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# Une approche par les groupes amples pour l'équivalence orbitale des actions minimales de $\mathbb{Z}$ sur l'espace de Cantor

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# Orbit equivalence of $\mathbb{Z}$ actions on the Cantor space : An approach by ample groups



**Simon ROBERT**  
Thèse de doctorat







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# Une approche par les groupes amples pour l'équivalence orbitale des actions minimales de $\mathbb{Z}$ sur l'espace de Cantor

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## Une approche par les groupes amples pour l'équivalence orbitale des actions minimales de $\mathbb{Z}$ sur l'espace de Cantor

**Résumé :** Cette thèse s'inscrit dans le cadre de la dynamique topologique, branche des systèmes dynamiques s'intéressant aux comportements qualitatifs asymptotiques de transformations continues provenant d'une action de groupe ou de semi-groupe sur un espace métrique usuellement compact. Par exemple, une question classique pourrait être de savoir si tel système dynamique admet des points récurrents, c'est à dire des points qui vont revenir arbitrairement proche de leur point de départ infiniment souvent sous la dynamique. Souvent, de par leur caractère qualitatif et asymptotique, ces propriétés ne dépendent pas précisément du système mais plutôt des *orbites* des points, i.e des positions qu'il vont atteindre. D'où la notion d'*équivalence orbitale* au cœur de cette thèse, qui consiste à considérer que, après identification des espaces sous-jacents, deux systèmes dont tous les points auraient les mêmes orbites seraient "qualitativement les mêmes".

Au cours des années 90, Giordano Putnam et Skau ont réussi à établir grâce à des outils d'algèbre homologique une classification à équivalence orbitale près des systèmes dynamiques minimaux provenant d'actions de  $\mathbb{Z}$  sur l'espace de Cantor en termes à la fois de groupes pleins et de mesures invariantes. Ce résultat montre en particulier qu'il existe une infinité non-dénombrable de tels systèmes différents à équivalence orbitale près, ce qui contraste assez fortement avec le cadre de la théorie ergodique, domaine très proche s'intéressant aux systèmes dynamiques mesurés, dans lequel la combinaison de deux célèbres résultats, l'un dû à Ornstein et Weiss et l'autre à Dye montre qu'il n'y a à équivalence orbitale près qu'une seule action de groupe moyennable sur un espace de probabilité standard.

Ma principale contribution à travers le présent manuscrit consiste à apporter un éclairage et des preuves dynamiques élémentaires aux classifications obtenues par Giordano, Putnam et Skau (celle sur l'équivalence orbitale susmentionnée ainsi qu'une autre traitant d'une variation nommée équivalence orbitale forte), tant afin de les comprendre sous une autre perspective que pour tenter de les étendre à d'autres contextes. Chemin faisant, je démontrerai également un résultat de complexité Borélienne, à savoir que la relation d'isomorphisme de groupes dénombrables, localement finis et simples et une relation universelle provenant d'une action Borélienne de  $S_\infty$ , et nous améliorerons un résultat de Krieger sur la conjugaison des groupes amples.

**Mots-clés :** Dynamique topologique, Équivalence Orbitale, Espace de Cantor, Groupes amples, Partitions de Kakutani-Rokhlin, Complexité Borélienne de la relation d'isomorphisme de groupes dénombrables localement finis et simples.

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## Orbit equivalence of minimal $\mathbb{Z}$ actions on the Cantor space : An approach by ample groups

**Abstract:** This thesis takes place in the context of topological dynamics, a branch of dynamical systems concerned with the asymptotic qualitative behavior of continuous transformations arising from a group or semigroup action on a usually compact metric space. For example, a classic question might be whether a dynamical system admits recurrent points, i.e. points that will return arbitrarily close to their starting point infinitely often under the dynamics. Often, because of their qualitative and asymptotic nature, these properties do not depend precisely on the system but rather on the *orbits* of the points, i.e. the positions they will reach. Hence the notion of orbit equivalence at the heart of this thesis, which consists in considering that, after identification of the underlying spaces, two systems whose points all have the same orbits would be "qualitatively the same".

In the 1990s, Giordano Putnam and Skau used homological algebra to establish a classification up to orbit equivalence of minimal dynamical systems arising from  $\mathbb{Z}$  actions on a Cantor space in terms of both full groups and invariant measures. This result shows in particular that there are non-countably many such different systems up to orbit equivalence, which contrasts quite strongly with the framework of ergodic theory, a very close field concerned with measured dynamical systems, in which the combination of two famous results, one due to Ornstein and Weiss and the other to Dye, shows that there is only one amenable group action on a standard probability space up to orbit equivalence.

My main contribution in the present manuscript is to bring an elementary perspective and dynamical proofs to the classifications obtained by Giordano, Putnam and Skau (the one on orbital equivalence mentioned above as well as another one dealing with a variation called strong orbital equivalence), both in order to understand them from another perspective and to try to extend them to other contexts. Along the way, I will also prove a result of Borelian complexity, namely that the isomorphism relation of countable, locally finite and simple groups and a universal relation arising from a Borelian action of  $S_\infty$ , and improve a result of Krieger about the conjugation of ample groups.

**Keywords:** Topological dynamics, Orbit equivalence, Cantor space, Ample groups, Kakutani-Rokhlin partitions, Borelian complexity of the isomorphism relation between countable locally finite simple groups.

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# Introduction en français

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## 1.1 Un bref détour historique

### 1.1.1 Origines de la dynamique moderne

Lorsqu'on entend dynamique, on pense très certainement en premier lieu aux équations différentielles. Et pour cause, les origines de la dynamique telle qu'elle se pratique actuellement peuvent être tracées jusqu'au travail de Poincaré en relation avec les problèmes de mécanique céleste, en particulier le célèbre problème à trois corps (pour une approche historique de la question, voir [BG]). Ce problème, qui posait de nouveaux défis, a encouragé le développement de nouveaux outils théoriques, et mené à la recherche d'informations dites *qualitatives* sur le comportement des solutions, comme le profil de comportement en temps long (est-ce qu'une solution s'éloigne vers l'infini, ou repasse arbitrairement proche d'un point donné..?). C'est à partir de là que la dynamique a commencé à se développer en tant que domaine à part entière, se détachant peu à peu des problèmes physiques sous-jacents pour devenir plus fondamentale, plus abstraite. Essayons de voir naître la dynamique

moderne en symboles : dans l'étude d'un système  $X$  évoluant au cours du temps sous l'impulsion d'une équation différentielle, on cherche à connaître la position  $\varphi(t, x)$  d'un point  $x$  au temps  $t$ . Un changement important de perspective consiste à découpler les variables de temps et d'espace, pour s'intéresser en particulier aux *orbites* des points : étant donné un point  $x \in X$ , on s'intéressera alors aux positions que va atteindre ce point particulier au cours du temps, les  $(\varphi(t, x))_{t \in \mathbb{R}}$ . D'un autre côté, en fixant un temps  $t$ , on peut définir l'application  $\varphi^t$  par  $\varphi^t(x) = \varphi(t, x)$  pour tout point  $x$  de l'espace, qui correspond à "faire avancer le système d'un pas de temps  $t$  fixé". En supposant que l'évolution est indépendante du temps (et définie), on obtient alors l'égalité suivante

$$\forall t, t' \quad \varphi^{t+t'} = \varphi^t \circ \varphi^{t'}.$$

Vue ainsi, la famille  $(\varphi^t)_{t \in \mathbb{R}}$  devient alors un groupe de transformations de  $X$  à un paramètre. Et c'est là le point de vue moderne adoptée par la dynamique : un *système dynamique* consiste désormais en l'action d'un groupe  $G$  sur un espace des phases  $X$ . En fait, cette notation est un peu abusive, car l'espace des phases porte le plus souvent une structure, le plus fréquemment (on se restreindra à ces cas-là) d'espace topologique ou d'espace mesurable, voire mesuré, et l'on note alors  $X$  pour désigner l'espace et sa structure, en lieu et place des plus formels  $(X, \tau)$  ou  $(X, \mathcal{B})$ . Dans ce cas, il sera naturel de demander que l'action du groupe préserve la structure. On voit alors se dessiner les contours de plusieurs dynamiques, certaines intrinsèquement liées, en fonction de la structure à préserver. Lorsque l'espace  $X$  est une variété différentielle, et que l'action du groupe s'effectue par difféomorphismes plus ou moins réguliers, on parle de *dynamique différentielle* ; lorsque l'espace des phases est un espace de probabilité standard (éventuellement muni d'une mesure) et que les transformations sont bi-mesurables (et le cas échéant préservent la mesure), on parle de *dynamique mesurée* ou encore de *théorie ergodique* ; enfin, *la dynamique topologique*, qui est le cadre de cette thèse, traite d'actions par homéomorphismes sur un espace topologique compact.

