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# Error Basis, Quantum Channel and Semigroups of Various Positive Operators

Base d'Erreur, Canal Quantique, Semigroupe de Divers Opérateurs Positifs

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THÈSE DE DOCTORAT DE L'ÉTABLISSEMENT UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ  
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Ecole doctorale n° 553

CARNOT PASTEUR

Doctorat de Mathématiques

Par

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## 0.1 Notations and Conventions

In the following discussion we will often use Dirac's bra-ket notation to denote rank-one operators. To define it in short, we understand  $|x\rangle$  to be a vector in some Hilbert space  $\mathbb{C}^n$  and  $\langle x|$  to be its dual vector. Then for two vectors  $x, y \in \mathbb{C}^n$  we define the rank one operator  $|x\rangle\langle y|$  by-

$$|x\rangle\langle y|(z) := \langle y|z\rangle x$$

for any  $z \in \mathbb{C}^n$ . The space of  $n \times n$  and  $n \times m$  matrices will be denoted by  $M_n(\mathbb{C})$  and  $M_{n,m}(\mathbb{C})$  respectively. If  $\{e_i\}_{i=1}^n$  is the standard basis of  $\mathbb{C}^n$  and  $\{E_{ij}; 1 \leq i, j \leq n\}$  is the standard matrix units (i.e. 1 at  $ij$ -th position and zero everywhere else) then one can easily verify that  $E_{ij} = |e_i\rangle\langle e_j|$ . We will use  $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  to denote the set of all linear maps from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$ . If  $n = m$  we will abbreviate it to  $\text{Lin}(M_n(\mathbb{C}))$ . We will frequently use the notation  $M_n(\mathbb{C})^*$  which is defined by  $M_n(\mathbb{C})^* := \{A^* : A \in M_n(\mathbb{C})\}$ .

# Chapter 1

## Introduction

Two basic ingredients of any information theory whether it is classical or quantum, are the information we want to communicate and a medium or channel to communicate. The information is a set of elements and a channel is a map on that information set. Now in the formalism of quantum mechanics every system corresponds to a Hilbert space which is usually taken to be of finite dimension in quantum information theory. For a system corresponding to a Hilbert space of dimension  $n$ , say  $\mathbb{C}^n$ , all the information of a quantum system is encoded in a quantum state which is mathematically represented by a positive matrix  $\rho \in M_n(\mathbb{C})$  of trace 1. A quantum channel  $S$  is then a map which takes a state  $\rho$  as an input and produces another state  $S(\rho)$  in some system represented by the Hilbert space  $\mathbb{C}^m$  of certain dimension  $m$  (or the same system  $\mathbb{C}^n$ ). To formalise, we need a map  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  that takes a positive matrix of  $M_n(\mathbb{C})$  to a positive matrix of  $M_m(\mathbb{C})$  and which preserves the trace of the input matrix. There are sufficient reasons to assume this map  $S$  to be linear. A non-linear map will violate the notion of locality i.e. the idea that any spatially localised action does not immediately influence distant parts (see sec. 1.4, [Wol12]). So we see that it requires our channel  $S$  to be a linear positivity and trace preserving map. But it turns out that such assumptions are not sufficient. What we need a stronger notion of positivity for  $S$ . Suppose we adjoin another system of dimension  $k$  (e.g. ancilla, environment) to our initial system so that the joint system is represented by the Hilbert space  $\mathbb{C}^k \otimes \mathbb{C}^n$ . We want our channel  $S$  to act on the initial system without affecting the ancillary system i.e.  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is a linear, trace and positivity preserving map such that the augmented map  $(\text{Id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  is also positivity preserving. Such an  $S$  is called  $k$ -positive. The dimension of the ancillary system could be any natural number  $k$ . Thus, we need  $(\text{Id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow$

$M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  to be positivity preserving for any natural number  $k$ . Such linear maps are called completely positive maps. With all these above ideas the quantum channel turns out to be a completely positive trace preserving linear map. Among all the various ways to define positivity of a linear map completely positive maps are most well understood object both mathematically and physically [Kra70]. The  $k$  positive maps or just the positive map lack the direct physical interpretation. But nonetheless they are very much important yet not so well understood objects in mathematics or mathematical quantum information. Entanglement is a phenomenon lying at the heart of quantum mechanics. It is one of the key properties which makes quantum mechanics completely different from any classical theory. To put it in simple terms- if we have two systems, possibly spatially separated, then unlike the classical case it is not possible to treat each system as an individual subsystem if they share a joint entangled state. We need to treat the two systems as a whole. If we perform any measurement on one subsystem then it immediately affects our description of the other subsystem and this has surprising consequences that do not occur for classically correlated systems. Positive or  $k$ -positive maps are deeply related to the entanglement theory as the  $k$ -entangled states are dual objects of  $k$  positive maps if we define a suitable duality between the spaces  $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$  and  $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ .

Another crucial role that the positive linear maps plays is to describe the dynamics of open quantum system. If a sub-atomic particle is not interacting with the environment then we call the system to be closed. The dynamics of such system is described by the Schrödinger's wave equation. If  $\rho$  represents the quantum state of a closed system at the time  $t = 0$  then the state of the system at then after some time  $t > 0$  is given by  $\rho_t = U_t^* \rho U_t$  for some one parameter group of unitary operators  $\{U_t : t \in \mathbb{R}\}$ . Stone's theorem says that there is a unique Hermitian operator  $H$  (called the Hamiltonian) such that  $U_t = \exp(itH)$  (see theorem 4.3.11, [App19]). This  $H$  is the generator of the one parameter group of unitaries  $U_t$ . But if the system is interacting with the environment then we have to take into account the joint system of (environment + initial open system). Locally the dynamics of such open system is described by one-parameter semigroup of completely positive maps. Mathematically, this is a consistent description with respect to the Schrödinger's unitary evolution theory since a famous result of Stinespring, known as the Stinespring dilation theorem (see theorem 2.2, [Wol12]) says that such any such semigroup of completely positive map can be extended to a unitary evolution of a bigger space. To put it in other way, if we consider the (open system + environment) to be a

closed system, any dynamics of initial system described by the semigroup of completely positive maps corresponds to a dynamics of the bigger joint system described by unitary operators. It was proved by Lindblad, Gorini, Kossakowski, Sudarshan(LKGS) [Lin76] [GKS76] that a semigroup of linear maps  $(\exp(t\Phi))_{t \geq 0}$  on  $M_n(\mathbb{C})$  describes a semigroup of completely positive, identity preserving map if and only if

$$\Phi(X) = i[H, X] + \sum_{j=1}^k \left\{ V_j^* X V_j - \frac{1}{2}(V_j^* V_j X + X V_j^* V_j) \right\}$$

for some Hermitian matrix  $H$  (Hamiltonian) and a set of matrices  $(V_j)_{j=1}^k \subset M_n(\mathbb{C})$ .

So to summarise the two models- the dynamics of closed and open system respectively are connected by the Stinespring's dilation theorem and what Stone's theorem stands for closed quantum system i.e. to describe the generator, LKGS's theorem does the analogous thing for open quantum system.

## 1.1 Presentation of the Thesis

The second chapter briefly discusses the existing results on different cones of positive operators which are very much important to both operator theory and quantum information. These are mostly works of Choi[Choi75], Kraus[Kra70], De Pillis[Pil67], Jamiołkowski[Jam72], Skowronek, Størmer, Zyczkowski [SSZ09], Ranade, Ali [RA07]. We will begin with the notion of quantum state, separability and entanglement. Then we will start with the conjugate map  $Ad_S : X \mapsto S^* X S$ , the simplest of the positive maps and will gradually try to introduce different positive cones of operators. The third chapter discusses the nice error basis and its constructions which are mainly the results E. Knill, A. Klappenecker, M. Rötteler [Kni96], [KR02],[KR05].

The last two chapters of the present thesis i.e. the chapter 4 and the chapter 5 consist of the research works that we have done. Our main motivations of the work for this thesis are two questions. First, does having a good basis of the space  $\text{Lin}(M_n(\mathbb{C}))$  the space of linear maps on  $M_n(\mathbb{C})$ , help to understand positivity of a linear map? Second, can we characterise the semigroup of  $k$ -(super)positive maps in terms of its generator, similar to the LKGS result for the completely positive semigroup? Chapter 4 is devoted to the first question and Chapter 5 to the second.

Chapter 4 is based on our paper [BCF23(ii)]. In this chapter, we take advantage of the notion of nice error basis(NEB) of  $M_n(\mathbb{C})$  to construct a convenient basis of the space

$\text{Lin}(M_n(\mathbb{C}))$ . A *nice error basis* is a unitary projective representation  $\{\pi_g \in \mathcal{U}(n) : g \in G\}$  of a group  $G$  of order  $n^2$  such that the trace of each such representative  $\pi_g$  is zero except for  $g = 1$ , the group identity element. The projective representation is a twisted homomorphism by a 2-cocycle i.e.  $\pi_g \cdot \pi_h = \omega(g, h)\pi_{gh}$  for any  $g, h \in G$  and a 2-cocycle map  $\omega : G \times G \rightarrow \mathbb{C}$  (see def. 3.2.1). Such a set of unitary matrices forms an orthogonal basis of  $M_n(\mathbb{C})$  with respect to the Hilbert-Schmidt inner product. Now consider the map  $T : M_n(\mathbb{C}) \otimes M_n^*(\mathbb{C}) \ni A \otimes B^* \mapsto T_{A, B^*} \in \text{Lin}(M_n(\mathbb{C}))$  where the linear map  $T_{A, B^*}$  is given by  $T_{A, B^*} : X \mapsto AXB^*$  for any  $X \in M_n(\mathbb{C})$ . The map  $T$  can be extended linearly to a Hilbert space isomorphism between the space  $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^*$  and  $\text{Lin}(M_n(\mathbb{C}))$ . So if  $\{B_i : 1 \leq i \leq n^2\}$  is a basis of  $M_n(\mathbb{C})$  then  $\{T_{B_i, B_j^*} : 1 \leq i, j \leq n^2\}$  is a basis of  $\text{Lin}(M_n(\mathbb{C}))$ . In particular if we take an NEB  $\{\pi_g : g \in G\}$  as a basis of  $M_n(\mathbb{C})$  then  $\{T_{x, y} := \pi_x(\cdot)\pi_y^* : x, y \in G\}$  is a basis of  $\text{Lin}(M_n(\mathbb{C}))$ . What is more  $\{T_{x, y} : x, y \in G\}$  is also an NEB of  $\text{Lin}(M_n(\mathbb{C}))$  if we identify the space with  $M_{n^2}(\mathbb{C})$ . As an example of NEB we will do the computation with Weyl operators  $\{W_x : x \in \mathbb{Z}_n \times \mathbb{Z}_n\}$  (see subsection 3.2). We decompose a linear map  $\alpha$  on  $M_n(\mathbb{C})$  with respect to an NEB  $\{T_{x, y} : x, y \in G\}$  i.e.  $\alpha = \sum_{x, y \in G} D_\alpha(x, y)T_{x, y}$  to obtain a  $n^2 \times n^2$  decomposition coefficient matrix  $D_\alpha$  (or a kernel on  $G \times G$ ). We now try to characterise different positivity properties of  $\alpha$  with respect to  $D_\alpha$ . We will prove the following result on 1-positivity-

**Theorem 1.1.1.** *Let  $\{B_x\}_{x=1,2,\dots,n^2}$  be a basis of  $M_n(\mathbb{C})$ . Consider a linear map  $\alpha \in L(M_n(\mathbb{C}))$  of form  $\alpha(X) = \sum_{x, y=1}^{n^2} D_\alpha(x, y)B_x X B_y^*$ . Then  $\alpha$  is*

- i. Hermiticity preserving if and only if  $D_\alpha$  is Hermitian.*
- ii. positive if and only if for any  $v, w \in \mathbb{C}^n$ ,*

$$\langle v \otimes w, \tilde{\alpha}(v \otimes w) \rangle \geq 0$$

where  $\tilde{\alpha} = \tau \circ \sum_{x, y=1}^{n^2} D_\alpha(x, y)(B_x \otimes B_y^*)$  and  $\tau(u \otimes v) = v \otimes u$  is the flip operator.

We will show a characterisation of complete positivity-

**Theorem 1.1.2.** *A linear map  $\alpha \in \text{Lin}(M_n)$  is a completely positive map with Kraus rank  $r$  if and only if the corresponding coefficient matrix  $D_\alpha \in M_{n^2}(\mathbb{C})$  is positive semi-definite of rank  $r$ .*

One should observe the similarity of the properties of  $D_\alpha$  and Choi-Jamiołkowski matrix  $C_\alpha$ . We will also show the following correspondence between  $D_\alpha$  and Choi matrix  $C_\alpha$



**Proposition 1.1.1.** *If  $T_{x,y}$  is defined with respect to the Weyl operators and if  $\alpha$  is of the form  $\alpha = \sum_{x,y \in \mathbb{Z}_n \times \mathbb{Z}_n} D_\alpha(x,y) T_{x,y}$ , then its Choi-Jamiołkowski matrix is given by*

$$C_\alpha(v,w) = \sum_{x_2, y_2 \in \mathbb{Z}_n} \frac{\varkappa(x_2, v_1)}{\varkappa(y_2, w_1)} D_\alpha((v_2 - v_1, x_2), (w_2 - w_1, y_2)),$$

for  $v, w \in \mathbb{Z}_n \times \mathbb{Z}_n$ . Conversely,  $D_\alpha$  can be computed from the Choi-Jamiołkowski matrix via

$$D_\alpha(x,y) = \frac{1}{n} \sum_{a,b \in \mathbb{Z}_n} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} C_\alpha((a, a + x_1), (b, b + y_1))$$

for  $x, y \in \mathbb{Z}_n \times \mathbb{Z}_n$ .

We will also characterise complete co-positivity of  $\alpha$ .

**Theorem 1.1.3.** *A linear map  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is completely co-positive iff the convolution product*

$$\sum_{p,q \in \mathbb{Z}_n \times \mathbb{Z}_n} \frac{\chi(p, x - p)}{\chi(q, y - q)} \text{Tr}(\overline{W}_p W_q) D_\alpha(x - p, y - q)$$

is positive semi-definite.

We will show a characterisation of entanglement breaking map for the special case of dimension  $n = 2$ .

**Corollary 1.1.1.** *A linear map  $\alpha \in \text{Lin}(M_2)$  is 1-super positive iff  $D_\alpha = \sum_{j=1}^r |l_j\rangle\langle l_j|$  where  $l_j = (l_j(1), \dots, l_j(4))^t$  is a vector in  $\mathbb{C}^4$  satisfying  $l_j(1)^2 = \sum_{k=2}^4 l_j(k)^2$ .*

The final chapter i.e. chapter 5 is based on our paper [BCF23(i)]. This chapter of the present thesis is about the second question we posed on the topic of one parameter semigroup of positive operators. We were motivated by the Lindblad, Gorini, Kossakowski, Sudarshan's theorem on the characterisation of the generator of a semigroup of completely positive maps to ask if it is also possible to give a characterisation of semigroup of other positive maps e.g.  $k$ -(super)positive maps. For this we generalise a result of M. Schürmann [Sch85] on the Schoenberg type correspondence of unital semigroup to its non-unital version. We prove the following result

**Theorem 1.1.4.** *Let  $A$  be a real Banach algebra with a closed convex cone  $C \subseteq A$  with non-empty interior. Let  $a_0 \in C$  be an idempotent such that for any  $c \in C$ , we have  $a_0 c a_0 \in C$ .*

*We assume furthermore that for any  $c \in C$  we have  $c^n \in C$  for  $n \geq 1$ .*

*Then, for any  $b \in A$  such that  $ba_0 = a_0 b = b$ , the following statements are equivalent.*

- (i)  $b$  is  $a_0$ -conditionally positive on  $C^\circ$ , i.e.,  $\varphi(b) \geq 0$  for all  $\varphi \in C^\circ$  with  $\varphi(a_0) = 0$ .
- (ii)  $\exp_{a_0}(tb) := \lim_{n \rightarrow \infty} (a_0 + \frac{tb}{n})^n \in C$  for all  $t \geq 0$ .

The reason we couldn't use the Schürmann's original result directly was that in his version of the result the cone contained the unit of the Banach algebra and the semigroup started at the unit element. But if we wish to consider a general positive cone in  $\text{Lin}(M_n(\mathbb{C}))^{\text{sa}}$  then it does not necessarily contain the identity map. For example, the cone of  $k$ -super positive maps (or also known as  $k$ -partially entanglement breaking map) does not contain the identity map for  $k < n$ . So a semigroup of  $k$ -super positive maps can not start at the identity map at  $t = 0$ . With the non-unit version of the Schoenberg type correspondence mentioned above we will prove the following result.

**Theorem 1.1.5.** *Let*

$$A = \text{Lin}(M_n(\mathbb{C}))^{\text{sa}} = \{T \in L(M_n, M_n); T \circ * = * \circ T\}$$

and let  $C \subseteq A$  be one of the cones of  $k$ -positive, completely positive or  $k$ -super positive maps,

Fix an idempotent map  $\Phi_0 \in C$ . Then for  $\Psi \in A$  with  $\Psi \circ \Phi_0 = \Phi_0 \circ \Psi = \Psi$  the following are equivalent.

- (i) We have  $\exp_{\Phi_0}(t\Psi) = \Phi_0 + \sum_{n=1}^{\infty} \frac{t^n \Psi^{on}}{n!} \in C$  for all  $t \geq 0$ ;
- (ii)  $\Psi$  is  $\Phi_0$ -conditionally positive on  $C^\circ$ , i.e., we have

$$\forall v \in C^\circ, \quad \langle v, \Phi_0 \rangle = 0 \quad \Rightarrow \quad \langle v, \Psi \rangle \geq 0.$$

In particular we have the following characterisation of the generator of a semigroup of  $k$ -positive maps-

**Corollary 1.1.2.**  $\Psi = T_S$  with  $S = \sum A_i \otimes B_i \in (M_n \otimes M_n^{\text{op}})^{\text{sa}}$  generates a semigroup  $T_t = \exp(t\Psi)$ ,  $t \geq 0$ , of  $k$ -positive maps if and only if

$$\forall V \in M_n, \quad (\text{rank}(V) \leq k \text{ and } \text{Tr}(V) = 0) \quad \Rightarrow \quad \sum \text{Tr}(A_i V^*) \text{Tr}(B_i V) \geq 0.$$

We can also retrieve the LKGS theorem using this characterisation. We will end the present thesis by our discussion on few examples of semigroup of positive operators generated by the depolarising channel  $P$ , transposition  $T$ , conditional expectation onto diagonal  $C$  and identity  $\text{Id}$  i.e. the semigroup  $\exp t(\alpha P + \beta T + \gamma C + \delta \text{Id})$ , where  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ .

## 1.2 Notations and Conventions

In the following discussion we will often use Dirac's bra-ket notation to denote rank-one operators. To define it in short, we understand  $|x\rangle$  to be a vector in some Hilbert space  $\mathbb{C}^n$  and  $\langle x|$  to be its dual vector. Then for two vectors  $x, y \in \mathbb{C}^n$  we define the rank one operator  $|x\rangle\langle y|$  by-

$$|x\rangle\langle y|(z) := \langle y|z\rangle x$$

for any  $z \in \mathbb{C}^n$ . The space of  $n \times n$  and  $n \times m$  matrices will be denoted by  $M_n(\mathbb{C})$  and  $M_{n,m}(\mathbb{C})$  respectively. If  $\{e_i\}_{i=1}^n$  is the standard basis of  $\mathbb{C}^n$  and  $\{E_{ij}; 1 \leq i, j \leq n\}$  is the standard matrix units (i.e. 1 at  $ij$ -th position and zero everywhere else) then one can easily verify that  $E_{ij} = |e_i\rangle\langle e_j|$ . We will denote  $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  the set of all linear maps from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$ . If  $n = m$  we will abbreviate it to  $\text{Lin}(M_n(\mathbb{C}))$ . We will frequently use the notation  $M_n(\mathbb{C})^*$  which is defined by  $M_n(\mathbb{C})^* := \{A^* : A \in M_n(\mathbb{C})\}$ .



## Chapter 2

# Positive Cones

We recall some definitions and basic facts on cones.

**Definition 2.0.1.** A subset  $C$  of a topological vector space  $V$  over  $\mathbb{R}$  is called a convex cone or just cone if there is no confusion if for any two elements  $x, y \in C$  and  $\alpha \geq 0$  we have  $\alpha x + y \in C$ . Furthermore, we call a cone solid if it has nonempty interior and it is pointed if  $C \cap (-C) = \{0\}$ .

**Definition 2.0.2.** For a cone  $C \subseteq V$  we define its dual cone  $C^\circ$ ,

$$C^\circ := \{z \in V'; \langle z, x \rangle \geq 0, x \in C\}$$

where  $V'$  is the topological dual space of  $V$ .

The following results are well known so we mention without proof (see Lemma 3.2, Corollary 3.3 [AT07]).

**Proposition 2.0.1.** Let  $C$  be a closed convex cone in  $\mathbb{R}^n$ . Then the following statements are equivalent:

- a.  $C$  is pointed i.e.  $C \cap (-C) = \{0\}$ .
- b.  $C^\circ - C^\circ = \mathbb{R}^n$ .
- c.  $C^\circ$  has non-empty interior.
- d.  $\text{span}(C^\circ) = \mathbb{R}^n$ .

If we assume  $C$  is closed (which is the case for all cones we give as examples) the same results hold if we replace  $C$  by  $C^\circ$  in the above proposition via the Bipolar theorem  $(C^\circ)^\circ = C$  (see theorem 5.5, [Sim11]).

## 2.1 Quantum States In $M_n(\mathbb{C})$

Let  $\mathcal{H}$  be a Hilbert space of dimension  $n$ . A *pure state* of  $\mathcal{H}$  is a vector  $|x\rangle$  in  $\mathcal{H}$  such that  $\|x\| = 1$ . We can also associate to the unit vector  $x$  the rank one projection  $P_x \in M_n(\mathbb{C})$  onto the subspace generated by the unit vector  $x \in \mathcal{H}$ . Upto the bijection  $|x\rangle \leftrightarrow P_x$ , we use the term pure state to denote both  $|x\rangle$  and  $P_x$ . We say a state is *mixed* if it is convex combination of pure states i.e.  $\rho \in M_n(\mathbb{C})$  is a mixed state if

$$\rho = \sum_{k=1}^r \lambda_k P_{x_k} \quad \text{and} \quad \sum_{k=1}^r \lambda_k = 1,$$

where  $P_{x_k}$  is the projection onto the subspace generated by  $|x_k\rangle$ . In general, a quantum state  $\rho$  is described by a positive semi-definite operator on  $\mathcal{H} = \mathbb{C}^n$  of trace 1 which is also called a *density matrix*. This does not lead to confusion since any such positive matrix of trace 1 has the spectral decomposition of the above mentioned form. The trace condition is important for describing a physical system but as we can always normalise a positive matrix when ever it is convenient we can easily get rid of it. So by a state or density matrix (non-normalised)  $\rho \in M_n(\mathbb{C})$ , we usually understand it as a positive matrix  $\rho \geq 0$ . The collection of such non-normalised states will be denoted by  $M_n(\mathbb{C})^+$ . If we denote the set of all Hermitian ( or self-adjoint) matrices of  $M_n(\mathbb{C})$  by  $M_n^{\text{sa}}(\mathbb{C})$  then  $M_n(\mathbb{C})^+ \subset M_n^{\text{sa}}(\mathbb{C})$ . Observe that  $M_n^{\text{sa}}(\mathbb{C})$  is a vector space over  $\mathbb{R}$  of dimension  $n^2$ .

Now let's consider a composite quantum system of two subsystem  $\mathcal{H}_A = \mathbb{C}^n$  and  $\mathcal{H}_B = \mathbb{C}^m$ . The joint quantum system is described by the tensor product of individual subsystems i.e.  $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^n \otimes \mathbb{C}^m$ . We denote by  $(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^{\text{sa}}$  the real linear space of self-adjoint operators and by  $(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+$  the set of all positive operators on  $\mathbb{C}^n \otimes \mathbb{C}^m$ . One obvious way to construct a state or density operator on the joint system is to consider the convex combinations of tensor products of density operators of individual subsystems. But not all density operator of the joint system can be expressed in that manner.

**Definition 2.1.1** (Separable state/Entangled state). *A state  $\rho \in (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+$  for some  $n, m \in \mathbb{N}$  is called separable if and only if there exists finite sets of states  $\{\rho_i^{(1)}\}_{i \in I} \subset M_n(\mathbb{C})^+$ ,  $\{\rho_i^{(2)}\}_{i \in I} \subset M_m(\mathbb{C})^+$  and non-negative numbers  $\{p_i\}_{i \in I}$  such that  $\rho = \sum_{i \in I} p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$  and  $\sum_{i \in I} p_i = 1$ .*

*It is called entangled if it can not be expressed in the above manner.*

There is an obvious isomorphism between  $\mathbb{C}^n \otimes \mathbb{C}^m$  and the space of all linear maps

from  $\mathbb{C}^n$  to  $\mathbb{C}^m$  i.e.  $\text{Lin}(\mathbb{C}^n, \mathbb{C}^m) = M_{n,m}(\mathbb{C})$ . If  $\{e_i\}_{i=1}^n$  and  $\{f_j\}_{j=1}^m$  are two orthonormal bases of  $\mathbb{C}^n$  and  $\mathbb{C}^m$  respectively then for any vector  $|x\rangle = \sum_{i,j} x_{ij} |e_i\rangle \otimes |f_j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$  the corresponding matrix  $X = \sum_{i,j} x_{ij} |e_i\rangle \langle f_j| \in M_{n,m}(\mathbb{C})$  is called the *matrisation* of the vector  $x$ . Conversely, for any matrix  $X \in \text{Lin}(\mathbb{C}^n, \mathbb{C}^m)$  of the above form the corresponding vector  $x \in \mathbb{C}^n \otimes \mathbb{C}^m$  is called the *vectorisation* of  $X$ .

