissonGW[n_] := Block[{U = RandomReal[{0, 1}, n - 1]},
  Table[Min[i - 1, IntegerPart[n*U[[i]]]] -> i, {i, n - 1}]]
```

*Rémy's algorithm.* This algorithm, introduced by Rémy [110], gives binary Galton-Watson trees with any number $n$ of leaves.

```
Remy[n_] := Block[{p = Table[0, {2 n - 1}],
    U = 1 + IntegerPart[Table[2 i - 3, {i, n}]*RandomReal[{0, 1}, n]]},
  Do[
    p[[2 i - 2]] = p[[U[[i]]]];
    p[[U[[i]]]] = 2 i - 2;
    p[[2 i - 1]] = 2 i - 2;,
  {i, 2, n}];
  Table[p[[i]] -> i, {i, 2 n - 1}]]
```

*Conditioned Geometric Galton-Watson trees.* This algorithm uses a transformation of critical Galton-Watson trees with $n$ leaves to critical Galton-Watson tree with $n$ vertices and a different offspring distribution. It was used by Rizzolo [111] and Abraham and Delmas [2] for instance. In particular, using this transformation on binary Galton-Watson trees with $n$ leaves gives critical geometric Galton-Watson trees with $n$ vertices.

```
GeometricGW[n_] := Block[{remy = Remy[n],
    U = RandomInteger[{0, 1}, 2 n - 1],
    children = Table[{}, {2*n}],
    p = Table[0, {n - 1}],
    v = {0}, u = 0,
    labels = Table[0, {2 n}],
    current = 0},
  Do[If[U[[e[[2]]]] == 1,
    AppendTo[children[[1 + e[[1]]]], e[[2]]],
    PrependTo[children[[1 + e[[1]]]], e[[2]]];,
  {e, remy}];
  labels[[1]] = 0;
  While[Length[children[[1 + v[[-1]]]]] > 0,
    labels[[v[[-1]]]] = current;
    AppendTo[v, children[[1 + v[[-1]]]][[1]]];
    children[[1 + v[[-2]]]] = Drop[children[[1 + v[[-2]]]], 1];];
  current++;
  While[current < n,
    p[[current]] = labels[[remy[[v[[-1]]]][[2]]]];
    While[Length[children[[1 + v[[-1]]]]] > 0,
      labels[[v[[-1]]]] = current;
      AppendTo[v, children[[1 + v[[-1]]]][[1]]];
      children[[1 + v[[-2]]]] = Drop[children[[1 + v[[-2]]]], 1];];
    v = Drop[v, -1];
    current++;];
  Table[p[[i]] -> i, {i, n - 1}]]
```

*Ford's $\alpha$-model.* This algorithm was introduced by Ford in [52] as a generalisation of Rémy's algorithm. It yields binary trees with any number $n$ of leaves. Here $\alpha \in [0, 1]$.

```
Ford[alpha_, n_] := Block[{p = Table[0, {2 n - 1}],
  U = RandomReal[{0, 1}, n],
  V = RandomReal[{0, 1}, n]},
  If[n == 1,
    {0 -> 1},
    p[[2]] = 0;
    p[[1]] = 2;
    p[[3]] = 2;
    Do[
      If[(i - 1 - alpha)*U[[i]] < (i - 1)*(1 - alpha),
        p[[2 i - 2]] = p[[1 + 2*IntegerPart[(i - 1)*V[[i]]]]];
        p[[1 + 2*IntegerPart[(i - 1)*V[[i]]]]] = 2 i - 2;,
        p[[2 i - 2]] = p[[2 + 2*IntegerPart[(i - 2)*V[[i]]]]];
        p[[2 + 2*IntegerPart[(i - 2)*V[[i]]]]] = 2 i - 2;];
      p[[2 i - 1]] = 2 i - 2;,
    {i, 3, n}];
  Table[p[[i]] -> i, {i, 2 n - 1}]]]
```

*The $\alpha$-$\gamma$ model.* Chen, Ford and Winkel [36] generalised Ford's algorithm to be able to generate non binary trees with any number of leaves. The parameters are such that $0 \leq \gamma \leq \alpha \leq 1$.