En plus de cette diversité de structures étudiée, la dynamique moderne s'est également tournée vers une diversité de groupes, voire de semi-groupes, opérant sur ces espaces. En effet, si l'heuristique de la démarche qualitative est d'obtenir des informations sur le comportement du système "en temps long", il n'est nul besoin que le groupe agissant soit  $\mathbb{R}$ . En effet, en fixant une transformation  $\varphi$  (inversible pour nous, mais ce n'est pas systématiquement le cas), on peut étudier les itérations  $\varphi^n(x)$  pour  $n$  un grand entier. Dans cette thèse, nous nous intéresserons exclusivement à des actions discrètes, c'est à dire au cas où le groupe  $G$  porte la topologie discrète. Notre principale préoccupation concernera les actions du groupe  $\mathbb{Z}$  des entiers relatifs, choix que nous expliquerons un peu plus loin. Plus précisément, nous chercherons à comprendre à quelles conditions deux actions de  $\mathbb{Z}$  sont *orbitalement équivalentes*, au sens de la définition ci-dessous :

*Définition 1.1.* Deux actions  $\alpha_1$  et  $\alpha_2$  de  $\mathbb{Z}$  sur des compacts  $X_1$  et  $X_2$  respectivement sont dites *orbitalement équivalentes* s'il existe un homéomorphisme  $g: X_1 \rightarrow X_2$

tel que

$$\forall x \in X_1 \text{ Orb}_{\alpha_2}(g(x)) = g(\text{Orb}_{\alpha_1}(x)).$$

On appelle alors  $g$  une *équivalence orbitale* entre  $\alpha_1$  et  $\alpha_2$ .

En des termes moins formels, on considère que seules les *orbites* des points, c'est à dire les positions qu'ils vont atteindre, sont importantes, et non pas le "temps" qu'il leur a fallu pour les atteindre. Ainsi, deux actions dont on pourrait identifier les orbites (c'est exactement ce que fait une équivalence orbitale) seront considérées comme "les mêmes".

### 1.1.2 L'espace de Cantor et minimalité

Nous avons ainsi fixé le contexte des systèmes dynamiques topologiques : pour un groupe discret  $G$ , l'objet d'étude (souvent appelé  $G$ -flot) sera la donnée d'une action de  $G$  sur un espace topologique  $X$ , habituellement notée  $G \overset{\alpha}{\curvearrowright} X$ , ou simplement  $G \curvearrowright X$  lorsqu'il n'est pas nécessaire de préciser l'action. En ce qui nous concerne, nous nous concentrerons exclusivement sur des actions de  $\mathbb{Z}$ , et nous parlerons plutôt du système  $(\varphi, X)$  pour désigner l'action  $\mathbb{Z} \overset{\alpha}{\curvearrowright} X$  définie par  $\alpha(1) = \varphi$ . Dans ce paragraphe, nous discuterons du choix de l'espace des phases  $X$ . Nous avons vu qu'originellement, les espaces étudiés, qui venaient de la physique, étaient des espaces lisses. Ce sera tout sauf le cas ici, puisque nous nous intéressons presque exclusivement à des actions sur l'espace de Cantor  $2^{\mathbb{N}}$  (où  $2 = \{0; 1\}$ , notation classique pour les logiciens), muni de la topologie produit. En d'autres termes, les points de l'espace de Cantor sont des suites constituées de 0 et de 1, indexées par  $\mathbb{N}$ , et deux points sont "proches" s'ils coïncident sur "beaucoup" de digits. Une base de la topologie est donnée par la famille des  $U_s = \{x \in 2^{\mathbb{N}} : s \text{ est un préfixe de } x\}$ , où  $s \in 2^{<\mathbb{N}}$  parcourt l'ensemble des suites finies à valeurs dans  $\{0; 1\}$ . En remarquant que, pour  $s$  appartenant à  $2^n$ , le complémentaire de  $U_s$  est l'union des  $U_v$  pour  $v$  parcourant  $2^n$  excepté  $s$ , on se rend compte que cette base est constituée d'ouvert-fermés. Il est donc temps d'introduire le terme qui sera sans doute le plus utilisé de ce manuscrit : en référence à la contraction anglaise de *closed-open*, nous écrirons *clopen* en lieu et place d'ouvert-fermé. Notons que l'espace de Cantor est un espace compact<sup>12</sup>, métrisable (on peut poser par exemple  $d(x, y) = \frac{1}{2^n}$  si  $n$  est le premier digit auquel les suites  $x$  et  $y$  diffèrent), parfait (ce qui signifie sans point isolé) et totalement discontinu (terminologie étrange qui signifie : avec une base de la topologie constituée de clopens. Une meilleure traduction de l'anglais "totally disconnected" pourrait être "totalement déconnecté"). Un théorème de Brouwer dans [B2] garantit que tout espace (non-vide) vérifiant ces propriétés est homéomorphe à l'espace de Cantor, et tout tel espace sera appelé un espace de

<sup>1</sup>En bons français, les espaces compacts sont séparés !

<sup>2</sup>Dans cette thèse, nous utiliserons sans vergogne l'axiome du choix, en particulier sous sa version "tout produit de compacts est compact", bien que, l'espace de Cantor étant métrisable, la version la plus générale de celui-ci ne soit pas nécessaire. Nous laissons au lecteur pointilleux le loisir de déterminer précisément quelle est la version minimale.

Cantor. En toute rigueur,  $2^{\mathbb{N}}$  est lui-même seulement *un* espace de Cantor que nous affublons d'un article défini en reconnaissance de son statut d'exemple canonique.

Mais alors pourquoi diantre travailler sur un espace aussi biscornu, si loin des espaces lisses si familiers ? L'une des raisons est que la connexité est un facteur de rigidité. Par exemple, plus l'espace est connexe, plus contraintes seront les applications continues sur cet espace. Ainsi, choisir un espace totalement discontinu  $X$  permet une grande richesse de l'espace de ses homéomorphismes  $\text{Homeo}(X)$ , dont on pourra observer les comportements locaux indépendamment. En guise d'illustration, si  $X = A \sqcup B$  où  $A$  et  $B$  sont clopen, alors les combinaisons d'un homéomorphisme de  $A$  et d'un homéomorphisme de  $B$  se recollent toujours en un homéomorphisme de  $X$  : on peut choisir n'importe quel comportement local sur  $A$  et  $B$ , il sera réalisé par un homéomorphisme de  $X$ . Cette liberté (ou cette rigidité) se voit en particulier si l'on s'intéresse aux sous-groupes d'ordre fini de  $\text{Homeo}(X)$ . Pour  $X = [0; 1]$ , les seuls éléments d'ordre 2 envoient 0 sur 1 et 1 sur 0 (le comportement des points extrémaux est contraint) par exemple, car l'injectivité et la continuité imposent la monotonie stricte, ce qui est une condition très forte. De plus il n'y a aucun élément d'ordre  $n$  pour  $n \geq 3$ . À l'inverse, si  $X$  est un espace de Cantor, il existe énormément de sous-groupes d'ordre 2 (et plus généralement d'ordre fini) et ceci sera d'ailleurs la base de notre approche, dans laquelle on tentera d'approcher un homéomorphisme quelconque par des homéomorphismes d'ordre fini. Une dernière raison (mais pas des moindres !) pour laquelle nous délaierons les espaces connexes, et qui illustre bien leur rigidité, est due à Boyle et Tomiyama dans [BT]. L'un des théorèmes principaux de cet article (Théorème 3.6) montre en particulier que dans le cas d'espaces connexes, la notion d'équivalence orbitale pour des actions (topologiquement libres) de  $\mathbb{Z}$  se réduit à la notion de *flip-conjugaison*, bien plus restrictive. Vu ainsi, les raisons de travailler sur des espaces totalement discontinus s'éclaircissent. Parmi eux, l'espace de Cantor est un candidat "minimal" naturel en tant qu'espace métrisable, on peut par exemple citer le corollaire suivant du théorème de Cantor-Bendixson (voir [G], Théorèmes 1.3.5 et 1.3.6) : Il y a une injection continue de l'espace de Cantor dans tout espace Polonais non-dénombrable. Ainsi la dynamique sur l'espace de Cantor est devenue le sujet de nombreux articles ; pour approfondir le sujet on pourra consulter [A],[AGW],[E2],[E1], [M1],[M2], [M3]), ou encore [GW] et [GPS1],[GPS2],[GMPS2],[GMPS1], dont les contributions sont centrales dans notre travail.

