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Wanqing Wang

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Thèse de doctorat



INSTITUT  
POLYTECHNIQUE  
DE PARIS



# Approximation et simulation des équations différentielles stochastiques rétrogrades réfléchies, applications en finance

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)  
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 10/11/2023, par

**WANQING WANG**

Composition du Jury :

Aurélien Alfonsi Professeur, Ecole Nationale des Ponts et Chaussées (CERMICS)	Président
Saïd Hamadène Professeur, Le Mans Université (Laboratoire Manceau de Mathématiques)	Rapporteur
Chao Zhou Associate Professeur, National University of Singapore (Risk Management Institute)	Rapporteur
Caroline Hillairet Professeure, École nationale de la statistique et de l'administration économique (CREST)	Examinatrice
Emmanuel Gobet Professeur, Ecole polytechnique (CMAP)	Directeur de thèse
Christophette Blanchet-Scaillet Maîtresse de Conférence, Ecole Centrale Lyon (ICJ)	Examinatrice
Gilles Pagès Professeur, Sorbonne Université (LPSM)	Examineur
Mingyu Xu Associate Professeur, Fudan University (School of mathematical sciences)	Co-directrice de thèse
Nicole El Karoui Professeure émérite, Sorbonne Université (LPSM)	Invitée

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**Abstract.** In this thesis, one concentrates on solving the reflected backward stochastic differential equations via penalization approach and its applications in finance. A complete non-asymptotic convergence result is investigated around penalized BSDE. The rate of penalized solution converging to reflected solution is presented in the first place and it follows by the rate of discrete penalized solution converging to the continuous penalized one. For solving numerically PBSDEs, we provide an implicit scheme using least-squares regression Monte-Carlo method. The non-asymptotic error analysis is deduced for this numerical scheme, in which both linear and non-linear least-squares regression are considered as optimization method. The thesis is completed by an application of RBSDEs on American Put/Call options in the non-linear market. The change of numéraire is investigated on reflected diffusions. We find, as in the perfect market, the pricing equivalence between an American Put option and an American Call option is achieved by exchanging the interest rates with dividend rates, and swapping the spot price with the strike price, in a imperfect market with 2 interest rates and 2 dividend rates.

**Résumé.** Dans cette thèse, on se concentre sur la résolution des équations différentielles stochastiques rétrogrades réfléchies par l'approche de pénalisation et ses applications en finance. Un résultat complet de convergence non asymptotique est étudié autour des EDSR pénalisées. La vitesse de convergence de la solution pénalisée vers la solution réfléchie est présenté en premier lieu, suivi de la vitesse de convergence de la solution pénalisée discrète vers la solution pénalisée continue. Pour la résolution numérique des EDSR pénalisées, on propose un schéma implicite utilisant la méthode de Monte-Carlo avec régression des moindres carrés. Une analyse d'erreur non asymptotique est déduite pour ce schéma numérique, dans laquelle à la fois la régression linéaire et non linéaire des moindres carrés sont considérées comme méthodes d'optimisation. La thèse se conclut par une application des EDSR pénalisées aux options d'achat/vente américaines sur un marché non linéaire. Le changement de numéraire est étudié sur les diffusions réfléchies. On constate, comme sur le marché parfait, que l'équivalence de prix entre une option de vente américaine et une option d'achat américaine est obtenue en échangeant les taux d'intérêt avec les taux de dividende, et en échangeant le prix spot avec le prix d'exercice, sur un marché imparfait avec 2 taux d'intérêt et 2 taux de dividende.

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# Chapter 1

## Introduction

In this thesis, we focus on the penalization approach to approximate the solution of Reflected Backward Stochastic Differential Equations (RBSDEs in short). These equations typically arise in the context of obstacle control problem such as optimal stopping. We study the convergence of penalized BSDEs (PBSDEs in short) in the sense of continuous and discrete solutions, and of numerical approximation.

### 1.1 Reflected BSDE

#### 1.1.1 Review

Backward Stochastic Differential Equation (BSDE for short) is a type of stochastic differential equation where the evolution of a process is described backward in time. In BSDE, the solution is determined by specifying a terminal condition and a generator function that relates the solution to the driving processes. BSDEs have applications in various fields, including mathematical finance, optimal control, and stochastic control problems. One key feature of BSDEs which makes this type equation particularly suitable for problems where the objective is to evaluate contingent claims or derivative securities at a given terminal time, is that the backward time evolution allows for incorporating future information into the analysis. We start by recalling that a BSDE is an equation taking the following form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s - \int_t^T Z_s \mathbf{d}B_s, \quad \forall 0 \leq t \leq T, \quad (1.1.1)$$

where  $\{B_t : 0 \leq t \leq T\}$  is a  $d$ -dimensional Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . This probability space is endowed with the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  generated by  $B$  where  $\mathcal{F}_0$  contains all the  $\mathbb{P}$ -null sets. The data of such equation are given by the terminal condition  $\xi$  which is  $\mathcal{F}_T$ -measurable and a generator  $f$  which can depend on  $\omega$ . We say  $(Y, Z)$  is the solution to BSDE with respect to  $(\xi, f)$  if  $(Y, Z)$  satisfies (1.1.1). This kind of equation

with linear generator  $f$  was first introduced by Bismut ([13]) as the self-adjoint equation of the stochastic control problem. Then Pardoux and Peng ([114]) presented the well-posedness result of BSDEs with Lipschitz generator. Let us recall this result.

**Theorem 1.1.1** ([114]). *Suppose  $\xi$  is square integrable and  $\mathcal{F}_T$ -measurable. Suppose the generator  $f$  is  $C_{Lip}^f$ -Lipschitz continuous in  $(y, z)$ , namely,  $\mathbf{d}t \otimes \mathbf{d}\mathbb{P}$ , a.e.*

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq C_{Lip}^f(|y - y'| + |z - z'|), \forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d.$$

If  $\mathbb{E} \left[ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 \mathbf{d}s \right] < \infty$ , then there exists a unique solution to BSDE (1.1.1) in  $\mathbb{H}^2 \times \mathbb{H}^2$ , with

$$\mathbb{H}^2 = \left\{ \{\phi_t : 0 \leq t \leq T\} \text{ is a predictable process such that } \mathbb{E} \left[ \int_0^T |\phi_s|^2 \mathbf{d}s \right] < \infty \right\}$$

and  $|\cdot|$  denoting the Euclidean norm with respect to the dimension of process.

Since then, in this developing domain, numerous works have emerged, among which, El Karoui, Peng and Quenez ([54]) linked this equation to the pricing problem of European options in both complete and incomplete markets, with identifying the value process as  $Y$  and the hedging strategy as  $Z$ .

Reflected backward stochastic differential equations (RBSDEs in short) were introduced right after by El Karoui, Kapoudjian, Pardoux, Peng and Quenez ([52]). The word 'reflected' means that the value process cannot go below a certain lower boundary or exceed a certain upper boundary. Therefore, RBSDEs are frequently used to model situations where there are constraints or barriers on the evolution of a stochastic process, such as in finance applications where prices or values are lower-bounded. As different as non-reflected equations, RBSDEs admit an obstacle process  $S$  in inputs and we require that the value process  $Y$  is always superior or equal than obstacle  $S$ . A standard RBSDE writes as follows:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, & \forall 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) \mathbf{d}K_t = 0, \end{cases} \quad (1.1.2)$$

where  $K$  consisting part of solution of RBSDE  $(\xi, f, S)$ , is a non-decreasing process in order to push  $Y$  above  $S$  on  $[0, T]$ . The third equation in (1.1.2) is so-called "Skorohod condition" which demands that the increment of  $K$  can only happens on  $\{(t, \omega) : Y_t = S_t\}$ . We say that  $(Y, Z, K)$  is the solution of RBSDE  $(\xi, f, S)$  if  $(Y, Z, K)$  satisfies (1.1.2).

Except the standard RBSDE as (1.1.2), there are also many works inspired by the pioneering work [52]. The doubly reflected BSDEs are studied by Cvitanic and Karatzas ([40]) and by Hamadène and Lepeltier ([81]), see also [48, 79, 97] for related topics; the obliquely reflected equations are studied by Hamadène and Jeanblanc ([80]) for 2 processes and then by Hu and Tang ([87]) for multiple processes, see also [35, 125]; equations with quadratic

growth in  $Z$  are studied by Briand and Hu [21, 22]; recently, mean reflected equations are investigated by Briand, Elie and Hu ([19]) and by Hamadène and his co-authors ([36, 46]). The well-posedness of BSDEs under some weak conditions is also a popular topic. For example, the  $(\mathbb{L}^p, p < 2)$ -existence is first studied by Briand et al. ([23]) for BSDEs with monotone generator and this is developed for RBSDEs with Lipschitz generator by Hamadène and Popier ([82]).

One recalls the following principal results in [52].

**Well-posedness** One first defines the square-integrable space  $\mathcal{S}^2$  for processes:

$$\mathcal{S}^2 = \{ \{ \phi_t : 0 \leq t \leq T \} \text{ is a predictable process such that } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty \}.$$

**Theorem 1.1.2** ([52, Theorem 5.2 and Proposition 3.5]). *Suppose  $S$  is a continuous stochastic process with  $S \vee 0 =: S^+ \in \mathcal{S}^2$ , the generator  $f$  is  $C_{Lip}^f$ -Lipschitz continuous and  $f(\cdot, 0, 0) \in \mathbb{H}^2$ , the terminal condition  $\xi$  is  $\mathcal{F}_T$ -measurable, square-integrable and  $\xi \geq S_T$ , then there exists a unique solution  $(Y, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$  satisfying that  $Y, K$  are continuous. Moreover we have the following a priori estimate:*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |Z_t|^2 dt \right) + K_T^2 \right] \leq C \mathbb{E} \left[ |\xi|^2 + \left( \int_0^T |f(t, 0, 0)| dt \right)^2 + \sup_{0 \leq t \leq T} (S_t^+)^2 \right]$$

where  $C$  is a constant depending only on  $T, C_{Lip}^f$ .

### Equivalence to optimal stopping problem

**Proposition 1.1.1** ([52][Proposition 2.3]). *Under the same assumptions as Theorem 1.1.2, let  $(Y, Z, K)$  be the solution of RBSDE w.r.t.  $(\xi, f, S)$ , then*

$$Y_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left[ \int_t^\tau f(u, Y_u, Z_u) du + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} \middle| \mathcal{F}_t \right],$$

where  $\mathcal{T}_{t,T}$  contains all the stopping times which are lower bounded by  $t$  and upper bounded by  $T$ .

**Density of increasing process  $K$ .** Proposition 1.1.2 reveals the absolute continuity of  $K$  with respect to Lebesgue measure. It is particularly useful when one needs to handle some convex Markovian obstacle processes  $S$ , where a local time term (which is non-decreasing) will be produced due to Itô-Tanaka formula. See Chapter 3.

**Proposition 1.1.2** ([52, Proposition 4.2 and Remark 4.3]). *Under the same assumptions as in Theorem 1.1.2 and assume  $S$  is a generalized semi-martingale of the form*

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + A_t$$

where  $U, V, A$  are, respectively,  $\mathbb{R}, \mathbb{R}^d, \mathbb{R}$ -valued and  $\mathcal{F}_t$ -progressively measurable processes satisfying

$$\int_0^T (|U_t| + |V_t|^2) dt < \infty, \quad a.s.,$$

and  $A$  is continuous and non-decreasing. Let  $(Y, Z, K)$  be a solution of the RBSDE  $(\xi, f, S)$ , then

- $Z_t = V_t, dt \otimes d\mathbb{P}$  a.e. on the set  $\{Y_t = S_t\}$ ;
- $0 \leq dK_t \leq \mathbf{1}_{\{Y_t = S_t\}} (f(s, S_t, V_t) + U_t)^- dt$ .

**Increasing limit of penalized BSDEs.** Another important result is the existence of solution of RBSDE proved via penalization. The penalized version of (1.1.2) writes as, for any penalty parameter  $\lambda > 0$ ,

$$Y_t^\lambda = \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) ds + \lambda \int_t^T (Y_s^\lambda - S_s)^- ds - \int_t^T Z_s^\lambda dB_s, \quad (1.1.3)$$

which is a BSDE with Lipschitz constant depending on  $\lambda$ . We say  $(Y^\lambda, Z^\lambda)$  is the solution of penalized BSDE with respect to  $(\xi, f, S)$  and  $\lambda$ , if  $(Y^\lambda, Z^\lambda)$  satisfies (1.1.3). We denote by  $K_t^\lambda = \lambda \int_0^t (Y_s^\lambda - S_s)^- ds$  the penalized version of  $K$ . The well-posedness of  $(Y^\lambda, Z^\lambda)$  comes from Theorem 1.1.1.

**Theorem 1.1.3** ([52][Section 6]). *Under the same assumptions as Theorem 1.1.2, let  $(Y^\lambda, Z^\lambda)$  be the solution of (1.1.3), then there exists a constant depending on  $T, C_{Lip}^f$ , such that, for any  $\lambda > 0$ ,*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda|^2 + \int_0^T |Z_t^\lambda|^2 dt + |K_T^\lambda|^2 \right] \leq C \mathbb{E} \left[ |\xi|^2 + \int_0^T |f(t, 0, 0)|^2 dt + \sup_{0 \leq t \leq T} (S_t^+)^2 \right],$$

where  $(Y^\lambda, Z^\lambda)$  be the solution of PBSDE with respect to  $(\xi, f, S)$  and  $\lambda$ . Moreover, let  $(Y, Z, K)$  be the solution of RBSDE with respect to  $(\xi, f, S)$ , then,

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[ \int_0^T (|Y_t - Y_t^\lambda|^2 + |Z_t - Z_t^\lambda|^2) dt + \sup_{0 \leq t \leq T} |K_t - K_t^\lambda|^2 \right] = 0.$$

**Obstacle PDEs.** Let  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  satisfying the uniform Lipschitz continuity, i.e. for any  $x, x' \in \mathbb{R}^d$ ,

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C_{Lip}^X |x - x'|, \quad \text{for any } t \in [0, T].$$

For each  $(t, x) \in [0, T] \times \mathbb{R}^d$ , let  $X^{t,x}$  be the solution of following SDE:

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x}) du + \int_t^s \sigma(u, X_u^{t,x}) dB_u, \quad t \leq s \leq T.$$

Consider the Markovian data of RBSDE, namely,

$$\xi = g(X_T^{t,x}), \quad f(s, y, z) = f(s, X_s^{t,x}, y, z), \quad S = S(s, X_s^{t,x}), \quad t \leq s \leq T$$

and denote  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  the solution of related RBSDE.

**Theorem 1.1.4** ([52][Theorem 8.5 and Theorem 8.6]). *Under the same assumptions as in Theorem 1.1.2, assume*

- $b, \sigma$  are uniformly Lipschitz;
- $g \in \mathbb{C}(\mathbb{R}^d)$  and  $g, f(t, \cdot, 0, 0), S(t, \cdot)$  have at most polynomial growth at infinity uniformly in  $t$ ;
- $f(\cdot, 0, 0, 0)$  and  $S(\cdot, 0)$  are bounded on  $[0, T]$ .

For each  $(t, x) \in [0, T] \times \mathbb{R}^d$ , let  $u(t, x) = Y_t^{t,x}$ , then  $u$  is the unique continuous viscosity solution of following PDE,

$$\min \left\{ u(t, x) - S(t, x), \right. \\ \left. - \partial_t u(t, x) - \mathcal{L}_t u(t, x) - f(t, x, u(t, x), (\nabla u \sigma)(t, x)) \right\} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where  $\mathcal{L}_t := \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x) \frac{\partial}{\partial x_i}$ .

## 1.1.2 Discrete solution

Beside of the above beautiful properties admit by BSDEs, the numerical approximation is another essential topic where various techniques have been developed. The very first step of establishing an error analysis is providing a reasonable discrete solution and prove its convergence. In this subsection, we recall some classical results about the discrete solutions under a uniform time discretization as

$$0 = t_0 < \cdots < t_i = ih < t_N = T. \quad (1.1.4)$$

### 1.1.2.1 Review for BSDEs

The discrete solution of BSDE  $(Y^h, Z^h)$  usually writes as follows

$$\begin{aligned} Y_{t_N}^h &= \xi, \\ Z_{t_i}^h &= \frac{1}{h} \mathbb{E} \left[ Y_{t_{i+1}}^h \Delta B_{t_i} | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1, \\ Y_{t_i}^h &= \mathbb{E} \left[ Y_{t_{i+1}}^h | \mathcal{F}_{t_i} \right] + hf(t_i, Y_{t_i}^h, Z_{t_i}^h), \quad 0 \leq i \leq N-1, \end{aligned} \quad (1.1.5)$$

where  $\Delta B_{t_i} = B_{t_{i+1}} - B_{t_i}$ .

Zhang ([132]), Bouchard and Touzi ([16]) prove that the convergence from  $Y^h$  to  $Y$  is at order  $h^{1/2}$  under Markovian framework. Let us recall this result.

**Theorem 1.1.5** ([16][Theorem 3.1]). *Under the same assumptions as Theorem 1.1.4 for  $g$  and  $f$ , then*

$$\limsup_{h \rightarrow 0} h^{-1} \left\{ \sup_{0 \leq t \leq T} \mathbb{E} [|Y_t^h - Y_t|^2] + \mathbb{E} \left[ \int_0^T |Z_t^h - Z_t|^2 dt \right] \right\} < \infty,$$

where  $(Y_t^h, Z_t^h)$  is defined as (1.1.5) for any  $t_i \in \{t_0, \dots, t_{N-1}\}$  and for any  $t \in (t_i, t_{i+1}]$ ,  $(Y_t^h, Z_t^h)$  is solution of forward SDE w.r.t.  $(\xi = Y_{t_i}^h, f = f_{t_i})$ :

$$Y_t^h = Y_{t_i}^h - (t - t_i)f_{t_i} + \int_{t_i}^t Z_s^h dB_s,$$

and  $f_{t_i} = f(t_i, X_{t_i}^h, Y_{t_i}^h, Z_{t_i}^h)$ .

**Malliavin weight** The approximation for  $Z^h$  in (1.1.5) is deduced by Itô isometry, nevertheless, Fournié et al. ([64]) establish a formulation with Malliavin calculus under a financial background. Ma and Zhang [104] extend this to the standard BSDE. Generally, this is due to the similar consequence of Feynman-Kac formula, where  $Z$  can be considered as the first derivative of  $Y$  in the Markovian function sense. Let  $(Y, Z)$  be the solution of BSDE (1.1.1), we have the following result.

**Theorem 1.1.6** ([104][Theorem 4.2]). *Under Markovian assumptions in Theorem 1.1.4, assume in addition,*

- $\sigma$  is elliptic in the sense that, there exists a constant  $c > 0$  s.t.  $a^\top \sigma(t, x) \sigma(t, x)^\top a \geq c|a|^2$ , for any  $a, x \in \mathbb{R}^d, \forall t \in [0, T]$ ;
- $f$  and  $g$  are uniformly Lipschitz w.r.t.  $x$ ;
- $b(\cdot, 0), \sigma(\cdot, 0), f(\cdot, 0, 0, 0), g(\cdot)$  are uniformly bounded;

then we have

$$Z_t = \mathbb{E} \left[ g(X_T) w_T^t + \int_t^T f(u, X_u, Y_u, Z_u) w_u^t du \middle| \mathcal{F}_t \right] \sigma(t, X_t), \quad \forall t \in [0, T],$$

where the Malliavin weight  $(w_s^t : 0 \leq t < s \leq T)$  is defined by:

$$w_s^t = \left( \frac{1}{s-t} \int_t^s (\sigma^{-1}(u, X_u) D_t X_u)^\top dB_u \right)^\top, \quad 0 \leq t < s \leq T$$

with  $D_t X_s$  is the Malliavin derivative of  $X_s$  at time  $t$ .

The above representation is considered as a discrete solution of numerical approximations in [17], [71].

### 1.1.2.2 Review for RBSDEs

The most popular approach for the discrete solution of RBSDE should be Snell envelope. The Snell envelope is a concept from the field of stochastic control theory and optimal stopping theory. It is defined as the smallest super-martingale that dominates a given stochastic process, in other words, it represents the best lower bound on the process at each time step, considering all stopping times.

For RBSDE, Proposition 1.1.1 provides the continuous version of Snell envelope, while the discrete version, which incorporates the Dynamic Programming Principle, is typically formulated based on that of BSDEs in (1.1.5),

$$\begin{aligned}
Y_{t_N}^h &= \xi, \\
Z_{t_i}^h &= \frac{1}{h} \mathbb{E} \left[ Y_{t_{i+1}}^h \Delta B_{t_i} | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1, \\
\tilde{Y}_{t_i}^h &= \mathbb{E} \left[ Y_{t_{i+1}}^h + hf(t_i, Y_{t_{i+1}}^h, Z_{t_i}^h) | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1 \\
Y_{t_i}^h &= \max \left\{ S_{t_i}, \tilde{Y}_{t_i}^h \right\}, \quad 0 \leq i \leq N-1, \\
K_{t_i}^h &= \sum_{j=0}^i (Y_{t_j}^h - \tilde{Y}_{t_j}^h), \quad 0 \leq i \leq N-1.
\end{aligned} \tag{1.1.6}$$

Ma and Zhang ([105]) prove that the convergence rate from  $(Y^h, Z^h, K^h)$  to  $(Y, Z, K)$  is  $h^{1/4}$  in the following sense. See also the similar discrete scheme for doubly reflected BSDEs in [32, 47, 129] and the one for obliquely reflected equations in [33, 34].

**Theorem 1.1.7** ([105][Theorem 7.1]). *Under the Markovian assumptions in Theorem 1.1.4 and in Theorem 1.1.6, assume*

- $S \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^d)$  with bounded derivatives;
- $b, \sigma, f$  are  $\frac{1}{2}$ -Hölder continuous in  $t$ .

We have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (|Y_t - Y_t^h|^2 + |K_t - K_t^h|^2) + \int_0^T |Z_t - Z_t^h|^2 dt \right] \leq C\sqrt{h},$$

where for  $t \in (t_i, t_{i+1})$ ,  $(Y_t^h, Z_t^h)$  is defined as the solution of SDE w.r.t.  $(\xi = Y_{t_{i+1}}^h, f = f_{t_{i+1}})$ :

$$Y_t^h = Y_{t_{i+1}}^h + (t_{i+1} - t)f_{t_{i+1}} - \int_t^{t_{i+1}} Z_s^h dB_s,$$

and  $f_{t_{i+1}} = f(t_{i+1}, X_{t_{i+1}}^h, Y_{t_{i+1}}^h, Z_{t_{i+1}}^h)$ .

Bouchard and Chassagneux ([15]) get the same result without the condition of ellipticity assumed on  $\sigma$  and improve this rate to  $h^{1/2}$  for some  $f$  with bounded second order derivatives. The similar results can also be found in [34] for obliquely reflected BSDEs.

**Malliavin weight** Ma and Zhang ([105]) also provide a representation for  $Z$  in reflected case.

**Theorem 1.1.8** ([105][Theorem 4.2]). *Under the same assumptions as in Theorem 1.1.7 except the Hölder continuity, let  $(Y, Z, K)$  be the solution of RBSDE (1.1.2) w.r.t. the Markovian  $\xi, f, S$  as assumed in Theorem 1.1.4. The martingale integrand  $Z$  can be written as*

$$Z_t = \mathbb{E} \left[ g(X_T) w_T^t + \int_t^T f(u, X_u, Y_u, Z_u) w_u^t \mathbf{d}u + \int_t^T w_u^t \mathbf{d}K_u \right] \sigma(t, X_t), \quad \forall t \in [0, T],$$

where the Malliavin weight  $(w_s^t : 0 \leq t < s \leq T)$  is defined by:

$$w_s^t = \left( \frac{1}{s-t} \int_t^s (\sigma^{-1}(u, X_u) D_t X_u)^\top \mathbf{d}B_u \right)^\top, \quad 0 \leq t < s \leq T$$

with  $D_t X_s$  being the Malliavin derivative of  $X_s$  at time  $t$ .

As far as the author knows, there is no available result for the discrete solution of RBSDEs based on this representation.

### 1.1.3 Contributions

In the realm of mathematical optimization, penalization methods serve as techniques for integrating constraints into an optimization problem. This is achieved by augmenting the objective function with penalty terms, which penalize constraint violations and guide the optimization algorithm towards feasible solutions.

One is motivated by studying the convergence rate from the solution of PBSDE to the one of related RBSDE because of two main reasons. On the one hand, Theorem 1.1.3 shows that the solution of PBSDE is a natural approximation to the one of RBSDE, however, the utilization of this method as a practical computational approximation approach is rarely seen in works related to BSDEs. On the other hand, the penalty method is well studied in the domain of obstacle related linear parabolic PDEs (see [45, 63, 134]).

Denote  $(Y^\lambda, Z^\lambda)$  the solution to (1.1.3) and  $(Y, Z, K)$  the one to (1.1.2). Given a generator  $f$  not depending on  $z$ , we define the discrete solution  $Y_{t_i}^{\lambda, h}$  of PBSDE by implicit projection as follows:

$$\begin{aligned} Y_{t_N}^{\lambda, h} &= \xi, \\ Y_{t_i}^{\lambda, h} &= \mathbb{E} \left[ Y_{t_{i+1}}^{\lambda, h} | \mathcal{F}_{t_i} \right] + h f^\lambda(t_i, Y_{t_i}^{\lambda, h}), \quad 0 \leq i \leq N-1, \end{aligned} \tag{1.1.7}$$

where  $f^\lambda(t, y) := f(t, y) + \lambda(y - S_t)^-$  is the penalized generator.

Our main contributions in Chapter 3 are,



- taking advantages of the absolute continuity of  $K$  as in [52][Proposition 4.2], the convergence rate from  $Y_t^\lambda$  to  $Y_t$  can attain  $\frac{1}{\lambda}$  in  $\mathbb{L}^\infty$  norm under some boundedness conditions; or  $Y^\lambda \rightarrow Y$  at  $\frac{1}{\sqrt{\lambda}}$  in  $\mathcal{S}^2$  norm under more general conditions.
- the convergence rate for scheme (1.1.7) from the discrete solution  $Y_{t_i}^{\lambda,h}$  to the continuous one  $Y_{t_i}^\lambda$  is  $O(h^{1/2} + \lambda h^{3/4})$  in  $\mathbb{L}^2$  under mild regularities of  $f$  and  $S$ .

Precisely, we have following results. The complete assumptions are detailed later in Chapter 3 and some additional conditions may be required.

**Theorem 1.1.9** (Rate of convergence 1/2, Theorem 3.3.2 in Chapter 3). *Let  $(Y, Z, K)$  be the solution of (1.1.2) in  $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$ ,  $p \geq 2$  and  $(Y^\lambda, Z^\lambda)$  be the solution of (1.1.3) in  $\mathcal{S}^p \times \mathbb{H}^p$ . Under hypothesis of Theorem 3.3.2. If the barrier  $S$  satisfies some boundedness conditions and  $f(\cdot, 0, 0) \in \mathcal{S}^p$ , then*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (Y_t - Y_t^\lambda)^p + \left( \int_0^T |Z_t - Z_t^\lambda|^2 dt \right)^{p/2} + \sup_{0 \leq t \leq T} (K_t - K_t^\lambda)^p \right] \leq \frac{C}{\lambda^{p/2}},$$

where  $C$  is a constant depending on  $\xi, f, S, T, p$ .

**Theorem 1.1.10** (Rate of convergence 1, Theorem 3.3.5 in Chapter 3). *Assume hypothesis of Theorem 3.3.5 where the process  $S$  is supposed to have the same form as in Proposition 1.1.2, i.e.  $S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + A_t$ , a.s., if*

$$\bar{\kappa}_\infty := \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} (f(t, S_t, V_t) + U_t)^- = \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \kappa_t(\omega) < \infty,$$

then we have

$$0 \leq Y_t - Y_t^\lambda \leq \frac{\bar{\kappa}_\infty}{\lambda}, \quad \forall 0 \leq t \leq T.$$

**Remark 1.1.3.** The existence of  $\bar{\kappa}_\infty$  is not a demanding condition. It can be found in the linear or non-linear market for many types of options. A table is provided in Chapter 3.

**Theorem 1.1.11** (Convergence of discrete solution, Theorem 3.4.1). *Let  $(Y_t^\lambda, Z_t^\lambda)_{0 \leq t \leq T}$  be the continuous solution of PBSDE (1.1.3) and  $(Y_{t_i}^{\lambda,h}, Z_{t_i}^{\lambda,h})_{0 \leq i \leq N-1}$  be the discrete solution of scheme of PBSDE (1.1.7). Under hypothesis of Theorem 3.4.1, for the case where generator  $f$  does not depend on  $Z$ , we have*

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda,h} - Y_{t_i}^\lambda)^2 \right] = O(h + \lambda^2 h^{3/2}),$$

as  $h \rightarrow 0$ .

Combine with Theorem 1.1.10 and Theorem 1.1.11, we conclude with the following global error estimate of RBSDEs.

**Corollary 1.1.4** (global error estimations, Corollary 3.4.2 in Chapter 3). *Let  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  be the continuous solution of RBSDE (1.1.2). Under same hypotheses as Theorem 3.4.1, let  $\lambda = h^{-3/8}$ , for  $f$  not depending on  $Z$ , we get the estimation of global error as  $h \rightarrow 0$ ,*

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda,h} - Y_{t_i})^2 \right]^{1/2} = O(h^{3/8}).$$

## 1.2 Computational methods

### 1.2.1 Review

An efficient numerical scheme for solving BSDEs is always worthy to be studied due to the significant role played by BSDEs in risk management and option pricing under various market conditions. In order to solve discrete solutions (1.1.5), (1.1.6) and (1.1.7), one needs to compute the conditional expectation  $\mathbb{E}[Y_{t_{i+1}}|\mathcal{F}_{t_i}]$ , where Monte-Carlo method is widely applied.

#### 1.2.1.1 Random walk

Instead of simulating the i.i.d Gaussian variable as the increment of Brownian motion, the method random walk provides an easier idea with simulating binomial or other random variable to represent the 'finite' direction of increment. So the conditional expectation is estimated by the empirical average of realizations in finite possible directions. This simplify the implementation of algorithm but this method suffers heavily from the so-called curse of dimensionality, i.e. its space complexity increases exponentially as the dimension increases. Briand, Deylon and Mémin ([18]) and Ma, Protter, San Martín and Torres ([102]) apply this method in BSDEs, then Mémin, Peng and Xu ([109]) and Martinez, San Martín, Torres ([107]) extend this method to penalized BSDEs, see also [129] for doubly reflected equations and [20] for convergence rate analysis of BSDEs in Wasserstein distance.

#### 1.2.1.2 Quantization

Bally and Pagès ([3]) present a quantization tree algorithm for solving numerically RBSDEs in the case of generator  $f$  not depending on  $z$  based on the discrete solution (1.1.6). The quantization provide a spatial discretization both on time and on space. Based on the uniform discretization as supposed in (1.1.4), this method proposes to engage continuously some discrete time in special length as an epoch. Given the length of epochs, the space discretization are optimized in the sense that its Markovian realizations of  $X$  on this time-space grid have the minimal error in  $\mathbb{L}^p$ -spaces and the transition weight from each grid is given in the form of conditional expectation. Delarue and Menozzi ([42]) also utilize this method in coupled FBSDEs. See also [62, 99, 100, 113] for quantization method.

#### 1.2.1.3 Regression

**Kernel.** Bouchard and Touzi ([16]) considers the discrete solution as in (1.1.5) and they use kernel function to establish a ratio representation for conditional expectation. The nominator and denominator of ratio are estimated empirically by a Monte-Carlo Dirac mass. The rate

of convergence of the estimator of discrete solution is proved at  $h^{-(p+\frac{d}{4})}M^{-\frac{1}{2}}$  in [16][Theorem 6.2], where  $M$  is the number of i.i.d paths simulated.

**Least-squares projection.** The conditional expectation has an important property that it is minimizer of mean square errors in the following sense: given a square-integrable random variable  $\bar{\mathcal{R}}$  and a  $\sigma$ -algebra  $\mathcal{G}$ , then

$$\mathbb{E} [\bar{\mathcal{R}}|\mathcal{G}] = \arg \inf_{\mathcal{R} \in \mathbb{L}^2 \text{ is } \mathcal{G}\text{-measurable}} \mathbb{E} [|\bar{\mathcal{R}} - \mathcal{R}|^2].$$

So it is reasonable to consider the approximation of conditional expectation as a least-squares optimization problem.

Longstaff and Schwartz ([101]) popularize the application of Monte-Carlo regression method where they use the linear least-squares basis projection regressor for computing the conditional expectation. Gobet, Lemor and Warin ([69]) first generalize this idea to the standard BSDEs as (1.1.1). They propose a scheme based on iterative regression functions which are approximated by projections on a reduced dimension set of functions, with the coefficients of the projection being evaluated using Monte Carlo simulations. Recall their numerical scheme. Based on discrete solution described in (1.1.5) under Markovian framework as in Theorem 1.1.4, we denote by  $(\hat{Y}, \hat{Z})$  the numerical approximation to discrete BSDE solution (1.1.7). For simulating  $(\hat{Y}, \hat{Z})$ , in [69], one first prepares

- $M$  i.i.d trajectories of Brownian motions  $B$  along with the time partition  $0 = t_0 < t_1 < \dots < t_i = ih < t_N = T$  and its Markovian realizations  $X$  ;
- a reduced dimension of functions as basis of projection  $\Psi = \text{Span}\{\psi_j : 1 \leq j \leq k\}$ .

It follows by the numerical scheme in dynamic programming,

$$\begin{aligned} \hat{Y}_{t_N}^h &= g(X_{t_N}^h), \\ \hat{Z}_{t_i}^h &= \alpha_i^Z \cdot \psi(X_{t_i}^h), \quad \hat{Y}_{t_i}^h = \alpha_i^Y \cdot \psi(X_{t_i}^h), \quad 0 \leq i \leq N-1. \end{aligned} \quad (1.2.1)$$

The optimal parametrizations  $\alpha_i^Z \in \mathbb{R}^{k \times d}$ ,  $\alpha_i^Y \in \mathbb{R}^k$  are such that the empirical mean squares error is minimum, i.e.

$$\begin{aligned} &\alpha^Y, \alpha^Z \\ &= \arg \min_{(\alpha^Y, \alpha^Z) \in \mathbb{R}^d \times \mathbb{R}^{k \times d}} \frac{1}{M} \sum_{m=1}^M \left( \hat{Y}_{t_{i+1}}^{h,m} + hf(t_i, X_{t_i}^{h,m}, \hat{Y}_{t_{i+1}}^{h,m}, \hat{Z}_{t_i}^{h,m}) - \alpha^Y \cdot \psi(X_{t_i}^{h,m}) - \alpha^Z \cdot \psi(X_{t_i}^{h,m}) \Delta B_{t_i}^{h,m} \right)^2, \end{aligned} \quad (1.2.2)$$

which can be solved explicitly by Singular-Value-Decomposition (SVD in short). The explicit error order of scheme (1.2.1)-(1.2.2) is deduced in [69][Theorem 1-3]. The similar results can be found in [72] for multi-steps scheme.

The case where  $f$  does not depend on  $z$  is always considered as the first step when one wants to develop an error analysis. Particularly, in financial engineering, this corresponds

with the pricing problem under risk neutral market (see the next subsection). Gobet [66] presents the following results. Analogously, the optimal parametrization  $\alpha_i^Y \in \mathbb{R}^k$  is such that the empirical mean squares error is minimum, i.e.

$$\begin{aligned} \hat{Y}_{t_N}^h &= g(X_{t_N}^h), \quad \hat{Y}_{t_i}^h = \alpha_i^Y \cdot \psi(X_{t_i}^h), \quad 0 \leq i \leq N-1, \\ \alpha_i^Y &:= \arg \min_{\alpha^y \in \mathbb{R}^d} \frac{1}{M} \sum_{m=1}^M \left( \hat{Y}_{t_{i+1}}^{h,m} + hf(t_i, X_{t_i}^{h,m}, \hat{Y}_{t_{i+1}}^{h,m}) - \alpha^y \cdot \psi(X_{t_i}^{h,m}) \right)^2. \end{aligned} \quad (1.2.3)$$

Denote  $u(t, x) := Y_t^{t,x}$  for each pair  $(t, x) \in [0, T] \times \mathbb{R}^d$  as in Theorem 1.1.4 and respectively  $\hat{u}(t_i, x) := \hat{Y}_{t_i}^{t_i,x} = \alpha_i^Y \cdot \psi(x)$  for  $(t_i, x) \in \{t_0, \dots, t_{N-1}\} \times \mathbb{R}^d$ , we have the following results.

**Theorem 1.2.1** ([66][Theorem 8.3.2]). *Under the assumptions as in Theorem 1.1.4 and assume  $u$  is bounded by  $L$  uniformly in  $(t, x)$ , for enough small  $h$ , there exists a constant  $C$  depending on  $L, T, C_{Lip}^f, C_{Lip}^X$  such that,*

$$\mathbb{E} \left[ \|\hat{u}(t_i, \cdot) - u(t_i, \cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] \leq C \left( \sum_{j=i}^{N-1} \left[ (1+N)4L^2 \frac{k}{M} + \min_{\varphi \in \Psi} |\varphi - u(t_j, \cdot)|_{\mathbb{L}^2(\mu_{X_{t_j}^h})}^2 \right] \right),$$

where  $\|\varphi(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 = \frac{1}{M} \sum_{m=1}^M |\varphi(X_{t_i}^{h,m})|^2$ ,  $\|\varphi(\cdot)\|_{\mathbb{L}^2(\mu_{X_{t_j}^h})}^2 = \int_{\mathbb{R}^d} |\varphi(x)|^2 \mu_{X_{t_j}^h}(\mathbf{d}x)$  and  $\mu_{X_{t_j}^h}$  is the law of  $X_{t_j}^h$ .

**Theorem 1.2.2** ([66][Theorem 8.3.4]). *Under assumptions as in Theorem 1.1.4 and assume  $u$  is bounded by  $L$  uniformly in  $(t, x)$ , for enough small  $h$ , there exists a constant  $C$  depending on  $L, T, C_{Lip}^f, C_{Lip}^X$  such that,*

$$\begin{aligned} \mathbb{E} \left[ \|\hat{u}(t_i, \cdot) - u(t_i, \cdot)\|_{\mathbb{L}^2(\mu_{X_{t_j}^h})}^2 \right] &\leq C \left( \sum_{j=i}^{N-1} \left[ (1+N)4L^2 \frac{k}{M} + \min_{\varphi \in \Psi} |\varphi - u(t_j, \cdot)|_{\mathbb{L}^2(\mu_{X_{t_j}^h})}^2 \right] \right. \\ &\quad \left. + L^2 \sqrt{\frac{(k+1) \log(6M)}{M}} \right). \end{aligned}$$

**Neural network.** Györfi et al.([78]) present their result about non-linear regression. This result gives us insight on how does the error propagate especially when we do not have the concept of 'basis' to describe the size of regression space. Given random variables  $\mathcal{R}, \mathcal{O}$  where  $\mathcal{R}$  is bounded by  $L$  and given a set of data  $\{(\mathcal{R}^m, \mathcal{O}^m) : 1 \leq m \leq M\}$ , we set

$$\mathcal{M}(x) = \mathbb{E}[\mathcal{R} | \mathcal{O} = x], \quad \mathcal{M}^{M, \Psi}(x) = -L \vee \left( \arg \min_{f \in \Psi} \sum_{m=1}^M (\mathcal{R}^m - f(\mathcal{O}^m))^2 \right) (x) \wedge L$$

**Theorem 1.2.3** ([78, Theorem 11.5]). *Assume  $1 \leq L < \infty$  and let  $\mu$  be the law of  $\mathcal{O}$ , then*

$$\begin{aligned} &\mathbb{E} \left[ \int_{\mathbb{R}^d} |\mathcal{M}(x) - \mathcal{M}^{M, \Psi}(x)|^2 \mu(\mathbf{d}x) \right] \\ &\leq \frac{c_1}{M} + \frac{c_2 + c_3 \log(M) \nu_{\Psi+}}{M} + 2 \inf_{f \in \Psi} \int_{\mathbb{R}^d} |f(x) - \mathcal{M}(x)|^2 \mu(\mathbf{d}x), \end{aligned}$$

where  $c_1, c_2, c_3$  are constants depending on  $L$  and  $v_{\Psi^+}$  represents the Vapnik-Chervonenkis dimension (VC dimension in short) of set  $\Psi^+ = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \leq \psi(x)\} : \psi \in \Psi\}$ .

In Chapter 4, we have fully interpreted this theorem to neural network regression space with identifying  $v_{\Psi^+}$  and  $\inf_{f \in \Psi} \int_{\mathbb{R}^d} |f(x) - \mathcal{M}(x)|^2 \mu(\mathbf{d}x)$  by recent works in machine learning.

## 1.2.2 Contributions

We complete Chapter 3 by this error analysis for discrete solution of PBSDE (1.1.7) developed in Chapter 4. Denote our target function to be simulated by  $u^{\lambda, h}$  and

$$\begin{aligned} u_N^{\lambda, h}(x) &= g(x), \\ u_i^{\lambda, h}(x) &= (\mathcal{V}_i^\lambda)^{-1} \left( x, \mathbb{E} \left[ u_{t_{i+1}}^{\lambda, h}(X_{t_{i+1}}^{t_i, x}) \right] \right), \quad \forall 0 \leq i \leq N-1. \end{aligned} \quad (1.2.4)$$

Compared with (1.2.3), we present the implicit numerical scheme as follows

$$\begin{aligned} \hat{u}_N^{\lambda, h}(x) &= g(x), \\ \hat{u}_i^{\lambda, h}(x) &= (\hat{\mathcal{V}}_i^\lambda)^{-1} \left( x, \hat{\mathbb{E}} \left[ \hat{u}_{t_{i+1}}^{\lambda, h}(X_{t_{i+1}}^{t_i, x}) \right] \right), \quad \forall 0 \leq i \leq N-1, \end{aligned} \quad (1.2.5)$$

where the implicit projection  $\mathcal{V}_i^\lambda$  from the discrete solution in (1.1.7) is written as,

$$\mathcal{V}_i^\lambda : \begin{cases} \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, y) \mapsto y - f^\lambda(t_i, x, y)h = y - f(t_i, x, y)h - \lambda h(y - S(t_i, x))-, \end{cases}$$

and  $(\hat{\mathcal{V}}_i^\lambda)^{-1}$  is the approximation of the inverse of  $\mathcal{V}_i^\lambda$ . We also denote

$$\hat{\mathbb{E}} \left[ \hat{u}_{t_{i+1}}^{\lambda, h}(X_{t_{i+1}}^{t_i, x}) \right] = \Phi(\alpha_i^Y, x). \quad (1.2.6)$$

The function  $\Phi(\alpha, \cdot)$  in the context of linear regression equals to  $\alpha \cdot \psi(\cdot)$ , while in non-linear case, it would be a network with some trained weights and biases  $\alpha$ . Under some conditions,  $\mathcal{V}_i^\lambda$  is continuous, increasing and invertible in the sense that  $(\mathcal{V}_i^\lambda)^{-1}(x, w) := y$  if  $\mathcal{V}_i^\lambda(x, y) = w$ . For each discrete time  $t_i$ , the regression will still be applied in approximating conditional expectation  $\mathbb{E} \left[ \hat{u}_{t_{i+1}}^{\lambda, h}(X_{t_{i+1}}^{t_i, x}) \right]$  and the final estimation  $\hat{u}_i^{\lambda, h}$  is done by tracking  $(\mathcal{V}_i^\lambda)^{-1}$  using Picard fixed point argument. Our contributions are the non-asymptotic error estimations of (1.2.5)-(1.2.6):

- Extend Theorem 1.2.1 and Theorem 1.2.2 to penalized BSDEs in case of linear regression and prove that the penalty does not change the convergence rate for scheme (1.2.5);
- Develop the error analysis of non-linear regression realized by neural network and deduce the tuning of Monte-Carlo parameters and neural network parameters.

Let us first clarify our notations. For both linear and non-linear regression, we prepare  $M$  i.i.d simulations of  $d$ -Brownian motion and its realizations of  $X$  along with time partition. Assume the approximation space for simulating conditional expectation fulfills one of following conditions.

- $\Psi := \text{Span}\{\psi_j : 1 \leq j \leq k\}$  where for any  $\psi_j : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $k$  is the size of bases  $\{\psi_j : 1 \leq j \leq k\}$ ;
- the class of functions computed by a feed-forward neural network with the rectified linear unit (ReLU for short),

$$\Psi = \left\{ \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \mid \varphi = \psi_{\mathcal{L}} \circ \psi_{\mathcal{L}-1} \cdots \circ \psi_0, \right. \\ \left. \text{where } \forall 1 \leq j \leq \mathcal{L} - 1, \psi_j \in \mathcal{L}_{d_{j-1}, d_j}^{\rho} \text{ and } \psi_{\mathcal{L}} \in \mathcal{L}_{d_{\mathcal{L}-1}, 1} \right\}$$

with

$$\mathcal{L}_{d_j, d_{j+1}}^{\rho} := \left\{ \phi : \mathbb{R}^{d_j} \rightarrow \mathbb{R}^{d_{j+1}} \mid \forall 0 \leq i \leq d_{j+1}, \phi(x)_i = \rho(b_i + w_i^{\top} x), \right. \\ \left. \text{where } b \in \mathbb{R}^{d_{j+1}}, w_i \in \mathbb{R}^{d_j} \right\}, \\ \mathcal{L}_{d_j, d_{j+1}} \text{ being defined in the same way as } \mathcal{L}_{d_j, d_{j+1}}^{\rho} \text{ with } \rho(x) = x,$$

the function  $\rho$  being the ReLU activation function which is defined by  $\rho(x) = x^+$  and  $\mathcal{L}$  being the number of layers of neural networks.

In this way, we define our estimator  $\hat{u}_i^{\lambda, h}$  respectively. Define operators  $\{P_i : 0 \leq i \leq N - 1\}$  which satisfy

$$P_i \hat{u}_{i+1}^{\lambda, h}(\cdot) = \mathbb{E} \left[ \hat{u}_{i+1}^{\lambda, h}(X_{t_{i+1}}^{t_i, \cdot}) \right], \forall 0 \leq i \leq N - 1.$$

Given a priori estimator  $\hat{u}_{i+1}^{\lambda, h}$ , with (1.2.6), the function  $P_i \hat{u}_{i+1}^{\lambda, h}$  is approximated by regression on  $\Psi$ , namely,

$$P_i \hat{u}_{i+1}^{\lambda, h}(\cdot) \approx \Phi(\alpha_i^Y, \cdot) \text{ where } \alpha_i^Y := \arg \min_{\alpha \in \mathcal{A}} \sum_{m=1}^M \left( \hat{u}_{i+1}^{\lambda, h}(X_{t_{i+1}}^m) - \Phi(\alpha, X_{t_i}^m) \right)^2.$$

Since the choice of  $\alpha$  depends absolutely on regression space  $\Psi$  and Monte-Carlo parameters  $M, h$ , in general, we denote the simulation  $\hat{u}_i^{\lambda, h}$  by  $\hat{u}_i^{\lambda, h, M, \Psi}$ . Particularly for the linear regression, we use  $\hat{u}_i^{\lambda, h, M, k}$  to emphasize its dependance with the size of basis.

Precisely, we have following results for linear and non-linear regression.

**Theorem 1.2.4** (global in-sample error, Theorem 4.4.2 in Chapter 4). *Under assumptions of Theorem 4.4.2, let  $\hat{u}_i^{\lambda, h, M, k}$  be defined in (1.2.5) for linear regression space and  $u_i^{\lambda, h}$  be defined in (1.2.4). If  $h < 1 \wedge \frac{1}{2C_{Lip}^{\mathcal{F}}}$ , then we have,*

- the function sequence  $\{u_i^{\lambda,h} : 0 \leq i \leq N\}$  is uniformly bounded by  $\bar{L}$  on  $[0, T] \times \mathbb{R}^d$  with  $\bar{L}$  not depending on  $\lambda, i, h$ ;
- for the linear regression on  $k$ -dimensional basis functions, there exists a constant  $C$  depending on  $T, C_{Lip}^f, \bar{L}$ , such that, for any  $0 \leq i \leq N - 1$ ,

$$\begin{aligned} & \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,k}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] \\ & \leq C \sum_{j=i}^{N-1} \left[ \frac{k}{hM} + (1 + Ch) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})}^2 \right], \end{aligned}$$

where  $P_i$  is conditional expectation operator such that  $P_i u_{i+1}(x) = \mathbb{E} [u_{i+1}(X_{t_{i+1}}^{t_i, x})]$  and  $\mu_{X_{t_j}}$  is the law of  $X_{t_j}$ .

**Theorem 1.2.5** (global out-sample error, Theorem 4.4.3 in Chapter 4). *Under assumptions in Theorem 4.4.3, let  $u_i^{\lambda,h}$  be defined in (1.2.4) and  $\hat{u}_i^{\lambda,h,M,k}$  be estimator by linear regression defined in (1.2.5). If  $h < 1 \wedge \frac{1}{2C_{Lip}^f}$ , then we have, for the linear regression on  $k$ -dimensional basis, there exists a constant  $C$  depending on  $T, C_{Lip}^f$ , for any  $0 \leq i \leq N - 1$ ,*

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,k}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \right] & \leq C \sum_{j=i}^{N-1} \left[ \frac{k}{hM} + (1 + Ch) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})}^2 \right] \\ & \quad + 101(1 + Ch)\bar{L}^2 \sqrt{\frac{(k+1) \log(6M)}{M}}, \end{aligned}$$

where  $\bar{L}$  is defined as in Theorem 1.2.4.

**Theorem 1.2.6** (Tuning of parameters, Theorem 4.5.1 in Chapter 4). *Under hypothesis of Theorem 4.5.1 and assume the integrability for  $q \geq 2$  is satisfied by coefficients of  $X$ . Let  $\mathbf{NN}$  be a feedforward neural network which has the architecture as follows: there are  $W$  parameters (weights and biases) to estimate,  $\mathcal{L}$  layers, one input unit, one output unit and  $U$  non-zero computation units with ReLU activation. Let  $\Psi$  be the function class computed by  $\mathbf{NN}$  and denote  $\hat{u}_i^{\lambda,h,M,\Psi}$  the approximation of  $u_i^{\lambda,h}$  by non-linear regression. For any  $\varepsilon \in (0, 1)$ , if parameters of network  $\mathbf{NN}$  satisfies,*

- $\max(W, U) = \left( \varepsilon^{-d(1+\frac{d+2}{q})} \log \frac{1}{\varepsilon} \right)$  ;
- $\mathcal{L} \leq O(\log \frac{1}{\varepsilon})$ .

Then, for  $0 \leq i \leq N - 1$ , there exists a constant  $C$  depending only on  $C_{Lip}^f$ , such that,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,\Psi}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] & \leq (1 + Ch) \mathbb{E} \left[ \left\| u_{i+1}^{\lambda,h,M,\Psi}(\cdot) - u_{i+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_{i+1}}})} \right] \\ & \quad + O \left( \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^4 + \varepsilon \right), \end{aligned}$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d, q$ ); for the global error, we have, for  $0 \leq i \leq N - 1$ ,

$$\mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi}(\cdot) - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] \leq O \left( N \frac{\log M}{M} \varepsilon^{-d(1 + \frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^4 + N\varepsilon \right),$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d, q$ ).

### 1.3 Financial Applications

For a long period in history, the pricing problem for European options in a complete and perfect market is usually represented by some linear parabolic PDEs, if one assumes the underlying asset is in Black-Scholes models (BS model in short). Thanks to the Feynman-Kac formula, the solution of the related PDE has a probabilistic representation in the form of conditional expectation. This conditional expectation is always written as  $\mathbb{E}^{\mathbb{Q}} [e^{-r(T-t)}\xi | \mathcal{F}_t]$  for a contingent claim  $\xi$  and interest rate  $r$ , especially for European Call and Put options, the solution is in closed form. We call the probability measure  $\mathbb{Q}$  the risk neutral probability such that the price of European option with any payoff  $\xi$  at time  $t$  could be expressed as the conditional expectation of the discounted contingent claim.

For the American option, the pricing problem is posed in Snell envelop formulation with combining the risk neutral probability, namely, the price of an American option  $Y$  with payoff process  $S$  is usually written as

$$Y_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} [e^{-r(\tau-t)} S_\tau | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

The risk neutral probability is also called by the pricing probability, which is used in financial modeling, particularly in options pricing, to simplify calculations and make them consistent with observed market prices in the perfect and complete market. However, in the imperfect market, it is not reasonable to apply this. If one assumes in the market, there are 2 interest rates (deposit/loan) and 2 dividend rates (shorting/longing), the completeness of this market is not evident and the risk-free rate is not applicable. The case with 2 interest rates are first presented by [92] and [9]. Right after these pioneering works, many researchers investigate on the attainability of hedging strategy in this type of market, see [39] and [55]. Indeed, in the incomplete market, one can only define the so-called upper price corresponding to the super-hedging strategy. The super hedging strategy is usually composed by a self-financing portfolio  $(V, \pi)$  and a consumption fee process  $C$ , such that,  $V_T^\pi - C_T = \xi$ . The process  $C$  is positive and increasing on  $[0, T]$ . The upper price for European options is hence defined by  $\inf \{V \in \mathbb{R}^+ : \exists(\pi, C) \text{ such that } V_T^\pi - C_T = \xi\}$ . The attainability is used to identify if there exists a super-hedging strategy with  $C = 0$ , which is equivalent to the perfect hedging  $V_T = \xi$ .

With the development of BSDEs/RBSDEs, the attainability in the non-linear market is no longer a problem since it corresponds to the well-posedness of solutions, which can be



easily solved by verifying if the generator satisfies the Lipschitz condition. Moreover, the non-linear market corresponds with a non-linear parabolic PDEs which can not be solved by the standard Feynman-Kac formula. So the backward equations not only solve the attainability and also provide the form of solutions for the non-linear case.

### 1.3.1 Contributions

In Chapter 5, the Put/Call symmetry (PCS in short) is concerned. PCS is a fundamental relationship between the prices of put options and call options with the exchanged underlying asset, strike price, along with exchanged market coefficients such as interest rate and dividend rate. Under the Black-Scholes model, when the market coefficients such as the interest rate  $r$ , the dividend  $q$ , the volatility  $\sigma$ , are constants, one has the symmetry formula between the American Call option and Put option (e.g [93, Proposition 4.2.2]), i.e. with exchanging  $r$  and  $q$ ,

$$\text{Call}(t, x; T, K; r, q, \sigma) = x \text{Put}(t, K/x; T, 1; q, r, -\sigma).$$

Our contributions of Chapter 5 lie in developing the PCS of American options in the non-linear market with 2 interest rates and 2 dividend rates taking advantages of RBSDEs, when the underlying assets in the market are assumed to be Itô processes.

Precisely, in the market with 2 interest rates  $r, R$  and 2 dividend rates  $q, Q$ , the driver is written as

$$f(t, y, z) := -r_t y - z \sigma_t^{-1} (\mu_t + q_t - r_t \mathbf{1}_d) + (R_t - r_t) (y - z \sigma_t^{-1} \mathbf{1}_d)^- + (z \sigma_t^{-1})^- (Q_t - q_t). \quad (1.3.1)$$

Then our main result in this chapter is as follows.

