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**Antonin JACQUET**

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Discipline/Spécialité : Mathématiques

**Étude des propriétés des géodésiques en  
percolation de premier passage**

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La thèse porte sur les géodésiques en percolation de premier passage. On se place sur le graphe  $\mathbb{Z}^d$  et on considère une famille de variables aléatoires positives, indépendantes et identiquement distribuées indexées par l'ensemble des arêtes, appelées temps de passage. On définit le temps de tout chemin fini comme la somme du temps de passage de chacune de ses arêtes. Les géodésiques sont alors les chemins de temps minimaux. On s'intéresse à la question suivante. Considérons une propriété locale de l'environnement des temps de passage. On appelle cela un motif. Combien de fois une géodésique emprunte un translaté de ce motif?

Sans hypothèse de moment sur la loi des temps de passage, et en autorisant un atome en l'infini, on montre que pour un motif donné, en dehors d'un événement de probabilité exponentiellement faible en la distance entre les extrémités des géodésiques, le nombre de translatés de ce motif empruntés par les géodésiques est linéaire en la distance entre les extrémités. On donne également quelques applications de ce résultat.

**Mots clés : Percolation de premier passage; Géodésiques.**



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## Abstract

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The thesis focuses on geodesics in first-passage percolation. We consider the graph  $\mathbb{Z}^d$  and a family of nonnegative, independent, and identically distributed random variables indexed by the set of edges, called passage times. The time of a finite path is defined as the sum of the passage times of each of its edges. Geodesics are then the paths with minimal time. We are interested in the following question. Consider a local property of the time environment. We call it a pattern. How many times does a geodesic cross a translate of this pattern?

Without assuming any moment assumption on the distribution of passage times and with possibly infinite passage times, it is shown that, apart from an event with exponentially small probability in the distance between the geodesic endpoints, the number of translates of the pattern crossed by the geodesics is linear in the distance between the endpoints. Some applications of this result are also provided.

**Keywords:** First-passage percolation; Geodesics





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## 1.1 Le modèle de percolation de premier passage

La percolation de premier passage a été introduite par Hammersley et Welsh en 1965 dans [11]. Une interprétation de ce modèle est l'étude de l'écoulement d'un liquide dans un milieu poreux à travers le temps. On considère que les arêtes du graphe  $\mathbb{Z}^d$  représentent des canaux microscopiques et on leur associe un temps de passage aléatoire modélisant le temps que le liquide met pour s'écouler le long de ce canal. Une question naturelle qui mène à l'étude de la percolation de premier passage est alors : combien de temps faut-il attendre avant qu'une zone donnée du graphe ne soit atteinte par le liquide ?

Si on se place du point de vue plus formel des mathématiques, la percolation de premier passage est l'étude de distances aléatoires sur un graphe.

Parmi les références classiques sur le sujet, citons le cours de Saint-Flour de Kesten [14] et la revue plus récente de Auffinger, Damron et Hanson [2]. Cette dernière revue rassemble notamment de nombreuses questions et conjectures sur la percolation de premier passage.

### 1.1.1 Définition du modèle

Fixons  $d \geq 2$ . Dans ce manuscrit, le modèle est défini sur le graphe  $\mathbb{Z}^d$  classique. Notons  $\mathcal{E}$  l'ensemble des arêtes du graphe. Les arêtes de  $\mathcal{E}$  sont celles reliant les sommets  $x$  et  $y$  tels que  $\|x - y\|_1 = 1$ . L'objet aléatoire basique considéré est une famille  $T = \{T(e) : e \in \mathcal{E}\}$  de variables aléatoires à valeurs dans  $[0, \infty)$ , indépendantes et identiquement distribuées, définies sur un même espace probabilisé  $(\Omega, \mathcal{F}, \mathbb{P})$ . Pour une arête  $e$ , la variable aléatoire  $T(e)$  est appelée temps de passage de l'arête  $e$ . On note  $\mathcal{L}$  la loi commune des variables aléatoires de la famille  $T$  et on note  $F$  sa fonction de répartition. Une extension de ce modèle autorisant les temps de passage à pouvoir être égaux à  $\infty$  est présentée dans la

section 1.1.4.

**Temps de passage d'un chemin.** On appelle chemin une suite  $\pi = (x_0, \dots, x_k)$  de sommets adjacents du graphe  $\mathbb{Z}^d$ , c'est-à-dire tels que pour tout  $i = 0, \dots, k-1$ ,  $\|x_{i+1} - x_i\|_1 = 1$ . On dit que  $\pi$  va de  $x_0$  à  $x_k$ . En écrivant  $\pi = (e_1, \dots, e_k)$ , où pour tout  $i = 1, \dots, k$ ,  $e_i = \{x_{i-1}, x_i\}$  est l'arête reliant les sommets  $x_{i-1}$  et  $x_i$ , on peut identifier un chemin entre deux sommets avec la suite des arêtes qu'il emprunte. On dit que  $k$  est la taille du chemin défini précédemment et on note  $|\pi| = k$ . Un chemin est dit auto-évitant s'il visite chaque sommet au plus une fois. Le temps de passage d'un chemin  $\pi = (e_1, \dots, e_k)$  est défini par

$$T(\pi) = \sum_{i=1}^k T(e_i).$$

Dans l'exemple de l'écoulement d'un liquide dans un milieu poreux, cela peut s'interpréter comme le temps que met le liquide à traverser le chemin de canaux microscopiques considéré.

**Temps géodésique et géodésique entre deux sommets.** Pour deux sommets  $x$  et  $y$ , on définit le temps géodésique de  $x$  à  $y$  :

$$t(x, y) = \inf\{T(\pi) : \pi \text{ est un chemin de } x \text{ à } y\}. \quad (1.1.1)$$

En remarquant que le temps géodésique satisfait l'inégalité triangulaire, on obtient que  $t(\cdot, \cdot)$  est une pseudo-distance aléatoire sur le graphe  $\mathbb{Z}^d$ . Si de plus  $\mathcal{L}(0) = 0$ , c'est une distance aléatoire.

Si on cherche une nouvelle fois à illustrer ces premières définitions avec l'exemple donné en introduction de cette section, le temps géodésique entre deux sommets s'interprète comme le temps que le liquide met, partant de  $x$ , pour atteindre  $y$  la première fois.

Un chemin auto-évitant  $\gamma$  de  $x$  à  $y$  tel que  $T(\gamma) = t(x, y)$  est une géodésique entre  $x$  et  $y$ . Sans hypothèse supplémentaire sur la loi  $\mathcal{L}$ , on ne peut pas affirmer que les géodésiques existent entre deux sommets avec probabilité 1. Nous y reviendrons dans la section 1.2.

**Boules pour les temps géodésiques.** Pour  $x$  et  $y$  dans  $\mathbb{R}^d$ , on définit également  $t(x, y)$  le temps géodésique entre  $x$  et  $y$  comme étant égal à  $t(x', y')$  où  $x'$  (resp.  $y'$ ) est l'unique sommet de  $\mathbb{Z}^d$  tel que  $x \in x' + [0, 1)^d$  (resp.  $y \in y' + [0, 1)^d$ ). Pour  $z \in \mathbb{Z}^d$  et  $r$  un réel strictement positif, on définit alors les boules aléatoires

$$\tilde{B}(z, r) = \{y \in \mathbb{R}^d : t(z, y) \leq r\} \text{ et } B(z, r) = \mathbb{Z}^d \cap \tilde{B}(z, r).$$

La boule  $B(z, r)$  définie ci-dessus est la boule de rayon  $r$  et de centre  $z$  pour la pseudo-distance aléatoire  $t(\cdot, \cdot)$ .

## 1.1.2 Approximation au premier ordre du temps géodésique sous une hypothèse de moment

### 1.1.2.1 Constante de temps.

Une approximation au premier ordre du temps  $t(0, n\varepsilon_1)$ , où  $\varepsilon_1$  est le premier vecteur de la base canonique, est donnée par le théorème suivant.

**Théorème 1.1.1** (Théorème 2.18 in [14]). *Supposons que*

$$\mathbb{E} \min[T_1, \dots, T_{2d}] < \infty, \quad (1.1.2)$$

où les variables  $T_i$  sont des copies indépendantes de  $T(e)$ . Alors, il existe une constante  $\mu(\varepsilon_1) \in [0, \infty)$  (appelée la constante de temps) telle que

$$\lim_{n \rightarrow \infty} \frac{t(0, n\varepsilon_1)}{n} = \mu(\varepsilon_1) \text{ p.s. et dans } L^1.$$

Notons que l'hypothèse (1.1.2) est nécessaire pour avoir la convergence presque sûre ou dans  $L^1$ . En effet, sans cette hypothèse, on peut montrer rapidement que pour tout  $c > 0$ , avec probabilité 1,

$$\limsup_n \frac{t(0, n\varepsilon_1)}{n} > c.$$

La preuve du théorème 1.1.1 est une application du théorème ergodique sous-additif (le théorème 2.2 dans [2]) à la famille de variables aléatoires  $(X_{m,n})_{0 \leq m < n}$  définie pour tout  $0 \leq m < n$  par  $X_{m,n} = T(m\varepsilon_1, n\varepsilon_1)$ . Toujours sous l'hypothèse (1.1.2), la preuve s'adapte à tout  $x \in \mathbb{Q}^d$ , et on obtient une fonction homogène  $\mu : \mathbb{Q}^d \rightarrow \mathbb{R}$  telle que, pour tout  $x \in \mathbb{Q}^d$ ,

$$\lim_{n \rightarrow \infty} \frac{t(0, nx)}{n} = \mu(x) \text{ p.s. et dans } L^1. \quad (1.1.3)$$

Cette application peut être prolongée sur  $\mathbb{R}^d$  par continuité et ce prolongement, noté  $\mu$  également, est une semi-norme sur  $\mathbb{R}^d$  qui fournit une approximation de la distance aléatoire  $t(\cdot, \cdot)$ . Lorsque  $\mathcal{L}(0) = 0$ , les temps de passage sont tous strictement positifs et l'application  $\mu$  est une norme sur  $\mathbb{R}^d$ . La condition pour savoir si  $\mu$  est une norme ou simplement une semi-norme est donnée par le théorème suivant dû à Kesten. Notons  $p_c$  la probabilité critique de percolation sur  $\mathbb{Z}^d$  (on peut se référer à [10] pour plus d'informations sur la percolation).

**Théorème 1.1.2** (Théorème 6.1 dans [14]). *Si  $\mathcal{L}(0) \geq p_c$ , alors pour tout  $x \in \mathbb{Z}^d$ ,  $\mu(x) = 0$ . Si  $\mathcal{L}(0) < p_c$  alors  $\mu$  est une norme sur  $\mathbb{R}^d$ .*

### 1.1.2.2 Théorème de forme asymptotique.

Une question naturelle est de savoir si la convergence de (1.1.3) est uniforme dans toutes les directions. Le théorème suivant répond à cette question.

**Théorème 1.1.3** (Cox-Durrett [7]). *Supposons que*

$$\mathbb{E} \min[T_1^d, \dots, T_{2d}^d] < \infty, \quad (1.1.4)$$

*où les variables  $T_i$  sont des copies indépendantes de  $T(e)$ . Alors,*

$$t(0, x) = \mu(x) + o(x) \text{ quand } \|x\|_1 \rightarrow \infty \text{ p.s.}$$

Lorsque  $\mu$  est une norme, cela permet de décrire le comportement au premier ordre de la boule  $\tilde{B}(0, r)$  quand  $r$  tend vers l'infini. C'est l'objet du théorème de forme asymptotique ci-dessous. Notons  $\mathbf{B}_\mu = \{x \in \mathbb{R}^d : \mu(x) \leq 1\}$  et pour tout sous-ensemble  $S$  de  $\mathbb{R}^d$  et tout  $r \in \mathbb{R}$ ,  $rS = \{rs : s \in S\}$ .

**Théorème 1.1.4** (Cox-Durrett [7]). *Supposons que l'hypothèse (1.1.4) est satisfaite et que  $\mathcal{L}(0) < p_c$ . Alors, pour tout  $\varepsilon > 0$ ,*

$$\mathbb{P} \left( (1 - \varepsilon)\mathbf{B}_\mu \subset \frac{\tilde{B}(0, r)}{r} \subset (1 + \varepsilon)\mathbf{B}_\mu \text{ pour tout } r \text{ assez grand} \right) = 1.$$

Notons que dans cette thèse, on se placera dans le cadre où  $\mu$  est une norme.

### 1.1.3 La semi-norme $\mu$

#### 1.1.3.1 Continuité de la constante de temps comme fonction de la loi des temps de passage

La sous-additivité permet d'obtenir l'existence de la constante de temps définie ci-dessus mais elle ne donne aucune idée de la valeur de  $\mu$ . Déterminer le rôle de la loi  $\mathcal{L}$  dans la description du temps géodésique, et en particulier ici déterminer  $\mu(\varepsilon_1)$  en fonction de  $\mathcal{L}$ , est un problème difficile et fondamental en percolation de premier passage. Ainsi, aucune loi non triviale permettant un calcul exact de la constante de temps n'est connue. Une question reliée à ce sujet est la suivante : pour deux lois  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  données, quelle est la différence entre leurs constantes de temps  $\mu(\varepsilon_1)$  et  $\tilde{\mu}(\varepsilon_1)$ ? Plusieurs résultats existent à ce propos et l'un des plus importants est le suivant.

**Théorème 1.1.5** (Cox-Kesten [8], Kesten [14]). *Si une suite  $(\mathcal{L}_n)_n$  de lois converge faiblement vers une loi  $\mathcal{L}$ , en notant  $(\mu_n(\varepsilon_1))$ ,  $\mu(\varepsilon_1)$  leurs constantes de temps respectives, on a*

$$\lim_{n \rightarrow \infty} \mu_n(\varepsilon_1) = \mu(\varepsilon_1).$$

Notons que dans ce résultat, aucune hypothèse de moment n'est faite sur les lois. La constante de temps est définie dans ce cadre élargi comme la limite en probabilité de  $\frac{t(0, n\varepsilon_1)}{n}$ . L'existence et le caractère fini de cette constante sont établis par Cox et Kesten dans [7].

### 1.1.3.2 Principe de comparaison de van den Berg-Kesten.

Un autre résultat permettant de comparer les constantes de temps de deux lois elles-mêmes comparables selon un ordre partiel précisé ci-dessous a été prouvé par van den Berg et Kesten dans [18]. Nous reviendrons sur ce résultat dans la section 1.3.4.4 puisque le résultat principal de cette thèse, le théorème 1.3.5, permet d'en donner une preuve simple ainsi qu'une extension dans le cadre plus général de la section 1.1.4.

**Une motivation.** Pour motiver le résultat de van den Berg et Kesten énoncé ci-dessous dans le théorème 1.1.9, commençons par l'observation suivante. Considérons deux lois  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  distinctes telles que pour tout  $t \in \mathbb{R}_+$ ,  $F(t) \leq \tilde{F}(t)$  où  $F$  (resp.  $\tilde{F}$ ) est la fonction de répartition de la loi  $\mathcal{L}$  (resp.  $\tilde{\mathcal{L}}$ ). En couplant les temps de passage des arêtes du graphe de sorte que pour tout  $e \in \mathbb{Z}^d$ ,  $\tilde{T}(e) \leq T(e)$ , on obtient immédiatement que  $\tilde{\mu}(\varepsilon_1) \leq \mu(\varepsilon_1)$  (où  $\tilde{\mu}(\varepsilon_1)$  est la constante de temps pour la loi  $\tilde{\mathcal{L}}$ ). Une question naturelle est : a-t-on stricte inégalité? Sous les hypothèses précisées ci-dessous, le principe de comparaison de van den Berg-Kesten permet de répondre par l'affirmative à cette question.

**Lois utiles et ordre partiel sur les lois sur  $[0, \infty)$ .** Deux définitions sont nécessaires pour énoncer le résultat. La première fait partie des hypothèses sur la loi et la deuxième donne un ordre partiel entre les lois sur  $[0, \infty)$ .

**Définition 1.1.6.** Une loi  $\mathcal{L}$  sur  $[0, \infty)$  est dite utile si les deux conditions suivantes sont satisfaites :

$$\begin{aligned} \mathcal{L}(t_{\min}) &< p_c \text{ quand } t_{\min} = 0, \\ \mathcal{L}(t_{\min}) &< \vec{p}_c \text{ quand } t_{\min} > 0, \end{aligned}$$

où  $t_{\min}$  désigne le minimum du support de la loi,  $p_c$  est la probabilité critique de percolation sur  $\mathbb{Z}^d$  et  $\vec{p}_c$  est la probabilité critique en percolation orienté sur  $\mathbb{Z}^d$  (voir section 12.8 de [10]).

**Définition 1.1.7.** Soient  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  deux lois sur  $[0, \infty)$ . On dit que  $\tilde{\mathcal{L}}$  est plus variable que  $\mathcal{L}$  si pour toute fonction croissante et concave  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\int \varphi d\tilde{\mathcal{L}} \leq \int \varphi d\mathcal{L},$$

quand les deux intégrales existent. Si de plus  $\mathcal{L} \neq \tilde{\mathcal{L}}$ , on dit que  $\tilde{\mathcal{L}}$  est strictement plus variable que  $\mathcal{L}$ .

**Exemple 1.1.8.** Si  $\mathcal{L}$  domine  $\tilde{\mathcal{L}}$ , c'est-à-dire si pour tout  $x$ ,  $F(x) \leq \tilde{F}(x)$ , alors  $\tilde{\mathcal{L}}$  est plus variable que  $\mathcal{L}$ .

Le théorème 2.6 de [18] et le lemme 6.1 de [16] garantissent que  $\tilde{\mathcal{L}}$  est plus variable que  $\mathcal{L}$  si et seulement si l'hypothèse suivante est satisfaite il existe un couple de variables aléatoires



$(\tau, \tilde{\tau})$  définies sur un même espace probabilisé, de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  et telles que

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau.$$

### Résultat de van den Berg-Kesten.

**Théorème 1.1.9** (Théorème 2.9 de [18]). *Soient  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  deux lois sur  $[0, \infty)$ , toutes deux d'espérances finies, telles que  $\mathcal{L}$  est utile,  $\mathcal{L} \neq \tilde{\mathcal{L}}$  et telles qu'il existe un couple de variables aléatoires  $(\tau, \tilde{\tau})$  définies sur un même espace probabilisé, de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  telles que*

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau.$$

*Alors, en notant  $\mu(\varepsilon_1)$  (resp.  $\tilde{\mu}(\varepsilon_1)$ ) la constante de temps pour la loi  $\mathcal{L}$  (resp.  $\tilde{\mathcal{L}}$ ),*

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

**Remarque 1.1.10.** *Le théorème 1.1.9 est énoncé pour  $\varepsilon_1$  mais la preuve est identique lorsque  $\varepsilon_1$  est remplacé par n'importe quel  $x \in \mathbb{Z}^d$  tel que  $x \neq 0$ .*

**Remarque 1.1.11.** *L'inégalité large dans le résultat du théorème 1.1.9 est élémentaire. Elle est montrée dans la section 1.3.4.4.*

Une conséquence de ce résultat est l'observation suivante. Soit  $\mathcal{L}$  une loi sur  $[0, \infty)$  utile et d'espérance finie. Supposons pour ce paragraphe que le support de la loi  $\mathcal{L}$  est non borné et notons pour tout  $x \in \mathbb{R}_+$ ,  $\mathcal{L}_x$  la loi obtenue en tronquant la loi  $\mathcal{L}$  à  $x$ . Autrement dit, si  $T$  est de loi  $\mathcal{L}$ , alors  $\mathcal{L}_x$  est la loi de  $\min(T, x)$ . Pour tout  $x \in \mathbb{R}_+$ , notons  $\mu_x(\varepsilon_1)$  la constante de temps pour la loi  $\mathcal{L}_x$ . La continuité de la constante de temps comme fonction de la loi des temps de passage donne que

$$\mu_x(\varepsilon_1) \xrightarrow{x \rightarrow \infty} \mu(\varepsilon_1).$$

Une question naturelle est alors de savoir si pour un  $x$  assez grand on peut avoir

$$\mu_x(\varepsilon_1) = \mu(\varepsilon_1).$$

Sous les hypothèses du théorème 1.1.9, le principe de comparaison de van den Berg-Kesten prouve que ce n'est pas possible et qu'on a pour tout  $x \in \mathbb{R}_+$ ,

$$\mu_x(\varepsilon_1) < \mu(\varepsilon_1). \tag{1.1.5}$$

**Généralisation de Marchand [16].** Deux conditions sont présentes dans la définition de loi utile. Comme cela est précisé dans la remarque 2.14 de [18], la condition  $\mathcal{L}(0) < p_c$  est nécessaire. En effet, sinon  $\mathcal{L}(0) \geq p_c$  et  $\mu(\varepsilon_1) = 0$  par le théorème 1.1.2. On peut alors en déduire que  $\tilde{\mu}(\varepsilon_1) = 0$ . En revanche, le résultat de Marchand dans [16] énoncé ci-dessous

permet, en dimension 2, de supprimer la condition

$$\mathcal{L}(t_{\min}) < \overrightarrow{p}_c \text{ quand } t_{\min} > 0. \quad (1.1.6)$$

Lorsque  $t_{\min} > 0$  et que  $\mathcal{L}(t_{\min}) > \overrightarrow{p}_c$ , le problème fait intervenir la percolation orientée. Dans ce cas, la plus grande partie de chaque géodésique reliant l'origine à un point très éloigné appartenant au cône de percolation est un chemin orienté empruntant des arêtes de temps minimal. Pour un sommet  $x$  dans ce cône, on a alors  $\mu(x) = t_{\min}\|x\|_1$  et si de plus  $t_{\min} = \tilde{t}_{\min}$  (où  $\tilde{t}_{\min}$  est le minimum du support de  $\tilde{\mathcal{L}}$ ) on a aussi  $\tilde{\mu}(x) = t_{\min}\|x\|_1$ . On ne peut alors pas espérer de généralisation du principe de comparaison de van den Berg et Kesten. Pour plus d'explications, on peut se référer à [16].

La généralisation de Marchand énoncée ci-dessous est plus forte que le théorème 1.1.9 sur deux aspects : en plus de l'aspect développé ci-dessus, l'hypothèse de moment sur  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  est supprimée.

**Théorème 1.1.12** (Théorème 1.2 de [16]). *Supposons que  $d = 2$ . Soit  $\mathcal{L}$  une loi sur  $[0, \infty)$  telle que  $\mathcal{L}(0) < p_c$ . Soit  $\tilde{\mathcal{L}}$  une loi sur  $[0, \infty)$  telle que  $\mathcal{L} \neq \tilde{\mathcal{L}}$  et telle qu'il existe un couple de variables aléatoires  $(\tau, \tilde{\tau})$  définies sur un même espace probabilisé, de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  telles que*

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau.$$

Alors

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

### 1.1.4 Extension du modèle en autorisant un atome en l'infini

Supposons maintenant que l'on autorise que  $\mathcal{L}(\infty) > 0$ . On peut alors définir le temps de passage d'un chemin et le temps géodésique entre deux sommets exactement de la même façon que dans la section 1.1.1. Une des premières questions qui intervient est également de savoir si on peut obtenir une approximation au premier ordre du temps géodésique.

Pour commencer, si le poids de l'atome en l'infini est trop grand, alors les arêtes dont le temps de passage est fini ne percolent pas et pour tout sommet  $x \in \mathbb{Z}^d$ , pour tout  $y \in \mathbb{Z}^d$  tel que  $\|x - y\|_1$  est assez grand,  $t(x, y) = \infty$ . C'est pourquoi l'hypothèse suivante, que l'on suppose satisfaite dans toute la suite, est naturelle :

$$\mathcal{L}([0, \infty)) > p_c. \quad (1.1.7)$$

En disant qu'une arête  $e$  est ouverte si son temps de passage  $T(e)$  est fini et fermée sinon, alors (1.1.7) nous donne que ce modèle de percolation est sucritique. Ainsi, presque sûrement, il existe un unique cluster infini d'arêtes dont le temps de passage est fini que l'on note  $\mathcal{C}_\infty$ .

Ce cadre est étudié par Cerf et Théret dans [5]. Nous décrivons ci-dessous quelques uns de leurs résultats. Commençons par quelques notations. Par (1.1.7), il existe  $M > 0$  tel que  $\mathcal{L}([0, M]) > p_c$ . Fixons un tel  $M$ . On note  $\mathcal{C}_M$  le cluster infini pour la percolation  $(\mathbf{1}_{T(e) \leq M}, e \in \mathcal{E})$  sur  $\mathbb{Z}^d$ , qui existe et est unique p.s. Pour tout  $x \in \mathbb{R}^d$ , on note  $\varphi(x)$  le

sommet aléatoire de  $\mathbb{Z}^d$  tel que  $\varphi(x) \in \mathcal{C}_M$  et tel que  $\|x - \varphi(x)\|_1$  est minimal. S'il existe plusieurs sommets de  $\mathcal{C}_M$  à une distance minimale de  $x$ , on choisit  $\varphi(x)$  en suivant une règle déterministe. Le théorème suivant permet alors de définir la constante de temps dans ce cadre.

**Théorème 1.1.13** (Théorème 1 de [5]). *Supposons que  $\mathcal{L}([0, \infty)) > p_c$ . Alors il existe une fonction déterministe  $\mu : \mathbb{R}^d \rightarrow [0, \infty)$  telle que*

$$\forall x \in \mathbb{Z}^d, \lim_{n \rightarrow \infty} \frac{t(\varphi(0), \varphi(nx))}{n} = \mu(x) \text{ p.s. et dans } L^1.$$

Puis le théorème suivant permet de déterminer dans quel cas la fonction  $\mu$  donnée par le théorème 1.1.13 est une norme.

**Théorème 1.1.14** (Théorème 2 de [5]). *Supposons que  $\mathcal{L}([0, \infty)) > p_c$ .*

- *Si  $\mathcal{L}(0) \geq p_c$ , alors  $\mu$  est nulle.*
- *Si  $\mathcal{L}(0) < p_c$ , alors  $\mu$  est une norme.*

La fonction  $\mu$  dépend de la dimension  $d$  et de la loi  $\mathcal{L}$  des temps de passages des arêtes mais ne dépend pas du réel  $M$  fixé tel que  $\mathcal{L}([0, M]) > p_c$ . Cela découle du théorème 4 de [5] énoncé ci-dessous. Pour tout  $x \in \mathbb{R}^d$ , on note  $\varphi_\infty(x)$  le sommet aléatoire de  $\mathbb{Z}^d$  tel que  $\varphi_\infty(x) \in \mathcal{C}_\infty$  et tel que  $\|x - \varphi_\infty(x)\|_1$  soit minimal. S'il existe plusieurs sommets de  $\mathcal{C}_\infty$  à une distance minimale de  $x$ , on choisit  $\varphi_\infty(x)$  en suivant une règle déterministe.

**Théorème 1.1.15** (Théorème 4 de [5]). *Supposons que  $\mathcal{L}([0, \infty)) > p_c$ . Alors*

$$\forall x \in \mathbb{Z}^d, \lim_{n \rightarrow \infty} \frac{t(\varphi_\infty(0), \varphi_\infty(nx))}{n} = \mu(x) \text{ en probabilité.}$$

*Par conséquent, la fonction  $\mu$  du théorème 1.1.13 ne dépend pas de la constante  $M$  vérifiant  $\mathcal{L}([0, M]) > p_c$  choisie dans la définition de  $\varphi$ .*

Le théorème 1.1.15 garantit également que lorsque l'hypothèse (1.1.2) est vérifiée, la semi-norme obtenue grâce au théorème 1.1.1 coïncide avec celle obtenue dans la section 1.1.2.1. Dans [5], Cerf et Théret établissent également différents résultats de forme asymptotique.

**Continuité de la constante de temps comme fonction de la loi des temps de passage dans ce cadre.** Dans [9], Garet, Marchand, Procaccia et Théret montrent une généralisation du théorème 1.1.5 dans ce cadre.

**Théorème 1.1.16** (Théorème 1.2 de [9]). *Soient  $\mathcal{L}$  et  $(\mathcal{L}_n)_n$  des lois sur  $[0, \infty]$  telles que pour tout  $n \in \mathbb{N}$ ,  $\mathcal{L}_n([0, \infty)) > p_c$  et  $\mathcal{L}([0, \infty)) > p_c$ . Notons  $(\mu_n)$ ,  $\mu$  les semi-normes respectives données par le théorème 1.1.13. Si  $(\mathcal{L}_n)_n$  converge faiblement vers  $\mathcal{L}$ , alors*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{S}^{d-1}} |\mu_n(x) - \mu(x)| = 0,$$

où  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ .

### Caractère lipschitzien de la constante de temps pour la percolation de Bernoulli.

Pour tout  $p > p_c$ , notons  $\mathcal{L}_p = p\delta_1 + (1-p)\delta_\infty$  et  $\mu_p$  la norme donnée par le théorème 1.1.13 pour la loi  $\mathcal{L}_p$ . Comme corollaire du théorème 1.1.16, on a que l'application  $p \mapsto \mu_p$  est continue sur  $(p_c, 1]$ . Dans [4], Cerf et Dembin obtiennent un meilleur résultat de régularité énoncé ci-dessous.

**Théorème 1.1.17** (Théorème 1.2 de [4]). *Soit  $p_0 > p_c$ . Alors il existe une constante  $\kappa_0$  qui dépend seulement de  $d$  et  $p_0$ , telle que*

$$\forall p, q \in [p_0, 1], \quad \sup_{x \in \mathbb{S}^{d-1}} |\mu_p(x) - \mu_q(x)| \leq \kappa_0 |q - p|.$$

## 1.2 Géodésiques en percolation de premier passage

Cette thèse porte sur les géodésiques, qui font partie des objets centraux en percolation de premier passage. Commençons par rappeler la définition. Pour  $x$  et  $y$  deux sommets de  $\mathbb{Z}^d$ , une géodésique de  $x$  à  $y$  est un chemin auto-évitant  $\gamma$  de  $x$  à  $y$  tel que  $T(\gamma) = t(x, y)$ . On présente dans cette section quelques résultats sur les géodésiques.

### 1.2.1 Existence et unicité

**Existence des géodésiques.** Revenons pour le moment au cadre défini en section 1.1.1, autrement dit supposons que  $\mathcal{L}(\infty) = 0$ . La toute première question qui vient après la définition de géodésique est celle de son existence. Une partie de la réponse est donnée par le théorème ci-dessous.

**Théorème 1.2.1** ([14],[19],[20]). *Lorsque  $\mathcal{L}(\infty) = 0$ , presque sûrement, pour tout  $x, y \in \mathbb{Z}^d$ , il existe au moins une géodésique entre  $x$  et  $y$  dès que l'une des conditions suivantes est satisfaite :*

1.  $d = 2$ ,
2.  $\mathcal{L}(0) < p_c$ ,
3.  $\mathcal{L}(0) > p_c$ ,
4.  $\mathcal{L}((0, h)) = 0$  pour un certain  $h > 0$ .

Prouver l'existence presque sûre des géodésiques en dimension  $d \geq 3$  lorsque  $\mathcal{L}(0) = p_c$  reste encore une question ouverte. Le résultat du théorème 1.2.1 ci-dessus est donné par le corollaire 1.3 de l'article [19] de Wierman et Reh dans le cas de la dimension  $d = 2$  et par le théorème 2 de l'article [20] de Zhang dans le cas  $\mathcal{L}(0) > p_c$ . Pour le cas  $\mathcal{L}(0) < p_c$ , le lecteur peut se référer à la section 9.23 de [14] ou à la proposition 4.4 de [2]. Enfin, la preuve dans le cas où  $\mathcal{L}((0, h)) = 0$  pour un certain  $h > 0$  est assez courte et écrite dans la remarque 1.3 de [3].

Lorsque  $\mathcal{L}(\infty) > 0$ ,  $\mathcal{L}([0, \infty)) > p_c$  et  $\mathcal{L}(0) < p_c$ , la preuve de [2] s'adapte facilement. Une preuve est donnée dans l'annexe 3.A.

**Unicité des géodésiques.** Une condition suffisante pour qu'entre tout couple de sommets, il existe une unique géodésique est que la loi  $\mathcal{L}$  soit une loi continue. En revanche, lorsque la loi possède un atome, il n'y a pas forcément unicité des géodésiques. Le théorème 1 de [17] affirme que lorsque  $\mathcal{L}$  est une loi utile sur  $[0, \infty)$  et possède un atome, le nombre de géodésiques entre deux sommets a une croissance exponentielle en la distance entre les sommets. On revient sur ce résultat dans la section 1.3.4.1.

## 1.2.2 Taille des géodésiques.

La taille des géodésiques est contrôlée par le résultat suivant lorsque  $\mathcal{L}(0) < p_c$ .

**Théorème 1.2.2** (Théorème 4.6 de [2]). *Supposons que  $\mathcal{L}(0) < p_c$  et  $\mathcal{L}(\infty) = 0$ . Alors il existe deux constantes strictement positives  $M$  et  $C$  telles que*

$$\mathbb{P}(m(x) \geq M\|x\|_1) \leq \exp\left(-C\|x\|_1^{\frac{1}{d}}\right) \text{ pour tout } x \in \mathbb{Z}^d,$$

où  $m(x) = \max\{|\gamma| : \gamma \text{ est une géodésique de } 0 \text{ à } x\}$ .

L'étude de la différence de taille entre les géodésiques entre deux sommets dans un cadre où il n'y a pas unicité est l'objet de la section 1.3.4.2.

## 1.2.3 Résultat d'Andjel et Vares

Rappelons (1.1.5). Lorsque la loi  $\mathcal{L}$  est une loi sur  $[0, \infty)$ , utile et d'espérance finie, en notant  $\mu_x(\varepsilon_1)$  la constante de temps pour la loi obtenue en tronquant la loi  $\mathcal{L}$  à  $x$ , on obtient pour tout  $x \in \mathbb{R}_+$

$$\mu_x(\varepsilon_1) < \mu(\varepsilon_1).$$

Cela suggère qu'il est plus efficace pour aller d'un sommet à un autre sommet distant d'emprunter une certaine proportion strictement positive d'arêtes avec un temps de passage très grand plutôt que d'essayer de toujours les éviter. Le résultat suivant d'Andjel et Vares [1] confirme et précise cette vision.

**Théorème 1.2.3** (Théorème 2.3 de [1]). *Supposons que  $\mathcal{L}$  soit une loi utile sur  $[0, \infty)$  avec un support non borné. Alors, pour tout  $M$  strictement positif, il existe  $\varepsilon = \varepsilon(M) > 0$  et  $\alpha = \alpha(M) > 0$  tel que pour tout  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P}\left(\exists \text{ une géodésique } \gamma \text{ de } 0 \text{ à } x \text{ telle que } \sum_{e \in \gamma} \mathbf{1}_{T(e) \geq M} \leq \alpha\|x\|_1\right) \leq e^{-\varepsilon\|x\|_1}.$$

## 1.2.4 Mesure empirique des temps le long des géodésiques

Bates [3] établit plusieurs énoncés du type suivant. Fixons une direction  $\xi \in \mathbb{S}^{d-1}$ . Pour une loi  $\mathcal{L}$  générique et raisonnable, il existe une mesure de probabilité  $\nu(\xi)$  telle que, avec

probabilité 1, la mesure empirique

$$\mu_n = \frac{1}{|\gamma_n|} \sum_{e \in \gamma_n} \delta_{T(e)}$$

converge vers  $\nu(\xi)$  où, pour tout  $n$ ,  $\gamma_n$  est l'une des géodésiques de 0 à  $\lfloor n\xi \rfloor$ .

Les mesures  $\nu(\xi)$  que l'on obtient ainsi sont nécessairement absolument continues par rapport à  $\mathcal{L}$ . Le résultat principal de cette thèse (le théorème 1.3.5) donne immédiatement que  $\mathcal{L}$  est absolument continue devant toute mesure  $\nu(\xi)$  apparaissant ainsi.

## 1.3 Résultats de la thèse

### 1.3.1 Cadre

Soit  $\mathcal{L}$  une loi sur  $[0, \infty)$ . Lorsque le support de la loi  $\mathcal{L}$  est non borné, le théorème 1.2.3 d'Andjel et Vares garantit sous certaines hypothèses qu'en dehors d'un événement de probabilité exponentiellement faible en la distance entre les sommets qu'elles relient, les géodésiques empruntent une proportion positive d'arêtes plus grandes que n'importe quel seuil qu'on s'est fixé. Supposons maintenant que la loi  $\mathcal{L}$  possède un atome que l'on note  $\kappa$ . Alors, presque sûrement, la configuration de 4 arêtes de temps égaux à  $\kappa$  de la figure 1.1 apparaît dans le graphe une infinité de fois. On peut se poser la question suivante.

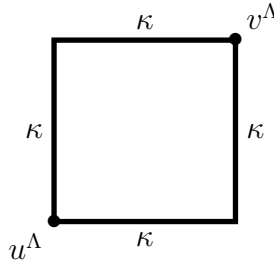


FIGURE 1.1 : Motif utilisé pour prouver le théorème 1 de [17] à l'aide du théorème 1.3.5. Les temps de passage des quatre arêtes du carré sont tous égaux à  $\kappa$ .

(Q) : A-t-on également qu'en dehors d'un événement de probabilité exponentiellement faible, toute géodésique emprunte un nombre linéaire de configurations identiques à celle de la figure 1.1 ?

Pour répondre à cette question, il faut commencer par donner un sens à "emprunter la configuration". Lorsque l'on dit qu'un chemin emprunte une arête  $e$ , la signification est claire : cela signifie que l'arête apparaît dans la suite d'arêtes qui compose le chemin. En revanche, pour la configuration de la figure 1.1, cela doit être précisé. Dans cet exemple, disons qu'un chemin emprunte cette configuration s'il passe par  $u^\Lambda$  et  $v^\Lambda$  et si la portion

de ce chemin entre  $u^\Lambda$  et  $v^\Lambda$  emprunte consécutivement l'arête du bas puis celle de droite ou consécutivement l'arête de gauche puis celle du haut. Supposons que la réponse à la question ( $\mathcal{Q}$ ) soit positive. Alors, puisqu'à chacune de ces configurations empruntée par une géodésique  $\gamma$  on obtient une autre géodésique en remplaçant la partie de  $\gamma$  reliant  $u^\Lambda$  et  $v^\Lambda$  par l'autre chemin de la configuration, on obtient que le nombre de géodésique entre deux sommets croît exponentiellement en la distance entre ces sommets. C'est le résultat du théorème 1 de Nakajima dans [17] et cela est présenté plus en détail dans la section 1.3.4.1.

D'autres résultats de ce type pour d'autres configurations particulières ont été montrés par van den Berg et Kesten dans [18] et par Krishnan, Rassoul-Agha et Seppäläinen dans [15]. L'un des objectifs de cette thèse est d'unifier et de généraliser ces résultats. La première étape pour cela est de définir la notion de motifs.

**Motifs.** De façon informelle, un motif est composé de quatre éléments :

- une boîte, c'est-à-dire l'ensemble de sommets (ou d'arêtes) dans lequel on spécifie l'environnement. Dans l'exemple précédent : l'ensemble des quatre arêtes formant le carré de la figure 1.1,
- un événement qui ne dépend que des arêtes de la boîte donnée ci-dessus. Dans l'exemple précédent : l'événement sur lequel toutes les arêtes du carré de la figure 1.1 ont un temps de passage égal à  $\kappa$ ,
- deux sommets qui appartiennent au bord de la boîte et que l'on appelle points d'entrée et de sortie du motif. Ces derniers servent à donner un sens à "emprunter le motif". Dans l'exemple précédent : les sommets  $u^\Lambda$  et  $v^\Lambda$  sur la figure 1.1.

Formellement, la définition d'un motif est donnée ci-dessous. On se place dans cette section dans le cadre où on autorise  $\mathcal{L}(\infty)$  à éventuellement être strictement positif. Pour un ensemble  $B$  de sommets, on note  $\partial B$  son bord, c'est-à-dire l'ensemble des sommets qui sont dans  $B$  et qui sont reliés par une arête à un sommet qui n'est pas dans  $B$ . On fait un abus de langage en disant qu'une arête  $e = \{u, v\}$  appartient à un ensemble de sommets si  $u$  et  $v$  sont dans cet ensemble.

**Définition 1.3.1.** *On dit que  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  est un motif si les trois conditions suivantes sont satisfaites :*

- *il existe  $L_1, \dots, L_d$  des entiers positifs tels qu'au moins l'un d'entre eux est non nul (pour éviter les trivialisés) et tels que*

$$\Lambda = \prod_{i=1}^d \{0, \dots, L_i\},$$

- *$u^\Lambda$  et  $v^\Lambda$  sont deux sommets distincts appartenant au bord de  $\Lambda$ ,*
- *$\mathcal{A}^\Lambda$  est un événement qui ne dépend que des temps de passage des arêtes de  $\Lambda$ .*

**Chemin empruntant un motif.** Un motif est par définition ancré en 0. On s'intéresse à l'ensemble de ses translatés. Les translations portent à la fois sur la boîte, les points d'entrée et de sortie et l'événement. De façon informelle, on dit qu'un chemin emprunte un translaté donné du motif s'il passe par ses points d'entrée et de sortie (dans n'importe quel ordre) et si entre ces deux sommets, il est contenu dans la boîte. Dans la suite, par commodité, lorsque le motif original est fixé, on désigne également par "le motif" n'importe quel translaté de celui-ci. De façon formelle, on le définit de la manière suivante.

Soit  $x \in \mathbb{Z}^d$ . On définit :

- pour  $y \in \mathbb{Z}^d$ ,  $\theta_x y = y - x$ ,
- pour  $e = \{u, v\}$  une arête reliant deux sommets  $u$  et  $v$ ,  $\theta_x e = \{\theta_x u, \theta_x v\}$ .

De façon similaire, si  $\pi = (x_0, \dots, x_k)$  est un chemin, on définit  $\theta_x \pi = (\theta_x x_0, \dots, \theta_x x_k)$ . Enfin, on définit  $\theta_x T$  comme l'environnement  $T$  translaté de  $-x$ , c'est-à-dire la famille de variables aléatoires indexée par les arêtes de  $\mathbb{Z}^d$  définie pour tout  $e \in \mathcal{E}$  par

$$(\theta_x T)(e) = T(\theta_{-x} e).$$

**Définition 1.3.2.** Soit  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  un motif. Soit  $\pi$  un chemin auto-évitant et  $x \in \mathbb{Z}^d$ . On dit que  $x$  satisfait la condition  $(\pi; \mathfrak{P})$  si les deux conditions suivantes sont satisfaites :

1.  $\theta_x \pi$  visite les sommets  $u^\Lambda$  et  $v^\Lambda$ , et le sous-chemin de  $\theta_x \pi$  entre  $u^\Lambda$  et  $v^\Lambda$  est contenu dans  $\Lambda$ ,
2.  $\theta_x T \in \mathcal{A}^\Lambda$ .

Quand le motif est fixé et qu'il n'y a pas d'ambiguïté, on dit aussi " $\pi$  emprunte le motif dans  $\theta_{-x} \Lambda$ " au lieu de " $x$  satisfait la condition  $(\pi; \mathfrak{P})$ ". Notons que lorsque le chemin  $\pi$  est une géodésique, s'il emprunte le motif dans  $\theta_{-x} \Lambda$ , alors le sous-chemin de  $\theta_x \pi$  entre  $u^\Lambda$  et  $v^\Lambda$  est l'un des chemins optimaux entre  $u^\Lambda$  et  $v^\Lambda$  dans l'environnement  $\theta_x T$  parmi les chemins contenus dans  $\Lambda$ .

Pour tout chemin fini auto-évitant  $\pi$ , on note :

$$N^{\mathfrak{P}}(\pi) = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{x \text{ satisfait la condition } (\pi, \mathfrak{P})}. \quad (1.3.1)$$

Autrement dit,  $N^{\mathfrak{P}}(\pi)$  est le nombre de fois que  $\pi$  emprunte le motif. Quand le motif  $\mathfrak{P}$  est fixé et quand  $N^{\mathfrak{P}}(\pi) \geq 1$ , on dit simplement que le chemin  $\pi$  emprunte le motif. Avec ces définitions, l'objectif est alors d'étudier le comportement de  $N^{\mathfrak{P}}(\gamma)$  pour toutes les géodésiques  $\gamma$  de 0 à  $x$  quand  $\|x\|_1$  devient grand.

**Motif valable.** Pour commencer, remarquons qu'il existe des motifs que les géodésiques ne peuvent pas emprunter. Un exemple d'un tel motif, représenté dans la figure 1.2, est le



suisant. Prenons  $d = 2$  et  $\mathcal{L} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_4$ . Considérons le motif  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  où  $\Lambda = \{0, 1\} \times \{0, 1, 2, 3\}$ ,  $u^\Lambda = (0, 2)$ ,  $v^\Lambda = (0, 1)$  et  $\mathcal{A}^\Lambda$  est l'événement sur lequel toute arête de  $\Lambda$  adjacente à  $u^\Lambda$  ou à  $v^\Lambda$  a un temps de passage égal à 4 et toute autre arête de  $\Lambda$  a un temps de passage égal à 1. En utilisant les notations de la figure 1.2, on remarque que le seul chemin optimal entre  $u^\Lambda$  et  $v^\Lambda$  contenu dans le motif est le chemin  $(u^\Lambda, v^\Lambda)$ . Or, aucun chemin parmi  $(a, u^\Lambda, v^\Lambda)$ ,  $(b, u^\Lambda, v^\Lambda)$ ,  $(u^\Lambda, v^\Lambda, e)$  et  $(u^\Lambda, v^\Lambda, d)$  n'est une géodésique. Ainsi, toute géodésique qui emprunterait ce motif devrait contenir le chemin  $(c, u^\Lambda, v^\Lambda, f)$  mais ce chemin n'est pas une géodésique puisque le chemin  $(c, f)$  a un temps de passage plus court.

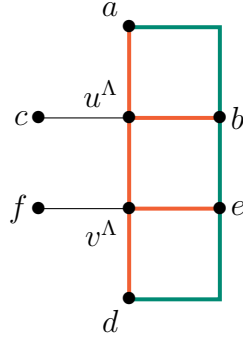


FIGURE 1.2 : Exemple d'un motif non valable. Les temps de passage des arêtes rouges valent 4 et ceux des arêtes vertes valent 1.

Cela nous conduit à définir la notion de motif valable qui sont des motifs ne présentant pas une incompatibilité à être empruntés par une géodésique. Nous avons choisi une définition simple à vérifier, peu contraignante et assurant que le motif valable peut être emprunté par une géodésique. Signalons néanmoins que ce n'est pas une condition nécessaire et suffisante.

**Définition 1.3.3.** Notons  $\{\varepsilon_1, \dots, \varepsilon_d\}$  les vecteurs de la base canonique. Un vecteur normal unitaire extérieur associé à un sommet  $z$  du bord de  $\Lambda$  est un élément  $\alpha$  de l'ensemble  $\{\pm\varepsilon_1, \dots, \pm\varepsilon_d\}$  tel que  $z + \alpha$  n'appartienne pas à  $\Lambda$ .

**Définition 1.3.4.** Un motif  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  est valable si les trois conditions suivantes sont satisfaites :

- $\mathcal{A}^\Lambda$  est réalisé avec probabilité strictement positive,
- quand  $\mathcal{A}^\Lambda$  est réalisé, il existe un chemin entre  $u^\Lambda$  et  $v^\Lambda$ , contenu dans  $\Lambda$ , dont le temps de passage est fini,
- l'une des deux conditions suivantes est satisfaite :
  - $\mathcal{L}$  a un support non borné,
  - il existe deux vecteurs normaux unitaires extérieurs distincts, l'un associé à  $u^\Lambda$ , l'autre à  $v^\Lambda$ .

Notons que la deuxième condition est toujours satisfaite lorsque  $\mathcal{L}(\infty) = 0$ . La troisième condition permet d'éliminer les motifs qui ne pourraient pas être empruntés par une géodésique, à l'image de celui de la figure 1.2.

### 1.3.2 Résultat principal de la thèse

Nous pouvons maintenant énoncer le résultat principal de la thèse. On se place dans le cadre où  $\mathcal{L}$  est une loi sur  $[0, \infty]$ . On étend la définition de loi utile à ce cadre : une loi  $\mathcal{L}$  sur  $[0, \infty]$  est utile si les deux conditions de la définition 1.1.6 sont satisfaites. Puisqu'on n'exclut pas le cas  $\mathcal{L}(\infty) > 0$ , il peut exister des sommets entre lesquels tous les chemins ont un temps de passage infini. On définit donc l'ensemble aléatoire suivant :

$$\mathfrak{C} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \exists \text{ un chemin } \pi \text{ entre } x \text{ et } y \text{ tel que } T(\pi) < \infty.\}$$

Le résultat principal de la thèse est le suivant.

**Théorème 1.3.5.** *Soit  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  un motif valable. Supposons que  $\mathcal{L}$  est utile (définition 1.1.6) et que  $\mathcal{L}([0, \infty)) > p_c$ . Alors il existe  $\alpha > 0$ ,  $\beta_1 > 0$  et  $\beta_2 > 0$  tels que pour tout  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P} \left( (0, x) \in \mathfrak{C} \text{ et } \exists \text{ une géodésique } \gamma \text{ de } 0 \text{ à } x \text{ telle que } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1 \right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

On retrouve comme cas particulier le théorème 1.2.3 en prenant le motif constitué d'une seule arête  $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  où  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  et  $\mathcal{A}^\Lambda$  est l'événement sur lequel l'unique arête a un temps plus grand que  $M$ .

**Comparaison avec d'autres résultats existant.** Montrer l'existence d'une constante  $c > 0$  telle que, pour tout  $n$  assez grand,

$$\mathbb{E} \left[ N^{\mathfrak{P}}(\gamma(n)) \right] \geq cn, \tag{1.3.2}$$

où  $\mathfrak{P}$  est un motif particulier fixé et où  $\gamma(n)$  est la première géodésique de 0 à  $n\varepsilon_1$  (où les géodésiques sont ordonnées dans un ordre quelconque donné), a été un résultat intermédiaire clé pour montrer plusieurs propriétés en percolation de premier passage. Le premier résultat établi suivant ce plan est le principe de comparaison de van den Berg et Kesten [18] énoncé dans le théorème 1.1.9. C'est donc également dans cet article que le premier résultat de type (1.3.2) est établi. Puis, dans [17], Nakajima prouve une version de (1.3.2) pour le motif de la figure 1.1 pour montrer que le nombre de géodésiques entre deux sommets croît exponentiellement en la distance entre les sommets si la loi possède un atome. Enfin, les résultats les plus récents de ce type apparaissent dans un article de Krishnan, Rassoul-Agha et Seppäläinen [15] (théorèmes 5.4 et 6.2 de [15]). Ils montrent (1.3.2) pour différents motifs et pour des géodésiques spécifiques puis utilisent ces résultats pour obtenir des propriétés sur la taille des géodésiques et sur la stricte concavité de la constante de temps vue comme

une fonction du décalage des temps de passage. Ces résultats sont exposés plus en détails dans la section 1.3.4 et le théorème 1.3.5 y est utilisé pour les renforcer.

Le théorème 1.3.5 est un résultat plus fort que les résultats du type (1.3.2) de [15], [17] et [18] sur trois aspects développés ci-dessous :

1. Il donne un résultat pour des motifs généraux alors que les résultats dans [15], [17] et [18] sont prouvés dans chaque cas pour un motif spécifique.
2. Dans le cas où il n'y a pas unicité des géodésiques, il donne un résultat pour toutes les géodésiques et pas seulement pour une géodésique particulière.
3. Il donne une croissance au moins linéaire du nombre de motifs empruntés par les géodésiques en dehors d'un événement de probabilité exponentiellement faible, et pas uniquement un résultat en espérance.

Depuis la preuve donnée par van den Berg et Kesten, il est envisagé que (1.3.2) doit être vrai pour tout motif raisonnable. Comme cela a été expliqué ci-dessus, (1.3.2) a en effet été démontré pour plusieurs motifs spécifiques dans [15], [17] et [18]. Dans une partie de la preuve de (1.3.2), on crée un nouvel environnement dans lequel les géodésiques sont contraintes d'emprunter le motif. Lorsque le support de  $\mathcal{L}$  est non borné, l'argument est relativement simple. Cependant, lorsque le support de  $\mathcal{L}$  est borné, cela devient beaucoup plus complexe. Ainsi, dans [15], [17] et [18], lorsque le support de  $\mathcal{L}$  est borné, chaque preuve est technique et utilise des propriétés spécifiques du motif considéré. La généralisation à tout motif valable, bien que naturellement attendue, présente des difficultés significatives et nécessite de développer de nouveaux arguments. Grâce au théorème 1.3.5, on peut par exemple, en exploitant la souplesse sur le choix du motif fournie par la théorème 1.3.5, généraliser le théorème 6.2 de [15] (voir le théorème 1.3.10 et en particulier la remarque à la fin de la section 1.3.4.2).

Dans [15], [17] et [18], (1.3.2) est seulement prouvée pour une géodésique spécifique. Cela n'a pas de conséquence sur les résultats principaux de [17] et [18]. Cependant, obtenir le résultat pour toute géodésique permet de renforcer un des résultats principaux de [15] dans le cas où le support de  $\mathcal{L}$  est borné. En effet, Krishnan, Rassoul-Agha et Seppäläinen s'intéressent au nombre minimal ou maximal d'arêtes des géodésiques mais ne peuvent pas prouver (1.3.2) pour les géodésiques ayant un nombre maximal d'arêtes. Avec le théorème 1.3.5, on peut renforcer le théorème 5.4 de [15] (voir le théorème 1.3.12).

La dernière différence avec les résultats de [15], [17] et [18] est que le résultat du théorème 1.3.5 est plus fort qu'un résultat en espérance. Il faut noter qu'un résultat en espérance est suffisant pour les applications de [17] et [18]. En revanche, il permet d'améliorer le théorème 6.2 de [15], ce qui est détaillé dans la section 1.3.4.2

### 1.3.3 Quelques remarques sur la preuve

**Renormalisation.** Donnons-nous un motif  $\mathfrak{P}$  et  $x \in \mathbb{Z}^d$  avec  $\|x\|_1$  grand. Considérons l'événement

$$\mathcal{M} = \{(0, x) \in \mathfrak{C} \text{ et il existe une géodésique de } 0 \text{ à } x \text{ qui n'emprunte pas le motif}\}.$$

L'objectif est de prouver que

$$\mathbb{P}(\mathcal{M}) \ll \frac{1}{\|x\|_1^{d-1}}. \quad (1.3.3)$$

Le théorème 1.3.5 se déduit facilement de ce résultat par un argument standard de renormalisation.

**Idée générale.** Pour simplifier les explications, plaçons-nous dans le cas où  $\mathcal{L}(\infty) = 0$ . Pour obtenir (1.3.3), l'idée est de définir une suite bien choisie d'événements  $\mathcal{M}(\ell)$  pour  $0 \leq \ell \leq q$ , telle que, pour une certaine constante positive  $c < 1$ ,

1.  $q \geq c\|x\|_1$ ,

2.  $\mathcal{M} \subset \mathcal{M}(q) \cup \mathcal{B}$  où  $\mathbb{P}(\mathcal{B}) \ll \frac{1}{\|x\|_1^{d-1}}$ ,

3. pour tout  $\ell \geq 1$ ,

$$\mathbb{P}(\mathcal{M}(\ell)) \leq c\mathbb{P}(\mathcal{M}(\ell - 1)). \quad (1.3.4)$$

Si les trois propriétés ci-dessus sont vraies, on obtient  $\mathbb{P}(\mathcal{M}) \leq c^{\|x\|_1} + \mathbb{P}(\mathcal{B})$ , puis (1.3.3). Le complémentaire de l'événement  $\mathcal{B}$  est approximativement "chaque géodésique emprunte suffisamment de bonnes boîtes" et ces bonnes boîtes sont des boîtes dans lesquelles l'environnement et les géodésiques se comportent de façon typique. Cela nous permet d'essayer de modifier l'environnement à l'intérieur de ces bonnes boîtes pour forcer toutes les géodésiques de 0 à  $x$  à y emprunter le motif. L'événement  $\mathcal{M}(\ell)$  est approximativement "il existe une géodésique de 0 à  $x$  qui n'emprunte pas le motif jusqu'à une distance d'ordre  $\ell$ ", avec un sens à préciser pour "distance d'ordre  $\ell$ ". Pensons pour le moment à une définition du type

$$\mathcal{M}(\ell) \approx \{\text{la première géodésique (dans l'ordre lexicographique}^1) \text{ n'emprunte pas le motif dans les } \ell \text{ premières bonnes boîtes qu'elle emprunte.}\} \quad (1.3.5)$$

La définition précise des événement  $\mathcal{M}(\ell)$  est choisie de sorte à avoir  $\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1)$  pour tout  $\ell$ . Ainsi (1.3.4) est équivalent à l'existence d'une constante  $\eta > 0$  (en prenant  $\eta = \frac{1}{c} - 1$ ) telle que

$$\mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) \geq \eta\mathbb{P}(\mathcal{M}(\ell)). \quad (1.3.6)$$

---

<sup>1</sup>L'ordre lexicographique est basé sur les directions des arêtes consécutives empruntées par les géodésiques.

Pour obtenir (1.3.6), nous faisons une modification dans un environnement dans lequel  $\mathcal{M}(\ell)$  est réalisé pour obtenir un environnement dans lequel  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  est réalisé.

Il est naturel de faire la modification pour obtenir (1.3.6) dans la  $\ell$ -ième bonne boîte empruntée par l'une des géodésiques de 0 à  $x$  et ainsi de forcer toutes les géodésiques de 0 à  $x$  à emprunter le motif dans cette  $\ell$ -ième boîte. Lorsque les temps de passage des arêtes sont non bornés, on peut jouer sur des écarts arbitrairement grands entre petits et grands temps pour forcer une géodésique à suivre un chemin précis. En revanche, lorsque le support de  $\mathcal{L}$  est borné, la modification est beaucoup plus complexe. C'est une des principales difficultés de la preuve du théorème 1.3.5 et les idées de la modification dans ce cas sont exposées dans la section 2.3.4.1.

Revenons maintenant à la définition des événements  $\mathcal{M}(\ell)$ . Une définition du type de (1.3.5) permet d'obtenir un résultat pour la première géodésique dans l'ordre lexicographique, mais ne permet pas d'obtenir un résultat pour toutes les géodésiques. La définition de  $\mathcal{M}(\ell)$  est donc nécessairement différente. De plus, il ne suffit pas de s'assurer que, dans l'environnement modifié, toutes les géodésiques de 0 à  $x$  empruntent le motif dans la boîte dans laquelle on a modifié l'environnement. Il s'agit de s'assurer que l'événement  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  est réalisé. Cela nécessite de contrôler l'influence de la modification sur l'ensemble des géodésiques et des boîtes qu'elles croisent. Cette difficulté n'apparaît pas lorsque l'on cherche à montrer seulement un résultat en espérance du type (1.3.2). Elle apparaît d'une part pour obtenir que le nombre de motifs empruntés par une géodésique est linéaire en dehors d'un événement de probabilité exponentiellement faible mais aussi pour obtenir un résultat pour toutes les géodésiques.

**Résultats des chapitres 2 et 3.** La preuve du théorème 1.3.5 est en fait séparée en deux preuves distinctes selon les hypothèses que l'on met sur la loi  $\mathcal{L}$ . Ces deux preuves mettent en place la stratégie décrite ci-dessus de deux manières différentes et sont indépendantes. Les chapitres 2 et 3 correspondent à deux articles ([13] et [12]) et sont autonomes.

Une de ces deux preuves est mise en place dans le chapitre 2 et donne le résultat suivant. Les idées spécifiques à la stratégie développée dans cette preuve sont exposées dans la section 2.1.5.

**Théorème 1.3.6.** *Soit  $\mathcal{L}$  une loi sur  $[0, \infty)$ . Soit  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  un motif valable, supposons que la loi  $\mathcal{L}$  est utile et que l'une des deux conditions suivantes est satisfaite :*

- (I) *le support de  $\mathcal{L}$  est borné,*
- (II) *le support de  $\mathcal{L}$  est non borné et*

$$\mathbb{E} \min [T_1^d, \dots, T_{2d}^d] < \infty, \quad (1.3.7)$$

*où  $T_1^d, \dots, T_{2d}^d$  sont des variables aléatoires indépendantes de loi  $\mathcal{L}$ .*

*Alors, il existe  $\alpha > 0$ ,  $\beta_1 > 0$  et  $\beta_2 > 0$  tels que pour tout  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P} \left( \exists \text{ une géodésique } \gamma \text{ de } 0 \text{ à } x \text{ telle que } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1 \right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

L'autre preuve est donnée dans le chapitre 3 et permet d'obtenir un résultat lorsque l'une des deux conditions suivantes est satisfaite :

$$\begin{aligned} & \mathcal{L}(\infty) > 0 \text{ et } \mathcal{L}([0, \infty)) > p_c, \\ \text{ou } & \mathcal{L}(\infty) = 0 \text{ et le support de } \mathcal{L} \text{ est non borné,} \end{aligned} \tag{1.3.8}$$

On obtient le théorème suivant. Les idées de preuve spécifiques de ce cas sont exposées dans la section 3.1.4.

**Théorème 1.3.7.** *Soit  $\mathcal{L}$  une loi sur  $[0, \infty]$ . Soit  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  un motif valable. Supposons que  $\mathcal{L}$  est utile et vérifie (1.3.8). Alors il existe  $\alpha > 0$ ,  $\beta_1 > 0$  et  $\beta_2 > 0$  tels que pour tout  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P} \left( (0, x) \in \mathfrak{C} \text{ et } \exists \text{ une géodésique } \gamma \text{ de } 0 \text{ à } x \text{ telle que } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1 \right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

## 1.3.4 Applications

On détaille quelques résultats des articles [15], [17] et [18] qui sont ici renforcés par l'utilisation du théorème 1.3.5.

### 1.3.4.1 Nombre de géodésiques lorsque la loi $\mathcal{L}$ possède un atome

Dans [17], lorsque la loi  $\mathcal{L}$  est utile, possède un atome et vérifie  $\mathcal{L}(\infty) = 0$ , Nakajima prouve une version de (1.3.2) pour montrer que le nombre de géodésiques entre deux sommets croît exponentiellement en la distance entre les sommets. On se place ici dans le cadre où  $\mathcal{L}(\infty) \geq 0$ ,  $\mathcal{L}([0, \infty)) > p_c$  et  $\mathcal{L}$  possède un atome fini. On a alors le résultat suivant.

**Théorème 1.3.8** (Renforcement du théorème 1 de [17]). *Supposons que  $\mathcal{L}$  est utile, possède un atome fini et que  $\mathcal{L}([0, \infty)) > p_c$ . Alors il existe  $\alpha > 1$ ,  $\beta_1 > 0$  et  $\beta_2 > 0$  tels que pour tout  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P} \left( (0, x) \in \mathfrak{C} \text{ et il existe moins de } \alpha^{\|x\|_1} \text{ géodésiques de } 0 \text{ à } x \right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

On donne une preuve de ce théorème à l'aide du théorème 1.3.5. Supposons que  $\mathcal{L}$  vérifie les hypothèses du théorème 1.3.8. Notons  $\kappa$  un atome fini de  $\mathcal{L}$ . On considère le motif à quatre arêtes  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ , représenté par la figure 1.1, où

- $\Lambda = \{0, 1\} \times \{0, 1\} \times \prod_{i=3}^d \{0\}$ ,
- $u^\Lambda = (0, \dots, 0)$  et  $v^\Lambda = (1, 1, 0, \dots, 0)$ ,
- $\mathcal{A}^\Lambda$  est l'événement sur lequel le temps de passage de chaque arête de  $\Lambda$  est égal à  $\kappa$ .

Le théorème 1.3.5 assure que, en dehors d'un événement de probabilité exponentiellement faible en la distance entre les sommets qu'elle relie, toute géodésique emprunte un nombre

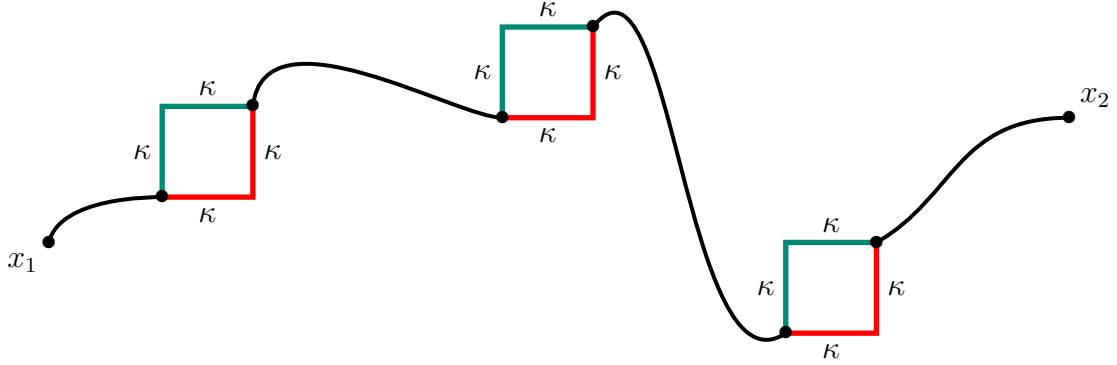


FIGURE 1.3 : Schéma illustrant l'idée de la preuve du théorème 1.3.8 à l'aide du théorème 1.3.5. Les translatés du motif sont en vert et rouge. On a ici 8 géodésiques entre  $x_1$  et  $x_2$ . Chacune d'entre elles est obtenue en suivant le chemin noir issu de  $x_1$  puis le chemin rouge ou le chemin vert dans le premier carré puis le chemin noir entre le premier et le deuxième carré, et ainsi de suite.

linéaire de ce motif. Cette situation est représentée par la figure 1.3. La clé est la remarque suivante : à chaque fois qu'une géodésique emprunte le motif, elle peut choisir, pour traverser ce motif, de suivre le chemin qui passe par l'arête du bas (le chemin rouge sur la figure 1.3) ou celui qui passe par l'arête du haut (le chemin vert sur la figure 1.3). Ainsi, si le nombre de motifs empruntés par les géodésiques est au moins linéaire, le nombre de géodésiques est au moins exponentiel. Le résultat s'en déduit.

### 1.3.4.2 Taille des géodésiques.

Considérons les deux hypothèses suivantes concernant la loi  $\mathcal{L}$  :

(H1) Il existe des entiers strictement positifs  $k$  et  $\ell$  et des atomes de  $\mathcal{L}$  finis  $r'_1, \dots, r'_{k+2\ell}$  et  $s'_1, \dots, s'_k$  (non nécessairement distincts) tels que

$$\sum_{i=1}^{k+2\ell} r'_i = \sum_{j=1}^k s'_j. \quad (1.3.9)$$

(H2)  $\mathcal{L}$  a un atome en 0 ou il existe des entiers strictement positifs  $k$  et  $\ell$  et deux atomes finis  $r < s$  tels que  $(k + 2\ell)r = ks$ .

Notons que l'hypothèse (H2) est strictement plus restrictive que (H1). Pour  $x \in \mathbb{Z}^d$ , on note  $\underline{L}_{0,x}$  (resp.  $\overline{L}_{0,x}$ ) la taille minimale (resp. maximale) des géodésiques de 0 à  $x$ . Dans [15], Krishnan, Rassoul-Agha et Seppäläinen prouvent le théorème suivant.

**Théorème 1.3.9** (Théorème 6.2 de [15]). *Supposons que  $\mathcal{L}(t_{\min}) < p_c$ ,  $\mathcal{L}(\infty) = 0$  et que pour des variables aléatoires indépendantes  $T_1, \dots, T_{2d}$  de loi  $\mathcal{L}$ , il existe  $p > 1$  tel que  $\mathbb{E} \min[T_1^p, \dots, T_{2d}^p] < \infty$ . De plus, supposons que l'une des deux hypothèses suivantes est satisfaite :*

- *le support de  $\mathcal{L}$  est non borné et (H1) est satisfaite,*
- *le support de  $\mathcal{L}$  est borné et (H2) est satisfaite.*

Alors, il existe des constantes  $0 < D, \delta, M < \infty$  telles que

$$\mathbb{P}(\bar{L}_{0,x} - \underline{L}_{0,x} \geq D\|x\|_1) \geq \delta \text{ pour } \|x\|_1 \geq M. \quad (1.3.10)$$

Le théorème 1.3.5 permet d'obtenir l'amélioration suivante.

**Théorème 1.3.10.** *Supposons que  $\mathcal{L}$  est utile, que  $\mathcal{L}([0, \infty)) > p_c$  et que l'hypothèse (H1) est satisfaite. Alors il existe des constantes  $0 < \beta_1, \beta_2, D < \infty$  telles que*

$$\mathbb{P}\left((0, x) \in \mathfrak{C} \text{ et } \bar{L}_{0,x} - \underline{L}_{0,x} < D\|x\|_1\right) \leq \beta_1 e^{-\beta_2 \|x\|_1}. \quad (1.3.11)$$

Cela améliore le théorème 1.3.9 puisque la majoration dans (1.3.11) est exponentiellement proche de 0 en la distance au lieu d'une minoration uniforme dans (1.3.10), puisque l'hypothèse de moment est supprimée, puisque dans le cas où le support de la loi  $\mathcal{L}$  est borné, on a une hypothèse moins restrictive, et puisque l'hypothèse  $\mathcal{L}$  utile est moins forte que l'hypothèse  $\mathcal{L}(t_{\min}) < p_c$ .

Lorsque  $\mathbb{E} \min[T_1^d, \dots, T_{2d}^d] < \infty$ , le théorème 1.3.10 est prouvé dans la section 2.4.1. On se propose de donner ici, en guise d'illustration des idées de preuve, une esquisse de la preuve dans un autre cadre qui n'est pas détaillé par la suite.

**Esquisse de preuve** Cette preuve est basée sur celle du théorème 6.2 de [15]. On suppose que l'une des deux hypothèses suivantes est satisfaite :

$$\begin{aligned} & \mathcal{L}(\infty) > 0 \text{ et } \mathcal{L}([0, \infty)) > p_c, \\ \text{ou } & \mathcal{L}(\infty) = 0 \text{ et le support de } \mathcal{L} \text{ est non borné,} \end{aligned} \quad (1.3.12)$$

On suppose que  $\mathcal{L}$  est utile et que l'hypothèse (H1) est satisfaite. On a donc des entiers  $k$  et  $\ell$  strictement positifs et des atomes finis  $r'_1, \dots, r'_{k+2\ell}, s'_1, \dots, s'_k$  tels qu'on ait l'égalité (1.3.9). L'idée est de créer un motif rectangulaire  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  de longueur  $k$  et de hauteur  $\ell$  à l'image de celui de la figure 1.4 puis de se servir de (1.3.9) pour créer l'événement  $\mathcal{A}^\Lambda$ . Si on reprend les notations de la figure 1.4, on peut définir l'événement  $\mathcal{A}^\Lambda$  comme l'événement sur lequel

$$T(\pi^+) = T(\pi^{++}) = \sum_{i=1}^{k+2\ell} r'_i = \sum_{j=1}^k s'_j,$$

et sur lequel toutes les arêtes en noir sur la figure



- ont un temps strictement plus grand que  $\sum_{i=1}^{k+2\ell} r'_i$  lorsque  $\mathcal{L}(\infty) = 0$ ,
- ont un temps égal à l'infini lorsque  $\mathcal{L}(\infty) > 0$ .

Par (1.3.9) et par (1.3.12), l'événement  $\mathcal{A}^\Lambda$  défini ci-dessus a une probabilité strictement positive. Le motif est valable.