**Theorem 2.1.1** (Schmidt decomposition). *For each non-zero vector  $x \in \mathbb{C}^n \otimes \mathbb{C}^m$  there exists positive numbers  $\lambda_1, \lambda_2, \dots, \lambda_s$  for some  $1 \leq s \leq \min(n, m)$  and orthonormal systems  $(\phi_i)_{1 \leq i \leq s} \subset \mathbb{C}^n$ ,  $(\psi_i)_{1 \leq i \leq s} \subset \mathbb{C}^m$  such that*

$$|x\rangle = \sum_{1 \leq i \leq s} \lambda_i |\phi_i\rangle \otimes |\psi_i\rangle,$$

where  $\lambda_i$  satisfies  $\sum_{i=1}^s \lambda_i^2 = \|x\|^2$ . The number  $s$  is uniquely determined and so are numbers  $(\lambda_1, \dots, \lambda_s)$  if we require that  $\lambda_1 \geq \dots \geq \lambda_s$ .

*Proof.* Let  $\{|e_i\rangle : 1 \leq i \leq n\}$  and  $\{|f_j\rangle : 1 \leq j \leq m\}$  be two bases of  $\mathbb{C}^n$  and  $\mathbb{C}^m$  respectively. Any vector  $x \in \mathbb{C}^n \otimes \mathbb{C}^m$  can be written in the form  $|x\rangle = \sum_{i=1}^n \sum_{j=1}^m x_{ij} |e_i\rangle \otimes |f_j\rangle$  with respect to these bases. Let  $X = \sum_{i,j=1}^{n,m} x_{ij} |e_i\rangle \langle f_j|$  be the matrisation of the vector  $x$  and let  $r$  be its rank. Consider the singular value decomposition  $X = UDV^*$ , where  $U = (u_{ij})_{n \times n}$  is an unitary matrix of order  $n$ ,  $D = (\lambda_{ij})_{n \times m}$  is a diagonal matrix of rank  $s$  and  $V = (v_{ij})_{m \times m}$  is an unitary matrix of order  $m$ . If we vectorise  $X$  now with respect to this decomposition then we have

$$\begin{aligned} |x\rangle &= \sum_{i,j,k} u_{ik} \lambda_k v_{jk} |e_i\rangle \otimes |f_j\rangle \\ &= \sum_{k=1}^s \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle, \end{aligned}$$

where the set of vectors  $\{|\phi_k\rangle : 1 \leq k \leq s\}$  and  $\{|\psi_k\rangle : 1 \leq k \leq s\}$  constitute orthonormal set in  $\mathbb{C}^n$  and  $\mathbb{C}^m$  respectively, since  $U, V$  are unitaries. That  $\|x\|^2 = \sum_{k=1}^s \lambda_k^2$ , follows directly now from the computation of  $\langle x|x\rangle$ .  $\square$

The number  $s$  is called the *Schmidt rank* of the vector  $x$  and denoted by  $\text{SR}(x)$  and  $\lambda_1, \dots, \lambda_s$  are called *Schmidt coefficients* of  $|x\rangle$ .

**Corollary 2.1.1.** *Let  $(u_i)_{1 \leq i \leq s}$  and  $(v_i)_{1 \leq i \leq s}$  be two systems of not necessarily orthogonal vectors in  $\mathbb{C}^n$  and  $\mathbb{C}^m$ . Then  $\text{SR}(\sum_{i=1}^s u_i \otimes v_i) \leq s$ .*

*Proof.* The Hilbert spaces spanned by the systems  $(u_i)_{1 \leq i \leq s}$  and  $(v_i)_{1 \leq i \leq s}$  are of dimension at most  $s$ . So the result follows directly from the Schmidt decomposition theorem.  $\square$

**Corollary 2.1.2** (Purification lemma). *Let  $\rho \in M_n(\mathbb{C})^+$  be a density operator. Then there always exists  $k \geq 1$  and a pure state  $|\Phi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^k$ ,  $k \geq \text{rank}(\rho)$ , such that  $\rho = \text{Tr}_2(|\Phi\rangle\langle\Phi|)$ , where  $\text{Tr}_2$  is the partial trace with respect to the second tensor factor.*

*Proof.* Suppose the spectral decomposition of the density operator  $\rho \in M_n(\mathbb{C})^+$  is of the form

$$\rho = \sum_{i=1}^r \lambda_i |\phi_i\rangle\langle\phi_i|,$$

where  $r$  is the rank of  $\rho$ . Now the recipe to cook up  $|\psi\rangle$  follows directly from the Schmidt decomposition theorem. We take  $\{\psi_i\}_{i=1}^r$  an orthonormal system in the space  $\mathbb{C}^k$ , where  $k \geq r$ . We define

$$|\Phi\rangle := \sum_{i=1}^r \sqrt{\lambda_i} |\phi_i\rangle \otimes |\psi_i\rangle.$$

Therefore we have

$$\begin{aligned} \text{Tr}_2(|\Phi\rangle\langle\Phi|) &= \text{Tr}_2 \left( \sum_{i,j=1}^r \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j| \otimes |\psi_i\rangle\langle\psi_j| \right) \\ &= \sum_{i,j=1}^r \delta_{ij} \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j| \\ &= \sum_{i=1}^r \lambda_i |\phi_i\rangle\langle\phi_i| = \rho. \end{aligned}$$

□

## 2.2 Positive Cones In $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$

It is easy to see that the set of all separable states of  $\mathbb{C}^n \otimes \mathbb{C}^m$  form a closed convex cone. We denote this cone  $\text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$ . If  $\rho = \sum_{i \in I} p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$  is separable then we can find the spectral decomposition of each  $\rho_i^{(1)}$  and  $\rho_i^{(2)}$  in terms of rank one projections  $\rho_i^{(1)} = \sum_x |x^{(i)}\rangle\langle x^{(i)}|$  and  $\rho_i^{(2)} = \sum_y |y^{(j)}\rangle\langle y^{(j)}|$ . So we can write  $\rho = \sum |x^{(i)}\rangle\langle x^{(i)}| \otimes |y^{(j)}\rangle\langle y^{(j)}|$ . Thus we see that the cone  $\text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$  is generated by the elements of the form  $|x\rangle\langle x| \otimes |y\rangle\langle y| \cong |x \otimes y\rangle\langle x \otimes y|$ . So we find that

$$\text{Sep}(\mathbb{C}^n, \mathbb{C}^m) = \text{Conv} \{ |x \otimes y\rangle\langle x \otimes y| : x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m \}.$$

Since  $x \otimes y$  has Schmidt rank 1 we can say that the cone of separable states is the convex hull of the projections onto vectors of Schmidt rank 1. So if a bipartite state  $\rho \in (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+$  is entangled then we can spectral decompose the state into



convex combination of rank one projections onto vectors such that at least one of the vectors has Schmidt rank  $> 1$ . We can use the maximal Schmidt rank of such vectors in the spectral decomposition of a bipartite state to categorise different entangled states. This motivates to define the cone of  $k$ -entangled states.

**Definition 2.2.1** ( $k$ -Entangled operator). *Let  $k$  be a positive integer. A state  $\rho \in (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+$  is called  $k$ -entangled operator iff it belongs to the set*

$$k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) = \text{Conv} \left\{ \sum_{ij=1}^k |x_i \otimes y_i\rangle\langle x_j \otimes y_j| : \{x_i\}_{i=1}^k \subset \mathbb{C}^n, \{y_j\}_{j=1}^k \subset \mathbb{C}^m \right\}.$$

Observe that for  $k = 1$ ,  $1\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) = \text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$  the cone of separable states. Since  $\text{SR}(\sum_{ij=1}^k x_i \otimes y_j) \leq k$ , equivalently  $\rho$  is  $k$ -entangled iff the Schmidt rank of its eigen vectors  $\leq k$  i.e.

$$k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) = \text{Conv} \{ |v\rangle\langle v| : v \in \mathbb{C}^n \otimes \mathbb{C}^m, \text{SR}(v) \leq k \}$$

We can check that  $k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$  is a closed convex cone for each  $k \in \mathbb{N}$ . It is obvious that if  $k = \min\{n, m\}$  then  $k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) = (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+$ . It also follows from the definition that  $k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) \subseteq (k+1)\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$ . In fact, the inclusion is strict. Let  $v \in \mathbb{C}^n \otimes \mathbb{C}^m$  be a vector with  $\text{SR}(v) = k + 1$ . Obviously,  $|v\rangle\langle v| \in (k + 1)\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$ . Let's assume we can write  $|v\rangle\langle v| = \sum_{i=1}^l |v_i\rangle\langle v_i|$ . This equality can only hold if each  $v_i$  is a scalar multiple of the vector  $v$ . Therefore  $\text{SR}(v_i)$  can not be less than  $k + 1$  and thus  $|v\rangle\langle v| \notin k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$ .

For any two elements  $A, B \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$  we can define the Hilbert-Schmidt inner product of  $A$  and  $B$  by

$$\langle A, B \rangle := \text{Tr}(AB^*).$$

If we restrict our attention to the real linear space  $(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^{\text{sa}}$  then we can define

$$\langle A, B \rangle := \text{Tr}(AB^t)$$

We now try to find the dual cone of  $k\text{-Ent}$  with respect to the Hilbert-Schmidt inner product on  $(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^{\text{sa}}$ . Since the positive rank 1 operators with range vector having Schmidt rank  $\leq k$ , generate the cone of  $k$ -entangled states, it suffices to work with such generators. Let  $\rho = \sum_{i,j=1}^k |x_i \otimes y_i\rangle\langle x_j \otimes y_j| \in k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$  and  $\phi \in M_n(\mathbb{C}) \otimes$

$M_m(\mathbb{C})$ . Then the scalar product of two

$$\begin{aligned} 0 \leq \langle \rho, \phi \rangle &= \left\langle \sum_{i,j=1}^k |x_i \otimes y_i\rangle \langle x_j \otimes y_j|, \phi \right\rangle = \sum_{i,j=1}^k \text{Tr}(|x_i \otimes y_i\rangle \langle \phi(x_j \otimes y_j)|) \\ &= \sum_{i,j=1}^k \langle x_i \otimes y_i | \phi(x_j \otimes y_j) \rangle = \left\langle \sum_{i=1}^k x_i \otimes y_i \middle| \phi \left( \sum_{j=1}^k x_j \otimes y_j \right) \right\rangle. \end{aligned}$$

This motivates us to define the  $k$ -block positive operators. The set of all such operators forms the dual cone of the cone of  $k$ -entangled operators.

**Definition 2.2.2** ( $k$ -Block positive operator). *An operator  $\rho \in (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))$  is called  $k$ -Block positive if for any  $\{x_i\}_{i=1}^k \subset \mathbb{C}^n$  and  $\{y_j\}_{j=1}^k \subset \mathbb{C}^m$  we have*

$$\left\langle \sum_{i=1}^k x_i \otimes y_i \middle| \rho \left( \sum_{j=1}^k x_j \otimes y_j \right) \right\rangle \geq 0.$$

Equivalently,  $\langle v | \rho | v \rangle \geq 0$  for all  $v \in \mathbb{C}^n \otimes \mathbb{C}^m$  with  $\text{SR}(v) \leq k$ . We denote the set of all such operators by  $k\text{-BP}(\mathbb{C}^n, \mathbb{C}^m)$ .

Note that the defining property of  $k$ -Block positivity implies that such operators are Hermitian i.e.  $k\text{-BP}(\mathbb{C}^n, \mathbb{C}^m) \subset (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^{\text{sa}}$  (see A2, [Sko08]). It is clear from the definition that if  $k = \min\{n, m\}$  then  $k\text{-Block}(\mathbb{C}^n, \mathbb{C}^m) = (M_n \otimes M_m)^+$  and we have the following inclusion

$$1\text{-BP}(\mathbb{C}^n, \mathbb{C}^m) \supseteq 2\text{-BP}(\mathbb{C}^n, \mathbb{C}^m) \supseteq \dots \supseteq \min(m, n)\text{-BP}(\mathbb{C}^n, \mathbb{C}^m) = (M_n \otimes M_m)^+.$$

In fact, all the above inclusions are strict since its dual cones follow strict inclusion. We can write

$$k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)^\circ = k\text{-BP}(\mathbb{C}^n, \mathbb{C}^m),$$

and since all these cones are closed, by the Bipolar theorem (theorem 5.5, [Sim11]) we have

$$k\text{-BP}(\mathbb{C}^n, \mathbb{C}^m)^\circ = k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m).$$

We summarise the chain of inclusions of positive cones in  $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ :

$$\text{Sep}(\mathbb{C}^n, \mathbb{C}^m) \subset k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m) \subset (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^+ \subset k\text{-BP}(\mathbb{C}^n, \mathbb{C}^m) \subset 1\text{-BP}(\mathbb{C}^n, \mathbb{C}^m).$$

## 2.3 Positive Cones In $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$

**Definition 2.3.1** (Hermitianity preserving map). *A linear map  $S \in \text{Lin}(M_n, M_m)$  is called Hermitianity preserving iff  $S(M_n^{\text{sa}}(\mathbb{C})) \subset M_m^{\text{sa}}(\mathbb{C})$ . We denote the set of all Hermitianity preserving linear maps from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$  by  $\text{Lin}(M_n, M_m)^{\text{sa}}$ .*

It is easy to see that  $\text{Lin}(M_n, M_m)^{\text{sa}}$  is a  $\mathbb{R}$  linear space (but not  $\mathbb{C}$  linear).

**Lemma 2.3.1.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}))$  is Hermitianity preserving if and only if  $S(X^*) = S(X)^*$  for any  $X \in M_n(\mathbb{C})$ . In particular, if  $(E_i)_{1 \leq i \leq n^2}$  is a basis of  $M_n(\mathbb{C})$ , this is the case, if and only if  $S(E_i^*) = S(E_i)^*$  for all  $1 \leq i \leq n^2$ .*

*Proof.* Suppose the condition  $S(X^*) = S(X)^*$  holds for any  $X \in M_n(\mathbb{C})$ . Then for an  $H \in M_n^{\text{sa}}(\mathbb{C})$  we have  $S(H) = S(H^*) = S(H)^*$  which shows that  $S(H) \in M_m^{\text{sa}}(\mathbb{C})$ . Conversely, we assume that  $S(M_n^{\text{sa}}(\mathbb{C})) \subset M_m^{\text{sa}}(\mathbb{C})$ . For any  $X \in M_n(\mathbb{C})$  we can decompose it into sum of two Hermitian elements i.e.  $X = H_1 + iH_2$ . So we have

$$\begin{aligned} S(X^*) &= S(H_1 - iH_2) = S(H_1) - iS(H_2) = S(H_1)^* - iS(H_2)^* \\ &= (S(H_1) + iS(H_2))^* = (S(H_1 + iH_2))^* = S(X)^* \end{aligned}$$

□

**Example 2.3.1.** *Consider the linear map  $M_n(\mathbb{C}) \ni X \mapsto A^*X^tA \in M_m(\mathbb{C})$  for some  $n \times m$  matrix  $A$ . Then it is easy to see that the map is Hermitianity preserving. In fact, De Pillis showed that any Hermitianity preserving linear map  $S \in \text{Lin}(M_n, M_m)$  will be of the form  $S(X) = \sum_{i=1}^r c_i A_i^* X^t A_i$  for any  $X \in M_n(\mathbb{C})$ , where  $c_i$ 's are real numbers and  $A_i$ 's certain  $n \times m$  matrices [Pil67].*

Similar to the Hermitianity preserving maps, we define the positivity preserving maps.

**Definition 2.3.2.** *A linear map  $T \in \text{Lin}(M_n, M_m(\mathbb{C}))$  is called positive iff  $T(M_n(\mathbb{C})^+) \subset M_m(\mathbb{C})^+$ . We denote the set of all positive map from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$  by  $\mathcal{P}_1(n, m)$*

It is easy to see that  $\mathcal{P}_1(n, m)$  is a convex cone in  $\text{Lin}(M_n, M_m)$ ,

**Example 2.3.2.** *i. The transposition map  $T : M_n(\mathbb{C}) \ni X \mapsto X^t \in M_n(\mathbb{C})$  with respect to a fixed orthonormal basis of  $\mathbb{C}^n$  is a positive map.*

*ii. For any  $n \times m$  matrix  $V$  the conjugate map  $\text{Ad}_V : M_n(\mathbb{C}) \ni X \mapsto V^*XV \in M_m(\mathbb{C})$  is a positive map. Indeed, if  $X \in M_n(\mathbb{C})^+$  then there exists  $C \in M_n(\mathbb{C})$  such that  $X = C^*C$ . Therefore  $\text{Ad}_V(X) = V^*XV = V^*C^*CV = (CV)^*CV$ , which is positive.*

If  $H$  is a Hermitian matrix in  $M_n(\mathbb{C})$  then we can write it as  $H = P_1 - P_2$  for some positive matrices  $P_1, P_2 \in M_n(\mathbb{C})$ . Indeed, we can define  $|H| := (H^*H)^{\frac{1}{2}}$  using functional calculus. Then  $P_1 := \frac{|H|+H}{2}$  and  $P_2 := \frac{|H|-H}{2}$ . If  $S \in \mathcal{P}_1(n, m)$  then  $S(H)^* = S(P_1)^* -$

$S(P_2)^* = S(P_1) - S(P_2) = S(H)$ . So we see that any positive map is also Hermiticity preserving. So we can think of  $\mathcal{P}_1(n, m)$  as a convex cone inside the  $\mathbb{R}$  linear space  $\text{Lin}(M_n, M_m)^{\text{sa}}$ .

### 2.3.1 Choi-Jamiolkowski Isomorphism

Let's equip the space  $M_n(\mathbb{C})$  with Hilbert-Schmidt inner product  $\langle A, B \rangle := \text{Tr}(AB^*)$ . De Pillis [Pil67] and Jamiolkowski [Jam72] considered a map  $\mathcal{J}_1 : \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C})) \rightarrow M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$  characterised by the property that

$$\langle \mathcal{J}_1(S), A^* \otimes B \rangle_{M_n \otimes M_m} = \langle S(A), B \rangle_{M_m}$$

holds for any  $A \in M_n(\mathbb{C}), B \in M_m$  and  $T \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ . De Pillis proved that

**Lemma 2.3.2.** *The map  $\mathcal{J}_1$  is uniquely defined and an isometric isomorphism between the spaces  $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  and  $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ . Moreover, for any orthonormal basis  $(E_i)_{1 \leq i \leq n^2}$  of  $M_n(\mathbb{C})$  and every operator  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  we have*

$$\mathcal{J}_1(S) = \sum_{i=1}^{n^2} E_i^* \otimes S(E_i).$$

**Theorem 2.3.3.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is*

- i. Hermiticity preserving iff  $\mathcal{J}_1(S) \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$  is Hermitian.*
- ii. positive iff  $\langle (v \otimes w), \mathcal{J}_1(S)(v \otimes w) \rangle \geq 0$  holds for any  $v \otimes w \in \mathbb{C}^n \otimes \mathbb{C}^m$ .*

*Proof.* The first claim was proved by De Pillis [Pil67] and Jamiolkowski [Jam72] showed the second one. □

A more useful variant of the above isomorphism was proposed by Choi in his paper [Choi75]. Instead of taking any orthonormal basis of  $M_n(\mathbb{C})$  we take the standard basis  $(E_{ij})_{1 \leq i, j \leq n}$ , where  $E_{ij}$  is the matrix whose  $ij^{\text{th}}$  is 1 and zero elsewhere. We consider the modified Jamiolkowski isomorphism or popularly known as Choi-Jamiolkowski isomorphism  $\mathcal{J}_2 : \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C})) \rightarrow M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$  defined by

$$\mathcal{J}_2(S) = \sum_{1 \leq i, j \leq n} E_{ij} \otimes S(E_{ij})$$

**Theorem 2.3.4.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is*

- i. Hermiticity preserving iff  $\mathcal{J}_2(S)$  is Hermitian.*

ii. positive iff  $\langle (v \otimes w), \mathcal{J}_2(S)(v \otimes w) \rangle \geq 0$  holds for any  $v \otimes w \in \mathbb{C}^n \otimes \mathbb{C}^m$ .

*Proof.*  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is Hermitianity preserving if and only if

$$\mathcal{J}_2(S)^* = \sum_{i,j=1}^n E_{ij}^* \otimes S(E_{ij})^* = \sum_{i,j=1}^n E_{ji} \otimes S(E_{ij}^*) = \sum_{i,j=1}^n E_{ji} \otimes S(E_{ji}) = \mathcal{J}_2(S).$$

It proves the claim of the first part of the theorem.

The proof of the second part is similar to the original proof of theorem 2.3.3 by Jamiolkowski. The proof below is taken from K. S. Ranade and M. Ali's paper [RA07]. Let  $S$  be a linear map in  $\text{Lin}(M_n, M_m)$ . It is positive iff for any  $X \in (M_n(\mathbb{C}))^+$ ,  $S(X)$  is positive in  $M_m(\mathbb{C})$ . Since any positive matrix  $X$  has the spectral decomposition into real linear combinations of rank one projections,  $S$  is positive is equivalent to the fact that  $S(|x\rangle\langle x|)$  is positive for any  $x \in \mathbb{C}^n$ . We write the basis decomposition of the projection  $|x\rangle\langle x|$  with respect to the standard basis  $\{E_{ij}\}_{i,j=1}^n$  of  $M_n(\mathbb{C})$ -

$$|x\rangle\langle x| = \sum_{ij} \langle (|x\rangle\langle x|)E_{ij} \rangle E_{ij} = \sum_{ij} \text{Tr}(|x\rangle\langle x|E_{ij}) E_{ij} = \sum_{ij} \langle x, E_{ij}x \rangle E_{ij}.$$

So  $S(|x\rangle\langle x|) = \sum_{ij} \langle x, E_{ij}x \rangle S(E_{ij})$ . Therefore we have that  $S(|x\rangle\langle x|)$  is positive if and only if  $\sum_{ij} \langle x, E_{ij}x \rangle \langle y, S(E_{ij})y \rangle$  is positive for any  $y \in M_m(\mathbb{C})$ .

$$\begin{aligned} \sum_{ij} \langle x, E_{ij}x \rangle \langle y, S(E_{ij})y \rangle &= \left\langle x \otimes y \left| (\text{id} \otimes S) \left( \sum_{ij} E_{ij} \otimes E_{ij} \right) x \otimes y \right. \right\rangle \\ &= \langle x \otimes y, \mathcal{J}_2(S)(x \otimes y) \rangle \end{aligned}$$

Thus,  $S$  is positive iff  $\langle x \otimes y, \mathcal{J}_2(S)(x \otimes y) \rangle \geq 0$  for any  $x \in \mathbb{C}^n$  and  $y \in \mathbb{C}^m$ .  $\square$

We have isomorphism between real linear spaces  $\text{Lin}(M_n, M_m)^{\text{sa}}$  and  $(M_n \otimes M_m)^{\text{sa}}$ .

### 2.3.2 Positive Cones In $\text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$

For any operator  $V \in M_{n,m}(\mathbb{C})$  we define the elementary positive map or also known as the conjugate map  $\text{Ad}_V : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  by  $\text{Ad}_V(X) := V^*XV$  for any  $X \in M_n(\mathbb{C})$ . It is known that such maps generate an extreme ray of the cone of positive maps from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$  [Sto63].

**Definition 2.3.3** (k-Super Positive Map). *Let  $k$  be a positive integer. The convex cone generated by the conjugate maps  $\text{Ad}_V$  for  $V \in M_{n,m}(\mathbb{C})$  of  $\text{rank}(V) \leq k$ , is called the cone of  $k$ -super positive maps i.e. A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is  $k$ -super positive if*

and only if it is of the form  $S = \sum_{i=1}^r \text{Ad}_{V_i}$  for some  $\{V_i\}_{i=1}^r \subset M_{n,m}(\mathbb{C})$  with  $\text{rank}(V_i) \leq k$  for each  $i$ . We denote the cone by  $\mathcal{S}_k(n, m)$ . In case  $n = m$  and there is no confusion about the dimension, we denote it simply by  $\mathcal{S}_k$ .

It is clear that  $\mathcal{S}_k(n, m) \subseteq \mathcal{S}_{k+1}(n, m) \subset \mathcal{P}_1 \subset \text{Lin}(M_n, M_m)^{\text{sa}}$  for any integer  $k \geq 1$ .

**Proposition 2.3.1.**  $\mathcal{J}_2(\text{Ad}_V) = C_{\text{Ad}_V} = |v\rangle\langle v|$ , where  $v \in \mathbb{C}^n \otimes \mathbb{C}^m$  is the vectorisation of the matrix  $V \in M_{n,m}(\mathbb{C})$  i.e. if  $V = \sum_{ij} v_{ij}|i\rangle\langle j| \in M_{n,m}(\mathbb{C})$  then its vectorisation is given by  $|v\rangle := \sum_{ij} v_{ij}|i\rangle \otimes |j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ .

*Proof.* Suppose the matrix  $V$  is given by  $V = \sum_{i,j} v_{ij}|i\rangle\langle j| \in M_{n,m}(\mathbb{C})$ , where  $|i\rangle$ s (or  $|j\rangle$ s) are standard basis of  $\mathbb{C}^n$  (or  $\mathbb{C}^m$ ). We have

$$\begin{aligned} C_{\text{Ad}_V} &= \sum_{p,q} |p\rangle\langle q| \otimes \left( \sum_{i,j} \bar{v}_{ij}|j\rangle\langle i| \right) |p\rangle\langle q| \left( \sum_{k,l} v_{kl}|k\rangle\langle l| \right) \\ &= \sum_{p,q} |p\rangle\langle q| \otimes \sum_{i,j} \sum_{k,l} \delta_{ip}\delta_{qk}\bar{v}_{ij}v_{kl}|j\rangle\langle l| \\ &= \sum_{p,q} |p\rangle\langle q| \otimes \sum_{j,l} \bar{v}_{pj}v_{ql}|j\rangle\langle l| \\ &= \left( \sum_{p,j} \bar{v}_{pj}|p\rangle \otimes |j\rangle \right) \left( \sum_{q,l} v_{ql}\langle q| \otimes \langle l| \right) = |v\rangle\langle v|. \end{aligned}$$

□

Since the rank of  $V$  is equal to the Schmidt rank of its vectorisation  $|v\rangle$  we see that the image of the cones of  $k$ -super positive operators under the Choi-Jamiolkowski isomorphism are the states (unnormalised) which are convex sum of rank one projections onto vectors in  $\mathbb{C}^n \otimes \mathbb{C}^m$  having Schmidt rank at most  $k$ . Thus we have the following corollary.

**Corollary 2.3.1.**  $\mathcal{J}_2(\mathcal{S}_k(n, m)) = k\text{-Ent}(\mathbb{C}^n, \mathbb{C}^m)$ . In particular,  $\mathcal{J}_2(\mathcal{S}_1(n, m)) = \text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$ .