**Theorem 1.3.1** (change of numéraire for RBSDEs, Theorem 5.4.1 in Chapter 5). *Let  $(Y, Z, K)$  be the solution of RBSDE w.r.t.  $(\xi = S_T, f, S)$  where  $f$  is as (1.3.1) and  $S_t = g(t, (X_s)_{0 \leq s \leq t})$ , under hypothesis in Theorem 5.4.1, we have that*

$$(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t) := \left( \frac{Y_t}{X_t^1}, \frac{Z_t}{X_t^1} - \sigma_t^{(1)} \frac{Y_t}{X_t^1}, \int_0^t \frac{dK_s}{X_s^1} \right)$$

is the solution of RBSDE w.r.t.  $(\tilde{S}_T, \tilde{f}, \tilde{S})$  with  $\tilde{S}_t = \frac{S_t}{X_t^1}$  and

$$\begin{aligned} \tilde{f}(t, y, z) := & -\tilde{r}_t y - z \tilde{\sigma}_t^{-1} (\tilde{\mu}_t + \tilde{q}_t - \tilde{r}_t \mathbf{1}_d) \\ & + (\tilde{R}_t - \tilde{r}_t) (y - z \tilde{\sigma}_t^{-1} \mathbf{1}_d)^- + (z \tilde{\sigma}_t^{-1})^- (\tilde{Q}_t - \tilde{q}_t). \end{aligned}$$

Here  $\tilde{r} \in \mathbb{R}$ ,  $\tilde{R} \in \mathbb{R}$ ,  $\tilde{\mu} \in \mathbb{R}^d$ ,  $\tilde{\sigma} \in \mathbb{M}_{d \times d}$ ,  $\tilde{q} \in \mathbb{R}^d$ ,  $\tilde{Q} \in \mathbb{R}^d$  are defined by  $\tilde{r}_t := q_t^{(1)}$ ,  $\tilde{R}_t := Q_t^{(1)}$ , and

$$\tilde{\mu}_t := \begin{bmatrix} -\mu_t^{(1)} \\ \mu_t^{(2)} - \mu_t^{(1)} \\ \vdots \\ \mu_t^{(d)} - \mu_t^{(1)} \end{bmatrix}_{d \times 1} - \tilde{\sigma}_t (\sigma_t^1)^\top, \quad \tilde{\sigma}_t := \begin{bmatrix} -\sigma_t^{(1)} \\ \sigma_t^{(2)} - \sigma_t^{(1)} \\ \vdots \\ \sigma_t^{(d)} - \sigma_t^{(1)} \end{bmatrix}_{d \times d}, \quad \tilde{q}_t := \begin{bmatrix} r_t \\ q_t^{(2)} \\ \vdots \\ q_t^{(d)} \end{bmatrix}, \quad \tilde{Q}_t := \begin{bmatrix} R_t \\ Q_t^{(2)} \\ \vdots \\ Q_t^{(d)} \end{bmatrix}.$$

From the above result, we deduce PCS for multidimensional basket options.

**Corollary 1.3.1** (Multidimensional PCS). *Under assumptions in Corollary 5.5.2, in the non-linear market described by (1.3.1), for American Options, we have the symmetry relation for the Call option with the pay-off  $S_t = (C^\top X_t - K)^+$  and the Put Option with pay-off  $\tilde{S}_t = (C^1 - \tilde{C}^\top \tilde{X}_t)$ , where  $C, \tilde{C} \in \mathbb{R}^d$  are constants. Precisely*

$$\mathbf{Call}(t, x; C, T, K; \mu, r, R, q, Q, \sigma) = x^1 \mathbf{Put}(t, \tilde{x}; \tilde{C}, T, C_1; \tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma}),$$

and

$$\mathbf{Call}(t, x; C, T, K; \mu, r, R, q, Q, \sigma) = \mathbf{Put}(t, x^1 \tilde{x}; \tilde{C}, T, C_1 x^1; \tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma}),$$

where  $\tilde{X} := \left(\frac{1}{X^1}, \frac{X^2}{X^1}, \dots, \frac{X^d}{X^1}\right)$ ,  $\tilde{x} = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \dots, \frac{x^d}{x^1}\right)$ ,  $\tilde{C} = (K, -C_2, \dots, -C_d)$ , and the  $X^1$ -market coefficients  $(\tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$  are as defined in Theorem 1.3.1.

More examples of PCS for American type options can be founded in Chapter 5.

## 1.4 Conclusions and Perspectives

In Chapter 3, the convergence rate from penalized BSDE (1.1.3) to reflected BSDE (1.1.2) is studied and this can attain at order 1 with respect to penalty parameter. When the generator  $f$  does not depend on  $z$ , the convergence from the discrete solution of PBSDE to the continuous one is also given at  $O(h^{1/2} + \lambda h^{3/4})$ .

Chapter 4 completes Chapter 3 as error analysis for solving numerically penalized BSDE by least-squares Monte-Carlo regression method, where both linear and non-linear regression are discussed. For the linear case, we have the error estimations in empirical norm and in distribution measure; for non-linear regression, only the estimations in distribution measure is available.

Chapter 5 presents the change of numéraire of RBSDEs and Put/Call symmetry in the non-linear market as the financial applications, where various types of American options are provided as examples of applications.

**Prospectives.** For Chapter 3, the convergence rate of  $Z^\lambda \rightarrow Z$  is only available at  $\frac{1}{\sqrt{\lambda}}$  under the general assumptions while the rate  $\frac{1}{\lambda}$  is applicable to  $Y^\lambda \rightarrow Y$  under stronger assumptions. A better convergence rate for  $Z$  can lead to an extension of our actual results about the convergence of discrete solution because we do not assume  $f$  depending on  $z$  for instant.

For Chapter 4, the tuning of parameters of neural networks still suffers from the curse of dimensionality, i.e. the size of network should increase exponentially when the dimension

$d$  increases. The empirical error for non-linear regression is also worthy to be investigated, which is difficult because dominating the covering number of the multi-layers neural network is complicated.

For Chapter 5, the applications of RBSDEs are not evident in some non-Markovian payoff cases, such as Lookback option, barrier options with sequential barriers etc.

# Chapter 2

## Introduction (en Français)

Dans cette thèse, nous nous concentrons sur l'approche de pénalisation pour approximer la solution des équations différentielles stochastiques à rebours réfléchies (EDSRs en abrégé). Ces équations apparaissent typiquement dans le contexte de problèmes de contrôle d'obstacles tels que l'arrêt optimal. Nous étudions la convergence des EDSRs pénalisées (EDSRPs en abrégé) dans le sens des solutions continues et discrètes, et de l'approximation numérique.

### 2.1 EDSR Réfléchie

#### 2.1.1 Révision

L'Équation Différentielle Stochastique Retrograde (EDSR) est un type d'équation différentielle stochastique dans laquelle l'évolution d'un processus est décrite à retrograde dans le temps. Dans une EDSR, la solution est déterminée en spécifiant une condition terminale et une fonction génératrice qui relie la solution aux processus moteurs. Les EDSRs ont des applications dans divers domaines, notamment la finance mathématique, le contrôle optimal et les problèmes de contrôle stochastique. L'une des principales caractéristiques des EDSR, qui rend ce type d'équation particulièrement adapté aux problèmes dont l'objectif est d'évaluer des créances conditionnelles ou des titres dérivés à un moment terminal donné, est que l'évolution temporelle retrograde permet d'incorporer des informations futures dans l'analyse.

Nous commençons par rappeler qu'une EDSR est une équation de la forme suivante :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s - \int_t^T Z_s \mathbf{d}B_s, \quad \forall 0 \leq t \leq T, \quad (2.1.1)$$

où  $\{B_t : 0 \leq t \leq T\}$  est un mouvement Brownien de dimension  $d$  défini sur un espace de probabilité  $(\Omega, \mathcal{F}, \mathbb{P})$ . Cet espace de probabilité est doté de la filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  générée par  $B$  où  $\mathcal{F}_0$  contient tous les ensembles  $\mathbb{P}$  nuls. Les données d'une telle équation sont données par la condition terminale  $\xi$  qui est  $\mathcal{F}_T$ -mesurable et un générateur  $f$  qui peut dépendre

de  $\omega$ . On dit que  $(Y, Z)$  est la solution d'une EDSR par rapport à  $(\xi, f)$  si  $(Y, Z)$  satisfait (2.1.1). Ce type d'équation avec un générateur linéaire  $f$  a été introduit pour la première fois par Bismut ([13]) en tant qu'équation auto-adjointe du problème de contrôle stochastique. Ensuite, Pardoux et Peng ([114]) ont présenté le résultat de bien-posé des EDSRs avec un générateur Lipschitz. Rappelons ce résultat.

**Théorème 2.1.1** ([114]). *Supposons que  $\xi$  soit carrément intégrable et  $\mathcal{F}_T$ -mesurable. Supposons que le générateur  $f$  soit  $C_{Lip}^f$ -Lipschitz continu en  $(y, z)$ , à savoir,  $\mathbf{d}t \otimes \mathbf{d}\mathbb{P}$ , p.p.*

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq C_{Lip}^f(|y - y'| + |z - z'|), \forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d.$$

Si  $\mathbb{E} \left[ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 \mathbf{d}s \right] < \infty$ , alors il existe une solution unique à la EDSR (2.1.1) dans  $\mathbb{H}^2 \times \mathbb{H}^2$ , avec

$$\mathbb{H}^2 = \left\{ \{ \phi_t : 0 \leq t \leq T \} \text{ est un processus prévisible tel que } \mathbb{E} \left[ \int_0^T |\phi_s|^2 \mathbf{d}s \right] < \infty \right\}$$

et  $|\cdot|$  représentant la norme euclidienne par rapport à la dimension du processus.

Depuis lors, de nombreux travaux ont vu le jour dans ce domaine, parmi lesquels El Karoui, Peng et Quenez ([54]) ont lié cette équation au problème de l'évaluation des options européennes sur les marchés complets et incomplets, en identifiant le processus de valeur comme  $Y$  et la stratégie de couverture comme  $Z$ .

Les Équations Différentielles Stochastiques Retrogrades Réfléchies (EDSRR en abrégé) ont été introduites juste après par El Karoui, Kapoudjian, Pardoux, Peng et Quenez ([52]). Le terme "réfléchi" signifie que le processus de valeur ne peut pas descendre en dessous d'une certaine limite inférieure ou dépasser une certaine limite supérieure. Par conséquent, les EDSRRs sont fréquemment utilisées pour modéliser des situations où l'évolution d'un processus stochastique est soumise à des contraintes ou à des barrières, comme dans les applications financières où les prix ou les valeurs sont soumis à des limites inférieures. Contrairement aux équations non réfléchies, les EDSRR admettent un processus obstacle  $S$  en entrée et nous exigeons que le processus valeur  $Y$  soit toujours supérieur ou égal à l'obstacle  $S$ . Une EDSRR standard s'écrit comme suit :

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, & \forall 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) \mathbf{d}K_t = 0, \end{cases} \quad (2.1.2)$$

où  $K$ , qui fait partie de la solution de l'EDSRR  $(\xi, f, S)$ , est un processus non décroissant qui pousse  $Y$  au-dessus de  $S$  sur  $[0, T]$ . La troisième équation de (2.1.2) est ce que l'on appelle la "condition de Skorohod" qui exige que l'incrément de  $K$  ne puisse se produire que sur  $\{(t, \omega) : Y_t = S_t\}$ . Nous disons que  $(Y, Z, K)$  est la solution de l'EDSRR  $(\xi, f, S)$  si  $(Y, Z, K)$  satisfait (2.1.2).

Hormis l'EDSRR standard comme (2.1.2), il existe également de nombreux travaux inspirés par le travail pionnier [52]. Les EDSR doublement réfléchies sont étudiées par Cvitanic et Karatzas ([40]) et par Hamadène et Lepeltier ([81]), voir aussi [48, 79, 97] pour des sujets connexes; les équations à réflexion oblique sont étudiées par Hamadène et Jeanblanc ([80]) pour 2 processus et ensuite par Hu et Tang ([87]) pour des processus multiples, voir aussi [35, 125]; les équations à croissance quadratique en  $Z$  sont étudiées par Briand et Hu [21, 22]; récemment, les équations à réflexion moyenne sont étudiées par Briand, Elie et Hu ([19]) et par Hamadène et ses co-auteurs ([36, 46]). La possibilité de bien poser les EDSRs sous certaines conditions faibles est également un sujet populaire. Par exemple, l'existence de  $(\mathbb{L}^p, p < 2)$  a été étudiée pour la première fois par Briand et al. ([23]) pour les EDSR avec un générateur monotone, puis développée pour les EDSRs avec un générateur Lipschitz par Hamadène et Popier ([82]).

On rappelle les principaux résultats suivants dans [52].

**Bien-posé.** On définit d'abord l'espace carré-intégrable  $\mathcal{S}^2$  pour les processus :

$$\mathcal{S}^2 = \{ \{ \phi_t : 0 \leq t \leq T \} \text{ est un processus prévisible tel que } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty \}.$$

**Théorème 2.1.2** ([52, Theorem 5.2 and Proposition 3.5]). *Supposons que  $S$  est un processus stochastique continu avec  $S \vee 0 =: S^+$  dans  $\mathcal{S}^2$ , le générateur  $f$  est  $C_{Lip}^f$ -Lipschitz continu et  $f(\cdot, 0, 0)$  dans  $\mathbb{H}^2$ , la condition terminale  $\xi$  est  $\mathcal{F}_T$ -mesurable, carré-intégrable et  $\xi \geq S_T$ , alors il existe une solution unique  $(Y, Z, K)$  dans  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$  satisfaisant que  $Y, K$  sont continus. De plus, nous disposons de l'estimation a priori suivante :*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |Z_t|^2 dt \right) + K_T^2 \right] \leq C \mathbb{E} \left[ |\xi|^2 + \left( \int_0^T |f(t, 0, 0)| dt \right)^2 + \sup_{0 \leq t \leq T} (S_t^+)^2 \right]$$

où  $C$  est une constante dépendant uniquement de  $T, C_{Lip}^f$ .

## Équivalence avec le problème de l'arrêt optimal

**Proposition 2.1.1** ([52][Proposition 2.3]). *Sous les mêmes hypothèses que le théorème 2.1.2, supposons que  $(Y, Z, K)$  soit la solution de RBSDE par rapport à  $(\xi, f, S)$ , alors*

$$Y_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left[ \int_t^\tau f(u, Y_u, Z_u) du + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right],$$

où  $\mathcal{T}_{t,T}$  contient tous les temps d'arrêt dont la borne inférieure est  $t$  et la borne supérieure est  $T$ .

**Densité du processus croissant  $K$ .** La proposition 2.1.2 révèle la continuité absolue de  $K$  par rapport à la mesure de Lebesgue. Elle est particulièrement utile lorsqu'il s'agit de traiter des processus d'obstacles markoviens convexes  $S$ , pour lesquels un terme temps locaux (non décroissant) sera produit grâce à la formule d'Itô-Tanaka. Voir le Chapter 3.

**Proposition 2.1.2** ([52, Proposition 4.2 and Remark 4.3]). *Sous les mêmes hypothèses que dans le théorème 2.1.2 et en supposant que  $S$  est une semi-martingale généralisée de la forme*

$$S_t = S_0 + \int_0^t U_s \mathbf{d}s + \int_0^t V_s \mathbf{d}B_s + A_t$$

où  $U, V, A$  sont, respectivement,  $\mathbb{R}, \mathbb{R}^d, \mathbb{R}$  processus mesurables progressivement et  $\mathcal{F}_t$  satisfaisant à

$$\int_0^T (|U_t| + |V_t|^2) \mathbf{d}t < \infty, \quad p.s.,$$

et  $A$  est continu et non-décroissant. Soit  $(Y, Z, K)$  une solution de la RBSDE  $(\xi, f, S)$ , alors

- $Z_t = V_t, \mathbf{d}t \otimes \mathbf{d}\mathbb{P}$  p.p. sur l'ensemble  $\{Y_t = S_t\}$  ;
- $0 \leq \mathbf{d}K_t \leq \mathbf{1}_{\{Y_t = S_t\}} (f(s, S_t, V_t) + U_t)^- \mathbf{d}t$ .

**Limite croissante des EDSR pénalisés.** Un autre résultat important est l'existence d'une solution de l'EDSRP prouvée par pénalisation. La version pénalisée de (2.1.2) s'écrit, pour tout paramètre de pénalité  $\lambda > 0$ ,

$$Y_t^\lambda = \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (Y_s^\lambda - S_s)^- \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \quad (2.1.3)$$

qui est une EDSR avec une constante de Lipschitz dépendant de  $\lambda$ . Nous disons que  $(Y^\lambda, Z^\lambda)$  est la solution d'une EDSR pénalisée par rapport à  $(\xi, f, S)$  et  $\lambda$ , si  $(Y^\lambda, Z^\lambda)$  satisfait (2.1.3). Nous désignons par  $K_t^\lambda = \lambda \int_0^t (Y_s^\lambda - S_s)^- \mathbf{d}s$  la version pénalisée de  $K$ .

Le fait que  $(Y^\lambda, Z^\lambda)$  soit bien posé provient du théorème 2.1.1.

**Théorème 2.1.3** ([52][Section 6]). *Sous les mêmes hypothèses que le théorème 2.1.2, soit  $(Y^\lambda, Z^\lambda)$  la solution de (2.1.3), il existe une constante dépendant de  $T, C_{Lip}^f$ , telle que, pour tout  $\lambda > 0$ , il existe une constante dépendant de  $T, C_{Lip}^f$ , telle que, pour tout  $C_{Lip}^f > 0$ ,*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda|^2 + \int_0^T |Z_t^\lambda|^2 \mathbf{d}t + |K_T^\lambda|^2 \right] \leq C \mathbb{E} \left[ |\xi|^2 + \int_0^T |f(t, 0, 0)|^2 \mathbf{d}t + \sup_{0 \leq t \leq T} (S_t^+)^2 \right],$$

où  $(Y^\lambda, Z^\lambda)$  est la solution de EDSRP par rapport à  $(\xi, f, S)$  et  $\lambda$ . De plus, si  $(Y, Z, K)$  est la solution de EDSRR par rapport à  $(\xi, f, S)$ , alors,

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[ \int_0^T (|Y_t - Y_t^\lambda|^2 + |Z_t - Z_t^\lambda|^2) \mathbf{d}t + \sup_{0 \leq t \leq T} |K_t - K_t^\lambda|^2 \right] = 0.$$

**EDP d'obstacles.** Soit  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  satisfaisant à la continuité Lipschitz uniforme, i.e. pour tout  $x, x' \in \mathbb{R}^d$ ,

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C_{Lip}^X |x - x'|, \quad \text{pour chaque } t \in [0, T].$$

Pour chaque  $(t, x) \in [0, T] \times \mathbb{R}^d$ , soit  $X^{t,x}$  la solution de l'EDS suivante :

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x}) \mathbf{d}u + \int_t^s \sigma(u, X_u^{t,x}) \mathbf{d}B_u, \quad t \leq s \leq T.$$

Considérons les données markoviennes de la EDSRR, à savoir,

$$\xi = g(X_T^{t,x}), \quad f(s, y, z) = f(s, X_s^{t,x}, y, z), \quad S = S(s, X_s^{t,x}), \quad t \leq s \leq T$$

et notons  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  la solution de l'EDSR correspondante.

**Théorème 2.1.4** ([52][Theorem 8.5 and Theorem 8.6]). *Sous les mêmes hypothèses que dans le théorème 2.1.2, supposons que*

- $b, \sigma$  sont uniformément Lipschitz ;
- $g \in \mathbb{C}(\mathbb{R}^d)$  and  $g, f(t, \cdot, 0, 0), S(t, \cdot)$  ont une croissance au plus polynomiale à l'infini uniformément en  $t$  ;
- $f(\cdot, 0, 0, 0)$  et  $S(\cdot, 0)$  sont bornées sur  $[0, T]$ .

Pour chaque  $(t, x) \in [0, T] \times \mathbb{R}^d$ , soit  $u(t, x) = Y_t^{t,x}$ , alors  $u$  est l'unique solution continue de viscosité de l'EDP suivante,

$$\min \left\{ u(t, x) - S(t, x), \right. \\ \left. - \partial_t u(t, x) - \mathcal{L}_t u(t, x) - f(t, x, u(t, x), (\nabla u \sigma)(t, x)) \right\} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

$$\text{où } \mathcal{L}_t := \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x) \frac{\partial}{\partial x_i}.$$

## 2.1.2 Solution discrete

En plus des propriétés admises par les EDSRs, l'approximation numérique est un autre sujet essentiel pour lequel de nombreuses techniques ont été développées. La toute première étape de l'établissement d'une analyse d'erreur consiste à fournir une solution discrète raisonnable et à prouver sa convergence. Dans cette sous-section, nous rappelons quelques résultats classiques concernant les solutions discrètes sous une discrétisation temporelle uniforme sous la forme suivante

$$0 = t_0 < \dots < t_i = ih < t_N = T.$$



### 2.1.2.1 Révision pour EDSRs

La solution discrète de la BSDE  $(Y^h, Z^h)$  s'écrit généralement comme suit

$$\begin{aligned} Y_{t_N}^h &= \xi, \\ Z_{t_i}^h &= \frac{1}{h} \mathbb{E} \left[ Y_{t_{i+1}}^h \Delta B_{t_i} | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1, \\ Y_{t_i}^h &= \mathbb{E} \left[ Y_{t_{i+1}}^h | \mathcal{F}_{t_i} \right] + hf(t_i, Y_{t_i}^h, Z_{t_i}^h), \quad 0 \leq i \leq N-1, \end{aligned} \quad (2.1.4)$$

où  $\Delta B_{t_i} = B_{t_{i+1}} - B_{t_i}$ . Zhang ([132]), Bouchard et Touzi ([16]) prouvent que la convergence de  $Y^h$  vers  $Y$  est d'ordre  $h^{1/2}$  dans un cadre markovien. Rappelons ce résultat.

**Théorème 2.1.5** ([16][Theorem 3.1]). *Sous les mêmes hypothèses que le théorème 2.1.4 pour  $g$  et  $f$ , alors*

$$\limsup_{h \rightarrow 0} h^{-1} \left\{ \sup_{0 \leq t \leq T} \mathbb{E} [|Y_t^h - Y_t|^2] + \mathbb{E} \left[ \int_0^T |Z_t^h - Z_t|^2 dt \right] \right\} < \infty,$$

où  $(Y_t^h, Z_t^h)$  est défini comme (2.1.4) pour tout  $t_i \in \{t_0, \dots, t_{N-1}\}$  et pour tout  $t \in (t_i, t_{i+1}]$ ,  $(Y_t^h, Z_t^h)$  est la solution de l'EDS par rapport à  $(\xi = Y_{t_i}^h, f = f_{t_i})$  :

$$Y_t^h = Y_{t_i}^h - (t - t_i)f_{t_i} + \int_{t_i}^t Z_s^h dB_s,$$

et  $f_{t_i} = f(t_i, X_{t_i}^h, Y_{t_i}^h, Z_{t_i}^h)$ .

### 2.1.2.2 Révision pour EDSRRs

L'approche la plus populaire pour la solution discrète de l'EDSRR devrait être l'enveloppe de Snell. L'enveloppe de Snell est un concept issu de la théorie du contrôle stochastique et de la théorie de l'arrêt optimal. Elle est définie comme la plus petite sur-martingale qui domine un processus stochastique donné. En d'autres termes, elle représente la meilleure limite inférieure du processus à chaque pas de temps, en tenant compte de tous les temps d'arrêt.

Pour les EDSRRs, la proposition 2.1.1 fournit la version continue de l'enveloppe de Snell, tandis que la version discrète, qui incorpore le principe de programmation dynamique, est typiquement formulée sur la base de celle des EDSRs dans (2.1.4),

$$\begin{aligned} Y_{t_N}^h &= \xi, \\ Z_{t_i}^h &= \frac{1}{h} \mathbb{E} \left[ Y_{t_{i+1}}^h \Delta B_{t_i} | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1, \\ \tilde{Y}_{t_i}^h &= \mathbb{E} \left[ Y_{t_{i+1}}^h + hf(t_i, Y_{t_{i+1}}^h, Z_{t_i}^h) | \mathcal{F}_{t_i} \right], \quad 0 \leq i \leq N-1 \\ Y_{t_i}^h &= \max \left\{ S_{t_i}, \tilde{Y}_{t_i}^h \right\}, \quad 0 \leq i \leq N-1, \\ K_{t_i}^h &= \sum_{j=0}^i (Y_{t_j}^h - \tilde{Y}_{t_j}^h), \quad 0 \leq i \leq N-1. \end{aligned} \quad (2.1.5)$$

Ma et Zhang ([105]) prouvent que la vitesse de convergence de  $(Y^h, Z^h, K^h)$  vers  $(Y, Z, K)$  est  $h^{1/4}$  dans le sens suivant. Voir aussi le schéma discret similaire pour les EDSR à double réflexion dans [32, 47, 129] et celui pour les équations à réflexion oblique dans [33, 34].

**Théorème 2.1.6** ([105][Theorem 7.1]). *Sous les hypothèses markoviennes de Theorem 2.1.4 and in Theorem 1.1.6, supposons que*

- $S \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^d)$  avec dérivées bornées ;
- $b, \sigma, f$  sont  $\frac{1}{2}$ -Hölder continus en  $t$ .

Nous avons

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (|Y_t - Y_t^h|^2 + |K_t - K_t^h|^2) + \int_0^T |Z_t - Z_t^h|^2 dt \right] \leq C\sqrt{h},$$

où pour  $t \in (t_i, t_{i+1})$ ,  $(Y_t^h, Z_t^h)$  est défini comme la solution de EDS par rapport à  $(\xi = Y_{t_{i+1}}^h, f = f_{t_{i+1}})$  :

$$Y_t^h = Y_{t_{i+1}}^h + (t_{i+1} - t)f_{t_{i+1}} - \int_t^{t_{i+1}} Z_s^h dB_s,$$

et  $f_{t_{i+1}} = f(t_{i+1}, X_{t_{i+1}}^h, Y_{t_{i+1}}^h, Z_{t_{i+1}}^h)$ .

Bouchard et Chassagneux ([15]) obtiennent le même résultat sans la condition d'ellipticité supposée sur  $\sigma$  et améliorent ce taux à  $h^{1/2}$  pour certains  $f$  avec des dérivées du second ordre bornées. Des résultats similaires peuvent également être trouvés dans [34] pour les EDSR à réflexion oblique.

## 2.2 Méthodes numériques

Un schéma numérique efficace pour résoudre les EDSRs mérite toujours d'être étudié en raison du rôle important joué par les EDSRs dans la gestion du risque et l'évaluation des options dans diverses conditions de marché. Afin de résoudre les solutions discrètes (2.1.4), (2.1.5), il est nécessaire de calculer l'espérance conditionnelle  $\mathbb{E}[Y_{t_{i+1}} | \mathcal{F}_{t_i}]$ , où la méthode de Monte-Carlo est largement appliquée.

### 2.2.0.1 Marche aléatoire

Au lieu de simuler la variable gaussienne i.i.d comme l'incrément du mouvement Brownien, la méthode de la marche aléatoire offre une idée plus simple en simulant une variable binomiale ou une autre variable aléatoire pour représenter la direction "finie" de l'incrément. Ainsi, l'espérance conditionnelle est estimée par la moyenne empirique des réalisations dans les directions finies possibles. Cela simplifie la mise en œuvre de l'algorithme, mais cette méthode souffre fortement de ce que l'on appelle la malédiction de la dimensionnalité, c'est-à-dire que sa complexité spatiale augmente de manière exponentielle à mesure que la

dimension augmente. Briand, Deylon et Mémin ([18]) et Ma, Protter, San Martín et Torres ([102]) appliquent cette méthode aux EDSR, puis Mémin, Peng et Xu ([109]) et Martinez, San Martín, Torres ([107]) étendent cette méthode aux EDSB pénalisées, voir aussi [129] pour les équations à double réflexion et [20] pour l'analyse du taux de convergence des EDSR dans la distance de Wasserstein.

### 2.2.0.2 Quantization

Bally et Pagès ([3]) présentent un algorithme d'arbre de quantification pour résoudre numériquement les EDSRRs dans le cas du générateur  $f$  ne dépendant pas de  $z$ , basé sur la solution discrète (1.1.6). La quantification fournit une discrétisation spatiale à la fois dans le temps et dans l'espace. Sur la base de la discrétisation uniforme telle que supposée dans (1.1.4), cette méthode propose d'engager continuellement un temps discret d'une longueur spéciale en tant qu'époque. Étant donné la longueur des époques, la discrétisation de l'espace est optimisée dans le sens où les réalisations markoviennes de  $X$  sur cette grille spatio-temporelle ont l'erreur minimale dans  $\mathbb{L}^p$ -espaces et le poids de transition de chaque grille est donné sous la forme d'une espérance conditionnelle. Delarue et Menozzi ([42]) utilisent également cette méthode dans les FBSDE couplées. Voir aussi [62, 99, 100, 113] pour la méthode de quantification.

### 2.2.0.3 Régression

**Kernel.** Bouchard et Touzi ([16]) considèrent la solution discrète comme dans (2.1.4) et ils utilisent la fonction noyau pour établir une représentation de rapport pour l'espérance conditionnelle. Le numérateur et le dénominateur du rapport sont estimés empiriquement par une masse de Dirac de Monte-Carlo. Le taux de convergence de l'estimateur de la solution discrète est prouvé à  $h^{-(p+\frac{d}{4})}M^{-\frac{1}{2}}$  dans [16][Théorème 6.2], où  $M$  est le nombre de chemins i.i.d simulés.

**Projection des moindres carrés.** L'espérance conditionnelle a la propriété importante d'être un minimiseur de l'erreur quadratique moyenne au sens suivant : étant donné une variable aléatoire intégrable au carré  $\bar{\mathcal{R}}$  et une  $\sigma$ -algèbre  $\mathcal{G}$ , alors

$$\mathbb{E} [\bar{\mathcal{R}}|\mathcal{G}] = \arg \inf_{\mathcal{R} \in \mathbb{L}^2 \text{ est mesurable } \mathcal{G}} \mathbb{E} [|\bar{\mathcal{R}} - \mathcal{R}|^2].$$

Il est donc raisonnable de considérer l'approximation de l'espérance conditionnelle comme un problème d'optimisation des moindres carrés.

Longstaff et Schwartz ([101]) popularisent l'application de la méthode de régression de Monte-Carlo dans laquelle ils utilisent le régresseur de projection linéaire des moindres carrés pour calculer l'espérance conditionnelle. Gobet, Lemor et Warin ([69]) généralisent d'abord cette idée aux EDSR standardes comme (2.1.1). Ils proposent un schéma basé sur des fonctions

de régression itératives qui sont approximées par des projections sur un ensemble de fonctions de dimension réduite, les coefficients de la projection étant évalués à l'aide de simulations de Monte Carlo.

## 2.3 Applications financières

Depuis longtemps, le problème de l'évaluation des options européennes sur un marché complet et parfait est généralement représenté par des EDP paraboliques linéaires, si l'on suppose que l'actif sous-jacent est un modèle de Black-Scholes (modèle BS en abrégé). Grâce à la formule de Feynman-Kac, la solution de l'EDP correspondante a une représentation probabiliste sous la forme d'une espérance conditionnelle. Cette espérance conditionnelle s'écrit toujours  $\mathbb{E}^{\mathbb{Q}} [e^{-r(T-t)}\xi|\mathcal{F}_t]$  pour une créance contingente  $\xi$  et un taux d'intérêt  $r$ , en particulier pour les options Européennes d'achat et de vente, la solution est en forme fermée. Nous appelons la mesure de probabilité  $\mathbb{Q}$  la probabilité de risque neutre telle que le prix de l'option Européenne avec n'importe quel gain  $\xi$  au temps  $t$  peut être exprimé comme l'espérance conditionnelle de la créance contingente actualisée.

Pour l'option Américaine, le problème d'évaluation est posé dans la formulation de l'enveloppe de Snell en combinant la probabilité de risque neutre, c'est-à-dire que le prix d'une option américaine  $Y$  avec un processus de paiement  $S$  s'écrit généralement comme suit

$$Y_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} [e^{-r(\tau-t)} S_{\tau} | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

La probabilité de risque neutre est également appelée probabilité de tarification, qui est utilisée dans la modélisation financière, en particulier dans la tarification des options, pour simplifier les calculs et les rendre cohérents avec les prix observés sur le marché dans le cas d'un marché parfait et complet. Cependant, dans un marché imparfait, il n'est pas raisonnable de l'appliquer. Si l'on suppose qu'il existe sur le marché deux taux d'intérêt (dépôt/prêt) et deux taux de dividende, l'exhaustivité de ce marché n'est pas évidente et le taux d'intérêt sans risque n'est pas applicable. Le cas avec 2 taux d'intérêt a été présenté pour la première fois par [92] et [9]. Juste après ces travaux pionniers, de nombreux chercheurs ont étudié la possibilité d'une stratégie de couverture dans ce type de marché, voir [39] et [55]. En effet, sur le marché incomplet, on ne peut définir que le prix dit supérieur correspondant à la stratégie de super-couverture. La stratégie de super-couverture est généralement composée d'un portefeuille autofinancé  $(V, \pi)$  et d'un processus de frais de consommation  $C$ , tel que,  $V_T^{\pi} - C_T = \xi$ . Le processus  $C$  est positif et croissant sur  $[0, T]$ . Le prix supérieur des options européennes est donc défini par  $\inf\{V \in \mathbb{R}^+ : \exists(\pi, C) \text{ tel que } V_T^{\pi} - C_T = \xi\}$ .

L'accessibilité est utilisée pour identifier s'il existe une stratégie de super-couverture avec  $C = 0$ , qui est équivalente à la couverture parfaite  $V_T = \xi$ .

Avec le développement des EDSRs/EDSRRs, l'accessibilité dans le marché non linéaire n'est plus un problème puisqu'elle correspond à l'aspect bien-posé des solutions, qui peut

être facilement résolu en vérifiant si le générateur satisfait la condition de Lipschitz. De plus, le marché non linéaire correspond à une EDP parabolique non-linéaire qui ne peut pas être résolue par la formule standard de Feynman-Kac. Les équations rétrospectives permettent donc non seulement de résoudre la question de l'accessibilité, mais aussi de fournir la forme des solutions pour le cas non-linéaire.

## 2.4 Conclusions et perspectives

Dans le chapitre 3, le taux de convergence de l'EDSR pénalisée (1.1.3) vers l'EDSR réfléchie (1.1.2) est étudié et peut atteindre l'ordre 1 en ce qui concerne le paramètre de pénalité. Lorsque le générateur  $f$  ne dépend pas de  $z$ , la convergence de la solution discrète de la EDSRP vers la solution continue est également donnée à  $O(h^{1/2} + \lambda h^{3/4})$ .

Le chapitre 4 complète le chapitre 3 en tant qu'analyse d'erreur pour la résolution de l'EDSR pénalisée numériquement par la méthode de régression de Monte-Carlo des moindres carrés, où les régressions linéaires et non linéaires sont discutées. Pour le cas linéaire, nous disposons des estimations d'erreur dans la norme empirique et dans la mesure de distribution ; pour la régression non linéaire, seules les estimations dans la mesure de distribution sont disponibles.

Le chapitre 5 présente le changement de numéraire des EDSRR et la symétrie Put/Call sur le marché non-linéaire en tant qu'applications financières, où divers types d'options américaines sont fournis en tant qu'exemples d'applications.

**Perspectives.** Pour le chapitre 3, le taux de convergence de  $Z^\lambda \rightarrow Z$  n'est disponible qu'à  $\frac{1}{\sqrt{\lambda}}$  sous les hypothèses générales alors que le taux  $\frac{1}{\lambda}$  est applicable à  $Y^\lambda \rightarrow Y$  sous des hypothèses plus fortes. Un meilleur taux de convergence pour  $Z$  peut conduire à une extension de nos résultats actuels sur la convergence de la solution discrète car nous ne supposons pas que  $f$  dépend de  $z$  pour l'instant.

Pour le chapitre 4, le réglage des paramètres des réseaux neuronaux souffre toujours de la malédiction de la dimensionnalité, c'est-à-dire que la taille du réseau devrait augmenter de manière exponentielle lorsque la dimension  $d$  augmente. L'erreur empirique pour la régression non-linéaire mérite également d'être étudiée, ce qui est difficile car il est compliqué de dominer le nombre de recouvrement du réseau neuronal multicouche.

Pour le chapitre 5, les applications des EDSRR ne sont pas évidentes dans certains cas non-markoviens, tels que l'option Lookback, les options à barrière avec des barrières séquentielles, etc.

# Chapter 3

## Improved convergence rate for Reflected BSDEs by penalization method

**Abstract** We investigate the convergence of numerical solution of Reflected Backward Stochastic Differential Equations (RBSDEs) using the penalization approach in a general non-Markovian framework. We prove the convergence between the continuous penalized solution and the reflected one, in full generality, at order  $1/2$  as a function of the penalty parameter; the convergence order becomes 1 when the increasing process of the RBSDE has a bounded density, which is a mild condition in practice. The convergence is analyzed in *a.s.*-sense and  $\mathbb{L}^p$ -sense ( $p \geq 2$ ). To achieve these new results, we have developed a refined analysis of the behavior of the process close to the barrier. Then we propose an implicit scheme for computing the discrete solution of the penalized equation and we derive that the global convergence order is  $3/8$  as a function of time discretization under mild regularity assumptions. This convergence rate is verified in the case of American Put options and some numerical tests illustrate these results.<sup>1</sup>

### 3.1 Introduction

**Context and model.** The Backward Stochastic Differential Equations, BSDEs in short, were first introduced in the case of linear generator by Bismut in [13] as the adjoint equations of some stochastic control problem and later in the general case by Pardoux and Peng [115]. It was then widely studied in stochastic finance (see [54]) in view of its natural formulation to model the problem of option pricing and hedging. See [103], [116], [133] for a broad overview on BSDE and applications in stochastic control. The Reflected BSDE is one of the most

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1. This chapter is based on the paper "Improved convergence rate for Reflected BSDEs by penalization method", [73], joint work with E. Gobet.

important branch of BSDE, which was first introduced by El Karoui et al. [52] as follows: given a terminal condition  $\xi$ , a generator  $f$  and a barrier  $(S_t)_{0 \leq t \leq T}$ , the RBSDE is written as

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, & \forall 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) \mathbf{d}K_t = 0 \end{cases} \quad (3.1.1)$$

where the process  $(K_t)_{0 \leq t \leq T}$  is continuous and non-decreasing,  $K_0 = 0$ . Here,  $Y, K$  are scalar processes and  $Z$  is a  $d$ -dimensional process (written as a row vector). The triple of processes  $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$  is solution of RBSDE with respect to the data  $(\xi, f, S)$  where the relative assumptions will be clarified later.

The RBSDE topic was quickly developed since it provides a new framework to study stochastic control problems and optimal stopping problems involving one obstacle (the barrier). After the pioneer work [52], the Dynkin game and its related doubly RBSDE is studied in [40]; the switching problem is discussed in [83]; the starting and stopping problem is addressed in [80]. But among all of its various applications, the pricing of American option – which writes as an optimal stopping problem in the probabilistic language or as a variational inequality in the PDE terminology – is considered as one of the most fundamental applications. Similarly to the link between BSDE solutions and European option prices, there also exists a relation between RBSDE and American option: when the market is complete without market imperfection, the relation is standard using a linear generator. But this relation can also be extended in several cases with modified equation or even reformulated problem, accounting for imperfections, or for market incompleteness or for other financial specificities. For example, when the market is complete, different settings have been studied for the pricing problem: using a convex risk measure in [58]; with asymmetric information in [59]; in the presence of 2 interest rates (for borrowing and lending) and default in [49]. Moreover, when the market is incomplete, the American option pricing turns out to be a problem of superhedging (see [53]) leading to the design of a special RBSDE. The equation, with superlinear and quadratic  $f$  is presented in [91], with an extra constraint in [117], or with imperfect market and default in [75].

Besides, regarding the mathematical analysis of existence and uniqueness of solutions, there are many theoretical results for RBSDE under either standard or general assumptions. Here we refer to the hypotheses in [52] as the standard ones, i.e. with Lipschitz generator  $f$ , continuous barrier  $S$  and under  $\mathbb{L}^p$  ( $p \geq 2$ ) integrability properties. First, under standard assumptions, the existence and uniqueness are usually deduced either by the Picard's fixed point argument or by the penalization method which builds a monotone convergent sequence of approximations. The Penalized BSDE (PBSDE in short) related to (4.1.1) writes as, for

the penalty parameter  $\lambda > 0$ ,

$$\begin{cases} Y_t^\lambda &= \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (Y_s^\lambda - S_s)^- \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \\ K_t^\lambda &= \lambda \int_0^t (Y_s^\lambda - S_s)^- \mathbf{d}s. \end{cases} \quad (3.1.2)$$

The variational inequality is also studied in [2]. Second, under more general assumptions, the well-posedness is widely studied, using the similar approaches based on fixed point argument or penalization. The reader can find the well-posedness of RBSDE via penalization, with càdlàg  $S$  in [96], with driving Lévy process in [56], with  $\mathbb{L}^p$  ( $p \in [1, 2)$ ) integrability in [82]. The well-posedness via fixed point argument of RBSDE, with superlinear and quadratic  $f$  is identified in [91], the case with monotone  $f$  is addressed in [95], the model with resistance is given in [120]. As we can see, the penalization method is usually applied as a tool for solving some existence problem, and seldom for effective numerical approximations. In this paper, we focus on the numerical solution of the one barrier's RBSDEs under standard assumptions, via the penalization method, and our goal is to establish some convergence rate results.

**State of the art on numerical methods.** Before we introduce our contributions to the penalization approach, let us have a look at the existing results in the literature. Due to the important and various applications of RBSDE, particularly in optimal stopping, it is essential to investigate effective numerical schemes. However, in comparison with the numerous contributions to the theoretical aspects of RBSDE, there are only a few available results about the numerical methods. Indeed, because of the barrier and as a big difference with BSDE, the RBSDE can hardly admit an explicit representation even in the case with linear generator. As a consequence, designing an efficient numerical solution turns out to be quite challenging.

Among all the available approaches, presumably the most usual one is the Snell envelop, consisting in taking the maximum between the barrier and a conditional expectation, the latter being the main difficulty regarding computations. In the Markovian framework, Bally and Pagès [3, Theorem 4] proposed a quantization tree algorithm for RBSDE with  $f$  not depending on  $z$  and proved the discrete approximation  $\hat{Y}$  converges to the continuous solution  $Y$  with a  $\mathbb{L}^p$  error of order  $h^{1/2} + h^{-(1+1/d)}N^{-1/d}$  where  $N$  denotes the size of quantization grid and  $h$  denotes the time step; Bouchard and Touzi [16, Theorem 7.2] proposed a kernel regression algorithm and deduced a rate in  $\mathbb{L}^p$  of order  $h^{1/2} + h^{-(1+d/(4p))}M^{-1/(2p)}$  for RBSDE with  $f$  not depending on  $Z$ , where  $M$  denotes the number of Monte-Carlo simulations. Ma and Zhang [105, Theorem 4.2 and Corollary 3.4] provided a semi- explicit representation for  $Z$ , using Malliavin calculus, and derived that, if the barrier  $S := \phi(X)$  with  $\phi \in C^2$  and the forward process  $X$  is an uniformly elliptic diffusion, their pseudo time-discretization  $(Y^h, Z^h, K^h)$  converges in  $\mathbb{L}^2$  to the continuous solution  $(Y, Z, K)$  is at rate  $h^{1/4}$ . Bouchard and Chassagneux [15, Theorem 4.1] extended the previous result without uniform ellipticity condition, and improved the rate to  $h^{1/2}$  (for  $Y$  only) if  $\phi \in C^2$  with a Lipschitz second derivative. Gobet and Lemor [68, Theorem 4] investigated empirical regression algorithms for solving discretized RBSDEs.



The regularization approach is an alternative to Snell Envelop techniques. Bally et al. [2] designed a new way to simulate the process  $K$ , more precisely,  $K$  admits a semi-explicit density of the form  $\alpha_t \kappa_t$  where  $\alpha$  is an unknown process valued in  $[0, 1]$  and  $\kappa$  is a known function of  $(t, S_t)$ . An empirical regression method based on this principle is investigated in [68], and gives good numerical results, but without theoretical guarantee. Last, in another series of works, numerical methods based on penalization like (4.1.2) have been developed. Mémin et.al [109] developed a discrete random walk solution with penalization, they showed  $(Y^\lambda, Z^\lambda) \rightarrow (Y, Z)$  and  $(Y^{\lambda,h}, Z^{\lambda,h}) \rightarrow (Y^h, Z^h)$  at  $\lambda^{-1/4}$  in  $\mathbb{L}^2$ , the convergence rate with respect to the time step  $h$  is lacking. Martinez et al. [107, Theorem 1 & 2] proposed a numerical random walk scheme combining Picard iteration and penalization and they proved the convergence  $(\hat{Y}^{\lambda,h,i}, \hat{Z}^{\lambda,h,i}) \rightarrow (Y, Z)$  in  $\mathbb{L}^2$  without identifying convergence order, where  $i$  denotes  $i$ -th Picard iteration. Nevertheless, the complexity of random walk schemes is known to increase quite fast with the Brownian motion dimension and with the number of time steps, and in many situations, we may prefer to avoid random walk approximations. As an alternative, it is reasonable to take into consideration the Monte-Carlo method (a.k.a. empirical regression methods), which is one of the ingredients of this work. For BSDE without reflection, see [70, 71, 94], for an account on empirical regression methods, including error analyses. Last, Bernhart et al. [11, Theorem 3.1] presented a discrete scheme for BSDE with constrained jump and proved the quadratic convergence w.r.t.  $\lambda$  and  $h$ , but the global rate is unfortunately quite slow (logarithmic rate).

Last, for the sake of having a complete state of the art, let us briefly discuss the use of penalization approach to derive schemes for other equations. Let us start with Reflected Forward SDEs (RSDEs in short). For RSDE in a convex set, Menaldi [110, Remark 3.1] deduced that the penalized continuous solution converges in  $\mathbb{L}^2$  to the reflected one at rate  $\lambda^{-1/2+\varepsilon}$  for any  $\varepsilon > 0$ . Liu [98] showed the penalized discrete solution converges to the continuous reflected one at  $h^{1/4-\varepsilon}$  if  $\lambda \sim O(h^{-1/2})$ . Petterson [118, Theorem 3.1] derived the rate  $(h \log \frac{1}{h})^{1/4}$  in  $\mathbb{L}^2$  provided that  $\lambda \sim O(1/h)$  and Slominski [124, Theorem 4.2] proved the same rate but uniformly in time. These results show that the convergence order is more or less  $1/4$  w.r.t. the time step  $h$  for appropriate choice of penalization parameter  $\lambda$ , this shows also the necessary entanglement between  $h$  and  $\lambda$ ; however, though inspiring, establishing error bounds for backward equation is usually more difficult than for forward one, due to the adaptedness condition. In this paper, we will show that, if the barrier is a generalized Itô process under mild conditions, our backward discrete penalized solution converges in  $\mathbb{L}^2$  to the reflected continuous one at rate  $h^{3/8}$  if  $\lambda \sim ch^{-3/8}$ : this is strictly better than for forward equation.

Regarding PDEs with variational inequality (related to optimal stopping problem), the penalty approach is also commonly used when it comes to design numerical schemes. This is restricted to Markovian cases, as a difference with our general setting. The discretization of PDE is both in space (with a mesh size  $\Delta x$ ) and in time (with a time step  $h$ ), but as a usual difference with stochastic equations, the spatial mesh is deterministic. Zvan et.al. [134, Section 5.3] designed a  $\theta$ -scheme using penalty method for identifying the early exercise region in a stochastic volatility model for the forward component and showed at each time step,

the penalization term where the approximation is smaller than the barrier is at  $O(\frac{1}{\lambda})$ , *a.e.* Forsyth and Vetzal [63, Theorem 4.1] discussed the quadratic convergence of  $\theta$ -schemes for pricing Put option as  $h, \Delta x \rightarrow 0$  for a fixed  $\lambda$  and proved a similar result to [134] but with the discretization constraint  $h = o(\Delta x)$ . They also showed numerically that the uniform discretization can not ensure the quadratic convergence near the exercise boundary, hence advocating for a time step selector. d'Halluin et.al. [45, Theorem 4.2] designed a iterative scheme for pricing American derivatives in jump models and proved the global convergence of the iterative procedure as the number of iteration goes to infinity at each time step for a fixed  $\lambda$  and time discretization. Howison et.al [86, Theorem 4.3 & Section 6] showed in the convex Markovian barrier, the penalized PDE converges to the exact solution at  $1/\lambda$  and presented a detailed error analysis about the convergence order of the penalized approximation to the penalized PDE solution, which is at  $\sqrt{\lambda}(h + (\Delta x)^2)$  in  $\mathbb{L}^2([0, T] \times \mathbb{R})$ .

**Our contributions.** In this paper, since the PBSDE provides a natural continuous approach to the RBSDE, we first concentrate on the rate of  $Y^\lambda \rightarrow Y$  and we show that this convergence can hold at  $1/\lambda$  a.s. or  $1/\sqrt{\lambda}$  in  $\mathbb{L}^p(\Omega)$  depending on the behavior of  $(\xi, S)$ , see Theorem 3.3.5 and Theorem 3.3.2 for a precise statement. As far as we know (see the previous state-of-the-art), this is the first time that the rate of continuous PBSDE converging to RBSDE is investigated. Second, when  $f$  does not depend on  $Z$ , we design an implicit discrete representation for  $Y^{\lambda, h}$  and show theoretically that the discretization error,  $Y^{\lambda, h} - Y^\lambda$ , converges to 0 at order  $h^{1/2} + \lambda h^{3/4}$  in  $\mathbb{L}^2$ , see Theorem 3.4.1. Finally, we conclude that the global discretization error between  $Y^{\lambda, h}$  and  $Y$  is at  $h^{3/8}$  if  $\lambda \sim ch^{-3/8}$ , which is a significant improvement compared to [11]. The order  $h^{3/8}$  is faster than  $h^{1/4}$  in [15] for another RBSDE scheme obtained when  $S = \phi(X)$  with  $\phi \in C^2$ , which is stronger than our mild Hölder-continuity condition (**H**<sub>2</sub>-ii); but our rate is lower than  $h^{1/2}$  in [15] obtained under the even stronger assumption  $\phi$  is  $C^3$ . The numerical experiment based on Monte-Carlo empirical regression is attached in the end and we show numerically the simulation of  $Y^{\lambda, h}$  may converge quicker to  $Y$  than the rate upper bound, at least in the example of pricing one-dimensional Put option in the linear market.

**Plan of the paper.** In Section 3.2, we define the model, state standing assumptions and exemplify some applications. In Section 3.3, a priori estimates in  $\mathbb{L}^p(\Omega)$ ,  $p \geq 2$  and the rate of  $(Y_t^\lambda - S_t)^- \rightarrow 0$  are given. The discretization error of  $Y^{\lambda, h} - Y$  in  $\mathbb{L}^2(\Omega)$  is well investigated in Section 3.4. The numerical results are presented at the end of Section 3.4.

## 3.2 Model, assumptions, examples

### 3.2.1 Notation

- For a vector  $x \in \mathbb{R}^d$ , we simply denote its transpose by  $x^\top$  and its  $i$ -th component by  $x^i$ . Its Euclidean norm is denoted by  $|x|$ .

- Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the probability space where a  $d$ -dimensional Brownian motion  $(B_t)_{0 \leq t \leq T}$  is defined and  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is the canonical filtration of  $(B_t)_{0 \leq t \leq T}$  where  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Denote the expectation under  $\mathbb{P}$  by  $\mathbb{E}$  and the conditional expectation given  $\mathcal{F}_t$  by  $\mathbb{E}_t$ .
- The following set of random variables and stochastic processes are useful for our subsequent study. Let  $p \geq 2$ :

$$\begin{aligned} \mathbb{L}^\infty &= \left\{ \text{random variables } \zeta \text{ s.t. } \operatorname{ess\,sup}_{\omega \in \Omega} |\zeta(\omega)| < \infty \right\}, \\ \mathbb{L}^p &= \left\{ \text{random variables } \zeta \text{ s.t. } \mathbb{E} [|\zeta|^p] < \infty \right\}, \\ \mathcal{S}^p &= \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are predictable} \right. \\ &\quad \left. \text{with } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^p \right] < \infty \right\}, \\ \mathcal{S}_{cont}^p &= \left\{ \text{stochastic processes } \phi \in \mathcal{S}^p \text{ that are continuous} \right\}, \\ \mathbb{H}^p &= \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are predictable} \right. \\ &\quad \left. \text{with } \mathbb{E} \left[ \left( \int_0^T |\phi_t|^2 dt \right)^{p/2} \right] < \infty \right\}. \end{aligned}$$

The above random objects take values in an Euclidean space  $\mathbb{R}$  or  $\mathbb{R}^d$  whose dimension is not indicated for the sake of simplicity, this will be clear in the context.

- The set of  $(\mathcal{F}_t)_{0 \leq t \leq T}$  stopping times larger than  $t$  and bounded by  $T$  is defined by:

$$\mathcal{T}_{t,T} = \{\tau; \tau \text{ is a stopping time s.t. } t \leq \tau \leq T\}.$$

### 3.2.2 Reflected BSDE

We work on the model of RBSDE (4.1.1) and its penalized version (4.1.2).

### 3.2.3 Assumptions

In our paper, we always consider  $p \geq 2$ .

**(H<sub>1</sub>): General Assumptions of Equation (4.1.1).**

**(H<sub>1</sub>-i)** The scalar random variable  $\xi$  is  $\mathcal{F}_T$ -measurable and  $\xi \in \mathbb{L}^p$ ;

**(H<sub>1</sub>-ii)** The generator  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is such that  $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, f(\cdot, y, z) \in \mathbb{H}^p$ ;

**(H<sub>1</sub>-iii)** The generator  $f$  is  $C_{Lip}^f$ -Lipschitz continuous with respect to  $(y, z)$  uniformly in  $(\omega, t)$ , i.e.  $\forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$ , a.s.

$$|f(t, y, z) - f(t, y', z')| \leq C_{Lip}^f (|y - y'| + |z - z'|);$$

**(H<sub>1-iv</sub>)** The barrier  $S := \{S_t, 0 \leq t \leq T\}$  is a  $\mathcal{F}_t$ -adapted scalar process s.t.  $S_T \leq \xi$  a.s. and  $S \in \mathcal{S}_{cont.}^p$ .

From [52, Theorem 5.2], for  $p = 2$ , under **(H<sub>1</sub>)**, the equation (4.1.1) admits a unique solution  $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$  in the space  $\mathcal{S}_{cont.}^2 \times \mathbb{H}^2 \times \mathcal{S}_{cont.}^2$ . The case when  $p > 2$  is proved in Appendix 3.5.4.1.

We now justify that, up to a simple change of variable, we can still assume that the generator satisfies, in addition to **(H<sub>1-iii</sub>)**, the monotone property **(H<sub>m</sub>)**, the latter will be widely used in our work (especially for better controlling the penalization term, see Theorem 3.3.4, Theorem 3.3.5 and Section 3.4).

**(H<sub>m</sub>)** The generator  $f$  is non-increasing with respect to  $y$  i.e.  $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(t, y_1, z) \leq f(t, y_2, z), \quad \forall y_1 \geq y_2.$$

Indeed, under **(H<sub>1</sub>)**, for each  $\nu \in \mathbb{R}$ , define

$$\xi^\nu := \xi e^{\nu T}, \quad f^\nu(t, y, z) := e^{\nu t} f(t, e^{-\nu t} y, e^{-\nu t} z) - \nu y, \quad S_t^\nu := e^{\nu t} S_t.$$

The assumption **H<sub>1</sub>** is also fulfilled by  $(\xi^\nu, f^\nu, S^\nu)$ , then the solution of RBSDE  $Y^\nu \times Z^\nu \times K^\nu$  associated with  $(\xi^\nu, f^\nu, S^\nu)$  is also well and uniquely defined in  $(\mathcal{S}_{cont.}^p, \mathbb{H}^p, \mathcal{S}_{cont.}^p)$  and it can be expressed as following,

$$(Y_t^\nu, Z_t^\nu, K_t^\nu)_{0 \leq t \leq T} = \left( e^{\nu t} Y_t, e^{\nu t} Z_t, \int_0^t e^{\nu s} \mathbf{d}K_s \right)_{0 \leq t \leq T}. \quad (3.2.1)$$

We now justify that a proper choice of  $\nu$  allows  $f^\nu$  to fulfill **(H<sub>m</sub>)**.

**Proposition 3.2.1.** *Assume **(H<sub>1-iii</sub>)** holds for  $f$ : for any  $\nu \geq C_{Lip}^f$ ,  $f^\nu$  satisfies **(H<sub>m</sub>)**.*

*Proof.* Let  $\nu$  as above, then for all  $y_1 \geq y_2$  and  $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$\begin{aligned} f^\nu(t, y_1, z) &= e^{\nu t} f(t, e^{-\nu t} y_1, e^{-\nu t} z) - \nu y_1 \leq C_{Lip}^f (y_1 - y_2) + e^{\nu t} f(t, e^{-\nu t} y_2, e^{-\nu t} z) - \nu y_1 \\ &\leq \nu (y_1 - y_2) + e^{\nu t} f(t, e^{-\nu t} y_2, e^{-\nu t} z) - \nu y_1 \\ &= f^\nu(t, y_2, z). \end{aligned}$$

□

Hence, for each  $f$  satisfying **(H<sub>1-iii</sub>)**, one can always identify  $f$  by  $f^\nu$  and assume **(H<sub>m</sub>)** for  $f$  without loss of generality.