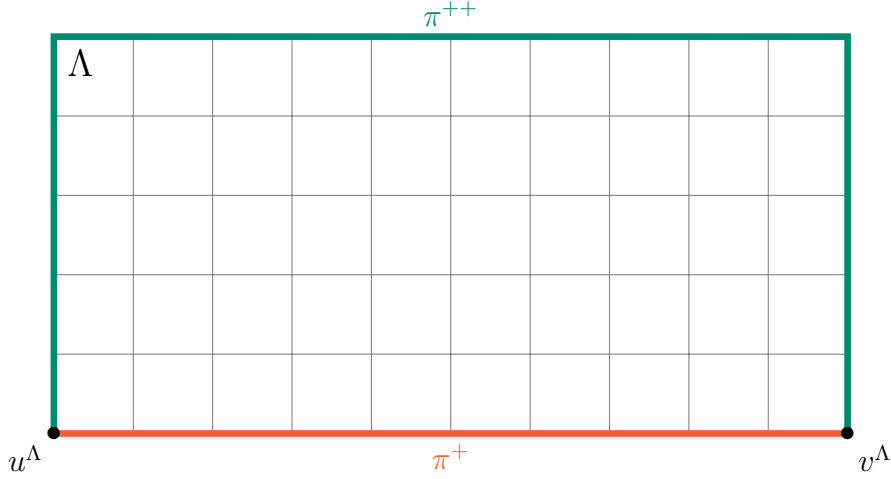


FIGURE 1.4 : Motif utilisé pour montrer le théorème 1.3.10. Le rectangle  $\Lambda$  est de longueur  $k$  et de hauteur  $\ell$ . Le chemin  $\pi^+$  en rouge emprunte un côté du rectangle et est de longueur  $k$ . Le chemin  $\pi^{++}$  en vert emprunte les trois autres côtés et est de longueur  $k + 2\ell$ . Ces deux chemins relient  $u^\Lambda$  et  $v^\Lambda$ .

Pour tout  $x \in \mathbb{Z}^d$ , lorsque  $(0, x) \in \mathfrak{C}$ , on note  $\gamma(x)$  la première géodésique de  $0$  à  $x$  dans l'ordre lexicographique parmi celles qui ont un nombre d'arêtes minimal. On note  $\mathcal{N}^{\mathfrak{P}}(\gamma(x))$  le nombre maximal de motifs disjoints empruntés par la géodésique  $\gamma(x)$ . Deux motifs sont dits disjoints s'ils correspondent à deux translatés du motif original n'ayant aucun sommet en commun. On fixe une telle liste de  $\mathcal{N}^{\mathfrak{P}}(\gamma(x))$  motifs disjoints. De simples considérations géométriques fournissent une constante  $c > 0$  telle que pour tout chemin  $\pi$ ,  $\mathcal{N}^{\mathfrak{P}}(\pi) \geq c\mathcal{N}^{\mathfrak{P}}(\pi)$ .

D'après la définition du motif  $\mathfrak{P}$ , lorsque l'événement  $\mathcal{A}^\Lambda$  est réalisé,  $\pi^+$  et  $\pi^{++}$  sont les deux seuls chemins de  $u^\Lambda$  à  $v^\Lambda$  entièrement contenus dans  $\Lambda$  dont le temps de passage est optimal. Par conséquent,  $\pi^+$  est l'unique chemin optimal en temps et en nombre d'arêtes entièrement contenu dans  $\Lambda$  et reliant  $u^\Lambda$  et  $v^\Lambda$ . Ainsi, pour chaque motif rencontré par  $\gamma(x)$ ,  $\gamma(x)$  emprunte le chemin  $\pi^+$ .

Lorsque  $(0, x) \in \mathfrak{C}$ , on peut alors définir, à partir de  $\gamma(x)$ , un nouveau chemin de  $0$  à  $x$ , que l'on note  $\hat{\gamma}(x)$ , en remplaçant dans chaque motif disjoint emprunté par  $\gamma(x)$  le chemin  $\pi^+$  par le chemin  $\pi^{++}$ . Supposons de plus que la propriété suivante soit vraie :

$\mathcal{P}$  : Lorsque  $\gamma(x)$  emprunte le motif,  $\gamma(x)$  n'emprunte aucun sommet du chemin  $\pi^{++}$  autre que  $u^\Lambda$  et  $v^\Lambda$ .

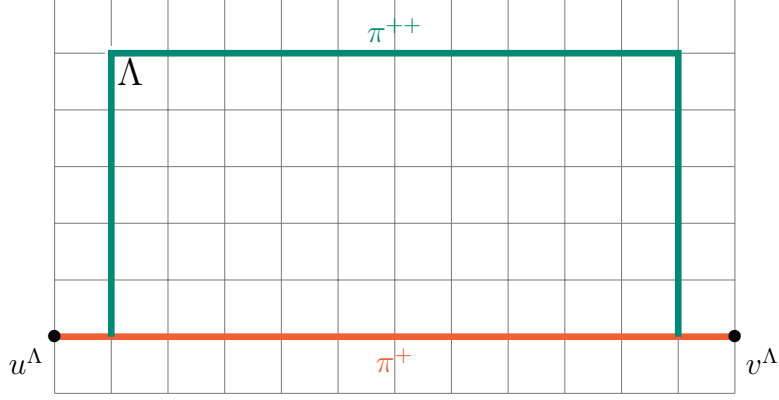


FIGURE 1.5 : Motif utilisé en dimension 2 pour obtenir la propriété  $\mathcal{P}$  dans la preuve du théorème 1.3.10.

Alors, le chemin  $\hat{\gamma}(x)$  est auto-évitant. De plus,  $\hat{\gamma}(x)$  a le même temps de passage que  $\gamma(x)$ . Par conséquent,  $\gamma(x)$  et  $\hat{\gamma}(x)$  sont deux géodésiques et on obtient

$$\bar{L}_{0,x} \geq |\hat{\gamma}(x)| \geq |\gamma(x)| + 2\mathcal{N}^{\mathfrak{P}}(\gamma(x)) \geq \underline{L}_{0,x} + 2\mathcal{N}^{\mathfrak{P}}(\gamma(x)). \quad (1.3.13)$$

Puis appliquant le théorème 1.3.5, on obtient  $\alpha > 0$ ,  $\beta_1 > 0$  et  $\beta_2 > 0$  tels que pour tout  $x \in \mathbb{Z}^d$ ,

$$\mathbb{P}((0, x) \in \mathfrak{C} \text{ et } \exists \text{ une géodésique } \gamma \text{ de } 0 \text{ à } x \text{ telle que } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

Alors, en prenant  $D = 2\alpha c$  et en utilisant (1.3.13), on obtient

$$\begin{aligned} \mathbb{P}\left((0, x) \in \mathfrak{C} \text{ et } \bar{L}_{0,x} - \underline{L}_{0,x} < D \|x\|_1\right) &\leq \mathbb{P}\left((0, x) \in \mathfrak{C} \text{ et } \mathcal{N}^{\mathfrak{P}}(\gamma(x)) < c\alpha \|x\|_1\right) \\ &\leq \mathbb{P}\left((0, x) \in \mathfrak{C} \text{ et } N^{\mathfrak{P}}(\gamma(x)) \leq \alpha \|x\|_1\right) \leq \beta_1 e^{-\beta_2 \|x\|_1}. \end{aligned}$$

**Propriété  $\mathcal{P}$ .** Lorsque  $\mathcal{L}(0) = 0$ , la propriété  $\mathcal{P}$  est vraie car dans ce cas un chemin de temps optimal est automatiquement auto-évitant. En revanche, lorsque  $\mathcal{L}(0) > 0$ , cela n'est plus vrai si certaines arêtes du chemin  $\pi^{++}$  ont un temps nul.

Supposons  $\hat{\gamma}(x)$  n'est pas un chemin auto-évitant, alors il existe un sommet  $w$  appartenant au chemin  $\pi^{++}$  d'un motif de la liste des motifs empruntés par  $\gamma(x)$  qui est emprunté deux fois par  $\hat{\gamma}(x)$ . Il existe alors un sous-chemin de  $\hat{\gamma}(x)$  empruntant au moins une arête reliant  $w$  à  $w$ . D'une part, ce chemin est de temps nul puisque  $\hat{\gamma}(x)$  est de temps optimal et d'autre part, il emprunte une arête reliant  $\pi^{++}$  à un sommet hors de  $\pi^+ \cup \pi^{++}$ .

Ainsi, si on s'assure que toutes les arêtes d'un sommet de  $\pi^{++}$  à un sommet hors de  $\pi^+ \cup \pi^{++}$  sont de temps non nul,  $\hat{\gamma}(x)$  est auto-évitant. Pour avoir cette propriété, on inclut le motif précédent dans un sur-motif (voir la figure 1.5).

Voici une description formelle. On définit le motif  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  de la manière suivante. On fixe  $L_1 = k + 2$ ,  $L_2 = \ell + 2$  et si  $d \geq 3$ , pour tout  $i \in \{3, \dots, d\}$ ,  $L_i = 2$ . On

prend  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ ,  $u^\Lambda = \sum_{i=2}^d \varepsilon_i$  et  $v^\Lambda = v^\Lambda + (k+2)\varepsilon_1$ . On note  $\pi^+$  le chemin allant de  $u^\Lambda$  à  $v^\Lambda$  en faisant  $k+2$  pas dans la direction  $\varepsilon_1$  et  $\pi^{++}$  le chemin allant de  $u^\Lambda$  à  $u^\Lambda + \varepsilon_1$  en faisant un pas dans la direction  $\varepsilon_1$ , puis jusqu'à  $u^\Lambda + \varepsilon_1 + \ell\varepsilon_2$  en faisant  $\ell$  pas dans la direction  $\varepsilon_2$ , puis jusqu'à  $u^\Lambda + (k+1)\varepsilon_1 + \ell\varepsilon_2$  en faisant  $k$  pas dans la direction  $\varepsilon_1$ , puis jusqu'à  $u^\Lambda + (k+1)\varepsilon_1$  en faisant  $\ell$  pas dans la direction  $-\varepsilon_2$  et enfin jusqu'à  $v^\Lambda$  en faisant un pas dans la direction  $\varepsilon_1$ . On définit  $\mathcal{A}^\Lambda$  comme l'événement sur lequel :

- $T(\pi^+) = T(\pi^{++}) = 2r'_1 + \sum_{i=1}^{k+2\ell} r'_i = 2r'_1 + \sum_{j=1}^k s'_j$ ,
- pour tout  $e \in \Lambda$  qui n'est pas dans  $\pi^+ \cup \pi^{++}$ ,  $T(e) = \infty$  si  $\mathcal{L}(\infty) > 0$  et  $T(e) > \sum_{i=1}^{k+2\ell} r'_i$  si  $\mathcal{L}(\infty) = 0$ .

On a bien que  $\mathcal{A}^\Lambda$  a une probabilité positive puisqu'on a supposé que (1.3.9) et (1.3.12) étaient satisfaites, et on a bien que  $\pi^+$  et  $\pi^{++}$  sont les deux seuls chemins de  $u^\Lambda$  à  $v^\Lambda$  entièrement contenus dans  $\Lambda$  dont le temps de passage est optimal. De plus, toutes les arêtes d'un sommet de  $\pi^{++}$  à un sommet hors de  $\pi^+ \cup \pi^{++}$  sont de temps non nul, ce qui permet d'obtenir la propriété  $\mathcal{P}$  pour ce nouveau motif et donc de conclure.

**Remarque sur le cas où le support de  $\mathcal{L}$  est borné.** Définir un événement  $\mathcal{A}^\Lambda$  sur lequel  $\pi^+$  et  $\pi^{++}$  font partie des chemins de  $u^\Lambda$  à  $v^\Lambda$  entièrement contenus dans  $\Lambda$  dont le temps de passage est optimal est plus compliqué lorsque le support de la loi  $\mathcal{L}$  est borné. C'est l'absence de preuve pour une propriété de ce type qui donne le résultat seulement avec l'hypothèse (H2) lorsque le support de  $\mathcal{L}$  est borné dans le théorème 1.3.9 de [15]. Pour obtenir cette propriété dans le cas borné avec l'hypothèse (H1) dans le théorème 1.3.10, on modifie la taille du motif et l'événement  $\mathcal{A}^\Lambda$ . Pour plus d'explications sur ce cas, on renvoie au lemme 2.4.1 et à la figure 2.5.

### 1.3.4.3 Stricte concavité de l'espérance des temps de passage comme fonction du décalage des temps de passages.

Pour  $b \in \mathbb{R}$ , on définit les temps de passage  $b$ -décalés par

$$T^{(b)} = \{T^{(b)}(e) : e \in \mathcal{E}\} \text{ avec } T^{(b)}(e) = T(e) + b \text{ pour tout } e \in \mathcal{E}.$$

En suivant les notations de [15] (et plus précisément celles de la section 2.2 de [15]), on note toutes les quantités associées aux temps de passage  $T^{(b)}$  avec  $(b)$  en exposant. Pour  $b$  négatif, certains temps de passage peuvent être négatifs. Cela conduit à adapter la définition du temps géodésique  $t^{(b)}(x, y)$  entre deux sommets  $x$  et  $y$  (donnée en (1.1.1)) en ne prenant l'infimum que sur la famille des chemins auto-évitant de 0 à  $x$ . Nous renvoyons à [15] et notamment à son annexe A pour les propriétés des temps  $t^{(b)}(0, x)$  lorsque les temps de

passage des arêtes peuvent être négatifs. Ci-dessous, la constante  $\varepsilon_0$  est la constante  $\delta$  pour laquelle (4.10) de [15] est vérifiée et la constante  $\delta_0$  est celle qui apparaît dans le lemme 5.2 de [15] (ou dans le lemme 5.5 de [18]). Dans [15], Krishnan, Rassoul-Agha et Seppäläinen prouvent le résultat suivant.

**Théorème 1.3.11** (Théorème 5.4 de [15]). *Supposons que  $\mathcal{L}(t_{\min}) < p_c$ . De plus supposons que l'une des deux hypothèses suivantes est satisfaite :*

- (a) *le support de  $\mathcal{L}$  est non borné et  $\mathbb{E}[\min(T_1, \dots, T_{2d})] < \infty$ , où  $T_1, \dots, T_{2d}$  sont des variables aléatoires i.i.d. de loi  $\mathcal{L}$ ,*
- (b) *le support de  $\mathcal{L}$  est borné et il existe un réel  $r_1$  dans le support de  $\mathcal{L}$  qui vérifie*

$$0 < r_1 < t_{\min} + \delta_0,$$

*Alors il existe une constante finie strictement positive  $M$  et une fonction  $D(b) > 0$  de  $b > 0$  telles que pour tout  $b \in (0, t_{\min} + \varepsilon_0)$  et tout  $\|x\|_1 \geq M$  :*

- (i)  *dans le cas (a) ci-dessus,*

$$\mathbb{E} [t^{(-b)}(0, x)] \leq \mathbb{E}[t(0, x)] - b\mathbb{E} [\bar{L}_{0,x}] - D(b)b\|x\|_1, \quad (1.3.14)$$

- (ii)  *dans le cas (b) ci-dessus,*

$$\mathbb{E} [t^{(-b)}(0, x)] \leq \mathbb{E}[t(0, x)] - b\mathbb{E} [\underline{L}_{0,x}] - D(b)b\|x\|_1. \quad (1.3.15)$$

Ce résultat est utilisé dans [15] pour montrer la stricte concavité de la constante de temps vue comme fonction du décalage  $b$  (voir le théorème 2.2 de [15]). La preuve du théorème 1.3.11 dans [15] repose sur un résultat du type (1.3.2). Dans le cas où le support de la loi est borné, en combinant le théorème 1.3.5 et la stratégie de [15], on obtient immédiatement le léger renforcement suivant. Notamment (1.3.14) est vraie dans les deux cas.

**Théorème 1.3.12.** *Supposons que  $\mathcal{L}(t_{\min}) < p_c$  et que*

$$\mathbb{E}[\min(T_1, \dots, T_{2d})] < \infty,$$

*où  $T_1, \dots, T_{2d}$  sont des variables aléatoires i.i.d. de loi  $\mathcal{L}$ . Supposons de plus que le support de  $\mathcal{L}$  contient au moins deux réels distincts strictement positifs. Alors, il existe une constante strictement positive finie  $M$  et une fonction  $D(b) > 0$  de  $b > 0$  telle qu'on ait l'inégalité suivante pour tout  $b \in (0, t_{\min} + \varepsilon_0)$  et tout  $\|x\|_1 \geq M$ ,*

$$\mathbb{E} [t^{(-b)}(0, x)] \leq \mathbb{E}[t(0, x)] - b\mathbb{E} [\bar{L}_{0,x}] - D(b)b\|x\|_1. \quad (1.3.16)$$

Le renforcement ci-dessus est rendu possible par le fait que le théorème 1.3.5 donne un résultat pour toutes les géodésiques, et donc en particulier pour la géodésique de 0 à  $x$  qui

possède le plus d'arêtes. Dans [15], la géodésique pour laquelle un résultat du type (1.3.2) est montré doit vérifier certaines conditions, ce qui ne permet pas de choisir celle qui possède le plus d'arêtes.

Le théorème 1.3.5 permet également de sortir du cadre où l'hypothèse

$$\mathbb{E}[\min(T_1, \dots, T_{2d})] < \infty$$

est satisfaite. Cependant, dans ce cas l'énoncé ne s'exprime plus en terme d'espérance du temps géodésique  $t(0, x)$ .

**Schéma de preuve.** Nous exposons dans ce paragraphe les idées de la preuve dans le cadre où le support de la loi  $\mathcal{L}$  est borné. Nous reprenons plus précisément l'étape 3 de la preuve du théorème 5.4 de [15]. On se place dans un cadre où  $\mathcal{L}(t_{\min}) < p_c$  et pour simplifier ce schéma de preuve, on suppose de plus que  $\mathcal{L}$  possède deux atomes strictement positifs  $r < s$ . Fixons  $b \in (0, t_{\min}]$ . Autrement dit on se limite, pour simplifier, au cas où les temps de passage restent positifs. Le lemme 5.5 dans [15] fournit deux entiers strictement positifs  $k$  et  $\ell$ , fixés pour le reste de la preuve, tels que

$$ks < (k + 2\ell)r < ks + (2\ell - 1)b. \quad (1.3.17)$$

L'idée est encore de créer un motif  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  rectangulaire de longueur  $k$  et de largeur  $\ell$  comme celui de la figure 1.4. Le chemin  $\pi^+$ , en rouge, possède  $k$  arêtes et le chemin  $\pi^{++}$ , en vert, en possède  $k + 2\ell$ . Ici, l'événement  $\mathcal{A}^\Lambda$  est l'événement sur lequel les arêtes du chemin  $\pi^{++}$  ont un temps de passage égal à  $r$  et toutes les autres arêtes ont un temps de passage égal à  $s$ . Dans ce cas, en utilisant (1.3.17), on peut montrer que  $\pi^+$  est l'unique chemin optimal en temps reliant  $u^\Lambda$  et  $v^\Lambda$  et étant entièrement contenu dans  $\Lambda$ .

Pour tout  $x$ , notons  $\gamma(x)$  la première géodésique de 0 à  $x$  dans l'ordre lexicographique parmi celles qui possèdent le plus d'arêtes. Notons  $\mathcal{N}^{\mathfrak{P}}(\gamma(x))$  le nombre maximal de motifs disjoints empruntés par  $\gamma(x)$ . On fixe une telle liste de  $\mathcal{N}^{\mathfrak{P}}(\gamma(x))$  motifs disjoints. On construit alors un nouveau chemin  $\hat{\gamma}(x)$  de 0 à  $x$  en remplaçant dans chacun des  $\mathcal{N}^{\mathfrak{P}}(\gamma(x))$  motifs disjoints empruntés par  $\gamma(x)$  le chemin  $\pi^+$  par le chemin  $\pi^{++}$ . Considérons maintenant les temps de passage  $(-b)$ -décalés. Dans chaque motif, par la définition du motif et en utilisant (1.3.17), on obtient les inégalités suivantes concernant les temps des chemins  $\pi^+$  et  $\pi^{++}$  :

$$T^{(-b)}(\pi^{++}) = T(\pi^{++}) - b|\pi^{++}| < T(\pi^+) + (2\ell - 1)b - b|\pi^{++}| = T^{(-b)}(\pi^+) - b. \quad (1.3.18)$$

On en déduit

$$\begin{aligned} T^{(-b)}(\hat{\gamma}(x)) &\leq T^{(-b)}(\gamma(x)) - b\mathcal{N}^{\mathfrak{P}}(\gamma(x)) \\ &= T(\gamma(x)) - b|\gamma(x)| - b\mathcal{N}^{\mathfrak{P}}(\gamma(x)) \\ &= t(0, x) - b\bar{L}_{0,x} - b\mathcal{N}^{\mathfrak{P}}(\gamma(x)), \end{aligned} \quad (1.3.19)$$

car  $|\gamma(x)| = \bar{L}_{0,x}$  puisque  $\gamma(x)$  fait partie des géodésiques de 0 à  $x$  qui possèdent le plus d'arêtes. Puisqu'on a fixé  $b$  de sorte que  $t_{\min} - b \geq 0$ , même si  $\hat{\gamma}(x)$  n'est pas auto-évitant, on a  $t^{(-b)}(0, x) \leq T^{(-b)}(\hat{\gamma}(x))$ , ce qui, combiné avec (1.3.19), donne

$$t^{(-b)}(0, x) \leq t(0, x) - b\bar{L}_{0,x} - b\mathcal{N}^{\mathfrak{P}}(\gamma(x)). \quad (1.3.20)$$

Puisque  $r$  et  $s$  sont des atomes pour la loi  $\mathcal{L}$ , le motif est valable. Par conséquent, on peut appliquer le théorème 1.3.5 et on obtient, en utilisant un argument géométrique pour se ramener à des motifs disjoints, l'existence d'une constante  $C > 0$  telle que

$$\mathbb{E} [\mathcal{N}^{\mathfrak{P}}(\gamma(x))] \geq C\|x\|_1 \text{ pour tout } \|x\|_1 \text{ assez grand.}$$

Ainsi, en prenant l'espérance dans (1.3.20), on obtient (1.3.16) pour  $D(b) = C$ .

Lorsqu'on n'a pas  $t_{\min} - b \geq 0$ , on modifie le motif donné ici pour obtenir un chemin  $\hat{\gamma}(x)$  auto-évitant. Pour plus de détails et pour un cadre plus général, on renvoie à la section 2.4.2.

#### 1.3.4.4 Application au principe de comparaison de van den Berg-Kesten

Le premier résultat de type (1.3.2) a été démontré par van den Berg et Kesten dans [18]. Leur objectif était de montrer le théorème 1.1.9 qui est le résultat principal de [18]. Le théorème 1.3.5 permet de prouver aisément le théorème 1.1.9 et d'en étendre le cadre d'application. Cette extension permet de comparer les constantes de temps de deux lois  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  données par le théorème 1.1.13 et est énoncée dans le théorème suivant.

**Théorème 1.3.13.** *Soient  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  deux lois sur  $[0, \infty]$  telles que*

- $\mathcal{L}$  est utile,
- $\mathcal{L}([0, \infty)) > p_c$  et  $\tilde{\mathcal{L}}([0, \infty)) > p_c$ ,
- $\mathcal{L} \neq \tilde{\mathcal{L}}$ ,
- il existe un couple de variables aléatoires  $(\tau, \tilde{\tau})$  définies sur un même espace probabilisé, de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$ , et telles que  $\mathbb{E}[\tilde{\tau}|\tau] \leq \tau$ .

Alors, en notant  $\mu$  (resp.  $\tilde{\mu}$ ) la constante de temps donnée par le théorème 1.1.13 pour la loi  $\mathcal{L}$  (resp. pour la loi  $\tilde{\mathcal{L}}$ ), on a pour tout  $x \in \mathbb{Z}^d$  tel que  $x \neq 0$ ,

$$\tilde{\mu}(x) < \mu(x). \quad (1.3.21)$$

Rappelons que Marchand [16] a étendu le théorème 1.1.9 dans une autre direction en affaiblissant l'hypothèse  $\mathcal{L}$  utile (voir le théorème 1.1.12). Pour obtenir ce résultat, Marchand a utilisé des résultats spécifiques, en particulier des grandes déviations pour la percolation orientée surcritique. Nous renvoyons aux commentaires du paragraphe sur la généralisation

de Marchand dans la section 1.1.3.2. Nous avons choisi pour l'instant de ne pas explorer cette direction.

La preuve du théorème 1.3.13 est donnée dans la section 3.3. Le schéma général de la preuve est le même que dans [18]. Signalons néanmoins que les motifs utilisés sont plus élémentaires et permettent une preuve plus directe.

Dans la suite de cette section, nous esquissons cette preuve en nous plaçant, pour simplifier, dans le cadre de [18]. Ainsi, on se donne deux lois  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  sur  $[0, \infty)$  vérifiant les hypothèses suivantes :

(A1)  $\mathcal{L}$  est utile,

(A2)  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  ont une espérance finie,

(A3)  $\mathcal{L} \neq \tilde{\mathcal{L}}$

(A4) il existe un couple de variables aléatoires  $(\tau, \tilde{\tau})$  définies sur un même espace probabilisé, de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$  et telles que

$$\mathbb{E} [\tilde{\tau} | \tau] \leq \tau. \quad (1.3.22)$$

En notant  $\mu(\varepsilon_1)$  (resp.  $\tilde{\mu}(\varepsilon_1)$ ) la constante de temps pour la loi  $\mathcal{L}$  (resp. pour la loi  $\tilde{\mathcal{L}}$ ), l'objectif est de montrer que

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

Notons que la preuve est identique en remplaçant  $\varepsilon_1$  par n'importe quel  $x \in \mathbb{Z}^d$  tel que  $x \neq 0$ .

En utilisant (A4), on peut se donner un couple  $(\tau, \tilde{\tau})$  de variables aléatoires de lois marginales respectives  $\mathcal{L}$  et  $\tilde{\mathcal{L}}$ , satisfaisant (1.3.22). On considère ensuite une famille  $(T, \tilde{T}) = \{(T(e), \tilde{T}(e)) : e \in \mathcal{E}\}$  de variables aléatoires i.i.d. définies sur un même espace probabilisé telles que, pour tout  $e \in \mathcal{E}$ ,  $(T(e), \tilde{T}(e))$  suit la même loi que  $(\tau, \tilde{\tau})$ .

Commençons par montrer l'inégalité large  $\tilde{\mu}(\varepsilon_1) \leq \mu(\varepsilon_1)$ . Pour tout  $n \in \mathbb{N}^*$ , notons  $\gamma_n$  la première géodésique de 0 à  $n\varepsilon_1$  dans l'environnement  $T$  dans un ordre arbitraire fixé. Notons  $\mathcal{G}$  la tribu engendrée par la famille  $(T(e))_{e \in \mathcal{E}}$ . Alors  $\gamma_n$  est  $\mathcal{G}$ -mesurable et

$$\begin{aligned} \mathbb{E} [\tilde{t}(0, n\varepsilon_1) | \mathcal{G}] &\leq \mathbb{E} [\tilde{T}(\gamma_n) | \mathcal{G}] \\ &= \sum_{e \in \gamma_n} \mathbb{E} [\tilde{T}(e) | T(e)] \\ &\leq \sum_{e \in \gamma_n} T(e) \text{ car pour tout } e \in \mathcal{E}, (T(e), \tilde{T}(e)) \text{ satisfait (1.3.22),} \\ &= T(\gamma_n) = t(0, n\varepsilon_1). \end{aligned}$$

En prenant l'espérance, on obtient

$$\mathbb{E} [\tilde{t}(0, n\varepsilon_1)] \leq \mathbb{E} [t(0, n\varepsilon_1)]. \quad (1.3.23)$$

Puis, puisque

$$\mu(\varepsilon_1) = \lim_{n \rightarrow \infty} \frac{\mathbb{E}[t(0, n\varepsilon_1)]}{n} \text{ et } \tilde{\mu}(\varepsilon_1) = \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\tilde{t}(0, n\varepsilon_1)]}{n},$$

on obtient que  $\tilde{\mu}(\varepsilon_1) \leq \mu(\varepsilon_1)$ .

Pour avoir l'inégalité stricte dans (1.3.23), une solution serait d'obtenir, par un raisonnement se basant sur ce qui vient d'être fait, une constante  $\eta > 0$  telle que

$$\mathbb{E}[\tilde{t}(0, n\varepsilon_1)] \leq \mathbb{E}[t(0, n\varepsilon_1)] - n\eta. \quad (1.3.24)$$

Pour cela, l'idée est d'utiliser un motif pour créer, à partir de  $\gamma_n$ , un chemin qui soit presque identique à  $\gamma_n$  sauf en un nombre linéaire en  $n$  d'endroits distincts où il y a un raccourci. Chacun de ces raccourcis, emprunté dans l'environnement  $\tilde{T}$ , permet de gagner du temps par rapport au temps dans l'environnement  $T$  de la portion de  $\gamma_n$  qui n'emprunte pas le raccourci.

Dans le cas où l'hypothèse

$$\mathbb{E}[\tilde{\tau}|\tau] = \tau \text{ p.s.} \quad (1.3.25)$$

n'est pas satisfaite, il existe un borélien  $I \subset [0, \infty)$  tel que  $\mathbb{P}(\tau \in I) > 0$  et tel que sur l'événement  $\{\tau \in I\}$ ,

$$\mathbb{E}[\tilde{\tau}|\tau] < \tau - \eta.$$

La preuve du théorème 1.1.9 est alors beaucoup plus simple et utilise le motif qui n'a qu'une seule arête  $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  où  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  et  $\mathcal{A}^\Lambda$  est l'événement sur lequel le temps de passage de l'unique arête du motif appartient à  $I$ . On se concentre dans la suite sur le cas le moins évident, c'est-à-dire que l'on suppose que l'hypothèse (1.3.25) est satisfaite. On obtient le lemme suivant.

**Lemma 1.3.14.** *Lorsque (1.3.25) est satisfaite, il existe un borélien  $I \subset (0, \infty)$  et une constante  $\eta > 0$  tels que  $\mathbb{P}(\tau \in I) > 0$  et*

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\tau_1, \tau_2, \tau_3, \tau_4] < \min(\tau_1 + \tau_2, \tau_3 + \tau_4) - \eta$$

sur l'événement  $\mathcal{I} = \{\tau_1, \dots, \tau_4 \in I\}$ , où  $(\tau_1, \tilde{\tau}_1), \dots, (\tau_4, \tilde{\tau}_4)$  sont des copies indépendantes de  $(\tau, \tilde{\tau})$ .

L'idée de la preuve du lemme 1.3.14 est d'exploiter le fait que la loi  $\tilde{\mathcal{L}}$  est plus variable que  $\mathcal{L}$ . La preuve de ce lemme est simple mais légèrement technique. Elle est donnée dans la section 3.3.1.3. Nous nous contentons ici de la donner dans un cas particulier très simple qui permet de comprendre la clé du lemme et, par là, le résultat de [18]. Supposons qu'on ait  $\mathbb{P}(\tau = 2) > 0$  et de plus

$$\mathbb{P}(\tilde{\tau} = 1|\tau = 2) = \mathbb{P}(\tilde{\tau} = 3|\tau = 2) = \frac{1}{2}.$$

On prend dans ce cas  $I = \{2\}$ . Un rapide calcul permet de montrer que sur  $\mathcal{I} = \{\tau_1 = \dots =$



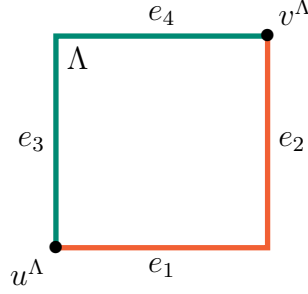


FIGURE 1.6 : Motif utilisé pour montrer le théorème 1.1.9 à l'aide du théorème 1.3.5 lorsque  $\mathbb{E}[\tilde{\tau}|\tau] = \tau$  p.s.

$\tau_4 = 2\}$ , on a

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\tau_1, \tau_2, \tau_3, \tau_4] = \frac{13}{4} < 4 = \min(\tau_1 + \tau_2, \tau_3 + \tau_4).$$

Fixons le borélien  $I \subset (0, \infty)$  et la constante  $\eta > 0$  donnés par le lemme 1.3.14. On définit le motif  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  où  $\Lambda$  est l'ensemble de 4 sommets défini par  $\Lambda = \{0, 1\} \times \{0, 1\} \times \prod_{j=3}^d \{0\}$ ,  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = \{1, 1, 0, \dots, 0\}$  et  $\mathcal{A}^\Lambda$  l'événement sur lequel  $T(e) \in I$  pour chaque arête  $e \in \Lambda$ . Ce motif est représenté par la figure 1.6. Les trois conditions pour que  $\mathfrak{P}$  soit valable sont satisfaites puisque :

- $u^\Lambda$  et de  $v^\Lambda$  ont été choisis sur des faces différentes,
- $I \subset [0, \infty)$ ,
- l'événement  $\mathcal{A}^\Lambda$  est réalisé avec probabilité strictement positive puisque  $\mathbb{P}(\tau \in I) > 0$ .

Pour tout  $n \in \mathbb{N}^*$ , notons  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  le nombre maximal de motifs disjoints empruntés par  $\gamma_n$ . Le théorème 1.3.5 s'applique. En effet,  $\mathcal{L}$  est utile d'après (A1),  $\mathcal{L}(\infty) = 0$  et le motif  $\mathfrak{P}$  est valable. Des arguments géométriques simples donnent alors l'existence d'une constante  $c > 0$  telle que pour tout  $n$  assez grand,

$$\mathbb{E}[\mathcal{N}^{\mathfrak{P}}(\gamma_n)] \geq cn. \quad (1.3.26)$$

Notons  $e_1 = \{u^\Lambda, u^\Lambda + \varepsilon_1\}$ ,  $e_2 = \{u^\Lambda + \varepsilon_1, v^\Lambda\}$ ,  $e_3 = \{u^\Lambda, u^\Lambda + \varepsilon_2\}$  et  $e_4 = \{u^\Lambda + \varepsilon_2, v^\Lambda\}$ . Ces arêtes sont représentées sur la figure 1.6. Fixons une liste de  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  motifs disjoints empruntés par  $\gamma_n$ . Puis, pour  $i \in \{1, \dots, \mathcal{N}^{\mathfrak{P}}(\gamma_n)\}$  et  $j \in \{1, \dots, 4\}$ , notons  $e_j^i$  le translaté de l'arête  $e_j$  appartenant au  $i$ -ième des  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  motifs disjoints empruntés par  $\gamma_n$ .

Pour  $i \in \{1, \dots, \mathcal{N}^{\mathfrak{P}}(\gamma_n)\}$ ,  $\gamma_n$  emprunte successivement  $e_1^i$  et  $e_2^i$ , ou successivement  $e_3^i$  et  $e_4^i$  (c'est-à-dire soit le chemin rouge, soit le chemin vert sur la figure 1.6). On note  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  l'ensemble des arêtes qui appartiennent à  $\gamma_n$  mais pas à l'un des chemins précédents emprun-

tés<sup>2</sup> par  $\gamma_n$ . L'ensemble  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  est l'ensemble des arêtes de  $\gamma_n$  qui ne sont pas remplacées par un raccourci dans notre argument. Pour tout  $n \in \mathbb{N}^*$ , on obtient d'une part

$$t(0, n\varepsilon_1) = T(\gamma_n) = \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{i=1}^{\mathcal{N}^{\mathfrak{P}}(\gamma_n)} \min \left( T(e_1^i) + T(e_2^i), T(e_3^i) + T(e_4^i) \right), \quad (1.3.27)$$

et d'autre part

$$\tilde{t}(0, n\varepsilon_1) \leq \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \tilde{T}(e) + \sum_{i=1}^{\mathcal{N}^{\mathfrak{P}}(\gamma_n)} \min \left( \tilde{T}(e_1^i) + \tilde{T}(e_2^i), \tilde{T}(e_3^i) + \tilde{T}(e_4^i) \right). \quad (1.3.28)$$

Alors, pour tout  $n \in \mathbb{N}^*$ , puisque  $\gamma_n$ ,  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  et  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  sont  $\mathcal{G}$ -mesurables,

$$\begin{aligned} \mathbb{E} \left[ \tilde{t}(0, n\varepsilon_1) | \mathcal{G} \right] &\leq \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[ \tilde{T}(e) | \mathcal{G} \right] + \sum_{i=1}^{\mathcal{N}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[ \min \left( \tilde{T}(e_1^i) + \tilde{T}(e_2^i), \tilde{T}(e_3^i) + \tilde{T}(e_4^i) \right) | \mathcal{G} \right] \\ &= \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{i=1}^{\mathcal{N}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[ \min \left( \tilde{T}(e_1^i) + \tilde{T}(e_2^i), \tilde{T}(e_3^i) + \tilde{T}(e_4^i) \right) | \mathcal{G} \right] \\ &\leq \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{i=1}^{\mathcal{N}^{\mathfrak{P}}(\gamma_n)} \min \left( T(e_1^i) + T(e_2^i), T(e_3^i) + T(e_4^i) \right) - \eta \\ &= T(\gamma_n) - \eta \mathcal{N}^{\mathfrak{P}}(\gamma_n) \\ &= t(0, n\varepsilon_1) - \eta \mathcal{N}^{\mathfrak{P}}(\gamma_n). \end{aligned}$$

Notons que

- l'inégalité de la première ligne vient de (1.3.28),
- l'égalité de la deuxième ligne du fait que pour tout  $e \in \mathcal{E}$ , le couple  $(T(e), \tilde{T}(e))$  a la même loi que  $(\tau, \tilde{\tau})$  qui vérifie (1.3.25),
- l'inégalité de la troisième ligne du fait que, pour tout  $i \in \{1, \dots, \mathcal{N}^{\mathfrak{P}}(\gamma_n)\}$ , le translaté de l'événement  $\mathcal{A}^{\Lambda}$  est réalisé dans l'ensemble des sommets reliés par les arêtes  $e_1^i$ ,  $e_2^i$ ,  $e_3^i$  et  $e_4^i$ , et donc par le lemme 1.3.14,

$$\mathbb{E} \left[ \min \left( \tilde{T}(e_1^i) + \tilde{T}(e_2^i), \tilde{T}(e_3^i) + \tilde{T}(e_4^i) \right) | \mathcal{G} \right] < \min \left( T(e_1^i) + T(e_2^i), T(e_3^i) + T(e_4^i) \right) - \eta,$$

- les égalités de la dernière ligne viennent de (1.3.27).

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<sup>2</sup>La géodésique  $\gamma_n$  emprunte un chemin si elle emprunte chacune des arêtes du chemin.

En prenant l'espérance, on obtient pour tout  $n$  assez grand,

$$\begin{aligned}\mathbb{E} [\tilde{t}(0, n\varepsilon_1)] &\leq \mathbb{E}[t(0, n\varepsilon_1)] - \eta \mathbb{E} [\mathcal{N}^{\mathfrak{P}}(\gamma_n)] \\ &\leq \mathbb{E}[t(0, n\varepsilon_1)] - c\eta n \text{ d'après (1.3.26)}.\end{aligned}$$

En divisant par  $n$  et en prenant la limite, on trouve

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1),$$

et le théorème 1.1.9 est prouvé.

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## Geodesics in first-passage percolation cross any pattern

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Ce chapitre correspond à l'article [13].

### Abstract

In first-passage percolation, one places nonnegative i.i.d. random variables  $(T(e))$  on the edges of  $\mathbb{Z}^d$ . A geodesic is an optimal path for the passage times  $T(e)$ . Consider a local property of the time environment. We call it a pattern. We investigate the number of times a geodesic crosses a translation of this pattern. Under mild conditions, we show that, apart from an event with exponentially small probability, this number is linear in the distance between the endpoints.

## 2.1 Introduction and main result

### 2.1.1 Settings

Fix an integer  $d \geq 2$ . In this article, we consider the model of first passage percolation on the hypercubic lattice  $\mathbb{Z}^d$ . We denote by  $0$  the origin of  $\mathbb{Z}^d$  and by  $\mathcal{E}$  the set of edges in this lattice. The edges in  $\mathcal{E}$  are those connecting two vertices  $x$  and  $y$  such that  $\|x - y\|_1 = 1$ . A finite path  $\pi = (x_0, \dots, x_k)$  is a sequence of adjacent vertices of  $\mathbb{Z}^d$ , i.e. for all  $i = 0, \dots, k-1$ ,  $\|x_{i+1} - x_i\|_1 = 1$ . We say that  $\pi$  goes from  $x_0$  to  $x_k$ . Sometimes we identify a path with the sequence of the edges that it visits, writing  $\pi = (e_1, \dots, e_k)$  where for  $i = 1, \dots, k$ ,  $e_i = \{x_{i-1}, x_i\}$ . We say that  $k$  is the length of  $\pi$  and we denote  $|\pi| = k$ .

The basic random object consists of a family  $T = \{T(e) : e \in \mathcal{E}\}$  of i.i.d. non-negative random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $T(e)$  represents the passage

time of the edge  $e$ . Their common distribution is denoted by  $F$ . The passage time  $T(\pi)$  of a path  $\pi = (e_1, \dots, e_k)$  is the sum of the variables  $T(e_i)$  for  $i = 1, \dots, k$ .

For two vertices  $x$  and  $y$ , we define the geodesic time

$$t(x, y) = \inf\{T(\pi) : \pi \text{ is a path from } x \text{ to } y\}. \quad (2.1.1)$$

A self-avoiding path  $\gamma$  between  $x$  and  $y$  such that  $T(\gamma) = t(x, y)$  is called a geodesic between  $x$  and  $y$ .

For the following and for the existence of geodesics, we need some assumptions on  $F$ . Let  $t_{\min}$  denote the minimum of the support of  $F$ . We recall a definition introduced in [18]. A distribution  $F$  with support in  $[0, \infty)$  is called useful if the following holds:

$$\begin{aligned} F(t_{\min}) &< p_c \text{ when } t_{\min} = 0, \\ F(t_{\min}) &< \vec{p}_c \text{ when } t_{\min} > 0, \end{aligned}$$

where  $p_c$  denotes the critical probability for the Bernoulli bond percolation model on  $\mathbb{Z}^d$  and  $\vec{p}_c$  the critical probability for the oriented Bernoulli bond percolation.

In the whole article, we assume that  $F$  has support in  $[0, \infty)$ , is useful, and that

$$\mathbb{E} \min [T_1^d, \dots, T_{2d}^d] < \infty, \quad (2.1.2)$$

where  $T_1, \dots, T_{2d}$  are independent with distribution  $F$ .

As  $F$  is useful,  $F(0) < p_c$ . By Proposition 4.4 in [2], we thus know that geodesics between any points exist with probability one.

## 2.1.2 Patterns

For a set  $B$  of vertices, we denote by  $\partial B$  its boundary, this is the set of vertices which are in  $B$  and which can be linked by an edge to a vertex which is not in  $B$ . We say that an edge  $e = \{u, v\}$  is contained in a set of vertices if  $u$  and  $v$  are in this set.

Let  $L_1, \dots, L_d$  be non-negative integers. To avoid trivialities we assume that at least one of them is positive. We fix  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$  and two distinct vertices  $u^\Lambda$  and  $v^\Lambda$  on the boundary of  $\Lambda$ . These points  $u^\Lambda$  and  $v^\Lambda$  are called endpoints. Then we fix an event  $\mathcal{A}^\Lambda$ , with positive probability, depending only on the passage times of the edges joining two vertices of  $\Lambda$ . We say that  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  is a pattern. Let  $x \in \mathbb{Z}^d$ . Define:

- for  $y \in \mathbb{Z}^d$ ,  $\theta_x y = y - x$ ,
- for an edge  $e = \{u, v\}$  connecting two vertices  $u$  and  $v$ ,  $\theta_x e = \{\theta_x u, \theta_x v\}$ .

Similarly, if  $\pi = (x_0, \dots, x_k)$  is a path, we define  $\theta_x \pi = (\theta_x x_0, \dots, \theta_x x_k)$ . Then  $\theta_x T$  denotes the environment  $T$  translated by  $-x$ , i.e. the family of random variables indexed by

the edges of  $\mathbb{Z}^d$  defined for all  $e \in \mathcal{E}$  by

$$(\theta_x T)(e) = T(\theta_{-x} e).$$

Let  $\pi$  be a self-avoiding path and  $x \in \mathbb{Z}^d$ . We say that  $x$  satisfies the condition  $(\pi; \mathfrak{P})$  if the following two conditions are satisfied:

1.  $\theta_x \pi$  visits  $u^\Lambda$  and  $v^\Lambda$ , and the subpath of  $\theta_x \pi$  between  $u^\Lambda$  and  $v^\Lambda$  is entirely contained in  $\Lambda$ ,
2.  $\theta_x T \in \mathcal{A}^\Lambda$ .

Note that, if  $x$  satisfies the condition  $(\pi; \mathfrak{P})$  and  $\pi$  is a geodesic, then  $(\theta_x \pi)_{u^\Lambda, v^\Lambda}$  is one of the optimal paths from  $u^\Lambda$  to  $v^\Lambda$  entirely contained in  $\Lambda$  in the environment  $\theta_x T$ . When the pattern is given, we also say " $\pi$  takes the pattern in  $\theta_{-x} \Lambda$ " for " $x$  satisfies the condition  $(\pi; \mathfrak{P})$ ". We denote:

$$N^{\mathfrak{P}}(\pi) = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\{x \text{ satisfies the condition } (\pi; \mathfrak{P})\}}. \quad (2.1.3)$$

Note that the number of terms in this sum is actually bounded from above by the number of vertices in  $\pi$ . If  $N^{\mathfrak{P}}(\pi) \geq 1$ , we say that  $\pi$  takes the pattern. The aim of the article is to investigate, under reasonable conditions on  $\mathfrak{P}$ , the behavior of  $N^{\mathfrak{P}}(\gamma)$  for all geodesics  $\gamma$  from 0 to  $x$  with  $\|x\|_1$  large. The first step is to determine these reasonable conditions, that is why we define the notion of valid patterns.

**Definition 2.1.1.** Denote by  $\{\varepsilon_1, \dots, \varepsilon_d\}$  the vectors of the canonical basis. An external normal unit vector associated to a vertex  $z$  of the boundary of  $\Lambda$  is an element  $\alpha$  of the set  $\{\pm\varepsilon_1, \dots, \pm\varepsilon_d\}$  such that  $z + \alpha$  does not belong to  $\Lambda$ .

**Definition 2.1.2.** We say that a pattern is valid if the following two conditions hold:

- $\mathcal{A}^\Lambda$  has a positive probability,
- the support of  $F$  is unbounded or there exist two distinct external normal unit vectors, one associated with  $u^\Lambda$  and one associated with  $v^\Lambda$ .

**Remark 2.1.3.** The existence of the two distinct vectors in the second condition of Definition 2.1.2 is equivalent to the fact that the endpoints of the pattern belong to two different faces. Note that a real obstruction can appear when this second condition is not satisfied. For example, take  $d = 2$ ,  $F = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_4$ ,  $\Lambda = \{0, 1\} \times \{0, 1, 2, 3\}$ ,  $u^\Lambda = (0, 2)$ ,  $v^\Lambda = (0, 1)$ , and  $\mathcal{A}^\Lambda$  the event on which for all edges  $e \in \Lambda$  such that  $e$  is adjacent to  $u^\Lambda$  or  $v^\Lambda$ ,  $T(e) = 4$  and for all other edges  $e$  of  $\Lambda$ ,  $T(e) = 1$ . This is the pattern of Figure 2.1 and in what follows in this remark, we use the notations of this figure. The only geodesic from  $u^\Lambda$  to  $v^\Lambda$  entirely contained in the pattern is  $(u^\Lambda, v^\Lambda)$ . However, neither  $(a, u^\Lambda, v^\Lambda)$ , nor  $(b, u^\Lambda, v^\Lambda)$ , nor  $(u^\Lambda, v^\Lambda, e)$ , nor  $(u^\Lambda, v^\Lambda, d)$  is a geodesic. Hence, every geodesic taking the pattern would contain the path  $(c, u^\Lambda, v^\Lambda, f)$  but this path is not a geodesic.



- $\Lambda = \{0, 1\} \times \{0, 1\} \times \prod_{i=3}^d \{0\}$ ,
- $u^\Lambda = (0, \dots, 0)$  and  $v^\Lambda = (1, 1, 0, \dots, 0)$ ,
- $\mathcal{A}^\Lambda$  the event on which the passage time of every edge of  $\Lambda$  is equal to  $\kappa$ .

The key fact about this pattern is the following: each time a geodesic takes the pattern, the geodesic can chose any of the two optimal paths between the endpoints.

Then, one of the most recent results of this kind appears in an article by Krishnan, Rassoul-Agha and Seppäläinen in [15] (see Theorem 5.4 and Theorem 6.2). They use (2.1.4) for some specific geodesics in order to get results about the Euclidean length of geodesics and the strict concavity of the expected passage times as a function of the weight shifts.

We explain in this section the differences between our result and the ones in [15], [17] and [18] but we give more details about [15] below in Section 2.1.4 as we wish to strengthen some of their results to illustrate the use of Theorem 2.1.4. Theorem 2.1.4 is stronger on three aspects that are commented below:

1. It deals with general patterns while the results in [15], [17] and [18] are stated for specific patterns.
2. In the case of non-uniqueness of geodesics, it gives the result for all geodesics and not only for a specific one.
3. It provides an at least linear growth of the number of crossed patterns out of an event of exponentially small probability.

Since the proof given by van den Berg and Kesten, it has been clear that (2.1.4) should be true for any reasonable pattern. As explained above, (2.1.4) has indeed been proven for several specific patterns in [15], [17] and [18]. In some part of the proof of (2.1.4), one needs to design a new environment in which the geodesics have to cross the pattern. When the support of  $F$  is unbounded, the argument is relatively straightforward. However, when the support of  $F$  is bounded, this is more involved. Actually, in [15], [17] and [18], when the support of  $F$  is bounded, each of the proofs is technical and makes use of specific properties of the considered pattern. The extension to any reasonable pattern, while naturally expected, actually requires new arguments and is a significant difficulty in the proof of Theorem 2.1.4. Thanks to Theorem 2.1.4, we can for example generalize Theorem 6.2 in [15]. See Theorem 2.1.8 below. Let us note however that the strategy developed in the bounded case in Section 2.4.1 to remove the restriction in Assumption 6.1 in [15] could be used in the proof of Theorem 6.2 in [15].

In [15], [17] and [18], (2.1.4) is proven only for a specific geodesic. This has no consequence on the main results of [17] and [18]. However obtaining a result for all geodesics enables to strengthen one of the main results of [15] in the bounded case. See Remark 2.1.10. Dealing with all geodesics is obtained thanks to a new idea using concentric annuli to define and localize good boxes (see Section 2.2.1).



The last difference with the results of [15], [17] and [18] is that our result is stronger than a result in expectation. However, notice that the result in expectation is sufficient for the applications in [17] and [18]. We refer to Section 2.1.4 for comments on [15].

A result fulfilling items 2 and 3 above appears in an article by Andjel and Vares [1] for the number of edges with large time crossed by a geodesic.

**Theorem 2.1.5** (Theorem 2.3 in [1]). *Let  $F$  be a useful distribution on  $[0, +\infty)$  with unbounded support. Then, for each  $M$  positive there exists  $\varepsilon = \varepsilon(M) > 0$  and  $\alpha = \alpha(M) > 0$  so that for all  $x$ , we have*

$$\mathbb{P} \left( \exists \text{ geodesic } \pi \text{ from } 0 \text{ to } x \text{ such that } \sum_{e \in \pi} \mathbf{1}_{T(e) \geq M} \leq \alpha \|x\|_1 \right) \leq e^{-\varepsilon \|x\|_1}. \quad (2.1.5)$$

Theorem 2.1.4 is a generalization of this theorem since, to get this result, we can take the pattern (reduced to one edge)  $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  where  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  and  $\mathcal{A}^\Lambda$  is the event on which the passage time of the only edge of the pattern is greater than  $M$ . The proof of Theorem 2.1.4 is partly inspired by the proof of this theorem and by the proof of (2.1.4) in [18].

Even if it is stated for distributions with unbounded support, one can check that Theorem 2.3 in [1] holds for  $F$  with bounded support with the same proof. As we need this extension in the proof of Theorem 2.1.4 we state it below.

**Theorem 2.1.6.** *Let  $F$  be a useful distribution on  $[0, +\infty)$  with bounded support. Then, for each  $M$  positive such that  $F([M, +\infty)) > 0$ , there exists  $\varepsilon = \varepsilon(M) > 0$  and  $\alpha = \alpha(M) > 0$  so that for all  $x$ , we have (2.1.5).*

## 2.1.4 Some applications

Several of the main results recently obtained in [15] are based on modification arguments leading to results of the type (2.1.4). We take advantage of Theorem 2.1.4 to slightly improve some of these results. The purpose of this section is primarily to illustrate the use of Theorem 2.1.4, the details of the proofs are postponed to Section 2.4.

### Euclidean length of geodesics

Consider the following two assumptions on the distribution  $F$ :

(H1) There exist strictly positive integers  $k$  and  $\ell$  and atoms  $r'_1, \dots, r'_{k+2\ell}, s'_1, \dots, s'_k$  (not necessarily distinct) such that

$$\sum_{i=1}^{k+2\ell} r'_i = \sum_{j=1}^k s'_j. \quad (2.1.6)$$

(H2) There exist strictly positive integers  $k$  and  $\ell$  and atoms  $r < s$  such that  $(k + 2\ell)r = ks$ , or  $F$  has an atom in 0.

Note that (H2) is strictly stronger than (H1). For  $x \in \mathbb{Z}^d$ , we denote by  $\underline{L}_{0,x}$  (resp.  $\overline{L}_{0,x}$ ) the minimal (resp. maximal) Euclidean length of self-avoiding geodesics from 0 to  $x$ . In [15], Krishnan, Rassoul-Agha and Seppäläinen prove the following theorem.

**Theorem 2.1.7** (Theorem 6.2 in [15]). *Assume that  $\mathbb{P}(T(e) = t_{\min}) < p_c$  and that for independent random variables  $T_1, \dots, T_{2d}$  with distribution  $F$ ,  $\mathbb{E} \min [T_1^p, \dots, T_{2d}^p] < \infty$  with  $p > 1$ . Furthermore, assume one of the following two assumptions:*

- *the support of  $F$  is unbounded and (H1) is satisfied,*
- *the support of  $F$  is bounded and (H2) is satisfied.*

*Then, there exist constants  $0 < D, \delta, M < \infty$  such that*

$$\mathbb{P} \left( \overline{L}_{0,x} - \underline{L}_{0,x} \geq D \|x\|_1 \right) \geq \delta \text{ for } \|x\|_1 \geq M. \quad (2.1.7)$$

We use Theorem 2.1.4 to prove the following result. It generalizes in a way Theorem 2.1.7 since in the case of bounded support, we have a less restrictive assumption and since the lower bound in (2.1.8) is exponentially close to one in the distance instead of the uniform lower bound in (2.1.7). However, the assumption on the moment is less restrictive in Theorem 2.1.7.

**Theorem 2.1.8.** *Assume that  $F$  is useful and  $\mathbb{E} \min [T_1^d, \dots, T_{2d}^d] < \infty$ . Furthermore, assume (H1). Then there exist constants  $0 < \beta_1, \beta_2, D < \infty$  such that*

$$\mathbb{P} \left( \overline{L}_{0,x} - \underline{L}_{0,x} \geq D \|x\|_1 \right) \geq 1 - \beta_1 e^{-\beta_2 \|x\|_1}. \quad (2.1.8)$$

The proof of this theorem is the aim of Section 2.4.1.

## Strict concavity of the expected passage times as a function of the weight shifts

For  $b \in \mathbb{R}$ , define the  $b$ -shifted weights by

$$T^{(b)} = \{T^{(b)}(e) : e \in \mathcal{E}\} \text{ with } T^{(b)}(e) = T(e) + b \text{ for all } e \in \mathcal{E}.$$

Following the notations of [15] (see Section 2.2 in [15]), all the quantities associated with the passage times  $T^{(b)}$  acquire the superscript. Further, the geodesic time  $t^{(b)}(x, y)$  between  $x$  and  $y$  is defined similarly to (2.1.1) but taking the infimum only over the family of self-avoiding paths from  $x$  to  $y$ . Theorem A.1 in [15] gives the existence of a constant  $\varepsilon_0 > 0$  with which we have an extension of the Cox-Durrett shape theorem for the shifted weights  $T^{(-b)}$  for  $b < t_{\min} + \varepsilon_0$  (note that here the weights can be negative). Note that (ii) in Theorem A.1 in [15] guarantees that  $\mathbb{E}[t^{(-b)}(0, x)]$  is finite if  $b \in (0, t_{\min} + \varepsilon_0)$ .

**Theorem 2.1.9.** *Assume  $F$  useful. Furthermore, assume that the support of  $F$  is bounded and that it contains at least two strictly positive reals. Then, there exists a finite positive*

constant  $M$  and a function  $D(b) > 0$  of  $b > 0$  such that the following bounds hold for all  $b \in (0, t_{\min} + \varepsilon_0)$  and all  $\|x\|_1 \geq M$ :

$$\mathbb{E}[t^{(-b)}(0, x)] \leq \mathbb{E}[t(0, x)] - b\mathbb{E}[\bar{L}_{0,x}] - D(b)b\|x\|_1. \quad (2.1.9)$$

**Remark 2.1.10.** In Theorem 2.1.9, we slightly strengthen Theorem 5.4 in [15] in the bounded case. Indeed,  $\mathbb{E}[\underline{L}_{0,x}]$  in [15] is replaced by  $\mathbb{E}[\bar{L}_{0,x}]$  in (2.1.9). This strengthening is made possible by the fact that Theorem 2.1.4 gives a result for all geodesics and thus, in particular, for the geodesic of maximal Euclidean length. We focus on the bounded case in Theorem 2.1.9 since Theorem 5.4 in [15] already contains (2.1.9) in the unbounded case.

## 2.1.5 Sketch of the proof

In what follows, we give an informal sketch of proof of Theorem 2.1.4. Fix a pattern  $\mathfrak{P}$  and  $x \in \mathbb{Z}^d$  with  $\|x\|$  large. Consider the event:

$$\mathcal{M} = \{\text{there exists a geodesic from } 0 \text{ to } x \text{ which does not take the pattern}\}.$$

The aim is to prove that  $\mathcal{M}$  has a probability small enough in  $\|x\|$ . More precisely, we want to prove

$$\mathbb{P}(\mathcal{M}) \ll \frac{1}{\|x\|^{d-1}}. \quad (2.1.10)$$

From this result, by a standard re-normalization argument, we easily get that, out of a very low probability event, every geodesic from 0 to  $x$  takes a number of patterns linear in  $\|x\|$  (see Proposition 2.1.11 in Section 2.1.6 for a formal statement of (2.1.10)).

**General idea.** The idea is to define a suitable sequence of events  $\mathcal{M}(\ell)$  for  $0 \leq \ell \leq q$  such that, for some positive constant  $c < 1$ ,

1.  $q \geq c\|x\|$ ,

2.  $\mathcal{M} \subset \mathcal{M}(q) \cup \mathcal{B}$  where  $\mathbb{P}(\mathcal{B}) \ll \frac{1}{\|x\|^{d-1}}$ ,

3. for all  $\ell \geq 1$ ,

$$\mathbb{P}(\mathcal{M}(\ell)) \leq c\mathbb{P}(\mathcal{M}(\ell - 1)). \quad (2.1.11)$$

If the above holds, we get  $\mathbb{P}(\mathcal{M}) \leq c^{\|x\|_1} + \mathbb{P}(\mathcal{B})$ , which allows us to get (2.1.10). The event  $\mathcal{M}(\ell)$  is approximately "there exists a geodesic from 0 to  $x$  which does not take the pattern until a distance of order  $\ell$ ", where we have to precise the sense of "distance of order  $\ell$ ". The complementary event of  $\mathcal{B}$  is approximately "each geodesic crosses enough good boxes" and these good boxes are the boxes in which the environment and the geodesics behave in a typical way. This enables us to try to modify the environment to ensure that all geodesics from 0 to  $x$  take the pattern inside. We take a good definition for the event  $\mathcal{M}(\ell)$  to have

$\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1)$  and thus (2.1.11) is equivalent to the existence of a constant  $\eta > 0$  (by taking  $\eta = \frac{1}{c} - 1$ ) such that

$$\mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) \geq \eta \mathbb{P}(\mathcal{M}(\ell)). \quad (2.1.12)$$

To get (2.1.12), we would like to make a modification in an environment where  $\mathcal{M}(\ell)$  occurs to get a new environment in which the event  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  occurs. This requires some stability in the definition of the events  $\mathcal{M}(\cdot)$  under the modification. This will be made clearer later.

**Associated geodesics.** We need the notion of associated paths. For the remaining of the sketch of the proof, "geodesic" means "geodesic from 0 to  $x$ ". Let  $B$  be a set of vertices (it is intended to be the "selected box", i.e. the box where we make the modification). Two paths  $\pi^1$  and  $\pi^2$  from 0 to  $x$  are  $B$ -associated if there exist  $a, b \in B$  such that:

1.  $\pi^1$  and  $\pi^2$  visit successively  $a$  and  $b$ ,
2.  $\pi_{0,a}^1 = \pi_{0,a}^2$  and  $\pi_{b,x}^1 = \pi_{b,x}^2$ ,
3.  $\pi_{a,b}^1$  and  $\pi_{a,b}^2$  are entirely contained in  $B$ ,

where  $\pi_{0,a}^i$  (resp.  $\pi_{a,b}^i, \pi_{b,x}^i$ ) denotes the subpath of  $\pi^i$  between 0 and  $a$  (resp.  $a$  and  $b$ ,  $b$  and  $x$ ), for  $i = 1, 2$ . In particular the two paths coincide outside  $B$ . With this definition, we can clearly enumerate the properties we want after the modification. Imagine we have a geodesic  $\gamma$  "selected" in a certain way and a "good" box  $B$  (where we want to make the modification) such that  $\gamma$  crosses  $B$ . The aim is to modify the environment in  $B$  such that:

1. Every geodesic in the new environment takes the pattern in  $B$ .
2. Every geodesic in the new environment is  $B$ -associated with a geodesic in the original environment.
3. The geodesic  $\gamma$  is  $B$ -associated with at least one geodesic in the new environment.

If we identify two geodesics  $B$ -associated, the last two properties can be rephrased as follows: we have not won new geodesics, we have not lost the geodesic  $\gamma$ . It is one of the keys of the stability. Getting these properties in the unbounded case is elementary but it is a significant difficulty in the bounded case (see Section 2.3.4.1 where we give the main ideas of the modification).

**Definition of the sequence  $\mathcal{M}(\ell)$ .** We would like to get a result on every geodesic (in the case where there is no uniqueness, which is a case we do not want to eliminate). A definition of the type:

$$\mathcal{M}(\ell) \approx \left\{ \begin{array}{l} \text{the first geodesic (in the lexicographical order) does not take the pattern} \\ \text{in its first } \ell \text{ good boxes} \end{array} \right\}$$

does not provide a result for all geodesics. However, it is possible to get a result for the first geodesic in the lexicographical order with this definition.

If every box crossed by geodesics were a good box, we could define a sequence of large concentric annuli centered in the origin and use a definition of the type:

$$\mathcal{M}(\ell) \approx \{\text{there exists a geodesic which does not take the pattern in the } \ell \text{ first annuli}\}.$$

Thus, we could choose one of these geodesics (let us denote it by  $\gamma$ ) and choose a box crossed by  $\gamma$  in the  $\ell$ -th annulus (let us denote it by  $B$ .) By making a modification giving the three properties stated above in the paragraph about the associated geodesics in the box  $B$ , in the new environment, the event  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  would occur. Indeed,  $\mathcal{M}(\ell)$  would not occur since every geodesic in the new environment would take the pattern in  $B$  and thus in the  $\ell$ -th annulus. However,  $\mathcal{M}(\ell - 1)$  would occur: this crucially uses the fact that  $\gamma$  is  $B$ -associated with a geodesic in the new environment (and the fact that the environment outside  $B$ , and thus outside the  $\ell$ -annulus, is not modified).

We could use the above definition if every geodesic crossed a good box in every annulus. Since it is not the case, we have to use a definition of the type:

$$\mathcal{M}(\ell) \approx \{\text{there exists a geodesic } \gamma \text{ which does not take the pattern in the union of the first } a_\ell(\gamma) \text{ annuli}\}$$

where  $a_\ell(\gamma)$  is the index of the  $\ell$ -th annulus in which  $\gamma$  crosses a good box (see Section 2.2.1 and Section 2.3.2).

**Modification.** Fix a positive integer  $\ell$ . Now, the aim is to define a general plan to prove (2.1.12). Here we do not discuss the modification itself (see Section 2.3.4.1 in the bounded case). Recall that we define the sequence  $\mathcal{M}(\ell)$  to have

$$\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1). \tag{2.1.13}$$

We denote by  $T$  the environment (i.e. the family of passage times on the edges) and by  $T'$  an independent copy of  $T$ . The basic idea consists in creating a modified environment  $T^* = \varphi(T, T')$  where  $T^*(e) = T(e)$  for some edges (whose passage times do not change) and  $T^*(e) = T'(e)$  for the other edges (whose passage times are re-sampled). In other words, we define a random set of edges  $R(T)$  (the edges we want to re-sample) and we set  $T^* = \varphi(T, T') = \varphi_{R(T)}(T, T')$  where, for every set of edges  $r$ ,  $\varphi_r(T, T')(e)$  is equal to  $T'(e)$  if  $e \in r$  and to  $T(e)$  else.

In a utopian situation, imagine that we could define a new environment  $T^* = \varphi(T, T')$  and an event  $\mathcal{A}$  (ensuring the success of the modification) such that:

- $T^*$  and  $T$  have the same distribution,
- $\eta := \mathbb{P}(T' \in \mathcal{A}) > 0$ ,
- and  $\{T \in \mathcal{M}(\ell) \text{ and } T' \in \mathcal{A}\} \subset \{T^* \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)\}$ .

Then we would have

$$\begin{aligned} \mathbb{P}(T \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) &= \mathbb{P}(T^* \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) \\ &\geq \mathbb{P}(T \in \mathcal{M}(\ell) \cap T' \in \mathcal{A}) = \eta \mathbb{P}(T \in \mathcal{M}(\ell)), \end{aligned}$$

which is (2.1.12).

However, this situation is unrealistic since the set of re-sampled edges  $R(T)$  depends on  $T$ , and thus the distribution of  $T^*$  is different from the distribution of  $T$ . But when  $r$  is fixed,  $\varphi_r(T, T')$  and  $T$  have the same distribution. It is possible to rely on this fact as soon as, observing only the modified environment, we can guess approximately in which box we performed the modification (see the use of the  $S^1$ -variables and  $S^2$ -variables in Lemma 2.2.3 and Lemma 2.3.3).

In the case where the passage times are bounded, we use two independent copies of  $T$  and we make a two-steps modification. The way we actually perform the modification in the bounded case is sketched in Section 2.3.4.1. The ideas described in this paragraph can be adapted to this two-steps modification without difficulties.

**Comparison with the plan of the proof of Proposition 5.22 in [18].** Let us compare the above strategy with the plan used in [18] to prove Proposition 5.22. Fix  $x$  in  $\mathbb{Z}^d$ . In [18], van den Berg and Kesten also start by associating with some specific geodesic  $\gamma$  some sequence of  $q = C\|x\|_1$  good boxes. By simple geometric arguments, they then get some family  $\mathcal{B}$  of boxes such that

$$\mathbb{E}[\text{number of boxes of } \mathcal{B} \text{ which are good and crossed by } \gamma] \geq c\|x\|_1$$

where  $c$  is a positive constant. Fix some box  $B \in \mathcal{B}$ . Then they also define a new environment  $T^*$  by resampling the times of the edges in  $B$ . It is then sufficient (this is the technical part of the proof in the bounded case) to prove

$$\mathbb{P}(\text{every geodesic in } T^* \text{ crosses the pattern in } B \mid \text{in the environment } T, \\ \gamma \text{ crosses } B \text{ and } B \text{ is good}) \geq \eta$$

for some positive constant  $\eta > 0$ . In particular, and contrary to what happens in our framework, it is not necessary in this setting to control what happens to geodesic(s) outside the considered box when we resample the times of the edges in the box. This is not a problem if the geodesic in the new environment completely changes.

**Comparison with the plan of the proof of Theorem 2.3 in [1].** In [1], the main difference with the strategy described above is the use of penalized geodesics. Indeed, Andjel and Vares only consider geodesics which do not take edges whose passage time is greater than  $M$  and it allows them to get a result on all geodesics from 0 to  $x$  thanks to the modification argument. However, it seems difficult to use penalized geodesics with the patterns, that is why we use the strategy of concentric annuli developed in Section 2.2.1.

## 2.1.6 Reduction

One can check that, using a standard re-normalization argument, Theorem 2.1.4 is a simple consequence of the following proposition (see for example the proof of Theorem 2.3 in [1]).

**Proposition 2.1.11.** *Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern and assume that  $F$  is useful and satisfies (2.1.2). Then there exist  $C > 0$  and  $D > 0$  such that for all  $n \geq 0$ , for all  $x$  such that  $\|x\|_1 = n$ ,*

$$\mathbb{P}\left(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0\right) \leq De^{-Cn^{\frac{1}{d}}}. \quad (2.1.14)$$

Thus, the aim of the paper is now to prove Proposition 2.1.11. Although they share some similarities, the proofs of Proposition 2.1.11 differ according to whether the support of  $F$  is bounded or unbounded. As the proof is easier in the unbounded case, we decide to first give the proof in the unbounded case in Section 2 and then give the proof in the bounded case in Section 3.

## 2.1.7 Some tools and notations

In this subsection, we recall some results and fix some notations. First, we denote by  $\mathbb{N}$  the set of all non-negative integers, by  $\mathbb{N}^*$  the set  $\mathbb{N} \setminus \{0\}$ , and by  $\mathbb{R}_+$  the set of all  $x \in \mathbb{R}$  such that  $x \geq 0$ .

For a self-avoiding<sup>2</sup> path  $\pi = (x_0, \dots, x_k)$  going from  $x_0$  to  $x_k$ , we say that  $x_i$  is visited by  $\pi$  before  $x_j$  if  $i < j$ ; we say that an edge  $\{x_i, x_{i+1}\}$  is visited before an edge  $\{x_j, x_{j+1}\}$  if  $i < j$ . A subpath of  $\pi$  going from  $x_i$  to  $x_j$  (where  $i, j \in \{0, \dots, k\}$  and  $i < j$ ) is the path  $(x_i, \dots, x_j)$  and is denoted by  $\pi_{x_i, x_j}$ .

For a set  $B$  of vertices, we denote by  $\partial B$  its boundary, this is the set of vertices of  $B$  which can be linked by an edge to a vertex which is not in  $B$ . Note that when we define a set of vertices of  $\mathbb{Z}^d$ , sometimes we also want to say that an edge is contained in this set. So, we say that an edge  $e = \{u, v\}$  is contained in a set of vertices if  $u$  and  $v$  are in this set. Since now a subset  $B$  of  $\mathbb{Z}^d$  can be seen as a set of vertices or as a set of edges, we denote by  $|B|_v$  the number of vertices of  $B$  and by  $|B|_e$  its number of edges.

Then, we define different balls in  $\mathbb{Z}^d$  or  $\mathbb{R}^d$ . For all  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ , we denote

$$\begin{aligned} B_\infty(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_\infty \leq r\}, \\ B_1(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_1 \leq r\}, \end{aligned}$$

and for  $n \in \mathbb{N}^*$ , we denote by  $\Gamma_n$  the boundary of  $B_1(0, n)$ , i.e.

$$\Gamma_n = \{u \in \mathbb{Z}^d : \|u\|_1 = n\}. \quad (2.1.15)$$

---

<sup>2</sup>The definition can be extended to not necessarily self-avoiding paths by saying that a vertex  $x$  is visited by  $\pi$  before  $y$  if there exists  $i_0 \in \{0, \dots, k\}$  such that  $x_{i_0} = x$  and for all  $j \in \{0, \dots, k\}$ ,  $x_j = y$  implies that  $j > i_0$ .