### 1.1.3 Pourquoi des actions du groupe des entiers ?

Il reste l'un des axes de cette thèse à éclaircir : y a-t-il des choses intéressantes à dire sur les actions de  $\mathbb{Z}$  sur l'espace de Cantor ? Après tout, il n'est guère de groupe infini plus simple que  $\mathbb{Z}$ ... En fait, c'est une remarque tout à fait pertinente,  $\mathbb{Z}$  étant un groupe bien compris et très simple (au sens engendré par un seul générateur), il existe des méthodes particulières à l'étude de ses actions (notamment les partitions de Kakutani-Rokhlin, dont nous reparlerons, qui sont une version simplifiée des partitions de Følner). Et pourtant, ses actions sur l'espace de Cantor

sont nombreuses, et les comprendre est déjà notoirement difficile. Dans [GPS1] et [GPS2], Giordano, Putnam et Skau sont parvenus à donner une caractérisation de l'équivalence orbitale de deux telle actions :

**Théorème 1.1** ([GPS2], Corollaire 4.6). *Soient  $(\varphi_1, X_1)$  et  $(\varphi_2, X_2)$  des systèmes dynamiques minimaux sur l'espace de Cantor. Alors les conditions suivantes sont équivalentes :*

- (i) *Les deux systèmes  $(\varphi_1, X_1)$  et  $(\varphi_2, X_2)$  sont orbitalement équivalents*
- (ii) *Il existe un homéomorphisme  $g: X_1 \rightarrow X_2$  qui envoie les mesures (de probabilité) invariantes de  $\varphi_1$  sur celles de  $\varphi_2$ , i.e  $g_*M(\varphi_1) = M(\varphi_2)$ , où*

$$M(\varphi_i) = \{ \mu \text{ mesure de probabilité sur } X_i : \varphi_{i*} \mu = \mu \}$$

Ce résultat, qui sera au coeur de cette thèse (une troisième condition équivalente, qui s'énonce en termes de groupes pleins, sera donnée plus tard -cf Section 3.2.1 du Chapitre 3), montre en particulier qu'il existe une infinité de systèmes dynamiques topologiques deux à deux non orbitalement équivalents<sup>3</sup>.

Cette situation contraste assez fortement avec le cadre mesuré : en théorie ergodique, les résultats suivants ont réglé depuis longtemps le cas des actions de  $\mathbb{Z}$ , et même de celles de tous les groupes moyennables :

**Théorème 1.2** ([D2]). *Il n'y a, à équivalence orbitale près, qu'une seule action ergodique préservant une mesure de probabilités de  $\mathbb{Z}$  sur un espace de probabilité standard.*

**Théorème 1.3** ([OW]). *Toute action ergodique, préservant une mesure de probabilité d'un groupe moyennable sur un espace de probabilité standard est orbitalement équivalente à une action de  $\mathbb{Z}$ .*

Ainsi, en dynamique mesurée, les actions d'une large variété de groupes sont donc depuis longtemps déjà comprise à équivalence orbitale près : il n'y en a qu'une seule !

Une différence importante pouvant expliquer cela est, dit grossièrement, que la dynamique topologique prend en compte les points, alors que dans le cadre mesuré, on ne travaille qu'à mesure nulle près, et des comportements étranges comme recoller deux orbites tout en laissant les autres inchangées, qui arrivent en dynamique topologique (cf section 1.2.4.3) et sont source de complexité, restent invisibles en théorie ergodique.

Depuis les années 90, des progrès ont été faits dans la compréhension des actions minimales de groupes discrets, en utilisant des outils élaborés d'algèbre homologique, mais une caractérisation pour tout les groupes moyennables ne semble pas encore à l'ordre du jour. À ce sujet on pourra consulter l'article [GMPS2],

<sup>3</sup>En effet, dans [D1], Downarowicz montre que l'on peut construire tout simplexe de Choquet compact métrisable comme l'espace de mesures invariantes d'un certain système dynamique minimal, et il existe une infinité non dénombrable de tels simplexes non homéomorphes deux à deux.



qui montre en particulier que toute action libre minimale de  $\mathbb{Z}^d$  sur l'espace de Cantor est orbitalement équivalente à une action de  $\mathbb{Z}$ .

L'un des objectifs de cette thèse est d'aborder le cas des actions de  $\mathbb{Z}$  sous un angle différent, plus dynamique, en espérant qu'une telle approche puisse ouvrir de nouveaux horizons pour la compréhension d'actions de groupes plus complexes.

## 1.2 Préliminaires de dynamique topologique sur l'espace de Cantor

Cette section se veut une entrée en matière abordable pour quiconque souhaite commencer à étudier la dynamique topologique. Les seuls pré-requis sont les bases de topologie générale (quoique l'espace de Cantor étant métrisable, quasiment tout puisse être fait avec le cadre métrique en tête, il est simplement plus facile parfois de donner les arguments généraux). Cette section est fortement inspirée de [S], qui a été pour moi une introduction très appréciable au domaine. Seules certaines preuves seront incluses, pour ne pas surcharger cette introduction. De plus, les lecteur.ices qui souhaiteraient une introduction à la dynamique topologique plus en lien avec la logique, qui présente notamment la notion de flot minimal universel (que nous n'aborderons pas ici), pourront consulter [dV] (en particulier le chapitre IV).

**À partir de maintenant, sauf mention contraire explicite,  $X$  dénote un espace de Cantor,  $\varphi$  se réfère à un homéomorphisme de  $X$ , et  $(\varphi, X)$  est le système dynamique faisant référence à l'action de  $\mathbb{Z}$  donnée par  $1.x = \varphi(x)$  pour tout  $x$  dans  $X$ . On se permettra même de parler uniquement de  $\varphi$  pour désigner ce système. On écrira  $CO(X)$  pour désigner l'algèbre Booléenne des clopens de  $X$ . De plus, l'ensemble des entiers  $\mathbb{N}$  commence par 0.**

### 1.2.1 Objets fondamentaux et leurs propriétés

Commençons par introduire un peu de vocabulaire :

*Définition 1.2.* Une partie  $P$  de  $X$  est dite *invariante* par un homéomorphisme  $\varphi$  si  $\varphi(P) = P$ .

Un *sous-système* du système  $(\varphi, X)$  consiste en la restriction  $(\varphi, F)$  de  $\varphi$  à un fermé invariant  $F \subset X$ .

Un homéomorphisme  $\varphi$  est dit

- *périodique* si toutes ses orbites sont finies,
- *apériodique* si toutes ses orbites sont infinies,
- *minimal* si l'orbite de chaque point est dense :  $\forall x \in X, \overline{\text{Orb}_\varphi(x)} = X$
- *topologiquement transitif* s'il existe une orbite dense. De tels systèmes n'interviendront pas ici, mais ils sont fréquents dans d'autres contextes.

On appellera *période* d'une orbite finie son cardinal, et dans le cas où toutes les orbites sont de période  $n$ , on dira que  $\varphi$  est de période  $n$  (dans ce cas on a  $\varphi^n = \text{Id}$ , et  $\varphi$  est d'ordre  $n$  dans le groupe  $(\text{Homeo}(X), \circ)$ ).

Notre principale préoccupation concerne les systèmes minimaux. Notons qu'un système minimal est automatiquement apériodique. Nous dirons parfois que certaines propriétés du système sont vérifiées du fait de sa minimalité, alors qu'en fait l'apériodicité suffirait. Les systèmes périodiques interviendront quant à eux assez naturellement, car l'une des stratégies employées consistera précisément à approcher un homéomorphisme par une suite d'homéomorphismes périodiques.

La minimalité admet plusieurs caractérisations très pratiques. Le point ii) ci-dessous garantit en particulier que le "temps de premier retour" qui apparaîtra dans la construction des partitions de Kakutani-Rokhlin (Section 1.2.3) est bien défini.

