



# Multi-choice games and applications to pollution issues

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# **THÈSE de DOCTORAT DE L'UNIVERSITÉ JEAN MONNET SAINT-ÉTIENNE**

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## **Multi-choice games and applications to pollution issues**

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## Résumé

Cette thèse porte sur l'analyse des institutions économiques et des comportements de coopération sous-jacents à ces institutions. Cette analyse de la coopération est conduite en considérant une approche dite "par la justice". Il s'agit de considérer des agents économiques qui produisent une valeur monétaire en coopérant, puis d'identifier des mécanismes de partage de cette valeur. En partant de l'hypothèse que la coopération est effective, on propose des arbitrages entre plusieurs principes désirables et raisonnables à partir desquels ces institutions sont construites. Cette méthode, qualifiée d'axiomatique, permet d'aborder de manière conceptuelle ces institutions coopératives. Elle consiste à traduire ces principes désirables en axiomes, puis d'étudier les conséquences de la combinaisons de différents ensembles d'axiomes. Ce travail s'inscrit dans le cadre de la théorie des jeux coopératifs à choix multiples (ou jeux multi-choix). Contrairement aux jeux coopératifs traditionnels, les jeux à choix multiples supposent que les agents peuvent choisir plusieurs niveaux de coopération. De ce point de vue, une institution coopérative est une solution pour les jeux coopératifs à choix multiples qui décrit la manière dont la valeur engendrée par la coopération de tous est partagée entre ses membres. En particulier, ce travail vise à souligner que cette approche est pertinente pour analyser des problèmes économiques en présence d'externalités négatives (la pollution). Après avoir introduit les notions fondamentales qui seront utilisées dans ce document, quatre contributions originales sont présentées.

Dans une première partie constituée de deux chapitres, l'étude porte sur la théorie des jeux coopératifs à choix multiples. Comme suite à une introduction et à un chapitre préliminaire, le troisième chapitre, co-écrit avec David Lowing, examine l'arbitrage entre les principes marginaliste et égalitariste. Le principe marginaliste repose sur l'idée que l'évaluation de la coopération d'un agent doit résulter des contributions marginales de cet agent en matière de coopération. Par contribution marginale, on entend la contribution de l'agent lorsqu'il rejoint une coalition. En ce sens la valeur de Shapley est-elle la plus populaire des solutions marginalistes dans les jeux coopératifs. L'égalitarisme renvoie à la fois à l'idée que tous les agents doivent être traités de la même façon et que les inégalités entre ces agents doivent être réduites au maximum au sein de l'institution. Un cas extrême de ce principe revient à distribuer à chaque agent la même part de la valeur engendrée par la coopération de tous. Le compromis entre marginalisme et égalitarisme est ensuite appréhendé via les combinaisons convexes entre la valeur de Shapley et la valeur de division égale. Dans le cadre des jeux à choix multiples il existe plusieurs manières d'étendre les principes marginaliste et égalitariste. Nous proposons une nouvelle façon d'étendre ces principes dans les jeux à choix multiples. Puis nous réalisons un compromis entre ces deux principes. Enfin, nous procédons à l'étude axiomatique.

Le quatrième chapitre, co-écrit avec David Lowing, considère des situations de coopération entre des agents hétérogènes. Deux types d'hétérogénéité sont prises en compte. Premièrement, chaque agent peut avoir différentes possibilités de coopération représentées par des ensembles de niveaux d'activité différents. Deuxièmement, les agents sont différenciés par rapport à une structure de priorité qui reflète les asymétries entre les agents au-delà des asymétries inhérentes au processus de coopération. Ces asymétries peuvent refléter des droits exogènes, des besoins différents, le mérite ou des contraintes hiérarchiques. Pour analyser ces situations, nous enrichissons le modèle des jeux à choix multiples d'une structure de priorité. Une

nouvelle valeur sur la classe des jeux à choix multiples avec une structure de priorité est introduite. Pour tenir compte à la fois des différents niveaux d'activité et des asymétries entre les agents, cette valeur est construite à l'aide d'une procédure lexicographique. Nous introduisons de nouveaux axiomes pour les jeux à choix multiples avec une structure de priorité. Ces axiomes déterminent de manière endogène la procédure lexicographique utilisée pour définir la valeur. Deux caractérisations axiomatiques de cette valeur sont fournies.

Dans une seconde partie constituée de deux chapitres, le travail porte sur des applications qui traitent des problèmes de pollution. Précisément, le cinquième chapitre considère le transport des déchets dangereux. En raison de leur nature dangereuse, le transport de ces déchets implique un risque d'incident ayant des conséquences irréversibles sur l'environnement. Ce problème a conduit à la mise en place d'un ensemble de réglementations qui encadrent la production et le transport des déchets dangereux. Ces réglementations mettent en œuvre des mesures qui visent à réduire le risque associé aux déchets dangereux. En supposant que ces mesures sont coûteuses, ce chapitre étudie le partage de ce coût entre les agents. Plus précisément, un ensemble d'agents, localisés sur un réseau, transportent une quantité maximale de déchets (qui peut varier d'un agent à l'autre) vers une installation de traitement. Afin de maintenir le réseau aussi fiable que possible, une autorité centrale définit certaines mesures préventives qui entraînent un coût de maintenance et d'opération du réseau. L'objectif est d'identifier la responsabilité de chaque agent sur le risque qu'il fait peser au réseau, et d'allouer le coût du réseau en conséquence. Pour ce faire, on introduit plusieurs axiomes inspirés de différents principes de droit environnemental, dont le principe du pollueur-payeur. En conformité avec ces principes de droit environnemental, les axiomes décrivent la responsabilité de chaque agent par rapport au risque qu'il fait peser sur le réseau. Ensuite, on montre qu'il existe une unique règle d'allocation qui satisfait à ces axiomes. Cette règle répartit la variation du coût de chaque portion du réseau (lorsque la quantité de déchets augmente) de manière égale entre les agents responsables de cette portion.

Le dernier chapitre s'intéresse aux situations dans lesquelles l'activité d'un agent a des conséquences néfastes sur d'autres agents. Ce problème, identifié par Ronald Coase, est celui du "coût social". La solution préconisée par Coase écarte le principe du pollueur-payeur pour laisser la place à des négociations des droits à polluer entre le pollueur et une victime potentielle. Dans ce chapitre, on reprend l'intuition de Coase en supposant qu'il existe un grand nombre de victimes potentielles. Le pollueur dispose de plusieurs niveaux d'activité qui correspondent à des niveaux de pollution différents. Une nouveauté de ce modèle est de distribuer des droits non seulement aux agents, mais à des groupes d'agents. Si une coalition d'agents incluant le pollueur possède des droits, alors ces membres peuvent négocier avec le pollueur une réduction du niveau d'activité. Dans le cas où cette coalition possède des droits et ne contient pas le pollueur, alors les victimes peuvent se prémunir de toute pollution, et c'est au pollueur de proposer un accord avec les victimes potentielles. A partir de ce problème, on considère un jeu à choix multiples, puis on fournit une caractérisation de la distribution des droits qui assurent que le cœur de ce jeu est non vide.

## *Abstract*

This thesis analyses economic institutions and the underlying cooperative behaviors of these institutions. This analysis of cooperation is conducted by considering a "justice" approach. The aim is to consider economic agents who produce monetary value by cooperating, and then to identify mechanisms for sharing this value. Assuming that cooperation is effective, we propose trade-offs between several desirable and reasonable principles from which these institutions are built. This method, called axiomatic, allows for a conceptual approach to these cooperative institutions. It consists of translating these desirable principles into axioms, and then studying the consequences of combining different sets of axioms. This work fits in the framework of multi-choice cooperative game theory. In contrast to traditional cooperative games, multi-choice games assume that agents can choose different levels of cooperation. From this point of view, a cooperative institution is a solution for multi-choice cooperative games that describes how the value generated by the cooperation of all is shared among its members. Specifically, this work aims to emphasize that this approach is relevant for analyzing economic problems in the presence of negative externalities (pollution). After introducing the basic concepts that will be used in this thesis, four original contributions are presented.

A first part studies multi-choice cooperative games. Following an introduction and a preliminary chapter, the third chapter, co-authored with David Lowing, examines the trade-off between marginalism and egalitarianism. Marginalism is based on the idea that the evaluation of an agent's cooperation must result from that agent's marginal contributions to cooperation. In this sense the Shapley value is the most popular marginalist solution in cooperative games. Egalitarianism refers both to the idea that all agents should be treated equally and that inequalities among those agents should be minimized within the institution. An extreme case of this principle is to distribute to each agent the same amount of the value generated by the cooperation of all. The trade-off between marginalism and egalitarianism is then obtained via the convex combinations between the Shapley value and the equal division value. In the context of multi-choice games there are several ways to extend the marginalist and egalitarian principles. We propose a new way to extend these principles in multi-choice games. Then we realize a compromise between these two principles. Finally, we proceed to the axiomatic study.

The fourth chapter, co-written with David Lowing, considers situations of cooperation between heterogeneous agents. Two types of heterogeneity are considered. First, each agent may have different cooperation options represented by different sets of activity levels. Second, agents differ with respect to a priority structure that reflects the asymmetries between agents beyond the asymmetries inherent in the cooperative process. These asymmetries may reflect exogenous rights, different needs, merit, or hierarchical constraints. To analyze these situations, we enrich the model of multi-choice games with a priority structure. A new value on the class of multi-choice games with a priority structure is introduced. To account for both the different activity levels and the asymmetries between agents, this value is constructed using a lexicographic procedure. We introduce new axioms for multi-choice games with a priority structure and show that they endogenously determine the lexicographic procedure used to define the value. Two axiomatic characterizations of this value are provided.

In a second part consisting of two chapters, the work focuses on applications that deal with pollution problems. Specifically, the fifth chapter deals with the transportation of hazardous waste. Because of their hazardous nature, the transportation

of such waste involves the risk of an incident with irreversible environmental consequences. This problem has led to the establishment of a set of regulations governing the generation and transport of hazardous waste. These regulations implement measures to reduce the risk associated with hazardous waste. Assuming that these measures are costly, this chapter explores how to share this cost among the agents. More precisely, a set of agents, located on a network, transport a maximum quantity of waste (which may vary from one agent to another) to a treatment facility. In order to keep the network as safe as possible, a central authority defines certain preventive measures that result in a cost to maintain and operate the network. The goal is to identify each agent's responsibility for the risk it poses to the network, and to allocate the cost accordingly. This is done by introducing several axioms inspired by different principles of environmental law, including the polluter-pays principle. Accordingly, the axioms describe the responsibility of each agent with respect to the risk it poses to the network. Then we show that there is a single allocation rule which satisfies these axioms. This allocation rule distributes the variation of the cost of each portion of the network (when the quantity of waste increases) equally among the agents responsible for that portion.

The last chapter is concerned with situations in which the activity of one agent has adverse effects on other agents. This problem, identified by Ronald Coase, is called the problem of the "social cost". The solution proposed by Coase rejects the polluter-pays principle and leaves room for negotiation of rights to pollute between the polluter and a potential victim. In this chapter, we take up Coase's intuition by assuming that there are a large number of potential victims. A novelty of this model is to distribute the rights not only to agents, but to groups of agents. If a coalition of agents including the polluter has the rights, then these members can negotiate with the polluter in order to reduce the activity level. In the case where this coalition has the rights and does not contain the polluter, then the victims can protect themselves from any pollution, and it is up to the polluter to propose an agreement with the potential victims. From this problem, we consider a multi-choice game, and then we provide a characterization of the distribution of rights which ensures that the core of this game is non-empty.

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## Chapter 1

# General introduction

Most of economic institutions can be thought through a cooperative point of view. The exchange or production of a private commodity involves a cooperative institution (see [Moulin \[1995b\]](#)) formed by the concerned parties. Such an institution arises as a market in which exchanging a commodity can be thought as a cooperative agreement between the agents. This is even more concrete if one considers a public commodity with common property rights. When choosing the appropriate level of a public good, one often requires a consensus, which can be seen as a cooperative institution. For the provision of such good, it is desirable that the concerned parties coordinate themselves regarding their contributions to the public good. A cooperative institution is also involved when it comes to share some scarce resource (when there is not enough to fulfill each agent's demand). For instance, consider a firm going bankrupt that has several creditors. Each creditor may have a claim on the remaining asset of the bankrupt firm, which is not enough to fulfill each claim. In such a case, an appropriate cooperative institution provides a consensus by assigning a share to each creditor. Such a cooperative institution also arises when one wants to share the surplus (respectively the cost) generated by a set of agents. For instance, consider several agents involved in a common venture. By cooperating, the set of agents generates some surplus. A cooperative institution then describes the share that each agent receives for its cooperation.

In the recent years, our understanding of cooperative institutions has steadily increased, highlighting the main patterns that underlie the cooperative processes in economics. According to [Moulin \[1995b\]](#), three modes of cooperation mainly take place in economic situations. The *direct agreement* mode in which agents voluntarily engage in face-to-face agreement; the *decentralized* mode in which the agents interact strategically and a collective authority rules the interactions; and the *justice* mode in which a collective authority enforces a cooperative agreement.

This thesis is concerned exclusively with the justice mode. In this mode of cooperation, a cooperative institution takes the form of a systematic method that select one or several cooperative agreements that should overcome any potential conflict of interests. Actually, the justice mode of cooperation appears to be particularly suitable in distributive (or allocation) problems. Accordingly, an agreement usually consists of an allocation method that should be understandable for the concerned parties, and that allows to evaluate the participation of each agent in the cooperation. However, the way one operates in mediating conflicts of interest and reconciling conflicting values is far from consensual, leading to several competing cooperative institutions that deserve careful analysis. This is remarkably highlighted by the words of Amartya Sen:<sup>1</sup>

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<sup>1</sup>See the foreword in [Moulin \[1991\]](#).

*"Most problems of social interactions involve both conflict and congruence of interests. The elements of congruence make it important to have cooperation, but there are many forms that cooperation can take, and the interests between the parties may diverge in the choice of the different cooperative arrangements."*

From this point of view, a benevolent central authority, involved in the justice mode of cooperation, aims to enforce a cooperative agreement considering principles as "equity", "consistency", "priority", or "efficiency" as essential requirements for such agreement. Thus, the justice mode of cooperation leaves aside the problem of setting up cooperation. Instead, it analyzes cooperative institutions that are based on the arbitration of several ethical notions.

In analyzing cooperation through the scope of the justice mode, the strategic dimension of the concerned parties is usually set aside. Therefore, it is often assumed that cooperation holds, so that the agents can only take part in the cooperation or not. Yet, when one takes a more concrete look at cooperative situations, it appears that the agents can choose the intensity (or effort) with which they cooperate. When several agents are involved in a common venture, the intensity with which each agent cooperates influences the outcome of the venture. Indeed, the agents can participate in the venture by working a certain number of hours, which can vary according to the agent. Thus, the worth generated by cooperation depends on the number of hours worked by the cooperating agents. Such aspect is particularly relevant for cooperative institutions dealing with pollution issues. To illustrate the situation, consider a waste treatment facility owned by some community. Each member of the community may generate a certain amount of waste to be treated by the facility at a certain cost. Let say that such cost depends on the amount of waste to be treated rather than only on the members of the community that are cooperating. The presence of such different cooperation intensities then involves a new dimension to consider while trying to overcome a potential conflict of interest. Indeed, how should a cooperative institution take into account the different intensities that agents can put in cooperation? How should such institution evaluates the agents' participation and cooperation intensities?

These questions constitute the main concern of the thesis. In order to consider a broader scope of the justice mode of cooperation, the analysis is developed around a formal framework (briefly introduced in the next section) in which agents can cooperate at several activity levels. Here, the cooperation intensities (or activity levels) are not considered as part of a strategic interaction between the agents. Instead, one can think of cooperation intensities as actions that an agent would implement if it received the decision-making power. Since I stick to the justice mode of cooperation, it is assumed that the decision-making power is given to the community or to a planner. Thus, this thesis has a twofold objective. First, it aims at designing understandable cooperative institutions that define explicit agreements regarding each agent's intensity of cooperation. The second goal is to investigate the potential of such cooperative institutions in concrete situations related to pollution issues. To meet these goals, this thesis takes place within the framework of game theory and more precisely cooperative game theory.

## 1.1 Game theory

Game theory has emerged as one of the most fruitful tools when it comes to analyze economic behavior, social cooperation, and the resulting conflicts of interest. In its

broadest sense, game theory provides a mathematical framework to study the situations of conflict and cooperation. It provides general mathematical techniques for analyzing situations in which several agents interact and may influence each other. The foundations of the field were laid by [Morgenstern and Von Neumann \[1944\]](#). Since then, game theoretic analysis have been getting more and more popular, being used in several and diverse research fields and specifically in economics. There are two branches of game theory that allow to address situations of conflict in two distinct ways: the branch of non-cooperative games and the branch of cooperative games. In non-cooperative games, the agents are the basic decision making units. Each agent has a set of actions it can choose, and each combination of actions involves a corresponding payoff vector. The objective is then to study the strategic interactions between the agents. In cooperative games, the agents aim to accomplish a common task and generate some worth by forming a coalition. The basic decision units become the coalitions that agents can form. The objective is then to determine a reasonable way to allocate the worth generated by cooperation among the agents. In this way, cooperative game theory naturally fits in the justice mode of cooperation described in the previous section.

## 1.2 Cooperative game theory

Consider a group of “agents” (that can be individuals, firms, or states) allowed to cooperate by forming coalitions. By cooperating, it is assumed that the agents join their abilities and coordinate their actions (in an efficient way) in order to accomplish some common task. In cooperative game theory, it is assumed that such cooperation can be enforced by any binding agreement among the agents. Thus, the agents are supposed to cooperate whatever the conflicting interests that may exist among them. By cooperating, each coalition can generate some worth, usually referred to as utility. It is interpreted as the worth a coalition can generate on its own, i.e., regardless of the agents outside the coalition. This worth generated by each coalition is represented by a characteristic function. It is generally supposed that utility is interpersonally comparable across the agents, and can be transferred between them. Thus, the agents in a coalition can make side payments to each other. Thereof, such cooperative game is called a transferable utility game (henceforth TU-game). The analysis of cooperative games is then centered around two main issues: what coalition will form? How to divide the worth generated by cooperation?

A solution concept is a map that assigns to each game (of a certain set of games) a set of payoff vectors, i.e., vectors that describe the payoff of each agent. It can be considered as a systematic method for selecting payoff vectors. Whenever a solution concept is single-valued, it is simply called a value. This term, introduced by [Shapley \[1953\]](#), refers to the evaluation of agents’ participation in cooperation. There are a multiplicity of solution concepts that have been studied in cooperative game theory. This multiplicity is mainly due to the diversity of conflict situations that cooperative game theory deals with, and the several possibilities to evaluate the agents’ cooperation as reasonable. The distinction between the several solution concepts can be made through the axiomatic analysis that we further discuss in Section 1.5. Such analysis focuses on the properties that are desirable for solution concepts. Thus, the choice or recommendation for a solution concept is based on the trade-off between desirable properties for the solutions.

### 1.3 Extending the model of cooperative games

Depending on the situation at hand, the model of TU-game may appear unrealistically simple. Still, it enables to improve our understanding of situations of conflict and cooperation in a broad sense. In order to consider more realistic situations, various extensions of the base model of TU-games have been proposed. Below, I briefly presents some of these extension, and conclude the section with the model of multi-choice games investigated in this thesis.

A first extension of the model of TU-games was introduced by [Thrall and Lucas \[1963\]](#). This model considers that the worth of a coalition can depend on the actions of players outside the coalition. More specifically, it takes into account how the agents outside this coalition are partitioned. Because the worth of a coalition depends on the behavior of outsiders, these games are often considered to model situations with externalities.

A second extension, introduced by [Aumann and Dreze \[1974\]](#) and [Owen \[1977\]](#), enriches the model of TU-games with a partition of the player set, called a coalition structure. This model allows to investigate situations where agents belong to certain a priori alliances, which may represent some affinities among the agents.

Another extension, introduced by [Myerson \[1977\]](#), enriches the model of TU-games with a graph. Such graph structure aims to represent partial communications. Each link relating two agents in the graph indicates that direct communication is possible between them. Accordingly, communication can occur between agents that are related through some path in the graph. Then, the evaluation of a coalition (its worth) depends on whether its members are connected or not.

At least three other notable extensions of TU-games have been proposed that enrich the model with a partial order on the player set. Each extension can be differentiated with respect to the interpretation of the partial order. Specifically, [Gilles et al. \[1992\]](#) consider hierarchical relations represented by a permission structures. This structure, generally represented by a directed graph, is mathematically equivalent to a partial order on the player set whenever it is acyclic. It describes situations where some agents need the permission from other agents before they cooperate. Then, the evaluation of a coalition depends on whether this coalition needs the permission of some agents outside the coalition or not. Alternatively, [Faigle and Kern \[1992\]](#) interpret the partial order on the player set as a precedence constraint. In this model, a coalition is feasible, i.e., it can form, only if it contains all its down set. Notice that unfeasible coalitions are not considered in the analysis. Hence, the precedence constraint influences the formation of the coalition. [Béal et al. \[2022\]](#) interpret the partial order on the player set as a priority relation. These priority relations aim to describe the most deserving agents in an allocation problem. Accordingly, the priority structure influences the allocation process, but neither the formation nor the evaluation of coalitions.

[Hsiao and Raghavan \[1992\]](#) introduce the model of multi-choice cooperative games. This extension considers that the agents can cooperate at several intensities (or activity levels) within a coalition. Then, the worth of a coalition depends on the intensity of cooperation of each agent. Since the purpose of the thesis is to evaluate agents cooperation when they have different intensity of cooperation, multi-choice cooperative game appears to be a suitable framework for the analysis.<sup>2</sup>

<sup>2</sup>Another notable extension, introduced by [Aubin \[1974, 1981\]](#), considers that players have gradual degree of membership in a coalition. This model, called cooperative game with fuzzy coalitions, can be viewed as the continuous counterpart of cooperative multi-choice games.

It is clear that the class of multi-choice games contains the class of TU-games as a subclass, a TU-game being a multi-choice game in which the agents have only two activity levels (being active in the cooperation or not). Thus, most of the concepts and notions that have been developed in TU-games can be extended in multi-choice games. However, because of the multi-dimensional nature of cooperation in multi-choice games, extending some notions from TU-games to multi-choice becomes a non-trivial exercise since numerous extensions can be discussed. Such discussion appears, for instance, when one wants to evaluate the agents' cooperation by means of a payoff. While it is desirable to distinguish each agent's activity level, the question arises on how to interpret such payoff for each agent's activity level. Another discussion appears when one wants to compare the performance of two agents in a multi-choice game: on what basis should we distinguish the agents' performance when they cooperate? Is it relevant to compare two different agents cooperating at different activity levels? More importantly, such a discussion arises when one wants to extend a solution from TU-games to multi-choice games. I address these discussions in Chapter 2 in which the formal material of the thesis is presented.

## 1.4 Applications related to pollution issues

Since its beginning, cooperative game theory has aimed to shed lights to concrete situations by developing several applications. Here, by applications, it is meant a situation that arises from some activities and that is formally modeled. As argued by [Forgó et al. \[1999\]](#):

*"If a game theoretic analysis helps to shed lights on the nature of a problem coming from a broad spectrum of activities (economics, business, military, politics, etc.), leads to grasping its essential elements and helps to pinpoint the difficulties involved, then it deserves to be called a successful application."*

Well known applications of cooperative game theory encompass, among others, exchange or production economies with a market structure, political interactions, firms going bankrupt, cost allocation, or common pool resource management. Of a particular interest in this thesis, are the applications of cooperative game theory related to pollution issues, explored since at least the last 1960's. To cite a few: [Shapley and Shubik \[1969\]](#) investigate several situations such as discharging garbage in some agent's yard, or sharing the cost of purifying a polluted lake. [Chander and Tulkens \[1997\]](#) study an economy in which the agents can be both polluter and recipient of the pollution. Such situation can be thought as a transboundary pollution problem.<sup>3</sup> [Ni and Wang \[2007\]](#) investigate the cost sharing problem of cleaning a polluted river. Recently, [Ambec and Kervinio \[2016\]](#) consider the problem of sitting an undesirable facility that generates pollution spreading among its neighborhood. Finally, [Gonzalez et al. \[2019\]](#) investigate the property rights regimes that allow to resolve a negotiation problem between one polluter and several potential victims.

As awareness of environmental problems has grown in the recent decades, the literature on pollution issues (in its broadest sense) has steadily increased.<sup>4</sup>

Actually, economists have been concerned with situations involving pollution since at least the end of the 19<sup>th</sup> century. Several studies starting by [Sidgwick \[1887\]](#), [Pigou \[1920\]](#), [Meade \[1952\]](#) have recognized the implications of the pollution (more

<sup>3</sup>Earlier related models can be found in [Starrett \[1973\]](#) and [Laffont \[1977\]](#).

<sup>4</sup>To illustrate this aspect, the number of items labeled as pollution-related scientific articles on ScienceDirect has increased from 4,844 in 2000 to 57,068 in 2022.



generally of externalities) arising from economic activities. Among these implications, one important point is the gap between the cost of the polluting activity for the society and the cost of this activity for the polluter. This gap leads to an inefficient allocation (of goods) in competitive economies, and is known as the problem of the social cost. For the illustration, [Stigler \[1942\]](#) inventories several activities whose costs to the society are different to the costs to the agents undertaking them.

In order to solve such situations, two longstanding traditions have emerged. On the one hand, [Pigou \[1920\]](#) argues that a central authority should impose a tax paid by the polluter, which result in the well known polluter pays principle. This approach has been the basis for the regulation of polluting activities for quite some time. On the other hand, [Coase \[1960\]](#) advocates negotiated solutions to the problem. Indeed, Coase argued that the concerned parties can solve the problem of the social cost through a bargaining process provided that there is a clear and well defined regime of rights. This approach has been the basis of market for liability rights proposed in the “Kyoto protocol”, or in the “European Emissions Trading Schemes”.

The main point in Coase’s reasoning is the necessary consideration of pollution liability. This liability is depicted through property rights. Thus, two interpretations of the rights arise depending on the concerned agent. From the polluter point of view, property rights describe whether or not a polluter is allowed to pollute, in which case it is not liable for the pollution. From a victim point of view, property rights describe whether or not a potential victim can legitimately prevent pollution. In this view, Pigou’s tradition considers that the liability for the pollution lies to the polluter. Coase’s tradition, instead, argues that the liability should be clearly defined, but may lie to the polluter or to the victim.

These two approaches are focal points of two dedicated chapters (Chapter 5 and 6). Both chapters develop applications related to multi-choice games and incorporate the dimension of liability towards pollution. The first application considers the transport of hazardous waste on a given network following Pigou’s tradition. It is known that tons of hazardous waste are generated and traded both internationally and locally each year, involving massive waste movements.<sup>5</sup> Because of the hazardous nature of these waste, a wide range of environmental regulations have emerged over the past decades. Among these regulations, one can cite: the “Basel convention” that regulates the international movements of hazardous waste and their disposal; the “Comprehensive Environmental Response, Compensation, and Liability Act” (CERCLA or Superfund Act) that regulates the storage sites of hazardous waste in the United States; the “Treaty on the Functioning of the European Union” that incorporates a regulation on the movements of hazardous waste within its borders. Each of these regulations relies on the polluter pays principle, and provide the legal framework defining the liabilities towards hazardous waste. Based on these environmental regulations, several costly measures to reduce the risk associated to hazardous waste can be considered. The question then becomes how to allocates this cost among the concerned parties.

The second application deals with the problem of the social cost in a broad sense (as described above) and investigates the Coase theorem. Introduced by [Coase \[1960\]](#) and formalized by [Stigler \[1966\]](#), the Coase theorem asserts that: *if there is no transaction costs and property rights are well-defined, then the agent can always bargain to reach an optimal allocation of resources, and this allocation is independent of the initial distribution of rights*. Therefore, a cooperative institutions can always assign the rights and foster an efficient allocation of conflicting resources. This conclusion

<sup>5</sup>A recent study of the international waste trade network can be found in [Martínez et al. \[2022\]](#).

of Coase constitutes one of these economic concepts with an extensive use outside of economics.<sup>6</sup> It is often used as a justification to apply economic principles in judicial decision making. It is considered as one of the foundation of Law and Economics movement. It has also been investigated through the scope of political theory.<sup>7</sup> It can also be found as a justification of tradable emission permits markets.

In economics, the Coase theorem has fostered a broad literature, mainly focused on three types of models. It has been considered in terms of competitive equilibrium (see for instance [Hurwicz \[1995\]](#) and [Chipman and Tian \[2012\]](#)), in terms of Nash equilibrium of a strategic/bargaining game (see for instance [Anderlini and Felli \[2001\]](#)), and in terms of cooperative game theory (see for instance [Aivazian and Callen \[1981\]](#) and [Gonzalez et al. \[2019\]](#)). Following this latter literature, the problem is to investigate cooperative institutions that assign property rights (in the sense of Coase) to the agents based on some desirable properties.

## 1.5 Methodology: the axiomatic analysis

The method or approach used all along the thesis is the axiomatic method. This method has been fruitfully applied in many economic situations, and specifically in fair allocation problems. Fair allocation problems arise in many situations in which several agents with conflicting interests compete for a certain resource. For a century, mathematicians, game theorists, and economists have come up with normative solutions to the question of how to avoid litigious allocation/division. The primary method that has emerged to solve these problems is the axiomatic method, or axiomatic analysis. The origins of axiomatic analysis in allocation problems go back to at least three seminal models: the axiomatic theory of bargaining due to [Nash \[1950\]](#), the value for  $n$ -person games due to [Shapley \[1953\]](#), the model of fair division introduced by [Foley \[1966\]](#). Since then, a broad literature involving the axiomatic analysis has flourished encompassing several generalizations of the seminal models and new problems. To cite a few examples one can refer to: bankruptcy problems introduced in [O'Neill \[1982\]](#) (for a recent survey see [Thomson \[2015\]](#)); fair allocation in economic environment developed in [Roemer \[1988\]](#), [Fleurbaey and Maniquet \[1996\]](#) among others (see [Moulin and Thomson \[1997\]](#) for an overview); the different extensions of TU-games discussed in Section 1.3. For each of these models, the axiomatic analysis appears to be a guide helping for collective decision making (see [Moulin \[1991\]](#) for a broad overview). Indeed, each model allows for several acceptable solutions. The axiomatic analysis constitutes a mean to distinguish among these solutions, and provides a formal justification for a solution by eliciting the desirable properties that characterize it. In the following, we provide a description of the axiomatic analysis of allocation problems.

### The axiomatic program

[Thomson \[2001\]](#) highlights the main features of what is called “the axiomatic program”. Following this description, an axiomatic analysis starts with the definition of the domain or class of the problems to be considered. The representation of the class of the problems can range from an abstract and general domain, to context specific domains. A problem consists of a set of agents, some of their relevant characteristics

<sup>6</sup>The reader is referred to [Medema \[2020\]](#) for a recent discussion on the Coase theorem and its developments since 1960.

<sup>7</sup>Such investigation can be found in [Acemoglu \[2003\]](#).

(e.g. agents' preferences or utilities, rights entitlements), and some contextual data (e.g. some exogenous rights, a network structure). As an example, a multi-choice cooperative game consists of a set of agents, a set of activity levels for each agents, and a set of attainable utility vectors (regarding each combination of the agents' activity level). Given a class of problems, a solution on this class is a map that assigns a set of feasible alternatives (a set of payoff vectors in multi-choice cooperative games).

Once the domain of the problems has been rigorously defined, the axiomatic analysis proceeds by setting out a list of desirable properties, called axioms, for solutions on the domain. These axioms are mathematical expressions of norms that a solution should follow. They can be based on some ethical or fairness condition translated into an appropriate formalization, or on an appealing technical condition. The objective of the axiomatic analysis is then to understand and describe the possible implications of the listed properties, and to distinguish between the plausible combination of properties. This is done by investigating the logical relations between the properties, considering alternative specification of the class of the problems, or studying whether some properties can be replaced by natural variants of them. By doing so, one obtains a description of solutions satisfying diverse combinations of the axioms of interest. The analysis further investigates that no axioms is redundant for describing the solution, i.e., the axioms are logically independent. These steps are mainly followed in all the chapters.

## 1.6 Outline of the thesis

This thesis is divided into two parts. The first part, Chapter 2, 3 and 4, is concerned with cooperative games in a quite general setting, and more specifically multi-choice cooperative games. Part two, Chapter 5 and 6, investigates some applications related to pollution problems.

Chapter 2 provides a formal overview of cooperative game theory notions that are used along the thesis. First, I present the mathematical model of transferable utility game (TU-games for short), and discuss several solution concepts for this class of games. This is followed by the axiomatic foundations of several solution concepts. The main focus is on values for TU-games presented before. Subsequently, I present TU-games enriched with a priority structure. The notion of a priority structure, the Priority value introduced by Béal et al. [2022] and its axiomatic characterizations are briefly discussed.

Finally, I present the model of multi-choice games. Specifically, I present several notions that have their counterpart in the base model of TU-games. Several solution concepts that generalize the same named concepts from TU-games are presented. A specific attention is paid to the different extensions of the Shapley value. These different extensions are compared based on the axioms they satisfy. This enables to highlight the different visions that have been explored to evaluate the performance of the agent's cooperation in the multi-choice setting.

Chapter 3 is based on a joint work with David Lowing (Lowing and Techer [2022a]). In this chapter, we investigate the trade-off between marginalism and egalitarianism in multi-choice games. Marginalism relies on the idea that the evaluation of an agent's cooperation should result from that agent's marginal contributions to the cooperation. By marginal contribution, it is meant the contribution that agent makes when it joins a coalition. In this sense, the Shapley value is often considered as

the embodiment of marginalism in cooperative games. Egalitarianism relies on the idea that each agent should obtain the same evaluation in the cooperation process. Thus, the equal division value is considered as the embodiment of egalitarianism in cooperative games. The compromise between marginalism and egalitarianism is then obtained by considering the convex combinations between the Shapley value and the equal division value.

While extending this compromise to the multi-choice game setting, one might wonder what extension of the Shapley value and the equal division value to consider. To do so, we propose to consider a necessary condition for a payoff vector to be in the core of à la [Grabisch and Xie \[2007\]](#) that we call multi-efficiency. Accordingly, we introduce new extensions of the Shapley value (the multi-choice Shapley value) and the equal division value (the multi-choice Equal division value) that satisfy this condition. With a particular interest on the multi-choice Shapley value, we provide two expressions of this solution. We also relate it to the serial cost sharing method introduced in the context of discrete cost sharing problems. Then, we compromise between these two solutions by considering their convex combinations. Finally, we investigate the axiomatic foundations of these new multi-efficient solutions.

Chapter 4 is based on [Lowing and Techer \[2022b\]](#). We investigate the influence of a priority structure on allocations in multi-choice games. In many allocation problems, there are exogenous asymmetries between the agents, which may reflect some exogenous rights, different needs or merit, hierarchical constraints or a combinations of several factors. The priority principle is an ordinal equity principle often required in such situations. This principle usually results in lists describing the most deserving agents regarding the circumstances and the allocation problem. Based on this principle, a priority structure is a partial order on the agent set that reflect the asymmetries between the agents.

It appears that the use of such partial order brings several possibilities in the multi-choice games setting. First, the different activity levels of the agents provide an “intra-agent” information. This information comes from the fact that the activity levels of a given players are ordered (linearly). Then, the priority structure provides an “inter-agents” information. Depending on the cooperative situation at hand, these two pieces of information can be aggregated in order to define a relevant allocation process. In this chapter, we propose to combine these two pieces of information in a lexicographic manner. This enables us to compare the agents at different activity levels. We consider the activity level as the most significant criterion. When considering two agents with different activity levels, the one with a higher activity level ends up with the higher position in the lexicographic order. Then, two agents at the same activity level are compared according to their position in the priority structure.

Accordingly, we introduce a new value called the multi-choice Priority value, and provide two axiomatic characterizations of it. This value shares the net surplus generated by a coalition equally among the agents with the highest position in the lexicographic order. Such allocation process seems particularly relevant in situations dealing with scarce resources. Finally, we consider priority relations in which the set of agents can be partitioned into several classes. Precisely, each class contains incomparable agents (with respect to the priority relation) that have the priority over agents in the next classes. When considering such priority structured by classes, we show that the multi-choice Priority value can be interpreted as a sequential procedure involving specific TU-games. We provide an algorithm to compute

the multi-choice Priority value in these cases.

The last two chapters constitute the second part of the thesis. Both chapters investigate applications related to a pollution problem.

Chapter 5 studies the problem of transporting hazardous waste. The generation of these wastes has been steadily increasing in recent years, involving extensive movements of wastes on a local and global scale. Due to their dangerous nature, the transportation of these waste implies a risk of incident having irreversible consequences on the environment. This problem has lead to a body of legal statutes that monitor the generation and transportation of hazardous waste (examples of such regulations are presented in Section 1.4). These regulations all aim at implementing measures to reduce the risk associated with hazardous waste, and to manage it appropriately. Assuming that the transport of hazardous waste is done in a cooperative manner, this chapter investigates how to share the cost of maintaining and operating the network among the involved agents.

To my knowledge, there is no study on how to share the cost of the network maintenance and operation in the context of hazardous waste. Chapter 5 aims to investigate cooperative institutions (formalized through a cost sharing method) grounded on certain principles required by environmental regulations. Such a cooperative institution, based on the polluter pays principle, clearly describes the liability of each agent with respect to the risk it poses to the network. It also assigns to each agent a cost share accordingly.

More precisely, the chapter considers a finite set of agents involved in a hazardous waste transport network. Each agent is shipping a maximal amount of waste (which may differ from one agent to another) to a treatment facility. I assume that a central authority is in charge of maintaining the network as safe as possible and operating it. This entails a cost defined for each portion of the network, which depends on the amount of waste passing through the portion. By providing an axiomatic analysis, the objective is then to identify each agent's responsibility on the risk it poses to the network, and to allocate the cost for maintaining the network accordingly. To do so, I introduce several axioms derived from different environmental law principles, with the polluter pays principle as a center piece. As a result of interpreting environmental rights principles, the axioms describe the responsibility of each agent with respect to the risk it poses to the network. Then, it is shown that there is a unique allocation rule that satisfies the axioms, called the responsibility rule. The responsibility rule allocates the variation in cost of each portion (when the amount of waste increase) equally among the agents responsible for this portion. Finally, I show that the responsibility rule is related to the multi-choice Shapley value introduced in Chapter 3.

Chapter 6 is based on [Techer \[2021\]](#), and investigates the problem of the social cost. A social cost problem is a situation in which the activity of some agents has harmful effects on others. Such situations, that highlights market imperfections, have fostered two traditions while it comes to their resolution (see Section 1.4). In this chapter, I follow the approach introduced by Coase who advocates negotiated solutions, and generalizes the framework introduced by [Gonzalez et al. \[2019\]](#) to the context of multi-choice games.

Accordingly, this chapter considers a class of social cost problems in which one polluter interacts with an arbitrary number of potential victims. The polluter is endowed with a finite set of activity levels it wishes to implement. Each victim can

choose whether or not it cooperates. The whole set of agents is supposed to cooperate and negotiate an optimal pollution level together with monetary transfers with respect to a mapping of rights.

This mapping of rights describes which coalition is allowed to negotiate an agreement or not. Specifically, a mapping of rights is a function that assigns to each (multi-choice) coalition either the value 0, meaning that the coalition is not allowed to negotiate, or the value 1 if the coalition is allowed to negotiate an agreement. These mappings of rights rely on four reasonable conditions. The two first conditions, effectivity of rights and sovereignty of the grand coalition, can be found in [Gonzalez et al. \[2019\]](#). The main novelty here, comes from the following new conditions. First, if a coalition in which the polluter is active at a certain activity level receives the rights, then the coalition retains the rights whenever the polluter decreases its activity level. Second, whenever a coalition is allowed to form and negotiate an agreement, it retains the rights while the number of cooperating victims increases. These two new conditions allow that the rights depend on the activity level of the polluter.

Then, the study considers the multi-choice cooperative games associated with a social cost problem and an assignment (or mapping) of rights. I introduce a new class of mappings of rights that assigns the rights to the polluter up to a fixed and regulated level. These mapping of rights reflect the existence of a pollution quota regulating the activity of the polluter. Three properties on mappings of rights are considered: core compatibility which requires that the core of the associated multi-choice game be non-empty, Kaldor-Hicks core compatibility which requires that a payoff in the core is non-negative for each agent, and no veto power for a victim which requires that no victim can individually veto an agreement reach by the other agents. The analysis then aims to characterize the mappings of rights that satisfy different combinations of these properties.





*Part 1. Cooperative games and extensions*





## Chapter 2

# Preliminaries

This chapter provides the formal material: concepts and notation used along the thesis. Section 2.1 presents the standard model of transferable utility games. Besides the basic concepts, I present some well-known solutions and discuss some of their properties. Section 2.2 presents the transferable utility games enriched with a priority structure. I also present the Priority value and its axiomatic foundations introduced by Béal et al. [2022]. Section 2.3 presents the model of multi-choice games, introduced by Hsiao and Raghavan [1993]. Despite most of the concepts for multi-choice games has its counterpart in transferable utility games (especially solution concepts), it appears that several extensions of these concepts are possible. I discuss the different approaches that extend a particular concept whenever it seems necessary. I also present several solutions for multi-choice games and compare them based on the properties they have.

## 2.1 TU-games and solution concepts

Cooperative game theory studies situations in which a (finite) set of agents cooperate in order to collectively generate payoffs, whatever the conflicting interests that could exist among them. Since it is assumed that agents cooperate, the basic decision making units are the coalitions that agents can form. Each coalition then generates a worth that can be interpreted as the utility that the agents in the coalition can obtain on their own. If utility is transferable, so that agents can make side payments to each other, then one speaks about transferable utility games. The objective of cooperative game theory is then to study the coordination between coalitions with respect to their actions and the distribution of the gains generated by cooperation among the agents. This section presents the model of transferable utility games and several solutions that arise in the literature.

### 2.1.1 Basic definitions

Let  $\mathcal{U}$  be a countably infinite universe of agents, and let  $\mathcal{N}$  be the set of all finite subsets of  $\mathcal{U}$ . For  $N \in \mathcal{N}$ , denote by  $2^N$  the power set of  $N$ . An element  $S \in 2^N$  is referred to as a coalition. For each coalition  $S \in 2^N$ , denote by  $|S| \in \mathbb{N}$  its cardinality. Observe that each finite set  $N \in \mathcal{N}$  endowed with the inclusion relation  $\supseteq$  forms a (complete) lattice denoted by the pair  $(N, \supseteq)$ , where the greatest element is the grand coalition  $N$ , and the least element is the empty coalition  $\emptyset$ . Consider the sign function  $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ , given by  $\text{sign}(x) = -1$  if  $x < 0$ ,  $\text{sign}(0) = 0$ , and  $\text{sign}(x) = 1$  if  $x > 0$ . Let  $A$  be any finite set. For each  $B \subseteq A : B \neq \emptyset$ , let  $e^B \in \mathbb{R}^{|A|}$  be the vector such that  $e_i^B = 1$  if  $i \in B$  and  $e_i^B = 0$  otherwise.

A *transferable utility game* (henceforth TU-game) on  $N \in \mathcal{N}$  is a pair  $(N, v)$  where  $N$  is a (finite) set of agents and  $v : 2^N \rightarrow \mathbb{R}$  is a *characteristic function*. By convention,  $v(\emptyset) = 0$ . For a coalition of agents  $S \subseteq N$ , the real number  $v(S) \in \mathbb{R}$  is interpreted as the worth the members of the coalition can generate by themselves. Let  $\mathcal{G}_N^{TU}$  denote the set of all TU-games on  $N$ . The set of all TU-games is denoted by

$$\mathcal{G}^{TU} = \bigcup_{N \in \mathcal{N}} \mathcal{G}_N^{TU}.$$

Below, I present several subsets of TU-games. A TU-game  $(N, v) \in \mathcal{G}^{TU}$  is called:

*Non-negative* if,

$$\forall S \subseteq N, \quad v(S) \geq 0.$$

*Monotonic* if,

$$\forall S \subseteq N, T \subseteq N : S \subseteq T, \quad v(S) \leq v(T).$$

*Simple* if,

$$\forall S \subseteq N, \quad v(S) \in \{0, 1\}.$$

*Essential* if,

$$\forall S \subseteq N, \quad v(N) > \sum_{i \in N} v(\{i\}).$$

*Additive* if,

$$\forall S \subseteq N, T \subseteq N : S \cap T = \emptyset, \quad v(S \cup T) = v(S) + v(T).$$

*Super-additive* if,

$$\forall S \subseteq N, T \subseteq N : S \cap T = \emptyset, \quad v(S \cup T) \geq v(S) + v(T).$$

*Sub-additive* if,

$$\forall S \subseteq N, T \subseteq N : S \cap T = \emptyset, \quad v(S \cup T) \leq v(S) + v(T).$$

*Super-modular* (or *Convex*) if,

$$\forall S \subseteq N, T \subseteq N, \quad v(S \cup T) + v(S \cap T) \geq v(S) + v(T),$$

or equivalently (see [Shapley \[1971\]](#), [Ichiishi \[1981\]](#)),

$$\forall i \in N, \forall S \subseteq N \setminus \{i\}, \forall T \subseteq N \setminus \{i\} : S \subseteq T, \quad v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

*Sub-modular* (or *Concave*) if,

$$\forall S \subseteq N, T \subseteq N, \quad v(S \cup T) + v(S \cap T) \leq v(S) + v(T).$$

Observe that a super-modular (respectively sub-modular) TU-game is super-additive (respectively sub-additive). A TU-game  $(N, v) \in \mathcal{G}^{TU}$  is additive if it is both super-modular and sub-modular. Given a TU-game  $(N, v) \in \mathcal{G}^{TU}$  and a coalition  $S \subseteq N$ , the *sub-game* of  $(N, v)$  induced by  $S$  is the pair  $(S, v^S) \in \mathcal{G}^{TU}$  such that

$$\forall T \subseteq S, \quad v^S(T) = v(T).$$

The TU-game  $(N, v_0) \in \mathcal{G}^{TU}$  such that for each  $S \subseteq N$ ,  $v_0(S) = 0$  is called the *null game*.

Let  $N \in \mathcal{N}$  be a finite set of agents. For any two TU-games  $(N, v) \in \mathcal{G}_N^{TU}$  and  $(N, w) \in \mathcal{G}_N^{TU}$ , and any  $\alpha \in \mathbb{R}$ , the TU-game  $(N, \alpha v + w) \in \mathcal{G}_N^{TU}$  is given by:

$$\forall S \subseteq N, \quad (\alpha v + w)(S) = \alpha v(S) + w(S).$$

Moreover, since  $2^N$  is a finite set, for any TU-game  $(N, v) \in \mathcal{G}_N^{TU}$  the characteristic function  $v$  is fully described by the vector  $(v(S))_{S \in 2^N} \in \mathbb{R}^{|2^N|}$ . Since  $v(\emptyset) = 0$ , any characteristic function  $v$ , on  $2^N$ , can be generated by a linear combination of  $|2^N| - 1$  linearly independent elements. So,  $\mathcal{G}_N^{TU}$  corresponds to a linear subspace of  $\mathbb{R}^{|2^N|}$  of dimension  $2^{|N|} - 1$ . A first basis for  $\mathcal{G}_N^{TU}$  is given by the set of specific simple games, called *Dirac TU-games*,  $\{\delta_S : S \subseteq N, S \neq \emptyset\}$ , such that

$$\forall S \subseteq N, \forall T \subseteq N, \quad \delta_S(T) = \begin{cases} 1 & \text{if } T = S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

For each TU-game  $(N, v) \in \mathcal{G}_N^{TU}$ , the characteristic function  $v$  admits a unique linear decomposition in terms of Dirac TU-games given by:

$$v = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} v(S) \delta_S. \quad (2.2)$$

A second basis of  $\mathcal{G}_N^{TU}$  that has been recognized particularly useful to analyze solutions for TU-games consists of the set of well-known *unanimity games* (Shapley [1953]),  $\{u_S : S \subseteq N, S \neq \emptyset\}$ , such that

$$\forall S \subseteq N, \forall T \subseteq N, \quad u_S(T) = \begin{cases} 1 & \text{if } T \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

For each TU-game  $(N, v) \in \mathcal{G}_N^{TU}$ , the characteristic function  $v$  admits a unique linear decomposition in terms of unanimity games given by:

$$v = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} \Delta_S(v) u_S, \quad (2.4)$$

where the real number  $\Delta_S(v)$  called the *dividend* of coalition  $S$ , which was introduced by Harsanyi [1959], is computed recursively as follows:

$$\Delta_S(v) = v(S) - \sum_{T \subset S} \Delta_T(v). \quad (2.5)$$

This recursive formula allows to interpret the dividends as the surplus a coalition generates net of the surpluses generated by sub-coalitions. An equivalent expression of the dividend is given by:<sup>1</sup>

$$\forall S \subseteq N, S \neq \emptyset, \quad \Delta_S(v) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T). \quad (2.5')$$

Given a TU-game  $(N, v) \in \mathcal{G}_N^{TU}$  and its sub-game  $(N, v^S)$ , each dividend  $\Delta_T(v^S)$  in  $(S, v^S)$  coincides with the dividend  $\Delta_T(v)$ ,  $T \subseteq S$ , in  $(N, v)$ .

Let  $(N, v) \in \mathcal{G}_N^{TU}$  be a TU-game. Each agent may have different contributions in

<sup>1</sup>Note that the dividend corresponds to the Möbius transform (or inverse) of the characteristic function  $v$ .

the cooperation. These contributions are measured by the agent's marginal contributions to coalitions and assess the individual performance of the agent while cooperating. Formally, the *marginal contribution* of agent  $i \in N$  to a coalition  $S \subseteq N \setminus \{i\}$  in  $(N, v)$  is given by:

$$v(S \cup \{i\}) - v(S).$$

Two distinct agents  $i \in N$  and  $i' \in N$  are called *equals* in  $(N, v)$  if they have the same marginal contributions to coalitions not containing them, i.e.,

$$\forall S \subseteq N \setminus \{i, i'\}, \quad v(S \cup \{i\}) = v(S \cup \{i'\}).$$

An agent  $i \in N$  is called *null* in  $(N, v)$  if it has null marginal contributions to coalitions in this game, i.e.,

$$\forall S \subseteq N \setminus \{i\}, \quad v(S \cup \{i\}) = v(S).$$

It is clear that two null agents in  $(N, v)$  are equals. An agent  $i \in N$  is called *nullifying* in  $(N, v)$  if each coalition  $i$  joins has a null worth, i.e.,

$$\forall S \subseteq N \setminus \{i\}, \quad v(S \cup \{i\}) = 0.$$

An agent  $i \in N$  is called *necessary* in  $(N, v)$  if each coalition that does not contain this agent generates a null worth, i.e.,

$$\forall S \subseteq N \setminus \{i\}, \quad v(S) = 0.$$

It should be noted that any two necessary agents in  $(N, v)$  are also equal agents in this game. The reverse implication is not true.

### 2.1.2 Solution concepts for TU-games

When analyzing a TU game, the main objective is to determine whether there are reasonable ways of sharing the worth generated from the cooperation of all the agents, i.e., the worth of the grand coalition. Given a TU-game  $(N, v) \in \mathcal{G}^{TU}$ , a *payoff vector* is a  $|N|$ -dimensional vector  $x \in \mathbb{R}^{|N|}$  representing the payoffs that each agent  $i \in N$  can obtain when cooperating.<sup>2</sup> A payoff vector  $x \in \mathbb{R}^{|N|}$  is *efficient* if it fully allocates the worth of the grand coalition  $N$  between all the agents, i.e.,  $\sum_{i \in N} x_i = v(N)$ . A payoff vector is *individually rational* if no agent can be better off by itself, i.e., for each  $i \in N$ ,  $x_i \geq v(\{i\})$ . A payoff vector is *coalitionally rational* if no coalition can be better off by itself, i.e., for each  $S \subseteq N$ ,  $\sum_{i \in S} x_i \geq v(S)$ .

A *set-valued solution* on  $\mathcal{G}^{TU}$  is a mapping  $F : \mathcal{G}^{TU} \rightrightarrows \mathbb{R}^{|N|}$  that assigns a (possibly empty) set of payoff vectors  $F(N, v) \subseteq \mathbb{R}^{|N|}$  to each TU-game  $(N, v) \in \mathcal{G}^{TU}$ . A *single-valued solution*  $f$  on  $\mathcal{G}^{TU}$  is a function  $f : \mathcal{G}^{TU} \rightarrow \mathbb{R}^{|N|}$ . A single-valued solution is also called a *value*. Historically, the first main solution that arises in the literature is the *Imputation set*, introduced by [Morgenstern and Von Neumann \[1944\]](#). This solution refers to the (possibly empty) set of payoff vectors that are efficient and individually rational. Formally, the Imputation set,  $I$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$I(N, v) = \left\{ x \in \mathbb{R}^{|N|} : \sum_{i \in N} x_i = v(N) \text{ and } \forall i \in N, x_i \geq v(\{i\}) \right\}. \quad (2.6)$$

<sup>2</sup>Observe that a payoff vector  $x \in \mathbb{R}^{|N|}$  can be thought as an additive TU-game  $(N, v^x)$ , where the characteristic function  $v^x$  is defined as:  $\forall S \subseteq N, v^x(S) = \sum_{i \in S} x_i$ .

Obviously, the Imputation set is non-empty if and only if  $\sum_{i \in N} v(\{i\}) \leq v(N)$ . Below, I present other well-known solutions on  $\mathcal{G}^{TU}$  and elaborate on some of their properties.

### The core of TU-games

The most popular set-valued solution on  $\mathcal{G}^{TU}$  is probably the *Core*, introduced independently by Gillies [1953] and Shapley [1955]. This solution refines the Imputation set by considering only coalitionally rational imputations. Thus, the core embodies a stability principle since no coalition can generate by itself a payoff greater than that assigned by a Core element. The Core appears as a very appealing solution in many economic situations. Whenever non-empty it is then interpreted as the set of agreements that are stable and likely to occur. Among the wide range of articles analyzing the Core in economic situations, one can cite Shapley and Shubik [1969], who study the Core of market games in which the agents exchange a set of infinitely divisible goods; Scarf [1961] studies the relation between the Core and the Walrasian equilibria in a competitive market environment; Funaki and Yamato [1999] study an economy with a common pool resource and interpret the *tragedy of the commons* (see Hardin [1968]) as the emptiness of the Core of such an economy. A broad study of the Core and its application to economics can be found in Moulin [1995b] or in Telser [2012].

Formally, the Core,  $C$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$C(N, v) = \left\{ x \in \mathbb{R}^{|N|} : \sum_{i \in N} x_i = v(N) \text{ and } \forall S \subseteq N, \sum_{i \in S} x_i \geq v(S) \right\} \quad (2.7)$$

As the Imputation set, the Core of a TU-game may be empty.<sup>3</sup> A necessary and sufficient condition which ensures the non-emptiness of the Core relies on the concept of balancedness (see Bondareva [1962], Shapley [1967]). Let  $N \in \mathcal{N}$  be a finite set of agents, and let  $\mathcal{B} \subseteq 2^N \setminus \emptyset$  be a collection of non-empty sets. The collection  $\mathcal{B}$  is called a *balanced collection* if for each  $S \in \mathcal{B}$  there exists  $\mu_S \in \mathbb{R}_+$  such that

$$\forall i \in N, \sum_{\substack{S \in \mathcal{B}: \\ i \in S}} \mu_S = 1.$$

The real numbers  $\mu_S$  are called balancing weights. A TU-game  $(N, v) \in \mathcal{G}^{TU}$  is called *balanced* if for each balanced collection  $\mathcal{B} \subseteq 2^N \setminus \emptyset$  it holds that:

$$\sum_{S \in \mathcal{B}} \mu_S v(S) \leq v(N). \quad (2.8)$$

**Theorem 2.1.1 (Bondareva [1962], Shapley [1967]).** *A TU-game  $(N, v) \in \mathcal{G}^{TU}$  has a non-empty core if and only if it is balanced.*

### The Weber set of TU-games

A second well-known set valued solution on  $\mathcal{G}^{TU}$  is the *Weber set*, introduced by Weber [1988]. To define this solution, I first present the notion of marginal vectors. Recall that the main hypothesis formulated by the cooperative game theory is that the agents form the grand coalition. Shapley [1953] first interpreted the formation of the grand coalition as the following bargaining process. The agents are admitted one

<sup>3</sup>Observe that the Core is a convex closed polyhedron.

by one, following a given order of the agent set, until the grand coalition is formed. On its admission to the coalition, each agent is assigned its marginal contribution to the coalition formed by the already admitted agents. Given an order on the agent set, the payoff vector resulting from this bargaining process is called the *marginal vector*. Formally, let  $N \in \mathcal{N}$  be a finite set of agents. An order on  $N$ , is a bijection  $\sigma : N \rightarrow \{1, \dots, |N|\}$ . For each  $i \in N$ ,  $\sigma(i)$  is agent  $i$ 's position in the order  $\sigma$ . Denote by  $O_N$  the set of all orders on  $N$ . For each  $\sigma \in O_N$  and each  $k \in \{0, \dots, |N|\}$ , the coalition  $S^{\sigma,k}$  is the coalition formed by the agents joining up to  $k$  in  $\sigma$ , i.e.,

$$S^{\sigma,k} = \{i \in N : \sigma(i) \leq k\}.$$

Given a TU-game  $(N, v) \in \mathcal{G}^{TU}$  and an order  $\sigma \in O_N$ , the marginal vector  $m^\sigma(N, v) \in \mathbb{R}^{|N|}$  is defined as:

$$\forall i \in N, \quad m_i^\sigma(N, v) = v(S^{\sigma, \sigma(i)}) - v(S^{\sigma, \sigma(i)-1}). \quad (2.9)$$

For each TU-game  $(N, v) \in \mathcal{G}^{TU}$ , the Weber set,  $W$ , is defined as the convex hull of all marginal vectors, i.e.,

$$W(N, v) = \text{co}\{m^\sigma(N, v) : \sigma \in O_N\}.$$

It is known that the core of a game is included in the Weber set.

**Theorem 2.1.2 (Weber [1988] and Derks [1992]).** For each TU-game  $(N, v) \in \mathcal{G}^{TU}$ ,

$$C(N, v) \subseteq W(N, v).$$

The converse inclusion holds whenever the game is super-modular, which makes explicit the structural properties of the core of super-modular games. In particular it is known that the core of a super-modular TU-game is large, and its extreme rays coincide with the set of marginal vectors.