Also for  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ , we denote by  $B(c, r)$  the random ball

$$B(c, r) = \{u \in \mathbb{Z}^d : t(c, u) \leq r\}.$$

Then, for  $x$  and  $y$  in  $\mathbb{R}^d$ , we define  $t(x, y)$  as  $t(x', y')$  where  $x'$  is the unique vertex in  $\mathbb{Z}^d$  such that  $x \in x' + [0, 1)^d$  (similarly for  $y'$ ). For  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ , we denote by  $\tilde{B}(c, r)$  the random ball

$$\tilde{B}(c, r) = \{y \in \mathbb{R}^d : t(c, y) \leq r\}. \quad (2.1.16)$$

Let  $x$  in  $\mathbb{R}^d$ . Thanks to (2.1.2), we can define

$$\mu(x) = \lim_{n \rightarrow \infty} \frac{t(0, nx)}{n} \quad a.s. \quad (2.1.17)$$

Thanks to the hypothesis (2.1.2) and since  $F(0) < p_c(\mathbb{Z}^d)$ , for all  $x \in \mathbb{R}^d \setminus \{0\}$ , we have  $\mu(x) \in (0, \infty)$ . Furthermore,  $\mu$  is a norm on  $\mathbb{R}^d$  and describes the first order of approximation of  $\tilde{B}(0, r)$  when  $r$  goes to infinity. For  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ , we denote

$$B_\mu(c, r) = \{y \in \mathbb{R}^d : \mu(c - y) \leq r\}.$$

Fix  $\mathbf{B} = B_\mu(0, 1)$ , then the Cox-Durrett shape theorem (Theorem 2.16 in [2]) guarantees that for each  $\varepsilon > 0$ ,

$$\mathbb{P} \left( (1 - \varepsilon)\mathbf{B} \subset \frac{\tilde{B}(0, t)}{t} \subset (1 + \varepsilon)\mathbf{B} \text{ for all large } t \right) = 1. \quad (2.1.18)$$

One can easily prove that the result (2.1.18) is equivalent to

$$\lim_{\mu(x) \rightarrow \infty} \frac{t(0, x) - \mu(x)}{\mu(x)} = 0 \quad a.s. \quad (2.1.19)$$

Since  $\mu$  is a norm, we can fix two constants  $c_\mu > 0$  and  $C_\mu > 0$  such that for all  $y$  in  $\mathbb{R}^d$ ,

$$c_\mu \|y\|_1 \leq \mu(y) \leq C_\mu \|y\|_1.$$

Finally, since  $F$  is useful, by Lemma 5.5 in [18], there exist  $\delta = \delta(F) > 0$  and  $D_0 = D_0(F)$  fixed for the remaining of the article such that for all  $u, v \in \mathbb{Z}^d$ ,

$$\mathbb{P}(t(u, v) \leq (t_{\min} + \delta)\|u - v\|_1) \leq e^{-D_0\|u-v\|_1}. \quad (2.1.20)$$

Notice that, using the Borel-Cantelli lemma with this result, we get that for all  $u \in \mathbb{Z}^d$  different from 0,

$$\mu(u) \geq (t_{\min} + \delta)\|u\|_1. \quad (2.1.21)$$



## 2.2 Unbounded case

### 2.2.1 Proof of Proposition 2.1.11 in the unbounded case

Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern. It is convenient to reduce to the case where there exists an integer  $\ell^\Lambda > 0$ , fixed for the remaining of the proof, such that<sup>3</sup>  $\Lambda = B_\infty(0, \ell^\Lambda)$ . There is no loss of generality (see Lemma 2.A.1 in Appendix 2.A.1). Let us begin with the definitions of a typical box and of a successful box. To this end, we have to fix some constants.

**Boxes and constants.** Recall that the minimum of the support of  $F$  is denoted by  $t_{\min}$  and that  $\delta$  comes from (2.1.20). Since

$$\lim_{M \rightarrow \infty} \mathbb{P} \left( \mathcal{A}^\Lambda \text{ is realized and for all edges } e \in \Lambda, T(e) \leq M \right) = \mathbb{P}(\mathcal{A}^\Lambda) > 0,$$

there exists a positive constant  $M^\Lambda$  fixed for the rest of the proof such that

$$\mathbb{P} \left( \mathcal{A}^\Lambda \text{ is realized and for all edges } e \in \Lambda, T(e) \leq M^\Lambda \right) > 0.$$

Although we may need to replace  $\mathcal{A}^\Lambda$  by  $\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq M^\Lambda\}$ , we can now assume that

$$\mathcal{A}^\Lambda \subset \{\forall e \in \Lambda, T(e) \leq M^\Lambda\}. \quad (2.2.1)$$

We fix

$$\tau^\Lambda = M^\Lambda \|u^\Lambda - v^\Lambda\|_1, \quad (2.2.2)$$

which is an upper bound for the travel time of an optimal path (for the passage time) going from  $u^\Lambda$  to  $v^\Lambda$  and entirely contained in  $\Lambda$  on the event  $\mathcal{A}^\Lambda$ .

For  $i \in \{1, 2, 3\}$  and  $s \in \mathbb{Z}^d$ ,  $B_{i,s,N}$  is the ball in  $\mathbb{Z}^d$  of radius  $r_i N$  for the norm  $\|\cdot\|_1$  centered at the point  $sN$  where the constants  $r_i$  are defined as follows. We fix  $r_1 = d$ . Denote by  $K$  the number of edges in  $B_\infty(0, \ell^\Lambda + 3)$ . Then, fix  $r_2$  an integer such that

$$r_2 \delta - r_1(t_{\min} + \delta) - K t_{\min} - \tau^\Lambda > 0. \quad (2.2.3)$$

Let  $r_{2,3}$  be a positive real such that

$$B_{2,0,1} \subset B_\mu \left( 0, \frac{r_{2,3}}{2} \right) \cap \mathbb{Z}^d,$$

then we fix  $r_3$  an integer such that

$$B_\mu(0, 9r_{2,3}) \cap \mathbb{Z}^d \subset B_{3,0,1}.$$

These choices for  $r_2$  and  $r_3$  will become transparent in the proofs.

---

<sup>3</sup>We make a very slight abuse of notation: we also consider patterns where 0 is in the center of  $\Lambda$ .

We use the word "box" to talk about  $B_{3,s,N}$ . Recall that we denote by  $\partial B_{i,s,N}$  the boundary of  $B_{i,s,N}$ , that is the set of points  $z \in \mathbb{Z}^d$  such that  $\|z - sN\|_1 = r_i N$ . For  $u$  and  $v$  two vertices contained in  $B_{3,s,N}$ , we denote by  $t_{3,s,N}(u, v)$  the minimum of the times of all paths entirely contained in  $B_{3,s,N}$  and going from  $u$  to  $v$ .

**Crossed boxes and weakly crossed boxes.** We say that a path

- crosses a box  $B_{3,s,N}$  if it visits a vertex in  $B_{1,s,N}$ ,
- weakly crosses a box  $B_{3,s,N}$  if it visits a vertex in  $B_{2,s,N}$ .

**Paths associated in a box.** We say that two paths  $\gamma$  and  $\gamma'$  from 0 to the same vertex  $x$  are associated in a box  $B_{3,s,N}$  if there exist two distinct vertices  $s_1$  and  $s_2$  such that the following conditions hold:

- $\gamma$  and  $\gamma'$  visit  $s_1$  and  $s_2$ ,
- $\gamma_{0,s_1} = \gamma'_{0,s_1}$ ,
- $\gamma_{s_1,s_2}$  and  $\gamma'_{s_1,s_2}$  are entirely contained in  $B_{3,s,N}$ ,
- $\gamma_{s_2,x} = \gamma'_{s_2,x}$ .

In particular, these two paths coincide outside  $B_{3,s,N}$ .

**Typical boxes.** We define a sequence  $(\nu(N))_{N \in \mathbb{N}^*}$  such that for all  $N \in \mathbb{N}^*$ ,  $\nu(N) > M^\Lambda$  and

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \sum_{e \in B_{2,0,N}} T(e) \geq \nu(N) \right) = 0. \quad (2.2.4)$$

Note that  $F((\nu(N), +\infty)) > 0$  for all  $N \in \mathbb{N}^*$  since the support of  $F$  is unbounded.

We can now define typical boxes. A box  $B_{3,s,N}$  is typical if it verifies the following properties:

- (i)  $\mathcal{T}(s; N)$  is realized, where  $\mathcal{T}(s; N)$  is the following event:

$$\left\{ \sup_{z \in B_{2,s,N}} t_{3,s,N}(Ns, z) \leq r_{2,3} N \right\} \cap \left\{ \inf_{z \in \partial B_{3,s,N}} t_{3,s,N}(Ns, z) \geq 4r_{2,3} N \right\},$$

- (ii) every path  $\pi$  entirely contained in  $B_{3,s,N}$  from  $u_\pi$  to  $v_\pi$  with  $\|u_\pi - v_\pi\|_1 \geq (r_2 - r_1)N$  has a passage time verifying:

$$t(\pi) \geq (t_{\min} + \delta) \|u_\pi - v_\pi\|_1, \quad (2.2.5)$$

- (iii)  $\sum_{e \in B_{2,0,N}} T(e) < \nu(N)$ .

**Lemma 2.2.1.** *We have the following three properties about typical boxes.*

1. *Let  $s \in \mathbb{Z}^d$  and  $N \in \mathbb{N}^*$ . If  $B_{3,s,N}$  is a typical box, for all points  $u_0$  and  $v_0$  in  $B_{2,s,N}$ , every geodesic from  $u_0$  to  $v_0$  is entirely contained in  $B_{3,s,N}$ .*
2. *Let  $s \in \mathbb{Z}^d$  and  $N \in \mathbb{N}^*$ . The typical box property only depends on the passage times of the edges in  $B_{3,s,N}$ .*
3. *We have*

$$\lim_{N \rightarrow \infty} \mathbb{P}(B_{3,0,N} \text{ is a typical box}) = 1.$$

This lemma guarantees that the properties of a typical box are indeed typical ones and that they are also local ones. Its proof is in Section 2.2.2. Let us now introduce some further definitions.

**Successful boxes.** For a fixed  $x \in \mathbb{Z}^d$ , a box  $B_{3,s,N}$  is successful if every geodesic from 0 to  $x$  takes a pattern which is entirely contained in  $B_{2,s,N}$ , i.e. if for every geodesic  $\gamma$  going from 0 to  $x$ , there exists  $x_\gamma \in \mathbb{Z}^d$  satisfying the condition  $(\gamma; \mathfrak{P})$  and such that  $B_\infty(x_\gamma, \ell^\Lambda)$  is contained in  $B_{2,s,N}$ . Note that the notion of successful box depends on some fixed  $x \in \mathbb{Z}^d$ .

**Annuli.** Now, we define the annuli mentioned in Section 2.1.5. Fix

$$r = 2(r_1 + r_3 + 1), \tag{2.2.6}$$

and for all integers  $i \geq 1$ , let us define

$$A_{i,N} = \left\{ y \in \mathbb{Z}^d : \|y\|_1 \in [(i-1)rN, irN) \right\}.$$

For any annulus  $A_{i,N}$ , we call  $\{y \in \mathbb{Z}^d : \|y\|_1 = (i-1)rN\}$  its inner sphere and  $\{y \in \mathbb{Z}^d : \|y\|_1 = irN\}$  its outer sphere. Then, we give two definitions about these annuli which are useful in the proof of Lemma 2.2.3.

- For  $i > 1$ , we say that a path from 0 to a vertex of  $\mathbb{Z}^d$  crosses (resp. weakly crosses) a box  $B_{3,s,N}$  in the annulus  $A_{i,N}$  if the following two conditions are satisfied:
  - it crosses (resp. weakly crosses) this box before it visits for the first time the outer sphere of  $A_{i,N}$ ,
  - $B_{3,s,N}$  is entirely contained in the annulus, i.e. every vertex of  $B_{3,s,N}$  is in  $A_{i,N}$ .
- We also say that a path takes a pattern in the annulus  $A_{i,N}$  if it takes a pattern which is entirely contained in  $A_{i,N}$ , i.e. if every vertex of this pattern is in  $A_{i,N}$ . Here, we do not require that the path takes a pattern before it visits the outer sphere of  $A_{i,N}$  for the first time.

Note that the choice of  $r$  guarantees that every path passing through an annulus has to cross a box in this annulus.

For all integer  $p \geq 2$  and all  $N \in \mathbb{N}^*$ , we denote by  $\mathcal{G}^p(N)$  the event on which for all  $x$  in the outer sphere of the  $p$ -th annulus, every geodesic from 0 to  $x$  crosses a typical box in at least  $\lfloor \frac{p}{2} \rfloor$  annuli. The following lemma, whose proof is given in Section 2.2.2, gives us a useful bound on the probability of the complement of  $\mathcal{G}^p(N)$ .

**Lemma 2.2.2.** *There exist two positive constants  $C_1$  and  $D_1$ , and an integer  $N_0 \geq 1$  such that for all  $p \geq 2$  and  $N \geq N_0$ ,*

$$\mathbb{P}(\mathcal{G}^p(N)^c) \leq D_1 e^{-C_1 p^{\frac{1}{d}}}.$$

**Setup for the proof of Proposition 2.1.11.** For the rest of the proof, we fix  $C_1$ ,  $D_1$  and  $N_0$  given by Lemma 2.2.2. Recall that  $K$  is the number of edges in  $B_\infty(0, \ell^\Lambda + 3)$ . Then we fix  $\delta' > 0$  such that

$$r_2(\delta - \delta') - r_1(t_{\min} + \delta) - K(t_{\min} + \delta') - \tau^\Lambda > 0. \quad (2.2.7)$$

Note that it is possible since we have taken  $r_2$  large enough (see (2.2.3)). Then, fix

$$N \geq \max(N_0, \ell^\Lambda + 3), n \geq 2rN \text{ and } x \in \Gamma_n, \quad (2.2.8)$$

(where  $\Gamma_n$  is defined at (2.1.15)). Fix  $p = \lfloor \frac{n}{rN} \rfloor$  and  $q = \lfloor \frac{p}{2} \rfloor$ . Note that  $x$  belongs to the  $(p+1)$ -th annulus.

**$M$ -sequences.** Let us now define some random sets and variables which are useful for stability questions for the modification argument. Unless otherwise specified, in the remaining of this section, we write geodesic as a shorthand for geodesic from 0 to  $x$ .

First, let us associate a sequence of 0 to  $p-1$  typical boxes to every geodesic from 0 to  $x$ . For a geodesic  $\gamma$ , the deterministic construction is what follows.

Initialize the sequence as an empty sequence. For  $j$  from 1 to  $p-1$ , do:

- let  $a_j(\gamma)$  be the index of the first annulus such that  $\gamma$  crosses a typical box in this annulus and such that  $a_j(\gamma) > a_{j-1}(\gamma)$  (where  $a_0(\gamma) = 1$ ). If there is no such annulus, then we stop the algorithm.
- Add the first typical box crossed<sup>4</sup> by  $\gamma$  in the annulus  $A_{a_j(\gamma), N}$  to the sequence. Note that, since this typical box is crossed before  $\gamma$  leaves  $A_{a_j(\gamma), N}$  by the outer sphere for the first time, the  $j$ -th box of the sequence is crossed by  $\gamma$  after the  $(j-1)$ -th one.

So, we get a sequence of at most  $p-1$  boxes crossed by the geodesic. These boxes are all in different annuli. Furthermore, every box of this sequence is crossed by  $\gamma$  before  $\gamma$  leaves

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<sup>4</sup>If a path crosses two boxes  $B_{3,s_1,N}$  and  $B_{3,s_2,N}$ , we say that it crosses  $B_{3,s_1,N}$  before  $B_{3,s_2,N}$  if it visits a vertex of  $B_{1,s_1,N}$  before one of  $B_{1,s_2,N}$ .

the annulus containing it for the first time by the outer sphere. If the event  $\mathcal{G}^p(N)$  occurs, we know that all these sequences have at least  $q$  elements. For  $j \in \{1, \dots, p-1\}$ , we define a set of geodesics  $\Gamma^j$ . A geodesic  $\gamma$  from 0 to  $x$  belongs to  $\Gamma^j$  if:

- the lengths of its sequences defined above is greater than or equal to  $j$ ,
- $\gamma$  does not take the pattern in any annuli  $A_{k,N}$  with  $k \leq a_j(\gamma)$ .

We call the sequences defined above the  $M$ -sequences.

**Selected geodesic and  $S^1$ -variables.** Then, for  $j \in \{1, \dots, p-1\}$ , if  $\Gamma^j$  is not empty, we define the selected geodesic among the geodesics of  $\Gamma^j$  as the one which minimizes the index of the annulus containing the  $j$ -th box of its sequence. If there are several such geodesics, the selected one is the first in the lexicographical order<sup>5</sup>. Then, the random variable  $S_j^1$  is equal to the vertex  $s$  such that the box  $B_{3,s,N}$  is the  $j$ -th box in the  $M$ -sequence of the selected geodesic. When  $j$  is fixed, we say that the box  $B_{3,S_j^1,N}$  is the selected box. Finally, if  $\Gamma^j$  is empty, set  $S_j^1 = 0$  and there is no selected geodesic.

**$S^2$ -variables.** Let  $j \in \{1, \dots, p-1\}$ , we define  $S_j^2$  as follows. For every geodesic  $\gamma \in \Gamma^{j-1}$ , we define, if it is possible,  $S_j^2(\gamma)$  as the vertex  $s$  corresponding to the box  $B_{3,s,N}$  where  $B_{3,s,N}$  is the first successful box crossed by  $\gamma$  in an annulus (in the sense given with the definition of the annuli above). If it is not possible,  $S_j^2(\gamma) = 0$ . We denote by  $a'_j(\gamma)$  the index of the annulus containing  $B_{3,S_j^2(\gamma),N}$ . Then,  $S_j^2$  is equal to the vertex  $S_j^2(\bar{\gamma})$  where  $\bar{\gamma}$  satisfies the following three conditions:

- $a'_j(\bar{\gamma}) > 1$ ,
- for all geodesic  $\gamma \in \Gamma^{j-1}$  such that  $a'_j(\gamma) \neq 1$ , we have  $a'_j(\gamma) \geq a'_j(\bar{\gamma})$ ,
- $\bar{\gamma}$  is the first geodesic in the lexicographical order among the geodesics  $\gamma$  such that  $a'_j(\gamma) = a'_j(\bar{\gamma})$ .

If it is not possible,  $S_j^2 = 0$ .

**Modification argument.** Finally, for  $j \in \{1, \dots, p-1\}$ , we define  $\mathcal{M}(j)$  as the event on which every geodesic  $\gamma$  from 0 to  $x$  has at least  $j$  typical boxes in its  $M$ -sequence and there exists one geodesic which does not take the pattern in any annuli  $A_{k,N}$  with  $k \leq a_j(\gamma)$ . We also define  $\mathcal{M}(0)$  as the event on which there exists a geodesic from 0 to  $x$ . Its probability is equal to 1 (see Section 2.1.1). Now, the aim is to bound from above  $\mathbb{P}(\mathcal{M}(q))$  independently of  $x$  since we have:

$$\begin{aligned} \mathbb{P}\left(\text{there exists a geodesic } \gamma \text{ from 0 to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0\right) \\ \leq \mathbb{P}(T \in \mathcal{G}^p(N)^c) + \mathbb{P}(T \in \mathcal{M}(q)). \end{aligned} \quad (2.2.9)$$

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<sup>5</sup>The lexicographical order is based on the directions of the consecutive edges of the geodesics.

In the sequel, we introduce an independent copy  $T'$  of the environment  $T$ , the two being defined on the same probability space. It is thus convenient to refer to the considered environment when dealing with the objects defined above. To this aim, we shall use the notation  $\{T \in \mathcal{M}(j)\}$  to denote that the event  $\mathcal{M}(j)$  holds with respect to the environment  $T$ . In other words,  $\mathcal{M}(j)$  is now seen as a subset of  $(\mathbb{R}_+)^{\mathcal{E}}$ , where  $\mathcal{E}$  is the set of all the edges. Similarly, for  $i \in \{1, 2\}$  we denote by  $S_j^i(T')$  the random variables defined above but in the environment  $T'$ .

Fix  $\ell \in \{1, \dots, q\}$ . On  $\{T \in \mathcal{M}(\ell)\}$ ,  $\Gamma^\ell \neq \emptyset$  and  $B_{3, S_\ell^1(T), N}$  is a typical box crossed by the selected geodesic. We get a new environment  $T^*$  defined for all edge  $e$  by:

$$T^*(e) = \begin{cases} T(e) & \text{if } e \notin B_{2, S_\ell^1(T), N} \\ T'(e) & \text{else.} \end{cases}$$

For  $y$  and  $z$  in  $\mathbb{Z}^d$ , we denote by  $t^*(y, z)$  the geodesic time between  $y$  and  $z$  in the environment  $T^*$ . Note that  $T$  and  $T^*$  do not have the same distribution.

**Lemma 2.2.3.** *There exists  $\eta = \eta(N) > 0$  such that for all  $\ell$  in  $\{1, \dots, q\}$ , there exist measurable functions  $E_+^* : \mathbb{R}_+^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$  and  $E_-^* : \mathbb{R}_+^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$  such that:*

- (i)  $E_+^*(T) \cap E_-^*(T) = \emptyset$  and  $E_+^*(T) \cup E_-^*(T) \subset B_{2, S_\ell^1(T), N}$ ,
- (ii) on the event  $\{T \in \mathcal{M}(\ell)\}$ , we have  $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$  where  $\{T' \in \mathcal{B}^*(T)\}$  is a shorthand for

$$\{\forall e \in E_+^*(T), T'(e) \geq \nu(N), \forall e \in E_-^*(T), T'(e) \leq t_{\min} + \delta', \theta_{NS_\ell^1(T)} T' \in \mathcal{A}^\Lambda\},$$

- (iii)  $\{T \in \mathcal{M}(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\} \subset \{T^* \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell)\}$  and  $\|S_\ell^2(T^*) - S_\ell^1(T)\|_1 \leq 2r_3$ .

Lemma 2.2.3 is a consequence of Lemma 2.2.4 below whose proof is given in Section 2.2.3. Recall the definition of associated paths given page 47.

**Lemma 2.2.4.** *There exists  $\eta = \eta(N) > 0$  such that for all  $\ell$  in  $\{1, \dots, q\}$ , there exist measurable functions  $E_+^* : \mathbb{R}_+^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$  and  $E_-^* : \mathbb{R}_+^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$  such that (i) and (ii) of Lemma 2.2.3 are satisfied and such that if the event  $\{T \in \mathcal{M}(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\}$  occurs, then we have the following properties:*

- (i) in the environment  $T^*$ , every geodesic from 0 to  $x$  takes the pattern inside  $B_{2, S_\ell^1(T), N}$ ,
- (ii) for all geodesic  $\bar{\gamma}^*$  from 0 to  $x$  in the environment  $T^*$ , there exists a geodesic  $\bar{\gamma}$  from 0 to  $x$  in the environment  $T$  such that  $\bar{\gamma}$  and  $\bar{\gamma}^*$  are associated in  $B_{3, S_\ell^1(T), N}$ ,
- (iii) there exists a geodesic  $\gamma^*$  in the environment  $T^*$  from 0 to  $x$  such that  $\gamma^*$  and the selected geodesic  $\gamma$  in the environment  $T$  are associated in  $B_{3, S_\ell^1(T), N}$ .

*Proof of Lemma 2.2.3.* Let  $\ell \in \{1, \dots, q\}$ . Consider  $E_+^*$  and  $E_-^*$  given by Lemma 2.2.4. Let  $s \in \mathbb{Z}^d$  and assume that the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\}$  occurs. To prove that the event  $\{T^* \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)\}$  occurs and that  $\|S_\ell^2(T^*) - s\|_1 \leq 2r_3$ , it is sufficient to prove that we have the following four points in the environment  $T^*$ :

1. every geodesic from 0 to  $x$  has at least  $\ell - 1$  typical boxes in its  $M$ -sequence,
2. there exists a geodesic from 0 to  $x$  which does not take the pattern in the annuli up to and including the one containing its  $(\ell - 1)$ -th box,
3. every geodesic from 0 to  $x$  whose  $M$ -sequence contains at least  $\ell$  elements takes the pattern in an annulus whose index is smaller than or equal to the one containing its  $\ell$ -th box,
4. there exists  $s'$  such that  $S_\ell^2(T^*) = s'$  and  $\|s - s'\|_1 \leq 2r_3$ .

Let us start with a few remarks. We denote by  $a_\ell$  the index of the annulus which contains  $B_{3,S_\ell^1(T),N}$ .

- (a) The environments  $T$  and  $T^*$  coincides outside the box  $B_{2,S_\ell^1(T),N}$ . As this box is included in the annulus  $A_{a_\ell,N}$ , the environments  $T$  and  $T^*$  are the same in all the other annuli. In particular, any box contained in an annulus  $A_{i,N}$  for  $i \neq a_\ell$  is typical in  $T$  if and only if it is typical in  $T^*$ .
- (b) Similarly, every path  $\pi$  takes a pattern which is outside the box  $B_{3,S_\ell^1(T),N}$  in the environment  $T$  if and only if it takes this pattern in the environment  $T^*$ . In particular, for any  $i \neq a_\ell$ ,  $\pi$  takes the pattern in the annulus  $A_{i,N}$  in the environment  $T$  if and only if it takes the pattern in the annulus  $A_{i,N}$  in the environment  $T^*$ .
- (c) Let  $\bar{\gamma}$  and  $\bar{\gamma}^*$  be as in item (ii) of Lemma 2.2.4. By definition of associated paths, there exist two vertices  $s_1$  and  $s_2$  in  $B_{3,S_\ell^1(T),N}$  visited by the two geodesics, such that  $\bar{\gamma}$  and  $\bar{\gamma}^*$  coincide except maybe for the part between  $s_1$  and  $s_2$ . To sum up:

$$\begin{aligned} \bar{\gamma}_{0,s_1} &= \bar{\gamma}_{0,s_1}^* \quad \text{and} \quad \bar{\gamma}_{s_2,x} = \bar{\gamma}_{s_2,x}^* \quad \text{and} \quad \bar{\gamma}_{s_1,s_2} \subset B_{3,S_\ell^1(T),N} \subset A_{a_\ell,N} \\ &\quad \text{and} \quad \bar{\gamma}_{s_1,s_2}^* \subset B_{3,S_\ell^1(T),N} \subset A_{a_\ell,N}. \end{aligned} \tag{2.2.10}$$

Furthermore, by remark (b), we have that for any  $i \neq a_\ell$ ,  $\bar{\gamma}$  takes the pattern in the annulus  $A_{i,N}$  in the environment  $T$  if and only if  $\bar{\gamma}^*$  takes the pattern in the annulus  $A_{i,N}$  in the environment  $T^*$ . The same property holds for the selected geodesic  $\gamma$  and for the associated geodesic  $\gamma^*$  (in the environment  $T^*$ ) given by item (iii) of Lemma 2.2.4.

- (d) Let again  $\bar{\gamma}$  and  $\bar{\gamma}^*$  be as in item (ii) of Lemma 2.2.4. Let us compare the  $M$ -sequence of  $\bar{\gamma}$  (which is built in the environment  $T$ ) with the  $M$ -sequence of  $\bar{\gamma}^*$  (which is built in the environment  $T^*$ ). By (a) and (2.2.10), we get that any box which belongs to the  $M$ -sequence of  $\bar{\gamma}$ , with the possible exception of a box contained in  $A_{a_\ell,N}$ , also belongs

to the  $M$ -sequence of  $\bar{\gamma}^*$ . The same property holds for the selected geodesic  $\gamma$  and for the associated geodesic  $\gamma^*$  (in the environment  $T^*$ ) given by item (iii) of Lemma 2.2.4. In particular the first  $\ell - 1$  elements of the  $M$ -sequence of  $\gamma$  (which is built in the environment  $T$ ) are the same as the first  $\ell - 1$  elements of the  $M$ -sequence of  $\gamma^*$  (which is built in the environment  $T^*$ ).

Let us now proceed to the proof of item 1. We assume  $\ell \geq 2$  otherwise there is nothing to prove. Let  $\bar{\gamma}^*$  be a geodesic from 0 to  $x$  in the environment  $T^*$ . Let  $\bar{\gamma}$  be the associated geodesic in the environment  $T$  given by item (ii) of Lemma 2.2.4. Since the event  $\{T \in \mathcal{M}(\ell)\}$  occurs, the  $M$ -sequence (in the environment  $T$ ) of  $\bar{\gamma}$  contains at least  $\ell$  typical boxes. By remark (d) above, the  $M$ -sequence (in the environment  $T^*$ ) of  $\bar{\gamma}^*$  contains at least  $\ell - 1$  typical boxes.

Let us consider item 2. We can again assume  $\ell \geq 2$ . Let  $\gamma^*$  be the geodesic given by item (iii) of Lemma 2.2.4. Recall that  $\gamma$  is the selected geodesic and that  $\gamma \in \Gamma^\ell$ . In particular, we have the following properties: its  $M$ -sequence contains at least  $\ell$  boxes; the  $\ell$ -th box of its  $M$ -sequence belongs to  $A_{a_\ell, N}$ ;  $\gamma$  does not take the pattern in any annulus whose index is smaller than or equal to  $a_\ell$ . Therefore, by remark (d) above, the first  $\ell - 1$  boxes of the  $M$ -sequence of  $\gamma$  and  $\gamma^*$  are the same. By remark (c) above, we conclude that  $\gamma^*$  does not take the pattern (in the environment  $T^*$ ) in any annulus whose index is smaller than the one containing its  $(\ell - 1)$ -th box.

Let us prove item 3. Let  $\bar{\gamma}^*$  be such a geodesic. Assume that the  $\ell$ -th box of the  $M$ -sequence of  $\bar{\gamma}^*$  is in an annulus whose index is strictly smaller than  $a_\ell$ . Let  $\bar{\gamma}$  be a geodesic in the environment  $T$  given by item (ii) of Lemma 2.2.4. Assume, aiming at a contradiction, that  $\bar{\gamma}^*$  does not take the pattern in an annulus until the one containing its  $\ell$ -th box. By remark (d), the  $\ell$  first boxes of the  $M$ -sequences of  $\bar{\gamma}^*$  and  $\bar{\gamma}$  are the same. By remarks (b) and (c),  $\bar{\gamma}$  does not take the pattern until the annulus containing its  $\ell$ -th box. This contradicts the definition of  $S_\ell^1$ , so it is impossible. Thus the  $\ell$ -th box of the  $M$ -sequence of  $\bar{\gamma}^*$  is in an annulus whose index is greater than or equal to  $a_\ell$ . By item (i) of Lemma 2.2.4,  $\bar{\gamma}^*$  takes the pattern in the annulus whose index is  $a_\ell$ . Therefore it takes the pattern in an annulus whose index is smaller than or equal to the one containing its  $\ell$ -th box and the third point is satisfied.

Finally, let us prove item 4. Note that since  $B_{3,s,N}$  is a successful box,  $S_\ell^2(T^*) \neq \emptyset$ . There are two steps. First, we prove that  $NS_\ell^2(T^*)$  is the center of a box contained in the annulus  $A_{a_\ell, N}$  and then we prove that  $B_{3,s,N} \cap B_{3,S^2(T^*),N} \neq \emptyset$ , which gives the result. Assume that  $NS_\ell^2(T^*)$  is not the center of a box contained in  $A_{a_\ell, N}$ . Let again  $\gamma^*$  be the geodesic given by item (iii) of Lemma 2.2.4 and recall that  $\gamma$  is the selected geodesic in the environment  $T$ . Since  $B_{3,s,N}$  is successful, we have  $a'_\ell(\gamma) \leq a_\ell$  and thus the definition of  $S_\ell^2(T^*)$  implies that  $NS_\ell^2(T^*)$  is the center of a box contained in an annulus  $A_{k,N}$  such that  $k < a_\ell$ . So  $\gamma^*$  takes the pattern in this annulus in the environment  $T^*$ . By remark (c),  $\gamma$  takes the pattern in this annulus in the environment  $T^*$  and then in the environment  $T$  by remark (b). This is impossible since  $\gamma$  does not take the pattern in any annuli  $A_{k,N}$  with  $k \leq a_\ell$ .

To conclude, assume, aiming at a contradiction, that  $B_{3,s,N} \cap B_{3,S^2(T^*),N} = \emptyset$ . It implies that  $\gamma^*$  takes a pattern outside  $B_{3,s,N}$  in the environment  $T^*$  and the portion of  $\gamma^*$  taking



this pattern is a portion of  $\gamma_{0,s'_1}^*$  or  $\gamma_{s'_2,x}^*$ . By remark (b), the portion of  $\gamma^*$  taking this pattern also takes this pattern in the environment  $T$ . By remark (c), it implies that  $\gamma$  takes the pattern in the environment  $T$  in the annulus  $A_{a_\ell,N}$ , which is a contradiction.  $\square$

Now, thanks to Lemma 2.2.3, we can adapt Lemma 3.8 from [1].

**Lemma 2.2.5.** *There exists  $\lambda \in (0, 1)$ , which does not depend on  $n$  and on  $x \in \Gamma_n$ , such that*

$$\mathbb{P}(T \in \mathcal{M}(q)) \leq \lambda^q.$$

*Proof.* Let  $\ell$  be in  $\{1, \dots, q\}$ . For every  $s \in \mathbb{Z}^d$ , let us consider the environment  $T_s^*$  defined for all edge  $e$  by:

$$T_s^*(e) = \begin{cases} T(e) & \text{if } e \notin B_{2,s,N} \\ T'(e) & \text{else.} \end{cases}$$

Thus  $T_s^*$  and  $T$  have the same distribution and on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\}$ ,  $T^* = T_s^*$ . So, using this environment and writing with indicator functions the result of Lemma 2.2.3, we get:

$$\mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \leq \mathbb{1}_{\{T_s^* \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell)\}} \mathbb{1}_{\bigcup_{s' \approx s} \{S_\ell^2(T_s^*) = s'\}},$$

where  $s' \approx s$  if  $\|s - s'\|_1 \leq 2r_3$ . We compute the expectation on both sides. The right side yields

$$\begin{aligned} & \mathbb{P} \left( T_s^* \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell), \bigcup_{s' \approx s} \{S_\ell^2(T_s^*) = s'\} \right) \\ &= \mathbb{P} \left( T \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell), \bigcup_{s' \approx s} \{S_\ell^2(T) = s'\} \right). \end{aligned}$$

For the left side, we have

$$\mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \right] = \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{E} \left[ \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \middle| T \right] \right].$$

Since on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\}$ , we have  $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$ , the left side is bounded from below by  $\eta \mathbb{P}(T \in \mathcal{M}(\ell), S_\ell^1(T) = s)$ . Then, by summing on all  $s \in \mathbb{Z}^d$  and writing  $K$  a constant which bounds from above for all  $s' \in \mathbb{Z}^d$  the number of vertices  $s \in \mathbb{Z}^d$  such that  $s' \approx s$ , we get

$$\frac{\eta}{K} \mathbb{P}(T \in \mathcal{M}(\ell)) \leq \mathbb{P}(T \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell)).$$

Now, since  $\mathcal{M}(\ell) \subset \mathcal{M}(\ell-1)$ ,

$$\mathbb{P}(T \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell)) = \mathbb{P}(T \in \mathcal{M}(\ell-1)) - \mathbb{P}(T \in \mathcal{M}(\ell)).$$

Thus,

$$\mathbb{P}(T \in \mathcal{M}(\ell)) \leq \lambda \mathbb{P}(T \in \mathcal{M}(\ell-1)),$$

where  $\lambda = \frac{1}{1 + \frac{\eta}{K}} \in (0, 1)$  does not depend on  $x$ . Hence, using  $\mathbb{P}(T \in \mathcal{M}(0)) = 1$ , we get by induction

$$\mathbb{P}(T \in \mathcal{M}(q)) \leq \lambda^q.$$

□

*Proof of Proposition 2.1.11.* Recall that  $N$ ,  $x$  (and then  $n$  and  $p$ ) are fixed at (2.2.8) but that  $C_1$ ,  $D_1$  and  $\lambda$  does not depend on  $x$ ,  $n$  and  $p$ . Then, by Lemma 2.2.2 and Lemma 2.2.5, using the inequality (2.2.9),

$$\begin{aligned} & \mathbb{P}\left(\text{there exists a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0\right) \\ & \leq \mathbb{P}(T \in \mathcal{G}^p(N)^c) + \mathbb{P}(T \in \mathcal{M}(q)) \\ & \leq D_1 e^{-C_1 p^{\frac{1}{d}}} + \lambda \lfloor \frac{p}{2} \rfloor. \end{aligned}$$

As  $C_1 > 0$  and  $\lambda \in (0, 1)$ , and as this inequality holds for any  $n \geq 2rN$  and any  $x \in \Gamma_n$ , we get the existence of two constants  $C > 0$  and  $D > 0$  such that for all  $n$ , for all  $x \in \Gamma_n$ ,

$$\mathbb{P}\left(\text{there exists a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0\right) \leq D \exp(-Cn^{\frac{1}{d}}).$$

□

## 2.2.2 Typical boxes crossed by geodesics

Let us first begin with the proof of the lemma stated in the paragraph of typical boxes in Section 2.2.1.

*Proof of Lemma 2.2.1.*

1. Let  $B_{3,s,N}$  be a typical box. Then the event  $\mathcal{T}(s, N)$  occurs. Let  $u_0$  and  $v_0$  be two vertices in  $B_{2,s,N}$ . We have

$$t_{3,s,N}(u_0, v_0) \leq 2 \sup_{z \in B_{2,s,N}} t_{3,s,N}(Ns, z) \leq 2r_{2,3}N.$$

Let  $\pi_0$  be a path from  $u_0$  to  $v_0$  which is not entirely contained in  $B_{3,s,N}$ . Let  $z_0$  denote the first vertex on the boundary of  $B_{3,s,N}$  visited by  $\pi_0$ . Then

$$\begin{aligned} T(\pi_0) & \geq t_{3,s,N}(u_0, z_0) \geq t_{3,s,N}(z_0, Ns) - t_{3,s,N}(u_0, Ns) \\ & \geq \inf_{z \in \partial B_{3,s,N}} t_{3,s,N}(Ns, z) - \sup_{z \in B_{2,s,N}} t_{3,s,N}(Ns, z) \geq 3r_{2,3}N > 2r_{2,3}N \\ & \geq t_{3,s,N}(u_0, v_0). \end{aligned}$$

Hence, every geodesic from  $u_0$  to  $v_0$  has to be entirely contained in  $B_{3,s,N}$ .

2. The properties (ii) and (iii) only depend on the time of edges in  $B_{2,s,N}$ . The event  $\mathcal{T}(s; N)$  only depends on edges in  $B_{3,s,N}$  by the definition of  $t_{3,s,N}$ .

3. First,

$$\lim_{N \rightarrow \infty} \mathbb{P}(\mathcal{T}(0; N)) = 1.$$

Indeed, recall the definition of the random ball  $\tilde{B}(c, r)$  given at (2.1.16) for any  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ . Then, by (2.1.18),

$$\mathbb{P}\left(B_\mu\left(0, \frac{r_{2,3}}{2}N\right) \subset \tilde{B}(0, r_{2,3}N) \text{ for all large } N\right) = 1,$$

and

$$\mathbb{P}\left(\tilde{B}(0, 4r_{2,3}N) \subset B_\mu(0, 8r_{2,3}N) \text{ for all large } N\right) = 1.$$

Thus, since  $B_{2,0,1} \subset B_\mu\left(0, \frac{r_{2,3}}{2}\right) \cap \mathbb{Z}^d$  and  $B_\mu(0, 9r_{2,3}) \cap \mathbb{Z}^d \subset B_{3,0,1}$ , almost surely there exists  $N_0 \in \mathbb{N}^*$  such that for all  $N \geq N_0$ ,

$$B_{2,0,N} \subset B(0, r_{2,3}N), \quad B(0, 4r_{2,3}N) \subset B_{3,0,N} \text{ and for all } y \in \partial B_{3,0,N}, \quad y \notin B(0, 4r_{2,3}N).$$

So, for all  $N \geq N_0$ ,

$$\sup_{z \in \partial B_{2,0,N}} t_{3,0,N}(0, z) \leq r_{2,3}N \quad \text{and} \quad \inf_{z \in \partial B_{3,0,N}} t_{3,0,N}(0, z) \geq 4r_{2,3}N.$$

Note that, for the first inequality, we use the fact that for all  $z \in B_{2,0,N}$ ,  $t_{3,0,N}(0, z) = t(0, z)$  thanks to the first point of Lemma 2.2.1 proved above.

The probability that (iii) is satisfied by  $B_{3,0,N}$  goes to 1 by (2.2.4). Then, let us prove that the probability that (ii) is satisfied by  $B_{3,0,N}$  goes to 1. Let  $|B_{3,0,N}|$  denote the number of vertices in  $B_{3,0,N}$  and  $\Pi_0$  denote the set of self-avoiding paths entirely contained in  $B_{3,0,N}$ . Then, using (2.2.3), we have that  $r_2 > r_1$ , and by (2.1.20),

$$\begin{aligned} & \mathbb{P}(B_{3,0,N} \text{ does not satisfy (ii)}) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq (r_2 - r_1)N}} \mathbb{P}((2.2.5) \text{ is not satisfied by a path of } \Pi_0 \text{ from } u_\pi \text{ to } v_\pi) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq (r_2 - r_1)N}} \mathbb{P}((2.2.5) \text{ is not satisfied by a path from } u_\pi \text{ to } v_\pi) \\ & \leq |B_{3,0,N}|^2 e^{-D_0(r_2 - r_1)N} \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

since  $|B_{3,0,N}|$  is bounded by a polynomial in  $N$ .

□

*Proof of Lemma 2.2.2.* To begin this proof, one needs an upper bound on the Euclidean length of geodesics. Using Theorem 4.6 in [2], we have two positive constants  $K_1$  and  $C_2$  such that for all  $y \in \mathbb{Z}^d$ ,

$$\mathbb{P}(m(y) \geq K_1 \|y\|_1) \leq e^{-C_2 \|y\|_1^{\frac{1}{d}}},$$

where  $m(y) = \max \{|\sigma|_e : \sigma \text{ is a geodesic from } 0 \text{ to } y\}$  and where for a path  $\sigma$ ,  $|\sigma|_e$  means the number of different edges taken by  $\sigma$ . For all  $p \in \mathbb{N}^*$ , we define the event  $\mathcal{N}^p(N)$  on which every geodesic from 0 to the outer sphere of the  $p$ -th annulus takes less than  $K_1 prN$  distinct edges. Note that  $r = 2(r_1 + r_3 + 1)$  is fixed at (2.2.6) and  $rN$  corresponds to the width of the annuli. Then,

$$\mathbb{P}(\mathcal{N}^p(N)^c) \leq \sum_{y: \|y\|_1 = prN} \mathbb{P}(m(y) \geq K_1 \|y\|_1) \leq (2prN + 1)^d e^{-C_2 (prN)^{\frac{1}{d}}}.$$

Hence, we obtain two positive constants  $C_3$  and  $D_3$  depending only on  $r$ ,  $d$  and  $F$  such that for all  $p \in \mathbb{N}^*$ , for all  $N \in \mathbb{N}^*$ ,

$$\mathbb{P}(\mathcal{N}^p(N)^c) \leq D_3 e^{-C_3 p^{\frac{1}{d}}}.$$

Now, we assume that the event  $\mathcal{N}^p(N) \cap \mathcal{G}^p(N)^c$  occurs. So, every geodesic from 0 to the outer sphere of the  $p$ -th annulus takes a number of distinct edges which is between  $prN$  and  $K_1 prN$ . Let us consider a re-normalized model. We introduce the meta-cubes

$$B_{s,N}^\infty = \left\{ w \in \mathbb{Z}^d : \left( s - \frac{1}{2} \right) N \leq w < \left( s + \frac{1}{2} \right) N \right\}, \text{ for all } s \in \mathbb{Z}^d,$$

(where  $v \leq w$  means  $v_i \leq w_i$  for  $1 \leq i \leq d$  and  $v < w$  means  $v_i < w_i$  for  $1 \leq i \leq d$ .) These meta-cubes form a partition of  $\mathbb{Z}^d$ . Furthermore, the meta-cubes and the boxes defined above have the same centers (which are the vertices  $Ns$  for  $s \in \mathbb{Z}^d$ ), and for all  $s \in \mathbb{Z}^d$ ,  $B_{s,N}^\infty \subset B_{1,s,N}$ . So, we can define typical meta-cubes. A meta-cube  $B_{s,N}^\infty$  is typical if  $B_{3,s,N}$  is a typical box.

For a geodesic  $\gamma$  from 0 to the outer sphere of the  $p$ -th annulus, we associate the set of meta-cubes visited by  $\gamma$ , that is

$$\mathfrak{A}(\gamma) = \{B_{s,N}^\infty \mid \gamma \text{ visits at least one vertex of } B_{s,N}^\infty\}.$$

This set can be identified with the subset of the re-normalized graph  $N\mathbb{Z}^d$ :

$$\mathfrak{A}_v^R(\gamma) = \{sN \mid B_{s,N}^\infty \in \mathfrak{A}(\gamma)\}.$$

Note that, if we consider the set  $\mathfrak{A}_e^R(\gamma)$  of edges of  $N\mathbb{Z}^d$  linking vertices which are both in  $\mathfrak{A}_v^R(\gamma)$ , then the pair of sets  $(\mathfrak{A}_v^R(\gamma), \mathfrak{A}_e^R(\gamma))$  forms a lattice animal, denoted by  $\mathfrak{A}^R(\gamma)$ . Recall

that a lattice animal  $\mathfrak{A}$  in  $N\mathbb{Z}^d$  is a finite connected sub-graph of  $N\mathbb{Z}^d$  that contains 0. We denote by  $\mathcal{A}^R$  the set of lattice animals in  $N\mathbb{Z}^d$  associated with a geodesic going from 0 to the outer sphere of the  $p$ -th annulus.

Let us bound the size of these lattice animals. By the size of a lattice animal  $\mathfrak{A}^R$ , denoted by  $|\mathfrak{A}^R|_v$ , we mean its number of vertices in the re-normalized model. Recall that, since the event  $\mathcal{N}^p(N)$  occurs, every geodesic from 0 to the outer sphere of the  $p$ -th annulus takes a number of distinct edges which is between  $prN$  and  $K_1prN$ . Then, in the meta-cube set  $\mathfrak{A}(\gamma)$  associated to such a geodesic  $\gamma$ , since  $r_1 = d$  and thanks to the choice of  $r$ , there are  $p - 1$  meta-cubes associated to boxes crossed by  $\gamma$  in distinct annuli. In particular (considering also the meta-cube centered at the origin), the size of every lattice animal  $\mathfrak{A}^R \in \mathcal{A}^R$  is bounded from below by  $p$ . For an upper bound, Lemma 3.4 in [9] gives

$$|\mathfrak{A}^R(\gamma)|_v \leq 3^d \left( 1 + \frac{|\gamma|_e + 1}{N} \right), \quad (2.2.11)$$

for all geodesic  $\gamma$  from 0 to the outer sphere of the  $p$ -th annulus, where  $|\gamma|_e$  still denotes the number of edges taken by  $\gamma$  and where  $|\mathfrak{A}^R(\gamma)|_v$  is the number of vertices of  $\mathfrak{A}^R(\gamma)$ . Thus, writing  $K_2 = \lceil 3^d(K_1r + 2) \rceil$  (which does not depend on  $p$  and  $N$ ), for every lattice animal  $\mathfrak{A}^R \in \mathcal{A}^R$ ,  $|\mathfrak{A}^R|_v$  is bounded from above by  $K_2p$ . Furthermore, for  $j \in \{p, \dots, K_2p\}$ , using (4.24) in [10], we have that

$$\left| \left\{ \mathfrak{A}^R \in \mathcal{A}^R : |\mathfrak{A}^R|_v = j \right\} \right| \leq |\{\text{lattice animals in } \mathbb{Z}^d \text{ of size } j\}| \leq 7^{dj}. \quad (2.2.12)$$

Now, let us consider the random variables  $(X_\ell^N)_{\ell \in \mathbb{Z}^d}$  such that  $X_\ell^N = 1$  if the meta-cube  $B_{\ell, N}^\infty$  is typical and  $X_\ell^N = 0$  otherwise. By Lemma 2.2.1,

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_\ell^N = 1) = 1.$$

Thus, we can fix  $N$  large enough such that

$$2 \left( 1 - \mathbb{P}(X_\ell^N = 1) \right)^{\frac{1}{2}} 7^{dK_2} < 1. \quad (2.2.13)$$

Denote  $\eta_1 = 1 - \mathbb{P}(X_\ell^N = 1)$ .

Then, the width of the annuli is such that every path passing through an annulus has to cross a box in this annulus. Furthermore, using again Lemma 2.2.1, for every  $s_1 \in \mathbb{Z}^d$  and  $s_2 \in \mathbb{Z}^d$  such that  $B_{3, s_1, N} \cap B_{3, s_2, N} = \emptyset$ , the random variables  $X_{s_1}^N$  and  $X_{s_2}^N$  are independent. Thus, for every lattice animal  $\mathfrak{A}^R \in \mathcal{A}^R$  associated with a geodesic from 0 to the outer sphere of the  $p$ -th annulus, we can find a sequence of  $p - 1$  i.i.d. random variables  $(Y_i^{\mathfrak{A}^R})_{i=2, \dots, p}$  such that for every  $i \in \{2, \dots, p\}$ , there exists  $s \in \mathbb{Z}^d$  such that  $Y_i^{\mathfrak{A}^R} = X_s^N$ ,  $sN \in \mathfrak{A}^R$  and  $B_{3, s, N}$  is entirely contained in the  $i$ -th annulus.

Now, for every  $p \geq 2$ , when  $\mathcal{G}^p(N)^c$  occurs, there exists a geodesic  $\gamma$  from 0 to the outer sphere of the  $p$ -th annulus which crosses a typical box in strictly less than  $\left\lfloor \frac{p}{2} \right\rfloor$  annuli, and

thus there are strictly more than  $\left\lceil \frac{p}{2} \right\rceil - 1$  annuli  $A_{i,N}$  with  $i > 1$  such that  $\gamma$  does not cross a typical box in them. Hence, there exists a lattice animal  $\mathfrak{A}^R \in \mathcal{A}^R$  such that at least  $\left\lceil \frac{p}{2} \right\rceil - 1$  random variables of the family  $(Y_i^{\mathfrak{A}^R})_{i=2,\dots,p}$  are equal to 0. Denote by  $\mathfrak{F}$  the set of subsets of  $\{2, \dots, p\}$  containing  $\left\lceil \frac{p}{2} \right\rceil - 1$  elements. We have  $|\mathfrak{F}| \leq 2^{p-1}$ . Then we get for every  $p \geq 2$ ,

$$\begin{aligned}
& \mathbb{P}(\mathcal{N}^p(N) \cap \mathcal{G}^p(N)^c) \\
& \leq \mathbb{P}(\exists \mathfrak{A}^R \in \mathcal{A}^R \text{ such that } p \leq |\mathfrak{A}^R|_v \leq K_2 p, \exists \mathcal{F} \in \mathfrak{F}, \forall i \in \mathcal{F}, Y_i^{\mathfrak{A}^R} = 0) \\
& \leq \sum_{p \leq j \leq K_2 p} |\{\text{lattice animals in } \mathbb{Z}^d \text{ of size } j\}| |\mathfrak{F}|(\eta_1)^{\left\lceil \frac{p}{2} \right\rceil - 1} \\
& \leq K_2 p 7^{dK_2 p} 2^{p-1} (\eta_1)^{\left\lceil \frac{p}{2} \right\rceil - 1} \text{ (by (2.2.12))} \\
& \leq \frac{K_2 p}{2\eta_1} \left[ 2(\eta_1)^{\frac{1}{2}} 7^{dK_2} \right]^p \\
& \leq D_4 \exp(-C_4 p),
\end{aligned}$$

for some constants  $C_4 > 0$  and  $D_4 > 0$  since  $2(\eta_1)^{\frac{1}{2}} 7^{dK_2} < 1$  by (2.2.13).

Thus, we get a constant  $N_0$  such that for all  $p \geq 2$ , for all  $N \geq N_0$ ,

$$\mathbb{P}(\mathcal{G}^p(N)^c) \leq \mathbb{P}(\mathcal{N}^p(N) \cap \mathcal{G}^p(N)^c) + \mathbb{P}(\mathcal{N}^p(N)^c) \leq e^{-C_4 p} + D_3 e^{-C_3 p^{\frac{1}{d}}}.$$

So, there exist two positive constants  $C_1$  and  $D_1$  such that for all  $p \geq 2$ , for all  $N \geq N_0$ ,

$$\mathbb{P}(\mathcal{G}^p(N)^c) \leq D_1 e^{-C_1 p^{\frac{1}{d}}}.$$

□

### 2.2.3 Modification argument

The aim of this subsection is to prove Lemma 2.2.4. Let  $\ell \in \{1, \dots, q\}$ . On  $\{T \notin \mathcal{M}(\ell)\}$ , we set  $E_+^*(T) = \emptyset$  and  $E_-^*(T) = \emptyset$ . Let  $s$  be in  $\mathbb{Z}^d$ . We now define  $E_+^*$  and  $E_-^*$  on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\}$ . So assume that this event occurs. On the event  $\{T \in \mathcal{M}(\ell)\}$ ,  $\Gamma^\ell$  is not empty and thus there is a selected geodesic. We denote this selected geodesic by  $\gamma$ . We define the entry point (resp. the exit point) of a self-avoiding path in a set of vertices as the first (resp. the last) vertex of this path belonging to this set. Let  $u$  denote the entry point of  $\gamma$  in  $B_{2,s,N}$  and  $v$  the exit point.

We call entry point and exit point of the pattern (centered at 0) the endpoints denoted by  $u^\Lambda$  and  $v^\Lambda$  in the introduction. Note that, if a self-avoiding path takes the pattern, its entry and exit points in the set  $B_\infty(0, \ell^\Lambda)$  are not necessarily the entry and exit points of the pattern (as it can visit the set before and after taking the pattern).

Here, we want to put the pattern centered at  $sN$ . The vertex  $s$  being fixed, we keep the

notation  $u^\Lambda$  and  $v^\Lambda$  to designate the entry and the exit points of the pattern centered at  $sN$ .

**Construction of  $\pi$ .**

We have the following inclusions:

- $B_\infty(sN, \ell^\Lambda) \subset B_\infty(sN, \ell^\Lambda + 3) \subset B_{1,s,N}$  since  $r_1 = d$  and  $N \geq \ell^\Lambda + 3$  (see (2.2.8)),
- $B_{1,s,N} \subset B_{2,s,N}$  since  $r_2 > r_1$  by (2.2.3).

For the modification, we need a path  $\pi$ , constructed in a deterministic way and satisfying several properties, whose existence is guaranteed by the following lemma.

**Lemma 2.2.6.** *We can construct a path  $\pi$  in a deterministic way such that:*

- (i)  $\pi$  goes from  $u$  to  $u^\Lambda$  without visiting a vertex of  $B_\infty(sN, \ell^\Lambda)$ , then goes from  $u^\Lambda$  to  $v^\Lambda$  in a shortest way for the norm  $\|\cdot\|_1$  (and thus being contained in  $B_\infty(sN, \ell^\Lambda)$ ) and then goes from  $v^\Lambda$  to  $v$  without visiting a vertex of  $B_\infty(sN, \ell^\Lambda)$ ,
- (ii)  $\pi$  is entirely contained in  $B_{2,s,N}$  and does not have vertices on the boundary of  $B_{2,s,N}$  except  $u$  and  $v$ ,
- (iii)  $\pi$  is self-avoiding,
- (iv) the length of  $\pi_{u,u^\Lambda} \cup \pi_{v^\Lambda,v}$  is bounded from above by  $2r_2N + K$ , where  $K$  is the number of edges in  $B_\infty(0, \ell^\Lambda + 3)$ .

The proof of this lemma is given in Appendix 2.A.2 but the idea is to construct two paths, one from  $u$  to  $sN$  and the other from  $sN$  to  $v$  which minimize the distance for the norm  $\|\cdot\|_1$  and such that the only vertex belonging to both paths is  $sN$ . Then, we denote by  $u_0$  the first vertex of  $B_\infty(sN, \ell^\Lambda + 3)$  visited by the path from  $u$  to  $sN$  and  $v_0$  the last vertex of  $B_\infty(sN, \ell^\Lambda + 3)$  visited by the path from  $sN$  to  $v$ . We construct two paths entirely contained in  $B_\infty(sN, \ell^\Lambda + 3)$  from  $u_0$  to  $u^\Lambda$  and from  $v^\Lambda$  to  $v_0$  which do not take vertices of  $B_\infty(sN, \ell^\Lambda)$  except  $u^\Lambda$  and  $v^\Lambda$  and which have no vertices in common and we consider the concatenation of the path from  $u$  to  $u_0$ , the one from  $u_0$  to  $u^\Lambda$ , a path from  $u^\Lambda$  to  $v^\Lambda$  in a shortest way, the path from  $v^\Lambda$  to  $v_0$  and the one from  $v_0$  to  $v$  (see Figure 2.2).

Let  $\pi$  be the path given by Lemma 2.2.6.

**Definition of  $E_+^*$ ,  $E_-^*$  and  $\mathcal{B}^*$ .** Define  $E_-^*(T)$  as the set of edges  $e$  such that  $e \in \pi \setminus B_\infty(sN, \ell^\Lambda)$  and  $E_+^*(T)$  as the set of edges which are in  $B_{2,s,N}$  but which are not in  $B_\infty(sN, \ell^\Lambda) \cup \pi$ . Recall that  $\{T' \in \mathcal{B}^*(T)\}$  is a shorthand for

$$\{\forall e \in E_+^*(T), T'(e) \geq \nu(N), \forall e \in E_-^*(T), T'(e) \leq r + \delta', \theta_{NS_1^1(T)} T' \in \mathcal{A}^\Lambda\}.$$

Fix  $\eta = \tilde{p}^{|B_{2,s,N}|} \mathbb{P}(T \in \mathcal{A}^\Lambda)$ , where  $\tilde{p} = \min(F([t_{\min}, t_{\min} + \delta']), F([\nu(N), t_{\max}]))$ . Thus,  $\eta$  only depends on  $F$ , the pattern and  $N$  and we have

$$\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \tilde{p}^{|B_{2,s,N}|} \mathbb{P}(T \in \mathcal{A}^\Lambda) = \eta,$$

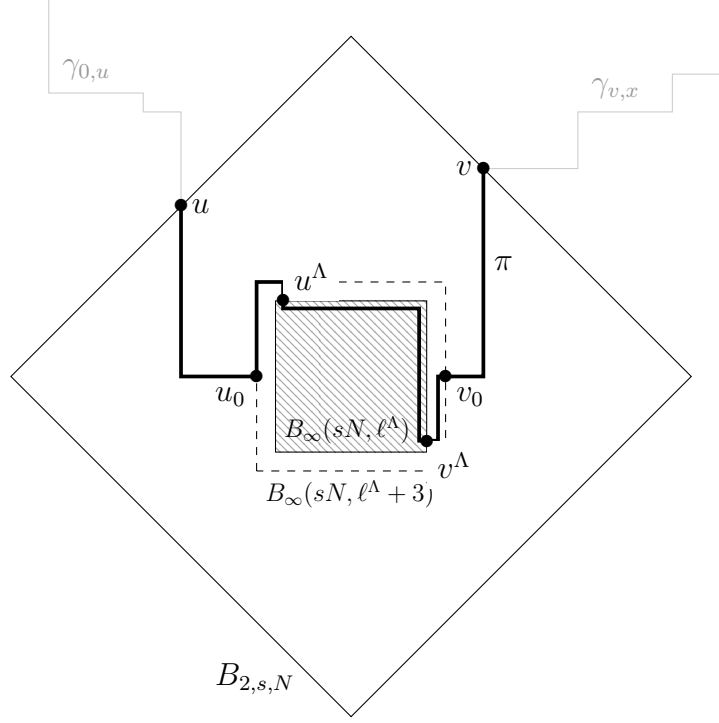


Figure 2.2: Example of the construction of  $\pi$  in dimension 2.

**Consequences of the modification.** We denote by  $\gamma^*$  the path  $\gamma_{0,u} \cup \pi \cup \gamma_{v,x}$ . Note that  $\gamma^*$  is a self-avoiding path.

**Lemma 2.2.7.** *We have  $T^*(\gamma^*) < T(\gamma)$ .*

*Proof.* We have that  $\gamma_{u,v}$  visits at least one vertex in  $B_{1,s,N}$ . Denote by  $w$  the first of these vertices. Then,  $\gamma_{u,w}$  and  $\gamma_{w,v}$  are two geodesics, both between two vertices in  $B_{2,s,N}$ . Using item 1 in Lemma 2.2.1,  $\gamma_{u,w}$  and  $\gamma_{w,v}$  are entirely contained in  $B_{3,s,N}$ . Thus, since  $B_{3,s,N}$  is a typical box, using (2.2.5) and the fact that  $\|u - w\|_1 \geq (r_2 - r_1)N$  and  $\|v - w\|_1 \geq (r_2 - r_1)N$ , we have

$$T(\gamma_{u,v}) \geq 2N(r_2 - r_1)(t_{\min} + \delta).$$

Then, by the construction of  $\pi$  and of  $\mathcal{B}^*(T)$ ,

$$T^*(\pi) \leq (2r_2N + K)(t_{\min} + \delta') + \tau^{\Lambda},$$

where  $\tau^{\Lambda}$  is fixed at (2.2.2). Thus,

$$T(\gamma) - T^*(\gamma^*) \geq 2N(r_2(\delta - \delta') - r_1(t_{\min} + \delta)) - K(t_{\min} + \delta') - \tau^{\Lambda}.$$

By (2.2.7) and since  $2N \geq 1$ , we get  $T(\gamma) - T^*(\gamma^*) > 0$ . □

**Lemma 2.2.8.** *Let  $\bar{\gamma}^*$  be a geodesic from 0 to  $x$  in the environment  $T^*$ . Then  $\bar{\gamma}^*$  weakly*



crosses the box  $B_{3,s,N}$  and the first vertex of  $B_{2,s,N}$  visited by  $\bar{\gamma}^*$  is  $u$  and the last is  $v$ . Furthermore,  $\bar{\gamma}^*$  takes the pattern in  $B_\infty(sN, \ell^\Lambda)$ ,  $\bar{\gamma}_{u,u^\Lambda}^* = \pi_{u,u^\Lambda}$  and  $\bar{\gamma}_{v^\Lambda,v}^* = \pi_{v^\Lambda,v}$ .

*Proof.* Let  $\bar{\gamma}^*$  be a geodesic from 0 to  $x$  in the environment  $T^*$ . By Lemma 2.2.7,  $T^*(\bar{\gamma}^*) < T(\bar{\gamma}^*)$ . Thus  $\bar{\gamma}^*$  takes an edge of  $B_{2,s,N}$  and by item (iii) of the definition of a typical box and since there is no edge whose time has been modified outside  $B_{2,s,N}$ ,  $\bar{\gamma}^*$  cannot take any edge of time greater than  $\nu(N)$  in  $B_{2,s,N}$ . Indeed, assume that  $\bar{\gamma}^*$  takes an edge  $e$  such that  $T^*(e) \geq \nu(N)$ . Then, denoting by  $\mathcal{E}(\bar{\gamma}^*)$  the edges of  $\bar{\gamma}^*$  and using property (iii) of typical boxes,

$$\begin{aligned} T^*(\bar{\gamma}^*) &= \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}} T^*(f) + \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}^c} T^*(f) \geq \nu(N) + \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}^c} T(f) \\ &> \sum_{f \in B_{2,s,N}} T(f) + \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}} T(f) \\ &\geq \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}} T(f) + \sum_{f \in \mathcal{E}(\bar{\gamma}^*) \cap B_{2,s,N}^c} T(f) = T(\bar{\gamma}^*), \end{aligned}$$

which is impossible. Hence,  $\bar{\gamma}^*$  has to take edges of  $\pi$  or of the pattern and cannot take other edges of  $B_{2,s,N}$ .

Since  $\pi$  does not visit any vertex on the boundary of  $B_{2,s,N}$  except  $u$  and  $v$ ,  $\bar{\gamma}^*$  has to visit  $u$  and  $v$  and to follow  $\pi$  between  $u$  and  $u^\Lambda$  and between  $v^\Lambda$  and  $v$ . If  $\bar{\gamma}_{u^\Lambda,v^\Lambda}^*$  leaves the pattern, it takes an edge whose time is greater than  $\nu(N)$ , which is impossible. So,  $\bar{\gamma}_{u^\Lambda,v^\Lambda}^*$  is a path entirely contained in  $B_\infty(sN, \ell^\Lambda)$  and is optimal for the passage time since  $\bar{\gamma}^*$  is a geodesic.

To conclude, let us show that  $u$  is visited by  $\bar{\gamma}^*$  before  $v$ . Assume that it is not the case. Then, there exists  $\bar{\gamma}_1^*$  a geodesic from 0 to  $v$  and  $\bar{\gamma}_2^*$  a geodesic from  $u$  to  $x$  in the environment  $T^*$  which does not take any edge in  $B_{2,s,N}$ . Thus, there are also geodesics in the environment  $T$ . Then,

$$T(\bar{\gamma}_1^*) + T(\bar{\gamma}_2^*) \leq t^*(0, x) < t(0, x) \text{ by Lemma 2.2.7.}$$

By concatenating  $\gamma_{0,u}$  and  $\bar{\gamma}_2^*$ , we obtain a path from 0 to  $x$ . Thus,

$$T(\gamma_{0,u}) + T(\bar{\gamma}_2^*) \geq t(0, x).$$

So  $T(\bar{\gamma}_1^*) < T(\gamma_{0,u})$ , which implies

$$T(\bar{\gamma}_1^*) + T(\gamma_{v,x}) < T(\gamma_{0,u}) + T(\gamma_{v,x}) \leq t(0, x),$$

which is impossible since  $\bar{\gamma}_1^* \cup \gamma_{v,x}$  is a path from 0 to  $x$ .  $\square$

Using Lemma 2.2.8, we can prove Lemma 2.2.4. Indeed, by this previous lemma, every geodesic from 0 to  $x$  takes the pattern inside  $B_{2,s,N}$  and the first item holds. For the third item, one can check that the concatenation of  $\gamma_{0,u}$ ,  $\pi_{u,u^\Lambda}$ , one of the optimal paths for the passage time between  $u^\Lambda$  and  $v^\Lambda$  entirely contained in  $B_\infty(sN, \ell^\Lambda)$ ,  $\pi_{v^\Lambda,v}$  and  $\gamma_{v,x}$  gives a geodesic  $\gamma^*$  which is associated to  $\gamma$  in  $B_{3,s,N}$  (with  $s_1 = u$  and  $s_2 = v$ ). Finally, let us prove

the second item. If  $\bar{\gamma}^*$  is a geodesic from 0 to  $x$  in the environment  $T^*$ , then  $\bar{\gamma}_{u,v}^*$  is contained in  $B_{2,s,N}$ . Furthermore,

$$T(\gamma_{0,u}) = T^*(\gamma_{0,u}) \geq T^*(\bar{\gamma}_{0,u}^*) = T(\bar{\gamma}_{0,u}^*) \geq T(\gamma_{0,u}),$$

so  $T(\bar{\gamma}_{0,u}^*) = T(\gamma_{0,u})$  and  $\bar{\gamma}_{0,u}^*$  is a geodesic in the environment  $T$ . Similarly,  $\bar{\gamma}_{v,x}^*$  is a geodesic in the environment  $T$ . Hence, we get a not necessarily self-avoiding optimal path for the passage time by considering  $\pi' = \bar{\gamma}_{0,u}^* \cup \gamma_{u,v} \cup \bar{\gamma}_{v,x}^*$ . We get a geodesic  $\gamma'$  which satisfies the properties of this item by cutting the loops of  $\pi'$  using a standard process<sup>6</sup>.

## 2.3 Bounded case

In this section, we assume that the support of  $F$  is bounded. In this case, Theorem 2.1.4 also follows from Proposition 2.1.11. Our proof of Proposition 2.1.11 still follows the strategy given in the preceding section, but the modification argument is more involved. Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern. We set  $t_{\max} = \sup(\text{support}(F))$ . Remark that, because of (2.1.20), we have  $t_{\min} + \delta < t_{\max}$ .

### 2.3.1 Oriented pattern

The proof in the bounded case uses a modification argument in which we have to connect the pattern to a straight path in a given direction. It is convenient to show the feasibility of this construction before starting the modification. The following lemma, whose proof is in Appendix 2.B, shows that it is indeed feasible by proving that a pattern can be associated to  $d$  patterns (with a larger size), each having endpoints aligned in a distinct direction, and each having the original pattern as a sub-pattern. By direction, we mean one of the  $d$  directions of the canonical basis which is denoted by  $\{\varepsilon_1, \dots, \varepsilon_d\}$ . Recall  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ .

**Lemma 2.3.1.** *There exists  $\ell_0 > \max(L_1, \dots, L_d)$  such that the following holds. Set  $\Lambda_0 = \{-\ell_0, \dots, \ell_0\}^d$ . For all  $j \in \{1, \dots, d\}$ , there exists a pattern  $\mathfrak{P}^j = (\Lambda_0, -\ell_0\varepsilon_j, \ell_0\varepsilon_j, \mathcal{A}_j^{\Lambda_0})$  such that:*

- $\mathbb{P}(\mathcal{A}_j^{\Lambda_0})$  is positive,
- on  $\mathcal{A}_j^{\Lambda_0}$ , any path from  $-\ell_0\varepsilon_j$  to  $\ell_0\varepsilon_j$  optimal for the passage time among the paths entirely inside  $\Lambda_0$  contains a subpath from  $u^\Lambda$  to  $v^\Lambda$  entirely inside  $\Lambda$ ,
- $\mathcal{A}_j^{\Lambda_0} \subset \mathcal{A}^\Lambda$ .

---

<sup>6</sup>Note that  $\bar{\gamma}_{0,u}^*$ ,  $\gamma_{u,v}$  and  $\bar{\gamma}_{v,x}^*$  are three self-avoiding paths and that  $\bar{\gamma}_{0,u}^*$  and  $\bar{\gamma}_{v,x}^*$  do not have vertices in common. So we can consider  $s_1$  the last vertex belonging to both path  $\bar{\gamma}_{0,u}^*$  and  $\gamma_{u,v}$  in the order in which they are visited by  $\gamma_{u,v}$  and  $s_2$  the first vertex belonging to both paths  $\gamma_{s_1,v}$  and  $\bar{\gamma}_{v,x}^*$  in the order in which they are visited by  $\gamma_{u,v}$ . Thus, since  $\gamma_{u,v}$  is entirely contained in  $B_{3,s,N}$ ,  $s_1$  and  $s_2$  are two vertices contained in  $B_{3,s,N}$  and we can take  $\gamma' = \bar{\gamma}_{0,s_1}^* \cup \gamma_{s_1,s_2} \cup \bar{\gamma}_{s_2,x}^*$ .

We fix  $\ell_0$ ,  $\Lambda_0$  and the patterns  $\mathfrak{P}_j$ , for  $j = 1, \dots, d$ , for the remaining of the proof. By definition, for all  $j \in \{1, \dots, d\}$ ,  $N^{\mathfrak{P}}(\pi) \geq N^{\mathfrak{P}_j}(\pi)$ , and actually

$$N^{\mathfrak{P}}(\pi) \geq \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{\text{there exists } j \in \{1, \dots, d\}, x \text{ satisfies the condition } (\pi; \mathfrak{P}_j)\}}.$$

From now on, we forget the original pattern and only consider the oriented patterns  $\mathfrak{P}_j$  for all directions  $j \in \{1, \dots, d\}$ . We talk about oriented patterns when their orientations are specified and we simply say "pattern" when talking about one of the oriented patterns  $\mathfrak{P}_j$ . Consistently with these conventions and to lighten notations, we write  $\ell^\Lambda$  instead of  $\ell_0^\Lambda$ ,  $\Lambda$  instead of  $\Lambda_0$  and  $\mathcal{A}_j^\Lambda$  instead of  $\mathcal{A}_j^{\Lambda_0}$ .

Now, several parameters related to the distribution  $F$  have to be introduced. First, we fix a positive real  $\nu$  such that:

- $t_{\min} + \delta \leq \nu \leq t_{\max}$ ,
- $F([\nu, +\infty)) > 0$ ,
- the event  $\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq \nu\}$  has a positive probability.