**(H<sub>2</sub>): Extra assumptions for the barrier  $S$**

**(H<sub>2-i</sub>)** The barrier  $S$  admits a generalized semi-martingale decomposition as

$$S_t = S_0 + \int_0^t U_s \mathbf{d}s + \int_0^t V_s \mathbf{d}B_s + A_t$$

where  $U, V \in \mathcal{S}^p$ ,  $A$  is continuous, non-decreasing and  $A_T \in \mathbb{L}^p$ ,  $A_0 = 0$ .

**(H<sub>2</sub>-ii)** The barrier  $S$  satisfies a 1/2-Hölder continuity condition:

$$C_{(3.2.2)} = \mathbb{E} \left[ \sup_{0 \leq t < s \leq T} \left( \frac{\mathbb{E}_t [|S_s - S_t|]}{\sqrt{s-t}} \right)^p \right] < \infty. \quad (3.2.2)$$

We know that, from [52, Proposition 4.2 and Remark 4.3], if **(H<sub>2</sub>-i)** is satisfied by  $S$ , then  $K$  is absolutely continuous with respect to Lebesgue measure, namely,

$$\begin{cases} \mathbf{d}K_t \leq \kappa_t \mathbf{1}_{\{Y_t = S_t\}} \mathbf{d}t, \\ \kappa_t := (f(t, S_t, V_t) + U_t)^-. \end{cases} \quad (3.2.3)$$

The process  $(\kappa_t)_{0 \leq t \leq T}$  is essential to derive the rate of  $Y^\lambda \rightarrow Y$  a.s. and in  $\mathbb{L}^p$ .

**Lemma 3.2.2.** *Assume the barrier  $S$  is of the form  $S_t = g(t, X_t)$ , where  $(X_t)_{0 \leq t \leq T}$  is an Itô process valued in  $\mathbb{R}^d$  satisfying the following SDE:  $\mathbf{d}X_t = \mu_t \mathbf{d}t + \sigma_t \mathbf{d}B_t$ , with predictable, bounded stochastic processes  $(\mu, \sigma)$ , and where  $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\frac{1}{2}$ -Hölder continuous in  $t$  and locally Lipschitz continuous in  $x$ :  $|g(t, x) - g(t', x')| \leq C (|t - t'|^{1/2} + |x - x'| (e^{C|x|} + e^{C|x'|}))$ , with some finite constant  $C$ . Then **(H<sub>2</sub>-ii)** is fulfilled by  $S$ .*

*Proof.* See Section 3.5.1. □

**Remark 3.2.3.** The immediate application in finance is American option pricing problem. If the barrier is the payoff of American Put Option and  $X$  is the asset log-price process (Itô process as in Lemma 3.2.2), i.e.  $S_t = (K^{stk} - e^{X_t})^+$  with  $K^{stk}$  represents the strike price, then it satisfies both **(H<sub>2</sub>-i)** and **(H<sub>2</sub>-ii)**. In fact, applying Itô-Tanaka formula on  $S_t$ :

$$\mathbf{d}S_t = -\mathbf{1}_{\{e^{X_t} \leq K^{stk}\}} \left[ \left( \mu_t + \frac{1}{2} |\sigma_t|^2 \right) e^{X_t} \mathbf{d}t + \sigma_t e^{X_t} \mathbf{d}B_t \right] + \frac{1}{2} \mathbf{d}L_t^{K^{stk}}(e^X)$$

where  $L^a(e^X)$  is the local time process of  $e^X$  at level  $a$ . Thanks to the boundedness of  $\mu, \sigma$  and  $\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{p|X_t|} \right] < \infty, \forall p \geq 2$ , we can easily deduce that the corresponding processes  $(U, V) \in \mathcal{S}^p$ . The arguments and results are similar for American Call Option.

### 3.3 Penalization

In this section, we present our main result about the rate of convergence  $Y^\lambda \rightarrow Y$  with respect to the penalty parameter  $\lambda$ , where the order relies critically on the convergence rate of penalization term, namely,  $(Y_t^\lambda - S_t)^- \rightarrow 0$ . Using several times the absolute continuity as (3.2.3), we first introduce a priori estimate for all general  $\mathcal{S}_{cont}^p$  barrier and derive the first bound for  $\lambda \int_0^T (Y_t^\lambda - S_t)^- \mathbf{d}t$  in  $\mathbb{L}^p$  which gives a order 1/2 for  $Y^\lambda \rightarrow Y$  in  $\mathcal{S}^p$ . Then we focus on the uniform boundedness in  $t$  of  $(Y_t^\lambda - S_t)^-$  in  $\mathbb{L}^\infty$  and in  $\mathcal{S}_{cont}^p$  where we can derive  $Y_t^\lambda \rightarrow Y_t$  at order 1 in  $\lambda$ .

### 3.3.1 The convergence rate in continuous case

We first investigate the convergence order between the PBDSE and the RBSDE, this is based on the following a priori estimate.

**Theorem 3.3.1.** *Under  $(\mathbf{H}_1)$ , let  $(Y, Z, K)$  be the solution of (4.1.1) in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p \times \mathcal{S}_{cont.}^p$  and  $(Y^\lambda, Z^\lambda)$  be the solution of (4.1.2) in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p$ . First, the norm of  $(Y^\lambda, Z^\lambda, K^\lambda)$  in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p \times \mathcal{S}_{cont.}^p$  is bounded uniformly in  $\lambda$ , in particular*

$$\sup_{\lambda > 0} \mathbb{E} \left[ \left( \lambda \int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}s \right)^p \right] < +\infty. \quad (3.3.1)$$

Second, we have the following a priori estimate

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t - Y_t^\lambda|^p + \left( \int_0^T |Z_t - Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} + \sup_{0 \leq t \leq T} |K_t - K_t^\lambda|^p \right] \\ \leq C \mathbb{E} \left[ \left( \int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}K_s \right)^{p/2} \right], \end{aligned} \quad (3.3.2)$$

where  $C$  is a constant depending only on  $\xi, f, S, T, p$ .

The proof is postponed to Appendix 3.5.4. Whenever necessary, a priori  $\mathbb{L}^p$ -estimates on  $(Y, Z, K)$  are available in Proposition 3.5.2.

**Theorem 3.3.2.** *Under  $(\mathbf{H}_1)$ , let  $(Y, Z, K)$  be the solution of (4.1.1) in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p \times \mathcal{S}_{cont.}^p$  and  $(Y^\lambda, Z^\lambda)$  be the solution of (4.1.2) in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p$ . If the barrier  $S$  satisfies  $(\mathbf{H}_2-i)$  and  $f(\cdot, 0, 0) \in \mathcal{S}^p$ , then*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (Y_t - Y_t^\lambda)^p + \left( \int_0^T |Z_t - Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} + \sup_{0 \leq t \leq T} (K_t - K_t^\lambda)^p \right] \leq \frac{C}{\lambda^{p/2}},$$

where  $C$  is a constant depending on  $\xi, f, S, T, p$ .

*Proof.* Let  $(\tilde{Y}^\lambda, \tilde{Z}^\lambda)$  be the solution (in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p$ ) of the linear BSDE

$$\begin{aligned} \tilde{Y}_t^\lambda &= \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (S_s - \tilde{Y}_s^\lambda) \mathbf{d}s - \int_t^T \tilde{Z}_s^\lambda \mathbf{d}B_s \\ &= \mathbb{E}_t \left[ e^{-\lambda(T-t)} \xi + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T e^{-\lambda(s-t)} S_s \mathbf{d}s \right]. \end{aligned} \quad (3.3.3)$$

By the classical comparison theorem as [54, Theorem 2.2], for  $0 \leq t \leq T$ ,  $Y_t^\lambda \geq \tilde{Y}_t^\lambda$ , *a.s.*, we just need to prove

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \left( \tilde{Y}_t^\lambda - S_t \right)^- \right|^p \right] \leq \frac{C}{\sqrt{\lambda}}.$$

▷ *Step 1:* We rewrite the formula of  $\tilde{Y}_t^\lambda - S_t$  with term  $e^{-\lambda(T-t)}S_T$ ,

$$\begin{aligned} \tilde{Y}_t^\lambda - S_t &= \mathbb{E}_t \left[ e^{-\lambda(T-t)}(\xi - S_T) + e^{-\lambda(T-t)}(S_T - S_t) + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s \right] \\ &\quad + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} (S_s - S_t) \mathbf{d}s \right]. \end{aligned}$$

Taking negative part on both side, and applying the Jensen inequality, it gives

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)}(\xi - S_T)^-] + \mathbb{E}_t [e^{-\lambda(T-t)}|S_T - S_t|] \\ &\quad + \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right] + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right], \end{aligned}$$

since  $\xi \geq S_T$ , we have

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)}|S_T - S_t|] + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right] \\ &\quad + \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]. \end{aligned} \tag{3.3.4}$$

▷ *Step 2:* We take supremum on these conditional expectations in (3.3.9), and consider the  $\mathbb{L}^p$ -norms of terms, which follows

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t [e^{-\lambda(T-t)}|S_T - S_t|]^p \right]^{1/p} &= \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ e^{-\lambda(T-t)} \sqrt{T-t} \frac{|S_T - S_t|}{\sqrt{T-t}} \right]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( e^{-\lambda(T-t)} \sqrt{T-t} \right)^p \times \sup_{0 \leq t \leq T} \left( \frac{\mathbb{E}_t [|S_T - S_t|]}{\sqrt{T-t}} \right)^p \right]^{1/p} \\ &\leq \frac{1}{\sqrt{\lambda}} \sup_{x \geq 0} (e^{-x} \sqrt{x}) C_{(3.2.2)}^{1/p}. \end{aligned} \tag{3.3.5}$$

Taking again the advantage of Hölder continuity of  $S$  in Assumption ( $\mathbf{H}_2$ -ii), we get

$$\begin{aligned} &\lambda \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right]^p \right]^{1/p} \\ &= \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \int_t^T \lambda e^{-\lambda(s-t)} \sqrt{s-t} \frac{|S_s - S_t|}{\sqrt{s-t}} \mathbf{d}s \right]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[ \sup_{0 \leq t < s \leq T} \mathbb{E}_t \left[ \frac{|S_s - S_t|}{\sqrt{s-t}} \right]^p \left( \int_t^T \lambda e^{-\lambda(s-t)} \sqrt{s-t} \mathbf{d}s \right)^p \right]^{1/p} \\ &\leq C_{(3.2.2)}^{1/p} \frac{\left( \int_0^{+\infty} e^{-x} \sqrt{x} \mathbf{d}x \right)}{\sqrt{\lambda}}. \end{aligned} \tag{3.3.6}$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]^p \right]^{1/p} \\
& \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \sqrt{\int_t^T e^{-2\lambda(s-t)} \mathbf{d}s} \times \mathbb{E}_t \left[ \sqrt{\int_t^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right] \right)^p \right]^{1/p} \\
& \leq \frac{1}{\sqrt{2\lambda}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right]^p \right]^{1/p}. \tag{3.3.7}
\end{aligned}$$

▷ *Step 3:* Set  $M_T := \sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s}$ . Under  $(\mathbf{H}_1)$ -(ii-iii) and combined with  $\lambda$ -uniform a priori estimate for  $(Y^\lambda, Z^\lambda)$  in (3.5.3), we claim that  $M_T \in \mathbb{L}^p$  uniformly in  $\lambda$ . Indeed,

$$\begin{aligned}
\mathbb{E} [|M_T|^p] &= \mathbb{E} \left[ \left( \int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s \right)^{p/2} \right] \\
&\leq C_{p,f,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda|^p + \left( \int_0^T |Z_s^\lambda|^2 \mathbf{d}s \right)^{p/2} + \left( \int_0^T |f(s, 0, 0)|^2 \mathbf{d}s \right)^{p/2} \right] < \infty,
\end{aligned}$$

for some constant  $C_{p,f,T}$  which does not depend on  $\lambda$ . Hence,  $M_t := \mathbb{E}_t [M_T]$  defines a martingale in  $\mathbb{L}^p$  and the right-hand side of (3.3.12) becomes

$$\frac{1}{\sqrt{2\lambda}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t|^p \right]^{1/p} \leq \frac{1}{\sqrt{2\lambda}} \left( \frac{p}{p-1} \right) \mathbb{E} [|M_T|^p]^{1/p}$$

in view of the Doob inequality. All in all, the right-hand side of (3.3.12) is of order  $1/\sqrt{\lambda}$ . Finally, combine (3.3.9), (3.3.10), (3.3.11), (3.3.12), we have the desired result.  $\square$

### 3.3.2 Advanced estimation for $Y^\lambda$ below the semi-martingale barrier

In the previous subsection, we used the convergence of integral of penalization term, namely,  $\int_0^T (Y_t^\lambda - S_t)^- \mathbf{d}t \rightarrow 0$  in  $\mathbb{L}^p$ . In this subsection, we focus on the uniform convergence in  $t$  i.e.  $\sup_{0 \leq t \leq T} (Y_t^\lambda - S_t)^- \rightarrow 0$ , a.s. and in  $\mathbb{L}^p$ .

**Theorem 3.3.3.** *Under  $(\mathbf{H}_1)$ , let  $(Y^\lambda, Z^\lambda)$  be the solution of PBSDE (4.1.2) in  $\mathcal{S}_{cont}^p \times \mathbb{H}^p$ , if the barrier  $S$  satisfies  $(\mathbf{H}_2$ -ii), then there exists a constant  $C$  which depends only on  $\xi, f, S, T, p$  such that*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right]^{1/p} \leq \frac{C}{\sqrt{\lambda}}.$$



*Proof.* Let  $(\tilde{Y}^\lambda, \tilde{Z}^\lambda)$  be the solution (in  $\mathcal{S}_{cont.}^p \times \mathbb{H}^p$ ) of the linear BSDE

$$\begin{aligned} \tilde{Y}_t^\lambda &= \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (S_s - \tilde{Y}_s^\lambda) \mathbf{d}s - \int_t^T \tilde{Z}_s^\lambda \mathbf{d}B_s \\ &= \mathbb{E}_t \left[ e^{-\lambda(T-t)} \xi + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T e^{-\lambda(s-t)} S_s \mathbf{d}s \right]. \end{aligned} \quad (3.3.8)$$

By the classical comparison theorem as [54, Theorem 2.2], for  $0 \leq t \leq T$ ,  $Y_t^\lambda \geq \tilde{Y}_t^\lambda$ , *a.s.*, it suffices to prove

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(\tilde{Y}_t^\lambda - S_t)^-| \right]^{1/p} \leq \frac{C}{\sqrt{\lambda}}.$$

▷ *Step 1:* We rewrite the formula of  $\tilde{Y}_t^\lambda - S_t$  with term  $e^{-\lambda(T-t)} S_T$ ,

$$\begin{aligned} \tilde{Y}_t^\lambda - S_t &= \mathbb{E}_t \left[ e^{-\lambda(T-t)} (\xi - S_T) + e^{-\lambda(T-t)} (S_T - S_t) + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s \right] \\ &\quad + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} (S_s - S_t) \mathbf{d}s \right]. \end{aligned}$$

Taking negative part on both side, and applying the Jensen inequality, it gives

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)} (\xi - S_T)^-] + \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|] \\ &\quad + \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right] + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right], \end{aligned}$$

since  $\xi \geq S_T$ , we have

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|] + \lambda \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right] \\ &\quad + \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]. \end{aligned} \quad (3.3.9)$$

▷ *Step 2:* We take supremum on these conditional expectations in the right side of (3.3.9), and consider the  $\mathbb{L}^p$ -norm of each term, which follows

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( e^{-\lambda(T-t)} \sqrt{T-t} \right)^p \times \sup_{0 \leq t \leq T} \left( \frac{\mathbb{E}_t [|S_T - S_t|]}{\sqrt{T-t}} \right)^p \right]^{1/p} \\ &\leq \frac{1}{\sqrt{\lambda}} \sup_{x \geq 0} (e^{-x} \sqrt{x}) C_{(3.2.2)}^{1/p}. \end{aligned} \quad (3.3.10)$$

Taking again the advantage of Hölder continuity of  $S$  in Assumption  $(\mathbf{H}_2\text{-ii})$ , we get

$$\begin{aligned} & \lambda \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right]^p \right]^{1/p} \\ & \leq \mathbb{E} \left[ \sup_{0 \leq t < s \leq T} \mathbb{E}_t \left[ \frac{|S_s - S_t|}{\sqrt{s-t}} \right]^p \left( \int_t^T \lambda e^{-\lambda(s-t)} \sqrt{s-t} \mathbf{d}s \right)^p \right]^{1/p} \\ & \leq C_{(3.2.2)}^{1/p} \frac{\left( \int_0^{+\infty} e^{-x} \sqrt{x} \mathbf{d}x \right)}{\sqrt{\lambda}}. \end{aligned} \quad (3.3.11)$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]^p \right]^{1/p} \\ & \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \sqrt{\int_t^T e^{-2\lambda(s-t)} \mathbf{d}s} \times \mathbb{E}_t \left[ \sqrt{\int_t^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right] \right)^p \right]^{1/p} \\ & \leq \frac{1}{\sqrt{2\lambda}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E}_t \left[ \sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right]^p \right]^{1/p}. \end{aligned} \quad (3.3.12)$$

▷ *Step 3:* Set  $M_T := \sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s}$ . Under  $(\mathbf{H}_1)$ -(ii-iii) and combined with  $\lambda$ -uniform a priori estimate for  $(Y^\lambda, Z^\lambda)$  in (3.5.3), we claim that  $M_T \in \mathbb{L}^p$  uniformly in  $\lambda$ . Indeed,  $\mathbb{E}[|M_T|^p]$  is bounded by

$$C_{p,f,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda|^p + \left( \int_0^T |Z_s^\lambda|^2 \mathbf{d}s \right)^{p/2} + \left( \int_0^T |f(s, 0, 0)|^2 \mathbf{d}s \right)^{p/2} \right] < \infty,$$

for some constant  $C_{p,f,T}$  which does not depend on  $\lambda$ . Hence,  $M_t := \mathbb{E}_t[M_T]$  defines a martingale in  $\mathbb{L}^p$  and the right-hand side of (3.3.12) becomes

$$\frac{1}{\sqrt{2\lambda}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t|^p \right]^{1/p} \leq \frac{1}{\sqrt{2\lambda}} \left( \frac{p}{p-1} \right) \mathbb{E}[|M_T|^p]^{1/p}$$

in view of the Doob inequality. All in all, the right-hand side of (3.3.12) is of order  $1/\sqrt{\lambda}$ . Finally, combine (3.3.9), (3.3.10), (3.3.11), (3.3.12), we have the desired result.  $\square$

**Theorem 3.3.4.** *Suppose  $(\mathbf{H}_1), (\mathbf{H}_2\text{-i})$  are fulfilled; in addition, suppose  $(\mathbf{H}_m)$  holds true, i.e.  $f(t, y, z)$  is non-increasing on  $y$ , assume*

$$\bar{\kappa}_\infty := \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} (f(t, S_t, V_t) + U_t)^- = \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \kappa_t(\omega) < \infty \quad (3.3.13)$$

then the penalization term is upper bounded as follows

$$(Y_t^\lambda - S_t)^- \leq \frac{\bar{\kappa}_\infty}{\lambda}, \quad \forall 0 \leq t \leq T, \quad a.s..$$

*Proof.* The strategy of proof relies on comparison arguments between BSDEs but the fact that  $S$  can not be decomposed as a standard BSDE, forbids us to apply standard comparison results (where drivers are compared only along one solution, see [54, Theorem 2.2]). Instead, we could invoke a comparison between RBSDEs but it usually requires to compare the generator everywhere ([52, Theorem 4.1]) which is a too stringent requirement in our case. Hence, we proceed with a direct proof.

Write the dynamic of  $S_t - \frac{1}{\lambda}\bar{\kappa}_\infty$  and the one of  $Y_t^\lambda$  as follows:

$$\begin{aligned} S_t - \frac{1}{\lambda}\bar{\kappa}_\infty &= S_T - \frac{1}{\lambda}\bar{\kappa}_\infty - \int_t^T U_s \mathbf{d}s - \int_t^T \mathbf{d}A_s - \int_t^T V_s \mathbf{d}B_s, \\ Y_t^\lambda &= \xi + \int_t^T f^\lambda(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \end{aligned}$$

where  $f^\lambda(s, y, z) := f(s, y, z) + \lambda(y - S_s)^-$ . Set  $\Delta Y_t = S_t - \frac{1}{\lambda}\bar{\kappa}_\infty - Y_t^\lambda$  and  $\Delta Z_t = V_t - Z_t^\lambda$ : we have

$$\begin{aligned} \Delta Y_t &= \Delta Y_T - \int_t^T (U_s + f^\lambda(s, Y_s^\lambda, Z_s^\lambda)) \mathbf{d}s - \int_t^T \mathbf{d}A_s - \int_t^T \Delta Z_s \mathbf{d}B_s \\ &= \Delta Y_T - \int_t^T (U_s + f^\lambda(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, V_s) + \Delta_y f^\lambda(s) \Delta Y_s + \Delta_z f^\lambda(s) \cdot \Delta Z_s) \mathbf{d}s \\ &\quad - \int_t^T \mathbf{d}A_s - \int_t^T \Delta Z_s \mathbf{d}B_s, \end{aligned}$$

here  $\Delta_y f^\lambda(s) = \frac{f^\lambda(s, Y_s^\lambda, Z_s^\lambda) - f^\lambda(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, Z_s^\lambda)}{\Delta Y_s} \mathbf{1}_{\Delta Y_s \neq 0}$  and similarly for  $\Delta_z f^\lambda(s)$  (see e.g. [54, Proof of Theorem 2.5]). Because  $f$  is Lipschitz in  $y$  and  $z$ , the coefficients  $\Delta_y f^\lambda(s)$  and  $\Delta_z f^\lambda(s)$  are bounded. Thus, using  $\Gamma_t^T := \exp(\int_t^T (\Delta_y f^\lambda(s) - \frac{1}{2} |\Delta_z f^\lambda(s)|^2) \mathbf{d}s + \int_t^T \Delta_z f^\lambda(s) \mathbf{d}B_s)$  and Itô formula, we easily get

$$\Delta Y_t = \mathbb{E}_t \left[ \Gamma_t^T \Delta Y_T - \int_t^T \Gamma_t^s (U_s + f^\lambda(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, V_s)) \mathbf{d}s - \int_t^T \Gamma_t^s \mathbf{d}A_s \right].$$

We claim that

$$U_s + f^\lambda(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, V_s) \geq 0, \mathbf{d}s \otimes \mathbf{d}\mathbb{P} \text{ a.e. } . \quad (3.3.14)$$

Since  $\Delta Y_T = S_T - \frac{1}{\lambda}\bar{\kappa}_\infty - \xi \leq -\frac{1}{\lambda}\bar{\kappa}_\infty \leq 0$  and  $A$  is non-decreasing, we get  $\Delta Y_t \leq 0$ , which implies the desired result. To justify (3.3.14), we write

$$\begin{aligned} U_s + f^\lambda(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, V_s) &= U_s + f(s, S_s - \frac{1}{\lambda}\bar{\kappa}_\infty, V_s) + \lambda(S_s - \frac{1}{\lambda}\bar{\kappa}_\infty - S_s)^- \\ &\geq U_s + f(s, S_s, V_s) + \bar{\kappa}_\infty \geq 0, \end{aligned}$$

where, in the last line, we used that  $f$  is non-increasing on  $y$  together with the definition of  $\bar{\kappa}_\infty$ .  $\square$

**Theorem 3.3.5.** *Assume the same hypotheses hold as in Theorem 3.3.4, we have*

$$0 \leq Y_t - Y_t^\lambda \leq \frac{\bar{\kappa}_\infty}{\lambda}, \quad \forall 0 \leq t \leq T,$$

where  $\bar{\kappa}_\infty$  is defined as in Theorem 3.3.4.

*Proof.* The left hand side of inequality is an immediate result from comparison theorem (see [52, Section 6]). For the right hand side, our strategy is to apply again a comparison result to  $Y$  and  $Y' := Y^\lambda + \frac{\bar{\kappa}_\infty}{\lambda}$ . Denote

$$K_t^\lambda = \lambda \int_0^t (Y_s^\lambda - S_s)^- \mathbf{d}s.$$

Then  $(Y', Z^\lambda, K^\lambda)$  can be considered as the solution of the following BSDE with barrier  $S$  and increasing process  $K^\lambda$ ,

$$\begin{cases} Y'_t = \xi' + \int_t^T f(s, Y'_s - \frac{\bar{\kappa}_\infty}{\lambda}, Z_s^\lambda) \mathbf{d}s + \int_t^T \mathbf{d}K_s^\lambda - \int_t^T Z_s^\lambda \mathbf{d}B_s, \\ Y'_t \geq S_t, \\ \int_0^T (Y'_t - S_t) \mathbf{d}K_t^\lambda \geq 0 \end{cases} \quad (3.3.15)$$

with  $\xi' := \xi + \frac{\bar{\kappa}_\infty}{\lambda}$ . The second and the third conditions of (3.3.15) hold thanks to Theorem 3.3.4 where the third can't be replaced by the equality, which means that (3.3.15) is not a real RBSDE because the Skorokhod condition is not fulfilled by  $(Y', S, K^\lambda)$ .

Nevertheless, following the arguments of [52, Theorem 4.1], we are going to establish a comparison theorem dedicated to our specific setting. Apply Itô formula on  $|(Y_t - Y'_t)^+|^2$ , we get, since  $\xi' \geq \xi$ ,

$$\begin{aligned} & \mathbb{E} [|(Y_t - Y'_t)^+|^2] + \mathbb{E} \left[ \int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right] \\ & \leq 2\mathbb{E} \left[ \int_t^T (Y_s - Y'_s)^+ [f(s, Y_s, Z_s) - f(s, Y'_s - \frac{\bar{\kappa}_\infty}{\lambda}, Z_s^\lambda)] \mathbf{d}s \right] \\ & \quad + 2\mathbb{E} \left[ \int_t^T (Y_s - Y'_s)^+ (\mathbf{d}K_s - \mathbf{d}K_s^\lambda) \right]. \end{aligned}$$

It still holds that

$$\int_t^T (Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda})^+ (\mathbf{d}K_s - \mathbf{d}K_s^\lambda) = - \int_t^T (Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda})^+ \mathbf{d}K_s^\lambda \leq 0$$

because on the set  $\{s : Y_s = S_s\}$  where  $K$  increases,  $Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda} = S_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda} \leq 0$  thanks to Theorem 3.3.4. Moreover, since  $\bar{\kappa}_\infty \geq 0$  and  $f$  is non-increasing in  $y$ , we get,  $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$ ,

$$f\left(t, y - \frac{\bar{\kappa}_\infty}{\lambda}, z\right) \geq f(t, y, z), \quad \mathbf{d}t \otimes \mathbf{d}\mathbb{P}, \quad \text{a.e. .}$$

Hence,

$$\begin{aligned} & \mathbb{E} \left[ |(Y_t - Y'_t)^+|^2 \right] + \mathbb{E} \left[ \int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right] \\ & \leq 2\mathbb{E} \left[ \int_t^T (Y_s - Y'_s)^+ [f(s, Y_s, Z_s) - f(s, Y'_s, Z_s^\lambda)] \mathbf{d}s \right] \\ & \leq (2C_{Lip}^f + 2(C_{Lip}^f)^2) \mathbb{E} \left[ \int_t^T |(Y_s - Y'_s)^+|^2 \mathbf{d}s \right] + \mathbb{E} \left[ \int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right]. \end{aligned}$$

Conclude that  $|(Y_t - Y'_t)^+|^2 = 0, 0 \leq t \leq T, a.s.$  using Gronwall's lemma.  $\square$

**Remark 3.3.1.** Theorems 3.3.4 and 3.3.5 are valid without assuming  $(\mathbf{H}_m)$  up to modifying the upper bounds in their statements. Namely, if  $\bar{\kappa}_\infty$  in (3.3.13) is finite, then we have

$$(Y_t^\lambda - S_t)^- \leq \frac{\bar{\kappa}_\infty e^{C_{Lip}^f T}}{\lambda}, \quad \forall 0 \leq t \leq T, \quad a.s., \quad (3.3.16)$$

$$0 \leq Y_t - Y_t^\lambda \leq \frac{\bar{\kappa}_\infty e^{C_{Lip}^f T}}{\lambda}, \quad \forall 0 \leq t \leq T, \quad a.s.. \quad (3.3.17)$$

Indeed, due to the change of variable in Proposition 3.2.1, we can always assume the generator to be non-increasing in  $y$ . For  $\nu = C_{Lip}^f$ , with  $(Y^\nu, Z^\nu, K^\nu)$  as defined in (3.2.1), the generator  $f^\nu$  satisfies  $(\mathbf{H}_m)$ , then Theorems 3.3.4 and 3.3.5 yield

$$e^{\nu t} (Y_t^\lambda - S_t)^- = (Y_t^{\lambda, \nu} - S_t^\nu)^- \leq \frac{\bar{\kappa}_\infty^\nu}{\lambda}, \quad e^{\nu t} (Y_t - Y_t^\lambda) = Y_t^\nu - Y_t^{\nu, \lambda} \leq \frac{\bar{\kappa}_\infty^\nu}{\lambda},$$

where after a few computations, we easily justify

$$\bar{\kappa}_\infty^\nu = \operatorname{ess\,sup}_{(t, \omega) \in [0, T] \times \Omega} e^{\nu t} (f(t, S_t, V_t) + U_t)^- \leq e^{\nu T} \bar{\kappa}_\infty.$$

Then we can conclude with (3.3.16) and (3.3.17).

**Remark 3.3.2.** Let us exemplify the above results in the case of the pricing of an American put in finance with two interest rates ( $r_t$  for lending,  $R_t$  for borrowing), see [54, Example 1.1]. Indeed, assume in the complete and perfect market,  $d$ -risky assets  $X = (X^1, \dots, X^d)^\top$  satisfy the following SDE:

$$\frac{\mathbf{d}X_t^i}{X_t^i} = \mu_t^i \mathbf{d}t + \sigma_t^i \mathbf{d}B_t,$$

where the stochastic process  $\mu = (\mu^1, \dots, \mu^d)^\top$  is bounded in  $\mathbb{R}^d$  and the matrix of volatility  $\sigma \in \mathbb{R}^{d \times d}$  is bounded, invertible and elliptic a.s.

$$\sigma = \begin{bmatrix} \sigma_t^1 \\ \vdots \\ \sigma_t^n \end{bmatrix},$$

type	pay-off	$\bar{\kappa}_\infty$	order $(\frac{1}{\lambda})$
Call	$(X_t - K)^+$	$ r^- _\infty K$	1 ( $r < 0$ )
Put	$(K - X_t)^+$	$ r^+ _\infty K$	1 ( $r > 0$ )
Call on spread	$(X_t^1 - X_t^2 - K)^+$	$ r^- _\infty K$	1
Put on spread	$(K - (X_t^1 - X_t^2))^+$	$ r^+ _\infty K$	1
Call on max	$(X_t^1 \vee X_t^2 - K)^+$	$ r^- _\infty K$	1
Put on min	$(K - X_t^1 \wedge X_t^2)^+$	$ r^+ _\infty K$	1
Call on basket	$(\sum^d X_t^i - K)^+$	$ r^- _\infty K$	1

Table 3.1 –  $\bar{\kappa}_\infty$  in linear market

where elliptic is in the sense that there exists a strictly positive constant  $c$  such that  $a^\top \sigma_t \sigma_t^\top a \geq c|a|^2$ , for any  $a \in \mathbb{R}^d, t \in [0, T]$ . Assume the interest rates  $r, R$  are bounded scalar stochastic processes and there is no dividend paid by  $X$ , then in the market with 2 interest rates  $r$  and  $R$ , the driver

$$f(t, y, z) = -r_t y - z \sigma_t^{-1} (\mu_t - r_t \mathbf{1}_d) + (R_t - r_t)(y - z \sigma_t^{-1} \mathbf{1}_d)^-$$

and  $S_t = (K^{stk} - X_t \mathbf{1}_d)^+$  where  $\mathbf{1}_d$  is a column vector filled by 1 in  $\mathbb{R}^d$ . From Remark 3.2.3, in unidimensional market ( $d = 1$ ), a direct computation leads to  $\kappa_t = r_t^+ K^{stk} \mathbf{1}_{\{X_t \leq K^{stk}\}}$  (see Appendix 3.5.5). Obviously, if we assume  $r$  is upper bounded, then  $\bar{\kappa}_\infty < +\infty$  and therefore

$$0 \leq Y_t - Y_t^\lambda \leq \frac{C \bar{\kappa}_\infty}{\lambda}$$

for some constant  $C$ .

On the other hand, if  $r_t \leq 0$  for any  $t$  then  $\bar{\kappa}_\infty = 0$ : combining (3.3.16) and (3.3.17) gives  $Y_t = Y_t^\lambda \geq S_t$ ; thus we retrieve that the increasing process  $K$  is zero, i.e. American and European put prices coincide when the interest rate remains non-positive.

The rate of convergence  $\frac{1}{\lambda}$  can also be founded in the perfect market, i.e. in the market where the self-financing portfolio is a linear combination of assets. In this case, under the same Itô models assumed on  $X$  and the boundedness conditions, the generator writes as

$$f(t, y, z) := -r_t y - z \sigma^{-1} (\mu_t - r_t \mathbf{1}_d).$$

Table 3.1 present some examples (see its proof in Appendix 3.5.5).

### 3.4 Error of implicit scheme

In this section, we aim at assessing the discretization error of the numerical solution of PBSDE using the estimations from the previous section. When applying a time discretization to  $Y^\lambda$  in (4.1.2), the difficulty lies in the fact that the Lipschitz constant of the generator

part  $f^\lambda$  goes to infinity as  $\lambda \rightarrow \infty$ . To overcome this, we use an implicit scheme  $Y^{\lambda,h}$  defined in (3.4.1). In our analysis, we first quantify the error of  $Y^{\lambda,h} - Y^\lambda$  w.r.t. the time step  $h$  and  $\lambda$ ; and second, we use the order 1-bound of  $Y - Y^\lambda$  in Theorem 3.3.5 to get a global convergence rate of  $Y^{\lambda,h} \rightarrow Y$ . Since we do not have a tight enough rate for  $Z - Z^\lambda$  (order  $\frac{1}{2}$  in  $\mathbb{L}^p$  norm, see Theorem 3.3.2), we restrict our study to a case where the generator  $f$  does not depend on  $z$ .

### 3.4.1 Main result

Consider the equidistant time discretization  $0 = t_0 < t_1 < \dots < t_N = T$  with  $t_i = i\frac{T}{N}, \forall 0 \leq i \leq N$ . Let  $h := \frac{T}{N}$ . For all  $0 \leq i \leq N - 1$ , we propose the piecewise implicit scheme for the discrete solution of  $Y^\lambda$  as follows,

$$\begin{aligned} Y_{t_i}^{\lambda,h} &= \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^{\lambda,h} + f^\lambda(t_i, Y_{t_i}^{\lambda,h})h \right], \\ Y_t^{\lambda,h} &= Y_{t_i}^{\lambda,h}, \forall t \in [t_i, t_{i+1}), \end{aligned} \quad (3.4.1)$$

where  $f^\lambda(t, y) := f(t, y) + \lambda(y - S_t)^-$  and  $\mathbb{E}_{t_i}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{t_i}]$ . The continuous solution of (4.1.2)  $Y^\lambda$ , at each  $t_i$ , can be represented as,

$$Y_{t_i}^\lambda = \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^\lambda + f^\lambda(t_i, Y_{t_i}^\lambda)h + \mathcal{E}_i^{f,\lambda} \right],$$

where the perturbation is defined by

$$\mathcal{E}_i^{f,\lambda} := \int_{t_i}^{t_{i+1}} f^\lambda(s, Y_s^\lambda) \mathbf{d}s - f^\lambda(t_i, Y_{t_i}^\lambda)h. \quad (3.4.2)$$

Before presenting the result of discretization error of PBSDE, let us denote  $D$  the Malliavin derivative operator (see [112]) and define the following integral space (see [54, p.58]):

$$\begin{aligned} \mathbb{D}^{1,2} &:= \left\{ \text{random variables } \zeta \text{ that are Malliavin differentiable with} \right. \\ &\quad \left. \zeta \in \mathbb{L}^2, \mathbb{E} \left[ |\zeta|^2 + \int_0^T |D_\theta \zeta|^2 \mathbf{d}\theta \right] < \infty \right\}, \\ \mathbb{H}^{1,2} &:= \left\{ \text{scalar predictable process } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ s.t.} \right. \\ &\quad \text{for a.e. } t \in [0, T], \quad \phi_t \in \mathbb{D}^{1,2}, \\ &\quad t \in [0, T] \mapsto D\phi_t \text{ has a progressively measurable version} \\ &\quad \text{in } L_2([0, T], \mathbb{R}^d), \text{ and} \\ &\quad \left. \mathbb{E} \left[ \int_0^T |\phi_t|^2 \mathbf{d}t + \int_0^T \int_0^T |D_\theta \phi_t|^2 \mathbf{d}\theta \mathbf{d}t \right] < \infty \right\}. \end{aligned}$$

**Theorem 3.4.1.** *Suppose  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_m)$  and  $(\mathbf{H}_2)$  are satisfied. In addition, assume*

- the terminal condition  $\xi$  satisfies  $\xi \in \mathbb{D}^{1,2}$  and  $\operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_t [|D_t \xi|] < \infty$ ;
- the generator  $f$  satisfies:
  - a.  $f(t, y, z) = f(t, y)$ ;
  - b.  $f(\cdot, 0) \in \mathcal{S}^2$ ;
  - c.  $\forall t \in [0, T]$ ,  $f(t, \cdot)$  is differentiable with uniformly bounded and continuous derivatives;
  - d.  $f(\cdot, y)$  is  $\frac{1}{2}$ -Hölder continuous s.t.  $\mathbb{E} \left[ \sup_{0 \leq t < s \leq T, y \in \mathbb{R}} \frac{|f(s, y) - f(t, y)|^2}{s-t} \right] < \infty$ ;
  - e.  $\forall y \in \mathbb{R}$ ,  $f(\cdot, y) \in \mathbb{H}^{1,2}$  and  $\operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_t \left[ \sup_{(s,y) \in [t,T] \times \mathbb{R}} |D_t f(s, y)|^2 \right] < \infty$ ;
  - f.  $y \mapsto D_\theta f(t, y)$  is Lipschitz continuous uniformly on  $t, \theta, \omega$ ;
- the barrier  $S$  satisfies:
  - a.  $\operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |V_t| < \infty$ ;
  - b.  $S \in \mathbb{H}^{1,2}$  and  $\operatorname{ess\,sup}_{(t,\theta,\omega) \in [0,T] \times [0,T] \times \Omega} |D_\theta S_t| < \infty$ ;
- $\bar{\kappa}_\infty$  defined by (3.3.13) is finite.

Then, we have

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right] = O(h + \lambda^2 h^{3/2}).$$

**Remark 3.4.1.** Notice that if  $A = 0$  in  $(\mathbf{H}_2\text{-i})$ , then  $V_t = D_t S_t$ , so the hypothesis of the boundedness of  $V$  on the barrier  $S$  can be removed.

Combining the above Theorem 3.4.1 with Theorem 3.3.5, we immediately deduce a bound on the global error.

**Corollary 3.4.2.** Under same hypotheses hold as Theorem 3.4.1, let  $\lambda = h^{-3/8}$ , we get the estimation of global error

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right]^{1/2} = O(h^{3/8}).$$

Notice that this convergence is better than that in [105] (order 1/4) and compared with [15], although this convergence holds at a slightly slower rate (order 3/8 instead of 1/2): however, instead of assuming high regularity of the Markovian barrier, we require milder regularity conditions for the barrier and for generator, which allows for a wider scope of applications in option pricing. The tightness of the upper bound in Corollary 3.4.2 is an open question: numerical tests on American Put option in the following section shows that the rate may be faster than  $h^{3/8}$  in some situations.

**Corollary 3.4.3.** Assume the same assumptions as Theorem 3.4.1 except  $\bar{\kappa}_\infty = +\infty$ , then

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right] = O(\lambda h + \lambda^{5/2} h^2 + \lambda^2 h^{3/2}).$$



Letting  $\lambda = h^{-1/2}$ , the global error is upper bounded as follows:

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{h,\lambda} - Y_{t_i})^2 \right]^{1/2} = O(h^{1/4}).$$

We retrieve the global rate from [105], but using this time a penalized scheme.

### 3.4.2 Numerical experiments

In this section, we provide a numerical test built for American Put in a risk neutral setting. Precisely, we lock the parameters of Put as following: the underlying asset is under Black-Scholes model with the constant interest rate  $r = 3\%$  and volatility  $\sigma = 0.2$ , the strike price is fixed at  $K = 100$ , the maturity  $T = 1$ . About the approximation approach, we apply Monte-Carlo method with empirical regression as in [69]. The number of trajectories  $M = 10^4$ , the number of discretization times  $N = 10^3$ , the degree  $l = 7$  of global polynomials as basis functions.

In the first subsection, we focus on simulating the penalized price function  $u^\lambda(t, \cdot)$  given by  $u^\lambda(t, X_t) = Y_t^{\lambda,h}$  and the behavior of penalization error with increasing  $\lambda$ . We also give 2 different illustrations with constant initial process value  $X_0 = 100$  and random one  $X_0 \sim U(50, 150)$ , where the latter brings robustness to the numerical scheme. Here the time step is  $h = T/N$  with  $N = 10^3$ .

In the second subsection, we show that, with fixed order of penalty  $\lambda = O(h^{-3/8})$ , changing only the time discretization step  $h$ , the error of MC simulation decreases in  $h$  more rapidly than the rate proved in Corollary 3.4.2. The reference value is given by a 1000-steps Binomial tree.

**Price function of American Put** In Figure 3.1, the global simulation of  $u^\lambda$  is very close to the reference value for large  $\lambda$ . Theorems 3.3.5 and 3.3.4 are verified numerically in the following sense: the RMSE at bottom-left shows effectively the tracking error decreases as  $\lambda$  increases; at the top-right, the penalisation term remains bounded (with small oscillations around the exercise boundary), it explodes for small  $x$  but this is due to the lack of samples in that region (see the sample distribution at bottom-right).

In Figure 3.2, we report the same quantities but with random  $X_0$ : in comparison with Figure 3.1, we observe that the penalisation term behaves much better, for a wide range of values of  $x$ , and we have noticed that the simulation scheme is globally more robust.

**Global discretization error** We take  $X_0 = 100$ ,  $M = 10^4$ ,  $\lambda = h^{-3/8}$ , and we change time discretization step  $h = T/N$ . We monitor the simulation error by MC regression, namely, we track

$$\sup_{0 \leq i \leq N} \frac{1}{M} \sum_{m=1}^M (\hat{Y}_{t_i}^{h,\lambda(h),m} - Y_{t_i}^{Bin,m})^2 \quad (3.4.3)$$

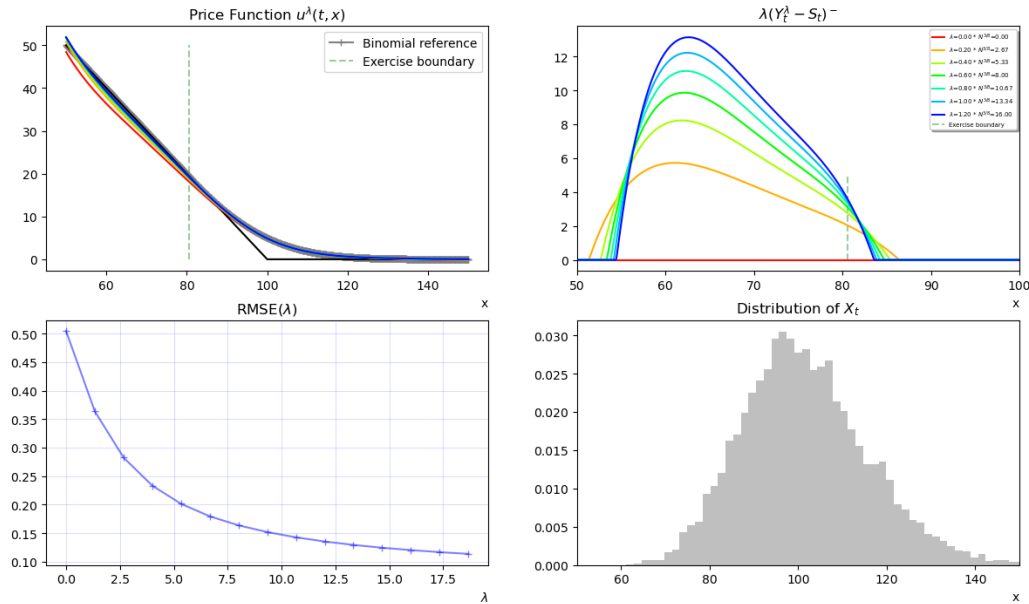


Figure 3.1 – The 4 panels are as following: at  $t = 0.5$ , the price function  $u^\lambda(t, \cdot)$  at top-left, the penalisation term  $\lambda[u^\lambda(t, \cdot) - (K - \cdot)^+]^-$  at top-right, the RMSE (compared with binomial reference) as a function of  $\lambda$  at bottom-left, the distribution of MC samples of  $X_t$  at  $t = 0.5$  at bottom-right.

as a function of  $h$ . The reference value is given by a 1000-steps Binomial tree. The numerical results are reported in Figure 3.3. This shows, in this example, our penalized scheme converges to the reference value more rapidly than what we have proved theoretically in Corollary 3.4.2; the error turns out to be roughly constant below  $h = 0.01$  due to the limit of capacity of other chosen simulation parameters (e.g.  $M$  and  $l$ ). On the one hand, these experiments show that our numerical scheme is efficient and accurate. On the other hand, the convergence order seems to be close to 1 in this specific case, which can not be explained by our current analysis. Further improvement is left to future works.

### 3.4.3 Proof of Theorem 3.4.1

In this subsection, we focus on the error w.r.t. the time step  $h$ , while the penalty parameter  $\lambda$  is fixed. To alleviate notations, we remove the notation  $\lambda$  in the processes, i.e. we write  $Y, Z, Y^h, \mathcal{E}_i^f$  instead of  $Y^\lambda, Z^\lambda, Y^{\lambda, h}, \mathcal{E}_i^{f, \lambda}$ , it will be clear in the context. Besides, it is essential to keep track of the impact of  $\lambda$  on the different constants arising in the error analysis. For this, we will use a  $C$  as a generic constant (changing from line to line), whose values do depend on  $T, f, \xi, S$  and other universal parameters, but not on  $\lambda$ . W.l.o.g, we assume in the following  $h \leq 1$ .

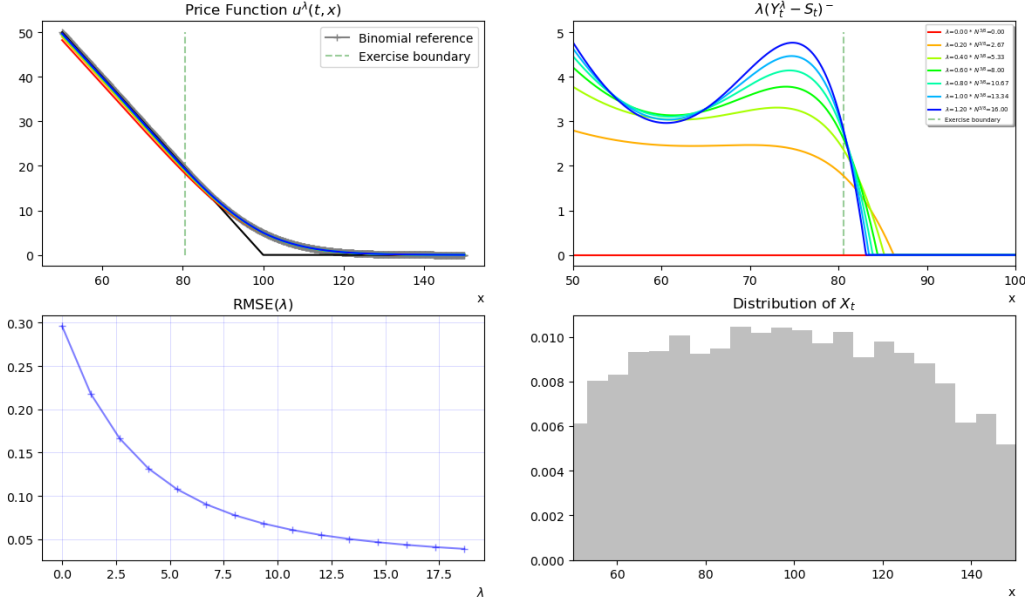


Figure 3.2 – The 4 panels are as following: at  $t = 0.5$ , the price function  $u^\lambda(t, \cdot)$  at top-left, the penalization term  $\lambda[u^\lambda(t, \cdot) - (K - \cdot)^+]^-$  at top-right, the RMSE (compared with binomial reference) as a function of  $\lambda$  at bottom-left, the distribution of MC samples of  $X_t$  at  $t = 0.5$  at bottom-right.

▷ *Step 1:* For any  $0 \leq i \leq N - 1$ , define the function

$$\mathcal{V}_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}, y \mapsto y - f^\lambda(t_i, y)h.$$

To avoid any confusion, we will keep writing  $f^\lambda$  to insist on the dependence of  $\lambda$ . Observed that  $\mathcal{V}_i$  is a  $\mathcal{F}_{t_i}$ -adapted stochastic mapping since the generator  $f^\lambda$  is random. Then one has

$$\begin{aligned} \mathcal{V}_i(Y_{t_i}^h) &= \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^h \right], & \mathcal{V}_i(Y_{t_i}) &= \mathbb{E}_{t_i} \left[ Y_{t_{i+1}} + \mathcal{E}_i^f \right], \\ Y_{t_i}^h &= \mathcal{V}_i^{-1} \left( \mathbb{E}_{t_i} \left[ Y_{t_{i+1}}^h \right] \right), & Y_{t_i} &= \mathcal{V}_i^{-1} \left( \mathbb{E}_{t_i} \left[ Y_{t_{i+1}} + \mathcal{E}_i^f \right] \right). \end{aligned} \quad (3.4.4)$$

The invertibility of  $\mathcal{V}_i$  (in the  $y$ -variable) is justified in the following lemma.

**Lemma 3.4.4.** *Suppose  $f$  fulfills  $(\mathbf{H}_1$ -iii) and  $(\mathbf{H}_m)$ , then  $\mathcal{V}_i$  is invertible. In addition,  $\forall h \leq \frac{1}{2C_{Lip}^f}$ , the function  $\mathcal{V}_i^{-1}$  is  $(1 + 2C_{Lip}^f h)$ -Lipschitz continuous uniformly in  $i$ , namely  $\forall 0 \leq i \leq N - 1$  and  $\forall w, w' \in \mathbb{R}$ ,*

$$|\mathcal{V}_i^{-1}(w) - \mathcal{V}_i^{-1}(w')| \leq (1 + 2C_{Lip}^f h)|w - w'|.$$

*Proof.* We first consider the simple case where  $f \equiv 0$  and denote  $\mathcal{V}_i^0(y) = y - \lambda(y - S_{t_i})^- h$ . Then  $(\mathcal{V}_i^0)^{-1}$  is well defined since  $\mathcal{V}_i^0$  is continuously increasing in  $y$ . Without loss of generality,

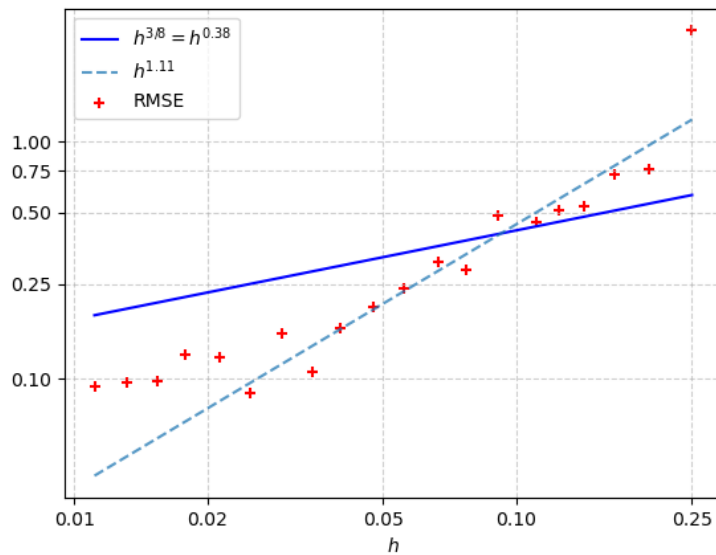


Figure 3.3 – For  $4 \leq N \leq 100$ , we report the estimation error (3.4.3). The solid blue line represents the expected rate  $h^{3/8}$  and the dashed blue line is fitted by linear regression based on the tracked errors, the later is at around  $C h^{1.11}$ .

suppose  $y \geq y'$ :

$$0 \leq \mathcal{V}_i^0(y) - \mathcal{V}_i^0(y') = \begin{cases} y - y', & \text{if } y, y' \geq S_{t_i}, \\ (1 + \lambda h)(y - y'), & \text{if } y, y' < S_{t_i}, \\ y - (1 + \lambda h)y' + \lambda h S_{t_i}, & \text{if } y \geq S_{t_i} \geq y'. \end{cases}$$

In the third case,  $y - (1 + \lambda h)y' + \lambda h S_{t_i} \geq y - (1 + \lambda h)y' + \lambda h y' = y - y'$ . It follows that

$$\mathcal{V}_i^0(y) - \mathcal{V}_i^0(y') \geq y - y', \forall y \geq y',$$

so  $(\mathcal{V}_i^0)^{-1}$  is 1-Lipschitz uniformly in  $\lambda$ .

Now let  $f^\lambda(t, y) = f(t, y) + \lambda(y - S_{t_i})^-$  with  $C_{Lip}^f$ -Lipschitz continuous  $f$ . Under  $(\mathbf{H}_m)$ ,  $\mathcal{V}_i$  preserves the same monotonicity as  $\mathcal{V}_i^0$ , so  $\mathcal{V}_i^{-1}$  is also well defined. Observe that the advertised result is equivalent to prove that for any  $y, y' \in \mathbb{R}$ ,

$$(1 + 2C_{Lip}^f h) |\mathcal{V}_i(y) - \mathcal{V}_i(y')| \geq |y - y'|.$$

We start by writing the decomposition of  $\mathcal{V}_i$ :

$$\begin{aligned} |\mathcal{V}_i(y) - \mathcal{V}_i(y')| &= |y - f(t_i, y)h - \lambda(y - S_{t_i})^- h - (y' - f(t_i, y')h - \lambda(y' - S_{t_i})^- h)| \\ &\geq |\mathcal{V}_i^0(y) - \mathcal{V}_i^0(y')| - |f(t_i, y) - f(t_i, y')| h \\ &\geq (1 - C_{Lip}^f h) |y - y'|. \end{aligned}$$

Now it's easy to show that,  $(1 + 2C_{Lip}^f h)(1 - C_{Lip}^f h) \geq 1$ , for all  $h \leq \frac{1}{2C_{Lip}^f}$ .  $\square$

▷ *Step 2:* From (3.4.4) and the previous lemma, the sequence  $(\mathbb{E}[(Y_{t_i}^h - Y_{t_i})^2])_{0 \leq i \leq N}$  satisfies

$$\begin{aligned} \mathbb{E}[(Y_{t_i}^h - Y_{t_i})^2] &= \mathbb{E} \left[ \left( \mathcal{V}_i^{-1} \left( \mathbb{E}_{t_i} [Y_{t_{i+1}}^h] \right) - \mathcal{V}_i^{-1} \left( \mathbb{E}_{t_i} [Y_{t_{i+1}} + \mathcal{E}_i^f] \right) \right)^2 \right] \\ &\leq (1 + Ch)^2 \mathbb{E} \left[ \left( \mathbb{E}_{t_i} [Y_{t_{i+1}}^h - Y_{t_{i+1}}] - \mathbb{E}_{t_i} [\mathcal{E}_i^f] \right)^2 \right] \\ &\leq (1 + Ch)^2 \left\{ (1 + h) \mathbb{E} [|Y_{t_{i+1}}^h - Y_{t_{i+1}}|^2] + \left( 1 + \frac{1}{h} \right) \mathbb{E} [\mathbb{E}_{t_i} [\mathcal{E}_i^f]^2] \right\} \\ &\leq (1 + Ch) \mathbb{E} [(Y_{t_{i+1}}^h - Y_{t_{i+1}})^2] + \frac{C}{h} \mathbb{E} [\mathbb{E}_{t_i} [\mathcal{E}_i^f]^2] \end{aligned}$$

where we have used Young inequality at the third line and  $h \leq 1 \wedge \frac{1}{2C_{Lip}^f}$  at the last line. Gronwall lemma combined with  $Y_{t_N}^h = Y_{t_N} = \xi$  gives,

$$\mathbb{E}[(Y_{t_i}^h - Y_{t_i})^2] \leq C \sum_{j=i}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_j} [\mathcal{E}_j^f]^2].$$

Hence, for  $h$  small enough,

$$\sup_{0 \leq i \leq N} \mathbb{E}[(Y_{t_i}^h - Y_{t_i})^2] \leq C \sum_{j=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_j} [\mathcal{E}_j^f]^2]. \quad (3.4.5)$$

▷ *Step 3:* In this step, we prove the above upper bound is  $O(h + \lambda^2 h^{3/2})$ . Globally, we will separate the sum into several parts and then investigate each of them. The general integrals are easy to deal with. The term with  $(Y_s - S_s)^-$  will give rise to a local time contribution which is the hardest to analyze. We first divide the perturbation  $\mathcal{E}_i^f$  into  $I_i$  and  $II_i$ : from (3.4.2),

$$\begin{aligned} \mathcal{E}_i^f &= \int_{t_i}^{t_{i+1}} (f(s, Y_s) - f(t_i, Y_{t_i})) \mathbf{d}s + \lambda \int_{t_i}^{t_{i+1}} ((Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-) \mathbf{d}s \\ &=: I_i + \lambda II_i. \end{aligned}$$

In the following calculus, we will deduce that,

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [I_i]^2] \leq O(h), \quad (3.4.6)$$

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] \leq O(h^{3/2}). \quad (3.4.7)$$

With (3.4.6) and (3.4.7) at hand and in view of (3.4.5), we directly complete the proof of Theorem 3.4.1.