**Theorem 2.1.3 (Shapley [1971], Ichiishi [1981]).** For each  $(N, v) \in \mathcal{G}^{TU}$  super-modular game,

$$C(N, v) = W(N, v).$$

### The Shapley value and the Equal division value

Regarding single-valued solutions (henceforth value) on  $\mathcal{G}^{TU}$ , one of the most popular solutions is the *Shapley value*, introduced by Shapley [1953]. This value has been extensively studied highlighting the several desirable properties that this value has. Moreover, the scope of applications of the Shapley value is very broad and keeps expanding. For instance: Aumann and Shapley [1974] apply the Shapley value to exchange economies and shows that it coincides with the Walrasian equilibrium of such economy; Shubik [1962], Moulin [1992], or Sprumont [2005], among others, apply the Shapley value to the so-called cost sharing model and add robustness to the original work of Shapley. A recent detailed analysis of the Shapley value and its applications in various fields can be found in Algaba et al. [2019].

The Shapley value has several mathematical expressions along with several interpretations that I describe below. Interpreted within the bargaining process used to define the marginal vectors (see (2.9)), the Shapley value relies on the assumption that each order on the agent set can occur with the same probability. Thus, the expected payoff of an agent is exactly its payoff assigned by the Shapley value.

Therefore, [Shapley \[1953\]](#) argues that “the value is best regarded as an a priori assessment of the situation, based on either ignorance or disregard of the social organization of the agents.”

Formally, the Shapley value,  $Sh$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{|N|!} \sum_{\sigma \in O_N} m_i^\sigma(N, v). \quad (2.10)$$

A second mathematical expression of the Shapley value relies on the marginal contribution. According to this expression, the Shapley value assigns to each agent a weighted average of its marginal contributions to the coalitions this agent can join. Formally, the Shapley value,  $Sh$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\forall i \in N, \quad Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[ v(S \cup \{i\}) - v(S) \right]. \quad (2.11)$$

The weight  $\frac{|S|!(|N| - |S| - 1)!}{|N|!}$ , called the Shapley coefficient, can be interpreted as the probability for an agent to join a coalition  $S$  containing  $|S|$  agents.

A third mathematical expression, proposed by [Shapley \[1953\]](#), is as follows: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\forall i \in N, \quad Sh_i(N, v) = \sum_{\substack{S \subseteq N: \\ i \in S}} \frac{\Delta_S(v)}{|S|}. \quad (2.12)$$

Thus, the Shapley value divides the net surplus of a coalition equally among its members. From the expression (2.10), it is clear that the Shapley value is the centroid of the Weber set. Thus, it belongs to the Core of super-modular games. This point is of particular interest in many economic applications that can often be described by a super-modular TU-game. Some notable examples are bankruptcy games introduced by [O'Neill \[1982\]](#), cooperative games associated with a river sharing problem introduced by [Ambec and Sprumont \[2002\]](#), or its pollution counter-part introduced by [Ni and Wang \[2007\]](#). In such cases, the Shapley value ensures that no coalition has an interest to split off from the grand coalition.

Another well-known value on  $\mathcal{G}^{TU}$  is the Equal division value that equally divides the worth of the grand coalition between the agents. While the Shapley value reflects the individual performance of the agents, the Equal division value only refers to the grand coalition. This value conveys the fact that each agent has an equal claim on the value generated by the cooperation. In this sense, it is the embodiment of egalitarianism in TU-games.

The Equal division value,  $ED$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\forall i \in N, \quad ED_i(N, v) = \frac{v(N)}{|N|}. \quad (2.13)$$

### The Egalitarian Shapley values

The search for a compromise between individual based performance and egalitarianism is one of the main issues in economic allocation problems. In TU-games, this compromise takes the form of convex combinations between the Shapley value and the Equal division value. This class of solutions, called the Egalitarian Shapley values, has been introduced by [Joosten \[1996\]](#).



Let  $\alpha \in [0, 1]$ , the Egalitarian Shapley value,  $ESh^\alpha$ , is defined as: for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\forall i \in N, \quad ESh_i^\alpha(N, v) = \alpha Sh_i(N, v) + (1 - \alpha) ED_i(N, v). \quad (2.14)$$

### 2.1.3 Axiomatic foundations of solutions

In this section, I present several axioms for solutions on  $\mathcal{G}^{TU}$ . I discuss the relations between these axioms and present some axiomatic characterization of solutions that result from their combinations.

#### General properties

A value  $f$  on  $\mathcal{G}^{TU}$  satisfies:

*Efficiency* if, for each  $(N, v) \in \mathcal{G}^{TU}$ ,

$$\sum_{i \in N} f_i(N, v) = v(N).$$

*Additivity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$ ,

$$f(N, v + w) = f(N, v) + f(N, w).$$

*Linearity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  and each  $\alpha \in \mathbb{R}$ ,

$$f(N, \alpha v + w) = \alpha f(N, v) + f(N, w).$$

The three above axioms are very common in axiomatic analysis. Efficiency asserts that the worth of the grand coalition is fully shared among the agents. Therefore, the payoff assigned by the value is feasible, i.e.,  $\sum_{i \in N} f_i(N, v) \leq v(N)$ , and no agent can increase its payoff without decreasing another agent's payoff (while keeping the payoff feasible). In this sense, the Efficiency axiom can be seen as the well-known Pareto optimality. The Additivity (respectively Linearity) axiom simply asserts that the value is additive (respectively linear) with respect to the characteristic functions.

#### Equal treatment principles

The next five axioms are related to a basic equal treatment principle widely invoked in axiomatic analysis. Let  $\sigma \in O_N$  be an order on the agent set and define  $\sigma v \in \mathcal{G}^{TU}$  such that for each  $S \subseteq N$ ,  $\sigma v(S) = v(\bigcup_{i \in S} \sigma(i))$ . A value  $f$  on  $\mathcal{G}^{TU}$  satisfies:

*Anonymity* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each order  $\sigma \in O_N$ ,

$$\forall i \in N, \quad f_i(N, \sigma v) = f_{\sigma(i)}(N, v).$$

*Equal treatment of equals* if, for each  $(N, v) \in \mathcal{G}^{TU}$ , and each two distinct equal agents  $i, i' \in N$ ,

$$f_i(N, v) = f_{i'}(N, v).$$

*Equal treatment of necessary agents* if, for each  $(N, v) \in \mathcal{G}^{TU}$ , and each two distinct necessary agents  $i, i' \in N$ ,

$$f_i(N, v) = f_{i'}(N, v).$$

*Sign symmetry of equals* if, for each  $(N, v) \in \mathcal{G}^{TU}$ , and each two distinct equal agents  $i, i' \in N$ ,

$$\text{sign}(f_i(N, v)) = \text{sign}(f_{i'}(N, v)).$$

*Desirability* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each two distinct agents  $i, i' \in N$  such that for each  $S \subseteq N \setminus \{i, i'\} : v(S \cup \{i\}) \geq v(S \cup \{i'\})$ ,

$$f_i(N, v) \geq f_{i'}(N, v).$$

Anonymity is a strong requirement that a solution be invariant to the agents' renaming. Observe that Equal treatment of equals, Sign symmetry of equals and Desirability all consider agents' marginal contributions to coalitions. This feature enables to formulate punctual axioms that compare the agents' payoffs. Precisely, Equal treatment of equals is a weaker requirement than Anonymity relying on the individual performance of the agents. It ensures that two agents with the same individual performance in the game (measured by their marginal contributions to coalitions) have the same payoff. Equal treatment for necessary agents, introduced by Béal and Navarro [2020] relaxes Equal treatment of equals by strengthening its hypothesis. It requires that two necessary agents receives the same payoff. Sign symmetry of equals, introduced by Casajus [2018], relaxes Equal treatment of equals by weakening its implication. It requires that equal agents obtain a payoff of the same sign instead of the same payoff. Finally, Desirability, introduced by Maschler and Peleg [1966], ensures that an agent with a greater individual performance than another agent receives a greater payoff.

The relations between the axioms are as follows. It is clear that Anonymity implies Equal treatment of equals. While the converse is not true, Malawski [2020] shows that the combination of Efficiency, Additivity and Equal treatment of equals implies Anonymity. Moreover Equal treatment of equals implies both Equal treatment of necessary agents and Sign symmetry of equals. Finally, Desirability implies Equal treatment of equals.

### Fairness principles related to null and nullifying agents

Next, I present four axioms related to fairness concepts. Each axiom takes into account the individual performance of the agents. Specifically, the axioms are concerned with null and nullifying agents. A value  $f$  on  $\mathcal{G}^{TU}$  satisfies:

*The null agent axiom* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each null agent  $i \in N$ ,

$$f_i(N, v) = 0.$$

*The null agent out axiom* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each null agent  $i \in N$ ,

$$\forall i' \in N \setminus \{i\}, \quad f_{i'}(N, v) = f_{i'}(N \setminus \{i\}, v^{N \setminus \{i\}}).$$

*Null agent in productive environment* if for each  $(N, v) \in \mathcal{G}^{TU}$  such that  $v(N) \geq 0$ , and each null agent  $i \in N$ ,

$$f_i(N, v) \geq 0.$$

*The nullifying agent axiom* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each nullifying agent  $i \in N$ ,

$$f_i(N, v) = 0.$$

The null agent axiom requires that an agent which does not contribute receives a null payoff. Thus, a null agent does not benefit from more than its marginal contributions. The null agent out axiom, introduced by [Derks and Haller \[1999\]](#), requires that the withdrawal of a null agent should not impact the payoff of the remaining agents. Observe that there is no implication between the null agent axiom and the null agent out axiom. However, the combination of Efficiency and the null agent out axiom implies the null agent axiom. Null agent in a productive environment, introduced by [Casajus and Huettner \[2013\]](#), requires that a null agent can benefit from more than its marginal contributions whenever the worth of the grand coalition is non-negative. Finally, the nullifying agent axiom, introduced by [van den Brink \[2007\]](#), considers nullifying agents instead of null agents. Thus, an agent that drops the worth of each coalition containing it to zero receives a null payoff.

### Monotonicity principles

Monotonicity is a general principle widely used in the axiomatic analysis. This principle hypothesizes a change in some parameter (or data) of the game between two situations, and requires a change of the same fashion in the solution. Below, I present some relational axioms for values on  $\mathcal{G}^{TU}$  that relies on this principle. A value  $f$  on  $\mathcal{G}^{TU}$  satisfies:

*Strong monotonicity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  and each  $i \in N$  such that for each  $S \subseteq N \setminus \{i\} : v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ , then

$$f_i(N, v) \geq f_i(N, w).$$

*Marginality* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  and each  $i \in N$  such that for each  $S \subseteq N \setminus \{i\} : v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ ,

$$f_i(N, v) = f_i(N, w).$$

*Aggregate monotonicity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  such that for each  $S \subset N : v(S) = w(S)$  and  $v(N) \geq w(N)$ ,

$$\forall i \in N, \quad f_i(N, v) \geq f_i(N, w).$$

*Coalitional monotonicity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  and each  $i \in N$  such that for each  $S \subseteq N \setminus \{i\} : v(S \cup \{i\}) \geq w(S \cup \{i\})$ ,

$$f_i(N, v) \geq f_i(N, w).$$

*Weak monotonicity* if, for each  $(N, v), (N, w) \in \mathcal{G}^{TU}$  and each  $i \in N$  such that for each  $S \subseteq N \setminus \{i\} : v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$  and  $v(N) \geq w(N)$ ,

$$f_i(N, v) \geq f_i(N, w).$$

Strong monotonicity, introduced by [Young \[1985\]](#), hypothesizes an increase in an agent's marginal contributions from one game to another. Then, this agent should receive at least the same payoff in the game in which it has a greater individual performance. Obviously, Strong monotonicity implies Marginality, so that the payoff of an agent depends only on its marginal contributions to coalitions. Aggregate monotonicity, introduced by [Megiddo \[1974\]](#), states that the payoff of each agent should not decrease whenever the worth of the grand coalition increases (the worth of the sub-coalitions remaining the same). Coalitional monotonicity, introduced by

van den Brink [2007], hypothesizes a change in the worth of each coalition containing an agent. The payoff this agent obtains should not decrease whenever the coalitions containing it better perform. Finally, Weak monotonicity, introduced by van den Brink et al. [2013] weakens Strong monotonicity by strengthening its hypothesis. It requires that the worth of the grand coalition should not decrease in addition to the hypothesis of Strong monotonicity. Recently, Yokote and Funaki [2017] focus on several parameters of a TU-game and formulate the corresponding monotonicity axioms.

### Balanced contributions principles

The principle of balanced contributions is based on how an agent's payoff varies when another agent leaves the game. Such variation in payoffs is often interpreted as a claim of an agent on a remaining agent. For instance, let  $f$  be a value on  $\mathcal{G}^{TU}$  and  $i, i' \in N$  be two distinct agents. The claim of agent  $i$  against  $i'$  is given by  $f_i(N, v) - f_{i'}(N \setminus \{i\}, v^{N \setminus \{i\}})$ , which can be interpreted as  $i$ 's contribution to the payoff of  $i'$ . A value  $f$  on  $\mathcal{G}^{TU}$  satisfies:

*Balanced contributions* if, for each  $(N, v) \in \mathcal{G}^{TU}$  and each two agents  $i, i' \in N$ ,

$$f_i(N, v) - f_i(N \setminus \{i'\}, v^{N \setminus \{i'\}}) = f_{i'}(N, v) - f_{i'}(N \setminus \{i\}, v^{N \setminus \{i\}}).$$

The axiom of Balanced contributions, introduced by Myerson [1980], states that for any two agents the amount that each agent gains or losses by the other withdrawing from the game should be equal. Thus, this axiom expresses the fairness requirement that any two agents should have balanced claims on each other. In other words, two agents should affect each other's payoff in the same way. Recently, Kamijo and Kongo [2010] study a weaker axiom called Balanced cycle contributions. This weaker axiom requires that the balancedness of the claims be achieved across the members of  $N$ . Recently, Casajus [2017] introduces the Weak balanced contributions axiom that weakens the Balanced contributions by weakening its implication. Instead of imposing that any two agents have equal claim on each other, it imposes that the claims should be of the same sign. Other relaxations of Balanced contributions can be found in Yokote and Kongo [2017] and Yokote et al. [2018].

Based on the above list of axioms, I list some well-known characterizations resulting from their combination. I first present the most well-known axiomatic characterizations of the Shapley value, then those of the Equal division value, and finally those of the Egalitarian Shapley values.

The first axiomatic result presented is due to Shubik [1962] and provides a characterization of the Shapley value.

**Theorem 2.1.4 (Shapley [1953], Shubik [1962]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Additivity, the null agent axiom and Equal treatment of equals if and only if it is the Shapley value.*

Recently, Casajus [2019] shows that Equal treatment of equals can be replaced by Sign symmetry of equals in the above characterization.

**Theorem 2.1.5 (Casajus [2019]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Additivity, the null agent axiom and Sign symmetry of equals if and only if it is the Shapley value.*

In the same line, Béal and Navarro [2020] show that Equal treatment of equals can be replaced by Equal treatment of necessary agents.

**Theorem 2.1.6 (Béal and Navarro [2020]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Additivity, the null agent axiom and Equal treatment of necessary agents if and only if it is the Shapley value.*

The Additivity axiom has been often criticized in the literature as being too technical and lacking in economic interpretation. I should point out that many other characterizations of Shapley value exist that replace this axiom with less controversial ones. In this line, Young [1985] characterizes the Shapley value by replacing Additivity and the null agent property by Strong monotonicity.

**Theorem 2.1.7 (Young [1985]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Equal treatment of equals and Strong monotonicity if and only if it is the Shapley value.*

Recently, Casajus [2018] and Béal and Navarro [2020] respectively show that Equal treatment of equals can be replaced by Sign symmetry of equals or Equal treatment of necessary agents in the above characterization.

Another approach to characterize the Shapley value is proposed in Myerson [1980] and Hart and Mas-Colell [1989]. This approach consists in using Balanced contributions.

**Theorem 2.1.8 (Myerson [1980], Hart and Mas-Colell [1989]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency and Balanced contributions if and only if it is the Shapley value.*

Since Balanced contributions can be interpreted as a fair requirement regarding what agents can claim to the other agents, the Shapley value is then the unique efficient value on  $\mathcal{G}^{TU}$  that satisfies this fairness requirement.

While the Shapley value captures the individual performance of the agents, the Equal division value reflects a strong egalitarian principle concerned with the grand coalition only. Despite this conceptual difference, van den Brink [2007] highlights the similarities (and differences) between the Shapley value and the Equal division value by providing parallel axiomatic characterizations. Specifically, van den Brink [2007] shows that replacing the null agent axiom by the nullifying agent axiom in Theorem 2.1.4 yields a characterization of the Equal division value.

**Theorem 2.1.9 (van den Brink [2007]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Additivity, the nullifying agent axiom and Equal treatment of equals if and only if it is the Equal division value.*

In the same line, van den Brink [2007] shows that replacing Strong monotonicity by Coalitional monotonicity in Theorem 2.1.7 yields a characterization of the Equal division value.

**Theorem 2.1.10 (van den Brink [2007]).** *A value on  $\mathcal{G}^{TU}$  satisfies Efficiency, Coalitional monotonicity and Equal treatment of equals if and only if it is the Equal division value.*

Actually, Equal treatment of equals can be weakened by imposing the same payoff for all agents in TU-games where all agents are equals.

Several studies focus on the class of Egalitarian Shapley values and reveal the desirable properties that characterize it. Originally, Joosten [1996] proposes three axiomatic characterizations of the Egalitarian Shapley values. Each characterization is in line with the main characterizations of the Shapley value. The main difference lies

in how to treat null agents, requiring a weak egalitarian (or solidarity) principle that allows a null agent to benefit from the worth generated cooperatively. Thus, Joosten [1996] replaces the null agent axiom by an axiom called the  $\alpha$ -egalitarian in Theorem 2.1.4 yielding a characterization of the Egalitarian Shapley values. In the same way, replacing Strong monotonicity by  $\alpha$ -marginality in Theorem 2.1.7 yields a second characterization of the Egalitarian Shapley values. One of the main drawback of these characterizations is the explicit use of a parameter  $\alpha$  in both  $\alpha$ -egalitarian and  $\alpha$ -marginality. Other characterizations of the Egalitarian Shapley values exist that overcome this drawback. Casajus and Huettner [2013] show that, replacing the null agent axiom and Equal treatment of equals by Null agent in a productive environment and Desirability in Theorem 2.1.4 yields a characterization of the Egalitarian Shapley values.

**Theorem 2.1.11 (Casajus and Huettner [2013]).** *A value  $f$  on  $\mathcal{G}^{TU}$  satisfies Efficiency, Additivity, Desirability and Null agent in a productive environment if and only if there exists  $\alpha \in [0, 1]$  such that  $f = ESh^\alpha$ .*

Moreover replacing Additivity and Null agent in a productive environment by Weak monotonicity, yields a characterization of the Egalitarian Shapley value for games with more than two agents. Let  $\mathcal{G}_{\neq 2}^{TU} = \{(N, v) \in \mathcal{G}^{TU} : |N| \neq 2\}$ .

**Theorem 2.1.12 (Casajus and Huettner [2014]).** *A value  $f$  on  $\mathcal{G}_{\neq 2}^{TU}$  satisfies Efficiency, Desirability and Weak monotonicity if and only if there exists  $\alpha \in [0, 1]$  such that  $f = ESh^\alpha$ .*

**Theorem 2.1.13 (van den Brink et al. [2013]).** *A value  $f$  on  $\mathcal{G}^{TU}$  satisfies Efficiency, Linearity, Anonymity and Weak monotonicity if and only if there exists  $\alpha \in [0, 1]$  such that  $f = ESh^\alpha$ .*

Other recent insights into the analysis of the Egalitarian Shapley values can be found in Wang et al. [2017], Yokote and Funaki [2017], Béal et al. [2021].

## 2.2 TU-games with a priority structure

The model of TU-games can be enriched in a variety of ways aiming at describing more accurate situations. In several situations, there exists asymmetries between the agents that are not captured by the parameters of a TU-game. These asymmetries may reflect exogenous rights, different needs, merit, or hierarchical constraints that should be conveyed by an allocation process. To account for such situations, Béal et al. [2022] enriched the model of TU-game with a priority structure. A priority structure, modeled by a partial order on the agent set, reflects the fact that some agents may have the priority over other agents in the allocation process. It should be noted that several studies have considered a partial order on the agent set under different interpretation (see for instance Faigle and Kern [1992], and Gilles et al. [1992]). In addition, Béal et al. [2022] introduces the Priority value for TU-games with a priority structure. This value shares the dividend of each coalition between the priority agents in the coalition.

In this section, I present the model of TU-games enriched with a priority structure along with the Priority value. I also present the axiomatic foundations of the Priority value provided by Béal et al. [2022].



### 2.2.1 The model

Let  $N \in \mathcal{N}$  be any finite set of agents, and let  $\succeq$  be a binary relation on  $N$ . The relation  $\succeq$  is called: *reflexive* if, for each  $i \in N$ ,  $i \succeq i$ ; *transitive* if, for each  $i, j, k \in N$  such that  $i \succeq j$  and  $j \succeq k$  then  $i \succeq k$ ; *antisymmetric* if, for each  $i, j \in N$  such that  $i \succeq j$  and  $j \succeq i$  then  $i = j$ ; *complete* if, for each  $i, j \in N$ , either  $i \succeq j$  or  $j \succeq i$ .

A priority structure on  $N$  is a *partially ordered set* (henceforth a poset)  $\succeq$  on  $N$ . Recall that a poset  $(N, \succeq)$  is a reflexive, antisymmetric and transitive binary relation. The relation  $i \succeq i'$  means that agent  $i$  has priority over agent  $i'$ . Two distinct agents  $i$  and  $i'$  are *incomparable* in  $(N, \succeq)$  if neither  $i \succeq i'$  nor  $i' \succeq i$ . The poset  $(N, \succeq^0)$  containing no priority relation between two distinct agents is called the *trivial poset*. In some situations, each pair of agents in  $N$  are comparable. In this case, the poset  $(N, \succeq)$  is a *linear order*, i.e.,  $(N, \succeq)$  is complete. The poset  $(N, \succeq')$  contains  $(N, \succeq)$  if for each  $i, i' \in N$ ,  $i \succeq i'$  implies  $i \succeq' i'$ . Then,  $(N, \succeq')$  is referred to as an extension of  $(N, \succeq)$ .

A poset  $(N, \succeq)$  gives rise to the asymmetric binary relation  $(N, \succ)$ :  $i \succ i'$  if  $i \succeq i'$  and not  $i' \succeq i$ . For an agent  $i \in N$ , define the *priority group* of  $i$ , denoted by  $\uparrow_{\succ} i$ , as the set of agents having priority over  $i$  in  $(N, \succeq)$ , i.e.,

$$\uparrow_{\succ} i = \left\{ i' \in N : i' \succ i \right\}.$$

For each nonempty  $A \subseteq N$ , the subposet  $(A, \succeq^A)$  of  $(N, \succeq)$  induced by  $A$  is such that for each  $i \in A$  and  $i' \in A$ ,  $i \succeq^A i'$  if  $i \succeq i'$ . An agent  $i \in A$  is a *priority agent* in  $(A, \succeq^A)$  if, for  $k \in A$ ,  $k \succeq i$  implies  $i = k$ . Denote by  $T(A, \succeq^A)$  the nonempty subset of priority agents in  $(A, \succeq^A)$ . Denote by  $\mathfrak{P}_N$  the set of all posets (on  $N$ ) and subposets that one can generate from  $N$ . Given  $(N, \succeq) \in \mathfrak{P}_N$ , and two distinct agents  $i, i' \in N$  such that  $i' \in N \setminus \uparrow_{\succ} i$ , let  $(N, \succeq_{i \rightarrow i'})$  be the poset defined as:

$$\forall k, l \in N, \quad l \succeq_{i \rightarrow i'} k \iff \begin{cases} \text{either } l \in \uparrow_{\succ} i \cup \{i\} \text{ and } k \in \downarrow_{\succ} i' \cup \{i'\}, \\ \text{or } l \succeq k. \end{cases} \quad (2.15)$$

Given  $N \in \mathcal{N}$ , a *TU-game enriched with a priority structure* is the triple  $(N, v, \succeq)$ , where  $(N, v) \in \mathcal{G}^{TU}$  and  $(N, \succeq) \in \mathfrak{P}_N$ . For  $S \subseteq N$ , denote by  $(S, v^S, \succeq^S)$  the subgame of  $(N, v, \succeq)$  induced by  $S$ . Denote by  $\mathcal{GP}_N^{TU}$  the set of all TU-games with a priority structure that can be constructed from  $\mathcal{G}^{TU}$  and  $\mathfrak{P}_N$ .

### 2.2.2 The Priority value and its axiomatic foundations

Whenever agents are heterogeneous, it seems reasonable that a solution account for these differences when allocating the value generated by cooperation. Below, I present the *Priority value*, introduced by Béal et al. [2022], which shares the dividend of each coalition equally between the priority agents in the coalition.

Formally, the Priority value,  $Pr$ , is defined as: for each  $(N, v, \succeq) \in \mathcal{GP}_N^{TU}$ ,

$$\forall i \in N, \quad Pr_i(N, v, \succeq) = \sum_{\substack{S \subseteq N: \\ i \in T(S, \succeq^S)}} \frac{\Delta_S(v)}{|T(S, \succeq^S)|}. \quad (2.16)$$

Observe that the Priority value is close in spirit to the Shapley value. Whenever the priority structure is the trivial poset, the Priority value coincides with the Shapley

value. Béal et al. [2022] study the influence of the priority structure on TU-games through an axiomatic study. The authors invoke both standard axioms (see Section 2.1.3) and new axioms related to the priority structure that I present below. A value  $f$  on  $\mathcal{GP}_N^{TU}$  satisfies:

*Equal treatment of necessary agents with equal priority group* if, for each  $(N, v, \succeq) \in \mathcal{GP}_N^{TU}$ , each two distinct necessary agents  $i, i' \in N$  such that  $\uparrow_{\succeq} i = \uparrow_{\succeq} i'$ ,

$$f_i(N, v, \succeq) = f_{i'}(N, v, \succeq).$$

*Necessary and priority agent* if, for each  $(N, v, \succeq) \in \mathcal{GP}_N^{TU}$ , and each two necessary agents  $i, i' \in N$  such that  $i' \succ i$ ,

$$f_i(N, v, \succeq) = 0.$$

*Priority agent out* if, for each  $(N, v, \succeq) \in \mathcal{GP}_N^{TU}$ , each two agents  $i, i' \in N$  such that  $i' \succ i$ ,

$$f_i(N, v, \succeq) = f_i(N \setminus \{i'\}, v^{N \setminus \{i'\}}, \succeq).$$

*Invariance to unproductive priority extension* if, for each  $(N, v, \succeq) \in \mathcal{GP}_N^{TU}$ , each two incomparable agents  $i, i' \in N$  such that each  $k \in \uparrow_{\succeq} i \cup \{i\}$  is a null agent,

$$f(N, v, \succeq) = f(N, v, \succeq_{i \rightarrow i'}).$$

Equal treatment of necessary agents with the same priority group weakens the axiom of Equal treatment of necessary agents by considering the priority structure. It requires that both necessary agents should have the same priority groups in order to obtain the same payoff. Necessary and priority agent asserts that a first agent that has the priority over a second agent can force the payoff of the second agent to be zero. This indicates the fact that the priority relation comes over the economic relation between two necessary agents. Priority agent out relies on the same idea that the priority comes over the economic performance. It indicates that the payoff of an agent is invariant to the presence of an agent having the priority over it. Finally, Invariance to unproductive priority extension indicates that the payoff of an agent is not affected if a group of unproductive agents takes priority over it.

The relations between the axioms are as follows. The authors show that the combination of Priority agent out and the null game axiom implies Necessary and priority agent. The combination of Efficiency and the null agent out axiom implies Invariance to unproductive priority extension.

Based on the above axioms and straightforward generalizations of standard axioms from TU-games to TU-games enriched with a priority structure, Béal et al. [2022] provides the two following characterization of the Priority value.

**Theorem 2.2.1 (Béal et al. [2022]).** *A value on  $\mathcal{GP}_N^{TU}$  satisfies Efficiency, Additivity, Priority agent out, Equal treatment of necessary agents with equal priority group and Invariance to unproductive priority extension if and only if it is the Priority value.*

A second characterization of the Priority value can be obtained by replacing Priority agent out and Invariance to unproductive priority extension by the null agent out axiom and Necessary and priority agent.

**Theorem 2.2.2 (Béal et al. [2022]).** *A value on  $\mathcal{GP}_N^{TU}$  satisfies Efficiency, Additivity, the null agent out axiom, Equal treatment of necessary agents with equal priority group and Necessary and priority agent if and only if it is the Priority value.*



## 2.3 Multi-choice games

In TU-games (as presented in Section 2.1), agents can either join a coalition and fully cooperate or abstain. Multi-choice games, introduced by Hsiao and Raghavan [1992, 1993], constitute a natural extension of TU-games in which agents can cooperate at different intensities (or activity levels) within a coalition.<sup>4</sup> Then the worth of a coalition depends on the intensity of cooperation of each agent. This section presents the model of multi-choice games and several solutions.

### 2.3.1 The model

Fix  $N \in \mathcal{N}$  the set of agents and  $K \in \mathbb{N}$ . Each agent  $i \in N$  has a finite set of pairwise distinct activity levels  $M_i := \{0, 1, \dots, m_i\}$  where  $m_i \leq K$ . For each agent  $i \in N$ , the set  $M_i$  is linearly ordered from its lowest activity level 0 ( $i$  does not cooperate) to its maximal activity level  $m_i$ . Moreover, denote  $M_i^+ = M_i \setminus \{0\}$  the set of  $i$ 's positive activity levels. For each  $S \subseteq N$ ,  $S \neq \emptyset$ , let  $\mathcal{M}_S$  be the cartesian product  $\prod_{i \in S} M_i$ . I will sometimes write  $\mathcal{M}$  instead of  $\mathcal{M}_N$ , and  $-i$  instead of  $N \setminus \{i\}$ . An element  $s = (s_1, \dots, s_n) \in \mathcal{M}$  is referred to as a (multi-choice) coalition, which indicates each agent's activity level when cooperating. The notation  $(s_S, s_{N \setminus S}) \in \mathcal{M}_S \times \mathcal{M}_{N \setminus S}$  will be used to specify the activity levels played by the set of agents  $S$  and  $N \setminus S$  respectively. The coalition  $m = (m_1, \dots, m_n) \in \mathcal{M}$  stands for the grand coalition in which each agent cooperates at its maximal activity level, whereas  $\mathbf{0} = (0, \dots, 0)$  stands for the empty coalition in which no agent cooperates. Observe that the set  $\mathcal{M}$  endowed with the usual binary relation  $\geq$  on  $\mathbb{N}^{|N|}$  forms a (complete) lattice  $(\mathcal{M}, \geq)$  where the greatest element is  $m$  and the least element is  $\mathbf{0}$ . For any two coalitions  $s, t \in \mathcal{M}$ ,  $s \vee t$  and  $s \wedge t$  denote their least upper bound and their greatest lower bound over  $\mathcal{M}$ , respectively. Given the grand coalition  $m$ , one sometimes denotes  $s \leq m$  instead of  $s \in \mathcal{M}$ . For  $S \subseteq N$ , define  $e^S = (1_S, 0_{N \setminus S}) \in \mathcal{M}_S \times \mathcal{M}_{N \setminus S}$  as the coalition in which only agents in  $S$  cooperate at activity level 1.

A (cooperative) multi-choice game on  $N$  and  $m$  is a pair  $(m, v)$ , where  $m \leq (K, \dots, K) \in \mathbb{N}^{|N|}$  is the vector describing the maximal activity level of each agent, and  $v : \mathcal{M} \rightarrow \mathbb{R}$  is a characteristic function. By convention, one sets  $v(\mathbf{0}) = 0$ . For each  $s \in \mathcal{M}$ , the real number  $v(s) \in \mathbb{R}$  specifies the worth that agents generate by cooperating at  $s$ . Denote by  $\mathcal{G}_m$  the set of all multi-choice games on  $N$  and  $m \leq K$ . Observe that the set of TU-games  $\mathcal{G}_N^{TU}$  on  $N$  can be viewed as the subset of multi-choice games on  $N$  and  $m$  where  $m = (1, \dots, 1)$ . The set of all multi-choice games on  $N$  is given by  $\mathcal{G} = \bigcup_{m \leq (K, \dots, K)} \mathcal{G}_m$ .<sup>5</sup> I sometimes consider the subset of multi-choice games on  $N$  in which each agent has the same maximal activity level, denoted by  $\bar{\mathcal{G}}$ . Below, I present several subsets of multi-choice games. Each subset generalizes the corresponding concept from TU-game to multi-choice games, therefore I retain the same name.<sup>6</sup>

A multi-choice game  $(m, v) \in \mathcal{G}$  is called:

<sup>4</sup>Originally, Hsiao and Raghavan [1992] refers to an activity level as the action of cooperating at a certain level. I use indifferently action and activity level since the two terms have the same meaning.

<sup>5</sup>Observe that  $(K, \dots, K) \in \mathbb{N}^{|N|}$  is an upper bound on the vector of maximal activity, where  $K \in \mathbb{N}$  is arbitrarily chosen.

<sup>6</sup>Calvo and Santos [2000] introduce another extension of the TU-games called multi-level cooperative games. The main difference with multi-choice games lies in the assumption that in a multi-choice game  $(m, v)$ ,  $\mathcal{M}$  is a comprehensive subset of  $\mathbb{N}^{|N|}$ , i.e., if  $s \in \mathcal{M}$  then for all  $\mathbf{0} \leq t \leq s$  it holds that  $t \in \mathcal{M}$ . This assumption is relaxed in the model of multi-level games.

Non-negative if,

$$\forall s \leq m, \quad v(s) \geq 0.$$

Monotonic if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M} : s \leq t, \quad v(s) \leq v(t).$$

Simple if,

$$\forall s \in \mathcal{M}, \quad v(s) \in \{0, 1\}.$$

Additive if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M} : s \wedge t = \mathbf{0}, \quad v(s \vee t) = v(s) + v(t).$$

Super-additive if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M} : s \wedge t = \mathbf{0}, \quad v(s \vee t) \geq v(s) + v(t).$$

Sub-additive if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M} : s \wedge t = \mathbf{0}, \quad v(s \vee t) \leq v(s) + v(t).$$

Super-modular (or Convex) if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M}, \quad v(s \vee t) + v(s \wedge t) \geq v(s) + v(t).$$

Sub-modular (or Concave) if,

$$\forall s \in \mathcal{M}, t \in \mathcal{M}, \quad v(s \vee t) + v(s \wedge t) \leq v(s) + v(t).$$

The *null game*,  $(m, v_0) \in \mathcal{G}$  is the multi-choice game such that for each  $s \in \mathcal{M}$ ,  $v_0(s) = 0$ .

Given a multi-choice game  $(m, v) \in \mathcal{G}$  and a coalition  $t \in M$ , the *sub-game* of  $(m, v)$  induced by  $t$  is the multi-choice game  $(t, v^t) \in \mathcal{G}$  such that  $v^t$  is the restriction of  $v$  to the subset of coalitions  $\mathcal{M}^t = \{s \in \mathcal{M} \mid \forall i \in N, s_i \leq t_i\}$ , such that

$$\forall s \in \mathcal{M}^t, \quad v^t(s) = v(s).$$

When no confusion arises, simply denote the sub-game  $(t, v^t)$  by  $(t, v)$ . It should be noted that the sub-game  $(t, v^t)$  describes the situation in which the agents' maximal activity reduces to  $t_i \leq m_i$ .

Consider  $m \leq (K, \dots, K)$ . For any two multi-choice games  $(m, v) \in \mathcal{G}_m$ ,  $(m, w) \in \mathcal{G}_m$  and any  $\alpha \in \mathbb{R}$ , the multi-choice game  $(\alpha v + w) \in \mathcal{G}_m$  is given by:

$$\forall s \in \mathcal{M}, \quad (\alpha v + w)(s) = \alpha v(s) + w(s).$$

Moreover, observe that  $\mathcal{M}$  is a finite set whose cardinal is  $|\mathcal{M}| = \prod_{i \in N} (m_i + 1)$ . Thus, for any multi-choice game  $(m, v) \in \mathcal{G}_m$  the characteristic function  $v$  is fully described by the vector  $(v(s))_{s \in \mathcal{M}} \in \mathbb{R}^{|\mathcal{M}|}$ . Since  $v(\mathbf{0}) = 0$ , any characteristic function  $v$  on  $\mathcal{M}$  can be generated by a linear combination of  $|\mathcal{M}| - 1$  linearly independent elements. Thus, the set  $\mathcal{G}_m$  is a linear subspace of  $\mathbb{R}^{|\mathcal{M}|}$  of dimension  $|\mathcal{M}| - 1$ . A first basis for  $\mathcal{G}_m$  is given by the analogue of Dirac TU-games for multi-choice games,  $\{\delta_s : \mathbf{0} < s \leq m\}$ , such that

$$\forall s \in \mathcal{M} \setminus \{\mathbf{0}\}, t \in \mathcal{M} \setminus \{\mathbf{0}\}, \quad \delta_s(t) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases} \quad (2.17)$$

For each  $(m, v) \in \mathcal{G}_m$ , the characteristic function  $v$  admits a unique linear decomposition in terms of Dirac games given by:

$$v = \sum_{\substack{s \in \mathcal{M}: \\ s \neq \mathbf{0}}} v(s) \delta_s. \quad (2.18)$$

A second basis for  $\mathcal{G}_m$  is given by an analogue of unanimity games in multi-choice games called *minimal (effort) games* (Hsiao and Raghavan [1993]),  $\{u_s : \mathbf{0} < s \leq m\}$ , such that

$$\forall s \in \mathcal{M} \setminus \{\mathbf{0}\}, t \in \mathcal{M} \setminus \{\mathbf{0}\}, \quad u_s(t) = \begin{cases} 1 & \text{if } t \geq s, \\ 0 & \text{otherwise.} \end{cases} \quad (2.19)$$

For each  $(m, v) \in \mathcal{G}_m$ , the characteristic function  $v$  admits a unique linear decomposition in terms of minimal games given by:

$$v = \sum_{\substack{s \in \mathcal{M}: \\ s \neq \mathbf{0}}} \Delta_s(v) u_s. \quad (2.20)$$

The real number  $\Delta_s(v)$ , called the *dividend* of  $s$ , is a straightforward generalization of the concept of dividend from TU-games to multi-choice games. The following definition arises from (2.5). For each coalition  $s \in \mathcal{M}$ , the dividend of  $s$  is computed recursively as follows:

$$\Delta_s(v) = v(s) - \sum_{t < s} \Delta_t(v). \quad (2.21)$$

Just as in TU-games, expression (2.21) allows to interpret the Harsanyi dividend of a coalition as the net surplus obtained by subtracting the surpluses generated by its smaller coalitions. An alternative definition of the dividend is provided in Hsiao and Raghavan [1992] as follows. For each coalition  $s \in \mathcal{M}$ , the dividend of  $s$  is given by:

$$\Delta_s(v) = \sum_{T \subseteq \mathcal{S}(s)} (-1)^{|T|} v(s - e^T), \quad (2.22)$$

where  $\mathcal{S}(s) = \{i \in N : s_i \neq 0\}$  denotes the (possibly empty) set of active agents in the coalition  $s$ . Given a multi-choice game  $(m, v) \in \mathcal{G}$  and its sub-game  $(t, v^t)$ , each dividend  $\Delta_s(v^t)$  in  $(t, v^t)$  coincides with the dividend  $\Delta_s(v)$  in  $(m, v)$ .

Since each agent has multiple actions when cooperating in a multiple-choice game, it is desirable, when evaluating an agent's performance, to distinguish between the agent's different actions. For this purpose, consider the set of pairs composed of an agent and one of its (positive) activity levels. Let  $(m, v) \in \mathcal{G}$  be a multi-choice game. Define  $M = \bigcup_{i \in N} (\{i\} \times M_i)$  and  $M^+ = \bigcup_{i \in N} (\{i\} \times M_i^+)$ . A pair  $(i, j) \in M$  represents an agent and one of its activity levels. The individual performance of an agent cooperating at a certain (non-null) activity level is assessed by its marginal contribution when playing that activity level. This marginal contribution is computed as the surplus generated in a coalition when the agent increase its activity level by one unit, all else being equal. Formally, let  $(m, v) \in \mathcal{G}$ ,  $(i, j) \in M^+$ ,

and  $s \in \mathcal{M}$  such that  $s_i = j - 1$ . The *marginal contribution* of  $(i, j)$  to  $s$  is given by:<sup>7</sup>

$$v(s + e^i) - v(s),$$

or equivalently:

$$v((s_{-i}, j)) - v((s_{-i}, j - 1)).$$

Two distinct pairs containing the same activity level  $(i, j) \in M^+$  and  $(i', j) \in M^+$  are called *equal pairs* if they have the same marginal contribution to coalitions, i.e.,

$$\forall s \in \mathcal{M} : s_i = s_{i'} = j - 1, \quad v(s + e^i) = v(s + e^{i'}). \quad (2.23)$$

A pair  $(i, j) \in M^+$  is called *level null* in  $(m, v)$  if it has a null marginal contribution to coalitions, i.e.,

$$\forall s \in \mathcal{M} : s_i = j - 1, \quad v(s + e_i) = v(s), \quad (2.24)$$

or equivalently:

$$\forall s_{-i} \in \mathcal{M}_{-i}, \quad v((s_{-i}, j)) = v((s_{-i}, j - 1)).$$

A pair  $(i, j) \in M^+$  is called *null* if agent  $i$  becomes unproductive beyond the activity level  $j$ , i.e.,

$$\forall j \leq l \leq m_i, \forall s \in \mathcal{M} : s_i = j - 1, \quad v(s + (l - j + 1)e^i) = v(s), \quad (2.25)$$

or equivalently:

$$\forall j \leq l \leq m_i, \forall s_{-i} \in \mathcal{M}_{-i}, \quad v((s_{-i}, l)) = v((s_{-i}, j - 1)).$$

Thus, if a pair  $(i, j) \in M^+$  is null in  $(m, v)$ , then each pair containing  $i$  and a greater activity level is also null. It is clear that any null pair is a level null pair, the converse is not true. A pair  $(i, j) \in M^+$  is *necessary* if each coalition in which  $i$  plays an activity level lower than  $j$  generates zero worth, i.e.,

$$\forall s \in \mathcal{M} : s_i < j, \quad v(s) = 0.$$

For  $(m, v) \in \mathcal{G}$  and  $i \in N$ , define the (possibly empty) set of necessary pairs containing agent  $i$ ,

$$Nc_i(m, v) = \{(i, j) \in M^+ : \forall s \in \mathcal{M} : s_i < j, \quad v(s) = 0\}.$$

### 2.3.2 Solutions for multi-choice games

The main objective in multi-choice games remains to determine reasonable ways of sharing the worth generated by cooperation. Since the agents have several activity levels, each agent should be aware of what it might obtain by cooperating at a certain level. Therefore, a solution allocates a payoff to each activity level of each agent. There exist two interpretations of a payoff in multi-choice game. First, [Hsiao and Raghavan \[1992\]](#) interpret a payoff as an agent's gain for a certain activity level. This interpretation can also be found in [Grabisch and Lange \[2007\]](#) and [Grabisch and Xie \[2007\]](#). A second interpretation, provided in [van den Nouweland et al. \[1995\]](#), considers the variation of gain when an agent increases its activity level from the

<sup>7</sup>One can imagine different extensions of the marginal contribution from TU-games to multi-choice games. For instance, one can measure the surplus generated when an agent increases its activity level from 0, i.e., for  $s \in \mathcal{M} : s_i = 0$ . An alternative definition of the marginal contribution is then given by:  $v(s + je_i) - v(s)$ , where  $s \in \mathcal{M}$  is such that  $s_i = 0$ .

level just below. I will mainly rely on this second interpretation, which is most often used in the literature. To avoid confusion, I distinguish both interpretations by referring to a payoff à la [Hsiao and Raghavan \[1992\]](#) as a *level payoff*, while a payoff à la [van den Nouweland \[1993\]](#) is referred to as a *payoff*.

Given a multi-choice game  $(m, v) \in \mathcal{G}$ , a *payoff vector* is a  $|M^+|$ -dimensional vector  $x \in \mathbb{R}^{|M^+|}$ , where  $x_{ij} \in \mathbb{R}$  is the payoff distributed to the pair  $(i, j) \in M^+$ . By convention, for each  $i \in N$  set  $x_{i0} = 0$ . Following the interpretation of [van den Nouweland \[1993\]](#),  $x_{ij} \in \mathbb{R}$  corresponds to a change in payoff to agent  $i$  corresponding to a change of activity level from  $j - 1$  to  $j$ . Given a payoff vector  $x \in \mathbb{R}^{|M^+|}$ , the corresponding *level payoff vector* (à la [Hsiao and Raghavan \[1992\]](#)),  $\bar{x} \in \mathbb{R}^{|M^+|}$ , is such that for each  $(i, j) \in M^+$ ,  $\bar{x}_{ij} = \sum_{l \leq j} x_{il}$ .<sup>8</sup> A payoff vector  $x \in \mathbb{R}^{|M^+|}$  is *efficient* if it fully allocates the worth of the grand coalition  $v(m)$  among all the agents' activity levels in the game, i.e.,  $\sum_{i \in N} \sum_{j \leq m_i} x_{ij} = v(m)$ . Since one assumes that each agent cooperates at its maximal activity level, one obtains  $\sum_{i \in N} \bar{x}_{im_i} = v(m)$ . A payoff vector is called *increase rational* if no agent can be better off when it increases its activity level from one unit, i.e., for each  $i \in N$  and each  $j \in M_i^+$ ,  $x_{ij} \geq v(je^i) - v((j-1)e^i)$ , or equivalently  $x_{ij} \geq v((0_{-i}, j)) - v((0_{-i}, j-1))$ . It is clear that increase rationality implies that an agent obtains a better level payoff than the worth it generates alone, i.e.,  $\bar{x}_{ij} \geq v(je^i)$ . A payoff vector is *coalitionally rational* if no coalition can be better off by itself, i.e., for each  $s \in \mathcal{M}$ ,  $\sum_{i \in N} \sum_{j \leq s_i} x_{ij} \geq v(s)$ . While coalitional rationality is a straightforward generalization from TU-games to multi-choice games, increase rationality generalizes the notion of individual rationality by relying on the interpretation of a payoff in multi-choice games.

A *set-valued solution* on  $\mathcal{G}$  is a mapping  $F : \mathcal{G} \rightrightarrows \mathbb{R}^{|M^+|}$  that assigns a (possibly empty) set of payoff vectors  $F(m, v) \subseteq \mathbb{R}^{|M^+|}$  to each game  $(m, v) \in \mathcal{G}$ . A *single-valued solution*, or a *value*, on  $\mathcal{G}$  is a function  $f : \mathcal{G} \rightarrow \mathbb{R}^{|M^+|}$ . A first set-valued solution concept aims at generalizing the *Imputation set* from TU-games to multi-choice games. This solution, introduced by [van den Nouweland et al. \[1995\]](#), refers to the (possibly empty) set of payoff vectors that are efficient and increase rational. Formally, the *Imputation set*,  $\mathfrak{I}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\mathfrak{I}(m, v) = \left\{ x \in \mathbb{R}^{|M^+|} : \begin{array}{l} \sum_{i \in N} \sum_{j \in M_i^+} x_{ij} = v(m), \\ \forall i \in N, \forall j \in M_i^+, x_{ij} \geq v(je^i) - v((j-1)e^i) \end{array} \right\}. \quad (2.26)$$

It can be shown that the Imputation set is non-empty if and only if  $\sum_{i \in N} v(m_i e^i) \leq v(m)$ . It appears that each solution presented in Section 2.1.2 has several reasonable extensions to multi-choice games. Below, I present some of these extensions on  $\mathcal{G}$  and elaborate on their properties.

### Extensions of the Core

As developed in Section 2.1.2, the stability principle underlying the Core makes this solution particularly appealing for several economic situations. Below, I present three extensions with a special attention to the last one. A first natural extension, called the *pre-Core*, is introduced by [Grabisch and Xie \[2007\]](#). This solution refers to the (possibly empty) set of efficient and coalitionally rational payoff vectors.

<sup>8</sup>It is clear that each payoff vector determines a unique level payoff vector.

Formally, the pre-Core,  $p\mathfrak{C}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$p\mathfrak{C}(m, v) = \left\{ x \in \mathbb{R}^{|M^+|} : \begin{array}{l} \sum_{i \in N} \sum_{j \in M_i^+} x_{ij} = v(m), \\ \forall s \in \mathcal{M}, \sum_{i \in N} \sum_{j \leq s_i} x_{ij} \geq v(s) \end{array} \right\}. \quad (2.27)$$

Whenever non-empty, the pre-Core may be an unbounded convex polyhedron. This point is the major drawback of this solution. It means that a pre-Core element may assign an arbitrarily large payoff to some agent's activity level.

A second extension of the core, which avoids this drawback, is introduced by [van den Nouweland et al. \[1995\]](#). This solution refines the Imputation set by imposing coalitional rationality. Formally, the Core à la [van den Nouweland et al. \[1995\]](#),  $\mathfrak{C}^{vN}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\mathfrak{C}^{vN}(m, v) = \left\{ x \in \mathbb{R}^{|M^+|} : \begin{array}{l} \sum_{i \in N} \sum_{j \in M_i^+} x_{ij} = v(m), \\ \forall i \in N, \forall j \in M_i^+, x_{ij} \geq v(je^i) - v((j-1)e^i), \\ \forall s \in \mathcal{M}, \sum_{i \in N} \sum_{j \leq s_i} x_{ij} \geq v(s) \end{array} \right\}. \quad (2.28)$$

Observe that, the Core à la [van den Nouweland et al. \[1995\]](#) is closed thanks to the specification of increase rationality. [van den Nouweland et al. \[1995\]](#) provide a necessary and sufficient condition that ensures the non-emptiness of their Core.<sup>9</sup>

A third extension of the core, introduced by [Grabisch and Xie \[2007\]](#), also avoids the drawback of the pre-Core by considering a normalization (provided by the efficiency condition) at each activity level. Formally, the Core à la [Grabisch and Xie \[2007\]](#),  $\mathfrak{C}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\mathfrak{C}(m, v) = \left\{ x \in \mathbb{R}^{|M^+|} : \begin{array}{l} \forall h \leq \max_{i \in N} m_i, \sum_{i \in N} \sum_{j \leq h} x_{ij} = v((h \wedge m_i)_{i \in N}), \\ \forall s \in \mathcal{M}, \sum_{i \in N} \sum_{j \leq s_i} x_{ij} \geq v(s) \end{array} \right\}. \quad (2.29)$$

Despite its technical use, the normalization at each activity level also has an appealing interpretation. Assume that all agents agree on forming a coalition in which they all play the same activity level, for instance  $h \leq \max_{i \in N} m_i$ . The agents that are not able to cooperate at this level play their maximal activity level. Such coalition is called a *h-synchronized* coalition. The normalization requires that the worth of each synchronized coalition is fully allocated among the agents' activity level needed to reach this coalition. This condition drives the agents to align on their activity level as far as possible. A necessary and sufficient condition that ensures the non-emptiness of the Core relies on the extension of the concept of balancedness from TU-games to multi-choice games. Let  $\mathcal{M} \setminus \{0\}$  be the set of coalitions and let  $\mathcal{B} \subseteq \mathcal{M} \setminus \{0\}$  be a collection of coalitions. The collection  $\mathcal{B}$  is called a *balanced collection* if, for each  $s \in \mathcal{B}$  there exists  $\mu_s \in \mathbb{R}_+$  such that

$$\begin{aligned} \forall i \in N, \quad & \sum_{\substack{s \in \mathcal{B}: \\ s_i = m_i}} \mu_s = (\max_{k \in N} m_k) - m_i + 1, \\ \forall i \in N, \forall j < m_i \quad & \sum_{\substack{s \in \mathcal{B}: \\ s_i = j}} \mu_s = 1. \end{aligned}$$

A natural extension of the standard interpretation of a balanced collection can be given as follows. Suppose there are  $\max_{k \in N} m_k$  working days, where each day is

<sup>9</sup>This condition extends the Bondareva-Shapley Theorem (see Theorem 2.1.1). I refer the reader to [van den Nouweland et al. \[1995\]](#) for further details.

dedicated to one activity level  $j = 1, \dots, \max_{i \in N} m_i$ . For each day  $j < m_i$ ,  $\sum_{s \in \mathcal{B}: s_i = j} \mu_s = 1$  holds. Agent  $i$  allocates one unit of time in coalitions  $s \in \mathcal{M} \setminus \{\mathbf{0}\}$  such that  $s_i = j$ . From day  $j = m_i$ ,  $\sum_{s \in \mathcal{B}: s_i = m_i} \mu_s = (\max_{k \in N} m_k) - m_i + 1$  holds and it remains  $(\max_{k \in N} m_k) - m_i + 1$  days to work. Then, agent  $i$  allocates the rest of its time in the coalitions  $s \in \mathcal{M} \setminus \{\mathbf{0}\}$  such that  $s_i = m_i$ . A collection of elements of  $\mathcal{M} \setminus \{\mathbf{0}\}$  is balanced if each agent  $i \in N$  fully allocates these  $\max_i m_i$  units of time as described. A multi-choice game  $(m, v) \in \mathcal{G}$  is called *balanced* if for each balanced collection  $\mathcal{B} \subseteq \mathcal{M} \setminus \{\mathbf{0}\}$  it holds that:

$$\sum_{s \in \mathcal{B}} \mu_s v(s) \leq \sum_{h=1}^{\max_{i \in N} m_i} v((h \wedge m_i)_{i \in N}). \quad (2.30)$$

**Theorem 2.3.1 (Grabisch and Xie [2007]).** *A multi-choice game  $(m, v) \in \mathcal{G}$  has a non-empty Core if and only if it is balanced.*

The relations between the three above extensions of the Core can be described as follows.<sup>10</sup> It can be shown that the intersection of the Imputation set with the Core à la Grabisch and Xie [2007] is included in the Core à la van den Nouweland et al. [1995], which is itself equal to the intersection between the Imputation set and the pre-Core:

$$\mathfrak{C} \cap \mathfrak{I} \subseteq \mathfrak{C}^{vN} = p\mathfrak{C} \cap \mathfrak{I}.$$

### Extensions of the Weber set

Two extensions of the Weber set have been proposed in the literature. The first one, the *pre-Weber set*, is a natural extension introduced by van den Nouweland et al. [1995].<sup>11</sup> This solution relies on the concepts of *admissible orders* and the corresponding *marginal vectors* described as follows. Assume that the grand coalition  $m \in \mathcal{M}$  forms step by step starting from the empty coalition  $\mathbf{0}$ . At each step, one agent increases its activity level by one unit, and demands the marginal contribution generated at this level. Given an admissible order, the payoff vector resulting from this process is called the *marginal vector*. Formally, let  $m \leq (K, \dots, K)$  be the vector describing the agents' maximal activity levels, and let  $M^+$  be the set of pairs. An *admissible order* on  $M^+$  is a bijection  $\sigma : M^+ \rightarrow \{1, \dots, \sum_{i \in N} m_i\}$  such that for each  $i \in N$  and  $j \in M_i^+$ ,  $\sigma((i, j)) < \sigma((i, j+1))$ . For each  $(i, j) \in M^+$ ,  $\sigma(i, j)$  denotes the pair  $(i, j)$ 's position in the order  $\sigma$ . Denote by  $O_m$  the set of all admissible orders on  $M^+$ . For each  $\sigma \in O_m$  and each  $k \in \{1, \dots, \sum_{i \in N} m_i\}$ , the coalition  $s^{\sigma, k}$  is the coalition formed by the agents that reached a certain activity level before step  $k$  in  $\sigma$ , i.e.,

$$\forall i \in N, \quad s_i^{\sigma, k} = \max\{j \in M_i^+ : \sigma(i, j) \leq k\} \cup \{0\}.$$

Given a multi-choice game  $(m, v) \in \mathcal{G}$  and an admissible order  $\sigma \in O_m$ , the marginal vector,  $m^\sigma(m, v) \in \mathbb{R}^{|M^+|}$  is defined as:

$$\forall (i, j) \in M^+, \quad m_{ij}^\sigma(m, v) = v(s^{\sigma, \sigma(i, j)}) - v(s^{\sigma, \sigma(i, j)-1}). \quad (2.31)$$

<sup>10</sup> Another extension of the Core, called the unit-level Core, can be found in Hwang and Liao [2010].

<sup>11</sup> Originally, van den Nouweland et al. [1995] called this solution the Weber set. The term pre-Weber set is introduced by Grabisch and Xie [2007]. I retain this term to distinguish it with the second extension presented below.