Notice that, if  $F$  has an atom, one could have  $\nu = t_{\max}$ . Although we may need to replace  $\mathcal{A}_j^\Lambda$  by  $\mathcal{A}_j^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq \nu\}$ , we can now assume that

$$\mathcal{A}_j^\Lambda \subset \{\forall e \in \Lambda, T(e) \leq \nu\}.$$

Further, we set  $\tau^\Lambda = 2\ell^\Lambda\nu$ , which can be interpreted as an upper bound for the passage time of an optimal path from  $\ell^\Lambda\varepsilon_j$  to  $-\ell^\Lambda\varepsilon_j$  on the event  $\mathcal{A}_j^\Lambda$ . Finally, we denote by  $T^\Lambda$  the constant  $K^\Lambda(t_{\max} - t_{\min})$  where  $K^\Lambda$  is the number of edges in an oriented pattern. We will use it as an upper bound for the time that a path can save using edges of a pattern after a modification.

### 2.3.2 Proof of Proposition 2.1.11 in the bounded case

We keep the overall plan of the unbounded case. Unlike in the unbounded case, we cannot use edges of prohibitive time and thus the modification argument is more elaborate here. This section follows the structure of Section 2.2.1 but the one step modification is replaced by a two-steps modification. To this aim, we slightly change the structure of our boxes and our definition of typical boxes. Let us begin by fixing some constants.

**Constants.** Note that we keep the notations introduced in Section 2.1.7 and that  $\tau^\Lambda$  and  $T^\Lambda$  are fixed in Section 2.3.1. In the next two paragraphs, we introduce the constants used in the proof. Before looking at their definitions below, one can keep in mind that we fix  $\varepsilon \ll 1$  and  $r_1 \ll \frac{1}{\varepsilon} \ll \nabla \ll r_2 \ll r_3 \ll r_4$ , where " $\ll$ " means that the ratio is large enough and only depends on the dimension  $d$  and on the distribution  $F$ .

- We fix  $\delta' = \min\left(\frac{\delta}{4}, \frac{\delta}{1+d}\right)$ .
- We fix  $L_1$  given by Lemma 2.3.11 depending only on  $d$  and  $\ell^\Lambda$ , and  $L_2 = L_1 + (10+d)\ell^\Lambda$ .
- Using Theorem 2.1.6 with  $M = t_{\min} + \delta$ , we get two constants  $\alpha > 0$  and  $C$ , fixed for the rest of the proof, such that for all  $n \in \mathbb{N}^*$  and  $u, v \in \mathbb{Z}^d$  such that  $\|u - v\|_1 = n$ ,

$$\mathbb{P}\left(\exists \text{ a geodesic } \bar{\gamma} \text{ from } u \text{ to } v \text{ such that } \sum_{e \in \bar{\gamma}} \mathbf{1}_{\{T(e) \geq t_{\min} + \delta\}} \leq \alpha n\right) \leq e^{-Cn}. \quad (2.3.1)$$

- Fix  $\varepsilon > 0$  such that

$$\varepsilon < \min\left(\frac{1}{11}, \frac{\delta}{24C_\mu}\right).$$

- Fix  $\nabla$  such that

$$\nabla > \max\left(\frac{4(1+t_{\max})C_\mu}{\varepsilon c_\mu}, 6dL_2C_\mu, \frac{8C_\mu T^\Lambda}{3\delta}, 4C_\mu(2t_{\max} + \tau^\Lambda)\right). \quad (2.3.2)$$

We give here other lower bounds for  $\nabla$  that we need for the sequel and which are consequences of (2.3.2).

- Using the fact that  $c_\mu \leq C_\mu$ , we get  $\nabla > \frac{4C_\mu}{\varepsilon c_\mu} > \frac{1}{\varepsilon}$ .
- From the inequality  $\nabla > \frac{4(1+t_{\max})C_\mu}{\varepsilon c_\mu}$ , using the fact that  $\varepsilon < \frac{1}{11}$ , we have  $1 - \varepsilon > 1 - 3\varepsilon > 1 - 10\varepsilon > \varepsilon$  and then  $\nabla > \frac{1+2t_{\max}}{1-\varepsilon}$ ,  $\nabla > \frac{4(1+t_{\max})}{1-10\varepsilon}$  and  $\nabla > \frac{3+2t_{\max}}{1-3\varepsilon}$ .
- Since  $\varepsilon < \frac{\delta}{24C_\mu}$  and  $\nabla > \frac{1}{\varepsilon}$ , we get  $\nabla > \frac{24C_\mu}{\delta}$ .
- Finally, since  $\delta - \delta' \geq \frac{3\delta}{4}$ , we have from  $\nabla > \frac{8C_\mu T^\Lambda}{3\delta}$  that  $\nabla > \frac{2C_\mu T^\Lambda}{\delta - \delta'}$ .

**Boxes.** With these constants, we can now define boxes. For  $i \in \{1, 2, 3, 4\}$ , as in the unbounded case,  $B_{i,s,N}$  is the ball of radius  $r_i$  for the norm  $\|\cdot\|_1$  centered at the point  $sN$  with:

- $r_1 = d$ ,
- $r_2$  an integer such that

$$r_2 > \max\left(r_1 + \frac{2(\nabla + 2)}{c_\mu}, r_1 + L_1 + \frac{3\nabla}{c_\mu} + \frac{2t_{\max}(1 + (1+d)\ell^\Lambda)}{\nu}\right), \quad (2.3.3)$$

- $r_3$  an integer such that

$$r_3 > \frac{7r_2(4t_{\max} + \alpha\delta)}{\alpha\delta},$$

Note that  $r_3 \geq r_2 + 1$ .

- $r_4$  an integer such that

$$r_4 > \frac{r_3(t_{\min} + \delta + t_{\max})}{t_{\min} + \delta}. \quad (2.3.4)$$

Note that  $r_4 \geq r_3 + 1$ .

We use the word "box" to talk about  $B_{4,s,N}$ . Recall that we denote by  $\partial B_{i,s,N}$  the boundary of  $B_{i,s,N}$ , that is the set of points  $z \in \mathbb{Z}^d$  such that  $\|z - sN\|_1 = r_i N$ .

**Crossed boxes and weakly crossed boxes.** We say that a path

- crosses a box  $B_{4,s,N}$  if it visits a vertex in  $B_{1,s,N}$ ,
- weakly crosses a box  $B_{4,s,N}$  if it visits a vertex in  $B_{3,s,N}$ .

**Paths associated in a box.** We say that two paths  $\gamma$  and  $\gamma'$  from 0 to the same vertex  $x$  are associated in a box  $B_{4,s,N}$  if there exist two distinct vertices  $s_1$  and  $s_2$  such that the following conditions hold:

- $\gamma$  and  $\gamma'$  visit  $s_1$  and  $s_2$ ,
- $\gamma_{0,s_1} = \gamma'_{0,s_1}$ ,
- $\gamma_{s_1,s_2}$  and  $\gamma'_{s_1,s_2}$  are entirely contained in  $B_{4,s,N}$ ,
- $\gamma_{s_2,x} = \gamma'_{s_2,x}$ .

In particular, these two paths coincide outside  $B_{4,s,N}$ .

**Typical boxes.**  $B_{4,s,N}$  is called a typical box if it verifies the following properties:

- (i) every geodesic  $\gamma_{u,v}$  from  $u$  to  $v$  entirely contained in  $B_{3,s,N}$  with  $\|u - v\|_1 \geq N$  has at least  $\alpha\|u - v\|_1$  edges whose time is greater than or equal to  $t_{\min} + \delta$ ,
- (ii) every path  $\pi$  from  $u$  to  $v$  entirely contained in  $B_{4,s,N}$  with  $\|u - v\|_1 \geq N$  has a passage time verifying:

$$t(\pi) \geq (t_{\min} + \delta) \|u - v\|_1, \quad (2.3.5)$$

- (iii) for all  $u$  and  $v$  in  $B_{3,s,N}$ , we have

$$(1 - \varepsilon)\mu(u - v) - N \leq t(u, v) \leq (1 + \varepsilon)\mu(u - v) + N.$$

As in the unbounded case, we need properties which are guaranteed with the definition of typical boxes. We state them in the following lemma whose proof is given in Section 2.3.3.

**Lemma 2.3.2.** *We have the following three properties about typical boxes.*

1. *If  $B_{4,s,N}$  is a typical box, for all points  $u_0$  and  $v_0$  in  $B_{3,s,N}$ , every geodesic from  $u_0$  to  $v_0$  is entirely contained in  $B_{4,s,N}$ .*
2. *The typical box property only depends on the time of the edges in  $B_{4,s,N}$ .*
3. *We have*

$$\lim_{N \rightarrow \infty} \mathbb{P}(B_{4,0,N} \text{ is a typical box}) = 1.$$

**Successful boxes.** For a fixed  $x \in \mathbb{Z}^d$ , a box  $B_{4,s,N}$  is successful if every geodesic from 0 to  $x$  takes a pattern which is entirely contained in  $B_{2,s,N}$ , i.e. if for every geodesic  $\gamma$  going from 0 to  $x$ , there exist  $j \in \{1, \dots, d\}$  and  $x_\gamma \in \mathbb{Z}^d$  satisfying the condition  $(\gamma; \mathfrak{P}^j)$  such that  $B_\infty(x_\gamma, \ell^\Lambda)$  is contained in  $B_{2,s,N}$ .

**Annuli.** Following the proof in the unbounded case, we define the annuli  $A_{i,N}$  with  $r = 2(r_1 + r_4 + 1)$  and  $\mathcal{G}^p(N)$  as in Section 2.2.1 but with the definitions of crossed and typical boxes defined here in Section 2.3.2. The bound on  $\mathbb{P}(\mathcal{G}^p(N)^c)$  of Lemma 2.2.2 also holds here. The proof is exactly the same in this case thanks to Lemma 2.3.2. For the rest of the proof, we fix  $C_1$ ,  $D_1$  and  $N_0$  given by Lemma 2.2.2.

**Modification argument.** Fix  $K' = T^\Lambda + 2(C_\mu L_1 + t_{\max}(\ell^\Lambda + 1))$ . Then, fix

$$N > \max\left(N_0, \frac{12C_\mu K'}{\delta \nabla}\right), \quad n \geq 2rN \text{ and } x \in \Gamma_n, \quad (2.3.6)$$

(where  $\Gamma_n$  is defined at (2.1.15)). Fix  $p = \lfloor \frac{n}{rN} \rfloor$  and  $q = \lfloor \frac{p}{2} \rfloor$ . For  $j \in \{1, \dots, q\}$ , we define  $\Gamma^j$ ,  $S_j^1$ ,  $S_j^2$  and  $\mathcal{M}(j)$  as in Section 2.2.1 but with the notions of typical and successful boxes defined here in Section 2.3.2. As in the unbounded case, the aim is to bound from above  $\mathbb{P}(T \in \mathcal{M}(q))$  independently of  $x$ . For the sequel, we use a two-steps modification. So we introduce two independent copies  $T'$  and  $T''$  of the environment  $T$ , the three being defined on the same probability space.

Fix  $\ell \in \{1, \dots, q\}$ . On  $\{T \in \mathcal{M}(\ell)\}$ ,  $B_{4,S_\ell^1(T),N}$  is a typical box crossed by the selected geodesic. From this configuration, as a first step, we shall associate a set of edges  $E_+^*(T)$  which is contained in  $B_{3,S_\ell^1(T),N} \setminus B_{2,S_\ell^1(T),N}$ . It corresponds to the edges for which we want to reduce the time. Then, we get a new environment  $T^*$  defined for all edge  $e$  by:

$$T^*(e) = \begin{cases} T'(e) & \text{if } e \in E_+^*(T) \\ T(e) & \text{else.} \end{cases} \quad (2.3.7)$$

From this environment, as a second step, we get three new subsets  $E_+^{**}(T, T')$ ,  $E_-^{**}(T, T')$  and  $E_P^{**}(T, T')$  of edges of  $B_{2, S_\ell^1(T), N}$  which are respectively the edges for which we want to reduce the time, to increase the time and the edges of the location where we want to put the pattern. We get a third environment  $T^{**}$  defined for all edge  $e$  by:

$$T^{**}(e) = \begin{cases} T''(e) & \text{if } e \in E_+^{**}(T, T') \cup E_-^{**}(T, T') \cup E_P^{**}(T, T') \\ T^*(e) & \text{else.} \end{cases} \quad (2.3.8)$$

Note that  $T^{**}$  and  $T$  do not have the same distribution. For  $y$  and  $z$  in  $\mathbb{Z}^d$ , we denote by  $t^*(y, z)$  (resp.  $t^{**}(y, z)$ ) the geodesic time between  $y$  and  $z$  in the environment  $T^*$  (resp.  $T^{**}$ ). Similarly, we define for  $c \in \mathbb{Z}^d$  and  $t \in \mathbb{R}_+$ :

$$B^*(c, t) = \{u \in \mathbb{Z}^d : t^*(c, u) \leq t\} \text{ and } B^{**}(c, t) = \{u \in \mathbb{Z}^d : t^{**}(c, u) \leq t\}. \quad (2.3.9)$$

We formalize this modification in the next lemma and we will describe precisely the construction of  $E_+^*$ ,  $E_+^{**}$ ,  $E_-^{**}$  and  $E_P^{**}$  in the next subsection.

**Lemma 2.3.3.** *There exists  $\eta = \eta(N)$  such that for all  $\ell$  in  $\{1, \dots, q\}$ , there exist measurable functions  $E_+^* : (\mathbb{R}_+)^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_+^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_-^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_P^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$  and  $\mathcal{O} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \{1, \dots, d\}$  such that:*

- (i)  $E_+^*(T)$ ,  $E_+^{**}(T, T')$ ,  $E_-^{**}(T, T')$  and  $E_P^{**}(T, T')$  are pairwise disjoint and are contained in  $B_{3, S_\ell^1(T)}$ ,
- (ii) on the event  $\{T \in \mathcal{M}(\ell)\}$ ,  $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$  and on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\}$ ,  $\mathbb{P}(T'' \in \mathcal{B}^{**}(T, T') | T, T') \geq \eta$ , where  $\{T' \in \mathcal{B}^*(T)\}$  is a shorthand for

$$\{\forall e \in E_+^*(T), T'(e) \leq t_{\min} + \delta'\},$$

and  $\{T'' \in \mathcal{B}^{**}(T, T')\}$  is a shorthand for

$$\left\{ \forall e \in E_+^{**}(T, T'), T''(e) \leq t_{\min} + \delta', \forall e \in E_-^{**}(T, T'), T''(e) \geq \nu, \theta_{NS_\ell^1(T)} T'' \in \mathcal{A}_{\mathcal{O}(T, T')}^\Lambda \right\},$$

- (iii)  $\{T \in \mathcal{M}(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{T'' \in \mathcal{B}^{**}(T, T')\} \subset \{T^{**} \in (\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell))\}$  and  $\|S_\ell^2(T^{**}) - S_\ell^1(T)\|_1 \leq 2r_4$ .

The proof of Lemma 2.3.3 is left to the reader. It is the same as the proof of Lemma 2.2.3, replacing the use of Lemma 2.2.4 by the following one.

**Lemma 2.3.4.** *There exists  $\eta = \eta(N)$  such that for all  $\ell$  in  $\{1, \dots, q\}$ , there exist measurable functions  $E_+^* : (\mathbb{R}_+)^{\mathcal{E}} \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_+^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_-^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$ ,  $E_P^{**} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \mathcal{P}(\mathcal{E})$  and  $\mathcal{O} : (\mathbb{R}_+^{\mathcal{E}})^2 \mapsto \{1, \dots, d\}$  such that items (i) and (ii) of Lemma 2.3.3 are satisfied and such that if the event  $\{T \in \mathcal{M}(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{T'' \in \mathcal{B}^{**}(T, T')\}$  occurs, then we have the following properties:*

(i) in the environment  $T^{**}$ , every geodesic from 0 to  $x$  takes the pattern inside  $B_{2,S_\ell^1(T),N}$ ,

(ii) for all geodesic  $\bar{\gamma}^{**}$  from 0 to  $x$  in the environment  $T^{**}$ , there exists a geodesic  $\bar{\gamma}$  from 0 to  $x$  in the environment  $T$  such that  $\bar{\gamma}$  and  $\bar{\gamma}^{**}$  are associated in  $B_{4,S_\ell^1(T),N}$ ,

(iii) there exists a geodesic  $\gamma^{**}$  in the environment  $T^{**}$  from 0 to  $x$  such that  $\gamma^{**}$  and the selected geodesic  $\gamma$  in the environment  $T$  are associated in  $B_{4,S_\ell^1(T),N}$ .

The proof of Lemma 2.3.4 is the aim of Section 2.3.4. We now conclude the proof of Proposition 2.1.11 in the bounded case. The following lemma is the counterpart of Lemma 2.2.5 in this case.

**Lemma 2.3.5.** *There exists  $\lambda \in (0, 1)$  which does not depend on  $n$  and on  $x \in \Gamma_n$  such that*

$$\mathbb{P}(T \in \mathcal{M}(q)) \leq \lambda^q.$$

*Proof.* Let  $\ell$  be in  $\{1, \dots, q\}$ . For every  $s \in \mathbb{Z}^d$  and  $\mathcal{E}^{**}$  subset of edges of  $B_{3,s,N}$ , let us consider the environment  $T_{s,\mathcal{E}^{**}}^{**}$  defined for all edge  $e$  by:

$$T_{s,\mathcal{E}^{**}}^{**} = \begin{cases} T''(e) & \text{if } e \in \mathcal{E}^{**} \cap B_{2,s,N} \\ T'(e) & \text{if } e \in \mathcal{E}^{**} \cap (B_{3,s,N} \setminus B_{2,s,N}) \\ T(e) & \text{else.} \end{cases}$$

We define  $E^{**}(T, T') = E_+^*(T) \cup E_+^{**}(T, T') \cup E_-^{**}(T, T') \cup E_P^{**}(T, T')$ . Thus, for every  $s$  and  $\mathcal{E}^{**}$ ,  $T_{s,\mathcal{E}^{**}}^{**}$  and  $T$  have the same distribution and on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{E^{**}(T, T') = \mathcal{E}^{**}\}$ ,  $T^{**} = T_{s,\mathcal{E}^{**}}^{**}$ . Using this environment and writing with indicator functions the result (iii) of Lemma 2.3.3, we have:

$$\begin{aligned} & \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mathbb{1}_{\{E^{**}(T, T') = \mathcal{E}^{**}\}} \mathbb{1}_{\{T'' \in \mathcal{B}^{**}(T, T')\}} \\ & \leq \mathbb{1}_{\{T_{s,\mathcal{E}^{**}}^{**} \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell)\}} \mathbb{1}_{\bigcup_{s' \approx s} \{S_\ell^2(T_{s,\mathcal{E}^{**}}^{**}) = s'\}}. \end{aligned}$$

We take the expectation on both sides. The right side yields

$$\begin{aligned} & \mathbb{P} \left( T_{s,\mathcal{E}^{**}}^{**} \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell), \bigcup_{s' \approx s} \{S_\ell^2(T_{s,\mathcal{E}^{**}}^{**}) = s'\} \right) \\ & = \mathbb{P} \left( T \in \mathcal{M}(\ell-1) \setminus \mathcal{M}(\ell), \bigcup_{s' \approx s} \{S_\ell^2(T) = s'\} \right). \end{aligned}$$



For the left side, we have

$$\begin{aligned}
& \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mathbb{1}_{\{E^{**}(T, T') = \mathcal{E}^{**}\}} \mathbb{1}_{\{T'' \in \mathcal{B}^{**}(T, T')\}} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mathbb{1}_{\{E^{**}(T, T') = \mathcal{E}^{**}\}} \mathbb{1}_{\{T'' \in \mathcal{B}^{**}(T, T')\}} \middle| T, T' \right] \middle| T \right] \right] \\
&= \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{E} \left[ \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mathbb{1}_{\{E^{**}(T, T') = \mathcal{E}^{**}\}} \mathbb{P}(T'' \in \mathcal{B}^{**}(T, T') | T, T') \middle| T \right] \right].
\end{aligned}$$

Since on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{E^{**}(T, T') = \mathcal{E}^{**}\}$ ,  $\mathbb{P}(T'' \in \mathcal{B}^{**}(T, T') | T, T')$  is bounded from below by  $\eta$ , the left side is bounded from below by

$$\eta \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{E} \left[ \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mathbb{1}_{\{E^{**}(T, T') = \mathcal{E}^{**}\}} \middle| T \right] \right].$$

Then, by summing on all  $\mathcal{E}^{**}$  subsets of edges of  $B_{3,s,N}$  and writing  $K$  a constant which does not depend on  $s$  and which bounds from above the number of different subsets of edges of  $B_{3,s,N}$ , we get

$$\begin{aligned}
& \frac{\eta}{K} \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}(\ell)\}} \mathbb{1}_{\{S_\ell^1(T) = s\}} \mathbb{P}(T' \in \mathcal{B}^*(T) | T) \right] \\
& \leq \mathbb{P} \left( T \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell), \bigcup_{s' \approx s} \{S_\ell^2(T) = s'\} \right).
\end{aligned}$$

Now, on the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\}$ ,  $\mathbb{P}(T' \in \mathcal{B}^*(T) | T)$  is bounded from below by  $\eta$ , so the left side is bounded from below by

$$\frac{\eta^2}{K} \mathbb{P}(T \in \mathcal{M}(\ell), S_\ell^1(T) = s).$$

Then, by summing on all  $s \in \mathbb{Z}^d$  and writing  $K'$  a constant which bounds from above for all  $s \in \mathbb{Z}^d$  the number of vertices  $s' \in \mathbb{Z}^d$  such that  $s' \approx s$ , we get

$$\frac{\eta^2}{KK'} \mathbb{P}(T \in \mathcal{M}(\ell)) \leq \mathbb{P}(T \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)).$$

Now, since  $\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1)$ ,

$$\mathbb{P}(T \in \mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) = \mathbb{P}(T \in \mathcal{M}(\ell - 1)) - \mathbb{P}(T \in \mathcal{M}(\ell)).$$

Thus,

$$\mathbb{P}(T \in \mathcal{M}(\ell)) \leq \lambda \mathbb{P}(T \in \mathcal{M}(\ell - 1)),$$

where  $\lambda = \frac{1}{1 + \frac{\eta^2}{KK'}} \in (0, 1)$  does not depend on  $x$ . Hence, using  $\mathbb{P}(T \in \mathcal{M}(0)) = 1$ , we get by induction

$$\mathbb{P}(T \in \mathcal{M}(q)) \leq \lambda^q.$$

□

From Lemma 2.3.5, the proof of Proposition 2.1.11 is the same as in the unbounded case.

### 2.3.3 Properties of a typical box

In this section, we state and prove the following lemma, which gives us properties of typical boxes useful for the modification argument, and the proof of Lemma 2.3.2.

**Lemma 2.3.6.** *If  $B_{4,s,N}$  is a typical box, we have the following properties.*

(i) *For all  $u$  and  $v$  in  $B_{3,s,N}$  with  $\|u - v\|_1 \geq K''N$  where  $K'' = \frac{1}{\varepsilon c_\mu} > 0$ ,*

$$(1 - 2\varepsilon)\mu(u - v) \leq t(u, v) \leq (1 + 2\varepsilon)\mu(u - v).$$

(ii) *For all  $z \in B_{3,s,N}$ , for all  $\bar{r} > 0$ , for all  $N \in \mathbb{N}^*$ ,*

$$B_{3,s,N} \cap B(z, N\bar{r}) \subset B_{3,s,N} \cap B_\mu\left(z, \frac{N(\bar{r} + 1)}{1 - \varepsilon}\right),$$

*and if  $\bar{r} \geq \frac{1}{\varepsilon} - 2$ ,*

$$B_{3,s,N} \cap B_\mu\left(z, \frac{N(\bar{r} + 1)}{1 - \varepsilon}\right) \subset B_{3,s,N} \cap B_\mu\left(z, \frac{N\bar{r}}{1 - 2\varepsilon}\right).$$

(iii) *For all  $z \in B_{3,s,N}$ , for all  $\bar{r} > 0$ , for all  $N \in \mathbb{N}^*$ ,*

$$B_{3,s,N} \cap B_\mu(z, N\bar{r}) \subset B_{3,s,N} \cap B(z, N((1 + \varepsilon)\bar{r} + 1)),$$

*and if  $\bar{r} \geq \frac{1}{\varepsilon}$ ,*

$$B_{3,s,N} \cap B(z, N((1 + \varepsilon)\bar{r} + 1)) \subset B_{3,s,N} \cap B(z, (1 + 2\varepsilon)N\bar{r}).$$

*Proof.* (i) Let  $u$  and  $v$  be in  $B_{3,s,N}$  with  $\|u - v\|_1 \geq K''N$ . Then, since  $B_{4,s,N}$  is a typical box, we have

$$(1 - \varepsilon)\mu(u - v) - N \leq t(u, v) \leq (1 + \varepsilon)\mu(u - v) + N.$$

The requirement on  $u$  and  $v$  implies

$$\frac{N}{c_\mu} \leq \varepsilon\|u - v\|_1,$$

so  $\varepsilon\mu(u - v) \geq N$ .

- (ii) and (iii) In both cases, it is easy to check that the first inclusion follows from property (iii) of a typical box and that the second inclusion is a consequence on the bound on  $\bar{r}$ .

□

*Proof of Lemma 2.3.2.*

1. Let  $B_{4,s,N}$  be a typical box and  $u_0$  and  $v_0$  two points of  $B_{3,s,N}$ . Then, taking paths minimizing the distance for the norm  $\|\cdot\|_1$  between  $u_0$  and  $sN$  and between  $v_0$  and  $sN$ , we have  $t(u_0, v_0) \leq 2r_3 N t_{\max}$ . Then, if a geodesic  $\gamma_{u_0, v_0}$  takes an edge which is not in  $B_{4,s,N}$ , since  $r_4 \geq r_3 + 1$ , using the item (ii) of the definition of a typical box leads to

$$T(\gamma_{u_0, v_0}) \geq 2(r_4 - r_3)N(t_{\min} + \delta),$$

which is impossible since  $r_4 > \frac{r_3(t_{\min} + \delta + t_{\max})}{t_{\min} + \delta}$ . Note that we only use item (ii) of the definition of a typical box to prove this property.

2. It is clear that the first property only depends on the time of edges in  $B_{3,s,N}$  and the second only depends on the time of edges in  $B_{4,s,N}$ . For the third property, by the preceding item, we know that for all points  $w_1$  and  $w_2$  in  $B_{3,s,N}$ , the knowledge of the time of all edges in  $B_{4,s,N}$  allows us to determine  $t(w_1, w_2)$ , so to know if the two inequalities are satisfied.
3. For each item of the definition of a typical box, we show that the probability that  $B_{4,0,N}$  satisfies this item goes to 1. To show that the probability that item (ii) of the definition of a typical box is satisfied goes to 1, we use the same proof as for (ii) in the proof of Lemma 2.2.1, replacing  $r_2 - r_1$  by 1 and  $B_{3,s,N}$  by  $B_{4,s,N}$ . Further, using (2.3.1) and a similar computation as for item (ii) in the proof of Lemma 2.2.1, we get:

$$\mathbb{P}(B_{4,0,N} \text{ satisfies (i)}) \xrightarrow{N \rightarrow \infty} 1.$$

To prove that

$$\mathbb{P}(B_{4,0,N} \text{ satisfies (iii)}) \xrightarrow{N \rightarrow \infty} 1,$$

recall that  $\varepsilon$  is fixed in Section 2.3.2 and fix  $\rho = \frac{1}{2d((1 + \varepsilon)C_\mu + t_{\max})}$ . Let us consider the following property for  $N$  large enough to have  $\lfloor \rho N \rfloor \neq 0$ :

$$\forall u', v' \in \lfloor \rho N \rfloor \mathbb{Z}^d \cap B_{3,0,N}, |t(u', v') - \mu(u' - v')| \leq \varepsilon \mu(u' - v'). \quad (2.3.10)$$

By (2.1.19), by stationarity, and since  $|\lfloor \rho N \rfloor \mathbb{Z}^d \cap B_{3,0,N}|$  is uniformly bounded in  $N$ , we get

$$\mathbb{P}((2.3.10) \text{ holds}) \xrightarrow{N \rightarrow \infty} 1.$$

Finally, the proof is completed by showing that (2.3.10) implies that  $B_{4,0,N}$  satisfies (iii). Assume (2.3.10) and let  $u$  and  $v$  be two vertices in  $B_{3,0,N}$ . The aim is to show

$$|t(u, v) - \mu(u - v)| \leq \varepsilon\mu(u - v) + N. \quad (2.3.11)$$

Let  $u', v' \in \lfloor \rho N \rfloor \mathbb{Z}^d \cap B_{3,0,N}$  such that  $\|u - u'\|_1 \leq d\rho N$  and  $\|u - v'\|_1 \leq d\rho N$ . Thus,

$$|t(u, v) - \mu(u - v)| \leq |t(u, v) - t(u', v')| + |t(u', v') - \mu(u' - v')| + |\mu(u' - v') - \mu(u - v)|.$$

By (2.3.10),  $|t(u', v') - \mu(u' - v')| \leq \varepsilon\mu(u' - v')$ . Furthermore,

$$|\mu(u - v) - \mu(u' - v')| \leq \mu(u - u') + \mu(v - v') \leq (\|u - u'\|_1 + \|v - v'\|_1)C_\mu \leq 2d\rho NC_\mu.$$

Similarly

$$|t(u, v) - t(u', v')| \leq t(u, u') + t(v, v') \leq (\|u - u'\|_1 + \|v - v'\|_1)t_{\max} \leq 2d\rho Nt_{\max},$$

Thus, we get

$$|t(u, v) - \mu(u - v)| \leq \varepsilon\mu(u - v) + 2d\rho N(t_{\max} + (1 + \varepsilon)C_\mu).$$

We get (2.3.11) thanks to the choice of  $\rho$ .

□

## 2.3.4 Modification argument

In this section, we prove Lemma 2.3.4.

### 2.3.4.1 Mains ideas of the proof

**2.3.4.1.1 Framework.** Before proving in detail Lemma 2.3.4, we give the main ideas of the proof of the modification argument. We consider a geodesic  $\gamma$  crossing a typical box. Recall that this box is composed of four concentric balls of different sizes, denoted<sup>7</sup> by  $B_1$ ,  $B_2$ ,  $B_3$  and  $B_4$ , and that  $\gamma$  visits at least one vertex in  $B_1$ . Recall also that the radii of the balls satisfy  $r_1 \ll r_2 \ll r_3 \ll r_4$ . The aim is to modify the environment to get the following properties:

1. Every geodesic in the new environment takes the pattern in  $B_2$ .
2. Every geodesic in the new environment is associated in  $B_4$  with a geodesic in the first environment.
3. The geodesic  $\gamma$  is associated in  $B_4$  with at least one geodesic in the new environment.

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<sup>7</sup>We denote  $B_i$  instead of  $B_{i,s,N}$  to lighten the notations in this subsection which is less formal than the proof.

**2.3.4.1.2 First modification.** Let us begin with some notations.

- We denote by  $\mathcal{D}_{\text{before}}$  the set of edges  $e$  verifying the following three conditions:
  - $\gamma$  takes  $e$  between its first entry in  $B_3$  and its first entry in  $B_2$ .
  - $e$  belongs to  $B_3$  but  $e$  does not belong to  $B_2$ .
  - $T(e) > t_{\min} + \delta$ .

We denote by  $s_1$  the vertex visited by  $\gamma$  just before its first edge of  $\mathcal{D}_{\text{before}}$ .

- $\mathcal{D}_{\text{after}}$  and  $s_2$  are symmetrically defined. In particular,  $s_2$  is the vertex visited by  $\gamma$  just after its last edge of  $\mathcal{D}_{\text{after}}$ .
- We set  $\mathcal{D} = \mathcal{D}_{\text{before}} \cup \mathcal{D}_{\text{after}}$ .

The first modification simply consists in reducing the passage times of edges in  $\mathcal{D}$  below  $t_{\min} + \delta'$  (with  $0 < \delta' < \delta$  properly chosen). This provides a localization of the geodesics in the new environment  $T^*$ : they all take all edges of  $\mathcal{D}$  (see Lemma 2.3.7 where  $\mathcal{D}$  is called  $E_+^*(T)$ ). Note also that  $\gamma$  remains a geodesic in the environment  $T^*$ . One could easily deduce that any geodesic in this new environment visits  $s_1$  and  $s_2$  and then that properties 2 and 3 about associated geodesics hold. The point is that these properties will be preserved by the second modification whose influence on time is negligible with respect to the first modification.

**2.3.4.1.3 Second modification.**

**Framework and overall plan.** The selected geodesic  $\gamma$  visits at least one vertex of  $B_1$ , let us denote by  $c_0$  the first of them. We denote by  $u_1$  the first vertex of  $\gamma$  such that  $\mu(u_1 - c_0) \leq \nabla N$  and  $v_1$  the last vertex of  $\gamma$  such that  $\mu(c_0 - v_1) \leq \nabla N$  (see (2.3.18)). Since  $r_1$  and  $\nabla$  are small enough compared with  $r_2$ , all that we consider (here and in what follows) takes place in  $B_2$  where the environments  $T$  and  $T^*$  coincide (see Lemma 2.3.8).

The rough plan is to modify the environment between  $u_1$  and  $v_1$ :

1. We consider an oriented path  $\pi$  from  $u_1$  to  $v_1$  and we make the passage times on its edges very small: this is our highway.
2. We put the pattern somewhere on  $\pi$ .
3. We make the passage times on the other edges in a certain neighborhood (to be defined) very large: these are our walls.

**Requirements for  $\pi$ .** It is quite simple and it is stated in Lemma 2.3.11. We want it to be close to the segment  $[u_1, v_1]$  of  $\mathbb{R}^d$ : this will allow us to have good estimates on the relevant times. We also require that the path contains towards its middle a sequence of steps in the same direction: this will allow us to place the pattern there. We call this sequence of steps the central segment of  $\pi$ . There is, however, a small difficulty: we cannot choose the orientation of the central segment. That is why we need to be able to place the original pattern in an overlapping pattern of arbitrary orientation (this is the purpose of Lemma 2.3.1).

**Requirements for the neighborhood of the walls.** This point is less simple. We do not want to put walls on edges of  $\gamma \setminus \gamma_{u_1, v_1}$  but we also do not want that these edges affect our estimates. Ideally, we would like the only relevant edges, apart those from  $\pi$  and the pattern-location, to be walls. This can be done in a non technical way as follows. Set

$$B^*(0) = B^*(0, t^*(0, u_1)) \text{ and } B^*(x) = B^*(x, t^*(x, v_1)).$$

There are three types of edges:

- The "before" edges: the ones belonging to  $B^*(0)$ .
- The "after" edges: the ones belonging to  $B^*(x)$ .
- The "intermediate" edges: all other edges.

Since  $\gamma$  is a geodesic in the environment  $T^*$ ,  $\gamma$  takes "before" edges, then "intermediate" edges and then "after" edges. Note that the edges of  $\mathcal{D}_{\text{before}}$  are "before" edges and that the edges of  $\mathcal{D}_{\text{after}}$  are "after" edges.

We can then define the neighborhood on which we put walls: this is the set of "intermediate" edges of  $B_2$  which do not belong to  $\pi$  and to the pattern-location. We call these edges the "wall" edges. The idea is that, as explained at (2.3.12), if  $\bar{\pi}^{**}$  is a part of a geodesic from 0 to  $x$  in the environment  $T^{**}$  linking two vertices outside  $B^*(0)$  and  $B^*(x)$ , then  $\bar{\pi}^{**}$  only takes "intermediate" edges. Hence, if in addition  $\bar{\pi}^{**}$  does not take edges of  $\pi$  and is entirely contained in  $B_2$ , then  $\bar{\pi}^{**}$  only takes "wall" edges or edges of the pattern-location: this will ensure in the end that every geodesic takes the pattern.

**It makes sense.** For any  $z \in B^*(0)$ , by the triangle inequality,

$$t^*(z, c_0) \geq t^*(0, c_0) - t^*(0, z).$$

Furthermore, since  $\gamma$  is still a geodesic in the environment  $T^*$  and visits  $u_1$  and then  $c_0$ ,  $t^*(0, c_0) = t^*(0, u_1) + t^*(u_1, c_0)$ . Using the definition of  $B^*(0)$ ,  $t^*(0, z) \leq t^*(0, u_1)$ . Thus, we get

$$t^*(B^*(0), c_0) \geq t^*(u_1, c_0) = t(u_1, c_0) \approx \nabla N.$$

In particular, the distance for the norm  $\|\cdot\|_1$  between  $B^*(0)$  and  $c_0$  is at least of order  $\nabla N$ . The same applies to the distance for the norm  $\|\cdot\|_1$  between  $B^*(x)$  and  $c_0$ . Thus there is enough room for the central segment and the pattern since their size is of order 1 (see Lemma 2.3.13).

**The second modification.**

- We put the pattern somewhere on the "central segment".
- With the exception of the edges of  $\pi$  connected to  $B^*(0)$  or  $B^*(x)$ , we put the time of every "intermediate" edge of  $\pi$  which is not in the pattern below  $t_{\min} + \delta'$ .
- We put the time of every other "intermediate" edge of  $B_2$  which does not belong to  $\pi$  and whose time is lower than  $\nu$  greater than or equal to  $\nu$  (recall that in particular we take  $\nu$  such that, in the pattern, the passage time of every edge is lower than or equal to  $\nu$ ).

**Consequences.** In what follows, we denote by  $\bar{\gamma}^{**}$  a geodesic in the environment  $T^{**}$ .

1. Since the passage times of the "before" and "after" edges have not been modified and since the passage times of the "intermediate" edges touching  $B^*(0)$  or  $B^*(x)$  have not been reduced,  $B^*(0)$  remains a ball centered in 0 for the passage times of the environment  $T^{**}$  and similarly for  $B^*(x)$  (see again Lemma 2.3.13). Hence,

$$\bar{\gamma}^{**} \text{ takes "before" edges, then "intermediate" edges and then "after" edges.} \quad (2.3.12)$$

2. The creation of the highway on  $\pi$  makes the passage time from 0 to  $x$  lower (as stated in Lemma 2.3.14, it allows to save a time of order  $\nabla N$ ; it can be seen by taking a geodesic from 0 to the last vertex of  $\pi$  belonging to  $B^*(0)$ , then following  $\pi$ , then taking a geodesic from the first vertex of  $\pi$  belonging to  $B^*(x)$  to  $x$ ). Hence  $\bar{\gamma}^{**}$  takes an edge whose time has been reduced during the second modification and thus

$$\begin{aligned} \bar{\gamma}^{**} \text{ takes an edge of } \pi \text{ whose time has been reduced or an edge} \\ \text{of the pattern (and thus an "intermediate" edge of } B_2). \end{aligned} \quad (2.3.13)$$

3. The time saved during the first modification (of order  $r_3$ ) is such that every geodesic in the environment  $T^{**}$  from 0 to a vertex  $w$  in  $B_2$  has to take an edge of  $\mathcal{D}$  (i.e. an edge whose time has been reduced during the first modification). If it was not the case, a path following  $\gamma$  until  $B_2$  and then taking edges of  $B_2$  to go to  $w$  would have a passage time smaller (we use here that  $r_3 \gg r_2$ ). This is similar for a geodesic between a vertex of  $B_2$  and  $x$ . Hence, as stated in Lemma 2.3.17,

$$\bar{\gamma}^{**} \text{ takes an edge of } \mathcal{D} \text{ before and after visiting } B_2. \quad (2.3.14)$$

4. Recall that  $\mathcal{D} = \mathcal{D}_{\text{before}} \cup \mathcal{D}_{\text{after}}$ , that the edges of  $\mathcal{D}_{\text{before}}$  are "before" edges and that those of  $\mathcal{D}_{\text{after}}$  are "after" edges. Combining (2.3.12), (2.3.13) and (2.3.14) we get that

$$\begin{aligned} \bar{\gamma}^{**} \text{ takes an edge of } \mathcal{D}_{\text{before}} \text{ then an edge of } \pi \text{ whose time has been reduced} \\ \text{or an edge of the pattern and then an edge of } \mathcal{D}_{\text{after}}. \end{aligned} \tag{2.3.15}$$

We can easily deduce from this that  $\bar{\gamma}^{**}$  visits  $s_1$  just before taking the first edge whose time has been modified and visits  $s_2$  just after taking the last edge whose time has been modified (see Lemma 2.3.18). Then we can deduce the desired properties about associated geodesics. It remains to prove that  $\bar{\gamma}^{**}$  takes the pattern in  $B_2$ .

5. Let us now prove that

$$\begin{aligned} \bar{\gamma}^{**} \text{ takes an edge of } \pi \text{ whose time has been reduced before (with respect to } \pi) \\ \text{the pattern and after (with respect to } \pi) \text{ the pattern.} \end{aligned} \tag{2.3.16}$$

This is the purpose of Lemma 2.3.20. By symmetry it is sufficient to prove the first part of this property. To this aim, it is sufficient to consider a vertex  $w$  in the pattern or on  $\pi$  between the pattern and  $B^*(x)$  and to prove

$$t^{**}(B^*(0), w) < t^*(B^*(0), w).$$

But  $t^*(B^*(0), w) \geq t^*(B^*(0), B^*(x)) - t^*(B^*(x), w)$  and

$$t^*(B^*(0), B^*(x)) = t^*(u_1, v_1) = t(u_1, v_1) \approx \mu(u_1 - v_1)$$

while

$$t^*(B^*(x), w) \leq t^*(v_1, w) = t(v_1, w) \approx \mu(v_1 - w).$$

Thus (since the three vertices are roughly aligned in the correct order):

$$t^*(B^*(0), w) \gtrsim \mu(u_1 - v_1) - \mu(v_1 - w) \approx \mu(u_1 - w).$$

But (following  $\pi$ ) we have

$$t^{**}(B^*(0), w) \lesssim (t_{\min} + \delta') \|w - u_1\|$$

and the result holds.

6. By (2.3.16) and the remark made after the definition of walls, we can easily deduce that  $\bar{\gamma}^{**}$  takes the pattern we have put on  $\pi$  (see Lemma 2.3.21).



### 2.3.4.2 First modification

Let us go back to the proof. For the rest of Section 2.3.4, we fix  $\ell \in \{1, \dots, q\}$  and  $s \in \mathbb{Z}^d$ , and we assume that the event  $\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\}$  occurs. In particular,  $B_{4,s,N}$  is a typical box in the environment  $T$ .

As explained above, the first modification is used to get that any geodesic from 0 to  $x$  in the new environment goes from 0 to a specific vertex in  $B_{4,s,N}$ , then follows  $\gamma$  to another specific vertex in  $B_{4,s,N}$  and then goes to  $x$ . It is useful to get the properties about associated geodesics after the second modification. We denote by  $\gamma$  the selected geodesic in the environment  $T$ . Recall the definition of entry and exit points given at the beginning of Section 2.2.3. Let  $u$  denote the entry point of  $\gamma$  in  $B_{3,s,N}$  and  $v$  the exit point, and let  $u_0$  denote the entry point of  $\gamma$  in  $B_{2,s,N}$  and  $v_0$  its exit point. Then, we set

$$E_+^*(T) = \{\text{edges of } B_{3,s,N} \text{ that belong to } \gamma_{u,u_0} \text{ or } \gamma_{v_0,v} \text{ and satisfy } T(e) > t_{\min} + \delta\}. \quad (2.3.17)$$

We denote the first edge of  $\gamma$  that belongs to  $E_+^*(T)$  by  $e_1$  and the last one by  $e_2$ . Moreover,  $s_1$  denotes the first vertex of  $e_1$  visited by  $\gamma$  and  $s_2$  the last vertex of  $e_2$  visited by  $\gamma$ . Assume that the event

$$\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \text{ occurs,}$$

where  $\mathcal{B}^*(T)$  is defined in (ii) of Lemma 2.3.3. Recall the definition of  $T^*$  in (2.3.7). Note that:

- only the time of the edges of  $E_+^*(T)$  have been modified,
- these edges belong to  $\gamma$  and to  $B_{3,s,N} \setminus B_{2,s,N}$ ,
- for all edges in  $E_+^*(T)$ , we have  $T^*(e) \leq t_{\min} + \delta' < t_{\min} + \delta \leq T(e)$ ,
- for all other edges  $e$ , we have  $T^*(e) = T(e)$ .

**Lemma 2.3.7.** *We have the following properties.*

(i) *There are at least  $\alpha(r_3 - r_2)N$  edges of  $\gamma_{u,u_0}$  and  $\alpha(r_3 - r_2)N$  edges of  $\gamma_{v_0,v}$  that belong to  $E_+^*(T)$ . Thus,*

$$\min(T(\gamma_{0,u_0}) - T^*(\gamma_{0,u_0}), T(\gamma_{v_0,x}) - T^*(\gamma_{v_0,x})) \geq \alpha(r_3 - r_2)N(\delta - \delta').$$

(ii) *In the environment  $T^*$ , every geodesic from 0 to  $x$  visits every edge of  $E_+^*(T)$ .*

(iii) *In the environment  $T^*$ ,  $\gamma$  is a geodesic from 0 to  $x$ .*

*Proof.* (i) This item follows from the first property of a typical box applied to the portion of  $\gamma_{u,u_0}$  entirely contained in  $B_{3,s,N}$  going from  $\partial B_{3,s,N}$  to  $u_0$  and to the portion of  $\gamma_{v_0,v}$  entirely contained in  $B_{3,s,N}$  going from  $v_0$  to  $\partial B_{3,s,N}$  (note that these two portions does not have edges in common).

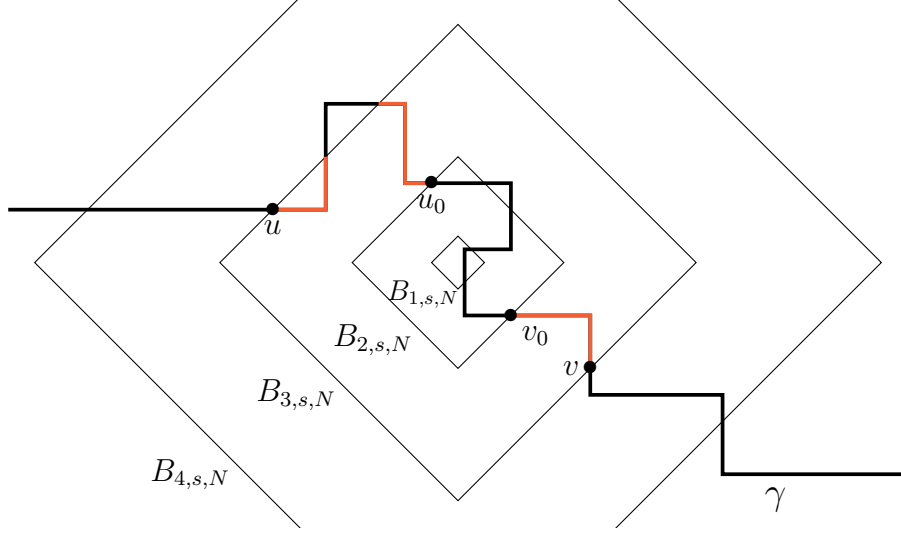


Figure 2.3: Elements involved in the first modification. The edges whose time can be modified by the first modification are represented in red.

- (ii) To prove the second point, let  $\gamma^*$  be a geodesic from 0 to  $x$  in the environment  $T^*$ . Assume that there exists an edge of  $E_+^*(T)$  which is not an edge of  $\gamma^*$ . Then

$$T(\gamma^*) - T^*(\gamma^*) < T(\gamma) - T^*(\gamma).$$

Since  $\gamma$  is a geodesic from 0 to  $x$  in the environment  $T$ , we have  $T(\gamma) \leq T(\gamma^*)$ , which implies

$$0 \leq T(\gamma^*) - T(\gamma) < T^*(\gamma^*) - T^*(\gamma).$$

Thus  $T^*(\gamma) < T^*(\gamma^*)$ , which contradicts the fact that  $\gamma^*$  is a geodesic from 0 to  $x$  in the environment  $T^*$ .

- (iii) Let us now assume that  $\gamma$  is not a geodesic in the environment  $T^*$ . Hence, if we denote by  $\gamma^*$  a geodesic from 0 to  $x$ , we have  $T^*(\gamma^*) < T^*(\gamma)$ . By item (ii),

$$T(\gamma^*) - T^*(\gamma^*) = T(\gamma) - T^*(\gamma).$$

Thus,  $T(\gamma^*) < T(\gamma)$ , which contradicts the fact that  $\gamma$  is a geodesic in the environment  $T$ . □

### 2.3.4.3 Construction of $\pi$

Here, we shall identify an oriented path  $\pi$  between two vertices of  $\gamma$  in  $B_{2,s,N}$ . This oriented path is later used to place the pattern. Let  $c_0$  denote the entry point of  $\gamma$  in  $B_{1,s,N}$  and recall the definition of  $\nabla$  in (2.3.2). Since by (2.3.3),  $r_2 \geq r_1 + \frac{\nabla}{c_\mu}$ ,  $B_\mu(c_0, N\nabla) \cap \mathbb{Z}^d$  is entirely

contained in  $B_{2,s,N}$ . We introduce

$$u_1 \text{ the entry point of } \gamma \text{ in } B_\mu(c_0, N\nabla) \cap \mathbb{Z}^d \text{ and } v_1 \text{ the exit point.} \quad (2.3.18)$$

**Lemma 2.3.8.** *We have  $\mu(u_1 - v_1) \geq N\nabla$  and  $\gamma_{u_1, v_1}$  is contained in  $B_{2,s,N}$ .*

**Remark 2.3.9.** *The idea of the proof is that  $\mu(u_1 - v_1)$  is roughly equal to  $t(u_1, v_1)$ . Furthermore, since  $\gamma$  is a geodesic visiting  $u_1$ ,  $c_0$  and  $v_1$  in this order, we have  $t(u_1, v_1) = t(u_1, c_0) + t(c_0, v_1)$ . So  $\mu(u_1 - v_1) \approx \mu(u_1 - c_0) + \mu(v_1 - c_0) \approx 2N\nabla$ . We have a sufficient control over the approximations to guarantee a lower bound by  $N\nabla$ .*

*Proof.* Using item (ii) of Lemma 2.3.6 with  $z = c_0$  and  $\bar{r} = \nabla(1 - \varepsilon) - 1$  leads to

$$B(c_0, N(\nabla(1 - \varepsilon) - 1)) \cap B_{3,s,N} \subset B_\mu(c_0, N\nabla) \cap B_{3,s,N} = B_\mu(c_0, N\nabla) \cap \mathbb{Z}^d. \quad (2.3.19)$$

Since  $u_1$  is the entry point of a path in  $B_\mu(c_0, N\nabla)$ , there exists a vertex  $\bar{u}_1 \in B_{3,s,N}$  such that  $\|\bar{u}_1 - u_1\|_1 = 1$  and  $\bar{u}_1 \notin B_\mu(c_0, N\nabla)$ . By (2.3.19),  $t(c_0, \bar{u}_1) \geq N(\nabla(1 - \varepsilon) - 1)$ , so

$$t(c_0, u_1) \geq t(c_0, \bar{u}_1) - \underbrace{t(\bar{u}_1, u_1)}_{\leq \|\bar{u}_1 - u_1\|_1 t_{\max}}.$$

The same argument holds for  $t(c_0, v_1)$ . Hence

$$t(u_1, v_1) = t(u_1, c_0) + t(c_0, v_1) \geq 2N(\nabla(1 - \varepsilon) - 1) - 2t_{\max}. \quad (2.3.20)$$

Thus, since  $\nabla > \frac{3 + 2t_{\max}}{1 - 3\varepsilon}$ , by the third item of the definition of a typical box,

$$\mu(u_1 - v_1) \geq \frac{N(2\nabla(1 - \varepsilon) - 3) - 2t_{\max}}{1 + \varepsilon} \geq N\nabla.$$

For the second part of the proof, using the third item of Lemma 2.3.6 with  $z = c_0$  and  $\bar{r} = \nabla$  leads to

$$B_\mu(c_0, N\nabla) \cap \mathbb{Z}^d \subset B(c_0, ((1 + \varepsilon)\nabla + 1)N).$$

Then, by (2.3.3) we have  $r_2 > r_1 + \frac{2(\nabla + 2)}{c_\mu}$ . So, for all  $z \in B_\mu(c_0, 2N(\nabla + 2))$ , we have

$$\|z - sN\|_1 \leq \|z - c_0\|_1 + \|c_0 - sN\|_1 \leq \left( \frac{2(\nabla + 2)}{c_\mu} + r_1 \right) N < r_2 N.$$

So, we have  $B_\mu(c_0, 2N(\nabla + 2)) \cap \mathbb{Z}^d \subset B_{2,s,N}$  and since  $\varepsilon < \frac{1}{3}$ , we have

$$B_\mu \left( c_0, N \frac{(1 + \varepsilon)\nabla + 2}{1 - \varepsilon} \right) \subset B_\mu(c_0, 2N(\nabla + 2)).$$

Thus, by the second item of Lemma 2.3.6 with  $z = c_0$  and  $\bar{r} = \nabla$ , we get

$$B(c_0, ((1 + \varepsilon)\nabla + N)) \subset B_\mu \left( c_0, N \frac{(1 + \varepsilon)\nabla + 2}{1 - \varepsilon} \right) \cap \mathbb{Z}^d \subset B_\mu(c_0, 2N(\nabla + 2)) \cap \mathbb{Z}^d.$$

To sum up,

$$B_\mu(c_0, N\nabla) \cap \mathbb{Z}^d \subset B(c_0, (1 + \varepsilon)N\nabla + N) \subset B_{2,s,N}.$$

Now, assume that  $\gamma_{u_1, v_1}$  visits a vertex which is not in  $B_{2,s,N}$ . Let denote by  $z$  such a vertex and assume for example that  $z$  is visited by  $\gamma_{u_1, c_0}$ . Then, we have, thanks to these inclusions,  $t(c_0, z) > t(c_0, u_1)$ , which is impossible since  $\gamma_{u_1, c_0}$  is a geodesic.  $\square$

**Lemma 2.3.10.** *Recall the definition of  $B^*$  given at (2.3.9) and that  $T^*(\gamma_{0, u_1}) = t^*(0, u_1)$  and  $T^*(\gamma_{v_1, x}) = t^*(v_1, x)$ .*

(i) *We have the following inclusions:*

$$B^*(0, T^*(\gamma_{0, u_1})) \subset B(0, t(0, u_1)) \text{ and } B^*(x, T^*(\gamma_{v_1, x})) \subset B(x, t(v_1, x)).$$

(ii) *We have*

$$B^*(0, T^*(\gamma_{0, u_1})) \cap B^*(x, T^*(\gamma_{v_1, x})) = \emptyset.$$

*Proof.* (i) Let  $s'$  be a vertex in  $B^*(0, T^*(\gamma_{0, u_1}))$ . The aim is to show

$$t(0, s') \leq t(0, u_1).$$

Let  $\bar{\gamma}^*$  be a geodesic from 0 to  $s'$  in the environment  $T^*$ . We denote by  $s^*$  the last vertex visited by  $\bar{\gamma}^*$  among those visited by  $\gamma$  (note that 0 is such a vertex). First, since  $\bar{\gamma}_{s^*, s'}$  does not take an edge of  $\gamma$ ,  $T^*(\bar{\gamma}_{s^*, s'}) = T(\bar{\gamma}_{s^*, s'})$ . Then, by Lemma 2.3.8,  $\gamma_{u_1, v_1}$  is entirely contained in  $B_{2,s,N}$ , so  $T^*(\gamma_{u_1, v_1}) = T(\gamma_{u_1, v_1})$ . Thus, also by Lemma 2.3.8,  $T^*(\gamma_{u_1, v_1}) > 0$ . So  $\bar{\gamma}^*$  does not take an edge of  $\gamma_{v_1, x}$ . Otherwise, since  $\gamma$  is a geodesic in the environment  $T^*$ ,  $t^*(0, s') \geq T^*(\gamma_{0, v_1}) > T^*(\gamma_{0, u_1})$ . Hence, the time saved by  $\bar{\gamma}^*$  after the modification comes only from the edges of  $\gamma_{0, u_1}$ . So,

$$T(\bar{\gamma}_{0, s^*}^*) - T^*(\bar{\gamma}_{0, s^*}^*) \leq T(\gamma_{0, u_1}) - T^*(\gamma_{0, u_1}). \quad (2.3.21)$$

Hence,

$$\begin{aligned}
t(0, s') &\leq T(\bar{\gamma}_{0, s^*}^*) + T(\bar{\gamma}_{s^*, s'}^*) \\
&\leq T(\gamma_{0, u_1}) - T^*(\gamma_{0, u_1}) + \underbrace{T^*(\bar{\gamma}_{0, s^*}^*) + T^*(\bar{\gamma}_{s^*, s'}^*)}_{=t^*(0, s')} \text{ by (2.3.21),} \\
&= t^*(0, s') + T(\gamma_{0, u_1}) - T^*(\gamma_{0, u_1}) \\
&\leq T^*(\gamma_{0, u_1}) + T(\gamma_{0, u_1}) - T^*(\gamma_{0, u_1}) \text{ since } s' \in B^*(0, T^*(\gamma_{0, u_1})), \\
&= t(0, u_1),
\end{aligned}$$

which proves the inclusion and the same proof gives us the second inclusion.

(ii) Let  $s'$  be a point of  $B^*(0, T^*(\gamma_{0, u_1})) \cap B^*(x, T^*(\gamma_{v_1, x}))$ . Then

$$\begin{aligned}
t(0, x) &\leq t(0, s') + t(s', x) \\
&\leq t(0, u_1) + t(v_1, x) \text{ by (i)} \\
&< t(0, x)
\end{aligned}$$

since  $\gamma$  is a geodesic visiting  $0, u_1, v_1$  and  $x$  in this order and since  $t(u_1, v_1) > 0$ , which is a contradiction. □

Now, we can make the construction of  $\pi$ , which is the path on which we would like to put the pattern that the geodesics have to take. We begin by two definitions:

- We say that a path between two vertices  $y_1$  and  $y_2$  is oriented if its number of edges is equal to  $\|y_1 - y_2\|_1$ .
- A step of length  $\ell$  is a path of  $\ell$  consecutive edges in the same direction.

We state the following lemma whose proof is left to the reader and where  $[u_1, v_1]$  is the segment in  $\mathbb{R}^d$  and  $\|\cdot\|_1$  is the norm on  $\mathbb{R}^d$ .

**Lemma 2.3.11.** *We can construct a path  $\pi$  with a deterministic rule such that:*

- (i)  $\pi$  is an oriented path connecting  $u_1$  and  $v_1$ ,
- (ii)  $\pi$  is the concatenation of steps of length  $10\ell^\Lambda$  except in  $B_\infty(v_1, 10\ell^\Lambda)$ , where  $\pi$  takes steps of length 1,
- (iii) there exists a constant  $L_1 \in \mathbb{R}_+$  depending only on  $\ell^\Lambda$  and  $d$  such that for all  $z \in \mathbb{Z}^d$  which is in  $\pi$ , the distance for the norm  $\|\cdot\|_1$  between  $z$  and  $[u_1, v_1]$  is bounded by  $L_1$ , and for all  $y \in [u_1, v_1]$ , the distance for the norm  $\|\cdot\|_1$  between  $y$  and  $\pi$  (seen as a set of vertices in  $\mathbb{Z}^d$ ) is bounded by  $L_1$ .

Let  $\pi$  be the path given by Lemma 2.3.11. We introduce

$$u_2 \text{ the first vertex of } \pi \text{ starting from } v_1 \text{ in } B^*(0, T^*(\gamma_{0, u_1})), \quad (2.3.22)$$

$$v_2 \text{ the first vertex of } \pi \text{ starting from } u_1 \text{ in } B^*(x, T^*(\gamma_{v_1, x})). \quad (2.3.23)$$

Remark that  $t^*(0, u_2) \leq t^*(0, u_1)$  and  $t^*(v_2, x) \leq t^*(v_1, x)$ .

**Lemma 2.3.12.** *The path  $\pi$  is contained in  $B_{2, s, N}$ .*

*Proof.* Let  $z$  be a vertex of  $\pi$ . Using Lemma 2.3.11, there exists  $y \in [u_1, v_1]$  such that  $\|z - y\|_1 \leq L_1$ . The vertices  $u_1$  and  $v_1$  belong to  $B_\mu(c_0, N\nabla)$  which is convex, thus  $y \in B_\mu(c_0, N\nabla)$ . So, we have

$$\begin{aligned} \|z - sN\|_1 &\leq \underbrace{\|z - y\|_1}_{\leq L_1} + \underbrace{\|y - c_0\|_1}_{\leq \frac{\mu(y - c_0)}{c_\mu}} + \underbrace{\|c_0 - sN\|_1}_{\leq r_1 N} \\ &\leq L_1 + \frac{N\nabla}{c_\mu} + r_1 N \text{ since } y \in B_\mu(c_0, N\nabla), \\ &\leq r_2 N \text{ by (2.3.3),} \end{aligned}$$

which proves that  $z \in B_{2, s, N}$ . □

#### 2.3.4.4 Second modification

Now, let us define  $E_P^{**}$ ,  $E_+^{**}$ ,  $E_-^{**}$  and  $\mathcal{O}$ . Let  $c_1 \in \mathbb{R}^d$  denote the midpoint of  $[u_1, v_1]$  and let us consider the set of all vertices of  $\pi_{u_2, v_2}$  at distance at most  $L_1$  for the norm  $\|\cdot\|_1$  from  $c_1$ . This set is not empty, hence we can choose one such vertex with a deterministic rule. Recall that  $L_2 = L_1 + (10 + d)\ell^\Lambda$ . Since  $\nabla \geq 6dL_2C_\mu$  (see (2.3.2)), we have that the distance for the norm  $\|\cdot\|_\infty$  between the chosen vertex and  $v_1$  is greater than  $10\ell^\Lambda$ . So, using item (ii) of Lemma 2.3.11, there exist one or two steps of the path  $\pi$  of length  $10\ell^\Lambda$  that contain the chosen vertex. We chose one step (between these one or two steps) with a deterministic rule to put the pattern. We denote the midpoint of this chosen step by  $c_P$ . Then,

$$\mathcal{O}(T, T') \text{ is equal to the direction of this step,} \quad (2.3.24)$$

$$\text{and } E_P^{**}(T, T') = \left\{ \text{edges connecting vertices belonging to } B_\infty(c_P, \ell^\Lambda) \right\}. \quad (2.3.25)$$

Note that by the direction of a step, we mean the integer  $j$  such that for every distinct vertices  $z_1$  and  $z_2$  of this step,  $z_1 - z_2 = \pm \|z_1 - z_2\|_1 \varepsilon_j$ . Note also that for all vertex  $z$  in  $B_\infty(c_P, \ell^\Lambda)$ ,

$$\|z - c_1\|_1 \leq L_2.$$

We define  $u_3$  and  $v_3$  as the endpoints of the oriented pattern such that  $\pi$  visits  $u_2, u_3, v_3$  and  $v_2$  in this order. Recall that for a pattern, the entry and exit points correspond to the endpoints given by the definition of this pattern. Note that, thanks to the construction,  $u_3$

and  $v_3$  are two vertices of the chosen step of length  $10\ell^\Lambda$ . Then, we have to define  $E_+^{**}(T, T')$  and  $E_-^{**}(T, T')$ :

$$E_+^{**}(T, T') = \{\text{edges of } \pi_{u_2, u_3} \cup \pi_{v_3, v_2} \text{ except the one connected to } u_2 \text{ and} \\ \text{the one connected to } v_2\}, \quad (2.3.26)$$

$$E_-^{**}(T, T') = \{\text{edges } e \text{ in } B_{2,s,N} \text{ such that } T(e) < \nu \text{ and which are not} \\ \text{in } E_P^{**}(T, T'), \pi_{u_2, u_3} \cup \pi_{v_3, v_2}, B^*(0, T^*(\gamma_{0, u_1})) \text{ or } B^*(x, T^*(\gamma_{v_1, x}))\}. \quad (2.3.27)$$

For all the sequel, we assume that the event

$$\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{T'' \in \mathcal{B}^{**}(T, T')\} \text{ occurs.}$$

Recall the definition of the environment  $T^{**}$  given at (2.3.8) and of  $\mathcal{B}^{**}(T, T')$  given in Lemma 2.3.3. In particular,

- for all  $e \in E_+^*(T)$ ,  $T^{**}(e) = T^*(e) < T(e)$ ,
- for all  $e \in E_+^{**}(T, T')$ ,  $T^{**}(e) \leq t_{\min} + \delta'$ ,
- for all  $e \in E_-^{**}(T, T')$ ,  $T^{**}(e) \geq \nu > T(e) = T^*(e)$ ,
- since by Lemma 2.3.12,  $\pi$  is contained in  $B_{2,s,N}$ , we have that  $E_+^{**}(T, T')$ ,  $E_-^{**}(T, T')$  and  $E_P^{**}(T, T')$  are pairwise disjoint and are contained in  $B_{2,s,N}$ . Thus, since  $E_+^*(T)$  is contained in  $B_{3,s,N} \setminus B_{2,s,N}$ , item (i) of Lemma 2.3.3 is satisfied.

In what follows, when we talk about edges whose time has been reduced, it means the edges  $e$  such that  $T^{**}(e) < T(e)$ . Before proving item (ii) of Lemma 2.3.3, we state the following lemma which completes the vision of the sets  $E_+^{**}(T, T')$ ,  $E_-^{**}(T, T')$  and  $E_P^{**}(T, T')$ . Recall the definition of  $\mathcal{B}^{**}$  given at (2.3.9).

**Lemma 2.3.13.** *We have*

$$B^{**}(0, T^*(\gamma_{0, u_1})) = B^*(0, T^*(\gamma_{0, u_1})) \text{ and } B^{**}(x, T^*(\gamma_{v_1, x})) = B^*(x, T^*(\gamma_{v_1, x})),$$

and there is no edge of  $E_P^{**}(T, T')$  in any of these balls.

*Proof.* We begin by proving that there is no edge of  $E_P^{**}(T, T')$  in  $B^*(0, T^*(\gamma_{0, u_1}))$ . To this aim, we prove that there is no vertex of  $B_\infty(c_P, \ell^\Lambda)$  in this ball. Let  $z$  be a vertex of  $B_\infty(c_P, \ell^\Lambda)$ . The idea is the following. Since  $\gamma$  is a geodesic in the environment  $T^*$  by item (iii) of Lemma 2.3.7, since the time of the edges of  $\gamma_{u_1, v_1}$  is not modified by the first modification and since  $B_{4,s,N}$  is a typical box in the environment  $T$ , for every  $w$  vertex of  $\gamma_{u_1, v_1}$ ,

$$t^*(0, w) = t^*(0, u_1) + t^*(u_1, w) = t^*(0, u_1) + t(u_1, w) \approx t^*(0, u_1) + \mu(u_1 - w).$$

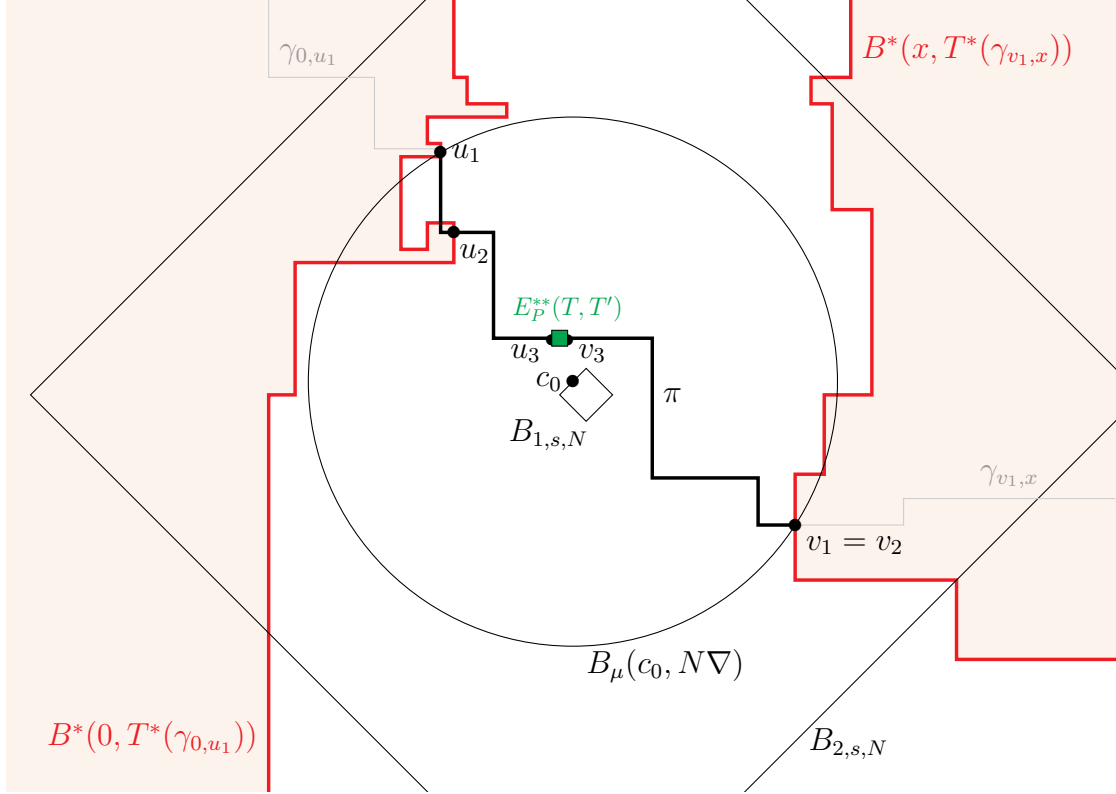


Figure 2.4: The construction of  $\pi$ , of the vertices  $u_1, u_2, u_3, v_1, v_2, v_3$  and of the pattern-location in dimension 2.

We cannot guarantee that  $z$  is a vertex of  $\gamma_{u_1, v_1}$  but using a similar argument:

$$t^*(0, z) \geq t^*(0, v_1) - t^*(v_1, z) = t^*(0, u_1) + t^*(u_1, v_1) - t^*(v_1, z),$$

since  $\gamma$  is a geodesic in the environment  $T^*$ . But

$$t^*(u_1, v_1) = t(u_1, v_1) \approx 2N\nabla \text{ and } t^*(v_1, z) \leq t^*(v_1, c_1) + t^*(c_1, z) \approx N\nabla,$$

since  $t^*(c_1, z)$  is negligible compared to  $N\nabla$ , which gives that  $t^*(0, z)$  is roughly greater than  $t^*(0, u_1) + N\nabla$ .

Now, we make the proof rigorous. Since  $z \in B_\infty(c_P, \ell^\Lambda)$ , we have  $z \in B_\mu(c_1, C_\mu L_2)$ . Since by (2.3.2),  $\nabla \geq 2C_\mu L_2$ ,  $z \in B_\mu\left(c_1, \frac{N\nabla}{2}\right)$ . Furthermore, since  $c_1$  is the midpoint of  $[u_1, v_1]$  and  $\mu(u_1 - v_1) \leq 2N\nabla$ , we have  $\mu(v_1 - c_1) \leq N\nabla$ . Thus, since  $\nabla \geq \frac{1}{\varepsilon}$  (by (2.3.2)), using the third item of Lemma 2.3.6 with  $z = v_1$  and  $\bar{r} = \frac{3\nabla}{2}$ , we have

$$z \in B_\mu\left(v_1, \frac{3N\nabla}{2}\right) \cap \mathbb{Z}^d \subset B\left(v_1, \frac{3N\nabla(1+2\varepsilon)}{2}\right).$$



Hence, using the lower bound (2.3.20) for  $t(u_1, v_1)$  of the proof of Lemma 2.3.8 and since by (2.3.2),  $\nabla > \frac{4(1+t_{\max})}{1-10\varepsilon}$ , we have

$$t(0, v_1) - t(0, u_1) = t(u_1, v_1) \geq 2N(\nabla(1-\varepsilon) - 1) - 2t_{\max} > \frac{3N\nabla(1+2\varepsilon)}{2},$$

and thus, using also that  $z \in B\left(v_1, \frac{3N\nabla(1+2\varepsilon)}{2}\right)$ ,

$$t(0, z) \geq t(0, v_1) - t(v_1, z) \geq t(0, v_1) - \frac{3N\nabla(1+2\varepsilon)}{2} > t(0, u_1).$$

The first item of Lemma 2.3.10 allows us to conclude.