▷ *Step 4: Estimation for  $I_i$ .* In fact, we have

$$I_i = \int_{t_i}^{t_{i+1}} [f(t_i, Y_s) - f(t_i, Y_{t_i})] \mathbf{d}s + \int_{t_i}^{t_{i+1}} [f(s, Y_s) - f(t_i, Y_s)] \mathbf{d}s.$$

For all  $s \in [t_i, t_{i+1})$ ,

$$\begin{aligned} & |f(t_i, Y_s) - f(t_i, Y_{t_i})| \\ & \leq C_{Lip}^f \left( \int_{t_i}^s |f(u, Y_u)| \mathbf{d}u + \lambda \int_{t_i}^s (Y_u - S_u)^- \mathbf{d}u + \left| \int_{t_i}^s Z_u \mathbf{d}B_u \right| \right) \\ & \leq C \left( \sup_{t_i \leq s < t_{i+1}} (|f(s, 0)| + |Y_s| + \lambda(Y_s - S_s)^-) h + \sup_{t_i \leq s < t_{i+1}} \left| \int_{t_i}^s Z_u \mathbf{d}B_u \right| \right). \end{aligned}$$

Thanks to the bound of  $f(\cdot, 0)$ , the estimate (3.5.3) and Theorem 3.3.4, we have

$$\sup_{\lambda > 0} \mathbb{E} \left[ \sup_{0 \leq s \leq T} (|f(s, 0)| + |Y_s| + \lambda(Y_s - S_s)^-)^2 + \int_0^T |Z_s|^2 \mathbf{d}s \right] < \infty. \quad (3.4.8)$$

Therefore, by Burkholder-Davis-Gundy (BDG in short) inequality, we obtain

$$\begin{aligned} & \sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} \left[ \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} |f(t_i, Y_s) - f(t_i, Y_{t_i})| \mathbf{d}s \right]^2 \right] \\ & \leq C \sum_{i=0}^{N-1} h \mathbb{E} \left[ \sup_{t_i \leq s < t_{i+1}} |f(t_i, Y_s) - f(t_i, Y_{t_i})|^2 \right] \\ & \leq C \sum_{i=0}^{N-1} h \left( h^2 + \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} |Z_u|^2 \mathbf{d}u \right] \right) \\ & \leq Ch \left( 1 + \mathbb{E} \left[ \int_0^T |Z_t|^2 \mathbf{d}t \right] \right) = O(h). \end{aligned}$$

In addition, thanks to  $\frac{1}{2}$ -Hölder continuous in  $t$  of  $f$ , clearly

$$\sum_i \frac{1}{h} \mathbb{E} \left[ \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} |f(s, Y_s) - f(t_i, Y_s)| \mathbf{d}s \right]^2 \right] = O(h).$$

We have proved (3.4.6).

▷ *Step 5: Estimation for  $II_i$ .* We apply Tanaka's formula and denote  $L$  the local time of  $Y - S$  at level 0, by taking the conditional expectation, it follows

$$\begin{aligned} & \mathbb{E}_{t_i} [(Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-] \\ & = \mathbb{E}_{t_i} \left[ \int_{t_i}^s \mathbf{1}_{\{Y_u \leq S_u\}} (f(u, Y_u) + U_u) \mathbf{d}u + \lambda \int_{t_i}^s (Y_u - S_u)^- \mathbf{d}u \right] \\ & \quad + \mathbb{E}_{t_i} \left[ \frac{1}{2} [L_s - L_{t_i}] + \int_{t_i}^s \mathbf{1}_{\{Y_u \leq S_u\}} \mathbf{d}A_u \right] \end{aligned}$$

where  $A$  is the non-decreasing process of  $S$  in  $(\mathbf{H}_2\text{-i})$ . Consider the integral on small interval  $[t_i, t_{i+1})$ , we get:

$$\begin{aligned} & \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} [(Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-] \mathbf{d}s \right] \\ & \leq h \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} |f(u, Y_u) + U_u| \mathbf{d}u \right] \\ & \quad + \lambda h \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} (Y_u - S_u)^- \mathbf{d}u \right] + \frac{h}{2} \mathbb{E}_{t_i} [\Delta L_i] + h \mathbb{E}_{t_i} [\Delta A_i], \end{aligned}$$

where  $\Delta L_i := L_{t_{i+1}} - L_{t_i}$ ,  $\Delta A_i := A_{t_{i+1}} - A_{t_i}$ . We denote, for  $0 \leq i \leq N-1$ ,

$$F_i := \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} |f(u, Y_u) + U_u| \mathbf{d}u, \quad E_i := \int_{t_i}^{t_{i+1}} (Y_u - S_u)^- \mathbf{d}u.$$

All in all, we get

$$\begin{aligned} \sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] & \leq 4 \sum_{i=0}^{N-1} \frac{1}{h} \left\{ h^2 \mathbb{E} [\mathbb{E}_{t_i} [F_i]^2] + \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] \right. \\ & \quad \left. + \frac{h^2}{4} \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \right\}. \end{aligned} \quad (3.4.9)$$

For the first term of right hand side, by Jensen's inequality and Cauchy-Schwarz inequality, it follows

$$\begin{aligned} \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [F_i]^2] & \leq Ch^2 \sum_{i=0}^{N-1} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} [|f(u, 0)|^2 + |Y_u|^2 + |U_u|^2] \mathbf{d}u \right] \\ & \leq Ch^2 (\|f(\cdot, 0)\|_{\mathbb{H}^2}^2 + \|Y\|_{\mathbb{H}^2}^2 + \|U\|_{\mathbb{H}^2}^2) \leq Ch^2 \end{aligned} \quad (3.4.10)$$

using again (3.4.8). For the second sum in (3.4.9), we use again the bound in Theorem 3.3.4:

$$\mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] \leq h^2 \mathbb{E} \left[ \left( \sup_{u \in [t_i, t_{i+1}]} (Y_u - S_u)^- \right)^2 \right] \leq h^2 \frac{\bar{\kappa}_\infty^2}{\lambda^2},$$

which implies

$$\sum_{i=0}^{N-1} \frac{1}{h} \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] = O(h^2). \quad (3.4.11)$$

Now we deal with the third term and the fourth in sum (3.4.9), applying Lemma 3.6.3:

$$\begin{aligned}
& \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \\
& \leq \sqrt{6} h \mathbb{E} [L_T^2]^{1/2} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta L_i]^2 \right]^{1/2} \\
& \quad + \sqrt{6} h \mathbb{E} [A_T^2]^{1/2} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right]^{1/2}. \tag{3.4.12}
\end{aligned}$$

An easy computation leads to  $\mathbb{E} [L_T^2 + A_T^2] < \infty$  thanks to  $f(\cdot, 0), U, V, S \in \mathcal{S}^2$  and Theorem 3.3.1. Thus it remains to estimate  $\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|\Delta L_i|^2] \right]^{1/2} + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|\Delta A_i|^2] \right]^{1/2}$ . Denote  $\Delta(Y_i - S_i) := Y_{t_{i+1}} - S_{t_{i+1}} - (Y_{t_i} - S_{t_i})$ . From Itô-Tanaka formula, we have,  $\forall 0 \leq i \leq N-1$ ,

$$\begin{aligned}
\frac{1}{2} \mathbb{E}_{t_i} [\Delta L_i] &= \mathbb{E}_{t_i} [\Delta(Y_i - S_i)^-] - \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{Y_u \leq S_u} [f(u, Y_u) + U_u] \mathbf{d}u \right] \\
&\quad - \lambda \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} (Y_u - S_u)^- \mathbf{d}u \right] - \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{Y_u \leq S_u} \mathbf{d}A_u \right] \\
&\leq \mathbb{E}_{t_i} [|\Delta(Y_i - S_i)|] - \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{Y_u \leq S_u} [f(u, Y_u) + U_u] \mathbf{d}u \right], \\
\Delta(Y_i - S_i) &= - \int_{t_i}^{t_{i+1}} (f(s, Y_s) + U_s) \mathbf{d}s - \lambda \int_{t_i}^{t_{i+1}} (Y_s - S_s)^- \mathbf{d}s \\
&\quad - \Delta A_i + \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s.
\end{aligned}$$

So, using Theorem 3.3.4, we deduce

$$\begin{aligned}
\mathbb{E}_{t_i} [\Delta L_i] &\leq C \left\{ h \mathbb{E}_{t_i} \left[ \sup_{0 \leq t \leq T} (|f(t, 0)| + |Y_t| + |U_t|) \right] + h + \mathbb{E}_{t_i} [\Delta A_i] \right. \\
&\quad \left. + \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right| \right] \right\}.
\end{aligned}$$

Thanks to the assumptions  $f(\cdot, 0), U, \xi, S \in \mathcal{S}^2$ , the random variable  $\zeta := \sup_{0 \leq t \leq T} (|f(t, 0)| + |Y_t| + |U_t|)$  belongs to  $\mathbb{L}^2$  and  $\{\mathbb{E}_{t_i} [\zeta], 0 \leq i \leq N-1\}$  is a  $\mathbb{L}^2$ -martingale, so

$$\begin{aligned}
\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta L_i]^2 \right] &\leq C \left\{ h^2 + h^2 \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\zeta]^2 \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right] \right. \\
&\quad \left. + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right|^2 \right] \right] \right\}. \tag{3.4.13}
\end{aligned}$$



We handle the terms at the right-hand side of (3.4.13) separately. First, applying Doob's inequality and Jensen's inequality,

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\zeta]^2 \right] \leq 4 \sup_{0 \leq i \leq N-1} \mathbb{E} [\mathbb{E}_{t_i} [\zeta]^2] \leq 4\mathbb{E} [\zeta^2] < \infty. \quad (3.4.14)$$

Second, denote  $\Delta S_i := S_{t_{i+1}} - S_{t_i}$ , then  $\Delta A_i = \Delta S_i - \int_{t_i}^{t_{i+1}} U_s \mathbf{d}s - \int_{t_i}^{t_{i+1}} V_s \mathbf{d}B_s$ . With  $(\mathbf{H}_2\text{-ii})$ , we get similarly,

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right] &\leq 2 \left( \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|\Delta S_i|^2] \right] \right. \\ &\quad \left. + h^2 \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \sup_{0 \leq t \leq T} |U_t| \right]^2 \right] \right) \\ &\leq C(h + h^2 \|U\|_{\mathcal{G}}). \end{aligned} \quad (3.4.15)$$

Third, introduce the following notation of regularized PBSDE solution:

For any fixed  $\lambda > 0$ , assume  $(Y^{\lambda, \varepsilon}, Z^{\lambda, \varepsilon})$  be the solution of (4.1.2) with respect to  $(\xi, f^{\lambda, \varepsilon}, S)$  and

$$f^{\lambda, \varepsilon}(t, y, z) = f(t, y) + \lambda \delta_\varepsilon(y - S_t) \quad (3.4.16)$$

where  $\varepsilon > 0$ ,  $\delta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+$  s.t.  $\delta_\varepsilon(x) = \frac{(x-\varepsilon)^2}{4\varepsilon}$  for  $x \in [-\varepsilon, \varepsilon]$  and  $\delta_\varepsilon(x) = x^-$  for  $|x| > \varepsilon$ .

In the following context, we use  $(Y^\varepsilon, Z^\varepsilon)$  to denote  $(Y^{\lambda, \varepsilon}, Z^{\lambda, \varepsilon})$  for the fixed  $\lambda$ . Notice that the bound of  $Z^\varepsilon$  in Proposition 3.6.4 holds uniformly in  $t, \omega, \lambda, \varepsilon$ , thus applying BDG inequality we obtain

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right|^2 \right] \right] \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s^\varepsilon - V_s) \mathbf{d}B_s \right|^2 \right] \right] \\ &\leq C \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left( \int_{t_i}^{t_{i+1}} |Z_s^\varepsilon - V_s|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \leq Ch. \end{aligned} \quad (3.4.17)$$

So, with (3.4.13), (3.4.14), (3.4.15), (3.4.17), we get,

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta L_i]^2 \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right] \leq Ch. \quad (3.4.18)$$

Back to (3.4.12), we finally have

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \leq Ch^{3/2}. \quad (3.4.19)$$

All in all, combining with (3.4.10), (3.4.11), (3.4.19), we prove successfully (3.4.7). Wrapping up our arguments, the proof of Theorem 3.4.1 is finished.  $\square$

**Proof of Corollary 3.4.3.** Using the estimation  $\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^2 \right] = O(\frac{1}{\lambda})$  in Theorem 3.3.3, the estimation of  $I_i$  in (3.4.6) becomes

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [I_i]^2] = O(\lambda h).$$

Regarding (3.4.7) we obtain

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] \leq O(\sqrt{\lambda} h^2 + h^{3/2})$$

because the estimation in (3.4.11) becomes

$$\sum_{i=0}^{N-1} \frac{1}{h} \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] = O(\sqrt{\lambda} h^2);$$

and (3.4.19) turns to

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] = O(\sqrt{\lambda} h^2 + h^{3/2}).$$

□

### 3.4.4 Alternative approach for estimating martingales

There are also some alternative estimations for (3.4.13) in Step 5 of Proof of Theorem 3.4.1 using different method, namely, we can also prove,

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right)^2 \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} V_s \mathbf{d}B_s \right)^2 \right] \leq Ch \log \left( 1 + \frac{T}{h} \right)$$

which yields a slower convergence rate compared with (3.4.18) in  $\mathbb{L}^2$  as follows

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta L_i]^2 \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right] \leq C \left( h + h^2 + h \log \left( 1 + \frac{T}{h} \right) \right) \quad (3.4.20)$$

This proof is given by a changing view: Maximal inequality and the Orlicz norm (which is also known as the extended  $\mathbb{L}^p$  norm). In fact, in the calculation of (3.4.13), instead of focusing on  $\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right]^2 \right]$ , we dominate, by Jensen inequality,

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right)^2 \right].$$

For estimating the above martingale term, we introduce the following result:

**Proposition 3.4.5.** For some  $\chi \in \mathbb{H}^2$ , define  $\mathcal{M} := \left\{ \left( \int_{t_i}^{t_{i+1}} \chi_s \mathbf{d}B_s \right)^2, 0 \leq i \leq N-1 \right\}$ . If one of the following conditions is satisfied:

(a):  $\operatorname{ess\,sup}_{(t,\omega) \in [0,T]} |\chi_t| \leq \bar{\chi}$  for some constant  $\bar{\chi}$ ;

(b): there exists a family  $((\chi_t^\varepsilon)_{0 \leq t \leq T})_{\varepsilon > 0}$  s.t.  $\forall \varepsilon > 0, \chi^\varepsilon \in \mathbb{H}^2, \lim_{\varepsilon \rightarrow 0^+} \|\chi - \chi^\varepsilon\|_{\mathbb{H}^2} \rightarrow 0$  and

$$\sup_{\varepsilon > 0} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |\chi_t^\varepsilon| \leq \bar{\chi},$$

then we have,

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathcal{M}_i \right] \leq \frac{6\bar{\chi}^2 \log(2)}{\log(3/2)} h \log \left( 1 + \frac{T}{h} \right).$$

Thus, we apply first, (a) for estimating  $\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} V_s \mathbf{d}B_s \right)^2 \right]$  which is direct thanks to the uniform bound of  $V$ . Second, with notation (3.4.16), we apply (b) for estimating the other integral of  $Z$ . In Proposition 3.6.4, we have proved the convergence in  $\mathbb{H}^2$  of  $Z^\varepsilon \rightarrow Z$  and the uniform boundedness of  $Z^\varepsilon$  on  $[0, T] \times \Omega$ . Again with the boundedness of  $V$ , let us denote  $\bar{\chi} := \sup_{\varepsilon > 0} \operatorname{ess\,sup}_{[0,T] \times \Omega} |Z_t^\varepsilon - V_t| < \infty$ . It follows

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} V_s \mathbf{d}B_s \right)^2 \right] + \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left( \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right)^2 \right] = O \left( h \log \left( 1 + \frac{T}{h} \right) \right).$$

Notice that  $\bar{\chi}$  is a uniform upper bound on  $\lambda$  due to the uniform boundedness of  $Z^\varepsilon$ , so in general we apply Proposition 3.4.5 for each fixed  $\lambda > 0$  and the independence of  $\bar{\chi}$  in  $\lambda$  allows us to show a uniform estimate for all  $\lambda > 0$ . As a consequence, combine with (3.4.13), (3.4.14), (3.4.15), for some constant  $C$  depending on  $T, f, S$ , we have (3.4.20). Back to (3.4.12),

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \leq C \left( h^{3/2} \sqrt{\log \left( 1 + \frac{T}{h} \right)} \right). \quad (3.4.21)$$

All in all, combining with (3.4.10), (3.4.11), (3.4.21), we can prove  $\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] = O \left( h^{3/2} \sqrt{\log \left( 1 + \frac{T}{h} \right)} \right)$ . Wrapping up our arguments, we can have

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right] = O \left( h + \lambda^2 h^{3/2} \sqrt{\log \left( 1 + \frac{T}{h} \right)} \right)$$

as an alternative result of Theorem 3.4.1.  $\square$

**Proof of Proposition 3.4.5.** Here we give the proof for (b) since one can always use  $\chi$  as an approximation for itself.

▷ *Step 1:* Prove  $\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} |\mathcal{M}_i| \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} |\mathcal{M}_i^\varepsilon| \right]$ . In fact,  $\forall \varepsilon > 0$ , since  $\chi^\varepsilon \rightarrow \chi$  in  $\mathbb{H}^2$ , applying Cauchy-Schwarz and Jensen inequality and Itô isometry, we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} |\mathcal{M}_i - \mathcal{M}_i^\varepsilon| \right] &\leq \mathbb{E} \left[ \sum_{i=0}^{N-1} \left| \left( \int_{t_i}^{t_{i+1}} \chi_s \mathbf{d}B_s \right)^2 - \left( \int_{t_i}^{t_{i+1}} \chi_s^\varepsilon \mathbf{d}B_s \right)^2 \right| \right] \\ &\leq \mathbb{E} \left[ \sum_{i=0}^{N-1} \left( \int_{t_i}^{t_{i+1}} (\chi_s + \chi_s^\varepsilon) \mathbf{d}B_s \right)^2 \right]^{1/2} \mathbb{E} \left[ \sum_{i=0}^{N-1} \left( \int_{t_i}^{t_{i+1}} (\chi_s - \chi_s^\varepsilon) \mathbf{d}B_s \right)^2 \right]^{1/2} \\ &= \|\chi + \chi^\varepsilon\|_{\mathbb{H}^2} \|\chi - \chi^\varepsilon\|_{\mathbb{H}^2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

▷ *Step 2:* Orlicz norm estimation.

**Lemma 3.4.6** ([31, Section 4]). *For  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+, x \mapsto e^x - 1$ , define the Orlicz norm  $\|\cdot\|_\psi := \inf \left\{ c > 0 : \mathbb{E} \left[ \psi \left( \frac{|\cdot|}{c} \right) \right] \leq 1 \right\}$ , then for any family  $(X_i)_{1 \leq i \leq n}$  of measurable positive random variables s.t.  $\sup_{1 \leq i \leq n} \|X_i\|_\psi < \infty$ , we have,*

$$\mathbb{E} \left[ \sup_{1 \leq i \leq n} |X_i| \right] \leq \frac{\log 2}{\log(3/2)} \log(1+n) \sup_{1 \leq i \leq n} \|X_i\|_\psi.$$

Applying Lemma 3.4.6 to  $\left( \mathcal{M}_i^\varepsilon := \left( \int_{t_i}^{t_{i+1}} \chi_s^\varepsilon \mathbf{d}B_s \right)^2 \right)_{0 \leq i \leq N-1}$ , we have

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathcal{M}_i^\varepsilon \right] \leq \frac{\log 2}{\log(3/2)} \log \left( 1 + \frac{T}{h} \right) \sup_{0 \leq i \leq N-1} \|\mathcal{M}_i^\varepsilon\|_\psi.$$

▷ *Step 3:* It only leaves us to compute  $\sup_{0 \leq i \leq N-1} \|\mathcal{M}_i^\varepsilon\|_\psi$ . Since  $\left( \int_{t_i}^t \chi_s^\varepsilon \mathbf{d}B_s \right)_{t_i \leq t \leq t_{i+1}}$  is a continuous martingale and thanks to the bound  $\bar{\chi}$ , the Bernstein inequality implies (see [122, p. 153 Exercise 3.16]),

$$\mathbb{P}(\mathcal{M}_i^\varepsilon \geq m) \leq \mathbb{P} \left( \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^t \chi_s^\varepsilon \mathbf{d}B_s \right| \geq \sqrt{m} \right) \leq 2 \exp \left( -\frac{m}{2\bar{\chi}^2 h} \right).$$

Then, with choosing  $c = 6\bar{\chi}^2 h$ , we have

$$\mathbb{E} \left[ \psi \left( \frac{\mathcal{M}_i^\varepsilon}{c} \right) \right] = \mathbb{E} \left[ e^{\frac{\mathcal{M}_i^\varepsilon}{c}} - 1 \right] = \frac{1}{c} \int_{\mathbb{R}^+} e^{\frac{m}{c}} \mathbb{P}(\mathcal{M}_i^\varepsilon \geq m) \mathbf{d}m \leq \frac{2}{\frac{c}{2\bar{\chi}^2 h} - 1} = 1.$$

The Orlicz norm of  $\mathcal{M}_i$  is dominated by a constant  $c$  s.t.  $\mathbb{E} \left[ \psi \left( \frac{\mathcal{M}_i}{c} \right) \right] \leq 1$  which implies  $\sup_{0 \leq i \leq N-1} \|\mathcal{M}_i^\varepsilon\|_\psi \leq 6\bar{\chi}^2 h, \forall \varepsilon > 0$  since  $\bar{\chi}$  is a uniform in  $\varepsilon$ . Therefore, we get the desired result

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathcal{M}_i \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathcal{M}_i^\varepsilon \right] \leq \frac{6\bar{\chi}^2 \log 2}{\log(3/2)} h \log \left( 1 + \frac{T}{h} \right).$$

□

## 3.5 Appendix

### 3.5.1 Proof of Lemma 3.2.2

Let us denote by  $C$  any finite positive constant occurring in the proof: it will depend only on  $T$ , the bounds on  $\mu, \sigma$  and the regularity constant of  $g$ . For any  $0 \leq t < s \leq T$ ,

$$\mathbb{E}_t [|S_s - S_t|] \leq C \left( |s - t|^{\frac{1}{2}} + \mathbb{E}_t [(e^{C|X_s|} + e^{C|X_t|}) |X_s - X_t|] \right).$$

Since  $\mu$  and  $\sigma$  are bounded, it is standard to show that

$$\mathbb{E}_t \left[ \sup_{t \leq r \leq T} e^{C|X_r|} \right] \leq C e^{C|X_t|},$$

with a possible increase of  $C$ . Besides,

$$\left( \mathbb{E}_t [|X_t - X_s|^2] \right)^{\frac{1}{2}} \leq C |t - s|^{\frac{1}{2}}$$

for some new constant  $C$ . Therefore, using Cauchy-Schwarz inequality gives

$$\sup_{0 \leq t < s \leq T} \frac{\mathbb{E}_t [|S_s - S_t|]}{\sqrt{s - t}} \leq C \sup_{0 \leq t \leq T} e^{C|X_t|}.$$

Take the power  $p$  and the expectation gives the advertised result. □

### 3.5.2 A priori estimates for general BSDEs

In this subsection, we will give some result of a priori estimate for general BSDEs.

**Proposition 3.5.1** ([116, Proposition 5.2, p.358, with  $p \geq 2$ ]). *Suppose  $(Y, Z) \in \mathcal{S}^2 \times \mathbb{H}^2$  satisfying the scalar equation*

$$Y_t = \xi + \int_t^T \mathbf{d}F_s - \int_t^T Z_s \mathbf{d}B_s$$

where  $\xi \in \mathbb{L}^2$ ,  $F \in \mathcal{S}^2$  and  $F$  is a scalar process with bounded variation a.s..

Let  $p \geq 2$ , we assume there exist stochastic processes  $(D, R, N, \Upsilon)$  satisfying

$$\mathbf{d}D_t + Y_t \mathbf{d}F_t \leq \mathbf{d}R_t + |Y_t| \mathbf{d}N_t + |Y_t|^2 \mathbf{d}\Upsilon_t + \frac{\eta}{2} |Z_t|^2 \mathbf{d}t, \quad (3.5.1)$$

where

- $D, R, N$  are progressively measurable, increasing, continuous processes with  $D_0 = R_0 = N_0 = 0$ ;
- $\Upsilon$  is a progressively measurable process with bounded variation and  $\Upsilon_0 = 0$ ;
- $\eta < 1$ .

If  $\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t e^{\Upsilon_t}|^p \right] < \infty$ , then one has the a priori estimate,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq t \leq T} |e^{\Upsilon_t} Y_t|^p \right] + \mathbb{E} \left[ \left( \int_0^T e^{2\Upsilon_s} \mathbf{d}D_s \right)^{p/2} \right] + \mathbb{E} \left[ \left( \int_0^T e^{2\Upsilon_s} |Z_s|^2 \mathbf{d}s \right)^{p/2} \right] \\ & \leq C_{p,\eta} \mathbb{E} \left[ |e^{\Upsilon_T} Y_T|^p + \left( \int_0^T e^{2\Upsilon_s} \mathbf{d}R_s \right)^{p/2} + \left( \int_0^T e^{\Upsilon_s} \mathbf{d}N_s \right)^p \right], \end{aligned}$$

for some constant  $C_{p,\eta}$  depending only on  $p, \eta$ .

### 3.5.3 A priori $\mathbb{L}^p$ -estimates for RBSDE

**Proposition 3.5.2** (A priori estimate for  $p \geq 2$ ). *Let  $p \geq 2$  and let  $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$  be a solution in  $\mathcal{S}_{cont}^p \times \mathbb{H}^p \times \mathcal{S}_{cont}^p$  of the RBSDE w.r.t.  $(\xi, f, S)$ . If  $(\mathbf{H}_1)$  is fulfilled, then there exists a constant  $C$  (depending only on  $T, f, p$ ) such that*

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p + \left( \int_0^T |Z_t|^2 \mathbf{d}t \right)^{p/2} + K_T^p \right] \\ & \leq C \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |f(t, 0, 0)| \mathbf{d}t \right)^p + \sup_{0 \leq t \leq T} (S_t^+)^p \right]. \end{aligned}$$

*Proof.* The proof will be done in several steps.

▷ *Step 1:* Apply Proposition 3.5.1 on (4.1.1) using obvious notations, we get

$$\begin{aligned} Y_t \mathbf{d}F_t &= Y_t f(t, Y_t, Z_t) \mathbf{d}t + Y_t \mathbf{d}K_t \\ &\leq S_t \mathbf{d}K_t + |Y_t| \left( |f(t, 0, 0)| + C_{Lip}^f |Y_t| + C_{Lip}^f |Z_t| \right) \mathbf{d}t \\ &\leq (S_t)^+ \mathbf{d}K_t + |Y_t| |f(t, 0, 0)| \mathbf{d}t + (C_{Lip}^f + (C_{Lip}^f)^2) |Y_t|^2 \mathbf{d}t + \frac{1}{4} |Z_t|^2 \mathbf{d}t, \end{aligned}$$

so (3.5.1) holds with

$$D = 0, \quad \mathbf{d}R_t = (S_t)^+ \mathbf{d}K_t, \quad \mathbf{d}N_t = |f(t, 0, 0)| \mathbf{d}t, \quad \mathbf{d}\Upsilon_t = (C_{Lip}^f + \frac{1}{2} (C_{Lip}^f)^2) \mathbf{d}t, \quad \eta = \frac{1}{2}.$$

For the rest of proof, let us denote by  $C$  any generic constant which value may change from line to line, but it depends only on  $T, f, p$ . Therefore, from Proposition 3.5.1, we get the intermediate a priori estimate for  $Y$  and  $Z$ , with  $p \geq 2$ , and for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] + \mathbb{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |f(t, 0, 0)| dt \right)^p + \left( \int_0^T (S_t)^+ dK_t \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |f(t, 0, 0)| dt \right)^p \right] + \frac{C^2}{4\varepsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |S_t^+|^p \right] + \varepsilon \mathbb{E} [|K_T|^p]. \end{aligned} \quad (3.5.2)$$

▷ *Step 2:* We now estimate  $K_T$ , simply by rewriting the formulae,

$$\begin{aligned} K_T &= Y_0 - \xi - \int_0^T f(t, Y_t, Z_t) dt + \int_0^T Z_t dB_t \\ &\leq \int_0^T |f(t, 0, 0)| dt + C \sup_{0 \leq t \leq T} |Y_t| + C \int_0^T |Z_t| dt + \left| \int_0^T Z_t dB_t \right|. \end{aligned}$$

Then we take the expectation on both sides and get

$$\mathbb{E} [|K_T|^p] \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] + C \mathbb{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} \right] + C \mathbb{E} \left[ \left( \int_0^T |f(t, 0, 0)| dt \right)^p \right],$$

where we use BDG inequality at the last step. Combining with (3.5.2) and the above, and taking  $\varepsilon$  small enough, we get the desired result.  $\square$

### 3.5.4 Proof of Theorem 3.3.1

#### 3.5.4.1 Existence, uniqueness for $(Y, Z, K)$ and a priori estimates for $(Y^\lambda, Z^\lambda, K^\lambda)$

We follow a standard routine to prove the existence of penalized BSDE with increasing penalty parameter  $\lambda$ . The case of  $p = 2$  is given in [52, Section 6] and the case of  $1 \leq p < 2$  is stated in [82]. Here we extend their results to  $p \geq 2$ .

▷ *Step 0:* Define  $f^\lambda(t, y, z) := f(t, y, z) + \lambda(y - S_t)^-$ . The existence and uniqueness of  $(Y^\lambda, Z^\lambda)$  in  $\mathbb{L}^p$  is obvious thanks to the Lipschitz generator  $f^\lambda$ . The difficult part is related to prove  $\lambda$ -uniform bounds. We could try to use estimates from [23]: on the one hand,  $f^\lambda$  satisfies the monotonicity condition [23, Assumption (H3)]

$$(y - y')(f^\lambda(t, y, z) - f^\lambda(t, y', z)) \leq C_{Lip}^f (y - y')^2$$

uniformly in  $\lambda$ , but on the other hand, the condition [23, Assumption (A)] writes as

$$\text{Sign}(y) f^\lambda(t, y, z) \leq |f^\lambda(t, 0, 0)| + C_{Lip}^f |y| + C_{Lip}^f |z|$$

where  $|f^\lambda(t, 0, 0)|$  does strongly depend on the penalty parameter  $\lambda$ . So although the monotonicity condition is satisfied, we can not get an uniform upper bound in  $\lambda$  for PBSDE using the argument of [23]. Instead, we establish a simple proof as in Proposition 3.5.2 using Proposition 3.5.1.

▷ *Step 1:* Given  $(Y^\lambda, Z^\lambda, K^\lambda)$  the solution of (4.1.2), since

$$Y_t^\lambda \mathbf{d}K_t^\lambda \leq S_t \mathbf{d}K_t^\lambda$$

( $K^\lambda$  increases only when  $Y_t^\lambda < S_t$ ) we can use again the argument in Proposition 3.5.2 by replacing the process  $K$  by  $K^\lambda$ . Note that all generic constants  $C$  do not depend in  $\lambda$ . Then we get the desired a priori estimate

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda|^p + \left( \int_0^T |Z_s^\lambda|^2 \mathbf{d}s \right)^{p/2} + |K_T^\lambda|^p \right] \leq c, \quad (3.5.3)$$

where  $c$  depends only on  $f, S, \xi, T, p$ .

From the comparison theorem for standard BSDE (see [52, Theorem 2.2]), we have  $Y_t^\lambda \leq Y_t^{\lambda'}, 0 \leq t \leq T, a.s.$  for any  $0 < \lambda \leq \lambda'$ ; so there exists a process  $Y$  s.t.  $Y_t^\lambda \rightarrow Y_t$ , as  $\lambda \rightarrow \infty, 0 \leq t \leq T, a.s..$  By Fatou's lemma and estimation (3.5.3), we easily get

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] \leq C$$

using again the convention of generic constant  $C$  independent on  $\lambda$ . Then, from the dominated convergence, we have  $\mathbb{E} \left[ \int_0^T |Y_t^\lambda - Y_t|^p \mathbf{d}t \right] \rightarrow 0$  as  $\lambda \rightarrow +\infty$ .

▷ *Step 2:* Let us justify that  $(Y^\lambda, Z^\lambda, K^\lambda)$  is a Cauchy sequence in  $\mathcal{S}_{cont}^p \times \mathbb{H}^p \times \mathcal{S}^p$ . We apply again Proposition 3.5.1 on  $Y^{\lambda'} - Y^\lambda$ . In fact, define

$$\Delta Y_t^\lambda := Y_t^{\lambda'} - Y_t^\lambda, \Delta Z_t^\lambda := Z_t^{\lambda'} - Z_t^\lambda, \Delta f_t := f(t, Y_t^{\lambda'}, Z_t^{\lambda'}) - f(t, Y_t^\lambda, Z_t^\lambda), \Delta K_t^\lambda = K_t^{\lambda'} - K_t^\lambda,$$

then, we have

$$\begin{aligned} \Delta Y_t^\lambda (\Delta f_t \mathbf{d}t + \mathbf{d}\Delta K_t^\lambda) &\leq \left( C_{Lip}^f + (C_{Lip}^f)^2 \right) |\Delta Y_t^\lambda|^2 \mathbf{d}t + \frac{1}{4} |\Delta Z_t^\lambda|^2 \mathbf{d}t \\ &\quad + (Y_t^\lambda - S_t)^- \mathbf{d}K_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- \mathbf{d}K_t^\lambda. \end{aligned} \quad (3.5.4)$$

Let  $\eta = 1/2$  and

$$D = 0, \quad \mathbf{d}R_t = (Y_t^\lambda - S_t)^- \mathbf{d}K_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- \mathbf{d}K_t^\lambda, \quad N = 0, \quad \mathbf{d}\Upsilon_t = \left( C_{Lip}^f + (C_{Lip}^f)^2 \right) \mathbf{d}t.$$



Then we deduce the a priori estimate for  $(\Delta Y_t^\lambda, \Delta Z_t^\lambda)$ :

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[ \left( \int_0^T |\Delta Z_t^\lambda|^2 dt \right)^{p/2} \right] \\
& \leq C \mathbb{E} \left[ \left( \int_0^T (Y_t^\lambda - S_t)^- dK_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- dK_t^\lambda \right)^{p/2} \right] \\
& \leq C \sqrt{\mathbb{E} [|K_T^\lambda|^p]} \sqrt{\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(Y_t^{\lambda'} - S_t)^-|^p \right]} \\
& \quad + C \sqrt{\mathbb{E} [|K_T^{\lambda'}|^p]} \sqrt{\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right]}. \tag{3.5.5}
\end{aligned}$$

Assume for a while that the following limit holds, its proof is postponed afterwards.

**Lemma 3.5.3.** *Under the assumptions of Theorem 3.3.1, we have, for  $p \geq 2$ ,*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right] = 0.$$

Then, from (3.5.5), we get, as  $\lambda, \lambda' \rightarrow \infty$ , the following convergence:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[ \left( \int_0^T |\Delta Z_t^\lambda|^2 dt \right)^{p/2} \right] \rightarrow 0.$$

Moreover, set

$$\Delta K_t^\lambda = \Delta Y_0^\lambda - \Delta Y_t^\lambda - \int_0^t \Delta f_s ds + \int_0^t \Delta Z_s^\lambda dB_s,$$

we have the following estimation for  $\Delta K^\lambda$ ,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta K_t^\lambda|^p \right] \leq C \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[ \left( \int_0^T |\Delta Z_t^\lambda|^2 dt \right)^{p/2} \right] \right\}. \tag{3.5.6}$$

So we deduce that  $(Y^\lambda, Z^\lambda, K^\lambda)_\lambda$  is a Cauchy sequence in  $\mathcal{S}_{cont}^p \times \mathbb{H}^p \times \mathcal{S}^p$ , denote by  $(Y, Z, K)$  its limit (for  $Y$ , it coincides with the previous monotone limit).

▷ *Step 3:* it remains to verify  $(Y, Z, K)$  is the solution to (4.1.1), this can be done by applying exactly the same arguments as [52, p.722]. We are done with the first result (3.3.1) of Theorem 3.3.1.

### 3.5.4.2 Proof of the bound (3.3.2) on penalisation error

We use again Proposition 3.5.1 as for Inequality (3.5.4). Before, we analysed the difference  $Y^{\lambda'} - Y^\lambda$ ; now we handle  $Y - Y^\lambda$ , the decomposition of which is really similar just doing as if  $\lambda' = +\infty$  in the algebra of equations. Replacing  $(Y^{\lambda'}, Z^{\lambda'}, K^{\lambda'})$  by  $(Y, Z, K)$  in the formulas (3.5.4) of  $(\Delta Y^\lambda, \Delta Z^\lambda, \Delta K^\lambda)$ , we get

$$\Delta Y_t^\lambda (\Delta f_t \mathbf{d}t + \mathbf{d}\Delta K_t^\lambda) \leq \left( C_{Lip}^f + (C_{Lip}^f)^2 \right) |\Delta Y_t^\lambda|^2 \mathbf{d}t + \frac{1}{4} |\Delta Z_t^\lambda|^2 \mathbf{d}t + (Y_t^\lambda - S_t)^- \mathbf{d}K_t.$$

Then, applying Proposition 3.5.1 with  $\eta = 1/2$  and  $D = 0$ ,  $\mathbf{d}R_t = (Y_t^\lambda - S_t)^- \mathbf{d}K_t$ ,  $N = 0$ ,  $\mathbf{d}\Upsilon_t = \left( C_{Lip}^f + (C_{Lip}^f)^2 \right) \mathbf{d}t$ , we get

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[ \left( \int_0^T |\Delta Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} \right] \leq C \mathbb{E} \left[ \left( \int_0^T (Y_t^\lambda - S_t)^- \mathbf{d}K_t \right)^{p/2} \right]$$

for some constant  $C$  independent on  $\lambda$ . We obtain a similar estimate on  $\Delta K^\lambda$  by considering the analogous of (3.5.6). The proof of (3.3.2) is complete.  $\square$

### 3.5.4.3 Proof of Lemma 3.5.3

We extend the proof of [52, Lemma 6.1] when  $p = 2$  to the case  $p \geq 2$ . As in the proof of Theorem 3.3.3, let  $(\tilde{Y}^\lambda, \tilde{Z}^\lambda)$  be the solution (in  $\mathcal{S}_{cont}^p \times \mathbb{H}^p$ ) of the linear BSDE (3.3.8) which has the explicit representation

$$\tilde{Y}_t^\lambda = \mathbb{E}_t \left[ e^{-\lambda(T-t)} \xi + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T e^{-\lambda(s-t)} S_s \mathbf{d}s \right].$$

$\triangleright$  *Step 1:* We first prove that, as  $\lambda \rightarrow \infty$ ,  $\tilde{Y}_\tau^\lambda \rightarrow \xi \mathbf{1}_{\{\tau=T\}} + S_\tau \mathbf{1}_{\{\tau < T\}}$  in  $\mathbb{L}^p$ , for any  $\tau \in \mathcal{T}_{0,T}$ . In fact, we consider the following convergences, which hold in both *a.s.* and in  $\mathbb{L}^p$ ,

$$\lim_{\lambda \rightarrow +\infty} e^{-\lambda(T-\tau)} \xi = \xi \mathbf{1}_{\{\tau=T\}}, \quad (3.5.7)$$

$$\lim_{\lambda \rightarrow +\infty} \lambda \int_\tau^T e^{-\lambda(t-\tau)} S_t \mathbf{d}t = S_\tau \mathbf{1}_{\{\tau < T\}}. \quad (3.5.8)$$

The first identity (3.5.7) is trivial since  $\xi \in \mathbb{L}^p$ . To prove (3.5.8), write

$$\begin{aligned} S_\tau &= (1 - e^{-\lambda(T-\tau)}) S_\tau + e^{-\lambda(T-\tau)} S_\tau = S_\tau \lambda \int_\tau^T e^{-\lambda(t-\tau)} \mathbf{d}t + e^{-\lambda(T-\tau)} S_\tau, \\ \lambda \int_\tau^T e^{-\lambda(t-\tau)} S_t \mathbf{d}t - S_\tau \mathbf{1}_{\{\tau < T\}} &= \lambda \int_\tau^T e^{-\lambda(t-\tau)} (S_t - S_\tau) \mathbf{d}t + e^{-\lambda(T-\tau)} S_\tau \mathbf{1}_{\{\tau < T\}}. \end{aligned}$$

Obviously, since  $S \in \mathcal{S}_{cont.}^p$ , from the dominated convergence theorem,  $\lim_{\lambda \rightarrow +\infty} e^{-\lambda(T-\tau)} S_\tau \mathbf{1}_{\{\tau < T\}} \rightarrow 0$  holds a.s. and in  $\mathbb{L}^p$ . Moreover, on  $\{\tau < T\}$ , we have

$$\lambda \int_\tau^T e^{-\lambda(t-\tau)} |S_t - S_\tau| \mathbf{d}t \leq \int_0^\infty e^{-s} |S_{T \wedge (\tau+s/\lambda)} - S_\tau| \mathbf{d}s.$$

Therefore, thanks to the continuity of  $S$ , letting  $\lambda \rightarrow +\infty$ , the right-hand converges to 0 a.s.; thus, from the dominated convergence theorem, with  $S \in \mathcal{S}_{cont.}^p$ , we get

$$\lim_{\lambda \rightarrow +\infty} \lambda \int_\tau^T e^{-\lambda(t-\tau)} |S_t - S_\tau| \mathbf{d}t = 0, \text{ in } \mathbb{L}^p.$$

We have proved (3.5.7)-(3.5.8). Hence, Jensen inequality ensures that the conditional expectation of  $e^{-\lambda(T-\tau)} \xi + \lambda \int_\tau^T e^{-\lambda(t-\tau)} S_t \mathbf{d}t$  given  $\mathcal{F}_\tau$  converges to  $\xi \mathbf{1}_{\{\tau=T\}} + S_\tau \mathbf{1}_{\{\tau < T\}}$  in  $\mathbb{L}^p$ .

To complete our proof, we only have to justify that  $\int_\tau^T e^{-\lambda(t-\tau)} f(t, Y_t^\lambda, Z_t^\lambda) \mathbf{d}t$  converges to 0 in  $\mathbb{L}^p$ . Indeed, from the Hölder inequality,

$$\mathbb{E} \left[ \left| \int_\tau^T e^{-\lambda(t-\tau)} f(t, Y_t^\lambda, Z_t^\lambda) \mathbf{d}t \right|^p \right] \leq \left( \frac{1}{q\lambda} \right)^{p/q} \mathbb{E} \left[ \int_0^T |f(t, Y_t^\lambda, Z_t^\lambda)|^p \mathbf{d}t \right]$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ . The expectation on the right-hand side is bounded uniformly in  $\lambda$ , because of the uniform bounds (3.5.3) and the assumptions on  $f$ ; thus the expectation on the left-hand goes to 0 as announced.

▷ *Step 2:* From [43, p. 220], we pass from "Y<sub>τ</sub> ≥ S<sub>τ</sub>, a.s. for any stopping time τ" to "Y<sub>t</sub> ≥ S<sub>t</sub>, for all times t ∈ [0, T], a.s." .

Notice that from the comparison theorem [54, Theorem 4.1], we have  $Y_t \geq Y_t^\lambda \geq \tilde{Y}_t^\lambda, 0 \leq t \leq T, a.s.$ , so  $(Y_t^\lambda - S_t)^- \downarrow 0$  a.s.; Dini's theorem yields the convergence uniformly in  $t$ , and owing to the dominated convergence theorem, we obtain the statement of Lemma 3.5.3.  $\square$

### 3.5.5 Appendix: Proof of Table 3.1

#### 3.5.5.1 Example: Call and Put in the market with 2 interest rates

Suppose the lending rate  $r$  and the borrowing rate  $R$  with  $r_t < R_t, 0 \leq t \leq T$ . The generator  $f$  is defined as following

$$f(t, y, z) = -r_t y - z \sigma_t^{-1} (\mu_t - r_t) + (R_t - r_t) (y - z \sigma_t^{-1})^-.$$

For unidimensional Call (resp. Put) option, we have the following Itô-Tanaka expansion: a.s.

$$\begin{aligned} \mathbf{d}S_t &= \mathbf{d}(X_t - K^{stk})^+ = \mathbf{1}_{\{X_t > K^{stk}\}} \mu_t X_t \mathbf{d}t + \mathbf{1}_{\{X_t > K^{stk}\}} X_t \sigma_t \mathbf{d}B_t + \frac{1}{2} \mathbf{d}L_t^{K^{stk}}(X), \\ &\left( \text{resp. for Put option,} \right. \\ \mathbf{d}S_t &= \mathbf{d}(K^{stk} - X_t)^+ = -\mathbf{1}_{\{X_t \leq K^{stk}\}} \mu_t X_t \mathbf{d}t - \mathbf{1}_{\{X_t \leq K^{stk}\}} X_t \sigma_t \mathbf{d}B_t + \frac{1}{2} \mathbf{d}L_t^{K^{stk}}(X), \end{aligned}$$

where  $L^{K^{stk}}(X)$  is the local time process of  $X$  at level  $K^{stk}$ . So the decompositions of Call (resp. Put) option are as follows:

$$\begin{aligned} U_t &= \mathbf{1}_{\{X_t > K^{stk}\}} \mu_t X_t, \quad V_t = \mathbf{1}_{\{X_t > K^{stk}\}} X_t \sigma_t, \quad A_t = \frac{1}{2} L_t^{K^{stk}}(X). \\ &\left( \text{resp. for Put option,} \right. \\ U_t &= -\mathbf{1}_{\{X_t \leq K^{stk}\}} \mu_t X_t, \quad V_t = -\mathbf{1}_{\{X_t \leq K^{stk}\}} X_t \sigma_t, \quad A_t = \frac{1}{2} L_t^{K^{stk}}(X). \end{aligned} \quad (3.5.9)$$

Denote  $\bar{\kappa}_\infty^C, \bar{\kappa}_\infty^P$  the bound in Theorem 3.3.5 for Call, Put option respectively and Let  $\nu = |R|_\infty \vee |\sigma^{-1}(\mu - r)|_\infty$ , then

$$\begin{aligned} \bar{\kappa}_\infty^C &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} (f(t, S_t, V_t) + U_t)^- \\ &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} \left( -r_t (X_t - K^{stk})^+ - \mathbf{1}_{\{X_t > K^{stk}\}} (\mu_t - r_t) X_t \right. \\ &\quad \left. + (R_t - r_t) \left( (X_t - K^{stk})^+ - X_t \mathbf{1}_{\{X_t > K^{stk}\}} \right)^- + \mathbf{1}_{\{X_t > K^{stk}\}} \mu_t X_t \right)^- \\ &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} \left( r_t K^{stk} \mathbf{1}_{\{X_t > K^{stk}\}} + (R_t - r_t) \left( (X_t - K^{stk})^+ - X_t \mathbf{1}_{\{X_t > K^{stk}\}} \right)^- \right)^- \\ &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} (R_t K^{stk} \mathbf{1}_{\{X_t > K^{stk}\}})^- \leq e^{\nu T} |R^-|_\infty K^{stk}. \end{aligned}$$

For Put Option,

$$\begin{aligned} \bar{\kappa}_\infty^P &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} \left( -r_t (K^{stk} - X_t)^+ + \mathbf{1}_{\{X_t \leq K^{stk}\}} (\mu_t - r_t) X_t \right. \\ &\quad \left. + (R_t - r_t) \left( (K^{stk} - X_t)^+ + X_t \mathbf{1}_{\{X_t \leq K^{stk}\}} \right)^- - \mu_t X_t \mathbf{1}_{\{X_t \leq K^{stk}\}} \right)^- \\ &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} (-r_t K^{stk} \mathbf{1}_{\{X_t \leq K^{stk}\}} + (R_t - r_t) \left( (K^{stk} - X_t)^+ + X_t \mathbf{1}_{\{X_t \leq K^{stk}\}} \right)^-)^- \\ &= \text{ess sup}_{[0, T] \times \Omega} e^{\nu t} (-r_t K^{stk} \mathbf{1}_{\{X_t \leq K\}})^- \leq e^{\nu T} |r^+|_\infty K^{stk}. \end{aligned}$$

### 3.5.5.2 Example in $dim = 1$ : Call and Put in perfect market

Suppose  $d = 1$ . Here we discuss the case with  $r \geq 0$  and  $r < 0$ , so for convenience, we will work with  $Y^\nu$  with  $\nu = \text{ess sup}_{[0, T] \times \Omega} |r_t| \vee |\sigma_t^{-1}(\mu_t - r_t)|$  and the linear self-financing generator

$$f(t, y, z) = -r_t y - z \sigma_t^{-1}(\mu_t - r_t).$$

The decompositions of Call and Put option as (3.5.9) remain true in the perfect market. So,

for Call option,  $U_t = \mu_t X_t \mathbf{1}_{X_t > K^{stk}}$ ,  $V_t = \sigma_t X_t \mathbf{1}_{X_t > K^{stk}}$ ,

$$\begin{aligned} \bar{\kappa}_\infty^C &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} (f(t, S_t, V_t) + U_t)^- \\ &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} (-r_t (X_t - K^{stk})^+ - \mathbf{1}_{\{X_t > K^{stk}\}} X_t (\mu_t - r_t) + \mathbf{1}_{\{X_t > K^{stk}\}} \mu_t X_t)^- \\ &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} (r_t K^{stk} \mathbf{1}_{\{X_t > K^{stk}\}})^- \\ &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} |(r_t)^-| K^{stk} \mathbf{1}_{\{X_t > K^{stk}\}} \leq e^{\nu T} |r^-|_\infty K^{stk}. \end{aligned}$$

For Put option,  $U_t = -\mu_t X_t \mathbf{1}_{X_t \leq K^{stk}}$ ,  $V_t = -\sigma_t X_t \mathbf{1}_{X_t \leq K^{stk}}$

$$\begin{aligned} \bar{\kappa}_\infty^P &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} (-r_t (K^{stk} - X_t)^+ + \mathbf{1}_{\{X_t \leq K^{stk}\}} X_t (\mu_t - r_t) - \mathbf{1}_{\{X_t \leq K^{stk}\}} \mu_t X_t)^- \\ &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} (-r_t K^{stk} \mathbf{1}_{\{X_t \leq K^{stk}\}})^- \\ &= \operatorname{ess\,sup}_{[0,T] \times \Omega} e^{\nu t} |(r_t)^+| K^{stk} \mathbf{1}_{\{X_t \leq K^{stk}\}} \leq e^{\nu T} |r^+|_\infty K^{stk} \end{aligned}$$

### 3.5.5.3 Examples in multi dimension: Call and Put in perfect market

Define

- the inner product of column vectors  $x \cdot y := x^\top y$ ;
- the product of column vectors  $(ax) = (a^1 x^1, \dots, a^d x^d)^\top$  where  $a, x \in \mathbb{R}^d$  and  $a^i$  is  $i$ -th component of  $a$ .

Denote the vector  $\mathbf{1}_d = (1, \dots, 1)_d^\top \in \mathbb{R}^d$ ,  $\mathbf{d}X_t = (\mathbf{d}X_t^1, \dots, \mathbf{d}X_t^d)^\top \in \mathbb{R}^d$ . See more multidimensional options in [24, 127].

**Call on spread.** Consider the pay-off  $S_t = (X_t^1 - X_t^2 - K^{stk})^+$ , then  $S$  is a generalised semi-martingale with a local time  $L$  such that

$$\mathbf{d}S_t = \mathbf{1}_{\{X_t^1 - X_t^2 > K\}} a \cdot \mathbf{d}X_t + \frac{1}{2} \mathbf{d}L_t^{K^{stk}} (X^1 - X^2)$$

with vector  $a := (1, -1)^\top \in \mathbb{R}^2$ . Then

$$U_t := \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} (a X_t) \cdot \mu_t, \quad V_t := \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} (a X_t) \cdot \sigma_t$$

and with this  $(f, S)$ ,  $\bar{\kappa}_\infty$  is well defined. Indeed,

$$\begin{aligned} f(t, S_t, V_t) + U_t &= -r_t S_t - V_t \sigma_t^{-1} (\mu_t - r_t \mathbf{1}_2) + U_t \\ &= -r_t (a \cdot X_t - K^{stk}) \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} - \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} (a X_t) \cdot (\mu_t - r_t \mathbf{1}_2) \\ &\quad + \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} (a X) \cdot \mu_t \\ &= -r_t (a \cdot X_t - K^{stk}) \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} + r_t \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}} (a X_t) \cdot \mathbf{1}_2 \\ &= r_t K^{stk} \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}}. \end{aligned}$$

Hence,  $\bar{\kappa}_\infty := \text{ess sup}_{[0,T] \times \Omega} e^{\nu t} (r_t K^{stk} \mathbf{1}_{\{X_t^1 - X_t^2 > K^{stk}\}})^- \leq e^{\nu T} |r^-|_\infty K^{stk}$ .