For each multi-choice game  $(m, v) \in \mathcal{G}$ , the pre-Weber set,  $p\mathfrak{W}$ , is defined as the convex hull of all marginal vectors, i.e.,

$$p\mathfrak{W}(m, v) = \text{co}\{m^\sigma(m, v) : \sigma \in O_m\}. \quad (2.32)$$

A second extension of the Weber set is introduced by [Grabisch and Xie \[2007\]](#). This solution relies on the concept of *restricted orders* and the corresponding marginal vectors. A restricted order is an admissible order such that an agent can increase its activity level provided that all the agents that are able to play the level just below have reached it. Formally, let  $m \leq (K, \dots, K)$  be the vector describing the agents' maximal activity levels, and let  $M^+$  be the set of pairs. A *restricted order* on  $M^+$  is a bijection  $\bar{\sigma} : M^+ \rightarrow \{1, \dots, \sum_{i \in N} m_i\}$  such that for each  $(i, j) \in M^+$  and  $(i', j') \in M^+$ ,  $j' < j$  implies  $\bar{\sigma}(i', j') < \bar{\sigma}(i, j)$ . Denote by  $\bar{O}_m$  the set of all restricted orders on  $M^+$ . For each  $\bar{\sigma} \in \bar{O}_m$  and each  $k \in \{1, \dots, \sum_{i \in N} m_i\}$ , the coalition  $s^{\bar{\sigma}, k}$  is the coalition formed by the agents that reached a certain activity level before step  $k$  in  $\bar{\sigma}$ , i.e.,

$$\forall i \in N, \quad s_i^{\bar{\sigma}, k} = \max\{j \in M_i^+ : \bar{\sigma}(i, j) \leq k\} \cup \{0\}.$$

I use the convention  $s^{\bar{\sigma}, 0} = \mathbf{0}$ . Given a multi-choice game  $(m, v) \in \mathcal{G}$  and an admissible order  $\bar{\sigma} \in \bar{O}_m$ , the marginal vector,  $m^{\bar{\sigma}}(m, v) \in \mathbb{R}^{|M^+|}$  is defined as:

$$\forall (i, j) \in M^+, \quad m_{ij}^{\bar{\sigma}}(m, v) = v(s^{\bar{\sigma}, \bar{\sigma}(i, j)}) - v(s^{\bar{\sigma}, \bar{\sigma}(i, j)-1}). \quad (2.33)$$

For each multi-choice game  $(m, v) \in \mathcal{G}$ , the Weber set,  $\mathfrak{W}$ , is defined as the convex hull of all marginal vectors, i.e.,

$$\mathfrak{W}(m, v) = \text{co}\{m^{\bar{\sigma}}(m, v) : \bar{\sigma} \in \bar{O}_m\}. \quad (2.34)$$

[Grabisch and Xie \[2007\]](#) extend Theorem 2.1.3 ([Shapley \[1971\]](#), [Ichiishi \[1981\]](#)), showing that the Core coincides with the Weber set on the class of super-modular multi-choice games. Such result does no longer holds if one invokes the Core à la [van den Nouweland et al. \[1995\]](#).

**Theorem 2.3.2 ([Grabisch and Xie \[2007\]](#)).** *For each super-modular multi-choice game  $(m, v) \in \mathcal{G}$ ,*

$$\mathfrak{C}(m, v) = \mathfrak{W}(m, v).$$

Observe that there is no similar coincidence when one considers the Core à la [van den Nouweland et al. \[1995\]](#) and the pre-Weber set.

### Extensions of the Shapley value

Regarding single-valued solutions, there is a significant part of the literature that focuses on extensions of the Shapley value. To paraphrase [van den Nouweland \[1993\]](#), “there is more than one reasonable extension of the Shapley value to multi-choice games”. Below, I present four commonly studied extensions. Each of these extensions has desirable properties that I will discuss in the next section.

While considering the process defining the marginal vectors (see Section 2.1.2, definition (2.9)), one of the most natural extension of the Shapley value to multi-choice games is probably the one introduced by [van den Nouweland et al. \[1995\]](#), denoted by  $\phi^{v^N}$ . Consider a multi-choice game, the set of admissible orders in this game, and the process that defines the marginal vectors (see (2.31)). Assume that each admissible order can occur with the same probability. The value  $\phi^{v^N}$  assigns



to each agent's activity level (or to each pair composed of an agent and one of its activity level) its expected payoff with respect to the probability on the admissible orders.

Formally, the value  $\phi^{vN}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{vN}(m, v) = \frac{\prod_{i \in N} (m_i!)}{(\sum_{i \in N} m_i)!} \sum_{\sigma \in O_m} m_{ij}^{\sigma}(m, v). \quad (2.35)$$

Below, I present a second mathematical expression of  $\phi^{vN}$ . According to this expression,  $\phi^{vN}$  assigns to each agent's activity level a weighted average of its marginal contributions to the coalitions this agent joins at this activity level. In order to define this alternative expression, one needs the following definition. Given a multi-choice game  $(m, v) \in \mathcal{G}$ , for each  $s \in \mathcal{M}$  and each  $(i, j) \in M^+$  such that  $j \leq s_i$ , the *hierarchical strength*  $h_{ij}^{vN}(s)$  is defined as:

$$h_{ij}^{vN}(s) = \frac{\prod_{k \in N} (m_k!)}{(\sum_{k \in N} m_k)!} |\{ \sigma \in O_m : \sigma(i, j) = \max_{\substack{(k, l) \in M^+ \\ l \leq s_k}} \sigma(k, l) \}|. \quad (2.36)$$

One interpretation of  $h_{ij}^{vN}(s)$  is as follows. Consider the notion of admissible orders introduced in the previous section. The weight  $h_{ij}^{vN}(s)$  is the average number of admissible orders  $\sigma \in O_m$  such that the pair  $(i, j) \in M^+$  has the highest position in the order  $\sigma$  among all the pairs  $(k, l) \in M^+$  with  $l \leq s_k$ . Observe that this number depends on the total number of activity levels leading to the coalition  $s$ , i.e.,  $\sum_{k \in N} s_k$  but also on the total number of activity levels of an agent.

Formally, the value  $\phi^{vN}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{vN}(m, v) = \sum_{\substack{s \leq m \\ s_i = j}} h_{ij}^{vN}(s) \left[ v(s) - v(s - e^i) \right]. \quad (2.37)$$

**Remark 1.** One interesting property of the hierarchical strength arises when one considers the decisive activity levels in a minimal effort game.<sup>12</sup> In this case, [Faigle and Kern \[1992\]](#) show that the hierarchical strength depends monotonically on the activity level. Let  $(m, u_s) \in \mathcal{G}$  be a minimal effort game with  $s \in \mathcal{M}$  and  $i, i' \in \mathcal{S}(s)$ . It holds that

$$\begin{aligned} h_{is_i}^{vN}(s) &= h_{i's_{i'}}^{vN}(s) & \text{if } s_i = s_{i'}, \\ h_{is_i}^{vN}(s) &> h_{i's_{i'}}^{vN}(s) & \text{if } s_i > s_{i'}. \end{aligned}$$

**Remark 2.** An alternative interpretation of the weight  $h_{ij}^{vN}(s)$  is as follows. Consider a pair  $(i, j) \in M^+$  and a coalition  $s \in \mathcal{M}$  such that  $s_i = j$ . Let  $I(s) := \sum_{k \in N} s_k$  be the “total intensity” in coalition  $s$ . Then,  $h_{ij}^{vN}(s)$  corresponds to the probability that  $i$  joins a coalition with the total intensity  $I(s)$  by playing its activity level  $j$ . Whenever for each agent  $i \in N$ ,  $m_i = 1$ ,  $h_{ij}^{vN}(s)$  is the Shapley coefficient (see (2.11)).

Three other extensions of the Shapley value stand out in the literature. I first present the value introduced by [Peters and Zank \[2005\]](#). The authors consider the subclass of multi-choice games in which each agent has the same activity level. They

<sup>12</sup>By decisive, it is meant a necessary pair such that the player featuring this pair becomes null when playing a higher activity level.

propose a value that shares the dividend of each coalition equally among the active agents. Below, I consider a definition of the extension proposed by [Peters and Zank \[2005\]](#) on  $\mathcal{G}$ .

Formally, the value  $\phi^{PZ}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{PZ}(m, v) = \sum_{\substack{s \leq m \\ s_i = j}} \frac{\Delta_s(v)}{|\mathcal{S}(s)|}. \quad (2.38)$$

An alternative expression of this extension relies on the marginal contributions. In order to introduce this second expression, a definition is needed. Consider any multi-choice game  $(m, v) \in \mathcal{G}$ , and any pair  $(i, j) \in M^+$ . Define the subset of coalitions  $C_{ij}^m \subseteq \mathcal{M}$  in which  $i$  is playing at the level  $j - 1$  and the remaining agents are either active at their maximal activity or inactive, i.e.,

$$C_{ij}^m = \{s \in \mathcal{M} : s_i = j - 1, \forall k \neq i, s_k \in \{0, m_k\}\}.$$

The value  $\phi^{PZ}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{PZ}(m, v) = \sum_{s \in C_{ij}^m} \gamma_s \left[ v(s + e^i) - v(s) \right], \quad (2.39)$$

where

$$\gamma_s = \frac{(|\{k \in \mathcal{S}(s) : k \neq i\}|)!(|N| - \mathcal{S}(s) - 1)!}{|N|!}.$$

Thus,  $\phi_{ij}^{PZ}$  can be interpreted as a weighted average of the marginal contributions to the coalitions in  $C_{ij}^m$ . By considering only coalitions in  $C_{ij}^m$ , the value  $\phi^{PZ}$  considers less information than  $\phi^{vN}$ .

Another extension is due to [Hsiao and Raghavan \[1992, 1993\]](#). The authors also consider the subclass of multi-choice games in which each agent has the same maximal activity level. Then, they consider an exogenous weight system on the activity levels. This value shares the dividend of a coalition proportionally to these weights. Below, I consider a definition of the value introduced by [Hsiao and Raghavan \[1992\]](#) on  $\mathcal{G}$ .

Let  $m \leq (K, \dots, K)$  be a vector of maximal activity levels and  $(w_j)_{j=1}^{\max_{i \in N} m_i} \in \mathbb{R}^{\max_{i \in N} m_i + 1}$  be a weight system on the activity levels with  $w_0 = 0$ . The value  $\phi^{HR, w}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{HR, w}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ s_i = j}} \frac{w_j}{\sum_{k \in N} w_{s_k}} \Delta_s(v). \quad (2.40)$$

Whenever  $w_j = w_{j'}$  for each  $j \leq \max_{i \in N} m_i$  and  $j' \leq \max_{i \in N} m_i$ ,  $\phi^{HR, w}$  coincides with  $\phi^{PZ}$ . Thus, the value  $\phi^{HR, w}$  generalizes the value  $\phi^{PZ}$ .

A second expression of this extension relies on the marginal contributions. Given a coalition  $s \in \mathcal{M}$  and an agent  $i \in \mathcal{S}(s)$ , define the set of agents (different from  $i$ ) that are not playing their maximal activity level in  $s$  as:

$$L_i^{HR}(s) = \{k \in N : s_k \neq m_k \text{ and } k \neq i\}.$$

Given a weight system on the activity levels  $(w_j)_{j=1}^{\max_{i \in N} m_i} \in \mathbb{R}^{\max_{i \in N} m_i + 1}$ , the value  $\phi^{HR,w}$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+,$$

$$\phi_{ij}^{HR,w}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ s_i = j}} \left[ \sum_{T \subseteq L_i^{HR}(s)} (-1)^{|T|} \frac{w_j}{\sum_{k \in N} w_{s_k} + \sum_{h \in T} w_{s_h}} \right] \left[ v(s + e^i) - v(s) \right]. \quad (2.41)$$

**Remark 3.** It should be noted that the weights  $(w_j)_{j=1}^{\max_{i \in N} m_i} \in \mathbb{R}^{\max_{i \in N} m_i + 1}$  are independent of the agents. When considering a multi-choice game in which each agent  $i \in N$  has a maximal activity level  $m_i = 1$ , the value coincides with the Shapley value (see Section 2.1.2, definition (2.12)). This constitutes a main departure from the idea of the weighted Shapley values introduced by Shapley [1953] and studied in Kalai and Samet [1987].

The last extension of the Shapley value that I present is proposed by Derks and Peters [1993] and further studied by Klijn et al. [1999]. This value strongly relies on the linearly ordered sets of activity levels. Consider any agent and one of its activity level  $j \in M_i$ . Derks and Peters [1993] argue that this agent need to pass each lower activity level in order to reach its level  $j$ . Based on this interpretation, the value  $\phi^{DP}$  shares the dividend of each coalition among the agent's activity levels that have lead to this coalition.

Formally, the value  $\phi^{DP}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{DP}(m, v) = \sum_{\substack{s \leq m \\ s_i \geq j}} \frac{\Delta_s(v)}{\sum_{i \in N} s_i}. \quad (2.42)$$

A second expression of the value  $\phi^{DP}$  relies on an alternative definition of the marginal vectors in multi-choice games. Consider any multi-choice game  $(m, v) \in \mathcal{G}$ . An order on  $M^+$  is a bijection  $\pi : M^+ \rightarrow \{1, \dots, \sum_{i \in N} m_i\}$ . Let  $O$  be the set of all orders on  $M^+$ . Denote by  $S^{\pi, k} := \pi^{-1}(\{1, \dots, k\}) \subseteq M^+$  the subset of pairs present after  $k$  steps according to  $\pi$ . Moreover, define the map  $\rho$  that assigns to each subset  $S \subseteq M^+$  the maximal feasible coalition  $\rho(S) = (t_1, \dots, t_n)$  where

$$t_i = \begin{cases} \max\{k \in M_i^+ : (i, 1), \dots, (i, k)\} & \text{if } (i, 1) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The value  $\phi^{DP}$  is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}^{DP} = \frac{1}{(\sum_{i \in N} m_i)!} \sum_{\pi \in O} v(\rho(S^{\pi, \pi(i, j)})) - v(\rho(S^{\pi, \pi(i, j) - 1})). \quad (2.43)$$

### 2.3.3 Axiomatic foundations of extensions of the Shapley value on $\mathcal{G}$

In this section, I present several axioms for solutions on  $\mathcal{G}$ . I discuss the relations between these axioms and some axiomatic characterizations resulting from their combinations. While some axioms can be seen as generalizing some axioms from TU-games to multi-choice games, others do not have their counterpart in TU-games.

### General properties

A value  $f$  on  $\mathcal{G}$  satisfies:

*Carrier* if, for each  $(m, v) \in \mathcal{G}$  and each  $t \in \mathcal{M}$  carrier, i.e., for each  $s \in \mathcal{M}$ :  $v(t) = v(s \wedge t)$ ,

$$\sum_{i \in \mathcal{S}(t)} \sum_{j=1}^{t_i} f_{ij}(m, v) = v(t).$$

*Efficiency* if, for each  $(m, v) \in \mathcal{G}$ ,

$$\sum_{i \in N} \sum_{j=1}^{m_i} f_{ij}(m, v) = v(m).$$

*Additivity* if, for each  $(m, v), (m, w) \in \mathcal{G}$ ,

$$f(m, v + w) = f(m, v) + f(m, w).$$

*Linearity* if, for each  $(m, v), (m, w) \in \mathcal{G}$  and each  $\alpha \in \mathbb{R}$ ,

$$f(m, \alpha v + w) = \alpha f(m, v) + f(m, w).$$

*The null game axiom* if, for each  $(m, v_0) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v_0) = 0.$$

The five above axioms are straightforward generalizations of the same named axioms from TU-games to multi-choice games (see Section 2.1.3). Observe that Efficiency relies on the specific definition of a payoff vector in multi-choice games. In this sense, the worth of the grand coalition,  $v(m)$ , is fully allocated among all the agents' activity levels. Obviously, Carrier implies Efficiency.

### Equal treatment principles

The next axioms are related to the basic equal treatment principle. First, I present an axiom that considers the subset of multi-choice games in which each agent has the same maximal activity level, i.e., on  $\overline{\mathcal{G}}$ . The other axioms presented consider the set of all multi-choice games.

Given a multi-choice game  $(m, v) \in \overline{\mathcal{G}}$ , an order on the agent set  $\sigma \in O_N$ , and a multi-choice coalition  $s \in \mathcal{M}$  define  $\sigma s \in \mathcal{M}$  as:  $\sigma s_{\sigma(i)} = s_i$  for each  $i \in N$ , and  $\sigma v(\sigma s) = v(s)$  for each  $s \in \mathcal{M}$ . A value  $f$  on  $\overline{\mathcal{G}}$  satisfies:

*Anonymity* if, for each  $(m, v) \in \overline{\mathcal{G}}$  and each  $\sigma \in O_N$ ,

$$\forall i \in N, \quad f_{ij}(m, \sigma v) = f_{\sigma(i)j}(m, v).$$

*Hierarchical strength* if, for each  $(m, \beta u_s) \in \mathcal{G}$  multiple of a minimal effort game, with  $\beta \in \mathbb{R}$ ,

$$\forall (i, j), (i', j') \in M^+, \quad h_{ij}^{vN}(s) f_{ij}(m, \beta u_s) = h_{i'j'}^{vN}(s) f_{i'j'}(m, \beta u_s).$$

*Weight axiom* if, for a given weight system  $(w_j)_{j=1}^{\max_{i \in N} m_i} \in \mathbb{R}^{\max_{i \in N} m_i}$ , for each  $(m, \beta u_s) \in \mathcal{G}$  multiple of a minimal effort game, with  $\beta \in \mathbb{R}$ ,

$$w_{s_i} f_{ij}(m, \beta u_s) = w_{s_i} f_{i'j'}(m, \beta u_s).$$

Given a multi-choice game  $(m, v) \in \mathcal{G}$ , a coalition  $t \in \mathcal{M}$  is called a *veto coalition* if,  $v(s) = v(t)$  for each  $s \geq t$ , and  $v(s) = 0$  otherwise. A value  $f$  on  $\mathcal{G}$  satisfies: *Hierarchical symmetry* if, for each  $(m, v) \in \mathcal{G}$ , each veto coalition  $t \in \mathcal{M}$ , and each two distinct pairs  $(i, t_i), (i', t_{i'}) \in M^+$ ,

$$\sum_{\substack{s \geq t \\ s_{i'} = t_{i'}}} h_{i't_{i'}}^{vN}(s) f_{it_i}(m, v) = \sum_{\substack{s \geq t \\ s_i = t_i}} h_{it_i}^{vN}(s) f_{i't_{i'}}(m, v).$$

The *necessary pair axiom* if, for each  $(m, v) \in \mathcal{G}$  and each two distinct necessary pairs  $(i, j), (i', j') \in M^+$ ,

$$f_{ij}(m, v) = f_{i'j'}(m, v).$$

Consider  $i \in N$ , and two of its activity levels  $j, j' \in M_i^+$ . The pairs  $(i, j)(i, j')$  are called *level symmetric* if, for each  $s, s' \in \mathcal{M}$  such that  $s_i = j - 1, s'_i = j' - 1, s_k = s'_k$  for  $k \neq i$ ,

$$v(s + e^i) - v(s) = v(s' + e^i) - v(s').$$

A value  $f$  on  $\mathcal{G}$  satisfies:

*Level symmetry* if, for each  $(m, v) \in \mathcal{G}$ , for each  $i \in N$  and  $j, j' \in M_i^+$  such that  $(i, j), (i, j')$  are level symmetric,

$$f_{ij}(m, v) = f_{ij'}(m, v).$$

Anonymity, introduced by Peters and Zank [2005], generalizes the same named axiom from TU-games to multi-choice games. It states that a solution should be invariant to the agent's renaming. However, such renaming would make sense only if all the agents have the same maximal activity level, i.e., one have to restrict the domain to  $\bar{\mathcal{G}}$ . Hierarchical strength, introduced by van den Nouweland [1993], asserts that a pair's payoff depends only on its hierarchical strength in minimal games. Whenever two pairs have the same hierarchical strength they should obtain the same payoff. This idea can be found in the same named axiom introduced by Faigle and Kern [1992] in the context of TU games with hierarchical constraints. The weight axiom, introduced by Hsiao and Raghavan [1993] relies on a close idea. In this case, the payoff of the pairs depend on the exogenous weight system  $w$ . The necessary pair axiom, introduced by Klijn et al. [1999], requires that two necessary pairs have the same payoff. If one considers a minimal game  $(m, u_s) \in \mathcal{G}$ , any pair  $(i, j) \in M^+$  such that  $j \leq s_i$  is a necessary pair. The necessary pair axiom implies that each necessary pair that leads to the coalition  $s$  are treated equally. Level symmetry, introduced by Peters and Zank [2005], compares two pairs featuring the same agent. It requires that two activity levels of an agent that have the same impact on the coalitions, i.e., the same marginal contributions, should have the same payoff.

### Fairness principles related to null contributions

Next, I present three axioms related to fairness requirements. Each axiom relies on the individual performance of an agent at a certain activity level. Specifically, two axioms are concerned with level null pairs, and null pairs. A value  $f$  on  $\mathcal{G}$  satisfies: *Minimal effort axiom* if, for each  $(m, v) \in \mathcal{G}$ , each  $t \in \mathcal{M}$  such that  $v(s) = 0$  for each  $s \not\geq t$ ,

$$\forall (i, j) \in M^+ : j < t_i, \quad f_{ij}(m, v) = 0.$$

*Zero contributions* if, for each  $(m, v) \in \mathcal{G}$ , each level null pair  $(i, j) \in M^+$ ,

$$f_{ij}(m, v) = 0.$$

*The null pair axiom* if, for each  $(m, v) \in \mathcal{G}$ , each null pair  $(i, j) \in M^+$ ,

$$f_{ij}(m, v) = 0.$$

The minimal effort axiom, introduced by Hsiao and Raghavan [1993], considers multi-choice games that require a minimal exertion from agents to generate some worth. Then, an agent that does not meet this required activity level should receive a null payoff. Zero contributions, introduced by Peters and Zank [2005], requires that an agent which does not contribute at a certain activity level receives a null payoff for this level. Thus, a pair does not benefit from more than its marginal contributions. The null pair axiom, introduced by Klijn et al. [1999], weakens Zero contributions by strengthening its hypothesis. Let  $(i, j) \in M^+$  be a pair. Instead of considering only the performance of agent  $i$ 's activity level  $j$ , it considers the performance of each of its activity levels above  $j$ . Therefore, such condition leaves room for an agent's activity level to benefit from the contributions of that agent's higher activity levels. It should be noted that the combination of Efficiency and the null pair axiom implies Carrier.

### Monotonicity principle

Next, I present two axioms that rely on the marginal contributions of an agent's activity level. A value  $f$  on  $\mathcal{G}$  satisfies:

*Strong monotonicity* if, for each  $(m, v), (m, w) \in \mathcal{G}$  and each  $(i, j) \in M^+$  such that for each  $s \in \mathcal{M}$  where  $s_i = j - 1$ :  $v(s + e^i) - v(s) \geq w(s + e^i) - w(s)$ ,

$$f_{ij}(m, v) \geq f_{ij}(m, w).$$

*Marginality* if for each  $(m, v), (m, w) \in \mathcal{G}$  and each  $(i, j) \in M^+$  such that for each  $s \in \mathcal{M}$  where  $s_i = j - 1$ :  $v(s + e^i) - v(s) = w(s + e^i) - w(s)$ ,

$$f_{ij}(m, v) = f_{ij}(m, w).$$

Strong monotonicity, introduced by van den Nouweland [1993], generalizes the same named axiom from TU-game to multi-choice games. Obviously, Strong monotonicity implies Marginality, introduced by Peters and Zank [2005]. It can be shown that the combination of Efficiency and Marginality implies Zero contributions.

### Fairness and balanced contribution principles

The principle of balanced contributions has been extensively studied in the multi-choice games literature. Because an agent has several activity levels, it can influence the payoff of another agent from different viewpoints, which gives rise to several axioms that are briefly discussed below. A value  $f$  on  $\mathcal{G}$  satisfies:

*Upper balanced contributions* if, for each  $(m, v) \in \mathcal{G}$ , and each distinct pairs  $(i, m_i), (i', m_{i'}) \in M^+$ ,

$$f_{im_i}(m, v) - f_{im_i}(m - e^{i'}, v) = f_{i'm_{i'}}(m, v) - f_{i'm_{i'}}(m - e^i, v).$$

*Lower balanced contributions* if, for each  $(m, v) \in \mathcal{G}$ , and each distinct pairs  $(i, 1), (i', 1) \in M^+$ ,

$$f_{i1}(m, v) - f_{i1}(m - m_{i'}e^{i'}, v) = f_{i'1}(m, v) - f_{i'1}(m - m_i e^i, v).$$

*General balanced contributions* if, for each  $(m, v) \in \mathcal{G}$  and each distinct pairs  $(i, j), (i', j') \in M^+$ ,

$$f_{ij}(m, v) - f_{ij}(m - (m_{i'} - j' + 1)e^{i'}, v) = f_{i'j'}(m, v) - f_{i'j'}(m - (m_i - j + 1)e^i, v).$$

*Equal loss axiom* if, for each  $(m, v) \in \mathcal{G}$ , each agent  $i \in N$ ,

$$\forall j < m_i, \quad f_{ij}(m, v) - f_{ij}(m - e^i, v) = f_{im_i}(m, v).$$

Recall that the balanced contributions principle states that any two agents should have the same claim on each other payoff. Although each above axioms relies on this principle, each one considers a different claim. Upper balanced contributions states that two agents should have the same claim on each other at their maximum activity level. Lower balanced contributions consider the impact of an agent's maximal activity level on the payoff of another agent's first activity level. General balanced contributions weakens this requirement by considering an equal claim for any activity level, say  $j$ , on another agent's activity level,  $j'$ . Such claim is measured when one removes all activity levels above  $j$ . Equal loss axiom focuses on a given agent. It implies that the presence of the maximal activity level of agent influences its lower activity level's payoff in the same manner. Obviously, General balanced contributions implies Upper balanced contributions and Lower balanced contributions.

### Independence principle

A value  $f$  on  $\mathcal{G}$  satisfies:

*Independence of the maximal activity level* if, for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = f_{ij}(m - e^i, v).$$

This axiom, introduced independently by [Hwang and Liao \[2009\]](#) and [Béal et al. \[2012\]](#), implies that the variation of an agent's maximal activity level should have no impact on the payoff of its remaining activity levels.

Based on the above axioms, I present some characterization results that can be found in the literature on multi-choice games. The first axiomatic characterization is due to [van den Nouweland \[1993\]](#).

**Theorem 2.3.3 ([van den Nouweland \[1993\]](#)).** *A value  $f$  on  $\mathcal{G}$  satisfies Carrier, Additivity, and Hierarchical strength if and only if  $f = \phi^{vN}$ .*

Recently, [Tang et al. \[2019\]](#) provide a second characterization of  $\phi^{vN}$  by replacing Carrier by Efficiency and the null pair axiom, and by replacing Hierarchical strength by Hierarchical symmetry.

**Theorem 2.3.4 ([Tang et al. \[2019\]](#)).** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, Additivity, the null pair axiom and Hierarchical symmetry if and only if  $f = \phi^{vN}$ .*

[Peters and Zank \[2005\]](#) provides an axiomatic characterization of the value  $\phi^{PZ}$  mainly inspired by the characterization of the Shapley value for TU-games proposed by [Shapley \[1953\]](#) and [Shubik \[1962\]](#). However, this characterization holds on the



subset of multi-choice games in which each agent has the same maximal activity level.

**Theorem 2.3.5 (Peters and Zank [2005]).** *A value  $f$  on  $\overline{\mathcal{G}}$  satisfies Efficiency, Additivity, Anonymity, Zero contributions, and Level symmetry if and only if  $f = \phi^{PZ}$ .*

By replacing Additivity and Zero contributions by Marginality, Peters and Zank [2005] propose a second characterization of the value  $\phi^{PZ}$ .<sup>13</sup>

**Theorem 2.3.6 (Peters and Zank [2005]).** *A value  $f$  on  $\overline{\mathcal{G}}$  satisfies Efficiency, Marginality, Anonymity, and Level symmetry if and only if  $f = \phi^{PZ}$ .*

Similarly, Hsiao and Raghavan [1993] propose a characterization of the value  $\phi^{HR}$  that relies on Additivity and Carrier. As above, this characterization holds on the subset of multi-choice games in which each agent has the same maximal activity level.

**Theorem 2.3.7 (Hsiao and Raghavan [1993]).** *Let  $m \leq (K, \dots, K)$  be a vector of maximal activity levels and  $(w_j)_{j=1}^{\max_{i \in N} m_i} \in \mathbb{R}^{\max_{i \in N} m_i + 1}$  be a given weight system on the activity levels and  $w_0 = 0$ . A value  $f$  on  $\overline{\mathcal{G}}$  satisfies Carrier, Additivity, the minimal effort axiom and the weight axiom if and only if  $f = \phi^{HR, w}$ .*

Klijn et al. [1999] characterize the extension of the Shapley value introduced in Derks and Peters [1993]. They consider two approaches: the first one relies on Additivity and the null pair axiom, the second one relies on the Balanced contributions principle.

**Theorem 2.3.8 (Klijn et al. [1999]).** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, Additivity, the null pair axiom and the necessary pair axiom if and only if  $f = \phi^{DP}$ .*

**Theorem 2.3.9 (Klijn et al. [1999]).** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, the Equal loss axiom and Upper balanced contributions if and only if  $f = \phi^{DP}$ .*

**Theorem 2.3.10 (Klijn et al. [1999]).** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, the Equal loss axiom and Lower balanced contributions if and only if  $f = \phi^{DP}$ .*

The last axiomatic characterization holds only on the subset of multi-choice games such that for each  $s \in \mathcal{M}$  with  $v(s) \neq 0$ ,  $|\mathcal{S}(s)| \geq 2$ . Denote by  $\mathcal{G}^*$  this subset of multi-choice games.

**Theorem 2.3.11 (Klijn et al. [1999]).** *A value  $f$  on  $\mathcal{G}^*$  satisfies Efficiency, the null game property and General balanced contributions if and only if  $f = \phi^{DP}$ .*

Based on the above characterization results, one can observe that the main difference between different extensions of the Shapley values in multi-choice games lies in the equity axiom they satisfy. Below, I summarize the axiomatic comparison between the different extensions of the Shapley values.

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<sup>13</sup>Observe that such characterization may be generalized on  $\mathcal{G}$  by replacing Anonymity by an appropriate axiom.



Axioms	$\phi^{vN}$	$\phi^{PZ}$	$\phi^{HR}$	$\phi^{DP}$
Carrier	✓	✓	✓	✓
Efficiency	✓	✓	✓	✓
Additivity	✓	✓	✓	✓
Linearity	✓	✓	✓	✓
The null game axiom	✓	✓	✓	✓
Anonymity (on $\mathcal{G}$ )	✓	✓	✓	✓
Hierarchical strength	✓	○	○	○
Weight axiom	○	○	✓	○
Hierarchical symmetry	✓	○	○	○
The necessary pair axiom	○	○	○	✓
Level symmetry	○	✓	○	○
Minimal effort	✓	✓	✓	○
Zero contribution	✓	✓	✓	○
the null pair axiom	✓	✓	✓	✓
Strong monotonicity	✓	✓	✓	○
Upper balanced contributions	○	✓	○	✓
Lower balanced contributions		○	○	✓
General balanced contributions	○	○	○	✓
Equal loss axiom	○	○	○	✓
Independence of the maximal activity level	○	✓	✓	○

## Chapter 3

# Marginalism, Egalitarianism and Efficiency in Multi-Choice Games

### 3.1 Introduction

One of the main issues in economic allocation problems is the trade-off between marginalism and egalitarianism. Marginalism supports allocations based on an agent's individual performances, while egalitarianism is in favor of an equal allocation at the expense of the differences between the agents' performances. In the context of TU-games, this trade-off can be seen as a compromise between the Shapley value and the Equal division value since the two values are often seen as the embodiment of marginalism and egalitarianism, respectively (see Chapter 2, Section 2.1.2).

This chapter aims to investigate the trade-off between marginalism and egalitarianism in the context of multi-choice games. To capture marginalism, one can think of the several values that extend the Shapley value from TU-games to multi-choice games (see Chapter 2, Section 2.3.2). In contrast, the Equal division value did not receive the same attention. To our knowledge, the only single-valued solution extending the Equal division value from TU-games to multi-choice games is the multi-choice constrained egalitarian solution introduced by [Branzei et al. \[2014\]](#). In addition, we study a specific efficiency condition defining the Core à la [Grabisch and Xie \[2007\]](#) (see Chapter 2, (2.29)). Precisely, we focus on a necessary condition for a payoff vector to be in the Core called *Multi-efficiency*. This condition extends Efficiency from TU-games to multi-choice games by considering each *synchronized* coalition (see Chapter 2, Section 2.3.2). It requires that the worth of each synchronized coalition is fully allocated among the agents' activity levels needed to reach this coalition. A solution on multi-choice games satisfies the Multi-Efficiency axiom if it assigns a multi-efficient payoff vector to each game in this class. It appears that Multi-Efficiency can be obtained by combining two axioms for solutions on the class of all multi-choice games: Efficiency and Independence of higher activity levels. On the one hand, Efficiency is a classical axiom weaker than Multi-Efficiency (see Chapter 2, Section 2.3.3). On the other hand, Independence of higher activity levels is a new axiom. It ensures that the payoff distributed to an agent's activity level is independent of any higher activity levels. In particular, this axiom protects agents with low activity levels from being influenced by agents with high activity levels. Axioms similar to Independence of higher activity levels already exist in the economic literature. The serial cost sharing method for discrete cost sharing problems introduced by [Moulin and Shenker \[1992\]](#) satisfies a similar axiom if we interpret activity levels as demands. Recently, [Albizuri et al. \[2020\]](#) study solutions for bargaining problems that satisfy a similar axiom if we interpret activity levels as claims. We show that if a value satisfies Independence of higher activity levels and Efficiency, then it satisfies Multi-Efficiency (Proposition 3.4.1). Therefore, Multi-Efficiency can be seen as a

desirable axiom since: it is implied by two desirable axioms; and from a technical point of view, it is a necessary condition to be in the Core.

Among all the single-valued solutions introduced in the literature, none of them satisfies Multi-Efficiency. For this reason, we propose several solution concepts for multi-choice games satisfying this axiom. This allows us to discuss the trade-off between marginalism and egalitarianism by means of a compromise between multi-efficient solutions.

We introduce a multi-efficient extension of the Shapley value, which we call the *multi-choice Shapley value*. This value is computed by means of restricted orders (see Chapter 2, Section 2.3.2). It assigns to each agent's activity level its expected marginal contribution (see (2.33)) assuming that each restricted order occurs with equal probability. This value is the centroid of the Weber set (see (2.34)) and therefore belongs to the Core of super-modular multi-choice games. As an additional remark, we show that the multi-choice Shapley value is closely related to the discrete serial cost sharing method for discrete cost sharing problems introduced by [Moulin and Shenker \[1992\]](#).

Next, we introduce the *multi-choice Equal division value*. This value divides the variation in worth between two consecutive synchronized coalitions (e.g. the  $j$ -synchronized and the  $(j + 1)$ -synchronized coalitions) equally among the agents able to play the required activity levels. Whenever there are only two activity levels (0 and 1), this value coincides with the Equal Division value for TU-games.

Then, we address the trade-off between marginalism and egalitarianism by compromising between the multi-choice Shapley value and the multi-choice Equal division value. To that end, we introduce the *multi-choice Egalitarian Shapley values* for multi-choice games. This family of values is composed of convex combinations of the multi-choice Shapley value and the multi-choice Equal division value. Obviously, the multi-choice Egalitarian Shapley values are multi-efficient. Since we consider multi-choice games, we can define a specific convex combination at each activity level. This allows for different types of compromise, depending on the activity level. To our knowledge, this work is the first to address this trade-off in the context of multi-choice games.

We provide several axiomatic characterizations of these new multi-efficient solution concepts. To that end, we invoke classical axioms as well as new axioms for multi-choice games. Among the new axioms, we introduce Sign symmetry for equal pairs, which is an extension of Sign symmetry for equal agents originally introduced by [Casajus \[2018\]](#) for TU-games. Additionally, we propose Equal treatment for equal pairs, which strengthens Sign symmetry for equal pairs. Furthermore, we introduce Weak monotonicity. This axiom relaxes the axiom of Strong monotonicity for multi-choice games originally introduced by [Klijn et al. \[1999\]](#), but also boils down to the axiom of Weak monotonicity as introduced by [van den Brink et al. \[2013\]](#) for TU-games. Combining classical and new axioms for multi-choice games, we provide two characterizations of the multi-choice Shapley value, one that relies on a classical Additivity axiom for multi-choice games (Theorem 3.4.1) and another one that does not (Theorem 3.4.2). Furthermore, we show that the multi-choice Shapley value can be obtained by a dividends sharing process (Corollary 3.4.1). Next, we provide an axiomatic characterization of the multi-choice Equal Division value (Theorem 3.4.3). Finally, we provide an axiomatic characterization of the Egalitarian Shapley values (Theorem 3.4.4).

The remainder of the chapter is as follows. In Section 3.2 we introduce some additional notations for multi-choice games. We introduce multi-efficient solution concepts in Section 3.3. Subsection 3.3.2 introduces the multi-choice Shapley value, Subsection 3.3.3 the multi-choice Equal division value and Subsection 3.3.4 the multi-choice Egalitarian Shapley values. We provide the axiomatic characterizations in Section 3.4. We make some additional remarks regarding the multi-choice Shapley value in Section 3.5. Finally, Section 3.6 concludes the chapter.

## 3.2 Notation

Let  $N = \{1, \dots, n\}$  be a fixed set of agents and  $K \in \mathbb{N}$ . We consider  $\mathcal{G}$  the class of all multi-choice games on  $N$ . Denote by  $Q(j) \subseteq N$  the set of *agents able to play activity level  $j$* . Formally, it is defined as

$$\forall j \leq K, \quad Q(j) = \{i \in N : m_i \geq j\}.$$

Without loss of generality, we assume that  $Q(1) = N$ . We introduce the set of *top pairs*,  $T(s)$ , containing agents playing the highest activity levels in a coalition  $s \in \mathcal{M}$ . Formally, the set of top pairs is defined as

$$\forall s \in \mathcal{M}, \quad T(s) = \{(i, s_i) \in M^+ : s_i \geq s_k, \forall k \in N\}. \quad (3.1)$$

Finally, for each  $s \in \mathcal{M}$ , denote by  $s^T$  the highest activity level played in  $s$ , i.e.,  $s^T = \max_{i \in N} s_i$ .

**Remark 4.** For  $t \in \mathcal{M}, t \neq \mathbf{0}$ , we formulate two distinct remarks regarding minimal effort games (see (2.19)). Take any  $(m, u_t) \in \mathcal{G}$ . Each pair  $(i, j) \in M^+$ , such that  $j > t_i$ , is a null pair in  $(m, u_t)$ . Let  $(i, j), (i', j) \in M^+$  be two distinct pairs such that  $j \leq t_i$  and  $j \leq t_{i'}$ . Both pairs are equal (see (2.23)) in  $(m, u_t)$ .

**Remark 5.** For  $t \in \mathcal{M}, t \neq \mathbf{0}$ , we formulate two distinct remarks regarding Dirac games (see (2.17)). Take any  $(m, \delta_t) \in \mathcal{G}$ . Each pair  $(i, j) \in M^+$ , such that  $j > t_i + 1$ , is a null pair in  $(m, \delta_t)$ . If there exists two distinct agents  $i, i' \in N$  such that  $t_i = t_{i'}$ , then  $(i, t_i)$  and  $(i', t_{i'})$  are equal (see (2.23)) in  $(m, \delta_t)$ .

## 3.3 Multi-efficient solution concepts

In this section, we discuss a necessary condition for a payoff vector to be in the Core of multi-choice games à la Grabisch and Xie [2007], which we call *Multi-Efficiency*. We propose new multi-efficient solution concepts for multi-choice games. We first provide a new extension of the Shapley value (Shapley [1953]) from TU-games to multi-choice games. Next, we provide new extensions of the Equal division value and the Egalitarian Shapley values from TU-games to multi-choice games.

### 3.3.1 Multi-Efficiency

Among the several solutions introduced on the class of multi-choice games, the Core à la Grabisch and Xie [2007] is of particular interest. Beyond embodying the appealing stability principle, it also considers a normalization that ensures the Core to be unbounded. This normalization constitutes the main departure with the other extensions of the core in multi-choice games. It can be seen as requiring the efficiency

condition at each activity level. Let  $(m, v) \in \mathcal{G}$  be a multi-choice game. Recall that the Core (see Chapter 2, Section 2.3.2, definition (2.29)) is defined as the set of coalitionally rational payoff vectors satisfying

$$\forall h \leq \max_{i \in N} m_i, \sum_{i \in N} \sum_{j \leq h} x_{ij} = v((h \wedge m_i)_{i \in N}). \quad (3.2)$$

Let us reformulate (3.2) as an axiom for solutions on  $\mathcal{G}$ . A value  $f$  on  $\mathcal{G}$  satisfies: *Multi-Efficiency (ME)* if, for each  $(m, v) \in \mathcal{G}$ ,

$$\forall h \leq \max_{k \in N} m_k, \sum_{i \in N} \sum_{j=1}^{h \wedge m_i} f_{ij}(m, v) = v((h \wedge m_i)_{i \in N}). \quad (3.3)$$

**Remark 6.** For each  $(m, v) \in \mathcal{G}$ , one can reformulate (3.3) as

$$\forall h \leq \max_{k \in N} m_k, \sum_{i \in Q(h)} f_{ih}(m, v) = v((h \wedge m_k)_{k \in N}) - v(((h-1) \wedge m_k)_{k \in N}). \quad (3.4)$$

The sum of the payoffs of all pairs  $(i, h)$  containing activity level  $h$  is equal to the surplus generated between the  $h$ -synchronized coalition and the  $(h-1)$ -synchronized coalition.

### 3.3.2 The multi-choice Shapley value

In this section, we define a multi-efficient value that extends the Shapley value from TU-games to multi-choice games. This value belongs to the Core of super-modular multi-choice games. Take any  $(m, v) \in \mathcal{G}$ , and consider  $\bar{O}$  the set of all restricted orders over the set of pairs  $M^+$  (see Chapter 2, Section 2.3.2). Observe that, the number of restricted orders over the set of pairs is given by

$$\prod_{j \leq \max_{k \in N} m_k} |Q(j)|!$$

Based on the process defining the marginal vectors according to the restricted orders, we introduce the *multi-choice Shapley value*. This value assigns to each pair  $(i, j) \in M^+$  its expected marginal contribution assuming that each restricted order over the set of pairs occurs with equal probability.

Formally, the multi-choice Shapley value,  $\varphi$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{\prod_{j \leq \max_{k \in N} m_k} |Q(j)|!} \sum_{\bar{\sigma} \in \bar{O}} m_{ij}^{\bar{\sigma}}(m, v). \quad (3.5)$$

Whenever  $m = (1, \dots, 1)$ , this value coincides with the Shapley value on  $\mathcal{G}^{TU}$ .

**Remark 7.** Following Grabisch and Xie [2007], for each  $(m, v) \in \mathcal{G}$ , the Weber set  $\mathfrak{W}$  is the convex hull of all marginal vectors (see Chapter 2, Section 2.3.2, definition (2.34)). The multi-choice Shapley value is the centroid of the Weber set. By Grabisch and Xie [2007], the Weber set coincides with the Core on the class of super-modular multi-choice games. Therefore, for each super-modular multi-choice game  $(m, v) \in \mathcal{G}$ , it holds that  $\varphi(m, v) \in \mathfrak{C}(m, v)$ .

Next results highlights that the multi-choice Shapley value admits an alternative expression that requires less restricted orders to be computed. For each  $j \leq$

$\max_{k \in N} m_k$ , denote by  $M^{+,j} = \{(i, j) \in M^+ : i \in Q(j)\}$  the subset of pairs containing the activity level  $j$ . We define orders over the set of pairs  $M^{+,j}$ . An order over  $M^{+,j}$  is a bijection  $\bar{\sigma}_j : M^{+,j} \rightarrow \{1, \dots, |Q(j)|\}$ . Denote by  $\bar{O}_j$  the set of all orders over  $M^{+,j}$ . These orders can also be interpreted as orders over the set of agents in  $Q(j)$ . For each  $\bar{\sigma}_j \in \bar{O}_j$  and  $h \in \{0, \dots, |Q(j)|\}$ , define  $s_{\bar{\sigma}_j, h}$  as

$$\forall i \in N, \quad s_i^{\bar{\sigma}_j, h} = \begin{cases} j & \text{if } i \in Q(j) \text{ and } \bar{\sigma}_j(i, j) \leq h, \\ j-1 & \text{if } i \in Q(j) \text{ and } \bar{\sigma}_j(i, j) > h, \\ m_i & \text{if } i \notin Q(j). \end{cases} \quad (3.6)$$

Observe that  $s^{\bar{\sigma}_j, |Q(j)|} = (j \wedge m_k)_{k \in N}$  and  $s^{\bar{\sigma}_j, 0} = ((j-1) \wedge m_k)_{k \in N}$ . The coalition  $s^{\bar{\sigma}_j, h} \in \mathcal{M}$  represents a situation in which each agent able to play at  $j$  and ordered prior to step  $h$ , with respect to  $\bar{\sigma}_j$ , participates at its activity level  $j$ , whereas each agent able to play  $j$  but not ordered prior to step  $h$ , with respect to  $\bar{\sigma}_j$ , participates at its activity level  $j-1$ . Agents unable to play  $j$  participate at their maximal activity level.

**Proposition 3.3.1.** For each  $(m, v) \in \mathcal{G}$ , the multi-choice Shapley value  $\varphi$  admits an alternative expression given by

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{|Q(j)|!} \sum_{\bar{\sigma}_j \in \bar{O}_j} \left[ v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}) - v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)-1}) \right]. \quad (3.7)$$

*Proof.* We show that the multi-choice Shapley value admits an alternative expression given by (3.7). Observe that there are  $|\bar{O}_l| = |Q(l)|!$  ways to order the pairs in  $M^{+,l}$ , for each  $l \leq \max_{k \in N} m_k$ . Additionally, there are  $\prod_{l < j} |Q(l)|!$  ways to order the pairs in  $M^{+,1}$ , then the pairs in  $M^{+,2}$ , and so on, until the pairs in  $M^{+,j-1}$ . Similarly, there are  $\prod_{l > j} |Q(l)|!$  ways to order the pairs in  $M^{+,j+1}$ , then the pairs in  $M^{+,j+2}$ , and so on. Observe that, for each  $\bar{\sigma} \in \bar{O}$ , there exists exactly one order  $\bar{\sigma}_j \in \bar{O}_j$  such that  $s^{\bar{\sigma}, \bar{\sigma}(i, j)} = s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}$ . Additionally, for each  $\bar{\sigma}_j \in \bar{O}_j$ , there are  $\prod_{l < j} |Q(l)|! \times \prod_{l > j} |Q(l)|!$  orders  $\bar{\sigma} \in \bar{O}$  such that  $s^{\bar{\sigma}, \bar{\sigma}(i, j)} = s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}$ . It follows that, for each  $(m, v) \in \mathcal{G}$  and  $(i, j) \in M^+$ ,

$$\begin{aligned} \varphi_{ij}(m, v) &= \frac{1}{\prod_{l \leq \max_{k \in N} m_k} |Q(l)|!} \sum_{\bar{\sigma} \in \bar{O}} \left[ v(s^{\bar{\sigma}, \bar{\sigma}(i, j)}) - v(s^{\bar{\sigma}, \bar{\sigma}(i, j)-1}) \right] \\ &= \frac{(\prod_{l < j} |Q(l)|!) (\prod_{l > j} |Q(l)|!)}{\prod_{l \leq \max_{k \in N} m_k} |Q(l)|!} \sum_{\bar{\sigma}_j \in \bar{O}_j} \left[ v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}) - v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)-1}) \right]. \end{aligned}$$

The first line comes from the definition of the multi-choice Shapley value, the second line follows from (3.6) and the fact that there are  $\prod_{l < j} |Q(l)|! \times \prod_{l > j} |Q(l)|!$  orders  $\bar{\sigma} \in \bar{O}$  such that  $s^{\bar{\sigma}, \bar{\sigma}(i, j)} = s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}$ , for each  $\bar{\sigma}_j \in \bar{O}_j$ . Since  $\prod_{l \leq \max_{k \in N} m_k} |Q(l)|! = \prod_{l < j} |Q(l)|! \times |Q(j)|! \times \prod_{l > j} |Q(l)|!$ , we obtain the desired result:

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{|Q(j)|!} \sum_{\bar{\sigma}_j \in \bar{O}_j} \left[ v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}) - v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)-1}) \right].$$

■

In the sequel, we will retain expression (3.7) of the multi-choice Shapley value.

**Example 1.** To illustrate the multi-choice Shapley value, consider the following application borrowed from [van den Nouweland et al. \[1995\]](#). Consider a large firm with several departments, where the firm's revenue depends on the performance of the departments. Let  $N = \{1, 2\}$  be the finite and fixed set of departments. Each  $i \in N$  has a finite set of activity levels, which can be interpreted as workdays. For the sake of simplicity, assume that each department can work at most 2 days, i.e.,  $m_i = 2$  for each  $i \in N$ . A characteristic function  $v$  describes the revenue generated when each department works during a certain number of days. Such characteristic function is given by

$s$	(1,0)	(2,0)	(0,1)	(0,2)	(1,1)	(2,1)	(1,2)	(2,2)
$v(s)$	3	5	3	3	4	5	4	5

Clearly, the couple  $(m, v)$  forms a multi-choice game. Assuming that both departments work during 2 days, the firm's revenue is given by  $v(2, 2) = 5$ . The problem is then to divide this revenue among each workday of each department. The payoff obtained by a department for a given workday can be interpreted as its budget for this specific day. The multi-choice Shapley value applied to  $(m, v)$  is given by

$$\varphi_{11}(m, v) = 2, \quad \varphi_{21}(m, v) = 2, \quad \varphi_{12}(m, v) = 1, \quad \varphi_{22}(m, v) = 0.$$

Observe that both departments are equally productive for their first workday, so the pairs (1, 1) and (2, 1) are equal in  $(m, v)$ . It appears that, the multi-choice Shapley value allocates an equal budget to both departments for their first workday, i.e.,  $\varphi_{11}(m, v) = \varphi_{21}(m, v)$ . Department 2 does not produce anything during its second workday, so the pair (2, 2) is a null pair in  $(m, v)$ . It turns out that the multi-choice Shapley value allocates it a null budget for this workday, i.e.,  $\varphi_{22}(m, v) = 0$ . All in all, the multi-choice Shapley value leads to a budget allocation based on the individual performances of the departments. Finally, the total budget allocated to the two departments for a given workday is equal to the revenue generated cooperatively on this workday, e.g.,  $\varphi_{11}(m, v) + \varphi_{21}(m, v) = v(1, 1)$ . This comes from the multi-efficiency of the multi-choice Shapley value.

### 3.3.3 The multi-choice Equal division value

In this section, we propose a new multi-efficient value that extends the Equal division value from TU-games to multi-choice games. This value is referred to as the *multi-choice Equal division value*. This value shares the surplus generated between two consecutive synchronized coalitions (3.4) equally among the pairs containing the activity level on which the agents in the larger of the two coalitions are synchronized.

Formally, the multi-choice Equal division value,  $\xi$ , is defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \xi_{ij}(m, v) = \frac{1}{|Q(j)|} \left[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right]. \quad (3.8)$$

Whenever  $m = (1, \dots, 1)$ , the multi-choice Equal division value boils down to the Equal division value on TU-games.



**Example 2.** Consider again the revenue distribution problem introduced in Example 1. The multi-choice Equal division value applied to  $(m, v)$  is given by

$$\xi_{11}(m, v) = 2, \quad \xi_{21}(m, v) = 2, \quad \xi_{12}(m, v) = 0.5, \quad \xi_{22}(m, v) = 0.5.$$

Department 2 receives a budget of  $\xi_{22}(m, v) = 0.5$  for its second workday, which is higher than  $\varphi_{22}(m, v) = 0$ . This increase is at the expense of the budget allocated to department 1 for its second work day since  $\xi_{12}(m, v) = 0.5$ , while  $\varphi_{12}(m, v) = 1$ . To put it simply, the multi-choice Equal division value is in favor of an egalitarian allocation of the budget per workday. The budget of a department for a given workday depends on the overall performance of the firm to date, rather than on the individual performance of that department.

### 3.3.4 The multi-choice Egalitarian-Shapley values

In this section, we propose a trade-off between marginalism and egalitarianism by considering convex combinations of the multi-choice Shapley value and the multi-choice Equal division value.

Let  $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$  be a parameter system such that  $\alpha^j \in [0, 1]$  for each  $1 \leq j \leq K$ . For each  $(m, v) \in \mathcal{G}$ , a *multi-choice Egalitarian Shapley value*,  $\chi^\alpha$ , is defined as

$$\forall (i, j) \in M^+, \quad \chi_{ij}^\alpha(m, v) = \alpha^j \varphi_{ij}(m, v) + (1 - \alpha^j) \xi_{ij}(m, v). \quad (3.9)$$

Whenever  $m = (1, \dots, 1)$ , these values boil down to the Egalitarian Shapley values on TU-games. We illustrate the possibilities offered by multiple convex combinations with an example.

**Example 3.** In the context of revenue distribution problems (Example 1), a multi-choice Egalitarian Shapley value proposes a compromise between a budget allocation based on individual performances and an egalitarian allocation of the budget. Moreover, such value allows for different types of compromises depending on the workday one considers. For instance, take a parameter system  $\alpha$  such that  $\alpha^1 = 0.1$  and  $\alpha^2 = 0.9$ . In this case, the multi-choice Egalitarian Shapley value  $\chi^\alpha$  proposes a budget allocation closer to the multi-choice Equal division value for the first workday of the departments. Then, the budget allocated to the departments for their second workday becomes closer to the multi-choice Shapley value. A firm may have an interest in choosing such parameter system  $\alpha$  if it wants to favor an egalitarian allocation of the budget on the first workday, but also wants to encourage individual performance on the second workday.

## 3.4 Axiomatic characterizations

In this section, we discuss new axioms for multi-choice games. We also provide axiomatic characterizations of each solution introduced in Section 3.3.

### 3.4.1 Characterizations of the multi-choice Shapley value

We provide two axiomatic characterizations of the multi-choice Shapley value. The first characterization relies on a Linearity axiom, whereas the second does not. We also provide an expression of the multi-choice Shapley value in terms of dividends.

First, we introduce an axiom based on the independence principle. This axiom requires that if the maximal activity level of each agent reduces to a certain level,



then the payoff of each agent for this activity level remains unchanged. In the context of Example 1, this axiom guarantees that the budget of a department for a given workday does not depend on future workdays. A value  $f$  on  $\mathcal{G}$  satisfies:

*Independence of higher activity levels (IH)* if, for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = f_{ij}((j \wedge m_k)_{k \in N}, v).$$

It turns out that the combination of Independence of higher activity levels and Efficiency (see Chapter 2 Section 2.3.3) implies Multi-Efficiency.

**Proposition 3.4.1.** *If a value  $f$  on  $\mathcal{G}$  satisfies Efficiency and Independence of higher activity levels, then it satisfies Multi-Efficiency.*

*Proof.* Let  $(m, v) \in \mathcal{G}$ ,  $h \leq \max_{k \in N} m_k$  and  $f$  a value satisfying Efficiency and Independence of higher activity levels. Consider the sub-game  $((h \wedge m_k)_{k \in N}, v)$ . By Efficiency, it holds that

$$\sum_{i \in N} \sum_{j=1}^{h \wedge m_i} f_{ij}((h \wedge m_k)_{k \in N}, v) = v((h \wedge m_k)_{k \in N}). \quad (3.10)$$

By Independence of higher activity levels, it holds that

$$\sum_{i \in N} \sum_{j=1}^{h \wedge m_i} f_{ij}((h \wedge m_k)_{k \in N}, v) = \sum_{i \in N} \sum_{j=1}^{h \wedge m_i} f_{ij}(m, v). \quad (3.11)$$

Combining (3.10) with (3.11), we obtain the desired result.  $\blacksquare$

**Remark 8.** The converse of Proposition 3.4.1 is not true. Indeed, consider the value  $d$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+,$$

$$d_{ij}(m, v) = \begin{cases} \frac{v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})}{|\{h \in N : m_h \geq m_k, \forall k \in N\}|} & \text{if } m_i \geq m_k, \forall k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

The value  $d$  satisfies Multi-Efficiency, but does not verify Independence of higher activity levels. To see this, consider  $N = \{1, 2, 3\}$  and  $(m, v) \in \mathcal{G}$  such that  $m = (3, 2, 3)$ . Observe that  $d_{1,1}((2, 2, 2), v) = \frac{1}{3}v(1, 1, 1) \neq d_{1,1}(m, v)$  which shows that  $d$  violates Independence of higher activity levels.

The next two axioms compare the payoffs of equal pairs (see Chapter 2, Section 2.3.1, definition (2.23)). First, we introduce the *Equal treatment for equal pairs* axiom, which states that two equal pairs should receive the same payoff. In the context of Example 1, two equal pairs may represent two departments in a firm that are equally productive on a given workday. Equal treatment for equal pairs ensures that these departments receive the same budget allocation for such a workday. We also suggest a relaxation of Equal treatment for equal pairs into Sign symmetry for equal pairs. This axiom states that two equal pairs should receive a payoff of the same sign. As a generalization of the Sign symmetry for equal agents axiom introduced by Casajus [2018] (see Chapter 2, Section 2.1.3). A value  $f$  on  $\mathcal{G}$  satisfies:

*Equal treatment for equal pairs (ET)* if, for each  $(m, v) \in \mathcal{G}$  and two distinct equal pairs  $(i, j), (i', j) \in M^+$ ,

$$f_{ij}(m, v) = f_{i'j}(m, v).$$

Whenever  $m = (1, \dots, 1)$ , Equal treatment for equal pairs conforms to the classical axiom of Equal Treatment of Equal agents for TU-games.

*Sign symmetry for equal pairs (SS)* if, for each  $(m, v) \in \mathcal{G}$  and two distinct equal pairs  $(i, j), (i', j) \in M^+$ ,

$$\text{sign}(f_{ij}(m, v)) = \text{sign}(f_{i'j}(m, v)).$$

Whenever  $m = (1, \dots, 1)$ , Sign symmetry for equal pairs conforms to the Sign symmetry of equal agents axiom originally introduced by Casajus [2018].

As our first main result, we obtain the following axiomatic characterization of the multi-choice Shapley value that relies on Linearity and the null pair property (see Chapter 2, Section 2.3.3).

**Theorem 3.4.1.** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, Independence of higher activity levels, Linearity, Sign symmetry for equal pairs and the null pair axiom if and only if  $f = \varphi$ .*

*Proof.* The proof is divided in two steps.