If we define the Hilbert-Schmidt scalar product on the real linear space  $\text{Lin}(M_n, M_m)^{\text{sa}}$  i.e. for any two linear map  $R, S \in \text{Lin}(M_n, M_m)^{\text{sa}}$  by  $\langle R, S \rangle := \text{Tr}(C_R C_S^t)$  then it is easy to see that for any  $A \otimes B \in (M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))^{\text{sa}}$  we have  $\langle A \otimes B, C_S \rangle = \langle B, S(A) \rangle$ . We now find the dual cone of  $\mathcal{S}_k(n, m)$  in  $\text{Lin}(M_n, M_m)^{\text{sa}}$ .

**Theorem 2.3.5.** A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))^{\text{sa}}$  belongs to the dual cone  $\mathcal{S}_k(n, m)^\circ$  if and only if  $(\text{id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  is positive.

*Proof.* Let  $\text{Ad}_V$  be a map in  $\mathcal{S}_k(n, m)$  so that the Choi matrix of it is of the form  $C_{\text{Ad}_V} = |v\rangle\langle v|$  where  $v = \sum_{i=1}^k x_i \otimes y_i \in \mathbb{C}^n \otimes \mathbb{C}^m$  whose Schmidt rank is at most  $k$ . For any  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  Hermitianity preserving linear map, we have

$$\begin{aligned} \langle \text{Ad}_V, S \rangle &= \langle C_{\text{Ad}_V}, C_S \rangle = \left\langle \sum_{i,j=1}^k |x_i\rangle\langle x_j| \otimes |y_i\rangle\langle y_j|, C_S \right\rangle \\ &= \sum_{i,j=1}^k \left\langle |y_i\rangle\langle y_j|, S(|x_i\rangle\langle x_j|) \right\rangle \\ &= \sum_{i,j=1}^k \left\langle |i\rangle\langle j| \otimes |y_i\rangle\langle y_j|, |i\rangle\langle j| \otimes S(|x_i\rangle\langle x_j|) \right\rangle \\ &= \left\langle \sum_{ij=1}^k |i\rangle\langle j| \otimes |y_i\rangle\langle y_j|, \sum_{ij=1}^k |i\rangle\langle j| \otimes S(|x_i\rangle\langle x_j|) \right\rangle. \end{aligned}$$

Putting  $\phi = \sum_{i=1}^k |i\rangle \otimes |y_i\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m$  and  $\psi = \sum_{i=1}^k |i\rangle \otimes |x_i\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$  we see that

$$\langle \text{Ad}_V, S \rangle = \left\langle |\phi\rangle\langle\phi|, (\text{id}_k \otimes S)|\psi\rangle\langle\psi| \right\rangle = \left\langle \phi, (\text{id}_k \otimes S)(|\psi\rangle\langle\psi|)\phi \right\rangle.$$

It shows that  $S$  is in the dual cone  $\mathcal{S}_k(n, m)$  if and only if  $(\text{id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  is positive (see [Kye22]).  $\square$

The previous proposition motivates to define the dual cone of  $\mathcal{S}_k(n, m)$  in the following manner

**Definition 2.3.4** (*k*-Positive map). *A linear map  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is called *k*-positive if and only if the map  $(\text{id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  is positive. The set of all such linear maps will be denoted by  $\mathcal{P}_k(n, m)$ .*

**Example 2.3.6.** *See lemma 2.5.2 and 2.5.1.*

Considering  $M_k(\mathbb{C})$  as embedded in  $M_{k+1}(\mathbb{C})$  it follows that we have the inclusion  $\mathcal{P}_{k+1}(n, m) \subseteq \mathcal{P}_k(n, m)$ . In fact, Chruściński and Kossakowski showed that the inclusion is strict for  $k < \min(m, n)$  i.e.  $\mathcal{P}_{k+1}(n, m) \subset \mathcal{P}_k(n, m)$  [CK09]. It is also not difficult to see that  $\mathcal{P}_k(n, m)$  is a convex cone in  $\text{Lin}(M_n, M_m)^{\text{sa}}$ . For  $k = 1$  it is simply the cone of positive operators  $\mathcal{P}_1(n, m)$  which we introduced earlier.

**Theorem 2.3.7.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is *k*-positive if and only if  $\langle v, \mathcal{J}_2(S)v \rangle \geq 0$  for any vector  $v \in \mathbb{C}^n \otimes \mathbb{C}^m$  such that  $\text{SR}(v) \leq k$ .*

*Proof.* The proof is taken from the M. Ali and K. S. Ranade's paper [RA07]. By the definition, a linear map  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is *k*-positive for some positive integer  $k$

iff  $(\text{id}_k \otimes S) : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$  is positive. The idea is to apply Jamiolkowski's result- theorem 2.3.4 to verify the positivity of  $(\text{id}_k \otimes S)$ . Consider the Hilbert spaces  $\mathbb{C}^k \otimes \mathbb{C}^n$  and  $\mathbb{C}^k \otimes \mathbb{C}^m$  corresponding to the matrix algebras  $M_k \otimes M_n \cong M_k(M_n(\mathbb{C}))$  and  $M_k \otimes M_m \cong M_k(M_m(\mathbb{C}))$ . Theorem 2.3.4 tells us that  $(\text{id}_k \otimes S)$  is positive if and only if  $\langle x \otimes y, \mathcal{J}_{2,k}(\text{id}_k \otimes S)(x \otimes y) \rangle \geq 0$  for any  $x \in \mathbb{C}^k \otimes \mathbb{C}^n$  and  $y \in \mathbb{C}^k \otimes \mathbb{C}^m$ , where  $\mathcal{J}_{2,k} : \text{Lin}(M_k \otimes M_n, M_k \otimes M_m) \rightarrow (M_k \otimes M_n) \otimes (M_k \otimes M_m)$  is the modified Jamiolkowski isomorphism on the respective spaces. Let  $(e_i)_{i=1}^n$  and  $(f_g)_{g=1}^k$  be the standard orthonormal bases of  $\mathbb{C}^n$  and  $\mathbb{C}^k$  respectively. Therefore we have  $(|e_i\rangle\langle e_j|)_{i,j=1}^n$  and  $(|f_g\rangle\langle f_h|)_{g,h=1}^k$  the corresponding standard orthonormal bases of  $M_n(\mathbb{C})$  and  $M_k(\mathbb{C})$  respectively. With respect to these bases the Choi matrix or the image of the Jamiolkowski isomorphism of  $(\text{id}_k \otimes S)$  is given by

$$\mathcal{J}_{2,k}(\text{id}_k \otimes S) = \sum_{g,h=1}^k \sum_{i,j=1}^n |f_g\rangle\langle f_h| \otimes |e_i\rangle\langle e_j| \otimes \left[ \text{id}_k \otimes S \right] \left( |f_g\rangle\langle f_h| \otimes |e_i\rangle\langle e_j| \right).$$

Any vector  $x \in \mathbb{C}^k \otimes \mathbb{C}^n$  can be written as  $x = \sum_{p=1}^k f_p \otimes x_p$  with elements  $x_p \in \mathbb{C}^n$ . Similarly for  $y \in \mathbb{C}^k \otimes \mathbb{C}^m$ , we write  $y = \sum_{q=1}^k f_q \otimes y_q$ , where  $y_q \in \mathbb{C}^m$ . Therefore,  $x \otimes y = \sum_{p,q=1}^k f_p \otimes x_p \otimes f_q \otimes y_q$ . We compute

$$\begin{aligned} & \langle x \otimes y, \mathcal{J}_{2,k}(\text{id}_k \otimes S)(x \otimes y) \rangle \\ &= \sum_{i,j,g,h,p,q} \left\langle (f_p \otimes x_p) \otimes (f_q \otimes y_q) \left| \left( \langle f_h, f_r \rangle f_g \otimes \langle e_j, x_r \rangle e_i \right) \otimes \left( \langle f_h, f_s \rangle f_g \otimes S(|e_i\rangle\langle e_j|) y_s \right) \right. \right\rangle \\ &= \sum_{i,j,g,h,p,q,r,s} \delta_{hr} \delta_{hs} \langle f_p, f_g \rangle \langle e_j, x_r \rangle \langle x_p, e_i \rangle \langle f_q, f_g \rangle \langle y_q, S(|e_i\rangle\langle e_j|) y_s \rangle \\ &= \sum_{i,j,g,h,p,q,r,s} \delta_{hr} \delta_{hs} \delta_{pg} \delta_{qg} \langle x_p \otimes y_q, |e_i\rangle\langle e_j| \otimes S(|e_i\rangle\langle e_j|) x_r \otimes y_s \rangle \\ &= \sum_{p,r} \langle x_p \otimes y_q, \mathcal{J}_2(S) x_r \otimes y_s \rangle = \langle v, \mathcal{J}_2(S) v \rangle, \end{aligned}$$

where  $v = \sum_{p=1}^k x_p \otimes y_q$ . By the corollary 2.1.1  $\text{SR}(v) \leq k$ . Hence the claim is proved.  $\square$

**Definition 2.3.5** (Completely positive map). *A linear map  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is called completely positive if and only if it is  $k$ -positive for any positive integer  $k$ . We denote the set of completely positive map from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$  by  $\mathcal{CP}(n, m)$ .*

**Example 2.3.8.** *Consider the conjugate map  $\text{Ad}_V \in \text{Lin}(M_n, M_m)$  for some  $n \times m$  matrix  $V$ . We already know that it is a positive map. For any positive integer  $k$ ,  $(\text{id}_k \otimes \text{Ad}_V) = \text{Ad}_{I_k \otimes V}$ , where  $I_k$  is the  $k \times k$  identity matrix. It shows that  $\text{id}_k \otimes \text{Ad}_V$  is positive on*



$M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$  for any positive integer  $k$ . Thus  $\text{Ad}_V$  is completely positive. In fact, any convex combination conjugates maps are completely positive.

**Corollary 2.3.2.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is completely positive if and only if it is  $\min\{n, m\}$ -positive.*

*Proof.* By the definition of complete positivity a linear map  $S$  is completely positive iff it is  $k$ -positive for all  $k \in \mathbb{N}$ . Again by the previous theorem a linear map is  $k$ -positive iff it is positive on all vectors of Schmidt rank at most  $k$ . But any vector of  $\mathbb{C}^n \otimes \mathbb{C}^m$  has Schmidt rank at most  $\min\{n, m\}$ . Thus if  $S$  is  $\min\{n, m\}$  positive then it is completely positive. The other direction is obvious.  $\square$

**Corollary 2.3.3** (Choi's 1st theorem). *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is completely positive if and only if the corresponding Choi matrix  $C_S := \mathcal{J}_2(S)$  is positive semi-definite.*

*Proof.* It follows directly from the theorem 2.3.7 that a linear map  $S : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  is completely positive if and only if its Choi matrix  $C_S := \mathcal{J}_2(S)$  is positive semi-definite on vectors of arbitrary Schmidt rank in  $\mathbb{C}^n \otimes \mathbb{C}^m$ .  $\square$

This result is known as the *channel-state duality* was proved by M. D. Choi (see theorem 2, [Choi75]). We have seen in the example 2.3.8 that any convex combination of conjugate map  $\text{Ad}_V$  is completely positive. Now we can ask the converse question i.e. if any completely positive map is a convex combination of some conjugate maps. Choi and Kraus proved that the answer is positive [Choi75], [Kra70].

**Theorem 2.3.9** (Kraus Decomposition). *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is completely positive if and only if there exists operators  $V_1, V_2, \dots, V_r \in M_{n,m}(\mathbb{C})$  such that*

$$S = \sum_{i=1}^r \text{Ad}_{V_i}.$$

*Proof.* By the corollary 2.3.2 and theorem 2.3.5

$$S \in \mathcal{CP}(n, m) \iff S \text{ is } \min\{n, m\}\text{-positive} \iff S \in \mathcal{S}_{\min\{n, m\}}(n, m)^\circ.$$

So we have  $\mathcal{CP}(n, m) = \mathcal{S}_{\min\{n, m\}}(n, m)^\circ$ . Since the Choi matrix corresponding to a completely positive map is positive semi-definite (corollary 2.3.3), for any two map  $R, S \in \mathcal{CP}(n, m)$

$$\langle R, S \rangle = \langle C_R, C_S \rangle \geq 0.$$

It shows that the cone of completely positive map  $CP$  is actually self dual implying that  $CP(n, m) = S_{\min\{n, m\}}(n, m)$ . Hence, the claim is proved.  $\square$

**Remark 2.3.10.** *An alternate way to prove the theorem without using the duality is to utilise the Choi's 1st theorem directly and the spectral theorem.*

The operators  $V_i$ 's in the Kraus decomposition are known as Kraus operators corresponding to the CP map  $S$ . But they are not unique. For example, consider a CP map  $S$  on  $M_n(\mathbb{C})$  represented by two Kraus operators  $\{V_1, V_2\}$  i.e.  $S = \text{Ad}_{V_1} + \text{Ad}_{V_2}$ . Let  $(u_{ij})_{i, j=1}^2$  be a unitary matrix. If we define two new operators  $\tilde{V}_1 = u_{11}V_1 + u_{12}V_2$  and  $\tilde{V}_2 = u_{21}V_1 + u_{22}V_2$  then it can be easily checked that  $\text{Ad}_{V_1} + \text{Ad}_{V_2} = \text{Ad}_{\tilde{V}_1} + \text{Ad}_{\tilde{V}_2}$ .

**Theorem 2.3.11** (Choi's second theorem). *Two sets of Kraus operators  $\{V_j\}_{j \in J}$  and  $\{\tilde{V}_l\}_{l \in L}$  represent the same map  $S = \sum_{j \in J} \text{Ad}_{V_j} = \sum_{l \in L} \text{Ad}_{\tilde{V}_l}$  iff there exists a unitary  $U = (u_{ij})$  such that  $V_j = \sum_l u_{jl} \tilde{V}_l$ , where the smaller set is padded with zeros.*

We need the following proposition to prove the theorem.

**Proposition 2.3.2.** *Let  $\rho \in M_n(\mathbb{C})^+$  with two decomposition  $\rho = \sum_{i=1}^p |v_i\rangle\langle v_i| = \sum_{j=1}^q |w_j\rangle\langle w_j|$ . Then there exists an isometry  $U = (u_{ij}) \in M_{q,p}(\mathbb{C})$  i.e.  $U^*U = Id_p$  such that*

$$w_i = \sum_{j=1}^p u_{ij} v_j.$$

*Proof.* Without loss of generality let's assume  $p \leq q$ . We construct two purifications of  $\rho$ , say  $\psi_1$  and  $\psi_2$  of the form  $\psi_1 = \sum_{i=1}^p |v_i\rangle \otimes |i\rangle$  and  $\psi_2 = \sum_{j=1}^q |w_j\rangle \otimes |j\rangle$ , where  $\{|i\rangle : 1 \leq i \leq p\}$  is an orthonormal basis of  $\mathbb{C}^p$ - the dilation space of  $\rho$  of dimension  $p$  and since  $p \leq q$  we can extend the same orthonormal set to an orthonormal basis of  $\mathbb{C}^q$ , the dilation space of  $\rho$  of dimension  $q$ , considering  $\mathbb{C}^p$  is embedded in  $\mathbb{C}^q$ . We know that two such purifications differ by isometry i.e.  $\psi_2 = (\text{id} \otimes U)\psi_1$  where  $U : \mathbb{C}^p \rightarrow \mathbb{C}^q$  is an isometry. Taking the scalar product with a basis vector  $|i\rangle$  on the second tensor factor gives  $w_i = \sum_{j=1}^p u_{ij} v_j$   $\square$

*Proof of Choi's 2nd theorem.* If  $S$  has two Kraus representations

$$S = \sum_{j \in J} \text{Ad}_{V_j} = \sum_{l \in L} \text{Ad}_{\tilde{V}_l},$$

then the corresponding Choi matrices is given by  $C_S = \sum_{j \in J} |v_j\rangle\langle v_j| = \sum_{l \in L} |\tilde{v}_l\rangle\langle \tilde{v}_l|$ . The previous proposition implies that there is an isometry  $U = (u_{ij})$  such that  $\tilde{v}_l =$

$\sum_j u_{lj} v_j$ . Since by the proposition 2.3.1  $V_j$  is just the matrisation of  $|v_j\rangle$  we obtain that  $\tilde{V}_l = \sum_j u_{lj} V_j$ .  $\square$

The minimum number of Kraus operators required to express a completely positive map in its Kraus decomposition forms is called the Choi-Kraus rank of the corresponding CP map. More precisely,

**Definition 2.3.6** (Choi-Kraus rank). *Let  $S$  be a completely positive map on  $M_n(\mathbb{C})$ . Then the Choi-Kraus rank of  $S$  is defined as*

$$ckr(S) := \min \left\{ q \in \mathbb{N} : S = \sum_{i=1}^q \text{Ad}_{V_i}, V_i \in M_n(\mathbb{C}) \right\}.$$

**Theorem 2.3.12.** *Let  $S$  be a completely positive map on  $M_n(\mathbb{C})$  and  $C_S := \mathcal{J}_2(S)$  be its Choi matrix. Then  $ckr(S) = \text{rank}(C_S)$ .*

**Proposition 2.3.3.** *Let  $S$  be a 1-super positive map from  $M_n(\mathbb{C})$  to  $M_m(\mathbb{C})$  i.e.  $S \in \mathcal{S}_1(n, m)$  if and only if for any  $k \geq 1$  and any bipartite state (non-normalised)  $\rho \in (M_k(\mathbb{C}) \otimes M_n(\mathbb{C}))^+$ ,  $(\text{Id}_k \otimes S)(\rho)$  is separable.*

*Proof.* Let  $\rho \in (M_k(\mathbb{C}) \otimes M_n(\mathbb{C}))^+$  for some fixed  $k \geq 1$ . By the Choi-Jamiołkowski isomorphism and Choi's theorem there is a unique completely positive map  $R : M_k(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  such that  $C_R = \mathcal{J}_2(R) = (\text{Id}_k \otimes R)(\sum_{i,j=1}^k E_{ij} \otimes E_{ij}) = \rho$ , where  $E_{ij}$ 's are matrix units of  $M_k(\mathbb{C})$ . Therefore,

$$(\text{Id}_k \otimes S)(\rho) = (\text{Id}_k \otimes S) \circ (\text{Id}_k \otimes R) \left( \sum_{i,j=1}^k E_{ij} \otimes E_{ij} \right) = (\text{Id} \otimes S \circ R) \left( \sum_{i,j=1}^k E_{ij} \otimes E_{ij} \right).$$

It is easy to see that  $S \circ R \in \mathcal{S}_1(k, m)$  for if  $S = \sum_{j=1}^p \text{Ad}_{S_j}$  and  $R = \sum_{i=1}^q \text{Ad}_{R_i}$  then  $S \circ R = \sum_{i,j=1}^{p,q} \text{Ad}_{R_i S_j}$ , where  $\text{rank}(R_i S_j) \leq \min\{\text{rank}(R_i), \text{rank}(S_j)\} = 1$ . Since we know by the corollary 2.3.1 that  $\mathcal{J}_2(\mathcal{S}_1(n, m)) = \text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$  therefore,  $(\text{Id}_k \otimes S \circ R)(\sum_{i,j} E_{ij} \otimes E_{ij}) = \mathcal{J}_2(S \circ R)$  which is separable and thus  $(\text{Id}_k \otimes S)(\rho)$  is separable.

Conversely, suppose for any  $k \geq 1$  and  $\rho \in (M_k(\mathbb{C}) \otimes M_n(\mathbb{C}))^+$  we have that  $(\text{Id}_k \otimes S)(\rho)$  is separable. We need to prove that  $S \in \mathcal{S}_1(n, m)$ . So it suffices to show that  $\mathcal{J}_2(S) \in \text{Sep}(\mathbb{C}^n, \mathbb{C}^m)$ . For  $k = n$  we take the maximally entangled state  $\rho = \sum_{i,j=1}^n E_{ij} \otimes E_{ij} \geq 0$ , where  $E_{ij}$ 's are matrix units of  $M_n(\mathbb{C})$ . By our assumption  $(\text{Id}_n \otimes S)(\sum_{i,j=1}^n E_{ij} \otimes E_{ij}) = \mathcal{J}_2(S)$  is separable.  $\square$

This motivates to define the *entanglement breaking* channel or more generally the *k-entanglement breaking* map.

**Definition 2.3.7** (Entanglement Breaking Map). *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is called entanglement breaking if  $(\text{Id}_k \otimes S)\rho$  is separable for any  $\rho \in (M_k(\mathbb{C}) \otimes M_n(\mathbb{C}))^+$  and for any  $k \geq 1$ . We denote the set of all entanglement breaking maps by  $\mathcal{EB}(n, m)$ .*

Proposition 2.3.3 shows that  $\mathcal{EB}(n, m) = \mathcal{S}_1(n, m)$ .

**Definition 2.3.8** ( $k$ -Entanglement Breaking Map). *A linear map  $S \in \text{Lin}(M_n, M_m)$  is called  $k$ -entanglement breaking if it is  $k$ -positive and  $(\text{id}_k \otimes S)X$  is separable whenever  $X \in (M_k \otimes M_n)^+$ . We denote the cone of  $k$ -entanglement breaking maps by  $\mathcal{EB}_k(n, m)$ .*

With the cones defined above we have the following chain of inclusions

- (i)  $\mathcal{EB}(n, m) = \mathcal{S}_1(n, m) \subset \mathcal{S}_k(n, m) \subset \mathcal{S}_{\min\{n, m\}}(n, m) = \mathcal{CP}(n, m) = \mathcal{P}_{\min\{n, m\}}(n, m) \subset \mathcal{P}_k(n, m) \subset \mathcal{P}_1(n, m)$ ,
- (ii)  $\mathcal{EB}(n, m) = \mathcal{EB}_{\min\{m, n\}}(n, m) \subset \mathcal{EB}_k(n, m) \subset \mathcal{P}_k(n, m) \subset \mathcal{P}_1(n, m)$ ,
- (iii)  $\mathcal{EB}(n, m) = \mathcal{S}_1(n, m) \subset \mathcal{S}_k(n, m) \subset \mathcal{EB}_k(n, m)^\circ \subset \mathcal{P}_1(n, m)$ .

**Proposition 2.3.4.** *The cones  $\mathcal{P}_k, \mathcal{S}_k, \mathcal{EB}_k, \mathcal{EB}_k^\circ \subseteq \text{Lin}(M_n, M_m)^{\text{her}}$ ,  $k = 1, \dots, n$ , are closed, pointed and solid. Furthermore they are stable under composition.*

*Proof.* Stability under composition is easy to check for  $k$ -positive maps and therefore for CP maps. For  $k$ -superpositive maps, if  $S, T \in \mathcal{S}_k$  have Kraus representations

$$S(X) = \sum_{i=1}^p L_i^* X L_i \quad \text{and} \quad T(X) = \sum_{j=1}^q K_j^* X K_j,$$

with Kraus operators of rank less than or equal to  $k$ , then so does their composition  $S \circ T$ ,

$$S \circ T(X) = \sum_{i=1}^p \sum_{j=1}^q (K_j L_i)^* X K_j L_i,$$

since  $\text{rank}(K_j L_i) \leq \min\{\text{rank}(L_i), \text{rank}(K_j)\} \leq k$ . For the stability of composition of  $k$ -entanglement breaking maps, see [DMS21, Theorem 5.4]. For the dual cone  $\mathcal{EB}_k^\circ$ , it follows from the characterisation given in [DMS21, Equation (3.5), Theorem 3.11]. Indeed, suppose that  $S_i$  are limits of convex combinations of the form given in [DMS21], i.e.,  $S_i = \lim \sum \lambda_p^{(i)} \Gamma_p^{(i)} \circ \Psi_p^{(i)}$ , with  $\Gamma_p^{(i)} : M_k \rightarrow M_n$  positive and  $\Psi_p^{(i)} : M_n \rightarrow M_k$  completely positive for  $i = 1, 2$  (where we suppressed the index for the limit). Then  $\Gamma_p^{(1)} \circ \Psi_p^{(1)} \circ \Gamma_p^{(2)} : M_k \rightarrow M_n$  is again positive, and so

$$S_1 \circ S_2 = \lim_{p, p'} \lambda_p^{(1)} \lambda_{p'}^{(2)} \left( \Gamma_p^{(1)} \circ \Psi_p^{(1)} \circ \Gamma_{p'}^{(2)} \right) \circ \Psi_{p'}^{(2)}$$

is again of the form given in [DMS21, Equation (3.5), Theorem 3.11].

To show that all these cones are solid (i.e.,  $C + (-C) = \text{Lin}(M_n, M_n)$ , or, equivalently, they have non-empty interior) and pointed (i.e.,  $C \cap (-C) = \{0\}$ ), we will show below in Lemma 2.3.3 that the depolarising channel  $P : M_n \rightarrow M_n$ ,  $P(X) = \text{Tr}(X) \frac{1}{n} I_n$  is in the interior of  $\mathcal{S}_1 = \mathcal{EB}$ . Since all other cones contain  $\mathcal{EB}$ , they are also solid. Which implies that they are pointed, as duals of solid cones.  $\square$

**Lemma 2.3.3.** *The depolarising channel  $P : M_n \rightarrow M_n$ ,  $P(X) = \text{Tr}(X) \frac{1}{n} I_n$  belongs to the interior of the cone  $\mathcal{S}_1 = \mathcal{EB}$  of entanglement breaching maps.*

*Proof.* We will use the duality between  $\mathcal{EB} = \mathcal{S}_1$  and  $\mathcal{PM} = \mathcal{P}_1$ . To show that  $P$  belongs to the interior of  $\mathcal{EB}$ , it is sufficient to show that

$$\forall T \in \mathcal{PM} = \mathcal{EB}^\circ, T \neq 0 \quad \Rightarrow \quad \langle T, P \rangle > 0.$$

Let  $T \in \mathcal{PM}$  such that  $\langle T, P \rangle = 0$ . Choose an orthonormal bases  $(u_i)_{i=1, \dots, n}$  of  $\mathbb{C}^n$ . Then the set of rank 1 operators  $(|u_i\rangle\langle u_j|)_{i, j=1, \dots, n}$  is an orthonormal basis of  $M_n$  and we have

$$0 = \langle T, P \rangle = \frac{1}{n^2} \sum_{i, j=1}^n \left\langle T(|u_i\rangle\langle u_j|), \underbrace{P(|u_i\rangle\langle u_j|)}_{= \frac{\delta_{ij}}{n} I_n} \right\rangle = \frac{1}{n^3} \sum_{i=1}^n \text{Tr} \left( T(|u_i\rangle\langle u_i|)^* \right)$$

Since  $T$  is positive map, each term in this sum is positive, and therefore has to vanish.