**Call on maximum.** Consider the pay-off  $S_t = (\max(X_t^1, X_t^2) - K^{stk})^+$ . Apply Itô-Tanaka,

$$\begin{aligned} \mathbf{d}(X_t^1 \vee X_t^2) &= \mathbf{d}X_t^1 + \mathbf{d}(X_t^1 - X_t^2)^- = \mathbf{d}X_t^1 - \mathbf{1}_{\{X_t^1 \leq X_t^2\}} \mathbf{d}(X_t^1 - X_t^2) + \frac{1}{2} \mathbf{d}L_t^0(X^1 - X^2) \\ &= (\mathbf{1}_{\{X_t^1 > X_t^2\}}, \mathbf{1}_{\{X_t^1 \leq X_t^2\}}) \mathbf{d}X_t + \frac{1}{2} \mathbf{d}L_t^0(X^1 - X^2) \end{aligned}$$

and

$$\begin{aligned} \mathbf{d}S_t &= \mathbf{1}_{\{X_t^1 \vee X_t^2 > K^{stk}\}} \mathbf{d}(X_t^1 \vee X_t^2) + \frac{1}{2} \mathbf{d}L_t^{K^{stk}}(X^1 \vee X^2) \\ &= (a_t X) \cdot (\mu_t \mathbf{d}t + \sigma_t \mathbf{d}B_t) + \frac{1}{2} [\mathbf{1}_{\{X_t^1 \vee X_t^2 > K^{stk}\}} \mathbf{d}L_t^0(X^1 - X^2) + \mathbf{d}L_t^{K^{stk}}(X^1 \vee X^2)] \end{aligned}$$

with matrix  $a_t := \left( \mathbf{1}_{\{X_t^1 \vee X_t^2 > K^{stk}, X_t^1 > X_t^2\}}, \mathbf{1}_{\{X_t^1 \vee X_t^2 > K^{stk}, X_t^1 \leq X_t^2\}} \right)^\top$ . So  $S$  is also a generalised semi-martingale with an increasing process where

$$U_t = (a_t X_t) \cdot \mu_t, \quad V_t = (a_t X_t) \cdot \sigma_t.$$

Hence

$$\begin{aligned} f(t, S_t, V_t) + U_t &= -r_t (X_t^1 \vee X_t^2 - K^{stk})^+ - V_t \sigma^{-1} (\mu_t - r_t \mathbf{1}_2) + U_t \\ &= -r_t (a_t \cdot X_t - K^{stk}) - (a_t X) \cdot (\mu_t - r_t \mathbf{1}_2) + (a_t X_t) \cdot \mu_t \\ &= -r_t a_t \cdot (X_t - K^{stk} \mathbf{1}_2) + r_t (a_t X_t) \cdot \mathbf{1}_2 \\ &= r_t K^{stk} a_t \cdot \mathbf{1}_2 = r_t K^{stk} \mathbf{1}_{X_t^1 \vee X_t^2 > K^{stk}}. \end{aligned}$$

It follows by  $\bar{\kappa}_\infty := \text{ess sup}_{[0,T] \times \Omega} e^{\nu t} (r_t K^{stk} \mathbf{1}_{\{X_t^1 \vee X_t^2 > K^{stk}\}})^- \leq e^{\nu T} K^{stk} |r^-|_\infty$ .

**Put on minimum.** Consider  $S_t = (K^{stk} - \min(X_t^1, X_t^2))^+$ . Similarly,

$$\begin{aligned} \mathbf{d}(X_t^1 \wedge X_t^2) &= \mathbf{d}X_t^1 - \mathbf{d}(X_t^1 - X_t^2)^+ = \mathbf{d}X_t^1 - \mathbf{1}_{\{X_t^1 > X_t^2\}} \mathbf{d}(X_t^1 - X_t^2) - \frac{1}{2} \mathbf{d}L_t^0(X^1 - X^2) \\ &= \mathbf{1}_{\{X_t^1 \leq X_t^2\}} \mathbf{d}X_t^1 + \mathbf{1}_{\{X_t^1 > X_t^2\}} \mathbf{d}X_t^2 - \frac{1}{2} \mathbf{d}L_t^0(X^1 - X^2), \end{aligned}$$

then

$$\begin{aligned} \mathbf{d}S_t &= -\mathbf{1}_{\{X_t^1 \wedge X_t^2 \leq K^{stk}\}} \mathbf{d}(X_t^1 \wedge X_t^2) + \frac{1}{2} L_t^{K^{stk}}(X^1 \wedge X^2) \\ &= -(a_t X_t) \cdot (\mu_t \mathbf{d}t + \sigma_t \mathbf{d}B_t) + \frac{1}{2} [\mathbf{1}_{\{X_t^1 \wedge X_t^2 \leq K^{stk}\}} \mathbf{d}L_t^0(X^1 - X^2) + L_t^{K^{stk}}(X^1 \wedge X^2)] \end{aligned}$$

with matrix  $a_t := \left( \mathbf{1}_{\{X_t^1 \wedge X_t^2 \leq K^{stk}, X_t^1 \leq X_t^2\}}, \mathbf{1}_{\{X_t^1 \wedge X_t^2 \leq K^{stk}, X_t^1 > X_t^2\}} \right)^\top$ . So  $U_t = -(a_t X_t) \cdot \mu_t$  and  $V_t = -(a_t X_t) \cdot \sigma_t$ . Hence,

$$\begin{aligned} f(t, S_t, V_t) + U_t &= -r_t(K^{stk} - X_t^1 \wedge X_t^2)^+ - V_t \sigma^{-1}(\mu_t - r_t \mathbf{1}_2) + U_t \\ &= -r_t a_t \cdot (K^{stk} \mathbf{1}_2 - X_t) + (a_t X_t) \cdot (\mu_t - r_t \mathbf{1}_2) - (a_t X_t) \cdot \mu_t \\ &= -r_t K^{stk} a_t \cdot \mathbf{1}_2 + r_t a_t \cdot X_t - r_t (a_t X_t) \cdot \mathbf{1}_2 \\ &= -r_t K^{stk} a_t \cdot \mathbf{1}_2 = -r_t K^{stk} \mathbf{1}_{X_t^1 \wedge X_t^2 \leq K^{stk}}; \end{aligned}$$

it follows  $\bar{\kappa}_\infty := \operatorname{ess\,sup}_{[0, T] \times \Omega} e^{\nu t} (-r_t K^{stk} \mathbf{1}_{X_t^1 \wedge X_t^2 \leq K^{stk}})^- \leq e^{\nu T} K^{stk} |r^+|_\infty$ .

□

### 3.6 Appendix: Proof of Section 4

We first introduce some results from [54, Proposition 5.3] and we restrict to the case where  $f$  does not depend on  $z$ .

**Lemma 3.6.1.** *Assume  $\xi \in \mathbb{D}^{1,2}$ ,  $f$  is continuously differentiable in  $y$ , with uniformly bounded and continuous derivatives and such that*

- for each  $y \in \mathbb{R}$ ,  $f(\cdot, y)$  is in  $\mathbb{H}^{1,2}$ ;
- $\int_0^T \mathbb{E} \left[ \int_0^T |D_\theta f(t, Y_t)|^2 dt \right] d\theta < \infty$  and there exists a constant  $C$  s.t. for any  $\theta \in [0, T]$  a.e., for any  $t \in [0, T]$  and any  $(y_1, y_2) \in \mathbb{R}^2$ ,

$$|D_\theta f(t, \omega, y_1) - D_\theta f(t, \omega, y_2)| \leq C |y_1 - y_2|.$$

Then for  $\forall 1 \leq i \leq d$ , a version of the Malliavin derivatives  $\{(D_\theta^i Y_t, D_\theta^i Z_t)\}_{0 \leq \theta, t \leq T}$  satisfies the following linear BSDE,

$$\left\{ \begin{array}{l} D_\theta^i Y_t = D_\theta^i \xi + \int_t^T [\partial_y f(u, Y_u) D_\theta^i Y_u + D_\theta^i f(u, Y_u)] du \\ \quad - \int_t^T D_\theta^i Z_u dB_u, \quad 0 \leq \theta \leq t \leq T; \\ D_\theta^i Y_t = 0, \quad D_\theta^i Z_t = 0, \quad 0 \leq t < \theta \leq T. \end{array} \right. \quad (3.6.1)$$

Moreover,  $\{D_t Y_t\}_{0 \leq t \leq T}$  defined by (3.6.1) gives a version of  $\{Z_t\}_{0 \leq t \leq T}$ .

From [54, Proposition 2.2], (3.6.1) has closed form.

**Lemma 3.6.2.** *Let  $(\beta, \gamma)$  be bounded  $(\mathbb{R}, \mathbb{R}^d)$ -valued predictable processes,  $\phi$  be an element of  $\mathbb{H}^2$ , and  $\xi$  be an element of  $\mathbb{L}^2$ . Then the linear BSDE*

$$-dY_t = [\phi_t + Y_t \beta_t + Z_t \gamma_t] dt - Z_t dB_t, \quad Y_T = \xi,$$

has a unique solution  $(Y, Z)$  in  $\mathbb{H}^2 \times \mathbb{H}^2$  and  $Y_t$  is given by the following formula

$$Y_t = \mathbb{E} \left[ \xi \Gamma_T^t + \int_t^T \Gamma_s^t \phi_s \mathbf{d}s \mid \mathcal{F}_t \right], \text{ a.s.},$$

where  $\Gamma_s^t$  is the adjoint process defined for  $s \geq t$  by the linear FSDE

$$\mathbf{d}\Gamma_s^t = \Gamma_s^t [\beta_s \mathbf{d}s + \gamma_s^\top \mathbf{d}B_s], \quad \Gamma_t^t = 1.$$

**Lemma 3.6.3.** *Given an adapted and non-decreasing process  $\mathcal{L}$  starting from  $\mathcal{L}_0 = 0$ , for some time discretization  $0 = t_0 < t_1 < \dots < t_N = T$ , define  $\Delta \mathcal{L}_i := \mathcal{L}_{t_{i+1}} - \mathcal{L}_{t_i}, \forall 0 \leq i \leq N-1$ . Assume that  $\mathcal{L}_T$  is square integrable. Then we have*

$$\sum_{i=0}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta \mathcal{L}_i]^2] \leq \sqrt{6} \mathbb{E} [\mathcal{L}_T^2]^{1/2} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i]^2 \right]^{1/2}.$$

*Proof.* Because  $\mathcal{L}$  is non-decreasing, the square integrability of  $\mathcal{L}_T$  easily propagates to each  $\mathcal{L}_{t_i}$ . By Cauchy-Schwarz inequality, the above left hand side satisfies

$$\begin{aligned} \sum_{i=0}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta \mathcal{L}_i]^2] &\leq \mathbb{E} \left[ \left( \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right) \left( \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right) \right] \\ &\leq \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i]^2 \right]^{1/2} \mathbb{E} \left[ \left( \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right)^2 \right]^{1/2}. \end{aligned}$$

From Jensen's inequality and Young's inequality, and using that  $\mathcal{L}$  is non-decreasing, we get

$$\begin{aligned} \mathbb{E} \left[ \left( \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right)^2 \right] &= \mathbb{E} \left[ \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i]^2 \right] + 2 \mathbb{E} \left[ \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \mathbb{E}_{t_j} [\Delta \mathcal{L}_j] \right] \\ &\leq \mathbb{E} \left[ \sum_{i=0}^{N-1} (\Delta \mathcal{L}_i)^2 \right] + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \Delta \mathcal{L}_j] \\ &\leq \mathbb{E} \left[ \left( \sum_{i=0}^{N-1} \Delta \mathcal{L}_i \right)^2 \right] + 2 \mathbb{E} \left[ \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] (\mathcal{L}_T - \mathcal{L}_{t_{i+1}}) \right] \\ &\leq \mathbb{E} [\mathcal{L}_T^2] + 2 \mathbb{E} \left[ \mathcal{L}_T \left( \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right) \right] \\ &\leq 3 \mathbb{E} [\mathcal{L}_T^2] + \frac{1}{2} \mathbb{E} \left[ \left( \sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right)^2 \right] \end{aligned}$$

from where we get the desired result.  $\square$



**Proposition 3.6.4.** *Under  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$ , let  $(Y_t^\lambda, Z_t^\lambda)$  be solution of PBSDE (4.1.2) and  $(Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon})$  be the solution of BSDE with  $f^{\lambda,\varepsilon}$  as defined in (3.4.16). First, we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^\lambda - Y_t^{\lambda,\varepsilon}|^2 + \int_0^T |Z_t^\lambda - Z_t^{\lambda,\varepsilon}|^2 dt \right] = 0. \quad (3.6.2)$$

Second, for any fixed  $\lambda > 0$  and for all  $0 \leq i \leq N-1$ , denote  $M_i^\lambda := \int_{t_i}^{t_{i+1}} (Z_s^\lambda - V_s) dB_s$  and  $M_i^{\lambda,\varepsilon} := \int_{t_i}^{t_{i+1}} (Z_s^{\lambda,\varepsilon} - V_s) dB_s$ , then we have

$$\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [ |M_i^\lambda|^2 ] \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [ |M_i^{\lambda,\varepsilon}|^2 ] \right]. \quad (3.6.3)$$

Moreover, suppose the same assumptions as Theorem 3.4.1, then there exists a constant  $C$  which depends only on  $\xi, f, T$  s.t.

$$\sup_{\lambda > 0} \sup_{\varepsilon > 0} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |Z_t^{\lambda,\varepsilon}| \leq C. \quad (3.6.4)$$

*Proof.* Since  $\lambda$  is fixed, we simplify the notations  $(Y^\lambda, Z^\lambda), (Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon}), M^\lambda, M^{\lambda,\varepsilon}$  into  $(Y, Z), (Y^\varepsilon, Z^\varepsilon), M, M^\varepsilon$ .

**Proof of (3.6.2).** The existence and uniqueness of  $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{S}_{cont}^2 \times \mathbb{H}^2$  come from the Lipschitz continuity of  $f^{\lambda,\varepsilon}$  since the regularization does not change the bound of derivatives ( $\delta_\varepsilon$  is 1-Lipschitz continuous). The proof of (3.6.2) is straightforward from a priori estimate (see [54, Proposition 2.1]), combined with the easy bound  $\sup_{x \in \mathbb{R}} |\delta_\varepsilon(x) - x^-| \leq \frac{\varepsilon}{4}$ : it gives

$$\begin{aligned} \|Y - Y^\varepsilon\|_{\mathcal{S}^2}^2 + \|Z - Z^\varepsilon\|_{\mathbb{H}^2}^2 &\leq C_\lambda \mathbb{E} \left[ \int_0^T \lambda^2 |(Y_t - S_t)^- - \delta_\varepsilon(Y_t - S_t)|^2 dt \right] \\ &\leq \lambda^2 C_\lambda \varepsilon^2 \end{aligned}$$

with changing constants  $C_\lambda$  depending on  $\lambda$  and  $T, f, S, \xi$ .

**Proof of (3.6.3).** We have

$$\begin{aligned} &\left| \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [ |M_i|^2 ] - \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [ |M_i^\varepsilon|^2 ] \right] \right| \\ &\leq \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left| \mathbb{E}_{t_i} [ |M_i|^2 ] - \mathbb{E}_{t_i} [ |M_i^\varepsilon|^2 ] \right| \right] \\ &\leq \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \left\{ \mathbb{E}_{t_i} [ |M_i - M_i^\varepsilon| ] \mathbb{E}_{t_i} [ |M_i| + |M_i^\varepsilon| ] \right\} \right] \\ &\leq \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s - Z_s^\varepsilon) dB_s \right|^2 \right] \right]^{1/2} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [ |M_i| + |M_i^\varepsilon| ]^2 \right]^{1/2}. \quad (3.6.5) \end{aligned}$$

To the first factor in (3.6.5), apply BDG's inequality and Doob's inequality, it gives

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} (Z_s - Z_s^\varepsilon) \mathbf{d}B_s \right|^2 \right] \right] \\
& \leq C \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left( \int_{t_i}^{t_{i+1}} |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \\
& \leq C \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[ \left( \int_0^T |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \\
& \leq C \mathbb{E} \left[ \int_0^T |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

We now handle the second factor in (3.6.5), by showing that it is uniformly bounded in  $\varepsilon$ . We only need to consider the term  $M_i^\varepsilon$ . Proceeding with similar arguments as before, combined with (3.6.4) and (3.5.3), we get

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i^\varepsilon|^2] \right] \leq C \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \int_0^T |Z_s^\varepsilon - V_s|^2 \mathbf{d}s \right] < +\infty$$

from where we deduce (3.6.3).

**Proof of (3.6.4).** Apply Lemma 3.6.1 and Lemma 3.6.2 to  $(Y^\varepsilon, Z^\varepsilon)$ , so we can solve  $Z_t^\varepsilon$  in an explicit form. For any  $1 \leq i \leq d$ , a version of the Malliavin derivatives  $\{(D_\theta^i Y_t^\varepsilon, D_\theta^i Z_t^\varepsilon)\}_{0 \leq \theta, t \leq T}$  satisfies the following linear BSDE,

$$\begin{cases} D_\theta^i Y_t^\varepsilon = D_\theta^i \xi + \int_t^T [\partial_y f^{\lambda, \varepsilon}(u, Y_u^\varepsilon) D_\theta^i Y_u^\varepsilon + D_\theta^i f^{\lambda, \varepsilon}(u, Y_u^\varepsilon)] \mathbf{d}u \\ \quad - \int_t^T D_\theta^i Z_u^\varepsilon \mathbf{d}B_u, & 0 \leq \theta \leq t \leq T; \\ D_\theta^i Y_t^\varepsilon = 0, & D_\theta^i Z_t^\varepsilon = 0, & 0 \leq t < \theta \leq T. \end{cases}$$

Observe that the formula is similar for each coordinate  $i$ , thus we can suppose  $d = 1$ . We have

$$\begin{aligned}
\partial_y f^{\lambda, \varepsilon}(u, Y_u^\varepsilon) &= \partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u), \\
D_\theta f^{\lambda, \varepsilon}(u, y) &= D_\theta f(u, y) + \lambda D_\theta \delta_\varepsilon(y - S_u) = D_\theta f(u, y) - \lambda \delta_\varepsilon'(y - S_u) D_\theta S_u.
\end{aligned}$$

Let  $\beta_u := \partial_y f^{\lambda, \varepsilon}(u, Y_u^\varepsilon)$ ,  $\phi_u^\theta := D_\theta f^{\lambda, \varepsilon}(u, Y_u^\varepsilon)$  and let

$$\Gamma_s^t = \exp \left\{ \int_t^s [\partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u)] \mathbf{d}u \right\},$$

then

$$Z_t^\varepsilon = D_t Y_t^\varepsilon = \mathbb{E}_t \left[ \Gamma_T^t D_t \xi + \int_t^T \Gamma_s^t \phi_s^t \mathbf{d}s \right].$$

Under the assumption  $\text{ess sup}_{(t,\omega)} |\partial_y f(t, Y_t)| < \infty$ , since  $\forall x \in \mathbb{R}, \delta_\varepsilon'(x) \leq 0$ ,  $\Gamma_s^t$  is bounded uniformly in  $\lambda, t, s, \omega$ . In the following, without dedicated explanation, the generic constant  $C$  can vary from line to line and it does not depend on  $\lambda$ . Moreover, as for  $\int_t^T \Gamma_s^t \phi_s \mathbf{d}s$ , we first write

$$\int_t^T -\lambda \delta_\varepsilon'(Y_s^\varepsilon - S_s) e^{\int_t^s \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u) \mathbf{d}u} \mathbf{d}s = 1 - e^{\int_t^T \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u) \mathbf{d}u} \leq 1,$$

then we get,

$$\begin{aligned} \left| \mathbb{E}_t \left[ \int_t^T \Gamma_s^t \phi_s^t \mathbf{d}s \right] \right| &\leq \mathbb{E}_t \left[ \int_t^T |D_t f(s, Y_s^\varepsilon) - \lambda \delta_\varepsilon'(Y_s^\varepsilon - S_s) D_t S_s| e^{\int_t^s [\partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u)] \mathbf{d}u} \mathbf{d}s \right] \\ &\leq C \left( \mathbb{E}_t \left[ \sup_{(s,y) \in [t,T] \times \mathbb{R}} |D_t f(s, y)| \right] + \mathbb{E}_t \left[ \sup_{s \in [t,T]} |D_t S_s| \right] \right). \end{aligned}$$

In addition, we have

$$\left| \mathbb{E}_t [\Gamma_T^t D_t \xi] \right| \leq \mathbb{E}_t \left[ e^{\int_t^T \partial_y f(u, Y_u^\varepsilon) \mathbf{d}u} |D_t \xi| \right] \leq C \mathbb{E}_t [|D_t \xi|].$$

As a consequence,  $Z_t^\varepsilon$  is dominated by some bounded terms independent on  $\lambda$ , thus

$$\sup_{\lambda > 0} \sup_{\varepsilon > 0} \text{ess sup}_{(t,\omega) \in [0,T] \times \Omega} |Z_t^\varepsilon| < \infty.$$

□

# Chapter 4

## Error analysis of backward Implicit scheme using Empirical Regression Monte-Carlo method

**abstract** In this paper, we investigate on the numerical analysis of implicit scheme for solving penalized Backward Stochastic Differential Equations (BSDEs in short). We focus particularly on the empirical regression method where both the linear and the non-linear least squares regressions are studied. For the linear case, we show that, even with the penalty term, the convergence of the Monte-Carlo implicit scheme for penalized BSDE has the same rate as that for solving the standard BSDE without penalty via explicit scheme in  $L^2$  space. For the non-linear case, the fixed-architecture Neural Network with piecewise linear activation function (ReLU) is investigated and we establish the convergence rate with respect to the parameters of neural networks and of the Monte-Carlo method, from which we deduce the optimal tuning of parameters.

### 4.1 Introduction

In financial engineering, the Monte-Carlo method (MC method in short) plays an essential role thanks to its natural advantage for computing expectations arising in option pricing. Then Carrière [30] employs MC method in backward induction schemes to compute the early exercise premium for American option pricing; Longstaff and Schwarz [101] popularize the application of regression MC method in option pricing and they use the linear least-squares basis projection regressor for computing the conditional expectation. A lot of works follow afterwards since MC method suffers less the so called curse of dimensionality compared to traditional PDE method.

**State of the art.** Bally and Pagès [3] develop the error analysis for the quantization method; Bouchard and Touzi [16] study the convergence rate of approximations by regression considering the kernel and the Malliavin approaches; Del Moral et al. [41] present an estimator preserving the convexity and monotonicity for American options; Gobet and Labart [67] propose a BSDE solver where an adaptive control variate is added making use of the link between linear BSDEs and linear parabolic PDEs; Guyon and Henry-Labordère [77] provide a regression-based approximation for the optimal covariance matrix in the uncertain volatility model.

Among all above different approaches for applying MC method, the most popular one is presumably the ordinary linear least-squares regression (OLS in short) as in [101] (see also [38]) thanks to its facility of computing optimizer explicitly. Under this framework, a finite time horizon is considered, one first discretizes the time and then simulates some paths by MC method. The estimator will be computed backwardly as a linear combination of a reduced set of functions where the coefficients are optimal in the empirical least-squares sense; Gobet, Lemor and Warin [69] first extend OLS for the standard BSDE and give the convergence rate of dynamic programming explicit scheme depending on the simulation parameters; Gobet and Lemor [68] show the approximation of BSDE with Jump process and of Reflected BSDE; Bender and Steiner [7] use a 'martingale basis function' to reduce the propagation error when computing conditional expectation; Bouchard and Warin [17] suggest to use an adaptive local basis where the support of the function basis is adapted to the density of samplings; Bernhart et al. [11] apply this procedure to approximate the penalized solution of BSDE with constrained jump and they show a slow logarithmic convergence rate; Gobet and its co-authors present the error analysis of many different numerical estimators: the multi-step explicit scheme is studied in [72], the stratification sampling with parallelization on GPU is considered to improve the computational performance in [70], the hedging strategy with Malliavin weight is presented in [71].

As the computational capacity of computer develops, many researchers tend to employ deep learning method in order to solve some high dimension problems. E, Han and Jentzen [50] first apply the so-called 'deep BSDE' method where a forward scheme is considered and the parameters of network are optimized subject to minimize Mean Square Error (MSE in short) on terminal condition by stochastic gradient descent; then Han and Long [84] extend the deep BSDE method to solve the coupled Forward BSDE; recently Carmona and Laurière [26] present their result in Mean field control problem where the related approximation of a specific decoupled FBSDE of McKean-Vlasov type is computed by deep BSDE, see also [119] for the suggestion of network architecture.

The so-called 'expression rates' in the context of the curse of dimensionality, namely, the complexity of neural network w.r.t.  $\varepsilon$ -accuracy such as the number of parameters  $W$  and the number of layers  $\mathcal{L}$ , is a popular topic. There are already some results for PDEs. Reisinger and Zhang [121][Theorem 2.1] prove the expression rate  $W = cd^c\varepsilon^{-c}$  with  $c$  a constant not depending on  $d, \varepsilon$  for the game options in  $\mathbb{L}^2$ ; Gonon and Schwab [74][Theorem 5.1] get the similar expression result for linear BSDE driven by Lévy process in  $\mathbb{L}^\infty$ ; E, Han and Jentzen [51][Theorem 4] propose a multilevel Picard approximation with 3 types of

optimizers and give the expression rate  $W = O(\varepsilon^{-d})$  on compacts up to a polynomial growth factor in dimension  $d$  in  $\mathbb{L}^2$ ; Jentzen et al. [57][Theorem 1.1] get  $W = O(d^{2+c}\varepsilon^{-c})$  for the rates on compacts in  $\mathbb{L}^\infty$  where  $c$  is an arbitrary small positive constant.

Nevertheless, the Monte-Carlo least-squares regression method particularly profit from its nature as empirical minimizer, but few research works are concerned with the expression rate combining with MC method. Berner et al. [10][Theorem 1.1] show the lower bound of MC paths  $M \geq O(d\varepsilon^{-4}(1 + \log(d\varepsilon^{-1}\delta^{-1}))$  and  $W = O(d\varepsilon^{-2})$  with high probability  $1 - \delta$  for the one-step regression from the terminal time  $T$  without performing the dynamic program on non-reflected backward scheme.

**Our method.** In our case, we are given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\mathcal{F}$  contains all  $\mathbb{P}$ -null sets and where  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is the natural filtration generated by a  $\mathbb{R}^d$ -Brownian motion  $B$ . Then we define a Reflected BSDE as the solution of

$$\begin{cases} Y_t = \xi + \int_t^T f(s, X_s, Y_s) \mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, & \forall 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) \mathbf{d}K_t = 0, \end{cases} \quad (4.1.1)$$

where  $(S_t)_{0 \leq t \leq T}$  is the obstacle. We want to solve it numerically by penalization method, namely, PBSDE:

$$Y_t^\lambda = \xi + \int_t^T f(s, X_s, Y_s^\lambda) \mathbf{d}s + \lambda \int_t^T (Y_s^\lambda - S_s)^- \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \quad (4.1.2)$$

where  $\lambda$  is the penalization parameter ( $\lambda \rightarrow \infty$ ). Due to the fact that the solution of PBSDE  $(Y^\lambda, Z^\lambda)$  converges increasingly to  $(Y, Z)$  the solution of RBSDE (see [52, Section 6]), the explicit scheme is not a good choice for developing the convergence analysis since it requires a much finer time step to achieve convergence. Therefore, we aim at solving numerically PBSDE problem by some implicit schemes. We first study Markovian equations i.e. the terminal condition is in form of  $\xi = g(X_T)$ , the generator  $f$  is Lipschitz in  $y$  and Markovian in  $x$ , the barrier process  $S_t = S(t, X_t)$ , where the process  $X$  satisfies SDE:

$$\mathbf{d}X_t = b(t, X_t) \mathbf{d}t + \sigma(t, X_t) \mathbf{d}B_t. \quad (4.1.3)$$

Consider an uniform time discretization  $0 = t_0 < \dots < t_i := ih < t_N = T$ . Then, for  $\lambda > 0$ , define the discrete solution  $Y^{\lambda, h}$  by implicit scheme:

$$\begin{aligned} Y_{t_N}^\lambda &= g(X_{t_N}), \\ Y_{t_i}^{\lambda, h} &= \mathbb{E} \left[ Y_{t_{i+1}}^{\lambda, h} + f^\lambda(t_i, Y_{t_i}^{\lambda, h})h \middle| \mathcal{F}_{t_i} \right], \forall 0 \leq i \leq N-1. \end{aligned} \quad (4.1.4)$$

If we define the implicit projection function  $\mathcal{V}_i^\lambda$  as follows:

$$\mathcal{V}_i^\lambda : \begin{cases} \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, y) \mapsto y - f^\lambda(t_i, x, y)h = y - f(t_i, x, y)h - \lambda h(y - S(t_i, x))^- , \end{cases}$$

then the implicit scheme (4.1.4) can be written as:

$$\begin{aligned} Y_{t_N}^{\lambda,h} &= g(X_{t_N}^h), \\ \mathcal{V}_i^\lambda(X_{t_i}, Y_{t_i}^{\lambda,h}) &= \mathbb{E} \left[ Y_{t_{i+1}}^{\lambda,h} | \mathcal{F}_{t_i} \right] = \mathbb{E} \left[ Y_{t_{i+1}}^{\lambda,h} | X_{t_i} \right], \forall 0 \leq i \leq N-1. \end{aligned} \quad (4.1.5)$$

We will see later that  $\mathcal{V}_i^\lambda$  is continuous, increasing and invertible, therefore, one can define, for  $0 \leq i \leq N-1$ ,

$$Y_{t_i}^{\lambda,h} = (\mathcal{V}_i^\lambda)^{-1} \left( X_{t_i}, \mathbb{E} \left[ Y_{t_{i+1}}^{\lambda,h} | X_{t_i} \right] \right), \quad (4.1.6)$$

where

$$(\mathcal{V}_i^\lambda)^{-1}(x, w) := y \text{ s.t. } \mathcal{V}_i^\lambda(x, y) = w.$$

It has been proved the convergence rate of  $Y^{\lambda,h} \rightarrow Y$  in [73, Theorem 4.1],

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ (Y_{t_i}^\lambda - Y_{t_i}^{\lambda,h})^2 \right]^{1/2} = O(h + \lambda h^{3/8})$$

and  $(\mathcal{V}_i^\lambda)^{-1}$  is  $(1 + Ch)$ -Lipschitz, see Lemma 4.6.1.

Our main ingredient is to use regression to approximate the conditional expectation. At time  $t_i$ ,  $\mathbb{E} \left[ Y_{t_{i+1}}^{\lambda,h} | X_{t_i} \right]$  is simulated by linear or non-linear regression with the observations  $X_{t_i}$ , then the solution of PBSDE  $Y_{t_i}^{\lambda,h}$  can be represented by some Markovian functions  $u_i^{\lambda,h}$  valued on  $X_{t_i}$ . Indeed, denoting the conditional expectation at time  $t_i$  by  $P_i$ , precisely,

$$P_i \varphi(x) = \mathbb{E} \left[ \varphi(X_{t_{i+1}}) | X_{t_i} = x \right],$$

then, by denoting  $Y_{t_i}^{\lambda,h} = u_i^{\lambda,h}(X_{t_i})$  for  $0 \leq i \leq N-1$ , from (4.1.5) and (4.1.6), it follows,

$$\begin{aligned} u_N^{\lambda,h}(X_{t_N}) &= g(X_{t_N}), \\ u_i^{\lambda,h}(X_{t_i}) &= (\mathcal{V}_i^\lambda)^{-1} \left( X_{t_i}, P_i u_{i+1}^{\lambda,h}(X_{t_i}) \right), \forall 0 \leq i \leq N-1, \end{aligned}$$

where each  $u_i^{\lambda,h}(\cdot)$  depends on the conditional distribution of  $X_{t_{i+1}} | X_{t_i}$ . Let us define the sequence of functions  $\left\{ u_i^{\lambda,h}(\cdot) := (\mathcal{V}_i^\lambda)^{-1} \left( \cdot, P_i u_{i+1}^{\lambda,h}(\cdot) \right), 0 \leq i \leq N \right\}$ , we have generally

$$\begin{aligned} u_N^{\lambda,h} x &= g(x), \\ u_i^{\lambda,h}(x) &= (\mathcal{V}_i^\lambda)^{-1} \left( x, P_i u_{i+1}^{\lambda,h}(x) \right), \forall 0 \leq i \leq N-1. \end{aligned} \quad (4.1.7)$$

Hence, for solving PBSDE, one only needs to solve  $(u_i^{\lambda,h})_{0 \leq i \leq N-1}$ . We are going to analyze the error occurring with linear regression and that with non-linear regression by Neural Networks. In the linear case, the coefficient of regression is computed by Singular-Value-Decomposition.

**Contributions.** As far as we know, there is no available convergence result for computing the obstacle-related penalized equation using MC regression method. We propose a backward implicit scheme for the penalized reflected BSDE whose Lipschitz constant tends to infinity and we give the error analysis for this scheme both for linear and non-linear regression. The main difficulty lies in how to handle the implicit projection and how to eliminate the effect of penalty in the analysis. Our results complete the previous work [73] in which a theoretical analysis is done for the discrete solution without MC scheme. We compare our result with [69] and [10] (all these references are for pure BSDE).

**Organization.** The discrete scheme of penalized BSDE is presented in Section 4.2. In Section 4.3, we introduce some properties of our target function. In Section 4.4, for the linear regression, we give the empirical and distributional estimation of error propagation and also the global error estimates. In Section 4.5, we present our results about non-linear regression by neural networks.

## 4.2 Notations and Assumptions

### 4.2.1 Notations

- Denote the terminal time  $T$ ; denote  $(X_t)_{0 \leq t \leq T}$  the stochastic process  $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^d$  which is  $\mathcal{F}_t$ -adapted;
- Denote the number of path simulations by  $M$ ; the uniform time discretization step by  $h$ ; the approximation space by  $\Psi$ ; the degree used for linear regression  $K$ ;
- For any  $0 \leq i \leq N - 1$ ,  $X_{t_i}^{(1:M)} := (X_{t_i}^m)_{1 \leq m \leq M}$  is the MC sampling at time  $t_i$ ;
- Denote the conditional expectation  $\mathbb{E}[\cdot | \mathcal{F}_t]$  by  $\mathbb{E}_t[\cdot]$ ; set  $\mathbb{E}[\cdot | X_{t_i}^{(1:M)}] = \mathbb{E}^{(1:M)}[\cdot]$ ;
- Denote  $X_t^{s,x}$  the value of  $X$  at time  $t$  starting from  $X_s = x$  with  $s \leq t$ ;
- Define the spaces: (for  $p \geq 2$ ):

$$\mathbb{L}^\infty = \left\{ \text{random variables } \zeta \text{ s.t. } \operatorname{ess\,sup}_{\omega \in \Omega} |\zeta(\omega)| < \infty \right\},$$

$$\mathbb{L}^p = \left\{ \text{random variables } \zeta \text{ s.t. } \mathbb{E}[|\zeta|^p] < \infty \right\},$$

$$\mathcal{S}^p = \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are predictable} \right. \\ \left. \text{with } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^p \right] < \infty \right\},$$

$$\mathbb{H}^p = \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are predictable} \right. \\ \left. \text{with } \mathbb{E} \left[ \left( \int_0^T |\phi_t|^2 dt \right)^{p/2} \right] < \infty \right\},$$



- Given a measure space  $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l), \mu)$ , for any  $\mathcal{B}(\mathbb{R}^l)$ -measurable deterministic function  $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}$ , set  $\|\varphi\|_{\mathbb{L}^2(\mu)}^2 := \int_{\mathbb{R}^l} |\varphi(x)|^2 \mu(\mathbf{d}x)$ ; given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , for any  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^l)$ -measurable random variable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\|\varphi(\omega, \cdot)\|_{\mathbb{L}^2(\mu)}^2 = \int_{\mathbb{R}^l} |\varphi(\omega, x)|^2 \mu(\mathbf{d}x)$ . For example, if one considers a 2-steps discrete scheme, then  $l = 2d$ ;
- **OLS**: Ordinary Least Squares for linear regression and we say  $\alpha^*$  solves **OLS** $(\mathcal{R}, \Psi, \mu)$ , i.e.  $\alpha^*$  solves the linear least squares regression w.r.t. the response  $\mathcal{R} : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ , the observation  $\mathcal{O}$ , the space of linear approximation  $\Psi$  and the law  $\mu$  of  $\mathcal{O}$ , if, a.s.  $\forall \omega \in \Omega$ ,

$$\mathbf{OLS}(\mathcal{R}(\omega, \cdot), \Psi, \mu) := \arg \inf_{\varphi \in \Psi} \|\mathcal{R}(\omega, \cdot) - \varphi(\cdot)\|_{\mathbb{L}^2(\mu)} = \Phi(\alpha^*(\omega), \cdot) = \alpha^*(\omega) \cdot \psi(\cdot)$$

with  $\Psi = \text{Span}(\psi)$ , where  $\alpha^* = \arg \inf_{\alpha \in \mathcal{A}} \|\mathcal{R}(\omega, \cdot) - \alpha \cdot \psi(\cdot)\|_{\mathbb{L}^2(\mu)}$  with  $\mathcal{A} = \mathbb{R}^K$ ,  $K$  being the number of elements of  $\psi$ . If there are several minimizers for  $\alpha^*$ , we choose the one with minimum Euclidean norm (it is obtained using Singular-Value-Decomposition), see also [72] and the SVD-optimal  $\alpha^*$  can be computed explicitly.

- **GLS**: General Least Squares for non-linear regression. Analogously, we say  $\alpha^*$  solves **GLS** $(\mathcal{R}, \Psi, \mu)$ , i.e.  $\alpha^*$  solves the general least squares regression w.r.t. the response  $\mathcal{R} : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ , the observation  $\mathcal{O}$ , the space of non-linear approximation  $\Psi$  and the law  $\mu$  of  $\mathcal{O}$ , if, a.s.  $\forall \omega \in \Omega$ ,

$$\mathbf{GLS}(\mathcal{R}(\omega, \cdot), \Psi, \mu) := \arg \inf_{\varphi \in \Psi} \|\mathcal{R}(\omega, \cdot) - \varphi(\cdot)\|_{\mathbb{L}^2(\mu)} = \Phi(\alpha^*(\omega), \cdot)$$

with  $\Psi = \{\Phi(\alpha, \cdot) : \alpha \in \mathcal{A}\}$ , where  $\alpha^* = \arg \inf_{\alpha \in \mathcal{A}} \|\mathcal{R}(\omega, \cdot) - \Phi(\alpha, \cdot)\|_{\mathbb{L}^2(\mu)}$  with  $\mathcal{A} \subset \mathbb{R}^\theta$  and  $\theta$  is to be defined. For example, in the case of Neural Network,  $\Phi(\alpha, \cdot) = \mathbf{NN}(\alpha, \cdot)$  where **NN** is a neural network with a given architecture. Similarly, if there are many choices for  $\alpha^*$ , we choose the one with minimum Euclidean norm: we are aware that it might be difficult in practice.

- Regression measures, for any  $0 \leq i \leq N - 1$ :
  - $\mu_{i,i+1}^M$ : the empirical measure on  $(\Omega, \mathcal{B}(\mathbb{R}^{2d}))$  at time  $t_i$ ,  $\mu_{i,i+1}^M = \frac{1}{M} \sum_{m=1}^M \delta_{(X_{t_i}^m, X_{t_{i+1}}^m)}$ ;
  - $\mu_{i,i+1}$ : the distribution of  $X_{t_i}$  and  $X_{t_{i+1}}$ ,  $\mu_{i,i+1} = \mathbb{P} \circ (X_{t_i}, X_{t_{i+1}})^{-1}$ ;
  - $\mu_i^M$ : the empirical measure on  $(\Omega, \mathcal{B}(\mathbb{R}^d))$  at  $t_i$ ,  $\mu_i^M = \frac{1}{M} \sum_{m=1}^M \delta_{(X_{t_i}^m)}$ ;
  - $\mu_{X_{t_i}}$ : the distribution of  $X_{t_i}$ ,  $\mu_{X_{t_i}} = \mathbb{P} \circ X_{t_i}^{-1}$ ;
- Coefficients: for  $0 \leq i \leq N - 1$ , ignoring  $\lambda, h$  in the superscripts of  $\alpha$ ,
  - $\hat{\alpha}^{M,i}$  is the optimal coefficient of **GLS** with predicted response  $\hat{u}_{i+1}^{\lambda,h,M,\Psi}(\cdot)$  and with the empirical regression measure  $\mu_{i,i+1}^M$ , i.e.  $\Phi(\hat{\alpha}^{M,i}, \cdot) = \mathbf{GLS}(\hat{u}_{i+1}^{\lambda,h,M,\Psi}(\cdot), \Psi, \mu_{i,i+1}^M)$ ;
  - $\bar{\alpha}^{M,i}$  is the optimal coefficient of **GLS** with the true response  $u^{\lambda,h}(\cdot)$  and with the empirical measure  $\mu_{i,i+1}^M$ , i.e.  $\Phi(\bar{\alpha}^{M,i}, \cdot) = \mathbf{GLS}(u_{i+1}^{\lambda,h}(\cdot), \Psi, \mu_{i,i+1}^M)$ ;
- For some positive constant  $L$ , define the Clipping operator  $\mathcal{C}_L$ : for any real function  $\phi$ ,  $\mathcal{C}_L \phi(\cdot) := -L \wedge \phi(\cdot) \vee L$ .

**Procedure 4.2.1.** In this procedure, we drop  $\lambda, h$  in superscripts which are fixed in the context (e.g.  $u, u^{M, \Psi}$  for  $u^{\lambda, h}, u^{\lambda, h, M, \Psi}$ ). For  $0 \leq i \leq N - 1$ , given  $u_{i+1}(\cdot)$ , we want to solve  $u_i(\cdot)$ .

▷ *Step 1: Regression for simulating  $P_i$ .* Find the minimizer of following GLS problem:

$$\begin{aligned} \mathbf{GLS} \left( \hat{u}_{i+1}^{M, \Psi}(\cdot), \Psi, \mu_{i, i+1}^M \right) &= \arg \inf_{\varphi \in \Psi} \frac{1}{M} \sum_{m=1}^M \left( \hat{u}_{i+1}^{M, \Psi} \left( X_{t_{i+1}}^m \right) - \varphi \left( X_{t_i}^m \right) \right)^2 \\ &=: \Phi(\hat{\alpha}^{M, i}, \cdot). \end{aligned}$$

Since **OLS** is a special case of **GLS**, the same minimizing process works for **OLS**.

▷ *Step 2: Clipping.* Assume  $\sup_{0 \leq i \leq N-1} |u_i|_\infty \leq L$  as in Lemma 4.3.3, for the stability of

simulation, we take  $\mathcal{C}_L \Phi(\hat{\alpha}^{M, i}, \cdot)$  as the approximation of  $P_i \hat{u}_{i+1}^{M, \Psi}(\cdot)$  for any  $0 \leq i \leq N - 1$ .

▷ *Step 3: Implicit step by fixed point.* Denote the final estimation of  $u_i(\cdot)$  by  $\hat{u}_i^{M, \Psi}(\cdot)$ :

$$\hat{u}_i^{M, \Psi}(x) = \mathcal{V}_i^{-1} \left( x, \mathcal{C}_L \mathbf{GLS} \left( \hat{u}_{i+1}^{M, \Psi}(\cdot), \Psi, \mu_{i, i+1}^M \right) (x) \right) = \mathcal{V}_i^{-1} \left( x, \mathcal{C}_L \Phi(\hat{\alpha}^{M, i}, x) \right) \quad (4.2.1)$$

where  $\mathcal{V}_i^{-1}$  is tracked by some fixed point argument. Indeed, for some given  $x_0 \in \mathbb{R}^d$  and  $w := \mathcal{C}_L \Phi(\hat{\alpha}^{M, i}, x_0)$ , we want to find  $y_0$  s.t.  $\mathcal{V}_i^\lambda(x_0, y_0) = w_0$ , which is equivalent to solve the fixed point problem  $F(x_0, y_0) = y_0$  where

$$F(x_0, y) = f^\lambda(t_i, x_0, y)h + w.$$

Without doubt,  $F$  is a contraction map in  $y$  i.e.

$$|F(x_0, y) - F(x_0, y')| \leq (C_{Lip}^f + \lambda)h|y - y'|$$

with the proper choice of  $\lambda, h$  such that  $(C_{Lip}^f + \lambda)h < 1$ . Notice that for ensuring the convergence of discrete penalized solution to the continuous reflected one, we require  $\lambda = O(h^{-3/8})$  (see [73, Corollary 4.3]). Hence, for small enough  $h$ ,  $(C_{Lip}^f + \lambda)h < 1$  and we can track  $y_0$  by Picard's fixed point argument.

## 4.2.2 Assumptions

( $H_1$ ) For RBSDE:

$H_1$ -1:  $\xi = g(X_T)$  with  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  bounded;

$H_1$ -2: the generator  $f$  is Markovian in  $x$ ,  $f$  is non-increasing and  $C_{Lip}^f$ -Lipschitz in  $y$ ;

$H_1$ -3:  $\|f_0\|_\infty := \sup_{0 \leq t \leq T} |f(t, \cdot, 0)|_\infty < \infty$ ;

$H_1$ -4: the barrier process  $S_t = S(t, X_t)$  is bounded with  $|S|_\infty := \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |S(t, x)| < \infty$ ;

$H_1$ -5: the coefficients  $b, \sigma$  are uniformly Lipschitz continuous in  $x$ , i.e.

$$|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq C_{Lip}^X |x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}^d, \forall t \in [0, T],$$

and  $b(\cdot, 0), \sigma(\cdot, 0)$  are bounded on  $[0, T]$ .

$H_1$ -6: the function  $g$  is Lipschitz on  $\mathbb{R}^d$  and  $S(t, \cdot)$  is uniformly Lipschitz on  $\mathbb{R}^d$ .

The monotonicity assumed in  $H_1$ -2 is not a restricted condition once the Lipschitz continuity is given (see [73, Proposition 2.1]).

( $H_2$ ) For regression:

$H_2$ -1: For OLS, assume the function space  $\Psi = \text{Span}(\psi)$  where  $\psi := \{\psi_j \in \mathbb{L}^2(\mu_{X_{t_i}}) : \mathbb{R}^d \rightarrow \mathbb{R}, 1 \leq j \leq K\}$  with  $K$  being the size of  $\Psi$ .

$H_2$ -2: Consider the following **GLS** regression problem:

- we are given a set of  $M$  i.i.d copies of  $\mathcal{O}^1, \mathcal{O}^2$ , which we denote  $\mathcal{O}^{1,(1:M)}, \mathcal{O}^{2,(1:M)}$ ;
- the set of real functions to be regressed  $\mathbb{L}_b^0(\Omega \times \mathbb{R}^d)$  i.e. all measurable and bounded functions  $\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ ;
- we want to approximate  $\mathcal{R}(\mathcal{O}^2)$  based on the observations  $\mathcal{O}^1$  with  $\mathcal{R} \in \mathbb{L}_b^0(\Omega \times \mathbb{R}^d)$ .

The approximation space  $\Psi$  is such that, for any  $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{L}_b^0(\Omega \times \mathbb{R}^d)$ , for any square integrable random variables  $\mathcal{O}^1, \mathcal{O}^2$ , we have,

$$\begin{aligned} \mathbb{E} \left[ \left\| \mathbf{GLS}(\mathcal{R}_1(\omega, \cdot), \Psi, \nu_{1,2}^M) - \mathbf{GLS}(\mathcal{R}_2(\omega, \cdot), \Psi, \nu_{1,2}^M) \right\|_{\mathbb{L}^2(\nu_1)} \right] \\ \leq \mathbb{E} \left[ \left\| \mathcal{R}_1(\omega, \cdot) - \mathcal{R}_2(\omega, \cdot) \right\|_{\mathbb{L}^2(\nu_2)} \right], \end{aligned}$$

where

- $\nu_{1,2}^M$  is the joint empirical measure of  $(\mathcal{O}^{1,(1:M)}, \mathcal{O}^{2,(1:M)})$ , i.e.  $\nu_{1,2}^M = \frac{1}{M} \sum_{m=1}^M \delta_{(\mathcal{O}^{1,m}, \mathcal{O}^{2,m})}$ ;
- $\nu_i$  is the law of  $\mathcal{O}^i$ , i.e.  $\nu_i = \mathbb{P} \circ (\mathcal{O}^i)^{-1}$ , for  $i \in \{1, 2\}$ .

The non-expanding error assumed in  $H_2$ -2 is especially verified in **OLS** (see [72, Proposition 4.12 (ii)]), in the following sense,

$$\begin{aligned} \mathbb{E} \left[ \left\| \mathbf{OLS}(\mathcal{R}_1(\omega, \cdot), \Psi, \nu_{1,2}^M) - \mathbf{OLS}(\mathcal{R}_2(\omega, \cdot), \Psi, \nu_{1,2}^M) \right\|_{\mathbb{L}^2(\nu_1^M)} \right] \\ \leq \mathbb{E} \left[ \left\| \mathcal{R}_1(\omega, \cdot) - \mathcal{R}_2(\omega, \cdot) \right\|_{\mathbb{L}^2(\nu_2^M)} \right], \\ \mathbb{E} \left[ \left\| \mathbf{OLS}(\mathcal{R}_1(\omega, \cdot), \Psi, \nu_{1,2}) - \mathbf{OLS}(\mathcal{R}_2(\omega, \cdot), \Psi, \nu_{1,2}) \right\|_{\mathbb{L}^2(\nu_1)} \right] \\ \leq \mathbb{E} \left[ \left\| \mathcal{R}_1(\omega, \cdot) - \mathcal{R}_2(\omega, \cdot) \right\|_{\mathbb{L}^2(\nu_2)} \right], \end{aligned}$$

where

- $\nu_{1,2}$  is the joint law of  $\mathcal{O}^1, \mathcal{O}^2$ , i.e.  $\nu_{1,2} := \mathbb{P} \circ (\mathcal{O}^1, \mathcal{O}^2)^{-1}$ ;
  - $\nu_i^M$  is the empirical measure of  $\mathcal{O}^i$ , i.e.  $\nu_i^M := \sum_{m=1}^M \delta_{\mathcal{O}^{i,m}}$  for  $i \in \{1, 2\}$ ;
- which is an easy consequence of Pythagoras Theorem. Proving  $H_2$ -2 for GLS is part of our future investigation.

### 4.3 Preliminary estimates

**Lemma 4.3.1.** *Assume  $(H_1)$ - $(1-4)$  and  $h \leq \frac{1}{2C_{Lip}^f} \wedge 1$  and  $L \geq (1 + C_{Lip}^f h)|S|_\infty + \|f_0\|_\infty h$ , we have,  $\forall 0 \leq i \leq N - 1$ ,*

$$\begin{aligned} |S|_\infty &\leq \frac{L - \|f_0\|_\infty h}{1 + C_{Lip}^f h} \leq (\mathcal{V}_i^\lambda)^{-1}(x, L) \leq \frac{L + \|f_0\|_\infty h}{1 - C_{Lip}^f h}, \\ \frac{-L - \|f_0\|_\infty h}{1 - C_{Lip}^f h} &\leq (\mathcal{V}_i^\lambda)^{-1}(x, -L) \leq |S|_\infty. \end{aligned} \quad (4.3.1)$$

Here  $C_{Lip}^f$  is the Lipschitz constant of  $f$ .

*Proof.* Notice that

$$\begin{aligned} \mathcal{V}_i^\lambda(x, |S|_\infty) &= |S|_\infty - f(t_i, x, |S|_\infty)h \leq (1 + C_{Lip}^f h)|S|_\infty + |f(t_i, x, 0)|h \leq L, \\ \mathcal{V}_i^\lambda(x, |S|_\infty) &\geq (1 - C_{Lip}^f h)|S|_\infty - |f(t_i, x, 0)|h \geq -L, \end{aligned}$$

which implies  $(\mathcal{V}_i^\lambda)^{-1}(x, -L) \leq |S|_\infty \leq (\mathcal{V}_i^\lambda)^{-1}(x, L)$  since  $\mathcal{V}_i^\lambda$  is continuous and non-decreasing. Moreover, for any  $y > |S|_\infty$ , we have

$$(1 - C_{Lip}^f h)y - |f(t_i, x, 0)|h \leq \mathcal{V}_i^\lambda(x, y) \leq (1 + C_{Lip}^f h)y + |f(t_i, x, 0)|h$$

which yields

$$\frac{L - |f(t_i, x, 0)|h}{1 + C_{Lip}^f h} \leq (\mathcal{V}_i^\lambda)^{-1}(x, L) \leq \frac{L + |f(t_i, x, 0)|h}{1 - C_{Lip}^f h}.$$

So we have proved the first inequalities in (4.3.1) and the left one in the second. For the inequality at the right side in the second ones, use the argument that  $\mathcal{V}_i^0(x, \cdot) \geq \mathcal{V}_i^\lambda(x, \cdot)$  on  $\mathbb{R}$ . Notice that

$$\mathcal{V}_i^0(x, 0) = -f(t_i, x, 0)h \geq -L \implies 0 \geq (\mathcal{V}_i^0)^{-1}(x, -L),$$

and for any  $y \leq 0$ , we have

$$(1 + C_{Lip}^f h)y - |f(t_i, x, 0)|h \leq \mathcal{V}_i^0(x, y) \leq (1 - C_{Lip}^f h)y + |f(t_i, x, 0)|h.$$

Thus

$$(\mathcal{V}_i^\lambda)^{-1}(x, -L) \geq (\mathcal{V}_i^0)^{-1}(x, -L) \geq \frac{-L - \|f_0\|_\infty h}{1 - C_{Lip}^f h}$$

where we get the desired result. □

An immediate result from the previous lemma is the following:

**Lemma 4.3.2.** *Under the same assumptions as Lemma 4.3.1, we have, for any  $\varphi \in \Psi$ ,*

$$\sup_{x \in \mathbb{R}^d} |(\mathcal{V}_i^\lambda)^{-1}(x, \mathcal{C}_L \varphi(x))| \leq \frac{L + \|f_0\|_\infty h}{1 - C_{Lip}^f h}.$$

*Proof.* For fixed  $x \in \mathbb{R}^d$ ,  $\mathcal{V}_i^\lambda(x, \cdot)$  is non-decreasing, then clearly

$$(\mathcal{V}_i^\lambda)^{-1}(x, -L) \leq (\mathcal{V}_i^\lambda)^{-1}(x, \mathcal{C}_L \varphi(x)) \leq (\mathcal{V}_i^\lambda)^{-1}(x, L).$$

□

**Lemma 4.3.3.** *Under  $(H_1)$ -(1-4), if  $h < \frac{1}{2C_{Lip}^f} \wedge 1$ , there exists a scalar sequence  $\{L_i, 0 \leq i \leq N\}$  and a constant  $L$  depending only on  $T, f, S, g$  s.t.  $\forall 0 \leq i \leq N - 1$ ,*

$$\sup_{x \in \mathbb{R}^d} |u_i^{\lambda, h}(x)| \leq L_i \leq L,$$

*Proof.*  $\triangleright$  *Step 1: Comparison.* Given  $\xi_1 := g_1(X_T)$  and  $\xi_2 := g_2(X_T)$ , consider the discrete solution of penalized BSDEs  $Y^{1, \lambda, h}$  and  $Y^{2, \lambda, h}$  w.r.t.  $(\xi^1, f, S)$  and  $(\xi^2, f, S)$  respectively; we denote the functions  $u^{1, \lambda, h}, u^{2, \lambda, h}$  s.t. for any  $0 \leq i \leq N - 1$ ,

$$Y_{t_i}^{j, \lambda, h} = u_i^{j, \lambda, h}(X_{t_i}) = (\mathcal{V}_i^\lambda)^{-1}(X_{t_i}, P_i u_{i+1}^{j, \lambda, h}(X_{t_i})), \text{ for } j \in \{1, 2\}.$$

A easy induction shows that, if  $g_1 \leq g_2$  on  $\mathbb{R}^d$ , then we have a comparison-like theorem in the discrete case, namely,  $\forall 0 \leq i \leq N$ ,  $u_i^{1, \lambda, h} \leq u_i^{2, \lambda, h}$  on  $\mathbb{R}^d$  because  $P_i$  and  $(\mathcal{V}_i^\lambda)^{-1}$  are non-decreasing.