Then, let us prove the first equality. The proof of the second one is the same. The inclusion  $B^*(0, T^*(\gamma_{0, u_1})) \subset B^{**}(0, T^*(\gamma_{0, u_1}))$  is easy to check. Let us take  $z \in B^*(0, T^*(\gamma_{0, u_1}))$  and  $\bar{\gamma}^*$  a geodesic from 0 to  $z$  in the environment  $T^*$ . Then  $\bar{\gamma}^*$  is entirely contained in  $B^*(0, T^*(\gamma_{0, u_1}))$  and there are no edges of  $E_-^{**}(T, T')$  or  $E_P^{**}(T, T')$  in  $B^*(0, T^*(\gamma_{0, u_1}))$ , so

$$t^{**}(0, z) \leq T^{**}(\bar{\gamma}^*) \leq T^*(\bar{\gamma}^*) \leq T^*(\gamma_{0, u_1}).$$

For the other inclusion, assume that there exists a vertex  $z \in B^{**}(0, T^*(\gamma_{0, u_1})) \setminus B^*(0, T^*(\gamma_{0, u_1}))$  and let  $\bar{\gamma}^{**}$  be a geodesic from 0 to  $z$  in the environment  $T^{**}$ . Let  $\omega$  be the first vertex of  $\bar{\gamma}^{**}$  which is not in  $B^*(0, T^*(\gamma_{0, u_1}))$ . By construction, there is no edge of  $\bar{\gamma}_{0, \omega}^{**}$  in  $E_+^{**}(T, T')$  or  $E_P^{**}(T, T')$  and thus:

$$t^{**}(0, z) \geq t^{**}(0, \omega) \geq t^*(0, \omega) > T^*(\gamma_{0, u_1}),$$

which is a contradiction. □

Now, to get item (ii) of Lemma 2.3.3, fix  $\eta = \min_{j \in \{1, \dots, d\}} \mathbb{P}(T \in \mathcal{A}_j^\Lambda) \tilde{p}^{|B_{3, s, N}|}$ , where

$$\tilde{p} = \min(F([t_{\min}, t_{\min} + \delta']), F([\nu(N), t_{\max}])) > 0.$$

Thus,  $\eta$  only depends on  $F$ , the pattern and  $N$  and we have that

$$\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \tilde{p}^{|B_{3, s, N}|} \geq \eta$$

and

$$\mathbb{P}(T'' \in \mathcal{B}^{**}(T, T') | T, T') \geq \min_{j \in \{1, \dots, d\}} \mathbb{P}(T \in \mathcal{A}_j^\Lambda) \tilde{p}^{|B_{3, s, N}|} = \eta.$$

We denote by  $\gamma^\pi$  the path composed by the first geodesic in the lexicographical order from 0 to  $u_2$  in the environment  $T^*$ , then  $\pi_{u_2, v_2}$  and then the first geodesic in the lexicographical order from  $v_2$  to  $x$  in the environment  $T^*$ . We end this section with two lemmas, one giving a

lower bound on the time saved by the geodesics from 0 to  $x$  thanks to the second modification and the other on the distance between  $u_1$  and any vertex of  $\pi_{v_3, v_2}$ .

**Lemma 2.3.14.** *For all  $N \in \mathbb{N}^*$ , we have*

$$T^*(\gamma) - T^{**}(\gamma^\pi) \geq \frac{N\nabla}{2C_\mu}(\delta - \delta') > T^\Lambda.$$

*Proof.* This result is an easy consequence of Lemma 2.3.8. Since  $\nabla \geq C_\mu$  (by (2.3.2)) and all edges of  $\gamma_{u_1, v_1}$  are contained in  $B_{2, s, N}$ , which implies that there is no edge of  $\gamma_{u_1, v_1}$  whose time has been reduced between the environment  $T$  and  $T^*$ ,

$$T^*(\gamma) \geq T^*(\gamma_{0, u_1}) + \|u_1 - v_1\|_1(t_{\min} + \delta) + T^*(\gamma_{v_1, x}),$$

where we used (i) of the definition of a typical box. Further, recall the definitions of  $u_2$  and  $v_2$  given at (2.3.22) and (2.3.23). We have

$$T^{**}(\gamma^\pi) \leq T^{**}(\gamma_{0, u_2}^\pi) + \|u_2 - v_2\|_1(t_{\min} + \delta') + T^{**}(\gamma_{v_2, x}^\pi) + 2t_{\max} + \tau^\Lambda,$$

where the term  $2t_{\max}$  is an upper bound for the time for both, the edge connecting  $u_2$  to  $E_+^{**}(T, T')$  and the one connecting  $v_2$  to  $E_+^{**}(T, T')$ , and the term  $\tau^\Lambda$  is an upper bound for the time collected by  $\gamma^\pi$  in  $E_P^{**}(T, T')$ . Since we have  $T^{**}(\gamma_{0, u_2}^\pi) = T^*(\gamma_{0, u_2}^\pi) \leq T^*(\gamma_{0, u_1})$ ,  $T^{**}(\gamma_{v_2, x}^\pi) = T^*(\gamma_{v_2, x}^\pi) \leq T^*(\gamma_{v_1, x})$ ,  $\|u_2 - v_2\|_1 \leq \|u_1 - v_1\|_1$  since  $\pi$  is an oriented path, and  $\nabla \geq 4C_\mu(2t_{\max} + \tau^\Lambda)$  (by (2.3.2)), and using Lemma 2.3.8, we obtain for all  $N \in \mathbb{N}^*$ ,

$$T^*(\gamma) - T^{**}(\gamma^\pi) \geq \|u_1 - v_1\|_1(\delta - \delta') - \tau^\Lambda - 2t_{\max} \geq \frac{N\nabla}{2C_\mu}(\delta - \delta').$$

Finally, since by (2.3.2),  $\nabla > \frac{2C_\mu T^\Lambda}{\delta - \delta'}$  we have the result.  $\square$

**Lemma 2.3.15.** *For all  $N \in \mathbb{N}^*$ , for all  $w$  vertex of  $\pi_{v_3, v_2}$ , we have*

$$\|u_1 - w\|_1 \geq \frac{\|u_1 - v_1\|_1}{3}.$$

*Proof.* Let us note that since  $\pi$  is an oriented path, for all  $w$  vertex of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$  after  $E_P^{**}(T, T')$ , we have  $\|u_1 - w\|_1 \geq \|u_1 - v_3\|_1$ . So

$$\|u_1 - w\|_1 \geq \|u_1 - c_1\|_1 - \|c_1 - v_3\|_1 \geq \frac{1}{2}\|u_1 - v_1\|_1 - L_2.$$

Then, since by (2.3.2),  $\nabla \geq 6L_2C_\mu$ , we have  $\frac{N\nabla}{6C_\mu} \geq L_2$  and using Lemma 2.3.8 leads to the result.  $\square$

### 2.3.4.5 Proof of the second and third items of Lemma 2.3.4

Recall that we assume that

$$\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{T'' \in \mathcal{B}^{**}(T, T')\} \text{ occurs.}$$

The aim of this section is to prove the following properties (which are the second and third items of Lemma 2.3.4):

- (ii) for all geodesic  $\bar{\gamma}^{**}$  from 0 to  $x$  in the environment  $T^{**}$ , there exists a geodesic  $\bar{\gamma}$  from 0 to  $x$  in the environment  $T$  such that  $\bar{\gamma}$  and  $\bar{\gamma}^{**}$  are associated in  $B_{4, S_\ell^1(T), N}$ ,
- (iii) there exists a geodesic  $\gamma^{**}$  in the environment  $T^{**}$  from 0 to  $x$  such that  $\gamma^{**}$  and the selected geodesic  $\gamma$  in the environment  $T$  are associated in  $B_{4, S_\ell^1(T), N}$ .

To prove this, we use the following sequence of lemmas.

**Lemma 2.3.16.** *Every geodesic from 0 to  $x$  in the environment  $T^{**}$  takes at least one edge of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ .*

*Proof.* Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . Since  $\gamma$  is a geodesic in the environment  $T^*$ , we have the following inequalities

$$T^{**}(\gamma^{**}) \leq T^{**}(\gamma^\pi) < T^*(\gamma) \leq T^*(\gamma^{**}).$$

So, using Lemma 2.3.14,

$$T^*(\gamma^{**}) - T^{**}(\gamma^{**}) > T^\Lambda.$$

It means that  $\gamma^{**}$  has to take at least one edge whose time has been reduced during the second modification which is not in  $E_P^{**}(T, T')$ . Hence, since the only edges which are not in  $E_P^{**}(T, T')$  whose time has been reduced are edges of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ , the result follows.  $\square$

**Lemma 2.3.17.** *Every geodesic from 0 to  $x$  in the environment  $T^{**}$  takes at least one edge of  $\gamma$  whose time has been reduced before taking its first edge of  $B_{2, s, N}$ , and takes at least one edge of  $\gamma$  whose time has been reduced after taking its last edge of  $B_{2, s, N}$ .*

*Proof.* The idea is to use that the time saved by the geodesics from 0 to  $x$  after the first modification is much greater than the geodesic time between any two vertices of  $B_{2, s, N}$  in any environment. Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . Let  $u^{**}$  be the first vertex in  $B_{2, s, N}$  that  $\gamma^{**}$  visits. Its existence is guaranteed by Lemma 2.3.16. The aim of the proof is to show

$$T^{**}(\gamma_{0, u^{**}}^{**}) < T(\gamma_{0, u^{**}}^{**}).$$

Indeed, the definition of  $u^{**}$  and the fact that the only edges whose time has been reduced which are in  $B_{3, s, N}$  but not in  $B_{2, s, N}$  are edges of  $\gamma$  gives us the result. Recall that  $u_0$  is the entry point of  $\gamma$  in  $B_{2, s, N}$ . First, since  $\gamma^{**}$  is a geodesic in the environment  $T^{**}$ ,

$$T^{**}(\gamma_{0, u^{**}}^{**}) \leq T^{**}(\gamma_{0, u_0}) + 2r_2 N t_{\max}.$$

Then, using the first item of Lemma 2.3.7, we obtain

$$T^{**}(\gamma_{0,u^{**}}^{**}) \leq T(\gamma_{0,u_0}) + 2r_2 N t_{\max} - \alpha(r_3 - r_2)N(\delta - \delta').$$

Finally, using the fact that  $\gamma$  is a geodesic in the environment  $T$  leads to

$$T^{**}(\gamma_{0,u^{**}}^{**}) \leq T(\gamma_{0,u^{**}}^{**}) + 4r_2 N t_{\max} - \alpha(r_3 - r_2)N(\delta - \delta').$$

To conclude, it is sufficient to observe that the condition  $r_3 > \frac{7r_2(4t_{\max} + \alpha\delta)}{\alpha\delta}$  implies that  $\alpha(r_3 - r_2)N(\delta - \delta') - 4r_2 N t_{\max} > 0$ , so we have the desired strict inequality. The same proof gives us the second part of the lemma.  $\square$

Recall that  $e_1, e_2, s_1$  and  $s_2$  are defined at the beginning of Section 2.3.4.2.

**Lemma 2.3.18.** *Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . Then the first edge of  $\gamma^{**}$  whose time has been reduced is  $e_1$  and the last is  $e_2$ . Moreover, the first vertex of  $e_1$  taking by  $\gamma^{**}$  is  $s_1$  and the last of  $e_2$  is  $s_2$ .*

*Proof.* Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . Let  $z^{**}$  denote the last vertex visited by  $\gamma^{**}$  before it takes for the first time an edge of  $\gamma$  whose time has been reduced. We know by the construction and by Lemma 2.3.17 that  $z^{**}$  is a vertex visited by  $\gamma_{u,u_0}$  or  $\gamma_{v_0,v}$ , and thus it is a vertex visited by  $\gamma_{u,u_1}$  or  $\gamma_{v_1,v}$ . Let us prove that it is a vertex visited by  $\gamma_{u,u_1}$ . Assume, aiming at a contradiction that  $z^{**}$  is a vertex visited by  $\gamma_{v_1,v}$ .

On the one hand, by Lemma 2.3.17,  $\gamma_{0,z^{**}}^{**}$  does not take edges in  $B_{2,s,N}$  and since by Lemma 2.3.12,  $\pi$  is contained in  $B_{2,s,N}$ ,  $\gamma_{0,z^{**}}^{**}$  does not take any edge of  $\pi_{u_2,u_3} \cup \pi_{v_3,v_2}$ .

On the other hand, all edges of  $\gamma_{v_1,x}$  are in  $B^*(x, T^*(\gamma_{v_1,x}))$ . So, if  $z^{**}$  is visited by  $\gamma_{v_1,x}$ , we have  $z^{**} \in B^*(x, T^*(\gamma_{v_1,x}))$ . But  $B^{**}(x, T^*(\gamma_{v_1,x})) = B^*(x, T^*(\gamma_{v_1,x}))$  by Lemma 2.3.13. Since  $\gamma^{**}$  is a geodesic in the environment  $T^{**}$ , it implies that  $\gamma_{z^{**},x}^{**}$  is entirely contained in  $B^*(x, T^*(\gamma_{v_1,x}))$ . Thus  $\gamma_{z^{**},x}^{**}$  does not take any edge of  $\pi_{u_2,u_3} \cup \pi_{v_3,v_2}$  since there is no edge of  $\pi_{u_2,u_3} \cup \pi_{v_3,v_2}$  in  $B^*(x, T^*(\gamma_{v_1,x}))$ .

Combining these two conclusions, we get that  $\gamma^{**}$  does not visit any edge of  $\pi_{u_2,u_3} \cup \pi_{v_3,v_2}$ , which contradicts Lemma 2.3.16. So  $z^{**}$  is a vertex visited by  $\gamma_{u,u_1}$ . Knowing this, we can complete the proof. By the definition of  $z^{**}$  and by Lemma 2.3.17, we have

$$T(\gamma_{0,z^{**}}^{**}) = T^{**}(\gamma_{0,z^{**}}^{**}).$$

Since  $\gamma^{**}$  is a geodesic,

$$T^{**}(\gamma_{0,z^{**}}^{**}) \leq T^{**}(\gamma_{0,z^{**}}^{**}).$$

Now, let us assume  $z^{**}$  is not  $s_1$ . Then, by the definition of  $s_1$ ,  $z^{**}$  is visited by  $\gamma_{u,u_1}$  after  $s_1$  and thus,

$$T^{**}(\gamma_{0,z^{**}}^{**}) < T(\gamma_{0,z^{**}}^{**}).$$

Combining these three inequalities yields

$$T(\gamma_{0,z^{**}}^{**}) < T(\gamma_{0,z^{**}}^{**}),$$

which is impossible because  $\gamma_{0,z^{**}}$  is a geodesic in the environment  $T$ . So, the result is proved and the same proof leads to the second part of this lemma.  $\square$

We can now prove the two properties stated at the beginning of this subsection. For item (ii), let  $\bar{\gamma}^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . By Lemma 2.3.17 and Lemma 2.3.18, like  $\gamma$ ,  $\bar{\gamma}^{**}$  visits  $s_1$  and before that  $\bar{\gamma}^{**}$  does not visit any edge whose time has been changed when replacing  $T$  by  $T^{**}$ . Hence,

$$T(\gamma_{0,s_1}) = T^{**}(\gamma_{0,s_1}) \geq T^{**}(\bar{\gamma}_{0,s_1}^{**}) = T(\bar{\gamma}_{0,s_1}^{**}) \geq T(\gamma_{0,s_1}). \quad (2.3.28)$$

This proves that  $T(\gamma_{0,s_1}) = T(\bar{\gamma}_{0,s_1}^{**})$  and  $\bar{\gamma}_{0,s_1}^{**}$  is a geodesic in the environment  $T$ . Similarly,  $\bar{\gamma}_{s_2,x}^{**}$  is a geodesic in the environment  $T$  and the path  $\bar{\gamma} = \bar{\gamma}_{0,s_1}^{**} \cup \gamma_{s_1,s_2} \cup \bar{\gamma}_{s_2,x}^{**}$  is a geodesic in the environment  $T$  that satisfies (ii) in Lemma 2.3.4 if we prove that it is a self-avoiding path and it is contained in  $B_{4,s,N}$ . Assume, aiming at a contradiction that  $\bar{\gamma}_{0,s_1}^{**}$  visits a vertex of  $\gamma_{s_1,s_2}$  which is not  $s_1$  and denote it by  $s_3$ . Then  $\bar{\gamma}_{0,s_1}^{**} \cup \gamma_{s_1,x}$  is an optimal path for the passage time in the environment  $T$  since  $\bar{\gamma}_{0,s_1}^{**}$  is a geodesic in this environment. It implies that  $\bar{\gamma}_{s_3,s_1}^{**} \cup \gamma_{s_1,s_3}$  has at least one edge which is  $e_1$  and that  $T(\bar{\gamma}_{s_3,s_1}^{**} \cup \gamma_{s_1,s_3}) = 0$ , which is impossible since  $T(e_1) > 0$ . The same proof gives that  $\bar{\gamma}_{s_2,x}^{**}$  does not visit a vertex of  $\gamma_{s_1,s_2}$  and thus  $\bar{\gamma}$  is self-avoiding. To get item (ii), it remains to prove that  $\bar{\gamma}_{s_1,s_2}^{**}$  is contained in  $B_{4,s,N}$ . This comes from the fact that, in any environment, the geodesic time between two vertices of  $B_{3,s,N}$  is bounded by  $2r_3t_{\max}$ , and, since the edges in  $B_{4,s,N} \setminus B_{3,s,N}$  have the same times in the environments  $T$  or  $T^{**}$ , by property (i) of a typical box, the geodesic time to reach a vertex outside  $B_{4,s,N}$  and to come back in  $B_{3,s,N}$  is bounded from below by  $2(r_4 - r_3)(t_{\min} + \delta)$ . The condition on  $r_4$  insures that  $(r_4 - r_3)(t_{\min} + \delta) > r_3t_{\max}$ .

For item (iii), observe that from (2.3.28), we also get

$$T^{**}(\bar{\gamma}_{0,s_1}^{**}) = T^{**}(\gamma_{0,s_1}) \text{ and } T^{**}(\bar{\gamma}_{s_2,x}^{**}) = T^{**}(\gamma_{s_2,x}).$$

Thus, the path  $\gamma_{0,s_1} \cup \bar{\gamma}_{s_1,s_2}^{**} \cup \gamma_{s_2,x}$  is an optimal path for the passage time in the environment  $T^{**}$ . If it is not self-avoiding, we get a geodesic that satisfies the requirement of (iii) in Lemma 2.3.4 by cutting its loops with the same process as in the proof of Lemma 2.2.4 in the unbounded case. If we denote by  $s'_1$  and  $s'_2$  the two vertices such that  $\gamma_{0,s'_1} \cup \bar{\gamma}_{s'_1,s'_2}^{**} \cup \gamma_{s'_2,x}$  is a geodesic in the environment  $T^{**}$  obtained by cutting the loops, we have to justify that  $\gamma_{s'_1,s'_2}$  is entirely contained in  $B_{4,s,N}$ . We know that  $\gamma_{s_1,s_2}$  is entirely contained in  $B_{4,s,N}$ . Let us show that  $\gamma_{s'_1,s_1}$  is also entirely contained in  $B_{4,s,N}$ , the proof for  $\gamma_{s_2,s'_2}$  is the same. Since  $\gamma_{0,s_1} \cup \bar{\gamma}_{s_1,s_2}^{**} \cup \gamma_{s_2,x}$  is an optimal path for the passage time in the environment  $T^{**}$ , we have  $T^{**}(\bar{\gamma}_{s_1,s'_1}^{**}) + T^{**}(\gamma_{s'_1,s_1}) = 0$  and thus in particular  $T^{**}(\gamma_{s'_1,s_1}) = 0$ . Since  $s_1$  belongs to  $B_{3,s,N}$ , if  $\gamma_{s'_1,s_1}$  visits a vertex outside  $B_{4,s,N}$ , there exist two vertices  $z$  and  $z'$  of  $\gamma_{s'_1,s_1}$  such that  $z \in \partial B_{3,s,N}$ ,  $z' \in \partial B_{4,s,N}$  and  $\gamma_{z,z'}$  except for  $z$  is contained in  $B_{4,s,N} \setminus B_{3,s,N}$ . So  $T^{**}(\gamma_{s'_1,s_1}) = 0$  implies that  $T^{**}(\gamma_{z,z'}) = 0$  but  $T(\gamma_{z,z'}) = T^{**}(\gamma_{z,z'}) = 0$  since the time of the edges of  $B_{4,s,N} \setminus B_{3,s,N}$  has not been changed. It makes a contradiction with (2.3.5) since  $r_4 - r_3 \geq 1$  by (2.3.4).

### 2.3.4.6 Every geodesic takes the pattern.

Recall that we assume that

$$\{T \in \mathcal{M}(\ell)\} \cap \{S_\ell^1(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \cap \{T'' \in \mathcal{B}^{**}(T, T')\} \text{ occurs.}$$

The aim of this last subsection is to show that every geodesic in the environment  $T^{**}$  takes the pattern in  $E_P^{**}(T, T')$ . The proof is decomposed in two steps. The first step is to show that every geodesic takes an edge of  $\pi_{u_2, u_3}$  and an edge of  $\pi_{v_3, v_2}$ . The second step is to show that every geodesic verifying this property takes the pattern in  $E_P^{**}(T, T')$ . We begin with a technical lemma.

**Lemma 2.3.19.** *For all  $w$  vertex of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ ,*

$$|\mu(u_1 - w) + \mu(v_1 - w) - \mu(u_1 - v_1)| \leq 2C_\mu L_1.$$

*Proof.* Let  $w$  be a vertex of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ . Then, by the construction of  $\pi$ , there exists a  $\bar{w} \in [u_1, v_1] \subset \mathbb{R}^d$  such that  $\|w - \bar{w}\|_1 \leq L_1$ . We have

$$\mu(u_1 - v_1) = \mu(u_1 - \bar{w}) + \mu(v_1 - \bar{w}).$$

Then

$$\begin{aligned} |\mu(u_1 - w) - \mu(u_1 - v_1) + \mu(v_1 - w)| &\leq |\mu(u_1 - w) - \mu(u_1 - \bar{w})| + |\mu(v_1 - w) - \mu(v_1 - \bar{w})| \\ &\leq 2C_\mu \|w - \bar{w}\|_1 \\ &\leq 2C_\mu L_1. \end{aligned}$$

□

**Lemma 2.3.20.** *Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . Then  $\gamma^{**}$  visits a vertex of  $\pi_{u_2, u_3}$  and one of  $\pi_{v_3, v_2}$ . More precisely, the first vertex of  $\gamma^{**}$  that belongs to  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$  belongs to  $\pi_{u_2, u_3}$  and the last belongs to  $\pi_{v_3, v_2}$ .*

*Proof.* Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . By Lemma 2.3.16, there exists at least one vertex of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$  visited by  $\gamma^{**}$ . Let  $w$  be the first vertex of  $\gamma^{**}$  that belongs to  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$  and assume that  $w$  belongs to  $\pi_{v_3, v_2}$ . Note that  $\gamma_{0, w}^{**}$  does not visit any other vertex of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ . The aim of the proof is to show that

$$T^{**}(\gamma_{0, w}^\pi) < T^{**}(\gamma_{0, w}^{**}),$$

which is impossible since  $\gamma^{**}$  is a geodesic in the environment  $T^{**}$ . We start with

$$\begin{aligned} T^*(\gamma_{0, w}^{**}) &\geq t^*(0, w) \\ &\geq t^*(0, v_1) - t^*(v_1, w) \\ &= t^*(0, u_1) + t^*(u_1, v_1) - t^*(v_1, w), \end{aligned}$$

since, by Lemma 2.3.7,  $\gamma$  is a geodesic in the environment  $T^*$ . By construction, there is no edge of  $\gamma_{u_1, v_1}$  whose time has been changed at the first modification, thus  $t^*(u_1, v_1) = t(u_1, v_1)$ . Furthermore, since there is no edge whose time has been increased at the first modification,  $t^*(v_1, w) \leq t(v_1, w)$ , so

$$T^*(\gamma_{0,w}^{**}) \geq t^*(0, u_1) + t(u_1, v_1) - t(v_1, w).$$

We now want to bound from below  $T^{**}(\gamma_{0,w}^{**})$ . The only edges whose time has been reduced at the second modification are among those of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$  and of  $E_P^{**}(T, T')$ . So, since  $\gamma_{0,w}^{**}$  does not take any edge of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ , it can only save time taking edges of  $E_P^{**}(T, T')$ . So,

$$T^{**}(\gamma_{0,w}^{**}) \geq t^*(0, u_1) + t(u_1, v_1) - t(v_1, w) - T^\Lambda.$$

Then, using the definition of a typical box, Lemma 2.3.19 and the inequality  $t^*(0, u_1) \geq t^*(0, u_2)$  (which comes from the definition of  $u_2$ ) leads to

$$T^{**}(\gamma_{0,w}^{**}) \geq t^*(0, u_2) + \mu(u_1 - w) - \varepsilon(\mu(u_1 - v_1) + \mu(v_1 - w)) - 2N - 2C_\mu L_1 - T^\Lambda.$$

On the other hand, note that,  $\pi$  being an oriented path,  $\|u_1 - w\|_1 \geq \|u_2 - w\|_1$ , so, using the knowledge of  $T^{**}$  on edges of  $\pi$ ,

$$\begin{aligned} T^{**}(\gamma_{0,w}^\pi) &= T^{**}(\gamma_{0,u_2}^\pi) + T^{**}(\gamma_{u_2,w}^\pi) \\ &\leq t^*(0, u_2) + 2t_{\max} + 2t_{\max}\ell^\Lambda + (t_{\min} + \delta')\|u_1 - w\|_1. \end{aligned}$$

To conclude, let us show that we have the inequality

$$\begin{aligned} &t^*(0, u_2) + 2t_{\max} + 2t_{\max}\ell^\Lambda + (t_{\min} + \delta')\|u_1 - w\|_1 \\ &< t^*(0, u_2) + \mu(u_1 - w) - \varepsilon(\mu(u_1 - v_1) + \mu(v_1 - w)) - 2N - 2C_\mu L_1 - T^\Lambda. \end{aligned} \quad (2.3.29)$$

First, combining Lemma 2.3.8 and Lemma 2.3.15 leads to

$$\|u_1 - w\|_1 \geq \frac{N\nabla}{3C_\mu}. \quad (2.3.30)$$

Then, by (2.1.21), we have

$$\mu(u_1 - w) \geq (t_{\min} + \delta)\|u_1 - w\|_1.$$

Recall that  $K' = T^\Lambda + 2(C_\mu L_1 + t_{\max} + t_{\max}\ell^\Lambda)$ , it is sufficient to have

$$\delta' < \delta - \varepsilon C_\mu \frac{\|u_1 - v_1\|_1 + \|v_1 - w\|_1}{\|u_1 - w\|_1} - \frac{2N}{\|u_1 - w\|_1} - \frac{K'}{\|u_1 - w\|_1}.$$

Then, by Lemma 2.3.15,  $\frac{\|u_1 - v_1\|_1 + \|v_1 - w\|_1}{\|u_1 - w\|_1} \leq 6$ . So, since  $\varepsilon < \frac{\delta}{24C_\mu}$ , we have

$$\varepsilon C_\mu \frac{\|u_1 - v_1\|_1 + \|v_1 - w\|_1}{\|u_1 - w\|_1} < \frac{\delta}{4}.$$

By (2.3.30) and since  $1 < \frac{\delta \nabla}{24C_\mu}$  (by (2.3.2)) and  $N > \frac{12C_\mu K'}{\delta \nabla}$  (by (2.3.6)), the condition  $\delta' \leq \frac{\delta}{4}$  gives us (2.3.29). Hence  $T^{**}(\gamma_{0,w}^\pi) < T^{**}(\gamma_{0,w}^{**})$ .  $\square$

Finally, let us prove the following lemma which completes the proof of Lemma 2.3.4.

**Lemma 2.3.21.** *Any geodesic from 0 to  $x$  takes the pattern at the pattern-location  $E_P^{**}(T, T')$ .*

*Proof.* Let  $\gamma^{**}$  be a geodesic from 0 to  $x$  in the environment  $T^{**}$ . By Lemma 2.3.20,  $\gamma^{**}$  visits a vertex of  $\pi_{u_2, u_3}$  and one of  $\pi_{v_3, v_2}$ . As a consequence, there exist a vertex  $u_4$  of  $\pi_{u_2, u_3}$  and a vertex  $v_4$  of  $\pi_{v_3, v_2}$  such that  $\gamma^{**}$  goes from  $u_4$  to  $v_4$  without taking edges of  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ . Let us remember that for all edge  $e$  of  $E_P^{**}(T, T')$ , we have  $T^{**}(e) \leq \nu$ . We prove successive properties.

- *The edges of  $\gamma_{u_4, v_4}^{**}$  which are not in  $E_P^{**}(T, T')$  have a passage time greater than or equal to  $\nu$ .*

Since there is no edge of  $\gamma_{u_4, v_4}^{**}$  in  $B^{**}(0, T^*(\gamma_{0, u_1}))$ ,  $B^{**}(x, T^*(\gamma_{v_1, x}))$  or  $\pi_{u_2, u_3} \cup \pi_{v_3, v_2}$ , it is sufficient to prove that  $\gamma_{u_4, v_4}^{**}$  is entirely contained in  $B_{2, s, N}$ . By convexity, we have that all points of  $[u_1, v_1]$  are contained in  $B_\mu(c_0, N\nabla)$ , so by Lemma 2.3.11,  $\|u_4 - c_0\|_1 \leq \frac{N\nabla}{c_\mu} + L_1$ , and thus,

$$\|u_4 - sN\|_1 \leq N \left( \frac{\nabla}{c_\mu} + r_1 \right) + L_1.$$

So, if  $\gamma_{u_4, v_4}^{**}$  is not entirely contained in  $B_{2, s, N}$ , the number of edges whose time is greater than or equal to  $\nu$  that  $\gamma_{u_4, v_4}^{**}$  has to travel to leave  $B_{2, s, N}$  is bounded from below by  $(r_2 - \frac{\nabla}{c_\mu} - r_1)N - 2d\ell^\Lambda - L_1$ , and we get

$$T^{**}(\gamma_{u_4, v_4}^{**}) \geq \left( \left( r_2 - \frac{\nabla}{c_\mu} - r_1 \right) N - 2d\ell^\Lambda - L_1 \right) \nu.$$

But

$$T^{**}(\gamma_{u_4, v_4}^\pi) \leq \frac{2N\nabla}{c_\mu} (t_{\min} + \delta') + 2t_{\max}(1 + \ell^\Lambda),$$

and since  $N \geq 1$ ,  $\frac{t_{\max}}{\nu} \geq 1$ ,  $\frac{t_{\min} + \delta}{\nu} \leq 1$  and by (2.3.3),

$$r_2 > r_1 + L_1 + \frac{3\nabla}{c_\mu} + \frac{2t_{\max}}{\nu} (1 + (1 + d)\ell^\Lambda),$$



we have  $T^{**}(\gamma_{u_4, v_4}^\pi) < T^{**}(\gamma_{u_4, v_4}^{**})$ , which is impossible since  $\gamma_{u_4, v_4}^{**}$  is a geodesic in the environment  $T^{**}$ .

- We have that  $\|u_4 - u_3\|_1 \leq 4\ell^\Lambda$  and  $\|v_4 - v_3\|_1 \leq 4\ell^\Lambda$ .

Assume that it is not the case. Let us show that  $T^{**}(\gamma_{u_4, v_4}^{**}) - T^{**}(\pi_{u_4, v_4}) > 0$ , which is a contradiction since  $\gamma_{u_4, v_4}^{**}$  is a geodesic in the environment  $T^{**}$ . To this aim, let us compare the time that can save each path compared with the other in any direction. Recall the notation introduced in the proof of Lemma 2.4.1: for  $i \in \{1, \dots, d\}$  and a path  $\tilde{\pi}$ ,  $T_i^{**}(\tilde{\pi})$  denotes the sum of the passage times of the edges of  $\tilde{\pi}$  which are in the direction  $\varepsilon_i$ . In the direction  $\mathcal{O}(T, T')$ , since  $\|u_4 - u_3\|_1 > 4\ell^\Lambda$  or  $\|v_4 - v_3\|_1 > 4\ell^\Lambda$ , and by the construction, we have that

$$T_{\mathcal{O}(T, T')}^{**}(\gamma_{u_4, v_4}^{**}) - T_{\mathcal{O}(T, T')}^{**}(\pi_{u_4, v_4}) \geq 4\ell^\Lambda \nu - 4\ell^\Lambda(t_{\min} + \delta') + 2\ell^\Lambda t_{\min} - 2\ell^\Lambda \nu,$$

where the term  $2\ell^\Lambda t_{\min} - 2\ell^\Lambda \nu$  comes from the time potentially saved by  $\gamma_{u_4, v_4}^{**}$  by taking edges of  $E_P^{**}(T, T')$ . Then, in any other direction,  $\gamma_{u_4, v_4}^{**}$  can save a time lower than or equal to  $2\ell^\Lambda \delta'$  compared with  $\pi_{u_4, v_4}$  thanks to the edges in  $E_P^{**}(T, T')$ . Hence,

$$\begin{aligned} T^{**}(\gamma_{u_4, v_4}^{**}) - T^{**}(\pi_{u_4, v_4}) &\geq 2\ell^\Lambda \nu + 2\ell^\Lambda t_{\min} - 4\ell^\Lambda(t_{\min} + \delta') - 2(d-1)\ell^\Lambda \delta' \\ &\geq 2\ell^\Lambda \nu - 2\ell^\Lambda t_{\min} - (4\ell^\Lambda + 2(d-1)\ell^\Lambda)\delta' > 0 \end{aligned}$$

since  $t_{\min} + (1+d)\delta' < t_{\min} + \delta \leq \nu$ .

- We have that  $u_4 = u_3$  and  $v_4 = v_3$ .

Assume that it is not the case. First, assume that  $\gamma_{u_4, v_4}^{**}$  does not take any edge of  $E_P^{**}(T, T')$ . According to the first property,  $T^{**}(\gamma_{u_4, v_4}^{**}) \geq \|u_4 - v_4\|_1 \nu$ . Then, since  $\pi$  is an oriented path,

$$T^{**}(\gamma_{u_4, v_4}^\pi) \leq (\|u_4 - v_4\|_1 - 2\ell^\Lambda)(t_{\min} + \delta') + 2\ell^\Lambda \nu.$$

Since  $u_4 \neq u_3$  or  $v_4 \neq v_3$ , we have that  $\|u_4 - v_4\|_1 > 2\ell^\Lambda$  and we obtain

$$T^{**}(\gamma_{u_4, v_4}^\pi) < T^{**}(\gamma_{u_4, v_4}^{**}),$$

which is a contradiction since  $\gamma_{u_4, v_4}^{**}$  is a geodesic in the environment  $T^{**}$ . So  $\gamma_{u_4, v_4}^{**}$  takes an edge of  $E_P^{**}(T, T')$ . Let  $u_0^{**}$  be the first entry point of  $\gamma_{u_4, v_4}^{**}$  in  $E_P^{**}(T, T')$  and consider the path  $\pi^{**}$  following  $\pi_{u_4, u_3}$ , then going from  $u_3$  to  $u_0^{**}$  in one of the shortest way for the norm  $\|\cdot\|_1$  and then following  $\gamma_{u_0^{**}, v_4}^{**}$ . Then the number of edges of  $\pi_{u_4, u_0^{**}}^{**}$  is lower than or equal to the number of edges of  $\gamma_{u_4, u_0^{**}}^{**}$ , for all  $e \in \pi_{u_4, u_0^{**}}^{**}$ ,  $T^{**}(e) \leq \nu$ , there exists  $e' \in \pi_{u_4, u_0^{**}}^{**}$  such that  $T^{**}(e') \leq t_{\min} + \delta'$  and for all  $e \in \gamma_{u_4, u_0^{**}}^{**}$ ,  $T^{**}(e) \geq \nu$ . So we have  $T^{**}(\pi^{**}) < T^{**}(\gamma_{u_4, v_4}^{**})$ , which is impossible since  $\gamma^{**}$  is a geodesic in the environment  $T^{**}$ . The same proof gives  $v_4 = v_3$ .

- $\gamma^{**}$  takes the pattern at the pattern-location  $E_P^{**}(T, T')$ .

Assume that  $\gamma_{u_4, v_4}^{**}$  is not entirely contained in  $E_P^{**}(T, T')$ . Let  $v_0^{**}$  be the first exit point from  $E_P^{**}(T, T')$  of  $\gamma_{u_4, v_4}^{**}$  and  $u_1^{**}$  the first entry point after  $v_0^{**}$ . Let us consider the shortcut  $\pi^{**}$  going from  $v_0^{**}$  to  $u_1^{**}$  in one of the shortest way for the norm  $\|\cdot\|_1$ . Then let  $v_{0,+}^{**}$  denote the first vertex visited by  $\gamma^{**}$  after  $v_0^{**}$ , then

$$\|u_1^{**} - v_{0,+}^{**}\|_1 = \|u_1^{**} - v_0^{**}\|_1 + 1.$$

Indeed, we have that  $\|u_1^{**} - v_{0,+}^{**}\|_1 - \|u_1^{**} - v_0^{**}\|_1$  is equal to 1 or  $-1$ , and if it is equal to  $-1$ , it implies that  $v_{0,+}^{**}$  is in  $E_P^{**}(T, T')$ , which is impossible. So,  $\gamma_{v_0^{**}, u_1^{**}}^{**}$  has strictly more edges than  $\pi^{**}$ . Furthermore, all edges of  $\gamma_{v_0^{**}, u_1^{**}}^{**}$  have a time greater than or equal to  $\nu$  although all edges of  $\pi^{**}$  have a time lower than or equal to  $\nu$ . So,  $T^{**}(\pi^{**}) < T^{**}(\gamma_{v_0^{**}, u_1^{**}}^{**})$ , which is a contradiction since  $\gamma_{v_0^{**}, u_1^{**}}^{**}$  is a geodesic in the environment  $T^{**}$ .

Thus  $\gamma_{u_4, v_4}^{**}$  is a path entirely contained in  $E_P^{**}(T, T')$ , going from  $u_3$  to  $v_3$  and with an optimal time. So, we have the result. □

## 2.4 Proofs of generalizations of modification arguments in [15]

### 2.4.1 Modification proof for the Euclidean length of geodesics

To prove Theorem 2.1.8, we begin by defining the valid pattern in three different cases. Recall that  $k$  and  $\ell$  are given by the assumptions of this theorem and that  $(\varepsilon_i)_{i \in \{1, \dots, d\}}$  are the vectors of the canonical basis.

**Case where zero is an atom.** In addition to the assumptions of Theorem 2.1.8, we assume that zero is an atom for  $F$ . We set  $L_1 = k + 2$ ,  $L_2 = \ell + 2$  and if  $d \geq 3$ , for all  $i \in \{3, \dots, d\}$ ,  $L_i = 2$ . We define a pattern in  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ . We take the endpoints  $u = \sum_{i=2}^d \varepsilon_i$  and  $v = u + (k + 2)\varepsilon_1$ . We denote by  $\pi^+$  the path going from  $u$  to  $v$  by  $k + 2$  steps in the direction  $\varepsilon_1$  and by  $\pi^{++}$  the path going from  $u$  to  $u + \varepsilon_1$  by one step in the direction  $\varepsilon_1$ , then to  $u + \varepsilon_1 + \ell\varepsilon_2$  by  $\ell$  steps in the direction  $\varepsilon_2$ , then to  $u + (k + 1)\varepsilon_1 + \ell\varepsilon_2$  by  $k$  steps in the direction  $\varepsilon_1$ , then to  $u + (k + 1)\varepsilon_1$  by  $\ell$  steps in the direction  $-\varepsilon_2$  and then to  $v$  by one step in the direction  $\varepsilon_1$ . We define  $\mathcal{A}^\Lambda$  as follows:

- for all  $e \in \pi^+ \cup \pi^{++}$ ,  $T(e) = 0$ ,
- for all  $e \in \Lambda$  which is not in  $\pi^+ \cup \pi^{++}$ ,  $T(e) > 0$ .

Note that  $\mathcal{A}^\Lambda$  has a positive probability. Then,  $\pi^+$  and  $\pi^{++}$  are the only two optimal self-avoiding paths from  $u$  to  $v$  entirely contained in the pattern. Furthermore, for every vertex  $z \in \pi^+ \cup \pi^{++}$  different from  $u$  and  $v$ , there exists no path from  $\partial\Lambda \setminus \{u, v\}$  to  $z$  whose passage time is equal to 0.

**Unbounded case.** Here, in addition to the assumptions of Theorem 2.1.8, we assume that zero is not an atom and that the support of  $F$  is unbounded. We set  $L_1 = k$ ,  $L_2 = \ell$  and if  $d \geq 3$ , for all  $i \in \{3, \dots, d\}$ ,  $L_i = 0$ . We define a pattern in  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ . We take the endpoints  $u = 0$  and  $v = k\varepsilon_1$ . We denote by  $\pi^+$  the path going from 0 to  $k\varepsilon_1$  by  $k$  steps in the direction  $\varepsilon_1$  and by  $\pi^{++}$  the path going from 0 to  $\ell\varepsilon_2$  by  $\ell$  steps in the direction  $\varepsilon_2$ , then to  $k\varepsilon_1 + \ell\varepsilon_2$  by  $k$  steps in the direction  $\varepsilon_1$  and then to  $v$  by  $\ell$  steps in the direction  $-\varepsilon_2$ . Then, we index the edges of  $\pi^+$  and the ones of  $\pi^{++}$  in the order in which they are taken by these paths. We respectively denote them by  $(e_i^1)_{i \in \{1, \dots, k\}}$  and  $(e_i^2)_{i \in \{1, \dots, k+2\ell\}}$ . We fix  $M > \sum_{j=1}^k s'_j$  and we define  $\mathcal{A}^\Lambda$  as follows:

- for all  $i \in \{1, \dots, k\}$ ,  $T(e_i^1) = s'_i$  and for all  $i \in \{1, \dots, k+2\ell\}$ ,  $T(e_i^2) = r'_i$ ,
- for all  $e \in \Lambda$  which are not in  $\pi^+$  or  $\pi^{++}$ ,  $T(e) > M$ .

Since the support of  $F$  is unbounded,  $\mathbb{P}(\mathcal{A}^\Lambda)$  is positive. Furthermore, we have

$$T(\pi^+) = T(\pi^{++}).$$

Since  $M > T(\pi^+)$ , the optimal paths from  $u$  to  $v$  entirely contained in the pattern cannot take other edges than those in  $\pi^+ \cup \pi^{++}$ . Hence  $\pi^+$  and  $\pi^{++}$  are the only two optimal paths from  $u$  to  $v$  entirely contained in the pattern.

**Bounded case.** In addition to the assumptions of Theorem 2.1.8, we assume that zero is not an atom and that the support of  $F$  is bounded. We set  $t_{\max} = \sup(\text{support}(F))$ . We denote  $a_{\max} = \max(r'_1, \dots, r'_{k+2\ell}, s'_1, \dots, s'_k)$ . Then, there are at least  $2\ell$  integers  $j \in \{1, \dots, k+2\ell\}$  such that  $r'_j < a_{\max}$ . Indeed, assume that this is not the case. Then, we have

$$\sum_{i=1}^{k+2\ell} r'_i \geq (k+1)a_{\max} > ka_{\max} \geq \sum_{j=1}^k s'_j,$$

and this contradicts (2.1.6). Thus, although we may need to change the indexes, we can assume that for all  $i \in \{1, \dots, 2\ell\}$ ,  $r'_i < a_{\max}$  and we denote  $t_w = a_{\max} - \max(r'_1, \dots, r'_{2\ell}) > 0$ . We fix  $\alpha > 0$  an integer such that:

$$\alpha > \max\left(\frac{k}{2\ell}, \frac{ka_{\max}}{\ell t_w}\right). \quad (2.4.1)$$

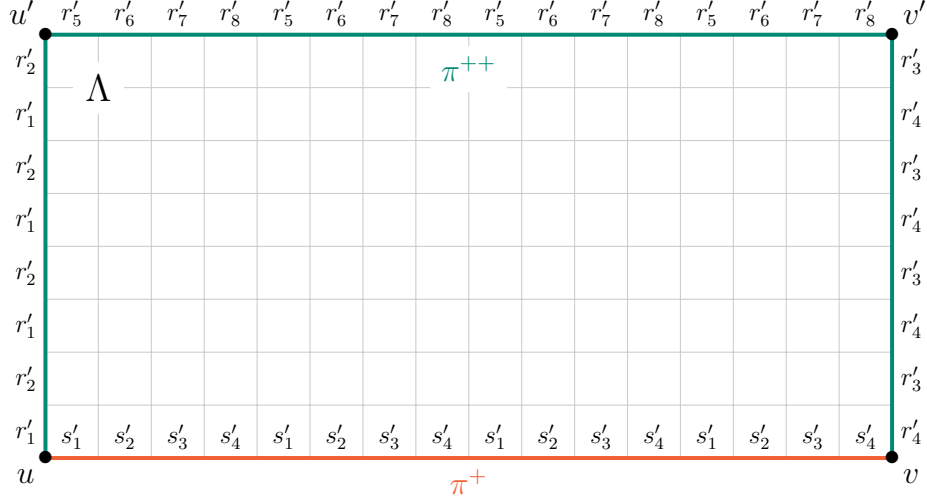


Figure 2.5: Paths  $\pi^+$  and  $\pi^{++}$  with their passage times in the pattern for the proof of Theorem 2.1.8 in the bounded case with  $d = 2$ ,  $k = 4$ ,  $\ell = 2$  and  $\alpha = 4$ .

We set  $k' = \alpha k$ ,  $\ell' = \alpha \ell$ ,  $L_1 = k'$ ,  $L_2 = \ell'$  and if  $d \geq 3$ , for all  $i \in \{3, \dots, d\}$ ,  $L_i = 0$ . We define a pattern in  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ . We take the endpoints  $u = 0$  and  $v = k'\varepsilon_1$ . Let  $\pi^+$  be the path going from  $u$  to  $v$  by  $k'$  steps in the direction  $\varepsilon_1$  and  $\pi^{++}$  be the path going from  $u$  to  $u' = \ell'\varepsilon_2$  by  $\ell'$  steps in the direction  $\varepsilon_2$ , then to  $v' = k'\varepsilon_1 + \ell'\varepsilon_2$  by  $k'$  steps in the direction  $\varepsilon_1$  and then to  $v$  by  $\ell'$  steps in the direction  $-\varepsilon_2$ . Then we index the edges of  $\pi^+$ ,  $\pi_{u,u'}^{++}$ ,  $\pi_{u',v'}^{++}$  and  $\pi_{v',v}^{++}$  in the order in which they are taken by these paths. We respectively denote them by  $(e_i^1)_{i \in \{1, \dots, k'\}}$ ,  $(e_i^2)_{i \in \{1, \dots, \ell'\}}$ ,  $(e_i^3)_{i \in \{1, \dots, k'\}}$  and  $(e_i^4)_{i \in \{1, \dots, \ell'\}}$ . The idea for the event  $\mathcal{A}^\Lambda$  is just to alternate the atoms on every boundary of the rectangle whose vertices are  $u$ ,  $u'$ ,  $v'$  and  $v$ . It allows us a better control of the time of a path taking both vertices of  $\pi^+$  and vertices of  $\pi_{u',v'}^{++}$  (see Figure 2.5).

So, we define  $\mathcal{A}^\Lambda$  as follows:

- for all  $i \in \{1, \dots, k'\}$ ,  $T(e_i^1) = s'_{i[k]}$  and  $T(e_i^3) = r'_{2\ell+i[k]}$  where  $i[k]$  is the integer in  $\{1, \dots, k\}$  such that  $i - i[k]$  is divisible by  $k$ ,
- for all  $i \in \{1, \dots, \ell'\}$ ,  $T(e_i^2) = r'_{i[\ell]}$  and  $T(e_i^4) = r'_{\ell+i[\ell]}$ ,
- for all other edges  $e \in \Lambda$ ,  $T(e) = a_{\max}$ .

Note that  $\mathcal{A}^\Lambda$  has a positive probability and that on this event,  $T(\pi^+) = T(\pi^{++})$ .

**Lemma 2.4.1.** *On the event  $\mathcal{A}^\Lambda$ , the paths  $\pi^+$  and  $\pi^{++}$  belong to the family of optimal paths from  $u$  to  $v$  which are entirely contained in the pattern.*

*Proof of Lemma 2.4.1.* Let us begin by introducing some notations. For  $i \in \{1, \dots, d\}$  and a path  $\pi$ ,  $T_i(\pi)$  denotes the sum of the passage times of the edges of  $\pi$  which are in the

direction  $\varepsilon_i$ . So we have  $T(\pi) = T_1(\pi) + \dots + T_d(\pi)$ . Furthermore, for  $j \in \{0, \dots, \ell' - 1\}$ , we denote by  $S_2^j$  the set of edges of  $\Lambda$  which can be written  $\{x, x + \varepsilon_2\}$  where the second coordinate of  $x$  is equal to  $j$ .

Now, to prove the lemma, assume for a contradiction that there exists an optimal self-avoiding path  $\pi$  entirely contained in  $\Lambda$  such that

$$T(\pi) < T(\pi^+) = T(\pi^{++}). \quad (2.4.2)$$

**First step:**  $\pi$  takes at least one edge in  $\pi_{u',v'}^{++}$ . Indeed, assume that it is not the case. Then

$$T(\pi) \geq T_1(\pi) \geq T_1(\pi^+) = T(\pi^+),$$

which contradicts (2.4.2). Note that the second inequality comes from the fact that edges in the direction  $\varepsilon_1$  whose passage time is strictly smaller than those in  $\pi^+$  must belong to  $\pi_{u',v'}^{++}$ .

**Second step:** We denote by  $x$  (resp.  $y$ ) the first (resp. last) vertex of  $\pi_{u',v'}^{++}$  visited by  $\pi$ . Let us prove that  $\pi_{x,y} = \pi_{x,y}^{++}$ . Note that, since  $\pi$  is a self-avoiding path from  $u$  to  $v$  and since every path entirely contained in the pattern can only take edges of directions  $\varepsilon_1$  and  $\varepsilon_2$ ,

- $x$  (resp.  $y$ ) is also the first (resp. last) vertex of  $\pi$  visited by  $\pi_{u',v'}^{++}$ ,
- $\pi_{x,y}$  does not take any edge of  $\pi_{u,x}^{++}$  and  $\pi_{y,v}^{++}$ .

Assume for a contradiction that  $\pi_{x,y} \neq \pi_{x,y}^{++}$ . Denote by  $x'$  and  $y'$  two distinct vertices of  $\pi_{x,y}^{++} \cap \pi_{x,y}$  such that  $\pi_{x',y'}$  does not take any edge of  $\pi^{++}$  and such that  $x'$  is visited by  $\pi_{u',v'}^{++}$  before  $y'$ . We have  $|\pi_{x',y'}| > |\pi_{x',y'}^{++}|$  and thus  $\pi_{x',y'}$  has to take at least one edge in  $\pi^+$ , else

$$T(\pi_{x',y'}) = |\pi_{x',y'}|a_{\max} > |\pi_{x',y'}^{++}|a_{\max} \geq T(\pi_{x',y'}^{++}),$$

which contradicts the fact that  $\pi$  is an optimal path. Since  $\pi_{x',y'}$  has to take edges in  $\pi^+$ , and since it cannot take edges of  $\pi^{++}$ , we get

$$T_2(\pi_{x',y'}) \geq 2\ell' a_{\max}.$$

Furthermore, for each edge  $e$  in  $\pi_{x',y'}^{++}$ ,  $\pi_{x',y'}$  has to take an edge in the direction  $\varepsilon_1$  such that this edge is the edge  $e - \ell'\varepsilon_2 \in \pi^+$  or such that its passage time is equal to  $a_{\max}$ . Hence

$$T_1(\pi_{x',y'}) \geq \left\lfloor \frac{\|x' - y'\|_1}{k} \right\rfloor \sum_{i=1}^k s'_i.$$

But,

$$T_1(\pi_{x',y'}^{++}) \leq \left\lceil \frac{\|x' - y'\|_1}{k} \right\rceil \sum_{i=1+2\ell}^{k+2\ell} r'_i.$$

Thus, since  $\sum_{i=1+2\ell}^{k+2\ell} r'_i \leq \sum_{i=1}^k s'_i \leq ka_{\max}$ , we have

$$T(\pi_{x',y'}^{++}) - T(\pi_{x',y'}) \leq \sum_{i=1}^k s'_i - 2\alpha\ell a_{\max} \leq (k - 2\alpha\ell)a_{\max} < 0 \text{ by (2.4.1),}$$

which contradicts the fact that  $\pi$  is an optimal path.

**Third step:** One of the keys of this step is planarity. We have just proven that  $\pi_{x,y} = \pi_{x,y}^{++}$ . Hence, (2.4.2) implies that  $T(\pi_{u,x}) < T(\pi_{u,x}^{++})$  or that  $T(\pi_{y,v}) < T(\pi_{y,v}^{++})$ . Assume for a contradiction that

$$T(\pi_{u,x}) < T(\pi_{u,x}^{++}), \quad (2.4.3)$$

the other case being the same. First, we have  $x \neq u'$  else, using again that  $\pi$  is a self avoiding path from  $u$  to  $v$  and the fact that every path entirely contained in the pattern can only take edges of directions  $\varepsilon_1$  and  $\varepsilon_2$ ,  $\pi_{u,x}$  cannot take any edge of  $\pi_{x,v}^{++}$ , we have  $T_2(\pi_{u,u'}) \geq T_2(\pi_{u,u'}^{++})$  and thus

$$T(\pi_{u,x}) = T(\pi_{u,u'}) \geq T_2(\pi_{u,u'}) \geq T_2(\pi_{u,u'}^{++}) = T(\pi_{u,u'}^{++}) = T(\pi_{u,x}^{++}),$$

which contradicts (2.4.3).

Denote by  $x''$  the last vertex of  $\pi_{u,u'}^{++}$  visited by  $\pi_{u,x}$  (note that we can have  $x'' = u$ ). We have that  $x'' \neq u'$ , else  $x'' = x = u'$ . Now, since  $\pi$  is a self-avoiding path and using the definitions of  $x''$  and  $x$ , we get that  $\pi_{x'',x}$  cannot take any edge of  $\pi_{u,x}^{++}$ . Hence,  $\pi_{x'',x}$  takes at least one edge of  $\pi^+$ . Indeed, if it is not the case, we have  $T_1(\pi_{x'',x}) \geq T_1(\pi_{x'',x}^{++})$  and we get (using that  $\|x'' - u'\|_1 > 0$ ):

$$\begin{aligned} T(\pi_{x'',x}) &= T_1(\pi_{x'',x}) + T_2(\pi_{x'',x}) \geq T_1(\pi_{x'',x}^{++}) + \|x'' - u'\|_1 a_{\max} \\ &> T_1(\pi_{x'',x}^{++}) + \|x'' - u'\|_1 \max(r'_1, \dots, r'_{2\ell}) \geq T_1(\pi_{x'',x}^{++}) + T_2(\pi_{x'',x}^{++}) = T(\pi_{x'',x}^{++}), \end{aligned}$$

which contradicts the fact that  $\pi$  is an optimal path.

Now, since  $\pi_{x'',x}$  takes edges of  $\pi^+$ ,  $\pi_{x'',x}$  has to take at least  $\ell'$  edges in the direction  $\varepsilon_2$  of passage time equal to  $a_{\max}$ . Thus,

$$T_2(\pi_{x'',x}) \geq \ell' a_{\max}.$$

Then, for each edge  $e$  in  $\pi_{u',x}^{++}$ ,  $\pi_{x'',x}$  has to take an edge in the direction  $\varepsilon_1$  such that this edge is the edge  $e - \ell'\varepsilon_2 \in \pi^+$  or such that its passage time is equal to  $a_{\max}$ . We get

$$T_1(\pi_{x',y'}) \geq \left\lfloor \frac{\|u' - x\|_1}{k} \right\rfloor \sum_{i=1}^k s'_i.$$

On the other hand, we have

$$T_1(\pi_{x'',x}^{++}) \leq \left\lceil \frac{\|u' - x\|_1}{k} \right\rceil \sum_{i=1+2\ell}^{k+2\ell} r'_i,$$

and

$$T_2(\pi_{x'',x}^{++}) \leq \|x'' - u'\|_1 \max(r'_1, \dots, r'_{2\ell}) \leq \ell' \max(r'_1, \dots, r'_{2\ell}) \leq \ell'(a_{\max} - t_w).$$

Hence, since  $\sum_{i=1+2\ell}^{k+2\ell} r'_i \leq \sum_{i=1}^k s'_i$ ,

$$\begin{aligned} T(\pi_{x'',x}^{++}) - T(\pi_{x'',x}) &\leq T_1(\pi_{x'',x}^{++}) - T_1(\pi_{x'',x}) + T_2(\pi_{x'',x}^{++}) - T_2(\pi_{x'',x}) \\ &\leq \underbrace{\sum_{i=1}^k s'_i}_{\leq k a_{\max}} - \ell' t_w \\ &\leq k a_{\max} - \alpha \ell t_w < 0 \text{ by (2.4.1),} \end{aligned}$$

which contradicts the fact that  $\pi$  is an optimal path. Thus, there is no optimal self-avoiding path  $\pi$  entirely contained in  $\Lambda$  such that (2.4.2) holds and the proof is complete.  $\square$

### Conclusion in the three cases.

*Proof of Theorem 2.1.8.* For  $x \in \mathbb{Z}^d$ , we denote by  $\pi(x)$  the first geodesic from 0 to  $x$  in the lexicographical order among those who have the minimal number of edges. We say that two patterns are disjoint if they have no vertex in common. We denote by  $\mathcal{N}^{\mathfrak{P}}(\pi(x))$  the maximum number of disjoint patterns defined above in the three cases visited by  $\pi(x)$ . Simple geometric considerations provide a constant  $c > 0$  such that for all path  $\pi$ ,  $\mathcal{N}^{\mathfrak{P}}(\pi) \geq c N^{\mathfrak{P}}(\pi)$ . Further, note that, in each pattern visited by  $\pi(x)$ ,  $\pi(x)$  takes the  $\pi^+$  segment since the  $\pi^+$  segment belongs to the set of optimal paths entirely contained in the pattern and is the only optimal path for the norm  $\|\cdot\|_1$ . We can define a self-avoiding path  $\hat{\pi}(x)$  from 0 to  $x$  by replacing each  $\pi^+$  segment of  $\pi(x)$  with the  $\pi^{++}$  segment in each disjoint pattern visited by  $\pi(x)$ . This path  $\hat{\pi}(x)$  has the same passage time and hence both  $\pi(x)$  and  $\hat{\pi}(x)$  are geodesics. Note that in the case where zero is not an atom,  $\hat{\pi}$  is obviously self-avoiding (since every geodesic is self avoiding), and in the case where zero is an atom,  $\hat{\pi}$  is self-avoiding since in the patterns, the passage times of all edges which are not in the  $\pi^+$  or  $\pi^{++}$  segments are strictly positive, and thus  $\pi(x)$  cannot visit vertices in the  $\pi^{++}$  segments except those which also belong to the corresponding  $\pi^+$  segment. In the three cases, we have:

$$|\hat{\pi}(x)| \geq |\pi(x)| + 2\mathcal{N}^{\mathfrak{P}}(\pi(x)).$$

Finally, we get:

$$\bar{L}_{0,x} \geq |\hat{\pi}(x)| \geq |\pi(x)| + 2\mathcal{N}^{\mathfrak{P}}(\pi(x)) \geq \underline{L}_{0,x} + 2\mathcal{N}^{\mathfrak{P}}(\pi(x)). \quad (2.4.4)$$

The hypothesis of Theorem 2.1.4 are satisfied, thus there exist  $\alpha > 0$ ,  $\beta_1 > 0$  and  $\beta_2 > 0$

such that for all  $x \in \mathbb{Z}^d$ ,

$$\mathbb{P}\left(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha\|x\|_1\right) \leq \beta_1 e^{-\beta_2\|x\|_1}.$$

Then, taking  $D = 2\alpha c$ , we get by (2.4.4),

$$\begin{aligned} \mathbb{P}\left(\bar{L}_{0,x} - \underline{L}_{0,x} \geq D\|x\|_1\right) &\geq \mathbb{P}\left(\mathcal{N}^{\mathfrak{P}}(\pi(x)) \geq c\alpha\|x\|_1\right) \geq \mathbb{P}\left(N^{\mathfrak{P}}(\pi(x)) \geq \alpha\|x\|_1\right) \\ &\geq 1 - \beta_1 e^{-\beta_2\|x\|_1}. \end{aligned}$$

□

## 2.4.2 Modification proof for the strict concavity of the expected passage times as a function of the weight shifts

*Proof of Theorem 2.1.9.* Recall that we assume that the support of  $F$  is bounded and that there are two different positive points in this support. We set  $t_{\max} = \sup(\text{support}(F))$ . Let  $0 < r < s$  be two points in the support of  $F$ . Fix  $b \in (0, r)$ . Applying Lemma 5.5 in [15], we get positive integers  $k, \ell$  fixed for the rest of the proof such that

$$k(s + \delta) < (k + 2\ell)(r - \delta) < (k + 2\ell)(r + \delta) < k(s - \delta) + (2\ell - 1)b \quad (2.4.5)$$

holds for sufficiently small  $\delta > 0$ . Fix  $L = k + \ell + 1$ ,  $L_1 = k + 2L$ ,  $L_2 = \ell + 2L$  and if  $d \geq 3$ , for all  $i \in \{3, \dots, d\}$ ,  $L_i = 2L$ . We define a pattern in  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ . We take the endpoints  $u^\Lambda = \sum_{i=2}^d L\varepsilon_i$  and  $v^\Lambda = u^\Lambda + (k + 2L)\varepsilon_1$ . Then we define

$$\Lambda_0 = \{L, \dots, k + L\} \times \{L, \dots, \ell + L\} \times \prod_{i=3}^d \{L\}. \quad (2.4.6)$$

With this definition, every path from the boundary of  $\Lambda$  to the boundary of  $\Lambda_0$  has to take at least  $L$  edges of  $\Lambda \setminus \Lambda_0$ . We take  $\pi^+$  the path going from  $u^\Lambda$  to  $v^\Lambda$  by  $k + 2L$  steps in the direction  $\varepsilon_1$ , and  $\pi^{++}$  the one going from  $u^\Lambda$  to  $u_1 = u^\Lambda + L\varepsilon_1 \in \Lambda_0$  by  $L$  steps in the direction  $\varepsilon_1$ , then going to  $u_2 = u_1 + \ell\varepsilon_2 \in \Lambda_0$  by  $\ell$  steps in the direction  $\varepsilon_2$ , then to  $u_3 = u_2 + k\varepsilon_1 \in \Lambda_0$  by  $k$  steps in the direction  $\varepsilon_1$ , then to  $u_4 = u_3 - \ell\varepsilon_2 \in \Lambda_0$  by  $\ell$  steps in the direction  $-\varepsilon_2$ , and then to  $v^\Lambda$  by  $L$  steps in the direction  $\varepsilon_1$ .

For a deterministic family  $(t_e)_{e \in \mathcal{E}_\Lambda}$  of passage times on the edges of  $\Lambda$  and for a path  $\pi$ , we use the abuse of writing  $T(\pi)$  to denote  $\sum_{e \in \pi} t_e$ . For all  $\delta \geq 0$ , we consider the set  $G(\delta)$  of families  $(t_e)_{e \in \mathcal{E}_\Lambda}$  of passage times on the edges of  $\Lambda$  which satisfy the following two conditions:

- for all  $e \in \pi^{++} \cap \Lambda_0$ ,  $t_e \in [r - \delta, r + \delta]$ ,



- for all other edges  $e$  in  $\Lambda$ ,  $t_e \in [s - \delta, s + \delta]$ .

Then, consider the set  $H$  of families  $(t_e)_{e \in \mathcal{E}_\Lambda}$  such that:

- (P1)  $\pi^+$  is the unique optimal path from  $u^\Lambda$  to  $v^\Lambda$  among the paths entirely contained in  $\Lambda$ ,
- (P2) for all path  $\pi_1$  from a vertex  $w_1$  of the boundary of  $\Lambda_0$  to a vertex of the boundary of  $\Lambda$ , for all  $w_2$  in  $\Lambda_0$ , for all path  $\pi_2$ , optimal for the norm  $\|\cdot\|_1$ , going from  $w_1$  to  $w_2$ , we have  $T(\pi_2) < T(\pi_1)$ .

Note that for all  $\delta > 0$ , since  $r$  and  $s$  belong to the support of  $F$ , we have that  $\mathbb{P}((T(e))_{e \in \mathcal{E}_\Lambda} \in G(\delta)) > 0$ . Let us prove that  $G(0) \subset H$ . Consider a family  $(t_e)_{e \in \mathcal{E}_\Lambda} \in G(0)$ . So there are only two different passage times in  $\Lambda$  which are  $r$  and  $s$ . Assume for a contradiction that (P1) does not hold. Then there exists an optimal path  $\pi$  going from  $u^\Lambda$  to  $v^\Lambda$ , different from  $\pi^+$ . Recall the notation  $T_i$  for  $i \in \{1, \dots, d\}$  introduced in the proof of Lemma 2.4.1. Since  $\pi^+$  is the unique path between  $u^\Lambda$  and  $v^\Lambda$  taking only edges in the direction  $\varepsilon_1$  and since there is no passage time equal to zero in  $\Lambda$ , we have  $T_1(\pi) < T(\pi)$ . Hence

$$T_1(\pi^+) = T(\pi^+) \geq T(\pi) > T_1(\pi).$$

Since the only edges in the direction  $\varepsilon_1$  whose passage time is smaller than  $s$  are in  $\pi_{u_2, u_3}^{++}$ ,  $\pi$  takes an edge of  $\pi_{u_2, u_3}^{++}$  and thus  $T_2(\pi) \geq 2\ell r$ . Furthermore

$$T_1(\pi) \geq T_1(\pi^+) - k(s - r).$$

Hence we get

$$T(\pi) \geq T_1(\pi) + T_2(\pi) \geq T_1(\pi^+) + 2\ell r - k(s - r) > T_1(\pi^+) = T(\pi^+),$$

where the strict inequality comes from (2.4.5). Thus, it contradicts the fact that  $\pi$  is an optimal path and (P1) holds. Now, to prove that (P2) holds, let  $\pi_1$  be a path from a vertex  $w_1$  of the boundary of  $\Lambda_0$  to a vertex of the boundary of  $\Lambda$  and let  $w_2$  a vertex of  $\Lambda_0$ . Let  $\pi_2$  be an optimal path for the norm  $\|\cdot\|_1$  going from  $w_1$  to  $w_2$ . Then, by the definition of  $\Lambda_0$  (see (2.4.6)),  $\pi_1$  has to take at least  $L$  edges whose passage time is equal to  $s$  although  $\pi_2$  takes at most  $k + \ell$  edges whose passage time is smaller than or equal to  $s$ . Thus

$$T(\pi_1) \geq Ls > (k + \ell)s \geq T(\pi_2),$$

where the strict inequality comes from the fact that  $L = k + \ell + 1$ . Hence (P2) holds and  $G(0) \subset H$ .

Furthermore,  $H$  is an open set since for a family  $(t_e)_{e \in \mathcal{E}_\Lambda}$  to belong to  $H$ , it is required that the time of a finite family of paths is strictly smaller than the time of every path of another finite family of paths. Hence, for  $\delta > 0$  small enough, we have

$$G(\delta) \subset H. \tag{2.4.7}$$

Fix  $\delta > 0$  such that (2.4.5) and (2.4.7) hold. Consider the pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  with  $\mathcal{A}^\Lambda = \{(T(e))_{e \in \mathcal{E}_\Lambda} \in G(\delta)\}$ . Now, we denote by  $\pi(x)$  the first geodesic from 0 to  $x$  in the lexicographical order among those of maximal Euclidean length and we denote by  $\mathcal{N}^{\mathfrak{P}}(\pi(x))$  the maximum number of disjoint patterns visited by  $\pi(x)$ . Recall the existence of a constant  $c > 0$  small enough such that for all path  $\pi$ ,  $\mathcal{N}^{\mathfrak{P}}(\pi) \geq cN^{\mathfrak{P}}(\pi)$ . Since  $\mathbb{P}(\mathcal{A}^\Lambda) > 0$ , we can apply Theorem 2.1.4. Let  $\alpha, \beta_1, \beta_2 > 0$  be the constants given by Theorem 2.1.4. Then, we have

$$\begin{aligned} \mathbb{E}[\mathcal{N}^{\mathfrak{P}}(\pi(x))] &\geq \lfloor c\alpha\|x\|_1 \rfloor \mathbb{P}(\mathcal{N}^{\mathfrak{P}}(\pi(x)) \geq c\alpha\|x\|_1) \\ &\geq \lfloor c\alpha\|x\|_1 \rfloor \mathbb{P}(N^{\mathfrak{P}}(\pi(x)) \geq \alpha\|x\|_1) \geq \lfloor c\alpha\|x\|_1 \rfloor (1 - \beta_1 e^{-\beta_2\|x\|_1}) \geq C\|x\|_1. \end{aligned}$$

Now, let us follow the end of Stage 3 of the proof of Theorem 5.4 in [15]. By (P1),  $\pi(x)$  takes the  $\pi^+$  segment in each pattern that it takes. Furthermore, by (P2),  $\pi(x)$  does not take any edge in the  $\pi^{++}$  segment which is not in the  $\pi^+$  segment. So, we can define a self-avoiding path  $\hat{\pi}(x)$  from 0 to  $x$  by replacing each  $\pi^+$  segment with the  $\pi^{++}$  segment in each pattern visited by  $\pi(x)$ . Reduce the weights on each edge  $e$  from  $T(e)$  to  $T^{(-b)}(e) = T(e) - b$ . By the definition of the pattern, the  $T^{(-b)}$ -passage times of the segments  $\pi^+$  and  $\pi^{++}$  obey the inequality:

$$T^{(-b)}(\pi^{++}) = T(\pi^{++}) - b|\pi^{++}| < T(\pi^+) + (2\ell - 1)b - b|\pi^{++}| = T^{(-b)}(\pi^+) - b.$$

Then, following the proof of Theorem 5.4 in [15], we get

$$\begin{aligned} t^{(-b)}(0, x) &\leq T^{(-b)}(\hat{\pi}(x)) < T^{(-b)}(\pi(x)) - b\mathcal{N}^{\mathfrak{P}}(\pi(x)) \\ &= T(\pi(x)) - b|\pi(x)| - b\mathcal{N}^{\mathfrak{P}}(\pi(x)) \\ &= t(0, x) - b\bar{L}_{0,x} - b\mathcal{N}^{\mathfrak{P}}(\pi(x)). \end{aligned} \tag{2.4.8}$$

Since  $b \in (0, t_{\min} + \varepsilon_0)$ ,  $\mathbb{E}[t^{(-b)}(0, x)]$  is finite. Thus, taking expectation in (2.4.8), we get the result.  $\square$

## 2.A Unbounded case

### 2.A.1 Assumptions on the pattern in the proof of the unbounded case

**Lemma 2.A.1.** *Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern. There exists a positive integer  $\ell^\Lambda$  and a valid pattern  $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$  such that:*

- $\Lambda_0 = B_\infty(0, \ell^\Lambda)$ ,
- for every self-avoiding path  $\pi$ ,  $N^{\mathfrak{P}}(\pi) \geq N^{\mathfrak{P}_0}(\pi)$ .