**Step 1:** We show that  $\varphi$  satisfies all the axioms of the statement of Theorem 3.4.1. For each  $(m, v) \in \mathcal{G}$ , it holds that

$$\begin{aligned} \sum_{i \in N} \sum_{j \in M_i^+} \varphi_{ij}(m, v) &= \sum_{j \leq \max_{k \in N} m_k} \sum_{i \in Q(j)} \varphi_{ij}(m, v) \\ &\stackrel{(3.7)}{=} \sum_{j \leq \max_{k \in N} m_k} \frac{1}{|Q(j)|!} \sum_{\bar{\sigma}_j \in \bar{O}_j} \sum_{i \in Q(j)} \left[ v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}) - v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)-1}) \right]. \end{aligned}$$

Observe that, for each  $\bar{\sigma}_j \in \bar{O}_j$ ,

$$\sum_{i \in Q(j)} \left[ v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)}) - v(s^{\bar{\sigma}_j, \bar{\sigma}_j(i, j)-1}) \right] = v(s^{\bar{\sigma}_j, |Q(j)|}) - v(s^{\bar{\sigma}_j, 0}).$$

By (3.6), for each  $\bar{\sigma}_j \in \bar{O}_j$ , it holds that

$$s^{\bar{\sigma}_j, |Q(j)|} = (j \wedge m_k)_{k \in N}, \quad \text{and} \quad s^{\bar{\sigma}_j, 0} = ((j-1) \wedge m_k)_{k \in N}.$$

It follows that

$$\sum_{i \in N} \sum_{j \in M_i^+} \varphi_{ij}(m, v) = \sum_{j \leq \max_{k \in N} m_k} \frac{1}{|Q(j)|!} \sum_{\bar{\sigma}_j \in \bar{O}_j} \left[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right]. \quad (3.12)$$

Since the quantity  $v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})$  is independent from any order  $\bar{\sigma}_j \in \bar{O}_j$ , it follows that it is summed as many times in (3.12) as there are orders

in  $\bar{O}_j$ . Therefore, it holds that

$$\begin{aligned} \sum_{i \in N} \sum_{j \in M_i^+} \varphi_{ij}(m, v) &= \sum_{j \leq \max_{k \in N} m_k} \frac{1}{|Q(j)|!} Q(j)! \left[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right] \\ &= \sum_{j \leq \max_{k \in N} m_k} \left[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right] \\ &= v(m), \end{aligned}$$

which shows that the value satisfies Efficiency. By the definition of the multi-choice Shapley value (see (3.7)), the payoff of a pair is independent of any activity level different from the activity level contained in this pair. Therefore, we have that  $\varphi$  satisfies Independence of higher activity levels. Linearity follows directly from (3.7). By the definition of equal pairs (see Chapter 2, Section 2.3.1, definition (2.23)),  $\varphi$  satisfies Equal treatment for equal pairs, which implies that  $\varphi$  satisfies Sign symmetry for equal pairs. By the definition of null pairs (see Chapter 2, Section 2.3.1, definition (2.25)), we have that  $\varphi$  satisfies the null pair axiom. This concludes Step 1.

**Step 2:** To complete the proof, it remains to show that there is at most one value satisfying all the axioms of the statement of Theorem 3.4.1. Take any  $f$  satisfying all the axioms of the statement of Theorem 3.4.1. Consider any  $(m, v) \in \mathcal{G}$ . We know that each multi-choice game admits a unique linear decomposition in terms of minimal effort games  $\{u_s\}_{s \in \mathcal{M}}$  (see Chapter 2, Section 2.3.1, definition (2.19)). Consider  $s \in \mathcal{M}$  such that  $s \neq \mathbf{0}$ . The set of top pairs  $T(s)$  (see the definition (3.1)) can be re-written as

$$T(s) = \{(i, s^T) \in M^{+s^T} : s_i = s^T\},$$

where  $s^T = \max_{i \in N} s_i$ . Let us show that  $f(m, u_s)$  is uniquely determined. We divide this Step into several sub-steps.

**Step 2.1.** Let us show that, for each  $(i, j) \in M^+$  such that  $j \neq s^T$ ,  $f_{ij}(m, u_s)$  is uniquely determined.

**Step 2.1.1.** If  $j < s^T$ , then  $(j \wedge m_k)_{k \in N} \not\geq s$ . It follows that  $((j \wedge m_k)_{k \in N}, u_s)$  is the null game since  $u_s(t) = 0$  for each  $t \leq (j \wedge m_k)_{k \in N}$ . Recall that each pair is a null pair in the null game. Combining these observations with the null pair axiom, for each  $(i, j) \in M^+$  such that  $j < s^T$ , we obtain

$$f_{ij}(m, u_s) \stackrel{(IH)}{=} f_{ij}((j \wedge m_k)_{k \in N}, u_s) \stackrel{(N)}{=} 0.$$

**Step 2.1.2.** If  $j > s^T$  then, by Remark 4,  $(i, j)$  is a null pair in  $(m, u_s)$ . By the null pair axiom, for each  $(i, j) \in M^+$  such that  $j > s^T$ , it holds that

$$f_{ij}(m, u_s) \stackrel{(N)}{=} 0.$$

We have shown that  $f_{ij}(m, u_s) = 0$ , and so is uniquely determined for each  $(i, j) \in M^+$  such that  $j \neq s^T$ .

**Step 2.2.** Now, we show that, for each pair  $(i, j) \in M^+$  such that  $j = s^T$  i.e. each pair  $(i, s^T) \in M^{+s^T}$ ,  $f_{is^T}(m, u_s)$  is uniquely determined. To that end, consider the game

$(m, w) \in \mathcal{G}$  defined as

$$\forall t \leq m, \quad w(t) = u_s(t) - \sum_{(i, s^T) \in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}(t). \quad (3.13)$$

**Step 2.2.1.** We show that

$$\sum_{(i, s^T) \in M^{+, s^T}} f_{is^T}(m, w) = 0.$$

We consider pairs in  $M^{+, s^T}$ . By definition of  $M^{+, s^T}$ , observe that

$$\sum_{i \in Q(s^T)} f_{is^T}(m, w) = \sum_{(i, s^T) \in M^{+, s^T}} f_{is^T}(m, w).$$

We have that any pair  $(i, s^T) \in M^{+, s^T}$  is either in  $T(s)$  or not. Since  $f$  satisfies Efficiency and Independence of higher activity levels, by Proposition 3.4.1,  $f$  also satisfies Multi-Efficiency. Therefore, it holds that

$$\begin{aligned} \sum_{(i, s^T) \in M^{+, s^T}} f_{is^T}(m, w) &\stackrel{(ME)}{=} w((s^T \wedge m_k)_{k \in N}) - w(((s^T - 1) \wedge m_k)_{k \in N}) \\ &\stackrel{(3.13)}{=} u_s((s^T \wedge m_k)_{k \in N}) - \sum_{(i, s^T) \in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}((s^T \wedge m_k)_{k \in N}) \\ &\quad - u_s(((s^T - 1) \wedge m_k)_{k \in N}) \\ &\quad + \sum_{(i, s^T) \in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}(((s^T - 1) \wedge m_k)_{k \in N}). \end{aligned} \quad (3.14)$$

Observe that  $((s^T \wedge m_k)_{k \in N}) \geq s \geq ((0_{-i}, s^T), (((s^T - 1) \wedge m_k)_{k \in N})) \not\geq s$  and  $((s^T - 1) \wedge m_k)_{k \in N} \not\geq (0_{-i}, s^T)$ , where  $(i, s^T) \in T(s)$ . By definition of a minimal effort game (2.19), it holds that

$$\begin{aligned} u_s((s^T \wedge m_k)_{k \in N}) &= 1, \quad \text{and,} \quad \forall (i, s^T) \in M^{+, s^T}, \quad u_{(0_{-i}, s^T)}((s^T \wedge m_k)_{k \in N}) = 1, \\ u_s(((s^T - 1) \wedge m_k)_{k \in N}) &= 0, \quad \text{and,} \quad \forall (i, s^T) \in M^{+, s^T}, \quad u_{(0_{-i}, s^T)}(((s^T - 1) \wedge m_k)_{k \in N}) = 0. \end{aligned}$$

It follows that (3.14) becomes

$$\sum_{(i, s^T) \in M^{+, s^T}} f_{is^T}(m, w) = 1 - \sum_{(i, s^T) \in T(s)} \varphi_{is^T}(m, u_s) - 0 + 0. \quad (3.15)$$

Observe that, since  $(i, s^T) \notin T(s)$  if and only if  $s^T > s_i$ , then each  $(i, s^T) \notin T(s)$  is a null pair in  $(m, u_s)$ . Since  $\varphi$  satisfies the null pair axiom, we have that  $\varphi_{is^T}(m, u_s) = 0$  for each  $(i, s^T) \notin T(s)$ . Since  $\varphi$  satisfies Efficiency, Independence of higher activity levels, by Proposition 3.4.1 the value satisfies Multi-Efficiency. Therefore, it holds

that

$$\begin{aligned}
\sum_{(i,s^T) \in T(s)} \varphi_{is^T}(m, u_s) &\stackrel{(N)}{=} \sum_{(i,s^T) \in T(s)} \varphi_{is^T}(m, u_s) + \sum_{(i,s^T) \notin T(s)} \varphi_{is^T}(m, u_s) \\
&= \sum_{(i,s^T) \in M^{+,s^T}} \varphi_{is^T}(m, u_s) \\
&\stackrel{(ME)}{=} u_s((s^T \wedge m_k)_{k \in N}) \\
&= 1.
\end{aligned}$$

Therefore, (3.15) becomes

$$\sum_{(i,s^T) \in M^{+,s^T}} f_{is^T}(m, w) = 1 - 1 = 0, \quad (3.16)$$

which concludes Step 2.2.1.

**Step 2.2.2.** We show that, for each  $(i, s^T) \in M^{+,s^T}$ ,

$$f_{is^T}(m, w) = 0.$$

We know that each pair  $(i, s^T) \notin T(s)$  is a null pair in  $(m, u_s)$ . Moreover, each pair  $(i, s^T) \notin T(s)$  is a null pair in each  $(m, u_{0_{-i'}, s^T})$ ,  $(i', s^T) \in T(s)$ . Indeed, in  $(m, u_{0_{-i'}, s^T})$ ,  $(i', s^T)$  is the only productive pair and all other pairs are null pairs. It follows that each pair  $(i, s^T) \notin T(s)$  is a null pair in  $(m, w)$ . By the null pair axiom, for each  $(i, s^T) \notin T(s)$ , it holds that

$$f_{is^T}(m, w) = 0. \quad (3.17)$$

It follows that

$$\begin{aligned}
\sum_{(i,s^T) \in M^{+,s^T}} f_{is^T}(m, w) &= \sum_{(i,s^T) \in T(s)} f_{is^T}(m, w) + \sum_{(i,s^T) \notin T(s)} f_{is^T}(m, w) \\
&\stackrel{(N)}{=} \sum_{(i,s^T) \in T(s)} f_{is^T}(m, w) + 0 \\
&\stackrel{(3.16)}{=} 0.
\end{aligned} \quad (3.18)$$

To complete the proof of Step 2.2.2, it remains to show that if there exist two distinct pairs  $(i, s^T), (i', s^T) \in T(s)$ , then these pairs are equal. By Remark 4 two distinct pairs  $(i, s^T), (i', s^T) \in T(s)$  are equal in  $(m, u_s)$ . Since  $\varphi$  satisfies Equal treatment for equal pairs, it follows that  $\varphi_{is^T}(m, u_s) = \varphi_{i's^T}(m, u_s)$ . By definition of a minimal effort game, for each  $t \in \mathcal{M}$  such that  $t_i = t_{i'} = s^T - 1$ , it holds that

$$u_{(0_{-i}, s^T)}(t) = u_{(0_{-i'}, s^T)}(t) = 0, \quad (3.19)$$

$$\text{and } u_{(0_{-i}, s^T)}(t + e_i) = u_{(0_{-i'}, s^T)}(t + e_{i'}) = 1. \quad (3.20)$$

Therefore, for each  $t \in \mathcal{M}$  such that  $t_i = t_{i'} = s^T - 1$ , we have that

$$\begin{aligned} \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}(t + e_i) &= \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}(t) + \varphi_{is^T}(m, u_s) \\ &= \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}(t) + \varphi_{i's^T}(m, u_s) \\ &= \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}(t + e_{i'}). \end{aligned}$$

Where the first equality and third equality follow from (3.19) and (3.20), and the second equality follows from  $\varphi_{is^T}(m, u_s) = \varphi_{i's^T}(m, u_s)$ , since  $\varphi$  satisfies Equal treatment for equal pairs. It follows that

$$w(t + e_i) = w(t + e_{i'}),$$

for each  $t \in \mathcal{M}$  such that  $t_i = t_{i'} = s^T - 1$ , showing that  $(i, s^T), (i', s^T) \in T(s)$  are equal pairs in  $(m, w)$ . By Sign symmetry for equal pairs, we have that

$$\text{sign}(f_{is^T}(m, w)) = \text{sign}(f_{i's^T}(m, w)).$$

It follows from (3.18) that, for each  $(i, s^T) \in T(s)$ ,

$$f_{is^T}(m, w) = 0. \quad (3.21)$$

Combining (3.17) with (3.21), the proof of Step 2.2.2 is complete.

**Step 2.2.3.** We show that, for each  $(i, s^T) \in M^{+, s^T}$ ,

$$f_{is^T}(m, u_s) = \varphi_{is^T}(m, u_s).$$

By (3.13), (3.21) and Linearity, for each  $(i, s^T) \in M^{+, s^T}$ , it holds that

$$\begin{aligned} f_{is^T}(m, w) &\stackrel{(3.13), (L)}{=} f_{is^T}(m, u_s) - f_{is^T}\left(m, \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}\right) \\ \iff f_{is^T}(m, u_s) &\stackrel{(3.21)}{=} f_{is^T}\left(m, \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k}, s^T)}\right) \\ &\stackrel{(L)}{=} \sum_{(k,s^T) \in T(s)} \varphi_{ks^T}(m, u_s) f_{is^T}\left(m, u_{(0_{-k}, s^T)}\right). \end{aligned}$$

Additionally, by the null pair axiom and Multi-Efficiency, we have that

$$f_{is^T}(m, u_{(0_{-i}, s^T)}) = 1$$

since  $(i, s^T)$  is the only productive pair in  $(m, u_{(0_{-i}, s^T)})$ . Therefore, for each  $(i, s^T) \in M^{+, s^T}$ , it holds that

$$\varphi_{ks^T}(m, u_s) f_{is^T}\left(m, u_{(0_{-k}, s^T)}\right) = \begin{cases} \varphi_{ks^T}(m, u_s) & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, for each  $(i, s^T) \in M^{+, s^T}$ ,

$$f_{is^T}(m, u_s) = \varphi_{is^T}(m, u_s).$$

Thus,  $f_{is^T}(m, u_s)$  is uniquely determined. This concludes Step 2.2.3.

From Step 2.1 and Step 2.2, we conclude that  $f(m, u_s)$  is uniquely determined. By Linearity, we have that  $f(m, v)$  is uniquely determined, which concludes the proof of Theorem 3.4.1. ■

**Logical independence** The axioms invoked in Theorem 3.4.1 are logically independent, as shown by the following alternative solutions.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = 0,$$

satisfies all the axioms except Efficiency.

- The value  $f$  defined as:

$$f_{ij}(m, v) = \begin{cases} \varphi_{ij}(m, v) + \frac{\Delta_{(2 \wedge m_k)_{k \in N}}(v)}{|Q(1)|} & \text{if } j = 1 \text{ and } m^T > 1, \\ \varphi_{ij}(m, v) - \frac{\Delta_{(2 \wedge m_k)_{k \in N}}(v)}{|Q(2)|} & \text{if } j = 2 \text{ and } m^T > 1, \\ \varphi_{ij}(m, v) & \text{otherwise,} \end{cases}$$

satisfies all the axioms except Independence of higher activity levels.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \\ f_{ij}(m, v) = \sum_{\substack{s \leq m \\ (i, j) \in T(s)}} \frac{(v((j-1 \wedge m_h)_{h \in N}) + e_i)^2 + 1}{\sum_{(k, s_k) \in T(s)} v((j-1 \wedge m_h)_{h \in N}) + e_k)^2 + 1} \Delta_s(v),$$

satisfies all the axioms except Linearity.

- The multi-choice Equal division value,  $\xi$ , satisfies all the axioms except the null pair axiom.
- Take any  $(m, v) \in \mathcal{G}$ , and for each  $(i, j) \in M^+$  fix any arbitrary  $\beta_{ij} \in \mathbb{R}_{++}$ . The value  $f^\beta$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}^\beta(m, v) = \sum_{\substack{s \leq m \\ (i, j) \in T(s)}} \frac{\beta_{ij}}{\sum_{(k, l) \in T(s)} \beta_{kl}} \Delta_s(v),$$

satisfies all the axioms except Sign symmetry for equal pairs. ■

By Theorem 3.4.1, we provide alternative expression of the multi-choice Shapley value in terms of dividends (see Chapter 2, Section 2.3.1, definition (2.21)).

**Corollary 3.4.1.** *For each game  $(m, v) \in \mathcal{G}$ , the multi-choice Shapley value,  $\varphi$ , can be written as*

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in T(s)}} \frac{\Delta_s(v)}{|T(s)|}. \quad (3.22)$$

*Proof.* By the the proof of Theorem 3.4.1,  $\varphi$  satisfies Efficiency, Independence of higher activity levels, Linearity, the null pair axiom and Equal treatment for equal pairs. Consider any minimal game  $(m, u_s) \in \mathcal{G}$ ,  $s \in \mathcal{M}$  such that  $s \neq \mathbf{0}$ . Similarly to (3.17), for each  $(i, j) \notin T(s)$ , the null pair axiom and Multi-Efficiency imply

$$\varphi_{ij}(m, u_s) = 0. \quad (3.23)$$

All pairs in  $T(s)$  are equal pairs in  $(m, u_s)$ . Thus, by Equal treatment for equal pairs, it holds that

$$\varphi_{isT}(m, u_s) = \dots = \varphi_{i'sT}(m, u_s). \quad (3.24)$$

By Efficiency and Linearity, we obtain the desired result. ■

In the following, we look for an alternative characterization of the multi-choice Shapley value without resorting to Linearity. In line with Young [1985] and Casajus [2018], we use the Strong monotonicity axiom for multi-choice games solutions (see Chapter 2, Section 2.3.3). As our second main result, we obtain an alternative characterization of the multi-choice Shapley value by replacing Linearity and the null pair axiom in Theorem 3.4.1 by Strong monotonicity (see Chapter 2, Section 2.3.3).

**Theorem 3.4.2.** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, Independence of higher activity levels, Strong monotonicity and Sign symmetry for equal pairs if and only if  $f = \varphi$ .*

*Proof.* From Theorem 3.4.1, we know that  $\varphi$  satisfies Efficiency, Independence of higher activity levels and Sign symmetry for equal pairs. By definition (see (3.7)), the multi-choice Shapley value satisfies Strong monotonicity.

Next, we show that  $\varphi$  is the unique value satisfying all the axioms of the statement of Theorem 3.4.2. Take any  $f$  as hypothesized and consider any  $(m, v) \in \mathcal{G}$ . Recall that  $(m, v) \in \mathcal{G}$  can be rewritten as  $(m, \sum_{t \in \mathcal{M}} \Delta_t(v) u_t)$ . We define the set of coalitions for which the Harsanyi dividend is non null as

$$\mathcal{T}(v) = \{t \in \mathcal{M} \mid \Delta_t(v) \neq 0\}.$$

By induction on the cardinality of  $\mathcal{T}(v)$ , we show that

$$f(m, v) = \varphi(m, v).$$

**Initialization:** If  $|\mathcal{T}(v)| = 0$ , then each dividend is null. The only game  $(m, v) \in \mathcal{G}$  such that  $|\mathcal{T}(v)| = 0$  is the null game. Recall that  $M^{+j} = \{(i, j) \in M^+ : i \in Q(j)\}$ . Since  $f$  satisfies Efficiency, Independence of higher activity levels, by Proposition 3.4.1, it satisfies Multi-Efficiency. It follows that, for each  $j \leq \max_{k \in N} m_k$ ,

$$\begin{aligned} \sum_{(i, j) \in M^{+j}} f_{ij}(m, v) &\stackrel{(ME)}{=} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &= 0. \end{aligned} \quad (3.25)$$



Recall that any two distinct pairs  $(i, j)(i', j) \in M^{+,j}$  are equal pairs in the null game  $(m, v)$ . Therefore, by Sign symmetry for equal pairs, we obtain

$$\text{sign}(f_{ij}(m, v)) = \text{sign}(f_{i'j}(m, v)). \quad (3.26)$$

Combining (3.25) and (3.26), for each  $j \leq \max_{k \in N} m_k$  and each  $(i, j) \in M^{+,j}$ , we obtain

$$f_{ij}(m, v) = 0.$$

Recall also that each pair is a null pair in the null game. Since  $\varphi$  satisfies the null pair axiom, for each  $j \leq \max_{k \in N} m_k$  and each  $(i, j) \in M^{+,j}$ , it holds that

$$\varphi_{ij}(m, v) \stackrel{(N)}{=} 0 = f_{ij}(m, v).$$

This concludes the initialization.

**Hypothesis:** Fix  $r \in \mathbb{N}$  such that  $r < |\mathcal{M}| - 1$ . We assume that, for each  $(m, v) \in \mathcal{G}$  such that  $|\mathcal{T}(v)| \leq r$ ,

$$f(m, v) = \varphi(m, v).$$

**Induction step:** Consider any  $(m, v) \in \mathcal{G}$  such that  $|\mathcal{T}(v)| = r + 1$ . Let us show that

$$f(m, v) = \varphi(m, v).$$

We define the minimum of the set  $\mathcal{T}(v)$  as

$$p = \bigwedge_{t \in \mathcal{T}(v)} t.$$

Two cases can be distinguished. First, assume that  $p \neq \mathbf{0}$ . Consider any pair  $(i, j) \in M^+$  such that  $j > p_i$ . By definition of  $p$ , there exists a  $t \in \mathcal{T}(v)$  such that  $j > t_i$ . For such  $t$ , consider the game  $(m, v - \Delta_t(v)u_t)$ . By definition of a minimal effort game (2.19) and Remark 1, we have that  $(i, j)$  is a null pair in  $(m, \Delta_t(v)u_t)$ . Therefore  $(i, j)$  has the same marginal contributions in  $(m, v)$  and in  $(m, v - \Delta_t(v)u_t)$ . Moreover, observe that  $|\mathcal{T}(v)| > |\mathcal{T}(v - \Delta_t(v)u_t)|$ . Therefore, we have that  $r \geq |\mathcal{T}(v - \Delta_t(v)u_t)|$ . By the induction hypothesis and Strong monotonicity, for each  $(i, j) \in M^+$  such that  $j > p_i$ , we obtain

$$f_{ij}(m, v) \stackrel{SM}{=} f_{ij}(m, v - \Delta_t(v)u_t) \stackrel{Hyp}{=} \varphi_{ij}(m, v - \Delta_t(v)u_t) \stackrel{SM}{=} \varphi_{ij}(m, v). \quad (3.27)$$

Next, we assume that  $p = \mathbf{0}$ . For each  $(i, j) \in M^+$ , there exists a  $t \in \mathcal{T}(v)$  such that  $j > t_i$ . In this case, (3.27) holds for each  $(i, j) \in M^+$  and the proof is complete.

It remains to show that, if  $p \neq \mathbf{0}$ , then for each  $(i, j) \in M^+$  such that  $j \leq p_i$ ,

$$f_{ij}(m, v) = \varphi_{ij}(m, v).$$

We proceed in two steps.

**Step 1.** We define the game  $(m, w) \in \mathcal{G}$  as

$$w = v - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v)u_{(0_{-i}j)}, \quad (3.28)$$

and we show that, for each  $(i, l) \in M^+$  such that  $l \leq p_i$ ,

$$f_{il}(m, w) = 0. \quad (3.29)$$

**Step 1.1.** To that end, we show that

$$\sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m, w) = 0.$$

By Proposition 3.4.1,  $f$  satisfies Multi-Efficiency. By Multi-Efficiency and (3.28), for each  $l \leq \max_{k \in N} m_k$ , it holds that

$$\begin{aligned} & \sum_{(i,l) \in M^{+,l}} f_{il}(m, w) \stackrel{(ME)}{=} w((l \wedge m_k)_{k \in N}) - w((l - 1 \wedge m_k)_{k \in N}) \\ \iff & \sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m, w) = w((l \wedge m_k)_{k \in N}) - w((l - 1 \wedge m_k)_{k \in N}) - \sum_{\substack{(i,l) \in M^{+,l} \\ l > p_i}} f_{il}(m, w) \\ & \stackrel{(3.28)}{=} v((l \wedge m_k)_{k \in N}) - v((l - 1 \wedge m_k)_{k \in N}) \\ & \quad - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0-i,j)}((l \wedge m_k)_{k \in N}) \\ & \quad + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0-i,j)}((l - 1 \wedge m_k)_{k \in N}) \\ & \quad - \sum_{\substack{(i,l) \in M^{+,l} \\ l > p_i}} f_{il}(m, w). \end{aligned} \quad (3.30)$$

Before proceeding further into the computation of (3.30), observe that

$$\begin{aligned} & - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0-i,j)}((l \wedge m_k)_{k \in N}) + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0-i,j)}((l - 1 \wedge m_k)_{k \in N}) \\ & = - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j < l}} \varphi_{ij}(m, v) u_{(0-i,j)}((l \wedge m_k)_{k \in N}) - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j = l}} \varphi_{ij}(m, v) u_{(0-i,j)}((l \wedge m_k)_{k \in N}) \\ & \quad - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j > l}} \varphi_{ij}(m, v) u_{(0-i,j)}((l \wedge m_k)_{k \in N}) \\ & \quad + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j < l}} \varphi_{ij}(m, v) u_{(0-i,j)}((l - 1 \wedge m_k)_{k \in N}) \\ & \quad + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j \geq l}} \varphi_{ij}(m, v) u_{(0-i,j)}((l - 1 \wedge m_k)_{k \in N}) \end{aligned}$$

By definition,  $p_i \leq m_i$ , for each  $i \in N$ . For each  $i \in N$  and  $j \leq p_i \leq m_i$ , it holds that

$$u_{(0_{-i,j})}((l \wedge m_k)_{k \in N}) = \begin{cases} 1 & \text{if } j \leq (l \wedge m_i), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} & - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0_{-i,j})}((l \wedge m_k)_{k \in N}) + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m, v) u_{(0_{-i,j})}((l-1 \wedge m_k)_{k \in N}) \\ = & - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j < l}} \varphi_{ij}(m, v) - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j = l}} \varphi_{ij}(m, v) + \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j < l}} \varphi_{ij}(m, v) \\ = & - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i \\ j = l}} \varphi_{ij}(m, v) = - \sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} \varphi_{il}(m, v). \end{aligned} \quad (3.31)$$

By (3.31), (3.30) becomes

$$\sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m, w) = v((l \wedge m_k)_{k \in N}) - v((l-1 \wedge m_k)_{k \in N}) - \sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} \varphi_{il}(m, v) \quad (3.32)$$

$$- \sum_{\substack{(i,l) \in M^{+,l} \\ l > p_i}} f_{il}(m, w). \quad (3.33)$$

By (3.27), for each  $(i, l) \in M^{+,l}$  such that  $l > p_i$ , it holds that

$$f_{il}(m, w) = \varphi_{il}(m, w). \quad (3.34)$$

Combining (3.32) and (3.34), we obtain

$$\sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m, w) = v((l \wedge m_k)_{k \in N}) - v((l-1 \wedge m_k)_{k \in N}) - \sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} \varphi_{il}(m, v) \quad (3.35)$$

$$a \quad - \sum_{\substack{(i,l) \in M^{+,l} \\ l > p_i}} \varphi_{il}(m, w). \quad (3.36)$$

Moreover, each  $(i, l) \in M^{+,l}$ , such that  $l > p_i$ , is a null pair in  $(m, u_{0_{-i,j}})$ , for each  $(i, j) \in M^+$  such that  $j \leq p_i$ . By definition of  $(m, w)$  (see (3.28)), it follows that each pair  $(i, l)$ , such that  $l > p_i$ , has the same marginal contributions in  $(m, w)$  and in  $(m, v)$ . Since  $\varphi$  satisfies Strong monotonicity, for each  $(i, l) \in M^{+,l}$  such that  $l > p_i$ , it holds that

$$\varphi_{il}(m, w) = \varphi_{il}(m, v). \quad (3.37)$$

Combining (3.35), (3.37) and the fact that  $\varphi$  satisfies Multi-Efficiency, we obtain

$$\begin{aligned} \sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m, w) &= v((l \wedge m_k)_{k \in N}) - v((l-1 \wedge m_k)_{k \in N}) - \sum_{(i,l) \in M^{+,l}} \varphi_{il}(m, v) \\ &\stackrel{(ME)}{=} 0. \end{aligned} \quad (3.38)$$

This concludes Step 1.1.

**Step 1.2.** We show that all the pairs  $(i, l) \in M^{+,l}$ , such that  $l \leq p_i$ , are equal in  $(m, w)$ .

By definition of  $M^{+,l}$ , we have that  $l \geq 1$ . Consider two pairs  $(i, l), (i', l) \in M^{+,l}$  such that  $l \leq p_i$  and  $l \leq p_{i'}$ . Since  $p = \bigwedge_{t \in \mathcal{T}(v)} t$ , we have that each  $t \in \mathcal{T}(v)$  verifies  $t_i \geq l$  and  $t_{i'} \geq l$ . In other words, for each  $s \in \mathcal{M}$ , such that  $s_i < l$  or  $s_{i'} < l$ , we have that  $\Delta_s(v) = 0$ . Therefore, for each  $s \in \mathcal{M}$  such that  $s_i = s_{i'} = l-1$ , we obtain

$$v(s + e_i) = v(s + e_{i'}) = 0. \quad (3.39)$$

Therefore,  $(i, l)$  and  $(i', l)$  are equal in  $(m, v)$ . Since  $\varphi$  satisfies Equal treatment for equal pairs and by (3.28), for each  $s \in \mathcal{M}$  such that  $s_i = s_{i'} = l-1$ , it holds that

$$\begin{aligned} w(s + e_i) &\stackrel{(3.28)}{=} v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h}} \varphi_{hj}(m, v) u_{(0-h,j)}(s + e_i) \\ &= v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0-h,j)}(s + e_i) - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0-i',j)}(s + e_i) \\ &\quad - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0-i,j)}(s + e_i) - \varphi_{il}(m, v) u_{(0-i,l)}(s + e_i) \\ &= v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0-h,j)}(s + e_i) - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0-i',j)}(s + e_i) \\ &\quad - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0-i,j)}(s + e_i) - \varphi_{il}(m, v) \\ &\stackrel{(ET), (3.39)}{=} v(s + e_{i'}) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0-h,j)}(s + e_{i'}) - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0-i',j)}(s + e_{i'}) \\ &\quad - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0-i,j)}(s + e_{i'}) - \varphi_{i'l}(m, v) \\ &\stackrel{(3.28)}{=} w(s + e_{i'}). \end{aligned}$$

Therefore, two pairs  $(i, l)$  and  $(i', l)$ , such that  $l \leq p_i$  and  $l \leq p_{i'}$ , are equal in  $(m, w)$ . This concludes Step 1.2.

By Sign symmetry for equal pairs, it holds that

$$\text{sign}(f_{il}(m, w)) = \text{sign}(f_{i'l}(m, w)). \quad (3.40)$$

Combining (3.38) and (3.40), for each  $(i, l) \in M^+$  such that  $l \leq p_i$ , we obtain

$$f_{il}(m, w) = 0,$$

which concludes Step 1.

**Step 2.** For each  $(i, j) \in M^+$ , such that  $0 < j \leq p_i$ , we define the game  $(m, w^{ij}) \in \mathcal{G}$  as

$$w^{ij} = v - \varphi_{ij}(m, v)u_{(0_{-i,j})}. \quad (3.41)$$

In this step, we first show that, for each  $(i, j) \in M^+$  such that  $j \leq p_i$ ,

$$\varphi_{ij}(m, v) = f_{ij}(m, v) - f_{ij}(m, w^{ij}).$$

The game  $(m, w^{ij})$  is defined in such a way that the pair  $(i, j)$  has the same marginal contribution in  $(m, w)$  as in  $(m, w^{ij})$ . Indeed, observe that the pair  $(i, j)$  has null marginal contributions to coalition in each game  $(m, u_{(0_{-i',j'})})$  such that  $i' \neq i$  or  $i' = i$  and  $j' \neq j$ . Therefore, by Strong monotonicity, for each  $(i, j) \in M^+$  such that  $j \leq p_i$ , it holds that

$$f_{ij}(m, w) = f_{ij}(m, w^{ij}). \quad (3.42)$$

Additionally, by Multi-Efficiency, (3.41) and the definition of a minimal effort game (see (2.19)), it holds that

$$\begin{aligned} \sum_{(k,j) \in M^{+,j}} f_{kj}(m, w^{ij}) &\stackrel{(ME)}{=} w^{ij}((j \wedge m_k)_{k \in N}) - w^{ij}(((j-1) \wedge m_k)_{k \in N}) \\ &\stackrel{(3.41)}{=} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &\quad - \varphi_{ij}(m, v)u_{(0_{-i,j})}((j \wedge m_k)_{k \in N}) \\ &\quad + \varphi_{ij}(m, v)u_{(0_{-i,j})}(((j-1) \wedge m_k)_{k \in N}) \\ &\stackrel{(2.19)}{=} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) - \varphi_{ij}(m, v). \end{aligned} \quad (3.43)$$

Each pair in  $M^{+,j} \setminus \{(i, j)\}$  is null in  $(m, u_{0_{-i,j}})$ . Therefore, by (3.41), each pair in  $M^{+,j} \setminus \{(i, j)\}$  has the same marginal contribution in  $(m, w^{ij})$  and in  $(m, v)$ . It follows that, by Strong monotonicity, each pair in  $M^{+,j} \setminus \{(i, j)\}$  receives the same payoff in  $(m, w^{ij})$  and in  $(m, v)$ . Then, we obtain

$$\begin{aligned} \sum_{(k,j) \in M^{+,j}} f_{kj}(m, w^{ij}) &= \sum_{\substack{(k,j) \in M^{+,j} \\ k \neq i}} f_{kj}(m, w^{ij}) + f_{ij}(m, w^{ij}) \\ &\stackrel{(SM)}{=} \sum_{\substack{(k,j) \in M^{+,j} \\ k \neq i}} f_{kj}(m, v) + f_{ij}(m, w^{ij}) \\ &\stackrel{(ME)}{=} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &\quad - f_{ij}(m, v) + f_{ij}(m, w^{ij}). \end{aligned} \quad (3.44)$$

Combining (3.43) and (3.44), for each  $(i, j) \in M^+$  such that  $j \leq p_i$ , we obtain

$$\varphi_{ij}(m, v) = f_{ij}(m, v) - f_{ij}(m, w^{ij}), \quad (3.45)$$

which concludes Step 2.

We have the material to conclude the proof of the Induction step. By (3.42), we have that  $f_{ij}(m, w^{ij}) = f_{ij}(m, w)$  and by (3.29) we have that  $f_{ij}(m, w) = 0$ , for each  $(i, j) \in M^+$  such that  $j \leq p_i$ . By (3.45), for each  $(i, j) \in M^+$  such that  $j \leq p_i$ , it holds that

$$f_{ij}(m, v) = \varphi_{ij}(m, v).$$

Therefore, for each  $(m, v) \in \mathcal{G}$  and each  $(i, j) \in M^+$ , it holds that  $f_{ij}(m, v) = \varphi_{ij}(m, v)$ . The proof of the theorem is complete. ■

**Logical independence** The axioms invoked in Theorem 3.4.2 are logically independent, as shown by the following alternative solutions.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = 0,$$

satisfies all the axioms except Efficiency.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+,$$

$$f_{ij}(m, v) = \begin{cases} \varphi_{ij}(m, v) + m^T & \text{if } (i, j) = (1, 1) \text{ and } \varphi_{ij}(m, v) > m^T, \\ \varphi_{ij}(m, v) - m^T & \text{if } (i, j) = (n, 1) \text{ and } \varphi_{ij}(m, v) > m^T, \\ \varphi_{ij}(m, v) & \text{otherwise,} \end{cases}$$

satisfies all the axioms except Independence of higher activity levels.

- The multi-choice Equal division value  $\xi$  satisfies all the axioms except Strong monotonicity.
- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \begin{cases} \varphi_{ij}(m, v) + 1 & \text{if } (i, j) = (1, 1), \\ \varphi_{ij}(m, v) - 1 & \text{if } (i, j) = (n, 1), \\ \varphi_{ij}(m, v) & \text{otherwise,} \end{cases}$$

satisfies all the axioms except Sign symmetry for equal pairs.

### 3.4.2 Characterization of the multi-choice Equal Division value

In this section, we characterize the multi-choice Equal Division values. To that end, we introduce the Sign preservation axiom. This axiom is a stronger version of Sign symmetry for equal pairs. It requires that two pairs featuring the same activity level receive a payoff of the same sign. In presence of Multi-Efficiency, this requirement can be seen as a solidarity condition with respect to the surplus generated between two successive synchronized coalitions. A value  $f$  on  $\mathcal{G}$  satisfies:

*Sign preservation (SP).* For each  $(m, v) \in \mathcal{G}$  and each  $(i, j), (i', j) \in M^+$ ,

$$\text{sign}(f_{ij}(m, v)) = \text{sign}(f_{i'j}(m, v)).$$

The next result provides an axiomatic characterization of the multi-choice Equal Division value.

**Theorem 3.4.3.** *A value  $f$  on  $\mathcal{G}$  satisfies Efficiency, Independence of higher activity levels, Linearity, Sign preservation and Equal treatment for equal pairs if and only if  $f = \xi$ .*

*Proof.* The proof is divided in two steps.

**Step 1:** We show that  $\xi$  satisfies all the axioms of the statement of Theorem 3.4.3. For each  $(m, v) \in \mathcal{G}$ , it holds that

$$\begin{aligned} \sum_{i \in N} \sum_{j \in M_i^+} \xi_{ij}(m, v) &= \sum_{j \leq \max_{k \in N} m_k} \sum_{i \in Q(j)} \frac{1}{|Q(j)|} \left[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right] \\ &= \sum_{j \leq \max_{k \in N} m_k} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &= v(m). \end{aligned}$$

This shows that the value satisfies Efficiency. By definition of the Equal Division value (see (3.8)), the payoff of a pair does not depend on activity levels different from the one contained in this pair. Therefore,  $\xi$  satisfies Independence of higher activity levels. By definition of  $\xi$ , it is straightforward to see that  $\xi$  satisfies Linearity, Sign preservation and Equal treatment for equal pairs. This concludes Step 1.

**Step 2:** We show the uniqueness part of the theorem. Let  $f$  be a value satisfying all the axioms of the statement of Theorem 3.4.3. We know that each characteristic function  $v$  admits a linear decomposition in terms of Dirac games (see 2.17). By Linearity, for each  $(m, v) \in \mathcal{G}$ , it holds that

$$f(m, v) = \sum_{s \leq m} v(s) f(m, \delta_s).$$

For each  $s \in \mathcal{M}$ , we show that

$$f(m, \delta_s) = \xi_{ij}(m, \delta_s).$$

We consider two cases.

**Case 1.** Suppose that  $s \in \mathcal{M}$  is not a synchronized coalition, that is  $s \neq ((l \wedge m_k)_{k \in N})$ , for each  $l \leq \max_{k \in N} m_k$ . Since  $f$  satisfies Efficiency and Independence of higher activity levels, by Proposition 3.4.1 it satisfies Multi-Efficiency. Therefore, by Multi-Efficiency, for each  $j \leq \max_{k \in N} m_k$ ,

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}).$$

Since  $s \neq ((j \wedge m_k)_{k \in N})$  and  $s \neq (((j-1) \wedge m_k)_{k \in N})$ , by definition of a Dirac game,

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = 0. \quad (3.46)$$

Since  $\delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}) \geq 0$ , by Sign preservation and (3.46), for each  $(i, j) \in M^{+,j}$ , it holds that

$$f_{ij}(m, \delta_s) \geq 0. \quad (3.47)$$

Combining (3.46) and (3.47), for each  $(i, j) \in M^{+,j}$ , we obtain

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 2.** Suppose that  $s \in \mathcal{M}$  is a synchronized coalition, that is  $s = (l \wedge m_k)_{k \in N}$ , where  $l \leq \max_{k \in N} m_k$ .

**Case 2.1.** Take any activity level  $j$  such that  $j < l$ . By Multi-Efficiency, it holds that

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}).$$

Since  $s \neq ((j \wedge m_k)_{k \in N})$  and  $s \neq (((j-1) \wedge m_k)_{k \in N})$ , by definition of a Dirac game, we have that

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = 0.$$

Observe that  $\delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}) = 0$ . Similarly to Case 1, by Sign preservation and (3.46), for each pair  $(i, j) \in M^{+,j}$  such that  $j < l$ ,

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 2.2.** Take any activity level  $j > l + 1$ . Similarly to Case 2.1, for each  $(i, j) \in M^+$  such that  $j > l + 1$

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 2.3.** Consider the pairs  $(i, j) \in M^+$  such that  $j = l$ , that is the pairs in  $M^{+,l}$ . By Multi-Efficiency and the definition of a Dirac game, it holds that

$$\begin{aligned} \sum_{(i,l) \in M^{+,l}} f_{il}(m, \delta_s) &= \delta_s((l \wedge m_k)_{k \in N}) - \delta_s((l-1 \wedge m_k)_{k \in N}) \\ &= 1. \end{aligned} \tag{3.48}$$

Two distinct pairs  $(i, l), (i', l) \in M^{+,l}$  are equal in  $(m, \delta_s)$ . By Equal treatment for equal pairs, one obtains

$$f_{il}(m, \delta_s) = f_{i'l}(m, \delta_s). \tag{3.49}$$

From (3.48) and (3.49), it follows that

$$f_{il}(m, \delta_s) = \frac{1}{|Q(l)|} = \xi_{ij}(m, \delta_s).$$

**Case 2.4.** Consider the pairs  $(i, l+1) \in M^{+,l+1}$ . By Multi-Efficiency and the definition of a Dirac game, it holds that

$$\sum_{(i,l+1) \in M^{+,l+1}} f_{i(l+1)}(m, \delta_s) = \delta_s(((l+1) \wedge m_k)_{k \in N}) - \delta_s((l \wedge m_k)_{k \in N}) = 0 - 1 = -1.$$

Similarly to Case 2.3, for each  $(i, l+1) \in M^{+,l+1}$ ,

$$f_{i(l+1)}(m, \delta_s) = -\frac{1}{|Q(l+1)|} = \xi_{i(l+1)}(m, \delta_s).$$



Therefore, for each  $s \in \mathcal{M}$ , we have that  $f(m, \delta_s) = \zeta(m, \delta_s)$ . By Linearity, we conclude the proof of Theorem 3.4.3. ■

**Logical independence** The axioms invoked in Theorem 3.4.3 are logically independent, as shown by the following alternative solutions.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = 0,$$

satisfies all the axioms except Efficiency.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \frac{1}{|Q(j)|} \sum_{k \geq j} \frac{v((k \wedge m_h)_{h \in N}) - v(((k-1) \wedge m_h)_{h \in N})}{k}$$

satisfies all the axioms except Independence of higher activity levels.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\begin{aligned} \forall (i, j) \in M^+, \quad f_{ij}(m, v) = & \frac{(v(((j-1) \wedge m_h)_{h \in N} + e_i)^2) + 1}{\sum_{k \in Q(k)} ((v(((j-1) \wedge m_h)_{h \in N} + e_k)^2) + 1)} \\ & \times [v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})], \end{aligned}$$

satisfies all the axioms except Linearity.

- The multi-choice Shapley value  $\varphi$  satisfies all the axioms except Sign preservation.
- Take any  $(m, v) \in \mathcal{G}$  and for each  $(i, j) \in M^+$  fix any arbitrary integer  $\beta^{ij} \in \{1, 2\}$ . The value  $f^\beta$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}^\beta(m, v) = \frac{\beta^{ij}}{\sum_{k \in Q(j)} \beta^{kj}} [v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})],$$

satisfies all the axioms except Equal treatment for equal pairs.

### 3.4.3 Characterization of the multi-choice Egalitarian Shapley values

In this subsection we provide an axiomatic characterization of the multi-choice Egalitarian Shapley values. To that end, we introduce two new axioms.

In presence of Multi-Efficiency, the surplus generated between two consecutive synchronized coalitions is what must be allocated among the pairs containing the required activity level (see Remark 6). This surplus can eventually be negative. Requiring that the payoff of a pair varies according to its marginal contributions to coalitions regardless of the surplus to be shared is then a strong requirement in Strong monotonicity. To the contrary, it seems reasonable that the payoff of a pair, let us say  $(i, j) \in M^{+j}$ , does not decrease from one game, let us say  $(m, v) \in \mathcal{G}$ , to another, let us say  $(m, w) \in \mathcal{G}$ , if the surplus generated between the  $j$ -synchronized coalition and the  $(j-1)$ -synchronized coalition does not decrease from  $(m, v)$  to

$(m, w)$ . The next axiom strengthens the hypothesis of Strong monotonicity by requiring that the surplus generated between two synchronized coalitions should not decrease from one game to another. A value  $f$  on  $\mathcal{G}$  satisfies:

*Weak monotonicity (WM)*. Take any two  $(m, v), (m, w) \in \mathcal{G}$ . If there exists a pair  $(i, j) \in M^+$  such that

$$v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \geq w((j \wedge m_k)_{k \in N}) - w(((j-1) \wedge m_k)_{k \in N}),$$

and such that

$$\forall s \in \mathcal{M}, s_i = j-1, \quad v(s + e_i) - v(s) \geq w(s + e_i) - w(s),$$

then, it holds that  $f_{ij}(m, v) \geq f_{ij}(m, w)$ .

Whenever  $m = (1, \dots, 1)$ , Weak monotonicity corresponds to the Weak Monotonicity axiom for TU-games introduced by [van den Brink et al. \[2013\]](#). Obviously, Strong monotonicity implies Weak monotonicity.

Consider  $(m, v) \in \mathcal{G}$  and two distinct pairs  $(i, j), (i', j) \in M^{+,j}$ . We say that the pair  $(i, j)$  is more *desirable* than the pair  $(i', j)$  in  $(m, v)$  if its has better marginal contributions to coalitions. Formally,  $(i, j)$  is more desirable than  $(i', j)$  if for each  $s \in \mathcal{M}$ , such that  $s_i = s_{i'} = j-1$ , we have that

$$v(s + e_i) \geq v(s + e_{i'}).$$

The next axiom requires that a pair receives a greater payoff than other less desirable pairs. In the context of Example 1, this axiom ensures that if one department performs better than the other department on a given workday, then the former receives a higher budget than the latter for this workday. A value  $f$  on  $\mathcal{G}$  satisfies:

*Level desirability (LD)*. Take any  $(m, v) \in \mathcal{G}$ . If there exists two distinct pairs  $(i, j), (i', j) \in M^+$ , such that  $(i, j)$  is more desirable than  $(i', j)$  in  $(m, v)$ , then, it holds that

$$f_{ij}(m, v) \geq f_{i'j}(m, v).$$

Whenever  $m = (1, \dots, 1)$ , Level desirability corresponds to the classical Desirability axiom for TU-games. We have the material to provide a characterization of the multi-choice Egalitarian Shapley values.

**Theorem 3.4.4.** *A solution  $f$  on  $\mathcal{G}$  satisfies Efficiency, Independence of higher activity levels, Linearity, Weak monotonicity and Level desirability if and only if  $f = \chi^\alpha$ , for some parameter system  $\alpha$ .*

*Proof.* Before starting the proof, which is divided in two steps, we provide a useful remark.

**Remark 9.** By definition, Level desirability implies Equal treatment for equal pairs. If  $(m, v) \in \mathcal{G}$  is the null game, then Multi-Efficiency and Level desirability imply  $f_{ij}(m, v) = 0$  for each  $(i, j) \in M^+$ .

Observe that  $\varphi$  and  $\xi$  both satisfy Level desirability and Weak monotonicity. Consider any parameter system  $\alpha$ . Since that multi-choice Egalitarian Shapley values are convex combinations of the multi-choice Shapley value and the multi-choice Equal division value (see (3.9)),  $\chi^\alpha$ , inherits all the axioms of the statement of Theorem 3.4.4.

Next, we show that the multi-choice Egalitarian Shapley values are the only values satisfying all the axioms of the statement of Theorem 3.4.4. Consider a value  $f$  satisfying all the axioms of the statement of Theorem 3.4.4. To prove the uniqueness part, we show that, for each  $(m, v) \in \mathcal{G}$ , there exists a parameter system  $\alpha$  such that

$$f(m, v) = \chi^\alpha(m, v).$$

By Linearity, for each  $(m, v) \in \mathcal{G}$ ,

$$f(m, v) = \sum_{t \leq m} \Delta_t(v) f(m, u_t).$$

Recall that  $t^T = \max_{i \in N} t_i$  denotes the highest activity level played in coalition  $t \in \mathcal{M}$ . Take any  $1 \leq l \leq m^T$ . We show that, for each  $(m, u_t)$  such that  $t^T = l$ ,  $f$  can be written as

$$\begin{aligned} \forall (i, j) \in M^+, \\ f_{ij}(m, u_t) = \begin{cases} \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t) & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for some  $0 \leq \alpha^l \leq 1$ . To that end, consider all pairs  $(i, j) \in M^+$  such that  $j < l$ . By Independence of higher activity levels,

$$f_{ij}(m, u_t) = f_{ij}((j \wedge m_k)_{k \in N}, u_t).$$

Since  $((j \wedge m_k)_{k \in N}, u_t)$  is the null game, by Remark 9, for each  $(i, j) \in M^+$  such that  $j < l$ ,

$$f_{ij}(m, u_t) = 0. \quad (3.50)$$

Consider all pairs  $(i, j) \in M^+$  such that  $j > l$ . Observe that these pairs are null pairs in  $(m, u_t)$  and thus are equal. From Remark 9 and Multi-Efficiency, for each  $(i, j) \in M^+$  such that  $j > l$ ,

$$f_{ij}(m, u_t) = 0. \quad (3.51)$$

Now, consider any pair  $(i, l) \in M^+$ . We show that  $f$  can be written as

$$f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t),$$

for some  $0 \leq \alpha^l \leq 1$ . We proceed by induction on  $q^t(l)$  the number of agents that play  $l$  in coalition  $t$ .

**Initialization:** Pick any minimal effort game  $(m, u_t) \in \mathcal{G}$  such that  $t^T = l$  and  $q^t(l) = 1$ . In such game, there is exactly one agent, say  $k \in N$ , that plays the activity level  $l$  in  $t$ .

Before proceeding further into the initialization step, we prove the following claim.

**Claim:** Pick any minimal effort game  $(m, u_t) \in \mathcal{G}$ , where  $t^T = l$  and  $q^t(l) = 1$ . Then, there exists  $c^l \in \mathbb{R}$  such that

$$\forall (i, l) \in M^+, i \neq k, \quad f_{il}(m, u_t) = c^l, \quad (3.52)$$

where  $k$  refers to the only agent that plays the activity level  $l$  in  $t$ . In other words, the

payoff assigned to the pair  $(i, l) \in M^+, i \neq k$ , does not depend on the agent  $k$  that plays  $t^T = l$  in  $(m, u_t)$ . To prove this claim, we distinguish three cases.

**Case 1.** If  $|Q(m^T)| \geq 3$ , then there is at least three agents in  $Q(l)$ . Consider any three distinct agents in  $Q(l)$  denoted by  $k, i$  and  $i'$ . Consider the three coalitions  $t, t', t'' \in \mathcal{M}$  defined as

$$\begin{aligned} t_k &= l, \text{ and } \forall h \in N \setminus \{k\}, \quad t_h < l, \\ t'_i &= l, \text{ and } \forall h \in N \setminus \{i\}, \quad t'_h < l, \\ t''_{i'} &= l, \text{ and } \forall h \in N \setminus \{i'\}, \quad t''_h < l. \end{aligned}$$

Consider the minimal effort games  $(m, u_t)$ ,  $(m, u_{t'})$  and  $(m, u_{t''})$ . Observe that:

- $(i, l) \in M^+$  is a null pair in  $(m, u_t)$  and  $(m, u_{t''})$ , and  $(i, l)$  is the only non null pair in  $(m, u_{t'})$ ;
- $(i', l) \in M^+$  is a null pair in  $(m, u_t)$  and  $(m, u_{t'})$ , and  $(i', l)$  is the only non null pair in  $(m, u_{t''})$ ;
- $(k, l) \in M^+$  is a null pair in  $(m, u_{t'})$  and  $(m, u_{t''})$ , and  $(k, l)$  is the only non null pair in  $(m, u_t)$ ;
- $(i, l), (i', l) \in M^+$  are equal pairs in  $(m, u_t)$ ;
- $(i, l), (k, l) \in M^+$  are equal pairs in  $(m, u_{t''})$ ;
- $(i', l), (k, l) \in M^+$  are equal pairs in  $(m, u_{t'})$ .

Moreover, observe that

$$\begin{aligned} u_t((l \wedge m_h)_{h \in N}) &= u_{t'}((l \wedge m_h)_{h \in N}) \\ &= u_{t''}((l \wedge m_h)_{h \in N}) \\ \text{and } u_t(((l-1) \wedge m_h)_{h \in N}) &= u_{t'}(((l-1) \wedge m_h)_{h \in N}) \\ &= u_{t''}(((l-1) \wedge m_h)_{h \in N}). \end{aligned} \quad (3.53)$$

By (3.53), the fact that  $(i, l) \in M^+$  is a null pair in  $(m, u_t)$  and  $(m, u_{t''})$ , and Weak monotonicity, one obtains

$$f_{il}(m, u_t) = f_{il}(m, u_{t''}). \quad (3.54)$$

Similarly,

$$f_{i'l}(m, u_t) = f_{i'l}(m, u_{t'}) \quad \text{and} \quad f_{kl}(m, u_{t'}) = f_{kl}(m, u_{t''}). \quad (3.55)$$

By the fact that  $(i, l), (i', l) \in M^+$  are equal pairs in  $(m, u_t)$  and Level desirability, one obtains

$$f_{il}(m, u_t) = f_{i'l}(m, u_t). \quad (3.56)$$

Similarly,

$$f_{i'l}(m, u_{t'}) = f_{kl}(m, u_{t'}) \quad \text{and} \quad f_{il}(m, u_{t''}) = f_{kl}(m, u_{t''}). \quad (3.57)$$

Combining (3.54), (3.55), (3.56) and (3.57), one obtains

$$\begin{aligned} f_{il}(m, u_t) &\stackrel{(3.54)}{=} f_{il}(m, u_{t''}) \stackrel{(3.57)}{=} f_{kl}(m, u_{t''}) \stackrel{(3.55)}{=} f_{kl}(m, u_{t'}) \\ &\stackrel{(3.57)}{=} f_{i'l}(m, u_{t'}) \stackrel{(3.55)}{=} f_{i'l}(m, u_t) = c^l, \end{aligned}$$

for some  $c^l \in \mathbb{R}$ .

We have shown that there exists a unique  $c^l \in \mathbb{R}$  such that, for any minimal effort game  $(m, u_t) \in \mathcal{G}$ ,  $|Q(m^T)| \geq 3$ ,  $t^T = l$ ,  $q^t(l) = 1$ , (3.52) holds.

**Case 2.** If  $|Q(m^T)| = 2$ , then there is at least two agents in  $Q(l)$ . If there are at least three agents in  $Q(l)$ , then the proof is identical to the one in Case 1. Therefore, let us assume that  $Q(l) = \{i, k\}$ . Consider the two coalitions  $t, t' \in \mathcal{M}$  defined as

$$\begin{aligned} t_i &= l, \quad \text{and} \quad \forall h \in N \setminus \{i\}, \quad t_h < l, \\ t'_k &= l, \quad \text{and} \quad \forall h \in N \setminus \{k\}, \quad t'_h < l. \end{aligned}$$

Consider the minimal games  $(m, u_t) \in \mathcal{G}$ ,  $(m, u_{t'}) \in \mathcal{G}$ , and  $(m, u_t + u_{t'}) \in \mathcal{G}$ . Observe that  $(i, l), (k, l) \in M^+$  are equal pairs in  $(m, u_t + u_{t'})$ . Therefore, by Level desirability, it holds that

$$f_{il}(m, u_t + u_{t'}) = f_{kl}(m, u_t + u_{t'}). \quad (3.58)$$

By Linearity, (3.58) becomes

$$\begin{aligned} f_{il}(m, u_t) + f_{il}(m, u_{t'}) &= f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) \\ \iff f_{il}(m, u_t) &= f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) - f_{il}(m, u_{t'}). \end{aligned} \quad (3.59)$$

Since  $f$  satisfies Efficiency and Independence of higher activity, by Proposition 3.4.1,  $f$  satisfies Multi-Efficiency. Since  $i$  and  $k$  are the only two agents in  $Q(l)$ , by Multi-Efficiency, it holds that

$$f_{il}(m, u_t) + f_{kl}(m, u_t) = 1 \quad \text{and} \quad f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) = 1.$$

It follows that

$$f_{il}(m, u_t) + f_{kl}(m, u_t) = f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) \quad (3.60)$$

Combining (3.59) and (3.60), one obtains

$$\begin{aligned} &f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) - f_{il}(m, u_{t'}) + f_{kl}(m, u_t) \\ &= f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) \\ \iff &f_{kl}(m, u_t) - f_{il}(m, u_{t'}) + f_{kl}(m, u_t) = f_{il}(m, u_{t'}) \\ \iff &f_{kl}(m, u_t) = f_{il}(m, u_{t'}) = c^l, \end{aligned}$$

for some  $c^l \in \mathbb{R}$ .

We have shown that there exists a unique  $c^l \in \mathbb{R}$  such that, for any minimal effort game  $(m, u_t) \in \mathcal{G}$ ,  $|Q(m^T)| = 2$ ,  $t^T = l$ ,  $q^t(l) = 1$ , (3.52) holds.

**Case 3.** Finally, assume that  $|Q(m^T)| = 1$ . If there is only one agent in  $Q(l)$ , then there is nothing to show. If there are two agents in  $Q(l)$ , then (3.52) holds according to Case 2. If there three agents or more in  $Q(l)$ , then (3.52) holds according to Case 1.

We have shown that there exists a unique  $c^l \in \mathbb{R}$  such that, for any minimal effort

game  $(m, u_t) \in \mathcal{G}$ ,  $t^T = l$  and  $q^t(l) = 1$ , (3.52) holds. This concludes the proof of the claim.