$\triangleright$  *Step 2:* Let  $|g_2|_\infty = |g|_\infty + e^{C_{Lip}^f T} \left[ \left(1 + \frac{1}{C_{Lip}^f}\right) \|f_0\|_\infty + (1 + C_{Lip}^f) |S|_\infty \right]$ ,

$$g_2(\cdot) \equiv |g_2|_\infty, \quad g_1(\cdot) = -|g_2|_\infty.$$

Therefore,

$$u_i^{1, \lambda, h}(x) \leq u_i^{\lambda, h}(x) \leq u_i^{2, \lambda, h}(x). \quad (4.3.2)$$

For proving  $u_i^{2, \lambda, h}(\cdot)$  is bounded: denote  $\forall 0 \leq i \leq N$ ,

$$\underline{y}_i := \frac{|g_2|_\infty - h \|f_0\|_\infty \sum_{j=i}^{N-1} (1 + C_{Lip}^f h)^{N-1-j}}{(1 + C_{Lip}^f h)^{N-i}}, \quad \bar{y}_i := \frac{|g_2|_\infty + h \|f_0\|_\infty \sum_{j=i}^{N-1} (1 - C_{Lip}^f h)^{N-1-j}}{(1 - C_{Lip}^f h)^{N-i}}.$$

We claim that, with the definition of  $g_2$ ,  $\forall 0 \leq i \leq N$ ,  $\forall x \in \mathbb{R}^d$ ,

$$(1 + C_{Lip}^f h) |S|_\infty + \|f_0\|_\infty h \leq \underline{y}_i \leq u_i^{2, \lambda, h}(x) \leq \bar{y}_i, \quad -\bar{y}_i \leq u_i^{1, \lambda, h}(x). \quad (4.3.3)$$

Hence, if  $h < \frac{1}{2C_{Lip}^f}$ , with (4.3.2),

$$|u_i^{\lambda,h}(x)| \leq \bar{y}_i \leq \frac{|g_2|_\infty + \frac{1}{C_{Lip}^f} \|f_0\|_\infty}{(1 - C_{Lip}^f h)^N} \leq e^{C_{Lip}^f T} \left( |g_2|_\infty + \frac{1}{C_{Lip}^f} \|f_0\|_\infty \right),$$

thus, we have the desired result by denoting

$$L_i := \bar{y}_i, \quad L := e^{C_{Lip}^f T} \left( |g_2|_\infty + \frac{1}{C_{Lip}^f} \|f_0\|_\infty \right).$$

▷ *Step 3:* For proving (4.3.3), with  $h \leq 1$ , an easy computation leads to

$$\inf_{0 \leq i \leq N} \underline{y}_i \geq (1 + C_{Lip}^f h) |S|_\infty + \|f_0\|_\infty h. \quad (4.3.4)$$

Indeed,

$$\begin{aligned} \underline{y}_i &\geq \frac{|g_2|_\infty + \frac{1}{C_{Lip}^f} \|f_0\|_\infty}{(1 + C_{Lip}^f h)^{N-i}} - \frac{\|f_0\|_\infty}{C_{Lip}^f} \geq \frac{|g_2|_\infty}{(1 + C_{Lip}^f h)^N} - \frac{\|f_0\|_\infty}{C_{Lip}^f} \\ &\geq e^{-C_{Lip}^f T} |g_2|_\infty - \frac{\|f_0\|_\infty}{C_{Lip}^f} \geq \|f_0\|_\infty + (1 + C_{Lip}^f) |S|_\infty \geq \|f_0\|_\infty h + (1 + C_{Lip}^f h) |S|_\infty. \end{aligned}$$

For another inequality in (4.3.3), we use the backward induction. First,

$$u_N^{2,\lambda,h}(x) = g_2(x) = |g_2|_\infty, \quad u_N^{2,\lambda,h}(x) = -|g_2|_\infty$$

which verifies (4.3.3) for  $i = N$ . Then, assume that (4.3.3) is true for some  $i + 1 \leq N$ . Applying Lemma 4.3.1, with (4.3.4), we obtain,

$$\begin{aligned} u_i^{2,\lambda,h}(x) &\geq (\mathcal{V}_i^\lambda)^{-1}(x, \underline{y}_{i+1}) \geq \frac{\underline{y}_{i+1} - \|f_0\|_\infty h}{1 + C_{Lip}^f h} = \underline{y}_i, \\ u_i^{2,\lambda,h}(x) &\leq (\mathcal{V}_i^\lambda)^{-1}(x, \bar{y}_{i+1}) \leq \frac{\bar{y}_{i+1} + \|f_0\|_\infty h}{1 - C_{Lip}^f h} = \bar{y}_i, \\ u_i^{1,\lambda,h}(x) &\geq (\mathcal{V}_i^\lambda)^{-1}(x, -\bar{y}_{i+1}) \geq \frac{-\bar{y}_{i+1} - \|f_0\|_\infty h}{1 - C_{Lip}^f h} = -\bar{y}_i, \end{aligned}$$

i.e. we have proved (4.3.3) for the index  $i$ . The proof is complete.  $\square$

**Corollary 4.3.4.** *Assume  $(H_1)$ -(1-4) and  $h < \frac{1}{2C_{Lip}^f} \wedge 1$ , letting  $L$  be the same as in Lemma 4.3.3, then there exists  $\bar{L}$  depending on  $T, S, g, f$  s.t.*

$$\sup_{x \in \mathbb{R}^d} (\mathcal{V}_i^\lambda)^{-1}(x, \mathcal{C}_L \varphi(x)) \leq \bar{L}, \quad \forall \varphi \in \Psi.$$

*Proof.* With the same argument  $\inf_{0 \leq i \leq N} L_i \geq (1 + C_{Lip}^f h) |S|_\infty + \|f_0\|_\infty h$ , we can apply Lemma 4.3.2 with

$$\bar{L} := \frac{L + h \|f_0\|_\infty \sum_{j=0}^{N-1} (1 - C_{Lip}^f h)^{N-1}}{(1 - C_{Lip}^f h)^N} \leq e^{C_{Lip}^f T} \left( L + \frac{1}{C_{Lip}^f} \|f_0\|_\infty \right).$$

□

**Proposition 4.3.5.** *Assume  $(H_1)$ -(1-6) and suppose  $f$  is  $C_{Lip}^f$ -Lipschitz in  $x$ , we have, for any finite  $\lambda$ ,  $P_i u_{i+1}^{\lambda, h}(\cdot)$  is Lipschitz on  $\mathbb{R}^d$ , for any  $0 \leq i \leq N - 1$ .*

*Proof.* From (4.1.4), we have, for any fixed  $\lambda > 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \left| Y_{t_{i+1}}^{h, t_i, x} - Y_{t_{i+1}}^{h, t_i, x'} \right| \right] &= \mathbb{E} \left[ \left| \mathbb{E}_{t_{i+1}} \left[ Y_{t_{i+2}}^{h, t_i, x} + f^\lambda(t_{i+1}, X_{t_{i+1}}^{t_i, x}, Y_{t_{i+1}}^{h, t_i, x}) h \right] \right. \right. \\ &\quad \left. \left. - \mathbb{E}_{t_{i+1}} \left[ Y_{t_{i+2}}^{h, t_i, x'} + f^\lambda(t_{i+1}, X_{t_{i+1}}^{t_i, x'}, Y_{t_{i+1}}^{h, t_i, x'}) h \right] \right| \right] \\ &\leq \mathbb{E} \left[ \left| Y_{t_{i+2}}^{h, t_i, x} - Y_{t_{i+2}}^{h, t_i, x'} \right| \right] \\ &\quad + h \mathbb{E} \left[ \left| f^\lambda(t_{i+1}, X_{t_{i+1}}^{t_i, x}, Y_{t_{i+1}}^{h, t_i, x}) - f^\lambda(t_{i+1}, X_{t_{i+1}}^{t_i, x'}, Y_{t_{i+1}}^{h, t_i, x'}) \right| \right] \\ &\leq \mathbb{E} \left[ \left| Y_{t_{i+2}}^{h, t_i, x} - Y_{t_{i+2}}^{h, t_i, x'} \right| \right] + C_{Lip}^f h \mathbb{E} \left[ \left| X_{t_{i+1}}^{t_i, x} - X_{t_{i+1}}^{t_i, x'} \right| \right] \\ &\quad + (C_{Lip}^f + \lambda) h \mathbb{E} \left[ \left| Y_{t_{i+1}}^{h, t_i, x} - Y_{t_{i+1}}^{h, t_i, x'} \right| \right] + \lambda h \mathbb{E} \left[ \left| S(t_i, X_{t_{i+1}}^{t_i, x}) - S(t_i, X_{t_{i+1}}^{t_i, x'}) \right| \right]; \end{aligned}$$

since  $S(t, \cdot)$  is assumed to be uniformly Lipschitz, so

$$\begin{aligned} (1 - (C_{Lip}^f + \lambda)h) \mathbb{E} \left[ \left| Y_{t_{i+1}}^{h, t_i, x} - Y_{t_{i+1}}^{h, t_i, x'} \right| \right] &\leq \mathbb{E} \left[ \left| Y_{t_{i+2}}^{h, t_i, x} - Y_{t_{i+2}}^{h, t_i, x'} \right| \right] \\ &\quad + (C_{Lip}^f + \lambda C_{Lip}^S) h \mathbb{E} \left[ \left| X_{t_{i+1}}^{t_i, x} - X_{t_{i+1}}^{t_i, x'} \right| \right], \\ \mathbb{E} \left[ \left| Y_{t_{i+1}}^{h, t_i, x} - Y_{t_{i+1}}^{h, t_i, x'} \right| \right] &\leq \frac{\mathbb{E} \left[ \left| Y_{t_{i+2}}^{h, t_i, x} - Y_{t_{i+2}}^{h, t_i, x'} \right| \right]}{1 - (C_{Lip}^f + \lambda)h} \\ &\quad + \frac{(C_{Lip}^f + \lambda C_{Lip}^S) h}{1 - (C_{Lip}^f + \lambda)h} \mathbb{E} \left[ \left| X_{t_{i+1}}^{t_i, x} - X_{t_{i+1}}^{t_i, x'} \right| \right]. \end{aligned}$$

The Gronwall's lemma tells us

$$\begin{aligned}
\mathbb{E} \left[ \left| Y_{t_{i+1}}^{h,t_i,x} - Y_{t_{i+1}}^{h,t_i,x'} \right| \right] &\leq e^{(C_{Lip}^f + \lambda)T} \mathbb{E} \left[ \left| g(X_{t_N}^{t_i,x}) - g(X_{t_N}^{t_i,x'}) \right| \right] \\
&\quad + \sum_{j=i+1}^{N-1} \frac{(C_{Lip}^f + \lambda C_{Lip}^S)h}{(1 - (C_{Lip}^f + \lambda)h)^{j-i}} \mathbb{E} \left[ \left| X_{t_j}^{t_i,x} - X_{t_j}^{t_i,x'} \right| \right] \\
&\leq \left( e^{(C_{Lip}^f + \lambda)T} C_{Lip}^g + \sum_{j=i+1}^{N-1} \frac{(C_{Lip}^f + \lambda C_{Lip}^S)h}{(1 - (C_{Lip}^f + \lambda)h)^{j-i}} \right) \sup_{i \leq j \leq N} \mathbb{E} \left[ \left| X_{t_j}^{t_i,x} - X_{t_j}^{t_i,x'} \right| \right] \\
&\leq \left( e^{(C_{Lip}^f + \lambda)T} C_{Lip}^g + (e^{(C_{Lip}^f + \lambda)T} - 1) \frac{C_{Lip}^f + \lambda C_{Lip}^S}{C_{Lip}^f + \lambda} \right) \sup_{i \leq j \leq N} \mathbb{E} \left[ \left| X_{t_j}^{t_i,x} - X_{t_j}^{t_i,x'} \right| \right].
\end{aligned}$$

From the a priori estimates of SDE (see [133, Theorem 3.2.4]), there exists a constant  $C_{SDE}$  depending only on  $T, C_{Lip}^X, d$  s.t.

$$\sup_{i \leq j \leq N} \mathbb{E} \left[ \left| X_{t_j}^{t_i,x} - X_{t_j}^{t_i,x'} \right| \right] \leq \mathbb{E} \left[ \sup_{i \leq j \leq N} \left| X_{t_j}^{t_i,x} - X_{t_j}^{t_i,x'} \right|^2 \right]^{1/2} \leq C_{SDE} |x - x'|.$$

We conclude with,  $\forall x, x' \in \mathbb{R}^d$ ,

$$\begin{aligned}
|P_i u_{i+1}(x) - P_i u_{i+1}(x')| &= \left| \mathbb{E} \left[ Y_{t_{i+1}}^{h,t_i,x} - Y_{t_{i+1}}^{h,t_i,x'} \right] \right| \leq \mathbb{E} \left[ \left| Y_{t_{i+1}}^{h,t_i,x} - Y_{t_{i+1}}^{h,t_i,x'} \right| \right] \\
&\leq C_{\lambda,T,g,X,f} |x - x'|.
\end{aligned}$$

□

**Remark 4.3.6.** In the above, we get the Lipschitz constant depends on the penalty  $\lambda$ . This result, though not optimal, serves as a starting point, and we intend to demonstrate this uniform Lipschitz continuity in our future investigations.

We give a result of exponential moment of the solution to (4.1.3) on finite horizon, see more in [90].

**Definition 4.3.7.** Define the BMO norm for the continuous martingale  $(\mathcal{X})_{0 \leq t \leq T}$  as follows,

$$\|\mathcal{X}\|_{\text{BMO}} := \sup_{\tau \in \mathcal{T}_{0,T}} \|\mathbb{E} [|\mathcal{X}_T - \mathcal{X}_\tau| | \mathcal{F}_\tau]\|_{\mathbb{L}^\infty},$$

and we say  $\mathcal{X}$  satisfies BMO condition if  $\|\mathcal{X}\|_{\text{BMO}} < \infty$ .

For any  $0 \leq i \leq N - 1$ , we define martingales  $\mathcal{X}^i := \{\mathbb{E} [X_{t_i} | \mathcal{F}_\tau] : \tau \in \mathcal{T}_{0,t_i}\}$ .



**Proposition 4.3.8.** *Under  $H_1$ -5, assume*

$$\sup_{0 \leq i \leq N-1} \|\mathcal{X}^i\|_{\text{BMO}} < \infty,$$

*then there exists a universal constant  $c$  such that,*

$$\sup_{0 \leq i \leq N-1} \mathbb{E} [e^{c|X_{t_i}|}] < \infty.$$

*Proof.* For a small enough positive constant  $c$  such that

$$\sup_{0 \leq i \leq N-1} \|c\mathcal{X}^i\|_{\text{BMO}} = c \sup_{0 \leq i \leq N-1} \|\mathcal{X}^i\|_{\text{BMO}} \leq \frac{1}{4},$$

apply John-Nirenberg inequality (see [90][Theorem 2.1]), we have,

$$\begin{aligned} \mathbb{E} [e^{c|\mathcal{X}_{t_i}^i|}] &\leq \mathbb{E} [e^{c|\mathcal{X}_0^i|} e^{c|\mathcal{X}_{t_i}^i - \mathcal{X}_0^i|}] \leq e^{c\mathbb{E}[|X_{t_i}|]} \mathbb{E} [e^{c|\mathcal{X}_{t_i}^i - \mathcal{X}_0^i|}] \\ &\leq e^{c\mathbb{E}[|X_{t_i}|]} \frac{1}{1 - 4\|c\mathcal{X}^i\|_{\text{BMO}}}, \end{aligned}$$

where we complete the proof. □

## 4.4 Linear Regression

In this section, we give the estimates of in-sample error (w.r.t.  $\mu_i^M$ ) and out-sample error (w.r.t.  $\mu_{X_{t_i}}$ ). Here the function space is linear:  $\Psi = \text{Span}(\psi)$ . To clarify the distinction from **GLS**, we will replace the approximation space  $\Psi$  in superscripts by its degree  $K$ , i.e. we denote  $\hat{u}^{\lambda, h, M, \Psi}$  by  $\hat{u}^{\lambda, h, M, K}$  etc.

### 4.4.1 In sample error

**Theorem 4.4.1.** *Under  $(H_1)$ -(1-4) and  $H_2$ -1, for the implicit scheme (4.1.5) and its approximation in (4.2.1), we have the following estimation of in-sample error: if  $h < 1 \wedge \frac{1}{2C_{Lip}^f}$ , then in  $\mathbb{L}^{2,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ , there exists a constant  $C$  depending only upon  $f, T, S, g, X$  such that,*

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, K}(\cdot) - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] &\leq (1 + Ch) \mathbb{E} \left[ \left\| \hat{u}_{i+1}^{\lambda, h, M, K}(\cdot) - u_{i+1}^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{i+1}^M)}^2 \right] \\ &\quad + \frac{CK}{hM} + (1 + C_{4.6.1}h) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_i u_{i+1}^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \end{aligned}$$

and in  $\mathbb{L}^{1,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ , let  $L$  be the bound as in Lemma 4.3.3,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] &\leq (1 + C_{4.6.1}h) \left\{ \mathbb{E} \left[ \left\| \hat{u}_{i+1}^{\lambda,h,M,K}(\cdot) - u_{i+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{i+1}^M)} \right] \right. \\ &\quad \left. + \frac{L\sqrt{K}}{\sqrt{M}} + \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_i u_{i+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right\}. \end{aligned}$$

*Proof.* In the proof, we drop the fixed  $\lambda, h$ . For any fixed  $M, \Psi, K$ , for  $0 \leq i \leq N-1$ , given  $u_{i+1}(\cdot)$ , we want to compute  $u_i(\cdot)$ . Recall that

$$u_i(x) = \mathcal{V}_i^{-1}(x, P_i u_{i+1}(x))$$

and denote  $\bar{\alpha}^{M,i}$  s.t.

$$\mathbf{OLS}(u_{i+1}(\cdot), \Psi, \mu_{i,i+1}^M) := \Phi(\bar{\alpha}^{M,i}, \cdot) = \bar{\alpha}^{M,i} \cdot \psi(\cdot).$$

Thanks to the linearity of SVD-optimal coefficient (see [72, Proposition 4.12]), we have

$$\begin{aligned} \bar{\alpha}^{M,i} \text{ solves } \mathbf{OLS}(u_{i+1}(\cdot), \Psi, \mu_{i,i+1}^M) &\implies \\ \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \text{ solves } \mathbf{OLS} \left( P_i u_{i+1}(\cdot), \Psi, \mu_i^M \right) &\quad (4.4.1) \end{aligned}$$

because

$$\begin{aligned} \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] &= \arg \inf_{\alpha \in \mathbb{R}^K} \frac{1}{M} \sum_{m=1}^M \left( \mathbb{E} \left[ u_{i+1}(X_{t_{i+1}}^m) | X_{t_i}^{(1:M)} \right] - \alpha \cdot \psi(X_{t_i}^m) \right)^2 \\ &= \arg \inf_{\alpha \in \mathbb{R}^K} \frac{1}{M} \sum_{m=1}^M \left( P_i u_{i+1}(X_{t_i}^m) - \alpha \cdot \psi(X_{t_i}^m) \right)^2. \end{aligned}$$

▷ *Step 1: Pythagoras decomposition.* With (4.2.1), since  $\mathcal{V}_i^{-1}$  is  $(1 + C_{4.6.1}h)$ -Lipschitz in  $w$  uniformly in  $x, \lambda$  (see Lemma 4.6.1) and the clipping function  $\mathcal{C}_L$  is 1-Lipschitz, we have

$$\begin{aligned} \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 &= \frac{1}{M} \sum_{m=1}^M \left( \mathcal{V}_i^{-1}(X_{t_i}^m, \mathcal{C}_L(\hat{\alpha}^{M,i} \cdot \psi(X_{t_i}^m))) - \mathcal{V}_i^{-1}(X_{t_i}^m, P_i u_{i+1}(X_{t_i}^m)) \right)^2 \\ &\leq (1 + C_{4.6.1}h)^2 \frac{1}{M} \sum_{m=1}^M \left( \mathcal{C}_L(\hat{\alpha}^{M,i} \cdot \psi(X_{t_i}^m)) - P_i u_{i+1}(X_{t_i}^m) \right)^2 \\ &\leq (1 + C_{4.6.1}h)^2 \left\| \hat{\alpha}^{M,i} \cdot \psi(\cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2. \end{aligned} \quad (4.4.2)$$

The Pythagoras decomposition provides, with (4.4.1),

$$\begin{aligned} \left\| \hat{\alpha}^{M,i} \cdot \psi(\cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 &= \left\| \left( \hat{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2. \end{aligned} \quad (4.4.3)$$

Thus, we have, for (4.4.2),

$$\begin{aligned} \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 &\leq (1 + C_{4.6.1}h)^2 \left\{ \left\| \left( \hat{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right. \\ &\quad \left. + \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 \right\}. \end{aligned} \quad (4.4.4)$$

▷ *Step 2: Young's inequality.* For the estimation of the right side of (4.4.4) in  $\mathbb{L}^{2,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ : using Young's inequality for the norm of  $\mathbb{L}^2(\mu_i^M)$ , we get

$$\begin{aligned} \left\| \left( \hat{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 &\leq \left( 1 + \frac{1}{h} \right) \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + (1 + h) \left\| \left( \hat{\alpha}^{M,i} - \bar{\alpha}^{M,i} \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2. \end{aligned}$$

So, with assuming  $h < 1 \wedge \frac{1}{C_{Lip}^f}$  and using the generic constant  $C$  depending on  $T, f, S, \xi$  but not on  $\lambda, h, i, M, K, \Phi$  which can vary from line to line, we have, from (4.4.4)

$$\begin{aligned} \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 &\leq (1 + C_{4.6.1}h)^2 \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + (1 + C_{4.6.1}h)^2 \left( 1 + \frac{1}{h} \right) \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + (1 + C_{4.6.1}h)^2 (1 + h) \left\| \left( \hat{\alpha}^{M,i} - \bar{\alpha}^{M,i} \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\leq (1 + Ch) \left\| \left( \hat{\alpha}^{M,i} - \bar{\alpha}^{M,i} \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + \frac{C}{h} \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \\ &\quad + (1 + C_{4.6.1}h) \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2. \end{aligned} \quad (4.4.5)$$

Alternatively, we can also have the estimation in  $\mathbb{L}^{1,2}(\Omega \times \mathcal{B}(\mathbb{R}^d), \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ . Indeed, from (4.4.2) and (4.4.3),

$$\begin{aligned} \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} &\leq (1 + C_{4.6.1}h) \left\{ \left\| \left( \hat{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right. \\ &\quad \left. + \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)} \right\}. \end{aligned}$$

Similarly, for the first term, applying triangular inequality, we get

$$\begin{aligned} \left\| \left( \hat{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} &\leq \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \\ &\quad + \left\| \left( \hat{\alpha}^{M,i} - \bar{\alpha}^{M,i} \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \end{aligned}$$

which yields

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] &\leq (1 + C_{4.6.1}h) \left\{ \mathbb{E} \left[ \left\| (\hat{\alpha}^{M,i} - \bar{\alpha}^{M,i}) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] \right. \\ &\quad + \mathbb{E} \left[ \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] \\ &\quad \left. + \mathbb{E} \left[ \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] \right\}. \end{aligned} \quad (4.4.6)$$

▷ *Step 3: Analyse each component in (4.4.5) and (4.4.6).* We start from the first term in (4.4.5) (respectively in (4.4.6)), due to the fact that the OLS coefficient is a linear map with respect to the response i.e.

$$\hat{\alpha}^{M,i} - \bar{\alpha}^{M,i} \text{ solves } \mathbf{OLS} \left( \hat{u}_{i+1}^{M,K}(\cdot) - u_{i+1}(\cdot), \Psi, \mu_{i,i+1}^M \right),$$

and the non-expanding assumption  $H_2$ -2 which is verified by **OLS**, we get

$$\left\| (\hat{\alpha}^{M,i} - \bar{\alpha}^{M,i}) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \leq \left\| \hat{u}_{i+1}^{M,K}(\cdot) - u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{i,i+1}^M)}. \quad (4.4.7)$$

For the second term in (4.4.5), we introduce the following result

**Lemma 4.4.1** ([78, Theorem 11.1]). *Consider the computation of  $\mathbb{E}[\mathcal{R}|\mathcal{O}]$  by linear regression. Let*

$$\tilde{\mathcal{M}}_M(\omega, \cdot) := \mathbf{OLS}(\mathcal{R}(\omega, \cdot), \Psi, \mu^M)$$

where  $\Psi$  is generated by at most  $K$  functions and  $\mu^M$  is the law of  $\mathcal{O}$ . If  $\sigma^2 := \sup_{o \in \mathbb{R}^d} \text{Var}[\mathcal{R}|\mathcal{O} = o] < \infty$ , then

$$\mathbb{E} \left[ \left\| \tilde{\mathcal{M}}_M - \mathbb{E}^{(1:M)} \left[ \tilde{\mathcal{M}}_M \right] \right\|_{\mathbb{L}^2(\mu^M)}^2 \middle| \mathcal{O}^{(1:M)} \right] \leq \sigma^2 \frac{K}{M}.$$

With Lemma 4.3.3, we have  $\sup_{0 \leq i \leq N-1} \sup_{x \in \mathbb{R}^d} \text{Var} \left[ u_{i+1}^{\lambda,h}(X_{t_{i+1}}^{t_i,x}) \right] \leq L^2$ , so applying the above result, we get

$$\frac{C}{h} \mathbb{E} \left[ \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] \leq \frac{C}{h} L^2 \frac{K}{M}. \quad (4.4.8)$$

( resp, with Cauchy-Schwarz inequality:

$$\begin{aligned} &\mathbb{E} \left[ \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E} \left[ \bar{\alpha}^{M,i} | X_{t_i}^{(1:M)} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] \\ &\leq \sqrt{\mathbb{E} \left[ \mathbb{E} \left[ \left\| \left( \bar{\alpha}^{M,i} - \mathbb{E}^{(1:M)} \left[ \bar{\alpha}^{M,i} \right] \right) \cdot \psi(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \middle| X_{t_i}^{(1:M)} \right] \right]} \leq L \sqrt{\frac{K}{M}}. \end{aligned} \quad (4.4.9)$$

To handle the last term in (4.4.5) (resp. in (4.4.6)), take expectation, since  $\inf$  is concave on  $\varphi \in \Psi$ ,

$$\begin{aligned} \mathbb{E} \left[ \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] &\leq \inf_{\varphi \in \Psi} \mathbb{E} \left[ \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] \\ &= \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2. \end{aligned} \quad (4.4.10)$$

$$\begin{aligned} \left( \text{resp. } \mathbb{E} \left[ \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)} \right] \right) &\leq \inf_{\varphi \in \Psi} \mathbb{E} \left[ \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_i^M)} \right] \\ &\leq \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_{X_{t_i}})}. \end{aligned} \quad (4.4.11)$$

All in all, combine (4.4.5),(4.4.7),(4.4.8), (4.4.10) (resp. (4.4.6), (4.4.7), (4.4.9), (4.4.11)) we have the desired results.  $\square$

## 4.4.2 Out sample error

**Proposition 4.4.2** (deviation between empirical mean and exact mean). *Under  $(H_1)$ -(1-4) and  $H_2$ -1 and let  $\bar{L}$  be the constant as in Corollary 4.3.4, we have*

$$\begin{aligned} \mathbb{E} \left[ \left| \left\| \hat{u}^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 - \left\| \hat{u}^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \right| \right] &\quad (4.4.12) \\ &\leq 101(1 + C_{4.6.1}h)\bar{L}^2 \sqrt{\frac{(K+1)\log(6M)}{M}} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[ \left| \left\| \hat{u}^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} - \left\| \hat{u}^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right| \right] &\quad (4.4.13) \\ &\leq 11\sqrt{(1 + C_{4.6.1}h)}\bar{L} \sqrt[4]{\frac{(K+1)\log(6M)}{M}}. \end{aligned}$$

*Proof.* For simplifying notations, we ignore  $\lambda, h$  universally in  $u_i^{\lambda,h,M,K}$  and  $u_i^{\lambda,h}$ . Denote

$$\begin{aligned} \eta_i &:= \left| \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 - \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \right|, \\ \phi_i^\Psi &:= \left\{ \phi_i^\varphi := (\mathcal{V}_i^{-1}(\cdot, \mathcal{C}_{L_i}\varphi) - \mathcal{V}_i^{-1}(\cdot, P_i u_{i+1}))^2 : \varphi \in \Psi \right\}. \end{aligned}$$

Then, since  $\eta_i$  is a positive random variable, we have

$$\mathbb{E} [\eta_i] = \int_{\mathbb{R}^+} \mathbb{P}(\eta_i > \varepsilon) \mathbf{d}\varepsilon$$

and from Corollary 4.3.4, we have,

$$\sup_{\varphi \in \Psi} |\phi_i^\varphi(\cdot)|_\infty \leq 4\bar{L}^2.$$

▷ *Step 1:* Estimate  $\mathbb{P}(\eta_i > \varepsilon)$  using the boundedness of space  $\mathcal{C}_L\Psi$ . We first have,

$$\mathbb{P}(\eta_i > \varepsilon) \leq \mathbb{P}\left(\exists \varphi \in \Psi \text{ s.t. } \left| \frac{1}{M} \sum_{m=1}^M \phi_i^\varphi(X_{t_i}^m) - \int_{\mathbb{R}^d} \phi_i^\varphi(x) \mu_{X_{t_i}}(\mathbf{d}x) \right| > \varepsilon\right).$$

Then, with assuming  $h \leq 1$ , use again the Lipschitz continuity of  $\mathcal{V}_i^{-1}$  and the bound as in Lemma 4.3.3, we have, for any  $\varphi, \varphi' \in \Psi, \mu \in \{\mu_i^M, \mu_{X_{t_i}}\}$ ,

$$\begin{aligned} \left\| \phi_i^\varphi(\cdot) - \phi_i^{\varphi'}(\cdot) \right\|_{\mathbb{L}^1(\mu)} &= \left\| [\mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi) - \mathcal{V}_i^{-1}(\cdot, P_i u_{i+1})]^2 - [\mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi') - \mathcal{V}_i^{-1}(\cdot, P_i u_{i+1})]^2 \right\|_{\mathbb{L}^1(\mu)} \\ &= \left\| |\mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi) + \mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi') - 2\mathcal{V}_i^{-1}(\cdot, P_i u_{i+1})| \times |\mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi) - \mathcal{V}_i^{-1}(\cdot, \mathcal{C}_L\varphi')| \right\|_{\mathbb{L}^1(\mu)} \\ &\leq 4\bar{L}(1 + C_{4.6.1}h) \|\mathcal{C}_L\varphi - \mathcal{C}_L\varphi'\|_{\mathbb{L}^1(\mu)}. \end{aligned}$$

If we compare the covering number of  $\phi_i^\Psi$  and the one of  $\mathcal{C}_L\Psi$  in  $\mathbb{L}^1(\mu)$ , we have, for any fixed  $0 < \varepsilon' \leq 2\bar{L}L(1 + C_{4.6.1}h)$ ,

$$\mathcal{N}_1(\varepsilon', \phi_i^\Psi, \mu) \leq \mathcal{N}_1\left(\frac{\varepsilon'}{4\bar{L}(1 + C_{4.6.1}h)}, \mathcal{C}_L\Psi, \mu\right) \leq \left(\frac{24\bar{L}L(1 + C_{4.6.1}h)}{\varepsilon'}\right)^{2(K+1)}$$

where the last inequality comes from Theorem 4.6.1: for any fixed  $0 < \varepsilon' \leq 2\bar{L}L(1 + C_{4.6.1}h)$ , there exists  $\{\mathcal{C}_L\varphi_j : 1 \leq j \leq n\}$  a  $\frac{\varepsilon'}{4\bar{L}(1 + C_{4.6.1}h)}$ -covering of  $\mathcal{C}_L\Psi$  and its covering number is dominated. Apply Theorem 4.6.2 (uniform deviation probability), for all  $\varepsilon \leq 16\bar{L}^2$ , then

$$\begin{aligned} \mathbb{P}\left(\exists \phi \in \phi_i^\Psi : \left| \frac{1}{M} \sum_{m=1}^M \phi(X_{t_i}^m) - \int_{\mathbb{R}^d} \phi(y) \mu_{X_{t_i}}(\mathbf{d}x) \right| > \varepsilon\right) \\ \leq 8\mathbb{E}\left[\mathcal{N}_1\left(\frac{\varepsilon}{8}, \phi_i^\Psi, \mu_i^M\right)\right] \exp\left(-\frac{\varepsilon^2 M}{128 \times (4\bar{L}^2)^2}\right) \\ \leq 8 \left(\frac{192\bar{L}L(1 + C_{4.6.1}h)}{\varepsilon}\right)^{2(K+1)} \exp\left(-\frac{\varepsilon^2 M}{2048\bar{L}^4}\right) \\ \leq 8 \left(\frac{192\bar{L}^2(1 + C_{4.6.1}h)}{\varepsilon}\right)^{2(K+1)} \exp\left(-\frac{\varepsilon^2 M}{2048\bar{L}^4}\right) \end{aligned}$$

since  $L \leq \bar{L}$ ; if this condition on  $\varepsilon$  is not satisfied, the upper bound still remains true since the probability to bound is then equal to 0 in view of the  $\Phi^\Psi$ -bounds equal to  $4\bar{L}^2$ .

▷ *Step 2: Adjusting  $\varepsilon_0$ .* Plugging this inequality into that for  $\mathbb{E}[\eta_i]$ , we obtain for any  $\varepsilon_0 > 0$ :

$$\mathbb{E}[\eta_i] \leq \varepsilon_0 + 8 \int_{\varepsilon_0}^{\infty} \left(\frac{192\bar{L}^2(1 + C_{4.6.1}h)}{\varepsilon}\right)^{2(K+1)} \exp\left(-\frac{\varepsilon^2 M}{2048\bar{L}^4}\right) \mathbf{d}\varepsilon.$$

For  $\varepsilon_0 \geq \frac{192\bar{L}^2(1+C_{4.6.1}h)}{\sqrt{6M}}$ , we deduce

$$\begin{aligned} \mathbb{E}[\eta_i] &\leq \varepsilon_0 + 8(6M)^{K+1} \int_{\varepsilon_0}^{\infty} \frac{\sqrt{6M}\varepsilon}{192\bar{L}^2(1+C_{4.6.1}h)} \exp\left(-\frac{\varepsilon^2 M}{2048\bar{L}^4}\right) \mathbf{d}\varepsilon \\ &= \varepsilon_0 + (6M)^{K+1} \frac{128\sqrt{6}\bar{L}^2}{3(1+C_{4.6.1}h)\sqrt{M}} \exp\left(-\frac{\varepsilon_0^2 M}{2048\bar{L}^4}\right). \end{aligned}$$

The choice  $\varepsilon_0 = \frac{\bar{L}^2(1+C_{4.6.1}h)}{\sqrt{M}} \sqrt{2048(K+1)\log(6M)}$  verifies  $(6M)^{K+1} \exp\left(-\frac{\varepsilon_0^2 M}{2048\bar{L}^4}\right) \leq 1$  and  $\varepsilon_0 \geq \frac{192\bar{L}^2(1+C_{4.6.1}h)}{\sqrt{6M}}$ . Hence,

$$\begin{aligned} \mathbb{E}[\eta_i] &\leq \varepsilon_0 + \frac{128\sqrt{6}\bar{L}^2}{3(1+C_{4.6.1}h)\sqrt{M}} \\ &= \frac{\bar{L}^2(1+C_{4.6.1}h)}{\sqrt{M}} \sqrt{(K+1)\log(6M)} \left( \sqrt{2048} + \frac{128\sqrt{6}}{3\sqrt{(K+1)\log(6M)}(1+C_{4.6.1}h)^2} \right) \\ &\leq 101\bar{L}^2(1+C_{4.6.1}h) \sqrt{\frac{(K+1)\log(6M)}{M}}. \end{aligned}$$

▷ *Step 3: Proof of (4.4.13).* The proof for (4.4.13) is quite similar with that of (4.4.12) because for any  $a, b \in \mathbb{R}^+$ ,

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}.$$

so, by Jensen's inequality, we have

$$\begin{aligned} &\mathbb{E} \left[ \left| \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} - \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right| \right] \\ &\leq \sqrt{\mathbb{E} \left[ \left| \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 - \left\| \hat{u}_i^{M,K}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \right| \right]} \\ &\leq 11\bar{L} \sqrt{(1+C_{4.6.1}h)} \sqrt[4]{\frac{(K+1)\log(6M)}{M}}. \end{aligned}$$

□

### 4.4.3 Global error estimation

We are ready to state our main results of this section in the case of **OLS**.

**Theorem 4.4.2** (global in-sample error). *Under  $(H_1)$ -(1-4) and  $H_2$ -1, let  $\hat{u}_i^{\lambda,h,M,K}$  be defined in (4.2.1) and  $u_i^{\lambda,h}$  be defined in (4.1.7), if  $h < 1 \wedge \frac{1}{2C_{Lip}^f}$ , let  $C$  be the constant as in Theorem*

4.4.1, then we have,  $\forall 0 \leq i \leq N-1$ , in  $\mathbb{L}^{2,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$

$$\mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] \leq e^{CT} \sum_{j=i}^{N-1} \left[ \frac{CK}{hM} + (1 + C_{4.6.1}h) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})}^2 \right];$$

and in  $\mathbb{L}^{1,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ , let  $L$  be the bound as in Lemma 4.3.3,

$$\mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)} \right] \leq e^{C_{4.6.1}T} (1 + C_{4.6.1}h) \sum_{j=i}^{N-1} \left[ L \sqrt{\frac{K}{M}} + \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})} \right].$$

*Proof.* From Theorem 4.4.1, applying the Gronwall's lemma,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_i^M)}^2 \right] &\leq e^{CT} \left( \mathbb{E} \left[ \left\| \hat{u}_N^{\lambda,h,M,K}(\cdot) - u_N^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_N^M)}^2 \right] \right. \\ &\quad \left. + \sum_{j=i}^{N-1} \left[ \frac{CK}{hM} + (1 + C_{4.6.1}h) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(X_{t_j})}^2 \right] \right), \end{aligned}$$

thanks to (4.1.7), the first term at the right side of the above inequality yields to 0, where we have the desired result. The proof for estimation in  $\mathbb{L}^{1,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$  also follows immediately from Gronwall's lemma.  $\square$

From Theorem 4.4.2 and Proposition 4.4.2, we immediately have the following results.

**Theorem 4.4.3** (global out-sample error). *Under  $(H_1)$ -(1-4) and  $H_2$ -1, let  $u_i^{\lambda,h}$  be defined in (4.1.7) and  $\hat{u}_i^{\lambda,h,M,K}$  be defined in (4.2.1), if  $h < 1 \wedge \frac{1}{2C_{4.6.1}^{Lip}}$ , let  $C$  be the constant as in Theorem 4.4.1, then we have,  $\forall 0 \leq i \leq N-1$ , in  $\mathbb{L}^{2,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ ,*

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}^2 \right] &\leq e^{CT} \sum_{j=i}^{N-1} \left[ \frac{CK}{hM} + (1 + C_{4.6.1}h) \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})}^2 \right] \\ &\quad + 101(1 + C_{4.6.1}h) \bar{L}^2 \sqrt{\frac{(K+1) \log(6M)}{M}}, \end{aligned}$$

and in  $\mathbb{L}^{1,2}(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ ,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda,h,M,K}(\cdot) - u_i^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] &\leq e^{C_{4.6.1}T} (1 + C_{4.6.1}h) \sum_{j=i}^{N-1} \left[ L \sqrt{\frac{K}{M}} + \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_j u_{j+1}^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_j}})} \right] \\ &\quad + 11 \sqrt{(1 + C_{4.6.1}h) \bar{L}^4 \frac{(K+1) \log(6M)}{M}}. \end{aligned}$$



## 4.5 Non-Linear Regression

In this section, we introduce the error analysis with respect to the probability measure  $\mu$  on  $\mathbb{R}^d$  for the non-linear regression as Procedure 4.2.1, namely, how to dominate

$$\mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi}(\cdot) - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right]$$

with respect to the  $M, N$  and the complexity of neural network. Differently from the linear case, we directly introduce the global error without analyzing the empirical error. We first give the following discussion:

**Discussion 4.5.1.** we drop  $\lambda, h$  in superscripts and recall that

$$\begin{aligned} u_i(x) &= \mathcal{V}_i^{-1}(x, P_i u_{i+1}(x)), \\ \hat{u}_i^{M, \Psi}(x) &= \mathcal{V}_i^{-1}\left(x, \mathcal{C}_L \mathbf{GLS}(\hat{u}_{i+1}^{M, \Psi}(\cdot), \Psi, \mu_{i, i+1}^M(x))\right) = \mathcal{V}_i^{-1}(x, \mathcal{C}_L \Phi(\hat{\alpha}^{M, i}, x)), \\ \bar{u}_i^{M, \Psi}(x) &= \mathcal{V}_i^{-1}\left(x, \mathcal{C}_L \mathbf{GLS}(u_{i+1}(\cdot), \Psi, \mu_{i, i+1}^M(x))\right) = \mathcal{V}_i^{-1}(x, \mathcal{C}_L \Phi(\bar{\alpha}^{M, i}, x)). \end{aligned}$$

For estimating out-sample error, at time  $t_i$ , we have

$$\begin{aligned} & \left\| \hat{u}_i^{M, \Psi}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \\ & \leq \left\| \hat{u}_i^{M, \Psi}(\cdot) - \bar{u}_i^{M, \Psi}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} + \left\| \bar{u}_i^{M, \Psi}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \\ & \leq (1 + C_{4.6.1}h) \left( \left\| \Phi(\hat{\alpha}^{M, i}, \cdot) - \Phi(\bar{\alpha}^{M, i}, \cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} + \left\| \Phi(\bar{\alpha}^{M, i}, \cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right) \\ & \leq (1 + C_{4.6.1}h) \left( \left\| \hat{u}_{i+1}^{M, \Psi}(\cdot) - u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_{i+1}}})} + \left\| \Phi(\bar{\alpha}^{M, i}, \cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right); \end{aligned}$$

the last inequality holds if Assumption  $H_2$ -2 is true. For the second term, [78, p. 201, Theorem 11.5] shows that,

$$\begin{aligned} \mathbb{E} \left[ \left\| \Phi(\bar{\alpha}^{M, i}, \cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] &= \mathbb{E} \left[ \left\| \mathbf{GLS}(u_{i+1}(\cdot), \Psi, \mu_{i, i+1}^M) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] \\ &\leq \frac{c_1}{M} + \frac{(c_2 + c_3 \log(M))v_{\Psi^+}}{M} + 2 \inf_{\varphi \in \Psi} \left\| \varphi(\cdot) - P_i u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})}, \end{aligned}$$

where

- $c_1, c_2, c_3$  are universal constants without depending on the simulation parameters (e.g.  $M, h, \lambda, \mu$ );
- $\Psi^+ := \{ \{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y \leq \varphi(x) \} : \varphi \in \Psi \}$ ;

- the response  $u_{i+1}$  is dominated by  $L \vee 1$  from Lemma 4.3.3 which verifies the assumption of Theorem 11.5;
- $v_{\Psi^+}$  defined in Definition 4.7.1 is the Vapnik-Chervonenkis dimension (VC dimension in short) of  $\Psi^+$  which is detailed in the following subsection.

Then we have,

$$\mathbb{E} \left[ \left\| \hat{u}_i^{M, \Psi}(\cdot) - u_i(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] \leq (1 + C_{4.6.1}h) \left\{ \mathbb{E} \left[ \left\| \hat{u}_{i+1}^{M, \Psi}(\cdot) - u_{i+1}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_{i+1}}})} \right] + C \frac{1 + (1 + \log M)v_{\Psi^+}}{M} + 2 \inf_{\varphi \in \Psi} \|\varphi(\cdot) - P_i u_{i+1}(\cdot)\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right\} \quad (4.5.1)$$

where the first term is going to be cumulated by Gronwall's lemma, and the complexity measure  $v_{\Psi^+}$  will be analyzed in Proposition 4.5.2 for the case of neural network and the approximation error will be stated in Proposition 4.5.3.

### 4.5.1 Complexity measure $v_{\Psi^+}$

**The linear approximation space:**

$$\Psi(K) := \left\{ \sum_{i=1}^K w_i \psi_i(x) : w_1, \dots, w_K \in \mathbb{R} \right\}. \quad (4.5.2)$$

**Proposition 4.5.1** ([78, Theorem 9.5]). *Assume  $(\mathbf{H}_2-i)$ , for the approximation space  $\Psi(K)$  defined in (4.5.2), we have,*

$$v_{\Psi(K)^+} \leq K + 1.$$

**The multi-layer neural network approximation space.** Let

$$\Psi(\mathcal{L}) = \left\{ \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \mid \varphi = \psi_{\mathcal{L}} \circ \psi_{\mathcal{L}-1} \cdots \circ \psi_0, \right. \\ \left. \text{where } \forall 1 \leq j \leq \mathcal{L} - 1, \psi_j \in \mathcal{L}_{d_{j-1}, d_j}^\rho \text{ and } \psi_{\mathcal{L}} \in \mathcal{L}_{d_{\mathcal{L}-1}, 1} \right\} \quad (4.5.3)$$

with

$$\mathcal{L}_{d_j, d_{j+1}}^\rho := \left\{ \phi : \mathbb{R}^{d_j} \rightarrow \mathbb{R}^{d_{j+1}} \mid \forall 0 \leq i \leq d_{j+1}, \phi(x)_i = \rho(b_i + w_i^\top x), \right. \\ \left. \text{where } b \in \mathbb{R}^{d_{j+1}}, w_i \in \mathbb{R}^{d_j} \right\},$$

$\mathcal{L}_{d_j, d_{j+1}}$  being defined in the same way as  $\mathcal{L}_{d_j, d_{j+1}}^\rho$  with  $\rho(x) = x$ ,

where  $d_0 = d$  and  $\rho$  is the activation functions such as ReLU, Sigmoid etc. In order to show how  $v_{\Psi(\mathcal{L})+}$  is dominated, we introduce the following notations of neural network: the depth of network  $\mathcal{L} + 2$  and the number of parameters to be estimated, namely the number of the weights and bias  $W$  where we use the convention that a network with one hidden layer has depth 3 (see also [1] for the use of these parameters).

We give our estimation of VC dimension, see its proof in Appendix 4.7.2.

**Proposition 4.5.2.** *Consider the architecture of neural network for computing  $\Psi(\mathcal{L})$  as (4.5.3) which has  $W$  parameters and  $\mathcal{L}$  layers, and suppose all the non-output units have the piecewise-linear activation function  $\rho$ , then*

$$v_{\Psi(\mathcal{L})+} \leq O(W\mathcal{L} \log(W)), \quad (4.5.4)$$

where  $O$  is up to a universal constant.

## 4.5.2 The approximation error by Neural Network in $\mathbb{L}^2(\mu)$

In this section, we handle the error  $\inf_{\varphi \in \Psi} \|\varphi - P_i u_{i+1}\|$  in (4.5.1). For the multi-layer neural network as (4.5.3) with piecewise linear activation  $\rho$ , many works have contributed to analyze this error especially on compact sets of  $\mathbb{R}^d$ . Hornik [85] first proves the approximation built on a compact set is dense in  $C(\mathbb{R}^d, \mathbb{R})$  under  $\mathbb{L}^p, p \geq 1$  metrics; Barron [4] shows the rate of convergence of the approximation on the compact with or without the constraint of parameters; Voigtlaender and Petersen [128] give a survey on recent results on compacts in  $\mathbb{L}^p$ -metrics. The  $\mathbb{L}^\infty$  error is also much investigated. Yarotsky [130, Theorem 1] illustrates the approximation on unit ball  $[0, 1]^d$  converges at  $O(W^{-n/d})$  up to a logarithmic factor with  $n$  denoting the differentiability order of target function; then he improves this result to  $O(W^{-2/d})$  in [131, Theorem 2] for a Lipschitz continuous target; Montanelli and Yang [111] also extend the result on any  $[0, R]^d$  by replacing the differentiability by the generalized 'Lipschitz'-kind condition. Some similar results can also be found in [10, Theorem 6.3] and [76, Theorem 3.14] for some special PDEs. For now, we use the  $\mathbb{L}^\infty$  norm on compact to get the estimation of norm  $\mathbb{L}^p(\mu)$ .

**Proposition 4.5.3** (Deep Network). *Assume  $(H_1)$  with  $H_1$ -5 being satisfied with some  $q \geq 2$  and assume  $P_i u_{i+1}^{\lambda, h}$  is uniformly Lipschitz with respect to  $\lambda$  and to  $i \in \{0, \dots, N-1\}$ . Suppose  $u_i^{\lambda, h}(\cdot)$  is bounded by  $L$  on  $\mathbb{R}^d$  as in Lemma 4.3.3, then for any fixed  $d > 0$ , for any  $\varepsilon \in (0, 1)$ , there exists a network **NN** with ReLU activation satisfying*

(a): *the number of weights and biases  $W$  and the number of computation units  $U$ :*

$$\max\{W, U\} \leq \left( \varepsilon^{-d(1+\frac{d+2}{q})} \log \frac{1}{\varepsilon} \right);$$

(b): *the depth  $\mathcal{L} \leq O(\log \frac{1}{\varepsilon})$ ,*

as  $\varepsilon \rightarrow 0$  (with  $O$  depending on  $d, q$ ), such that,

$$\inf_{\varphi \in \Psi} \left\| \varphi - P_i u_{i+1}^{\lambda, h} \right\|_{\mathbb{L}^2(\mu)} \leq \varepsilon$$

where  $\Psi$  contains all the functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  computed by **NN** which can be written as  $\varphi(\cdot) := \Phi(\alpha, \cdot)$  with  $\alpha := \{\alpha_j : 1 \leq j \leq W\}$  the weights and biases,  $\mu$  is the law of  $X_{t_i}$ .

*Proof.* See Appendix 4.7.3. □

An alternative analyze for the complexity of neural network is provided in the following, in which a 'very deep' network is considered, namely,  $\mathcal{L} = O(W)$ . The proof is provided in Appendix 4.7.4.

**Proposition 4.5.4** (Very Deep network). *Assume  $(H_1)$  with  $H_1$ -5 being satisfied with some  $q \geq 2$  and  $P_i u_{i+1}^{\lambda, h}$  is uniformly Lipschitz with respect to  $\lambda$  and to  $i \in \{0, \dots, N-1\}$ . Suppose  $u_i^{\lambda, h}(\cdot)$  is bounded by  $L$  on  $\mathbb{R}^d$  as in Lemma 4.3.3, then for any fixed  $d > 0$ , for any  $\varepsilon \in (0, 1)$ , there exists a network **NN** with ReLU activation satisfying*

(a): *the number of weights and biases  $W$  and the number of computation units  $U$ :*

$$\max\{W, U\} \leq O\left(\varepsilon^{-\frac{d}{2}(1+\frac{d+2}{q})}\right);$$

(b): *the depth  $\mathcal{L} \leq O\left(\varepsilon^{-\frac{d}{2}(1+\frac{d+2}{q})}\right)$ ,*

as  $\varepsilon \rightarrow 0$  (with  $O$  depending on  $d$ ), such that

$$\inf_{\varphi \in \Psi} \left\| \varphi - P_i u_{i+1}^{\lambda, h} \right\|_{\mathbb{L}^2(\mu)} \leq \varepsilon,$$

where  $\Psi$  contains all the functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  simulated by **NN** which can be written as  $\varphi(\cdot) := \Phi(\alpha, \cdot)$  with  $\alpha := \{\alpha_j : 1 \leq j \leq W\}$  the weights and biases,  $\mu$  is the law of  $X_{t_i}$ .

### 4.5.3 Approximation by Deep Neural Network

With Discussion 4.5.1, Proposition 4.5.2 and Proposition 4.5.3, we have the following estimate of approximation errors.

**Theorem 4.5.1** (Tuning of parameters). *Assume  $(H_1)$  with  $H_1$ -5 being satisfied with some  $q \geq 2$  and  $H_2$ -2; assume  $P_i u_{i+1}^{\lambda, h}$  is uniformly Lipschitz w.r.t.  $\lambda$  and to  $i \in \{0, \dots, N-1\}$ . Let **NN** be a feedforward neural network for computing  $\Psi(\mathcal{L})$  as (4.5.3), which has the architecture as follows: there are  $W$  parameters (weights and biases) to estimate,  $\mathcal{L}$  layers, one input unit, one output unit and  $U$  non-zero computation units with ReLU activation. For any  $\varepsilon \in (0, 1)$ , if parameters of network **NN** satisfies,*

$$(a) \max(W, U) = \left(\varepsilon^{-d(1+\frac{d+2}{q})} \log \frac{1}{\varepsilon}\right);$$

$$(b) \mathcal{L} \leq O\left(\log \frac{1}{\varepsilon}\right).$$

Then we get, for  $0 \leq i \leq N - 1$ ,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi} - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] &\leq (1 + C_{4.6.1}h) \mathbb{E} \left[ \left\| u_{i+1}^{\lambda, h, M, \Psi}(\cdot) - u_{i+1}^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_{i+1}}})} \right] \\ &\quad + O \left( \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 + \varepsilon \right), \end{aligned} \quad (4.5.5)$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d, q$ ); for the global error, we have, for  $0 \leq i \leq N - 1$ ,

$$\mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi}(\cdot) - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] \leq O \left( N \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 + N\varepsilon \right), \quad (4.5.6)$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d, q$ ).

*Proof.* From Proposition 4.5.3, for  $\varepsilon \in (0, 1)$ , the network **NN** is such that,

$$\inf_{\varphi \in \Psi(\mathcal{L})} \left\| \varphi(\cdot) - P_i u_{i+1}^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu)} \leq \varepsilon.$$

Combine the discussion in subsection (4.5.1) and Proposition 4.5.2, we immediately get, with  $O$  depending on  $d, q$ ,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi} - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] &\leq (1 + C_{4.6.1}h) \mathbb{E} \left[ \left\| u_{i+1}^{\lambda, h, M, \Psi}(\cdot) - u_{i+1}^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_{i+1}}})} \right] \\ &\quad + O \left( \frac{(\log M)W\mathcal{L} \log W}{M} + \varepsilon \right); \end{aligned}$$

since  $\log W \leq O(\log \frac{1}{\varepsilon})$ , we have,

$$W\mathcal{L} \log W \leq O \left( \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 \right),$$

then we prove (4.5.5). For the global error, apply Gronwall's lemma,

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{u}_i^{\lambda, h, M, \Psi}(\cdot) - u_i^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_i}})} \right] &\leq e^{C_{4.6.1}T} \left\{ \mathbb{E} \left[ \left\| \hat{u}_N^{\lambda, h, M, \Psi}(\cdot) - u_N^{\lambda, h}(\cdot) \right\|_{\mathbb{L}^2(\mu_{X_{t_N}})} \right] \right. \\ &\quad \left. + O \left( N \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 + N\varepsilon \right) \right\} \\ &= O \left( N \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 + N\varepsilon \right), \end{aligned}$$

where we prove (4.5.6). □

**Remark 4.5.5** (Tuning of parameters). We are given the expected tolerance  $tol$  for the approximation  $u_i^{\lambda,h,M,\Psi}$  for all  $0 \leq i \leq N-1$ , i.e. if one wants to ensure

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[ \left\| u_i^{\lambda,h,M,\Psi}(\cdot) - u^{\lambda,h}(\cdot) \right\|_{\mathbb{L}^2(\mu)} \right] \leq tol,$$

then (4.5.6) tells us,

$$O \left( N \frac{\log M}{M} \varepsilon^{-d(1+\frac{d+2}{q})} \left( \log \frac{1}{\varepsilon} \right)^3 \right) = tol, \quad O(N\varepsilon) = tol$$

which yields the tuning of parameters (ignoring the term  $\log M$ ):

- the MC paths:  $M \geq O \left( \left( \frac{N}{tol} \right)^{d(1+\frac{d+1}{q})+1} \left( \log \frac{N}{tol} \right)^3 \right)$ ;
- the size of network:  $\max(W, U) = O \left( \left( \frac{N}{tol} \right)^{-d(1+\frac{d+2}{q})} \log \frac{N}{tol} \right)$ ;
- the depth of network:  $\mathcal{L} \leq O \left( \log \frac{N}{tol} \right)$ .

## 4.6 Appendix of Section 3.4

**Lemma 4.6.1** ([73, Lemma 4.1]). Assume  $H_1$ -2, if  $h < \frac{1}{2C_{Lip}^f} \wedge 1$ , there exists a constant  $C_{4.6.1}$ , s.t. for all  $x \in \mathbb{R}^d$ ,  $\lambda > 0$ ,  $(\mathcal{V}_i^\lambda)^{-1}$  is uniformly  $(1 + C_{4.6.1}h)$ -Lipschitz in  $w \in \mathbb{R}$ , namely

$$|(\mathcal{V}_i^\lambda)^{-1}(x, w) - (\mathcal{V}_i^\lambda)^{-1}(x, w')| \leq (1 + C_{4.6.1}h)|w - w'|.$$

where  $C_{4.6.1}$  depends only on  $C_{Lip}^f$ .

**Definition 4.6.2** ( $\mathbb{L}^1$ -covering numbers). Let  $\mathcal{G}$  be a class of functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ . Consider  $M$  points  $(y_m)_{1 \leq m \leq M}$  in  $\mathbb{R}^d$ , and denote by  $\nu^M := \frac{1}{M} \sum_{m=1}^M \delta_{y_m}$  the associated empirical measure and  $\|\cdot\|_{\mathbb{L}^1(\nu^M)}$  the associated empirical  $\mathbb{L}^1$  norm: for any function  $\varphi$ ,

$$\|\varphi(\cdot)\|_{\mathbb{L}^1(\nu^M)} := \frac{1}{M} \sum_{m=1}^M |\varphi(y_m)|.$$

An  $\varepsilon$ -covering ( $\varepsilon > 0$ ) of  $\mathcal{G}$  with respect to the norm  $\|\cdot\|_{\mathbb{L}^1(\nu^M)}$  is a finite set of functions  $\varphi_1, \dots, \varphi_n : \mathbb{R}^d \rightarrow \mathbb{R}$  such that for any  $\varphi \in \mathcal{G}$ , we can find an index  $j \in \{1, \dots, n\}$  such that  $\|\varphi - \varphi_j\|_{\mathbb{L}^1(\nu^M)} \leq \varepsilon$ .