*Proof.* Denote by  $L_1, \dots, L_d$  the integers such that  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ . Fix

$$\ell^\Lambda = \max(L_1, \dots, L_d).$$

Let  $M^\Lambda > 0$  such that

$$\mathbb{P}(\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq M^\Lambda\}) > 0.$$

Consider the pattern  $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$  defined by:

- $\Lambda_0 = B_\infty(0, \ell^\Lambda)$
- $u_0^\Lambda$  (resp.  $v_0^\Lambda$ ) one vertex of  $\partial\Lambda_0$  such that there exists a path  $\pi_u$  (resp.  $\pi_v$ ) fixed for the remaining of the proof linking  $u_0^\Lambda$  and  $u^\Lambda$  (resp.  $v_0^\Lambda$  and  $v^\Lambda$ ) which satisfies the following two conditions:
  - $\pi_u$  (resp.  $\pi_v$ ) does not visit any vertex of  $\Lambda$  except  $u^\Lambda$  (resp.  $v^\Lambda$ ),
  - $\pi_u$  (resp.  $\pi_v$ ) only takes edges in the direction of one external normal unit vector associated with  $u^\Lambda$  (resp.  $v^\Lambda$ ) chosen arbitrarily if there are several such vectors,
- $\mathcal{A}_0^\Lambda$  the event such that:
  - $\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq M^\Lambda\}$  occurs,
  - for all  $e$  belonging to  $\pi_u \cup \pi_v$ , we have  $T(e) \leq M^\Lambda$ ,
  - for all  $e$  which does not belong to  $\Lambda \cup \pi_u \cup \pi_v$ ,

$$T(e) > |\Lambda_0|_e M^\Lambda,$$

where  $|\Lambda_0|_e$  is the number of edges in  $\Lambda_0$ .

Note that, since  $u^\Lambda$  and  $v^\Lambda$  are distinct,  $\pi_u$  and  $\pi_v$  are disjoint. We get:

- $\mathbb{P}(\mathcal{A}_0^\Lambda)$  is positive since  $\mathfrak{P}$  is valid and since the support of  $F$  is unbounded, and then  $\mathfrak{P}_0$  is a valid pattern.
- On  $\mathcal{A}_0^\Lambda$ , any path from  $u_0^\Lambda$  to  $v_0^\Lambda$  optimal for the passage time among the paths entirely inside  $\Lambda_0$  contains a subpath from  $u^\Lambda$  to  $v^\Lambda$  entirely inside  $\Lambda$ . Indeed, let  $\pi_0$  be a path from  $u_0^\Lambda$  to  $v_0^\Lambda$  which does not contain a subpath from  $u^\Lambda$  to  $v^\Lambda$  entirely inside  $\Lambda$ . It implies that  $\pi_0$  takes an edge whose time is greater than  $|\Lambda_0|_e M^\Lambda$ . Let  $\pi'_0$  be a path following  $\pi_u$ , then going from  $u^\Lambda$  to  $v^\Lambda$  inside  $\Lambda$ , and then following  $\pi_v$ . We have  $T(\pi'_0) \leq |\Lambda_0|_e M^\Lambda < T(\pi_0)$  and thus  $\pi_0$  is not an optimal path. Hence, for every path  $\pi$ , if a vertex  $x \in \mathbb{Z}^d$  satisfies the condition  $(\pi; \mathfrak{P}_0)$ ,  $x$  satisfies the condition  $(\pi; \mathfrak{P})$ .

Thus we get that  $N^{\mathfrak{P}}(\pi) \geq N^{\mathfrak{P}_0}(\pi)$ . □

## 2.A.2 Construction of the path $\pi$ for the modification in the unbounded case

Recall that, here,  $u^\Lambda$  and  $v^\Lambda$  are the vertices defined at the beginning of Section 2.2.3. The aim is to prove that we can construct a path  $\pi$  in a deterministic way such that:

- (i)  $\pi$  goes from  $u$  to  $u^\Lambda$  without visiting a vertex of  $B_\infty(sN, \ell^\Lambda)$ , then going from  $u^\Lambda$  to  $v^\Lambda$  in a shortest way for the norm  $\|\cdot\|_1$  (and thus being contained in  $B_\infty(sN, \ell^\Lambda)$ ) and then goes from  $v^\Lambda$  to  $v$  without visiting a vertex of  $B_\infty(sN, \ell^\Lambda)$ ,
- (ii)  $\pi$  is entirely contained in  $B_{2,s,N}$  and does not have vertices on the boundary of  $B_{2,s,N}$  except  $u$  and  $v$ ,
- (iii)  $\pi$  is self-avoiding,
- (iv) the length of  $\pi_{u,u^\Lambda} \cup \pi_{v^\Lambda,v}$  is bounded from above by  $2r_2N + K$ , where  $K$  is the number of edges in  $B_\infty(0, \ell^\Lambda + 3)$ .

As it is said in Section 2.2.3, we want to construct a path from  $u$  to  $sN$  and a path from  $sN$  to  $v$  which have no vertex in common except  $sN$  and such that their lengths are bounded from above by  $r_2N$ . To get them, we use the following lemma whose proof is left to the reader.

**Lemma 2.A.2.** *Let  $m \in \mathbb{N}^*$  and  $x, y$  two vertices of  $\mathbb{Z}^d$  such that  $\|x\|_1 = \|y\|_1 = m$  and  $x \neq y$ . Then we can build in a deterministic way two paths  $\pi_x$  and  $\pi_y$  linking respectively  $x$  and  $y$  to 0 and such that their length is equal to  $m$ , they have only 0 as a common vertex and they have respectively only  $x$  and  $y$  as vertices whose norm  $\|\cdot\|_1$  is greater than or equal to  $m$ .*

Using this lemma and replacing 0 by  $sN$  using a translation, we get two paths linking  $u$  to  $sN$  and  $v$  to  $sN$  with the stated properties. Recall that  $B_\infty(sN, \ell^\Lambda + 3) \subset B_{2,s,N}$  and let  $u_0$  (resp.  $v_0$ ) denote the first vertex in  $B_\infty(sN, \ell^\Lambda + 3)$  visited by the path going from  $u$  to  $sN$  (resp. the one going from  $v$  to  $sN$ ). Then we get two paths  $\pi_{u,u_0}$  and  $\pi_{v_0,v}$  respectively from  $u$  to  $u_0$  and from  $v_0$  to  $v$  both constructed in a deterministic way such that  $\pi_{u,u_0}$  and  $\pi_{v_0,v}$  do not have any vertex in common, are entirely contained in  $B_{2,s,N}$ , have only  $u$  or  $v$  as points on the boundary of  $B_{2,s,N}$ , and their lengths are bounded from above by  $r_2N - (\ell^\Lambda + 3)$ .

Then we build two paths  $\pi_{u_0,u^\Lambda}$  and  $\pi_{v^\Lambda,v_0}$  respectively from  $u_0$  to  $u^\Lambda$  and from  $v^\Lambda$  to  $v_0$  contained in  $B_\infty(sN, \ell^\Lambda + 3) \setminus B_\infty(sN, \ell^\Lambda)$  except for  $u^\Lambda$  and  $v^\Lambda$ , such that they do not have any vertex in common. By the definition of  $K$ , this implies that the sum of their lengths is bounded from above by  $K$  (recall that  $K$  is the number of edges in  $B_\infty(0, \ell^\Lambda + 3)$ ). To get these paths, assume first that  $\|u_0 - v^\Lambda\|_1 > 3$  or  $\|v_0 - u^\Lambda\|_1 > 3$ . Assume that we have  $\|v_0 - u^\Lambda\|_1 > 3$ , the other case being the same. We begin by considering a path  $\tilde{\pi}$  going in a shortest way from  $v_0$  to a vertex of the boundary of  $B_\infty(sN, \ell^\Lambda + 2)$ , denoted by  $v'_0$ . This path has at most  $d$  edges. Then, let  $u^{\Lambda,0}$  be a vertex on the boundary of  $B_\infty(sN, \ell^\Lambda + 3)$  such that  $\|u^\Lambda - u^{\Lambda,0}\|_1 = 3$ . We get  $\pi_{u_0,u^\Lambda}$  by going from  $u_0$  to  $u^{\Lambda,0}$  in a shortest way on the

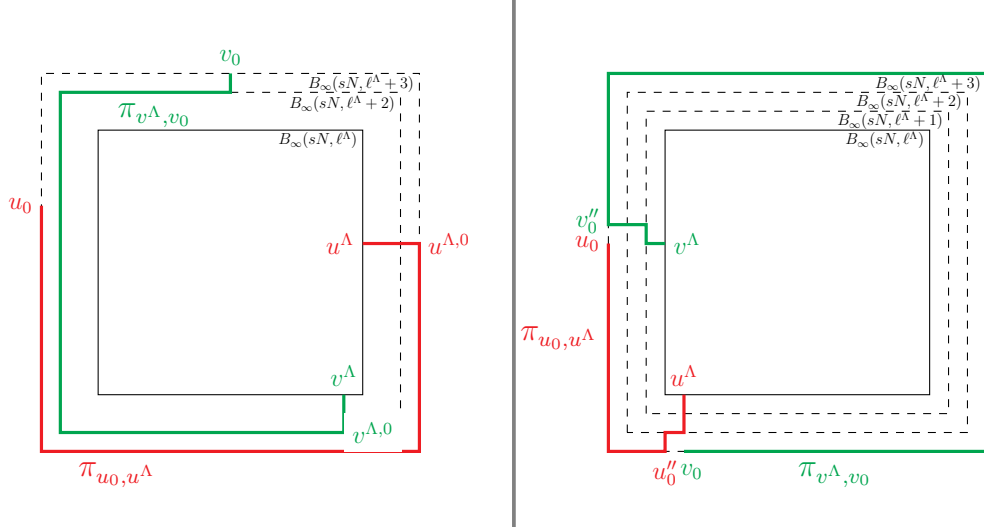


Figure 2.6: Example of the construction of the portion of  $\pi$  which is contained in  $B_\infty(sN, \ell^\Lambda + 3) \setminus B_\infty(sN, \ell^\Lambda)$  in dimension 2. On the left, this is an example of the case where  $\|u_0 - v^\Lambda\|_1 > 3$  or  $\|v_0 - u^\Lambda\|_1 > 3$  and on the right the other case.

boundary of  $B_\infty(sN, \ell^\Lambda + 3)$  avoiding all vertices of  $\tilde{\pi}$ , and then by going from  $u^{\Lambda,0}$  to  $u^\Lambda$  by three steps. This is possible since  $\|v_0 - u^\Lambda\|_1 > 3$ . To get  $\pi_{v_0, v^\Lambda}$ , we begin by following  $\tilde{\pi}$ . Let  $v^{\Lambda,0}$  be a vertex on the boundary of  $B_\infty(sN, \ell^\Lambda + 2)$  such that  $\|v^\Lambda - v^{\Lambda,0}\|_1 = 2$ . Then,  $\pi_{v_0, v^\Lambda}$  goes from  $v'_0$  to  $v^{\Lambda,0}$  in a shortest way on the boundary of  $B_\infty(sN, \ell^\Lambda + 2)$  avoiding the unique vertex of  $\pi_{u_0, v^\Lambda}$  belonging to  $B_\infty(sN, \ell^\Lambda + 2)$ . Finally,  $\pi_{v_0, v^\Lambda}$  goes from  $v^{\Lambda,0}$  to  $v^\Lambda$  by two steps (see the left side of Figure 2.6 for an example of this construction in dimension 2).

Now, if  $\|u_0 - v^\Lambda\|_1 = 3$  and  $\|v_0 - u^\Lambda\|_1 = 3$ , the construction has to be slightly different. From  $u_0$ ,  $\pi_{u_0, u^\Lambda}$  goes to a vertex  $u''_0$  belonging to the boundary of  $B_\infty(sN, \ell^\Lambda + 3)$  and such that  $\|u''_0 - v_0\|_1 = 1$  in a shortest way on the boundary of  $B_\infty(sN, \ell^\Lambda + 3)$ . After, it makes one step to go to the boundary of  $B_\infty(sN, \ell^\Lambda + 2)$  and goes to  $u^\Lambda$  in a shortest way by taking only one vertex in  $B_\infty(sN, \ell^\Lambda + 1)$ . Then,  $\pi_{v_0, v^\Lambda}$  goes on the boundary of  $B_\infty(sN, \ell^\Lambda + 3)$  to a vertex  $v''_0$  belonging to the boundary of  $B_\infty(sN, \ell^\Lambda + 3)$  and such that  $\|v''_0 - u_0\|_1 = 1$  by avoiding every vertex which belongs to  $\pi_{u_0, u^\Lambda}$ , and then makes two steps to go to the boundary of  $B_\infty(sN, \ell^\Lambda + 1)$  and goes to  $v^\Lambda$  in a shortest way (see the right side of Figure 2.6 for an example of this construction in dimension 2). In this case, we also have that the sum of their lengths is bounded from above by  $K$ .

Finally,  $\pi$  is the path obtained by concatenating  $\pi_{u, u_0}$ ,  $\pi_{u_0, u^\Lambda}$ , a path going from  $u^\Lambda$  to  $v^\Lambda$  in a shortest way for the norm  $\|\cdot\|_1$ ,  $\pi_{v^\Lambda, v_0}$  and  $\pi_{v_0, v}$  in this order. We have that  $\pi$  is a self-avoiding path contained in  $B_{2, s, N}$  and has only  $u$  and  $v$  on the boundary of  $B_{2, s, N}$ .

## 2.B Overlapping pattern in the bounded case

*Proof of Lemma 2.3.1.* Recall that  $\nu$  is fixed at the beginning of Section 2.3 and that  $\delta' = \min\left(\frac{\delta}{8}, \frac{\delta}{1+d}\right)$ . Then, we fix a positive real  $\nu_0$  such that:

- $t_{\min} + \delta \leq \nu_0 \leq \nu \leq t_{\max}$ ,
- the event  $\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, t_e \leq \nu_0\}$  has a positive probability,
- $F([\nu_0, \nu]) > 0$ .

Notice that, if  $F$  has an atom, one could have  $\nu_0 = \nu$ , or even  $\nu_0 = \nu = t_{\max}$ . Then, fix

$$\ell^\Lambda = \max(L_1, \dots, L_d), \quad (2.B.1)$$

$$\ell_1 > \frac{4d\ell^\Lambda \left(\nu_0 - t_{\min} - \frac{\delta'}{2}\right)}{\nu_0 - t_{\min} - \delta'}, \quad (2.B.2)$$

$$\text{and } \ell_0 > \frac{\ell^\Lambda((2d+1)\nu_0 + (2d-1)t_{\min} + 2d\delta') + \ell_1(\nu_0 + 3t_{\min} + 4\delta')}{\nu_0 - t_{\min} - 2\delta'}. \quad (2.B.3)$$

Fix also

$$\delta'' < \min\left(\delta', \frac{\nu_0 - t_{\min}}{2d\ell_0}\right). \quad (2.B.4)$$

Let  $j \in \{1, \dots, d\}$ . Let us construct the overlapping pattern in  $\Lambda_0 = \{-\ell_0, \dots, \ell_0\}^d$  with endpoints  $\ell_0\varepsilon_j$  and  $-\ell_0\varepsilon_j$ . We denote by  $u_3$  and  $v_3$  the endpoints of the original pattern. The first step is the construction of an oriented path going from the face of  $\Lambda_0$  containing  $\ell_0\varepsilon_j$  to the face containing  $-\ell_0\varepsilon_j$  and visiting  $u_3$  and  $v_3$ . We denote by  $\Lambda_1$  the set  $\prod_{i=1}^d \{-\ell_1, \dots, L_i + \ell_1\}$ . Then we define  $u_2$  (resp.  $v_2$ ) the vertex of  $\partial\Lambda_1$  which can be linked to  $u_3$  (resp.  $v_3$ ) by a path using exactly  $\ell_1$  edges in only one direction (the direction of the external normal vector associated to  $u_3$  (resp.  $v_3$ )). If  $u_3$  or  $v_3$  are associated to several external normal unit vectors, then the choice of the direction is not unique. We choose one external normal unit vector in an arbitrary way but we ensure that the two normal unit vectors chosen to build  $u_2$  and  $v_2$  are different. Note that this is possible since the pattern is valid (recall Definition 2.1.2). For a vertex  $z \in \mathbb{Z}^d$ , denote by  $z(j)$  its  $j$ -th coordinate. Although we may need to exchange the roles of  $u_3$  and  $v_3$ , we can assume that  $u_2(j) \geq v_2(j)$ . Then, we define  $u_1 = u_2 + (\ell_0 - u_2(j))\varepsilon_j$  and  $v_1 = v_2 - (\ell_0 + v_2(j))\varepsilon_j$ . Note that  $u_1$  (resp.  $v_1$ ) belongs to the face of  $\Lambda_0$  containing  $\ell_0\varepsilon_j$  (resp.  $-\ell_0\varepsilon_j$ ). We define a path  $\tilde{\pi}$  going from  $\ell_0\varepsilon_j$  to  $u_1$  in a shortest way, then to  $u_2$  in the shortest way, then to  $u_3$  in the shortest way, then to  $v_3$  in a shortest way, then to  $v_2$  in the shortest way, then to  $v_1$  in the shortest way and to  $v_0$  in a shortest way. Note that  $\tilde{\pi}_{u_1, v_1}$  is an oriented path. Indeed,  $\tilde{\pi}_{u_3, v_3}$  is an oriented path. Extending it outside  $\Lambda$  following the direction of an external normal unit vector preserves the fact that it is oriented. Thus  $\tilde{\pi}_{u_2, v_2}$

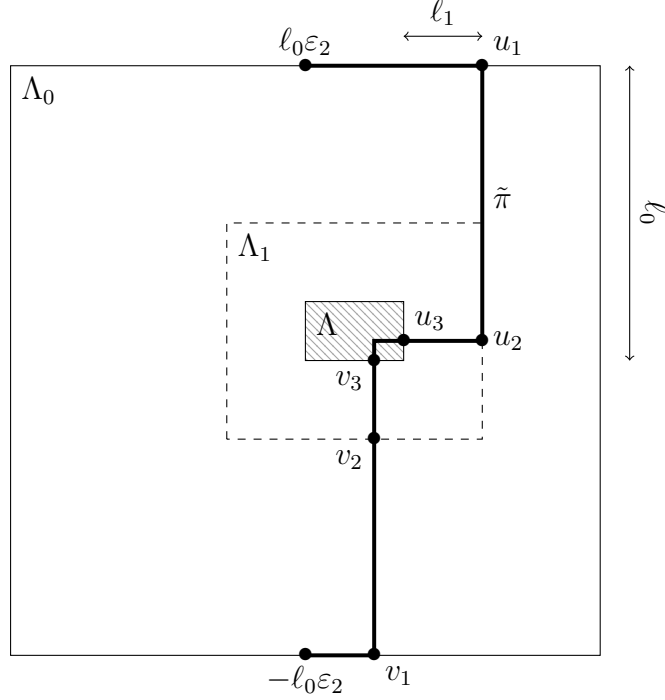


Figure 2.7: Example of the construction of  $\tilde{\pi}$  in an overlapping pattern in dimension 2.

is an oriented path. Then, we assume that  $u_2(j) \geq v_2(j)$  on purpose to guarantee that, with the definitions of  $u_1$  and  $v_1$ , the path  $\tilde{\pi}_{u_1, v_1}$  remains an oriented path.

The event (whose probability is positive) of the overlapping pattern, denoted by  $\mathcal{A}_j^{\Lambda_0}$  is the following:

- for all  $e \in \tilde{\pi}_{\ell_0 \varepsilon_j, u_3}$  and  $e \in \tilde{\pi}_{v_3, -\ell_0 \varepsilon_j}$ ,  $T(e) < t_{\min} + \delta''$ ,
- the event  $\mathcal{A}^\Lambda \cap \{\forall e \in \Lambda, T(e) \leq \nu_0\}$  is realized,
- for all  $e$  in  $\Lambda_0 \setminus (\tilde{\pi} \cup \Lambda)$ ,  $\nu_0 \leq T(e) \leq \nu$ .

On this event, let  $\bar{\gamma}$  be one of the fastest path from  $\ell_0 \varepsilon_j$  to  $-\ell_0 \varepsilon_j$  among the path entirely contained in  $\Lambda_0$  and let us show that  $\bar{\gamma}$  visits  $u_3$  and  $v_3$  and that  $\bar{\gamma}_{u_3, v_3}$  is entirely contained in  $\Lambda$ . We proceed by proving successive properties.

- *There exist one vertex  $a_0$  of  $\tilde{\pi}_{u_1, u_2}$  and one vertex  $b_0$  of  $\tilde{\pi}_{v_2, v_1}$  visited by  $\bar{\gamma}$ . Further,  $\bar{\gamma}_{\ell_0 \varepsilon_j, a_0} = \tilde{\pi}_{\ell_0 \varepsilon_j, a_0}$  and  $\bar{\gamma}_{b_0, -\ell_0 \varepsilon_j} = \tilde{\pi}_{b_0, -\ell_0 \varepsilon_j}$ .*

Let us assume that  $\bar{\gamma}$  does not visit any vertex of  $\tilde{\pi}_{u_1, u_2}$ , the other case being the same. The path  $\bar{\gamma}$  has to take at least  $\ell_0 - \ell^\Lambda - \ell_1$  edges connecting vertices such that at least one of them has its  $j$ -th coordinate strictly between  $\ell^\Lambda + \ell_1$  and  $\ell_0$ . The only edges in this set whose passage time is smaller than  $\nu_0$  are those of  $\tilde{\pi}_{u_1, u_2}$ . Hence

$$T(\bar{\gamma}) \geq (\ell_0 - \ell^\Lambda - \ell_1)\nu_0 + (\ell_0 + \ell^\Lambda + \ell_1)t_{\min}.$$

But, thanks to our construction, we have

$$\begin{aligned}
& T(\tilde{\pi}) \\
& \leq \underbrace{T(\tilde{\pi}_{\ell_0\varepsilon_j, u_1}) + T(\tilde{\pi}_{v_1, -\ell_0\varepsilon_j})}_{\leq (2d\ell^\Lambda + 2\ell_1)(t_{\min} + \delta'')} + \underbrace{T(\tilde{\pi}_{u_1, u_2}) + T(\tilde{\pi}_{v_2, v_1})}_{\leq 2\ell_0(t_{\min} + \delta'')} + \underbrace{T(\tilde{\pi}_{u_2, u_3}) + T(\tilde{\pi}_{v_3, v_2})}_{\leq 2\ell_1(t_{\min} + \delta'')} + \underbrace{T(\tilde{\pi}_{u_3, v_3})}_{\leq 2d\ell^\Lambda\nu_0} \\
& \leq 2(d\ell^\Lambda + 2\ell_1 + \ell_0)(t_{\min} + \delta') + 2d\ell^\Lambda\nu_0 \text{ since } \delta'' < \delta'.
\end{aligned}$$

So, (2.B.3) leads to  $T(\tilde{\pi}) < T(\bar{\gamma})$ , which is impossible.

To prove that  $\bar{\gamma}_{\ell_0\varepsilon_j, a_0} = \tilde{\pi}_{\ell_0\varepsilon_j, a_0}$ , we use the fact that by the construction, if  $\bar{\gamma}_{\ell_0\varepsilon_j, a_0} \neq \tilde{\pi}_{\ell_0\varepsilon_j, a_0}$ ,  $\bar{\gamma}_{\ell_0\varepsilon_j, a_0}$  has to take at least one edge whose time is greater than or equal to  $\nu_0$ . Hence,

$$T(\bar{\gamma}_{\ell_0\varepsilon_j, a_0}) \geq \nu_0 + (\|\ell_0\varepsilon_j - a_0\|_1 - 1)t_{\min},$$

although

$$T(\tilde{\pi}_{\ell_0\varepsilon_j, a_0}) \leq \|\ell_0\varepsilon_j - a_0\|_1(t_{\min} + \delta'').$$

Since  $\|\ell_0\varepsilon_j - a_0\|_1 \leq 2d\ell_0$ , (2.B.4) leads to  $T(\tilde{\pi}_{\ell_0\varepsilon_j, a_0}) < T(\bar{\gamma}_{\ell_0\varepsilon_j, a_0})$ , which is impossible since  $\bar{\gamma}$  is an optimal path.

- If  $a_1$  (resp.  $b_1$ ) is a vertex of  $\tilde{\pi}_{u_2, u_3}$  (resp.  $\tilde{\pi}_{v_3, v_2}$ ) visited by  $\bar{\gamma}$ , then  $\bar{\gamma}_{\ell_0\varepsilon_j, a_1} = \tilde{\pi}_{\ell_0\varepsilon_j, a_1}$  (resp.  $\bar{\gamma}_{b_1, -\ell_0\varepsilon_j} = \tilde{\pi}_{b_1, -\ell_0\varepsilon_j}$ ).

Indeed, let  $a_1$  be such a vertex and let  $a_0$  be the vertex of the preceding property. We only have to prove that  $\bar{\gamma}_{a_0, a_1} = \tilde{\pi}_{a_0, a_1}$  and the proof is the same as the one for  $\bar{\gamma}_{\ell_0\varepsilon_j, a_0} = \tilde{\pi}_{\ell_0\varepsilon_j, a_0}$ .

Among the vertices visited by  $\bar{\gamma}$ , we denote by  $a$  (resp.  $b$ ) the last vertex of  $\tilde{\pi}_{\ell_0\varepsilon_j, u_3}$  (resp. the first vertex of  $\tilde{\pi}_{v_3, -\ell_0\varepsilon_j}$ ). Then,  $\bar{\gamma}_{a, b}$  does not visit any vertex of  $\tilde{\pi}_{\ell_0\varepsilon_j, u_3} \cup \tilde{\pi}_{v_3, -\ell_0\varepsilon_j}$  (except  $a$  and  $b$ ). Indeed, by the two properties above, we have that  $\bar{\gamma}_{\ell_0\varepsilon_j, a} = \tilde{\pi}_{\ell_0\varepsilon_j, a}$  and  $\bar{\gamma}_{b, -\ell_0\varepsilon_j} = \tilde{\pi}_{b, -\ell_0\varepsilon_j}$  and  $\bar{\gamma}_{a, b}$  cannot visit a vertex of  $\tilde{\pi}_{a, u_3} \cup \tilde{\pi}_{v_3, b}$  thanks to the definition of  $a$  and  $b$ .

- The vertex  $a$  (resp.  $b$ ) belongs to  $\tilde{\pi}_{u_2, u_3}$  (resp.  $\tilde{\pi}_{v_3, v_2}$ ).

Assume that  $a$  or  $b$  does not satisfy this property. Then  $\|a - u_3\|_1 \geq \ell_1$  or  $\|b - v_3\|_1 \geq \ell_1$ . Since  $\tilde{\pi}_{a, b}$  is oriented and since  $\|u_3 - v_3\|_1 \leq 2d\ell^\Lambda$ ,

$$T(\tilde{\pi}_{a, b}) \leq (\|a - b\|_1 - 2d\ell^\Lambda)(t_{\min} + \delta'') + 2d\ell^\Lambda\nu_0 \leq (\|a - b\|_1 - 2d\ell^\Lambda)(t_{\min} + \delta') + 2d\ell^\Lambda\nu_0.$$

Then, since the edges whose time is smaller than  $\nu_0$  taken by  $\bar{\gamma}_{a, b}$  are those in  $\Lambda$ ,

$$T(\bar{\gamma}_{a, b}) \geq (\|a - b\|_1 - 2d\ell^\Lambda)\nu_0 + 2d\ell^\Lambda t_{\min}.$$

Using (2.B.2), we get  $T(\bar{\gamma}_{a, b}) > T(\tilde{\pi}_{a, b})$ , which is impossible.



- We have that  $a = u_3$  and  $b = v_3$ .

Assume that  $a \neq u_3$ , the other case being the same. We have

$$T(\tilde{\pi}_{a,b}) \leq (\|a - b\|_1 - \|u_3 - v_3\|_1)(t_{\min} + \delta') + \|u_3 - v_3\|_1 \nu_0,$$

and thus  $\bar{\gamma}_{a,b}$  takes at least one edge of  $\Lambda$  otherwise  $T(\bar{\gamma}_{a,b}) \geq \|a - b\|_1 \nu_0 > T(\tilde{\pi}_{a,b})$  since  $\|a - b\|_1 - \|u_3 - v_3\|_1 > 0$ . Let  $u_0$  be the first entry point of  $\bar{\gamma}_{a,b}$  in  $\Lambda$  and let us consider the path  $\pi$  following  $\tilde{\pi}_{a,u_3}$ , then going from  $u_3$  to  $u_0$  in a shortest way and then following  $\bar{\gamma}_{u_0,b}$ . Then, the number of edges of  $\pi_{a,u_0}$  is lower than or equal to the number of edges of  $\bar{\gamma}_{a,u_0}$ , for all  $e \in \pi_{a,u_0}$ ,  $T(e) \leq \nu_0$ , there exists  $e' \in \pi_{a,u_0}$  such that  $T(e') \leq t_{\min} + \delta'$  and for all  $e \in \bar{\gamma}_{a,u_0}$ ,  $T(e) \geq \nu_0$ . So, we have  $T(\pi) < T(\bar{\gamma}_{a,b})$  which is impossible since  $\bar{\gamma}$  is an optimal path among paths entirely contained in  $\Lambda_0$ .

- $\bar{\gamma}_{a,b}$  takes the original pattern.

Assume that  $\bar{\gamma}_{a,b}$  is not entirely contained in  $\Lambda$ . Let  $v_0$  be the first exit point from  $\Lambda$  of  $\bar{\gamma}_{a,b}$  and  $u_0$  the first entry point after  $v_0$ . Let us consider the shortcut  $\pi$  going from  $v_0$  to  $u_0$  in a shortest way. Then, using the same argument as in the proof of Lemma 2.3.21, we have that  $\bar{\gamma}_{v_0,u_0}$  has strictly more edges than  $\pi$ . Furthermore, all edges of  $\bar{\gamma}_{v_0,u_0}$  have a time greater than or equal to  $\nu_0$  although all edges of  $\pi$  have a time lower than or equal to  $\nu_0$ . So,  $T(\pi) < T(\bar{\gamma}_{v_0,u_0})$ , which is impossible since  $\bar{\gamma}$  is an optimal path among paths entirely contained in  $\Lambda_0$ . Since  $\bar{\gamma}_{a,b}$  is a path entirely contained in  $\Lambda$ , going from  $u_3$  to  $v_3$  and with an optimal time, so we have the result.

□

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## Geodesics cross any pattern in first-passage percolation without any moment assumption and with possibly infinite passage times

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Ce chapitre correspond à l'article [12].

### Abstract

In first-passage percolation, one places nonnegative i.i.d. random variables  $(T(e))$  on the edges of  $\mathbb{Z}^d$ . A geodesic is an optimal path for the passage times  $T(e)$ . Consider a local property of the time environment. We call it a pattern. We investigate the number of times a geodesic crosses a translate of this pattern. When we assume that the common distribution of the passage times satisfies a suitable moment assumption, it is shown in [Antonin Jacquet. Geodesics in first-passage percolation cross any pattern, arXiv:2204.02021, 2023] that, apart from an event with exponentially small probability, this number is linear in the distance between the extremities of the geodesic. This paper completes this study by showing that this result remains true when we consider distributions with an unbounded support without any moment assumption or distributions with possibly infinite passage times. The techniques of proof differ from the preceding article and rely on a notion of penalized geodesic.

## 3.1 Introduction

### 3.1.1 Settings

Fix an integer  $d \geq 2$ . In this paper, we consider first passage percolation on the hypercubic lattice  $\mathbb{Z}^d$ . We denote by  $0$  the origin of  $\mathbb{Z}^d$  and by  $\mathcal{E}$  the set of edges in this lattice. The edges in  $\mathcal{E}$  are those connecting two vertices  $x$  and  $y$  such that  $\|x - y\|_1 = 1$ . The basic

random object consists of a family  $T = \{T(e) : e \in \mathcal{E}\}$  of i.i.d. random variables taking values in  $[0, \infty]$  and defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The random variable  $T(e)$  represents the passage time of the edge  $e$ . Their common distribution is denoted by  $\mathcal{L}$ .

A finite path  $\pi = (x_0, \dots, x_k)$  is a sequence of adjacent vertices of  $\mathbb{Z}^d$ , i.e. for all  $i = 0, \dots, k-1$ ,  $\|x_{i+1} - x_i\|_1 = 1$ . We say that  $\pi$  goes from  $x_0$  to  $x_k$ . Sometimes we identify a path with the sequence of edges it visits, writing  $\pi = (e_1, \dots, e_k)$  where for  $i = 1, \dots, k$ ,  $e_i = \{x_{i-1}, x_i\}$ . We say that  $k$  is the length of  $\pi$  and we denote  $|\pi| = k$ . The passage time  $T(\pi)$  of a path  $\pi = (e_1, \dots, e_k)$  is the sum of the variables  $T(e_i)$  for  $i = 1, \dots, k$ .

We do not exclude the case  $\mathcal{L}(\infty) > 0$ . In this case, there are vertices between which all paths have an infinite passage time. Thus, we define the following random set:

$$\mathfrak{C} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \exists \text{ a path } \pi \text{ from } x \text{ to } y \text{ such that } T(\pi) < \infty\}.$$

Throughout the article, we assume that

$$\mathcal{L}([0, \infty)) > p_c, \tag{3.1.1}$$

where  $p_c$  denotes the critical probability for Bernoulli bond percolation model on  $\mathbb{Z}^d$ . We refer to [10] for background on percolation. Say that an edge  $e$  is open if its passage time  $T(e)$  is finite and closed otherwise. Thanks to (3.1.1), this percolation model is supercritical. Therefore there exists a unique infinite component which we denote by  $\mathcal{C}_\infty$ . When  $\mathcal{L}(\infty) = 0$ , note that every couple of vertices belongs to  $\mathfrak{C}$  and that  $\mathcal{C}_\infty$  is equal to the whole graph.

Now, for two vertices  $x$  and  $y$ , we define the geodesic time

$$t(x, y) = \inf\{T(\pi) : \pi \text{ is a path from } x \text{ to } y\}. \tag{3.1.2}$$

Note that  $\mathfrak{C} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : t(x, y) \text{ is finite}\}$ . A self-avoiding path  $\gamma$  from  $x$  to  $y$  such that  $T(\gamma) = t(x, y)$  is called a geodesic between  $x$  and  $y$ .

For the following and for the existence of geodesics, we need some assumptions on  $\mathcal{L}$ . Let  $t_{\min}$  denote the minimum of the support of  $\mathcal{L}$ . We extend a definition introduced in [18]. A distribution  $\mathcal{L}$  with support in  $[0, \infty]$  is called useful if the following holds:

$$\begin{aligned} \mathcal{L}(t_{\min}) &< p_c \text{ when } t_{\min} = 0, \\ \mathcal{L}(t_{\min}) &< \vec{p}_c \text{ when } t_{\min} > 0, \end{aligned} \tag{3.1.3}$$

where  $p_c$  has been introduced above and where  $\vec{p}_c$  is the critical probability for oriented Bernoulli bond percolation on  $\mathbb{Z}^d$  (see Section 12.8 in [10]). Throughout the article, we also assume that  $\mathcal{L}$  is useful.

Geodesics between any pair of vertices belonging to  $\mathfrak{C}$  exist with probability one. This is Proposition 4.4 in [2] when  $\mathcal{L}(\infty) = 0$  and Proposition 3.A.1 in Appendix 3.A when  $\mathcal{L}(\infty) > 0$ . Thus, geodesics between any pair of vertices belonging to  $\mathcal{C}_\infty$  exist with probability one.

### 3.1.2 Patterns

For a set  $B$  of vertices, we denote by  $\partial B$  its boundary, this is the set of vertices which are in  $B$  and which are linked by an edge to a vertex which is not in  $B$ . We make an abuse of notation by saying that an edge  $e = \{u, v\}$  belongs to a set of vertices if  $u$  and  $v$  are in this set.

Let  $L_1, \dots, L_d$  be non-negative integers. To avoid trivialities we assume that at least one of them is positive. We fix  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$  and two distinct vertices  $u^\Lambda$  and  $v^\Lambda$  on the boundary of  $\Lambda$ . These points  $u^\Lambda$  and  $v^\Lambda$  are called endpoints. Then we fix an event  $\mathcal{A}^\Lambda$ , with positive probability, only depending on the passage time of the edges of  $\Lambda$ . We say that  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  is a pattern. Let  $x \in \mathbb{Z}^d$ . Define:

- for  $y \in \mathbb{Z}^d$ ,  $\theta_x y = y - x$ ,
- for  $e = \{u, v\}$  an edge connecting two vertices  $u$  and  $v$ ,  $\theta_x e = \{\theta_x u, \theta_x v\}$ .

Similarly, if  $\pi = (x_0, \dots, x_k)$  is a path, we define  $\theta_x \pi = (\theta_x x_0, \dots, \theta_x x_k)$ . Then  $\theta_x T$  denotes the environment  $T$  translated by  $-x$ , i.e. the family of random variables indexed by the edges of  $\mathbb{Z}^d$  defined for all  $e \in \mathcal{E}$  by

$$(\theta_x T)(e) = T(\theta_{-x} e).$$

Let  $\pi$  be a self-avoiding path and  $x \in \mathbb{Z}^d$ . We say that  $x$  satisfies the condition  $(\pi; \mathfrak{P})$  if these two conditions are satisfied:

1.  $\theta_x \pi$  visits  $u^\Lambda$  and  $v^\Lambda$ , and the subpath of  $\theta_x \pi$  between  $u^\Lambda$  and  $v^\Lambda$  is entirely contained in  $\Lambda$ ,
2.  $\theta_x T \in \mathcal{A}^\Lambda$ .

Note that, if  $x$  satisfies the condition  $(\pi; \mathfrak{P})$  when  $\pi$  is a geodesic, then the subpath of  $\theta_x \pi$  between  $u^\Lambda$  and  $v^\Lambda$  is one of the optimal paths from  $u^\Lambda$  to  $v^\Lambda$  entirely contained in  $\Lambda$  in the environment  $\theta_x T$ . When the pattern is given, we also say " $\pi$  takes the pattern in  $\theta_{-x} \Lambda$ " for " $x$  satisfies the condition  $(\pi; \mathfrak{P})$ ". We denote:

$$N^{\mathfrak{P}}(\pi) = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\{x \text{ satisfies the condition } (\pi; \mathfrak{P})\}}. \quad (3.1.4)$$

Note that the number of terms in this sum is actually bounded from above by the number of vertices in  $\pi$ . If  $N^{\mathfrak{P}}(\pi) \geq 1$ , we say that  $\pi$  takes the pattern. The aim of the article is to investigate, under reasonable conditions on  $\mathfrak{P}$ , the behavior of  $N^{\mathfrak{P}}(\gamma)$  for all geodesics  $\gamma$  from 0 to  $x$  with  $\|x\|_1$  large. The first step is to determine these reasonable conditions, that is why we define the notion of valid patterns.

**Definition 3.1.1.** Denote by  $\{\varepsilon_1, \dots, \varepsilon_d\}$  the vectors of the canonical basis. An external normal unit vector associated to a vertex  $z$  of the boundary of  $\Lambda$  is an element  $\alpha$  of the set  $\{\pm \varepsilon_1, \dots, \pm \varepsilon_d\}$  such that  $z + \alpha$  does not belong to  $\Lambda$ .

**Definition 3.1.2.** We say that a pattern is valid if the following three conditions hold:

- $\mathcal{A}^\Lambda$  has a positive probability,
- when  $\mathcal{A}^\Lambda$  occurs, there exists a path between the two endpoints, entirely contained in  $\Lambda$ , whose passage time is finite,
- one of the following two conditions holds:
  - the support of  $\mathcal{L}$  is unbounded,
  - there exist two distinct external normal unit vectors, one associated with  $u^\Lambda$  and one associated with  $v^\Lambda$ .

**Remark 3.1.3.**

- The second condition is always satisfied when  $\mathcal{L}(\infty) = 0$ .
- The existence of the two distinct vectors in the third condition of Definition 3.1.2 is equivalent to the fact that the endpoints of the pattern belong to two different faces. As explained in Remark 1.3 in [13], a real obstruction can appear when the support of  $\mathcal{L}$  is bounded and this third condition is not satisfied.

### 3.1.3 Main result and applications

Here is our main result. We assume that one of the following two conditions is satisfied:

$$\begin{aligned} & \mathcal{L}(\infty) > 0 \text{ and } \mathcal{L}([0, \infty)) > p_c, \\ \text{or } & \mathcal{L}(\infty) = 0 \text{ and the support of } \mathcal{L} \text{ is unbounded.} \end{aligned} \tag{3.1.5}$$

**Theorem 3.1.4.** Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern, assume (3.1.5) and that  $\mathcal{L}$  is useful. Then there exist  $\alpha > 0$ ,  $\beta_1 > 0$  and  $\beta_2 > 0$  such that for all  $x \in \mathbb{Z}^d$ ,

$$\mathbb{P}\left((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1\right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

In [13] we proved the following result.

**Theorem A** (Theorem 1.4 in [13]). Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern, assume that  $\mathcal{L}$  is useful,  $\mathcal{L}(\infty) = 0$  and one of the following two conditions is satisfied:

- (I)  $\mathcal{L}$  has a bounded support,
- (II)  $\mathcal{L}$  has an unbounded support and we have

$$\mathbb{E} \min [T_1^d, \dots, T_{2d}^d] < \infty, \tag{3.1.6}$$

where  $T_1^d, \dots, T_{2d}^d$  are independent with distribution  $\mathcal{L}$ .

Then there exist  $\alpha > 0$ ,  $\beta_1 > 0$  and  $\beta_2 > 0$  such that for all  $x \in \mathbb{Z}^d$ ,

$$\mathbb{P}\left(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1\right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

Combining Theorems 3.1.4 and A we immediately get:

**Theorem 3.1.5.** *Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern, assume that  $\mathcal{L}$  is useful and  $\mathcal{L}([0, \infty)) > p_c$ . Then there exist  $\alpha > 0$ ,  $\beta_1 > 0$  and  $\beta_2 > 0$  such that for all  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P}\left((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1\right) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

Theorem 3.1.5 is a generalization of Theorem 2.3 in [1] (stated below as Theorem B) since, to get this result, we can take the pattern (reduced to one edge)  $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  where  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  and  $\mathcal{A}^\Lambda$  is the event on which the passage time of the only edge of the pattern is greater than  $M$ .

**Theorem B** (Theorem 2.3 in [1]). *Let  $\mathcal{L}$  be a useful distribution on  $[0, +\infty)$  with unbounded support. Then, for each  $M$  positive there exists  $\varepsilon = \varepsilon(M) > 0$  and  $\alpha = \alpha(M) > 0$  so that for all  $x$ , we have*

$$\mathbb{P}\left(\exists \text{ geodesic } \pi \text{ from } 0 \text{ to } x \text{ such that } \sum_{e \in \pi} \mathbf{1}_{T(e) \geq M} \leq \alpha \|x\|_1\right) \leq e^{-\varepsilon \|x\|_1}. \quad (3.1.7)$$

The proof of Theorem 3.1.4 is given in Section 3.2. It is partly inspired by the proof of Theorem 2.3 in [1]. The proofs of Theorems 3.1.4 and A are independent and the only intersection between these two theorems is the case (II) above. Theorem 3.1.5 is an extension of Theorem 1.4 in [13]. We refer to [13] for an account of the history of such results and for applications. As an example of application we prove the following result, which is a generalization of the main result of [18].

**The van den Berg-Kesten comparison principle without any moment assumption.**

Let  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  be two distributions taking values in  $[0, \infty]$  such that:

(H1)  $\mathcal{L}$  is useful,

(H2)  $\mathcal{L}([0, \infty)) > p_c$  and  $\tilde{\mathcal{L}}([0, \infty)) > p_c$ ,

(H3)  $\mathcal{L} \neq \tilde{\mathcal{L}}$ ,

(H4) there exists a couple of random variables  $\tau, \tilde{\tau}$  on some probability space, with marginal distributions  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ , respectively, and satisfying

$$\mathbb{E}[\tilde{\tau} | \tau] \leq \tau. \quad (3.1.8)$$

We consider a family  $T = \{T(e) : e \in \mathcal{E}\}$  of i.i.d. random variables with distribution  $\mathcal{L}$  and another family  $\tilde{T} = \{\tilde{T}(e) : e \in \mathcal{E}\}$  of i.i.d. random variables with distribution  $\tilde{\mathcal{L}}$ .

The geodesic time defined at (3.1.2) is denoted by  $t$  in the environment  $T$  and by  $\tilde{t}$  in the environment  $\tilde{T}$ . With these assumptions, a time constant for each distribution can be defined thanks to [5]. We refer to [5] for an extensive account. Here we recall what we need for our purpose. By  $(\mathcal{H}2)$ , there exists  $M \in \mathbb{R}$  such that

$$\mathcal{L}([0, M]) > p_c \text{ and } \tilde{\mathcal{L}}([0, M]) > p_c. \quad (3.1.9)$$

Fix such a  $M$ . Let  $\mathcal{C}_M$  (resp.  $\tilde{\mathcal{C}}_M$ ) be the infinite cluster for the Bernoulli percolation  $(\mathbb{1}_{\{T(e) \leq M\}}, e \in \mathcal{E})$  (resp.  $(\mathbb{1}_{\{\tilde{T}(e) \leq M\}}, e \in \mathcal{E})$ ) which exists and is unique a.s. To any  $x \in \mathbb{R}^d$ , we associate a random point  $\varphi(x)$  (resp.  $\tilde{\varphi}(x)$ ) in  $\mathcal{C}_M$  (resp. in  $\tilde{\mathcal{C}}_M$ ) such that  $\|x - \varphi(x)\|_1$  (resp.  $\|x - \tilde{\varphi}(x)\|_1$ ) is minimal, with a deterministic rule to break ties.

Theorem 1 in [5] gives the existence of two deterministic functions  $\mu : \mathbb{R}^d \rightarrow [0, \infty)$  and  $\tilde{\mu} : \mathbb{R}^d \rightarrow [0, \infty)$  such that

$$\begin{aligned} \forall x \in \mathbb{Z}^d, \lim_{n \rightarrow \infty} \frac{t(\varphi(0), \varphi(nx))}{n} &= \mu(x) \text{ a.s. and in } L^1, \\ \text{and } \lim_{n \rightarrow \infty} \frac{\tilde{t}(\tilde{\varphi}(0), \tilde{\varphi}(nx))}{n} &= \tilde{\mu}(x) \text{ a.s. and in } L^1. \end{aligned} \quad (3.1.10)$$

Theorem 4 in [5] ensures that the functions  $\mu$  and  $\tilde{\mu}$  do not depend on the choice of the constant  $M$  satisfying (3.1.9). Furthermore, when

$$\mathbb{E} \min[\tau_1, \dots, \tau_{2d}] < \infty, \quad (3.1.11)$$

where  $\tau_1, \dots, \tau_{2d}$  are i.i.d. copies of  $\tau$ , Theorem 4 in [5] also ensures that for all  $x \in \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} \frac{t(0, \lfloor nx \rfloor)}{n} = \mu(x) \text{ a.s. and in } L^1. \quad (3.1.12)$$

This is the usual definition of the time constant. We refer to Theorem 2.18 in [14] and Section 2.1 in [2] for more details on the result (3.1.12). The same holds for the environment  $\tilde{T}$  if (3.1.11) holds for  $2d$  i.i.d. copies of  $\tilde{\tau}$ .

**Remark 3.1.6.** *We warn the reader that notations  $\tilde{T}$  and  $\tilde{\mu}$  are used in [5] with a different meaning. We refer in particular to Remark 2 in [5] for explanations.*

We can now state the van den Berg-Kesten comparison principle for these time constants.

**Theorem 3.1.7** (Extension of the van den Berg-Kesten comparison principle). *Let  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  be two distributions taking values in  $[0, \infty]$  satisfying  $(\mathcal{H}1)$ ,  $(\mathcal{H}2)$ ,  $(\mathcal{H}3)$  and  $(\mathcal{H}4)$ . For all  $x \in \mathbb{Z}^d$  such that  $x \neq 0$ ,*

$$\tilde{\mu}(x) < \mu(x). \quad (3.1.13)$$

The proof of Theorem 3.1.7 is given in Section 3.3. In [18], van den Berg and Kesten prove the following theorem.

**Theorem C** (Theorem 2.9 in [18]). *Let  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  be two distributions taking values in  $[0, \infty)$ , having a finite first moment, satisfying<sup>1</sup>  $(\mathcal{H}1)$ ,  $(\mathcal{H}3)$  and  $(\mathcal{H}4)$ . Then,*

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

Theorem 3.1.7 is an extension of Theorem C. With Theorem 1.2 in [16], Marchand extends Theorem C in another direction.

**Theorem D** (Theorem 1.2 in [16]). *Assume that  $d = 2$  and let  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  be two distributions taking values in  $[0, \infty)$ , such that  $\mathcal{L}(0) < p_c$ , satisfying<sup>2</sup>  $(\mathcal{H}3)$  and  $(\mathcal{H}4)$ . Then,*

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

In dimension 2, the result of Marchand is stronger than Theorem C on two aspects: on the one hand, there is no moment assumption and on the other hand, the condition

$$\mathcal{L}(t_{\min}) < \vec{p}_c \text{ when } t_{\min} > 0 \tag{3.1.14}$$

is removed. When  $t_{\min} > 0$  and  $\mathcal{L}(t_{\min}) > \vec{p}_c$ , the problem involves oriented percolation, where the open edges correspond to those with the smallest time values. In this context, the largest part of each geodesic linking the origin to a distant point within the cone of percolation is a directed path made of minimal edges, highlighting a distinct behavior.

For a point  $x$  inside this cone,  $\mu(x) = t_{\min}\|x\|_1$ . Moreover, when  $\tilde{t}_{\min} = t_{\min}$  and the  $(\mathcal{H}2)$  condition is met,  $\tilde{\mu}(x) = t_{\min}\|x\|_1$  as well. Notably, (3.1.13) does not apply to such  $x$  values. Establishing (3.1.13) for  $x$  outside the cone, like  $\varepsilon_1$ , requires specific arguments, in particular large deviations for supercritical oriented percolation. We have opted not to explore this case in this article. We refer to [16], and more specifically to Theorem 1.5 in [16], for further explanations.

### 3.1.4 Sketch of the proof

In this section, we give an informal sketch of the proof of Theorem 3.1.4. Fix a pattern  $\mathfrak{P}$  and  $x \in \mathbb{Z}^d$  with  $\|x\|$  large. Consider the event:

$$\mathcal{M} = \{(0, x) \in \mathfrak{C} \text{ and there exists a geodesic from } 0 \text{ to } x \text{ which does not take the pattern}\}.$$

The aim is to prove that  $\mathcal{M}$  has a probability small enough in  $\|x\|$ . More precisely, we want to prove

$$\mathbb{P}(\mathcal{M}) \ll \frac{1}{\|x\|^{d-1}}. \tag{3.1.15}$$

---

<sup>1</sup>It is stated in [18] with the definition of a distribution more variable than another, but Theorem 2.6 in [18] ensures that the fact that  $\tilde{\mathcal{L}}$  is more variable than  $\mathcal{L}$  is equivalent to  $(\mathcal{H}4)$  when  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  have a finite first moment.

<sup>2</sup>It is also stated in [16] with the definition of a distribution more variable than another, but Lemma 6.1 in [16] ensures that this definition is also equivalent to  $(\mathcal{H}4)$  when  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  takes value in  $[0, \infty)$ .



From this result, by a standard re-normalization argument, we easily get Theorem 3.1.4 (see Proposition 3.1.8 in Section 3.1.5 for a formal statement of (3.1.15)).

**General idea.** As in [13], to get (3.1.15), the idea is to define a suitable event  $\mathcal{G}$  and a suitable sequence of events  $\mathcal{M}(\ell)$  for  $0 \leq \ell \leq q$  such that, for some positive constant  $c < 1$ ,

1.  $q \geq c\|x\|$ ,

2.  $\mathcal{M} \subset \mathcal{M}(q) \cup \mathcal{G}^c$  where  $\mathbb{P}(\mathcal{G}^c) \ll \frac{1}{\|x\|^{d-1}}$ ,

3. for all  $\ell \geq 1$ ,

$$\mathbb{P}(\mathcal{M}(\ell)) \leq c\mathbb{P}(\mathcal{M}(\ell - 1)). \quad (3.1.16)$$

If the above holds, we get  $\mathbb{P}(\mathcal{M}) \leq c^{\|x\|_1} + \mathbb{P}(\mathcal{G}^c)$ , which allows us to conclude.

**Penalized geodesics.** We now introduce the notion of penalized path. This is an idea which comes from the article [1] by Andjel and Vares in their proof of Theorem 3.1.7. A penalized path is a path which does not take the pattern. In other words, this is a path  $\pi$  such that  $N^{\mathfrak{P}}(\pi) = 0$ . This allows us to define the penalized passage time for every  $z \in \mathbb{Z}^d$ :

$$t_P(0, z) = \inf\{T(\pi) : \pi \text{ is a penalized path from } 0 \text{ to } z\},$$

with the convention  $\inf \emptyset = \infty$ . Then, for every  $z \in \mathbb{Z}^d$ , if it exists, a penalized geodesic from 0 to  $z$  is a penalized path  $\gamma$  from 0 to  $z$  such that  $T(\gamma) = t_P(0, z)$ . With these definitions, we have

$$\mathcal{M} \subset \{(0, x) \in \mathfrak{C} \text{ and } t_P(0, x) = t(0, x)\}. \quad (3.1.17)$$

**Shortcuts.** A good way to get that the event  $\{(0, x) \in \mathfrak{C} \text{ and } t_P(0, x) = t(0, x)\}$  does not occur is to prove that a penalized geodesic has a shortcut. The formal definition of a shortcut is given in Section 3.2.1.4. Informally, a shortcut for a penalized geodesic  $\gamma$  is a path going from a vertex  $u$  of  $\gamma$  to another vertex  $v$  of  $\gamma$  which takes the pattern and which has a passage time lower than the passage time of the subpath of  $\gamma$  going from  $u$  to  $v$ . Hence, if a penalized geodesic  $\gamma$  from 0 to  $x$  has a shortcut, it implies that there exists a path from 0 to  $x$  which is not penalized and such that its passage time is strictly lower than the passage time of  $\gamma$ . It gives  $t(0, x) < t_P(0, x)$ .

**Events  $\mathcal{G}$  and  $\mathcal{M}(\ell)$ .** A successful box for a path  $\pi$  is a box satisfying one of the following two conditions:

- it is a typical box,
- the path  $\pi$  has a shortcut taking the pattern inside the box.

We say that a box is shortcut-equipped for a path  $\pi$  or not shortcut-equipped for  $\pi$  depending on whether the second condition is satisfied or not. We define  $\mathcal{G}$  as the event on which  $(0, x) \in \mathfrak{C}$  and there exists a penalized geodesic  $\pi$  whose passage time is finite and which crosses at least  $q$  successful boxes for  $\pi$ . On this event, we define the selected penalized geodesic denoted by  $\gamma$ : it is the first (for an arbitrary deterministic order) of the penalized geodesics satisfying the condition which appears in the definition of  $\mathcal{G}$ . We define the sequence of successful boxes crossed by  $\gamma$  as the sequence of the first  $q$  successful boxes crossed by  $\gamma$  indexed in the order in which they are crossed by  $\gamma$ . It allows us to define, for every  $\ell \in \{1, \dots, q\}$  the event

$$\mathcal{M}(\ell) = \mathcal{G} \cap \{\text{the } \ell \text{ first successful boxes of the sequence of } \gamma \text{ are not shortcut-equipped for } \gamma\}.$$

This gives us  $\ell$  opportunities to modify the environment in each of these  $\ell$  typical boxes to create a shortcut for  $\gamma$ . The aim is now reduced to proving (3.1.16).

**$k$ -boxes.** We describe a small change of the plan above. This change does not create any complications. In particular, the entire plan describe above works with these new objects. The advantage is to avoid a number of complications, such as, for example, those related to what happens at the boundary of a box when we modify the environment in it.

The idea is to only consider a family of boxes (called the  $k$ -boxes, see Section 3.2.1.4) that partitions  $\mathbb{Z}^d$ . We also replace "penalized paths" -the paths which does not take the pattern- by " $k$ -penalized paths" -the paths which does not take any pattern contained in a  $k$ -box-. We similarly replace "penalized geodesics" by " $k$ -penalized geodesics" : the geodesics which does not take any pattern contained in a  $k$ -box. We say " $k$ -geodesic" instead of " $k$ -penalized geodesic" for short.

**Modification and stability.** For all  $\ell$ , we have  $\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1)$ . Thus (3.1.16) is equivalent to the existence of a constant  $\eta > 0$  (by taking  $\eta = \frac{1}{c} - 1$ ) such that

$$\mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) \geq \eta \mathbb{P}(\mathcal{M}(\ell)). \quad (3.1.18)$$

Fix  $\ell \in \{1, \dots, q\}$  and denote by  $B_s$  the  $\ell$ -th successful  $k$ -box crossed by  $\gamma$ . The aim is to prove (3.1.18). The idea is to resample the passage times of edges of  $B_s$  in an environment in which  $\mathcal{M}(\ell)$  occurs to get a new environment in which  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  occurs. When the resampled passage times satisfy good conditions (to be determined), the following properties are satisfied:

1. The event  $\mathcal{G}$  still occurs and the selected  $k$ -geodesic is still  $\gamma$  in the new environment.
2. The box  $B_s$  is shortcut-equipped for  $\gamma$  in the new environment.
3. The sequence of successful boxes crossed by  $\gamma$  is the same in the two environments.
4. The event  $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$  occurs in the new environment.

By the fourth property, we get roughly

$$\mathbb{P}(\mathcal{M}(\ell))\mathbb{P}(\text{good conditions on the resampled passage times}) \leq \mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)).$$

Since  $\eta$  is fixed according to the probability of the good conditions on the resampled passage times, which is positively bounded from below independently of the box, we get (3.1.18). See the proof of Lemma 3.2.12 using Lemma 3.2.13 in Section 3.2.2.1.

The third property follows from the first two. Indeed, the box  $B_s$  is typical in the first environment (since it is a successful box for  $\gamma$  and it is not shortcut-equipped for  $\gamma$  as  $\mathcal{M}(\ell)$  holds) and is shortcut-equipped for  $\gamma$  in the new environment. Furthermore, the other boxes have the same status (successful or not for  $\gamma$ ) in the two environments since the passage times of the edges of the other boxes have not been modified.

The fourth property follows from the first three by similar ideas. We thus see that, in order to get the fourth property, we do not only need to get the second one. We also need the first and the third ones. We call these two additional properties "stability properties". The proof is thus reduced to getting the first two properties.

**Some more details.** Recall that we assume (3.1.5). There are two cases to be considered differently:

(INF)  $\mathcal{L}(\infty) > 0$ ,

(FU)  $\mathcal{L}(\infty) = 0$  and the support of  $\mathcal{L}$  is unbounded.

(INF) stands for "infinite" and (FU) stands for "finite unbounded".

In what follows, when we say "after the modification" or "in the new environment", we mean "in the new environment where passage times of the edges in the box  $B_s$  have been resampled and on the event where the resampled passage times satisfy some good properties that we do not explicit here". In this paragraph, we focus on the first property of the previous paragraph. To get it, it is sufficient to prove the following properties:

- (i) The path  $\gamma$  still has a finite passage time in the new environment. There are no difficulties with this property.
- (ii) The path  $\gamma$  remains a  $k$ -penalized path in the new environment. Since the passage times of the  $k$ -boxes different from  $B_s$  have not been modified, it is sufficient to prove that  $\gamma$  does not take the pattern in  $B_s$  in the new environment. This is based on the two following ideas.
  - We identify *forbidden zones* which are subsets of  $B_s$  where  $\gamma$  can not go. In the case (INF) the forbidden zones are simply balls whose edges have infinite passage time. In the case (FU) we refer to Lemma 3.2.5. By definition, a typical box possesses many forbidden zones (see the third item of the definition of a typical box in Section 3.2.1.2).

- We make sure that, after a successful modification, there is a *unique* pattern inside  $B_s$ . The uniqueness is ensured by replacing the original pattern by a new larger pattern, containing the original one, and by requiring that the behavior of passage times in the boundary of the new pattern is very atypical (see Lemma 3.2.4 and in particular its last item). In the case (FU) we just require that the passage times on the boundary of the new pattern are very high and contained in a special interval (see Remark 3.2.3 and (AF-4')). These will be the unique edges with passage times in this interval after the modification. In the case (INF) we require the existence of a large connected component of edges with finite passage time (see Definition 3.2.1 and (AI-4)). This will be the unique such large component not touching the boundary of  $B_s$  after the modification.

We place the pattern in a forbidden zone. By this we mean that, after the modification, the pattern lies in what was a forbidden zone before the modification. Recall that the pattern is unique in  $B_s$  and that  $\gamma$  does not enter into forbidden zones. Therefore  $\gamma$  does not take the pattern in  $B_s$  after the modification.

- (iii) A  $k$ -penalized path  $\pi$  with finite passage time in the new environment is also a  $k$ -penalized path in the initial environment. Once again, it is sufficient to prove that  $\pi$  does not take a pattern entirely contained in  $B_s$  in the initial environment. The proof differs between the case (INF) and the case (FU).
  - In the case (FU), this is a consequence of the fact that  $B_s$  is a typical box and that there is no pattern in a typical box. Indeed, the passage times on the boundary of the pattern are bigger (in the case (FU)) than they can be in a typical box.
  - In the case (INF), it comes from the fact that, in the new environment, a path with a finite passage time taking edges in  $B_s$  is very constrained (see Figure 3.4 where a path with a finite passage time can only take edges of the green, red, blue and orange parts). If  $\pi$  does not take the pattern in the new environment, it can only take edges of  $\gamma$ . Since  $\gamma$  does not take a pattern entirely contained in  $B_s$  in the initial environment, neither does  $\pi$ .
- (iv) A  $k$ -penalized path in the new environment has a passage time greater than or equal to the passage time of  $\gamma$  in the new environment. The proof comes from the fact that a path with a reasonable passage time in the new environment is very constrained in  $B_s$ . The edges that do not belong to  $\gamma$ , the shortcut for  $\gamma$  or the unique pattern in  $B_s$  have a prohibitive passage time in the case (FU) and an infinite passage time in the case (INF). Based on this observation and on the fact that  $\gamma$  has a lower passage time in the new environment than in the initial environment, we simply prove that a  $k$ -penalized path can not save more time than  $\gamma$  during the modification.
- (v) With the same ideas as above, we also prove that a  $k$ -geodesic in the new environment is also a  $k$ -geodesic in the initial environment.

Indeed, by (i), (ii) and (iv) we get that  $\gamma$  is a  $k$ -geodesic with a finite passage time in the new environment. Using the same arguments as before, we get that  $\gamma$  crosses at least  $q$  successful  $k$ -boxes in the new environment. Hence, the event  $\mathcal{G}$  occurs in the new environment. Furthermore, we also get that every  $k$ -geodesic crossing at least  $q$  successful  $k$ -boxes in the new environment crosses at least  $q$  successful  $k$ -boxes in the initial environment. Adding (iii) and (v), we get that the set of potential selected  $k$ -geodesics in the new environment is contained in the set of potential selected  $k$  geodesic in the initial environment, and then  $\gamma$  remains the first geodesic (and thus the selected one) among the geodesics of this set.

**Advantages of a strategy using penalized geodesics.** In [13] the proof does not rely on penalized geodesics. Using penalized geodesics has two main advantages:

- In [13], proving the result for all geodesics (and not only for one selected geodesic) requires further technicalities (see the use of concentric annuli in Section 2.1 in [13]). Here, it comes for free from the fact that the existence of one  $k$ -geodesic having a shortcut implies that  $t(0, x) < t_P(0, x)$  and thus that every geodesic from 0 to  $x$  takes the pattern.
- It allows us to remove Assumption (3.1.6). Indeed, in item 3 in the paragraph on the modification and stability above, we need to have the same sequence of successful boxes crossed by  $\gamma$  in the two environments. Assume that we do not use penalized geodesics and, to make things easier, assume (only in this item) that we are in the case where there is a unique geodesic between any couple of vertices. Then the modification consists in replacing a subpath (denoted by  $\bar{\gamma}$ ) of the geodesic from 0 to  $x$  (denoted by  $\gamma$ ) by a path (denoted by  $\pi$ ) with a shorter passage time which takes the pattern. It implies that in the new environment,  $\bar{\gamma}$  does not belong to the new geodesic (which is the concatenation of the part of  $\gamma$  from 0 to  $\pi$ , then  $\pi$  and then the part of  $\gamma$  from  $\pi$  to  $x$ ). It can create a problem of stability if a box of the sequence of successful boxes crossed by  $\gamma$  was crossed by  $\bar{\gamma}$ : the sequence of successful boxes crossed by the geodesic from 0 to  $x$  would not be the same in the two environments. To avoid this problem when we do not use penalized geodesics, we use the Cox-Durrett shape theorem (Theorem 2.16 in [2]) in order to control the length of geodesics excursions from a box (see for example the proof of Lemma 2.1 in [13]). This is why we need Assumption (3.1.6) in the strategy developed in [13].

Hence, by making a modification which guarantees that the penalized geodesic is the same in the initial environment as in the modified one, we avoid this problem without using the Cox-Durrett shape theorem and thus without requiring Assumption (3.1.6).

### 3.1.5 Organization of the proof of Theorem 3.1.4

Recall that, in this article, we assume (3.1.5). One can check, using a standard re-normalization argument, that Theorem 3.1.4 is a simple consequence of the following proposition (see for example the proof of Theorem 2.3 in [1]).

**Proposition 3.1.8.** *Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern. Assume (3.1.5) and that  $\mathcal{L}$  is useful. Then there exist  $C > 0$  and  $D > 0$  such that for all  $n \geq 0$ , for all  $x$  such that  $\|x\|_1 = n$ ,*

$$\mathbb{P}\left((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0\right) \leq De^{-Cn}. \quad (3.1.19)$$

Thus, the aim is now to prove Proposition 3.1.8. Recall that there are two cases to be considered differently:

(INF)  $\mathcal{L}(\infty) > 0$ ,

(FU)  $\mathcal{L}(\infty) = 0$  and the support of  $\mathcal{L}$  is unbounded.

The proof of Proposition 3.1.8 is the aim of Section 3.2. This section is divided in two parts.

Section 3.2.1 is devoted to patterns and typical boxes. We replace the original pattern by a larger pattern containing the original one and which satisfies several assumptions. Some of the assumptions are simply convenient: they simplify some parts of the proof. The assumption on the boundary is more crucial as explained in item (ii) in the paragraph "Some more details" in Section 3.1.4. In the case (FU), the requirement on the passage times on the boundary depends on the size of the boxes we consider in the proof. But the size of the boxes depends on the notion of typical boxes which in turn depends on parts of the definition of the pattern we consider. The definitions are thus intertwined. This is why we first start defining the new pattern in Section 3.2.1.1 (postponing the boundary conditions in the case (FU)), we then define and study typical boxes in Section 3.2.1.2 and we finally choose the boundary of the pattern in the case (FU) in Section 3.2.1.3. We then introduce the notions of  $k$ -penalized paths, shortcuts and successful boxes in Section 3.2.1.4.

The second part of Section 3.2 is divided in four parts. In Section 3.2.2.1, the proof of Proposition 3.1.8 is reduced to the proof of a key lemma : Lemma 3.2.13. In this lemma, we introduce some sets of edges which correspond to the edges whose passage times have to be modified. The exact definitions of these sets are postponed to Section 3.2.2.2. It corresponds to the modification we want to make. The more difficult part in the proof of Lemma 3.2.13 is item (iii), which is the key to get (3.1.18). To get this item and also item (i), we state and prove several properties which are consequences of the modification in Section 3.2.2.3 before using them to conclude in Section 3.2.2.4.

### 3.1.6 Some tools and notations

In this subsection, we recall some results and fix some notations. First, we denote by  $\mathbb{N}$  the set of all non-negative integers, by  $\mathbb{N}^*$  the set  $\mathbb{N} \setminus \{0\}$ , and by  $\mathbb{R}_+$  the set of all  $x \in \mathbb{R}$  such that  $x \geq 0$ .

For a self-avoiding<sup>3</sup> path  $\pi = (x_0, \dots, x_k)$  going from  $x_0$  to  $x_k$ , we say that  $x_i$  is visited

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<sup>3</sup>The definition can be extended to not necessarily self-avoiding paths by saying that a vertex  $x$  is visited by  $\pi$  before  $y$  if there exists  $i_0 \in \{0, \dots, k\}$  such that  $x_{i_0} = x$  and for all  $j \in \{0, \dots, k\}$ ,  $x_j = y$  implies that  $j > i_0$ .

by  $\pi$  before  $x_j$  if  $i < j$ ; we say that an edge  $\{x_i, x_{i+1}\}$  is visited before an edge  $\{x_j, x_{j+1}\}$  if  $i < j$ . A subpath of  $\pi$  going from  $x_i$  to  $x_j$  (where  $i, j \in \{0, \dots, k\}$  and  $i < j$ ) is the path  $(x_i, \dots, x_j)$  and is denoted by  $\pi_{x_i, x_j}$ .

We say that a path, whose endpoints are denoted by  $u$  and  $v$ , is oriented if this path has exactly  $\|u - v\|_1$  edges. In other words, its number of edges is minimal among those of all the other paths linking  $u$  and  $v$ .

For a set  $B$  of vertices, we denote by  $\partial B$  its boundary, this is the set of vertices of  $B$  which can be linked by an edge to a vertex which is not in  $B$ . We denote by  $B^c$  the set of all vertices which does not belong to  $B$ . When we define a set of vertices of  $\mathbb{Z}^d$ , sometimes we also want to say that an edge belongs to this set. So we make an abuse of notation by saying that an edge  $e = \{u, v\}$  belongs to a set of vertices if  $u$  and  $v$  are in this set. Since now a subset  $B$  of  $\mathbb{Z}^d$  can be seen as a set of vertices or as a set of edges, we denote by  $|B|_v$  the number of vertices of  $B$  and by  $|B|_e$  its number of edges.

Then, for all  $c \in \mathbb{Z}^d$  and  $r \in \mathbb{R}_+$ , we denote

$$\begin{aligned} B_\infty(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_\infty \leq r\}, \\ B_1(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_1 \leq r\}, \end{aligned}$$

and for  $n \in \mathbb{N}^*$ , we denote by  $\Gamma_n$  the boundary of  $B_1(0, n)$ , i.e.

$$\Gamma_n = \{u \in \mathbb{Z}^d : \|u\|_1 = n\}. \quad (3.1.20)$$

**Constants related to the distribution.** One can check that Lemma 5.5 in [18] can be adapted for a useful distribution  $\mathcal{L}$  such that  $\mathcal{L}(\infty) > 0$  and  $\mathcal{L}([0, \infty)) > p_c$ . Thus there exist  $\delta = \delta(\mathcal{L}) > 0$  and  $D_0 = D_0(\mathcal{L})$  fixed for the remaining of the article such that for all  $u, v \in \mathbb{Z}^d$ ,

$$\mathbb{P}(\text{there exists a path } \pi \text{ from } u \text{ to } v \text{ such that } T(\pi) \leq (t_{\min} + \delta)\|u - v\|_1) \leq e^{-D_0\|u - v\|_1}. \quad (3.1.21)$$

Furthermore, when  $t_{\min} = 0$ ,

- even if it means reducing  $\delta$ , in the cases (INF) and (FU), we assume that  $\delta > 0$  is such that  $\mathcal{L}([0, \delta]) < p_c$ ,
- even if it means reducing  $\delta$ , in the case (INF), we can fix  $\nu_0$  such that

$$\nu_0 > \delta \text{ and } \mathcal{L}((\delta, \nu_0)) > 0. \quad (3.1.22)$$

Note that it is possible since in the case (INF), we have  $\mathcal{L}(0) + \mathcal{L}(\infty) < 1$ . Indeed, it comes from the fact that  $\mathcal{L}$  is useful and that  $\mathcal{L}([0, \infty)) > p_c$ .

Then, still in the case where  $t_{\min} = 0$ , we fix

$$\beta > 0, \beta' > 0 \text{ and } \rho > 0 \quad (3.1.23)$$

such that (3.1.24) below holds with  $\tau = \delta$ . The existence of such constants is guaranteed by Lemma 3.1.9 below whose proof is given in Appendix 3.B.