Since the orthonormal basis  $(u_i)_{i=1, \dots, n}$  is arbitrary, this implies

$$\forall u \in \mathbb{C}^n, \quad \text{Tr} \left( T(|u\rangle\langle u|) \right) = 0.$$

and therefore  $T = 0$ .  $\square$

## 2.4 Equivariance And $k$ -Positivity

Given a linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ , it is computationally easy to decide if it is completely positive but very difficult to decide if it is  $k$ -positive for some  $k < \min\{n, m\}$ .

Since  $\sum_{i, j=1}^k \tilde{E}_{ij} \otimes E_{ij} \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$  is positive, where  $\tilde{E}_{ij}$  and  $E_{ij}$  are matrix units of  $M_k(\mathbb{C})$  and  $M_n(\mathbb{C})$  respectively, we know that the  $km \times km$  matrix  $[S(E_{ij})]_{i, j=1}^k = \sum_{i, j=1}^k \tilde{E}_{ij} \otimes S(E_{ij})$  is positive. The converse is of course not true in general. B. Collins, H. Osaka and G. Sapiro showed that the converse holds for a special case- if the map  $S$  satisfies some equivariance property [COS18][BCS20].

**Definition 2.4.1.** *A linear map  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  is called*

- i. equivariant if for any unitary  $U \in M_n(\mathbb{C})$  there exists  $V \in M_m(\mathbb{C})$  (not necessarily unitary) such that following equality holds

$$S(UXU^*) = VS(X)V^*$$

for all  $X \in M_n(\mathbb{C})$ .

- ii. unitarily equivariant if  $V$  in the above definition can be chosen to be a unitary.
- iii.  $(a, b)$ -unitarily equivariant if  $m = n^{(a+b)}$  and  $M_m(\mathbb{C}) = M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b}$  such that for any unitary  $U \in M_n(\mathbb{C})$

$$S(UXU^*) = (\bar{U}^{\otimes a} \otimes U^{\otimes b})X(\bar{U}^{\otimes a} \otimes U^{\otimes b})^*$$

**Example 2.4.1.** i. Any  $*$ -homomorphism or  $*$ -anti-homomorphism on a finite dimensional matrix algebra is equivariant.

- ii. Let  $A \in M_n(\mathbb{C})$  be an invertible matrix. Define the linear map  $S_A : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  by

$$S_A(X) := A^*XA$$

for any  $X \in M_n(\mathbb{C})$ . Then,

$$\begin{aligned} S_A(UXU^*) &= A^*UXU^*A \\ &= A^*U(A^*)^{-1}(A^*XA)A^{-1}U^*A \\ &= (A^*U(A^*)^{-1})S_A(X)(A^*U(A^*)^{-1})^*. \end{aligned}$$

It shows that  $S_A$  is equivariant and if  $A$  is unitary then  $S_A$  is unitarily equivariant.

- iii. B. V. R. Bhat characterised all  $(0, 1)$ -unitarily equivariant maps [Bhat11]. More precisely, he showed that any linear map  $S$  on  $M_n(\mathbb{C})$  which verifies the following equality

$$S(UXU^*) = US(X)U^*$$

for any unitary  $U \in M_n(\mathbb{C})$  and any  $X \in M_n(\mathbb{C})$ , is of the form

$$S = \alpha Id + \beta P$$

for some  $\alpha, \beta \in \mathbb{C}$ , where  $Id$  is the identity map and  $P$  is the depolarising channel. It follows directly from this result that any  $(1, 0)$  unitarily equivariant map  $Q$  is of the form

$$Q = \alpha T + \beta P$$

for some  $\alpha, \beta \in \mathbb{C}$ , where  $T$  is the transposition map.

Collins, Osaka, Sapro proved the following results-

**Lemma 2.4.1.** *Let  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  be a map with equivariance property. If  $S(E_{11}) \geq 0$ , where  $E_{11}$  is a matrix unit in  $M_n(\mathbb{C})$  then  $S$  is positive.*

*Proof.* It suffices to show that  $S(p)$  is positive for any rank-one projection  $p \in M_n(\mathbb{C})$ . Since two Hermitian matrices are unitarily equivalent if and only if they have the same eigenvalues (counting multiplicities),  $E_{11}$  and  $p$  are unitarily equivalent. So there exists a unitary  $U \in M_n(\mathbb{C})$  such that  $p = UE_{11}U^*$ . This implies  $S(p) = VS(E_{11})V^*$  by the equivariance property of  $S$ . Since  $S(E_{11}) \geq 0$  we have  $VS(E_{11})V^* \geq 0$  which prove the claim.  $\square$

**Theorem 2.4.2** (Collins, Osaka, Sapro, 2018). *Let  $S \in \text{Lin}(M_n(\mathbb{C}), M_m(\mathbb{C}))$  be a self-adjoint linear map which verifies the equivariance property. Then for  $k \leq \min\{n, m\}$ ,  $S$  is  $k$ -positive if and only if the block matrix  $[S(E_{ij})]_{i,j=1}^k$  is positive, where  $(E_{ij})$ s are the matrix units of  $M_n(\mathbb{C})$ .*

*Proof.* We have already seen that if  $S$  is  $k$  positive then  $[S(E_{ij})]_{i,j=1}^k$  is positive. For the converse it suffices to prove that  $(id_k \otimes S)(p)$  is positive for any rank one projection  $p$  in  $M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$ . Let  $p = |x\rangle\langle x|$  for some  $|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$  with  $\|x\| = 1$ . We write the Schmidt decomposition of  $|x\rangle$ ,

$$|x\rangle = (U \otimes V^*) \sum_{i=1}^r c_i |i\rangle \otimes |i\rangle,$$

where  $\{|i\rangle : 1 \leq i \leq n\}$  is the standard basis of  $\mathbb{C}^n$ ,  $c_i$ 's are positive real numbers with  $\sum_{i=1}^r c_i^2 = 1$ ,  $U$  and  $V$  unitaries arising from the singular value decomposition of  $C$  the matrisation of the vector  $|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$ . Therefore,

$$p = |x\rangle\langle x| = (U \otimes V^*) \left( \sum_{i,j=1}^r c_i c_j |i\rangle\langle j| \otimes |i\rangle\langle j| \right) (U \otimes V^*)^*.$$

Denoting  $X = \sum_{i,j=1}^r c_i c_j |i\rangle\langle j| \otimes |i\rangle\langle j|$  we can write

$$p = (U \otimes V^*) X (U \otimes V^*)^*$$

It is easy to see that if  $S$  satisfies the equivariance property so does  $(id_k \otimes S)$ . Thus,

$$\begin{aligned} (id_k \otimes S)(p) &= (id_k \otimes S)[(U \otimes V^*) X (U \otimes V^*)^*] \\ &= (V_1 \otimes V_2)(id_k \otimes S)(X)(V_1 \otimes V_2)^* \end{aligned}$$

We now show that  $(id_k \otimes S)(X)$  is positive to conclude the argument. Suppose the block matrix  $[S(E_{ij})]_{i,j=1}^k$  is positive. Then,

$$\begin{aligned} (id_k \otimes S)(X) &= \sum_{i,j=1}^r c_i c_j |i\rangle\langle j| \otimes S(|i\rangle\langle j|) \\ &= [S(c_i c_j E_{ij})]_{i,j=1}^k \\ &= [c_i c_j S(E_{ij})]_{i,j=1}^k \\ &= [c_i c_j]_{i,j=1}^k \odot [S(E_{ij})]_{i,j=1}^k, \end{aligned}$$

where  $\odot$  denotes the Schur product of two matrices. The last term being the Schur product of two positive matrices, it is again positive. This proves the claim.  $\square$

## 2.5 A 4-Parameter Family Of Maps

We take the linear combinations of some well known maps e.g. depolarizing channel, transpose, or map which is conditional expectation onto diagonal and discuss the criterion of positivity,  $k$ -positivity or complete positivity of such combinations. Later we can use these examples to generate identity preserving semigroups, which are of our interest. We have already defined the *depolarising channel* in Lemma 2.3.3. as a linear map  $P : M_n \rightarrow M_n$  satisfying

$$P(X) = \frac{1}{n} \text{Tr}(X) I_n.$$

We have

$$P(X) = \frac{1}{n} \sum_{j,k=1}^n \langle e_j, X e_j \rangle |e_k\rangle\langle e_k| = \sum_{j,k=1}^n \frac{1}{\sqrt{n}} |e_k\rangle\langle e_j| X \left( \frac{1}{\sqrt{n}} |e_k\rangle\langle e_j| \right)^*,$$

which shows that  $P \in \mathcal{S}_1 = \mathcal{EB}$ . The Choi-Jamiołkowski matrix of  $P$  is

$$C_P = \sum_{j,k=1}^n E_{jk} \otimes P(E_{jk}) = \frac{1}{n} \sum_{j,k=1}^n E_{jj} \otimes E_{kk} = \frac{1}{n} I_n \otimes I_n.$$

We consider also the *transposition map*,  $T(X) = X^T$ . It is known that  $T$  is positive, but not 2-positive, i.e.

$$T \in \mathcal{P}_1, \quad T \notin \mathcal{P}_2,$$

for  $n \geq 2$ . The Choi-Jamiołkowski matrix of  $T$  is

$$C_T = \sum_{j,k=1}^n E_{jk} \otimes T(E_{jk}) = \sum_{j,k=1}^n E_{jk} \otimes E_{kj}.$$



*Conditional expectation onto the diagonal:* Consider the linear map  $D : M_n \rightarrow M_n$ ,  $D(X) = (\delta_{jk}x_{jk})_{1 \leq j,k \leq n}$  for  $X = (x_{jk})_{1 \leq j,k \leq n} \in M_n$ . This map is the conditional expectation onto the \*-subalgebra of diagonal matrices (w.r.t. the standard basis).

We have

$$D(X) = \sum_{j=1}^n |e_j\rangle\langle e_j|X|e_j\rangle\langle e_j|,$$

which shows that  $D \in \mathcal{S}_1 = \mathcal{EB}$ .

We can furthermore show that  $D$  belongs to the boundary of  $\mathcal{CP}$ , and therefore also to the boundary of  $\mathcal{EB}$ . Indeed, denote by  $C \in M_n$  that matrix that cyclically permutes the vectors of the standard basis,

$$Ce_j = e_{j \oplus 1} = \begin{cases} e_{j+1} & \text{if } 1 \leq j \leq n-1, \\ e_1 & \text{if } j = n, \end{cases}$$

where we use  $\oplus$  to denote the addition modulo  $n$  in  $\{1, \dots, n\}$ . Then the completely positive map  $T_C$  with  $T_C(X) = CXC^*$  acts as

$$T_C(E_{jk}) = E_{j \oplus 1, k \oplus 1}.$$

Thus we have

$$\langle T_C, D \rangle = \sum_{j,k=1}^n \text{Tr}(T_C(E_{jk})^* D(E_{jk})) = \sum_{j=1}^n \text{Tr}(E_{j \oplus 1, j \oplus 1} E_{jj}) = 0.$$

The Choi-Jamiolkowski matrix of  $D$  is

$$C_D = \sum_{j,k=1}^n E_{jk} \otimes D(E_{jk}) = \sum_{j=1}^n E_{jj} \otimes E_{jj}.$$

We have

$$C_{Id} = \sum_{j,k=1}^n E_{jk} \otimes E_{jk}.$$

We are interested in the 4 parameter family of linear maps-  $\alpha P + \beta D + \gamma T + \delta \text{Id}$  for  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ . We need the real parameters so that the family of maps is Hermiticity preserving. These are exactly the linear maps on  $M_n(\mathbb{C})$  that are invariant under the action of the hyperoctahedral group as signed permutations and have been considered from that perspective in [JPPY23]. We wish to decide to which cone it belongs depending on the real values of the coefficients. We can easily check that the Choi-Jamiolkowski matrices  $C_P, C_D, C_T, C_{Id} \in M_n \otimes M_n$  commute. Therefore, we can simultaneously diagonalise these four matrices. The minimal polynomials of  $C_D, C_T, C_{Id}$  have degree two,

and these matrices have two distinct eigenvalues. Computing the traces, we also get the multiplicities.

|  |                        |                        |
|--|------------------------|------------------------|
| Eigenvalues of $C_D$                       | $\rho = 0$             | $\rho = 1$             |
| $\dim \ker(C_D - \rho I_n \otimes I_n)$    | $n^2 - n$              | $n$                    |
| Eigenvalues of $C_T$                       | $\sigma = -1$          | $\sigma = 1$           |
| $\dim \ker(C_T - \sigma I_n \otimes I_n)$  | $\frac{1}{2}(n^2 - n)$ | $\frac{1}{2}(n^2 + n)$ |
| Eigenvalues of $C_{Id}$                    | $\tau = 0$             | $\tau = n$             |
| $\dim \ker(C_{Id} - \tau I_n \otimes I_n)$ | $n^2 - 1$              | $1$                    |

In particular,  $C_{Id}$  is a multiple of the orthogonal projection onto  $\Omega = \sum_{j=1}^n e_j \otimes e_j$ , which is also an eigenvector for the other matrices.

Denote by

$$V(\rho, \sigma, \tau) = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n : C_D v = \rho v, C_T v = \sigma v, C_{Id} v = \tau v\}$$

the joint eigenspaces of  $C_D, C_T, C_{Id}$ . We can verify that,  $V(\rho, \sigma, \tau)$  is non-null corresponding to the four triples  $(\rho, \sigma, \tau) = (0, -1, 0), (0, 1, 0), (1, 1, 0), (1, 1, n)$ .

**Proposition 2.5.1.** *Let  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  and set*

$$\Phi(\alpha, \beta, \gamma, \delta) = \alpha P + \beta D + \gamma T + \delta \text{Id}.$$

*Then  $\Phi$  is completely positive if and only if  $\alpha, \beta, \gamma, \delta$  satisfy the inequality*

$$\alpha \geq \max\{n|\gamma|, -n(\beta + \gamma), -n(\beta + \gamma) - n^2\delta\}.$$

*Proof.* We check that the Choi-Jamiołkowski matrix  $C_{\Phi(\alpha, \beta, \gamma, \delta)} = \alpha C_P + \beta C_D + \gamma C_T + \delta C_{Id}$  of  $C_{\Phi(\alpha, \beta, \gamma, \delta)}$  has eigenvalues

$$\frac{\alpha}{n} - \gamma, \quad \frac{\alpha}{n} + \gamma, \quad \frac{\alpha}{n} + \beta + \gamma, \quad \frac{\alpha}{n} + \beta + \gamma + n\delta,$$

corresponding to the four triples  $(\rho, \sigma, \tau) = (0, -1, 0), (0, 1, 0), (1, 1, 0), (1, 1, n)$  with non-trivial eigenspaces.  $\square$

Restricting to equivariant maps as in the terminology of [BCS20] (see also [COS18, Theorem 2.2]) we can also characterize the  $k$ -positivity criterion of the above mentioned family.

### (1,0)-Unitarily Equivariant Case - Linear Combinations Of Identity And Depolarising Channel

The identity map and the depolarising channel are (1,0)-unitarily equivariant map. For  $\Phi_{\alpha,\delta} = \alpha P + \delta \text{Id}$ ,  $\Phi_{\alpha,\delta}(X) = \delta X + \frac{\alpha}{n} \text{Tr}(X) I_n$  we

$$\Phi_{\alpha,\delta}(UXU^*) = U\Phi_{\alpha,\delta}(X)U^*,$$

for all  $U, X \in M_n$  with  $U$  unitary.

Let  $1 \leq k \leq n$ . By the theorem 2.4.2,  $\Phi_{\alpha,\delta}$  is  $k$ -positive, if and only if

$$C_{\Phi_{\alpha,\delta}}^{(k)} = \sum_{i,j=1}^k E_{ij} \otimes \Phi_{\alpha,\delta}(E_{ij}) = \sum_{i,j=1}^k E_{ij} \otimes \left( \frac{\alpha}{n} I_n^{(k)} \otimes I_n + \delta \sum_{i,j=1}^k E_{ij} \otimes E_{ij} \right) \in M_n \otimes M_n$$

is positive, where  $I_n^{(k)} = \sum_{i=1}^k E_{ii} \in M_n$ .

Note that  $C_{\Phi_{\alpha,\delta}}^{(k)} = \sum_{i,j=1}^k E_{ij} \otimes E_{ij}$  commutes with  $I_n^{(k)} \otimes I_n$ , satisfies  $\left( C_{\Phi_{\alpha,\delta}}^{(k)} \right)^2 = k C_{\Phi_{\alpha,\delta}}^{(k)}$ , and is a multiple of the orthogonal projection onto  $\Omega_k = \sum_{i=1}^k e_i \otimes e_i$ . One can show that the eigenvalues of  $C_{\Phi_{\alpha,\delta}}^{(k)} = \frac{\alpha}{n} I_n^{(k)} \otimes I_n + \delta C_{\Phi_{\alpha,\delta}}^{(k)}$  are given by

$$\text{spec}(C_{\Phi_{\alpha,\delta}}^{(k)}) = \left\{ 0, \frac{\alpha}{n}, \frac{\alpha}{n} + k\delta \right\}.$$

We summarize our results in the following Lemma.

**Lemma 2.5.1.** (*[Tom85, Theorem 2]*) *Let  $\alpha, \delta \in \mathbb{R}$ . The linear map  $\Phi_{\alpha,\delta}$  is  $k$ -positive iff the matrix  $C_{\Phi_{\alpha,\delta}}^{(k)}$  is positive iff  $\alpha, \delta$  satisfy the following two inequalities*

$$\alpha \geq 0 \quad \text{and} \quad \delta \geq -\frac{\alpha}{kn}.$$

### (0,1)-Unitarily Equivariant Case - Linear Combinations Of Transposition And Depolarising Channel

The transposition and the depolarising channel are (0,1)-unitarily equivariant (in the terminology of [BCS20, Definition 1.1 (iii)]). For  $\Psi_{\alpha,\gamma} = \alpha P + \gamma T$ ,  $\Psi_{\alpha,\gamma}(X) = \gamma X^T + \frac{\alpha}{n} \text{Tr}(X) I_n$ , we have

$$\Psi_{\alpha,\gamma}(UXU^*) = \bar{U} \Psi_{\alpha,\gamma}(X) U^T,$$

for all  $U, X \in M_n$  with  $U$  unitary.

By [BCS20, Theorem 2.4],  $\Psi_{\alpha,\gamma}$  is  $k$ -positive, if and only if

$$C_{\Psi_{\alpha,\gamma}}^{(k)} = \sum_{i,j=1}^k E_{ij} \otimes \Psi_{\alpha,\gamma}(E_{ij}) = \frac{\alpha}{n} I_n^{(k)} \otimes I_n + \gamma \sum_{i,j=1}^k E_{ij} \otimes E_{ji} \in M_n \otimes M_n$$

is positive, where  $I_n^{(k)} = \sum_{i=1}^k E_{ii} \in M_n$ .

Note that  $C_{\Psi_{0,1}}^{(k)} = \sum_{i,j=1}^k E_{ij} \otimes E_{ji}$  satisfies

$$I_n^{(k)} \otimes I_n C_{\Psi_{0,1}}^{(k)} = C_{\Psi_{0,1}}^{(k)} = C_{\Psi_{0,1}}^{(k)} I_n^{(k)} \otimes I_n,$$

and  $(C_{\Psi_{0,1}}^{(k)})^2 = \sum_{i,j=1}^k E_{ii} \otimes E_{jj} = I_n^{(k)} \otimes I_n^{(k)}$ . which implies

$$\text{spec}(C_{\Psi_{0,1}}^{(1)}) = \{0, 1\} \quad \text{and} \quad \text{spec}(C_{\Psi_{0,1}}^{(k)}) = \{-1, 0, 1\} \text{ for } k \geq 2.$$

We see that the eigenvalues of  $C_{\Psi_{\alpha,\gamma}}^{(k)} = \frac{\alpha}{n} I_n^{(k)} \otimes I_n + \gamma C_{\Psi_{0,1}}^{(k)}$  are

$$\text{spec}(C_{\Psi_{\alpha,\gamma}}^{(1)}) = \left\{ 0, \frac{\alpha}{n}, \frac{\alpha}{n} + \gamma \right\} \quad \text{and} \quad \text{spec}(C_{\Psi_{\alpha,\gamma}}^{(k)}) = \left\{ 0, \frac{\alpha}{n}, \frac{\alpha}{n} + \gamma, \frac{\alpha}{n} - \gamma \right\}$$

for  $k \geq 2$ .

Therefore we have the following Lemma.

**Lemma 2.5.2.** (*[Tom85, Theorem 3]*) *Let  $\alpha, \gamma \in \mathbb{R}$ . The linear map  $\Psi_{\alpha,\gamma}$  is*

*i. 1-positive iff  $\alpha \geq 0$  and  $\alpha \geq -n\gamma$ .*

*ii. k-positive for  $k \geq 2$ , iff  $\alpha \geq n|\gamma|$ .*

## Chapter 3

# Error Basis

The Pauli matrices are set of  $2 \times 2$  Hermitian, unitary matrices

$$\sigma_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They were first introduced by Pauli to describe quantum spin of a subatomic particle interacting with the external electromagnetic field. But they found a crucial role for quantum error correcting codes. They satisfy the following relations

$$\sigma_p^2 = \sigma_0^2, \quad \sigma_p \sigma_q = -\sigma_q \sigma_p, \quad \sigma_p \sigma_q = i \sigma_r \sigma_0$$

if  $(p, q, r) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ . Moreover, except for the identity each Pauli matrix has trace zero. Due to this trace condition, together with the above identities they form an orthonormal basis (up to a scaling) of  $M_2(\mathbb{C})$  with respect to the Hilbert-Schmidt scalar product. The Pauli matrices model the sign and bit flip errors in the case of 2 dimensional error correcting codes. To generalise them in higher dimension, Schwinger (1960), Knill (1996), Werner (2001) introduced the idea of unitary error basis.

**Definition 3.0.1** (Unitary Error Basis). *An Unitary Error Basis (UEB) of  $M_n(\mathbb{C})$  is a family of  $n \times n$  unitary matrices  $\{U_i\}_{i=1}^{n^2}$  such that*

$$\text{Tr}(U_i^* U_j) = \delta_{ij} n.$$

If we equip the vector space  $M_n(\mathbb{C})$  with the Hilbert-Schmidt inner product

$$\langle A, B \rangle := \text{Tr}(A^* B)$$

then the UEB becomes an orthonormal basis (upto a scaling) of  $M_n(\mathbb{C})$ .

**Definition 3.0.2.** Two UEB's  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are said to be equivalent iff

- i. there exist unitaries  $W_1, W_2 \in U(n)$
- ii. there exist constant  $c_U \in \mathbb{C}$  for  $U \in \mathcal{B}_1$

such that

$$\mathcal{B}_2 = \{c_U W_1 U W_2; U \in \mathcal{B}_1\}.$$

**Lemma 3.0.1.** Any unitary error basis in dimension 2 is equivalent to the Pauli basis.

*Proof.* Let  $\mathcal{E} = \{A_1, A_2, A_3, A_4\}$  be an arbitrary unitary error basis of dimension 2. This error basis is equivalent to a basis of the form  $\{I_n, \text{diag}(1, -1), B_3, B_4\}$ . Indeed, we can multiply all matrices in  $\mathcal{E}$  from the left with  $A_1^*$  to get an equivalent error basis which contain identity. By the orthogonality condition the rest of the non-identity matrices have trace zero. A change of basis and multiplication by scalar allows us to reduce the second matrix to  $\text{diag}(1, -1)$  without changing the identity matrix.

Again due to the orthogonality condition, the diagonal elements of  $B_3, B_4$  are zero. We can assume they are of the form  $B_3 = \text{anti-diag}(1, e^{i\theta})$  and  $B_4 = \text{anti-diag}(1, e^{-i\theta})$  for some  $\theta \in \mathbb{R}$  since we are allowed to multiply the matrices with scalars. Conjugating the matrices with  $\text{diag}(1, e^{-i\theta/2})$  yields the Pauli matrices up to scalar multiples. Hence  $\mathcal{E}$  is equivalent to the Pauli basis. □

Two fundamentally different constructions of UEB are known. One is combinatorial in nature known as *shift and multiply basis* (SMB) introduced by Werner [Wer01] and another is more algebraic in nature, known as *nice error basis* (NEB) [Kni96]. We will first briefly discuss the shift and multiply basis and mainly focus on the construction of nice error basis as we will use them later to study different positive maps and quantum channels.

### 3.1 Shift And Multiply Basis

We recall that a *Latin square* is a  $n \times n$  matrix such that each element of the set  $\mathbb{Z}_n$  occurs exactly once in each row and column.

**Example 3.1.1.**

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

A complex *Hadamard matrix*  $H$  of order  $n$  is a  $n \times n$  complex matrix such that

1.  $|H_{ij}| = 1$  for  $1 \leq i, j \leq n$ .
2.  $H^*H = nI_n$

For example, for any  $n \in \mathbb{N}$  we can define the following Hadamard matrix of order  $n$

$$(H)_{kl} = \exp\left(\frac{2\pi ikl}{n}\right).$$

**Definition 3.1.1.** Let  $\mathbf{H} = \{H^{(j)}; 1 \leq j \leq n\}$  be a finite sequence of Hadamard matrices and  $L$  be a Latin square of order  $n$ . A shift and multiply basis  $\mathcal{B}$  associated to  $L$  and  $\mathbf{H}$  is given by the unitary matrices

$$B_{ij} := P_j \text{diag}\left(H_{ik}^{(j)} : 0 \leq k < n\right) \quad \text{for } i, j \in \mathbb{Z}_n,$$

where  $P_j$  denotes the permutation matrix with entries given by  $P_j(L(j, k), k) = 1$  for  $k \in \mathbb{Z}_n$  and zero elsewhere.

If we label the standard basis of  $\mathbb{C}^n$  by the set  $\{|k\rangle; k \in \mathbb{Z}_n\}$  then we can verify the action of  $B_{ij}$  on the basic vectors of  $\mathbb{C}^n$ -

$$B_{ij}|k\rangle = H_{ik}^{(j)}|L(j, k)\rangle.$$

**Lemma 3.1.1** (Werner).  $\mathcal{B} = \{B_{ij}; 1 \leq i, j \leq n\}$  is a UEB (known as Shift and Multiply basis associated to  $L$  and  $\mathbf{H}$ ).