**Proposition 1.1.** *Soit  $\varphi \in \text{Homeo}(X)$ . Les conditions suivantes sont équivalentes :*

i)  $\varphi$  est minimal

ii) toute orbite positive de  $\varphi$  est dense :  $\forall x \in X \overline{\{\varphi^n(x)\}_{n \in \mathbb{N}}} = X$

iii)  $\varphi$  est minimal au sens des sous-systèmes : les seuls sous-espaces fermés  $\varphi$ -invariants sont  $\emptyset$  et  $X$ .

iv) Pour tout clopen non-vide  $U$ , il existe  $N \in \mathbb{N}$  tel que  $X = \bigcup_{k=0}^N \varphi^k(U)$

Notons que la condition iii) ci-dessus, couplée au Lemme de Zorn appliqué à la famille des sous-systèmes, permet de voir les systèmes minimaux comme les plus petits des systèmes dynamiques topologiques, au sens où tout système  $(\varphi, X)$  contiendra automatiquement au moins un sous-système minimal (et ce même si  $X$  est un compact quelconque plutôt qu'un espace de Cantor). En revanche, à la différence du cadre mesuré, un système dynamique topologique ne se décomposera pas nécessairement en sous-systèmes minimaux : on peut par exemple imaginer une dynamique qui s'enroule en spirale autour d'un point fixe, auquel cas le seul sous-système minimal sera la restriction de l'action à ce point fixe.

La notion principale que nous étudions est l'équivalence orbitale, dont nous rappelons la définition 1.1 donnée section précédente sous une forme légèrement différente :

**Définition 1.3.** Deux systèmes dynamiques  $(\varphi, X_1)$  et  $(\psi, X_2)$  sur l'espace de Cantor sont dits *orbitalement équivalents* s'il existe un homéomorphisme  $g: X_1 \rightarrow X_2$  et des applications  $n_1: X_1 \rightarrow \mathbb{Z}$  et  $n_2: X_2 \rightarrow \mathbb{Z}$  tels que

$$\forall x \in X_1 \quad \psi(g(x)) = g(\varphi^{n_1(x)}(x))$$

et

$$\forall x \in X_2 \quad \varphi(g^{-1}(x)) = g^{-1}(\psi^{n_2(x)}(x)).$$

On appelle alors  $g$  une *équivalence orbitale* entre  $\varphi$  et  $\psi$ . De plus, les applications  $n_1$  et  $n_2$  sont appelés *des cocycles associés à  $g$*  (les cocycles sont uniquement définis dans le cas d'une action libre).

Remarquons que le cas  $n_1 \equiv 1$  (et alors automatiquement  $n_2 \equiv 1$ ) correspond au fait que  $\varphi$  et  $\psi$  soient conjugués, et que par conséquent la notion d'équivalence orbitale est une relation d'équivalence moins fine que l'isomorphisme.

Une notion légèrement plus contraignante qui pourra nous intéresser, notamment au Chapitre 2, est la notion d'*équivalence orbitale forte* : une équivalence orbitale  $g$  sera dite forte s'il existe des cocycles admettant au plus un point de discontinuité.

Elle peut être vue à travers un groupe très important dans l'étude des systèmes dynamiques topologiques :

*Définition 1.4.* Le *groupe plein de  $\varphi$* , noté  $[\varphi]$ , est le sous-groupe de  $\text{Homeo}(X)$  défini par

$$[\varphi] = \left\{ g \in \text{Homeo}(X) : \forall x \in X, \exists n_g(x) \in \mathbb{Z}, g(x) = \varphi^{n_g(x)}(x) \right\}.$$

Pour  $g \in [\varphi]$ , une telle fonction  $n_g : X \rightarrow \mathbb{Z}$  est appelé *un cocycle de  $g$* . Notons que pour un système apériodique (et donc à fortiori pour un système minimal), chaque élément du groupe plein admet un unique cocycle.

Le *groupe plein topologique de  $\varphi$* , noté  $[[\varphi]]$ , est le sous-groupe de  $[\varphi]$  constitué des homéomorphismes admettant un cocycle continu (pour la topologie discrète sur  $\mathbb{Z}$ ).

Comme précisé dans la section Background and terminology (voir 3.2.1) du Chapitre 3, il existe une lecture naturelle de l'équivalence orbitale en termes de groupes pleins : deux systèmes  $\varphi$  et  $\psi$  sont orbitalement équivalents si et seulement si les groupes pleins  $[\varphi]$  et  $[\psi]$  sont conjugués. De plus, le deuxième item du Théorème 1.1 admet une lecture en termes d'adhérence du groupe plein (voir la discussion après le Théorème de Krieger dans l'introduction du Chapitre 3), et ces objets sont donc particulièrement important dans notre travail.

Le groupe plein topologique est caractérisé par la propriété suivante, qui le rend particulièrement facile à se représenter : il est constitué des homéomorphismes qui sont localement une puissance de  $\varphi$ .

**Proposition 1.2.** *Soit  $\varphi$  un homéomorphisme quelconque. Un homéomorphisme  $g$  appartient à  $[[\varphi]]$  si et seulement si il existe des clopens  $A_1, \dots, A_n$  et des entiers relatifs  $k_1, \dots, k_n$  tels que  $X = \bigsqcup_{i=1}^n A_i$  et  $g \upharpoonright_{A_i} = \varphi^{k_i} \upharpoonright_{A_i}$  pour tout  $i \in \llbracket 1, n \rrbracket$ .*

*Preuve.*

Soit  $g \in [[\varphi]]$  et  $n_g$  un cocycle continu. L'image continue d'un compact étant compacte,  $n_g(X)$  est un compact de  $\mathbb{Z}$ , donc fini :  $n_g(X) = \{k_i\}_{i \in \llbracket 1, n \rrbracket}$ . Il suffit alors de poser  $A_i = n_g^{-1}(\{k_i\})$ . Réciproquement, le cocycle  $n_g$  défini par  $n_g \upharpoonright_{A_i} = k_i$  pour tout  $i$  est continu, puisque les  $A_i$  sont clopen.  $\square$

*Définition 1.5.* Le *support d'un homéomorphisme  $\varphi \in \text{Homeo}(X)$*  est défini par

$$\text{supp}(\varphi) = \overline{\{x \in X : \varphi(x) \neq x\}}.$$

C'est le plus petit fermé contenant tous les points non fixés par  $\varphi$ .

En général le support d'un homéomorphisme n'est pas ouvert, par exemple n'importe quel  $f: 3^{\mathbb{N}} \rightarrow 3^{\mathbb{N}}$  qui induit les permutations  $0^n 1 \leftrightarrow 0^n 0$  pour tout  $n \in \mathbb{N}^*$  et est l'identité ailleurs aura un support non-ouvert. En effet pour tout  $n \in \mathbb{N}$ ,  $0^n 2$  est un point fixe de  $f$  mais la limite  $0^\infty$  de la suite appartient au support, ce qui montre que le complémentaire de celui-ci n'est pas fermé.

Cependant, ce sera le cas pour tout élément du groupe plein topologique d'un homéomorphisme minimal :

**Proposition 1.3.** *Soit  $\varphi$  un homéomorphisme minimal sur  $X$ . Alors pour tout  $g \in \llbracket \varphi \rrbracket$ ,  $\text{supp}(g)$  est un clopen.*

### 1.2.2 Quelques exemples classiques

Afin de fixer les idées, nous donnons trois exemples classiques d'homéomorphismes sur l'espace de Cantor :

- LES (SOUS-)DÉCALAGES :

Pour présenter cet exemple, notons tout d'abord que  $A^{\mathbb{Z}}$  est également un espace de Cantor pour  $A$  un ensemble fini (comme n'importe quel produit dénombrable d'ensembles finis de cardinal supérieur à 2).

*Définition 1.6.* On note  $S$  (en référence à l'anglais "shift"), et on appelle *décalage à gauche sur  $A$*  l'application

$$S: \begin{array}{ccc} A^{\mathbb{Z}} & \longrightarrow & A^{\mathbb{Z}} \\ (x_i)_{i \in \mathbb{Z}} & \longmapsto & (x_{i+1})_{i \in \mathbb{Z}} \end{array} .$$

Littéralement, le décalage à gauche décale d'un cran sur la gauche les suites auxquelles on l'applique. Cet homéomorphisme admet des points fixes (les suites constantes), et des orbites de tout ordre (une suite formée de répétitions de la séquence  $0^n 1$  pour  $A = \{0; 1\}$  -par exemple- aura une orbite de cardinal exactement  $n + 1$ ). De ce fait, il n'est pas minimal.