**Lemma 3.1.9.** *Assume that  $\mathcal{L}$  is useful and that  $t_{\min} = 0$ . Let  $\tau > 0$  such that  $\mathcal{L}([0, \tau]) < p_c$ . Then there exists  $\beta > 0$ ,  $\beta' > 0$  and  $\rho > 0$  such that for all  $v, w \in \mathbb{Z}^d$ ,*

$$\mathbb{P}(\text{there exists a self-avoiding path from } v \text{ to } w \text{ taking at most } \rho \|v - w\|_1 \text{ edges } e \text{ such that } T(e) > \tau) \leq \beta' e^{-\beta \|v - w\|_1}. \quad (3.1.24)$$

## 3.2 Proof of Proposition 3.1.8

Let  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  be a valid pattern. We assume (3.1.5) and that  $\mathcal{L}$  is useful. Thus one of the cases (INF) or (FU) stated in Section 3.1.5 is realized. The proofs in these two cases are almost the same. However, throughout this section, it will sometimes be necessary to distinguish the cases.

### 3.2.1 Settings for the proof

#### 3.2.1.1 Assumptions on the patterns

We begin by making some assumptions on  $\mathfrak{P}$  for the remaining of the proof. At first sight, these assumptions can be seen as a restriction but Lemma 3.2.4 guarantees that we can make them with no loss of generality.

**In the case (INF).**

**Definition 3.2.1** (Boundary condition). *For every  $s \in \mathbb{Z}^d$  and  $\ell^\Lambda \geq 3$ , define the set  $\mathcal{S}_{s, \ell^\Lambda}$  as the set of edges belonging to the path going from  $s - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  to  $s + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  in the shortest way by  $2(\ell^\Lambda - 1)$  steps in the direction  $\varepsilon_1$ .*

*Then, we say that  $B_\infty(s, \ell^\Lambda)$  satisfies the boundary condition in the environment  $T$  if for all edges  $e$  belonging to  $B_\infty(s, \ell^\Lambda)$  but not to  $B_\infty(s, \ell^\Lambda - 3)$ ,*

- *either  $e$  belongs to  $\mathcal{S}_{s, \ell^\Lambda} \cup (s + \mathbb{Z}\varepsilon_1)$  and  $T(e)$  is finite,*
- *or  $T(e)$  is infinite.*

**Remark 3.2.2.** *Let  $s \in \mathbb{Z}^d$  and  $\ell^\Lambda \geq 3$ . If  $B_\infty(s, \ell^\Lambda)$  satisfies the boundary condition in the environment  $T$ , then there is no path from  $\partial B_\infty(s, \ell^\Lambda)$  to  $\partial B_\infty(s, \ell^\Lambda)$  with finite passage time which takes an edge of  $\mathcal{S}_{s, \ell^\Lambda}$  (see Figure 3.1 for a representation of the boundary condition in two dimensions).*

Let us consider the following assumptions:



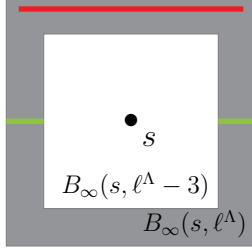


Figure 3.1: Example of a ball satisfying the boundary condition. The set  $\mathcal{S}_{s, \ell^\Lambda}$  is represented in red. The edges of the gray area have an infinite passage time and the edges in green and red have a finite passage time.

(AI-1) there exists an integer  $\ell^\Lambda \geq 3$ , fixed for the remaining of the proof, such that<sup>4</sup>  $\Lambda = B_\infty(0, \ell^\Lambda)$ ,

(AI-2)  $u^\Lambda = -\ell^\Lambda \varepsilon_1$  and  $v^\Lambda = \ell^\Lambda \varepsilon_1$ ,

(AI-3) there exist a constant  $T^\Lambda > 0$  and a path  $\pi_\infty$  from  $u^\Lambda$  to  $v^\Lambda$  entirely contained in  $\Lambda$  such that, when  $\mathcal{A}^\Lambda$  occurs,

$$T(\pi_\infty) < T^\Lambda. \quad (3.2.1)$$

(AI-4) if  $\mathcal{A}^\Lambda$  occurs,  $\Lambda$  satisfies the boundary condition

**In the case (FU).** Let us consider the following assumptions:

(AF-1) there exists an integer  $\ell^\Lambda > 0$ , fixed for the remaining of the proof, such that  $\Lambda = B_\infty(0, \ell^\Lambda)$ ,

(AF-2)  $u^\Lambda = -\ell^\Lambda \varepsilon_1$  and  $v^\Lambda = \ell^\Lambda \varepsilon_1$ ,

(AF-3) when  $\mathcal{A}^\Lambda$  occurs, there exists a constant  $M^\Lambda$  such that for every edge  $e$  belonging to  $\Lambda$  but not to  $\partial\Lambda$ ,  $T(e) \leq M^\Lambda$ ,

(AF-4) for all  $M > 0$ , the event  $\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) > M\}$  has a positive probability.

**Remark 3.2.3.** *The aim of wanting the pattern to satisfy the condition (AF-4) above is to be able to choose the passage times of the edges on its boundary once we have fixed some constants. Thus, from Section 3.2.1.3 onwards the condition (AF-4) is replaced by the condition (AF-4') stated at this point.*

<sup>4</sup>We make a very slight abuse of notation: we also consider patterns where 0 is in the center of  $\Lambda$ .

In the two cases.

**Lemma 3.2.4.** *Let  $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$  be a valid pattern. There exists a pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  such that:*

- $\Lambda_0 \subset \Lambda$ ,
- $\mathbb{P}(\mathcal{A}^\Lambda)$  is positive,
- on  $\mathcal{A}^\Lambda$ , any path from  $u^\Lambda$  to  $v^\Lambda$  optimal for the passage time among the paths entirely inside  $\Lambda$  contains a subpath from  $u_0^\Lambda$  to  $v_0^\Lambda$  entirely inside  $\Lambda_0$ ,
- $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$ ,
- in the case (INF),  $\mathfrak{P}$  satisfies the conditions (AI-1), (AI-2), (AI-3) and (AI-4), and in the case (FU),  $\mathfrak{P}$  satisfies the conditions (AF-1), (AF-2), (AF-3) and (AF-4).

Consider a valid pattern  $\mathfrak{P}_0$  and a pattern  $\mathfrak{P}$  satisfying the conditions of Lemma 3.2.4 above. Then, by this lemma, for every path  $\pi$ , if a vertex  $x$  satisfies the condition  $(\pi; \mathfrak{P})$ ,  $x$  satisfies the condition  $(\pi; \mathfrak{P}_0)$ . Thus we get  $N^{\mathfrak{P}_0}(\pi) \geq N^{\mathfrak{P}}(\pi)$  and to prove Proposition 3.1.8 for the pattern  $\mathfrak{P}_0$ , it is sufficient to prove it for the pattern  $\mathfrak{P}$ . That is why from now on, we can assume that the pattern  $\mathfrak{P}$  introduced at the beginning of Section 3.2 satisfies the conditions (AI-1), (AI-2), (AI-3), (AI-4) in the case (INF) and the conditions (AF-1), (AF-2), (AF-3) and (AF-4) in the case (FU). The proof of Lemma 3.2.4 is postponed in Appendix 3.C.

For the remaining of the proof, fix  $\ell^\Lambda$  given by (AI-1) and (AF-1). In the case (INF), fix  $T^\Lambda$  given by (AI-3) and in the case (FU), fix

$$M^\Lambda \text{ satisfying (AF-3) and } T^\Lambda > |\Lambda|_e M^\Lambda. \quad (3.2.2)$$

### 3.2.1.2 Typical boxes

Recall that  $\delta$  is fixed at (3.1.21) and that, when  $t_{\min} = 0$ ,  $\nu_0$  is fixed at (3.1.22).

**Technical lemma.** We state in this paragraph the lemma used to create a "forbidden zone" in the case (FU) (see item (ii) of the seventh paragraph of Section 3.1.4).

Fix  $r_P$  an integer such that

$$r_P > \max \left( \ell^\Lambda + 2, \frac{2(|B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + 1) + T^\Lambda)}{\delta}, \frac{|B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + 1) + T^\Lambda + 1}{2t_{\min}} \right). \quad (3.2.3)$$

**Lemma 3.2.5.** *In the case (FU), we can define an event  $\mathcal{T}$ , whose probability is positive, only depending on the edges of  $B_\infty(0, r_P)$  and such that for all  $x, y \in \partial B_\infty(0, r_P)$ , for every self-avoiding path  $\pi$  going from  $x$  to  $y$  and using only edges in  $\partial B_\infty(0, r_P)$  and every path  $\tilde{\pi}$*

going from  $x$  to  $y$  using edges of  $B_\infty(0, r_P)$  and at least one edge which is not in  $\partial B_\infty(0, r_P)$ , we have

$$T(\pi) < T(\tilde{\pi}).$$

*Proof.* Fix

$$\nu > |\partial B_\infty(0, r_P)|_e(t_{\min} + 1).$$

Define the event  $\mathcal{T}$  as the event on which for all edge  $e \in B_\infty(0, r_P)$ ,

- $T(e) < t_{\min} + 1$  if  $e \in \partial B_\infty(0, r_P)$ ,
- $T(e) > \nu$  else.

Since the support of  $F$  is unbounded, the event  $\mathcal{T}$  has a positive probability. Assume that  $\mathcal{T}$  occurs. Then, let  $x, y \in \partial B_\infty(0, r_P)$ . Let  $\pi$  be a self-avoiding path going from  $x$  to  $y$  and using only edges in  $\partial B_\infty(0, r_P)$  and  $\tilde{\pi}$  be a path going from  $x$  to  $y$  using edges of  $B_\infty(0, r_P)$  and at least one edge  $e'$  which is not in  $\partial B_\infty(0, r_P)$ . We get

$$T(\pi) \leq |\partial B_\infty(0, r_P)|_e(t_{\min} + 1),$$

and

$$T(\tilde{\pi}) \geq T(e') \geq \nu > |\partial B_\infty(0, r_P)|_e(t_{\min} + 1).$$

Hence,  $T(\pi) < T(\tilde{\pi})$ . □

**Boxes.** Recall that, when  $t_{\min} = 0$ ,  $\rho$  is fixed at (3.1.23). Fix

$$\begin{aligned} & r_1 = 1, r_2 > 4dr_P \text{ and} \\ & \text{when } t_{\min} > 0, r_3 > 2d(r_2 + 1) \\ & \text{when } t_{\min} = 0, r_3 > \max\left(2d(r_2 + 1), \frac{2\nu_0}{\rho\delta}\right). \end{aligned} \tag{3.2.4}$$

Note that, in particular, we have the following inequalities:

- $r_2 > 4r_1 > 2r_1$  since  $r_1 = 1$  and  $r_P \geq 1$ ,
- $r_3 > 2r_2$  and  $r_3 > r_2 + 1$ .

Then, for all  $N \geq 1$ , we define

$$B_{3,s,N} = \{v \in \mathbb{Z}^d : (s - r_3)N \leq z < (s + r_3)N\}.$$

and for  $i \in \{1, 2\}$ , we define

$$B_{i,s,N} = \{v \in \mathbb{Z}^d : (s - r_i)N \leq z \leq (s + r_i)N\}.$$

We use the word "box" to talk about  $B_{3,s,N}$ . For  $i \in \{1, 2, 3\}$ ,  $\partial B_{i,s,N}$  is the set of vertices of  $B_{i,s,N}$  having an adjacent vertex not contained in  $B_{i,s,N}$ .

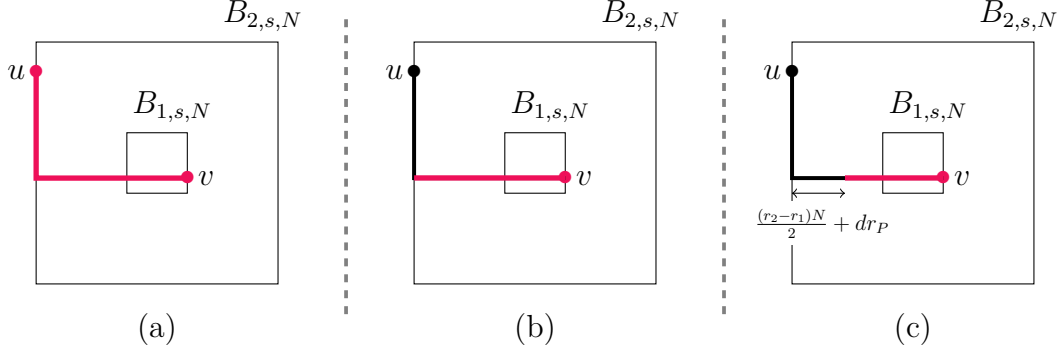


Figure 3.2: Representation of the objects defined in Definition 3.2.6 in two dimensions. (a) The directed path  $\vec{\pi}(u, v)$  is in red. (b) The straight segment  $\vec{\pi}[u, v]$  is in red. (c) The selected straight segment between  $u$  and  $v$  is in red.

**Definition 3.2.6** (Directed path and its selected straight segment). *Let  $u \in \partial B_{2,s,N}$  and  $v \in \partial B_{1,s,N}$ .*

- *We fix in an arbitrary way  $\vec{\pi}(u, v)$  an oriented<sup>5</sup> path from  $u$  to  $v$  with a subpath  $\vec{\pi}[u, v]$  between  $\partial B_{2,s,N}$  and  $v$  using only edges in the same direction. We say that  $\vec{\pi}(u, v)$  is the directed path between  $u$  and  $v$ . The subpath  $\vec{\pi}[u, v]$  is called the straight segment between  $u$  and  $v$  and its length is greater than or equal to  $(r_2 - r_1)N$ .*
- *We define the selected straight segment between  $u$  and  $v$  as the set of vertices  $c$  belonging to the straight segment between  $u$  and  $v$  and such that the distance for the norm  $\|\cdot\|_1$  between  $c$  and  $(B_{2,s,N})^c$  is at least  $\frac{(r_2 - r_1)N}{2} + dr_P$ .*

**Remark 3.2.7.** *The selected straight segment is not the empty set since  $v$  belongs to it. Indeed, the distance between  $v$  and  $(B_{2,s,N})^c$  is equal to  $(r_2 - r_1)N$  and  $(r_2 - r_1)N > \frac{(r_2 - r_1)N}{2} + dr_P$  since  $r_2 > 2r_1$  and  $r_2 > 4dr_P$  by (3.2.4).*

**Typical boxes in the case (INF).** In these cases, a box  $B_{3,s,N}$  is typical if it verifies the following properties:

- (i) if  $t_{\min} = 0$ , every path  $\pi$  entirely contained in  $B_{3,s,N}$  from  $u_\pi$  to  $v_\pi$  with  $\|u_\pi - v_\pi\|_1 \geq N$  has at least  $\rho\|u_\pi - v_\pi\|_1$  edges whose passage time is greater than  $\delta$ ,
- (ii) every path  $\pi$  entirely contained in  $B_{3,s,N}$  from  $u_\pi$  to  $v_\pi$  with  $\|u_\pi - v_\pi\|_1 \geq N$  has a passage time verifying:

$$T(\pi) \geq (t_{\min} + \delta)\|u_\pi - v_\pi\|_1, \quad (3.2.5)$$

<sup>5</sup>Recall that, as it is defined in Section 3.1.6, a path, whose endpoints are denoted by  $u$  and  $v$ , is oriented if this path has exactly  $\|u - v\|_1$  edges.

- (iii) for all vertices  $u \in \partial B_{2,s,N}$  and  $v \in \partial B_{1,s,N}$ , there exists a vertex  $c$  belonging to the selected straight segment between  $u$  and  $v$  such that for every edge  $e \in B_\infty(c, r_P)$ ,  $T(e) = \infty$ .

**Typical boxes in the case (FU).** Fix  $\mathcal{T}$  the event given by Lemma 3.2.5. We define a sequence  $(\nu_1(N))_{N \in \mathbb{N}^*}$  such that:

- for all  $N \in \mathbb{N}^*$ ,  $\nu_1(N) > T^\Lambda$  if  $t_{\min} > 0$  and  $\nu_1(N) > \max(\nu_0, T^\Lambda)$  if  $t_{\min} = 0$ ,
- we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \sum_{e \in B_{3,0,N}} T(e) \geq \nu_1(N) \right) = 0. \quad (3.2.6)$$

Note that by (3.2.2) and by the first item above, for all  $N \in \mathbb{N}^*$ , when  $\mathcal{A}^\Lambda$  occurs,  $\nu_1(N)$  is strictly greater than the passage time of every edge belonging to  $\Lambda$  but not to  $\partial\Lambda$ . Note also that  $\mathcal{L}((\nu_1(N), \infty)) > 0$  for all  $N \in \mathbb{N}^*$  since the support of  $\mathcal{L}$  is unbounded.

In this case, a box  $B_{3,s,N}$  is typical if it verifies the following properties:

- (i) if  $t_{\min} = 0$ , every path  $\pi$  entirely contained in  $B_{3,s,N}$  from  $u_\pi$  to  $v_\pi$  with  $\|u_\pi - v_\pi\|_1 \geq N$  has at least  $\rho \|u_\pi - v_\pi\|_1$  edges whose passage time is greater than  $\delta$ ,
- (ii) every path  $\pi$  entirely contained in  $B_{3,s,N}$  from  $u_\pi$  to  $v_\pi$  with  $\|u_\pi - v_\pi\|_1 \geq N$  has a passage time verifying:
 
$$T(\pi) \geq (t_{\min} + \delta) \|u_\pi - v_\pi\|_1, \quad (3.2.5)$$
- (iii) for all vertices  $u \in \partial B_{2,s,N}$  and  $v \in \partial B_{1,s,N}$ , there exists a vertex  $c$  belonging to the selected straight segment between  $u$  and  $v$  such that  $\theta_c T \in \mathcal{T}$ ,

(iv)  $\sum_{e \in B_{3,s,N}} T(e) < \nu_1(N)$ .

### Properties of typical boxes in both cases.

**Lemma 3.2.8.** *We have these two properties about typical boxes.*

1. Let  $s \in \mathbb{Z}^d$  and  $N \in \mathbb{N}^*$ . The typical box property only depends on the passage times of the edges in  $B_{3,s,N}$ .
2. We have

$$\lim_{N \rightarrow \infty} \mathbb{P}(B_{3,0,N} \text{ is a typical box}) = 1.$$

*Proof.*

1. Properties (i) and (ii) in the two cases and property (iv) in the case (FU) only depend on the edges of  $B_{3,s,N}$ . Then, for all vertices  $u \in \partial B_{2,s,N}$  and  $v \in \partial B_{1,s,N}$ , every vertex  $c$  belonging to the selected straight segment between  $u$  and  $v$  has a distance

with  $(B_{2,s,N})^c$  greater than or equal to  $\frac{(r_2 - r_1)N}{2} + dr_P$ . Hence, property (iii) only depends on the edges of  $B_{2,s,N}$  and  $B_{2,s,N} \subset \overset{2}{B}_{3,s,N}$ .

2. First, let us prove that, in the two cases, the probability that (i) is satisfied by  $B_{3,0,N}$  goes to 1. For this item, assume that  $t_{\min} = 0$ . Let  $\Pi$  denote the set of self-avoiding paths entirely contained in  $B_{3,0,N}$ . For a path  $\pi$  going from a vertex  $u_\pi$  to a vertex  $v_\pi$ , we say that  $\pi$  satisfies the property  $\mathcal{P}_\delta$  if  $\pi$  takes at least  $\rho\|u_\pi - v_\pi\|_1$  edges  $e$  such that  $T(e) > \delta$ . Then, using Lemma 3.1.9,

$$\begin{aligned} & \mathbb{P}(B_{3,0,N} \text{ does not satisfy (i)}) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq N}} \mathbb{P}(\mathcal{P}_\delta \text{ is not satisfied by a path of } \Pi \text{ whose endpoints are } u_\pi \text{ and } v_\pi) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq N}} \mathbb{P}(\mathcal{P}_\delta \text{ is not satisfied by a path whose endpoints are } u_\pi \text{ and } v_\pi) \\ & \leq |B_{3,0,N}|_v^2 \beta' e^{-\beta N} \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

since  $|B_{3,0,N}|_v$  is bounded by a polynomial in  $N$ .

Now, for the remaining of this proof,  $t_{\min}$  can be positive. Using (3.1.21) and a similar computation as above, we get that

$$\mathbb{P}(B_{3,0,N} \text{ does not satisfy (ii)}) \xrightarrow{N \rightarrow \infty} 0.$$

Recall Definition 3.2.6. To prove that the probability that (iii) is satisfied by  $B_{3,0,N}$  goes to 1, we begin by associating in a deterministic way to each couple of vertices  $(u, v) \in \partial B_{2,0,N} \times \partial B_{1,0,N}$  a set of vertices, denoted by  $V(\vec{\pi}(u, v))$  such that:

- every vertex of  $V(\vec{\pi}(u, v))$  belongs to the selected straight segment between  $u$  and  $v$ ,
- for all  $z_1, z_2 \in V(\vec{\pi}(u, v))$ ,  $B_\infty(z_1, r_P) \cap B_\infty(z_2, r_P) = \emptyset$ ,
- there can be no other set satisfying the two conditions above containing strictly more vertices than  $V(\vec{\pi}(u, v))$ .

Note that there exists a constant  $K_1$  only depending on  $r_1$ ,  $r_2$  and  $r_P$  such that  $|V(\vec{\pi}(u, v))| \geq K_1 N$ . In the case (INF), we denote by  $\mathcal{T}_\infty$  the event on which for all  $e \in B_\infty(0, r_P)$ ,  $T(e) = \infty$ . We use the notation  $\overline{\mathcal{T}}$  to designate the event  $\mathcal{T}$  in the case (FU) and to designate  $\mathcal{T}_\infty$  in the case (INF), which allows us to conclude this part

of proof in the two cases. We have

$$\begin{aligned} \mathbb{P}(B_{3,0,N} \text{ does not satisfy (iii)}) &\leq \sum_{\substack{u \in \partial B_{2,0,N} \\ v \in \partial B_{1,0,N}}} \mathbb{P}(\forall c \in \vec{\pi}(u, v), \theta_c \bar{\mathcal{T}} \text{ does not occur}) \\ &\leq \sum_{\substack{u \in \partial B_{2,0,N} \\ v \in \partial B_{1,0,N}}} \mathbb{P}(\forall c \in V(\vec{\pi}(u, v)), \theta_c \bar{\mathcal{T}} \text{ does not occur}). \end{aligned}$$

Fix  $u \in \partial B_{2,0,N}$  and  $v \in \partial B_{1,0,N}$ . Since for all  $z_1, z_2 \in V(\vec{\pi}(u, v))$ ,  $B_\infty(z_1, r_P) \cap B_\infty(z_2, r_P) = \emptyset$ , the family of events  $(\{\theta_c \bar{\mathcal{T}} \text{ does not occur}\})_{c \in V(\vec{\pi}(u, v))}$  are independent and thus

$$\mathbb{P}(\forall c \in V(\vec{\pi}(u, v)), \theta_c \bar{\mathcal{T}} \text{ does not occur}) \leq (1 - \mathbb{P}(\bar{\mathcal{T}}))^{|V(\vec{\pi}(u, v))|} \leq (1 - \mathbb{P}(\bar{\mathcal{T}}))^{K_1 N}.$$

Since  $\mathbb{P}(\bar{\mathcal{T}}) > 0$ , we get the existence of a constant  $K_2$  not depending on  $u, v$  and  $N$  such that:

$$\mathbb{P}(\forall c \in V(\vec{\pi}(u, v)), \theta_c \bar{\mathcal{T}} \text{ does not occur}) \leq e^{-K_2 N}.$$

Hence,

$$\mathbb{P}(B_{3,0,N} \text{ does not satisfy (iii)}) \leq |\partial B_{2,0,N}| |\partial B_{1,0,N}|_v e^{-K_2 N} \xrightarrow{N \rightarrow \infty} 0,$$

since  $|\partial B_{2,0,N}| |\partial B_{1,0,N}|_v$  is bounded by a polynomial in  $N$ .

Finally, in the case (FU), we get that the probability that (iv) is satisfied by  $B_{3,0,N}$  goes to 1 by (3.2.6). □

**Crossing a box.** A self-avoiding path crosses a box  $B_{3,s,N}$  if it visits one vertex of  $\partial B_{3,s,N}$ , then one of  $B_{1,s,N}$  and then another one of  $\partial B_{3,s,N}$ .

The following lemma is a consequence of Lemma 5.2 in [18] which applies using Lemma 3.2.8.

**Lemma 3.2.9.** *For any  $N$  sufficiently large, we can take  $D_1 > 0$  and  $\alpha > 0$  such that for all  $n \geq 1$ ,*

$$\mathbb{P}(\exists z \in \Gamma_n \text{ such that } (0, z) \in \mathfrak{C} \text{ and } \exists \text{ a path from } 0 \text{ to } z \text{ that crosses} \\ \text{at most } \lfloor \alpha n \rfloor \text{ typical boxes}) \leq e^{-D_1 n}. \quad (3.2.7)$$

Using Lemma 3.2.9 above, we fix

$$N \geq 1 \text{ large enough and } D_1 > 0, \alpha > 0 \text{ such that (3.2.7) holds.} \quad (3.2.8)$$

For the remaining of the proof, since  $N$  is fixed, we write  $\nu_1$  instead of  $\nu_1(N)$  in the case

(FU). We fix  $\delta' > 0$  such that

$$\delta' < \min\left(\frac{\delta}{2}, \frac{1}{N}\right). \quad (3.2.9)$$

Note that, in particular, since  $N \geq 1$ , we have  $\delta' < 1$ .

### 3.2.1.3 Boundaries of the patterns in the case (FU)

In the case (FU), we fix  $\nu_2 > \nu_1$  such that

$$\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) \in (\nu_1, \nu_2)\} \text{ has a positive probability.} \quad (3.2.10)$$

It is possible since by the condition (AF-4) in Section 3.2.1.1, for all  $M >$ ,  $\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) > M\}$  has a positive probability.

(AF-4') For the remaining of the proof, we now replace  $\mathcal{A}^\Lambda$  by

$$\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) \in (\nu_1, \nu_2)\}.$$

As announced in Remark 3.2.3, from now on, the event  $\mathcal{A}^\Lambda$  of the pattern  $\mathfrak{P}$  has been modified. The assumption (AF-4) is not satisfied by this new event but it satisfies the assumption (AF-4') above.

**Remark 3.2.10.** *In the case (FU), since there is no edge whose passage time is greater than  $\nu_1$  in a typical box, there can be no pattern in a typical box.*

### 3.2.1.4 $k$ -penalized paths, shortcuts and successful boxes

Recall that  $r_3$  is fixed at (3.2.4) and that  $N$  is fixed at (3.2.8). We partition  $\mathbb{Z}^d$  with boxes  $B_{3,s,N}$  in  $K = K(r_3) = |B_{3,0,1}|_v$  ways as follows. For each  $z \in B_{3,0,1}$ , the partition associated with  $z$  is

$$\{B_{3,s,N}, s - z \in 2r_3\mathbb{Z}^d\}.$$

For convenience, we index these different partitions from 1 to  $K$ . For  $k \in \{1, \dots, K\}$ , the boxes belonging to the  $k$ -th partition are called  $k$ -boxes.

Recall that we say that a self-avoiding path  $\pi$  takes the pattern if there exists  $z \in \mathbb{Z}^d$  satisfying the condition  $(\pi; \mathfrak{P})$ . For  $k \in \{1, \dots, K\}$ , we say that a self-avoiding path  $\pi$  takes a pattern entirely contained in a  $k$ -box if there exists  $z \in \mathbb{Z}^d$  such that:

- $z$  satisfies the condition  $(\pi, \mathfrak{P})$ ,
- there exists a  $k$ -box containing  $B_\infty(z, \ell^\Lambda)$ .

**$k$ -penalized paths.** For  $k \in \{1, \dots, K\}$ , a  $k$ -penalized path is a self-avoiding path which takes no pattern entirely contained in a  $k$ -box.



**Penalized passage time.** For all  $x \in \mathbb{Z}^d$ , and for  $k \in \{1, \dots, K\}$ , we define

$$t_k(0, x) = \inf\{T(\pi) : \pi \text{ is a } k\text{-penalized path from } 0 \text{ to } x\},$$

with the convention  $\inf \emptyset = \infty$ .

**$k$ -geodesics.** For all  $x \in \mathbb{Z}^d$ , for all  $k \in \{1, \dots, K\}$ , a  $k$ -geodesic from 0 to  $x$  is a  $k$ -penalized path  $\gamma$  from 0 to  $x$  such that  $T(\gamma) = t_k(0, x)$ .

**Shortcuts.** For all boxes  $B_{3,s,N}$ , we say that a path  $\pi$  has a shortcut in  $B_{3,s,N}$  if  $\pi$  crosses  $B_{3,s,N}$  and if there exist two vertices  $u$  and  $v$  of  $\pi$  and a path  $\pi'$  going from  $u$  to  $v$  such that:

- $\pi'$  is entirely contained in  $B_{3,s,N}$ ,
- $\pi_{u,v}$  and  $\pi'$  have only  $u$  and  $v$  as vertices in common,
- $\pi'$  takes a pattern entirely contained in  $B_{3,s,N}$ ,
- $T(\pi') < T(\pi_{u,v} \cap B_{3,s,N})$ .

**Successful boxes.** Let  $B_{3,s,N}$  be a box and  $\pi$  a self-avoiding path. We say that  $B_{3,s,N}$  is **successful for the path**  $\pi$  if the following two conditions hold:

- $\pi$  crosses  $B_{3,s,N}$ ,
- $B_{3,s,N}$  is a typical box or  $\pi$  has a shortcut in  $B_{3,s,N}$ .

**$S^k$ -sequences.** For every  $k \in \{1, \dots, K\}$ , for every  $k$ -geodesic  $\gamma$  between two vertices, the  $S^k$ -sequence of  $\gamma$  is the sequence of different  $k$ -boxes successful for  $\gamma$  by order of first visit by  $\gamma$ . Note that the boxes of this  $S^k$ -sequence are pairwise disjoint by the definition of  $k$ -boxes.

## 3.2.2 Proof

### 3.2.2.1 Reduction

We begin the proof with some definitions. Recall that  $\alpha$  is fixed at (3.2.8) and that  $K$  is fixed at the beginning of Section 3.2.1.4. For all  $n \geq 1$ , write

$$Q_n = \left\lfloor \frac{\alpha n}{K} \right\rfloor.$$

Fix

$$n \geq 1 \text{ and } x \in \Gamma_n \tag{3.2.11}$$

From now on, when we talk about a path, a geodesic or a  $k$ -geodesic without specifying its extremities, we mean that it is from 0 to  $x$ . For all  $k \in \{1, \dots, K\}$ , we define

$$\mathcal{G}^k = \{(0, x) \in \mathfrak{C} \text{ and there exists a } k\text{-geodesic whose passage time is finite having at least } Q_n \text{ boxes in its } S^k\text{-sequence}\}.$$

**Selected  $k$ -geodesic and  $S^k$ -variables.** For all  $k \in \{1, \dots, K\}$ , on the event  $\mathcal{G}^k$ ,

- we define the selected  $k$ -geodesic as the first  $k$ -geodesic in the lexicographical order<sup>6</sup> among those having at least  $Q_n$  boxes in their  $S^k$ -sequences,
- for all  $j \in \{1, \dots, Q_n\}$ , we define the random variable  $\mathcal{S}_j^k$  as the vertex  $s$  such that  $B_{3,s,N}$  is the  $j$ -th box of the  $S^k$ -sequence of the selected  $k$ -geodesic.

**Events  $\mathcal{M}^k$ .** For all  $k \in \{1, \dots, K\}$  and all  $j \in \{1, \dots, Q_n\}$ , we define

$$\mathcal{M}^k(j) = \mathcal{G}^k \cap \{\text{the selected } k\text{-geodesic does not have a shortcut in any of the first } j \text{ boxes of its } S^k\text{-sequence}\}.$$

To make the end of this proof easier to read, we define the events

$$A = \{(0, x) \in \mathfrak{C} \text{ and there exists a geodesic from 0 to } x \text{ which does not take the pattern}\},$$

$$B = \{\text{every path from 0 to } \Gamma_n \text{ crosses at least } \lfloor \alpha n \rfloor + 1 \text{ typical boxes}\},$$

Note that  $A$  is the event considered in Proposition 3.1.8 and  $B$  the complementary event to the one considered in Lemma 3.2.9. The proof of Proposition 3.1.8 is based on the following two lemmas.

**Lemma 3.2.11.** *We have  $A \cap B \subset \bigcup_{k=1}^K \mathcal{M}^k(Q_n)$ .*

**Lemma 3.2.12.** *There exists  $\lambda \in (0, 1)$  which does not depend on  $x$  and  $n$  such that for all  $k \in \{1, \dots, K\}$ ,*

$$\mathbb{P}(\mathcal{M}^k(Q_n)) \leq \lambda^{Q_n}.$$

*Proof of Proposition 3.1.8 using Lemma 3.2.11 and 3.2.12.* Recall that  $N$  is fixed at (3.2.8) and that  $n$  and  $x$  are fixed at (3.2.11) but that  $D_1$  and  $\lambda$  do not depend on  $x$  and  $n$ . We

---

<sup>6</sup>The lexicographical order is based on the directions of the consecutive edges of the geodesics.

have

$$\begin{aligned}
\mathbb{P}(A) &\leq \mathbb{P}(A \cap B) + \mathbb{P}(B^c) \\
&\leq \sum_{k=1}^K \mathbb{P}(\mathcal{M}^k(Q_n)) + \mathbb{P}(B^c) \text{ by Lemma 3.2.11,} \\
&\leq K\lambda^{Q_n} + \mathbb{P}(B^c) \text{ by Lemma 3.2.12,} \\
&\leq K\lambda^{Q_n} + e^{-D_1 n} \text{ by Lemma 3.2.9.}
\end{aligned}$$

As  $D_1 > 0$  and  $\lambda \in (0, 1)$ , and as this inequality holds for any  $n \geq 1$  and any  $x \in \Gamma_n$ , we get the existence of two constants  $C > 0$  and  $D > 0$  such that for all  $n$ , for all  $x \in \Gamma_n$ ,

$$\mathbb{P}(A) \leq D \exp(-Cn).$$

□

*Proof of Lemma 3.2.11.* Assume that  $A$  occurs. Then there exists a self-avoiding path  $\gamma$  from 0 to  $x$  such that:

- $\gamma$  does not take the pattern,
- $T(\gamma) = t(0, x) < \infty$ .

For all  $k \in \{1, \dots, K\}$ , we get that:

- $\gamma$  is a  $k$ -penalized path,
- $T(\gamma) = t(0, x) = t_k(0, x) < \infty$ .

Thus, for all  $k \in \{1, \dots, K\}$ ,  $\gamma$  is a  $k$ -geodesic from 0 to  $x$  and no  $k$ -geodesic has a shortcut in any box. Assume that  $B$  also occurs. Then  $\gamma$  crosses at least  $\lfloor \alpha n \rfloor + 1$  typical boxes. Hence, there exists  $k \in \{1, \dots, K\}$  such that  $\gamma$  crosses at least  $Q_n = \lfloor \frac{\alpha n}{K} \rfloor$  typical boxes. Since every typical  $k$ -box crossed by  $\gamma$  is a successful box for  $\gamma$ ,  $\gamma$  is a  $k$ -geodesic having at least  $Q_n$  boxes in its  $S^k$ -sequence. Hence the event  $\mathcal{G}^k$  occurs and the selected  $k$ -geodesic (which is not necessarily  $\gamma$ ) does not have a shortcut in any of the first  $Q_n$  boxes of its  $S^k$ -sequence since it does not have a shortcut in any box. So the event  $\mathcal{M}^k(Q_n)$  occurs. □

Now, for the remaining of the proof, the aim is to prove Lemma 3.2.12.

**Modification argument.** We introduce an independent copy  $T'$  of the environment  $T$ , the two being defined on the same probability space. It is thus convenient to refer to the considered environment when dealing with the objects defined above. To this aim, we shall use the notation  $\{T \in \mathcal{M}^k(j)\}$  to denote that the event  $\mathcal{M}^k(j)$  holds with respect to the environment  $T$ . In other words,  $\mathcal{M}^k(j)$  is now seen as a subset of  $[0, \infty]^\mathcal{E}$  where  $\mathcal{E}$  is the set of all the edges. Similarly, we denote by  $\mathcal{S}_j^k(T')$  the random variables defined above but in the environment  $T'$ .

Fix  $k \in \{1, \dots, K\}$  and  $\ell \in \{1, \dots, Q_n\}$ . On  $\{T \in \mathcal{M}^k(\ell)\}$ , the event  $\mathcal{G}^k$  occurs and  $B_{3, S_\ell^k(T), N}$  is the  $\ell$ -th box of the  $S^k$ -sequence of the selected  $k$ -geodesic. From this new environment, we associate a set of edges  $E_{\text{modif}}^*(T)$  which is contained in  $B_{3, S_\ell^k(T), N}$ . It corresponds to the edges for which we want to modify the time. We get a new environment  $T^*$  defined for all edges  $e$  by:

$$T^*(e) = \begin{cases} T(e) & \text{if } e \notin E_{\text{modif}}^*(T), \\ T'(e) & \text{else.} \end{cases}$$

For  $y$  and  $z$  in  $\mathbb{Z}^d$ , we denote by  $t^*(y, z)$  the geodesic time between  $y$  and  $z$  in the environment  $T^*$ . Note that  $T$  and  $T^*$  do not have the same distribution as the set  $E_{\text{modif}}^*(T)$  depends on  $T$ .

The proof of Lemma 3.2.12 relies on the following lemma whose proof is given in the next subsection. Recall that, in the case (FU),  $\nu_2$  is fixed at (3.2.10) and that, if  $t_{\min} = 0$  in the cases= (INF),  $\nu_0$  is fixed at (3.1.22).

**Lemma 3.2.13.** *There exists  $\eta = \eta(N) > 0$  such that for all  $\ell$  in  $\{1, \dots, Q_n\}$ , for all  $k \in \{1, \dots, K\}$ , there exist measurable functions  $E_-^*$ ,  $E_{\text{mid}}^*$ ,  $E_+^*$ ,  $E_\infty^*$  all from  $[0, \infty]^\mathcal{E}$  to  $\mathcal{P}(\mathcal{E})$  and a measurable function  $\mathcal{C} : [0, \infty]^\mathcal{E} \mapsto \mathbb{Z}^d$  such that:*

(i) *on the event  $\{T \in \mathcal{M}^k(\ell)\}$ ,  $E_-^*(T)$ ,  $E_{\text{mid}}^*(T)$ ,  $E_+^*(T)$ ,  $E_\infty^*(T)$  and  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  are pairwise disjoint and are contained in  $B_{3, S_\ell^k(T), N}$ ,*

(ii) *on the event  $\{T \in \mathcal{M}^k(\ell)\}$ , we have  $\mathbb{P}(T' \in \mathcal{B}^*(T)|T) \geq \eta$ , where  $\{T' \in \mathcal{B}^*(T)\}$  is a shorthand for the event on which:*

- $\forall e \in E_-^*(T)$ ,  $T'(e) \leq t_{\min} + \delta'$ ,
- $\forall e \in E_{\text{mid}}^*(T)$ ,  $T'(e) \in (t_{\min} + \delta, \nu_0)$ ,
- $\forall e \in E_+^*(T)$ ,  $T'(e) > \nu_2$ ,
- $\forall e \in E_\infty^*(T)$ ,  $T'(e) = \infty$ ,
- $\theta_{\mathcal{C}(T)} T' \in \mathcal{A}^\Lambda$ .

(iii)  $\{T \in \mathcal{M}^k(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\} \subset \{T^* \in \mathcal{M}^k(\ell - 1) \setminus \mathcal{M}^k(\ell)\}$  and  $S_\ell^k(T^*) = S_\ell^k(T)$ .

**Remark 3.2.14.** *Several of the functions of the previous lemma can be equal to the empty set depending on the cases. Thus this does not prevent us from having  $\mathbb{P}(T' \in \mathcal{B}^*(T)|T) \geq \eta$ . In particular, for every environment  $T$ :*

- *in the case (INF),  $E_+^*(T) = \emptyset$ ,*
- *in the case (FU),  $E_\infty^*(T) = \emptyset$ ,*
- *in the two cases, if  $t_{\min} > 0$ ,  $E_{\text{mid}}^*(T) = \emptyset$ .*

*Proof of Lemma 3.2.12 using Lemma 3.2.13.* Let  $\ell \in \{1, \dots, Q_n\}$  and  $k \in \{1, \dots, K\}$ . For every  $s \in \mathbb{Z}^d$  and  $\mathcal{E}^*$  subset of edges of  $B_{3,s,N}$ , let us consider the environment  $T_{s,\mathcal{E}^*}^*$  defined for all edges  $e$  by:

$$T_{s,\mathcal{E}^*}^*(e) = \begin{cases} T'(e) & \text{if } e \in \mathcal{E}^*, \\ T(e) & \text{else.} \end{cases}$$

We define  $E_{\text{modif}}^*(T) = E_-^*(T) \cup E_{\text{mid}}^*(T) \cup E_+^*(T) \cup E_\infty^*(T) \cup B_\infty(\mathcal{C}(T), \ell^\Lambda)$ . Thus, for every  $s$  and  $\mathcal{E}^*$ ,  $T_{s,\mathcal{E}^*}^*$  and  $T$  have the same distribution and on the event  $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\} \cap \{E_{\text{modif}}^*(T) = \mathcal{E}^*\}$ ,  $T^* = T_{s,\mathcal{E}^*}^*$ . So, using this environment and writing with indicator functions the result of Lemma 3.2.13, we get:

$$\mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \leq \mathbb{1}_{\{T_{s,\mathcal{E}^*}^* \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T_{s,\mathcal{E}^*}^*) = s\}}, \quad (3.2.12)$$

We compute the expectation on both sides. For the left side, we have

$$\begin{aligned} & \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{E} \left[ \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \middle| T \right] \right]. \end{aligned}$$

Since on the event  $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\}$ , we have  $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$ , the left side is bounded from below by  $\eta \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s, E_{\text{modif}}^*(T) = \mathcal{E}^*)$ . Since  $T_{s,\mathcal{E}^*}^*$  and  $T$  have the same distribution, using (3.2.12), we get:

$$\eta \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s, E_{\text{modif}}^*(T) = \mathcal{E}^*) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell), S_\ell^k(T) = s).$$

Then, by writing  $K'$  the number of subsets of edges of  $B_{3,0,N}$  and by summing on all subsets  $\mathcal{E}^*$  of edges of  $B_{3,s,N}$ , we get for all  $s \in \mathbb{Z}^d$ ,

$$\frac{\eta}{K'} \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell), S_\ell^k(T) = s).$$

Finally, by summing<sup>7</sup> on all  $s \in \mathbb{Z}^d$ , we get

$$\frac{\eta}{K'} \mathbb{P}(T \in \mathcal{M}^k(\ell)) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)).$$

Now, since  $\mathcal{M}^k(\ell) \subset \mathcal{M}^k(\ell-1)$ ,

$$\mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)) = \mathbb{P}(T \in \mathcal{M}^k(\ell-1)) - \mathbb{P}(T \in \mathcal{M}^k(\ell)).$$

Thus,

$$\mathbb{P}(T \in \mathcal{M}^k(\ell)) \leq \lambda \mathbb{P}(T \in \mathcal{M}^k(\ell-1)),$$

where  $\lambda = \frac{1}{1 + \frac{\eta}{K'}} \in (0, 1)$  does not depend on  $x$  and  $n$ . Hence, using  $\mathbb{P}(T \in \mathcal{M}^k(0)) = 1$ ,

---

<sup>7</sup>Note that here, we must have the event  $\{S_\ell^k(T_{s,\mathcal{E}^*}^*) = s\}$  on the right side of the inequality (3.2.12) to sum on all  $s \in \mathbb{Z}^d$ .

we get by induction

$$\mathbb{P}(T \in \mathcal{M}^k(Q_n)) \leq \lambda_n^Q.$$

□

### 3.2.2.2 Proof of Lemma 3.2.13: modification

Let  $\ell \in \{1, \dots, Q_n\}$ ,  $k \in \{1, \dots, K\}$  and  $s \in \mathbb{Z}^d$  such that  $B_{3,s,N}$  is a  $k$ -box. Assume that the event  $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\}$  occurs. Note that  $(0, x) \in \mathfrak{C}$ . We denote by  $\gamma$  the selected  $k$ -geodesic. We know that:

- (H1)  $\gamma$  has at least  $Q_n$  boxes in its  $S^k$ -sequence,
- (H2)  $\gamma$  does not have a shortcut in any of the first  $\ell$  boxes of its  $S^k$ -sequence,
- (H3)  $B_{3,s,N}$  is the  $\ell$ -th box of the  $S^k$ -sequence of  $\gamma$  and is a typical box.

**Construction of the forbidden zone and definition of  $\mathcal{C}(T)$ .** Let  $u_2$  (resp.  $u_1$ ) be the entry point of  $\gamma$  in  $B_{2,s,N}$  (resp.  $B_{1,s,N}$ ). Since  $B_{3,s,N}$  is a typical box, we can define:

- $\pi$  as the path  $\vec{\pi}(u_2, u_1)$  defined in Definition 3.2.6,
- $\mathcal{C}(T)$  the first vertex belonging to the selected straight segment between  $u_2$  and  $u_1$  and satisfying property (iii) of a typical box. Such a vertex exists by Remark 3.2.7 and since  $B_{3,s,N}$  is a typical box in the environment  $T$ . Recall that in the case (INF), for every edge  $e \in B_\infty(\mathcal{C}(T), r_P)$ ,  $T(e) = \infty$  and in the case (FU),  $\theta_{-\mathcal{C}(T)}T \in \mathcal{T}$ .

We use the expression "forbidden zone" to refer to  $B_\infty(\mathcal{C}(T), r_P) \setminus \partial B_\infty(\mathcal{C}(T), r_P)$ . This is the place where we want to place the pattern taken by the shortcut in the modified environment.

#### Properties of the forbidden zone.

**Lemma 3.2.15.** 1. *The path  $\gamma$  does not visit any vertex of the forbidden zone.*

2. *The ball  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  is contained in the forbidden zone.*

3. *The forbidden zone is contained in  $B_{2,s,N}$  and for every  $c$  in the forbidden zone and every  $z \in \partial B_{2,s,N}$ ,*

$$\|z - c\|_1 \geq \frac{(r_2 - r_1)N}{2}.$$

*Proof.* 1. In the case (INF), every edge of the forbidden zone has an infinite passage time although  $\gamma$  has a finite passage time. In the case (FU), it follows from Lemma 3.2.5 and the fact that  $\gamma$  crosses this box and is a geodesic in the environment  $T$ .

2. It comes from the inequality  $r_P > \ell^\Lambda + 1$  by (3.2.3).

3. By Definition 3.2.6,  $\mathcal{C}(T)$  belongs to  $B_{2,s,N}$  and the distance between  $\mathcal{C}(T)$  and  $(B_{2,s,N})^c$  is at least  $\|z - c\|_1 \geq \frac{(r_2 - r_1)N}{2}$ . We get the result using that, for every  $c$  in the forbidden zone,  $\|c - \mathcal{C}(T)\|_1 \leq dr_P$ . □

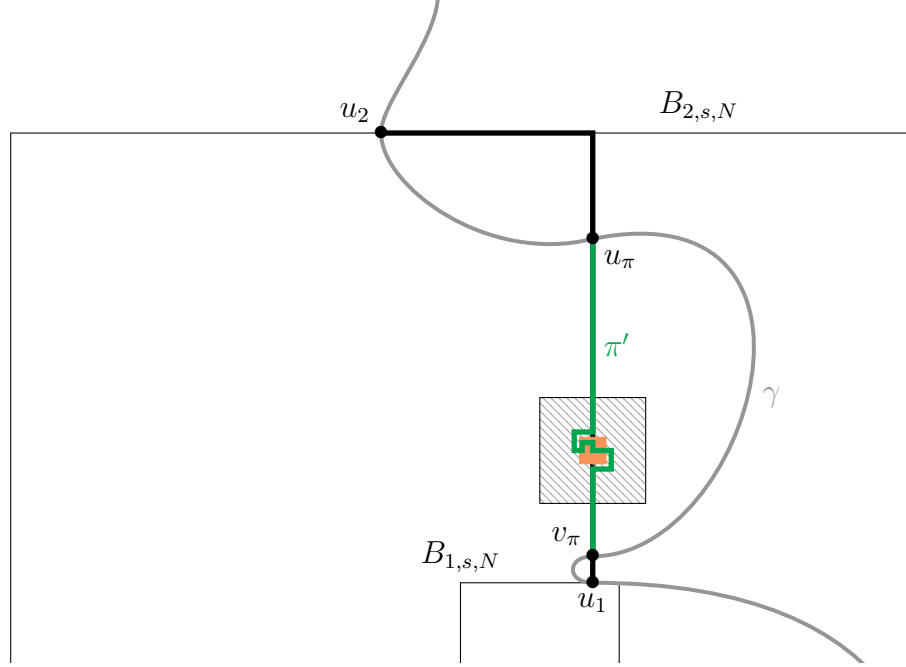


Figure 3.3: Example of construction of the shortcut  $\pi'$  in two dimensions. The shortcut  $\pi'$  is represented in green, the path  $\gamma$  in gray, the ball  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  in orange and the forbidden zone by the hatched area.

**Construction of the shortcut  $\pi'$ .** Let  $u_\pi$  be the last vertex of  $\pi$  belonging to  $\gamma$  before  $\pi$  visits the forbidden zone and  $v_\pi$  be the first vertex of  $\pi$  belonging to  $\gamma$  after the forbidden zone. One can check that we can build a path, denoted by  $\pi'$  for the remaining of the proof, such that:

- $\pi'$  is a self-avoiding path from  $u_\pi$  to  $v_\pi$ ,
- $\pi'$  is the concatenation of the subpath of  $\pi$  between  $u_\pi$  and the forbidden zone, then of a path entirely contained in the forbidden zone and then of the subpath of  $\pi$  between the forbidden zone and  $v_\pi$ ,
- $\pi'$  visits  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  for the first time in  $\theta_{-\mathcal{C}(T)}u^\Lambda$  and for the last time in  $\theta_{-\mathcal{C}(T)}v^\Lambda$ , and between these two vertices,  $\pi'$  is entirely contained in  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ . Furthermore, in the case (INF), between these two vertices,  $\pi'$  is equal to  $\theta_{\mathcal{C}(T)}\pi_\infty$  where  $\pi_\infty$  is defined

in Assumption (AI-3). Note that, if  $T' \in \mathcal{B}^*(T)$  (where  $\mathcal{B}^*(T)$  is defined in Lemma 3.2.13), then  $\theta_{\mathcal{C}(T)}\pi_\infty$  has finite passage time in the environment  $T^*$ ,

- we have an upper bound for the number of edges in  $\pi'$ :

$$|\pi'|_e \leq \|u_\pi - v_\pi\|_1 + |B_\infty(0, \ell^\Lambda + 1)|_e. \quad (3.2.13)$$

Note that, by the second item above and the definition of  $u_\pi$  and  $v_\pi$ ,  $\pi'$  only has  $u_\pi$  and  $v_\pi$  in common with  $\gamma$ . See Figure 3.3 for an example of construction of the shortcut  $\pi'$ .

**3.2.2.2.1 Beginning of the modification.** There are two cases for the beginning of the modification depending on whether  $t_{\min} = 0$  or  $t_{\min} > 0$  and on the number of edges in  $\gamma_{u_\pi, v_\pi}$ , denoted by  $|\gamma_{u_\pi, v_\pi}|_e$ . We have to distinguish two cases because we must be able to have a lower bound on the passage time of  $\gamma_{u_\pi, v_\pi}$ . To this aim, if  $\gamma_{u_\pi, v_\pi}$  takes enough edges, we can use the second property of a typical box and if it is not the case, we can have a lower bound using the number of edges of  $\gamma_{u_\pi, v_\pi}$  and  $t_{\min}$  if  $t_{\min} > 0$ . If  $t_{\min} = 0$  and if we can not use the second property of a typical box, then we use the modification to increase the passage times of  $\gamma_{u_\pi, v_\pi}$ . We describe the modification in each case. See Figure 3.4 for a representation of the objects involved in the modification.

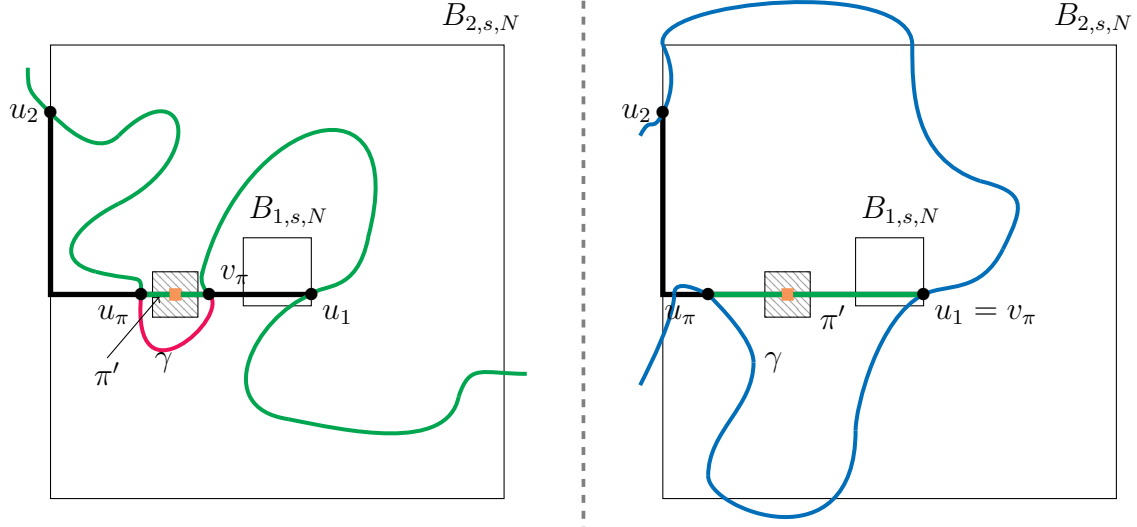
**Case A: assume  $t_{\min} = 0$  and  $|\gamma_{u_\pi, v_\pi}|_e < N$ .** The beginning of the modification is the following.

- The edges of  $E_{\text{mid}}^*(T)$  are the edges  $e$  belonging to  $\gamma_{u_\pi, v_\pi}$  and such that  $T(e) < t_{\min} + \delta$ .
- Recall that  $\mathcal{C}(T)$  is defined at the beginning of Section 3.2.2.2.
- The edges of  $E_-^*(T)$  are the edges  $e$  of  $B_{3,s,N}$  satisfying the following two conditions:
  - $e$  belongs to  $(\gamma \setminus \gamma_{u_\pi, v_\pi})$  or to  $\pi'$  but not to  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ ,
  - $T(e) \geq t_{\min} + \delta'$ .

**Case B: assume  $t_{\min} > 0$  or  $|\gamma_{u_\pi, v_\pi}|_e \geq N$ .** The beginning of the modification is the following.

- $E_{\text{mid}}^*(T) = \emptyset$ .
- Recall that  $\mathcal{C}(T)$  is defined at the beginning of Section 3.2.2.2.
- The edges of  $E_-^*(T)$  are the edges  $e$  of  $B_{3,s,N}$  satisfying the following two conditions:
  - $e$  belongs to  $\pi'$  but not to  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ ,
  - $T(e) \geq t_{\min} + \delta'$ .





**Case A:** when  $t_{\min} = 0$  and  $|\gamma_{u_\pi, v_\pi}|_e < N$ . **Case B:** when  $t_{\min} > 0$  or  $|\gamma_{u_\pi, v_\pi}|_e \geq N$ .

Figure 3.4: The modification. In the cases A and B, the pattern centered in  $\mathcal{C}(T)$  is represented in orange and the forbidden zone by the hatched area. In the case A,  $\gamma$  is the path composed by the edges in green up to  $u_\pi$ , then the edges in red and then the edges in green from  $v_\pi$ . In the case B,  $\gamma$  is the path composed by the edges in blue. In the cases A and B, every edge of  $E_-^*(T)$  belongs to the green part of the figure. When the modification is successful, the passage times of the edges in green which are greater than or equal to  $t_{\min} + \delta'$  are replaced by passage times smaller than  $t_{\min} + \delta'$ . In the case A, every edge of  $E_{\text{mid}}^*(T)$  belongs to the red part. When the modification is successful, the passage times of the edges in red which are smaller than  $t_{\min} + \delta$  are replaced by passage times belonging to  $(t_{\min} + \delta, \nu_0)$ . In the case B, the passage times of the edges in blue are not modified. The boundary of  $B_{3,s,N}$  is not represented even if  $B_{2,s,N}$  is included in  $B_{3,s,N}$ . The edges of  $E_\infty^*(T)$  or  $E_+^*(T)$  (depending on the case (INF) or (FU)) are all edges which are not in green, red, orange and blue.

**3.2.2.2.2 End of the modification.** Up to now, we have defined  $E_-^*(T)$ ,  $E_{\text{mid}}^*(T)$  and  $\mathcal{C}(T)$ . Note that  $E_-^*(T)$  and  $E_{\text{mid}}^*(T)$  disjoint sets included in  $B_{3,s,N} \cap (\gamma \cup \pi')$ . It remains to define  $E_+^*(T)$  and  $E_\infty^*(T)$ . There are two cases depending on whether there can be edges with infinite passage times or not.

**In the case (INF).** In this case,  $E_+^*(T) = \emptyset$  and the edges of  $E_\infty^*(T)$  are the edges of  $B_{3,s,N}$  which does not belong to  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ , to  $\pi'$  or to  $\gamma$ .

**In the case (FU).** In this case,  $E_\infty^*(T) = \emptyset$  and the edges of  $E_+^*(T)$  are the edges of  $B_{3,s,N}$  which does not belong to  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ , to  $\pi'$  or to  $\gamma$ .

### 3.2.2.3 Proof of Lemma 3.2.13: consequences of the modification

Assume for the remaining of the proof that the event

$$\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \text{ occurs,}$$

where  $\mathcal{B}^*(T)$  is defined in (ii) of Lemma 3.2.13. We now state some consequences of the modification useful for the following.

- (a) If  $|\gamma_{u_\pi, v_\pi}|_e < N$ , every edge in  $\gamma_{u_\pi, v_\pi}$  belongs to  $B_{3,s,N}$ . Thus, every edge belonging to  $E_{\text{mid}}^*(T)$  belongs to  $B_{3,s,N}$ .
- (b)  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  is entirely contained in  $B_{3,s,N}$ .
- (c) We have  $T^*(\gamma) \leq T(\gamma)$ . Furthermore, for all vertices  $u$  and  $v$  in  $\partial B_{3,s,N}$  visited by  $\gamma$ ,  $T^*(\gamma_{u,v}) \leq T(\gamma_{u,v})$ .
- (d)  $T^*(\pi') < T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N})$ .
- (e) There is only one pattern entirely contained in  $B_{3,s,N}$  in the environment  $T^*$ , which is the one centered in  $\mathcal{C}(T)$ .
- (f) Let  $\pi_0$  be a self-avoiding path from  $\partial B_{3,s,N}$  to  $\partial B_{3,s,N}$  entirely contained in  $B_{3,s,N}$  such that in the environment  $T^*$ :
  - in the case (INF), it has a finite passage time,
  - in the case (FU), it does not take any edge whose passage time is greater than  $\nu_1$ ,
  - in the cases (INF) and (FU), it does not take any pattern contained in  $B_{3,s,N}$ .

Then every edge of  $B_{3,s,N}$  belonging to  $\pi_0$  belongs to  $\gamma$ .

- (g) Let  $\tilde{\pi}$  be a  $k$ -penalized path from 0 to  $x$  in the environment  $T^*$  with  $T^*(\tilde{\pi})$  finite. Then

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) \leq T(\gamma) - T^*(\gamma).$$

In other words, no  $k$ -penalized path can save more time than  $\gamma$  during the modification.

*Proof of (a).* Since  $\pi$  is included in  $B_{2,s,N}$ ,  $u_\pi \in B_{2,s,N}$ . Thus

$$\|u_\pi - sN\|_1 \leq r_2 N. \tag{3.2.14}$$

Assume that  $|\gamma_{u_\pi, v_\pi}|_e < N$  and let  $z$  be a vertex visited by  $\gamma_{u_\pi, v_\pi}$ . We have

$$\|z - u_\pi\|_1 \leq N. \tag{3.2.15}$$

Combining (3.2.14) and (3.2.15) gives

$$\|z - sN\|_1 \leq (r_2 + 1)N < r_3 N,$$

since  $r_3 > r_2 + 1$  by (3.2.4). Hence  $z$  belongs to  $B_{3,s,N}$ . For the second part of the property, in the case B of the modification, there is no edge in  $E_{\text{mid}}^*(T)$  and in the case A of the modification all the edges of  $E_{\text{mid}}^*(T)$  belong to  $\gamma_{u_\pi, v_\pi}$ .  $\square$

*Proof of (b).* By Lemma 3.2.15,  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  is contained in the forbidden zone and the forbidden zone is contained in  $B_{2,s,N}$ . Since  $r_3 > r_2$ ,  $B_{2,s,N}$  is contained in  $B_{3,s,N}$ .  $\square$

*Proof of (c).* In the case B of the modification, the passage time of every edge of  $\gamma$  in  $T^*$  is equal to its passage time in  $T$ . Thus, property (c) holds. Now, assume the case A of the modification and recall that in this case  $t_{\min} = 0$ . The only edges of  $\gamma$  whose passage times in  $T^*$  are strictly greater than their passage times in  $T$  are those in  $\gamma_{u_\pi, v_\pi}$  and all of these edges are contained in  $B_{3,s,N}$  by property (a) above. Hence, to prove property (c), it is sufficient to prove that  $T^*(\gamma_{u,v}) \leq T(\gamma_{u,v})$  when  $u$  is the last vertex in  $\partial B_{3,s,N}$  visited by  $\gamma$  before it visits  $u_\pi$  and  $v$  is the first vertex in  $\partial B_{3,s,N}$  visited by  $\gamma$  after it visits  $v_\pi$ . So, let  $u$  and  $v$  be these vertices. First, since  $u_\pi$  belongs to  $B_{2,s,N}$  and  $u$  to  $\partial B_{3,s,N}$ , we have

$$\|u - u_\pi\|_1 \geq (r_3 - r_2)N \geq N,$$

since  $r_3 > r_2 + 1$  by (3.2.4). Hence, by the first property of a typical box,  $\gamma_{u, u_\pi}$  takes at least  $\rho(r_3 - r_2)N$  edges whose passage times is greater than  $\delta$ . Since  $\delta > \delta'$  by (3.2.9), all of these edges belongs to  $E_-^*(T)$  and there are no edges of  $\gamma_{u, u_\pi}$  whose passage times have been increased. Thus

$$T^*(\gamma_{u, u_\pi}) - T(\gamma_{u, u_\pi}) \leq -\rho(r_3 - r_2)N(\delta - \delta'). \quad (3.2.16)$$

The same arguments give

$$T^*(\gamma_{v_\pi, v}) - T(\gamma_{v_\pi, v}) \leq -\rho(r_3 - r_2)N(\delta - \delta'). \quad (3.2.17)$$

Furthermore, since the only edges of  $\gamma_{u_\pi, v_\pi}$  whose passage times have been modified belong to  $E_{\text{mid}}^*(T)$ ,

$$T^*(\gamma_{u_\pi, v_\pi}) - T(\gamma_{u_\pi, v_\pi}) \leq N\nu_0. \quad (3.2.18)$$

Thus, we get

$$\begin{aligned} T^*(\gamma_{u,v}) - T(\gamma_{u,v}) &= T^*(\gamma_{u, u_\pi}) - T(\gamma_{u, u_\pi}) + T^*(\gamma_{u_\pi, v_\pi}) - T(\gamma_{u_\pi, v_\pi}) + T^*(\gamma_{v_\pi, v}) - T(\gamma_{v_\pi, v}) \\ &\leq N(\nu_0 - 2\rho(\delta - \delta')(r_3 - r_2)) \quad \text{by (3.2.16), (3.2.17) and (3.2.18),} \\ &\leq 0, \end{aligned}$$

since  $r_3 > 2r_2$ ,  $\delta > 2\delta'$ ,  $r_3$  is large enough compared to  $\nu_0$  by (3.2.4) and  $N \geq 1$  by (3.2.8).  $\square$

*Proof of (d).* First, in all cases, by (3.2.13), and by (3.2.1) and (3.2.2),

$$T^*(\pi') \leq (\|u_\pi - v_\pi\|_1 + |B_\infty(0, \ell^\Lambda + 1)|_e)(t_{\min} + \delta') + T^\Lambda. \quad (3.2.19)$$

To conclude this proof, we distinguish three cases.

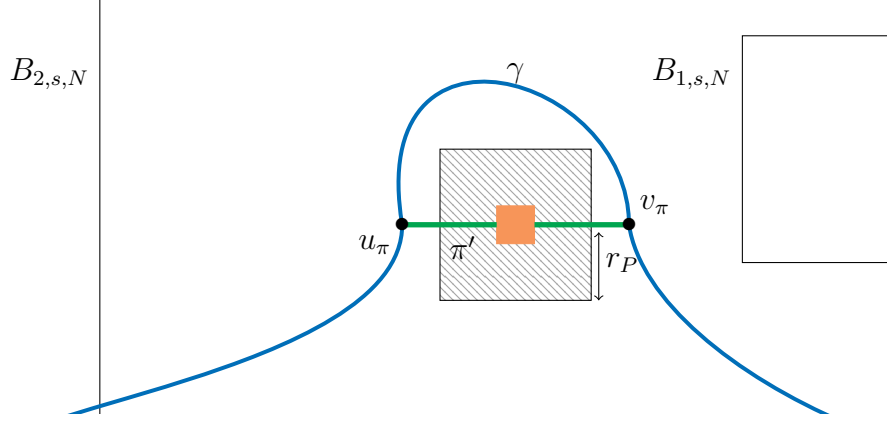


Figure 3.5: A picture to illustrate (3.2.22). The legend is the same as in Figure 3.4. In this example in two dimensions, when  $t_{\min} > 0$  and  $|\gamma_{u_\pi, v_\pi}|_e < N$  (a special case of case B of the modification),  $\gamma_{u_\pi, v_\pi}$  has to take  $\|u_\pi - v_\pi\|_1$  edges in the direction  $\varepsilon_1$  and also at least  $2r_P$  edges in the direction  $\varepsilon_2$  to avoid the forbidden zone.

**If  $|\gamma_{u_\pi, v_\pi}|_e \geq N$  (case B of the modification).** First, let us prove that

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq \|u_\pi - v_\pi\|_1 (t_{\min} + \delta). \quad (3.2.20)$$

Since this is the case B of the modification, the edges of  $\gamma_{u_\pi, v_\pi}$  have not been modified. Thus, if  $\gamma_{u_\pi, v_\pi}$  is entirely contained in  $B_{3,s,N}$  the second property of the typical boxes gives (3.2.20). If  $\gamma_{u_\pi, v_\pi}$  is not entirely contained in  $B_{3,s,N}$ , let  $u'$  be the first vertex of  $\partial B_{3,s,N}$  visited by  $\gamma_{u_\pi, v_\pi}$ . The vertex  $u_\pi$  is in  $B_{2,s,N}$ , so  $\|u' - u_\pi\|_1 \geq (r_3 - r_2)N \geq N$  since  $r_3 > r_2$  by (3.2.4). Using that  $\|u_\pi - v_\pi\|_1 \leq d(r_2 + r_1)N$ , the fact that  $r_3 - r_2 > d(r_2 + r_1)$  by (3.2.4) and the second property of a typical box gives:

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq \underbrace{\|u' - u_\pi\|_1}_{\geq (r_3 - r_2)N} (t_{\min} + \delta) \geq (r_3 - r_2)N (t_{\min} + \delta) \geq \|u_\pi - v_\pi\|_1 (t_{\min} + \delta).$$

This concludes the proof of (3.2.20). Then, combining (3.2.19) and (3.2.20), and using that  $\|u_\pi - v_\pi\|_1 \geq r_P$ , we have

$$\begin{aligned} T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) - T^*(\pi') &\geq \|u_\pi - v_\pi\|_1 (\delta - \delta') - |B_\infty(0, \ell^\Lambda + 1)|_e (t_{\min} + \delta') - T^\Lambda \\ &\geq r_P (\delta - \delta') - |B_\infty(0, \ell^\Lambda + 1)|_e (t_{\min} + \delta') - T^\Lambda > 0, \end{aligned}$$

by (3.2.3) and since  $\delta > 2\delta'$ .

**If  $t_{\min} = 0$  and  $|\gamma_{u_\pi, v_\pi}|_e < N$  (case A of the modification).** In the environment  $T^*$ , all the edges  $e$  belonging to  $\gamma_{u_\pi, v_\pi}$  have a time greater than  $t_{\min} + \delta$  and the property (a),  $\gamma_{u_\pi, v_\pi} \cap B_{3,s,N} = \gamma_{u_\pi, v_\pi}$ . Hence

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq \|u_\pi - v_\pi\|_1 (t_{\min} + \delta). \quad (3.2.21)$$

We conclude the proof of this case as the previous one combining (3.2.19) and (3.2.21).

**If  $t_{\min} > 0$  and  $|\gamma_{u_\pi, v_\pi}|_e < N$  (case B of the modification).** In this case, since  $\mathcal{C}(T)$  belongs to the selected straight segment between  $u_2$  and  $u_1$  and since  $v_\pi$  is visited by  $\pi$  after the forbidden zone,  $v_\pi$  belongs to the selected straight segment. Thus, the distance between  $v_\pi$  and  $(B_{2,s,N})^c$  is greater than or equal to  $\frac{(r_2 - r_1)N}{2}$ . Then, since  $|\gamma_{u_\pi, v_\pi}|_e < N$ , we have  $\|u_\pi - v_\pi\|_1 < N$ . Since  $r_2 > 4r_1$  by (3.2.4),  $u_\pi$  does not belong to  $\partial B_{2,s,N} \cap \pi$  and thus  $\pi_{u_\pi, v_\pi}$  takes edges in only one direction: the direction of the selected straight segment. Denote this direction by  $\varepsilon_i$ . Then  $\gamma_{u_\pi, v_\pi}$  has to take  $\|u_\pi - v_\pi\|_1$  edges in the direction  $\varepsilon_i$  but it can not take edges of the forbidden zone (see Figure 3.5). Hence

$$|\gamma_{u_\pi, v_\pi}|_e \geq \|u_\pi - v_\pi\|_1 + 2r_P. \quad (3.2.22)$$

By the property (a), we have  $|\gamma_{u_\pi, v_\pi}|_e = |\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}|_e$  and thus

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq (\|u_\pi - v_\pi\|_1 + 2r_P)t_{\min}. \quad (3.2.23)$$

Combining (3.2.19) and (3.2.23) gives

$$\begin{aligned} T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) - T^*(\pi') &\geq 2r_P t_{\min} - \|u_\pi - v_\pi\|_1 \delta' - |B_\infty(0, \ell^\Lambda + 1)|_e (t_{\min} + \delta') - T^\Lambda \\ &\geq 2r_P t_{\min} - N \delta' - |B_\infty(0, \ell^\Lambda + 1)|_e (t_{\min} + \delta') - T^\Lambda > 0, \end{aligned}$$

since  $N\delta' \leq 1$  by (3.2.9) and since  $r_P$  is large enough by (3.2.3).  $\square$

*Proof of (e). In the case (INF).* Recall that the pattern satisfies the boundary condition (see Definition 3.2.1). Thus, if there is a pattern entirely contained in  $B_{3,s,N}$  centered in a vertex  $z$ , it implies that there exists a path of length  $2(\ell^\Lambda - 1)$  such that:

- it goes from  $z - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  to  $z + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  only using edges in the direction  $\varepsilon_1$ ,
- its passage time is finite,
- there exists no path with finite passage time from  $\partial B_{3,s,N}$  to one of its vertices.