*Proof.* It is sufficient to show that  $\mathcal{B}$  is an orthogonal system with respect to the Hilbert-Schmidt inner product i.e.  $\text{Tr}(B_{ij}^* B_{kl}) = 0$  if  $(i, j) \neq (k, l)$ . If  $j \neq l$  then the matrix  $P_j^* P_l$  has vanishing diagonal which implied that  $\text{Tr}(B_{ij}^* B_{kl}) = 0$  for any choice of  $i$  and  $k$ . If  $j = l$  and  $i \neq k$  then  $\text{Tr}(B_{ij}^* B_{kl})$  is equal to the inner product of  $i$ -th and  $k$ -th row of the complex Hadamard matrix  $H^{(j)}$ , hence  $\text{Tr}(B_{ij}^* B_{kl}) = 0$ .  $\square$

**Example 3.1.2.** For  $n = 3$ , let's take the following Latin square  $L$  and Hadamard matrix  $H$ :

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix},$$

where  $\omega$  is a third root of unity. We can check that the three permutation matrices corresponding to the Latin square  $L$  are

$$P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now we can construct the shift and multiply basis according to the definition 3.1.1. For example the basis  $B_{01}$  and  $B_{12}$  are given by

$$B_{01} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}.$$

## 3.2 Nice Error Basis

The other type of unitary error basis was introduced by E. Knill [Kni96], called nice error basis (NEB). We first recall the definition of a NEB.

**Definition 3.2.1.** Let  $G$  be a group of order  $n^2$  for some natural number  $n$ . A nice error basis (NEB) on  $\mathbb{C}^n$  is a set of unitary operators  $E = \{\rho_g \in U(n) : g \in G\}$  such that

1.  $\rho_1$  is the identity matrix, where  $1$  denotes the identity element of the group  $G$ .
2.  $\text{Tr}(\rho_g) = n\delta_{g,1}$ .
3.  $\rho_g\rho_h = \omega(g,h)\rho_{gh}$ , where  $\omega(g,h)$  is a scalar.

For such a set of operators the labelling group  $G$  is called the index group of the corresponding NEB.

Conditions (1) and (3) simply tell us that the representation  $\rho$  is a projective representation. If we equip  $M_n(\mathbb{C})$  with the inner product  $\langle A, B \rangle := \text{Tr}(A^*B)$  then condition (2) ensures that  $E$  is an orthonormal set since

$$\begin{aligned} \langle \rho_g, \rho_h \rangle &= \text{Tr}(\rho_g^* \rho_h) = \omega(g^{-1}, g)^{-1} \text{Tr}(\rho_{g^{-1}} \rho_h) = \omega(g^{-1}, g)^{-1} \omega(g^{-1}, h) \text{Tr}(\rho_{g^{-1}h}) \\ &= \omega(g^{-1}, g)^{-1} \omega(g^{-1}, h) n \delta_{g,h}. \end{aligned}$$



Comparing the dimension it follows that  $E$  is an orthonormal basis (upto a scaling) of  $M_n(\mathbb{C})$ . It follows that  $\pi$  is an irreducible representation. Recall that a 2-cocycle is a function  $f : H \times H \rightarrow \mathbb{C}$ , where  $H$  is a group, verifying the relation

$$f(a, bc)f(b, c) = f(a, b)f(ab, c)$$

for any  $a, b, c \in G$ . In the case of a NEB the associativity of the index group  $G$  implies that function  $\omega : G \times G \rightarrow \mathbb{C}$  is a 2-cocycle. Indeed for any  $g, h, k \in G$  we have

$$\begin{aligned} (\rho_g \rho_h) \rho_k &= \rho_g (\rho_h \rho_k) \\ \implies \omega(g, h) \omega(gh, k) \rho_{ghk} &= \omega(g, hk) \omega(h, k) \rho_{ghk}. \end{aligned}$$

If we normalise each  $\pi_g$  so that  $\det(\pi_g) = 1$  then  $\omega(g, h)$  becomes an  $n$ -th root of unity for any  $g, h \in G$ .

Recall that the character corresponding to an irreducible representation of a group  $G$  is called an irreducible character of  $G$ . If the irreducible representation is projective with 2-cocycle  $\alpha$  then the corresponding character is called an irreducible  $\alpha$  character of  $G$ .

**Theorem 3.2.1.** *Let  $\alpha$  be a 2-cocycle and let  $\chi$  be a irreducible  $\alpha$  character of the group  $G$ . Then  $\chi(1)^2 \leq |G : \text{Ker}\chi|$  with equality iff  $\chi(g) = 0$  for all  $g \in G/\text{Ker}\chi$ .*

*Proof.* See corollary 11.13, [Kar94] □

**Theorem 3.2.2** (Theorem 1, [KR02]). *Let  $\mathcal{E} = \{\pi(g); g \in G\}$  be a set of unitary matrices indexed by the a finite group  $G$ . Then  $\mathcal{E}$  is a NEB iff  $\pi$  is a unitary faithful irreducible projective representation of order  $|G|^{1/2}$ .*

*Proof.* If  $\mathcal{E}$  is a nice error basis then we have already seen that  $\pi$  is an unitary irreducible projective representation of degree  $|G|^{1/2}$ . The condition (2) of the definition of NEB ensures that the projective representation is faithful i.e. no  $\rho_g$  is scalar multiple of identity except for  $g = 1$ . Conversely, if  $\rho$  is an irreducible faithful projective representation of order  $|G|^{1/2}$  then by the previous theorem the extremal degree condition  $\text{tr}(\rho(1)) = |G|^{1/2}$  implies that  $\text{tr}(\rho(g)) = 0$  holds for any  $g \in G/\text{Ker}(\rho)$ . It follows that  $\rho$  satisfies the condition (ii) of the definition of a NEB. □

### Characterisation Of Abelian Index Group

As a consequence of theorem 3.2.2, Klappenecker and Rötteler characterised all the index groups which are abelian [KR02]. They used a classical result of group representation theory by Frucht. A group  $G$  is called of *symmetric type* if  $G \cong H \times H$  for some group  $H$ .

**Theorem 3.2.3** (Frucht, 1931). *A finite abelian group admits a faithful unitary irreducible projective representation iff it is of symmetric type. The degree of the representation is  $|G|^{1/2}$ .*

*Proof.* See [Fruc31]. □

**Theorem 3.2.4.** *If a NEB  $\mathcal{E}$  has an abelian index group  $G$  then  $G$  is of symmetric type. Conversely, any finite abelian group of symmetric type is a index group of a NEB.*

*Proof.* We have seen in the theorem 3.2.2 that an NEB  $\mathcal{E}$  can be viewed as an irreducible faithful projective representation of a group  $G$ . By the theorem of Frucht a finite abelian group  $G$  admits such a representation if it is of symmetric type i.e.  $G \cong H \times H$  for some group  $H$  and conversely, any symmetric group  $G$  will have an irreducible, faithful, projective representation of degree  $|G|^{\frac{1}{2}}$ . □

### Characterisation of Non-abelian Index Group

The characterisation of an non-abelian index group is more difficult than the abelian case. To study this case Knill [Kni96] considered a slightly bigger group than the index group, known as the abstract error group.

**Definition 3.2.2.** *Let  $G$  be an index group and  $\mathcal{E} = \{\pi_g; g \in G\}$  be a NEB. Then the group  $H$  generated by  $\pi_g$ 's i.e.  $H := \langle \pi_g \rangle$  is called abstract error group.*

In general it may not be finite but if we multiply each  $\pi_g$  by a suitable scalar  $c_g$  such that  $\det(c_g \pi_g) = 1$  then we get an equivalent error basis

$$\mathcal{E}' = \{\pi'(g) := c_g \pi_g : g \in G\}$$

for which the abstract error basis  $H'$  becomes finite. Indeed, in such case the value of  $\omega(g, h)$  for each  $g, h \in G$  becomes an  $n$ -th root of the unity.

$$1 = \det(\pi'_g) \det(\pi'_h) = \omega(g, h)^n \det(\pi'_{gh}) = \omega(g, h)^n.$$

So after this normalisation the corresponding abstract error group  $H'$  can have at most order  $n \times n^2 = n^3$ . So without any loss of generality we can assume any abstract error group  $H$  to be of finite order. If  $\epsilon$  is a primitive  $n$ -th root of the unity then we can write the abstract error group  $H$  as

$$H = \{\epsilon^i \pi_g : g \in G, 0 \leq i \leq n - 1\},$$

with the group operation

$$(\epsilon^i \pi_g) \cdot (\epsilon^j \pi_h) = \epsilon^{i+j} \omega(g, h) \pi_{gh}.$$

If  $T$  is the cyclic group generated by the values of  $\omega$  then the abstract error group  $H$  can be identified with the group  $G_\omega := T \times G$  where the group operation is given by

$$(a, g) \circ (b, h) := (ab\omega(g, h), gh)$$

The map  $G_\omega : (a, g) \mapsto g \in G$  is a surjective homomorphism with kernel  $\{(a, 1); a \in T\} \cong T$ . The group  $G_\omega$  is called the  $\omega$ -**covering group** of  $G$  [Kar93]. Recall that if  $G$  and  $N$  are two groups then  $H$  is called an extension of  $G$  by  $N$  if there is a short exact sequence

$$1 \longrightarrow N \xrightarrow{g} H \xrightarrow{f} G \longrightarrow 1$$

i.e  $g$  is monomorphism,  $f$  is epimorphism and  $\text{Ker } f = \text{Im } g$ . When  $g(N)$  lies in the centre of the group  $H$  it is called a *central extension* of  $G$ . Following the previous discussion it is easy to see the abstract error group  $H \cong G_\omega := T \times G$  is a central extension of  $G$ , the index group, by  $T$

$$1 \longrightarrow T \longrightarrow G_\omega \longrightarrow G \longrightarrow 1$$

The main motivation of studying the abstract error group will be clear soon. The idea is that if we have a central extension of a group then any projective representation of a group lifts to an usual representation of its central extension. Since we have already seen that the abstract error group is a central extension of the index group, characterising the abstract error group will be sufficient to study an index group.

**Definition 3.2.3** (lifting a projective representation to central extension). *Let  $A, G$  and  $G^*$  be three groups such that  $G^*$  is an extension of  $G$  by  $A$ ,*

$$1 \longrightarrow A \longrightarrow G^* \xrightarrow{f} G \longrightarrow 1.$$

*Given a projective representation of  $G$ ,  $\pi : G \longrightarrow GL(V)$ , we say  $\pi$  lifts to an ordinary representation  $\pi^* : G^* \longrightarrow GL(V)$  of  $G^*$  if*

- i.  $\pi^*(a)$  is scalar matrix for any  $a \in A$ .*
- ii. there is a section  $\mu : G \longrightarrow G^*$  of  $f$  (i.e.  $\mu(1) = 1$  and  $f \circ \mu = \text{Id}$ ) such that for all  $g \in G$*

$$\pi(g) = \pi^*(\mu(g))$$

**Theorem 3.2.5.** *Let  $\omega$  be a 2-cocycle and  $G_\omega$  be the  $\omega$  covering group of  $G$ . Then every  $\omega$  projective representation of  $G$  lifts to a ordinary representation of  $G_\omega$ .*

*Proof.* If  $\pi$  is a projective representation of  $G$  then we define the ordinary representation  $\pi^*$  of  $G_\omega \cong T \times G$  by

$$\pi^*(g)(\epsilon^i, g) = \epsilon^i \pi(g).$$

We define the section map  $\mu : G \rightarrow G^*$  by  $\mu(g) := (1, g)$  for any  $g \in G$ . It is easy to see now that for any  $g \in G$ ,  $\pi(g) = \pi^*(\mu(g))$  and we have  $\pi^*(\epsilon^i, 1) = \epsilon^i I_n$ . Since  $T \cong \{(\epsilon^i, 1) : 1 \leq i \leq n-1\}$ , it follows that  $\pi^*$  is a lift of  $\pi$ .  $\square$

We see that characterising the index group is equivalent to the characterisation of the abstract error group which is nothing but the central extension of the index group. It was A. Klappenecker and M. Rötteler who characterised all the abstract error groups as a central extension of an index group by a group of central type with cyclic center [KR02].

**Definition 3.2.4.** *A group  $H$  is called group of central type if there exists an ordinary irreducible character  $\chi$  of  $H$  such that  $\chi(1)^2 = |H : Z(H)|$ , where  $Z(H)$  is the center of the group  $H$ .*

Before we discuss the result of Klappenecker and Rötteler we need the following well known results on finite group representation theory, which we will mention without any proof as they are out of the scope of this thesis.

Let  $G$  be a finite group and  $\chi$  a character corresponding to a representation of the group  $G$ . We call the center of the character the set

$$Z(\chi) := \{g \in G : |\chi(g)| = \chi(1)\}.$$

**Lemma 3.2.1.** *If  $\chi$  is an irreducible character of the group  $H$  then  $\chi(1)^2 \leq |H : Z(\chi)|$ . Equality occurs if and only if  $\chi$  vanishes on  $G - Z(\chi)$ .*

*Proof.* Corollary 2.30, [Iss76].  $\square$

**Remark 3.2.6.** *Since we know that  $Z(H) = \cap\{Z(\chi) : \chi \in \text{Irr}(H)\}$ , where  $\text{Irr}(H)$  denotes the set of all irreducible characters of  $H$ , it follows from the above mentioned lemma that  $\chi(1)^2 \leq |H : Z(H)|$ . Equality can occur and when it does we have  $Z(\chi) = Z(H)$*

**Theorem 3.2.7** (Klappenecker & Röttler, 2002). *A group  $H$  is an abstract error group if and only if it is a group of central type with cyclic centre  $Z(H)$ .*

*Proof.* If  $H$  is an abstract error group then it is isomorphic to a  $\omega$ -covering group of an index group  $G$  which has an irreducible faithful projective representation  $\pi$  of order  $|G|^{\frac{1}{2}}$  with 2-cocycle  $\omega$  (cf. theorem 3.2.2). In particular,  $G \cong H/T$  for some cyclic central subgroup  $T$  of  $H$  generated by a primitive  $n$ th root of the unity. Each projective representation of  $G$  lifts to an ordinary representation of  $H$  of same degree. So there exists an irreducible ordinary representation of  $H$  and the corresponding character  $\chi$  of  $H$  with  $\chi(1) = |H : T|^{\frac{1}{2}}$ . Since each irreducible character satisfies the inequality  $\chi(1)^2 \leq |H : Z(H)|$  (see remark 3.2.6) and we already know that  $T \subset Z(H)$  we have  $T = Z(H)$ . Thus,  $H$  is a group of central type with cyclic center.

Conversely, Suppose  $H$  is a group of central type with cyclic center. It was shown in a seminal work of Pahling [Pah70] that  $H$  has a faithful irreducible unitary representation  $\sigma$  of degree  $|H : Z(H)|^{\frac{1}{2}}$ . Let  $G = H/Z(H)$  and denote by  $W = \{x_g : g \in G\}$  a set of coset representative of  $Z(H)$  in  $H$ . Now define a projective representation

$$\pi(g) = \sigma(x_g).$$

The projective representation  $\pi$  is unitary, irreducible and faithful of degree  $|G|^{\frac{1}{2}}$ . Therefore, by theorem 3.2.2  $G$  is an index group of an NEB. Finally, since  $H$  is isomorphic to a  $\omega$  covering group of  $G$ ,  $H$  is an abstract error group.  $\square$

We present in the next result a recipe to obtain an index group from a group of central type. But we need the following lemma before moving on.

**Lemma 3.2.2.** *Let  $H$  be a group and  $\chi$  be an irreducible character of  $G$ . Then  $Z(H/\text{Ker}\chi) = Z(\chi)/\text{Ker}\chi$ . Moreover,  $Z(\chi)/\text{Ker}\chi$  is a cyclic group.*

*Proof.* Lemma 2.27, [Iss76]  $\square$

**Theorem 3.2.8.** *Let  $H$  be a group of central type. Then the group  $(H/\text{Ker}\chi)/(Z(H)/\text{Ker}\chi) \cong H/Z(H)$  is an index group.*

*Proof.* Let  $H$  be a group of central type with an irreducible character  $\chi$  which satisfies  $\chi(1)^2 = |H : Z(H)|$  then  $Z(\chi) = Z(H)$  (see remark 3.2.6). Therefore we have that  $Z(H/\text{Ker}\chi) = Z(H)/\text{Ker}\chi$  by Lemma 3.2.2. For each  $h \in (H/\text{Ker}\chi)/(Z(H)/\text{Ker}\chi)$  we choose a coset representative  $\phi(h)$  in  $H/\text{Ker}\chi$ . Let us denote  $\pi_h = \tilde{\chi}_\chi(\phi(h))$ . Therefore we have

$$\pi_h \pi_k = \tilde{\chi}(\phi(h)) \tilde{\chi}(\phi(k)) = \tilde{\chi}(\phi(h)\phi(k)) = \tilde{\chi}(\phi(hk)z_{h,k}),$$

where  $z_{h,k} \in Z(H)/\text{Ker}\chi$ .  $\tilde{\mathfrak{X}}_\chi(Z(H)/\text{Ker}\chi)$  consists of scalar multiples of identity only. So we obtain  $\pi_h \cdot \pi_k = \tilde{\mathfrak{X}}(\phi(hk))\tilde{\mathfrak{X}}(z_{h,k}) = \omega(h,k)\pi_{hk}$ , where  $\omega(h,k) \in \mathbb{C}$ . Since the representation is irreducible, all the  $\pi_h$  spans  $M_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ . Using isomorphism theorem we get  $(H/\text{Ker}\chi)/(Z(H)/\text{Ker}\chi) \cong H/Z(H)$ . As we know the character  $\chi$  vanishes outside  $Z(H)$ , it follows that  $\text{Tr}(\pi_h) = 0$  except at the identity. So we find that  $(H/\text{Ker}\chi)/(Z(H)/\text{Ker}\chi) \cong H/Z(H)$  is an index group if  $H$  is a group of central type.  $\square$

**Examples:** We present here two examples to briefly clarify the previous result- one with an Abelian index group and another with a non-Abelian index group.

- i. The group of unit quaternions  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$  (with multiplication as the group operation) has eight elements and five irreducible representations (up to equivalence), which we can choose as

$$\begin{aligned} \varepsilon & : \varepsilon(i) = \varepsilon(j) = 1, \\ \sigma_i & : \sigma_i(i) = 1, \quad \sigma_i(j) = -1, \\ \sigma_j & : \sigma_j(i) = -1, \quad \sigma_j(j) = 1, \\ \sigma_k & : \sigma_k(i) = -1 = \sigma_k(j), \\ \pi & : \pi(i) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \pi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

For the character table we get

|                                  | 1 | -1 | $i$ | $-i$ | $j$ | $-j$ | $k$ | $-k$ | dim |
|----------------------------------|---|----|-----|------|-----|------|-----|------|-----|
| $\chi_\varepsilon = \varepsilon$ | 1 | 1  | 1   | 1    | 1   | 1    | 1   | 1    | 1   |
| $\chi_{\sigma_i} = \sigma_i$     | 1 | 1  | 1   | 1    | -1  | -1   | -1  | -1   | 1   |
| $\chi_{\sigma_j} = \sigma_j$     | 1 | 1  | -1  | -1   | 1   | 1    | -1  | -1   | 1   |
| $\chi_{\sigma_k} = \sigma_k$     | 1 | 1  | -1  | -1   | -1  | -1   | 1   | 1    | 1   |
| $\chi_\pi$                       | 2 | -2 | 0   | 0    | 0   | 0    | 0   | 0    | 2   |

We see that

$$\begin{aligned} \ker(\chi_\pi) & = \{1\}, \\ Z(\chi_\pi) & = \{1, -1\} = Z(Q). \end{aligned}$$

$Q$  is a group of central type: its center is  $Z(Q) = \{-1, 1\}$  and it has a  $|Q/Z(Q)|^{1/2}$ -dimensional irreducible representation. We have  $Q/Z(Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

The 2-cocycle  $\omega : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{T}$  is given by the relations

$$\pi(g_1)\pi(g_2) = \omega(g_1, g_2)\pi(g_1g_2),$$

for  $g_1, g_2 \in \mathbb{Z}_2$ . If we write  $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\pm 1, \pm 1)\}$  multiplicatively and choose

$$\begin{aligned} \pi(+1, +1) &= \pi(1) = I_2, \\ \pi(+1, -1) &= \pi(i) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ \pi(-1, +1) &= \pi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \pi(-1, -1) &= \pi(k) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned}$$

We get

| $\omega$   | $(+1, +1)$ | $(+1, -1)$ | $(-1, +1)$ | $(-1, -1)$ |
|------------|------------|------------|------------|------------|
| $(+1, +1)$ | 1          | 1          | 1          | 1          |
| $(+1, -1)$ | 1          | -1         | 1          | -1         |
| $(-1, +1)$ | 1          | -1         | -1         | 1          |
| $(-1, -1)$ | 1          | 1          | -1         | -1         |

- ii. Klappenecker and Rötteler constructed an example of NEB corresponding to a non-commutative index group which we briefly mention here. Consider the group  $H_n$  for some  $n \in \mathbb{N}$ , generated by the composition of the maps

$$\tau : x \mapsto x + 1 \pmod{2^n} \quad \text{and} \quad \alpha : x \mapsto 5x \pmod{2^n}.$$

If  $A := \langle \tau \rangle$  and  $B := \langle \alpha \rangle$  then  $H_n = A \rtimes B$ .

**Theorem 3.2.9.** *The group  $H_n$  is a group of central type of order  $2^{2n-2}$  with cyclic center  $Z(H_n) = \langle \tau^{2^{n-2}} \rangle$ . The index group  $H_n/Z(H_n)$  is non-Abelian for  $n \geq 5$ .*

*Proof.* See theorem 5, [KR02]. □

Let  $\phi : \mathbb{Z}/2^n\mathbb{Z} \rightarrow \mathbb{C}$  be a map defined by

$$\phi(x) = \exp\left(\frac{2\pi i 5^x}{2^n}\right).$$

Then the diagonal matrix

$$\pi(\tau) = \text{diag}(\phi(0), \phi(1), \dots, \phi(2^{n-1} - 1))$$

and the shift

$$\pi(\alpha) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

define an ordinary faithful irreducible representation of  $H_n$ . The NEB corresponding to the index group  $H_n/Z(H_n)$  is given by

$$\mathcal{E} = \{\pi(\tau)^k \pi(\alpha)^l : 0 \leq k, l < 2^{n-2} - 1\}.$$

### NEB and SMB from Discrete Weyl Operators

An open problem posed by Schligemann and Werner was if each NEB is equivalent to a SMB. Klappenecker and Rötteler showed the answer to be negative [KR05]. We discuss here an important class of example called Weyl operators which is both SMB and NEB. We define  $\varkappa((\cdot, \cdot))$  a fixed symmetric nondegenerate bicharacter on  $\mathbb{Z}_n \times \mathbb{Z}_n$  i.e., a function  $\varkappa(\cdot, \cdot) : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{C}$  such that

- (i)  $|\varkappa(x, y)| = 1$ , for all  $x, y \in \mathbb{Z}_n$ ,
- (ii) Symmetry:  $\varkappa(x, y) = \varkappa(y, x)$ , for all  $x, y \in \mathbb{Z}_n$ ,
- (iii) Non-degeneracy:  $\varkappa(x, y) = 1$  for all  $y \in \mathbb{Z}_n$  iff  $x = 0$ ,
- (iv) Character:  $\varkappa(x, y + z) = \varkappa(x, y) \cdot \varkappa(x, z)$ .

In general  $\mathbb{Z}_n$  can be replaced by an arbitrary Abelian finite group

**Example 3.2.10.** *We can take*

$$\varkappa(k, \ell) = \exp\left(\frac{2\pi i k \ell}{N}\right), \quad k, \ell \in \mathbb{Z}_N.$$

Fix an orthonormal basis

$$\{|x\rangle; x \in \mathbb{Z}_n\}$$

of  $\mathbb{C}^n$ . We define two unitary representations  $U$  and  $V$  of  $\mathbb{Z}_n$  on  $\mathbb{C}^n$ , by the relations

$$\begin{aligned} U_a|x\rangle &= |x + a\rangle, \\ V_a|x\rangle &= \varkappa(a, x)|x\rangle, \end{aligned}$$



for  $x, a \in \mathbb{Z}_n$ .

The operators  $U_a, V_b$ ,  $a, b \in \mathbb{Z}_n$ , satisfy the *Weyl commutation relations*

$$U_a U_b = U_{a+b}, \quad V_a V_b = V_{a+b}, \quad V_b U_a = \varkappa(a, b) U_a V_b$$

for  $a, b \in \mathbb{Z}_n$ .

We define the *Weyl operators*

$$W_{a,b} = U_a V_b$$

for  $a, b \in \mathbb{Z}_n$  [Wat18]. The matrix coefficients of a Weyl operator  $W_{a,b}$  w.r.t. to the basis  $\{|x\rangle; x \in \mathbb{Z}_n\}$  are given by

$$\langle y | W_{a,b} | x \rangle = \varkappa(b, x) \delta_{y, x+a}, \quad x, y \in \mathbb{Z}_n,$$

or, equivalently,

$$W_{a,b} = \sum_{x \in G} \varkappa(b, x) |x+a\rangle \langle x|. \quad (3.2.1)$$

It is straightforward that they satisfy the following relations,

$$\begin{aligned} W_{a,b} W_{x,y} &= \varkappa(b, x) W_{a+x, b+y}, \\ W_{a,b}^* &= W_{a,b}^{-1} = \varkappa(a, b) W_{-a, -b}, \\ W_{a,b} W_{x,y} W_{a,b}^* &= \frac{\varkappa(b, x)}{\varkappa(a, y)} W_{x,y}, \end{aligned}$$

for  $a, b, x, y \in \mathbb{Z}_n$ .