Next, pick any minimal effort game  $(m, u_t) \in \mathcal{G}$  such that  $t^T = l$  and  $q^t(l) = 1$ . Observe that the pair  $(k, l)$  is the only pair featuring the activity level  $l$  which is non null in  $(m, u_t)$ , and  $u_t(m) \geq 0$ . By (3.52), the fact that null pairs featuring the activity levels  $l$  are equal pairs in  $(m, u_t)$  and Multi-Efficiency, it holds that

$$f_{kl}(m, u_t) = 1 - (|Q(l)| - 1)c^l.$$

Define  $\alpha^l = 1 - c^l|Q(l)|$  so that we obtain

$$c^l = \frac{1 - \alpha^l}{|Q(l)|}.$$

Now, we show that  $\alpha^l \leq 1$ . By Remark 9, each pair receives a zero payoff in the null game. Observe that each pair in  $M^+$  has better marginal contributions to coalitions in  $(m, u_t)$  than in the null game. Moreover, it holds that

$$u_t((l \wedge m_k)_{k \in N}) - u_t(((l-1) \wedge m_k)_{k \in N}) \geq 0.$$

Thus, by Weak monotonicity,  $f_{il}(m, u_t) \geq 0$  for each  $(i, l) \in M^+$ . It follows that

$$c^l = \frac{1 - \alpha^l}{|Q(l)|} \geq 0 \implies \alpha^l \leq 1.$$

Therefore, for each  $(i, l) \in M^+$ ,

$$f_{il}(m, u_t) = \begin{cases} \frac{1 - \alpha^l}{|Q(l)|} & \text{if } j = l \text{ and } i \neq k, \\ \frac{1 - \alpha^l}{|Q(l)|} + \alpha^l & \text{if } j = l \text{ and } i = k, \end{cases} \quad (3.61)$$

for some  $0 \leq \alpha^l \leq 1$ . Observe that, for each  $(i, l) \in M^+$ ,

$$\xi_{il}(m, u_t) = \frac{1}{|Q(l)|}, \quad \varphi_{il}(m, u_t) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Comparing  $\xi_{il}(m, u_t)$  and  $\varphi_{il}(m, u_t)$  with (3.61), one obtains

$$\forall (i, l) \in M^+ \quad \text{and} \quad f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t).$$

for some  $0 \leq \alpha^l \leq 1$ . This concludes the initialization.

**Hypothesis:** Consider  $r \in \mathbb{N}$  such that  $1 \leq r < |Q(l)|$ . Consider any  $t$  such that  $q^t(l) = r$ . In this case, there are  $r$  agents that play  $l$  in  $t$ . Assume that

$$\forall (i, l) \in M^+, \quad f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t).$$

**Induction step:** Consider any  $t$  such that  $q^t(l) = r + 1$ . Let  $s = t - e_h$ , for some  $h \in N$  such that  $t_h = l$ . Obviously, it holds that  $q^s(l) = r$ . Recall that  $(i, l) \notin T(t)$  if  $t_i < l$ . Observe that if  $(i, l) \notin T(t)$  then  $(i, l) \notin T(s)$ . If  $(i, l) \notin T(t)$  then  $(i, l)$  is a null pair in  $(m, u_t)$  and is also a null pair in  $(m, u_s)$ . Therefore, each  $(i, l) \notin T(t)$  has the same marginal contributions in both games  $(m, u_t)$  and  $(m, u_s)$ . Moreover, it holds

that

$$u_t((l \wedge m_h)_{h \in N}) - u_t(((l-1) \wedge m_h)_{h \in N}) = u_s((l \wedge m_h)_{h \in N}) - u_s(((l-1) \wedge m_h)_{h \in N}).$$

Thus, by double application of Weak monotonicity, the induction hypothesis and the definitions of  $\varphi$  and  $\xi$ , for each  $(i, l) \notin T(t)$  it holds that

$$\begin{aligned} f_{il}(m, u_t) &= f_{il}(m, u_s) \\ &\stackrel{Hyp}{=} \alpha^l \varphi_{il}(m, u_s) + (1 - \alpha^l) \xi_{il}(m, u_s) \\ &= \frac{(1 - \alpha^l)}{|Q(l)|}. \end{aligned} \quad (3.62)$$

By Multi-Efficiency, (3.62) and the definition of a minimal effort game,

$$\begin{aligned} \sum_{(i,l) \in T(t)} f_{il}(m, u_t) &= u_t((l \wedge m_h)_{h \in N}) - u_t(((l-1) \wedge m_h)_{h \in N}) \\ &\quad - \sum_{(i,l) \notin T(t)} f_{il}(m, u_t) \\ &= 1 - 0 - (|Q(l)| - |T(t)|) \frac{1 - \alpha^l}{|Q(l)|}. \end{aligned} \quad (3.63)$$

Additionally, any two distinct pairs  $(i, l), (i', l) \in M^+$  such that  $(i, l), (i', l) \in T(t)$ , are equal in  $(m, u_t)$ . By Remark 9 and by Level desirability, for each  $(i, l) \in T(t)$ ,

$$f_{il}(m, u_t) = c',$$

for some  $c' \in \mathbb{R}$ . It follows that

$$\sum_{(i,l) \in T(t)} f_{il}(m, u_t) = |T(t)|c'. \quad (3.64)$$

Therefore, combining (3.63) and (3.64), for each  $(i, l) \in T(t)$ , we obtain

$$c' = \frac{1 - (|Q(l)| - |T(t)|) \frac{1 - \alpha^l}{|Q(l)|}}{|T(t)|}.$$

It follows that, for each  $(i, l) \in T(t)$ ,

$$\begin{aligned} f_{il}(m, u_t) &= \frac{1 - (|Q(l)| - |T(t)|) \frac{1 - \alpha^l}{|Q(l)|}}{|T(t)|} \\ &= \frac{\alpha^l}{|T(t)|} + \frac{1 - \alpha^l}{|Q(l)|} \\ &= \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t). \end{aligned} \quad (3.65)$$

Combining (3.62) and (3.65), if  $t^T = l$  then, for each  $(i, l) \in M^+$ ,

$$f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t).$$

This concludes the induction step.

We have shown that there exists a parameter system  $\alpha$  such that  $f$  can be written, for each  $(m, u_t)$  such that  $t^T = l$ , as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, u_t) = \begin{cases} \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l) \xi_{il}(m, u_t) & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of multi-choice Egalitarian Shapley values (see (3.9)), for such parameter system  $\alpha$ , there is a  $\chi^\alpha$  such that, for each  $(m, u_t)$

$$f(m, u_t) = \chi^\alpha(m, u_t).$$

We conclude by Linearity that there exists a parameter system  $\alpha$  such that, for each  $(m, v) \in \mathcal{G}$

$$f(m, v) = \chi^\alpha(m, v).$$

This concludes the proof of the theorem. ■

**Logical independence** The five axioms of the statement of Theorem 3.4.4 are logically independent, as shown by the following alternative solutions.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = 0,$$

satisfies all the axioms except Efficiency.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+,$$

$$f_{ij}(m, v) = \frac{\max_{k \in N} m_k}{\sum_{k \in N} m_k} \varphi(m, v) + \left(1 - \frac{\max_{k \in N} m_k}{\sum_{k \in N} m_k}\right) \xi(m, v),$$

satisfies all the axioms except Independence of higher activity levels.

- Take any  $(m, v) \in \mathcal{G}$ . Denote the difference in worth between the grand coalition and the  $(m_T - 1)$ -synchronized coalition by

$$V = v(m) - v((m_T - 1) \wedge m_k)_{k \in N}.$$

Let  $f$  be the value defined as:

- $f(m, v) = \varphi(m, v)$  if  $|m^T| \neq 2$ ;
- otherwise, for each  $(i, j) \in M^+$  such that  $j \neq m_T$ ,  $f_{ij}(m, v) = \varphi_{ij}(m, v)$ , and for the two remaining pairs  $(i, m_T), (i', m_T) \in M^+$ ,

$$(f_{im_T}(m, v), f_{i'm_T}(m, v)) = \begin{cases} (\varphi_{im_T}(m, v), \varphi_{i'm_T}(m, v)) & \text{if } \varphi_{im_T}(m, v) \geq 0 \text{ and } \varphi_{i'm_T}(m, v) \geq 0, \\ (0, V) & \text{if } \varphi_{im_T}(m, v) < 0, \varphi_{i'm_T}(m, v) > 0 \text{ and } V \geq 0, \\ (V, 0) & \text{if } \varphi_{im_T}(m, v) < 0, \varphi_{i'm_T}(m, v) > 0 \text{ and } V < 0, \\ (\varphi_{im_T}(m, v), \varphi_{i'm_T}(m, v)) & \text{if } \varphi_{im_T}(m, v) \leq 0 \text{ and } \varphi_{i'm_T}(m, v) \leq 0, \\ (0, V) & \text{if } \varphi_{im_T}(m, v) > 0, \varphi_{i'm_T}(m, v) < 0 \text{ and } V \geq 0, \\ (V, 0) & \text{if } \varphi_{im_T}(m, v) > 0, \varphi_{i'm_T}(m, v) < 0 \text{ and } V < 0. \end{cases}$$



The value  $f$  satisfies all the axioms except Linearity.

- The value  $f$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\begin{aligned} \forall (i, j) \in M^+, \\ f_{ij}(m, v) = & v((j-1) \wedge m_k)_{k \in N} + e_i \\ & + \frac{(v((j \wedge m_k)_{k \in N}) - v((j-1) \wedge m_k)_{k \in N}))}{|Q(j)|} \\ & - \frac{\sum_{k \in Q(j)} v((j-1) \wedge m_k)_{k \in N} + e_k}{|Q(j)|}, \end{aligned}$$

satisfies all the axioms except Weak monotonicity. Observe that this value extends the Equal surplus division from TU-games to multi-choice games.

- Take any  $(m, v) \in \mathcal{G}$ . For each  $(i, j) \in M^+$  fix an arbitrary integer  $\beta^{ij} \in \{1, 2\}$ . The value  $f^\beta$  defined as: for each  $(m, v) \in \mathcal{G}$ ,

$$\begin{aligned} \forall (i, j) \in M^+, \\ f_{ij}^\beta(m, v) = \frac{\beta^{ij}}{\sum_{k \in Q(j)} \beta^{kj}} \left[ v((j \wedge m_k)_{k \in N}) - v((j-1) \wedge m_k)_{k \in N} \right], \end{aligned}$$

satisfies all the axioms except Level desirability.

**Remark 10.** In some economic situations, it may be interesting to consider families of specific parameter systems, for instance parameter systems that operate a progressive compromise between marginalism and egalitarianism. In this case, it is possible to refine Theorem 3.4.4 in order to characterize multi-choice Egalitarian Shapley values endowed with such parameter systems.

### 3.5 Additional remarks

In this section, we make two remarks regarding the multi-choice Shapley value. First, we discuss the relationship between the multi-choice Shapley value and the discrete serial cost sharing method introduced by [Moulin and Shenker \[1992\]](#) for discrete cost sharing problems. Then, we axiomatically compare the multi-choice Shapley value with the values introduced by [Derks and Peters \[1993\]](#) and [Peters and Zank \[2005\]](#) for multi-choice games.

The class of discrete cost sharing problems is introduced by [Moulin and Shenker \[1992\]](#) and studied by [Moulin \[1995a\]](#), [Albizuri et al. \[2003\]](#), [Sprumont \[2005\]](#) and [Bahel and Trudeau \[2013\]](#) to cite a few. Fix  $N = \{1, \dots, n\}$  a set of  $n$  different goods produced in indivisible units. A discrete cost sharing problem is a couple  $(q, C)$ , where  $q = (q_1, \dots, q_n)$ . Each  $q_i \in \mathbb{N}$  represents the demand in good  $i$ , and  $C$  is a non decreasing real-valued function on  $\prod_{i \in N} \{0, 1, \dots, q_i\}$  such that  $C(\mathbf{0}) = 0$ . The total cost to be shared is given by  $C(q)$ . As shown by [Calvo and Santos \[2000\]](#) and [Albizuri et al. \[2003\]](#), one can view discrete cost sharing problems as a sub-class of multi-choice games. Indeed,  $q$  can be interpreted as a vector of maximal activity levels and  $C$  can be interpreted as characteristic function. Since  $C$  is a non decreasing real-valued function, it follows that discrete cost sharing problems can be viewed as the subclass of multi-choice games with a non decreasing real-valued characteristic function. We denote by  $\mathcal{C} \subseteq \mathcal{G}$  the class of discrete cost sharing problems. In the cost sharing literature, a method on  $\mathcal{C}$  is a map  $S$  that associates to each problem  $(q, C) \in$

C a vector  $S(q, C) \in \mathbb{R}^n$  satisfying the budget balanced condition i.e.  $\sum_{i \in N} S_i(q, C) = C(q)$ . In this sense, a method on the class of discrete cost sharing problems differs from a value, which distributes a payoff to each pair in  $M^+$ . A popular cost sharing method for cost sharing problems is the *discrete serial cost sharing method* (denoted SCS afterward) introduced by [Moulin and Shenker \[1992\]](#).

In order to present the discrete serial cost sharing method, we define a specific TU-game on  $N$ . Consider  $(q, C) \in \mathcal{C}$  and  $j \leq \max_{k \in N} m_k$ . Define the TU-game  $(Q(j), w_j^{(q, C)}) \in \mathcal{G}_N^{TU}$  as

$$\forall E \subseteq Q(j), \quad w_j^{(q, C)}(E) = C\left(\left((j-1) \wedge q_k\right)_{k \in N} + e_E\right) - C\left(\left((j-1) \wedge q_k\right)_{k \in N}\right).$$

The worth  $w_j^{(q, C)}(E)$  can be interpreted as the additional costs generated when each agent (good) in  $E$  increases its activity level (demand) from  $j-1$  to  $j$  while all the other agents play either the activity level  $j-1$  or their maximal activity level if they are unable to do so. We denote by  $Sh$  the Shapley value (see [Shapley \[1953\]](#)) for TU-games. [Albizuri et al. \[2003\]](#) show that the discrete serial cost sharing admits the following expression

$$\forall i \in N, \quad SCS_i(q, C) = \sum_{j=1}^{q_i} Sh_i\left(Q(j), w_j^{(q, C)}\right). \quad (3.66)$$

The next proposition links the discrete serial cost sharing method proposed by [Moulin and Shenker \[1992\]](#) with the multi-choice Shapley value. It states that, for each cost sharing problem, the multi-choice Shapley value captures the variation of the joint cost distribution caused by a demand increase for each good.

**Proposition 3.5.1.** *For each cost sharing problem  $(q, C) \in \mathcal{C}$ , it holds that*

$$\forall i \in N, \quad SCS_i(q, C) - SCS_i(q - e_i, C) = \varphi_{iq_i}(q, C).$$

*Proof.* Observe that, for each  $i \in N$  and  $j < q_i$ , it holds that

$$\forall E \subseteq Q(j), \quad w_j^{(q, C)}(E) = w_j^{(q - e_i, C)}(E).$$

Therefore, we obtain

$$\forall i \in N, j < q_i, \quad Sh_i\left(N, w_j^{(q, C)}\right) = Sh_i\left(N, w_j^{(q - e_i, C)}\right). \quad (3.67)$$

Additionally, recall that, for each  $j \leq \max_{k \in N} m_k$ , the set of orders  $\overline{O}_j$  over  $M^{+j}$  can be interpreted as the set of orders over the set of agents in  $Q(j)$ . An order over  $Q(j)$  is a bijection  $\sigma_j^N : Q(j) \rightarrow \{1, \dots, |Q(j)|\}$ . We denote by  $\overline{Q}(j)$  the set of orders over  $Q(j)$ . Consider an order  $\sigma_j^N \in \overline{Q}_j$  and  $h \in \{1, \dots, |Q(j)|\}$ . Recall that, for each  $B \subseteq N$ , the vector  $e_B \in \mathbb{R}^{|N|}$  is defined by  $(e_B)_i = 1$  if  $i \in B$  and  $(e_B)_i = 0$  otherwise. We denote by

$$\left((j-1) \wedge q_k\right)_{k \in N} + e_{E_{\sigma_j^N, h}}$$

the coalition in which each agent in  $Q(j)$  ordered prior to step  $h$  with respect to  $\sigma_j^N$ , participates at its activity level  $j$ , whereas each agent in  $Q(j)$  ordered after step  $h$  with respect to  $\sigma_j^N$ , participates at its activity level  $j-1$ . Each agent not in  $Q(j)$

participates at its maximal activity level. Obviously, this coalition coincides with  $s_j^{\sigma_j^h}$ , where  $\sigma_j$  is the counterpart of  $\sigma_j^N$  among the orders in  $\bar{O}_j$ . We use the convention  $((j-1) \wedge q_k)_{k \in N} + e_{E_j^{\sigma_j^N, 0}} = ((j-1) \wedge q_k)_{k \in N}$ . Consider an order  $\sigma_j^N \in \bar{Q}_j$ . For each  $i \in Q(j)$ , we denote by

$$\mu_i^{\sigma_j^N}(q, C) = C(((j-1) \wedge q_k)_{k \in N} + e_{E_j^{\sigma_j^N, \sigma_j^N(i)}}) - C(((j-1) \wedge q_k)_{k \in N} + e_{E_j^{\sigma_j^N, \sigma_j^N(i)-1}}), \quad (3.68)$$

the marginal contribution of agent  $i$  for its activity level  $j$  with respect to the order  $\sigma_j^N$ . By (3.6), (3.7) and (3.68), for each  $(q, C) \in C$ , the multi-choice Shapley value can be re-written as

$$\forall (i, j) \in M^{+,j}, \quad \varphi_{ij}(q, C) = \frac{1}{|Q(j)|!} \sum_{\sigma_j^N \in \bar{Q}_j} \mu_i^{\sigma_j^N}(q, C).$$

By definition of the Shapley value for TU-games (see [Shapley \[1953\]](#)), for each  $j \leq \max_{k \in N} m_k$ , it holds that

$$\forall i \in Q(j), \quad Sh_i(N, w_j^{(q, C)}) = \frac{1}{|Q(j)|!} \sum_{\sigma_j^N \in \bar{Q}_j} \mu_i^{\sigma_j^N}(q, C) = \varphi_{ij}(q, C). \quad (3.69)$$

It follows that the multi-choice Shapley value is consistent with the discrete serial cost sharing method since, for each  $i \in N$ , it holds that

$$\begin{aligned} SCS_i(q, C) - SCS_i(q - e_i, C) &= \sum_{j=1}^{q_i} Sh_i(N, w_j^{(q, C)}) - \sum_{j=1}^{q_i-1} Sh_i(N, w_j^{(q-e_i, C)}) \\ &= Sh_i(N, w_{q_i}^{(q, C)}) \\ &= \varphi_{iq_i}(q, C), \end{aligned}$$

where the second equality follows from (3.66) and (3.67), and the third equality follows from (3.69).  $\blacksquare$

To conclude this section, we briefly compare the multi-choice Shapley value with the different extensions of the Shapley value for multi-choice games (see Chapter 2, Section 2.3.2). To do so, we consider the different axioms presented in Chapter 2, Section 2.3.3 and the new axioms introduced in this Chapter. This comparison is summarized in the table below.

First of all, observe that all the extensions of the Shapley value, i.e.,  $\varphi$ ,  $\phi^{vN}$ ,  $\phi^{PZ}$ ,  $\phi^{HR}$ ,  $\phi^{DP}$  satisfy Efficiency, Additivity (or Linearity), and the null pair axiom. However  $\phi^{vN}$ ,  $\phi^{PZ}$ ,  $\phi^{HR}$ ,  $\phi^{DP}$  do not satisfy Multi-efficiency. By Proposition 3.4.1, it follows that none of these extension satisfies Independence of the higher activity level. In the same way, it can be shown that none of these extension satisfies Equal treatment for equal pairs, and therefore Sign symmetry for equal pairs. It can be shown that the multi-choice Shapley value,  $\varphi$ , does not satisfy Anonymity, Hierarchical strength, the weight axiom, the necessary pair axiom, and Level symmetry. Thus, the two main difference between  $\varphi$  and the previous extensions of the Shapley value lies in the Independence of higher activity level axiom and the Equity principle they have.

Axioms	$\phi^{vN}$	$\phi^{PZ}$	$\phi^{HR}$	$\phi^{DP}$	$\phi$
Carrier	✓	✓	✓	✓	✓
Efficiency	✓	✓	✓	✓	✓
Additivity	✓	✓	✓	✓	✓
Linearity	✓	✓	✓	✓	✓
The null game axiom	✓	✓	✓	✓	✓
Anonymity (on $\bar{\mathcal{G}}$ )	✓	✓	✓	✓	○
Hierarchical strength	✓	○	○	○	○
Weight axiom	○	○	✓	○	○
Hierarchical symmetry	✓	○	○	○	○
The necessary pair axiom	○	○	○	✓	○
Level symmetry	○	✓	○	○	○
Equal treatment for equal pairs	○	○	○	○	✓
Sign symmetry for equal pairs	○	○	○	○	✓
Minimal effort	✓	✓	✓	○	✓
Zero contribution	✓	✓	✓	○	✓
the null pair axiom	✓	✓	✓	✓	✓
Strong monotonicity	✓	✓	✓	○	✓
Upper balanced contributions	○	✓	○	✓	○
Lower balanced contributions		○	○	✓	✓
General balanced contributions	○	○	○	✓	○
Equal loss axiom	○	○	○	✓	○
Independence of the maximal activity level	○	✓	✓	○	✓
Independence of higher activity levels	○	○	○	○	✓

TABLE 3.1: Axiomatic comparison of the Shapley like values.

### 3.6 Conclusion

In this chapter we proposed several multi-efficient values for multi-choice games. We introduced the multi-choice Shapley value and the multi-choice Equal Division values. This chapter is in line with the literature that deals with the trade-off between marginalism and egalitarianism using cooperative game theory since we introduce the multi-choice Egalitarian Shapley values for multi-choice games. These values are computed as the convex combination of the multi-choice Shapley value and the multi-choice Equal division value. We provided at least one axiomatic characterization for each of these solution concepts.

Some questions remain of interest. It would be interesting to characterize multi-efficient solutions for multi-choice games with a structure. Regarding multi-choice games with a structure, several studies have already been conducted. Albizuri [2009] study multi-choice games with a coalition structure, Béal et al. [2012] study multi-choice games with communication constraints, and Lowing [2022] studies multi-choice games with a permission structure. The solution concepts proposed in these studies are not multi-efficient. It can be interesting to look for multi-efficient values for such games, since non-multi-efficient values cannot belong to the Core.

## Chapter 4

# Priority relations and cooperation with multiple activity levels

### 4.1 Introduction

In many situations, there exist asymmetries between agents that are not captured by a standard multi-choice game. These asymmetries may reflect exogenous rights, different needs, merit, or hierarchical constraints that discriminate between agents. In such settings, it would be desirable for a value to reflect these exogenous asymmetries. To this end, this chapter enriches the model of multi-choice games with a priority structure modeled by a partial order on the agent set (see Chapter 2, Section 2.2).

In the framework of multi-choice games, the use of a partial order on the agent set brings additional possibilities that would not have been possible in the framework of TU-games. First, the set of activity levels is linearly ordered and provides an “intra-agent” information. Then, the priority structure provides an “inter-agent” information. Depending on the cooperative situation, these two pieces of information can be useful to define a relevant allocation process. These pieces of information are aggregated into a relationship on the set of pairs formed by an agent and one of its activity levels. This relationship is then used to define a new value for multi-choice games with a priority structure. The following example aims to illustrate this approach.

Consider a lake commonly owned by a fishing community. Each fisher has a certain maximal level of fishing effort. Assume that the fishers coordinate their effort and share the quantity of fish they catch.<sup>1</sup> Moreover, suppose that the fishers are partially ordered according to some exogenous rights each fisher has over the fishery. The problem is then to allocate the fishery among the fishers, taking into account both their different rights and their effort levels. A possible solution would be to consider an extension of the Priority value introduced by Béal et al. [2022] to multi-choice games. In the framework of TU-games, this value allocates the net surplus of a coalition, i.e., the dividend of a coalition, among the priority agents in the coalition (see Chapter 2, Section 2.2). A straightforward generalization of the Priority value to the above example amounts to favor fishers with the higher rights in the allocation process, regardless of their fishing effort. As a result, the catch may be grabbed by a fisher with a low effort, thereby reducing the incentive to fish.

We propose to combine the ordered set of the activity levels with the priority structure in a lexicographic manner. This lexicographic order enables to compare pairs formed by an agent and one of its activity levels in the following way. Consider

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<sup>1</sup>This assumption is in line with Funaki and Yamato [1999] who study the tragedy of the commons from a cooperative viewpoint.

any two agents with a certain activity level, i.e., any two pairs. Similarly to the common pool resource example, the activity levels of the agents are assumed to be the most significant criterion. If one of the two pairs has a higher position than the second in the lexicographic order, then it features a higher activity level than the second pair. If the two activity levels are equal, then the pair featuring the agent that has priority over the second agent has a higher position than the other pair in the lexicographic order. In line with this lexicographic order, a generalization of the Priority value from TU-games to multi-choice games is introduced: the *multi-choice Priority value*. This value equally divides the net surplus of each coalition among the agent(s) in the coalition with the highest position in the lexicographic order. Such allocation process especially makes sense when resources are scarce and trade-offs have to be made among different criteria.

Two axiomatic characterizations of the multi-choice Priority value are provided. Both characterizations rely on standard axioms for multi-choice games and specific axioms related to the priority structure.

Regarding the first characterization, the axioms invoked carry a relevant meaning in resources allocation problem. The standard axioms are Efficiency, Independence of the maximal activity level and Independence of level reductions. The two axioms of independence relate to the variation in the payoffs when an agent gets available a higher activity level. Independence of the maximal activity level requires that the fishes allocated to a fisher at a certain fishing effort do not depend on its highest effort level. Thus, the excess caught whenever the effort increases should be allocated to this new level. Independence of level reductions considers two fishers with different fishing effort. If the effort of the fisher with the higher effort level increases, the quantity of fish allocated to the other fisher should not vary. This ensures that the latter do not support fishers with a higher effort level. Two other axioms that take into account the priority structure are introduced. The axiom of Priority relation for the same maximal activity level considers two agents with the same maximal activity level and comparable in terms of priority. Regarding the common pool resource allocation problem, consider two fishers with the same maximal effort level, and such that one has more rights than the other. The axiom requires that the amount of fishes allocated to the fisher with less rights should not vary whenever the fishing effort of the other fishers, that has more rights, slightly decreases. Then, the axiom of Balanced contributions for the same prevailing group is introduced.<sup>2</sup> This axiom relies on a fairness requirement for indistinguishable agents regarding both the priority structure and their maximal activity level. It indicates that such agents affect each other payoff in the same manner. Regarding the common pool allocation problem, consider two indistinguishable fishers in terms of effort level and rights. If the effort of one of the two fishers increases, then the amount of resources allocated to the other fisher may vary. This variation is the same no matter which fisher's effort increases. As our first main result, we show that the multi-choice Priority value is characterized by the above list of axioms (Proposition 4.3.2).

The second characterization of the multi-choice Priority value is close to the one of the multi-choice Shapley value studied in Chapter 3. It invokes Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions as standard axioms. In addition, it invokes the axiom of Independence of null pair: removing an agent's maximal activity level does not alter the payoffs of the remaining agent's activity level if this agent is unproductive in each coalition in which it

<sup>2</sup>See Chapter 2, Section 2.3.3 for a brief discussion on the different extensions of the balanced contributions principle from TU-games to multi-choice games.



plays its maximal activity level. Yet, the multi-choice Priority value departs from the multi-choice Shapley value in one important aspect. Contrary to the latter, it does not satisfy the axiom of Equal treatment for equal pairs (see Chapter 3, Section 3.4). In the presence of exogenous asymmetry, this axiom becomes very strong. Therefore, it is replaced by two new axioms that take into account the priority structure. Both axioms deal with decisive agents. An agent is decisive if each coalition in which it does not play its maximal activity level generates zero worth. This notion generalizes the notion of a necessary agent from TU-games to multi-choice games. Consider two decisive agents with the same maximal activity level. First, the axiom of Priority relation for decisive agents requires that the payoff of the first decisive agent is zero if the second decisive agent has priority over it. Second, the axiom of Equal treatment for decisive agents with the same prevailing group deals with decisive agents that are incomparable in terms of both maximal activity level and priority. It requires that such agents should be treated equally for their maximal activity level. Therefore, this axiom is a relaxation of the axiom of Equal treatment for equal pairs. As our second main result, we show that the multi-choice Priority value is the unique value that satisfies this second set of axioms (Proposition 4.3.5).

Finally, we consider priority structures in which the set of agents can be partitioned into several priority classes. Precisely, each class contains incomparable agents that have priority over each agent in the next class. In such priority relations structured by classes, we show that the multi-choice Priority value can be interpreted as a sequential procedure involving specific TU-games. Consider a given activity level that is played by at least one agent, and a priority class  $p$ . Consider all agents in this priority class which are able to play the required activity level. These agents obtain their contribution to the coalition formed by all agents over which this class has priority (and that are able to play the required activity level). The final payoff of agents in this priority class for the required activity level is the Shapley value applied to a TU-game on the subset of agents in the priority class  $p$ , which are able to play the required activity level (Proposition 4.4.1).

## Related literature

This chapter relates to a large literature in cooperative game theory that integrates economical, communicational, or hierarchical constraints into the allocation process. The largest part of this literature relates to TU-games. To cite a few, Myerson [1977] introduces communication graph to model communication constraints; Aumann and Dreze [1974] and Owen [1977] model coalitional constraints by a partition of the agent set; Gilles et al. [1992] model hierarchical constraints by a permission structure, while Faigle and Kern [1992] consider precedence constraints. In all these approaches, the Shapley value (see Shapley [1953]) is adapted to account for the position of the agents in the structure. Recently, Béal et al. [2022] study priority structures modeled by a partial order on the agent set, and introduce the Priority value for TU-games with a priority structure. Contrary to the previous structures, a priority structure aims to influence the allocation process but has no impact either on the formation or on the evaluation of coalitions (see Chapter 2, Section 2.2).

Only few recent developments have focused on integrating such constraints in the multi-choice setting. Albizuri [2009] combines multi-choice games with a coalition structure. This model considers that agents meet together in coalitions and form a family of coalitions or a coalition structure. The author proposes a value that extends the Owen value (see Owen [1977]) from TU-games to multi-choice games. Further developments of this model have been provided by Jones and Wilson [2013].



Béal et al. [2012] consider multi-choice games with limited cooperation possibilities represented by an undirected communication forest on the agent set. This model considers that agents can cooperate if they are connected in the forest. They propose a value that extends the Average-tree solution (see Herings et al. [2008]) from forest TU-games to forest multi-choice games. Branzei et al. [2021] introduce the model of two-sided multi-choice market games and consider core like solutions for this class of games. Moreover, they introduce the Pairwise egalitarian set motivated by the market structure in which a buyer is paired with a seller. Lowing [2022] enriches multi-choice games with a permission structure. The author considers several combinations of the ordered set of activity levels with the permission structure and defines several values on the class of multi-choice games with a permission structure.

Finally, this chapter also relates to the literature that analyzes allocation processes among heterogeneous agents. A large part of this literature focus on rationing models. The simplest model describes situations in which agents demand a certain commodity, and the available resources fall short of the total demand. In particular, Moulin [2000] characterizes a large class of priority rules. Each rule in this class considers that agents are linearly ordered according to a priority relation. This model also arises in a variety of contexts in which a priority relation is relevant as inheritance context (see O'Neill [1982]), bankruptcy context (see Aumann and Maschler [1985], Thomson [2003, 2015]), the river sharing context (see Ansink and Weikard [2015]) or claim problems (see Thomson [2019]).

The rest of the chapter is organized as follows. Section 4.2 provides basic definitions for multi-choice games and priority structures. Section 4.3 introduces the axioms and proceeds to the axiomatic study. Section 4.4 discusses multi-choice games with a priority relation structured by classes. Finally, Section 4.5 concludes the chapter.

## 4.2 Notation

Let  $N = \{1, \dots, n\}$  be a fixed set of agents and  $K \in \mathbb{N}$ . We consider  $\mathcal{G}$  the class of all multi-choice games on  $N$ . For  $s \in \mathcal{M}$ , define the set of pairs featuring each agent and their (non-null) activity level within coalition  $s$  as

$$C(s) = \{(i, j) \in M^+ \mid j = s_i\}. \quad (4.1)$$

Let  $(m, v) \in \mathcal{G}$  be a multi-choice game. A pair  $(i, m_i)$  is *decisive* if each coalition in which  $i$  plays an activity level lower than  $m_i$  generates zero worth. Formally,  $i \in N$  is decisive if

$$\forall s \in \mathcal{M} : s_i < m_i, \quad v(s) = 0.$$

**Remark 11.** By the definition of the dividend (see Chapter 2, Section 2.3.1, definition (2.21)), if  $i$  is a decisive agent in  $(m, v)$ , then  $\Delta_s(v) = 0$  whenever  $s_i < m_i$ .

In addition, we consider  $\mathfrak{P}_N$  the set of all priority structures (i.e., all partial orders) that one can generate from  $N$ . The notation and notions related to priority structures are the same as introduced in Chapter 2, Section 2.2. A multi-choice game on  $N$  enriched with a priority structure on  $N$  is a triple  $(m, v, \succeq)$ . Denote by  $\mathcal{GP}_N$  the class of multi-choice games with a priority structure that one can construct from  $\mathcal{G}$  and  $\mathfrak{P}_N$ . A value on  $\mathcal{GP}_N$  is a map  $f : \mathcal{GP}_N \rightarrow \mathbb{R}^{|M^+|}$ .

### 4.3 Axiomatic study

There are several ways to consider the influence of a priority structure on multi-choice games. Here, this issue is approached through an axiomatic study. Both standard axioms and axioms related to the priority structure are considered.

Denote by  $f$  a value on  $\mathcal{GP}_N$ . The first three axioms are straightforward generalizations of standard axioms for multi-choice games to multi-choice games with a priority structure.

*Efficiency.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ ,

$$\sum_{(i,j) \in M^+} f_{ij}(m, v, \succeq) = v(m).$$

The next axiom is based on the principle stating that the variation of an agent's maximal activity level should have no impact on the payoff of its remaining activity levels. It was first introduced by [Hwang and Liao \[2009\]](#) and [Béal et al. \[2012\]](#).

*Independence of the maximal activity level.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ ,

$$\forall (i, j) \in M^+ : j < m_i, \quad f_{ij}(m, v, \succeq) = f_{ij}(m - e_i, v, \succeq).$$

The following axiom requires that the payoff of an agent's maximal activity level is not affected by another agent with a higher maximal activity level. This axiom is based on the principle stating that an agent should not be affected by activity levels that it is not able to perform. This principle is discussed by [Moulin and Shenker \[1992\]](#) in the context of cost sharing problems, and by [Albizuri et al. \[2020\]](#) in the context of bargaining problems with claims.

*Independence of level reductions.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , each  $i, i' \in N$  such that  $m_i > m_{i'}$ ,

$$f_{i'm_{i'}}(m, v, \succeq) = f_{i'm_{i'}}(m - e_i, v, \succeq).$$

The above axioms are independent of the priority structure. Below, we introduce two new axioms that take into account the priority structure.

#### 4.3.1 Balanced contributions principle and priority relations

The next two axioms relate to the payoff of agents' maximal activity. The first one considers comparable agents (with respect to the priority structure) with the same maximal activity level. It is an invariance axiom inspired by the Priority agent out axiom introduced by [Béal et al. \[2022\]](#). Originally, the Priority agent out axiom asserts that removing any agent does not change the payoff of agents over which it has priority. The idea is that removing an agent  $i$  has a double effect: first the situation of an agent over which  $i$  has priority improves since less agents have priority over it; second, its situation is getting worse since it has less possibilities to cooperate. The Priority agent out axiom states that these two effect neutralize each other. We propose a conceptually similar axiom that applies to agents' maximal activity level. Consider two agents  $i$  and  $i'$  such that  $m_i = m_{i'}$  and  $i \succ i'$ . The axiom states that the payoff of  $i'$ 's maximal activity level remains unchanged if agent  $i$ 's maximal activity level decreases by one unit.

*Priority relation for the same maximal activity level.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , each  $i, i' \in N$  such that  $m_i = m_{i'}$  and  $i \succ i'$ ,

$$f_{i'm_{i'}}(m, v, \succeq) = f_{i'm_{i'}}(m - e_i, v, \succeq).$$

To introduce the next axiom, a definition is needed. An agent  $i$  *prevails* on another agent  $i'$  if it has priority over  $i'$  and its maximal activity level  $m_i$  is not lower than  $m_{i'}$ . Given  $(m, v, \succeq) \in \mathcal{GP}_N$ , for each  $i \in N$ , the set of agents that prevail on  $i$  is defined as

$$L_i(m, v, \succeq) = \uparrow_{\succeq} i \cap \{k \in N \setminus \{i\} : m_k \geq m_i\}.$$

The next axiom is inspired by the well known Balanced contributions axiom introduced by Myerson [1980] for TU-games (see Chapter 2, Section 2.3.3). Due to the presence of priority relations between the agents, it seems reasonable that this requirement takes the priority structure into account. The next axiom accommodates this idea by requiring that two agents affect each other payoff in the same manner if they have the same maximal activity level and the same prevailing group.

*Balanced contributions for the same prevailing group.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , each  $i, i' \in N$  such that  $i \neq i'$ ,  $m_i = m_{i'}$  and  $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$ ,

$$f_{im_i}(m, v, \succeq) - f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq).$$

Our first result shows that the combination of Efficiency, Independence of the maximal activity level, Independence of level reductions, Priority relation for the same maximal activity level, and Balanced contributions for the same prevailing group yields a unique value for multi-choice games with a priority structure.

**Proposition 4.3.1.** *On the class  $\mathcal{GP}_N$  there is at most one value that satisfies Efficiency, Independence of the maximal activity level, Independence of level reductions, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group.*

*Proof.* Let  $f$  be a value on  $\mathcal{GP}_N$  that satisfies Efficiency, Independence of the maximal activity level, Independence of level reductions, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group. Let us show that  $f(m, v, \succeq)$  is uniquely determined, for each  $(m, v, \succeq) \in \mathcal{GP}_N$ . This is proved by induction on the number of activity levels in a game defined as  $\sum_{i \in N} m_i$ , for each  $(m, v) \in \mathcal{GP}_N$ .

**Initialization.** Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $\sum_{i \in N} m_i = 1$ . There exists a unique  $i \in N$  such that  $m_i = 1$  and  $m_k = 0$ , for each  $k \neq i$ . By Efficiency and by definition of a payoff vector, each payoff vector verifies  $f_{i1}(m, v, \succeq) = v(m)$ . This shows that  $f(m, v, \succeq)$  is uniquely determined.

**Induction hypothesis.** Assume that  $f(m, v, \succeq)$  is uniquely determined for each  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $\sum_{i \in N} m_i = W \geq 1$ .

**Induction step.** Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $\sum_{i \in N} m_i = W + 1$ . Let us show that  $f(m, v, \succeq)$  is uniquely determined. Denote by  $\bar{m} = \max_{k \in N} m_k$  the highest maximal activity level in  $(m, v, \succeq)$ , and denote by

$$N_{\bar{m}} = \{i \in N \mid m_i = \bar{m}\},$$

the set of agents that possess the highest maximal activity level. By Independence of the maximal activity level, for each agent  $i \in N$  and for each  $j < m_i$ ,

$$f_{ij}(m, v, \succeq) = f_{ij}(m - e_i, v, \succeq). \quad (4.2)$$

Observe that there are a total of  $W$  activity levels in  $(m - e_i, v, \succeq)$ . Therefore, by the induction hypothesis,  $f_{ij}(m - e_i, v, \succeq)$  is uniquely determined. By (4.2),  $f_{ij}(m, v, \succeq)$

is uniquely determined, for each  $i \in N$  and each  $j < m_i$ . It remains to show that  $f_{im_i}(m, v, \succeq)$  is uniquely determined, for each  $i \in N$ .

First, take any agent  $k \notin N_{\bar{m}}$ , i.e.  $m_k < \bar{m}$ . By definition of  $N_{\bar{m}}$ , there is at least one agent  $i \in N_{\bar{m}}$  such that  $m_i > m_k$ . By Independence of level reductions,

$$f_{km_k}(m, v, \succeq) = f_{km_k}(m - e_i, v, \succeq). \quad (4.3)$$

As above, observe that there are a total of  $W$  activity levels in  $(m - e_i, v, \succeq)$ . By the induction hypothesis,  $f_{km_k}(m - e_i, v, \succeq)$  is uniquely determined. Therefore, by (4.3),  $f_{km_k}(m, v, \succeq)$  is uniquely determined for each  $k \notin N_{\bar{m}}$ .

Second, take any  $i \in N_{\bar{m}}$ . Assume that  $|N_{\bar{m}}| = 1$ . It follows that  $m_k < m_i$ , for each  $k \in N$ . Denote by  $M_{-e_i}^+ = M^+ \setminus \{(i, m_i)\}$  the set of pairs associated with  $(m - e_i, v, \succeq)$ . By Independence of level reductions,

$$\begin{aligned} \forall k \in N, j < m_i \quad f_{kj}(m, v, \succeq) &= f_{kj}(m - e_i, v, \succeq) \\ \implies \sum_{(k,j) \in M_{-e_i}^+} f_{kj}(m - e_i, v, \succeq) &= \sum_{(k,j) \in M_{-e_i}^+} f_{kj}(m, v, \succeq). \end{aligned} \quad (4.4)$$

By Efficiency and (4.4),

$$\begin{aligned} \sum_{(k,j) \in M_{-e_i}^+} f_{kj}(m - e_i, v, \succeq) &= v(m - e_i) \\ \implies \sum_{(k,j) \in M_{-e_i}^+} f_{kj}(m, v, \succeq) &= v(m - e_i) \\ \implies f_{im_i}(m, v, \succeq) &= \sum_{(k,j) \in M^+} f_{kj}(m, v, \succeq) - \sum_{(k,j) \in M_{-e_i}^+} f_{kj}(m, v, \succeq) \\ &= v(m) - v(m - e_i). \end{aligned}$$

This shows that  $f_{im_i}(m, v, \succeq)$  is uniquely determined.

Now, assume that  $|N_{\bar{m}}| > 1$ . Consider the subposet  $(N_{\bar{m}}, \succeq^{N_{\bar{m}}})$  of  $(N, \succeq)$  and the priority group  $\uparrow_{\succeq^{N_{\bar{m}}}} k$  of agent  $k \in N_{\bar{m}}$ . Two separate cases are considered.

Case 1. Take any  $k \in N_{\bar{m}}$  such that  $\uparrow_{\succeq^{N_{\bar{m}}}} k \neq \emptyset$ . Then, there is at least one  $i \in N_{\bar{m}}$  such that  $i \succ k$ . By Priority relation for the same maximal activity level,

$$f_{k,m_k}(m, v, \succeq) = f_{k,m_k}(m - e_i, v, \succeq). \quad (4.5)$$

Observe that there are a total of  $W$  activity levels in  $(m - e_i, v, \succeq)$ . By the induction hypothesis,  $f_{k,m_k}(m - e_i, v, \succeq)$  is uniquely determined. Therefore, by (4.5),  $f_{k,m_k}(m, v, \succeq)$  is uniquely determined.

Case 2. Take any  $i \in N_{\bar{m}}$  such that  $\uparrow_{\succeq^{N_{\bar{m}}}} i = \emptyset$ . Define

$$\bar{N}_{\bar{m}} = \{i \in N_{\bar{m}} \mid \uparrow_{\succeq^{N_{\bar{m}}}} i = \emptyset\}$$

the set of agents with the highest maximal activity level and with an empty priority group in  $(N_{\bar{m}}, \succeq)$ . Denote by  $n_{\bar{m}} = |\bar{N}_{\bar{m}}|$  the number of agents in  $\bar{N}_{\bar{m}}$ . By definition, for each  $i \in \bar{N}_{\bar{m}}$ ,  $L_i(m, v, \succeq) = \emptyset$ . If  $n_{\bar{m}} > 1$ , then by Balanced contributions for the same prevailing group,

$$\forall i, i' \in \bar{N}_{\bar{m}}, \quad f_{im_i}(m, v, \succeq) - f_{i'm_{i'}}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq). \quad (4.6)$$

Observe that there are a total of  $W$  activity levels in  $(m - e_i, v, \succeq)$  and  $(m - e_{i'}, v, \succeq)$ .

The right hand-side of (4.6) is uniquely determined by the induction hypothesis. Applying the same argument for each agent in  $\bar{N}_{\bar{m}}$  one generates  $\binom{n_{\bar{m}}}{2}$  equations of type (4.6). Moreover, by Efficiency

$$\begin{aligned} \sum_{(i,j) \in M^+} f_{ij}(m, v, \succeq) &= \sum_{i \in N} \sum_{j=1}^{m_i} f_{ij}(m, v, \succeq) \\ &= v(m) \\ &= \sum_{i \in N} \sum_{j=1}^{m_i-1} f_{ij}(m, v, \succeq) + \sum_{k \notin N_{\bar{m}}} f_{km_k}(m, v, \succeq) + \sum_{i \in N_{\bar{m}}} f_{im_i}(m, v, \succeq). \end{aligned}$$

This can also be written as

$$\begin{aligned} \sum_{i \in \bar{N}_{\bar{m}}} f_{im_i}(m, v, \succeq) &= v(m) - \sum_{i \in N} \sum_{j=1}^{m_i-1} f_{ij}(m, v, \succeq) - \sum_{k \notin N_{\bar{m}}} f_{km_k}(m, v, \succeq) \\ &\quad - \sum_{\substack{k \in N_{\bar{m}} \\ k \notin \bar{N}_{\bar{m}}}} f_{km_k}(m, v, \succeq). \end{aligned} \tag{4.7}$$

Let  $b$  denote the right-hand side of (4.7). At this step, it has been shown that:

- $f_{ij}(m, v, \succeq)$  is uniquely determined for each  $j < m_i$ ;
- $f_{km_k}(m, v, \succeq)$  is uniquely determined for each  $k \notin N_{\bar{m}}$ ;
- $f_{km_k}(m, v, \succeq)$  is uniquely determined for each  $k \in N_{\bar{m}}$  and  $k \notin \bar{N}_{\bar{m}}$ .

Therefore,  $b$  is uniquely determined. From (4.7) and (4.6), one generates the system of  $\binom{n_{\bar{m}}}{2} + 1$  equations with  $n_{\bar{m}}$  unknowns

$$(A) = \begin{cases} \sum_{i \in \bar{N}_{\bar{m}}} f_{im_i}(m, v, \succeq) = b, \\ \forall i, i' \in \bar{N}_{\bar{m}}, \\ f_{im_i}(m, v, \succeq) - f_{i'm_{i'}}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq). \end{cases}$$

Without loss of generality, relabel each agent in  $\bar{N}_{\bar{m}}$  by  $i_1, i_2, \dots, i_{n_{\bar{m}}}$ . By considering the sequence of agents  $(i_1, i_2, \dots, i_{n_{\bar{m}}})$ , one can reduce (A) to the subsystem (B) of  $n_{\bar{m}}$  equations with  $n_{\bar{m}}$  unknowns

$$(B) = \begin{cases} \sum_{i \in \bar{N}_{\bar{m}}} f_{im_i}(m, v, \succeq) = b, \\ \forall k \in \{1, \dots, n_{\bar{m}} - 1\}, \\ f_{i_k m_{i_k}}(m, v, \succeq) - f_{i_{k+1} m_{i_{k+1}}}(m, v, \succeq) = f_{i_k m_{i_k}}(m - e_{i_{k+1}}, v, \succeq) - f_{i_{k+1} m_{i_{k+1}}}(m - e_{i_k}, v, \succeq). \end{cases}$$

System (B) can be represented by the equation  $X(f_{i_k m_k})_{k \in \{1, \dots, n_{\overline{m}}\}} = Y$ , where

$$X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

and  $Y = \begin{pmatrix} b \\ f_{i_1 m_{i_1}}(m - e_{i_{1+1}}, v, \succeq) - f_{i_{1+1} m_{i_{1+1}}}(m - e_{i_1}, v, \succeq) \\ \vdots \\ f_{i_{n_{\overline{m}}-1} m_{i_{n_{\overline{m}}-1}}}(m - e_{i_{n_{\overline{m}}}}, v, \succeq) - f_{i_{n_{\overline{m}}} m_{i_{n_{\overline{m}}}}}(m - e_{i_{n_{\overline{m}}-1}}, v, \succeq) \end{pmatrix}$

It is clear that  $X$  is full rank, and thus is invertible. Therefore,  $X(f_{i_k m_k})_{k \in \{1, \dots, n_{\overline{m}}-1\}} = Y$  admits a unique solution, which is equivalent to say that (X) admits a unique solution. We conclude that the original system (A) admits at most one solution. This allows to conclude the induction step. The proof is completed. ■

In many economic decision problems, a decision maker has to choose an option within a finite set of possible alternatives. Often, these alternatives are defined according to a finite number of criteria (see [Svenson \[1979\]](#)). A popular decision making process is to select the alternatives according to a lexicographic ordering based on these criteria (see [Fishburn \[1974\]](#)). In other words, one classifies the criteria according to their relative significance. Then, the alternative that performs the best on the first criterion is elected. If more than one alternative is elected, then the second criterion is used to tiebreak the alternatives. This process continues until there is only one alternative left, or no criterion left.

We introduce the *multi-choice Priority value* for multi-choice games with a priority structure that follows a lexicographic allocation process. The activity levels of the agents are assumed to be the most significant criterion. The multi-choice Priority value allocates the surplus of each coalition among a subset of agents within this coalition. These agents are chosen according to a lexicographic decision process that selects the agents with the highest activity level in the coalition. In case more than one agent is selected, the process sorts the agents according to their position in the priority structure.

Precisely, we derive a lexicographic partial order  $(M^+, \succeq^*)$  from the total order on the set of activity levels  $(M_i^+, \geq)$  of the agents  $i \in N$  and the priority structure  $(N, \succeq)$ . Given a multi-choice game with a priority structure  $(m, v, \succeq) \in \mathcal{GP}_N$ , define the lexicographic partial order  $\succeq^*$  on the set of pairs  $M^+$  as

$$\forall (i, j), (i', j') \in M^+, \quad (i, j) \succeq^* (i', j') \iff \left[ (j > j') \text{ or } (j = j' \text{ and } i \succeq i') \right]. \quad (4.8)$$

We now have the material to define the multi-choice Priority value,  $f^P$ . This value assigns to a pair  $(i, j)$  a share of the dividend of a coalition  $s$  if no other pair has a higher position in the lexicographic order induced by the coalition.

Formally, the multi-choice Priority value,  $f^P$ , is defined as: for each  $(m, v, \succeq) \in \mathcal{GP}_N$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}^P(m, v, \succeq) = \sum_{\substack{s \leq m \\ j = s_i \\ (i, j) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}, \quad (4.9)$$

where for each  $s \in \mathcal{M}$ ,  $T(C(s), \succeq^*)$  is the set of maximal elements of the subposet  $(C(s), \succeq^*)$ , and  $C(s)$  is the set of pairs featuring agents and their non-null activity level in  $s$  as defined in (4.1).

**Remark 12.** Whenever  $m = (1, \dots, 1)$ , this value coincides with the Priority value on TU-games introduced by Béal et al. [2022] (see 2, Section 2.2, definition 2.16). Whenever the priority structure is a trivial poset  $(N, \succeq^0)$ , this value coincides with the multi-choice Shapley value studied in Chapter 3.

Below, we provide a brief illustration of the computation of  $f^P$ .

**Example 4.** Let  $(m, v, \succeq) \in \mathcal{GP}_N$  be such that  $N = \{a, b, c\}$ ,  $a \succ b$ ,  $a \succ c$ , and  $m = (1, 2, 1)$ . By (4.8), the combination of these elements leads to the lexicographic partial order  $(M^+, \succeq^*)$  described by the following Hasse diagram.

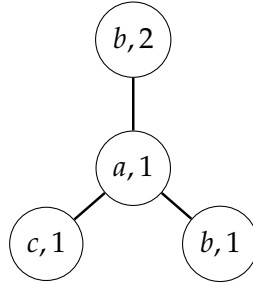


Figure 2: Hasse diagram of  $(M^+, \succeq^*)$ .

Consider the characteristic function  $v$  and its associated dividends described in the following tables.

$s = (s_a, s_b, s_c)$	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)
$v(s)$	1	1	1	2	4	4
$\Delta_s(v)$	1	1	1	0	2	2

$s = (s_a, s_b, s_c)$	(1,1,1)	(0,2,0)	(1,2,0)	(0,2,1)	(1,2,1)
$v(s)$	6	2	2	6	9
$\Delta_s(v)$	-1	1	-1	1	2

Pick  $s = (1, 1, 0)$ . According to (4.1),  $C(s) = \{(a, 1), (b, 1)\}$  and according to (4.8) and  $(M^+, \succeq^*)$ ,  $T(C(s), \succeq^*) = (a, 1)$ . Therefore,  $(a, 1)$  is the only pair to which the dividend  $\Delta_{(1,1,0)}(v)$  should be allocated. More generally, the coalitions  $s \in \mathcal{M}$  such



that  $(a, 1) \in T(C(s), \succeq^*)$  are  $(1, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 0, 1)$  and  $(1, 1, 1)$ . It follows that

$$\begin{aligned} f_{a1}^P(m, v, \succeq) &= \frac{\Delta_{(1,0,0)}(v)}{|T(C(1,0,0), \succeq^*)|} + \frac{\Delta_{(1,1,0)}(v)}{|T(C(1,1,0), \succeq^*)|} + \frac{\Delta_{(1,0,1)}(v)}{|T(C(1,0,1), \succeq^*)|} \\ &\quad + \frac{\Delta_{(1,1,1)}(v)}{|T(C(1,1,1), \succeq^*)|} \\ &= \frac{1}{1} + \frac{0}{1} + \frac{2}{1} + \frac{(-1)}{1} \\ &= 2, \\ f_{b1}^P(m, v, \succeq) &= \frac{\Delta_{(0,1,0)}(v)}{|T(C(0,1,0), \succeq^*)|} + \frac{\Delta_{(0,1,1)}(v)}{|T(C(0,1,1), \succeq^*)|} \\ &= \frac{1}{1} + \frac{2}{2} \\ &= 2. \end{aligned}$$

Similarly,  $f_{c1}^P(m, v, \succeq) = 2$  and  $f_{b2}^P(m, v, \succeq) = 3$ .

The next proposition provides an axiomatic characterization of  $f^P$  on the class of multi-choice games with a priority structure.

**Proposition 4.3.2.** *On the class  $\mathcal{GP}_N$  a value  $f$  satisfies Efficiency, Independence of the maximal activity level, Independence of level reductions, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group if and only if  $f = f^P$ .*

*Proof.* By Proposition 4.3.1, we know that there is at most one value that satisfies Efficiency, Independence of the maximal activity level, Independence of level reductions, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group. It remains to show that  $f^P$  satisfies these axioms.

*Efficiency.* Take any  $(m, v, \succeq) \in \mathcal{GP}_N$ . By definition of  $f^P$ , the dividend of a coalition  $s \in \mathcal{M}$  is equally allocated among the pairs in  $T(C(s), \succeq^*)$ . Therefore, it holds that

$$\begin{aligned} \sum_{(i,j) \in M^+} f_{ij}^P(m, v, \succeq) &= \sum_{s \leq m} \sum_{(i,j) \in T(C(s), \succeq^*)} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} \\ &= \sum_{s \leq m} \Delta_s(v) \\ &= v(m). \end{aligned}$$

This shows that  $f^P$  satisfies Efficiency.

*Independence of the maximal activity level.* Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  and any pair  $(i, j) \in M^+$  such that  $j < m_i$ . By definition of  $f^P$ , the payoff  $f_{ij}^P(m, v, \succeq)$  only depends on the coalitions  $s \in \mathcal{M}$  verifying  $s_i = j$ . Thus, it is straightforward that  $f^P$  satisfies Independence of the maximal activity level.

*Independence of level reductions.* Take any  $(m, v, \succeq) \in \mathcal{GP}_N$ , any  $s \in \mathcal{M}$ , and any two agents  $i, i' \in N$  such that  $m_i < m_{i'}$ . If  $(i, m_i) \in T(C(s), \succeq^*)$ , then we necessarily have  $s_{i'} \leq m_i$  since  $m_i < m_{i'}$ . Therefore, the payoff  $f_{im_i}^P(m, v, \succeq)$  depends only on coalitions  $s \in \mathcal{M}$  such that  $s_{i'} \leq m_i < m_{i'}$ . It follows that

$$f_{im_i}^P(m, v, \succeq) = f_{im_i}^P(m - e_{i'}, v, \succeq),$$



as desired.

*Priority relation for the same maximal activity level.* Take any  $(m, v, \succeq) \in \mathcal{GP}_N$ , any  $s \in \mathcal{M}$ , and any two agents  $i, i' \in N$  such that  $m_i = m_{i'}$  and  $i' \succ i$ . If  $(i, m_i) \in T(C(s), \succeq^*)$ , then we necessarily have  $s_{i'} < m_{i'}$  i.e  $s \leq m - e_{i'}$ . Therefore, it holds that

$$\begin{aligned} f_{im_i}^P(m, v, \succeq) &= \sum_{\substack{s \leq m \\ m_i = s_i \\ (i, m_i) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} \\ &= \sum_{\substack{s \leq m - e_{i'} \\ m_i = s_i \\ (i, m_i) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} \\ &= f_{im_i}(m - e_{i'}, v, \succeq). \end{aligned}$$

The last equality holds by the last point of Remark 11. This shows that  $f^P$  satisfies Priority relation for the same maximal activity level.