The minimal integer  $n$  for which an  $\varepsilon$ -covering exists is called the  $\varepsilon$ -covering number and it is denoted by  $\mathcal{N}_1(\varepsilon, \mathcal{G}, \nu^M)$ .

More generally, we can replace the empirical measure  $\nu^M$  by a probability measure  $\nu$  on  $\mathbb{R}^d$  and similarly define  $\mathcal{N}_1(\varepsilon, \mathcal{G}, \nu)$ .

**Theorem 4.6.1** (robust upper bound for the  $\mathbb{L}^1$  covering number (see [78, Lemma 9.2, Theorem 9.4 and 9.5])). *In a linear least squares regression problem, consider*

- $\Psi = \text{Span}(\psi_j : 1 \leq j \leq K)$  with  $\psi_i : \mathbb{R}^d \rightarrow \mathbb{R}$ ;
- $\mathcal{C}_L \Psi := \{\mathcal{C}_L \varphi : \varphi \in \Psi\}$  the clipping of  $\Psi$  (for  $L > 0$ );
- $\mu$  a probability measure on  $\mathbb{R}^d$ .

Then for any  $\varepsilon \leq \frac{L}{2}$ , we have,

$$\mathcal{N}_1(\varepsilon, \mathcal{C}_L \Psi, \mu) \leq \left( \frac{6L}{\varepsilon} \right)^{2(K+1)}.$$

**Theorem 4.6.2** (uniform deviation probability (see [78, Theorem 9.1])). *Let  $\mathcal{G}$  be a countable set of the functions  $\varphi : \mathbb{R}^d \rightarrow [0, L]$  with  $L > 0$ . Let  $(Y_m)_{1 \leq m \leq M}$  be a sample of independent random variables with the distribution  $\mu$  and denote  $\mu^M$  as the associated empirical measure. Suppose that  $\mathcal{G}$  can be covered for  $\mu^M$ . Then for all  $\varepsilon > 0$ , we have*

$$\begin{aligned} \mathbb{P} \left( \sup_{\varphi \in \mathcal{G}} \left| \frac{1}{M} \sum_{m=1}^M \varphi(X_m) - \int_{\mathbb{R}^d} \varphi(x) \mu(\mathbf{d}x) \right| > \varepsilon \right) \\ \leq 8 \mathbb{E} \left[ \mathcal{N}_1 \left( \frac{\varepsilon}{8}, \mathcal{G}, \mu^M \right) \right] \exp \left( -\frac{\varepsilon^2 M}{128 L^2} \right). \end{aligned}$$

## 4.7 Appendix of Section 3.5

We first list the definitions of VC dimension and give the proof of Proposition 4.5.2 in Section 4.7.2.

### 4.7.1 VC dimension and Pseudo dimension

We first give the different definitions around Vapnik-Chervonenkis Dimension (see also [126] for the original definition):

**Definition 4.7.1** ([78, Section 9.4] Version Intersection). *Let  $\mathcal{A}$  be a class of subsets of  $\mathbb{R}^l$  and*

$$v_{\mathcal{A}} := \max\{n \in \mathbb{N} : \mathcal{S}(\mathcal{A}, n) = 2^n\} \tag{4.7.1}$$

where the growth function  $\mathcal{S}$  is defined by:

$$\mathcal{S}(\mathcal{A}, n) := \max_{x_1, \dots, x_n \in \mathbb{R}^l} |\{A \cap \{x_1, \dots, x_n\} : A \in \mathcal{A}\}|$$

with  $|\cdot|$  being the cardinality of sets. If  $\mathcal{S}(\mathcal{A}, n) = 2^n$ , then we say that there exists a  $n$  points in  $\mathbb{R}^l$  which can be shattered by  $\mathcal{A}$ . If no maximum exists in (4.7.1),  $v_{\mathcal{A}} = \infty$ .

**Definition 4.7.2** ([5, Definition 1] Version Binary Classification). *Let  $\mathcal{H}$  be a class of binary functions from  $\mathbb{R}^l \rightarrow \{0, 1\}$  then we define  $\text{VCdim}(\mathcal{H})$  by*

$$\text{VCdim}(\mathcal{H}) := \max\{n \in \mathbb{N} : \Pi_{\mathcal{H}}(n) = 2^n\} \quad (4.7.2)$$

where the growth function  $\Pi$  is given by:

$$\Pi_{\mathcal{H}}(n) := \max_{x_1, \dots, x_n \in \mathbb{R}^l} |\{(h(x_1), \dots, h(x_n)) : h \in \mathcal{H}\}|.$$

If no such maximum exists in (4.7.2), then we define  $\text{VCdim}(\mathcal{F}) = \infty$ .

**Definition 4.7.3** ([1, Definition 11.1 and 11.2] Pseudo-Dimension). *Suppose  $\mathcal{F}$  is a set of functions  $\mathbb{R}^l \rightarrow \mathbb{R}$ , define the Pseudo dimension by*

$$\text{Pdim}(\mathcal{F}) := \max\{n \in \mathbb{N} : \Pi(\mathcal{F}, n) = 2^n\} \quad (4.7.3)$$

where the growth function  $\Pi$  is defined by:

$$\Pi(\mathcal{F}, n) := \max_{\substack{x_1, \dots, x_n \in \mathbb{R}^l, \\ y_1, \dots, y_n \in \mathbb{R}}} |\{(\mathbf{1}_{\{f(x_1) \geq y_1\}}, \dots, \mathbf{1}_{\{f(x_n) \geq y_n\}}) : f \in \mathcal{F}\}|.$$

If no such maximum exists in (4.7.3), then we define  $\text{Pdim}(\mathcal{F}) = \infty$ .

From Definition 4.7.1 and Definition 4.7.3, an easy computation leads to  $v_{\Psi(\mathcal{L})^+} \leq \text{Pdim}(\Psi(\mathcal{L}))$  where  $\text{Pdim}$  is the pseudo-dimension defined in [1, Definition 11.1 and 11.2] for the class of the real-valued functions. Then, the argument in [1, Theorem 14.1] shows that, if  $\Psi(\mathcal{L})$  is with fixed architecture (compared with adaptive one), then  $\text{Pdim}(\Psi(\mathcal{L})) \leq \text{VCdim}(\Psi(\mathcal{L})')$  where the class  $\Psi(\mathcal{L})'$  is defined in the following result.

**Lemma 4.7.4** ([1, Theorem 14.1]). *For any class of real functions  $\mathcal{F}$  computed by a Neural Network with the fixed architecture and the right continuous activation function, we have*

$$\text{VCdim}(\mathcal{F}) := \text{VCdim}(\text{sgn}(\mathcal{F})) \leq \text{Pdim}(\mathcal{F}) \leq \text{VCdim}(\mathcal{F}')$$

where  $\text{sgn}(\mathcal{F}) := \{\text{sgn}(f) : f \in \mathcal{F}\}$  and we use the convention  $\text{sgn}(x) := \mathbf{1}_{\{x \geq 0\}}$ . Here  $\mathcal{F}'$  is a class of binary functions:  $\mathbb{R} \times \mathbb{R} \rightarrow \{0, 1\}$  and it is computed by **NN'** constructed as follows: begin with **NN** the network for computing the function in  $\mathcal{F}$ , then consider

- Input unit: output unit of **NN** and one extra unit from  $\mathbb{R}$ ;
- Computation unit: linear threshold unit receiving input only from input unit;
- Output unit: the computation unit without activation.

*Proof.* See also the remark after [5, Definition 2]. The inequality  $\text{VCdim}(\text{sgn}(\mathcal{F})) \leq \text{Pdim}(\mathcal{F})$  is obvious because

$$\begin{aligned} \Pi_{\text{sgn}(\mathcal{F})}(n) &= \max_{x_1, \dots, x_n \in \mathbb{R}^l} |\{(\mathbf{1}_{\{f(x_1) \geq 0\}}, \dots, \mathbf{1}_{\{f(x_n) \geq 0\}}) : f \in \mathcal{F}\}| \\ &\leq \max_{\substack{x_1, \dots, x_n \in \mathbb{R}^l, \\ y_1, \dots, y_n \in \mathbb{R}}} |\{(\mathbf{1}_{\{f(x_1) \geq y_1\}}, \dots, \mathbf{1}_{\{f(x_n) \geq y_n\}}) : f \in \mathcal{F}\}| = \Pi(\mathcal{F}, n). \end{aligned}$$

As for  $\text{VCdim}(\mathcal{F}') \geq \text{Pdim}(\mathcal{F})$ , we refer to the argument of [1, Theorem 14.1].  $\square$



**Remark 4.7.5.** If the activation function  $\rho$  of the output of **NN** is non-decreasing and right-continuous, then  $\{f(x) \geq y\} = \{\rho(g(x)) \geq y\} = \{g(x) \geq \rho^{-1}(y)\}$  yields that the extra linear threshold computation unit in **NN'** can be offset by the activation of the output of **NN**, namely, we can use the same number of computation units as the network used for  $\mathcal{F}$ . Suppose in **NN**, one has used  $W$  parameters and  $U$  computation units, then in **NN'**,  $W' := W + 2, U' = U, \mathcal{L}' = \mathcal{L}$ .

From the above results, we can easily prove Proposition 4.5.2.

## 4.7.2 Proof of Proposition 4.5.2

We first prove  $v_{\mathcal{G}^+} = \text{Pdim}(\mathcal{G})$  for any class of real functions  $\mathcal{G} := \{g : \mathbb{R}^d \rightarrow \mathbb{R}\}$ . Recall that

$$\mathcal{G}^+ := \left\{ \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \leq g(x)\} : g \in \mathcal{G} \right\}.$$

Indeed, from Definition 4.7.1, we have, if  $\mathcal{S}(\mathcal{G}^+, n) = 2^n$ , then there exists a subset of  $(\mathbb{R}^d \times \mathbb{R})^n$   $\mathcal{X}_n := \{(x_1, y_1), \dots, (x_n, y_n)\}$  such that: for any subset of  $\mathcal{X}_n$ ,  $\mathcal{X}_{n_i} := \{(x_{i_1}, y_{i_1}), \dots, (x_{i_n}, y_{i_n})\}$ , there exists  $g \in \mathcal{G}$  s.t.

$$g(x_{i_j}) \geq y_{i_j}, \forall (x_{i_j}, y_{i_j}) \in \mathcal{X}_{n_i} \text{ and } g(x_k) < y_k, \forall (x_k, y_k) \in \mathcal{X}_n \cap \mathcal{X}_{n_i}^c,$$

which is equivalent to the argument: for any  $b \in \{0, 1\}^n$ , there exists  $g \in \mathcal{G}$  s.t.

$$\mathbf{1}_{\{g(x_i) \geq y_i\}} = b_i, \forall (x_i, y_i) \in \mathcal{X}_n$$

and this implies  $\Pi(\mathcal{G}, n) = 2^n$  from Definition 4.7.3 and vice versa. Thus  $v_{\Psi(\mathcal{L})^+} = \text{Pdim}(\Psi(\mathcal{L}))$ .

Applying Lemma 4.7.4, we immediately get  $v_{\Psi(\mathcal{L})^+} \leq \text{VCdim}(\Psi(\mathcal{L})')$  where  $\Psi(\mathcal{L})'$  is defined as in Lemma 4.7.4. As for (4.5.4), we refer to [5]. We can show, from [5, Theorem 8], there exists a universal constant  $C$  such that, for the piecewise linear activation function  $\rho$ ,  $v_{\Psi(\mathcal{L})^+} \leq CW\mathcal{L} \log W$ ; for the piecewise polynomial  $\rho$ ,  $v_{\Psi(\mathcal{L})^+} = CWU$  with  $U$  the number of computation units arranged in  $\mathcal{L}$  layers.  $\square$

## 4.7.3 Proof of Proposition 4.5.3

Before we head to the proof of Proposition 4.5.3, we first give the following lemma. The proof of Proposition 4.5.4 is at the end of this section.

**Lemma 4.7.6** ([130, Theorem 1]). *For any  $\delta \in (0, 1)$ , there is a ReLU network architecture satisfying*

- (a):  $\max\{W, U\} \leq C\delta^{-d} (\log \frac{1}{\delta} + 1)$ ;
- (b):  $\mathcal{L} = C (\log \frac{1}{\delta} + 1)$ ;

where the constant  $C_d$  depends only on  $d$ , such that, for any  $g \in C^0([-R, R]^d, \mathbb{R})$  with  $\|g\|_{1, [-R, R]^d} := \max\{\text{ess sup}_{x \in [-R, R]^d} |g'(x)|, \text{ess sup}_{x \in [-R, R]^d} |g(x)|\} < \infty$ , there exists an approximation  $\varphi$  computed by **NN** s.t.

$$\frac{1}{2R} \|g - \varphi\|_{\mathbb{L}^\infty([-R, R]^d)} \leq \|g\|_{1, [-R, R]^d} \delta.$$

*Proof.* We employ the result [130, Theorem 1]. For any  $0 < \delta < 1$ , there exists a neural network **NN** satisfying (a), (b) and we denote by  $\Psi$  the class of functions computed by **NN**, such that, for any Lipschitz continuous function  $g \in C^0([0, 1]^d, \mathbb{R})$  satisfying that  $\|g\|_{1, [0, 1]^d} := \max\{\text{ess sup}_{x \in [0, 1]^d} |g'(x)|, \text{ess sup}_{x \in [0, 1]^d} |g(x)|\} \leq 1$ , there exists  $\varphi \in \Psi$ , s.t.

$$\inf_{\varphi \in \Psi} \|g - \varphi\|_{\mathbb{L}^\infty([0, 1]^d)} \leq \delta.$$

This result can be easily extended to all compact sets of  $\mathbb{R}^d$  because the linear transformation on the input or the output does not change the complexity of network. We first show, for all  $g$  with  $\|g\|_{1, [0, 1]^d} < \infty$ : for the fixed  $\delta \in (0, 1)$ , there exists  $\varphi \in \Psi$ , s.t.

$$\|\varphi - g\|_{\mathbb{L}^\infty([0, 1]^d)} \leq \|g\|_{1, [0, 1]^d} \delta \quad (4.7.4)$$

because  $\frac{g}{\|g\|_{1, [0, 1]^d}}$  satisfies  $\left\| \frac{g}{\|g\|_{1, [0, 1]^d}} \right\|_{1, [0, 1]^d} \leq 1$  and for  $\varphi \in \Psi$ ,  $\varphi \|g\|_{1, [0, 1]^d} \in \Psi$ .

Second, we extend to the compact set  $[-R, R]^d$  with  $R > 1$ , suppose  $g \in C^0([-R, R]^d, \mathbb{R})$  and  $\|g\|_{1, [-R, R]^d} < \infty$ , due to a change of variable, there exists  $\varphi \in \Psi$ , s.t.

$$\frac{1}{2R} \|g - \varphi\|_{\mathbb{L}^\infty([-R, R]^d)} \leq \|g\|_{1, [-R, R]^d} \delta.$$

In fact, letting  $\tilde{g}(x) = g(2Rx - R), \forall x \in [0, 1]^d$ , from (4.7.4), there exists  $\tilde{\varphi} \in \Psi$  s.t.

$$\|\tilde{g} - \tilde{\varphi}\|_{1, [0, 1]^d} \leq \|\tilde{g}\|_{1, [0, 1]^d} \delta \leq 2R \|g\|_{1, [-R, R]^d} \delta$$

where the last inequality comes from  $\|\tilde{g}\|_{1, [0, 1]^d} \leq 2R \|g\|_{1, [-R, R]^d}$ . Use again  $\tilde{\varphi}(\frac{\cdot + R}{2R}) \in \Psi$  to get the desired result.  $\square$

*Proof of Proposition 4.5.3.* Apply the lemma 4.7.6 with  $g = P_i u_{i+1}$ , then, for any  $\delta \in (0, 1)$ , there exists a neural network **NN** with complexity  $W, \mathcal{L}$  s.t. there exists a function  $\varphi \in \Psi$  computed by **NN** who approximates  $P_i u_{i+1}$ :

$$\frac{1}{2R} \|\varphi - P_i u_{i+1}\|_{\mathbb{L}^\infty([-R, R]^d)} \leq L\delta.$$

Now let us consider the approximation  $\mathcal{C}_L \varphi$  of  $P_i u_{i+1}$  by adding one extra computation unit after the output unit of **NN** where we clip all the output of **NN** by  $L$ , since  $\mathcal{C}_L$  is 1-Lipschitz and  $P_i u_{i+1}$  is bounded by  $L$ , then

$$\|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^\infty([-R, R]^d)} \leq \|\varphi - P_i u_{i+1}\|_{\mathbb{L}^\infty([-R, R]^d)} \leq 2RL\delta.$$

If we call this 'clipped' neural network by  $\mathbf{NN}'$ , denote by  $\Psi'$  for the function class computed by  $\mathbf{NN}'$ , for the approximation given from  $\Psi'$ , we have

$$\inf_{\varphi' \in \Psi'} \|\varphi' - P_i u_{i+1}\|_{\mathbb{L}^\infty([-R, R]^d)} \leq \|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^\infty([-R, R]^d)} \leq 2RL\delta.$$

Next we dominate the error in  $\mathbb{L}^2$ , for any probability measure  $\mu$  on  $\mathbb{R}^d$ ,

$$\begin{aligned} \|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} &= \left( \int_{\mathbb{R}^d} |\mathcal{C}_L \varphi(x) - P_i u_{i+1}(x)|^2 \mu(\mathbf{d}x) \right)^{1/2} \\ &\leq \left( \int_{[-R, R]^d} |\mathcal{C}_L \varphi(x) - P_i u_{i+1}(x)|^2 \mu(\mathbf{d}x) \right)^{1/2} \\ &\quad + \left( \int_{([-R, R]^d)^c} |\mathcal{C}_L \varphi(x) - P_i u_{i+1}(x)|^2 \mu(\mathbf{d}x) \right)^{1/2} \\ &\leq (2R)^{1+\frac{d}{2}} L\delta + 2L\sqrt{\mathbb{P}(|X_{t_i}| > R)}. \end{aligned} \quad (4.7.5)$$

With assuming  $H_1$ -5 for some  $q \geq 2$ , then  $\mathbb{E} \left[ \sup_{0 \leq i \leq N-1} |X_{t_i}|^q \right] < \infty$  from the standard a priori estimates of SDE (see [133][Theorem 3.2.2]), it follows by, there exists a constant  $C_q$  depending on  $q, C_{Lip}^X, T$  s.t.

$$\mathbb{P}(|X_{t_i}| > R) \leq \mathbb{E} \left[ \frac{|X_{t_i}|^q}{R^q} \right] \leq C_q R^{-q}.$$

Let  $C$  be a generic constant depending on  $q, L, C_{Lip}^X, T, d$ , we have,

$$\|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq (2R)^{1+\frac{d}{2}} L\delta + 2L\sqrt{C_q R^{-q}} \leq C(R^{\frac{d}{2}+1}\delta + R^{-\frac{q}{2}}).$$

Letting

$$R = \left( \frac{2C}{\varepsilon} \right)^{\frac{2}{q}} \leq O(\varepsilon^{-\frac{2}{q}}), \quad \delta = \frac{\varepsilon}{2C} R^{-(1+\frac{d}{2})} \leq O\left(\varepsilon^{1+\frac{d+2}{q}}\right),$$

as  $\varepsilon \rightarrow 0$ , then

$$\inf_{\varphi' \in \Psi'} \|\varphi' - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq \|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq \varepsilon.$$

So the network  $\mathbf{NN}'$  with specific  $\delta = O\left(\varepsilon^{1+\frac{d+2}{q}}\right)$  is what we want, precisely, from Lemma 4.7.6, for  $U, W$  and  $\mathcal{L}$ ,

$$\begin{aligned} \mathcal{L} &= O\left(\log \frac{1}{\delta}\right) = O\left(\log \frac{1}{\varepsilon}\right), \\ \max(U, W) &= O\left(\delta^{-d}\left(\log \frac{1}{\delta} + 1\right)\right) \leq O\left(\varepsilon^{-d(1+\frac{d+2}{q})} \log \frac{1}{\varepsilon}\right), \end{aligned}$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d$  and  $q$ ). Therefore we complete the proof.  $\square$

**Remark 4.7.7.** In the above proof, if  $X_{t_i}$  satisfies the BMO condition in Proposition 4.3.8, then we can have a better estimation. Indeed, the universal constant  $c$  in Proposition 4.3.8 such that

$$\mathbb{P}(|X_{t_i}| > R) \leq \mathbb{E} \left[ \frac{e^{c|X_{t_i}|}}{e^{cR}} \right] = e^{-cR} \mathbb{E} [e^{c|X_{t_i}|}].$$

Let  $c$  be a generic constant depending on  $C_{Lip}^X, T$  and  $C$  be a generic constant depending on  $L, C_{Lip}^X, T$  up to a logarithmic factor of  $d$ , we have, from (4.7.5),

$$\|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq (2R)^{1+\frac{d}{2}} L \delta + 2L \sqrt{C_\mu e^{-cR}} \leq C(R^{\frac{d}{2}+1} \delta + e^{-cR}),$$

Letting

$$R = \frac{1}{c} \log \frac{2C}{\varepsilon} \leq O\left(\log \frac{1}{\varepsilon}\right), \quad \delta = \frac{\varepsilon}{2C} R^{-(1+\frac{d}{2})} \leq O\left(\varepsilon \left(\log \frac{1}{\varepsilon}\right)^{-(1+\frac{d}{2})}\right),$$

as  $\varepsilon \rightarrow 0$ , then

$$\inf_{\varphi' \in \Psi'} \|\varphi' - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq \|\mathcal{C}_L \varphi - P_i u_{i+1}\|_{\mathbb{L}^2(\mu)} \leq \varepsilon.$$

$$\begin{aligned} \mathcal{L} &= O\left(\log \frac{1}{\delta}\right) = O\left(\left(1 + \frac{d}{2}\right) \log \log \frac{1}{\varepsilon} + \log \frac{1}{\varepsilon}\right) \leq O\left(\log \frac{1}{\varepsilon}\right), \\ \max(U, W) &= O\left(\delta^{-d} \left(\log \frac{1}{\delta} + 1\right)\right) \leq O\left(\varepsilon^{-d} \left(\log \frac{1}{\varepsilon}\right)^{d+\frac{d^2}{2}} \left(1 + \frac{d}{2}\right) \log \frac{1}{\varepsilon}\right) \\ &\leq O\left(\varepsilon^{-d} \left(\log \frac{1}{\varepsilon}\right)^{d+\frac{d^2}{2}+1}\right), \end{aligned}$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d$ ).

#### 4.7.4 Proof of Proposition 4.5.4

We employ the result [131, Theorem 2] and [111, Corollary 3.2] where a  $\mathbb{L}^\infty([0, R])$ -error bound is proved for the approximation of Lipschitz function. For any  $\delta > 0$  and for any Lipschitz continuous function  $g$  in  $\mathbb{L}^2([-R, R]^d)$ , there exists a neural network **NN** satisfying

- $\max\{W, U\} \leq C_d \delta^{-d/2}$ ;
- $\mathcal{L} = C_d W$ ,

where the constant  $C_d$  depends only on  $d$ , s.t.

$$\frac{1}{2R} \inf_{\varphi \in \Psi} \|g - \varphi\|_{\mathbb{L}^\infty([-R, R]^d)} \leq \delta.$$

Taking exactly the same arguments used in Proposition 4.5.3 for estimating in  $\mathbb{L}^2(\mu)$  norm and choosing the same tuning for  $R$  and  $\delta$ , we get, for  $U, W$  and  $\mathcal{L}$ ,

$$\begin{aligned}\max(U, W) &= O\left(\delta^{-\frac{d}{2}}\right) \leq O\left(\varepsilon^{-\frac{d}{2}\left(1+\frac{d+2}{q}\right)}\right), \\ \mathcal{L} &= O\left(\delta^{-\frac{d}{2}}\right) \leq O\left(\varepsilon^{-\frac{d}{2}\left(1+\frac{d+2}{q}\right)}\right),\end{aligned}$$

as  $\varepsilon \rightarrow 0$  ( $O$  depending on  $d$  and  $q$ ). Therefore we complete the proof.  $\square$

# Chapter 5

## The Put/Call symmetry for the American option in the non-linear market

**Abstract** In this paper, the Put/Call Symmetry relation for the American options is investigated in the non-linear market. We assume in this imperfect market, there are 2 interest rates and 2 dividend rates and the assets are in diffusion models. Taking advantages of the link between RBSDEs and American options, after the change of numéraire, we find that as in linear pricing case, the exchange between the interest rate and dividend rate still hold in the numéraire market. The examples with Markovian and path-dependent payoff are provided. The classical change of measure technique is also applied on the optimal stopping representation in risk-neutral market in order to compare with the results from RBSDE's.

### 5.1 Introduction

The Put/Call symmetry (PCS in short) is a standard relation in the financial market for both European and American options which facilitates us to transfer from Call option to Put option. This relation establishes that the price of a Call option is equivalent to the price of a Put option when the positions of the strike price  $K$  and spot value  $X$  of the underlying asset are exchanged, along with the positions of the interest rate  $r$  and dividend rate  $q$ . The classical American PCS is talked in [6] and the relation of American Call/Put with asymmetric jumps is also studied; Carr and Chesney in [28] prove the American PCS under Black-Scholes model and provide a specific example of PCS when volatility is a particular deterministic function of the stock price by PDEs.

With the develop of mathematical finance, Geman, El Karoui and Rochet in [65] introduce the techniques of the change of numéraire and the related change of measure which become an efficient tool to prove PCS in the complete market. Schroder in [123] present a

wide range of examples for symmetry formula of American options with Markovian payoff where the model of underlying asset with jump are also considered; Detemple in [44] extend the results in [123] to the general diffusion models and to multidimensional assets case, where some path-dependent options are also discussed; Fajardo and Mordecki in [60] and [61] investigate European and American PCS in the market driven by Lévy process with high jump and with constant interest and dividend rate.

In addition to the technique employed in PCS, there are also other methods in use. Byun and Kim in [25] take advantage of the early exercise premium to prove the PCS under BS model; Carr and Lee in [29] make use of the symmetry of distribution of underlying asset to get the PCS for European options and a detailed discussion is provided for barrier options. See also [14], [108].

We know that under the Black-Scholes model, when the market coefficients the interest rate  $r$ , the dividend  $q$ , the volatility  $\sigma$ , are constants, one has the symmetry formula between the American Call option and Put option (e.g [93][Proposition 4.2.2]), i.e. with exchanging  $r$  and  $q$ ,

$$Call(t, x; T, K; r, q, \sigma) = xPut(t, K/x; T, 1; q, r, -\sigma).$$

Now we extend this relation to the non-linear market with the help of BSDE. We assume in the imperfect market, for the risk free asset, the deposit rate and the loan rate are different; for the risky asset, the repo rate gives an extra dividend when the investor is shorting asset. The pricing problem in the non linear market with 2 interest rates for European options are well studied, for example, the arbitrage is studied in [8], the existence of hedging strategy is discussed in [39][Theorem 9.1] and hedging strategy is presented in [92][Section 4], see also [54][Example 1.1]. The American pricing in the imperfect market with defaults is studied in [49] by non-linear expectation. In our case, when the deposit/loan rate are different, and the shorting/longing dividend are different too, we have the similar relation that the loan rate exchange positions with the shorting dividend.

**Contribution:** We focus on the symmetry of American option in the non-linear market from the point of view of BSDE. Most of the literatures rely on the representation of optimal stopping which limits their results under the risk-neutral probability framework and lose the symmetry in the non-linear market. We show the general result for different kind of payoff which can be easily deduced to the case of linear pricing.

**Organization:** This chapter is organized as follows. Some basic assumptions are in Section 5.2 and the well-posedness of non-linear market is discussed in Section 5.3. In Section 5.4, we first show our main results about the change of numéraire for American option in the non-linear market. In Section 5.5, we introduce our results of PCS including both vanilla option and options with path-dependent payoff. In Section 5.6, we recall the result obtained from optimal stopping in the complete linear market by change of probability.

## 5.2 Notations and Hypothesis

Without specific instructions, for simplification, sometimes we will drop the subscript  $t$  when we refer to processes.

### 5.2.1 Notations

Set  $(\Omega, \mathcal{F}, \mathbb{P})$  to be a probability space, let  $B$  be the  $d$ -Brownian motion defined on this space and  $\mathcal{F}_t$  is the natural filtration generated by  $B$  where  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets.

— The following spaces are needed,

$$\begin{aligned} \mathbb{L}^p &:= \left\{ \xi : \Omega \rightarrow \mathbb{R} \text{ is a random variable s.t. } \mathbb{E} [|\xi|^p] < \infty \right\}; \\ \mathcal{S}^p &:= \left\{ \phi : \{\phi_t : 0 \leq t \leq T\} \text{ is a predictable process s.t. } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^p \right] < \infty \right\}; \\ \mathbb{H}^p &:= \left\{ \phi : \{\phi_t : 0 \leq t \leq T\} \text{ is a predictable process s.t. } \mathbb{E} \left[ \left( \int_t^T |\phi_t|^2 dt \right)^{p/2} \right] < \infty \right\}, \end{aligned}$$

where  $|\cdot|$  is Euclidean norm whose dimension will be clear from the context.

- **Call** $(t, x; T, K; \mu, r, R, q, Q, \sigma)$  (resp. **Put** $(t, x; T, K; \mu, r, R, q, Q, \sigma)$ ) denotes the price of the American Call (resp. Put) Option at time  $t$  with strike price  $K$ , where the underlying satisfies (H-M) with  $(\mu, r, R, q, Q)$ ;
- Except the market coefficients  $(\mu, r, R, q, Q, \sigma)$ , for any vector  $x \in \mathbb{R}^d$ ,  $x^i$  is the  $i$ -th component of  $x$ ; for any matrix  $M \in \mathbb{R}^{n \times m}$  except  $\sigma$ ,  $M^i$  is the  $i$ -th row of  $M$ ;  $M^\top$  is the transpose matrix of  $M$ ;
- For the  $d$ -dimensional market coefficients  $\mu, q, Q, \sigma$ , we denote  $\mu^{(i)}, q^{(i)}, Q^{(i)}$  the  $i$ -th component of  $\mathbb{R}^d$  column vector; denote  $\sigma^{(i)}$  the  $i$ -th row of  $\mathbb{R}^{d \times d}$  matrix  $\sigma$ ;
- For any vector  $x \in \mathbb{R}^d$ , any matrix  $M \in \mathbb{R}^{d \times m}$ , the multiply of  $x$  and  $M$  is defined as

$$x \cdot M = \begin{bmatrix} x^1 M^1 \\ \dots \\ x^d M^d \end{bmatrix} \in \mathbb{R}^{d \times m},$$

where  $M = \begin{bmatrix} M^1 \\ \vdots \\ M^d \end{bmatrix}$ ,  $x^i M^i = (x^1 M^{11}, \dots, x^m M^{1m})$  and  $M^{ij}$  is the element in  $i$ -th row and  $j$ -th column.



## 5.2.2 Assumptions

### 5.2.2.1 Non-linear Market Assumptions (H-M)

(H-M) Suppose in the Itô market, there are 2 different interest rates processes, the deposit rate  $r$  and the loan rate  $R$  with  $r < R$ . Assume the risk-free saving account  $\pi^0$  satisfies

$$d\pi_t^0 = r_t(\pi_t^0)^+ dt - R_t(\pi_t^0)^- dt.$$

Suppose there are  $d$  risky assets  $X^1, \dots, X^d$  in the market, which satisfies for  $1 \leq i \leq d, \forall 0 \leq t \leq T$ ,

$$\frac{dX_t^i}{X_t^i} = \mu_t^{(i)} dt + \sigma_t^{(i)} dB_t.$$

And the risky assets pay continuous dividend at 2 levels: at  $q$  for longing asset and at  $Q$  for shorting asset,  $q^{(i)} < Q^{(i)}$ , for any  $1 \leq i \leq d$ ,  $dt \otimes d\mathbb{P}$  a.e.. Assume that  $r, R$  are the scalar stochastic processes;  $\mu, q, Q$  are the bounded stochastic processes valued in  $\mathbb{R}^d$ ;  $\sigma$  is the stochastic process valued in  $\mathbb{R}^{d \times d}$  and for a.e.  $t \in [0, T]$ ,  $\sigma_t$  is invertible and elliptic a.s.:

$$\mu_t = \begin{bmatrix} \mu_t^{(1)} \\ \mu_t^{(2)} \\ \vdots \\ \mu_t^{(d)} \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} \sigma_t^{(1)} \\ \sigma_t^{(2)} \\ \vdots \\ \sigma_t^{(d)} \end{bmatrix}, \quad q_t = \begin{bmatrix} q_t^{(1)} \\ q_t^{(2)} \\ \vdots \\ q_t^{(d)} \end{bmatrix}, \quad Q_t = \begin{bmatrix} Q_t^{(1)} \\ Q_t^{(2)} \\ \vdots \\ Q_t^{(d)} \end{bmatrix}.$$

**Definition 5.2.1.** *If assets  $X$  fulfill all above conditions in the market assumption (H-M), we call that the assets  $X$  satisfies (H-M) with coefficients  $(\mu, r, R, q, Q)$ .*

### 5.2.2.2 Related Reflected BSDE and its assumptions (H-E)

In order to solve the pricing problem of American option in the non-linear market (H-M), we introduce the Reflected BSDE as following

$$\begin{cases} Y_t = S_T + \int_t^T f(s, Y_s, Z_s) ds + \mathcal{K}_T - \mathcal{K}_t - \int_t^T Z_s dB_s, \\ Y_t \geq S_t, \forall 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) d\mathcal{K}_t = 0, \end{cases} \quad (5.2.1)$$

where we call  $(Y, Z, \mathcal{K})$  the solution with respect to the data  $(f, S)$ . The driver  $f$  characterizes the market and the obstacle  $S$  identifies the option by letting  $S$  be the payoff process. We use the following assumptions to guarantee the well-posedness of (5.2.1).

**Basic assumptions for RBSDE (H-E):** for some  $p \geq 2$ ,

(H-E-1):  $S \in \mathcal{S}^p$ ;

(H-E-2):  $f(\cdot, 0, 0) \in \mathbb{H}^p$  and  $f$  is uniformly Lipschitz in  $(y, z)$ ;

(H-E-3):  $Y, \mathcal{K}$  are continuous processes on  $[0, T]$ ,  $\mathcal{K}$  is non-decreasing and  $\mathcal{K}_0 = 0$ .

### 5.3 Preliminaries

In this section, we state that the pricing problem of American option in the market (H-M) is equivalent to a RBSDE problem as (5.2.1) w.r.t.  $(f, S)$  where  $S$  is the payoff process of option and its driver is given as follows

$$f(t, y, z) := -r_t y - z\sigma_t^{-1}(\mu_t + q_t - r_t \mathbf{1}_d) + (R_t - r_t)(y - z\sigma_t^{-1} \mathbf{1}_d)^- + (z\sigma_t^{-1})^-(Q_t - q_t). \quad (5.3.1)$$

It has been proved in [52] that, under (H-E)-(1-2), there exists a unique solution of (5.2.1)  $(Y, Z, \mathcal{K}) \in \mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$  satisfying (H-E-3). If (H-M) is fulfilled by the underlying asset, then the driver  $f$  can be deduced from the dynamic of the self-financing portfolio. It's easy to check that  $f$  satisfies (H-E-2). In order to show the relation between the pricing problem of American option and RBSDE, we introduce the following results.

#### 5.3.1 The driver $f$ of the market

In this subsection, we are going to explain the procedure how to deduce the market driver  $f$  shown in (5.3.1). Indeed, consider a self-financing portfolio consisted of the strategy (the amount invested in the risky asset)  $\pi$  which is a row vector valued in  $\mathbb{R}^d$  s.t. the portfolio is composed in linear way by the risk-free asset and  $d$  risky assets, from which, we derive its corresponding value process  $V$ , namely,

$$\forall t > 0, \quad V_t = \pi_t^0 + \sum_{i=1}^d \frac{\pi_t^i}{X_t^i} X_t^i = \pi_t^0 + \pi_t \mathbf{1}_d.$$

Write the dynamics of  $V_t$ , with the dividends mentioned in (H-M):

$$\begin{aligned} dV_t &= d\pi_t^0 + \sum_{i=1}^d (d\pi_t^i + q_t^{(i)}(\pi_t^i)^+ dt - Q_t^{(i)}(\pi_t^i)^- dt) \\ &= [r_t(\pi_t^0)^+ - R_t(\pi_t^0)^-] dt + \sum_{i=1}^d (q_t^i(\pi_t^i)^+ - Q_t^{(i)}(\pi_t^i)^-) dt + \sum_{i=1}^d (d\pi_t^i) \\ &= [r_t \pi_t^0 - (R_t - r_t)(\pi_t^0)^- + \pi_t q_t - (\pi_t^-)(Q_t - q_t)] dt + \sum_{i=1}^d \frac{\pi_t^i}{X_t^i} dX_t^i, \end{aligned}$$

substituting the dynamics of  $X_t$ , and the risk-free asset satisfying  $\pi_t^0 = V_t - \pi_t \mathbf{1}_d$ , we get,

$$\begin{aligned} \mathbf{d}V_t &= (r_t \pi_t^0 - (R_t - r_t)(\pi_t^0)^- + \pi_t q_t - (\pi_t^-)(Q_t - q_t)) \mathbf{d}t + \sum_{i=1}^d \pi_t^i [(\mu_t^{(i)} + q_t^{(i)}) \mathbf{d}t + \sigma_t^{(i)} \mathbf{d}B_t] \\ &= r_t (V_t - \pi_t \mathbf{1}_d) \mathbf{d}t - (R_t - r_t)(V_t - \pi_t \mathbf{1}_d)^- \mathbf{d}t + (\pi_t q_t - (\pi_t^-)(Q_t - q_t)) \mathbf{d}t + \pi_t \mu_t \mathbf{d}t + (\pi_t \sigma_t) \mathbf{d}B_t \\ &= [r_t V_t + \pi_t (\mu_t + q_t - r_t \mathbf{1}_d)] \mathbf{d}t - (R_t - r_t)(V_t - \pi_t \mathbf{1}_d)^- \mathbf{d}t - (\pi_t^-)(Q_t - q_t) \mathbf{d}t + (\pi_t \sigma_t) \mathbf{d}B_t \\ &= -f(t, V_t, \pi_t \sigma_t) \mathbf{d}t + (\pi_t \sigma_t) \mathbf{d}B_t, \end{aligned}$$

with  $f$  shown in (5.3.1). The decomposition of dynamics of self-financing portfolio reflects the evolution of the market which explains (5.3.1). Indeed, the term  $[r_t V_t + \pi_t (\mu_t + q_t - r_t \mathbf{1}_d)]$  is the driver in the market with only one interest rate  $r$  and one dividend rate  $q$ ; the term  $-(R_t - r_t)(V_t - \pi_t \mathbf{1}_d)^- \mathbf{d}t$  comes from the extra interest when the investor borrows the risk-free asset, i.e. when  $V_t - \pi_t \mathbf{1}_d < 0$ ; the term  $-(\pi_t^-)(Q_t - q_t) \mathbf{d}t$  is due to the extra dividend when the investor is shorting some assets, i.e.  $\pi_t^i < 0$ . In summary we call this  $f$  the market driver.

### 5.3.2 The pricing problem under our settings

In this subsection, we recall the definition of the pricing problem of American contingent claim in a given Itô market  $f$ , among which the American option is one of the special case. Once the driver covers all informations in the market, we can somehow ignore the dynamics of underlying assets. So we deviate slightly from the conventional use of superscripts in representing the process, i.e. we denote  $V_s^{t,v}$  the wealth process  $V$  at time  $s$  starting with  $V_t = v$  where  $v$  is  $\mathcal{F}_t$ -measurable.

**Definition 5.3.1** ([89, Definition 4.1]). *An American Contingent Claim (or ACC)( $T, \mathcal{S}, s_t$ ) is a financial instrument consisting of an expiration date  $T \in (0, +\infty]$ , the selection of an exercise time  $0 \leq \tau \leq T$ , a pay-off rate  $s_t$  per unit time on  $(0, \tau)$  and a terminal payoff  $\mathcal{S}_\tau$  at the exercise time  $\tau$ .*

Considering the case of American option with a finite expiration date, namely  $s_t \equiv 0, \forall t \in [0, T], T < +\infty$ , we have the following definition of hedging strategy. In [12][Definition 5.1], this hedging strategy for American contingent claim is also called the super-hedging strategy, from which the price is defined by the smallest endowment among all admissible super-hedging strategies. See also the definition for European option in [54][Definition 1.3].

**Definition 5.3.2** ([89, Definition 5.1 and 5.3]). *Given the financial market with driver  $f$ , for any  $t > 0, v \geq 0, (\pi, \mathcal{C}) \in \mathbb{H}^2 \times \mathcal{S}^2$ , the wealth process  $V$ , with initial capital  $v$  at time  $t$ , strategy  $\pi$  and the cumulative withdrawals for consumption  $\mathcal{C}$  writes as*

$$V_s^{t,v,\pi,\mathcal{C}} = v - \int_t^s f(u, V_u^{t,v,\pi,\mathcal{C}}, \pi_u \sigma_u) \mathbf{d}u + \int_t^s \pi_u \sigma_u \mathbf{d}B_u - \int_t^s \mathbf{d}\mathcal{C}_u. \quad (5.3.2)$$

*We call  $(\pi, \mathcal{C})$  is a hedging strategy against the ACC( $T, \mathcal{S}, 0$ ) and write  $(\pi, \mathcal{C}) \in \mathcal{H}_t(T, v)$  if*

- the consumption process  $(C_s)_{t \leq s \leq T}$  is a continuous, non-decreasing process;
- $V_s^{t,v,\pi,\mathcal{C}} \geq S_s$ , for  $t \leq s \leq T$ ;
- $V_T^{t,v,\pi,\mathcal{C}} = S_T$ ;
- $C_u(\omega) = C_{\tau_u(\omega)}(\omega)$  a.s. for any fixed  $u \in [t, T]$ , with the stopping time  $\tau_u := \inf\{s \geq u : V_s^{t,v,\pi,\mathcal{C}} = S_s\} \wedge T$ .

Namely,  $\mathcal{H}_t(T, v)$  contains all super-hedging strategies starting with capital  $v$  at time  $t$ . We define the fair price at time  $t$  for the claim  $\mathcal{S}$  by

$$V_t := \inf\{v \geq 0 : \exists(\pi, \mathcal{C}) \in \mathcal{H}_t(T, v)\}.$$

With the above definitions, we link the pricing problem of American option to the RBSDE. Let us consider a RBSDE with driver  $f$  and takes the payoff process  $S$  to be its reflected obstacle, we can identify the solution  $(Y, Z, \mathcal{K})$  of RBSDE  $(f, \mathcal{S})$  to the optimal hedging portfolio  $(V, \pi^*, \mathcal{C}^*)$  of  $ACC(T, \mathcal{S}, 0)$  in the market involved as (5.3.2):

$$V_t = Y_t, \pi_t^* = Z_t \sigma_t^{-1}, \mathcal{C}_t^* = \mathcal{K}_t, \quad a.s..$$

Indeed, notice that, for any  $(\pi, \mathcal{C}) \in \mathcal{H}_t(T, v)$ ,  $(V^{t,v,\pi,\mathcal{C}}, \pi\sigma, \mathcal{C})$  satisfies naturally the equation (5.2.1) except the Skorohod's condition since  $\int_t^T (V_s^{t,v,\pi,\mathcal{C}} - S_s) d\mathcal{C}_s \geq 0$  which implies, for any  $v \geq 0$  s.t.  $\exists(\pi, \mathcal{C}) \in \mathcal{H}_t(T, v)$ , we have  $v \geq Y_t$ . Taking inf on all endowments  $v$ , we get  $V_t \geq Y_t$ . Meanwhile, for all  $0 \leq t \leq T$ , we know the solution of RBSDE  $(Z\sigma^{-1}, \mathcal{K}) \in \mathcal{H}_t(T, Y_t)$ , so necessarily  $V_t = Y_t$ .

All in all, we have immediately the following result.

**Proposition 5.3.3.** *Assume (H-M) and (H-E-1) hold, the pricing problem of American option with payoff  $S$  in this market is equivalent to solving RBSDE w.r.t.  $(f, S)$  where  $f$  is defined by the market driver (5.3.1).*

**In this following, we say that  $(Y, Z, \mathcal{K})$  solves the pricing problem of American Option with payoff  $S$  in the market  $f$  if  $(Y, Z, \mathcal{K})$  is the solution of RBSDE (5.2.1) w.r.t.  $(f, S)$ .**

## 5.4 Change of numéraire

We introduce the our principle result about the change of numéraire. Assume we work always under the historical probability  $\mathbb{P}$ . Our proof is based on BSDE theories which without any doubt can be easily generalized to the case with non-linear driver. Some applications will be provided in the next section.

**Theorem 5.4.1.** *Suppose (H-M) is fulfilled by  $(\pi^0, X^1, \dots, X^d)$  and for some  $\varepsilon > 0$ , the payoff process  $S_t = g(t, (X_s)_{0 \leq s \leq t})$  satisfies (H-E-1) with  $p = 2 + \varepsilon$ , then there exists  $(Y, Z, \mathcal{K}) \in (\mathcal{S}^{2+\varepsilon} \times \mathbb{H}^{2+\varepsilon} \times \mathcal{S}^{2+\varepsilon})$  who solves the pricing problem of American option with payoff  $S$  written on  $X$  with the market driver  $f$  given as in (5.3.1). If one takes the first asset  $X^1$  as numéraire, then we have the following results:*

— Let

$$(\tilde{Y}_t, \tilde{Z}_t, \tilde{\mathcal{K}}_t) := \left( \frac{Y_t}{X_t^1}, \frac{Z_t}{X_t^1} - \sigma_t^{(1)} \frac{Y_t}{X_t^1}, \int_0^t \frac{d\mathcal{K}_s}{X_s^1} \right),$$

then the processes  $(\tilde{Y}, \tilde{Z}, \tilde{\mathcal{K}}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$  satisfies RBSDE (5.2.1) with  $(\tilde{f}, \tilde{S})$ , where

$$\begin{aligned} \tilde{f}(t, y, z) := & -\tilde{r}_t y - z \tilde{\sigma}_t^{-1} (\tilde{\mu}_t + \tilde{q}_t - \tilde{r}_t \mathbf{1}_d) \\ & + (\tilde{R}_t - \tilde{r}_t) (y - z \tilde{\sigma}_t^{-1} \mathbf{1}_d)^- + (z \tilde{\sigma}_t^{-1})^- (\tilde{Q}_t - \tilde{q}_t), \end{aligned} \quad (5.4.1)$$

and  $\tilde{S}_t := \frac{S_t}{X_t^1}$ . Here  $\tilde{r} \in \mathbb{R}$ ,  $\tilde{R} \in \mathbb{R}$ ,  $\tilde{\mu} \in \mathbb{R}^d$ ,  $\tilde{\sigma} \in \mathbb{M}_{d \times d}$ ,  $\tilde{q} \in \mathbb{R}^d$ ,  $\tilde{Q} \in \mathbb{R}^d$  are defined by  $\tilde{r}_t := q_t^{(1)}$ ,  $\tilde{R}_t := Q_t^{(1)}$ , and

$$\tilde{\mu}_t := \begin{bmatrix} -\mu_t^{(1)} \\ \mu_t^{(2)} - \mu_t^{(1)} \\ \vdots \\ \mu_t^{(d)} - \mu_t^{(1)} \end{bmatrix}_{d \times 1} - \tilde{\sigma}_t (\sigma_t^{(1)})^\top, \quad \tilde{\sigma}_t := \begin{bmatrix} -\sigma_t^{(1)} \\ \sigma_t^{(2)} - \sigma_t^{(1)} \\ \vdots \\ \sigma_t^{(d)} - \sigma_t^{(1)} \end{bmatrix}_{d \times d}, \quad \tilde{q}_t := \begin{bmatrix} r_t \\ q_t^{(2)} \\ \vdots \\ q_t^{(d)} \end{bmatrix}, \quad \tilde{Q}_t := \begin{bmatrix} R_t \\ Q_t^{(2)} \\ \vdots \\ Q_t^{(d)} \end{bmatrix}.$$

— In numéraire  $X^1$ , the new risk-free saving account  $\tilde{\pi}^0$  satisfies

$$d\tilde{\pi}_t^0 = \tilde{r}_t (\tilde{\pi}_t^0)^+ dt - \tilde{R}_t (\tilde{\pi}_t^0)^- dt,$$

and the  $d$ -risky assets  $(\tilde{X}_t^1, \tilde{X}_t^2, \dots, \tilde{X}_t^d) := \left( \frac{1}{X_t^1}, \frac{X_t^2}{X_t^1}, \dots, \frac{X_t^d}{X_t^1} \right)$  satisfy

$$\frac{d\tilde{X}_t^i}{\tilde{X}_t^i} = \tilde{\mu}_t^{(i)} dt + \tilde{\sigma}_t^{(i)} dB_t, \quad 0 \leq t \leq T,$$

and for each asset  $X^i$ , its continuous dividend is paid at different rate  $\tilde{q}_t^{(i)}$  (when longing asset) and  $\tilde{Q}_t^{(i)}$  (when shorting asset). In the following, we call this setting  $X^1$ -market, whose basic numéraire is the asset  $X^1$ .

Equivalently,  $(\tilde{Y}, \tilde{Z}, \tilde{\mathcal{K}})$  solves the non-linear pricing problem of American option with payoff  $\tilde{S}$  written on  $\tilde{X}$  in the  $X^1$ -market  $\tilde{f}$  where (H-M) is fulfilled with coefficients  $(\tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$ .

*Proof.* We prove this result in several steps.

▷ *Step 1:* Under the market setting (H-M), the driver  $f$  (5.3.1) satisfies automatically (H-E-2) for any  $p \geq 2$ . With the assumptions  $S \in \mathcal{S}^{2+\varepsilon}$ , from [52], we know (5.2.1) admits an unique solution  $(Y, Z, \mathcal{K}) \in (\mathcal{S}^{2+\varepsilon} \times \mathbb{H}^{2+\varepsilon} \times \mathcal{S}^{2+\varepsilon})$ . The boundedness condition in (H-M) implies that  $\frac{1}{X^1} \in \mathcal{S}^p$ , for  $p \geq 2$  and  $\tilde{f}$  satisfies Lipschitz condition (H-E-2). Thus by Hölder's inequality,  $\tilde{S} \in \mathcal{S}^2$ . Again from [52], there exists a unique solution to (5.2.1) in  $(\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2)$  with  $(\tilde{f}, \tilde{S})$  satisfying the continuity of the solution in (H-E-3). We now verify  $(\tilde{Y}, \tilde{Z}, \tilde{\mathcal{K}})$  is the solution to (5.2.1) with  $(\tilde{f}, \tilde{S})$ .

The extra integrability of order  $(2 + \varepsilon)$  of  $(Y, Z, \mathcal{K})$  ensures that  $(\tilde{Y}, \tilde{Z}, \tilde{\mathcal{K}}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$ , using again  $\frac{1}{X} \in \mathcal{S}^p$ , for  $p \geq 2$ . Notice that, if  $(Y, Z, \mathcal{K})$  is the solution to (5.2.1) w.r.t.  $(f, S)$ , then  $\frac{Y}{X^1} \geq \tilde{S} = \frac{S}{X^1}$  and since  $X_t > 0$ , for  $t \in [0, T]$ , a.s.,

$$\int_0^T (\tilde{Y}_t - \tilde{S}_t) d\tilde{\mathcal{K}}_t = \int_0^T \left( \frac{Y_t - S_t}{(X_t^1)^2} \right) d\mathcal{K}_t = 0, \quad a.s..$$

So the boundary condition and the Skorohod's condition are verified by  $\tilde{Y}, \tilde{\mathcal{K}}$ , thus it just leaves us to verify the dynamic equation. Apply Itô's lemma on  $Y/X^1$ ,

$$\begin{aligned} d\left(\frac{Y_t}{X_t^1}\right) &= -\left(\frac{f(t, Y_t, Z_t)}{X_t^1} + \mu_t^{(1)} \frac{Y_t}{X_t^1} + \left(\frac{Z_t}{X_t^1} - \sigma_t^{(1)} \frac{Y_t}{X_t^1}\right) (\sigma_t^{(1)})^\top\right) dt - \frac{1}{X_t^1} d\mathcal{K}_t \\ &\quad + \left(\frac{Z_t}{X_t^1} - \sigma_t^{(1)} \frac{Y_t}{X_t^1}\right) dB_t, \end{aligned}$$

equivalently,

$$d\tilde{Y}_t = -\left(\frac{f(t, Y_t, Z_t)}{X_t^1} + \mu_t^{(1)} \tilde{Y}_t + \tilde{Z}_t (\sigma_t^{(1)})^\top\right) dt - d\tilde{\mathcal{K}}_t + \tilde{Z}_t dB_t.$$

Notice that the driver  $f$  in (5.3.1) is homogeneous in  $(y, z)$ , i.e.  $cf(t, y, z) = f(t, cy, cz)$  holds for  $c > 0$ , so we can denote the driver of  $\tilde{Y}_t$  by  $\tilde{f}^1(t, \tilde{Y}_t, \tilde{Z}_t)$  which does not depend on  $X_t^1$ , namely

$$\tilde{f}^1(t, y, z) = f(t, y, z + y\sigma_t^{(1)}) + \mu_t^{(1)}y + z(\sigma_t^{(1)})^\top,$$

so from the uniqueness of solution of RBSDE (see [52][Section 6]),  $(\tilde{Y}, \tilde{Z}, \tilde{\mathcal{K}})$  is the unique solution w.r.t.  $(\tilde{f}, \tilde{S})$  if for any  $y \in \mathbb{R}, z \in \mathbb{R}^d, \tilde{f}^1(t, y, z) = \tilde{f}(t, y, z)$ , for any  $t \in [0, T]$ , a.s..