It follows that

$$\{n^{-1/2} W_{a,b}; a, b \in G\}$$

is an NEB of  $M_n(\mathbb{C})$ , since

$$\mathrm{Tr}(W_{a,b}) = \sum_{x \in \mathbb{Z}_n} \langle x | \underbrace{U_a V_b}_{=\varkappa(b,x)|x+a} | x \rangle = \begin{cases} n & \text{if } (a, b) = (0, 0), \\ 0 & \text{else,} \end{cases} \quad (3.2.2)$$

and

$$\begin{aligned} \mathrm{Tr}(W_{x,y}^* W_{a,b}) &= \mathrm{Tr}(((x, y)) W_{-x, -y} W_{a,b}) \\ &= ((x - a, y)) \mathrm{Tr}(W_{a-x, b-y}) \\ &= \begin{cases} N & \text{if } (a, b) = (x, y), \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Here we used the orthogonality of the characters,

$$\sum_{x \in \mathbb{Z}_n} \underbrace{\overline{\chi(a, x)} \chi(b, x)}_{=\chi(b-a, x)} = n \delta_{a, b}.$$

The following *Fourier expansion* shows how to express any element  $X \in M_n(\mathbb{C})$  in terms of the o.n.b.  $\{n^{-1/2}W_{a, b}; a, b \in G\}$ . For all  $X \in M_n(\mathbb{C})$  we have

$$X = \frac{1}{n} \sum_{a, b \in \mathbb{Z}_n} \text{Tr}(W_{a, b}^* X) W_{a, b}.$$

## Chapter 4

# Error Basis and Quantum Channel

In this chapter we will use nice error basis, in particular the Weyl operators to construct a convenient basis of  $\text{Lin}(M_n(\mathbb{C}))$ . We want to decompose any linear map with respect to such convenient basis to obtain an  $n^2 \times n^2$  coefficient matrix. We will then characterise the positivity, complete positivity, complete co-positivity in terms of that coefficient matrix. We will also establish the connection between the Choi matrix of a linear map on  $M_n(\mathbb{C})$  and the coefficient matrix that we have found. The discussion and the results of this chapter are based on our paper [BCF23(ii)], co-authored by Uwe Franz and B. V. Rajarama Bhat.

### 4.1 Convenient Basis Of $\text{Lin}(M_n(\mathbb{C}))$

For a pair of matrices  $A, B \in M_n(\mathbb{C})$  we define a linear map  $T_{A,B} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  by

$$T_{A,B}(X) = AXB^*, \quad \text{for } X \in M_n(\mathbb{C}).$$

**Proposition 4.1.1.** *The map  $M_n(\mathbb{C}) \times M_n(\mathbb{C}) \ni (A, B) \mapsto T_{A,B} \in \text{Lin}(M_n(\mathbb{C}))$  extends to a unique isomorphism of  $*$ -algebras  $T : M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^* \ni A \otimes B^* \rightarrow T_{A,B} \in \text{Lin}(M_n(\mathbb{C}))$ , which is also an isomorphism of Hilbert spaces.*

The above proposition shows that if  $\{B_i; 1 \leq i \leq n^2\}$  is a basis of  $M_n(\mathbb{C})$  and we define  $T_{ij}(X) := B_i X B_j^*$  for any  $X \in M_n(\mathbb{C})$  then  $\{T_{ij} \in \text{Lin}(M_n(\mathbb{C})); 1 \leq i, j \leq n^2\}$  is a basis of  $\text{Lin}(M_n(\mathbb{C}))$ . Taking an NEB as a basis of  $M_n(\mathbb{C})$  has the added advantage that in that case  $T_{ij}$  also becomes an NEB in  $\text{Lin}(M_n(\mathbb{C}))$ .

**Lemma 4.1.1.** *Let  $\{\frac{1}{\sqrt{n}}\pi_g; g \in G\}$  be an NEB of  $M_n(\mathbb{C})$  corresponding to an index group  $G$ . Define the linear map  $T_{x,y} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  by*

$$T_{x,y}(X) := \pi_x X \pi_y^*.$$

Then the  $\{\frac{1}{n}T_{x,y}; x, y \in G\}$  is an NEB of  $\text{Lin}(M_n(\mathbb{C}))$  with the index group  $G \times G$  and 2-cocycle  $\omega^L : (G \times G) \times (G \times G) \rightarrow \mathbb{T}$  given by

$$\omega^L((x', y'), (x, y)) = \frac{\omega(x', x)}{\omega(y', y)}.$$

*Proof.* It is trivial to check that  $T_{1,1} = \text{Id}$  where 1 is the identity element of  $G$ . To check the trace condition, we compute

$$\begin{aligned} \text{Tr}_{\text{Lin}(M_n(\mathbb{C}))}(T_{x,y}) &= \sum \langle |i\rangle\langle j|, T_{xy}|i\rangle\langle j| \rangle = \sum \text{Tr}_{M_n(\mathbb{C})}(|j\rangle\langle i|\pi_x|i\rangle\langle j|\pi_y^*) \\ &= \sum \langle i|\pi_x|i\rangle\langle j|\pi_y^*|j\rangle = \text{Tr}(\pi_x)\text{Tr}(\pi_y^*) = n^2\delta_{1,x}\delta_{1,y}, \end{aligned}$$

where  $\{|i\rangle\langle j|; 1 \leq i, j \leq n\}$  is the set of matrix units of  $M_n(\mathbb{C})$ . This shows that trace of each operator  $T_{x,y}$  is zero except for the identity. For any  $X \in M_n(\mathbb{C})$  and  $x, y, x', y' \in G$  we see that

$$T_{x',y'} \circ T_{x,y}(X) = \pi_{x'}\pi_x X \pi_y^* \pi_{y'}^* = \frac{\omega(x', x)}{\omega(y', y)} \pi_{x'} X \pi_{y'}^* = \frac{\omega(x', x)}{\omega(y', y)} T_{x',y'}(X),$$

which proves the claim about the 2-cocycle  $\omega^L$ .  $\square$

The next proposition follows immediately since any NEB forms an ONB of the associated space of linear maps.

**Proposition 4.1.2.** *The set  $\{\frac{1}{n}T_{x,y}; x, y \in G\}$  forms an orthonormal basis of  $\text{Lin}(M_n(\mathbb{C}))$  with respect to the Hilbert-Schmidt inner product.*

Let  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  and  $\{B_i; 1 \leq i \leq n^2\}$  be a basis of  $M_n(\mathbb{C})$ . Since  $T_{ij}$  (defined above) forms a basis of  $\text{Lin}(M_n(\mathbb{C}))$  we can decompose  $\alpha$  as

$$\alpha(X) = \sum_{1 \leq i, j \leq n^2} D_\alpha(i, j) B_i X B_j^*. \quad (4.1.1)$$

In particular, if we take an NEB  $\{\frac{1}{\sqrt{n}}\pi_x; x \in G\}$  as a basis of  $M_n(\mathbb{C})$  we can explicitly compute the coefficient matrix  $D_\alpha$ .

$$\alpha(X) = \frac{1}{n} \sum_{x,y} D_\alpha(x, y) T_{x,y}(X) = \frac{1}{n} \sum_{x,y} D_\alpha(x, y) \pi_x X \pi_y^*. \quad (4.1.2)$$

for all  $X \in M_n(\mathbb{C})$ . Using the orthonormality of the basis  $\frac{1}{n}T_{x,y}$  and NEB  $\{\frac{1}{\sqrt{n}}\pi_g; g \in G\}$  of  $M_n(\mathbb{C})$ , we have  $D_\alpha(x, y) = \frac{1}{n} \langle T_{x,y}, \alpha \rangle_{\text{Lin}(M_n(\mathbb{C}))}$  i.e.

$$\begin{aligned} D_\alpha(x, y) &= \frac{1}{n} \text{Tr}(T_{x,y}^\dagger \alpha) \\ &= \frac{1}{n^2} \sum_{g \in G} \langle T_{x,y}(\pi_g), \alpha(\pi_g) \rangle \\ &= \frac{1}{n^2} \sum_{g \in G} \text{Tr}(\pi_y \pi_g^* \pi_x^* \alpha(\pi_g)). \end{aligned} \quad (4.1.3)$$

Here  $T^\dagger$  denotes the involution applied on  $T$  w.r.t the Hilbert-Schmidt inner product on  $\text{Lin}(M_n(\mathbb{C}))$ . If we use the Weyl operators as the nice error basis for defining  $T_{x,y}$  i.e.  $T_{x,y}(X) = W_x X W_y^*$  for  $x, y \in \mathbb{Z}_n \times \mathbb{Z}_n$  then we can compute the coefficient  $D_\alpha$  in (4.1.3), using  $\{|a\rangle\langle b|; a, b \in \mathbb{Z}_n\}$  as an o.n.b of  $M_n(\mathbb{C})$

$$\begin{aligned} D_\alpha(x, y) &= \frac{1}{n} \sum_{a, b \in G} \text{Tr}\left(W_y |b\rangle\langle a| W_x^* \alpha(|a\rangle\langle b|)\right) \\ &= \frac{1}{n} \sum_{a, b \in G} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} \left\langle a + x_1 \left| \alpha(|a\rangle\langle b|) \right| b + y_1 \right\rangle, \end{aligned} \quad (4.1.4)$$

for  $x = (x_1, x_2), y = (y_1, y_2)$ .

**Lemma 4.1.2.** *Let  $\alpha, \beta \in \text{Lin}(M_n(\mathbb{C}))$  with coefficients  $(D_\alpha(x, y))_{x, y \in G}$  and  $(D_\beta(x, y))_{x, y \in G}$  respectively, as defined in the equation (4.1.2). Then the coefficients of their composition  $\alpha \circ \beta$  are given by*

$$D_{\alpha \circ \beta}(x, y) = \sum_{p, q \in G} \omega(p, xp^{-1}) \overline{\omega(q, yq^{-1})} D_\alpha(p, q) D_\beta(p^{-1}x, q^{-1}y),$$

for  $x, y \in G$

*Proof.* We have

$$\begin{aligned} \alpha \circ \beta(X) &= \sum_{p, q \in G} D_\alpha(p, q) \pi_p \left( \sum_{p', q' \in G} D_\beta(p', q') \pi_{p'} X \pi_{q'}^* \right) \pi_q^* \\ &= \sum_{p, p', q, q' \in G} \omega(p, p') \overline{\omega(q, q')} D_\alpha(p, q) D_\beta(p', q') \pi_{pp'} X \pi_{qq'}^* \\ &= \sum_{x, y \in G} \underbrace{\left( \sum_{p, q \in G} \omega(p, p^{-1}x) \overline{\omega(q, q^{-1}y)} D_\alpha(p, q) D_\beta(p^{-1}x, q^{-1}y) \right)}_{D_{\alpha \circ \beta}} \pi_x X \pi_y^*, \end{aligned}$$

which completes the proof.  $\square$

We can define two different involutions on  $\text{Lin}(M_n(\mathbb{C}))$ . The first comes from the Hilbert-Schmidt inner product and is characterised by the condition

$$\langle X, \alpha(Y) \rangle = \langle \alpha^\dagger(X), Y \rangle$$

for all  $X, Y \in M_n(\mathbb{C})$ .

The second is inherited from the involution in  $M_n(\mathbb{C})$  and defined by  $\alpha^\#(X) = \alpha(X^*)^*$ .

Both involutions are conjugate linear, but only the first is anti-multiplicative, whereas the second is multiplicative, i.e., we have

$$(\alpha \circ \beta)^\dagger = \beta^\dagger \circ \alpha^\dagger, \quad (\alpha \circ \beta)^\# = \alpha^\# \circ \beta^\#$$

for  $\alpha, \beta \in \text{Lin}(M_n(\mathbb{C}))$ .

**Proposition 4.1.3.** *Let  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$ . We have*

$$D_{\alpha^\dagger}(x, y) = \frac{\omega(x, x^{-1})}{\omega(y, y^{-1})} \overline{D_\alpha(x^{-1}, y^{-1})}$$

and

$$D_{\alpha^\#}(x, y) = \overline{D_\alpha(y, x)}$$

for  $x, y \in G$ .

*Proof.* It is easy to see that for any  $x, y \in G$  we have  $T_{x,y}^\dagger = \frac{\omega(x, x^{-1})}{\omega(y, y^{-1})} T_{x^{-1}, y^{-1}}$ . Applying the involution  $\dagger$  on the decomposition of  $\alpha$  in (4.1.2)

$$\alpha^\dagger = \sum_{x,y \in G} \overline{D_\alpha(x, y)} T_{x,y}^\dagger = \sum_{x,y \in G} \frac{\omega(x, x^{-1})}{\omega(y, y^{-1})} \overline{D_\alpha(x, y)} T_{x^{-1}, y^{-1}}$$

the first claim follows. Similarly, we can trivially check that  $T_{x,y}^\# = T_{y,x}$  for any  $x, y \in G$ . Then the second claim follows by applying  $\#$  on the decomposition (4.1.2)

$$\alpha^\# = \sum_{x,y \in G} \overline{D_\alpha(x, y)} T_{x,y}^\# = \sum_{x,y \in G} \overline{D_\alpha(x, y)} T_{y,x}$$

□

### 4.1.1 Examples

Here we compute the kernel or the  $n^2 \times n^2$  matrix  $D_\alpha$  corresponding to different positive maps  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  which are important in quantum information. We use  $\{\frac{1}{n}T_{x,y}; x, y \in G\}$  as an o.n.b of  $\text{Lin}(M_n(\mathbb{C}))$  where  $G$  is an index group corresponding to an NEB  $\{\frac{1}{\sqrt{n}}\pi_x; x \in G\}$ .

**Identity map:** The identity map corresponds to the kernel  $D_{\text{Id}}(x, y) = n\delta_{1,x}\delta_{1,y}$  for  $x, y \in G$  as we can write

$$X = \pi_1 X \pi_1^* = \frac{1}{n} \sum_{x,y \in G} n\delta_{1,x}\delta_{1,y} \pi_x X \pi_y^*$$

**Depolarising Channel:** Let  $P \in \text{Lin}(M_n(\mathbb{C}))$  be the diagonal sum  $P = \frac{1}{n} \sum_{g \in G} T_{g,g}$ . For any  $h \in G$  and  $X \in M_n(\mathbb{C})$  we have

$$\pi_h P(X) = \frac{1}{n} \sum_g \omega(h, g) \pi_{hg} X \pi_g^* \quad \text{and} \quad P(X) \pi_h = \frac{1}{n} \sum_g \frac{\omega(h, h^{-1})}{\omega(h^{-1}, g)} \pi_g X \pi_{h^{-1}g}^*$$

After a change of variable we can write  $P(X)\pi_h = \frac{1}{n} \sum_g \frac{\omega(h, h^{-1})}{\omega(h^{-1}, hg)} \pi_{hg} X \pi_g^*$ . Using the definition of 2-cocycle

$$\frac{\omega(h, h^{-1})}{\omega(h^{-1}, hg)} = \frac{\omega(h, h^{-1})}{\omega(h, h^{-1})\omega(1, g)\omega(h, g)} = \omega(h, g).$$

Thus we see that  $P(X)$  commutes with every basis element  $\pi_h$  on  $M_n(\mathbb{C})$ . So we conclude that  $P(X) = cI_n$  for some  $c \in \mathbb{C}$ . Computing the trace of both sides we find that  $c = \text{Tr}(X)$ . Therefore we see that the map  $P$ , defined as the diagonal sum of the operators  $T_{g,g}$ , is actually the depolarising channel which corresponds to the identity matrix  $D_P(x, y) = \frac{1}{n} \delta_{x,y}$  where  $x, y \in G$ .

**Transposition:** Let  $T \in \text{Lin}(M_n(\mathbb{C}))$  be the transposition map given by  $X \mapsto X^t$ . We compute the  $n^2 \times n^2$  matrix  $D_T$  corresponding to the transposition map. For any  $x, y \in G$  we have

$$\begin{aligned} D_T(x, y) &= \frac{1}{n} \langle T_{x,y} | T \rangle = \frac{1}{n} \sum_{1 \leq i, j \leq n} \text{Tr} \left( (T_{x,y} | i \rangle \langle j |)^* T | i \rangle \langle j | \right) \\ &= \frac{1}{n} \sum_{1 \leq i, j \leq n} \text{Tr} \left( \pi_y | j \rangle \langle i | \pi_x^* | j \rangle \langle i | \right) \\ &= \frac{1}{n} \sum_{1 \leq i, j \leq n} \langle i | \pi_x^* | j \rangle \text{Tr} \left( \pi_y | j \rangle \langle i | \right) \\ &= \frac{1}{n} \sum_{1 \leq i, j \leq n} \langle i | \pi_x^* | j \rangle \langle i | \pi_y | j \rangle = \frac{1}{n} \text{Tr}(\bar{\pi}_x \pi_y) \end{aligned}$$

In particular, if we take the Weyl operators  $\{W_{a,b}; a, b \in \mathbb{Z}_n\}$  as the chosen NEB and  $\{|i\rangle; i \in \mathbb{Z}_n\}$  as the standard basis of  $\mathbb{C}^n$  then

$$D_T((a, b), (c, d)) = \frac{1}{n} \sum_{i, j \in \mathbb{Z}_n} \langle i | W_{a,b}^* | j \rangle \langle i | W_{c,d} | j \rangle = \frac{1}{n} \sum_{i, j \in \mathbb{Z}_n} \chi(a, b) \chi(-b, j) \chi(d, j) \delta_{i, j-a} \delta_{i, c+j}.$$

So we have

$$D_T((a, b), (c, d)) = \begin{cases} \frac{1}{n} \sum_{i \in \mathbb{Z}_n} \chi(a, b) \chi(d - b, i + a) & \text{if } c = -a, \\ 0 & \text{otherwise.} \end{cases}$$

**Conditional Expectation onto Diagonal:** Consider the linear map  $C : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ ,  $C(X) = (\delta_{ij} x_{ij})_{1 \leq i, j \leq n}$  for  $X = (x_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbb{C})$ . This map is a conditional expectation onto the \*-subalgebra of diagonal matrices with respect to the standard basis.

We compute the coefficient matrix  $D_C$  of the map  $C$ . For any  $x, y \in G$

$$\begin{aligned} D_C(x, y) = \frac{1}{n} \langle T_{x,y} | C \rangle &= \frac{1}{n} \sum_{1 \leq i, j \leq n} \text{Tr} \left( (T_{x,y} | i \rangle \langle j |)^* C | i \rangle \langle j | \right) \\ &= \frac{1}{n} \sum_{1 \leq i, j \leq n} \delta_{ij} \text{Tr}(\pi_y | j \rangle \langle i | \pi_x^* | i \rangle \langle j |) \\ &= \frac{1}{n} \sum_{1 \leq j \leq n} \delta_{ij} \langle i | \pi_x^* | i \rangle \langle j | \pi_y | j \rangle = \frac{1}{n} \text{Tr}(C(\pi_x^*) C(\pi_y)). \end{aligned}$$

In particular, taking the Weyl operators as NEB gives

$$D_C((a, b), (c, d)) = \begin{cases} \frac{1}{n} \sum_{j \in \mathbb{Z}_n} \chi(d - b, j) & \text{if } c = a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

## 4.2 Correspondence between Choi matrix $C_\alpha$ and $D_\alpha$

Recall that the Choi-Jamiołkowski matrix of a map  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is the  $n^2 \times n^2$ -matrix defined by

$$C_\alpha = \sum_{j,k=1}^n E_{jk} \otimes \alpha(E_{jk}) \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \cong M_{n^2}(\mathbb{C}).$$

It is known that  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is completely positive (CP) iff  $C_\alpha$  is positive. Furthermore,  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is  $k$ -positive if and only if

$$\langle v, C_\alpha v \rangle \geq 0$$

for all  $v \in \mathbb{C}^n \otimes \mathbb{C}^n$  with Schmidt rank not more than  $k$  [RA07, SSZ09]. For any completely positive map  $\alpha$  we have the Kraus decomposition

$$\alpha = \sum_{j=1}^r \text{Ad}_{L_j}$$

for some matrices  $L_j \in M_n(\mathbb{C})$ , where for any  $X \in M_n(\mathbb{C})$  the conjugate map  $\text{Ad}_{L_j}$  is given by  $\text{Ad}_{L_j}(X) = L_j X L_j^*$ . The minimum number of  $\text{Ad}_{L_j}$  required to express  $\alpha$  as its Kraus decomposition is called the Kraus rank of  $\alpha$ . We call  $\alpha$  1-super positive or entanglement breaking iff  $\text{rank}(L_j) = 1$  for any  $j$ . The linear map  $\alpha$  is called completely co-positive iff  $T \circ \alpha$  is completely positive, where  $T$  is the transposition map.

We can switch from  $C_\alpha$  to  $D_\alpha$  by a change of basis.

**Proposition 4.2.1.** *If  $T_{x,y}$  is defined with respect to the Weyl operators then the Choi-Jamiołkowski matrix of  $T_{x,y}$  is given by*

$$C_{T_{x,y}}(v, w) = \frac{\varkappa(x_2, v_1)}{\varkappa(y_2, w_1)} \delta_{v_1+x_1, v_2} \delta_{w_1+y_1, w_2},$$



for  $x = (x_1, x_2), y = (y_1, y_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ . More generally, if  $\alpha$  is of the form  $\alpha = \sum_{x,y \in \mathbb{Z}_n \times \mathbb{Z}_n} D_\alpha(x, y) T_{x,y}$ , then its Choi-Jamiołkowski matrix is given by

$$C_\alpha(v, w) = \frac{1}{n} \sum_{x_2, y_2 \in G} \frac{\varkappa(x_2, v_1)}{\varkappa(y_2, w_1)} D_\alpha((v_2 - v_1, x_2), (w_2 - w_1, y_2)),$$

for  $v, w \in \mathbb{Z}_n \times \mathbb{Z}_n$ . Conversely, the coefficients from the equation (4.1.4) can be computed from the Choi-Jamiołkowski matrix via

$$D_\alpha(x, y) = \frac{1}{n} \sum_{a, b \in G} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} C_\alpha((a, a + x_1), (b, b + y_1))$$

for  $x, y \in \mathbb{Z}_n \times \mathbb{Z}_n$ .

*Proof.* Using Formula (3.2.1), we get

$$W_y^* = \sum_{d \in \mathbb{Z}_n} \frac{1}{\varkappa(y_2, d)} |d\rangle \langle d + y_1|$$

and

$$T_{x,y}(|a\rangle \langle b|) = \frac{\varkappa(x_2, a)}{\varkappa(y_2, b)} |a + x_1\rangle \langle b + y_1|,$$

for  $a, b \in \mathbb{Z}_n, x, y \in \mathbb{Z}_n \times \mathbb{Z}_n$ . So, if we choose  $\{|a\rangle; a \in \mathbb{Z}_n\}$  as a basis of  $\mathbb{C}^n$ , we can write the corresponding matrix units as  $|a\rangle \langle b|, a, b \in \mathbb{Z}_n$ , and we get

$$\begin{aligned} C_{T_{x,y}} &= \sum_{a, b \in \mathbb{Z}_n} |a\rangle \langle b| \otimes T_{x,y}(|a\rangle \langle b|) \\ &= \sum_{a, b \in \mathbb{Z}_n} \frac{\varkappa(x_2, a)}{\varkappa(y_2, b)} |(a, a + x_1)\rangle \langle (b, b + y_1)|, \end{aligned}$$

which proves the first claim of the proposition.

For  $\alpha = \frac{1}{n} \sum_{x,y \in \mathbb{Z}_n \times \mathbb{Z}_n} D_\alpha(x, y) T_{x,y}$ , this yields

$$C_\alpha = \frac{1}{n} \sum_{x,y \in \mathbb{Z}_n \times \mathbb{Z}_n} \sum_{a, b \in \mathbb{Z}_n} D_\alpha(x, y) \frac{\varkappa(x_2, a)}{\varkappa(y_2, b)} |(a, a + x_1)\rangle \langle (b, b + y_1)|$$

or

$$C_\alpha(v, w) = \frac{1}{n} \sum_{x_2, y_2 \in \mathbb{Z}_n} \frac{\varkappa(x_2, v_1)}{\varkappa(y_2, w_1)} D_\alpha((v_2 - v_1, x_2), (w_2 - w_1, y_2)).$$

For the converse we use the equation (4.1.4),

$$\begin{aligned} D_\alpha(x, y) &= \frac{1}{n} \sum_{a, b \in G} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} \left\langle a + x_1 \left| \alpha(|a\rangle \langle b|) \right| b + y_1 \right\rangle \\ &= \frac{1}{n} \sum_{a, b \in \mathbb{Z}_n} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} \left\langle a + x_1 \left| \left( \sum_{g, h \in \mathbb{Z}_n} C_\alpha((a, g), (b, h)) |g\rangle \langle h| \right) \right| b + y_1 \right\rangle \\ &= \frac{1}{n} \sum_{a, b \in \mathbb{Z}_n} \frac{\varkappa(y_2, b)}{\varkappa(x_2, a)} C_\alpha((a, a + x_1), (b, b + y_1)), \end{aligned}$$

where we used the identity

$$\alpha(|a\rangle\langle b|) = \sum_{g,h \in G} C_\alpha((a,g), (b,h)) |g\rangle\langle h|.$$

□

### 4.3 Characterisation of positive and completely positive maps

**Theorem 4.3.1.** *Let  $\{B_x\}_{x=1,2,\dots,n^2}$  be a basis of  $M_n(\mathbb{C})$ . Consider a linear map  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  of the form  $\alpha(X) = \sum_{x,y=1}^{n^2} D_\alpha(x,y) B_x X B_y^*$ . Then  $\alpha$  is*

- i. *hermiticity preserving if and only if  $D_\alpha$  is Hermitian.*
- ii. *positive if and only if for any  $v, w \in \mathbb{C}^n$ ,*

$$\langle v \otimes w, \tilde{\alpha}(v \otimes w) \rangle \geq 0$$

where  $\tilde{\alpha} = \tau \circ \sum_{x,y=1}^{n^2} D_\alpha(x,y) (B_x \otimes B_y^*)$  and  $\tau(u \otimes v) = v \otimes u$  is the flip operator.

*Proof.*  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is hermiticity preserving iff  $\alpha(X^*)^* = \alpha(X)$  i.e.  $\alpha^\# = \alpha$ . Comparing the coefficient matrix of both sides the first claim follows directly from the proposition 4.1.3.