```
AlphaGamma[alpha_, gamma_, n_] := If[n == 1, {0 -> 1},
  Block[{int = 0,
    p = Table[0, {2 n - 1}],
    leaves = Table[0, {n}],
    intern = Table[0, {n}],
    chosen = Table[0, {n}],
    U = RandomReal[{0, 1}, n],
    V = RandomReal[{0, 1}, n],
```

```
    W = RandomReal[{0, 1}, n]},
  p[[2]] = 0;
  p[[1]] = 2;
  p[[3]] = 2;
  leaves[[1]] = 1;
  intern[[1]] = 2;
  leaves[[2]] = 3;
  int = 1;
  Do[Which[
    (i - 1 - alpha)*U[[i]] <= (i - 1)*(1 - alpha),
      p[[i + int]] = p[[leaves[[1 + IntegerPart[(i - 1)*V[[i]]]]]]];
      p[[leaves[[1 + IntegerPart[(i - 1)*V[[i]]]]]] = i + int;
      p[[i + int + 1]] = i + int;
      intern[[int + 1]] = i + int;
      int++;,
    (i - 1 - alpha)*U[[i]] <= (i - 1)*(1 - alpha) +
      int*gamma,
      p[[i + int]] = p[[intern[[1 + IntegerPart[int*V[[i]]]]]]];
      p[[intern[[1 + IntegerPart[int*V[[i]]]]]] = i + int;
      p[[i + int + 1]] = i + int;
      intern[[int + 1]] = i + int;
      int++;,
    ((i - 2)*alpha - int*gamma)*V[[i]] <= int*(alpha - gamma),
      p[[i + int]] = intern[[1 + IntegerPart[int*W[[i]]]]];
      chosen[[i - int - 1]] = intern[[1 + IntegerPart[int*W[[i]]]]];,
    True,
      p[[i + int]] = chosen[[1 + IntegerPart[(i - int - 2)*W[[i]]]]];
      chosen[[i - int - 1]] =
        chosen[[1 + IntegerPart[(i - int - 2)*W[[i]]]]];
    ];
    leaves[[i]] = i + int;,
  {i, 3, n}];
  Table[p[[i]] -> i, {i, n + int}]]]
```

*Marchal's algorithm.* Marchal [95] gave an algorithm to generate Galton-Watson trees conditioned which have $n$ leaves and whose offspring distribution is in the domain of attraction of a stable distribution. which coincides with the $\alpha$-$\gamma$ algorithm for well chosen parameters More precisely, take $\beta \in ]1,2]$, Marchal's algorithm yields $\mathrm{GW}_{\xi}^{\mathcal{L},n}$ trees for any positive integer $n$ where the generating function of $\xi$ is $s \mapsto s + (1-s)^{\beta}/\beta$. For well chosen parameters, the said algorithm coincides with that of the $\alpha$-$\gamma$ model.

```
Marchal[beta_,n_] := AlphaGamma[1/beta,1-1/beta,n]
```

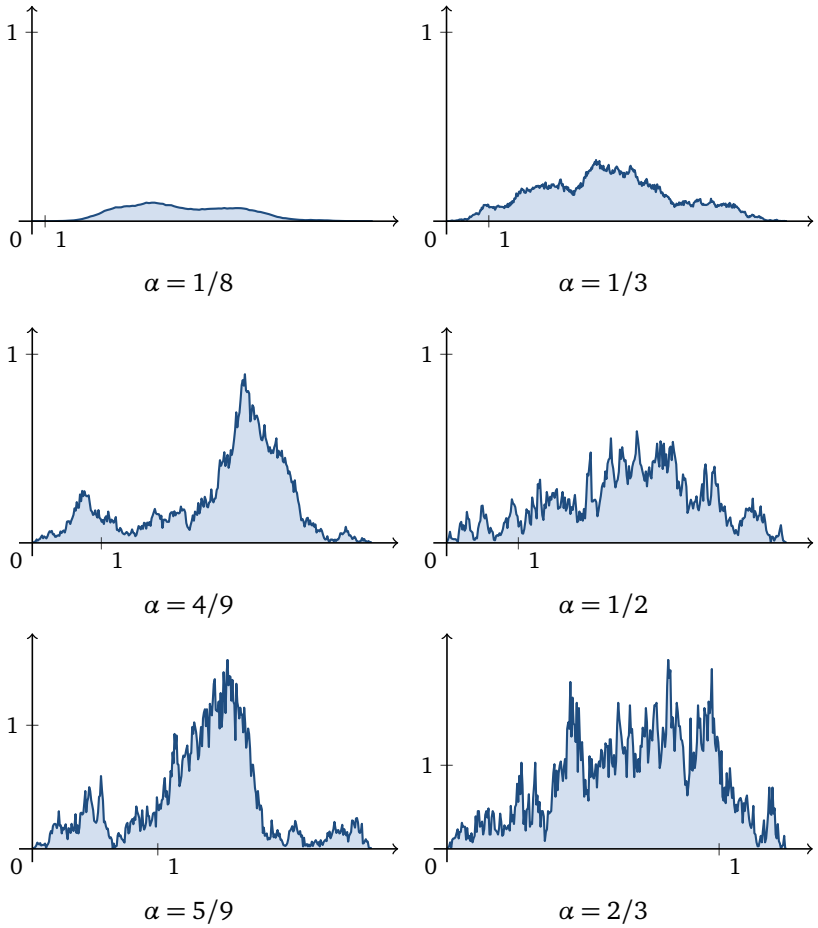
## Profiles of trees

Here we give two programs which respectively give the profile and external profile of trees obtained with the previous programs. The main principle here is to visit each successive layer of a given tree, starting from the root 0, and to count at each step how many vertices or leaves there are.