*Balanced contributions for the same prevailing group.* Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  and any two agents  $i, i' \in N$  such that  $m_i = m_{i'}$  and  $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$ . Observe that, for each  $s \in \mathcal{M}$ , if  $(i, m_i), (i, m_{i'}) \in C(s)$ , then

$$(i, m_i) \in T(C(s), \succeq^*) \iff (i', m_{i'}) \in T(C(s), \succeq^*). \quad (4.10)$$

Let us compute the payoff of the pair  $(i, m_i)$ . By definition of  $f^P$ ,

$$\begin{aligned} f_{im_i}^P(m, v, \succeq) &= \sum_{\substack{s \leq m \\ (i, m_i) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} \\ &= \sum_{\substack{s \leq m \\ (i, m_i) \in T(C(s), \succeq^*) \\ s_{i'} < m_{i'}}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} + \sum_{\substack{s \leq m \\ (i, m_i) \in T(C(s), \succeq^*) \\ s_{i'} = m_{i'}}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}. \end{aligned}$$

By (4.10),

$$f_{im_i}^P(m, v, \succeq) = \sum_{\substack{s \leq m \\ (i, m_i) \in T(C(s), \succeq^*) \\ s_{i'} < m_{i'}}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} + \sum_{\substack{s \leq m \\ (i, m_i), (i', m_{i'}) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}.$$

Moreover,

$$f_{im_i}^P((m - e_{i'}), v, \succeq) = \sum_{\substack{s \leq (m - e_{i'}) \\ (i, m_i) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} = \sum_{\substack{s \leq m \\ (i, m_i) \in T(C(s), \succeq^*) \\ s_{i'} < m_{i'}}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}.$$

The same reasoning holds for  $(i', m_{i'})$ , from which one concludes that

$$\begin{aligned} f_{im_i}^P(m, v, \succeq) - f_{im_i}^P((m - e_{i'}), v, \succeq) &= \sum_{\substack{s \leq m \\ (i, m_i), (i', m_{i'}) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} \\ &= f_{i'm_{i'}}^P(m, v, \succeq) - f_{i'm_{i'}}^P((m - e_i), v, \succeq), \end{aligned}$$

as desired. This concludes the proof. ■

**Logical independence** To illustrate our counter-examples, consider the following example. Pick  $N = \{a, b\}$  and  $m = (2, 2)$ . Consider  $(m, v, \succeq)$ , where  $(N, \succeq)$  is a poset and  $v$  is the characteristic function given by

$s = (s_a, s_b)$	(1,0)	(0,1)	(1,1)	(2,0)	(0,2)	(1,2)	(2,1)	(2,2)
$v(s)$	1	1	2	3	4	6	6	12
$\Delta_v(s)$	1	1	0	2	3	1	2	2

- Dropping Efficiency. The null value defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = 0,$$

satisfies all the axioms except Efficiency.

- Dropping Independence of the maximal activity level. The value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i \geq j > 0 \\ (i, s_i) \in T(C(s), \succeq^*)}} \frac{\Delta_v(s)}{s_i |T(C(s), \succeq^*)|},$$

satisfies all the axioms except Independence of the maximal activity level. To see this, let us apply  $f$  on the above game by fixing  $a \succ b$ . Observe that  $f_{a1}(m, v, \succeq) = 4$ , whereas  $f_{a1}(m - e_a, v, \succeq) = 3$ . This shows that  $f$  violates Independence of the maximal activity level.

- Dropping Independence of level reductions. The value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ \uparrow_{\succeq \mathcal{S}(s)} i = \emptyset}} \frac{\Delta_v(s)}{|\{i' \in \mathcal{S}(s) \mid \uparrow_{\succeq \mathcal{S}(s)} i' = \emptyset\}|},$$

where  $\mathcal{S}(s) = \{i \in N \mid s_i > 0\}$ , satisfies all the axioms except Independence of level reductions. To see this, let us apply  $f$  on the above game by fixing  $a \succ b$ . Observe that  $f_{a1}(m, v, \succeq) = 2$ , whereas  $f_{a1}(m - e_b, v, \succeq) = 1$ . This shows that  $f$  violates Independence of level reductions.

- Dropping Priority relation for the same maximal activity level. The multi-choice Shapley value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ j = s_i = \bar{s}}} \frac{\Delta_v(s)}{|\{i \in N \mid s_i = \bar{s}\}|},$$

where  $\bar{s} = \max_{i \in N} s_i$ , satisfies all the axioms except Priority relation for the same maximal activity level. To see this, let us apply  $f$  on the above game by fixing  $a \succ b$ . The value does not satisfy Priority relation for the same maximal activity level since  $f_{a2}(m, v, \succeq) = 5$  and  $f_{a2}(m - e_b, v, \succeq) = 4$ .

- Dropping Balanced contributions for the same prevailing group. Fix  $\omega = (\omega_{i,j})_{(i,j) \in M^+} \in \mathbb{R}_{++}^{|M^+|}$ . The value  $f^\omega$  defined as

$$\forall (i,j) \in M^+, \quad f_{ij}^\omega(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i, s_i) \in T(C(s), \succeq^*)}} \frac{\omega_{i,j}}{\sum_{(i', j') \in T(C(s), \succeq^*)} \omega_{i' j'}} \Delta_v(s),$$

satisfies all the axioms except Balanced contributions for the same prevailing group. To see this, let us apply  $f$  on the above game and assume that  $(N, \succeq)$  is the trivial poset  $(N, \succeq^0)$ . In this case,  $a$  and  $b$  have the same (empty) prevailing group. Let us fix  $\omega_{a,1} = \omega_{a,2} = 1$  and  $\omega_{b,1} = \omega_{b,2} = 2$ . Observe that

$$\begin{aligned} f_{a2}(m, v, \succeq^0) - f_{a2}(m - e_b, v, \succeq^0) &= \frac{2}{3} \\ \text{and } f_{b2}(m, v, \succeq^0) - f_{b2}(m - e_a, v, \succeq^0) &= \frac{4}{3}. \end{aligned}$$

This shows that  $f$  violates Balanced contributions for the same prevailing group.

**Remark 13.** Consider the games  $(m, v, \succeq) \in \mathcal{GP}_N$  such that for each  $i \in N$ ,  $m_i = 1$ . Let us investigate the implication of the axioms invoked in the statement of Proposition 4.3.2 for such games. The following three observations hold. First, the axioms of Independence of the maximal activity level and Independence of level reductions become useless. Second, Priority relations for the same maximal activity level boils down to Priority agent out (see Béal et al. [2022]), which states that if an agent leaves the game, then the payoff of the agents over whom it had priority are not affected. Finally, for each  $i \in N$ , the prevailing group coincides with the priority group. Consequently, Balanced contributions for the same prevailing group becomes Balanced contributions for the same priority group. Thus, Proposition 2 reduces to Efficiency, Priority agent out and Balanced contribution for the same priority group. By Remark 12, this leads to a new characterization of the Priority value on the class of TU-games with a priority structure.

Next, consider the games  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $(N, \succeq)$  is the trivial poset. Let us investigate the implication of the axioms invoked in the statement of Proposition 4.3.2 for such games. First, because there is no priority relations between the agents, Priority relation for the same maximal activity level becomes useless. Another consequence is that the prevailing group of each agent becomes empty so that Balanced contribution for the same prevailing group becomes Balanced contributions for agents with the same maximal activity level. Thus, Proposition 2 reduces to Efficiency, Independence of the maximal activity level, Independence of level reductions, and Balanced contributions for agents with the same maximal activity level. By Remark 12, this leads to a new characterization of the multi-choice Shapley value on the class of multi-choice games.

### 4.3.2 Equal treatment principle and priority relations

This section provides a second characterization of  $f^P$  for multi-choice games with a priority structure. The next four axioms are generalizations of standard axioms for multi-choice games to multi-choice games with a priority structure.

*Additivity.* For each  $(m, v, \succeq), (m, w, \succeq) \in \mathcal{GP}_N$ ,

$$f(m, v + w, \succeq) = f(m, v, \succeq) + f(m, w, \succeq).$$

*Null game.* For each  $(m, v_0, \succeq) \in \mathcal{GP}_N$ ,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v_0, \succeq) = 0.$$

*Independence of null pair.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , for each null pair  $(i, j) \in M^+$ ,

$$\forall (i', j') \in M^+ \setminus \{(i, m_i)\}, \quad f_{i'j'}(m, v, \succeq) = f_{i'j'}(m - e_i, v, \succeq).$$

*The null pair axiom.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , for each null pair  $(i, j) \in M^+$ ,

$$f_{ij}(m, v, \succeq) = 0.$$

By successive applications of Independence of null pair and Efficiency one obtains the following result.

**Lemma 4.3.1.** *On the class of multi-choice games with a priority structure  $\mathcal{GP}_N$ , Efficiency and Independence of null pair imply the null pair axiom.*

The next two axioms relate to decisive agents with the same maximal activity level. Recall that an agent is decisive if any coalition in which it does not play its maximal activity level generates zero worth. Let  $s \in \mathcal{M}$  be a coalition such that  $s_i = j - 1$ . The surplus  $v(s + e_i) - v(s)$  refers to the contribution of  $i$  for its activity level  $j$  to the coalition  $s$ . It is straightforward to see that two decisive agents  $i, i'$  such that  $m_i = m_{i'}$  have the same contributions to coalitions, i.e, for each  $s$  where  $s_i = s_{i'} = m_i - 1$  and  $s_k \leq m_k, k \neq i, i'$ , it holds that  $v(s + e_i) = v(s + e_{i'})$ .<sup>3</sup> Therefore, the pairs  $(i, m_i), (i', m_{i'})$  are equals. A standard requirement in cooperative games is that equal agents are treated equally. However, although decisive agents are equal from an economic perspective, they may not have the same position in the priority structure. Due to this asymmetry, it seems reasonable to favor the decisive agent with the highest priority. The next axiom accommodates this feature and requires that the payoff of a decisive agent is zero if another decisive agent has priority over it.

*Priority relation for decisive agents.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , for each  $i, i' \in N$  decisive agents such that  $m_{i'} = m_i$  and  $i' \succ i$ ,

$$f_{im_i}(m, v, \succeq) = 0.$$

Whenever two decisive agents are indistinguishable in terms of both maximal activity levels and priority, there is no reason to treat them differently. The next axiom follows this standpoint and ensures an equal payoff for such agents.

*Equal treatment for decisive agents with the same prevailing group.* For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , for each  $i, i' \in N$  decisive such that  $m_i = m_{i'}$  and  $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$ ,

$$f_{im_i}(m, v, \succeq) = f_{im_{i'}}(m, v, \succeq).$$

<sup>3</sup>This observation can be extended by considering two decisive agents with different maximal activity levels.

Interestingly, the combination of Priority relation for the same maximal activity level and Null game implies Priority relation for decisive agents. Moreover, the combination of Balanced contributions for the same prevailing group and Null game implies Equal treatment for decisive agents with the same prevailing group.

**Proposition 4.3.3.** *On the class of multi-choice games with a priority structure  $\mathcal{GP}_N$ ,*

- a. *Priority relation for the same maximal activity level and Null game implies Priority relation for decisive agents;*
- b. *Balanced contributions for the same prevailing group and Null game implies Equal treatment for decisive agents with the same prevailing group.*

*Proof.* Point a. Let  $f$  be a value on  $\mathcal{GP}_N$  that satisfies Priority relation for the same maximal activity level and Null game. Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  and any  $i' \in N$  decisive agent. By Priority relation for the same maximal activity level, for each decisive agent  $i \in N$  such that  $m_{i'} = m_i$  and  $i' \succeq i$ ,

$$f_{im_i}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq).$$

Since  $(i', m_{i'})$  is decisive in  $(m, v, \succeq)$ , the sub-game  $(m - e_{i'}, v, \succeq)$  is a null game. Therefore, by Null game,

$$f_{im_i}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq) = f_{im_i}(m - e_{i'}, v_0, \succeq) = 0.$$

This shows that  $f$  satisfies Priority relation for decisive agents.

Point b. Let  $f$  be a value on  $\mathcal{GP}_N$  that satisfies Balanced contributions for the same prevailing group and Null game. Take any  $(m, v, \succeq) \in \mathcal{GP}_N$  and any two decisive agents  $i, i' \in N$  such that  $m_i = m_{i'}$  and  $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$ . By Balanced contributions for the same prevailing group,

$$f_{im_i}(m, v, \succeq) - f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq).$$

Since  $(i, m_i)$  and  $(i', m_{i'})$  are decisive in  $(m, v, \succeq)$ , it holds that  $(m - e_{i'}, v, \succeq)$  and  $(m - e_i, v, \succeq)$  are null games. By Null game,

$$f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m - e_i, v, \succeq) = 0.$$

Thus, we conclude that

$$f_{im_i}(m, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq).$$

This shows that  $f$  satisfies Equal treatment for decisive agents with the same prevailing group. ■

The next result shows that the combination of Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, Priority relation for decisive agents, and Equal treatment for decisive agents with the same prevailing group yields a unique value for multi-choice games with a priority structure.

**Proposition 4.3.4.** *On the class  $\mathcal{GP}_N$ , there is at most one value that satisfies Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, Priority relation for decisive agents and Equal treatment for decisive agents with the same prevailing group.*

*Proof.* Let  $f$  be a value on  $\mathcal{GP}_N$  that satisfies Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, Priority relation for decisive agents and Equal treatment for decisive agents with the same prevailing group. Let us show that  $f$  is uniquely determined. Take any  $(m, v, \succeq) \in \mathcal{GP}_N$ . By Additivity, it is enough to show that, for each  $t \in \mathcal{M}$ ,  $f(m, \Delta_t(v)u_t, \succeq)$  is uniquely determined. Take any  $t \in \mathcal{M}$ , and denote by  $\bar{t} = \max_{i \in N} t_i$  the highest activity level in  $t$ .

Each pair  $(i, j) \in M^+$ , such that  $j > t_i$ , is a null pair in  $(m, \Delta_t(v)u_t, \succeq)$ . Since  $f$  satisfies Efficiency and Independence of null pair, by Lemma 4.3.1 it satisfies the null pair axiom. Therefore, for each  $(i, j) \in M^+$  such that  $j > t_i$ ,  $f_{ij}(m, \Delta_t(v)u_t, \succeq) = 0$ . By successive applications of Independence of null pair, for each  $(i, j) \in M^+$  such that  $j \leq t_i$ , we obtain

$$f_{ij}(m, \Delta_t(v)u_t, \succeq) = f_{ij}(t, \Delta_t(v)u_t, \succeq).$$

Take any pair  $(i, j) \in M^+$  such that  $j < t_i \leq \bar{t}$ . By successive application of Independence of the maximal activity level, we obtain

$$f_{ij}(t, \Delta_t(v)u_t, \succeq) = f_{ij}((t_{-i}, j), \Delta_t(v)u_t, \succeq).$$

Moreover, by successive application of Independence of level reductions, we obtain

$$f_{ij}((t_{-i}, j), \Delta_t(v)u_t, \succeq) = f_{ij}((t_k \wedge j)_{k \in N}, \Delta_t(v)u_t, \succeq).$$

Because  $j < t_i$ , the sub-game  $((t_k \wedge j)_{k \in N}, \Delta_t(v)u_t, \succeq)$  is a null game. Since Additivity implies Null game,

$$\begin{aligned} f_{ij}(t, \Delta_t(v)u_t, \succeq) &= f_{ij}((t_{-i}, j), \Delta_t(v)u_t, \succeq) \\ &= f_{ij}((t_k \wedge j)_{k \in N}, \Delta_t(v)u_t, \succeq) \\ &= f_{ij}((t_k \wedge j)_{k \in N}, v_0, \succeq) \\ &= 0. \end{aligned}$$

Take any pair  $(i, j) \in M^+$  such that  $j = t_i < \bar{t}$ . By successive applications of Independence of level reductions, we obtain

$$f_{ij}(t, \Delta_t(v)u_t, \succeq) = f_{ij}((t_k \wedge j)_{k \in N}, \Delta_t(v)u_t, \succeq).$$

Because  $j < \bar{t}$ , the sub-game  $((t_k \wedge j)_{k \in N}, \Delta_t(v)u_t, \succeq)$  is a null game. Since Additivity implies Null game,

$$f_{ij}(t, \Delta_t(v)u_t, \succeq) = 0.$$

Define the set  $N_{\bar{t}} = \{i \in N \mid t_i = \bar{t}\}$ . Observe that any  $i \in N_{\bar{t}}$  is a decisive agent in the game  $(t, \Delta_t(v)u_t, \succeq)$ . Take any pair  $(i, j) \in M^+$  such that  $j = t_i = \bar{t}$ , i.e.,  $i \in N_{\bar{t}}$ . Two cases are distinguished. Let  $L_i(t, \Delta_t(v)u_t, \succeq) \neq \emptyset$ . Since  $i$ 's maximal activity level in  $(t, \Delta_t(v)u_t, \succeq)$  is  $\bar{t}$ ,  $i' \in L_i(t, \Delta_t(v)u_t, \succeq)$  implies  $i' \succ i$  and  $t_{i'} = \bar{t}$ . Thus, there exists another agent  $i' \in N_{\bar{t}}$ , such that  $i' \succ i$ . By Priority relation for decisive agents,

$$f_{ij}(t, \Delta_t(v)u_t, \succeq) = 0.$$

Now suppose that  $L_i(t, \Delta_t(v)u_t, \succeq) = \emptyset$ . Consider the set of pairs

$$M_{L(t)}^+ = \{(i, j) \in M^+ \mid i \in N_{\bar{t}}, L_i(t, \Delta_t(v)u_t, \succeq) = \emptyset\}.$$

At this step, we know that  $f_{ij}(m, \Delta_t(v)u_t, \succeq) = 0$  for each  $(i, j) \in M^+ \setminus M_{L(t)}^+$ . If  $|M_{L(t)}^+| = 1$ , then  $f_{ij}(m, \Delta_t(v)u_t, \succeq)$ , for  $(i, j) \in M_{L(t)}^+$ , is uniquely determined by Efficiency. If  $|M_{L(t)}^+| > 1$ , then by Efficiency,

$$\sum_{(i,j) \in M^+} f_{ij}(m, \Delta_t(v)u_t, \succeq) = \sum_{(i,j) \in M_{L(t)}^+} f_{ij}(t, \Delta_t(v)u_t, \succeq) = \Delta_t(v). \quad (4.11)$$

By Equal treatment for decisive agents with the same prevailing group,

$$\forall (i, j), (i', j') \in M_{L(t)}^+, \quad f_{ij}(t, \Delta_t(v)u_t, \succeq) = f_{i'j'}(t, \Delta_t(v)u_t, \succeq) = c. \quad (4.12)$$

From (4.12) and (4.11),  $f_{ij}(m, \Delta_t(v)u_t, \succeq)$  is uniquely determined for each  $(i, j) \in M_{L(t)}^+$ . Additivity allows to conclude the proof. ■

The next result provides a second axiomatic characterization of  $f^P$  on the class of multi-choice games with a priority structure.

**Proposition 4.3.5.** *On the class  $\mathcal{GP}_N$ , a value  $f$  satisfies Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, Priority relation for decisive agents and Equal treatment for decisive agents with the same prevailing group if and only if  $f = f^P$ .*

*Proof.* Thanks to Proposition 4.3.4, it suffices to show that  $f^P$  satisfies Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, Priority relation for decisive agents and Equal treatment for decisive agents with the same prevailing group.

By Proposition 4.3.2, we know that  $f^P$  satisfies Efficiency, Independence of the maximal activity level, Independence of level reductions. Since  $f^P$  is a weighted sum of the dividend, it satisfies Additivity. Observe that, for each null pair  $(i, j) \in M^+$  and each  $s \in \mathcal{M}$  such that  $s_i \geq j$ , it holds that  $\Delta_s(v) = 0$ . From this observation, it follows that  $f^P$  satisfies Independence of null pair. Since Additivity implies Null game, by Proposition 4.3.3,  $f^P$  satisfies Priority relation for decisive agents and Equal treatment for decisive agents with the same prevailing group. ■

### Logical independence

- Dropping Efficiency. The null value defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = 0,$$

satisfies all the axioms except Efficiency.

- Dropping Additivity. The value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i,j) \in T(C(s), \succeq^*)}} \frac{(v(0_{-i}, s_i))^2 + 1}{\sum_{(k, s_k) \in T(C(s), \succeq^*)} (v(0_{-k}, s_k))^2 + 1} \Delta_v(s)$$

satisfies all the axioms except Additivity.

- Dropping Independence of the maximal activity level. The value defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i \geq j > 0 \\ (i, s_i) \in T(C(s), \succeq^*)}} \frac{\Delta_v(s)}{s_i |T(C(s), \succeq^*)|'}$$

satisfies all the axioms except Independence of the maximal activity level.

- Dropping Independence of level reductions. The value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ \uparrow_{\succeq S(s)} i = \emptyset}} \frac{\Delta_v(s)}{|\{i' \in \mathcal{S}(s) \mid \uparrow_{\succeq S(s)} i = \emptyset\}|'}$$

where  $\mathcal{S}(s) = \{i \in N \mid s_i > 0\}$ , satisfies all the axioms except Independence of level reductions.

- Dropping Independence of null pair. The value  $f$  defined as

$$f_{ij}(m, v, \succeq) = \begin{cases} \frac{v((m_k \wedge j)_{k \in N}) - v((m_k \wedge j - 1)_{k \in N})}{|T(C((m_k \wedge j)_{k \in N}), \succeq^*)|} & \text{if } (i, j) \in T(C((m_k \wedge j)_{k \in N}), \succeq^*), \\ 0 & \text{otherwise.} \end{cases}$$

satisfies all the axioms except Independence of null pair. To see this, pick  $N = \{a, b\}$ ,  $a \succ b$  and  $m = (1, 1)$ . The characteristic function  $v$  is defined in the following table.

$s = (s_a, s_b)$	(1,0)	(0,1)	(1,1)
$v(s)$	0	1	1

Observe that  $(a, 1)$  is a null pair. Since  $f_{b1}(m, v, \succeq) = 0$  and  $f_{b1}(m - e_a, v, \succeq) = 1$ ,  $f$  violates Independence of null pair.

- Dropping Priority relation for decisive players. The multi-choice Shapley value  $f$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ j = s_i = \bar{s}}} \frac{\Delta_v(s)}{|\{i \in N \mid s_i = \bar{s}\}|'}$$

satisfies all the axioms except Priority relation for decisive players. To see this, pick  $N = \{a, b\}$ ,  $a \succ b$  and  $m = (1, 1)$ . Consider the minimal effort game  $(m, u_m)$ . Observe that  $a$  and  $b$  are both decisive players in  $(m, u_m, \succeq)$ . The fact that  $f_{b1}(m, u_m, \succeq) = 0.5$  shows that  $f$  violates Priority relation for decisive players.

- Dropping Equal treatment for decisive players with the same prevailing group. Fix  $\omega = (\omega_{i,j})_{(i,j) \in M^+} \in \mathbb{R}_{++}^{|M^+|}$ . The value  $f^\omega$  defined as

$$\forall (i, j) \in M^+, \quad f_{ij}^\omega(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i, s_i) \in T(C(s), \succeq^*)}} \frac{\omega_{i,j}}{\sum_{(i', j') \in T(C(s), \succeq^*)} \omega_{i' j'}} \Delta_v(s),$$



satisfies all the axioms except Equal treatment for decisive players with the same prevailing group. To see this, pick  $N = \{a, b\}$ ,  $m = (1, 1)$  and the trivial poset  $(N, \succeq^0)$ . Since the priority structure is trivial, then  $a$  and  $b$  have the same (empty) priority group. Consider the minimal effort game  $(m, u_m)$ . Observe that  $a$  and  $b$  are both decisive players in  $(m, u_m, \succeq^0)$ . Let us fix  $\omega_{a,1} = 1$  and  $\omega_{b,1} = 2$ . We obtain  $f_{a1}(m, u_m, \succeq^0) = 0.33$  and  $f_{b1}(m, u_m, \succeq^0) = 0.66$ , which shows that  $f$  violates Equal treatment for decisive players with the same prevailing group.

**Remark 14.** Consider the games  $(m, v, \succeq) \in \mathcal{GP}_N$  such that for each  $i \in N$ ,  $m_i = 1$ . Let us investigate the implication of the axioms invoked in the statement of Proposition 4.3.5 for such games. The following observations hold. First, Independence of null pair axiom boils down to the null agent out axiom (see Derks and Haller [1999]), which states that the removal of a null agent from a TU-game does not impact the payoff of the remaining agents. Second, because the decisive agent generalizes the notion of necessary agent, Priority relation for decisive agents boils down to Priority and necessary agent (see Béal et al. [2022]). Finally, as observed in Remark 13, for each  $i \in N$  the prevailing group coincides with the priority group. Therefore, Equal treatment for decisive agents with the same prevailing group reduces to Equal treatment for necessary agents with the same priority group (see Béal et al. [2022]). Using Remark 12, Proposition 4.3.5 reduces to Theorem 2 in Béal et al. [2022].

Next, consider the games  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $(N, \succeq)$  is the trivial poset. Let us investigate the implication of the axioms invoked in the statement of Proposition 4.3.5 for such games. As observed in Remark 13, the prevailing group of each agent is empty. Consequently, Equal treatment for decisive agents with the same prevailing group becomes Equal treatment for decisive agents with the same maximal activity level. Thus, Proposition 4.3.5 reduces to Efficiency, Additivity, Independence of the maximal activity level, Independence of level reductions, Independence of null pair, and Equal treatment for decisive agents with the same maximal activity level. By Remark 12, this leads to a new characterization of the multi-choice Shapley value.

## 4.4 Priority relations structured by classes

This section presents situations in which the set of agents can be partitioned into several priority classes  $(N_1, \dots, N_q)$ . Each priority class contains incomparable agents that have priority over each agent in the next class. Such priority structures generalize the linear priority structures. Formally,

$$\forall i, i' \in N, \quad [i \succ i'] \iff [i \in N_p, i' \in N_{p'} \implies p < p'].$$

The set of agents over which agent  $i \in N_p$  has priority is

$$\downarrow_{\succ} i = \bigcup_{p' > p} N_{p'}.$$

In the following,  $N_{>p}$  stands for  $\bigcup_{p' > p} N_{p'}$ .

**Example 5.** Let  $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$  be the set of agents and consider  $(N, \succeq)$  a priority structured by classes. Suppose that the set of agents is partitioned into three cells:  $N_1 = \{4, 5, 6\}$ ,  $N_2 = \{2, 3\}$  and  $N_3 = \{1, 7, 8\}$ . Each agent in  $N_1$  has priority

over agents in  $N_2 \cup N_3$ , and each agent in  $N_2$  has priority over agents in  $N_3$ . Figure 2 represents the Hasse diagram of the poset  $(N, \succeq)$ .

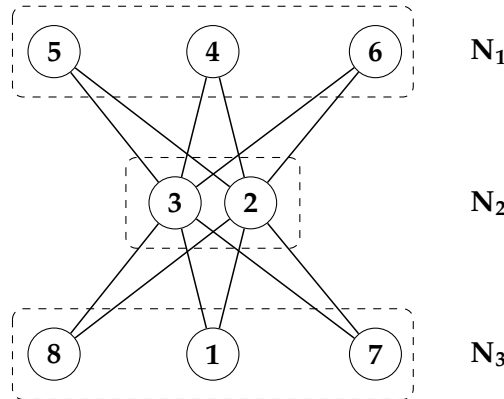


Figure 2: Hasse diagram of  $(N, \succeq)$  structured by classes.

To analyze the impact of priority relations structured by classes on multi-choice games, some useful notations and definitions are introduced. Consider a multi-choice game  $(m, v) \in \mathcal{G}$  and an activity level  $j \leq \max_{i \in N} m_i$ . Denote by  $Q(j) \subseteq N$  the set of agents able to play activity level  $j$ . Formally, the set  $Q(j)$  is defined as

$$Q(j) = \{i \in N : m_i \geq j\}.$$

Without loss of generality, we assume that  $Q(1) = N$ . Recall that, any TU-game  $(N, v)$  can be viewed as a multi-choice game  $(m, v)$  where for each  $i \in N$ ,  $m_i = 1$ . For each TU-game, the Shapley value (see [Shapley \[1953\]](#)) is defined as

$$\forall i \in N, \quad Sh_i(N, v) = Sh_{i1}((1, \dots, 1), v) = \sum_{\substack{s \leq (1, \dots, 1) \\ s_i = 1}} \frac{\Delta_s(v)}{\sum_{i \in N} s_i}.$$

### The multi-choice Priority value as a sequential allocation process

Consider the class of multi-choice games with priority relations structured by classes. We define a sequential allocation process denoted by  $f$ . First consider an activity level  $j \leq \max_{i \in N} m_i$  and the set of agents that are able to play this activity level, i.e.,  $Q(j)$ . Then, consider the priority class  $p \in \{1, \dots, q\}$ , and agents in  $N_p$  that are able to play  $j$ , i.e., the subset of agents  $N_p \cap Q(j)$ . Finally, one assigns to each pair  $(i, j) \in M^+$  such that  $i \in N_p$ , the Shapley value obtained by  $i$  in a TU-game on  $N_p \cap Q(j)$ . This sequential allocation process is detailed by the following procedure.

---

**Allocation process for multi-choice games with a priority relation structured by classes**


---

```

1: for  $j = 1$  to  $\max_{i \in N} m_i$  do
2:
3:   for  $t = 1$  to  $q$  do
4:      $p = q - t + 1$ 
5:     if  $N_p \cap Q(j) \neq \emptyset$  then
6:

```

$$\forall E \subseteq N_p \cap Q(j), \quad w_j^p(E) = v\left(\left((j-1) \wedge m_k\right)_{k \in N} + e_{(E \cup N_{>p}) \cap Q(j)}\right) - v\left(\left((j-1) \wedge m_k\right)_{k \in N} + e_{N_{>p} \cap Q(j)}\right) \quad (4.13)$$

$$\forall i \in N_p \cap Q(j), \quad f_{ij}(m, v, \succeq) = Sh_i(N_p \cap Q(j), w_j^p)$$

```

7:   else
8:
9:   end if
10: end for
11: end for

```

---

Line 1 highlights that the procedure applies from the smallest activity level  $j = 1$  to the highest activity level in the multi-choice game. Similarly, line 3 emphasizes that for each level of activity, the procedure applies from the last priority class  $N_q$  to the first priority class  $N_1$ . Line 6 of the above procedure allows to define the TU-game  $(N_p \cap Q(j), w_j^p)$ , whose interpretation is the following. Assume that all agents in  $N_p \cap Q(j)$  play the activity level  $j$ . They obtain their contribution to the coalition in which all agents in  $N_{>p} \cap Q(j)$  play  $j$ , while all other agents play  $j - 1$  (or their maximal activity level). To determine how this contribution is allocated, the TU-game  $(N_p \cap Q(j), w_j^p)$  is defined as in (4.13). For each coalition  $E \subseteq N_p \cap Q(j)$ , the worth  $w_j^p(E)$  corresponds to the surplus generated (in the game  $(m, v, \succeq)$ ) when each agent in  $E$  increases its activity level from  $j - 1$  to  $j$ . For an illustration of the procedure, see Example 6 below.

**Remark 15.** Observe that, for each priority class  $p \in \{1, \dots, q\}$  the TU-game  $(N_p \cap Q(j), w_j^p)$ , as defined in (4.13), is the sub-game of the TU-game  $(Q(j), w_j)$  where

$$\forall E \subseteq Q(j), \quad w_j(E) = v\left(\left((j-1) \wedge m_k\right)_{k \in N} + e_E\right) - v\left(\left((j-1) \wedge m_k\right)_{k \in N}\right). \quad (4.14)$$

One can interpret  $w_j(E)$  as follows. Suppose that each agent outside coalition  $E$  plays its activity level  $j - 1$  or its maximal activity level (if it is unable to play  $j - 1$ ). Then, the worth  $w_j(E)$  corresponds to the surplus generated (in the game  $(m, v, \succeq)$ ) when each agent in  $E$  increases its activity level from  $j - 1$  to  $j$ .<sup>4</sup>

Let  $E \subseteq Q(j)$ , by definition of the dividends given by (2.21),

$$w_j(E) = \sum_{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E} \Delta_t(v) - \sum_{t \leq ((j-1) \wedge m_k)_{k \in N}} \Delta_t(v),$$

---

<sup>4</sup>One can also find a similar construction of TU-games in the context of cost sharing problems (for instance, see Albizuri et al. [2003]).

that is

$$\forall E \subseteq Q(j), \quad w_j(E) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_t(v). \quad (4.15)$$

It should be observed that this sum takes in arguments each coalition  $t \leq ((j-1) \wedge m_k)_{k \in N} + e_E$  in which at least one agent in  $E$  plays activity level  $j$ . This observation will be useful to prove the next results.

**Example 6.** To illustrate the allocation process, consider  $(m, v, \succeq) \in \mathcal{GP}_N$  such that  $(N, \succeq)$  is the priority structured by classes given in Example 5. Let  $m_7 = 1$ ,  $m_1 = m_2 = m_6 = 2$ ,  $m_3 = m_4 = m_5 = m_8 = 3$ . It holds that  $Q(1) = N$ ,  $Q(2) = N \setminus \{7\}$  and  $Q(3) = \{3, 4, 5, 8\}$ . Apply the sequential procedure to determine the payoffs of the pairs in  $M^+$ .

1. Consider the activity level  $j = 1$ , and the subset of agents  $Q(1) = N$ .

1.1. Consider the priority class  $p = 3$  and agents in  $N_3 \cap Q(1) = \{1, 7, 8\} = N_3$ . All agents in this subset cooperate at activity level  $j = 1$ . The contribution  $v(e_{N_3}) - v(\mathbf{0})$  is allocated among all agents in  $N_3$  according to the TU-game  $(N_3, w_1^3)$  defined as,

$$\forall E \subseteq N_3, \quad w_1^3(E) = v(e_E).$$

Then, assign  $Sh_i(N_3, w_1^3)$  to each agent in  $i \in N_3$ , which corresponds to their payoff for their activity level  $j = 1$ .

1.2. Consider the priority class  $p = 2$  and agents in  $N_2 \cap Q(1) = \{3, 2\} = N_2$ . Define the TU-game  $(N_2, w_1^2)$  as

$$\forall E \subseteq N_2, \quad w_1^2(E) = v(e_E + e_{N_3}) - v(e_{N_3}).$$

Then, assign  $Sh_i(N_2, w_1^2)$  to each agent  $i \in N_2$ , which corresponds to their payoff for their activity level  $j = 1$ .

1.3. The procedure is applied similarly. Thus, one defines the TU-game  $(N_1, w_1^1)$  as

$$\forall E \subseteq N_1, \quad w_1^1(E) = v(e_E + e_{N_2 \cup N_3}) - v(e_{N_2 \cup N_3}).$$

Then, each agent  $i \in N_1$  receives  $Sh_i(N_1, w_1^1)$  which corresponds to their payoff for their activity level  $j = 1$ .

The total worth allocated to agents in  $Q(1)$  is

$$v(e_{N_1} + e_{N_2 \cup N_3}) + v(e_{N_2 \cup N_3}) + v(e_{N_3}) - v(e_{N_2 \cap N_3}) - v(e_{N_3}) = v(1, \dots, 1).$$

2. Consider the activity level  $j = 2$  and the subset of agents  $Q(2)$ .

2.1. Consider the priority class  $p = 3$  and agents in  $N_3 \cap Q(2)$ . Then, define the TU-game  $(N_3 \cap Q(2), w_2^3)$  as

$$\forall E \subseteq N_3 \cap Q(2), \quad w_2^3(E) = v((1, \dots, 1) + e_E) - v((1, \dots, 1)).$$

Then, assigns  $Sh_i(N_3 \cap Q(2), w_2^3)$  to each agent  $i \in N_3 \cap Q(2)$ .

The next steps are applied similarly.

The following lemma establishes that the dividend of a coalition  $E \subseteq Q(j)$  of the intermediary TU-game  $(Q(j), w_j)$ , defined as (4.14), has a relevant expression in terms of the dividends of multi-choice coalitions (of the multi-choice game  $(m, v, \succeq)$ ). Specifically, the dividend of a coalition  $E \subseteq Q(j)$  is the sum of the dividend of the multi-choice coalitions  $t$  such that for each  $i \notin E$ ,  $t_i \leq (j-1) \wedge m_i$  and for each  $i \in E$ ,  $t_i = j$ .

**Lemma 4.4.1.** *For each TU-game  $(Q(j), w_j)$  as defined by (4.14), it holds that*

$$\forall E \subseteq Q(j), \quad \Delta_E(w_j) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ \forall i \in E, t_i = j}} \Delta_t(v).$$

*Proof.* Let  $(Q(j), w_j)$  be the TU-game derived from  $(m, v, \succeq) \in \mathcal{GP}_N$  and the activity level  $j \leq \max_{i \in N} m_i$ . The proof proceeds by induction on the size of coalitions  $E \subseteq Q(j)$ .

**Initialization.** Take any coalition  $E \subseteq Q(j)$ , such that  $|E| = 1$ . Let say that  $E = \{i\} \subseteq Q(j)$ . By definition of the dividend of a TU-game,

$$\begin{aligned} \Delta_{\{i\}}(w_j) &= w_j(\{i\}) \\ &= \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_i \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_t(v). \end{aligned}$$

where the first equality comes from the definition of the dividend in TU-games and the definition of  $w_j$ ; and the second equality comes from (4.15). Observe that the set of coalitions  $t \leq ((j-1) \wedge m_k)_{k \in N} + e_i$  and  $t \not\leq ((j-1) \wedge m_k)_{k \in N}$  is equal to the set of coalitions  $t \leq ((j-1) \wedge m_k)_{k \in N} + e_i$  such that  $t_i = j$ . Therefore, it holds that

$$\Delta_{\{i\}}(w_j) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_i \\ t_i = j}} \Delta_t(v),$$

the desired result.

**Induction hypothesis.** Assume that for each  $E \subseteq Q(j)$ , such that  $|E| = r$  with  $1 \leq r < |Q(j)|$ , we have

$$\Delta_E(w_j) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ \forall i \in E: t_i = j}} \Delta_t(v).$$

**Induction step.** Take any coalition  $E \subseteq Q(j)$ , such that  $|E| = r+1$ . By definition of the dividend,

$$\Delta_E(w_j) = w_j(E) - \sum_{T \subset E} \Delta_T(w_j).$$

Observe that, for each  $T \subset E$ , it holds that  $|T| \leq r$ . By (4.15) and the induction hypothesis, it follows that

$$\Delta_E(w_j) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_t(v) - \sum_{T \subset E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_t(v). \quad (4.16)$$

Note that, each coalition  $t \leq ((j-1) \wedge m_k)_{k \in N} + e_E$  and  $t \not\leq ((j-1) \wedge m_k)_{k \in N}$  is such that

- for each agent  $i \in N \setminus E$ ,  $t_i \leq (j-1) \wedge m_i$ ,

- for some agents in  $E$ ,  $t_i = j$ .

Consider a coalition  $t \leq ((j-1) \wedge m_k)_{k \in N} + e_E$  and  $t \not\leq ((j-1) \wedge m_k)_{k \in N}$ . Denote by  $T \subseteq E$ , the subset of agents such that  $t_i = j$ . Summing over all coalitions  $t$  of this form, one obtains

$$\sum_{T \subseteq E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_t(v) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_t(v).$$

From this observation, (4.16) now becomes

$$\begin{aligned} \Delta_E(w_j) &= \sum_{T \subseteq E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_t(v) - \sum_{T \subseteq E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_t(v) \\ &= \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ \forall i \in E: t_i = j}} \Delta_t(v). \end{aligned}$$

This concludes the induction step and completes the proof. ■

The main result of this section shows that, on the class of multi-choice games with a priority relation structured by classes, the multi-choice Priority value assigns to each pair  $(i, j) \in M^+$  a payoff resulting from the above sequential procedure: the payoff assigned to an activity level  $j$  of the agent  $i \in N_p$  coincides with the Shapley value obtained by  $i$  in the TU-game  $(N_p \cap Q(j), w_j^p)$ .

**Proposition 4.4.1.** *For each  $(m, v, \succeq) \in \mathcal{GP}_N$ , where  $(N, \succeq)$  is a priority structured by the classes  $(N_1, \dots, N_q)$ , it holds that for each priority class  $p \in \{1, \dots, q\}$ , for each pair  $(i, j) \in M^+$  where  $i \in (N_p \cap Q(j))$ ,*

$$f_{ij}^p(m, v, \succeq) = Sh_i((N_p \cap Q(j)), w_j^p),$$

where the TU-game  $(N_p \cap Q(j), w_j^p)$  is defined as (4.13).

*Proof.* Take any multi-choice game with priority structure  $(m, v, \succeq) \in \mathcal{G}_p$ , where  $(N, \succeq)$  is structured by the classes  $(N_1, \dots, N_q)$ . Take any class  $p$ , any activity level  $j \leq \max_{i \in N} m_i$  and consider the TU-game  $(N_p \cap Q(j), w_j^p)$  given by (4.13). Adapting an intermediary result provided by Béal et al. [2022],<sup>5</sup> one can show that the dividends of the TU-game  $(N_p \cap Q(j), w_j^p)$  are given by

$$\forall E \subseteq N_p \cap Q(j), \quad \Delta_E(w_j^p) = \sum_{T \subseteq N_{>p} \cap Q(j)} \Delta_{T \cup E}(w_j^p).$$

From, Lemma 4.4.1, one can write

$$\forall E \subseteq N_p \cap Q(j), \quad \Delta_E(w_j^p) = \sum_{T \subseteq N_{>p} \cap Q(j)} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_{E \cup T} \\ \forall i \in E \cup T: t_i = j}} \Delta_t(v). \quad (4.17)$$

<sup>5</sup>The reader is referred to the proof of Proposition 3 by Béal et al. [2022].

Next, take any priority class  $p \in \{1, \dots, q\}$  and any pair  $(i, j) \in M^+$  such that  $i \in N_p$ . The payoff of the pair  $(i, j)$  through  $f^P$  is

$$f_{ij}^P(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i, j) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}.$$

Observe that, if a pair  $(i', j') \in C(s)$  is such that either  $i' \neq i$  and  $j' > j$  or  $i' = i$  and  $j' > j$ , then we necessarily have  $(i, j) \notin T(C(s), \succeq^*)$ . From this observation,  $f_{ij}^P(m, v, \succeq)$  can be written as

$$f_{ij}^P(m, v, \succeq) = \sum_{\substack{s \leq (j \wedge m_k)_{k \in N} \\ s_i = j \\ (i, j) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|} = \sum_{\substack{s \leq ((j-1) \wedge m_k)_{k \in N} + e_{Q(j)} \\ s_i = j \\ (i, j) \in T(C(s), \succeq^*)}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}.$$

Because  $i \in N_p$ , for each coalition  $s \leq ((j-1) \wedge m_k)_{k \in N} + e_{Q(j)}$  such that  $(i, j) \in T(C(s), \succeq^*)$ , it holds that

- $s_i = j$
- $\forall k \in N \setminus Q(j): s_k \leq m_k < j$ ;
- $\forall k \in N_{p'} \cap Q(j)$ , where  $p' > p$ :  $s_k < j$ ;
- $\forall k \in (N_p \cup N_{>p}) \cap Q(j): s_k \leq j$ .

These four points imply that the payoff of the pair  $(i, j)$  is obtained by summing the dividends of  $(m, v, \succeq)$  over the multi-choice coalitions  $s \leq ((j-1) \wedge m_k)_{k \in N} + e_{(N_p \cup N_{>p}) \cap Q(j)}$  such that  $s_i = j$ . Moreover, by summing over such multi-choice coalitions, one consider all sub-coalitions of agents  $T \subseteq (N_p \cup N_{>p}) \cap Q(j)$  containing agent  $i$ . Thus, we obtain

$$f_{ij}^P(m, v, \succeq) = \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \sum_{T \subseteq N_{>p} \cap Q(j)} \sum_{\substack{s \leq ((j-1) \wedge m_k)_{k \in N} + e_{E \cup T} \\ \forall k \in E \cup T: s_k = j}} \frac{\Delta_s(v)}{|T(C(s), \succeq^*)|}.$$

It should be observed that  $|T(C(s), \succeq^*)|$  does not depend on the coalitions  $T \subseteq N_{>p} \cap Q(j)$ . More precisely, it can be shown that, for each coalition  $s \leq ((j-1) \wedge m_k)_{k \in N} + e_{E \cup T}$  such that  $\forall k \in E \cup T: s_k = j$ ,  $|T(C(s), \succeq^*)| = |E|$ . Therefore, it holds that

$$\begin{aligned} f_{ij}^P(m, v, \succeq) &= \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \frac{1}{|E|} \sum_{T \subseteq N_{>p} \cap Q(j)} \sum_{\substack{s \leq ((j-1) \wedge m_k)_{k \in N} + e_{E \cup T} \\ \forall k \in E \cup T: s_k = j}} \Delta_s(v) \\ &= \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \frac{1}{|E|} \Delta_E(w_j^p) \\ &= Sh_i((N_p \cap Q(j)), w_j^p), \end{aligned}$$

where the second equality comes from (4.17), and the third comes from the definition of the Shapley value for TU-games. This completes the proof.  $\blacksquare$

## 4.5 Conclusion

This chapter provides an axiomatic study on the class of multi-choice games with a priority structure. A new value for this class is introduced and characterized: the multi-choice Priority value. This value takes advantage of a lexicographic partial order resulting from the combination of the ordered set of activity levels of the agents with the priority structure. The followed approach allows to endogenously determine the partial lexicographic order on the set of pairs (composed of agents and their activity levels) from the axioms.

We propose several directions for future research. First, the axioms discussed in this chapter are conceptually close to those satisfied by the (multi-choice) Shapley value(s) (Additivity, the null agent axiom and Balanced contributions). Another well-known axiom satisfied by the Shapley value is Strong monotonicity (see [Young \[1985\]](#)). In Chapter 3, we propose an axiomatic characterization of the multi-choice Shapley value that invokes a generalization of Strong monotonicity from TU-games to multi-choice games. It turns out that the multi-choice Priority value satisfies this generalization of Strong monotonicity. Thus, it might be interesting to investigate whether the multi-choice Priority value can be characterized by invoking this axiom.

Second, in this chapter, we combine the linearly ordered sets of activity levels with a priority structure to derive a lexicographic partial order on the set of pairs. It might be interesting to investigate what happens if the activity levels of the agents are not linearly ordered, but only partially ordered.





*Part 2. Applications to pollution issues*



## Chapter 5

# Allocation of hazardous waste transportation costs

### 5.1 Introduction

Over the last decades, the generation of hazardous waste from industrial activities has been steadily increasing. Tons of such waste are traded both internationally and locally, involving massive waste movements. Because of the nature of such waste, a threatening incident may occur during transportation causing damage that may be irreversible. To address the problems of hazardous waste, a variety of environmental regulations have emerged, establishing a body of legal statutes that monitor their generation and transportation. To cite a few: the “Basel convention” regulates the international movements of hazardous waste and their disposal; the “Comprehensive Environmental Response, Compensation, and Liability Act” (CERCLA or Superfund Act) regulates the storage sites of hazardous waste in the United States; the “Treaty on the Functioning of the European Union” regulates the movements of hazardous waste within its borders. Each of these regulations is based on a preventive approach aimed at implementing measures to reduce the risk associated with hazardous waste.

In this chapter, we consider a finite set of agents involved in a hazardous waste transport network.<sup>1</sup> The network is modeled by a directed sink tree graph divided into several portions, and each agent has a certain amount of waste it wishes to ship. We suppose that a central authority sets preventive measures regarding the environmental risk, which entails some costs of maintaining and operating the network. Two questions then arise: which agent is liable for the risk on the network? How should the cost be allocated?

To answer these questions, we define a hazardous waste transportation problem as a tuple consisting of a directed sink tree graph, a vector of maximal amount of waste and a list of cost functions (one for each portion of the network). The sink of the tree is a waste treatment facility where the total waste is shipped. The vector of maximal amount of waste corresponds to the total amount of waste each agent is shipping to the facility. The cost function of each portion describes the maintenance and operation cost of the portion when a certain amount of waste is conveyed to it. An allocation rule is defined as a mapping that associates to each hazardous waste transportation problem a list of payoffs describing the cost share of each agent for each of its amount of waste.

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<sup>1</sup>Recently, [Martínez et al. \[2022\]](#) study the hazardous waste trade by considering a network representation for the waste movements.

We analyze the hazardous transportation problem from the viewpoint of axiomatic analysis. We consider several axioms that are derived from different environmental law principles. These principles, that we further discuss in Section 5.3.2, can be distinguished into two subsets. The first subset is specific to environmental issues and contains: the prevention and precautionary principles, which set the duty of agents to take appropriate measures to prevent environmental risk; the polluter pays principle, which asserts that the cost associated with a pollution hazard should be born by those causing the hazard. The second subset of principles intersects with some tort law principles. It contains different liability regimes that set out the terms under which an agent is liable for certain hazards. Inspired by the Basel convention, we consider the strict liability, the joint liability and the several liability regimes. This allows us to clearly define which agent is liable for the risk caused on which portion of the network. Accordingly, we consider several axioms related to these environmental law principles. According to the prevention principle, the maintenance and operation cost of the hazardous waste transport network should be fully covered by the whole set of agents. We relate this interpretation to the Efficiency axiom widely used in axiomatic analysis in economics. The second axiom is related to the several liability principle. Several liability is a legal doctrine that depicts an agent liable in proportion to its contribution to a damage. Accordingly, we consider the axiom of Independence of higher waste amount, which requires that the cost share of an agent for any amount of waste does not depend on a higher waste amount of any agent, including itself. This implies that an agent does not pay for the risk associated with waste it is not able to generate. Then, we consider the axiom of Path consistency, which requires that an agent's cost share does not depend on the costs of portions not used by that agent. We relate this axiom to the strict liability whereby an agent's liability is tied to the portion it uses. Finally, we introduce the axiom of Upstream solidarity for a cost increase, which requires that each agent located upstream a portion should be equally impacted if the marginal cost of this portion varies. This axiom complements the joint liability principles by removing any ambiguity that may arise under this liability regime.

We show that there is a unique allocation rule satisfying Efficiency, Independence of higher waste amount, Path consistency and Upstream solidarity for a cost increase (Proposition 5.3.4). This rule, called the responsibility rule (Theorem 5.3.1), can be described as follows. Consider any portion  $k$  and suppose that each agent liable for this portion is shipping the same amount of waste  $j$ . If an agent is not able to ship this amount then it ships its maximum. In this case, we say that the agents form a  $j$ -synchronized waste profile. Then, the agents liable for the portion  $k$ , and that ship the amount  $j$ , share equally the variation in cost between shipping the  $j$ -synchronized and shipping the  $(j - 1)$ -synchronized waste profiles. By successively applying this procedure on each portion for which an agent is liable, we obtain its payoff (for its waste amount  $j$ ) according to the responsibility rule.

Then, we analyze hazardous waste transportation problems by adopting a cooperative game theoretic approach. Specifically, we define a multi-choice game from a hazardous waste transportation problem. We show that the responsibility rule coincides with the multi-choice Shapley value, introduced in Chapter 2, of this appropriate multi-choice game (Theorem 5.4.1).

### Related literature

This work is closely related to the wide literature on cost sharing problems in networks. Specifically, our model generalizes the model of cleaning a polluted river

introduced in Ni and Wang [2007]. The authors consider a polluted river as a line divided into several segments. The cost of cleaning each segment is exogenously given. Ni and Wang [2007] propose and characterize two allocation rules: the local responsibility rule and the upstream responsibility rule. Further, they show that both rules coincide with the Shapley value of an appropriate cooperative TU-game. This model has been extended by Dong et al. [2012] by considering a polluted river network modeled by a directed sink tree. Recently, van den Brink et al. [2018] study the polluter river network problem and show that a polluted river network can be interpreted as a permission tree structure. They provide different allocation rules based on solutions for TU-games with a permission structure. Based on the polluter pays principle, Gómez-Rúa [2013] propose new allocation rules for sharing the cost of a polluted river inspired by properties of water taxes. Alcalde-Unzu et al. [2015] consider a transfer rate that measures the proportion of pollutant transferred from one segment to another. They show that uncertainty on this transfer rate can be reduced by using information from the cleaning costs. This allows to define some limits to the liability of agents. Moreover they propose and characterize the Upstream responsibility rule that takes into account these limits. Although our model can be related to Ni and Wang [2007] and Dong et al. [2012], our study differs from them on two points. First, we allow the agents to have multiple amount of waste. Second, we explicitly consider that the amount of waste flows from one point of network to another. This leads to a cumulative effect of the waste in the cost of maintaining and operating a portion. Such effect is not taken into account by Dong et al. [2012].

This work can also be related to the literature on the transport of hazardous materials. This literature mainly focuses on the reduction of threatening incident that may arise from the transportation of hazardous materials. Using an operation research point of view, the problem is more about routing-scheduling and network design than on cost allocation. A recent survey of network design problem can be found in Mohri et al. [2021].

This chapter is organized as follows. Section 5.2 introduces the notation and provides a brief presentation of directed graphs. Section 5.3 introduces the hazardous waste transportation problem. Section 5.3.1 presents the model, and Section 5.3.2 discusses the different environmental principles. Our axiomatic analysis is contained in Section 5.3.3, while Section 5.4 develops the cooperative game approach. Finally, Section 5.5 concludes.

## 5.2 Preliminaries

Denote by  $\mathbb{N}$  the set of natural integers containing 0, and fix  $0 < K \in \mathbb{N}$ . Given an integer  $x \leq K$ , denote by  $[[0, x]]$  the set  $\{0, 1, \dots, x\}$  of successive integers from 0 to  $x$ , i.e.,  $[[0, x]] = \{z \in \mathbb{N} : 0 \leq z \leq x\}$ .

### 5.2.1 Directed (tree) graph

A directed graph (henceforth *digraph*) is a pair  $g = (N, E)$  where  $N$  is a finite set of nodes (representing the set of agents) and  $E \subseteq N \times N$  is a binary irreflexive relation. This binary relation  $E$  induces a collection of ordered pairs, i.e.,  $E \subseteq \{(i, i') \in N \times N : i \neq i'\}$ . The pair  $(i, i') \in E$  is interpreted as: there exists a directed link from  $i$  to  $i'$  in  $g$ . For each node  $i \in N$ , define  $U(i) = \{i' \in N : (i', i) \in E\}$  the set of *direct predecessors* of  $i$  in  $g$ . The set  $U^{-1}(i) = \{i' \in N : (i, i') \in E\} = \{i' \in N : i \in U(i')\}$

denotes the set of *direct successors* of  $i$  in  $g$ . Given a digraph  $g$ , a (directed) *path* from  $i$  to  $i'$  is a sequence of distinct nodes  $(i_1, \dots, i_h)$  such that  $i_1 = i$  and for each  $t = 1, \dots, h-1$ ,  $i_{t+1} \in U^{-1}(i_t)$ , and  $i_h = i'$ . For each  $i \in N$ , define  $\hat{U}(i) = \{i' \in N : \text{there exists a path from } i' \text{ to } i\}$  as the set of agents located *upstream*  $i$  in  $g$ . Similarly, for each  $i \in N$  define  $\hat{U}^{-1}(i) = \{i' \in N : i \in \hat{U}(i')\}$  as the set of agents located *downstream*  $i$  in  $g$ . Given a set of agents  $S \subseteq N$ , we denote by  $U(S) = \bigcup_{i \in S} U(i)$  and  $U^{-1}(S) = \bigcup_{i \in S} U^{-1}(i)$  the set of predecessors, respectively successors, of agents in  $S$ . Similarly, for  $S \subseteq N$  we denote  $\hat{U}(S) = \bigcup_{i \in S} \hat{U}(i)$  and  $\hat{U}^{-1}(S) = \bigcup_{i \in S} \hat{U}^{-1}(i)$ . For convenience, we denote  $\hat{U}_S := \hat{U}(S) \cup S$ , and  $\hat{U}_S^{-1} := \hat{U}^{-1}(S) \cup S$ , and  $\hat{U}_k$  instead of  $\hat{U}_{\{k\}}$ .

Let  $g = (N, E)$  be a digraph. A directed path  $(i_1, \dots, i_h)$ , with  $h > 2$ , is a cycle in  $g$  if  $(i_h, i_1) \in E$ . A digraph  $g$  is called *acyclic* if there is no cycle in  $g$ . A digraph  $g$  is a *sink tree* if there is exactly one node  $i_d \in N$  such that  $U^{-1}(i_d) = \emptyset$ ,  $\hat{U}(i_d) = N \setminus \{i_d\}$ , and for each  $i \in N \setminus \{i_d\}$ ,  $|U^{-1}(i)| = 1$ . Observe that a sink tree is an acyclic directed graph. Denote by  $\mathcal{E}$  the collection of all irreflexive digraphs and by  $\bar{\mathcal{E}}$  the collection of all directed sink trees.

### 5.3 Hazardous waste transportation problem

#### 5.3.1 The model

Consider a *transport network* connecting a finite set of agents  $N$  (directly or indirectly) to a special node  $d$  called the *delivery node*. Such transport network structure is modeled by a sink tree  $g = (N \cup \{d\}, E)$ , where  $d$  is the sink of the tree. Each element  $(i, i') \in E$  is called a *portion*. Since  $g$  is a sink tree, each agent  $i \in N$  has exactly one direct successor in  $g$ . When no confusion arises, we simply denote a portion  $(i, i')$  by  $i$ . All the notation introduced for sink trees remain valid for a hazardous waste transport network.

Each agent  $i \in N$  is endowed with a given amount of hazardous waste  $w_i \in \mathbb{N}$ ,  $0 < w_i \leq K$ , to be treated by a treatment facility. We assume that the treatment facility is located at the delivery node  $d$ . Each agent is able to ship any amount  $j \leq w_i$  of waste. However, the actual amount shipped is  $w_i$ . Consider the product set  $W = \prod_{i \in N} [0, w_i]$ . An element  $s = (s_i)_{i \in N} \in W$  is referred to a *waste profile*, which indicates each agent's amount of waste. The profile of maximal amount of waste is given by  $w = (w_i)_{i \in N}$ . Given the set of agents  $N$ , the profile of maximal amount of waste  $w$ , and an amount  $j \leq \max_{i \in N} w_i$ , define  $Q(j)$  as the set of agents able to ship  $j$  amount of waste, i.e.,  $Q(j) := \{i \in N : w_i \geq j\}$ . For any two distinct amount of waste  $j, j'$  such that  $j \geq j'$ , we have that  $Q(j) \subseteq Q(j')$ . Moreover, given a waste amount  $j \leq \max_{i \in N} w_i$ , define the  $j$ -synchronized waste profile,  $(j \wedge w_i)_{i \in N}$ , in which each agent send the waste amount  $j$ . If an agent is unable to ship  $j$ , then it sends its maximal amount of waste to the treatment facility.

**Example 7.** Let  $N = \{1, 2, 3, 4, 5\}$  be the set of agents involved in the transport of hazardous waste and  $d$  the delivery node. Consider the hazardous waste transport network  $g = (N \cup \{d\}, E)$  where  $E = \{(1, d), (3, 1), (2, 1), (5, 2), (4, 2)\}$ , depicted in Figure 1. The direct successor of agent 3 is  $U^{-1}(3) = \{1\}$ . The set of direct predecessors of agent 2 is  $U(2) = \{4, 5\}$ . The set of agents located upstream agent 1

is  $\hat{U}(1) = N \setminus \{1\}$ .