On appelle *sous-décalage minimal (de  $S$ )* toute restriction de  $S$  à un fermé invariant de  $A^{\mathbb{Z}}$  telle que cette restriction soit minimale. Nous avons vu qu'il existe des sous-décalages agissant sur des espaces à  $n$  points pour tout entier  $n$ , mais la classe des sous-décalages est beaucoup plus large que cela. En fait, de très nombreux exemples de systèmes minimaux peuvent être vus comme des composantes minimales du décalage. À titre d'exemple, les sous-décalages de Toeplitz forment une classe suffisamment riche pour que l'équivalence orbitale y soit "aussi complexe que possible" au sens de la complexité Borélienne (pour plus de détails, voir [M5]).

- LES ODOMÈTRES :

Les odomètres sont des homéomorphismes qui, moralement, “ajoutent un avec retenue à droite”. Pour comprendre leur fonctionnement, examinons plus en détails le 2-odomètre que nous noterons  $\sigma_2$  (cette notation n’est pas classique), défini comme suit :

$$\begin{aligned} \sigma_2: \quad 2^{\mathbb{N}} &\longrightarrow 2^{\mathbb{N}} && \text{et } \sigma_2(1^\infty) = 0^\infty. \\ 1^n 0 \frown x &\longmapsto 0^n 1 \frown x \end{aligned}$$

Pour une suite finie  $s$  et une suite infinie  $x$ , on a noté ici  $s \frown x$  pour désigner la concaténation de  $s$  et  $x$ . Les odomètres ont plusieurs bonnes propriétés. Notons tout d’abord qu’ils sont minimaux. Ils sont aussi particulièrement adaptés à la topologie de l’espace, puisqu’ils agissent transitivement sur les cylindres de taille  $n$  (que nous avons notés  $U_s$ , avec  $s \in 2^n$ , dans la section 1.1.2), ce qui les rend très simples à visualiser, notamment à l’aide de partitions de Kakutani-Rokhlin (voir section 1.2.3). Enfin, à l’exception du point  $1^\infty$ , l’image d’un point est déterminée de manière locale, puisque ces homéomorphismes ne changent qu’un nombre fini de digits de la suite. Ainsi, la relation d’équivalence induite par les orbites sous un odomètre est presque la relation de queue “être égal à partir d’un certain rang”, classiquement notée  $E_0$ . Ce “presque” contient en fait des subtilités assez fascinantes, que nous développerons plus en détails dans la section 1.2.4.3.

Le titre de cet exemple parlait d’odomètres au pluriel car on peut considérer des odomètres sur des espaces de Cantor présentés différemment, par exemple le 3-odomètre  $\sigma_3$  sur  $3^{\mathbb{N}}$ , ou l’odomètre universel  $\sigma_\infty$  sur  $\prod_{n \in \mathbb{N}} n$  (où encore une fois  $n = \llbracket 0; n - 1 \rrbracket$ ). Ces différents odomètres, construits sur le même modèle, pourront avoir des propriétés légèrement différentes. Il est par exemple remarquable que la classe d’isomorphisme de l’odomètre universel soit comeagre dans l’espace des homéomorphismes transitifs de l’espace de Cantor (voir [H] pour une preuve de ce résultat. Pour l’existence et la description d’un système dont la classe d’isomorphisme est comeagre dans  $\text{Homeo}(X)$  tout entier, on pourra consulter [KR] et [AGW]. Enfin, pour une preuve du fait que l’action diagonale de  $\text{Homeo}(X)$  sur  $\text{Homeo}(X)^m$  par conjugaison a une orbite comeagre pour tout  $m$ , voir [K2]).

- LES ROTATIONS IRRATIONNELLES SUR LE CERCLE :

Soit  $\alpha$  un irrationnel. La rotation irrationnelle d’angle  $\alpha$  sur le cercle  $\mathbb{R}/\mathbb{Z}$ , notée  $R_\alpha$ , est un homéomorphisme bien connu. Sa minimalité est un exercice classique. À première vue, cet exemple traite d’un espace lisse et n’a donc pas de lien avec le cadre de cette thèse. Toutefois, le procédé suivant permet de plonger  $\mathbb{R}/\mathbb{Z}$  dans un espace de Cantor, et ainsi d’étudier ces rotations sous l’angle de la dynamique topologique : pour tout point  $x$  appartenant à une orbite fixée (disons l’orbite de 0), on “éclate” le point  $x$  en deux points  $x^+$  et  $x^-$ . On obtient alors un espace  $X$  que l’on munit de la topologie naturelle de l’ordre sur le cercle, c’est à dire la topologie engendrée par

les intervalles  $[x^+; y^-]$  pour  $x, y \in \text{Orb}_{R_\alpha}(0)$ . Ceci fait de  $X$  un espace de Cantor, et  $R_\alpha$  s'étend naturellement en un homéomorphisme minimal de  $X$  en posant  $R_\alpha(x^+) = R_\alpha(x)^+$  et  $R_\alpha(x^-) = R_\alpha(x)^-$ . De plus la surjection naturelle  $s$  de  $X$  vers  $\mathbb{R}/\mathbb{Z}$  est équivariante (i.e elle préserve l'action, au sens de  $\forall x \in X, R_\alpha(s(x)) = s(R_\alpha(x))$ ), et toutes les propriétés dynamiques du système  $(R_\alpha, X)$  se répercuteront sur  $(R_\alpha, \mathbb{R}/\mathbb{Z})$ . Ceci nous donne donc un moyen d'étudier une action lisse en la considérant comme une action sur un espace de Cantor. Par exemple, dans [L], ce genre de procédé est utilisé pour voir certains sous-groupes d'IET (Interval Exchange Transformations) comme des groupes d'homéomorphismes d'actions sur un espace de Stone. Akin a également utilisé les rotations irrationnelles (ainsi que les odomètres) pour comprendre les mesures qui proviennent d'actions de systèmes minimaux uniquement ergodiques (i.e ayant uniquement une mesure invariante) sur l'espace de Cantor, qui ont des propriétés dynamiques intéressantes. Les lectrices intéressées par le sujet peuvent consulter [A].

### 1.2.3 Partitions de Kakutani-Rokhlin

Nous allons maintenant décrire une décomposition de l'espace qui joue un rôle central dans notre travail, et est un outil omniprésent dans l'étude des homéomorphismes minimaux. Les partitions de Kakutani-Rokhlin (que nous abrègerons en "partitions K-R") sont un cas particulier de partitions Følner dans le cas des actions de  $\mathbb{Z}$ , qui ont l'avantage de bien se raffiner. De plus, l'existence de l'application de premier retour est une spécificité qui permet de restreindre facilement le système à un clopen donné. Cette construction sera de nouveau expliquée (en anglais) dans le Chapitre 2 afin de garantir que chaque chapitre puisse être lu indépendamment.

L'idée est la suivante : étant donné un clopen non-vide  $A$ , on regarde la fonction de premier retour

$$\varphi_A: \begin{cases} A \longrightarrow A \\ x \longmapsto \varphi^{\tau_A(x)}(x) \end{cases}, \text{ où}$$

$$\tau_A: \begin{cases} A \longrightarrow \mathbb{N} \\ x \longmapsto \min\{k > 0: \varphi^k(x) \in A\} \end{cases}$$

est bien définie grâce au point ii) de la Proposition 1.1. En effet, l'homéomorphisme  $\varphi$  étant minimal, toute orbite positive est dense dans  $X$ , et donc en particulier intersecte  $A$ . De plus, cette application est continue, ce qui n'est peut-être pas intuitif à première vue : pour  $i \in \mathbb{Z}$ ,  $\varphi^i(A)$  et  $\varphi^i(X \setminus A)$  sont des ouverts, et si  $\tau_A(x) = k$ , on a alors un voisinage ouvert de  $x$

$$U = \varphi^{-k}(A) \cap \bigcap_{i \in \llbracket 1, k-1 \rrbracket} \varphi^{-i}(X \setminus A) \subset A$$

tel que  $\tau_A$  est constante égale à  $k$  sur  $U$ . Ainsi,  $\varphi_A$  est bien continue. Par ailleurs, on pourrait considérer la fonction de premier retour dans le passé, construite de

la même manière. C'est clairement la réciproque de  $\varphi_A$ , et cette dernière est donc bien un homéomorphisme de  $A$ . Il est également pratique de considérer  $\varphi_A$  comme un homéomorphisme de  $X$  en imposant  $\varphi_{A \uparrow A^c} = id_{\uparrow A^c}$ .