On the other hand,  $\alpha$  is positive if and only if it maps rank one projections to positive operators. i.e. for all  $v, w \in \mathbb{C}^n$

$$\begin{aligned} 0 \leq \langle v, \alpha(|u\rangle\langle u|)v \rangle &= \left\langle v, \sum_{x,y} D_\alpha(x,y) B_x |u\rangle\langle u| B_y^* v \right\rangle \\ &= \sum_{x,y} D_\alpha(x,y) \langle v, B_x u \rangle \langle u, B_y^* v \rangle \\ &= \left\langle u \otimes v, \tau \circ \sum_{x,y} D_\alpha(x,y) B_x \otimes B_y^* (u \otimes v) \right\rangle = \langle u \otimes v, \tilde{\alpha}(u \otimes v) \rangle. \end{aligned}$$

□

**Theorem 4.3.2.** *A linear map  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is a completely positive map with Kraus rank  $r$  if and only if the corresponding coefficient matrix  $D_\alpha \in M_{n^2}(\mathbb{C})$  as defined in (4.1.1), is positive semi-definite of rank  $r$ .*

*Proof.* Let  $\{B_x; x = 1, \dots, n^2\}$  be a basis of  $M_n(\mathbb{C})$ .  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is CP with Kraus rank  $r$  then there exists  $\{L_j \in M_n(\mathbb{C}); 1 \leq j \leq r\}$  such that  $\alpha$  can be written as Kraus

decomposition

$$\alpha = \sum_{j=1}^r \text{Ad}_{L_j},$$

where  $\text{Ad}_{L_j}$  is the conjugate map given by  $\text{Ad}_{L_j}(X) = L_j X L_j^*$  for any matrix  $X \in M_n(\mathbb{C})$ . Since the map  $\text{Lin}(M_n(\mathbb{C})) \ni \alpha \mapsto D_\alpha \in M_{n^2}(\mathbb{C})$  is a linear isomorphism, we have  $D_\alpha = \sum_{j=1}^r D_{\text{Ad}_{L_j}}$ . We can expand each  $L_j$  with respect to the basis  $\{B_x; x = 1, \dots, n^2\}$  and write  $L_j = \sum_{z=1}^{n^2} l_j(z) B_z$ . Therefore we get

$$\begin{aligned} \text{Ad}_{L_j}(X) &= L_j X L_j^* = \left( \sum_{z=1}^{n^2} l_j(z) B_z \right) X \left( \sum_{z'=1}^{n^2} l_j(z') B_{z'} \right)^* \\ &= \sum_{z, z'=1}^{n^2} l_j(z) \overline{l_j(z')} B_z X B_{z'}^* \end{aligned}$$

We find that  $D_{\text{Ad}_{L_j}}$  is a rank one operator given by

$$D_{\text{Ad}_{L_j}} = |l_j\rangle\langle l_j|$$

where  $l_j = (l_j(1), l_j(2), \dots, l_j(n^2))^t$  is a vector in  $\mathbb{C}^{n^2}$ . Thus  $D_\alpha = \sum_{j=1}^r |l_j\rangle\langle l_j|$  is a positive semi-definite operator of rank  $r$ .

Conversely, assume that  $D_\alpha$  is positive semi-definite with rank  $r$ . So there exists vectors  $v_1, v_2, \dots, v_r \in \mathbb{C}^{n^2}$  such that  $D_\alpha = \sum_{j=1}^r |v_j\rangle\langle v_j|$ . If we denote  $\{|x\rangle; x = 1, \dots, n^2\}$  the standard basis of  $\mathbb{C}^{n^2}$  then

$$D_\alpha(x, y) = \sum_{j=1}^r \langle x|v_j\rangle\langle v_j|y\rangle = \sum_{j=1}^r \langle x|v_j\rangle\overline{\langle y|v_j\rangle}.$$

Therefore we can write the equation (4.1.1) as

$$\begin{aligned} \alpha(X) &= \sum_{x, y=1}^{n^2} \sum_{j=1}^r \langle x|v_j\rangle\overline{\langle y|v_j\rangle} B_x X B_y^* \\ &= \sum_{x, y=1}^{n^2} \sum_{j=1}^r (\langle x|v_j\rangle B_x) X (\langle y|v_j\rangle B_y)^* \\ &= \sum_{j=1}^r \left( \sum_{x=1}^{n^2} \langle x|v_j\rangle B_x \right) X \left( \sum_{y=1}^{n^2} \langle y|v_j\rangle B_y \right)^*. \end{aligned}$$

If we denote  $L_j = \sum_{x \in G} \langle x|v_j\rangle B_x$  then we get  $\alpha(X) = \sum_{j=1}^r L_j X L_j^*$ , which shows that  $\alpha$  is completely positive with Kraus rank  $r$ .  $\square$

We remark here that the similar result was obtained by Poluikis and Hill from a different approach [PH81]. But our approach has the advantage that coefficient matrix

corresponding to the composition of two linear maps becomes the convolution type product of their individual coefficient matrices. This composition law can be used to characterise the completely co-positive maps too.

**Corollary 4.3.1.** *A linear map  $\alpha \in \text{Lin}(M_n(\mathbb{C}))$  is completely co-positive iff the convolution product*

$$\sum_{p,q \in \mathbb{Z}_n \times \mathbb{Z}_n} \frac{\chi(p, x-p)}{\chi(q, y-q)} \text{Tr}(\overline{W}_p W_q) D_\alpha(x-p, y-q)$$

*is positive semi-definite.*

*Proof.* The linear map  $\alpha$  is co-CP iff the composition with transposition  $T \circ \alpha$  is CP. Using the Theorem 4.3.2, we see that  $T \circ \alpha$  is CP iff the coefficient matrix  $D_{T \circ \alpha}$  is positive semi-definite. In particular, in the decomposition  $T \circ \alpha$  w.r.t the Weyl operators

$$T \circ \alpha(X) = \frac{1}{n} \sum_{x,y \in \mathbb{Z}_n \times \mathbb{Z}_n} D_{T \circ \alpha} W_x X W_y^*,$$

the coefficient matrix  $D_{T \circ \alpha}$  is positive semi-definite. We use the Lemma 4.1.2 and the coefficients we have found for transposition in example 4.1.1 to compute the coefficient  $D_{T \circ \alpha}$  which complete the claim.  $\square$

**Corollary 4.3.2.** *A linear map  $\alpha \in \text{Lin}(M_2(\mathbb{C}))$  is 1-super positive iff  $D_\alpha = \sum_{j=1}^r |l_j\rangle\langle l_j|$  where  $l_j = (l_j(1), \dots, l_j(4))^t$  is a vector in  $\mathbb{C}^4$  satisfying  $l_j(1)^2 = \sum_{k=2}^4 l_j(k)^2$ .*

*Proof.* Since  $\alpha$  is 1-super positive in  $M_2(\mathbb{C})$ , there exists matrices  $L_1, L_2, \dots, L_r \in M_2(\mathbb{C})$  of rank 1 such that  $\alpha = \sum_1^r \text{Ad}_{L_j}$ . We can decompose each  $L_j$  w.r.t the Pauli basis

$$\sigma_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_4 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

to obtain  $L_j = \sum_{k=1}^4 l_j(k) \sigma_k$ . After this decomposition  $L_j$  has the form

$$\frac{1}{2} \begin{bmatrix} l_j(1) + l_j(4) & l_j(2) - il_j(3) \\ l_j(2) + il_j(3) & l_j(1) - l_j(4) \end{bmatrix}.$$

$L_j$  has rank 1 iff  $\det(L_j) = 0$  i.e.  $l_j(1)^2 = l_j(2)^2 + l_j(3)^2 + l_j(4)^2$ . We have already seen that in the proof of 4.3.2 that  $D_{\text{Ad}_{L_j}} = |l_j\rangle\langle l_j|$ , which completes the claim.  $\square$

**Proposition 4.3.1.** *A linear map  $\alpha : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$*

(a) is trace preserving if and only if

$$\sum_{x \in G} \omega(x, g) D_\alpha(x, xg) = \delta_{1, g}$$

for all  $g \in G$ .

(b) is unit preserving if and only if

$$\sum_{x \in G} \frac{\omega(x, x^{-1}z)}{\omega(z^{-1}x, x^{-1}z)} D_\alpha(x, z^{-1}x) = \delta_{1, z}$$

for all  $z \in G$ .

*Proof.*

(a) Since  $\{\pi_g; g \in G\}$  forms a basis of  $M_n$  the map  $\alpha$  is trace preserving iff  $\text{Tr}(\alpha(\pi_g)) = \text{Tr}(\pi_g)$  for all  $g \in G$ . Now

$$\text{Tr}(\alpha(\pi_g)) = \sum_{x, y} D_\alpha(x, y) \frac{\omega(x, g)\omega(xg, y^{-1})}{\omega(y^{-1}, y)} \text{Tr}(\pi_{xgy^{-1}})$$

Substituting  $xg = y$  we get

$$\sum_{x \in G} \omega(x, g) D_\alpha(x, xg) = \text{Tr}(\pi_g) = \delta_{1, g}$$

for all  $g \in G$ .

(b) By definition  $\alpha$  is unit preserving iff  $\alpha(I_n) = I_n$ .

$$\begin{aligned} \alpha(I_n) &= \sum_{x, y \in G} D_\alpha(x, y) \pi_x \pi_y^* = \sum_{x, y \in G} \frac{\omega(x, y^{-1})}{\omega(y, y^{-1})} D_\alpha(x, y) \pi_{xy^{-1}} \\ &= \sum_{x, z \in G} \frac{\omega(x, x^{-1}z)}{\omega(z^{-1}x, x^{-1}z)} D_\alpha(x, z^{-1}x) \pi_z. \end{aligned}$$

Comparing the coefficients we get

$$\sum_{x \in G} \frac{\omega(x, x^{-1}z)}{\omega(z^{-1}x, x^{-1}z)} D_\alpha(x, z^{-1}x) = \delta_{1, z}$$

for all  $z \in G$ . □



## Chapter 5

# Semigroup Of Different Positive Maps

The following discussion is based on our paper [BCF23(i)], co-authored by Uwe Franz and B. V. Rajarama Bhat. Let  $V$  be a Banach space and  $\mathcal{L}(V)$  be the algebra of all bounded linear operators on  $V$ . We recall the definition of a semigroup of linear operators-

**Definition 5.0.1.** *A unital semigroup of operators on  $V$  is a family of bounded linear operators on  $(T_t)_{t \geq 0}$  on  $V$  such that*

- i.  $T_0 = \text{Id}$ ,*
- ii.  $T_{t+s} = T_t \circ T_s$  for  $t, s \geq 0$ ,*
- iii. the map  $t \mapsto T_t$  is continuous.*

**Remark 5.0.1.** *We remark here that the third condition which imposes certain continuity on the semigroup  $(T_t)_{t \geq 0}$  can be replaced by weaker conditions of continuity e.g. strong continuity ( $C_0$  semigroup), ultra weak continuity. However, as we will work on finite dimension it will be enough to restrict our discussion on norm continuity of the semigroup of linear maps.*

It is well known that there exists an operator  $A \in \mathcal{L}(V)$  such that  $T_t = \exp(tA)$  for  $t \geq 0$ . Such an operator  $A$  is called the generator of the semigroup of linear operators  $(T_t)_{t \geq 0}$  (see prop. 9.4, [BFR17]). Even if we weaken the continuity condition (iii) by replacing the norm continuity by strong continuity existence of closed densely defined generators follows from Hill-Yoshida theorem (see theorem 2.2.5, [App19]).

On finite dimension, let's consider a semigroup of operators  $T_t : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ . We call it an  $n$ -level quantum dynamical semigroup of linear maps if it preserves identity i.e.  $T_t(I_n) = I_n$  and each  $T_t$  is a completely positive map for  $t \geq 0$ . The following result was proved by Lindblad, Gorini, Kossakowski and Sudarshan [Lin76][GKS76].

**Theorem 5.0.2.** *A semigroup of linear operators  $T_t = \exp t\Psi$  is a quantum dynamical semigroup if and only if the generator  $\Psi$  has the form*

$$\Psi(X) = i[H, X] + \sum_{j=1}^k \{V_j^* X V_j - \frac{1}{2}(V_j^* V_j X + X V_j^* V_j)\}$$

Motivated by this result we were interested to see if it is possible to characterise the generator of semigroups of  $k$ -(super)positive maps.

Let  $A$  be a unital algebra with unit  $\mathbf{1}$ . Let  $C$  be a cone inside the topological dual space  $A'$ . Then an element  $a \in A$  is called conditionally positive on  $C$  if  $\phi(a) \geq 0$  for any  $\phi \in C$  such that  $\phi(\mathbf{1}) = 0$ . Michael Schürmann proved the following result [Sch85]

**Theorem 5.0.3.** *Let  $A$  be a real Banach algebra with unit  $\mathbf{1}$ . Denote by  $A'$  the topological dual of  $A$ . Let  $C$  be a closed convex cone with non-empty interior such that  $c \in C$  implies that  $c^n \in C$  for all  $n \geq 0$  (where  $c^0 = 1$ ). Then for an element  $a \in A$  the following statements are equivalent*

- i.  $a$  is conditionally positive on the dual cone  $C^\circ$ .*
- ii.  $\exp(ta) \in C$  for all  $t \geq 0$ .*

We could use apply this result to characterise the generator of semigroup of  $k$ -positive maps or  $k$ -super positive maps but the problem was that for  $k < n$  the identity map is not a  $k$  super positive map. So a semigroup of  $k$ -super positive maps can not be Id at time  $t = 0$ . It compels us to consider a non-unital semigroup of linear maps.

## 5.1 Non-unital Semigroups

We will be interested in semigroups of linear operators  $(T_t)_{t \geq 0}$  (on some Banach space  $V$ ), which do not start from the identity, i.e. we have  $T_s T_t = T_{s+t}$  for all  $s, t \geq 0$ , but not necessarily  $T_0 = \text{id}$ . We still want  $t \mapsto T_t$  to be continuous.

The semigroup property implies

$$T_0^2 = T_0$$



i.e.,  $T_0$  is idempotent. Then we can decompose  $V$  as  $V = \text{Im}(T_0) \oplus \text{Ker}(T_0)$ , where  $\text{Im}(T_0) = T_0(V) = \text{Ker}(\text{id}_V - T_0)$ , and  $\text{Ker}(T_0) = \text{Im}(\text{id}_V - T_0) = (\text{id}_V - T_0)(V)$ . We assume that  $T_0$  is bounded, so both subspaces are closed. With respect to this decomposition  $T_0$  has the form

$$T_0 = \begin{pmatrix} \text{id}_{\text{Im}(T_0)} & 0 \\ 0 & 0 \end{pmatrix}$$

Furthermore, the semigroup property implies  $T_0 T_t = T_t T_0 = T_t$  for all  $t \geq 0$ . Therefore

$$\text{Ker}(T_0) \subseteq \text{Ker}(T_t) \quad \text{and} \quad \text{Im}(T_t) \subseteq \text{Im}(T_0).$$

W.r.t. the decomposition  $V = \text{Im}(T_0) \oplus \text{Ker}(T_0)$  we can write the  $T_t$  as

$$T_t = \begin{pmatrix} \tilde{T}_t & 0 \\ 0 & 0 \end{pmatrix}$$

with some linear operators  $\tilde{T}_t \in B(\text{Im}(T_0))$ , which form a continuous semigroup  $(\tilde{T}_t)_{t \geq 0}$  with initial value  $\tilde{T}_0 = \text{Id}_{\text{Im}(T_0)}$ . This allows to extend classical result on unital semigroups to the non-unital case.

In our examples, if  $T_0$  is a conditional expectation onto some unital  $*$ -subalgebra, then we are lead to study semigroups  $(\tilde{T}_t)_{t \geq 0}$  that preserve the corresponding cones of  $T_0(M_n) \subseteq M_n$ .

But in general  $T_0$  need not be a conditional expectation, as the example in the following subsection shows.

### 5.1.1 Examples Of $k$ -Positive Semigroups

Since the identity map  $\text{id} : M_n \rightarrow M_n$  is not  $k$ -superpositive for  $k < n$ , there exist no  $k$ -superpositive semigroups  $(T_t : M_n \rightarrow M_n)_{t \geq 0}$  with  $T_0 = \text{id}$ . But there do exist semigroups of  $k$ -superpositive linear maps on  $M_n$  that start with an idempotent  $k$ -superpositive map  $T_0$ . Very simple examples are given by  $T_0(X) = PXP$  with  $P$  a  $k$ -dimensional orthogonal projection.

Another class of examples are semigroups  $T_t : M_2 \otimes M_n \rightarrow M_2 \otimes M_n$  of the form

$$T_t \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \begin{pmatrix} S_t(A) & 0 \\ 0 & \alpha(S_t(A)) \end{pmatrix}$$

with  $(S_t)_{t \geq 0}$  a  $k$ -superpositive semigroup acting on  $M_n$  and  $\alpha : M_n \rightarrow M_n$  any  $k$ -superpositive linear map. If  $\alpha$  is not a  $*$ -homomorphism, then  $T_0$  is a  $k$ -superpositive idempotent whose image is not a  $*$ -subalgebra, and so  $T_0$  is not a conditional expectation.

## 5.2 A Schoenberg Type Correspondence For General Non-unital Semigroups

We give a non-unital version of Schürmann's result [Sch85, Lemma 2.1].

For  $X$  a Banach space, we will denote by  $B_{X'}$  and  $S_{X'}$  the unit ball and the unit sphere of the dual space  $X'$ . If  $C \subseteq X$  is a cone in  $X$  then  $C^\circ$  denotes the dual cone in  $X'$ , i.e.

$$C^\circ = \{\varphi \in X'; \forall v \in C, \varphi(v) \geq 0\}.$$

Note that we have (for  $C$  a closed convex cone)

$$(C^\circ)^\circ = C.$$

**Theorem 5.2.1.** *Let  $A$  be a real Banach algebra with a closed convex cone  $C \subseteq A$  with non-empty interior. Let  $a_0 \in C$  be an idempotent such that for any  $c \in C$ , we have  $a_0ca_0 \in C$ .*

*We assume furthermore that for any  $c \in C$  we have  $c^n \in C$  for  $n \geq 1$ .*

*Then, for any  $b \in A$  such that  $ba_0 = a_0b = b$ , the following statements are equivalent.*

- (i)  *$b$  is  $a_0$ -conditionally positive on  $C^\circ$ , i.e.,  $\varphi(b) \geq 0$  for all  $\varphi \in C^\circ$  with  $\varphi(a_0) = 0$ .*
- (ii)  *$\exp_{a_0}(tb) := \lim_{n \rightarrow \infty} (a_0 + \frac{tb}{n})^n \in C$  for all  $t \geq 0$ .*

*Proof.* (ii) $\Rightarrow$ (i): This follows by differentiation at  $t = 0$ . If  $\varphi \in C^\circ$  is such that  $\varphi(a_0) = 0$ , then

$$\varphi(b) = \lim_{\substack{t \rightarrow 0 \\ t > 0}} \varphi \left( \frac{\exp_{a_0}(tb) - \exp_{a_0}(0)}{t} \right) = \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{1}{t} \varphi(\exp_{a_0}(tb)) \geq 0,$$

(i) $\Rightarrow$ (ii): We have

$$\exp_{a_0}(tb) := \lim_{n \rightarrow \infty} \left( a_0 + \frac{tb}{n} \right)^n = a_0 + \sum_{n=1}^{\infty} \frac{(tb)^n}{n!} \quad (5.2.1)$$

and we want to show that this quantity is positive for  $t \geq 0$  if  $b$  satisfies condition (i).

Without loss of generality we can take  $t = 1$ .

We will prove in four steps that  $\exp_{a_0}(b)$  is positive.

**Step I:** For any interior point  $c \in C$  there exists  $\delta > 0$  such that

$$\forall \varphi \in C^\circ, \quad \varphi(c) \geq \delta \|\varphi\|.$$

Indeed, let  $c \in C$  be an interior point. Then there exists  $\delta > 0$  such that  $c + \delta B_A \subseteq C$ , where  $B_A$  is the unit ball in  $A$ . Therefore for  $v \in B_A$ ,

$$\varphi(c \pm \delta v) \geq 0,$$

and

$$\varphi(c) \geq \delta \sup_{v \in B_A} |\varphi(v)| = \delta \|\varphi\|.$$

**Step II:** for any  $\rho > 0$  there exists  $\eta > 0$  such that

$$\forall \varphi \in C^\circ \cap B_{A'}, \quad \varphi(a_0) < \eta \Rightarrow \varphi(b) > -\rho.$$

Indeed, fix  $\rho > 0$  and set

$$V_n(\rho) = \left\{ \varphi \in C^\circ \cap B_{A'}; \varphi(a_0) \leq \frac{1}{n} \text{ and } \varphi(b) \leq -\rho \right\}$$

By the  $a_0$ -conditional positivity of  $b$  w.r.t.  $C^\circ$ , we have

$$\bigcap_{n \geq 1} V_n(\rho) = \emptyset.$$

Since  $C^\circ \cap B_{A'}$  is compact by the Banach-Alaoglu theorem in the weak-\* topology, and since the  $V_n(\rho) \subseteq C^\circ \cap B_{A'}$  are weak-\* closed, there exists  $n_0 \in \mathbb{N}$  such that  $V_{n_0}(\rho) = \emptyset$ . Take  $\eta = \frac{1}{n_0}$ .

**Step III:** For any  $\varepsilon > 0$  and  $c \in C$  an interior point, there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,

$$\forall \varphi \in C^\circ \cap S_{A'}, \quad \varphi\left(a_0 + \frac{b + \varepsilon c}{n}\right) \geq 0,$$

where  $S_{A'}$  denotes the unit sphere in  $A'$ . Let  $\delta > 0$  be the real number guaranteed by Step I such that  $\varphi(c) \geq \delta \|\varphi\|$  for all  $\varphi \in C^\circ$ . Let  $\eta > 0$  be the real number guaranteed by Step II such that for all  $\varphi \in C^\circ \cap B_{A'}$  with  $\varphi(a_0) < \eta$  we have  $\varphi(b) \geq -\varepsilon \delta$ .

Let  $\varphi \in C^\circ \cap S_{A'}$ . We distinguish two cases, according to the value of  $\varphi$  on  $a_0$ .

**Case  $\varphi(a_0) < \eta$ :** in this case we have

$$\varphi\left(a_0 + \frac{b + \varepsilon c}{n}\right) = \underbrace{\varphi(a_0)}_{\geq 0} + \frac{1}{n} \left( \underbrace{\varphi(b)}_{\geq -\varepsilon \delta} + \varepsilon \underbrace{\varphi(c)}_{\geq \delta} \right) \geq 0.$$

**Case  $\varphi(a_0) \geq \eta$ :** now we get

$$\varphi\left(a_0 + \frac{b + \varepsilon c}{n}\right) = \underbrace{\varphi(a_0)}_{\geq \eta} + \frac{1}{n} \varphi(b + \varepsilon c) \geq \eta - \frac{\|b + \varepsilon c\|}{n},$$

which is positive as soon as  $n \geq \frac{\|b + \varepsilon c\|}{\eta}$ .

**Step IV:** By the Bipolar theorem, this means that for any  $\varepsilon > 0$  and  $c \in C$  an interior point there exists an  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $a_0 + \frac{b+\varepsilon c}{n} \in C$ . Then we have  $a_0 \left( a_0 + \frac{b+\varepsilon c}{n} \right) a_0 = a_0 + \frac{b+\varepsilon a_0 c a_0}{n} \in C$ .

Since  $C$  is stable under taking powers and closed, we get

$$\exp_{a_0}(b + \varepsilon a_0 c a_0) = \lim_{n \rightarrow \infty} \left( a_0 + \frac{b + \varepsilon a_0 c a_0}{n} \right)^n \in C.$$

To conclude the proof we let  $\varepsilon \searrow 0$ . □

### 5.3 Application To The Semigroups Of $k$ - (Super)Positive Maps

Now we apply Theorem 5.2.1 to the algebra  $\text{Lin}(M_n(\mathbb{C}))^{\text{sa}}$  of hermitianity preserving linear maps from  $M_n$  to  $M_n$ .

**Theorem 5.3.1.** *Let*

$$A = \text{Lin}(M_n(\mathbb{C}))^{\text{sa}} = \{T \in \text{Lin}(M_n(\mathbb{C})); T \circ * = * \circ T\}$$

and let  $C \subseteq A$  be one of the cones  $\mathcal{P}_1 = \mathcal{PM}, \mathcal{P}_2, \dots, \mathcal{P}_{n-1}, \mathcal{P}_n = \mathcal{CP} = \mathcal{S}_n, \mathcal{S}_{n-1}, \dots, \mathcal{S}_2, \mathcal{S}_1 = \mathcal{EB}, \mathcal{EB}_{n-1}, \dots, \mathcal{EB}_1$ .

Fix an idempotent map  $\Phi_0 \in C$ . Then for  $\Psi \in A$  with  $\Psi \circ \Phi_0 = \Phi_0 \circ \Psi = \Psi$  the following are equivalent.

(i) We have  $\exp_{\Phi_0}(t\Psi) = \Phi_0 + \sum_{n=1}^{\infty} \frac{t^n \Psi^{\circ n}}{n!} \in C$  for all  $t \geq 0$ ;

(ii)  $\Psi$  is  $\Phi_0$ -conditionally positive on  $C^\circ$ , i.e., we have

$$\forall v \in C^\circ, \quad \langle v, \Phi_0 \rangle = 0 \quad \Rightarrow \quad \langle v, \Psi \rangle \geq 0.$$

*Proof.* The hermitianity preserving maps on  $M_n$  form a real Banach algebra, when we equip it with the norm induced by the operator norm on  $M_n \cong \text{Lin}(\mathbb{C}^n, \mathbb{C}^n)$ .

Proposition 2.3.4 ensures that all the cones are convex, solid, pointed and stable under composition. Furthermore, they are closed under composition, so we have  $S^{\circ n} \in C$  and  $T_0 \circ S \circ T_0 \in C$  for any  $S, T_0 \in C$  and  $n \geq 1$ .

Therefore, we can apply Theorem 5.2.1 to any pair  $(T_0, S)$ , with  $T_0$  an idempotent in  $C$  and  $S \in A$  such that  $S \circ T_0 = T_0 \circ S = S$ , and the result follows. □

In particular, if we take the cone of  $k$ -positive maps we obtain the the following result-

**Corollary 5.3.1.** *Take  $C = \mathcal{P}_k$  with  $k \in \{1, \dots, n\}$  and  $T_0 = \text{Id} = T_{I \otimes I}$ .*

*A semigroup  $T_t = \exp(tS)$  with generator  $S = T_W$  where  $W = \sum A_i \otimes B_i \in (M_n \otimes M_n^{op})^{\text{sa}}$ , consists of  $k$ -positive maps for all  $t \geq 0$  if and only if*

$$\forall V \in M_n, \quad (\text{rank}(V) \leq k \text{ and } \text{Tr}(V) = 0) \quad \Rightarrow \quad \sum \text{Tr}(A_i V^*) \text{Tr}(B_i V) \geq 0.$$

*Proof.* Here we consider unital semigroups, i.e.,  $T_0 = \text{Id}$ , which belongs to  $\mathcal{P}_k$  for all  $k \geq 1$ . Furthermore, the condition  $S \circ T_0 = T_0 \circ S = S$  now holds for any  $S \in \text{Lin}(M_n, M_n)^{\text{her}}$ .