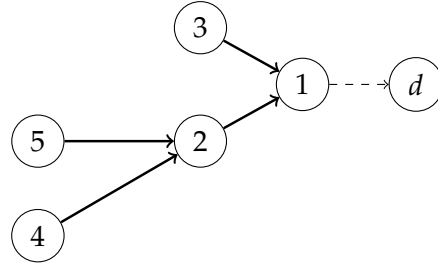


Figure 1: Hazardous waste transport network.

◇

Shipping any amount of waste through a portion carries risks and a cost. This cost is considered as the cost of maintaining and operating the network. For each portion  $i \in N$ , let  $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the cost function of this portion. We assume that, for each  $i \in N$  the cost function  $C_i$  is non-negative and non-decreasing over  $\mathbb{R}_+$ . Moreover, we use the convention  $C_i(0) = 0$  for any  $i \in N$ . For each portion  $i \in N$ , it is assumed that the cost function  $C_i$  depends on the total amount of waste passing through the portion, i.e., the sum of the waste sent by all agents located upstream  $i$  including its own waste. Thus, the cost of transporting the waste profile  $w$  through a portion  $i$  is given by  $C_i(\sum_{k \in \hat{U}_i} w_k)$ . The total cost of maintaining the network is then

given by

$$\sum_{i \in N} C_i\left(\sum_{k \in \hat{U}_i} w_k\right). \quad (5.1)$$

Let  $\mathcal{C}$  be the set of such cost functions. Denote by  $C = (C_i)_{i \in N} \in \mathcal{C}^N$  a profile of cost functions. For each  $i \in N$ , we denote by  $C_i^0 \in \mathcal{C}$  the null cost function, i.e.,  $C_i^0(x) = 0$  for all  $x \in \mathbb{R}_+$ . A *hazardous waste transportation problem* on a fixed agent set  $N$  and delivery node  $d$  is then a triplet

$$(g, w, C) \in \bar{\mathcal{E}} \times W \times \mathcal{C}^N.$$

Let  $\mathcal{P}^N = \bar{\mathcal{E}} \times W \times \mathcal{C}^N$  be the set of all hazardous waste transportation problems on  $N$ . Notice that polluted river network problems, introduced in [Dong et al. \[2012\]](#), can be viewed as a subclass of hazardous waste transportation problem where  $w_i = 1$  for each  $i \in N$  and each cost function is a constant function.<sup>2</sup>

Consider any hazardous waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . The question that arises is how to allocate the total maintenance and operation cost of the hazardous waste transport network. An *allocation rule*  $f$  on  $\mathcal{P}^N$  assigns a non-negative payoff (or cost share)  $f_{ij}(g, w, C) \in \mathbb{R}_+$  to each agent  $i \in N$  for each of its waste amount  $0 < j \leq w_i$ . Then, for each  $i \in N$  and each  $j \leq w_i$  the payoff  $f_{ij}(g, w, C) \in \mathbb{R}_+$  is interpreted as the variation of payoff when agent  $i$  increases its transported waste from  $j - 1$  to  $j$ . Such interpretation is consistent with the fact that agent are actually shipping the amount  $w_i$ .

<sup>2</sup>We refer the reader to [van den Brink et al. \[2018\]](#) for a recent study of polluted river network problems.



### 5.3.2 Principles from environmental legislation

The transport of hazardous waste is regulated by a set of legal rules and statutes established both at international and national levels. In this section, we present some principles invoked in the environmental issues and specifically in the transport of hazardous waste. We provide a brief interpretation of each principle within our model. While some principles can be related to tort law principles, others are very specific to environmental issues and deserve a specific interpretation.

**The prevention principle** The prevention principle is the most prevalent principle in environmental law. This principle is at the heart of the preventive approach to the environment, which aims at anticipating, minimizing and preventing environmental hazards. Indeed, the damage caused by pollution tends to be very expensive and even has irreversible consequences. The implementation of preventive measures is therefore crucial in the protection of the environment. Regarding hazardous waste management and transportation, the preventive approach is advocated by the Basel convention. Notably, the convention requires that anyone involved in the management or transportation of hazardous waste to take appropriate measures to prevent any potential pollution from the hazardous waste.

**The polluter pays principle** The polluter pays principle ensures that those causing pollution must bear the cost it generates. This principle, introduced by Arthur Cecil Pigou in the 1920's, aims at internalizing the external costs generated by polluting activities. It can be thought from two complementary standpoints. First, this principle has a preventive function, assigning the cost of prevention and precautionary measures to agents involved in environmentally hazardous activities. This is reflected in the recommendations on the pollution control made by the OECD or the European Union.<sup>3</sup> This principle also has a curative function by making the polluter liable in case of environmental damage. From this perspective, it can be seen as lying at the intersection between tort law principles and environmental statutes by assigning the liability of a damage to the polluter.

**Strict liability** Strict liability is a legal doctrine according to which an agent is liable for a damage regardless of its intent. Thus, in case of a damage, this legal regime does not require justification of fault or negligence from the damaging agent. This doctrine is enforced in situations considered as inherently dangerous. It is required by the Basel convention, the European Environmental liability Directive (2004/35/EC), or the CERCLA (Superfund act). It complements the polluter-pays principle (from the curative standpoint) by defining the conditions ruling the liability for environmental hazard.<sup>4</sup>

**Joint liability** Joint liability is a second legal doctrine ruling situations with combined causes of damage. According to this doctrine, each agent involved in a damage is responsible for it, and can therefore be charged for it.

<sup>3</sup>See OECD recommendations 1972, and the Treaty of Functioning of the European Union, Article 192(1).

<sup>4</sup>We refer the reader to Landes and Posner [1980] for a study of strict liability in economic analysis of law.

**Several liability** The several liability doctrine is another legal doctrine that holds an agent liable in proportion to its contribution to the damage. This doctrine can be found in the Basel convention asserting that each liable agent is liable in proportion to the contribution made by its waste to a damage.

### 5.3.3 Axiomatic analysis

In this section, we provide an axiomatic analysis on the class of hazardous waste transportation problems. We discuss several axioms for allocation rules inspired by the environmental principles introduced in Section 5.3.2.

We first consider the Efficiency axiom widely used in axiomatic analysis in economics. This axiom can be related to a major concern in most of the hazardous waste litigation, which is the damage remediation.<sup>5</sup> In our framework, this concern focuses on paying the cost of the network, leaving room for how to allocate this cost among the agents. Accordingly, the Efficiency axiom requires that the total cost for maintaining the network to be fully borne by all agents.

*Efficiency* For any  $(g, w, C) \in \mathcal{P}^N$ ,

$$\sum_{i \in N} \sum_{j \leq w_i} f_{ij}(g, w, C) = \sum_{i \in N} C_i \left( \sum_{k \in \hat{U}_i} w_k \right).$$

The second axiom can be thought as limiting an agent's cost share according to its waste's contribution to the risk of damage. In this sense, this axiom is weakly related to the several liability regime. This liability regime relies on the assumption that there exists a reasonable basis for determining each agent's contribution to the risk it causes to the network. In our framework, since each agent's waste is of the same nature, we use the waste amount of each agent to define such a basis. Accordingly, the axiom of Independence of higher waste amount axiom requires that an agent's payoff for a given amount of waste  $j$  does not depend on any amount higher than  $j$ . This axiom is related to the Independence of maximal activity level introduced in Chapter 3. A similar axiom can be found in Moulin and Shenker [1992] in the context of cost sharing problems. Regarding hazardous waste transportation problems, this axiom ensures that no one subsidizes a higher waste amount from any agent (including its own).

*Independence of higher waste amount* For any  $i \in N$ , and any  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall j < w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, C).$$

Next, we consider an intuitive axiom that relates an agent's cost share to its distance from the delivery node. Relying on the several liability regime, we consider a second basis for determining the agents' contribution to the risk on the network. We argue that the greater the number of portions used to ship a given amount of waste, the greater the risk to the network. Distance consistency then requires that the farther an agent is from the delivery node, the higher its cost share for a given waste amount  $j$ .

*Distance consistency.* For any  $(g, w, C) \in \mathcal{P}^N$ , any  $i \in N$ ,  $j \leq w_i$  and  $k \in \hat{U}^{-1}(i) \cap Q(j)$ ,

$$f_{ij}(g, w, C) \geq f_{kj}(g, w, C).$$

<sup>5</sup>For instance, see the hazardous waste litigation in United States v. Monsanto (1988).

The next axiom is inspired by the polluter pays principle when considering the strict liability regime. One interpretation of this liability regime (see [Posner \[1973\]](#)) implies that an agent insures the network against the risk it causes. In our framework, this risk is tied to the portion an agent uses. Conversely, an agent does not cause any risk on a portion it does not use. The next axiom is based on the above interpretation of strict liability, taking into account the portions an agent is liable for. Path consistency requires that an agent's cost share is independent of the cost of portions not used by that agent. This axiom is closely related to the axiom of Independence of upstream costs introduced in [Dong et al. \[2012\]](#) in the context of polluted river problems.

*Path consistency.* For any  $i \in N$  and any  $(g, w, C) \in \mathcal{P}^N$ ,  $(g, w, C') \in \mathcal{P}^N$  such that for each  $h \in \hat{U}_i^{-1}$ ,  $C_h = C'_h$ ,

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C').$$

Actually, Path consistency can be related to several axioms invoked in the literature on cost sharing problems in networks. To cite a few, one may consider the axioms of: Independence of unused edges introduced by [Bergantiños et al. \[2019\]](#) in the context of energy networks; Individual independence of outside changes introduced by [Sudhölter and Zarzuelo \[2017\]](#) in the context of highway problems; No free riding introduced by [Gopalakrishnan et al. \[2021\]](#) in the context of allocating responsibility for emissions in a supply chain; Marginal damage independence introduced by [Ferey and Dehez \[2016\]](#) and [Oishi et al. \[2023\]](#) in the context of liability problems.

The next axiom is based on a fairness principle describing how the cost share of certain agents vary when only one cost function varies. Under the joint liability regime, each agent located upstream of a portion is liable for the cost incurred on this portion. However, the extent to which an agent should be held liable remains to be determined. Upstream solidarity for a cost increase then requires that agents located upstream of a portion be equally impacted by a cost increase of this portion.

*Upstream solidarity for a cost increase.* For any  $i \in N$ , any  $(g, w, C), (g, w, C') \in \mathcal{P}^N$  such that

$$\begin{aligned} & \forall s \in W, \\ & C_i(\sum_{h \in \hat{U}_i} s_h) - C_i(\sum_{h \in \hat{U}_i} (s_h - 1) \vee 0) \geq C'_i(\sum_{h \in \hat{U}_i} s_h) - C'_i(\sum_{h \in \hat{U}_i} (s_h - 1) \vee 0), \end{aligned}$$

and, for each  $k \in N \setminus \{i\}$ ,  $C_k = C'_k$ , it holds that: for each  $j \in [[0, \max_{l \in \hat{U}_i} w_l]]$ , and each  $l, l' \in \hat{U}_i \cap Q(j)$ ,

$$f_{lj}(g, w, C) - f_{lj}(g, w, C') = f_{l'j}(g, w, C) - f_{l'j}(g, w, C').$$

Upstream solidarity for a cost increase can be seen as a strong application of the principle of solidarity discussed in [Thomson \[2016\]](#). Since the agents are jointly liable for a portion, any change in the cost function has the same impact on each liable agent's cost share.

Below, we analyze the implication of these axioms. We first show that the combination of Efficiency and Path consistency implies that the cost share of an agent is null whenever the maintenance and operation cost of each portion it is liable for is null.

**Proposition 5.3.1.** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Path consistency, then for any  $i \in N$ , any  $(g, w, C) \in \mathcal{P}^N$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C_k^0$  it holds that*

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = 0.$$

*Proof.* Let  $f$  be an allocation rule on  $\mathcal{P}^N$  that satisfies Efficiency and Path consistency. Consider the waste transportation problem  $(g, w, C^0) \in \mathcal{P}^N$ . By Efficiency, we have that

$$\sum_{i \in N} \sum_{j \leq w_i} f_{ij}(g, w, C^0) = 0.$$

Since an allocation rule  $f$  assigns a non-negative cost share  $f_{ij}(g, w, C) \in \mathbb{R}_+$  to each agent  $i \in N$  and each  $1 \leq j \leq w_i$ , it holds that

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C^0) = 0.$$

Consider any  $i \in N$  and any waste transportation problem  $(g, w, C)$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C_k^0$ . By Path consistency, one obtains

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C^0) = 0,$$

the desired result. This concludes the proof of the proposition.  $\blacksquare$

The next result shows the implication of combining Efficiency and Independence of higher waste amount.<sup>6</sup>

**Proposition 5.3.2.** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Independence of higher waste amount, then*

$$\forall j \leq \max_{i \in N} w_i, \quad \sum_{k \in Q(j)} f_{kj}(g, w, C) = \sum_{i \in N} \left( C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) - C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) \right).$$

*Proof.* Let  $f$  be an allocation rule as hypothesized and consider any waste amount  $0 < j \leq \max_{i \in N} w_i$ . By Independence of higher waste amount, we have that

$$\forall i \in Q(j), \quad f_{ij}(g, w, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, C), \quad (5.2)$$

$$\forall i \in Q(j-1) \quad f_{ij-1}(g, w, C) = f_{ij-1}(g, ((j-1) \wedge w_k)_{k \in N}, C). \quad (5.3)$$

By Efficiency and Independence of higher waste amount, it holds that

$$\sum_{i \in N} \sum_{l=1}^{j \wedge w_i} f_{il}(g, (j \wedge w_k)_{k \in N}, C) = \sum_{i \in N} C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) = \sum_{i \in N} \sum_{l=1}^{j \wedge w_i} f_{il}(g, w, C), \quad (5.4)$$

$$\sum_{i \in N} \sum_{l=1}^{j-1 \wedge w_i} f_{il}(g, ((j-1) \wedge w_k)_{k \in N}, C) = \sum_{i \in N} C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) = \sum_{i \in N} \sum_{l=1}^{j-1 \wedge w_i} f_{il}(g, w, C). \quad (5.5)$$

Thus, subtracting (5.5) to (5.4), one obtains

$$\sum_{k \in Q(j)} f_{kj}(g, w, C) = \sum_{i \in N} \left( C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) - C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) \right),$$

the desired result.  $\blacksquare$

Our next result shows that Efficiency, Independence of higher waste amount, Path consistency, and Upstream solidarity for a cost increase implies Distance consistency.

<sup>6</sup>This result is related to Proposition 3.4.1 (see Chapter 3, Section 3.4).

**Proposition 5.3.3.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , the combination of Efficiency, Independence of higher waste amount, Path consistency, and Upstream solidarity for a cost increase implies Distance consistency.*

*Proof.* Let  $f$  be an allocation rule on  $\mathcal{P}^N$  as hypothesized. Take any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . For each  $k \in N$ , define the cost profile  $\lambda^{k,C} \in \mathcal{C}^N$  such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \in N \setminus \{k\}$ . Take a waste amount  $j \leq \max_{i \in N} w_i$  such that  $|Q(j)| \geq 2$ , and fix two agents  $i, k \in N$  such that:  $i \in Q(j)$  and  $k \in \hat{U}_i^{-1} \cap Q(j)$ . By Path consistency, we have that

$$\begin{aligned} f_{kj}(g, w, C) &= f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}), \\ f_{ij}(g, w, C) &= f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}). \end{aligned} \quad (5.6)$$

We show that  $f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C})$ . To do so, consider the waste allocation problems  $(g, w, C^0) \in \mathcal{P}^N$  and  $(g, w, \lambda^{k,C}) \in \mathcal{P}^N$ . Observe that, for each  $h \in N \setminus \{k\}$ ,  $\lambda_h^{k,C} = C_h^0$ . Moreover, since  $\lambda_k^{k,C}$  is a non-decreasing function, it holds that

$$\begin{aligned} \forall s \in W, \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} s_h \right) - \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) &\geq C_k^0 \left( \sum_{h \in \hat{U}_k} s_h \right) - C_k^0 \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) \\ \iff \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} s_h \right) - \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) &\geq 0. \end{aligned} \quad (5.7)$$

Therefore, by Upstream solidarity for a cost increase one obtains

$$f_{kj}(g, w, \lambda^{k,C}) - f_{kj}(g, w, C^0) = f_{ij}(g, w, \lambda^{k,C}) - f_{ij}(g, w, C^0).$$

By Proposition 5.3.1,  $f_{kj}(g, w, C^0) = f_{ij}(g, w, C^0) = 0$ . Thus, one obtains

$$f_{kj}(g, w, \lambda^{k,C}) = f_{ij}(g, w, \lambda^{k,C}).$$

We apply the same reasoning by adding successively each  $\lambda_h^{h,C}$  where  $h \in \hat{U}_k^{-1}$  to the problem. Then, by successive applications of Upstream solidarity for a cost increase it holds that

$$f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}). \quad (5.8)$$

Thus, by (5.6) and (5.8), we know that

$$f_{kj}(g, w, C) = f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

It remains to show that

$$f_{ij}(g, w, C) = f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

By Independence of higher waste amount, this can be written as

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

Consider the directed path from  $i$  to  $k$  denoted by  $(i_1, \dots, i_t)$ , where  $i_1 = i$  and  $i_t = k$ . Pick  $i_{t-1} \in \hat{U}_i^{-1} \setminus \hat{U}_k^{-1}$  located just upstream  $k$  on the path from  $i$  to  $k$ . Consider the hazardous waste problem  $(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C})$ . Observe that, for each  $h' \in N \setminus \{i_{t-1}\}$ ,  $\lambda_{h'}^{i_{t-1},C} = C_{h'}^0$ . Since  $\lambda_{i_{t-1}}^{i_{t-1},C}$  is non-decreasing, by Upstream solidarity for a cost increase, there is  $c_j^{i_{t-1}} \in \mathbb{R}$  such that:

$$\begin{aligned} \forall l \in \hat{U}_{i_{t-1}} \cap Q(j), \\ f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) - f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = c_j^{i_{t-1}}. \end{aligned} \quad (5.9)$$

We now show that  $c_j^{i_{t-1}} \geq 0$ . Since  $f$  satisfies Efficiency and Independence of higher waste amount, by Proposition 5.3.2 we have that

$$\begin{aligned} \sum_{l \in Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) &= \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} j \wedge w_a \right) \\ &\quad + \lambda_{i_{t-1}}^{i_{t-1},C} \left( \sum_{a \in \hat{U}_{i_{t-1}}} j \wedge w_a \right) \\ &\quad - \left( \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} (j-1) \wedge w_a \right) \right. \\ &\quad \left. + \lambda_{i_{t-1}}^{i_{t-1},C} \left( \sum_{a \in \hat{U}_{i_{t-1}}} (j-1) \wedge w_a \right) \right), \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} \sum_{l \in Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) &= \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} j \wedge w_a \right) \\ &\quad - \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} (j-1) \wedge w_a \right). \end{aligned} \quad (5.11)$$

Observe that, for  $l \in Q(j) \setminus \hat{U}_{i_{t-1}}$  we have  $\lambda_{l'}^{i_{t-1},C} = C_{l'}^0$ , where  $l' \in \hat{U}_l^{-1}$ . This implies that  $\sum_{h \in \hat{U}_k^{-1}} \lambda_{l'}^{h,C} + \lambda_{l'}^{i_{t-1},C} = \sum_{h \in \hat{U}_k^{-1}} \lambda_{l'}^{h,C}$ . Therefore, by applying Path consistency to  $l \in Q(j) \setminus \hat{U}_{i_{t-1}}$  one obtains

$$f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) = f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

From this observation, and subtracting (5.10) and (5.11), one obtains

$$\begin{aligned}
& \sum_{l \in \hat{U}_{k'} \cap Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{k',C}) - f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) \\
&= \lambda_{k'}^{k',C} \left( \sum_{a \in \hat{U}_{h'}} j \wedge w_a \right) - \lambda_{k'}^{k',C} \left( \sum_{a \in \hat{U}_{h'}} (j-1) \wedge w_a \right) \\
&= |\hat{U}_{k'} \cap Q(j)| \times c_j^{k'},
\end{aligned} \tag{5.12}$$

where the last equality comes from (5.9). Since  $\lambda_{i-1}^{i-1,C}$  is non-decreasing, it holds that  $c_j^{k'} \geq 0$ . Therefore,

$$\forall l \in \hat{U}_{i-1} \cap Q(j), f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i-1,C}) \geq f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

In particular, it holds that

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i-1,C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

Repeating the same reasoning by considering each agent that belongs to the path from  $i$  to  $k$ , one obtains

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}). \tag{5.13}$$

Thus, by (5.6), (5.8), (5.13), and Independence of higher waste amount, one concludes that

$$f_{ij}(g, w, C) \stackrel{(5.6)}{=} f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \stackrel{(5.13)}{\geq} f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) \stackrel{(5.8)}{=} f_{kj}(g, w, C),$$

the desired result. ■

We now introduce the main result of this section. We show that the combination of Efficiency, Independence of higher waste amount, Path consistency and Upstream cost solidarity yields at most one allocation rule on  $\mathcal{P}^N$ .

**Proposition 5.3.4.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , there is at most one allocation rule satisfying Efficiency, Independence of higher waste amount, Path consistency and Upstream cost solidarity.*

*Proof.* Let  $f$  be a solution on  $\mathcal{P}^N$  satisfying Efficiency, Independence of higher waste amount, Path consistency and Upstream cost solidarity. We show that  $f$  is uniquely determined. Take any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . For each  $k \in N$ , define the cost profile  $\lambda^{k,C} \in \mathcal{C}^N$  such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \in N \setminus \{k\}$ . Observe that, for each waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ , the cost profile  $C$  can be written as

$$C = \sum_{k \in N} \lambda^{k,C}.$$

For each  $(g, w, C) \in \mathcal{P}^N$ , set

$$\mathcal{K}(g, w, C) = \{k \in N : C_k \neq C_k^0\}. \quad (5.14)$$

We proceed by induction on  $|\mathcal{K}(g, w, C)|$ .

**Induction basis.** If  $|\mathcal{K}(g, w, C)| = 0$ , then  $C = C^0$ . By Efficiency and the definition of an allocation rule, it holds that

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = 0.$$

**Induction hypothesis.** Assume that, for each  $(g, w, C)$  such that  $|\mathcal{K}(g, w, C)| = t$ , where  $0 \leq t \leq n - 1$ ,  $f(g, w, C)$  is uniquely determined.

**Induction step.** Let  $(g, w, C) \in \mathcal{P}^N$  be such that  $|\mathcal{K}(g, w, C)| = t + 1$ . Set

$$\bar{T} = \bigcap_{k \in \mathcal{K}(g, w, C)} \hat{U}_k, \quad (5.15)$$

and for each  $k \in \mathcal{K}(g, w, C)$  set

$$T_k = \hat{U}_k \setminus \bar{T}. \quad (5.16)$$

Observe that,  $\bar{T}$  is such that there is a unique agent  $k \in \bar{T}$  such that  $C_k \neq C_k^0$ . Indeed, if  $C_k = C_k^0$  for each agent  $k \in \bar{T}$ , then a contradiction arises and  $k \notin \mathcal{K}(g, w, C)$ . Any other agent  $k' \in \bar{T}$  is such that  $C_{k'} = C_{k'}^0$ . Two cases have to be considered with respect to  $\bar{T}$ .

**Case 1.** If  $\bar{T} = \emptyset$ , then for each agent  $i \in N$ , there is  $k \in \mathcal{K}(g, w, C)$  such that  $k \in N \setminus \hat{U}_i^{-1}$ . By Path consistency, it holds that

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C - \lambda^{k,C}).$$

By the induction hypothesis,  $f_{ij}(g, w, C - \lambda^{k,C})$  is uniquely determined. Thus, for each  $i \in N$  and each  $j \leq w_i$ ,  $f_{ij}(g, w, C)$  is uniquely determined.

**Case 2.** Now, suppose that  $\bar{T} \neq \emptyset$ . For each agent  $i \in N \setminus \bar{T}$ , there is  $k \in \mathcal{K}(g, w, C)$  such that  $k \in N \setminus \hat{U}_i^{-1}$ . By applying the same reasoning as in case 1, for each agent  $i \in N \setminus \bar{T}$  and each  $j \leq w_i$ ,  $f_{ij}(g, w, C)$  is uniquely determined.

Consider the subset of agents  $\bar{T}$ . Take any waste amount  $j \leq K$ . We show that any two agents  $i, i' \in \bar{T} \cap Q(j)$  have the same payoff in  $(g, w, C)$  for the waste amount  $j$ . Pick any  $k \in \mathcal{K}(g, w, C)$ . Notice that  $i, i' \in \hat{U}_k \cap Q(j)$ . By Upstream solidarity for a cost increase and the induction basis, it holds that

$$\begin{aligned} f_{ij}(g, w, \lambda^{k,C}) - f_{ij}(g, w, C^0) &= f_{i'j}(g, w, \lambda^{k,C}) - f_{i'j}(g, w, C^0) \\ f_{ij}(g, w, \lambda^{k,C}) &= f_{i'j}(g, w, \lambda^{k,C}). \end{aligned} \quad (5.17)$$

Now pick any other  $k' \in \mathcal{K}(g, w, C)$ ,  $k' \neq k$  (if any). Observe that

$$\forall s \in W, \quad \lambda_{k'}^{k',C} \left( \sum_{h \in \hat{U}_{k'}} s_h \right) - \lambda_{k'}^{k',C} \left( \sum_{h \in \hat{U}_{k'}} (s_h - 1) \vee 0 \right) \geq 0,$$



and for each  $h \in N \setminus \{k'\}$ ,  $\lambda_h^{k,C} + \lambda_h^{k',C} = \lambda_h^{k,C}$ . By the definition of  $\bar{T}$ , it holds that  $i, i' \in \hat{U}_k$ . Thus, by Upstream solidarity for a cost increase

$$\begin{aligned} f_{ij}(g, w, \lambda^{k,C} + \lambda^{k',C}) - f_{i'j}(g, w, \lambda^{k,C}) &= f_{i'j}(g, w, \lambda^{k,C} + \lambda^{k',C}) - f_{i'j}(g, w, \lambda^{k,C}) \\ f_{ij}(g, w, \lambda^{k,C} + \lambda^{k',C}) &= f_{i'j}(g, w, \lambda^{k,C} + \lambda^{k',C}), \end{aligned} \quad (5.18)$$

where the last equation comes from (5.17). We apply the same reasoning by adding successively each  $\lambda^{k,C}$  where  $k \in \mathcal{K}(g, w, C)$  to the problem. Therefore, by successive applications of Upstream solidarity for a cost increase, it holds that

$$\begin{aligned} f_{ij}(g, w, \sum_{k \in \mathcal{K}(g, w, C)} \lambda^{k,C}) &= f_{i'j}(g, w, \sum_{k \in \mathcal{K}(g, w, C)} \lambda^{k,C}) \\ f_{ij}(g, w, C) &= f_{i'j}(g, w, C). \end{aligned} \quad (5.19)$$

Thus, there exists  $\bar{c}_j \in \mathbb{R}_+$  such that for each  $i, i' \in \bar{T} \cap Q(j)$ :

$$f_{ij}(g, w, C) = f_{i'j}(g, w, C) = \bar{c}_j. \quad (5.20)$$

Now, we show that for each agent in  $\bar{T} \cap Q(j)$ ,  $f_{ij}(g, w, C)$  is uniquely determined. Since  $f$  satisfies Efficiency and Independence of higher waste amount, by Proposition 5.3.2 we have

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(g, w, C) &= \sum_{k \in N} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right) \\ &= \sum_{k \in \mathcal{K}(w, C)} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right). \end{aligned}$$

Equivalently, one can write

$$\begin{aligned} \sum_{i \in (Q(j) \cap \bar{T})} f_{ij}(g, w, C) &= \sum_{k \in \mathcal{K}(w, C)} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right) \\ &\quad - \sum_{l \in Q(j) \setminus \bar{T}} f_{lj}(g, w, C). \end{aligned} \quad (5.21)$$

Recall that  $f_{ij}(g, w, C)$  is uniquely determined for each agent in  $Q(j) \setminus \bar{T}$ . Therefore, by (5.17) and (5.21),  $f_{ij}(g, w, C)$  is uniquely determined for each  $i \in \bar{T} \cap Q(j)$ . This concludes the proof of the induction step. ■

### 5.3.4 The responsibility rule

In the previous section, we have highlighted that at most one allocation rule satisfies the combination of Efficiency, Independence of higher waste amount, Path consistency, and Upstream solidarity for a cost increase. Below, we investigate the *responsibility rule* and show that it is characterized by the above mentioned axioms. The responsibility rule allocates the variation in cost of portion  $k$  between the  $j$  and  $(j-1)$ -synchronized waste profiles equally among the agents located upstream of this portion.

Formally, for each hazardous waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ , the responsibility rule,  $f^R$ , is given by

$$\forall i \in N, \forall j \leq w_i,$$

$$f_{ij}^R(g, w, C) = \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|}. \quad (5.22)$$

Observe that formula (5.22) relies on the implicit assumption that each agent's waste is of the same nature and therefore has the same impact on a portion. Moreover, one can provide a specific allocation process underlying the responsibility rule. Suppose that all agents agree to ship at the same moment the same amount of waste. Then, the variation of cost generated by this shipment on each portion is split equally among the agents liable for the portion. The responsibility rule can be computed by repeating this reasoning for each amount of waste.

The next result shows that the responsibility rule is the unique allocation rule that matches our interpretation of the environmental principles presented in Section 5.3.2. Thus, the responsibility rule ensures that each agent pays a fair part of the cost of maintaining and operating the hazardous waste network according to its liability for the risk it poses to the network.

**Theorem 5.3.1.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , an allocation rule satisfies Efficiency, Independence of higher waste amount, Path consistency, and Upstream solidarity for a cost increase if and only if it is the responsibility rule.*

*Proof.* By Proposition 5.3.4, we know that there is a unique allocation rule that satisfies Efficiency, Independence of higher waste amount, Path consistency, and Upstream solidarity for a cost increase. It remains to show that  $f^R$  satisfies the axioms invoked in Theorem 5.3.1.

Consider any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . By the definition of  $f^R$  (see (5.22)), it holds that

$$\begin{aligned} \sum_{i \in N} \sum_{j \leq w_i} f_{ij}^R(g, w, C) &= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{i \in Q(j)} f_{ij}^R(g, w, C) \\ &= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{i \in Q(j)} \left[ \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \right] \\ &= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{k \in N} \left[ C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l) \right] \\ &= \sum_{k \in N} C_k(\sum_{l \in \hat{U}(k)} w_l). \end{aligned}$$

This shows that  $f^R$  satisfies Efficiency.

By the definition of the responsibility rule, the payoff of any agent for its waste amount  $j$  does not depend of any waste amount higher than  $j$ . Therefore,  $f^R$  satisfies Independence of higher waste amount.

Now consider any agent  $i \in N$  and any two hazardous waste problem  $(g, w, C) \in$

$\mathcal{P}^N, (g, w, C') \in \mathcal{P}^N$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C'_k$ . It holds that

$$\begin{aligned} f_{ij}^R(g, w, C) &= \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\ &= \sum_{k \in \hat{U}_i^{-1}} \frac{C'_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C'_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\ &= f_{ij}^R(g, w, C'), \end{aligned}$$

which shows that  $f^R$  satisfies Path consistency.

Next, consider any agent  $i \in N$  and any  $(g, w, C), (g, w, C') \in \mathcal{P}^N$  such that

$$\begin{aligned} \forall s \in W, \\ C_i(\sum_{k \in \hat{U}_i} s_k) - C_i(\sum_{k \in \hat{U}_i} (s_k - 1) \vee 0) &\geq C'_i(\sum_{k \in \hat{U}_i} s_k) - C'_i(\sum_{k \in \hat{U}_i} (s_k - 1) \vee 0), \end{aligned}$$

and, for each  $k \in N \setminus \{i\}$ ,  $C_k = C'_k$ . Pick any agent  $l \in \hat{U}_i$ . Observe that for any waste amount  $j \leq w_l$ , we have that

$$\begin{aligned} f_{lj}^R(g, w, C) - f_{lj}^R(g, w, C') &= \frac{C_i(\sum_{k \in \hat{U}_i} j \wedge w_k) - C_i(\sum_{k \in \hat{U}_i} (j-1) \wedge w_k)}{|\hat{U}_i \cap Q(j)|} \\ &\quad - \frac{C'_i(\sum_{k \in \hat{U}_i} j \wedge w_k) - C'_i(\sum_{k \in \hat{U}_i} (j-1) \wedge w_k)}{|\hat{U}_i \cap Q(j)|}, \end{aligned}$$

which does not depend on the chosen  $l \in \hat{U}_i$ . Thus,  $f^R$  satisfies Upstream solidarity for a cost increase. This concludes the proof of the theorem.  $\blacksquare$

**Logical independence** The axioms invoked in Theorem 5.3.1 are logically independent, as shown by the following alternative allocation rules on  $\mathcal{P}^N$ .

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = 0,$$

satisfies all the axioms except Efficiency.

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = \sum_{k \in \hat{U}_i} \frac{C_k(\sum_{l \in \hat{U}_k} w_l)}{\sum_{l \in \hat{U}_k} w_l},$$

satisfies all the axioms except Independence of higher waste amount.

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = \sum_{k \in N} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|Q(j)|},$$

satisfies all the axioms except Path consistency.

- Take any  $(g, w, C) \in \mathcal{P}^N$ . For each  $i \in N$  and each  $j \leq w_i$  fix an arbitrary  $\alpha^{ij} \in \mathbb{R}_{++}$ . The value  $f^\alpha$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i,$$

$$f_{ij}^\alpha(g, w, C) = \sum_{k \in \hat{U}_i^{-1}} \frac{\alpha^{ij}}{\sum_{h \in \hat{U}_k \cap Q(j)} \alpha^{hj}} \times \left[ C_k(\sum_{k \in \hat{U}_k} j \wedge w_k) - C_k(\sum_{k \in \hat{U}_k} (j-1) \wedge w_k) \right],$$

satisfies all the axioms except Upstream solidarity for a cost increase.

## 5.4 A cooperative game-theoretical interpretation

In this section, we analyze the class of hazardous waste transportation problems by adopting a cooperative game theoretic approach. For each hazardous waste transportation problem, we define an associate multi-choice game. The notation and notions related to multi-choice games are the same as introduced in Chapter 2 Section 2.3.1.

For each waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$ , define the associated multi-choice game  $(w, v_P)$  where  $w$  is the grand coalition and  $v_P : W \rightarrow \mathbb{R}_+$  is the associated characteristic function. Let  $s \leq w$  be a profile of hazardous waste. Recall that  $\mathcal{S}(s)$  is the set of agents that ship a positive amount of waste in the profile  $s$ . The worth  $v_P(s) \in \mathbb{R}_+$  then describes the total maintenance and operation cost whenever the profile  $s$  is shipped to the delivery node under the considered liability regime. Because the polluter pays principle and the strict liability principle (see Section 5.3.2) hold an agent liable for the costs of downstream portions, we define  $v_P$  as

$$\forall s \leq w, \quad v_P(s) = \sum_{i \in \hat{U}_{\mathcal{S}(s)}^{-1}} C_i \left( \sum_{k \in \hat{U}_i} s_k \right). \quad (5.23)$$

Observe that the worth of the grand coalition  $v_P(w)$  is equal to the total cost defined by (5.1). Actually, most properties of the cost functions  $C_i$  are inherited by the multi-choice game  $(w, v_P)$ . Since for each  $i \in N$ ,  $C_i(0) = 0$ , it holds that  $v_P(\mathbf{0}) = 0$ . Thus, for each hazardous waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$ , the associated multi-choice game  $(w, v_P)$  is well-defined. Finally, since each cost function  $C_i$ ,  $i \in N$ , is non-negative and non-decreasing, it follows that each multi-choice game  $(w, v_P)$  is a non-negative and monotonic game (see Chapter 2).

**Remark 16.** Suppose that the maximal waste amount of each agent  $i \in N$  is  $w_i = 1$ . Then, the game  $(w, v_P)$  is a TU-game that coincides with the upstream oriented game associated with a polluted river network problem introduced in Dong et al. [2012]. Following van den Brink et al. [2018], given  $P = (g, w, C) \in \mathcal{P}^N$ , one can interpret the network  $g$  as a permission structure. Moreover, van den Brink et al. [2018] shows that the upstream oriented game associated with a polluted river network problem

coincides with the dual of a TU-game with permission structure. However, the definition of the dual of a multi-choice game remains debatable. Thus, it is not so clear that the same relation holds for the whole class  $\mathcal{P}^N$ .<sup>7</sup>

The next result states that the responsibility rule coincides with the multi-choice Shapley value introduced in Chapter 3.

**Theorem 5.4.1.** *For each hazardous waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$  and its associated multi-choice game  $(w, v_P)$ , it holds that*

$$f^R(P) = \varphi(w, v_P).$$

*Proof.* Let  $P = (g, w, C) \in \mathcal{P}^N$  be any hazardous waste transportation problem. For each  $k \in N$ , define  $\lambda^{k,C} = (\lambda_i^{k,C})_{i \in N}$  the profile of cost functions such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \neq k$ . Observe that, for each  $(g, w, C) \in \mathcal{P}^N$ , the cost profile  $C$  can be written as

$$C = \sum_{k \in N} \lambda^{k,C}.$$

Fix any  $k \in N$  and consider the multi-choice game  $(w, u_P^k)$ , where  $u_P^k$  is the characteristic function defined as

$$\begin{aligned} \forall s \leq w, \quad u_P^k(s) &= \sum_{i \in \hat{U}_{S(s)}^{-1}} \lambda_i^{k,C} \left( \sum_{l \in \hat{U}_i} s_l \right), \\ &= \begin{cases} \lambda_k^{k,C}(\sum_{l \in \hat{U}_k} s_l) & \text{if } k \in \hat{U}_{S(s)}^{-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that for each  $P = (g, w, C) \in \mathcal{P}^N$  and each  $(w, v_P)$ , it holds that

$$v_P = \sum_{k \in N} u_P^k.$$

By computing the multi-choice Shapley value (see (3.7)) of the multi-choice game  $(w, u_P^k)$ , one can make the following observations. For each  $i \in N \setminus \hat{U}_k$  and each  $j \leq w_i$ , the pair  $(i, j) \in M^+$  is a null pair in  $(w, u_P^k)$ . Since the multi-choice Shapley value satisfies the null pair axiom (see Theorem 3.4.1), it holds that

$$\forall i \in N \setminus \hat{U}_k, \forall j \leq w_i, \quad \varphi_{ij}(w, u_P^k) = 0.$$

Moreover, any two pairs  $(i, j), (i', j)$ , such that  $i, i' \in \hat{U}_k \cap Q(j)$ , are equal pairs in  $(w, u_P^k)$ . Since the multi-choice Shapley value satisfies Equal treatment of equal pairs, we have that for each  $j \leq \max_{i \in N} w_i$ ,

$$\forall i, i' \in \hat{U}_k \cap Q(j), \quad \varphi_{ij}(w, u_P^k) = \varphi_{i'j}(w, u_P^k).$$

<sup>7</sup>For a recent study of multi-choice games with a permission structure, we refer the reader to [Lowing \[2022\]](#).

Finally, since the Shapley value satisfies Multi-efficiency (see (3.3)), for each  $j \leq \max_{i \in N} w_i$  it holds that

$$\begin{aligned} \sum_{i \in \hat{U}_k \cap Q(j)} \phi_{ij}(w, u_P^k) &= u_P^k((j \wedge w_k)_{k \in N}) - u_P^k(((j-1) \wedge w_k)_{k \in N}) \\ &= \lambda_k^{k,C} \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - \lambda_k^{k,C} \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right). \end{aligned}$$

By definition of  $\lambda_k^{k,C}$  and Equal treatment of equal pairs, we have that

$$\phi_{ij}(w, u_P^k) = \begin{cases} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} & \text{if } i \in \hat{U}_k \cap Q(j), \\ 0 & \text{otherwise.} \end{cases} \quad (5.24)$$

Recall that, for each  $P \in \mathcal{P}^N$  and multi-choice game  $(w, v_P)$ , it holds that

$$v_P = \sum_{k \in N} u_P^k.$$

Since the multi-choice Shapley value satisfies Additivity, one concludes that

$$\begin{aligned} \forall i \in N, \forall j \leq w_i, \quad \phi_{ij}(w, v_P) &= \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\ &= f_{ij}^R(g, w, C). \end{aligned}$$

This concludes the proof of the theorem. ■

## 5.5 Conclusion

In this chapter, we study the problem of sharing the maintenance and operation cost of a hazardous waste transport network modeled by a directed sink tree. We propose an axiomatic analysis on the class of hazardous waste transportation problems. Specifically, we propose several axioms related to environmental law principles. This allows us to precise the liability of each agent regarding the risk it causes to the network, and to characterize a specific cost sharing method called the responsibility rule. We also show that this method coincides with the multi-choice Shapley value of an appropriate multi-choice game.

This work can be extended in several directions. First, it should be noticed that the delivery node,  $d$ , is exogenously given and does not belong to the agent's set. Relaxing this hypothesis would mean to choose a location for the treatment facility among the set of agents. However, due to the nature of the treatment facility, the agents may be reluctant to host it. This gives rise to a so-called NIMBY problem that has been studied by [Shapley and Shubik \[1969\]](#), and [Ambec and Kervinio \[2016\]](#), using cooperative game theory. From a different viewpoint, [Sakai \[2012\]](#) conducts an axiomatic study for sharing the cost associated with a NIMBY facility. To the best of our knowledge, no work has addressed the cost allocation of a NIMBY facility in

a network structure using the axiomatic method. We suggest that such work can be conducted by adapting our model.

Another interesting direction for further research would be to study how the implementation of the responsibility rule may affect the incentive for agents to decrease their generation of waste. This could be an important aspect in meeting the goals of environmental law regarding environmental preservation.

## Chapter 6

# Stable agreements through liability rules: a multi-choice game approach to the social cost problem

### 6.1 Introduction

A large literature has been developed to describe and solve situations known as social cost problems, in which the activity of some agents has harmful effects on others. This chapter aims at analyzing precisely the conditions for resolving social cost problems involving one polluter and some potential victims. Two traditions stand out to solve a social cost problem: the Pigouvian and the Coasean traditions. The Pigouvian tradition (Pigou [1920]) advocates a central intervention by means of taxation on the externality. This results in the introduction of the polluter-pays principle whereby the polluter should bear the cost of pollution control and prevention measures. The Coasean tradition (Coase [1960]) challenges this polluter-pays principle. This tradition argues that agents can solve a social cost problem through a bargaining process provided that property rights are well assigned. The Coase theorem, first formulated by Stigler [1966], summarizes this approach in two properties: first, the *efficiency property* states that in the absence of transaction costs, and if property rights are well defined, agents will always reach an optimal agreement by bargaining; second, the *invariance* or *neutrality property* states that the outcome of the bargaining process is independent of the assignment of rights. Throughout this chapter, we follow the Coase perspective, and we use cooperative game theory to solve social cost problems.

The Coase theorem has been analyzed through the scope of cooperative games by Aivazian and Callen [1981] and more recently by Gonzalez et al. [2019]. These articles discuss social cost problems involving more than two agents and consider the Coase theorem in term of non-emptiness of the core. This chapter extends the framework introduced by Gonzalez et al. [2019], which investigates situations in which one polluter interacts with a set of at least two victims. The authors introduce mappings of rights that describe the legal structure of negotiations among agents. Specifically, a mapping of rights assigns to each coalition either the value 0, meaning the coalition is not allowed to negotiate, or 1 if the coalition is allowed to form and negotiate an agreement. This set of mappings of rights can be seen as the set of winning coalition of a proper voting game.<sup>1</sup> They propose three properties for mappings of rights: *core compatibility* which requires that the core of the cooperative

<sup>1</sup>Gonzalez et al. [2019] require three conditions for their mappings of rights: the grand coalition is sovereign; if a coalition receives the right then any larger coalition containing the former inherits the right (monotonicity); if a coalition receives the right then nonmembers cannot prevent the coalition to negotiate (effectivity of rights).



game associated with any social cost problem is non-empty; *Kaldor-Hicks core compatibility* which requires that a payoff vector in the core ensures a non-negative payoff to each agent; *no veto power for a victim* requires that no victim can individually veto an agreement reached by the rest of the society. The authors provide two main results. On the one hand, a mapping of rights satisfies core compatibility if and only if it assigns the rights either to the polluter or to the whole set of victims. On the other hand, no mapping of rights satisfies those three properties at the same time.

Nonetheless, the set of mappings of rights introduced by [Gonzalez et al. \[2019\]](#) is independent of the polluter's activity level. Either a coalition containing the polluter can negotiate and choose any activity level, or it cannot negotiate. As a result, the model does not take into account the more realistic issue of quotas, which may arise when one wishes to reduce the pollution level. For instance, the *US Clean Air Act* defines a pollution quota which is viewed as a limitation on the polluter's activity. This means that the rights may impose a restriction on the polluter's activity level as a right of use. In this regard, we introduce the possibility that the rights depend on the polluter's activity level. This has three main implications for the model: the conditions defining a mapping of rights must be adjusted; the cooperative game must take into account the different activity levels of the polluter; and the solution concept that captures the Coase theorem must be adapted in line with the new setting.

To address these points, we modify the conditions defining mappings of rights. To start with, we impose antitonicity on mappings of rights with respect to the activity level of the polluter. Precisely, if a coalition containing the polluter is allowed to negotiate an agreement for a certain activity level, then the coalition retains the rights whenever the polluter decreases its activity level. Second, we impose monotonicity of a mapping of rights with respect to the participation of victims. If a coalition is allowed to form and negotiate an agreement, it retains the rights when the number of cooperating victims increases. Notice that we can no longer see the set of mapping of rights as generating winning coalitions since a coalition may lose its rights if the activity level increases. Thus we obtain a new class of mappings of rights different from those introduced by [Gonzalez et al. \[2019\]](#). In this class, we pinpoint the mappings of rights that assign the rights to the polluter up to a fixed and regulated activity level. Those mappings of rights reflect the fact that the polluter has to produce up to a quota. When the pollution problem is severe, the only way for the polluter to override this quota is to reach an agreement with the set of victims. For the sake of simplicity, we suppose the polluter has a finite number of activity levels. Each victim can choose whether to participate or not in the negotiation. If the coalition has the right to negotiate, then the polluter proposes a certain activity level and negotiates a binding agreement with victims which agree to participate. Because agents have more than one way of acting within a coalition, we model the above situations using multi-choice games. Thus, from each social cost problem endowed with a mapping of rights, we can define a multi-choice game. In the same vein as [Aivazian and Callen \[1981\]](#) and [Gonzalez et al. \[2019\]](#), we consider the Coase theorem in terms of non-emptiness of the core of this game. Here we choose to retain the extension of the core introduced and characterized by [Grabisch and Xie \[2007\]](#) (see Chapter 2, Section 2.3.2). Finally, we naturally extend the properties of core compatibility and Kaldor-Hicks core compatibility from TU-games to multi-choice games. On the other hand, we consider no veto power for a victim regarding the highest activity level of the polluter to fit our new framework.

Our main result extends the subset of mappings of rights which ensure the non-emptiness of the core. We first show that these new mappings of rights that assign a quota on the activity of the polluter are core compatible. Each victim perceives the

activity of the polluter as a threat which reduces the incentive for victims to free-ride and ensures the stability of the agreement. Furthermore, any mapping of rights that exclusively assigns the rights either to a subset of victims or to the grand coalition (i.e. the set of victims and the polluter) is core compatible. Since not all mapping of rights are core compatible, this result invalidates the neutrality property of the Coase theorem. We show that the unique way to satisfy core compatibility and no veto power for a victim is to assign the rights to the polluter at its highest activity level. On the contrary, the only mappings of rights that satisfy Kaldor-Hicks core compatibility are those that assign the rights either to a subset of victims or to the grand coalition. Finally, we confirm the impossibility result regarding Kaldor-Hicks core compatibility and no veto power for a victim at the same time.

### Related literature:

The Coase theorem has fostered a broad literature, in particular because Coase considers only two agents and does not provide any formal model. That literature has mainly focused on three types of models. The first type considers the theorem in terms of competitive equilibrium. For instance [Hurwicz \[1995\]](#) and [Chipman and Tian \[2012\]](#) provide necessary and sufficient conditions for the validity of the Coase theorem for situations involving two agents. Both papers highlight the pivotal role of parallel preferences that ensure the validity of the neutrality property. The second type incorporates strategic interactions between agents and formulates the Coase theorem in terms of Nash equilibrium of a strategic/bargaining game: [Anderlini and Felli \[2001\]](#) analyze negotiations between two agents in the presence of transaction costs; [Ellingsen and Paltseva \[2016\]](#) highlight the possibility of free-riding behaviors when more than two agents are involved. This paper aligns with the third type of models which consider cooperative game theory. A famous instance is the [Shapley and Shubik \[1969\]](#) garbage game where a finite set of neighbors have to decide on where to locate their garbage. Using a simple construction authors show that the core of the associated garbage game is non-empty if and only if no more than two agents are involved. In the same spirit, [Ambec and Kervinio \[2016\]](#) analyze the Not In My Back Yard (NIMBY) problem. In comparison to the garbage game, the NIMBY problem consider the location of a locally undesirable but globally desirable facility. Authors propose a spatial model with externalities, and analyze the conditions under which the Coase theorem is likely to hold. They also provide an index for testing the non-emptiness of the core of a game associated with a NIMBY problem. [Zhao \[2018\]](#) proposes a solution for the empty-core problem introduced by [Aivazian and Callen \[1981\]](#). The author argues that the set of agents does not necessarily split the worth of the grand coalition but can organize themselves into the most beneficial collection of coalitions.

The rest of the chapter is organized as follows. Section 6.2 presents the formal framework within which we define the social cost problem and the mappings of rights along with the corresponding multi-choice game. Section 6.3 presents our main results, and Section 6.4 concludes the chapter.

## 6.2 Notation

Let  $N := \{1, \dots, n\}$  be a finite set of agents and  $K \in \mathbb{N}$ . Throughout this chapter  $N$  and  $m \leq K$  are fixed. Thus, we consider the set of multi-choice games  $\mathcal{G}_m$ . For simplicity, we will denote a multi-choice game by  $v$  instead of  $(m, v)$ . We also consider

the concept of level payoff vectors (see Chapter 2, Section 2.3.2). Let  $v \in \mathcal{G}_m$ , and  $\bar{x} \in \mathbb{R}^{|M^+|}$  be a level payoff vector. For any coalition  $s \in \mathcal{M}$ , the *total payoff*  $\bar{x}(s)$  is defined as:

$$\bar{x}(s) = \sum_{i \in N} \bar{x}_{i,s_i}.$$

### 6.2.1 Social cost problems

Let  $p$  be a polluter and denote by  $U$  a finite set of at least two potential victims. The set  $N = U \cup \{p\}$  denotes the finite set of agents involved in a social cost problem,  $N$  is supposed to be fixed. The polluter chooses a participation level it wishes to operate in the fixed set  $M_p := \{0, 1, 2, \dots, m_p\}$  which corresponds to its set of activity levels. Each victim  $i \in U$  decides whether or not to take part in the cooperation; hence the action set for each victim is  $M_i := \{0, 1\}$ . Let  $\mathcal{M} := \times_{i \in N} M_i$ , an element  $s := (s_1, \dots, s_u, s_p) \in \mathcal{M}$  is a multi-choice coalition/participation profile.

The activity of the polluter generates a private benefit. This private benefit is represented by a non-decreasing function  $B_p : M_p \rightarrow \mathbb{R}_+$ . Besides, this activity causes damage for each victim  $i \in U$ . This damage is represented by a non-decreasing function  $D_i : M_p \rightarrow \mathbb{R}_+$ . For each  $i \in N$ , we assume that  $B_p(0) = D_i(0) = 0$ . We define a social cost problem as  $P := (N, \mathcal{M}, B_p, (D_i)_{i \in U})$ . Let  $\mathcal{P}$  be the class of all social cost problems.

Given a social cost problem  $P \in \mathcal{P}$ , for each participation profile  $s \in \mathcal{M}$ , we denote by  $K_s^*$  the non-empty set of optimal activity levels. Formally,

$$K_s^* := \arg \max_{k \in [[0, s_p]]} \left\{ 1_p(s) B_p(k) - \sum_{\substack{i \in U \\ s_i = 1}} D_i(k) \right\},$$

where  $[[0, s_p]]$  denotes the set  $\{0, 1, \dots, s_p\} \subseteq M_p$ ,  $1_p(s) = 1$  if  $s_p > 0$  and  $1_p(s) = 0$  otherwise. We denote by  $k_s^*$  an element of  $K_s^*$ . Note that  $0 \in K_s^*$  whenever  $s \in \mathcal{M}$  is such that  $s_p = 0$ .

### 6.2.2 Mapping of rights

Given a participation profile  $s \in \mathcal{M}$ , we consider a *right* as an authorization for the group of active agents  $\mathcal{S}(s)$  to form and negotiate a binding agreement when each active agent chooses its participation level  $s_i \in M_i$ . In this way, rights are related to the participation profiles rather than to groups of agents exclusively.

A *mapping of rights*  $\phi : \mathcal{M} \rightarrow \{0, 1\}$  assigns to each participation profile  $s \in \mathcal{M}$  the value 0 or 1. The value  $\phi(s) = 0$  means that the group of active agents in this participation profile does not have the right to negotiate an agreement. The value  $\phi(s) = 1$  means that the group of active agents is allowed to form a coalition and negotiate an agreement. We impose the following conditions on mappings of rights:

(C1)  $\phi(0) = 0$  and  $\phi(1_{-p}, m_p) = 1$ ,

(C2) for each  $s_p \in M_p$ , it holds that:

$$[\forall s_{-p}, s'_{-p} \in \mathcal{M}_{-p}, s_{-p} \leq s'_{-p}] \Rightarrow [\phi(s_{-p}, s_p) \leq \phi(s'_{-p}, s_p)],$$

(C3) for each  $s_{-p} \in \mathcal{M}_{-p}$ , it holds that:

$$[\forall s_p, s'_p \in M_p \setminus \{0\}, s_p \leq s'_p] \Rightarrow [\phi(s_{-p}, s_p) \geq \phi(s_{-p}, s'_p)],$$

(C4) for each  $S \subseteq N$ , and each  $(0_{-S}, s_S) \in \mathcal{M}_{-S} \times \mathcal{M}_S$ , it holds that:

$$[\phi(0_{-S}, s_S) = 1] \Rightarrow [\forall s_{-S} \in \mathcal{M}_{-S}, \phi(s_{-S}, 0_S) = 0].$$

(C1) states that if agents do not participate they do not receive any rights. In contrast, if they fully participate, then they receive the rights. Therefore, full participation implies sovereignty.

(C2) translates the idea of monotonicity of  $\phi$  on  $\mathcal{M}_{-p}$  for any fixed participation level  $s_p \in M_p$  of the polluter. Any group of active agents that receives the rights should retain these rights when more victims become active. Notice that, in the model of [Gonzalez et al. \[2019\]](#), the mapping of rights is monotonic with respect to set inclusion. On the contrary, we propose that a group of active agents should preserve its rights only when new victims become active. However, it might be possible that a group of active agents loose its rights if the polluter becomes active or if it increases its activity level.

(C3) relates to participation profiles where the polluter is active and translates the idea of antitonicity of  $\phi$  on  $M_p \setminus \{0\}$ . If a group of active agents is not allowed to form a coalition and negotiate when  $s_p \in M_p \setminus \{0\}$ , then this remains true when the activity level of the polluter increases that is, if  $\phi(s_{-p}, s_p) = 0$  and  $s_p \leq s'_p$  then  $\phi(s_{-p}, s'_p) = 0$ . On the contrary, it may be the case that this group receives the rights when the polluter decreases its activity level. If a group of agents has the right to negotiate, then it retains the right when the activity level of the polluter decreases that is, if  $\phi(s_{-p}, s_p) = 1$  and  $s''_p \leq s_p$ , then  $\phi(s_{-p}, s''_p) = 1$ .

Finally, (C4) describes the effectivity of the rights. Consider a participation profile  $s \in \mathcal{M} \setminus \{0\}$  such that  $\mathcal{S}(s) = S \subseteq N$  and  $\phi(s) = 1$ . Whatever the participation level  $s_{-S} \in \mathcal{M}_{-S}$  of agents in  $-S$ , if all agents in  $S$  become inactive, then the corresponding participation profile does not receive the rights:  $\phi(s_{-S}, 0_S) = 0$ . Thus, it cannot hold that  $S$  and  $-S$  receive the rights at the same time.

Combining (C1) with (C3), the whole society  $N$  has the right to negotiate whatever the activity level  $s_p \in M_p \setminus \{0\}$  chosen by the polluter,  $\phi(1_{-p}, s_p) = 1$ . In other words, the whole society is always sovereign.

We denote by  $\Phi$  the class of mapping of rights that satisfy the above conditions.<sup>2</sup>

**Example 8.** Consider the mapping which gives the rights to each majority coalition. In the multi-choice game framework this mapping of rights is denoted  $\phi^{maj} \in \Phi$ , and is defined as follows:

$$\phi^{maj}(s) = \begin{cases} 1 & \text{if } |\mathcal{S}(s)| > n/2, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that  $\phi^{maj} \in \Phi$  satisfies each condition defining a mapping of rights.

■

### 6.2.3 Multi-choice games and mapping of rights

Given a social cost problem  $P = (N, \mathcal{M}, B_p, (D_i)_{i \in U}) \in \mathcal{P}$  and a mapping of rights  $\phi \in \Phi$ , we define a multi-choice game which describes the cooperation possibilities between the agents. Due to the presence of externalities and the distribution of

<sup>2</sup>One can verify that the class of mapping of rights introduced in [Gonzalez et al. \[2019\]](#) can be defined using our new conditions, therefore our class of mapping of rights constitutes a generalization.

rights in the society, the worth that a group of agents can obtain in a social cost problem depends crucially on the expected behavior of the agents outside this group. Suppose that a group of agents  $S$  agree to cooperate. They will form a participation profile  $s \in \mathcal{M}$ , where  $\mathcal{S}(s) = S$ , and seek to obtain their maximal social benefit

$$1_p(s)B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*), \quad k_s^* \in K_s^*.$$

However, the agents do not necessarily obtain this maximal social benefit because of externalities and the distribution of rights. If  $p$  does not belong to  $S$ , that is, if  $1_p(s) = 0$ , the damage  $D_i(k_s)$  suffered by each active victim  $i \in U$  depends crucially on the behavior of the agents outside  $S$ . Furthermore, this behavior is conditioned by the distribution of rights in the society. In the same manner, if  $p$  belongs to  $S$ , that is, if  $1_p(s) = 1$ , then the social benefit that the members of  $S$  can guarantee depends on whether members outside  $S$  have acquired the right to prevent them from reaching certain agreements. In others words, both the mapping of rights and the behavior of the agents outside  $S$  exert an externality on the worth that the active agents in  $S$  can reach. There are many possible cases. The members outside  $S$  may act non-cooperatively or they may cooperate and form one coalition; some subgroups outside  $S$  may have the right to sign agreements whereas some other subgroups may not have this right. In what follows, we assume that the members of  $S$  have a pessimistic view of the coalition formation of outsiders. Given the mapping of rights  $\phi$ , the members of coalition  $S$  pessimistically expect that the coalitions of outsiders form in the worst possible way for  $S$ .