Maintenant, cette application donne naturellement la décomposition de l'espace qui nous intéresse. Puisque  $A$  est compact et  $\tau_A$  est continue, l'image  $\tau_A(A)$  est finie, et  $A$  admet une partition en clopens finie qui consiste en les points qui reviennent dans  $A$  en un certain nombre d'étapes. Plus formellement,  $A = \bigsqcup_{k \in \mathbb{N}} A_k$ , où les

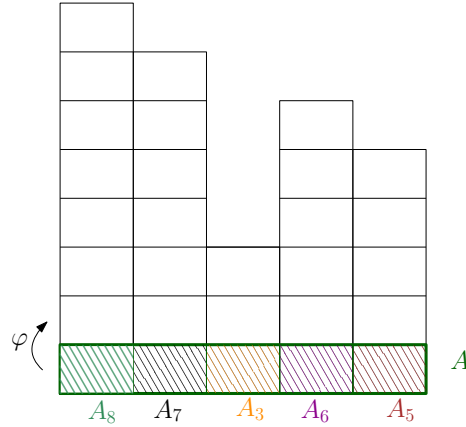


Figure 1.1: K-R partition construite à partir de  $A$

$A_k := \tau_A^{-1}(k)$  sont clopens, et tous sauf un nombre finis d'entre eux sont vides. On obtient alors une partition clopen de  $X$  qui peut être représentée comme sur la figure 1.1, en faisant apparaître les translats des  $A_k$  non vides par  $\varphi$ .

Plus généralement on définit une partition K-R comme suit :

*Définition 1.7.* Une *partition K-R associée à  $\varphi$*  est une partition clopen

$$\Xi = (D_{i,j})_{i \in \llbracket 1, N \rrbracket, j \in \llbracket 0, H_i - 1 \rrbracket}$$

de  $X$  telle que  $\varphi(D_{i,j-1}) = D_{i,j}$  pour tout  $i \in \llbracket 1, N \rrbracket$  et tout  $j \in \llbracket 1, H_i - 1 \rrbracket$ .

On appelle, pour  $i \in \llbracket 1, N \rrbracket$ ,  $i$ -ième *tour* de  $\Xi$  le clopen  $T_i = \bigsqcup_{j \in \llbracket 0, H_i - 1 \rrbracket} D_{i,j}$ , et *hauteur* de cette tour le nombre  $H_i$ . On parlera aussi du  $k$ -ième *étage* de  $\Xi$  pour désigner  $\bigsqcup_{i \in \llbracket 1, N \rrbracket} \varphi^k(D_{i,0})$ . L'étage 0 est appelé *la base* de la partition  $\Xi$ , et notée  $B(\Xi)$ . De plus, l'étage  $-1$  est appelé *le sommet* de la partition.

Dans le cas où la partition est indexée ( $\Xi_n$  à la place de  $\Xi$ ), ce qui sera le cas très rapidement, on reportera cet indice sur la notation de ces différents éléments, et on parlera ainsi de la tour  $T_i^n$ , de l'atome  $D_{i,j}^n$ , etc.

*Remarque 1.1.* Remarquons que par définition  $\bigsqcup_{i \in \llbracket 1, N \rrbracket} D_{i,H_i-1}$  est envoyé surjectivement par  $\varphi$  sur  $B(\Xi)$ , et c'est donc bien ce que nous avons appelé le sommet de la partition. Toutefois, nous attirons l'attention du lecteur sur le fait qu'un atome  $D_{i,H_i-1}$  peut être envoyé n'importe où dans  $B(\Xi)$ , et en général ne retourne pas dans l'atome de sa colonne.

*Exemple.* L'odomètre binaire (cf exemple 1.2.2) que nous avons défini précédemment possède des partitions de Kakutani-Rokhlin particulièrement simples puisqu'elles ne possèdent qu'une seule tour. Par exemple, la partition

$$\Xi = (N_{00}, N_{10}, N_{01}, N_{11})$$

est une telle partition.

**Notation.** On note  $\langle \Xi \rangle$  l'algèbre Booléenne engendrée par les atomes d'une partition  $\Xi$ .

L'idée associée aux partitions K-R est qu'elles représentent comment  $\varphi$  agit sur les clopens de  $\langle \Xi \rangle$  à l'exception des atomes appartenant au sommet de la partition. Dès lors, on veut obtenir une information de plus en plus fine sur la façon dont  $\varphi$  agit sur les clopens. Cette idée vient de la correspondance de Stone pour les algèbres Booléennes qui produit un lien très fort entre un compact totalement discontinu et l'algèbre Booléenne de ses ouverts fermés. Dans notre cas, et dans un esprit très proche, on peut montrer que connaître l'action de  $\varphi$  sur tous les clopens ne contenant pas un point précis suffit à déterminer  $\varphi$ . Ce point précis sera "le sommet" de nos partitions, au sens de l'intersection des sommets d'une suite de partitions de plus en plus fines, dont nous précisons la construction ci-dessous. Ce dont nous avons besoin, c'est d'être capable d'améliorer l'information donnée par une partition K-R, sans rien perdre de ce que celle-ci nous donnait. C'est l'objet de la propriété suivante :

**Proposition 1.4.** Soit  $A \in CO(X)$ , et  $\Xi$  une partition K-R. Alors il existe une partition K-R plus fine que  $\Xi$  (au sens des algèbres Booléennes), notée  $\Xi'$ , telle que  $A \in \langle \Xi' \rangle$ .

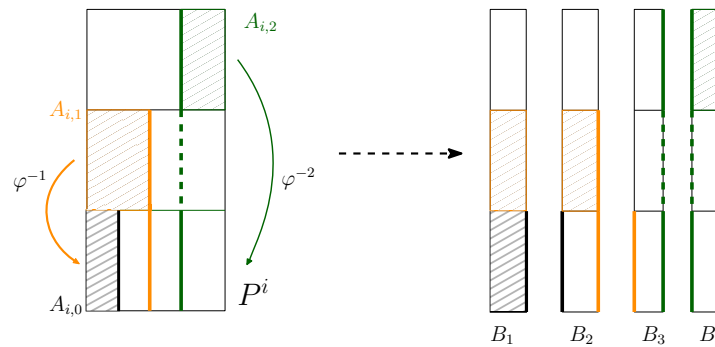


Figure 1.2: Procédé de découpage dans la  $i$ -ième tour de  $\Xi$ , avec  $A_{i,j} = A \cap D_{i,j}$

*Preuve.*

Pour chaque tour  $T_i$  de  $\Xi$ , on définit une relation d'équivalence  $\mathcal{R}_i$  sur  $D_{i,0}$  par

$$x \mathcal{R}_i y \Leftrightarrow \forall j < H_i \left( (\varphi^j(x) \in A \wedge \varphi^j(y) \in A) \vee (\varphi^j(x) \in A^c \wedge \varphi^j(y) \in A^c) \right).$$

Notons  $P^i$  la partition de  $D_{i,0}$  associée à  $\mathcal{R}_i$ . (cf figure 1.2)



En notant  $(B_l)_{l \in \llbracket 0, N_i - 1 \rrbracket}$  les atomes de  $P^i$ , on “coupe” alors simplement  $T_i$  en  $N_i$  tours dont les bases sont les  $B_l$ . En répétant cette technique dans chaque tour, on obtient bien une partition  $\Xi'$  plus fine que  $\Xi$  et telle que tout  $A \cap D_{i,j}$  est dans  $\langle \Xi' \rangle$ , de sorte que  $A$  est également dans  $\langle \Xi' \rangle$ .  $\square$

Soit maintenant  $(U_n)_{n \in \mathbb{N}}$  une base clopen de la topologie. En appliquant de manière répétée la Proposition 1.4 on obtient le corollaire suivant :

**Corollaire 1.4.** *Il existe une suite de partitions K-R  $(\Xi_n)_{n \in \mathbb{N}}$  telle que les conditions suivantes soient respectées pour tout  $n \in \mathbb{N}$  :*

- (i)  $\Xi_{n+1}$  est plus fine que  $\Xi_n$
- (ii)  $B(\Xi_{n+1}) \subset B(\Xi_n)$  et  $\bigcap_i B(\Xi_i) = \{x_0\}$
- (iii)  $U_n \in \langle \Xi_n \rangle$
- (iv) Toutes les tours de  $\Xi_n$  sont de hauteurs supérieures à  $n$ .