In the environment  $T^*$ , the only edges with finite passage times are edges belonging to  $\gamma$ ,  $\pi'$ ,  $\mathcal{S}_{\mathcal{C}(T), \ell^\Lambda}$  and  $B_\infty(\mathcal{C}(T), \ell^\Lambda - 3)$ . For every vertex  $z$  belonging to  $\gamma$  and  $\pi'$ , there exists a path from  $\partial B_{3,s,N}$  to  $z$  with finite passage time. Furthermore, there is no path of length  $2(\ell^\Lambda - 1)$  with finite passage time using only edges in the direction  $\varepsilon_1$  having at least one edge in  $B_\infty(\mathcal{C}(T), \ell^\Lambda - 3)$  and which does not visit any vertex of  $\pi'$ . Thus, the only path of length  $2(\ell^\Lambda - 1)$  satisfying the three conditions above is the one from  $\mathcal{C}(T) - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  to  $\mathcal{C}(T) + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$  and the only pattern entirely contained in  $B_{3,s,N}$  is the one centered in  $\mathcal{C}(T)$ .

**In the case (FU).** If, in the environment  $T^*$ , there is a pattern entirely contained in  $B_{3,s,N}$  centered in a vertex  $z$ , then for every edge  $e \in \partial B_\infty(z, \ell^\Lambda)$ ,  $T(e) \in (\nu_1, \nu_2)$  by the

assumption (AF-4') in Section 3.2.1.3. Since  $B_{3,s,N}$  is a typical box in the environment  $T$ , for every edge  $e \in B_{3,s,N}$ ,  $T(e) \leq \nu_1$ . The only edges  $e$  such that  $T^*(e) > T(e)$  are:

- the edges of  $E_{\text{mid}}^*(T)$  when  $t_{\text{min}} = 0$  but for every edge  $e \in E_{\text{mid}}^*(T)$ ,  $T^*(e) < \nu_0 < \nu_1$  since  $\nu_1 > \nu_0$ ,
- the edges of  $E_+^*(T)$  but for every edge  $e \in E_+^*(T)$ ,  $T^*(e) > \nu_2$ ,
- some edges in  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ .

Thus, if an edge  $e \in B_{3,s,N}$  is such that  $T^*(e) \in (\nu_1, \nu_2)$ , this edge belongs to  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ . Thus there is only one pattern entirely contained in  $B_{3,s,N}$  which is the one centered in  $\mathcal{C}(T)$ .  $\square$

*Proof of (f).* Let  $\pi_0$  be a path from  $\partial B_{3,s,N}$  to  $\partial B_{3,s,N}$  entirely contained in  $B_{3,s,N}$  such that in the environment  $T^*$ , it has a finite passage time in the case (INF) and it does not take any edge whose passage time is greater than  $\nu_1$  in the case (FU). Then the only edges of  $B_{3,s,N}$  that  $\pi_0$  can take are edges of  $\gamma$ ,  $\pi'$  and some edges of  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ . Furthermore, since by Lemma 3.2.15,  $\gamma$  does not take any edge of the forbidden zone, if  $\pi_0$  links two vertices of  $\gamma$  without taking edges of  $\pi'$ , then  $\pi_0$  is exactly  $\gamma$  between these two vertices.

Now, assume that  $\pi_0$  does not take any pattern entirely contained in  $B_{3,s,N}$ . Since  $\pi_0$  can not take edges of  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  without taking the pattern centered in  $\mathcal{C}(T)$ , it remains to prove that  $\pi_0$  does not take any edge of  $\pi'$ . But since  $\pi'$  is a self-avoiding path entirely contained in  $B_{2,s,N}$  which takes the pattern centered in  $\mathcal{C}(T)$ , and which has only two vertices in common with  $\gamma$ , if  $\pi_0$  takes an edge of  $\pi'$ ,  $\pi_0$  takes the pattern centered in  $\mathcal{C}(T)$ , which is impossible.  $\square$

*Proof of (g).* Let  $\tilde{\pi}$  be a  $k$ -penalized path from 0 to  $x$  in the environment  $T^*$  with finite passage time in the environment  $T^*$ . There are three cases.

**First case.** If  $\tilde{\pi}$  does not take edges of  $B_{3,s,N}$ ,  $T^*(\tilde{\pi}) = T(\tilde{\pi})$  since the only edges whose passage time have been modified are edges of  $B_{3,s,N}$ . Property (g) follows from (c).

**Second case.** Assume that we are in the case (FU) and that  $\tilde{\pi}$  takes an edge  $e' \in B_{3,s,N}$  such that  $T^*(e') \geq \nu_1$ . Then,

$$T^*(\tilde{\pi}) = \sum_{e \in \tilde{\pi}} T^*(e) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e).$$

Since  $e' \in \tilde{\pi} \cap B_{3,s,N}$  and since  $B_{3,s,N}$  is a typical box, we have using the fourth property of a typical box

$$\sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) \geq \nu_1 > \sum_{e \in B_{3,s,N}} T(e) \geq \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T(e).$$

Furthermore, the passage times of the edges outside  $B_{3,s,N}$  have not been modified. Hence,

$$\sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T(e).$$

Thus,

$$T^*(\tilde{\pi}) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e) > \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T(e) = T(\tilde{\pi}). \quad (3.2.24)$$

In this case, property (g) follows from (3.2.24) and from property (c).

**Third case.** Now assume that  $\tilde{\pi}$  takes at least one edge in  $B_{3,s,N}$  and that in the case (FU),  $\tilde{\pi}$  does not take any edge in  $B_{3,s,N}$  having a passage time greater than or equal to  $\nu_1$ . Since  $\tilde{\pi}$  has a finite passage time in the environment  $T^*$ , in the case (INF),  $\tilde{\pi}$  does not take any edge  $e'$  in  $B_{3,s,N}$  such that  $T^*(e') = \infty$ . Since  $\tilde{\pi}$  is a  $k$ -penalized path, it does not take any pattern entirely contained in  $B_{3,s,N}$ . Hence, using (f), the only edges in  $B_{3,s,N}$  that  $\tilde{\pi}$  can take are edges of  $\gamma$ . So, let  $u_1, v_1, \dots, u_\kappa, v_\kappa$  be the successive entry and exit points of  $\tilde{\pi}$  in  $B_{3,s,N}$ , we get for all  $i \in \{1, \dots, \kappa\}$ ,  $\tilde{\pi}_{u_i, v_i} = \gamma_{u_i, v_i}$ . Furthermore, we also get that  $T(\tilde{\pi})$  is finite. Indeed, since the only edges whose passage time have been modified are the edges in  $B_{3,s,N}$  and since  $T^*(\tilde{\pi})$  is finite, the only edges with infinite passage time in the environment  $T$  that  $\tilde{\pi}$  can take are edges in  $B_{3,s,N}$ . But these edges being edges of  $\gamma$  which has a finite passage time in the environment  $T$ ,  $T(\tilde{\pi})$  is finite.

Thus, using again that the only edges whose passage time have been modified are the edges of  $B_{3,s,N}$ , we have

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) = \sum_{i \in \{1, \dots, \kappa\}} T(\tilde{\pi}_{u_i, v_i}) - T^*(\tilde{\pi}_{u_i, v_i}) = \sum_{i \in \{1, \dots, \kappa\}} T(\gamma_{u_i, v_i}) - T^*(\gamma_{u_i, v_i}). \quad (3.2.25)$$

Now, using (c),

$$\sum_{i \in \{1, \dots, \kappa\}} T(\gamma_{u_i, v_i}) - T^*(\gamma_{u_i, v_i}) \leq T(\gamma) - T^*(\gamma). \quad (3.2.26)$$

Thus, combining (3.2.25) and (3.2.26), we also get in this last case that

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) \leq T(\gamma) - T^*(\gamma).$$

□

### 3.2.2.4 End of the proof of Lemma 3.2.13

We prove Lemma 3.2.13 with the sets  $E_-^*(T)$ ,  $E_{\text{mid}}^*(T)$ ,  $E_+^*(T)$  and  $E_\infty^*(T)$  and the vertex  $\mathcal{C}(T)$  defined in Section 3.2.2.2. Let us first prove item (i) of this lemma.  $E_{\text{mid}}^*(T)$  is contained in  $B_{3,s,N}$  by property (a) in Section 3.2.2.3. Using property (b) of Section 3.2.2.3, we get that  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$  is contained in  $B_{3,s,N}$ .  $E_-^*(T)$ ,  $E_\infty^*(T)$  and  $E_+^*(T)$  are contained in  $B_{3,s,N}$  by their definitions. To get that these sets are pairwise disjoint in both cases, we only have to prove that:

- $\gamma_{u_\pi, v_\pi}$  does not visit any edge of  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ . This comes from the fact that  $B_\infty(\mathcal{C}(T), \ell^\Lambda) \subset B_\infty(\mathcal{C}(T), r_P)$ , which comes from the fact that  $r_P > \ell^\Lambda + 1$  by (3.2.3). By property (iii) of a typical box, in the case (INF), in the environment  $T$ , for every edge  $e \in B_\infty(\mathcal{C}(T), r_P)$ ,  $T(e) = \infty$  but  $T(\gamma_{u_\pi, v_\pi}) < \infty$ . Hence,  $\gamma_{u_\pi, v_\pi}$  does not take any edge

in  $B_\infty(\mathcal{C}(T), r_P)$ . In the case (FU), by property (iii) of a typical box, the event  $\theta_{\mathcal{C}(T)}\mathcal{T}$  holds with respect to the environment  $T$ . Recall Remark 3.2.10: there is no pattern in  $B_{3,s,N}$ . Since  $\mathcal{T}$  satisfies the second condition in Lemma 3.2.5 and since  $\gamma_{u_\pi, v_\pi}$  is a  $k$ -geodesic,  $\gamma_{u_\pi, v_\pi}$  does not visit any edge of the forbidden zone. So  $\gamma_{u_\pi, v_\pi}$  does not visit any edge of  $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ .

- $\gamma_{u_\pi, v_\pi}$  and  $\pi'$  do not have any edge in common by the definition of  $u_\pi$ ,  $v_\pi$  and  $\pi'$ .

To get item (ii), fix  $\eta = \mathbb{P}(T \in \mathcal{A}^\Lambda) \tilde{p}^{|B_{3,s,N}|}$ , where, in the case (INF) and if  $t_{\min} = 0$ ,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((t_{\min} + \delta, \nu_0)), \mathcal{L}(\infty)),$$

in the case (INF) and if  $t_{\min} > 0$ ,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}(\infty)),$$

in the case (FU) and if  $t_{\min} = 0$ ,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((t_{\min} + \delta, \nu_0)), \mathcal{L}((\nu_2, \infty))),$$

and in the case (FU) and if  $t_{\min} > 0$ ,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((\nu_2, \infty))),$$

Thus,  $\eta$  only depends on  $\mathcal{L}$ , the pattern and  $N$  and we have that

$$\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \mathbb{P}(T \in \mathcal{A}^\Lambda) \tilde{p}^{|B_{3,s,N}|} = \eta.$$

Now, let us prove item (iii) of Lemma 3.2.13. Let  $\gamma^*$  be the selected  $k$ -geodesic in the environment  $T^*$  if it exists. The aim is to prove the following properties in the environment  $T^*$ :

- (C1)  $\gamma^*$  exists, i.e. there exists a  $k$ -geodesic having at least  $Q_n$  boxes in its  $S^k$ -sequence,
- (C2)  $\gamma^*$  does not have a shortcut in the first  $\ell - 1$  boxes of its  $S^k$ -sequence,
- (C3)  $\gamma^*$  has a shortcut in the  $\ell$ -th box of its  $S^k$ -sequence,
- (C4)  $B_{3,s,N}$  is the  $\ell$ -th box of the  $S^k$ -sequence of  $\gamma^*$ .

To get these four properties, we use the following ones:

- (P1) a  $k$ -box different from  $B_{3,s,N}$  is a typical box in the environment  $T$  if and only if it is a typical box in the environment  $T^*$ ,
- (P2) a path has a shortcut in a  $k$ -box different from  $B_{3,s,N}$  in the environment  $T$  if and only if it has a shortcut in this box in the environment  $T^*$ ,



(P3)  $\gamma$  has a shortcut in  $B_{3,s,N}$  in the environment  $T^*$  and then,  $B_{3,s,N}$  is successful in the environment  $T^*$  for  $\gamma$ ,

(P4)  $\gamma^*$  exists and  $\gamma^* = \gamma$ .

To conclude the proof, we have to prove (P1), (P2), (P3) and (P4). Indeed, by (P1) and (P2), a  $k$ -box different from  $B_{3,s,N}$  is successful for a path  $\tilde{\pi}$  in the environment  $T$  if and only if it is successful for  $\tilde{\pi}$  in the environment  $T^*$ . Thus, since  $B_{3,s,N}$  is a typical box for  $\gamma$  in the environment  $T$ , using (P3), the successful boxes crossed by  $\gamma$  in the environments  $T$  et  $T^*$  are the same. Furthermore, if we have (P4), then  $\gamma$  is a  $k$ -geodesic in the environment  $T^*$  and we can define its  $S^k$ -sequence in this environment. We get that the  $S^k$ -sequence of  $\gamma$  is the same in the environments  $T$  and  $T^*$ . Hence, using again (P4), we get (C1), we get (C2) by (H2) and we get (C3) and (C4) by (H3) using again (P3).

At this stage of the proof, we easily get (P1), (P2) and (P3) (which is the aim of the following paragraph) but the proof of (P4) is a bit longer (this is the aim of Section 3.2.2.4.2).

**3.2.2.4.1 Proof of properties (P1), (P2) and (P3).** We get (P1) using that the fact that a box is typical only depends on the edges of the box (by Lemma 3.2.8), that every  $k$ -box different from  $B_{3,s,N}$  does not have edges in common with  $B_{3,s,N}$  and that the only edges whose time has been modified are edges belonging to  $B_{3,s,N}$ .

(P2) uses the same arguments than above. The fact that a path has a shortcut in a  $k$ -box only depends on the edges of the box.

We get (P3) by considering the path  $\pi'$  defined at the beginning of Section 3.2.2.2. By construction  $\pi'$  is entirely contained in  $B_{3,s,N}$ ,  $\pi'$  and  $\gamma$  only have  $u_\pi$  and  $v_\pi$  in common,  $\pi'$  takes the pattern and  $T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) > T^*(\pi')$  by (d).

**3.2.2.4.2 Proof of (P4):  $\gamma$  is the selected  $k$ -geodesic in the environment  $T^*$ .** To prove this property, we prove the following ones in the last four lemmas of this section:

- $\gamma$  is a  $k$ -penalized path in the environment  $T^*$ ,
- every  $k$ -penalized path from 0 to  $x$  in the environment  $T^*$  has a passage time greater than or equal to the passage time of  $\gamma$ ,
- if a path is a  $k$ -geodesic from 0 to  $x$  in the environment  $T^*$ , it is also a  $k$ -geodesic in the environment  $T$ ,
- if a  $k$ -geodesic from 0 to  $x$  in the environment  $T^*$  has at least  $Q_n$  boxes in its  $S^k$ -sequence, it has also at least  $Q_n$  boxes in its  $S^k$ -sequence in the environment  $T$ .

We can conclude with these properties. Indeed, with the first two properties above,  $\gamma$  is a  $k$ -geodesic in the environment  $T^*$ . As a consequence of (P1), (P2) and (P3), it has the same  $S^k$ -sequence in the environments  $T$  and  $T^*$ . Thus  $\gamma$  has at least  $Q_n$  boxes in its  $S^k$ -sequence and it can be the selected  $k$ -geodesic. By the last two properties above, we have that the set of the  $k$ -geodesics having at least  $Q_n$  boxes in their  $S^k$ -sequences in the environment  $T^*$  is

included in the set of the  $k$ -geodesics having at least  $Q_n$  boxes in their  $S^k$ -sequences in the environment  $T$ . Since  $\gamma$  is the first path in the lexicographical order among the paths of this last set, it is also the first path in the lexicographical order in the first set.

It remains to prove the four properties above. Before proving them, we begin by the following lemma.

**Lemma 3.2.16.** *A  $k$ -penalized path in the environment  $T^*$  with finite passage time in the environment  $T^*$  is also a  $k$ -penalized path in the environment  $T$ .*

*Proof.* Let  $\tilde{\pi}$  be a  $k$ -penalized path in the environment  $T^*$  with  $T^*(\tilde{\pi})$  finite. Since the edges outside  $B_{3,s,N}$  have not been modified,  $\tilde{\pi}$  takes a pattern entirely contained in a  $k$ -box different from  $B_{3,s,N}$  in the environment  $T$  if and only if it takes a pattern entirely contained in this box in the environment  $T^*$ . It remains to prove that  $\tilde{\pi}$  does not take a pattern entirely contained in  $B_{3,s,N}$  in the environment  $T$ .

In the case (INF), since the time of  $\tilde{\pi}$  is finite and since  $\tilde{\pi}$  is a  $k$ -penalized path in the environment  $T^*$ , by property (f), the only edges of  $B_{3,s,N}$  that  $\tilde{\pi}$  can take are edges of  $\gamma$ . It implies that, in the environment  $T$ , if  $\tilde{\pi}$  takes a pattern entirely contained in  $B_{3,s,N}$ ,  $\gamma$  also takes this pattern, which is impossible since  $\gamma$  is a  $k$ -penalized path in the environment  $T$ .

In the case (FU), it is impossible since  $B_{3,s,N}$  is a typical box in the environment  $T$  and there is no pattern in a typical box by Remark 3.2.10.  $\square$

**Lemma 3.2.17.**  *$\gamma$  is a  $k$ -penalized path in the environment  $T^*$ .*

*Proof.* The fact that a path takes a pattern entirely contained in a  $k$ -box only depends on the passage times of the edges of this  $k$ -box. Since  $\gamma$  is a  $k$ -penalized path in the environment  $T$ , it does not take a pattern entirely contained in a  $k$ -box in this environment. Since the edges of the  $k$ -boxes different from  $B_{3,s,N}$  have not been modified,  $\gamma$  does not take a pattern entirely contained in a  $k$ -box different from  $B_{3,s,N}$  in the environment  $T^*$ . To conclude, it remains to prove that  $\gamma$  does not take a pattern entirely contained in  $B_{3,s,N}$  in the environment  $T^*$ . By (e), there is only one pattern entirely contained in  $B_{3,s,N}$  which is the one centered in  $\mathcal{C}(T)$ . By Lemma 3.2.15,  $\gamma$  does not take any edge of the forbidden zone and the pattern centered in  $\mathcal{C}(T)$  is entirely contained in the forbidden zone, which gives the result.  $\square$

**Lemma 3.2.18.** *Every  $k$ -penalized path from 0 to  $x$  in the environment  $T^*$  has a passage time for  $T^*$  greater than or equal to the passage time of  $\gamma$  for  $T^*$ .*

*Proof.* Let  $\bar{\gamma}^*$  be a  $k$ -penalized path in the environment  $T^*$ . By Lemma 3.2.16,  $\bar{\gamma}^*$  is also a  $k$ -penalized path in the environment  $T$ . Thus, since  $\bar{\gamma}^*$  is a  $k$ -penalized path in the environment  $T$  and since  $\gamma$  is a  $k$ -geodesic, we get  $T(\gamma) \leq T(\bar{\gamma}^*)$ . Hence, using (g) in Section 3.2.2.3,

$$T^*(\bar{\gamma}^*) \geq T^*(\gamma) + \underbrace{T(\bar{\gamma}^*) - T(\gamma)}_{\geq 0} \geq T^*(\gamma).$$

$\square$

**Lemma 3.2.19.** *If a path from 0 to  $x$  is a  $k$ -geodesic in the environment  $T^*$ , it is also a  $k$ -geodesic in the environment  $T$ .*

*Proof.* Let  $\bar{\gamma}^*$  be a  $k$ -geodesic in the environment  $T^*$  from 0 to  $x$ . By Lemma 3.2.17,  $\gamma$  is a  $k$ -penalized path in the environment  $T^*$ . Thus, since  $T^*(\gamma)$  is finite,  $T^*(\bar{\gamma}^*)$  is also finite and by Lemma 3.2.16,  $\bar{\gamma}^*$  is a  $k$ -penalized path in the environment  $T$ . Moreover, using (g) in Section 3.2.2.3, we get

$$T(\bar{\gamma}^*) \leq T(\gamma) + T^*(\bar{\gamma}^*) - T^*(\gamma).$$

Since  $\bar{\gamma}^*$  is a  $k$ -geodesic, by Lemma 3.2.17 and Lemma 3.2.18,  $T^*(\bar{\gamma}^*) = T^*(\gamma)$ . So  $T(\bar{\gamma}^*) \leq T(\gamma)$  and  $\bar{\gamma}^*$  is a  $k$ -geodesic in the environment  $T$ .  $\square$

**Lemma 3.2.20.** *If a  $k$ -geodesic from 0 to  $x$  in the environment  $T^*$  has at least  $Q_n$  boxes in its  $S^k$ -sequence, it has also at least  $Q_n$  boxes in its  $S^k$ -sequence in the environment  $T$ .*

*Proof.* Let  $\bar{\gamma}^*$  be a  $k$ -geodesic in the environment  $T^*$  from 0 to  $x$  having at least  $Q_n$  boxes in its  $S^k$ -sequence. Using (P2) and by the construction of the  $S^k$ -sequence, each box different from  $B_{3,s,N}$  belonging to the  $S^k$ -sequence of  $\bar{\gamma}^*$  in the environment  $T^*$  belongs to its  $S^k$ -sequence in the environment  $T$ . If  $B_{3,s,N}$  belongs to its  $S^k$ -sequence in the environment  $T^*$ , since  $B_{3,s,N}$  is a typical box in the environment  $T$ , then  $B_{3,s,N}$  belongs to its  $S^k$ -sequence in the environment  $T$ , which allows us to conclude.  $\square$

### 3.3 Extension of the van den Berg-Kesten comparison principle

This section is dedicated to the proof of Theorem 3.1.7. Let  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  be two distributions taking values in  $[0, \infty]$  such that:

(H1)  $\mathcal{L}$  is useful,

(H2)  $\mathcal{L}([0, \infty)) > p_c$  and  $\tilde{\mathcal{L}}([0, \infty)) > p_c$ ,

(H3)  $\mathcal{L} \neq \tilde{\mathcal{L}}$ ,

(H4) there exists a couple of random variables  $(\tau, \tilde{\tau})$  on some probability space, with marginal distributions  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ , respectively, and satisfying

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau. \tag{3.3.1}$$

In what follows,  $(\tau, \tilde{\tau})$  is a couple of random variables with marginal distributions  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ , and satisfying (3.3.1). Such a couple exists by (H4). Note that by (3.3.1), we have

$$\{\tau < \infty\} \subset \{\tilde{\tau} < \infty\} \text{ a.s.} \tag{3.3.2}$$

Then, we consider a family  $(T, \tilde{T}) = \{(T(e), \tilde{T}(e)) : e \in \mathcal{E}\}$  of i.i.d. random variables defined on the same probability space such that for all  $e \in \mathcal{E}$ ,  $(T(e), \tilde{T}(e))$  has the same distribution as  $(\tau, \tilde{\tau})$ .

The proof of Theorem 3.1.7 is an application of Theorem 3.1.5. We begin by defining a valid pattern in Section 3.3.1 and then, we apply Theorem 3.1.5 with this pattern in Section 3.3.2.

### 3.3.1 Definition of the valid pattern

The fact that a pattern is valid or not depends on the distribution of the passage times of the environment. Here, we use Theorem 3.1.5 only in the environment  $T$ . Thus, when we define a pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  below, we consider that the event  $\mathcal{A}^\Lambda$  only depends on the family  $(T(e))_{e \in \Lambda}$ .

Now, for a valid pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ , denote by  $\Pi^\mathfrak{P}$  the set of all self-avoiding paths going from  $u^\Lambda$  to  $v^\Lambda$  and which are contained in  $\Lambda$ . Denote by  $\mathcal{G}$  the  $\sigma$ -field generated by the family  $(T(e))_{e \in \mathcal{E}}$ . Section 3.3.1 is devoted to the proof of the following lemma.

**Lemma 3.3.1.** *There exist a valid pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  and a constant  $\eta > 0$  such that on the event  $\mathcal{A}^\Lambda$ ,*

$$\mathbb{E} \left[ \min_{\pi \in \Pi^\mathfrak{P}} \tilde{T}(\pi) | \mathcal{G} \right] < \min_{\pi \in \Pi^\mathfrak{P}} T(\pi) - \eta. \quad (3.3.3)$$

To prove Lemma 3.3.1, there are three different cases to be considered. Noting that, if  $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) = 0$ , using (3.3.2), we get

$$\{\tau < \infty\} = \{\tilde{\tau} < \infty\} \text{ a.s.},$$

these three cases can be written as follows:

- $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$ ,
- $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$  a.s. and  $\mathbb{P}(\mathbb{E}[\tilde{\tau} | \tau] < \tau) > 0$ ,
- $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$  a.s. and  $\mathbb{P}(\mathbb{E}[\tilde{\tau} | \tau] = \tau) = 1$ .

The most technical case is the third one.

#### 3.3.1.1 First case: when $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$

*Proof of Lemma 3.3.1 in the first case.*

Assume that  $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$ . Let  $\nu \in (0, \infty)$  such that

$$\mathcal{L}([\nu, \infty)) > 0. \quad (3.3.4)$$

Such a constant exists since we have  $\mathcal{L}(0) + \mathcal{L}(\infty) < 1$  since  $\mathcal{L}$  is useful by Assumption  $(\mathcal{H}1)$  and  $\mathcal{L}([0, \infty)) > p_c$  by Assumption  $(\mathcal{H}2)$ . Let  $M \in [0, \infty)$  such that

$$\mathbb{P}(\tilde{\tau} < M \text{ and } \tau = \infty) > 0. \quad (3.3.5)$$

Let  $\eta_0 > 0$ . Fix

$$m > \frac{M + \eta_0}{2\nu}. \quad (3.3.6)$$

Then, define the pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  where  $\Lambda = \{0, 1\} \times \{0, \dots, m\} \times \prod_{j=3}^d \{0\}$ ,  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  and  $\mathcal{A}^\Lambda$  is defined as follows. Denote by  $\pi^{\text{fin}}$  the path going from  $u^\Lambda$  to  $m\varepsilon_2$  by  $m$  steps in the direction  $\varepsilon_2$ , then to  $\varepsilon_1 + m\varepsilon_2$  by one step in the direction  $\varepsilon_1$  and finally to  $v_2^\Lambda$  by  $m$  steps in the direction  $\varepsilon_2$ . The event  $\mathcal{A}^\Lambda$  is the event on which for every  $e \in \pi^{\text{fin}}$ ,  $\nu \leq T(e) < \infty$  and for every edge  $e \in \Lambda$  but not in  $\pi^{\text{fin}}$ ,  $T(e) = \infty$ . This pattern is valid since the event  $\mathcal{A}^\Lambda$  has a positive probability by (3.3.4) and (3.3.5), and since the path  $\pi^{\text{fin}}$  is a path between  $u^\Lambda$  and  $v^\Lambda$  with a finite passage time in the environment  $T$  when the event  $\mathcal{A}^\Lambda$  occurs.

Now, denote

$$\beta = \mathbb{P}(\tilde{\tau} < M | \tau = \infty).$$

By (3.3.5),  $\beta > 0$ . Then, on the event  $\mathcal{A}^\Lambda$ , we have

$$\mathbb{E} \left[ \min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] \leq T(\pi^{\text{fin}}) - \beta(T(\pi^{\text{fin}}) - M). \quad (3.3.7)$$

Indeed, denote by  $e^\infty$  the edge  $\{u^\Lambda, v^\Lambda\}$  and by  $\pi^\infty$  the path going from  $u^\Lambda$  to  $v^\Lambda$  by taking only this edge. Then, on the event  $\mathcal{A}^\Lambda$ , since  $\mathcal{A}^\Lambda \subset \{T(e^\infty) = \infty\}$ ,

$$\begin{aligned} \mathbb{E} \left[ \min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] &\leq \mathbb{E} \left[ \min(\tilde{T}(\pi^{\text{fin}}), \tilde{T}(\pi^\infty)) | \mathcal{G} \right] \\ &\leq \mathbb{E} \left[ M \mathbf{1}_{\{\tilde{T}(e^\infty) < M\}} + \tilde{T}(\pi^{\text{fin}}) \mathbf{1}_{\{\tilde{T}(e^\infty) \geq M\}} | \mathcal{G} \right] \\ &\leq M\beta + \sum_{e \in \pi^{\text{fin}}} \mathbb{E} \left[ \tilde{T}(e) \mathbf{1}_{\tilde{T}(e^\infty) \geq M} | \mathcal{G} \right]. \end{aligned} \quad (3.3.8)$$

But, for every  $e \in \pi^{\text{fin}}$ ,

$$\begin{aligned} \mathbb{E} \left[ \tilde{T}(e) \mathbf{1}_{\tilde{T}(e^\infty) \geq M} | \mathcal{G} \right] &= \mathbb{E} \left[ \tilde{T}(e) | T(e) \right] \mathbb{P} \left( \tilde{T}(e^\infty) \geq M | T(e^\infty) \right) \\ &\leq \mathbb{E} \left[ \tilde{T}(e) | T(e) \right] (1 - \beta) \text{ since } \mathcal{A}^\Lambda \subset \{T(e^\infty) = \infty\}, \\ &\leq T(e)(1 - \beta), \end{aligned} \quad (3.3.9)$$

since  $(T(e), \tilde{T}(e))$  has the same distribution as  $(\tau, \tilde{\tau})$  which satisfies (3.3.1). Thus, combining (3.3.8) and (3.3.9), we get, on the event  $\mathcal{A}^\Lambda$ ,

$$\mathbb{E} \left[ \min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] \leq M\beta + (1 - \beta)T(\pi^{\text{fin}}) = T(\pi^{\text{fin}}) - \beta(T(\pi^{\text{fin}}) - M),$$

and (3.3.7) is proved.

Now, by the definition of the pattern, on the one hand,  $T(\pi^{\text{fin}}) = \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi)$  and on the

other hand,  $T(\pi^{\text{fin}}) \geq 2m\nu$ . This gives, using (3.3.6),

$$T(\pi^{\text{fin}}) - M > \eta_0.$$

Hence,

$$\mathbb{E} \left[ \min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] < \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi) - \beta\eta_0,$$

which allows us to conclude since  $\beta\eta_0 > 0$ .  $\square$

### 3.3.1.2 Second case: when $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) > 0$

*Proof of Lemma 3.3.1 in the second case.*

Assume that  $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) > 0$  and that  $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$  a.s. Then there exist  $\eta > 0$  and a Borel set  $I \subset [0, \infty)$  such that  $\mathbb{P}(\tau \in I) > 0$  and on the event  $\{\tau \in I\}$ ,

$$\mathbb{E}[\tilde{\tau}|\tau] < \tau - \eta. \quad (3.3.10)$$

Now, define the pattern  $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  where  $u^\Lambda = (0, \dots, 0)$ ,  $v^\Lambda = (1, 0, \dots, 0)$  and  $\mathcal{A}^\Lambda$  is the event on which the passage time of the only edge of the pattern, denoted by  $e$ , belongs to  $I$ . Then, this pattern is valid since the event  $\mathcal{A}^\Lambda$  has a positive probability since  $\mathbb{P}(\tau \in I) > 0$  and since the passage time of the path  $(u^\Lambda, v^\Lambda)$  in the environment  $T$  is finite when  $\mathcal{A}^\Lambda$  occurs since  $I \subset [0, \infty)$ . Furthermore, on the event  $\mathcal{A}^\Lambda$ ,

$$\begin{aligned} \mathbb{E} \left[ \min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] &= \mathbb{E} [\tilde{T}(e) | \mathcal{G}] = \mathbb{E} [\tilde{T}(e) | T(e)] < T(e) - \eta \text{ by (3.3.10),} \\ &= \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi) - \eta. \end{aligned}$$

$\square$

### 3.3.1.3 Third case: when $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] = \tau) = 1$

Assume that

$$\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] = \tau) = 1, \quad (3.3.11)$$

and that

$$\{\tau < \infty\} = \{\tilde{\tau} < \infty\} \text{ a.s.} \quad (3.3.12)$$

**Lemma 3.3.2.** *In this case, there exist  $\beta > 0$  and  $\delta > 0$  such that*

$$\mathbb{P} \left( \tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} \leq \tau - 2\delta | \tau) \geq \beta \text{ and } \mathbb{P}(\tilde{\tau} \geq \tau | \tau) \geq \beta \right) > 0. \quad (3.3.13)$$

*Proof.* Denote  $A = \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} < \tau | \tau) > 0\}$  and  $B = \{\mathbb{P}(\tilde{\tau} \geq \tau | \tau) > 0\}$ . To prove the lemma, it suffices to prove that

$$\mathbb{P}(A \cap B) > 0. \quad (3.3.14)$$

Let  $K$  be a transition probability kernel such that for every measurable function  $\varphi : [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$ ,

$$\mathbb{E} [\varphi(\tau, \tilde{\tau}) | \tau] = \int_{[0, \infty]} \varphi(\tau, \tilde{t}) K(\tau, d\tilde{t}). \quad (3.3.15)$$

The existence of such a  $K$  is given by Theorem 2.19 in Chapter 4 in [6].

First, let us prove that  $\mathbb{P}(B) = 1$ . We have

$$\begin{aligned} B^c &= \{\mathbb{E} [\mathbb{1}_{\tilde{\tau} < \tau} | \tau] = 1\} = \{K(\tau, [0, \tau]) = 1\} \text{ using (3.3.15),} \\ &= \{K(\tau, [0, \tau]) = 1 \text{ and } \tau < \infty\} \text{ using (3.3.12)} \\ &\subset \left\{ \int_{[0, \infty]} \tilde{t} K(\tau, d\tilde{t}) < \tau \right\} \\ &= \{\mathbb{E} [\tilde{\tau} | \tau] < \tau\}. \end{aligned}$$

Now, since  $\mathbb{P}(\mathbb{E} [\tilde{\tau} | \tau] < \tau) = 0$  by (3.3.11), we get  $\mathbb{P}(B^c) = 0$  and thus  $\mathbb{P}(B) = 1$ .

So, to get (3.3.14), it remains to prove that  $\mathbb{P}(A) > 0$ . To this aim, we shall prove that

$$A^c = \{\tau = \tilde{\tau}\} \text{ a.s.}, \quad (3.3.16)$$

which leads to  $\mathbb{P}(A^c) < 1$  by  $(\mathcal{H}3)$ , and thus  $\mathbb{P}(A) > 0$ . To prove (3.3.16), observe that

$$\begin{aligned} A^c &= \{\tau = \infty \text{ or } \mathbb{P}(\tilde{\tau} < \tau | \tau) = 0\} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} < \tau | \tau) = 0\}, \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } K(\tau, [\tau, \infty]) = 1\} \text{ using (3.3.15),} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } K(\tau, \{\tau\}) = 1\}, \end{aligned}$$

since

$$\{\tau < \infty, K(\tau, [\tau, \infty]) = 1 \text{ and } K(\tau, (\tau, \infty]) > 0\} \subset \{\mathbb{E}[\tilde{\tau} | \tau] > \tau\}$$

and  $\mathbb{P}(\mathbb{E}[\tilde{\tau} | \tau] > \tau) = 0$  by (3.3.11). Thus, using again (3.3.15), we get

$$\begin{aligned} A^c &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} = \tau | \tau) = 1\} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \tilde{\tau} = \tau\} \text{ a.s.} \end{aligned}$$

Now, since  $\{\tau = \infty\} = \{\tau = \tilde{\tau} = \infty\}$  a.s. by (3.3.12), we get (3.3.16). Hence, (3.3.14) holds and the lemma is proved.  $\square$

**Lemma 3.3.3.** *In this case, there exists a bounded Borel set  $I \subset (0, \infty)$  and  $\eta > 0$  such that*

- $\mathbb{P}(\tau \in I) > 0$ ,
- and

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) | \mathcal{F}] < \min(\tau_1 + \tau_2, \tau_3 + \tau_4) - \eta$$

on the event  $\mathcal{I} = \{\tau_1, \dots, \tau_4 \in I\}$ , where  $(\tau_1, \tilde{\tau}_1), \dots, (\tau_4, \tilde{\tau}_4)$  are independent copies of  $(\tau, \tilde{\tau})$  and  $\mathcal{F} = \sigma(\tau_1, \dots, \tau_4)$ .

*Proof.* Let  $(\tau_1, \tilde{\tau}_1), \dots, (\tau_4, \tilde{\tau}_4)$  be independent copies of  $(\tau, \tilde{\tau})$ . Denote  $\mathcal{F} = \sigma(\tau_1, \dots, \tau_4)$ . Fix  $\beta > 0$  and  $\delta > 0$  given by Lemma 3.3.2. Using (3.3.13), we can find a Borel set  $I_0 \subset (0, \infty)$ , fixed for the remaining of the proof, such that  $\mathbb{P}(\tau \in I_0) > 0$  and on the event  $\{\tau \in I_0\}$ ,

$$\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} \leq \tau - 2\delta | \tau) \geq \beta \text{ and } \mathbb{P}(\tilde{\tau} \geq \tau | \tau) \geq \beta. \quad (3.3.17)$$

Now, fix  $\eta > 0$  such that  $\eta < 2\delta\beta^4$  and then fix

$$0 < \delta_0 < \frac{2\delta\beta^4 - \eta}{4}. \quad (3.3.18)$$

Note that this gives  $\delta_0 < \frac{\delta}{2}$  as  $\beta \leq 1$ . Let  $y_0 \in I_0$  such that  $\mathbb{P}(\tau \in I_0 \cap (y_0 - \delta_0, y_0 + \delta_0)) > 0$ . Such a  $y_0$  exists because  $\mathbb{P}(\tau \in I_0) > 0$ . Set  $I = I_0 \cap (y_0 - \delta_0, y_0 + \delta_0)$ .

Then, we have

$$\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) + 2\delta \mathbb{1}_{\{\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta \text{ and } \tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta\}} \leq \tilde{\tau}_1 + \tilde{\tau}_2.$$

Thus, on the event  $\mathcal{I} = \{\tau_1, \dots, \tau_4 \in I\}$ , since  $(\tau_1, \tilde{\tau}_1)$  and  $(\tau_2, \tilde{\tau}_2)$  satisfy (3.3.1),

$$\begin{aligned} \mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) | \mathcal{F}] &\leq \mathbb{E}[\tilde{\tau}_1 + \tilde{\tau}_2 | \mathcal{F}] - 2\delta \mathbb{P}(\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta | \mathcal{F}) \mathbb{P}(\tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta | \mathcal{F}) \\ &\leq \tau_1 + \tau_2 - 2\delta \mathbb{P}(\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta | \mathcal{F}) \mathbb{P}(\tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta | \mathcal{F}) \\ &\leq 2 \sup I - 2\delta \mathbb{P}(\tilde{\tau}_3 \leq \tau_3 - 2\delta \text{ and } \tilde{\tau}_1 \geq \tau_1 | \mathcal{F}) \mathbb{P}(\tilde{\tau}_4 \leq \tau_4 - 2\delta \text{ and } \tilde{\tau}_2 \geq \tau_2 | \mathcal{F}), \end{aligned}$$

since the diameter of  $I$  is lower than  $\delta$  by (3.3.18). Then, by (3.3.17)

$$\begin{aligned} \mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) | \mathcal{F}] &\leq 2 \sup I - 2\delta\beta^4 \\ &< 2 \inf I - \eta, \end{aligned}$$

since  $2(\sup I - \inf I) \leq 4\delta_0 < 2\delta\beta^4 - \eta$  by (3.3.18). Hence, since on the event  $\mathcal{I}$ ,

$$\min(\tau_1 + \tau_2, \tau_3 + \tau_4) \geq 2 \inf I,$$

we get, on the event  $\mathcal{I}$ ,

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) | \mathcal{F}] < \min(\tau_1 + \tau_2, \tau_3 + \tau_4) - \eta.$$

□

*Proof of Lemma 3.3.1 in the third case.* Fix  $I \subset (0, \infty)$  and  $\eta > 0$  given by Lemma 3.3.3. Define the pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  where  $\Lambda$  is the set containing 4 vertices defined by  $\Lambda = \{0, 1\} \times \{0, 1\} \times \prod_{j=3}^d \{0\}$ ,  $u_1^\Lambda = (0, \dots, 0)$ ,  $v_1^\Lambda = (1, 1, 0, \dots, 0)$  and  $\mathcal{A}^\Lambda$  is the event on which for every edge  $e \in \Lambda$ ,  $T(e) \in I$ . This pattern is valid since the event  $\mathcal{A}^\Lambda$  has a positive probability since  $\mathbb{P}(\tau \in I) > 0$  by Lemma 3.3.3, and since on the event  $\mathcal{A}^\Lambda$ , every edge of the pattern has a finite passage time.



Now denote  $e_1 = \{u^\Lambda, u^\Lambda + \varepsilon_1\}$ ,  $e_2 = \{u^\Lambda + \varepsilon_1, v^\Lambda\}$ ,  $e_3 = \{u^\Lambda, u^\Lambda + \varepsilon_2\}$  and  $e_4 = \{u^\Lambda + \varepsilon_2, v^\Lambda\}$ . Since there are only two paths in  $\Pi^\mathfrak{B}$ , the one taking  $e_1$  and then  $e_2$  and the one taking  $e_3$  and then  $e_4$ , we get

$$\begin{aligned} \mathbb{E} \left[ \min_{\pi \in \Pi^\mathfrak{B}} \tilde{T}(\pi) | \mathcal{G} \right] &= \mathbb{E} \left[ \min \left( \tilde{T}(e_1) + \tilde{T}(e_2), \tilde{T}(e_3) + \tilde{T}(e_4) \right) | \mathcal{G} \right] \\ &= \mathbb{E} \left[ \min \left( \tilde{T}(e_1) + \tilde{T}(e_2), \tilde{T}(e_3) + \tilde{T}(e_4) \right) | \sigma(T(e_1), T(e_2), T(e_3), T(e_4)) \right] \\ &< \min(T(e_1) + T(e_2), T(e_3) + T(e_4)) - \eta \text{ by Lemma 3.3.3,} \\ &= \min_{\pi \in \Pi^\mathfrak{B}} T(\pi) - \eta. \end{aligned}$$

□

### 3.3.2 End of the proof of Theorem 3.1.7

By (H2) and (3.3.2), we can fix  $M > 0$  such that

$$\mathbb{P}(\tau \leq M \text{ and } \tilde{\tau} \leq M) > p_c. \quad (3.3.19)$$

Let  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  be the clusters defined respectively as the clusters  $\mathcal{C}_M$  and  $\tilde{\mathcal{C}}_M$  in Section 3.1.3 for  $M$  fixed above. Recall the definitions of  $\varphi$ ,  $\tilde{\varphi}$ ,  $\mu$  and  $\tilde{\mu}$  also given in Section 3.1.3, and the convergence given at (3.1.10).

Let  $x \in \mathbb{Z}^d \setminus \{0\}$ . For any  $n \in \mathbb{N}$ , we define the random path  $\gamma_n$  as the first geodesic in the lexicographical order from  $\varphi(0)$  to  $\varphi(nx)$  in the environment  $T$ . As stated in Section 3.1.1, almost surely, there exists at least one geodesic from  $\varphi(0)$  to  $\varphi(nx)$ . Recall that  $\mathcal{G}$  is the  $\sigma$ -field generated by the family  $(T(e))_{e \in \mathcal{E}}$ . Note that  $\gamma_n$  is  $\mathcal{G}$ -measurable.

**Lemma 3.3.4.** *We have  $\mathbb{E} [\tilde{t}(\varphi(0), \varphi(x))] < \infty$ .*

*Proof.* Since  $\gamma_1$  is  $\mathcal{G}$ -measurable,

$$\begin{aligned} \mathbb{E} [\tilde{t}(\varphi(0), \varphi(x)) | \mathcal{G}] &\leq \mathbb{E} [\tilde{T}(\gamma_1) | \mathcal{G}] \\ &= \sum_{e \in \gamma_1} \mathbb{E} [\tilde{T}(e) | T(e)] \\ &\leq \sum_{e \in \gamma_1} T(e) = T(\gamma_1) = t(\varphi(0), \varphi(x)). \end{aligned}$$

Note that the last inequality comes from the fact that for every  $e \in \mathcal{E}$ ,  $\mathbb{E} [\tilde{T}(e) | T(e)] \leq T(e)$ . Then, taking expectation, it gives

$$\mathbb{E} [\tilde{t}(\varphi(0), \varphi(x))] \leq \mathbb{E} [t(\varphi(0), \varphi(x))].$$

Since by Proposition 2 in [5],  $\mathbb{E} [t(\varphi(0), \varphi(x))] < \infty$ , we get  $\mathbb{E} [\tilde{t}(\varphi(0), \varphi(x))] < \infty$ . □

With Lemma 3.3.4, we can use the Subadditive Ergodic Theorem (see for example Theorem 2.2 in [2]), and thus we get the existence of a finite constant  $\bar{\mu}(x)$  such that

$$\lim_{n \rightarrow \infty} \frac{\tilde{t}(\varphi(0), \varphi(nx))}{n} = \bar{\mu}(x) \text{ a.s. and in } L^1.$$

**Remark 3.3.5.** *There are two main differences between the definitions of  $\mu(x)$  and  $\tilde{\mu}(x)$ :*

- $\mu(x)$  is defined with the passage times in the environment  $T$  although  $\tilde{\mu}(x)$  is defined with the passage times in the environment  $\tilde{T}$ ,
- $\mu(x)$  is defined with geodesic times between vertices obtained with  $\varphi$  although  $\tilde{\mu}(x)$  is defined with geodesic times between vertices obtained with  $\tilde{\varphi}$ .

In order to compare  $\mu(x)$  and  $\tilde{\mu}(x)$ , it is therefore natural to introduce  $\bar{\mu}(x)$ , an intermediate object which has one difference with  $\mu(x)$  and one difference with  $\tilde{\mu}(x)$  in its definition.

**Remark 3.3.6.** *The proof of Theorem 3.1.7 given in this section also holds in the case originally proven by van den Berg and Kesten in [18], i.e. when we assume that  $\tau$  has a finite first moment. However, it is simpler in this case since we do not need the clusters  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  to define  $\mu(x)$  and  $\tilde{\mu}(x)$ . In this case, for every  $y \in \mathbb{Z}^d$ , we can take  $\varphi(y) = \tilde{\varphi}(y) = y$  and  $\bar{\mu}(x) = \tilde{\mu}(x)$ , and thus in the sequel, we do not need Lemma 3.3.7 and the proof of Lemma 3.3.8 is much simpler.*

The following lemma is based on elementary arguments of percolation.

**Lemma 3.3.7.** *We have  $\tilde{\mu}(x) \leq \bar{\mu}(x)$ .*

*Proof.* Let  $\hat{\mathcal{C}}$  be the infinite cluster for the Bernoulli percolation  $(\mathbb{1}_{\{T(e) \leq M \text{ and } \tilde{T}(e) \leq M\}}, e \in \mathcal{E})$  which exists and is unique a.s. by (3.3.19). Note that  $\hat{\mathcal{C}}$  is included in  $\mathcal{C}$  and in  $\tilde{\mathcal{C}}$ . For any  $y \in \mathbb{R}^d$ , we define  $\hat{\varphi}(y)$  the random point of  $\hat{\mathcal{C}}$  such that  $\|y - \hat{\varphi}(y)\|_1$  is minimal, with a deterministic rule to break ties. For any  $n \in \mathbb{Z}$ , we have

$$\begin{aligned} \tilde{t}(\tilde{\varphi}(0), \tilde{\varphi}(nx)) &\leq \tilde{t}(\varphi(0), \hat{\varphi}(0)) + \tilde{t}(\hat{\varphi}(0), \tilde{\varphi}(0)) + \tilde{t}(\varphi(0), \varphi(nx)) + \tilde{t}(\varphi(nx), \hat{\varphi}(nx)) + \tilde{t}(\hat{\varphi}(nx), \tilde{\varphi}(nx)) \\ &= A(0) + \tilde{t}(\varphi(0), \varphi(nx)) + A(nx), \end{aligned} \tag{3.3.20}$$

by writing for any  $y \in \mathbb{Z}^d$ ,  $A(y) = \tilde{t}(\varphi(y), \hat{\varphi}(y)) + \tilde{t}(\hat{\varphi}(y), \tilde{\varphi}(y))$ .

Now, for any  $y \in \mathbb{Z}^d$ , there exists a path between  $\varphi(y)$  and  $\hat{\varphi}(y)$  contained in  $\mathcal{C}$ . Thus, it only takes edges with finite passage times in the environment  $T$ , and thus with finite passage times in the environment  $\tilde{T}$  by (3.3.2). Furthermore, there exists a path between  $\hat{\varphi}(y)$  and  $\tilde{\varphi}(y)$  contained in  $\tilde{\mathcal{C}}$ , which gives that  $\tilde{t}(\hat{\varphi}(y), \tilde{\varphi}(y))$  is also finite. Hence  $A(y)$  is a finite random variable. It gives that

$$\frac{A(0)}{n}$$

converges almost surely, and thus in probability, towards 0. Hence

$$\frac{A(nx)}{n}$$

also converges towards 0 in probability. We get that

$$\frac{A(0) + \tilde{t}(\varphi(0), \varphi(nx)) + A(nx)}{n}$$

converges towards  $\bar{\mu}(x)$  in probability. Using (3.3.20), this gives  $\tilde{\mu}(x) \leq \bar{\mu}(x)$ .  $\square$

Let  $\eta > 0$  and  $\mathfrak{P}$  the pattern given by Lemma 3.3.1. For any path  $\pi$  we denote by  $\mathcal{N}^{\mathfrak{P}}(\pi)$  the maximum number of disjoint translations of the pattern  $\mathfrak{P}$  crossed by  $\pi$ .

**Lemma 3.3.8.** *There exists a constant  $c > 0$  such that for any  $n$  sufficiently large,*

$$\mathbb{E} \left[ \mathcal{N}^{\mathfrak{P}}(\gamma_n) \right] \geq cn.$$

*Proof.* Recall the definition of  $N^{\mathfrak{P}}(\pi)$  given for any path  $\pi$  at (3.1.4). Since the pattern  $\mathfrak{P}$  is valid by Lemma 3.3.1,  $\mathcal{L}$  is useful by  $(\mathcal{H}1)$  and  $\mathcal{L}([0, \infty)) > p_c$  by  $(\mathcal{H}2)$ , we can use Theorem 3.1.5 and thus there exist  $\alpha > 0$ ,  $\beta_1 > 0$  and  $\beta_2 > 0$  such that for every  $n \in \mathbb{N}$  and for every  $y \in B_1 \left( 0, \frac{\|nx\|_1}{4} \right)$  and  $z \in B_1 \left( nx, \frac{\|nx\|_1}{4} \right)$ ,

$$\mathbb{P} \left( (y, z) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } y \text{ to } z \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha n \right) \leq \beta_1 e^{-\beta_2 n}. \quad (3.3.21)$$

Denote by  $\mathcal{V}_n$  the set  $B_1 \left( 0, \frac{\|nx\|_1}{4} \right) \times B_1 \left( nx, \frac{\|nx\|_1}{4} \right)$ . Then, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{P} \left( N^{\mathfrak{P}}(\gamma_n) < \alpha n \right) &\leq \mathbb{P} \left( (\varphi(0), \varphi(nx)) \notin \mathcal{V}_n \right) \\ &+ \sum_{(y,z) \in \mathcal{V}_n} \mathbb{P} \left( (y, z) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } y \text{ to } z \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha n \right). \end{aligned} \quad (3.3.22)$$

Since  $x \neq 0$ , by Theorem 8 in [5], there exist  $\beta_3 > 0$  and  $\beta_4 > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\mathbb{P} \left( (\varphi(0), \varphi(nx)) \notin \mathcal{V}_n \right) \leq \beta_3 e^{-\beta_4 n}.$$

Hence, using (3.3.21) and since  $|\mathcal{V}_n|$  is bounded by a polynomial in  $n$ , there exist  $\beta_5 > 0$  and  $\beta_6 > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\mathbb{P} \left( N^{\mathfrak{P}}(\gamma_n) < \alpha n \right) \leq \beta_5 e^{-\beta_6 n}.$$

Thus, there exists  $c' > 0$  such that for any  $n$  sufficiently large

$$\mathbb{E} \left[ \mathcal{N}^{\mathfrak{P}}(\gamma_n) \right] \geq c'n.$$

We conclude the proof by observing that simple geometric considerations provide a constant  $c'' > 0$  such that for all path  $\pi$ ,

$$\mathcal{N}^{\mathfrak{P}}(\pi) \geq c'' N^{\mathfrak{P}}(\pi).$$

□

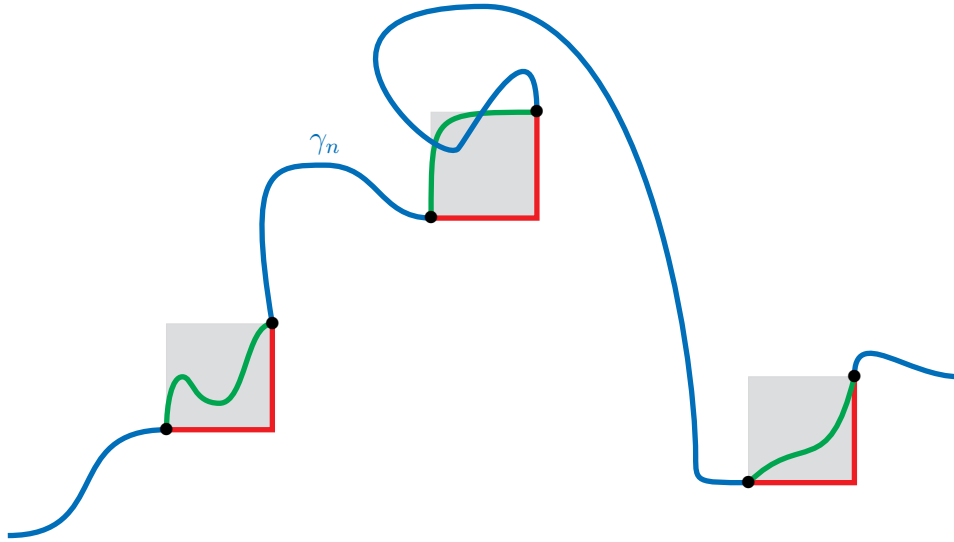


Figure 3.6: Illustration of the proof of Lemma 3.3.9. The translations of the pattern  $\mathfrak{P}$  are in gray. The path  $\gamma_n$  is the concatenation of the blue parts and of the red parts. In each pattern, the green part corresponds to the optimal path in the environment  $\tilde{T}$  among the paths between the endpoints of the pattern and entirely contained in the pattern. The idea is to bound from above the geodesic time between  $\varphi(0)$  and  $\varphi(nx)$  in the environment  $\tilde{T}$  by the passage time in the environment  $\tilde{T}$  of the concatenation of the blue parts and the green parts.

**Lemma 3.3.9.** *We have  $\bar{\mu}(x) < \mu(x)$ .*

*Proof.* For any  $n \geq 1$ , denote by  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$  the set, chosen according to a deterministic rule if there are several such sets, of  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  disjoint translations of  $\mathfrak{P}$  crossed by  $\gamma_n$ . Denote by  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  the set of all edges of  $\gamma_n$  which are not in a subpath of  $\gamma_n$  between the endpoints of a pattern of  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ . Recall that we denote by  $\Pi^{\mathfrak{P}}$  the set of all self-avoiding paths going from  $u^\Lambda$  to  $v^\Lambda$  and which are contained in  $\Lambda$ . For a pattern  $\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)$ , we can associate a unique  $s \in \mathbb{Z}^d$  such that  $s$  satisfies the condition  $(\gamma_n; \mathfrak{P})$  and such that  $\mathfrak{P}'$  is located at  $\theta_s \Lambda$ . Then, we denote by  $\Pi^{\mathfrak{P}'}$  the set of all self-avoiding paths  $\pi$  such that  $\theta_{-s} \pi \in \Pi^{\mathfrak{P}}$ . Denote by  $\Gamma^{\mathfrak{P}}(\gamma_n)$  the set of all paths from  $\varphi(0)$  to  $\varphi(nx)$  following  $\gamma_n$  outside all the patterns of  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$  and following a path of  $\Pi^{\mathfrak{P}'}$  for every pattern  $\mathfrak{P}'$  of  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ . With these definitions, we immediately get that

$$\min_{\pi \in \Gamma^{\mathfrak{P}}(\gamma_n)} \tilde{T}(\pi) = \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \tilde{T}(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \min_{\pi' \in \Pi^{\mathfrak{P}'}} \tilde{T}(\pi'). \quad (3.3.23)$$

Let  $n$  be sufficiently large such that Lemma 3.3.8 holds and let  $c > 0$  be the constant given by this lemma. Recall that  $\mathcal{G}$  is the  $\sigma$ -field generated by the family  $(T(e))_{e \in \mathcal{E}}$ . Then,

$\gamma_n$ ,  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$  and  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  are  $\mathcal{G}$ -measurable and we get

$$\begin{aligned} \mathbb{E} \left[ \tilde{t}(\varphi(0), \varphi(nx)) | \mathcal{G} \right] &\leq \mathbb{E} \left[ \min_{\pi \in \Gamma^{\mathfrak{P}}(\gamma_n)} \tilde{T}(\pi) | \mathcal{G} \right] \\ &= \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[ \tilde{T}(e) | T(e) \right] + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[ \min_{\pi' \in \Pi^{\mathfrak{P}'}} \tilde{T}(\pi') | \mathcal{G} \right] \text{ by (3.3.23),} \\ &\leq \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \left( \min_{\pi' \in \Pi^{\mathfrak{P}'}} T(\pi') - \eta \right). \end{aligned}$$

The first inequality comes from the definition of the geodesic time. For the first sum, the last inequality comes from the fact that for every edge  $e \in \mathcal{E}$ ,  $(T(e), \tilde{T}(e))$  has the same distribution as  $(\tau, \tilde{\tau})$  and thus satisfies  $E[\tilde{T}(e) | T(e)] \leq T(e)$ . For the second sum, it comes from Lemma 3.3.1.

Then, by the definitions of  $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$  and  $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ ,

$$\sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \min_{\pi' \in \Pi^{\mathfrak{P}'}} T(\pi') = T(\gamma_n) = t(\varphi(0), \varphi(nx)).$$

Furthermore, recall that  $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$  is the number of elements of  $\Pi^{\mathfrak{P}'}$ . Thus, we get

$$\mathbb{E} \left[ \tilde{t}(\varphi(0), \varphi(nx)) | \mathcal{G} \right] \leq t(\varphi(0), \varphi(nx)) - \eta \mathcal{N}^{\mathfrak{P}}(\gamma_n).$$

Now, taking expectation and dividing by  $n$  gives

$$\frac{\mathbb{E} \left[ \tilde{t}(\varphi(0), \varphi(nx)) \right]}{n} \leq \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} - \eta \frac{\mathbb{E} [\mathcal{N}^{\mathfrak{P}}(\gamma_n)]}{n} \leq \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} - \eta c,$$

by Lemma 3.3.8. We conclude the proof using that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ \tilde{t}(\varphi(0), \varphi(nx)) \right]}{n} = \bar{\mu}(x) \text{ and } \lim_{n \rightarrow \infty} \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} = \mu(x).$$

□

Now, we conclude the proof of Theorem 3.1.7 by combining Lemma 3.3.7 and Lemma 3.3.9.

### 3.A Existence of geodesics

**Proposition 3.A.1.** *Assume that  $\mathcal{L}(0) < p_c$ . With probability one, for all  $x, y$  such that  $(x, y) \in \mathfrak{C}$ , there exists a geodesic between  $x$  and  $y$ .*

To prove the above proposition, we begin by the following lemma.

**Lemma 3.A.2.** *Assume that  $\mathcal{L}(0) < p_c$ . There exists  $\beta > 0$ ,  $\beta' > 0$  and  $\rho > 0$  such that for all  $n \geq 1$ ,*

$$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } T(\pi) < \rho n) \leq \beta' e^{-\beta n}.$$

*Proof of Lemma 3.A.2.* For each environment  $T$ , we define a new environment  $\tilde{T}$  defined for all edges  $e$  by

$$\tilde{T}(e) = \begin{cases} T(e) & \text{if } T(e) < \infty \\ 1 & \text{else.} \end{cases}$$

Since  $\mathcal{L}(0) < p_c$ , we have  $\mathbb{P}(\tilde{T}(e) = 0) = \mathbb{P}(T(e) = 0) < p_c$ . Thus, we can use Proposition (5.8) in [14] and we get  $\beta > 0$ ,  $\beta' > 0$  and  $\rho > 0$  such that for all  $n \geq 1$ , for all  $z \in \mathbb{Z}^d$ ,

$$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } z \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n) \leq \beta' e^{-\beta n}.$$

Now, for every edge  $e$ ,  $\tilde{T}(e) \leq T(e)$ . Therefore,

$$\begin{aligned} & \mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } T(\pi) < \rho n) \\ & \leq \mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } z \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n), \end{aligned}$$

which allows us to conclude.  $\square$

*Proof of Proposition 3.A.1.* It is sufficient to prove that for every  $x$  and  $y$ , with probability one, there exists a geodesic between  $x$  and  $y$  if  $(x, y) \in \mathfrak{C}$ . Fix  $x$  and  $y$  in  $\mathbb{Z}^d$ . Fix  $\beta$ ,  $\beta'$  and  $\rho$  given by Lemma 3.A.2. For every  $n \geq 1$ , denote by  $A_n$  the event on which every path  $\pi$  from  $x$  which contains at least  $n$  edges has  $T(\pi) \geq \rho n$ . By the Borel-Cantelli Lemma and by Lemma 3.A.2, with probability one, for all  $n$  large enough,  $A_n$  occurs. We work on this probability one event. Assume that  $(x, y) \in \mathfrak{C}$ . Let  $\pi_{x,y}$  be a path between  $x$  and  $y$  such that  $T(\pi_{x,y}) < \infty$ . Fix  $n$  large enough such that  $A_n$  occurs and

$$n > \frac{T(\pi_{x,y})}{\rho}. \tag{3.A.1}$$

Then, every path from  $x$  to the boundary of  $B_1(x, n)$  has a passage time greater than or equal to  $\rho n$  since the event  $A_n$  occurs, and thus a passage time strictly greater than  $T(\pi_{x,y})$  by (3.A.1). Hence, the infimum in the definition of  $t(x, y)$  is over the finite set of paths contained in  $B_1(x, n)$ , and there must be a geodesic between  $x$  and  $y$ .  $\square$

## 3.B Edges with positive passage times taken by self-avoiding paths

*Proof of Lemma 3.1.9.* Assume that  $\mathcal{L}$  is useful and that  $t_{\min} = 0$ . Let  $\tau > 0$  such that  $\mathcal{L}([0, \tau]) < p_c$ . For each environment  $T$ , we define a new environment  $\tilde{T}$  defined for all edges

$e$  by

$$\tilde{T}(e) = \begin{cases} 0 & \text{if } T(e) \leq \tau \\ 1 & \text{else.} \end{cases}$$

We have  $\mathbb{P}(\tilde{T}(e) = 0) = \mathbb{P}(T(e) \leq \tau) < p_c$ . By Proposition (5.8) in [14], we get  $\beta > 0$ ,  $\beta' > 0$  and  $\rho > 0$  such that for all  $n \geq 1$ ,

$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n) \leq \beta' e^{-\beta n}$ .

Thus,

$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but contains at most } \rho n \text{ edges } e \text{ such that } T(e) > \tau) \leq \beta' e^{-\beta n}$ ,

and we get (3.1.24) for all  $v, w \in \mathbb{Z}^d$ . □

### 3.C Overlapping patterns

*Proof of Lemma 3.2.4.* Let  $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$  be a valid pattern. Denote by  $L_1, \dots, L_d$  the integers such that  $\Lambda_0 = \prod_{i=1}^d \{0, \dots, L_i\}$ . Fix

$$\ell^\Lambda = \max(L_1, \dots, L_d) + 4. \tag{3.C.1}$$

Let  $M_0^\Lambda > 0$  such that

$$\mathbb{P}(\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda \text{ or } T(e) = \infty\}) > 0. \tag{3.C.2}$$

**In the case (INF).** Consider the pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  defined as follows:

- $\Lambda = B_\infty(0, \ell^\Lambda)$ .
- $u^\Lambda = -\ell^\Lambda \varepsilon_1$  and  $v^\Lambda = \ell^\Lambda \varepsilon_1$ .
- Let  $\pi_\infty$  be a path from  $u^\Lambda$  to  $v^\Lambda$  such that:
  - $\pi_\infty$  is a self-avoiding path.
  - In  $\Lambda \setminus B_\infty(0, \ell^\Lambda - 3)$ ,  $\pi_\infty$  uses only 6 vertices, all in the set  $\{k\varepsilon_1, -\ell^\Lambda \leq k \leq \ell^\Lambda\}$ ,
  - $\pi_\infty$  visits  $u_0^\Lambda$  and  $v_0^\Lambda$ , and the portion of  $\pi_\infty$  between these two vertices, denoted by  $\pi_{\infty,0}$  is entirely contained in  $\Lambda_0$ . Furthermore, when  $\mathcal{A}_0^\Lambda$  occurs,  $T(\pi_{\infty,0}) < \infty$ . Note that this is possible since  $\mathfrak{P}_0$  is valid.
  - $\pi_\infty \setminus \pi_{\infty,0}$  does not take any edge of  $\Lambda_0$ .

Then,  $\mathcal{A}^\Lambda$  is the event such that:

- $\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda \text{ or } T(e) = \infty\}$  occurs,
- for all  $e$  belonging to  $\mathcal{S}_{s, \ell^\Lambda}$ ,  $T(e) < \infty$ ,
- for all  $e$  belonging to  $\pi_\infty \setminus \pi_{\infty, 0}$ ,  $T(e) \leq M_0^\Lambda$ ,
- for all  $e$  which does not belong to  $\Lambda_0 \cup \pi_\infty$ ,  $T(e) = \infty$ .

We get that  $\mathfrak{P}$  satisfies (AI-1), (AI-2) and (AI-4). Then, since  $\pi_\infty$  takes only edges whose passage time is smaller than or equal to  $M_0^\Lambda$ ,  $\mathfrak{P}$  satisfies (AI-3) by taking  $T^\Lambda > |B_\infty(0, \ell^\Lambda)|_e M_0^\Lambda$ .

Furthermore, we have  $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$  by the definition of  $\mathcal{A}^\Lambda$ ,  $\Lambda_0 \subset \Lambda$  by (3.C.1),  $\mathcal{A}^\Lambda$  has a positive probability by (3.C.2) and when  $\mathcal{A}^\Lambda$  occurs, every path from  $u^\Lambda$  to  $v^\Lambda$  whose passage time is finite is equal to  $\pi_\infty \setminus \pi_{\infty, 0}$  outside  $\Lambda_0$ , visits  $u_0^\Lambda$  and  $v_0^\Lambda$  and is entirely contained in  $\Lambda_0$ .

**In the case (FU).** Let  $M^\Lambda > |B_\infty(0, \ell^\Lambda)|_e M_0^\Lambda + 1$  such that

$$\mathcal{L}((M^\Lambda - 1, M^\Lambda)) > 0. \quad (3.C.3)$$

Consider the pattern  $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$  defined as follows:

- $\Lambda = B_\infty(0, \ell^\Lambda)$ .
- $u^\Lambda = -\ell^\Lambda \varepsilon_1$  and  $v^\Lambda = \ell^\Lambda \varepsilon_1$ .
- Let  $\pi_f$  be a path from  $u^\Lambda$  to  $v^\Lambda$  such that:
  - $\pi_f$  is a self-avoiding path.
  - $\pi_f$  does not visit any vertex in  $\partial\Lambda$  except  $u^\Lambda$  and  $v^\Lambda$ .
  - $\pi_f$  visits  $u_0^\Lambda$  and  $v_0^\Lambda$  and the portion of  $\pi_f$  between these two vertices, denoted by  $\pi_{f, 0}$  is entirely contained in  $\Lambda_0$ .
  - $\pi_f \setminus \pi_{f, 0}$  does not take any edge of  $\Lambda_0$ .

Then  $\mathcal{A}^\Lambda$  is the event such that:

- $\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda\}$  occurs,
- for all  $e$  belonging to  $\pi_f \setminus \pi_{f, 0}$ , we have  $T(e) \leq M_0^\Lambda$ ,
- for all  $e$  which does not belong to  $\partial\Lambda \cup \Lambda_0 \cup \pi_f$ ,  $T(e) \in (M^\Lambda - 1, M^\Lambda)$ ,
- for all  $e \in \partial\Lambda$ ,  $T(e) \geq M^\Lambda$ .

We get that  $\mathfrak{P}$  satisfies (AF-1), (AF-2), (AF-3) and (AF-4). Furthermore,

- $\Lambda_0 \subset \Lambda$  by (3.C.1).
- $\mathbb{P}(\mathcal{A}^\Lambda)$  is positive by (3.C.2) and (3.C.3), and then  $\mathfrak{P}$  is a valid pattern.



- On  $\mathcal{A}^\Lambda$ , any path from  $u^\Lambda$  to  $v^\Lambda$  optimal for the passage time among the paths entirely inside  $\Lambda$  contains a subpath from  $u_0^\Lambda$  to  $v_0^\Lambda$  entirely inside  $\Lambda_0$ . Indeed, let  $\pi$  be a path from  $u^\Lambda$  to  $v^\Lambda$  which does not contain a subpath from  $u^\Lambda$  to  $v^\Lambda$  entirely inside  $\Lambda_0$ . Since  $\pi_f$  is a self-avoiding path, it implies that  $\pi$  takes an edge whose time is greater than  $M^\Lambda - 1 > |\Lambda|_e M_0^\Lambda$ . But we have  $T(\pi_f) \leq |\Lambda|_e M_0^\Lambda < T(\pi)$  and thus  $\pi$  is not an optimal path. Hence, for every optimal path  $\pi$ , if a vertex  $x \in \mathbb{Z}^d$  satisfies the condition  $(\pi; \mathfrak{P}_0)$ ,  $x$  satisfies the condition  $(\pi; \mathfrak{P})$ .
- $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$  by the definition of  $\mathcal{A}^\Lambda$ .

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# Antonin JACQUET

##  tude des g od siques en percolation de premier passage

### R sum 

La th se porte sur les g od siques en percolation de premier passage. On se place sur le graphe  $\mathbb{Z}^d$  et on consid re une famille de variables al atoires positives, ind pendantes et identiquement distribu es index es par l'ensemble des ar tes, appel es temps de passage. On d finit le temps de tout chemin fini comme la somme du temps de passage de chacune de ses ar tes. Les g od siques sont alors les chemins de temps minimaux. On s'int resse   la question suivante. Consid rons une propri t  locale de l'environnement des temps de passage. On appelle cela un motif. Combien de fois une g od sique emprunte un translat  de ce motif?

Sans hypoth se de moment sur la loi des temps de passage, et en autorisant un atome en l'infini, on montre que pour un motif donn , en dehors d'un  v nement de probabilit  exponentiellement faible en la distance entre les extr mit s des g od siques, le nombre de translat s de ce motif emprunt s par les g od siques est lin aire en la distance entre les extr mit s. On donne  galement quelques applications de ce r sultat.

**Mots cl s :** Percolation de premier passage; G od siques.

### Abstract

The thesis focuses on geodesics in first-passage percolation. We consider the graph  $\mathbb{Z}^d$  and a family of nonnegative, independent, and identically distributed random variables indexed by the set of edges, called passage times. The time of a finite path is defined as the sum of the passage times of each of its edges. Geodesics are then the paths with minimal time. We are interested in the following question. Consider a local property of the time environment. We call it a pattern. How many times does a geodesic cross a translate of this pattern?

Without assuming any moment assumption on the distribution of passage times and with possibly infinite passage times, it is shown that, apart from an event with exponentially small probability in the distance between the geodesic endpoints, the number of translates of the pattern crossed by the geodesics is linear in the distance between the endpoints. Some applications of this result are also provided.

**Keywords:** First-passage percolation; Geodesics