▷ *Step 2:* It just leaves us to verify  $\tilde{f}^1 = \tilde{f}$ . We give some primary results to simplify the calculations: denote  $\Lambda_1 := (1, 0, \dots, 0)$  a row vector in  $\mathbb{R}^d$ ,

$$\sigma_t^{-1} = \tilde{\sigma}_t^{-1} \tilde{\sigma}_t \sigma_t^{-1} = \tilde{\sigma}_t^{-1} \left( \sigma_t - \begin{bmatrix} 2\sigma_t^{(1)} \\ \sigma_t^{(1)} \\ \vdots \\ \sigma_t^{(1)} \end{bmatrix} \right) \sigma_t^{-1} = \tilde{\sigma}_t^{-1} \left( I_d - \begin{bmatrix} 2\Lambda_1 \\ \Lambda_1 \\ \vdots \\ \Lambda_1 \end{bmatrix} \right), \quad \sigma_t^{(1)} (\sigma_t)^{-1} = \Lambda_1,$$

then the  $j$ -th component of the row vector  $(z + y\sigma_t^{(1)})\sigma_t^{-1}$  can be calculated by

$$\begin{aligned} ((z + y\sigma_t^{(1)})\sigma_t^{-1})^j &= [(y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)\Lambda_1^j + (z\tilde{\sigma}_t^{-1} - (z\tilde{\sigma}_t^{-1})^1\Lambda_1)^j] \\ &= \begin{cases} (z\tilde{\sigma}_t^{-1})^j, & \text{if } j \neq 1, \\ y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d, & \text{if } j = 1, \end{cases} \end{aligned}$$

thus, we can calculate the terms in  $f(t, y, z + y\sigma_t^{(1)})$ . For the extra term due to 2 interest rates, we get,

$$\begin{aligned} (R_t - r_t)[y - (z + y\sigma_t^{(1)})\sigma_t^{-1}\mathbf{1}_d]^- &= (R_t - r_t)[y - (y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d) - \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j]^- \\ &= (R_t - r_t)[z\tilde{\sigma}_t^{-1}\mathbf{1}_d - \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j]^- \\ &= (R_t - r_t)[(z\tilde{\sigma}_t^{-1})^1]^- . \end{aligned}$$

For the extra term from 2 dividends, we have

$$[(z + y\sigma_t^{(1)})\sigma_t^{-1}]^-(Q_t - q_t) = (y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)^-(Q_t^{(1)} - q_t^{(1)}) + \sum_{j=1}^d [(z\tilde{\sigma}_t^{-1})^j]^- (Q_t^{(j)} - q_t^{(j)}) .$$

It comes to the last term with  $(\mu, q, r)$ ,

$$\begin{aligned} (z + y\sigma_t^{(1)})\sigma_t^{-1}(\mu_t + q_t - r_t\mathbf{1}_d) &= (y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)(\mu_t^{(1)} + q_t^{(1)} - r_t) + \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(j)} + q_t^{(j)} - r_t) \\ &= y(\mu_t^{(1)} + q_t^{(1)} - r_t) - \sum_{j=1}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(1)} + q_t^{(1)} - r_t) + \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(j)} + q_t^{(j)} - r_t) \\ &= y(\mu_t^{(1)} + q_t^{(1)} - r_t) - (z\tilde{\sigma}_t^{-1})^1 (\mu_t^{(1)} + q_t^{(1)} - r_t) + \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(j)} - \mu_t^{(1)} + q_t^{(j)} - q_t^{(1)}) . \end{aligned}$$

Plugging all aboves into  $\tilde{f}^1$ , it follows by,

$$\begin{aligned} \tilde{f}^1(t, y, z) &= -r_t y - (z + y\sigma_t^{(1)})\sigma_t^{-1}(\mu_t + q_t - r_t\mathbf{1}_d) + (R_t - r_t)[y - (z + y\sigma_t^{(1)})\sigma_t^{-1}\mathbf{1}_d]^- \\ &\quad + ((z + y\sigma_t^{(1)})\sigma_t^{-1})^-(Q_t - q_t) + \mu_t^{(1)} y + z(\sigma_t^{(1)})^\top \\ &= -q_t^1 y + (z\tilde{\sigma}_t^{-1})^1 (\mu_t^{(1)} + q_t^{(1)} - r_t) - \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(j)} - \mu_t^{(1)} + q_t^{(j)} - q_t^{(1)}) + z(\sigma_t^{(1)})^\top \\ &\quad + (R_t - r_t)[(z\tilde{\sigma}_t^{-1})^1]^- + (y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)^-(Q_t^{(1)} - q_t^{(1)}) + \sum_{j=2}^d [(z\tilde{\sigma}_t^{-1})^j]^- (Q_t^{(j)} - q_t^{(j)}) , \end{aligned}$$

notice that  $(R_t - r_t)[(z\tilde{\sigma}_t^{-1})^1]^- + \sum_{j=2}^d [(z\tilde{\sigma}_t^{-1})^j]^- (Q_t^{(j)} - q_t^{(j)}) = (z\tilde{\sigma}_t^{-1})^- (\tilde{Q}_t - \tilde{q}_t)$ , thus

$$\begin{aligned} \tilde{f}^1(t, y, z) &= -\tilde{r}_t y - (z\tilde{\sigma}_t^{-1})^1 (-\mu_t^{(1)} + r_t - q_t^{(1)}) - \sum_{j=2}^d (z\tilde{\sigma}_t^{-1})^j (\mu_t^{(j)} - \mu_t^{(1)} + q_t^{(j)} - q_t^{(1)}) + z(\sigma_t^{(1)})^\top \\ &\quad + (\tilde{R}_t - \tilde{r}_t)(y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)^- + (z\tilde{\sigma}_t^{-1})^- (\tilde{Q}_t - \tilde{q}_t) \\ &= -\tilde{r}_t y - z\tilde{\sigma}_t^{-1}(\tilde{\mu}_t + \tilde{q}_t - \tilde{r}_t) + (\tilde{R}_t - \tilde{r}_t)(y - z\tilde{\sigma}_t^{-1}\mathbf{1}_d)^- + (z\tilde{\sigma}_t^{-1})^- (\tilde{Q}_t - \tilde{q}_t). \end{aligned}$$

So we prove  $\tilde{f}^1(t, y, z) = \tilde{f}(t, y, z)$ ,  $\mathbf{d}t \times \mathbf{d}\mathbb{P}$ , a.e.  $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$ . Applying the comparison theorem [52, Theorem 4.1], we know that  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  is the unique solution in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$  of (5.2.1) w.r.t.  $(\tilde{f}, \tilde{S})$ . As to the dynamics of  $\tilde{X}$ , it suffices to apply Itô's lemma.  $\square$

**Remark 5.4.1** (Well-posedness). In some special case, it suffices to have  $\varepsilon = 0$  for the well-posedness of the solution. For example, let  $Y$  be the price of an American Call option with  $S_t = (X_t - K)^+$ , then its price in  $X$ -market, i.e.  $Y/X$ , corresponds to the price of the Put option written on  $1/X$  with the pay-off process  $\tilde{S}_t = (1 - \frac{K}{X_t})^+$ , which is bounded, so  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  is square integrable as the solution of (5.2.1) w.r.t.  $(\tilde{f}, \tilde{S})$ . Inversely, if  $Y$  represents the price of American Put option with  $S_t = (K - X_t)^+$ , then  $Y \in \mathcal{S}^p$  for any  $p \geq 2$ , as a consequence,  $Y/X$ , the price of Call option with  $\tilde{S}_t = (\frac{K}{X_t} - 1)^+$ , is of course in  $\mathcal{S}^2$ , in view of the integrability of  $X$  and  $\frac{1}{X}$ .

**Remark 5.4.2.** The significant observation with the change of numéraire lies in the exchanged position of risk-free asset and risky assets. In fact, before the change of numéraire, the risk-less asset  $X_0$  and the  $d$ -risky assets  $(X^1, \dots, X^d)$  compose the money market, but after choosing  $X^1$  as the numéraire,  $X^1$  becomes the risk-less asset in  $X^1$ -market, and the risk-less asset  $X^0$  in the original market turns out to be a risky asset  $\tilde{X}^1 = \frac{1}{X_t^1}$  in  $X^1$ -market, so that  $X^1$ -market is composed by one risk-less account  $\tilde{\pi}_0$  and  $d$ -risky assets  $(\tilde{X}^1, \dots, \tilde{X}^d)$ .

This observation explains also the differences of the drivers between the original market and  $X^1$ -market. The return rate  $\tilde{\mu}$  is actually the relative returns with respect to  $X^1$  and thanks to this log-normal form of the components of  $X$ , this relative returns are represented in the affine terms; the term  $\tilde{\sigma}_t(\sigma_t^{(1)})^\top$  in  $\tilde{\mu}$  will be eliminated if one applies the change of measure; the matrix of volatility  $\tilde{\sigma}$  is exactly the difference between two volatilities which is not surprising due to Itô's formula; the changes of  $\tilde{r}, \tilde{R}, \tilde{q}^{(1)}, \tilde{Q}^{(1)}$  come from the changes of the risk-less asset and the risky one, which is quite easy to understand.

The techniques in the proof of Theorem 5.4.1 also can be applied to get the symmetry of European options which is closely related to the BSDEs (see [54, Example 2.1]). We say  $(Y, Z)$  the solution of BSDE w.r.t.  $(f, S_T)$  if

$$Y_t = S_T + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s - \int_t^T Z_s \mathbf{d}B_s. \quad (5.4.2)$$



**We call that  $(Y, Z)$  solves the pricing problem of European option with payoff  $S$  at maturity  $T$  with the market driver  $f$ , if  $(Y, Z)$  is the solution of BSDE (5.4.2) w.r.t.  $(f, S)$ .** Notice that, when (H-M) is supposed to be true, the driver  $f$  of BSDE is the same as (5.3.1), which is the market driver.

**Corollary 5.4.3.** *Suppose (H-M) is fulfilled by  $(\pi^0, X^1, \dots, X^d)$  and suppose at date  $T$ , the payoff  $S_T = g(T, (X_s)_{0 \leq s \leq T})$  which is  $\mathcal{F}_T$ -measurable and  $S_T \in \mathbb{L}^p$  with  $p = 2 + \varepsilon$  for some  $\varepsilon > 0$ , then there exists  $(Y, Z) \in \mathcal{S}^{2+\varepsilon} \times \mathbb{H}^{2+\varepsilon}$  who solves the pricing problem of European option with payoff  $S$  written on  $X$  with the market driver  $f$  as in (5.3.1) If one takes the first asset  $X^1$  as numéraire, then we have the following results:*

— Let

$$(\tilde{Y}_t, \tilde{Z}_t) := \left( \frac{Y_t}{X_t^1}, \frac{Z_t}{X_t^1} - \sigma_t^{(1)} \frac{Y_t}{X_t^1} \right),$$

then the processes  $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2 \times \mathbb{H}^2$  satisfy the BSDE (5.4.2) w.r.t.  $(\tilde{f}, \tilde{S}_T)$  with  $\tilde{f}$  as in (5.4.1).

— With numéraire  $X^1$ , the risk-free saving account  $\tilde{\pi}^0$  satisfies

$$d\tilde{\pi}_t^0 = \tilde{r}_t(\tilde{\pi}_t^0)^+ dt - \tilde{R}_t(\tilde{\pi}_t^0)^- dt,$$

the  $d$ -risky assets in the  $X^1$ -market  $(\tilde{X}_t^1, \tilde{X}_t^2, \dots, \tilde{X}_t^d) := \left( \frac{1}{X_t^1}, \frac{X_t^2}{X_t^1}, \dots, \frac{X_t^d}{X_t^1} \right)$  satisfy

$$\frac{d\tilde{X}_t^i}{\tilde{X}_t^i} = \tilde{\mu}_t^{(i)} dt + \tilde{\sigma}_t^{(i)} dB_t, \quad 0 \leq t \leq T,$$

and the continuous dividend is paid at different rate  $\tilde{q}_t$  (when longing asset) and  $\tilde{Q}_t$  (when shorting asset).

Equivalently,  $(\tilde{Y}, \tilde{Z})$  solves the non-linear pricing problem of European option with payoff  $\tilde{S}$  written on  $\tilde{X}$  in  $X^1$ -market which fulfills (H-M) with coefficients  $(\tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$ .

**Remark 5.4.4.** All the examples with different payoff of American options in Section 5.5 can be easily verified for the European-type options.

To simplify the explanations of the applications in the following sections, we introduce a result about the homogeneity of the driver which will help us to understand the symmetry of the spot value between Put and Call options in the case of the Markovian price function.

**Proposition 5.4.5** (Homogeneity). *Assume  $f(t, y, z)$  is positive homogeneous in  $y, z$  i.e.*

$$cf(t, y, z) = f(t, cy, cz) \quad \text{for any constant } c > 0,$$

and  $S \in \mathcal{S}^2$  is continuous.

Let  $(Y, Z, K)$  be the solution of RBSDE (5.2.1) w.r.t.  $(f, S)$  in  $(\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2)$ , then for any positive constant  $c > 0$ ,  $(cY, cZ, cK)$  solve (5.2.1) w.r.t.  $(f, cS)$  in  $(\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2)$ ; let  $(Y, Z)$  be the solution of BSDE (5.4.2) w.r.t.  $(f, S_T)$  in  $(\mathcal{S}^2 \times \mathbb{H}^2)$ , then for any positive constant  $c > 0$ ,  $(cY, cZ)$  solve (5.4.2) w.r.t.  $(f, cS_T)$  in  $(\mathcal{S}^2 \times \mathbb{H}^2)$ .

*Proof.* Notice that  $cS$  has the same integrability as  $S$ , so the unique solution of RBSDE w.r.t.  $(f, cS)$  exists in  $(\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2)$ . We only need to verify that  $(cY, cZ, cK)$  satisfying (5.2.1). An easy computation leads to the results thanks to the homogeneity of  $f$ . The proof for BSDEs is similar.  $\square$

**Remark 5.4.6.** Consider the case of pricing American option with Markovian payoff, then its price at time  $t$  can be formulated as a function of the price of underlying asset at time  $t$ . For example, consider the market (H-M) with  $d = 1$ , for the American Call option in this market, set  $S^{t,x} = \{(X_s^{t,x} - K)^+ : t \leq s \leq T\}$ , we know that  $Y$  to (5.2.1) is the value process of the American Call option, from Theorem 5.4.1,  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  solves the pricing problem of  $\tilde{S}^{t,\tilde{x}} = \{(1 - K\tilde{X}_s^{t,\tilde{x}}) : t \leq s \leq T\}$  where  $\tilde{X}_s^{t,\tilde{x}} = \frac{1}{X_s^{t,x}}$  and  $\tilde{x} = \frac{1}{x}$ . Thanks to Proposition 5.4.5, we can also say that  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  solves the pricing problem with  $\tilde{S}^{t,K\tilde{x}} = \left\{ \left(1 - \tilde{X}_s^{t,K\tilde{x}}\right)^+ : t \leq s \leq T \right\}$ . This will be used frequently afterwards.

## 5.5 Put/Call Symmetry and Applications

In this section, we are going to present the various applications of our main result in this chapter, Theorem 5.4.1. The most interesting part is that, with the help of RBSDE, we can get rid of the linear pricing framework and immediately reach the symmetry result between the different options without applying the change of measure.

### 5.5.1 Put/Call Symmetry formula

- Denote  $\{X_s^{t,x} : s \geq t\}$  the process  $X$  after  $t$  when  $X$  starts at  $t$  with  $X_t = x$  and  $\{S_s^{t,x} : t \leq s \leq T\}$  the payoff  $S$  from  $X_t = x$ ; denote  $(Y^{t,x}, Z^{t,x}, \mathcal{K}^{t,x})$  the solution of RBSDE (5.2.1) w.r.t.  $(f, S^{t,x})$  where  $f$  is the market driver as (5.3.1) and  $Y^{t,x} := \{Y_s^{t,x} : s \geq t\}$ ,  $Z^{t,x} := \{Z_s^{t,x} : t \leq s \leq T\}$  and  $\mathcal{K}^{t,x} = \{\mathcal{K}_s^{t,x} : t \leq s \leq T\}$ ;
- Denote  $\mathbf{Call}(t, x; C, T, K; \mu, r, R, q, Q, \sigma)$  the price at time  $t$  of the American Call option with payoff process  $S^{t,x} := \{(C^\top X_s^{t,x} - K)^+, t \leq s \leq T\}$  where the strike price is  $K$  and the maturity is  $T$ , in the non-linear market where  $C$  is a constant column vector in  $\mathbb{R}^d$  and the underlying assets satisfy (H-M) with coefficients  $(\mu, r, R, q, Q, \sigma)$ ; if we assume (H-M) with  $d = 1$ , we denote  $\mathbf{Call}(t, x; T, K; \mu, r, R, q, Q, \sigma)$  the price of American Call with payoff  $S^{t,x} := \{(X_s^{t,x} - K)^+, t \leq s \leq T\}$ ;
- Denote  $\mathbf{Put}(t, \tilde{x}; C, T, \tilde{K}; \tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$  the price at time  $t$  of the American Put Option with payoff process  $S^{t,\tilde{x}} := \{(\tilde{K} - C^\top \tilde{X}_s^{t,\tilde{x}})^+, t \leq s \leq T\}$  where the underlying assets  $\tilde{X}$  satisfy (H-M) with coefficients  $(\tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$ ; similar with the Call option, we drop  $C$  in  $\mathbf{Put}(t, \tilde{x}; T, \tilde{K}; \tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$  when  $d = 1$ ;
- Denote  $\Theta := (\mu, r, R, q, Q, \sigma)$  and  $\tilde{\Theta} := (\tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$  as defined in Theorem 5.4.1 for simplification.

We first show an application on the Put-Call symmetry (PCS in short) in uni-dimension and

then generalize to the multidimensional case, which are both applications of Theorem 5.4.1.

**Proposition 5.5.1** (PCS). *Suppose (H-M) is fulfilled to  $d = 1$ , we have, the relation between the pricing of American Call option and Put option in the non-linear market with market driver  $f$  as (5.3.1),*

$$\mathbf{Call}(t, x; T, K; \mu, r, R, q, Q, \sigma) = x \mathbf{Put}(t, \frac{K}{x}; T, 1; -\mu + \sigma^2, q, Q, r, R, -\sigma), \quad (5.5.1)$$

and the symmetry formula writes as follows

$$\mathbf{Call}(t, x; T, K; \mu, r, R, q, Q, \sigma) = \mathbf{Put}(t, K; T, x; -\mu + \sigma^2, q, Q, r, R, -\sigma). \quad (5.5.2)$$

*Proof.* Letting  $S^{t,x} := \{(X_s^{t,x} - K)^+ : t \leq s \leq T\}$ , with boundedness assumed in (H-M),  $S^{t,x} \in \mathcal{S}^4$ . Applying Theorem 5.4.1 with  $S^{t,x}$  and using the equivalence between the American Option and the RBSDE in Proposition 5.3.3,  $(Y^{t,x}, Z^{t,x}, K^{t,x})$ , the solution of RBSDE w.r.t.  $(f, S^{t,x})$  whose existence is guaranteed by (H-E), solves the pricing problem of American Call option  $S^{t,x}$  in the non-linear market (H-M) i.e.

$$Y_t^{t,x} = \mathbf{Call}(t, x; T, K; \Theta).$$

At same time,  $(\tilde{Y}^{t,\tilde{x}}, \tilde{Z}^{t,\tilde{x}}, \tilde{K}^{t,\tilde{x}})$  which is the solution of RBSDE w.r.t.  $(\tilde{f}, \tilde{S}^{t,\tilde{x}})$ , solves the pricing problem of American Put option

$$\tilde{S}^{t,\tilde{x}} := \{(\tilde{K} - K \tilde{X}_s^{t,\tilde{x}})^+ : t \leq s \leq T\} = \{(\tilde{K} - \tilde{X}_s^{t,K\tilde{x}})^+ : t \leq s \leq T\},$$

in  $X$ -market with market driver in (5.4.1), where  $\tilde{x} = \frac{1}{x}$ ,  $\tilde{K} = 1$ ,  $\tilde{X}_s^{t,\tilde{x}} = \frac{1}{X_s^{t,x}}$ , briefly,

$$\tilde{Y}_t^{t,\tilde{x}} = \mathbf{Put}(t, K\tilde{x}; T, \tilde{K}; \tilde{\Theta}).$$

Moreover,  $Y_t^{t,x} = x \tilde{Y}_t^{t,\tilde{x}}$ , when replace  $\tilde{\Theta}$  by  $(\tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma}) = (-\mu - \sigma^2, q, Q, r, R, -\sigma)$ , we prove successfully (5.5.1).

As for (5.5.2), we only need to justify that  $\tilde{f}$  in (5.4.1) is homogeneous as in Proposition 5.4.5, in fact, with an easy calculation, we can prove it. So for any fixed  $x > 0$ ,  $x \tilde{Y}_t^{t,\tilde{x}}$  is the pricing process of American Put with the payoff process

$$x \tilde{S}^{t,\tilde{x}} = \{(x \tilde{K} - x K \tilde{X}_s^{t,\tilde{x}})^+ : t \leq s \leq T\} = \left\{ \left( x - \tilde{X}_s^{t,K} \right)^+ : t \leq s \leq T \right\},$$

which gives us (5.5.2). □

With similar techniques, we can generalize this result to multidimensional market.

**Corollary 5.5.2** (Multidimensional PCS). *Suppose (H-M) is fulfilled with  $d > 1$ , for the American Option, in the non-linear market with market driver (5.3.1), one has the symmetry relation for the Call option with the pay-off  $S_t = (C^\top X_t - K)^+$  and the Put Option with pay-off  $\tilde{S}_t = (C^1 - \tilde{C}^\top \tilde{X}_t)$ , precisely*

$$\mathbf{Call}(t, x; C, T, K; \Theta) = x^1 \mathbf{Put}(t, \tilde{x}; \tilde{C}, T, C_1; \tilde{\Theta}),$$

and

$$\mathbf{Call}(t, x; C, T, K; \Theta) = \mathbf{Put}(t, x^1 \tilde{x}; \tilde{C}, T, C_1 x^1; \tilde{\Theta}),$$

where  $\tilde{X} := \left(\frac{1}{X^1}, \frac{X^2}{X^1}, \dots, \frac{X^d}{X^1}\right)$ ,  $\tilde{x} = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \dots, \frac{x^d}{x^1}\right)$ ,  $\tilde{C} = (K, -C_2, \dots, -C_d)$ ,  $\tilde{\Theta} = (\tilde{\mu}, \tilde{r}, \tilde{R}, \tilde{q}, \tilde{Q}, \tilde{\sigma})$  as defined in Theorem 5.4.1.

In the following, we turn to study the problem of the exercise boundary of American options. The exercise boundary is the crucial price (see [28, Section 2]) to determine if the holder of American option should exercise or keep the option at the moment. Unlike the price function who is tracked backwardly, the exercise boundary is a process on  $[0, T]$  when we are given the initial value  $X_0$ . To avoid confusion, we denote  $\mathbf{B}(t; x_0, T, K; \Theta)$  the exercise boundary at time  $t$  with  $X_0 = x_0$  for the American option with strike price  $K$  in the market with parameters  $\Theta$ . Denote  $\mathbf{B}^c(t; x_0, T, K; \Theta)$  the exercise boundary of  $\mathbf{Call}(t, X_t^{0,x_0}; T, K; \Theta)$  which satisfies

$$\mathbf{Call}(t, x; T, K; \Theta) = (x - K)^+, \text{ if } x = \mathbf{B}^c(t; x_0, T, K; \Theta)$$

with easy analysis we can see the crucial price and the exercise boundary play the same role in American pricing problem; and we denote  $\mathbf{B}^p(t; \tilde{x}_0, T, \tilde{K}; \tilde{\Theta})$  the exercise boundary of  $\mathbf{Put}(t, \tilde{X}_t^{0,\tilde{x}_0}; T, \tilde{K}; \tilde{\Theta})$  respectively which satisfies

$$\mathbf{Put}(t, x; T, \tilde{K}; \tilde{\Theta}) = (\tilde{K} - x)^+, \text{ if } x = \mathbf{B}^p(t; \tilde{x}_0, T, \tilde{K}; \tilde{\Theta}).$$

**Proposition 5.5.3** (Exercise boundary). *Assume (H-M) holds with  $d = 1$ . Then we have*

$$\mathbf{B}^c(t; x_0, T, K; \Theta) = \frac{K}{\mathbf{B}^p(t; \frac{K}{x_0}, T, 1; \tilde{\Theta})}.$$

*Proof.* We consider the relations among the events and the positive homogeneous property as Remark 5.4.6, for any fixed  $t \in [0, T]$ ,

$$\begin{aligned} \{X_t^{0,x_0} = \mathbf{B}^c(t; x_0, T, K; \Theta)\} &= \{\mathbf{Call}(t, X_t^{0,x_0}; T, K; \Theta) = (X_t^{0,x_0} - K)^+\} \\ &= \{X_t^{0,x_0} \mathbf{Put}(t, K \tilde{X}_t^{0,\tilde{x}_0}; T, 1; \tilde{\Theta}) = (X_t^{0,x_0} - K)^+\} \\ &= \{\mathbf{Put}(t, K \tilde{X}_t^{0,\tilde{x}_0}; T, 1; \tilde{\Theta}) = (1 - K \tilde{X}_t^{0,\tilde{x}_0})^+\} \\ &= \left\{ \mathbf{Put}(t, \tilde{X}_t^{0,K\tilde{x}_0}; T, 1; \tilde{\Theta}) = (1 - \tilde{X}_t^{0,K\tilde{x}_0})^+ \right\} \\ &= \{K \tilde{X}_t^{0,\tilde{x}_0} = \mathbf{B}^p(t; K \tilde{x}_0, T, 1; \tilde{\Theta})\}, \end{aligned}$$

where we apply the symmetry formula in Proposition 5.2.1 for the second equality. It follows that for any fixed  $t \in [0, T]$ , let  $x = \mathbf{B}^c(t; x_0, T, K; \Theta)$ , then  $\mathbf{B}^p(t; K\tilde{x}_0, T, 1; \tilde{\Theta}) = \frac{K}{x}$ , from where we get the desired result.  $\square$

We also consider the exchange option which is introduced by Magrabe in [106] as an example of this part. This option plays an important role in the financial market since it allows the holder to choose between two underlying assets. We will show that the dimension of this pricing problem can be reduced thanks to the change of numéraire. See also [27], [65][Section 3.2.(b)], [37] for related topic.

**Example 5.5.4** (Exchange Option). *Assume (H-M) holds for 2 underlying assets  $(X^1, X^2)$ . Consider a 2 factor exchange option  $S_t := (X_t^1 - X_t^2)^+$ , its price at time  $t$  in the non-linear market with market driver as (5.3.1) is denoted by  $EX(t, x_1, x_2; T; \Theta)$ , then*

$$EX(t, x_1, x_2; T; \Theta) = x_1 \mathbf{Put}\left(t, \frac{x_2}{x_1}; T, 1; \tilde{\Theta}^{(2)}\right)$$

where  $\mathbf{Put}\left(t, \frac{x_2}{x_1}; T, 1; \tilde{\Theta}^{(2)}\right)$  is the price of American Put option

$$\tilde{S}^{t, \tilde{x}_1, \tilde{x}_2} := \left\{ \left(1 - \tilde{X}_s^{t, \tilde{x}}\right)^+ : t \leq s \leq T \right\},$$

and  $\tilde{X}_s = \frac{X_s^2}{X_s^1}$ ,  $\tilde{x} = \frac{x_2}{x_1}$ ,

$$\begin{aligned} \tilde{\Theta}^{(2)} &= \left( \tilde{\mu}^{(2)}, \tilde{r}, \tilde{R}, \tilde{q}^{(2)}, \tilde{Q}^{(2)}, \tilde{\sigma}^{(2)} \right) \\ &= \left( \mu^{(2)} - \mu^{(1)} - (\sigma^{(2)} - \sigma^{(1)})(\sigma^{(1)})^\top, q^{(1)}, Q^{(1)}, q^{(2)}, Q^{(2)}, \sigma^{(2)} - \sigma^{(1)} \right) \end{aligned}$$

and  $\sigma^{(2)} - \sigma^{(1)} = \tilde{\sigma}^{(2)}$  is the second row of  $\tilde{\sigma}$ .

*Proof.* Apply directly Theorem 5.4.1. Notice that given the American exchange option with payoff  $S_t := (X_t^1 - X_t^2)^+$  in which the risk-less asset does not involve. If we take  $X^1$  as numéraire, the corresponding payoff is  $\tilde{S}_t = (1 - X_t^2/X_t^1)^+$  where the asset  $\frac{1}{X_t^1}$  is no longer in the replicating portfolio of  $\tilde{S}$ , which explains the reduction of dimension.  $\square$

## 5.5.2 American Exotic options

We apply Theorem 5.4.1 to the American binary options and barrier options, and present related results.

The *cash-or-nothing* option and *asset-or-nothing* option are classical binary options which allows the holder to have either 0 or some quantity of risk-free asset and risky asset respectively. For example, for American *cash-or-nothing* Call option, its payoff process is  $S = \{\mathbf{1}_{\{X_t \geq K\}} : 0 \leq t \leq T\}$ ; for American *asset-or-nothing* Put option, its payoff process is  $S = \{X_t \mathbf{1}_{\{X_t < K\}} : 0 \leq t \leq T\}$ .

**Corollary 5.5.5** (Cash-or-Nothing and Asset-or-Nothing). *If we set  $S_t$  to be the pay-off process of the American Cash-or-Nothing option whose price is denoted by  $\mathbf{Bin}^{CN}$ , namely  $S_t = \mathbf{1}_{\{X_t \geq K\}}$ , and let  $\tilde{S}_t = \frac{1}{X_t} \mathbf{1}_{\{\frac{1}{X_t} \leq \frac{1}{K}\}}$  which is the pay-off process of the American Asset-or-Nothing option, then we have*

$$\mathbf{Bin}^{CN} \mathbf{Call}(t, x; T, K; \Theta) = x \mathbf{Bin}^{AN} \mathbf{Put}(t, \frac{1}{x}; T, \frac{1}{K}; \tilde{\Theta}).$$

*Inversely,*

$$\mathbf{Bin}^{AN} \mathbf{Call}(t, x; T, K; \Theta) = x \mathbf{Bin}^{CN} \mathbf{Put}(t, \frac{1}{x}; T, \frac{1}{K}; \tilde{\Theta}).$$

**Barrier options** We consider the symmetry in the case of Barrier option with the strike price  $K$  and the constant barrier of activation  $H$ , namely, the option with the payoff process

*Out:*  $(X_t - K)^+ \mathbf{1}_{\{\tau_H > t\}}$  for Call option, if  $X_0 > H$ , this is a *down and out* Call  
(resp.  $(K - X_t)^+ \mathbf{1}_{\{\tau_H > t\}}$  for Put option, if  $X_0 < H$ , this is a *up and out* Put);

*In:*  $(X_t - K)^+ \mathbf{1}_{\{\tau_H \leq t\}}$  for Call Option, if  $X_0 > H$ , this is a *down and in* Call  
(resp.  $(K - X_t)^+ \mathbf{1}_{\{\tau_H \leq t\}}$  for Put option, if  $X_0 < H$ , this is a *up and in* Put),

where the stopping time  $\tau_H$  and  $\tau_{t,H}$  are given by

$$\tau_H := \inf\{u \geq 0 : X_u = H\} \wedge T, \quad \tau_{t,H} = \inf\{u \geq t : X_u^{t,X_t} = H\} \wedge T.$$

With assumption (H-M) with  $d = 1$  and assuming  $H < K$ , we know the payoff processes of Call Option for both *out* and *in* are continuous, so  $\tau_H$  and  $\tau_{t,H}, \forall t \in [0, T]$  are well-defined. We concentrate on the symmetry between *down and in* Call and *up and in* Put, and between *down and out* Call and *up and out* Put. See also [29][Section 5] for the European barrier options with multiple barriers.

Denote the price at time  $t$  knowing  $X_t = x > H$  of the *down and in* Call with  $S^{t,x} = \{(X_s^{t,x} - K)^+ \mathbf{1}_{\{\tau_{t,H} \leq s\}}, t \leq s \leq T\}$  by  $\mathbf{DICall}(t, x; T, K, H; \Theta)$ ; denote the price at time  $t$  of *down and out* Call with  $S^{t,x} = \{(X_s^{t,x} - K)^+ \mathbf{1}_{\{\tau_{t,H} > s\}}, t \leq s \leq T\}$  by  $\mathbf{DOCall}(t, x; T, K, H; \Theta)$ .

**Proposition 5.5.6** (*down and in* Call and *up and in* Put). *Suppose (H-M) holds with  $d = 1$ , the non-linear market with market driver  $f$  as (5.3.1), set  $H < K, H < x$ , we have*

$$\begin{aligned} \mathbf{DICall}(t, x; T, K, H; \Theta) &= x \mathbf{UIPut} \left( t, \frac{K}{x}; T, 1, \frac{K}{H}; \tilde{\Theta} \right) \\ &= \mathbf{UIPut} \left( t, K; T, x, \frac{Kx}{H}; \tilde{\Theta} \right), \end{aligned} \quad (5.5.3)$$

where  $\mathbf{UIPut}(t, \tilde{x}; T, \tilde{K}, \tilde{H}; \tilde{\Theta})$  is the price of the American up and in Put option written on  $\tilde{X}$  with payoff  $\tilde{S}^{t,\tilde{x}} = \{(\tilde{K} - \tilde{X}_s^{t,\tilde{x}})^+ \mathbf{1}_{\{\tilde{\tau}_{t,\tilde{H}} \leq s\}} : t \leq s \leq T\}$ ,  $\tilde{\tau}_{t,\tilde{H}} := \inf\{t \leq u : \tilde{X}_u^{t,\tilde{x}} = \tilde{H}\} \wedge T$ , and  $\tilde{X}^{t,\tilde{x}}$  satisfies

$$\begin{cases} \frac{d\tilde{X}_s}{\tilde{X}_s} = \tilde{\mu} dt + \tilde{\sigma}_t dB_t, & t \leq s \leq T, \\ \tilde{X}_t = \tilde{x} \end{cases}$$

with  $\tilde{\mu}$  and  $\tilde{\sigma}$  are components of  $\tilde{\Theta}$  as defined in Theorem 5.4.1.

*Proof.* Apply Theorem 5.4.1, we only need to treat the change of the stopping time  $\tau_{t,H}$ . Since  $X$  is a continuous process by assumptions, we can replace  $\inf\{t \leq u \leq T : X_u^{t,x} = H\}$  to the stopping time  $\inf\{t \leq u \leq T : X_u^{t,x} \leq H\}$ , then for the following events:

$$\begin{aligned} \{\tau_{t,H} \leq s\} &= \left\{ \inf_{t < u \leq s} X_u^{t,x} \leq H \right\} = \left\{ \sup_{t < u \leq s} \frac{1}{X_u^{t,x}} \geq \frac{1}{H} \right\} = \left\{ \sup_{t < u \leq s} \frac{K}{X_u^{t,x}} \geq \frac{K}{H} \right\} \\ &= \left\{ \sup_{t < u \leq s} K \tilde{X}^{t, \frac{1}{x}} \geq \frac{K}{H} \right\} = \left\{ \sup_{t < u \leq s} \tilde{X}^{t, \frac{K}{x}} \geq \frac{K}{H} \right\} = \{\tilde{\tau}_{t, \tilde{H}} \leq s\}, \end{aligned}$$

where we use homogeneity of  $X$  for initial value as in Remark 5.4.6,  $\tilde{H} = \frac{K}{H}$ . Apply again the homogeneity in Proposition 5.4.5, we get the second equality of (5.5.3).  $\square$

As a consequence, the pricing problem of American *down and in* Call option is symmetric to an American *up and in* Put option which has a barrier  $\frac{Kx}{H}$ . Thanks to the assumption  $K \geq H$ , we have  $x \leq \frac{Kx}{H}$  which yields, from Proposition 5.5.6, that the payoff process  $\tilde{S}^{t,K}$  of **UIPut** remains continuous because at the time of activation, the payoff  $(x - \frac{Kx}{H})^+ = 0$ ; similarly, for the relation between the spot value and the barrier, under the assumption  $x > H$ , we have  $K < \frac{Kx}{H}$ , which implies that the symmetry does not change the status (active or not) of the options.

Using the same arguments in Proposition 5.5.6, we can prove the similar result for the American *down and out* Call option and *up and out* Put option. Denote **DOC**all( $t, x; T, K, H; \Theta$ ) the price of *down and out* Call option with payoff  $S^{t,x} = \{(X_s - K)^+ \mathbf{1}_{\{\tau_{t,H} > s\}} : t \leq s \leq T\}$ .

**Example 5.5.7** (*down and out* Call and *up and out* Put). *Under the same assumptions as in Proposition 5.5.6, we have, for American down and out Call option*

$$\begin{aligned} \text{DOC}all(t, x; T, K, H; \Theta) &= x \text{UOP}ut \left( t, \frac{K}{x}; T, 1, \frac{K}{H}; \tilde{\Theta} \right) \\ &= \text{UOP}ut \left( t, K; T, x, \frac{Kx}{H}; \tilde{\Theta} \right), \end{aligned}$$

where **UOP**ut( $t, \tilde{x}; T, \tilde{K}, \tilde{H}; \tilde{\Theta}$ ) is the price of American *up and out* Put option with payoff  $\tilde{S}^{t, \tilde{x}} = \{(\tilde{K} - \tilde{X}_s^{t, \tilde{x}})^+ \mathbf{1}_{\{\tilde{\tau}_{t, \tilde{H}} > s\}} : t \leq s \leq T\}$ ,  $\tilde{\tau}_{t, \tilde{H}} := \inf\{t \leq u : \tilde{X}_u^{t, \tilde{x}} = \tilde{H}\} \wedge T$  and the underlying asset  $\tilde{X}$  starts from  $\tilde{X}_t = \tilde{x}$ .

### 5.5.3 American Quanto options

We present an application of change of numéraire in the international market as in [88, Section 2.7.3]. Denote  $X^{\mathbf{D}}$  the underlying in the domestic market, denote  $\mathcal{X}$  the exchange rate of the foreign currency price in domestic unit; denote  $X^{\mathbf{F}}$  the price of the underlying asset in the foreign market. We have always  $X_t^{\mathbf{D}} = X_t^{\mathbf{F}} \mathcal{X}_t$ , for  $t \in [0, T]$ .

**Foreign stock option with a strike in a foreign currency:** Considering an option written on  $X^f$  with the strike price  $K$  in foreign currency, for this contract, we investigate its pricing problem in domestic market, i.e. an option with payoff  $S_t = \mathcal{X}_t(X_t^f - K)^+$ . Write  $S_t = (X_t^D - K\mathcal{X}_t)^+$  and consider the market composed with  $(\mathcal{X}, X^D)$ . Notice that even though the currency  $\mathcal{X}$  is not a real asset in the domestic market, but regarding the payoff  $S$ , its hedging portfolio would be composed by  $X^D$  and the risk-free asset in the foreign market where the amount invested on the risk-free asset in the foreign market can be considered as the quantity invested on the 'asset'  $\mathcal{X}$ . As a consequence, we can still use the driver in (5.3.1) to describe the domestic market for pricing a contract in foreign currency.

From Theorem 5.4.1, we take  $\mathcal{X}$  as the numéraire then we will get the symmetry result similar to that of exchange option in Example 5.5.4.

**Corollary 5.5.8** (Quanto options). *Assume (H-M) holds for  $(\mathcal{X}, X^D)$ . Denote the price  $\mathbf{EX}^D(t, x, x^D; T, K; \Theta)$  of American option with payoff process*

$$S^{t,x,x^D} = \{((X^D)^{t,x^D} - K\mathcal{X}^{t,x})^+, 0 \leq t \leq T\}$$

where the underlying assets  $(\mathcal{X}, X^D)$  starting from  $(\mathcal{X}_t, X_t^D) = (x, x^D)$ . We have

$$\mathbf{EX}^D(t, x, x^D; T, K; \Theta) = x\mathbf{Call}^F(t, \frac{x^D}{x}; T, K; \tilde{\Theta}^{(2)}),$$

where  $\mathbf{Call}^F(t, \tilde{x}; T, K; \tilde{\Theta}^{(2)})$  is the price of the American Call option with payoff process  $\tilde{S} := (X^F - K)^+$  when the underlying asset  $X^F$  starting from  $X_t^F = \tilde{x}$ , which is exactly a pricing process of Call option written in the foreign market and in foreign currency.

An easy verification is performed under the domestic risk neutral probability where  $R = r, Q = q, \mu = r - q$  and we find the dynamics of  $X^F$ . Let  $r^D, r^F$  to be the domestic and foreign interest rate respectively, and  $q$  the dividend rate of underlying. The argument of *Absence of Opportunity of Arbitrage* yields that the drift of  $\mathcal{X}$  under the domestic risk neutral probability should be  $r^D - r^F$ . Indeed, assume under the domestic risk-neutral probability  $\mathbb{Q}$ ,  $X^D$  and  $X^F$  satisfy

$$\frac{dX_t^D}{X_t^D} = (r_t^D - q_t)dt + \sigma_t^D dW_t^1, \quad \frac{dX_t^F}{X_t^F} = \mu_t^F dt + \sigma_t^F(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2),$$

where  $W^1, W^2$  are 2 independent Brownian motions under  $\mathbb{Q}$ , the correlation  $\rho \in [-1, 1]$ ,  $\mu^F$  is the expected return rate of  $X^F$  under  $\mathbb{Q}$ , which will be determined afterwards. Assume  $\mathcal{X}$  satisfies

$$\frac{d\mathcal{X}_t}{\mathcal{X}_t} = (r_t^D - r_t^F)dt + \sigma^{\mathcal{X}} dW_t,$$

where  $W_t = (W_t^1, W_t^2)^\top$  and  $\sigma^{\mathcal{X}} = (\sigma_t^D - \sigma_t^F \rho, -\sigma_t^F \sqrt{1 - \rho^2})$ . Thus we can determine  $\mu^F$  since  $X_t^D = X_t^F \mathcal{X}_t$ , otherwise there will be an arbitrage in the market. Applying Itô's formula,



we get

$$\begin{aligned} \frac{dX_t^D}{X_t^D} &= \frac{dX_t^F}{X_t^F} + \frac{d\mathcal{X}_t}{\mathcal{X}_t} + \frac{d\langle X^F, \mathcal{X} \rangle_t}{X_t^F \mathcal{X}_t} \\ &= (\mu_t^F + r_t^D - r_t^F + \rho \sigma_t^D \sigma_t^F - (\sigma_t^F)^2) dt + \sigma_t^D dW_t^1, \end{aligned}$$

the argument that the drift of  $X^D$  should equals to  $r^D - q_t$  yields that  $\mu_t^F = r_t^F - q_t - \rho \sigma_t^D \sigma_t^F + (\sigma_t^F)^2$ . So we can apply Theorem 5.4.1, assuming (H-M) for  $(\mathcal{X}, X^D)$  with coefficient  $\Theta$ ,

$$\Theta = \begin{bmatrix} r^D - r^F, & r^D, & r^D, & r^F, & r^F, & \sigma^{\mathcal{X}} \\ r^D - q, & r^D, & r^D, & q, & q, & \sigma^{X^D} \end{bmatrix}$$

and take  $\mathcal{X}$  as numéraire, then we have, the  $\mathcal{X}$ -market coefficient  $\tilde{\Theta}^{(2)} = (r^F - q - [(\sigma^D, 0) - \sigma^{\mathcal{X}}](\sigma^{\mathcal{X}})^\top, r^F, r^F, q, q)$ . An easy computation leads to, for the return rate  $\tilde{\mu}^{(2)}$  in  $\mathcal{X}$ -market,

$$\begin{aligned} \tilde{\mu}^{(2)} &= r^F - q - [(\sigma^D, 0) - \sigma^{\mathcal{X}}](\sigma^{\mathcal{X}})^\top \\ &= r^F - q - \rho \sigma_t^D \sigma_t^F + (\sigma_t^F)^2 = \mu^F, \end{aligned}$$

which implies that  $\mathcal{X}$ -market is exactly foreign market composed by  $X^F$ . Hence, we can verify Corollary 5.5.8 in the domestic risk neutral market. Moreover, if we apply a change of measure on it, we enter the foreign risk-neutral market, see Section 5.6.1.

## 5.6 Change of probability and American-type options

In this section, we interpret our result of non-linear pricing in the previous section to the optimal stopping problem with changing of probability. The change of probability plays an important rule in the classical symmetric results where the Girsanov Theorem is usually applied and the dynamics of the underlyings under the new probability are computed explicitly.

Consider a simple case of (H-M) with  $R = r$ ,  $Q = q$ ,  $\mu = r - q$ , namely the risk neutral market with dividend, we also have the symmetry in the representation of optimal stopping problem so that we can give another time the symmetry of American option pricing. The risk-neutral pricing can be considered as a special case of this result, thanks to its simple linear driver  $f(t, y, z) = -r_t y$ . Let us start with the multidimensional case as in Corollary 5.5.2 and we work with the same notation, under the risk neutral probability  $\mathbb{Q}$ , the price of the American Call option  $\mathbf{Call}(t, x; C, T, K; r, q)$  admits the following representation:

$$\mathbf{Call}(t, X_t; C, T, K; r, q) = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s ds} (C^\top X_\tau - K)^+ | \mathcal{F}_t \right]. \quad (5.6.1)$$

Define the equivalent probability measure  $\mathbb{Q}^*$ , of which the Radon–Nikodym density process is strictly positive given as following:

$$d\mathbb{Q}^*|_{\mathcal{F}_t} = L_t d\mathbb{Q}|_{\mathcal{F}_t}, \quad L_t := \frac{e^{-\int_0^t (r_s - q_s^{(1)}) ds} X_t^1}{X_0^1} > 0, t \in [0, T].$$

Then Girsanov theorem and Baye's formula for the conditional expectation leads us to

$$\begin{aligned}
\mathbf{Call}(t, X_t; C, T, K; r, q) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \mathbf{d}s} X_\tau^1 \left( C_1 - \tilde{C}^\top \tilde{X}_\tau \right)^+ \middle| \mathcal{F}_t \right] \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} X_t^1 \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s^{(1)} \mathbf{d}s} \left( C_1 - \tilde{C}^\top \tilde{X}_\tau \right)^+ \middle| \mathcal{F}_t \right] \\
&= X_t^1 \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s^{(1)} \mathbf{d}s} \left( C_1 - \tilde{C}^\top \tilde{X}_\tau \right)^+ \middle| \mathcal{F}_t \right] \\
&= X_t^1 \mathbf{Put}(t, \tilde{X}_t; \tilde{C}, T, C_1; \tilde{r}, \tilde{q}),
\end{aligned}$$

where  $\tilde{r} = q^{(1)}$ ,  $\tilde{q} = (r, q^{(2)}, \dots, q^{(d)})$ ,  $\tilde{X}_t = \left( \frac{1}{X_t^1}, \frac{X_t^2}{X_t^1}, \dots, \frac{X_t^d}{X_t^1} \right)$ ,  $\tilde{C} = (K, -C_2, \dots, -C_d)$ . So in this simple case, we prove that the symmetry of optimal stopping problem by change of probability and we find again a similar result to Corollary 5.5.2.

Moreover, thanks to the relation between the optimal stopping and RBSDE as in [52, Proposition 2.3], we can verify the equation satisfied by the value process obtained from optimal stopping is the same as RBSDE shown in Theorem 5.4.1. Let's look at the equations satisfied by  $\mathbf{Call}(t, X_t; C, T, K; r, q)$  and  $\mathbf{Put}(t, \tilde{X}_t; \tilde{C}, T, C_1; \tilde{r}, \tilde{q})$ : in the right side of (5.6.1), for any fixed stopping time  $\tau \leq T$ , let  $Y_t = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \mathbf{d}s} (C^\top X_\tau - K)^+ \middle| \mathcal{F}_t \right]$ , then  $Y \in \mathcal{S}^2$  and there exists  $Z \in \mathbb{H}^2$  such that  $Y$  is the solution of the following linear BSDE on  $[0, \tau]$

$$\begin{cases} -\mathbf{d}Y_t = -r_t Y_t \mathbf{d}t - Z_t \mathbf{d}B_t, & 0 \leq t < \tau, \\ Y_\tau = (C^\top X_\tau - K)^+. \end{cases}$$

Thus

$$Y_t = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \mathbf{d}s} (C^\top X_\tau - K)^+ \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[ (C^\top X_\tau - K)^+ + \int_t^\tau -r_s Y_s \mathbf{d}s \middle| \mathcal{F}_t \right].$$

Notice that  $Y$  depends on  $\tau$ , so by taking the supremum on  $\tau \in \mathcal{T}_{t,T}$ , from [52, Proposition 2.3] and (5.6.1), we obtain that, the value process  $\mathbf{Call}(t, X_t; C, T, K; r, q)$  is the solution of RBSDE w.r.t.  $S = (C^\top X - K)^+$ ,  $f(t, x, y, z) = -r_t y$  under  $\mathbb{Q}$ . With the same arguments, we also deduce that the value process  $\mathbf{Put}(t, \tilde{X}_t; \tilde{C}, T, C_1; \tilde{r}, \tilde{q})$  is the solution of RBSDE w.r.t.  $\tilde{S} = (C_1 - \tilde{C}^\top \tilde{X})^+$ ,  $\tilde{f}(t, x, y, z) = -\tilde{r}_t y$  under  $\mathbb{Q}^*$ . So this identification allows us to verify Theorem 5.4.1 by applying Girsanov theorem for the change of probability. Without any doubt, the American exchange option as in Corollary 5.5.2 is one of special case of current discussion.

### 5.6.1 American Quanto Option

After applying Theorem 5.4.1 on the Quanto option in Section 5.5.3, we consider a different approach: applying the change of measure on the representation of American option by optimal stopping to get the symmetry, which provides another observation of the change of numéraire.

**American type foreign stock option with a strike in a foreign currency:** Consider a payoff  $S_t = \mathcal{X}_t(X_t^{\mathbf{F}} - K)^+$ , in this case, the American type option's price is given by

$$\begin{aligned} \mathbf{Call}(t, X_t^{\mathbf{D}}; T, K; r^{\mathbf{D}}, q) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s^{\mathbf{D}} \mathbf{d}s} \mathcal{X}_\tau (X_\tau^{\mathbf{F}} - K)^+ | \mathcal{F}_t \right] \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathcal{X}_t \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau r_s^{\mathbf{F}} \mathbf{d}s} \mathcal{X}_\tau (X_\tau^{\mathbf{F}} - K)^+ | \mathcal{F}_t \right] \\ &= \mathcal{X}_t \mathbf{Call}(t, X_t^{\mathbf{F}}; T, K; r^{\mathbf{F}}, q), \end{aligned} \quad (5.6.2)$$

with  $\mathbf{d}\mathbb{Q}^*|_{\mathcal{F}_t} = L_t \mathbf{d}\mathbb{Q}|_{\mathcal{F}_t}$ ,  $L_t = e^{-\int_0^t (r_s^{\mathbf{D}} - r_s^{\mathbf{F}}) \mathbf{d}s} \frac{\mathcal{X}_t}{\mathcal{X}_0}$ . Under  $\mathbb{Q}^*$ ,  $X_t^{\mathbf{F}}$  satisfies

$$\frac{\mathbf{d}X_t^{\mathbf{F}}}{X_t^{\mathbf{F}}} = (r_t^{\mathbf{F}} - q_t) \mathbf{d}t + \sigma_t^{\mathbf{F}} (\rho \mathbf{d}W_t^{*,1} + \sqrt{1 - \rho^2} \mathbf{d}W_t^{*,2}), \quad (5.6.3)$$

where  $W_t^* := (W_t^{*,1}, W_t^{*,2}) = (W_t^1 - \int_0^t (\sigma_s^{\mathbf{D}} - \sigma_s^{\mathbf{F}} \rho) \mathbf{d}s, W_t^2 + \sqrt{1 - \rho^2} \int_0^t \sigma_s^{\mathbf{F}} \mathbf{d}s)$  are  $\mathbb{Q}^*$ -Brownian motions from Girsanov theorem. Thus (5.6.3) implies that by the change of probability, we are now working with the foreign risk neutral market and  $\mathbf{Call}(t, X_t^{\mathbf{F}}; T, K; r^{\mathbf{F}}, q)$  on the right side of (5.6.2) is exactly the price of American Call option in foreign currency and in foreign risk neutral market.

## 5.6.2 American Barrier option

We apply the change of probability to prove the same conclusion as in Section 5.5.2 for the American barrier options and we take the same notations as in Section 5.5.2.

**Down and In Call** Consider the barrier option with payoff  $S_t = (X_t - K)^+ \mathbf{1}_{\{\tau_H \leq t\}}$ , the corresponding optimal stopping problem under the risk-neutral probability  $\mathbb{Q}$  gives the value process of American type options, which can be written as

$$\begin{aligned} \mathbf{DICall}(t, X_t; T, K, H; r, q) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \mathbf{d}s} (X_\tau - K)^+ \mathbf{1}_{\{\tau_H \leq \tau\}} | \mathcal{F}_t \right] \\ &= K X_t \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s \mathbf{d}s} \left( \frac{1}{K} - \frac{1}{X_\tau} \right)^+ \mathbf{1}_{\{\tau_H \leq \tau\}} | \mathcal{F}_t \right], \end{aligned} \quad (5.6.4)$$

where  $\frac{\mathbf{d}\mathbb{Q}^*}{\mathbf{d}\mathbb{Q}}|_{\mathcal{F}_t} = e^{-\int_0^t (r_s - q_s) \mathbf{d}s} \frac{X_t}{X_0}$ . Notice that under  $\mathbb{Q}^*$ ,  $\frac{1}{X_t}$  follows the dynamic

$$\mathbf{d} \frac{1}{X_t} = \frac{1}{X_t} (q_t - r_t) \mathbf{d}t - \sigma_t \frac{1}{X_t} \mathbf{d}W_t^*,$$

where  $W^*$  is the Brownian motion under  $\mathbb{Q}^*$ . From (5.6.4), using the arguments in the proof of Proposition 5.5.1, we get the same conclusion, i.e.

$$\begin{aligned} \mathbf{DICall}(t, X_t; T, K, H; \Theta) &= KX_t \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s \mathbf{d}s} \left( \frac{1}{K} - \frac{1}{X_\tau} \right)^+ \mathbf{1}_{\{\tilde{\tau}_{\frac{1}{H}} \leq \tau\}} \middle| \mathcal{F}_t \right] \\ &= KX_t \mathbf{UIPut} \left( t, \frac{1}{X_t}; T, \frac{1}{K}, \frac{1}{H}; \tilde{\Theta} \right), \end{aligned}$$

where  $\tilde{\tau}_{\frac{1}{H}} := \inf\{t \geq 0 : \frac{1}{X_t} = \frac{1}{H}\}$ .

**Down and Out Call** Consider the payoff process  $S_t = (X_t - K)^+ \mathbf{1}_{\{\tau_H > t\}}$ , the only difference compared to the previous case lies on the symmetry of stopping time: since

$$\{\tau_H > s\} = \left\{ \inf_{0 \leq u \leq s} X_u > H \right\} = \left\{ \sup_{0 \leq u \leq s} \frac{1}{X_u} < \frac{1}{H} \right\} = \{\tilde{\tau}_{\frac{1}{H}} > s\}.$$

So with the same arguments as in the above case, we get that

$$\begin{aligned} \mathbf{DOCall}(t, X_t; T, K, H; r, q) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \mathbf{d}s} (X_\tau - K)^+ \mathbf{1}_{\{\tau_H > \tau\}} \middle| \mathcal{F}_t \right] \\ &= KX_t \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s \mathbf{d}s} \left( \frac{1}{K} - \frac{1}{X_\tau} \right)^+ \mathbf{1}_{\{\tau_H > \tau\}} \middle| \mathcal{F}_t \right], \\ &= KX_t \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}^*} \left[ e^{-\int_t^\tau q_s \mathbf{d}s} \left( \frac{1}{K} - \frac{1}{X_\tau} \right)^+ \mathbf{1}_{\{\tilde{\tau}_{\frac{1}{H}} > \tau\}} \middle| \mathcal{F}_t \right] \\ &= KX_t \mathbf{UOPut} \left( t, \frac{1}{X_t}; T, \frac{1}{K}, \frac{1}{H}; \tilde{\Theta} \right). \end{aligned}$$

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**Titre :** Approximation et simulation des équations différentielles stochastiques rétrogrades réfléchies, applications en finance

**Mots clés :** équations rétrogrades, vitesse de convergence, borne d'erreur,

**Résumé :** Dans cette thèse, on se concentre sur la résolution des équations différentielles stochastiques rétrogrades réfléchies par l'approche de pénalisation et ses applications en finance. Un résultat complet de convergence non asymptotique est étudié autour des EDSR pénalisées. La vitesse de convergence de la solution pénalisée vers la solution réfléchie est présentée en premier lieu, suivi de la vitesse de convergence de la solution pénalisée discrète vers la solution pénalisée continue. Pour la résolution numérique des EDSR pénalisées, on propose un schéma implicite utilisant la méthode de Monte-Carlo avec régression des moindres carrés. Une analyse d'erreur non asymptotique est déduite pour ce schéma

numérique, dans laquelle à la fois la régression linéaire et non linéaire des moindres carrés sont considérées comme méthodes d'optimisation. La thèse se conclut par une application des EDSR pénalisées aux options d'achat/vente américaines sur un marché non linéaire. Le changement de numéraire est étudié sur les diffusions réfléchies. On constate, comme sur le marché parfait, que l'équivalence de prix entre une option de vente américaine et une option d'achat américaine est obtenue en échangeant les taux d'intérêt avec les taux de dividende, et en échangeant le prix spot avec le prix d'exercice, sur un marché imparfait avec 2 taux d'intérêt et 2 taux de dividende.

**Title :** Approximation and simulation of reflected Backward Stochastic Differential Equations, applications in Finance

**Keywords :** backward equations, convergence rate, error bound

**Abstract :** In this thesis, one concentrates on solving the reflected backward stochastic differential equations via penalization approach and its applications in finance. A complete non-asymptotic convergence result is investigated around penalized BSDE. The rate of penalized solution converging to reflected solution is presented in the first place and it follows by the rate of discrete penalized solution converging to the continuous penalized one. For solving numerically PBSDEs, we provide an implicit scheme using least-squares regression Monte-Carlo method. The non-asymptotic error analysis is deduced for this nu-

merical scheme, in which both linear and non-linear least-squares regression are considered as optimization method. The thesis is completed by an application of RBSDEs on American Put/Call options in the non-linear market. The change of numéraire is investigated on reflected diffusions. We find, as in the perfect market, the pricing equivalence between an American Put option and an American Call option is achieved by exchanging the interest rates with dividend rates, and swapping the spot price with the strike price, in a imperfect market with 2 interest rates and 2 dividend rates.