**À partir de maintenant, chaque fois que l'on considérera une suite de partitions K-R, elles seront supposées vérifier les conditions (i) à (iv).**

*Remarque 1.2.* Le point (iv) nécessite clairement l'apériodicité de  $\varphi$ , et celle-ci est suffisante :

les points  $x_0, \varphi(x_0), \dots, \varphi^n(x_0)$  sont tous distincts, donc pour un suffisamment petit voisinage  $U$  de  $x_0$ , les clopens  $U, \varphi(U), \dots, \varphi^n(U)$  sont deux à deux disjoints. En fait, l'hypothèse de minimalité peut être relaxée et on peut mener cette construction pour un homéomorphisme apériodique, bien que celle-ci soit plus technique. Ceci est fait dans [BDM].

*Définition 1.8.* On appelle *point base* d'une suite de partitions K-R  $(\Xi_n)$  le point  $x_0$  qui apparaît dans la propriété (ii), et *point-sommet* de la suite le point  $\varphi^{-1}(x_0)$ .

*Remarque 1.3.* Une chose importante à comprendre à propos de cette construction est qu'une tour de  $\Xi_{n+1}$  est obtenue en coupant (verticalement) les tours de  $\Xi_n$  et en en empilant certaines les unes par dessus les autres. C'est ce que l'on appelle le procédé de *découpage et empilement*. Pour éclairer cette remarque, se référer à la figure 1.3 ; les flèches verticales numérotées disent où un clopen situé au sommet de  $\Xi_n$  est envoyé par  $\varphi$ . S'il n'y en a pas, cela signifie qu'il est envoyé dans la base de

$\Xi_{n+1}$ ). En effet, pour tout  $i$ ,  $D_{i,0}^{n+1} \subset D_{i_0,0}^n \subset B(\Xi_n)$  pour un certain  $i_0$ , d'où  $\bigsqcup_{k=0}^{H_{i_0}^n - 1} D_{i,k}^{n+1}$

est obtenu exactement en coupant la  $i_0$ -ième tour de  $\Xi_n$ . Si  $D_{i,H_{i_0}^n - 1}^{n+1}$  n'est pas au sommet de  $\Xi_{n+1}$ , il suffit de regarder dans quelle tour de  $\Xi_n$  il est envoyé par  $\varphi$  pour savoir quelle “tour” empiler au-dessus de celle-ci (pour être plus précis ce n'est pas une tour de  $\Xi_n$  mais une tour de la partition plus fine obtenue en découpant  $\Xi_n$ , mais on confondra volontiers ces deux partitions pour le moment) : disons que

$\varphi(D_{i,H_{i_0}^{n+1}}^{n+1}) \subset D_{i_1,0}^n$  alors  $\bigsqcup_{k=H_{i_0}^n}^{H_{i_0}^n+H_{i_1}^n-1} D_{i,k}^{n+1}$  est obtenue en coupant la  $i_1$ -ième tour de  $\Xi_n$ , et ainsi de suite.

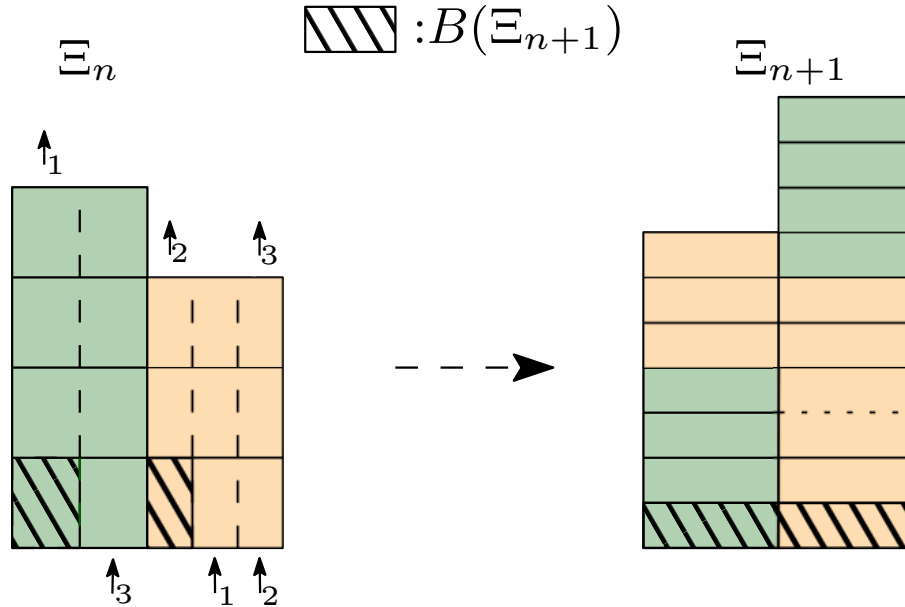


Figure 1.3: Procédé de découpage et empilement

### 1.2.4 Diagrammes de Bratteli

Une autre façon de représenter les informations comprises dans une suite de partitions K-R provient des diagrammes de Bratteli. Nous ne les utiliserons que pour présenter un exemple intéressant mettant en valeur les difficultés inhérentes à l'équivalence orbitale topologique, mais la lectrice intéressée par le sujet pourra se référer à [HPS].

#### 1.2.4.1 Diagrammes de Bratteli ordonnés

*Définition 1.9.* On appelle *diagramme de Bratteli* la donnée des éléments suivants :

- Un ensemble de sommets

$$V = \bigsqcup_{n \in \mathbb{N}} V_n$$

avec chaque  $V_n$  fini et non-vide, et de plus on exige que  $V_0$  soit un singleton :  $V_0 = \{R\}$ .

- Un ensemble d'arêtes

$$E = \bigsqcup_{n \in \mathbb{N}^*} E_n$$

avec chaque  $E_n$  fini et non-vide contenant des arêtes entre un sommet de  $V_{n-1}$  et un sommet de  $V_n$ .

- Une application source  $s: E \rightarrow V$  surjective (au moins une arête part de chaque sommet)
- Une application but (notée  $r$  en référence à "range")  $r: E \rightarrow V$  surjective sur  $V \setminus V_0$  (au moins une arête arrive à chaque sommet -mis à part la racine)

On appelle *diagramme de Bratteli ordonné* la donnée d'un diagramme de Bratteli  $(V, E)$  muni d'un ordre partiel  $\leq$  sur les arêtes tel que deux arêtes sont comparables si et seulement si elles ont le même but. Une arête  $e$  d'un diagramme de Bratteli ordonné est dite *minimale* (respectivement *maximale*) s'il s'agit du minimum (respectivement maximum) de  $r^{-1}(\{r(e)\})$ . On note respectivement  $E_{min}$  et  $E_{max}$  l'ensemble des arêtes minimales et maximales.

On associe à un diagramme de Bratteli  $B = (V, E)$  le *compact de Bratteli*  $X_B$  constitué de toutes les branches infinies (commençant à la racine  $R$ ). On voit naturellement  $X_B$  comme un sous-espace du produit  $\prod_{n \in \mathbb{N}^*} E_n$ , que l'on équipe de la topologie produit. Vu ainsi,  $X_B$  devient un compact métrisable totalement discontinu, et ce sera un espace de Cantor si et seulement s'il n'a pas de point isolé (et est non vide).

**Proposition 1.5.** *Les graphes  $(V, E_{min})$  et  $(V, E_{max})$  sont des arbres enracinés en  $R$ .*

*Preuve.*

Faisons la preuve pour  $(V, E_{max})$ , celle pour  $(V, E_{min})$  est identique. Puisqu'il n'y a qu'une seule arête maximale ayant pour but un sommet donné, il ne peut pas y avoir de cycle dans  $E_{max}$ . Fixons maintenant un sommet quelconque  $v \in V_n$ . Nous allons montrer que le graphe est connexe en exhibant un chemin de  $R$  à  $v$ . Si  $v = R$  il n'y a rien à faire. Sinon, il existe une unique arête  $e_n \in E_n$  qui soit maximale et avec  $r(e_n) = v$ . En répétant cet argument sur  $s(e_n)$ , si ce n'est pas  $R$ , on obtient  $e_{n-1} \in E_{n-1}$  maximale avec  $r(e_{n-1}) = s(e_n)$ , et ainsi de suite.  $\square$

Puisqu'il existe des arêtes de  $E_{min}$  et  $E_{max}$  à des étages arbitrairement lointains, et que  $(V, E_{min})$  et  $(V, E_{max})$  sont des arbres, ils ont des branches finies arbitrairement grandes, et un argument de compacité dans  $X_B$  (ou le Lemme de König pour les lecteurs plus familiers avec la théorie des graphes) assure qu'ils ont au moins une branche infinie  $e_{min}$  et  $e_{max}$  respectivement. Il est toutefois possible que  $e_{min} = e_{max}$ .