```
Profile[t_] :=
  Block[{children = Table[{}, {Length[t] + 1}], current = {0}},
    Do[
      children[[1 + e[[1]]]] = Append[children[[1 + e[[1]]]], e[[2]]];,
    {e, t}];
    Reap[
      While[Length[current] > 0,
        Sow[Length[current]];
        current = Catenate[Table[children[[1 + i]], {i, current}]];];
      Sow[0];][[2]][[1]]]
```

```
ExtProfile[t_] :=
  Block[{children = Table[{}, {Length[t] + 1}], current = {0}, leaves},
    Do[
      children[[1 + e[[1]]]] = Append[children[[1 + e[[1]]]], e[[2]]];,
    {e, t}];
    leaves = Reap[
      Do[If[Length[children[[i]]] == 0, Sow[i - 1]],
        {i, Length[children]}]][[2]][[1]];
    Reap[
      While[Length[current] > 0,
        Sow[Length[Intersection[current, leaves]]];
        current = Catenate[Table[children[[1 + i]], {i, current}]];];
      Sow[0];][[2]][[1]]]
```

What follows are simulations of the external profiles of trees obtained with Ford's algorithm with $2^{18}$ leaves and different choices of $\alpha$. Recall that results from Section 4.3.4, namely Proposition 4.3.13, ensure that when $\alpha < 1/2$, the profile of the corresponding limit $\mathbb{R}$-tree, i.e. a $(\alpha, \nu_\alpha^{(\mathrm{F})})$ fragmentation tree, is almost surely continuous. Moreover when $\alpha = 1/2$, the limit tree is a multiple of the Brownian tree, which is known to have a continuous profile. However, our results don't give any new information when $\alpha > 1/2$. These simulations were therefore intended to shed light on whether a $(\alpha, \nu_\alpha^{(\mathrm{F})})$ fragmentation tree can have a continuous profile when $\alpha > 1/2$, and whether the condition $\gamma < 1/2$ in Theorem 4.3.6 is necessary or merely technical.



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## Résumé

Dans cette thèse, on étudiera les propriétés asymptotiques de modèles d'arbres aléatoires satisfaisant la propriété dite de Markov branchante. En particulier, nous établirons un résultat de convergence de ces arbres au sens de la limite locale vers des arbres infinis qui satisfont une version de la propriété de Markov branchante. Nous donnerons de plus un résultat sur la convergence au sens des limites d'échelle pour cette famille d'arbres infinis. Nous nous intéresserons ensuite au comportement asymptotique du profil des arbres Markov branchants finis ainsi qu'à la régularité du profil limite. Enfin, nous donnerons des applications de nos résultats à divers modèles d'arbres aléatoires comme les arbres de Galton-Watson conditionnés par leur nombre de nœuds.

## Mots Clés

Arbres aléatoires; Arbres de Galton-Watson; Propriété de Markov branchante; Arbres de fragmentation auto-similaire; Topologie locale; Topologie GHP; Profils d'arbres

## Abstract

In this thesis, we will study asymptotic properties of a family of random trees satisfying the so-called Markov branching property. In particular, we will establish convergence for these trees under the local limit topology to infinite random trees which satisfy a version of the Markov branching property. We will also give a scaling limit result for this family of infinite trees. We will then turn our focus to the profiles of Markov branching trees as well as the regularity of the limit object. Finally, we will give applications of our results to various models of random trees such as Galton-Watson trees conditioned by their number of vertices.

## Keywords

Random trees; Galton-Watson trees; Markov branching property; Self-similar fragmentation trees; Local topology; GHP topology; Tree profiles