To be more precise, let  $-\mathcal{S}(s)$  be the complementary group of the group  $\mathcal{S}(s)$  of active agents in the participation profile  $s \in \mathcal{M}$ . Without any consideration of the mapping of rights  $\phi$ , the group  $-\mathcal{S}(s)$  can choose any participation profile of the form  $(0_{\mathcal{S}(s)}, y_{-\mathcal{S}(s)}) \in \mathcal{M}$ . From this observation, several cases arise. First, if the polluter is part of the active agents and the latter have the right to negotiate, then, by (C4), the complementary group  $-\mathcal{S}(s)$  does not have the right to negotiate. Therefore, active agents expect that the behavior of the agents in  $-\mathcal{S}(s)$  will have no consequences on the worth they can reach by cooperating. If  $\mathcal{S}(s)$  does not have the right to negotiate, then they can not sign a binding agreement. It results that they guarantee themselves a null worth. Second, if  $p$  does not belong to the group  $\mathcal{S}(s)$  and  $\mathcal{S}(s)$  does not have the right to negotiate, then active agents expect that the remaining agents will choose a participation profile of the form  $(0_{\mathcal{S}(s)}, y_{-\mathcal{S}(s)})$ . Two sub-cases arise. If no subgroups of  $-\mathcal{S}(s)$  has the right to negotiate, then, as above, active agents in  $\mathcal{S}(s)$  expect that the behavior of the agents in  $-\mathcal{S}(s)$  will have no consequences on the the worth they can reach by cooperating. If there are subgroups of  $-\mathcal{S}(s)$  that have the right to negotiate, then active agents expect the worst scenario for them. Thus, they expect the largest externality that the members of  $-\mathcal{S}(s)$  can generate by reorganizing and signing a binding agreement with the polluter.

The rationale behind this assumption about the expectations of active agents regarding the behavior of outsiders is as follows. Under pessimistic expectations, the worth of a participation profile will always be smaller than under any other assumption about expectations. Therefore, this worth corresponds to the worth that the group of active agents can guarantee regardless of the behavior of the remaining agents. Furthermore, if the core of a multi-choice game is empty under pessimistic expectations, then it will be empty under any other expectations. In other words, it

is the most favorable assumption to obtain a non-empty core.<sup>3</sup>

To formally define the worth of each coalition some definitions are in order. First, let  $\mathcal{M}' \subseteq \mathcal{M}$  be a non-empty subset of participation profiles.  $\mathcal{M}'$  is said to be *trivial* if for each  $s \in \mathcal{M}'$ ,  $\phi(s) = 0$ . Next, for each participation profile  $s = (0_{-S}, s_S) \in \mathcal{M}$ , where  $S = \mathcal{S}(s)$ , we associate the *set of complementary participation profiles* defined by:

$$\bar{\mathcal{C}}_s := \{(y_{-S(s)}, 0_{S(s)}) \in \mathcal{M} \mid y_{-S(s)} \in \times_{i \in -S(s)} M_i\}.$$

Each element of  $\bar{\mathcal{C}}_s$  corresponds to a participation profile where each agent  $i \in S$  is inactive. The *worst response* at profile  $s \in \mathcal{M}$ , denoted by  $w_s \in \bar{\mathcal{C}}_s \subseteq \mathcal{M}$ , is a participation profile such that:

$$k_{w_s}^* = \begin{cases} 0 & \text{if } \bar{\mathcal{C}}_s \text{ is trivial,} \\ \max_{y \in \bar{\mathcal{C}}_s: \phi(y)=1} \{k_y^* \mid k_y^* \in K_y^*\} & \text{otherwise.} \end{cases}$$

where  $k_y^* \in K_y^*$  is the optimal activity level at profile  $y \in \bar{\mathcal{C}}_s$ . If  $\phi(s) = 1$ , then by (C4) each agent involved in the participation profile  $s \in \mathcal{M}$  knows that the remaining agents cannot prevent them to negotiate ( $\bar{\mathcal{C}}_s$  is trivial), so that  $k_{w_s}^* = 0$ .

Finally, given a social cost problem  $P \in \mathcal{P}$  and a mapping of rights  $\phi \in \Phi$ , the associated multi-choice game  $v_{\phi, P} \in \mathcal{G}_m$  is defined as follows:

$$v_{\phi, P}(s) = \begin{cases} \max_{k \in [[0, s_p]]} \left( B_p(k) - \sum_{i \in \mathcal{S}(s) \setminus \{p\}} D_i(k) \right) & \text{if } \phi(s) = 1, \text{ and } \mathcal{S}(s) \ni p, \\ - \sum_{i \in \mathcal{S}(s)} D_i(k_{w_s}^*) & \text{if } \phi(s) = 0, \text{ and } \mathcal{S}(s) \not\ni p, \\ 0 & \text{otherwise.} \end{cases} \quad (6.1)$$

To interpret (6.1), let  $s \in \mathcal{M}$  be a participation profile such that  $\phi(s) = 1$  and  $\mathcal{S}(s) \ni p$ . Active agents can select any socially optimal activity level in  $K_s^*$ . By (C4) the rights are effective and active members,  $\mathcal{S}(s)$ , do not have to worry about the behavior of the remaining agents. Therefore,

$$v_{\phi, P}(s) = \max_{k \in [[0, s_p]]} \left( B_p(k) - \sum_{i \in \mathcal{S}(s) \setminus \{p\}} D_i(k) \right).$$

If  $s \in \mathcal{M}$  is such that  $\phi(s) = 0$  and  $\mathcal{S}(s) \not\ni p$ , that is if only victims participate  $\mathcal{S}(s) \subseteq U$  and do not receive the rights. By the pessimistic expectations assumption, active agents expect the worst scenario from the remaining agents, which includes the polluter. Therefore each agent in  $\mathcal{S}(s) \subseteq U$  expects the activity level  $k_{w_s}^*$ .

Finally, two last cases arise: either  $s \in \mathcal{M}$  is a participation profile such that  $\phi(s) = 0$  and  $\mathcal{S}(s) \ni p$ , or  $s \in \mathcal{M}$  is such that  $\phi(s) = 1$  and  $\mathcal{S}(s) \not\ni p$ . Regarding the former case, the active agents do not have the right to negotiate an agreement and the worst scenario active agents expect from the outsiders is  $k_{w_s}^* = 0$ . Hence the worth of the participation profile is  $v_{\phi, P}(s) = 0$ . For the latter case, by (C4) the remaining agents cannot prevent  $\mathcal{S}(s)$  to exercise the rights, so  $v_{\phi, P}(s) = 0$ .

<sup>3</sup>In a similar way, Funaki and Yamato [1999] analyze the tragedy of the commons through a cooperative game with externalities. The authors prove that the core of a common pool resource game is always non-empty under the assumption of pessimistic expectations. More recently, Abe and Funaki [2017] provide a necessary and sufficient condition for the non-emptiness of the core of games with externalities, and they apply this condition to the tragedy of the commons.



### 6.2.4 Properties for mappings of rights

Given a mapping of rights  $\phi \in \Phi$  and a social cost problem  $P \in \mathcal{P}$ , the outcome of the negotiation is given by a level payoff vector  $\bar{x} \in \mathbb{R}^M$  of the multi-choice game  $v_{\phi,P}$ . In this section, we present some desirable properties for mappings of rights. The first one is related to the existence of core elements for the multi-choice game derived from a mapping of rights  $\phi \in \Phi$  and a social cost problem  $P \in \mathcal{P}$ . The second property relates to the feasibility of payoff vectors in the core and the full compensation principle. This property strengthens the later imposing a positive payoff for each agent whatever the participation level. Finally, the last property indicates that no victim has the power to veto an agreement reached by the grand coalition.

**Definition 1.** A mapping of rights  $\phi \in \Phi$  satisfies core compatibility if, for each social cost problem  $P \in \mathcal{P}$ ,  $\mathfrak{C}(v_{\phi,P}) \neq \emptyset$ .

This first property is straightforward. The game associated with a mapping of rights that satisfies core compatibility always has a non-empty core (see Chapter 2, Section 2.3.2) regardless the social cost problem  $P \in \mathcal{P}$ . Therefore whatever the benefit function  $B_p$  or the damage functions  $D_i$  for each victim  $i \in U$ , the core of the social cost game is non-empty.

**Definition 2.** A mapping of rights  $\phi \in \Phi$  satisfies Kaldor-Hicks core compatibility if, for each social cost problem  $P \in \mathcal{P}$ ,  $\mathfrak{C}(v_{\phi,P}) \cap \mathbb{R}_+^M \neq \emptyset$ .

The Kaldor-Hicks core compatibility property imposes a non-negative payoff for each agent at each participation level. In our context, the polluter is the only agent which can have different activity levels. Hence Kaldor-Hicks core compatibility imposes that each victim should receive a non-negative payoff for its participation in order to be fully compensated for the damage. Also, the polluter should receive a non-negative payoff for each of its activity level.

**Definition 3.** A mapping of rights satisfies no veto power for a victim if, for each victim  $i \in U$ ,  $\phi(1_{-\{i,p\}}, 0_i, m_p) = 1$ .

According to this property, if a victim decides to leave the grand coalition, the remaining coalition retains the right to negotiate. Notice that, by (C3) if a mapping of rights  $\phi \in \Phi$  satisfies no veto power for a victim, each participation profile  $(1_{-\{i,p\}}, 0_i, s_p) \in \mathcal{M}$  such that  $\mathcal{S}(1_{-\{i,p\}}, 0_i, s_p) = N \setminus \{i\}$  retains the right:

$$\phi(1_{-\{i,p\}}, 0_i, s_p) = 1.$$

By (C4), each victim  $i \in U$  does not have the power to prevent negotiations among the remaining agents  $\phi(0_{-i}, 1_i) = 0$ . When we can consider a large set of victims, this property can be seen as a weak democratic principle across the set of victims. Indeed, one isolated victim cannot impose its choice to the remaining agents. No victim should have more power than other victims regarding the legal framework.

**Example 9.** Consider the following social cost problem  $P \in \mathcal{P}$  defined as follows. Let  $N := \{1, 2, p\}$  be the set of agents,  $M_p := \{0, 1, 2, 3, 4\}$  is the action set of the polluter,  $M_1 = M_2 = \{0, 1\}$ , and

$$B_p(s_p) = 4\sqrt{s_p}, \quad D_1(s_p) = \frac{6}{10} \times 2s_p, \quad D_2(s_p) = \frac{3}{10} \times 2s_p.$$

Consider the mapping of rights  $\phi^{maj} \in \Phi$  as introduced in Example 8. The associated multi-choice game  $v_{\phi^{maj},p} \in \mathcal{G}_m$  is as follows:

$$v_{\phi^{maj},p}(s) = \begin{cases} 4\sqrt{k_s^*} - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) & \text{if } \mathcal{S}(s) \ni p, |\mathcal{S}(s)| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

We can show that  $\mathfrak{C}(v_{\phi^{maj},p}) = \emptyset$  by considering the balanced collection  $\mathcal{B}$  defined as

$$\mathcal{B} := \{s \in \mathcal{M} : |\mathcal{S}(s)| = 2\},$$

with the balancing weights:

$$\mu(s) = \frac{4}{2}, \text{ if } s = (1, 1, 0), \text{ and } \mu(s) = \frac{1}{2} \text{ otherwise.}$$

It should be noticed that the worth of the grand coalition will necessarily be lower than the worth of any two agents coalition. By simple computations we have that:

$$\sum_{s \in \mathcal{B}} \mu(s) v_{\phi^{maj},p}(s) = 15.64 > 8.8 = \sum_{k=1}^4 v_{\phi^{maj},p}(1, 1, k).$$

By Theorem 2.3.1, we conclude that the core of the game is empty. Thus,  $\phi^{maj} \in \Phi$  does not satisfy core compatibility. Moreover, because Kaldor-Hicks core compatibility implies core compatibility  $\phi^{maj}$  cannot satisfy Kaldor-Hicks core compatibility. In contrast, because we consider social cost problems with one polluter and at least two victims, whenever a victim decides to leave the grand coalition we have  $n - 1 > n/2$ . Therefore,  $\phi^{maj} \in \Phi$  satisfies no veto power for a victim.

■

## 6.3 Core compatible mappings of rights

### 6.3.1 Restriction on the activity level of the polluter

In order to reduce the pollution level, it might be preferable to control the activity of the polluter. For instance, one can imagine that the agents have a first agreement on the pollution level that compels the polluter not to exceed a certain activity level. In this perspective, we consider mappings of rights that allow the polluter to produce up to a fixed “regulated activity level”  $\tilde{k}_p \in M_p \setminus \{0\}$ .

The rationale behind this family of mappings of rights is to assess the liability of the polluter as follows. If  $s_p$  is low,  $s_p \leq \tilde{k}_p$ , the polluter should not be held liable for the damage its activity creates. The pollution problem is considered to be less severe since the activity level is lower than the regulated level. Therefore, each coalition containing the polluter can negotiate an activity level lower than or equal to  $\tilde{k}_p$ . If  $s_p$  is high,  $s_p > \tilde{k}_p$ , the polluter is held liable for the resulting damage. Then, only the grand coalition has the right to negotiate. The reason is that the pollution problem becomes more severe and requires the participation of all agents in order to reach an agreement.



We formally define this type of mapping of rights  $\phi_{\tilde{k}_p}$  as follows:

$$\phi_{\tilde{k}_p}(s) = \begin{cases} 1 & \text{if either } \mathcal{S}(s) \ni p \text{ and } 0 < s_p \leq \tilde{k}_p, \text{ or } \mathcal{S}(s) = N, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by  $\tilde{\Phi} \subseteq \Phi$  this family of mapping of rights.

Given a social cost problem  $P \in \mathcal{P}$ , a mapping of rights  $\phi_{\tilde{k}_p} \in \tilde{\Phi}$ , from equation (6.1) we obtain the corresponding multi-choice game  $v_{\phi_{\tilde{k}_p}, P} \in \mathcal{G}_m$ , defined as follows:

$$v_{\phi_{\tilde{k}_p}, P}(s) = \begin{cases} B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) & \text{if } \mathcal{S}(s) \ni p \text{ and } 0 < s_p \leq \tilde{k}_p, \text{ or } \mathcal{S}(s) = N, \\ - \sum_{\substack{i \in U \\ s_i=1}} D_i(\tilde{k}_p) & \text{if } \mathcal{S}(s) \not\ni p, \\ 0 & \text{otherwise.} \end{cases} \quad (6.2)$$

Observe that the worst activity level a victim  $i \in U$  can face is  $\tilde{k}_p$ . Indeed, if a victim  $i \in U$  chooses to be active in a coalition that does not contain the polluter, it expects the remaining agents to organize themselves into the worst scenario for it. On the other hand, by construction of the game, the polluter is encouraged to be the only active agent among the remaining ones. Therefore the polluter would choose the activity level  $\tilde{k}_p$  which maximizes its private benefit.

**Proposition 6.3.1.** *For each  $\tilde{k}_p \in M_p \setminus \{0\}$ , the mapping of rights  $\phi_{\tilde{k}_p} \in \tilde{\Phi}$  satisfies core compatibility.*

*Proof.* In order to prove Proposition 6.3.1, observe that the conditions to have a balanced collection  $\mathcal{B}$  (see Chapter 2, Section 2.3.2, definition 2.30) can be rewritten as follows. A collection  $\mathcal{B}$  is balanced if there exists positive coefficients  $\mu(s), x \in \mathcal{B}$ , such that:

$$\begin{aligned} \forall i \in U, \quad \sum_{\substack{x \in \mathcal{B} \\ s_i=1}} \mu(s) &= m_p, \\ \forall k = 1, \dots, m_p, \quad \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) &= 1. \end{aligned}$$

Thus, a multi-choice game  $v_{\phi, P}$  derived from a mapping of rights  $\phi \in \Phi$  and a social cost problem  $P \in \mathcal{P}$ , is balanced if for every balanced collection  $\mathcal{B}$  of elements of  $\mathcal{M} \setminus \{0\}$  it holds

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi, P}(s) \leq \sum_{k_p=1}^{m_p} v_{\phi, P}(1_{-p}, k_p). \quad (6.3)$$

This observation allows us to derive the following intermediary result regarding balanced collections.

**Lemma 6.3.1.** *Let  $\mathcal{B}$  be a balanced collection and  $\mu(s), x \in \mathcal{B}$ , the associated balancing weights. Then, the following holds:*

$$\forall i \in U, \quad \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=0}} \mu(s) = \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=0, \\ s_p=k}} \mu(s).$$

We interpret this Lemma through time allocation in the same way as for balanced collections. This Lemma simply indicates that the amount of time each victim spends in coalitions where the polluter is not active is equal to the total amount of time the polluter spends in coalitions where the victim is not active. This constitutes the main argument in order to prove Proposition 6.3.1. Indeed, according to this Lemma we can consider a victim in terms of participation profiles in which it is not active but the polluter is.

*Proof.* By the definition of a balanced collection (see Chapter 2, Section 2.3.2, definition 2.30) it holds that

$$\begin{aligned} \forall i \in U, m_p &= \sum_{\substack{x \in \mathcal{B} \\ s_i=1}} \mu(s) \\ &= \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=0}} \mu(s) + \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=1}} \mu(s) + \dots + \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=m_p}} \mu(s) \\ &= \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=0}} \mu(s) + \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=1, \\ s_p=k}} \mu(s) \end{aligned}$$

So we can write

$$\sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) = m_p - \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=k}} \mu(s) \quad (6.4)$$

By definition, for each  $k = 1, \dots, m_p$  we have

$$\sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=k}} \mu(s) = 1 - \sum_{\substack{x \in \mathcal{B} \\ s_i=0 \\ s_p=k}} \mu(s) \quad (6.5)$$

By substituting (6.5) in (6.4) we obtain

$$\begin{aligned} m_p &= \sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) + \sum_{k=1}^{m_p} \left( 1 - \sum_{\substack{x \in \mathcal{B} \\ s_i=0 \\ s_p=k}} \mu(s) \right) \\ &= \sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) + m_p - \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=0 \\ s_p=k}} \mu(s) \end{aligned}$$

and so

$$\sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) = \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=0 \\ s_p=k}} \mu(s). \quad (6.6)$$

In order to prove Proposition 6.3.1, consider the modified Bondareva-Shapley theorem for multi-choice setting (Theorem 2.3.1). We have to show that for any mapping of rights  $\phi_{\tilde{k}_p} \in \tilde{\Phi}$  and for any social cost problem  $P \in \mathcal{P}$ , the corresponding game is balanced. Let  $\mathcal{B}$  be any balanced collection of elements of  $\mathcal{M} \setminus \{\mathbf{0}\}$  and  $\mu(s), s \in \mathcal{B}$ , the associated balancing weights. First of all, notice that by definition of

$v_{\phi_{\tilde{k}_p}, P}$  given by (6.2), for each participation profile  $s \in \mathcal{M}$  such that  $p \in \mathcal{S}(s) \subsetneq N$  and  $s_p > \tilde{k}_p$  we have:  $v_{\phi_{\tilde{k}_p}, P}(s) = 0$ . Therefore, we have to consider three types of participation profile:  $s \in \mathcal{M}$  such that  $0 < s_p \leq \tilde{k}_p$ ,  $s \in \mathcal{M}$  such that  $s_p > \tilde{k}_p$  and  $\mathcal{S}(s) = N$ , and finally  $s \in \mathcal{M}$  such that  $s_p = 0$ . From this point we have:

$$\begin{aligned} \sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &= \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} v_{\phi_{\tilde{k}_p}, P}(s) + \sum_{k'=\tilde{k}_p+1}^{l_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) \\ &+ \sum_{\substack{x \in \mathcal{B} \\ s_p=0}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s). \end{aligned} \quad (6.7)$$

Using (6.2) which gives the corresponding worth of  $v_{\phi_{\tilde{k}_p}, P}$ , we have:

$$\begin{aligned} \sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &= \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) \right) \\ &+ \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in U} D_i(k_s'^*) \right) \\ &- \sum_{\substack{x \in \mathcal{B} \\ s_p=0}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=1}} D_i(\tilde{k}_p) \right). \end{aligned} \quad (6.8)$$

Let us focus on the last part of the right hand-side of equation (6.8). In other words, we focus on the worth of participation profiles in which the polluter is inactive. It holds that

$$- \sum_{\substack{x \in \mathcal{B} \\ s_p=0}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=1}} D_i(\tilde{k}_p) \right) = - \sum_{i \in U} D_i(\tilde{k}_p) \left( \sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) \right).$$

Using Lemma 6.3.1 we obtain:

$$\begin{aligned} - \sum_{i \in U} D_i(\tilde{k}_p) \left( \sum_{\substack{x \in \mathcal{B} \\ s_i=1 \\ s_p=0}} \mu(s) \right) &= - \sum_{i \in U} D_i(\tilde{k}_p) \left( \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_i=0 \\ s_p=k}} \mu(s) \right) \\ &= - \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right). \end{aligned} \quad (6.9)$$

Therefore, by Lemma 6.3.1 we can replace the sum of balancing weights for participation profiles in which a victim is active while the polluter is inactive by the sum of balancing weights for participation profiles  $x \in \mathcal{B}$  in which the victim is inactive

while the polluter is active. By replacing (6.9) in (6.8) we obtain:

$$\begin{aligned}
\sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &= \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) \right) \\
&\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in U} D_i(k_s'^*) \right) \\
&\quad - \sum_{k=1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right) \\
&= \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) \right) \\
&\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in U} D_i(k_s'^*) \right) \\
&\quad - \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right) - \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k'}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right).
\end{aligned} \tag{6.10}$$

Notice that for each  $x \in \mathcal{B}$  such that  $s_p = k_p \leq \tilde{k}_p$ , the optimal level negotiated by agents given the participation profile  $x$  must be lower than or equal to the regulated level:  $k_s^* \leq \tilde{k}_p$ , where  $k_s^* \in K_s^*$ . Since for each  $i \in U$ ,  $D_i$  is a non-decreasing function, we have  $D_i(\tilde{k}_p) \geq D_i(k_s^*)$ . Therefore we obtain:

$$\begin{aligned}
\sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &\leq \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{\substack{i \in U \\ s_i=1}} D_i(k_s^*) \right) \\
&\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in U} D_i(k_s'^*) \right) \\
&\quad - \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(k_s^*) \right) - \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k'}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right).
\end{aligned} \tag{6.11}$$

Observe that,

$$- \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k'}} \mu(s) \left( \sum_{\substack{i \in U \\ s_i=0}} D_i(\tilde{k}_p) \right) \leq 0.$$

So we have:

$$\begin{aligned} \sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &\leq \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{\substack{i \in \mathcal{U} \\ s_i=1}} D_i(k_s^*) \right) - \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( \sum_{\substack{i \in \mathcal{U} \\ s_i=0}} D_i(k_s^*) \right) \\ &\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in \mathcal{U}} D_i(k_s'^*) \right). \end{aligned} \quad (6.12)$$

For participation profiles  $x \in \mathcal{B}$  such that  $s_p \leq \tilde{k}_p$ , the corresponding optimal activity level  $k_s^*$  considers only active victims, but we insert this activity level  $k_s^*$  for inactive victims rather than  $\tilde{k}_p$ . Hence, every agents face the activity level  $k_s^*$  decided by the active agents in  $x \in \mathcal{B}$ . Therefore we obtain:

$$\begin{aligned} \sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &\leq \sum_{k=1}^{\tilde{k}_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \left( B_p(k_s^*) - \sum_{i \in \mathcal{U}} D_i(k_s^*) \right) \\ &\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in \mathcal{U}} D_i(k_s'^*) \right). \end{aligned} \quad (6.13)$$

However, the activity level  $k_s^*$  is not necessarily optimal when we consider all agents. Therefore, we have

$$\begin{aligned} \sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) &\leq \sum_{k=1}^{\tilde{k}_p} \max_{h \in [0, k]} \left( B_p(h) - \sum_{i \in \mathcal{U}} D_i(h) \right) \left( \sum_{\substack{x \in \mathcal{B} \\ s_p=k}} \mu(s) \right) \\ &\quad + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p=k' \\ \mathcal{S}(s)=N}} \mu(s) \left( B_p(k_s'^*) - \sum_{i \in \mathcal{U}} D_i(k_s'^*) \right). \end{aligned} \quad (6.14)$$

Recall that, by definition of  $v_{\phi_{\tilde{k}_p}, P}$ , for each  $s \in \mathcal{M}$  such that  $s_p = k_p \leq \tilde{k}_p$ :

$$\max_{h \in [0, k_p]} \left( B_p(h) - \sum_{i \in \mathcal{U}} D_i(h) \right) = v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k_p),$$

and for each  $s \in \mathcal{M}$  such that  $s_p = k'_p > \tilde{k}_p$ , with  $\mathcal{S}(s) = N$ :

$$\left( B_p(k_s'^*) - \sum_{i \in \mathcal{U}} D_i(k_s'^*) \right) = v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k'_p).$$

Therefore, we have:

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) \leq \sum_{k=1}^{\tilde{k}_p} v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k_p) \left( \sum_{\substack{x \in \mathcal{B} \\ s_p = k}} \mu(s) \right) + \sum_{k'=\tilde{k}_p+1}^{m_p} \sum_{\substack{x \in \mathcal{B} \\ s_p = k' \\ \mathcal{S}(s)=N}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k'_p). \quad (6.15)$$

Finally, by definition of the balanced collection, for each  $k \in M_p$ ,  $\sum_{\substack{x \in \mathcal{B} \\ s_p = k}} \mu(s) = 1$ . Then we obtain:

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi_{\tilde{k}_p}, P}(s) \leq \sum_{k=1}^{\tilde{k}_p} v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k_p) + \sum_{k'=\tilde{k}_p+1}^{m_p} v_{\phi_{\tilde{k}_p}, P}(1_{-p}, k'_p).$$

Therefore, by equation (6.3) we conclude that the multi-choice game  $v_{\phi_{\tilde{k}_p}, P}$  is balanced. By Theorem 2.3.1,  $\mathfrak{C}(v_{\phi_{\tilde{k}_p}, P}) \neq \emptyset$ . ■

Proposition 6.3.1 ensures that it is possible to reach a stable agreement whatever the regulated level  $\tilde{k}_p$  imposed by the law on the activity of the polluter, and regardless the social cost problem. According to the definition of the core, even if the law imposes a regulated level on the polluter activity that is lower than the optimal level for the whole society, agents will participate in the negotiation through a participation profile  $(1, \dots, 1, k_p) \in \mathcal{M}$ .

Observe that this result is strongly related to the pessimistic hypothesis regarding the behavior of the remaining agents and the negative externality the polluter generates. Any mapping of rights in  $\tilde{\Phi}$  allows the polluter to generate an activity. By expecting the worst case, victim  $i \in U$  anticipates that the polluter will run its activity alone if it deviates from any agreement reached by a coalition containing  $p$ . The negative externality generated by the polluter typically reduces the incentives for victims to free-ride.

**Proposition 6.3.2.** *A mapping of rights  $\phi \in \Phi$  satisfies core compatibility and no-veto power for a victim if and only if  $\phi = \phi_{m_p} \in \tilde{\Phi}$ .*

*Proof.* By Proposition 6.3.1,  $\phi_{m_p} \in \tilde{\Phi}$  satisfies core compatibility. By definition of  $\phi_{m_p}$  and conditions (C2) and (C3) defining mappings of rights, for any victim  $i \in U$ , for any  $s \in \mathcal{M}$  such that  $\mathcal{S}(s) = N \setminus \{i\}$ ,  $\phi_{m_p}(1_{-\{i,p\}}, 0_i, s_p) = 1$ . Hence,  $\phi_{m_p}$  satisfies no-veto power for a victim. But this is not the case for other mappings of rights in  $\tilde{\Phi}$ . Indeed, for any  $\phi_{\tilde{k}_p} \in \tilde{\Phi} \setminus \{\phi_{m_p}\}$  defined with respect to any regulated level  $\tilde{k}_p < m_p$ , we must have  $\phi_{\tilde{k}_p}(1_{-\{i,p\}}, 0_i, m_p) = 0$ . So,  $\phi_{\tilde{k}_p} \in \tilde{\Phi} \setminus \{\phi_{m_p}\}$  violates no-veto power for a victim.

To show that  $\phi_{m_p}$  is the only mapping of rights satisfying core compatibility and no veto power, consider any mapping of rights  $\phi \in \Phi \setminus \tilde{\Phi}$  that satisfy no-veto power. By (C3), it must be the case that for each victim  $i \in U$ ,  $\phi(1_{-\{i,p\}}, 0_i, s_p) = 1$ , where  $\mathcal{S}(s) = N \setminus \{i\}$ . Combined with (C4), it implies that for each victim  $i \in U$ ,  $\phi(0_{-i}, 1_i) = 0$ . Furthermore, we have  $\phi(0_{-p}, s_p) = 0$ , otherwise  $\phi \in \tilde{\Phi}$ . From this, there exist only two types of mapping of rights for  $\phi \in \Phi \setminus \tilde{\Phi}$ :

- (a) for each  $i \in N$ , for each  $s \in \mathcal{M}$  such that  $\mathcal{S}(s) = N \setminus \{i\}$ ,  $\phi(s_{-i}, 0_i) = 1$  and  $\phi(1_{-p}, 0_p) = 1$ ,

- (b) for each  $i \in N \setminus \{p\}$ , for each  $s \in \mathcal{M}$  such that  $\mathcal{S}(s) = N \setminus \{i\}$ ,  $\phi(s_{-i}, 0_i) = 1$  and  $\phi(1_{-p}, 0_p) = 0$ .

To prove that  $\phi \in \Phi \setminus \tilde{\Phi}$  does not satisfy core compatibility consider the following social cost problem. Let  $N := \{1, 2, p\}$  be the set of agents. For each agent, the action sets are  $M_1 = M_2 = \{0, 1\}$ ,  $M_p = \{0, 1, 2, 3\}$ . The benefit function and damage function for each agent are such that, for each activity level  $k \in M_p$ :

$$B_p(k) = 7\sqrt{k}, \quad D_1(k) = 3k, \quad D_2(k) = 2k.$$

Given the social cost problem  $P \in \mathcal{P}$  and the mapping of rights  $\phi^{(a)}, \phi^{(b)}$ , we construct the multi-choice game  $(N, M, v_{\phi, P})$ . Notice that for both types of mapping of rights we have:  $v_{\phi, P}(1, 1, 0) = 0$ , for each  $k \in M_p$ ,  $v_{\phi, P}(0, 0, k) = 0$  and

$$\begin{aligned} v_{\phi, P}(1, 0, 1) &= 4, & v_{\phi, P}(1, 0, 2) &= 4, & v_{\phi, P}(1, 0, 3) &= 4, \\ v_{\phi, P}(0, 1, 1) &= 5, & v_{\phi, P}(0, 1, 2) &= 5.89, & v_{\phi, P}(0, 1, 3) &= 6.12, \\ v_{\phi, P}(1, 1, 1) &= 2, & v_{\phi, P}(1, 1, 2) &= 2, & v_{\phi, P}(1, 1, 3) &= 2. \end{aligned}$$

Consider the balanced collection

$$\mathcal{B} := \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 2), (0, 1, 2), (1, 0, 3), (0, 1, 3)\},$$

and the balancing weights  $\mu(1, 1, 0) = 3/2$  and for each  $x \in \mathcal{B} \setminus \{(1, 1, 0)\}$ ,  $\mu(s) = 1/2$ . Straightforward computation gives

$$\sum_{x \in \mathcal{B}} \mu(s) v(s) = 19.505 > 6 = \sum_{k=1}^3 v_{\phi, P}(1, 1, k).$$

By Theorem 2.3.1 the multi-choice game constructed from the mapping of rights  $\phi \in \Phi \setminus \tilde{\Phi}$  and the social cost problem  $P \in \mathcal{P}$  has an empty core. This concludes the proof of the proposition.  $\blacksquare$

Though all mappings of rights in  $\tilde{\Phi}$  satisfy core compatibility, only one satisfies no-veto power for a victim. This confirms the result from Gonzalez et al. [2019] that identifies only one possibility to conciliate the efficiency property with the requirement that no victim can veto an agreement reached by the rest of the society. Therefore, the only possibility to mitigate the power of isolated victims while imposing core compatibility is  $\phi_{m_p} \in \tilde{\Phi}$ .

### 6.3.2 Enhancing the role of victims

One can argue that agent  $i \in N$  has power in negotiations since there exists at least one participation profile  $s \in \mathcal{M}$  such that  $\phi(s_{-i}, 0_i) = 0$ . In other words, an agent  $i$  has power if it can veto at least one agreement. While mappings of rights in  $\tilde{\Phi}$  mainly rely on the involvement of the polluter, they may give power to victims in the negotiation only for participation profiles where the polluter decides to implement an activity level beyond the regulated level  $\tilde{k}_p$ .

In order to give more power to victims, we introduce a new family of mappings which assigns the rights to a subset of victims. Notice that conditions (C1)-(C4) do not precise the behavior of a mapping of rights when the polluter becomes active while a subset of victims  $S \subseteq U$  has already received the rights. Active victims may either lose the rights or retain them. From that point, we should accurately describe the behavior of a mapping which assigns the rights to a subset of victims whenever the polluter becomes active.

We introduce the subset of mappings which allows only subsets of victims and the whole society to negotiate. Let  $\emptyset \neq S \subseteq U$  be any group of victims, we define  $\phi_S^u \in \Phi$  as follows :

$$\phi_S^u(s) = \begin{cases} 1 & \text{if } S \subseteq \mathcal{S}(s) \subseteq U \text{ or } \mathcal{S}(s) = N, \\ 0 & \text{otherwise.} \end{cases}$$

Also, we introduce the mapping that gives the rights to the grand coalition only, denoted by  $\phi'^u \in \Phi$ :

$$\phi'^u(s) = \begin{cases} 1 & \text{if } \mathcal{S}(s) = N, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\Phi^u = \{\{\phi_S^u\}_{S \subseteq U} \cup \phi'^u\}$  be the family of mappings which assigns the rights either to victims or to the whole society.

From (6.1), for each a social cost problem  $P \in \mathcal{P}$ , and each  $\phi^u \in \Phi^u$ , the multi-choice game  $v_{\phi^u, P} \in \mathcal{G}_m$  is as follows:

$$v_{\phi^u, P}(s) = \begin{cases} B_p(k_s^*) - \sum_{i \in U} D_i(k_s^*) & \text{if } \mathcal{S}(s) = N, \\ 0 & \text{otherwise.} \end{cases} \quad (6.16)$$

Since the polluter can only negotiate with the whole set of victims, the worth of each participation profile  $s \in \mathcal{M}$  where  $\mathcal{S}(s) \subsetneq N$  is  $v_{\phi_S^u, P}(s) = 0$ . Finally, only participation profiles  $s \in \mathcal{M}$  in which each agent is active,  $\mathcal{S}(s) = N$ , have a non-negative worth  $v_{\phi_S^u, P}(s) \geq 0$ . From this, it is straightforward that each mapping of rights  $\phi^u \in \Phi^u$  satisfies core compatibility. It should be observed that this family of mapping of rights fully captures the idea behind the polluter-pays principle. Indeed, such assignment of rights always considers the polluter liable for the harmful damage. Besides, there is no configuration in which the polluter can be active except by contracting with the whole society and thus by compensating each victim.

By Propositions 6.3.1 and 6.3.2, we are now able to introduce our main result regarding core compatibility.

**Theorem 6.3.1.** *A mapping of rights  $\phi \in \Phi$  satisfies core compatibility if and only if  $\phi \in \tilde{\Phi} \cup \Phi^u$ .*

The intuition of this result is as follows. First, it is clear that any mapping of rights in  $\Phi^u$  satisfies core compatibility. By Proposition 6.3.1 we know that any  $\phi_{\tilde{k}_p} \in \tilde{\Phi}$  is core compatible. Then, we use Proposition 6.3.2 to show that if a mapping of rights  $\phi \in \Phi \setminus \tilde{\Phi}$  is core compatible. This forces  $\phi \in \Phi^u$ . This result aligns with the existing cooperative game literature which has analyzed the Coase theorem. It highlights the impossibility to satisfy both the efficiency and neutrality properties.

*Proof.* According to Proposition 6.3.1 each mapping of rights  $\phi_{\tilde{k}_p} \in \tilde{\Phi}$  satisfies core compatibility. Furthermore it is obvious that any mapping of rights  $\phi^u \in \Phi^u$  satisfies core compatibility. Indeed, as mentioned in section 6.3.2, only participation profiles  $s \in \mathcal{M}$  such that  $\mathcal{S}(s) = N$  may have positive worth. Then, for all balanced collections  $\mathcal{B}$  of elements of  $\mathcal{M} \setminus \{\mathbf{0}\}$ , it always holds that:

$$\sum_{s \in \mathcal{B}} \mu(s) v_{\phi^u, P}(s) \leq \sum_{k_p=1}^{m_p} v_{\phi^u, P}(1_{-p}, k_p).$$



To prove that there is no other mapping of rights satisfying core compatibility consider any mapping of rights  $\phi \in \Phi \setminus \tilde{\Phi}$  that satisfies core compatibility. We have to show that  $\phi \in \Phi^u$ .

First, for each  $s_p \in M_p \setminus \{0\}$ , we have  $\phi(0_{-p}, s_p) = 0$ , otherwise  $\phi \in \tilde{\Phi}$ . By Proposition 6.3.2 the only mapping of rights which satisfies core compatibility and no veto power for a victim is  $\phi_{m_p} \in \tilde{\Phi}$ . Define  $J \subseteq U$  the subset of victims which do not have any veto power as:

$$J = \{j \in U \mid \forall s_p \in M_p \setminus \{0\}, \phi(1_{-\{j,p\}}, 0_j, s_p) = 1\}.$$

Any mapping of rights  $\phi \in \Phi \setminus \tilde{\Phi}$  satisfying core compatibility does not satisfy no veto power for a victim, meaning  $J \neq U$ . Hence, there is at least one victim  $i \in U$  such that  $\phi(1_{-\{i,p\}}, 0_i, m_p) = 0$ . In other words there is at least one victim  $i \in U$  which can veto the highest activity level of the polluter. However it can be the case that there exists an activity level  $s_p \in M_p \setminus \{0, m_p\}$  such that  $\phi(1_{-\{i,p\}}, 0_i, s_p) = 1$ .

In order to analyze such mapping of rights, we consider the following partition of  $U$ . Define  $\tilde{I}$  the set of victims which hold a veto power as:

$$\tilde{I} := \{i \in U \mid \exists s_p^i \in M_p \setminus \{0\} \text{ such that } \forall k_p \geq s_p^i, \phi(1_{-\{i,p\}}, 0_i, k_p) = 0\}.$$

For each  $i \in \tilde{I}$ , let  $\tilde{s}_p^i \in M_p \setminus \{0\}$  be the minimum activity level of the polluter such that  $\phi(1_{-\{i,p\}}, 0_i, \tilde{s}_p^i) = 0$ . Either  $\tilde{s}_p^i = 1$ , then victim  $i$  has the power to veto any agreement for any activity level of the polluter; or  $\tilde{s}_p^i > 1$ , then for each activity level of the polluter  $1 \leq k_p < \tilde{s}_p^i$ ,  $\phi(1_{-\{i,p\}}, 0_i, k_p) = 1$ . Therefore, we decompose  $\tilde{I}$  into two subsets denoted by  $I$  and  $H$  defined by:

$$\begin{aligned} I &= \{i \in \tilde{I} \mid \tilde{s}_p^i > 1\}, \\ H &= \{h \in \tilde{I} \mid \tilde{s}_p^h = 1\}. \end{aligned}$$

$I \subseteq U$  is the subset of victims which can veto the highest participation level of the polluter but not a lower participation level  $k_p < \tilde{s}_p^i \in M_p \setminus \{0\}$ .  $H \subseteq U$  is the subset of victims which can veto any participation level of the polluter. The three subsets  $H, I, J$  allows us to describe every mappings of rights with respect to the veto power of victims.

1. Suppose that for each  $i \in U$ ,  $\phi(1_{-\{i,p\}}, 0_i, m_p) = 0$ , so  $J = \emptyset$ . We have to consider three cases:
  - (a)  $H = U$ , so for each  $i \in U$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $\phi(1_{-\{i,p\}}, 0_i, s_p) = 0$ . Then by monotonicity of  $\phi$  on  $M_{-p}$ , (C2) of the definition of mapping of rights, for any  $S \subset U$  we have  $\phi(1_{-S \cup p}, 0_S, s_p) = 0$ . Therefore for each participation profile  $s \in \mathcal{M}$  in which the polluter is active  $p \in \mathcal{S}(s)$ , we must have  $\phi(s) = 0$ . Therefore,  $\phi = \phi^u \in \Phi^u$ ;
  - (b)  $I \neq \emptyset$  and  $H \neq \emptyset$ , so for some  $i \in U$ , there exists  $s_p^i \in M_p \setminus \{0, m_p\}$  such that  $\phi(1_{-\{i,p\}}, 0_i, s_p^i) = 1$ ;
  - (c)  $I \neq \emptyset$  and  $H = \emptyset$ , so for each  $i \in U$ , there exists  $s_p^i \in M_p \setminus \{0, m_p\}$  such that  $\phi(1_{-\{i,p\}}, 0_i, s_p^i) = 1$ .
2. Suppose that for some  $i \in U$ ,  $\phi(1_{-\{i,p\}}, 0_i, m_p) = 1$ , so  $J \neq \emptyset$ . We have to consider three cases:
  - (d)  $H = U \setminus J$ , so for each  $i \in U \setminus J$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $\phi(1_{-i}, 0_i, s_p) = 0$ ;

- (e)  $I = U \setminus J$ , so for some  $i \in U \setminus J$ , there exists  $s_p^i \in M_p \setminus \{0, m_p\}$  such that  $\phi(1_{-\{i,p\}}, 0_i, s_p^i) = 1$ ;
- (f)  $I \neq \emptyset$  and  $H \neq \emptyset$ , so for some  $i \in U \setminus J$ , there exists  $s_p^i \in M_p \setminus \{0, m_p\}$  such that  $\phi(1_{-\{i,p\}}, 0_i, s_p^i) = 1$

To obtain a contradiction, we have to show that  $\phi$  is not core compatible for cases 1.(b)-2.(f). To this end, we have to consider each type of mapping of rights  $\phi^{1.(b)}$ - $\phi^{2.(f)}$  corresponding to each case. For each type of mapping of rights we have to find an instance  $P \in \mathcal{P}$  such that the core of the associated game is empty. We will consider the social cost problem  $P = (N, M_p, B_p, (D_i)_{i \in U}) \in \mathcal{P}$  defined by:

$$\forall k \in M_p = \{0, 1, 2, 3\}, \quad B_p(k) = 7\sqrt{k}, \quad D_1(k) = 5.5k, \quad D_2(k) = 2k, \\ D_i(k) = 0, \quad i \in U \setminus \{1, 2\}.$$

Also we will focus on participation profiles  $s \in \mathcal{M}$  where at least  $n - 1$  agents are active. Therefore we will consider the balanced collection  $\mathcal{B}$  of elements of  $L$  defined by:

$$\mathcal{B} := \{s \in \mathcal{M} : |\mathcal{S}(s)| = n - 1\},$$

with the balancing weights defined as follows:

$$\mu(s) = \frac{m_p}{n - 1}, \text{ if } \mathcal{S}(s) = U; \text{ and } \mu(s) = \frac{1}{n - 1} \text{ otherwise.}$$

Consider any mapping of rights  $\phi^{1.(b)}$  such that  $H \neq \emptyset$  and  $I \neq \emptyset$ . Assume that  $1 \in H$ ,  $2 \in I$  and for each  $i \in I$ ,  $\tilde{s}_p^i = 2$ . For participation profiles  $s \in \mathcal{M}$ , such that  $|\mathcal{S}(s)| = \{n - 1, n\}$ , we have:

1. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(b)}, P}(1_{-p}, s_p) = 0$ ,
2.  $v_{\phi^{1.(b)}, P}(1_{-p}, 0_p) = 0$ ,
3.  $v_{\phi^{1.(b)}, P}(1_{-\{2,p\}}, 0_2, 1_p) = 1.5$
4. for any  $s_p \in M_p \setminus \{0, 1\}$ ,  $v_{\phi^{1.(b)}, P}(1_{-\{2,p\}}, 0_2, s_p) = 0$ ,
5. for each  $h \in H$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(b)}, P}(1_{-\{h,p\}}, 0_h, s_p) = 0$ ,
6. for each  $i \in I \setminus \{2\}$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(b)}, P}(1_{-\{i,p\}}, 0_i, s_p) = v_{\phi^{1.(b)}, P}(1_{-p}, s_p) = 0$ .

Simple computations give:

$$\sum_{s \in \mathcal{B}} \mu(s) v_{\phi^{1.(b)}, P}(s) = \frac{1.5}{n - 1} > 0 = \sum_{s_p=1}^3 v(1_{-p}, s_p).$$

Therefore, by Theorem 2.3.1 we have  $\mathfrak{C}(v_{\phi^{1.(b)}, P}) = \emptyset$ .

Consider any mapping of rights  $\phi^{1.(c)}$ , then  $I = U$ . Furthermore assume that  $\tilde{s}_p^i = 2$  we have:

1. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(c)}, P}(1_{-p}, s_p) = 0$ ,
2.  $v_{\phi^{1.(c)}, P}(1_{-p}, 0_p) = 0$ ,

3.  $v_{\phi^{1.(c)},P}(1_{-\{2,p\}}, 0_2, 1_p) = 1.5$ ,
4.  $v_{\phi^{1.(c)},P}(1_{-\{1,p\}}, 0_1, 1_p) = 5$ ,
5. for each  $i \in I \setminus \{1, 2\}$ , for each  $s_p \in M_p \setminus \{0\}$ ,

$$v_{\phi^{1.(c)},P}(1_{-\{i,p\}}, 0_i, s_p) = v_{\phi^{1.(c)},P}(1_{-p}, s_p) = 0.$$

By taking the balanced collection  $\mathcal{B}$  as defined above, simple computations give us:

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi^{1.(c)},P}(s) = \frac{6.5}{n-1} > 0 = \sum_{s_p=1}^3 v(1_{-p}, s_p).$$

Again, by Theorem 2.3.1 we have  $\mathfrak{C}(v_{\phi^{1.(c)},P}) = \emptyset$ .

Consider any mapping of rights  $\phi^{2.(d)}$  such that  $J \neq \emptyset$  and  $H = U \setminus J$ . Assume that  $1 \in H$  and  $2 \in J$ . Then we obtain,

1. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(b)},P}(1_{-p}, s_p) = 0$ ,
2.  $v_{\phi^{2.(d)},P}(1_{-p}, 0_p) = 0$ ,
3. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{2.(d)},P}(1_{-\{2,p\}}, 0_2, s_p) = 1.5$ ,
4. for each  $h \in H$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{2.(d)},P}(1_{-\{h,p\}}, 0_h, s_p) = 0$ ,
5. for each  $j \in J \setminus \{2\}$ , for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{2.(d)},P}(1_{-\{j,p\}}, 0_j, s_p) = v_{\phi^{2.(d)},P}(1_{-p}, s_p) = 0$ .

By taking the balanced collection  $\mathcal{B}$  as defined above, simple computations give us:

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi^{2.(d)},P}(s) = \frac{4.5}{n-1} > 0 = \sum_{s_p=1}^3 v(1_{-p}, s_p).$$

Therefore, by Theorem 2.3.1 we have  $\mathfrak{C}(v_{\phi^{2.(d)},P}) = \emptyset$ .

Consider any mapping of rights  $\phi^{2.(e)}$  such that  $J \neq \emptyset$  and  $I = U \setminus J$ . Assume that  $2 \in I$  with  $\tilde{s}_p^i = 2$  and  $1 \in J$ . Then we obtain,

1. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{1.(c)},P}(1_{-p}, s_p) = 0$ ,
2.  $v_{\phi^{2.(e)},P}(1_{-p}, 0_p) = 0$ ,
3.  $v_{\phi^{2.(e)},P}(1_{-\{2,p\}}, 0_2, 1_p) = 1.5$ ,
4. for each  $s_p \in M_p \setminus \{0\}$ ,  $v_{\phi^{2.(e)},P}(1_{-\{1,p\}}, 0_1, s_p) = 5$ ,
5. for each  $i \in I \setminus \{2\}$ , for each  $s_p \in M_p \setminus \{0\}$ ,

$$v_{\phi^{2.(e)},P}(1_{-\{i,p\}}, 0_i, s_p) = v_{\phi^{2.(e)},P}(1_{-p}, s_p) = 0.$$

$$\sum_{x \in \mathcal{B}} \mu(s) v_{\phi^{2.(e)},P}(s) = \frac{16.5}{n-1} > 0 = \sum_{s_p=1}^3 v(1_{-p}, s_p).$$

Therefore, by Theorem 2.3.1 we have  $\mathfrak{C}(v_{\phi^{2.(e)},P}) = \emptyset$ .

Finally, consider any mapping of rights  $\phi^{2.(f)}$  such that  $J \neq \emptyset$ ,  $I \neq \emptyset$  and  $H \neq \emptyset$ . Assume that  $2 \in I$  and  $1 \in H$ . Then we obtain  $v_{\phi^{2.(f)},P} = v_{\phi^{1.(b)},P}$ , therefore by Theorem 2.3.1 we have  $\mathfrak{C}(v_{\phi^{2.(f)},P}) = \emptyset$ .

From 1(a)-2(f), we conclude that if  $\phi \in \Phi \setminus \tilde{\Phi}$  is core compatible then  $\phi \in \Phi^u$ . This concludes the proof.  $\blacksquare$

**Theorem 6.3.2.** *There is no mapping of rights in  $\Phi$  that satisfies Kaldor-Hicks core compatibility and no veto power for a victim.*

*Proof.* Notice that any mapping of rights  $\phi^u \in \Phi^u$  satisfies Kaldor-Hicks core compatibility. But, any of them satisfy no veto power for a victim.

By Proposition 6.3.2, the only mapping of rights satisfying core compatibility and no veto power for a victim is  $\phi_{m_p} \in \tilde{\Phi}$ . By definition of the corresponding game  $v_{\phi_{m_p},P}$  given by (6.2), it holds that:

$$v_{\phi_{m_p},P}(0_{-p}, k_p) = B_p(k_p), \text{ and } v_{\phi_{m_p},P}(1_{-p}, k_p) = B_p(k_s^*) - \sum_{i \in U} D_i(k_s^*), \text{ where } k_p \in M_p.$$

For each  $k_p \in M_p$ , we have  $v_{\phi_{m_p},P}(0_{-p}, k_p) \geq v_{\phi_{m_p},P}(1_{-p}, k_p)$ . However, the core constraints impose for each  $k_p \in M_p$ ,

$$\bar{x}_{p,k_p} \geq v_{\phi_{m_p},P}(0_{-p}, k_p), \text{ and } \sum_{i \in U} \bar{x}_{i,1} + \bar{x}_{p,k_p} = v_{\phi_{m_p},P}(1_{-p}, k_p).$$

We can easily construct a social cost problem  $P \in \mathcal{P}$  such that  $v_{\phi_{m_p},P}(0_{-p}, k_p) > v_{\phi_{m_p},P}(1_{-p}, k_p) \geq 0$ . In this case we have:

$$\forall k_p \in M_p, \bar{x}_{p,k_p} > \sum_{i \in U} \bar{x}_{i,1} + \bar{x}_{p,k_p} \geq 0.$$

This implies that:

$$\sum_{i \in U} \bar{x}_{i,1} < 0.$$

Therefore, the mapping of rights  $\phi_{m_p} \in \tilde{\Phi}$  does not satisfy Kaldor-Hicks core compatibility. This concludes the proof.

According to Theorem 6.3.2, it is not possible to conciliate the efficiency property while imposing full compensation and no veto power for a victim.

## 6.4 Concluding remarks

In this chapter, we have analyzed the legal structure ruling negotiations among a finite set of agents when the activity of one agent is harmful for the rest of the society. Examples include any situation in which agents with conflicting interests aim to negotiate compensations for the damage.

We contribute to the literature devoted to the Coase theorem by deriving a multi-choice cooperative game from a social cost problem and a liability rule (defined by mappings of rights). We identify two specific families of liability rules that satisfy the efficiency property formalized in terms of non-emptiness of the core. We show that

any regulation that imposes a quota on the activity level of the polluter is compatible with the efficiency property, provided that the whole set of potential victims can negotiate an agreement with the polluter. In the same way we show that any mapping that exclusively assigns the rights to a set of victim satisfies the efficiency property. However, we show that it is not possible to conciliate the efficiency property while requiring the full compensation property and no veto power for a victim. Let us mention that our construction of the multi-choice game relies on the assumption that agents form pessimistic expectations regarding the behavior of the remaining agents. Under other assumptions, a multi-choice game associated with the mapping of rights which considers quotas on the polluter's activity level may have an empty core. Indeed, the pessimistic expectation assumption restricts the incentives for isolate victims to benefit from an agreement reached by the rest of the society without contributing.

## Chapter 7

# Conclusion

This thesis has mainly investigated the model of multi-choice cooperative games, and two relevant applications dealing with pollution issues. As argued in Chapter 1, the model of multi-choice games seems to naturally fit the analysis of cooperative institutions in which the agents can cooperate at several intensities. Specifically, it allows to address the problem of evaluating the agents' participation and cooperation intensities. By sticking to the justice mode of cooperation presented in Chapter 1, the approach followed all along the thesis was the axiomatic method.

In summary, Chapter 3 proposed several values for multi-choice games: the multi-choice Shapley value, the multi-choice Equal Division values, and the Egalitarian Shapley values for multi-choice games. Each of these values satisfies a necessary condition to be in the core of a multi-choice games called multi-efficiency. The latter values allow to address the trade-off between marginalism and egalitarianism in the context of multi-choice games. These values are computed as the convex combination of the multi-choice Shapley value and the multi-choice Equal division value. For each value, at least one axiomatic characterization has been provided. Chapter 4 investigated the class of multi-choice games with a priority structure. The multi-choice Priority value has been introduced and characterized by two set of axioms. This value relies on a lexicographic allocation process, which mixes both the linear order of the agent's activity levels and the priority structure. Chapter 5 investigated the problem of transporting hazardous waste. By assuming that this transportation involves a cost on each portion of the network, the question was to allocate this cost among the involved agents. Several axioms have been proposed in this context that have been derived by interpreting environmental regulation regarding the transportation of hazardous waste. Then, an allocation rule has been characterized (the responsibility rule), which is related to the multi-choice Shapley value introduced in Chapter 3. Finally, Chapter 6 has investigated the Coase theorem in the context of multi-choice games. By considering a class of social cost problems in which one polluter interacts with several potential victims. It has identified new distribution of rights that ensure stable negotiations, as fostered by Coase. This chapter also confirms the incompatibility between the efficiency thesis of the Coase theorem (agents should always reach an optimal agreement if the property rights are well-defined and there is no transaction cost) and the neutrality thesis (this optimal agreement is independent of the assignment of rights).

**Discussion and future research** The contributions and the results in this thesis establish the basis for further investigations. First, I want to discuss some possibilities that can be explored by further axiomatic investigations. Second, I would discuss modifications of the model of multi-choice games and different interpretations that can arise from these modifications.

In Chapter 3, we provided further remarks on the Shapley like values for multi-choice games. Precisely, we have provided a comparison of the different extensions of the Shapley value introduced in the literature. As highlighted in this comparison, the main difference between the values comes from the axiom formalizing the concept of equal treatment of equals (see Table 3.1). From an axiomatic perspective, it would be enlightening to investigate a weaker axiom formalizing the concept of equal treatment in multi-choice games, which might be satisfied by each Shapley like value introduced in the literature. On this basis, one could explore the class of solutions satisfying efficiency, linearity, and such weaker condition of equal treatment. By doing so, the objective would be to provide a complete description of the solutions satisfying the combination of these axioms.

In the axiomatic analysis of Chapter 4, we considered two axioms that favor agents with a higher position in the priority structure (Priority relation for the same maximal activity level and Priority relation for decisive agents). Both axioms consider two agents such that one has the priority over the second. Actually, both axioms imply that adding one activity level to an agent with the highest position in the priority structure does not impact the second agent's payoffs. Such requirement can become specifically strong depending on the situation at hand. For example, consider agents involved in a sponsorship system. An agent having the priority over another represents a relation between a sponsor and sponsored. In this case, it can be too strong demanding that an additional activity level for the sponsor agent has no impact on the sponsored agent. Indeed, both agents may be affected by such an additional activity level but at different intensity. Thus, it would be interesting to investigate the implications of relaxing these axioms in the context of multi-choice games with a priority structure.

Regarding the model of multi-choice cooperative games, one possible direction would be to relax the hypothesis that the agents' activity levels are linearly ordered. A first relaxation would be to consider a partial order on the set of activity levels. Such case might be relevant to model situations in which the activity levels represent both qualitative and quantitative data. For example, in Chapter 5 on the transportation of hazardous waste, one can consider that the different activity levels represents different hazardous material generated by the agents. These hazardous materials can be related with respect to their potential for harm. However, two materials can have the same degree of hazard. Such relaxation can also fit the context of multi-choice games with a priority structure introduced in Chapter 4. In this case, this would allow for new combinations of the priority structure and the set of activity levels. Going further in this direction, one can focus on the case where there is no relation between the activity levels. In this case, the notion of an activity level would be closer to the different actions that the agents may have in a non-cooperative setting. While considering such a model, one can let the set of available actions vary. In this way, a solution can be interpreted as the evaluation of the agent's cooperation with respect to its set of actions.

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