We know that

$$C^\circ = \mathcal{P}_k^\circ = \mathcal{S}_k = \text{convex hull of } \{T_{V \otimes V^*}; V \in M_n, \text{rank}(V) \leq k\},$$

cf. [SSZ09].

Note that

$$\langle T_{V \otimes V^*}, \text{Id} \rangle = \text{Tr}(V^* \otimes V) = |\text{Tr}(V)|^2,$$

since, by Proposition 4.1.1,  $T$  is an isomorphism of Hilbert spaces.

By Theorem 5.3.1,  $\exp(tS) \in \mathcal{P}_k$  for all  $t \geq 0$ , iff  $S$  is  $\text{Id}$ -conditionally positive on  $\mathcal{S}_k = \mathcal{P}_k^\circ$ , i.e., if

$$\forall \varphi \in \mathcal{S}_k, \quad \langle \varphi, \text{Id} \rangle = 0 \quad \Rightarrow \quad \langle \varphi, S \rangle \geq 0.$$

Let us check that it is sufficient to verify this for  $\varphi \in \{T_{V \otimes V^*}; V \in M_n, \text{rank}(V) \leq k\}$ . Indeed, this set generates  $\mathcal{S}_k$ . And, since for a convex combination  $\varphi = \sum \lambda_i T_{V_i \otimes V_i^*}$  with  $\lambda_i > 0$ ,  $\sum \lambda_i = 1$ , we have

$$\left\langle \sum \lambda_i T_{V_i \otimes V_i^*}, \text{Id} \right\rangle = \sum \lambda_i \text{Tr}(V_i^* \otimes V_i) = \sum \lambda_i |\text{Tr}(V_i)|^2,$$

we see that the condition  $\langle \varphi, \text{Id} \rangle = 0$  is satisfied for a convex combination iff it is satisfied for each term.

If  $S = T_W$  with  $W = \sum A_i \otimes B_i \in (M_n \otimes M_n^{op})^{\text{sa}}$ , then

$$\langle T_{V \otimes V^*}, T_W \rangle = \langle V \otimes V^*, W \rangle = \text{Tr}(V^* A_i \otimes V B_i) = \text{Tr}(A_i V^*) \text{Tr}(B_i V),$$

which completes the proof. □

Let  $\{B_j\}_{j=1, \dots, n^2}$  be an orthonormal basis of  $M_n$  such that  $B_1 = I_n$  and  $\text{Tr}(B_j) = 0$  for  $2 \leq j \leq n^2$ . Taking  $\{B_i \otimes B_j\}_{p,q=1}^{n^2}$  as a basis of  $M_n \otimes M_n^{op}$ , or equivalently  $\{B_i \otimes B_j^*\}_{i,j=1}^{n^2}$  of  $M_n \otimes M_n^*$  further we write  $S = \sum_{i,j=1}^{n^2} D_{ij} B_i \otimes B_j^*$ , where  $D_{ij}$  is a  $n^2 \times n^2$  matrix in  $\mathbb{C}$ , so that the map  $T_S$  is given by

$$\Psi(X) = T_S(X) = \sum_{i,j=1}^{n^2} D_{ij} B_i X B_j^* \tag{5.3.1}$$

for all  $X \in M_n$ . The semigroup  $\exp(t\Psi)$  is Hermiticity preserving if and only if the generator  $T_S$  is hermiticity perserving if and only if the matrix  $(D_{ij})$  is hermitian.

**Proposition 5.3.1.** *A semigroup  $T_t = \exp t\Psi$  is completely positive if and only if the matrix  $(D_{ij})_{i,j=1}^{n^2}$  is hermitian and for any  $v = (0, v_2, v_3, \dots, v_{n^2}) \in \mathbb{C}^{n^2}$  we have*

$$\langle v | (D_{ij})_{i,j=1}^{n^2} v \rangle \geq 0,$$

*Proof.* Let  $V$  be a  $n \times n$  matrix with the basis decomposition  $V = \sum_{j=1}^{n^2} v_j B_j$ . The trace conditions on  $B_j$ 's implies that  $\text{Tr}(V) = 0$  if and only if  $v_1 = 0$ . By a direct application of Corollary 5.3.1 we see that the map  $T_S$  with  $S = \sum_{i,j=1}^{n^2} B_i \otimes B_j^*$  generates a completely positive semigroup (i.e.  $n$  positive) if and only if for any  $V \in M_n$  with  $\text{Tr}(V) = 0$ , we have  $\sum_{i,j=1}^{n^2} D_{ij} \text{Tr}(B_i V^*) \text{Tr}(B_j^* V) \geq 0$ . Writing the basis decomposition of  $V$  in this condition, we obtain

$$(\forall (0, v_2, v_3, \dots, v_{n^2}) \in \mathbb{C}^{n^2}) \implies \sum_{i,j=2}^{n^2} \sum_{k,l=2}^{n^2} D_{ij} \bar{v}_k v_l \text{Tr}(B_i B_k^*) \text{Tr}(B_j^* B_l) \geq 0.$$

Because of the orthonormality of the basis  $B_i$ 's,  $\text{Tr}(B_i B_k^*) = \delta_{ik}$  and  $\text{Tr}(B_j^* B_l) = \delta_{jl}$ . Thus the above condition becomes

$$\sum_{i,j=2}^{n^2} \sum_{k,l=2}^{n^2} D_{ij} \bar{v}_k v_l \delta_{ik} \delta_{jl} = \sum_{i,j=2}^{n^2} D_{ij} \bar{v}_i v_j \geq 0,$$

which is the desired result. □

From this result we can re-derive the Lindblad[Lin76], Gorini, Kossakowski, Sudarshan's theorem [GKS76] on generator of CP semigroup.

**Theorem 5.3.2.** *An identity preserving semigroup  $\exp(t\Psi)$  will be completely positive for all time  $t \geq 0$  if and only if the generator  $\Psi$  has the form*

$$\Psi(X) = i[H, X] + \sum_{j=1}^k \{V_j^* X V_j - \frac{1}{2}(V_j^* V_j X + X V_j^* V_j)\}$$

for all  $X \in M_n$ , where  $H$  is an hermitian matrix and  $V \in M_n$ .

*Proof.* We use the basis decomposition 5.3.1 of the generator  $\Psi$

$$\Psi(X) = D_{11}X + \sum_{i=2}^{n^2} D_{i1} B_i X + \sum_{j=2}^{n^2} D_{1j} X B_j^* + \sum_{i,j=2}^{n^2} D_{ij} B_i X B_j^*$$

As  $(D_{ij})$  is hermitian, if we denote  $W := \sum_{i=2}^{n^2} D_{i1} B_i$  and  $\kappa := D_{11}$  then  $W^* = \sum_{j=2}^{n^2} D_{1j} B_j^*$  and  $\kappa$  is a real number. From the previous proposition 5.3.1 we know that  $(D_{ij})_{i,j=2}^{n^2}$  is

a positive matrix. Therefore there exists  $A_1, A_2, \dots, A_k \in M_{n^2-1}$  such that  $(D_{ij})_{i,j=2}^{n^2} = A_1 A_1^* + A_2 A_2^* + \dots + A_k A_k^*$ . If we write  $A_r = (a_r(p, q))$  then in terms of the coefficients of the matrix  $A_r$  we have

$$D_{ij} = \sum_{r=1}^k \sum_{p=2}^{n^2} a_r(i, p) \overline{a_r(j, p)}.$$

Substituting these in the above expression of  $\Psi$  and regrouping the terms we obtain

$$\begin{aligned} \Psi(X) &= \kappa X + WX + XW^* + \sum_{r=1}^k \sum_{p=2}^{n^2} \left( \sum_{i=2}^{n^2} a_r(i, p) B_i \right) X \left( \sum_{j=2}^{n^2} a_r(j, p) B_j \right)^* \\ &= \kappa X + WX + XW^* + \sum_{p=2}^{n^2} \sum_{r=1}^k V_{p,r} X V_{p,r}^* \end{aligned}$$

where  $V_{p,r} := \sum_{i=2}^{n^2} a_r(i, p) B_i$ . That the semigroup preserves the identity is equivalent to the generator mapping it to zero i.e.  $\Psi(I_n) = 0$ . Plugging this condition we get

$$0 = \kappa + W + W^* + \sum_p \sum_r V_{p,r} V_{p,r}^*.$$

So we can set  $W = iH - \frac{1}{2}\kappa - \frac{1}{2} \sum_p \sum_r V_{p,r} V_{p,r}^*$ , where  $H$  is a Hermitian matrix and substituting it in the above expression of  $\Psi(X)$ , we have

$$\begin{aligned} \Psi(X) &= \kappa X + \left( iH - \frac{1}{2}\kappa - \frac{1}{2} \sum_p \sum_r V_{p,r} V_{p,r}^* \right) X + X \left( -iH - \frac{1}{2}\kappa - \frac{1}{2} \sum_p \sum_r V_{p,r}^* V_{p,r} \right) \\ &\quad + \sum_p \sum_r V_{p,r} X V_{p,r}^* \\ &= i[H, X] + \sum_p \sum_r \left\{ V_{p,r} X V_{p,r}^* - \frac{1}{2} \left( V_{p,r} V_{p,r}^* X + X V_{p,r}^* V_{p,r} \right) \right\} \end{aligned}$$

□

## 5.4 Positive Semigroups On $M_2(\mathbb{C})$

In [Car04] positive semigroup has been characterized in terms of the generator of the semigroup. Here we give another characterization of the same, following the discussion above. The Pauli matrices are unitary matrices

$$\sigma_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which form an orthogonal basis of  $M_2$  with respect to Hilbert-Schmidt inner product. Moreover, they satisfy the following relations

$$\sigma_p^2 = \sigma_0^2, \quad \sigma_p \sigma_q = -\sigma_q \sigma_p, \quad \sigma_p \sigma_q = i \sigma_r \sigma_0$$

if  $(p, q, r) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ . Decomposing a matrix  $V$  with respect to these basis i.e  $V = \sum_{p=0}^3 v_p \sigma_p$ , we observe that  $V$  has the form

$$\frac{1}{2} \begin{bmatrix} v_0 + v_3 & v_1 - i v_2 \\ v_1 + i v_2 & v_0 - v_3 \end{bmatrix}.$$

It is easy to see that

- (i)  $V$  has trace zero if and only if  $v_0 = 0$ ,
- (ii) it has rank one if and only if  $\det(V) = 0$  i.e.  $v_0^2 = v_1^2 + v_2^2 + v_3^2$ .

For a semigroup  $\exp(t\Phi)$  on  $M_2$  we can write the generator again in terms of Pauli basis as in 5.3.1-

$$\Phi = T_S, \quad \text{where } S = \sum_{p,q=0}^3 D_{p,q} \sigma_p \otimes \sigma_q.$$

**Proposition 5.4.1.** *A semigroup  $\exp(t\Phi)$  is positive if and only if the matrix  $(D_{pq})_{p,q=0}^3$  is hermitian and for all  $v = (v_1, v_2, v_3) \in \mathbb{C}^3$  with  $v_1^2 + v_2^2 + v_3^2 = 0$  we have*

$$\langle v | (D_{pq})_{p,q=1}^3 v \rangle \geq 0 \tag{5.4.1}$$

where  $(D_{pq})_{p,q=1}^3$  is the  $3 \times 3$  submatrix of the matrix  $(D_{ij})_{i,j=0}^3$ , defined above. Moreover, it is identity preserving if and only if the following relations are satisfied,

$$\sum_{p=0}^3 D_{pp} = 0, \quad \text{and } (D_{p0} + D_{0p}) + i(D_{qr} - D_{rq}) = 0$$

for  $(p, q, r) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ .

*Proof.* As a direct application of corollary 5.3.1 it follows that the semigroup  $\exp(t\Phi)$  is positive if and only if for any  $V \in M_2$  with  $\text{rank}(V) = 1$  and  $\text{Tr}(V) = 0$

$$\sum_{p,q=0}^3 D_{pq} \text{Tr}(\sigma_p V^*) \text{Tr}(\sigma_q V) \geq 0 \tag{5.4.2}$$

As we observed, expanding  $V$  in the Pauli basis  $V = \sum_{j=0}^3 v_j \sigma_j$  the rank and trace conditions translate into  $v_0 = 0$  and  $\sum_{j=1}^3 v_j^2 = 0$ . Plugging this decomposition of  $V$  in



the expression 5.4.2 it becomes

$$\begin{aligned} \sum_{p,q=0}^3 \sum_{j,k=0}^3 D_{pq} \bar{v}_j v_k \text{Tr}(\sigma_p \sigma_j) \text{Tr}(\sigma_q \sigma_k) &= \sum_{p,q=0}^3 \sum_{j,k=0}^3 \bar{v}_j v_k \delta_{pj} \delta_{qk} \\ &= \frac{1}{4} \sum_{p,q=1}^3 D_{pq} \bar{v}_p v_q \geq 0, \end{aligned}$$

which gives the desired inequality.

To obtain the conditions of hermitianity and identity preserving property, it is easy to see that

- (i)  $\exp(t\Phi)$  is hermitianity preserving if and only if the generator  $\Phi = T_S$  has the same property, which is equivalent to the matrix  $(D_{pq})_{p,q=0}^3$  being hermitian.
- (ii) The semigroup preserves identity if and only if the generator  $\Phi$  takes identity to zero i.e.  $T_S(\sigma_0) = 0$ .

□

## 5.5 A General 4-Parameter Family Of Semigroups

Now we discuss as an example the semigroup generated by 4 parameter family  $L = \alpha P + \beta D + \gamma T + \delta \text{Id}$  and time evolution for specific cases in  $M_2$ . As all the operators  $P, D, T, \text{Id}$  commute, the semigroup generated is given by

$$\begin{aligned} \exp(tL) &= e^{t(\beta+\gamma+\delta)}(e^{t\alpha} - 1)P + e^{t(\gamma+\delta)}(e^{t\beta} - 1)D + e^{t\delta} \frac{(e^{t\gamma} - e^{-t\gamma})}{2} T \\ &\quad + e^{t\delta} \frac{(e^{t\gamma} + e^{-t\gamma})}{2} \text{Id} \end{aligned}$$

### 5.5.1 The Depolarising Channel

We consider the semigroup  $T_{\alpha,\delta} = \exp t(\alpha P + \delta \text{Id})$ . This semigroup is again a linear combination of the operator  $P$  and identity,

$$T_{\alpha,\delta}(t) = \exp(t(\alpha P + \delta \text{Id})) = \underbrace{e^{\delta t}(e^{\alpha t} - 1)}_{=:\alpha(t)} P + \underbrace{e^{\delta t}}_{=:\delta(t)} \text{Id},$$

which, by Lemma 2.5.1, is  $k$ -positive iff

$$\alpha(t) \geq 0 \quad \text{and} \quad \delta(t) \geq -\frac{\alpha(t)}{kn}$$

This condition is satisfied for all  $t \geq 0$  iff  $e^{\alpha t} \geq 1$  for all  $t \geq 0$  iff  $\alpha \geq 0$ .

If we consider the identity preserving semigroup generated by the operator  $P$  and the identity operator i.e. the semigroup  $T_{\alpha,-\alpha}$ . we can see that it is a convex combination of  $P$  and the identity operator

$$T_{\alpha,-\alpha} = \exp t\alpha(P - \text{Id}) = (1 - e^{-t\alpha})P + e^{-t\alpha}\text{Id}.$$

which is completely positive for all  $t \geq 0$  iff  $\alpha \geq 0$ . What is more, in this case the semigroup converges to  $P$  as  $t \rightarrow \infty$ , which we know by Lemma 2.3.3 to be an interior point of the cone  $\mathcal{EB}$  or 1-superpositive. So if  $\alpha \geq 0$ , the semigroup  $T_{\alpha,-\alpha}$  enters the cone  $\mathcal{EB}$  in finite time  $t_1 > 0$ . The Choi matrix of the semigroup is given by

$$C_{T_{\alpha,-\alpha}} = \begin{bmatrix} \frac{1}{2}(1 + e^{-t\alpha}) & 0 & 0 & e^{-t\alpha} \\ 0 & \frac{1}{2}(1 - e^{-t\alpha}) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1 - e^{-t\alpha}) & 0 \\ e^{-t\alpha} & 0 & 0 & \frac{1}{2}(1 + e^{-t\alpha}) \end{bmatrix}.$$

To decide when the semigroup enters  $\mathcal{EB}$  we use Peres-Horodeci (or PPT) criterion. If  $T$  is the transpose map then

$$(\text{Id} \otimes T)C_{\exp(tL)} = \begin{bmatrix} \frac{1}{2}(1 + e^{-t\alpha}) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1 - e^{-t\alpha}) & e^{-t\alpha} & 0 \\ 0 & e^{-t\alpha} & \frac{1}{2}(1 - e^{-t\alpha}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1 + e^{-t\alpha}) \end{bmatrix}.$$

So  $T_{\alpha,-\alpha}$  is in  $\mathcal{EB}$  if and only if  $C_{T_{\alpha,-\alpha}}$  is separable if and only if  $(\text{Id} \otimes T)C_{T_{\alpha,-\alpha}}$  is positive (by PPT criterion, see [Per96],[HHH96]). We can easily check that the above matrix is positive iff the determinant

$$\begin{vmatrix} \frac{1}{2}(1 - e^{-t\alpha}) & e^{-t\alpha} \\ e^{-t\alpha} & \frac{1}{2}(1 - e^{-t\alpha}) \end{vmatrix} = -\frac{3}{4}e^{-2t\alpha} - 2e^{-t\alpha} + 1.$$

is positive. Replacing by  $x = e^{-t\alpha}$  in the above expression, we observe that the polynomial  $-\frac{3}{4}x^2 - 2x + 1$  has the positive root  $x_0 := -\frac{4}{3} + 2\frac{\sqrt{7}}{3}$  in the interval  $[0, 1]$ . The polynomial is positive on the interval  $[0, x_0]$  and negative on  $[x_0, 1]$ . Thus the semigroup enters the cone  $\mathcal{S}_1$  at the time

$$t_1 = -\frac{1}{\alpha} \ln \left( -\frac{4}{3} + \frac{2\sqrt{7}}{3} \right).$$

### 5.5.2 Transposition

We consider the identity preserving semigroup generated by the transposition and the depolarizing channel i.e. by the generator  $L = \alpha P + \gamma T - (\alpha + \gamma)\text{Id}$ , for  $\alpha, \gamma \in \mathbb{R}$ . Since all the quantities  $T, P, \text{Id}$  commutes we have

$$\begin{aligned} \exp(tL) &= \exp(t\gamma(T - \text{Id})) \exp(t\alpha(P - \text{Id})) \\ &= e^{-t\alpha} \frac{1 + e^{-2t\gamma}}{2} \text{Id} + e^{-t\alpha} \frac{1 - e^{-2t\gamma}}{2} T + (1 - e^{-t\alpha})P, \end{aligned}$$

which is a convex combination of the  $\text{Id}, T$  and  $P$ .

If we take  $\alpha, \gamma > 0$ , then in general it is a positive semigroup but not necessarily completely positive.

If the parameter  $\alpha = 0$  then the semigroup is just convex combination of  $\text{Id}$  and  $T$ , which converges to  $\frac{1}{2}(\text{Id} + T)$  as  $t \rightarrow \infty$ .

If  $\alpha > 0$ , then the semigroup eventually becomes completely positive, even 1-superpositive as it converges to  $P$ .

We compute the Choi matrix of of the semigroup  $\exp(tL)$ ,

$$C_{\exp(tL)} = \rho_t C_{\text{Id}} + \mu_t C_T + \nu_t C_P, \quad (5.5.1)$$

where  $\rho_t = e^{-t\alpha} \frac{1 + e^{-2t\gamma}}{2}$ ,  $\mu_t = e^{-t\alpha} \frac{1 - e^{-2t\gamma}}{2}$  and  $\nu_t = (1 - e^{-t\alpha})$ . Using the same arguments as in Proposition 2.5.1, the Choi matrix is positive iff the eigenvalues  $\frac{\nu_t}{n} - \mu_t$  and  $\frac{\nu_t}{n} + \mu_t$  are positive. Combining these two conditions, we conclude that the semigroup  $\exp(tL)$  becomes CP at time  $t$  iff

$$\frac{\nu_t}{n} \geq |\mu_t| \quad \text{i.e.} \quad 2(e^{\alpha t} - 1) \geq n|1 - e^{-2t\gamma}|.$$

The above inequality shows that even if  $\gamma \gg \alpha > 0$  it is possible that the semigroup not CP for certain time but ultimately becomes CP and then superpositive after finite time. If  $\gamma = \alpha > 0$  then substituting  $x = e^{\alpha t}$  in the above inequality gives the following criterion,

$$2x^3 - (2 + n)x^2 + n \geq 0.$$

The polynomial  $p(x) = 2x^3 - (2 + n)x^2 + n$  has two positive roots- 1 and  $\frac{n + \sqrt{n^2 + 8n}}{4}$ , and  $p(x) \geq 0$  for  $x \geq \frac{n + \sqrt{n^2 + 8n}}{4}$ . The root 1 corresponds to the time  $t = 0$ . So the semigroup becomes CP at time  $t_1 = \frac{1}{\alpha} \ln \frac{n + \sqrt{n^2 + 8n}}{4}$ .

We can find the time when the semigroup becomes 1-superpositive using again PPT criterion for the case  $M_2(\mathbb{C})$ .

$$(\text{Id} \otimes T)C_{\exp tL} = \begin{bmatrix} \rho_t + \mu_t + \frac{1}{2}\nu_t & 0 & 0 & \mu_t \\ 0 & \frac{1}{2}\nu_t & \rho_t & 0 \\ 0 & \rho_t & \frac{1}{2}\nu_t & 0 \\ \mu_t & 0 & 0 & \rho_t + \mu_t + \frac{1}{2}\nu_t \end{bmatrix}.$$

The above matrix is positive iff the determinant  $\begin{vmatrix} \frac{1}{2}\nu_t & \rho_t \\ \rho_t & \frac{1}{2}\nu_t \end{vmatrix} = \frac{1}{4}\nu_t^2 - \rho_t^2$  is positive, which gives the condition

$$2 \leq e^{t\alpha} - e^{-2t\gamma}. \quad (5.5.2)$$

If  $\gamma = \alpha > 0$  then substituting  $x = e^{t\alpha}$  in the above inequality gives that

$$x^3 - 2x^2 - 1 \geq 0.$$

If  $\xi$  is the positive root of the polynomial then we see that at the time  $t_2 = \frac{1}{\nu} \ln \xi$  the semigroup becomes 1-superpositive.

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**Abstract:** In the first three chapters of this thesis we discuss the existing notions of different positivity of a linear map between matrix algebras e.g.  $k$ -positivity, complete positivity,  $k$ -superpositivity, separable and entangled quantum state, unitary error basis and their constructions etc. We discuss the relation between different positive cones, and their duals in detail. In the last two chapters, we present our results on the topic. We try to characterise different positive maps using a suitable basis of the space of linear maps between matrix algebras. The Weyl operators give a convenient basis of matrix algebra which is also orthonormal with respect to the Hilbert-Schmidt inner product. The properties of such a basis can be generalised to the notion of a nice error basis (NEB), as introduced by E. Knill. We can use an NEB of matrix algebra to construct an NEB for the space of linear maps between matrix algebras. Any linear map will then correspond to a coefficient matrix in the basis decomposition with respect to such basis. Positivity, complete (co)positivity or other properties of a linear map can be characterised in terms of such coefficient matrix. Finally, we will characterise the one parameter semigroup of different positive maps in terms of its generators. We prove a Schoenberg-type correspondence for non-unital semigroups which generalises an analogous result for unital semigroups proved by Michael Schürmann. It characterises the generators of semigroups of linear maps on matrix algebra which are  $k$ -positive,  $k$ -superpositive, or  $k$ -entanglement breaking. We present some concrete examples of semigroup of operators and study how their positivity properties improve with time.

**Keywords:** Positive operators, Quantum information, Quantum channel, Semigroup of positive operators, Nice error basis, Semigroup of  $k$ -positive maps

**Résumé:** Dans les trois premiers chapitres de cette thèse, nous discutons des notions existantes de différentes positivités d'une application linéaire entre des algèbres de matrices, par exemple la  $k$ -positivité, la positivité complète ou le canal quantique, la  $k$ -superpositivité, état quantique séparable et intriqué, la base d'erreur unitaire et leurs constructions etc. Nous discutons en détail de la relation entre les différents cônes positifs et leurs duals. Dans les deux derniers chapitres, nous présentons nos résultats sur le sujet. Nous essayons de caractériser différentes applications positives en utilisant une base appropriée de l'espace des applications linéaires entre les algèbres de matrices. Les opérateurs de Weyl donnent une base pratique de l'algèbre des matrices qui est également orthonormée par rapport au produit scalaire de Hilbert-Schmidt. Les propriétés d'une telle base peuvent être généralisées à la notion de "nice error basis" (NEB), telle qu'introduite par E. Knill. Nous pouvons utiliser une NEB de l'algèbre des matrices pour construire une NEB pour l'espace des applications linéaires entre algèbres de matrices. Toute application linéaire correspondra alors à une matrice de coefficient dans la décomposition de la base par rapport à cette base. La positivité, la (co)positivité complète ou d'autres propriétés d'une application linéaire peuvent être caractérisées en termes de cette matrice de coefficients. Enfin, nous caractériserons le semigroupe à un paramètre de différentes applications positives en fonction de ses générateurs. Nous prouvons une correspondance de type Schoenberg pour les semigroupes non-unitaires qui généralise un résultat analogue pour les semigroupes unitaires prouvé par Michael Schürmann. Elle caractérise les générateurs des semigroupes de applications linéaires sur l'algèbre des matrices qui sont  $k$ -positifs,  $k$ -superpositifs, ou  $k$ -rupture d'enchevêtrement. Nous présentons quelques exemples concrets de semigroupes d'opérateurs et étudions comment leurs propriétés de positivité peuvent s'améliorer avec le temps.

**Mots-clés :** Opérateur Positif, Canal quantique, Information quantique, Semigroupe d'opérateurs positifs, Base d'erreur, Semigroupe d'opérateurs  $k$ -positifs

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