**Définition 1.10.** Un diagramme de Bratteli ordonné  $(V, E, \leq)$  est dit *essentiellement simple* si  $E_{min}$  et  $E_{max}$  ont une *unique* branche infinie  $e_{min}$  et  $e_{max}$  respectivement.

#### 1.2.4.2 Application de Vershik

Soit  $B = (V, E, \leq)$  un diagramme de Bratteli essentiellement simple.

Définition 1.11. On définit l'application de Vershik  $\varphi_B: X_B \rightarrow X_B$  comme suit :

- On pose  $\varphi_B(e_{max}) = e_{min}$
- Pour  $x \in X_B$  non-maximal, soit  $n$  le plus petit entier tel que  $x(n)$  soit une arête non-maximale. Soit alors  $e_n$  l'arête qui suit  $x(n)$  dans l'ensemble fini et totalement ordonné  $r^{-1}(\{r(x(n))\})$ . Comme  $E_{min}$  est un arbre, il existe un unique chemin  $e_1, \dots, e_{n-1}$  de  $R$  à  $s(e_n)$  dans  $E_{min}$ . On pose alors

$$\varphi_B(x)(m) = \begin{cases} e_m & \text{si } m \leq n \\ x(m) & \text{si } m > n \end{cases}.$$

Pour illustrer cette définition, la figure 1.4 donne l'image d'un chemin par l'application de Vershik (ici les arêtes sont ordonnées de gauche à droite) :

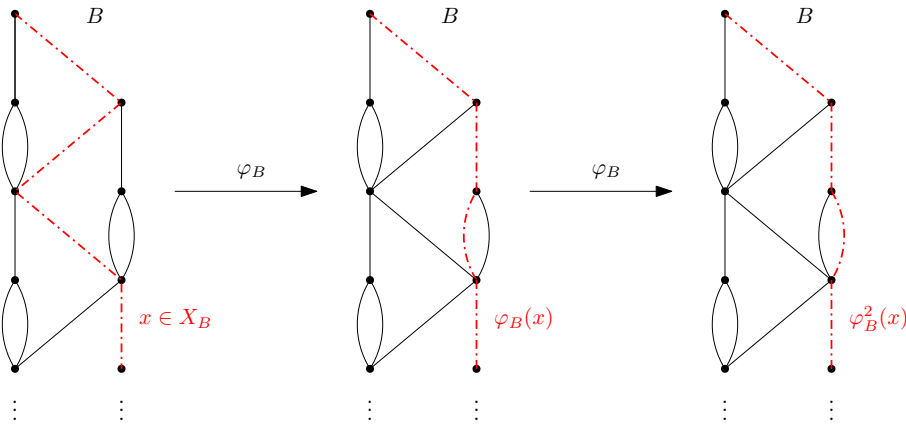


Figure 1.4: Illustration de comment fonctionne l'application de Vershik

**Proposition 1.6.** Soit  $B = (V, E, \leq)$  un diagramme de Bratteli essentiellement simple. Alors l'application de Vershik  $\varphi_B$  est un homéomorphisme.

*Preuve.*

Nous allons exhiber la réciproque de  $\varphi_B$ . On définit  $\psi_B: X_B \rightarrow X_B$  par  $\psi_B(e_{min}) = e_{max}$  et pour  $x \in X_B$  non-minimal, soit  $n$  le plus petit entier tel que  $x(n)$  soit une arête non-minimale. Soit alors  $e_n$  l'arête qui précède  $x(n)$  dans l'ensemble fini et totalement ordonné  $r^{-1}(\{r(x(n))\})$ . Comme  $E_{max}$  est un arbre, il existe un unique chemin  $e_1, \dots, e_{n-1}$  de  $R$  à  $s(e_n)$  dans  $E_{max}$ . On pose alors

$$\psi_B(x)(m) = \begin{cases} e_m & \text{si } m \leq n \\ x(m) & \text{si } m > n \end{cases}.$$

On peut facilement vérifier que  $\psi_B$  est la bijection réciproque de  $\varphi_B$ , et cette dernière est donc bien une bijection. De plus, pour tout  $x \neq e_{max}$ ,  $\varphi_B(x)$  est défini localement, à partir des  $n$  premiers digits uniquement, et donc tout  $y$  dans  $U_{x \uparrow n}$  (le voisinage

de  $x$  constitué des branches ayant les mêmes  $m$  premiers digits que  $x$ ) pour  $m \geq n$  sera envoyé dans  $U_{\varphi_B(x) \upharpoonright m}$ , ce qui prouve la continuité en  $x$ . La continuité en  $e_{max}$  est également directe à prouver : un élément commençant par  $n$  digits maximaux aura une image commençant par  $n$  digits minimaux.  $\square$

**Définition 1.12.** Un diagramme de Bratteli  $(V, E)$  est dit *simple* si pour tout entier  $m$  il existe  $n > m$  telle que depuis tout sommet de  $V_m$  il y ait un chemin vers tout sommet de  $V_n$ . Un diagramme de Bratteli ordonné  $(V, E, \leq)$  est dit *simple* s'il est essentiellement simple en tant que diagramme ordonné et simple en tant que diagramme non-ordonné.

Notons que si  $B = (V, E)$  est simple (et non trivial dans le sens où il n'y a pas ultimement un seul sommet par étage), alors  $X_B$  n'a pas de point isolé et est un espace de Cantor.

**Proposition 1.7.** Soit  $B = (V, E, \leq)$  un diagramme de Bratteli essentiellement simple, et  $v \in V$ . Soit de plus  $x \in X$  tel que  $x \upharpoonright_n$  soit un chemin minimal de  $R$  à  $v$ . Alors les  $(\varphi_B^k(x))_{k \in \llbracket 0; N-1 \rrbracket}$  énumèrent tous les chemins de  $R$  à  $v$  (ici  $N$  est donc le nombre de tels chemins).

La preuve peut se faire par récurrence, nous choisissons de ne pas l'inclure pour ne pas encombrer l'esprit du lecteur avec des technicités, ce résultat ne nous étant utile que dans la preuve du résultat suivant.

**Proposition 1.8.** Soit  $B = (V, E, \leq)$  un diagramme de Bratteli essentiellement simple. L'application de Vershik  $\varphi_B$  est minimale si et seulement si  $B$  est simple.

*Preuve.*

Supposons  $B$  simple. Soient  $x, y \in X_B$  et  $M \in \mathbb{N}$ . Il suffit de montrer qu'il existe  $n \in \mathbb{N}$  tel que  $\varphi_B^n(x) \upharpoonright_M = y \upharpoonright_M$ . Puisque  $B$  est simple, on peut trouver  $N > M$  tel que tout sommet de  $V_M$  est connecté à tout sommet de  $V_N$ . En connectant

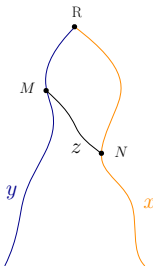


Figure 1.5: Un chemin  $z$  connectant  $r(y(M))$  à  $r(x(N))$

$u = r(y(M)) \in V_M$  à  $v = r(x(N)) \in V_N$ , on obtient un élément  $z \in X_B$  tel que

$$z(i) = \begin{cases} y(i) & \text{si } i \leq M \\ x(i) & \text{si } i \geq N \end{cases} .$$

Par la Proposition 1.7, on peut alors trouver  $n \in \mathbb{N}$  tel que  $\varphi_B^n(x)|_N = z|_N$ , et en particulier pour tout  $i \leq M$ , on a  $\varphi_B^n(x)(i) = y(i)$ , et  $\varphi_B$  est bien minimal.

Pour la réciproque, montrons la contraposée. Supposons que  $B$  n'est pas simple : il existe alors  $m$  tel que pour tout  $n > m$ , il existe  $u_n \in V_m$  et  $v_n \in V_n$  sans chemin entre eux. Comme  $V_m$  est fini, il existe un  $u \in V_m$  tel que ceci arrive pour une infinité d'entiers  $(n_k)_{k \in \mathbb{N}}$ . Pour tout  $k \in \mathbb{N}$ , soit  $y_k \in X_B$  tel que  $r(y_k(n_k)) = v_{n_k}$ . Par compacité de  $X_B$ , cette suite  $(y_k)_{k \in \mathbb{N}}$  admet une valeur d'adhérence  $y$ , et on suppose désormais que  $y$  est la limite de  $(y_k)_{k \in \mathbb{N}}$ . Notons que pour tout  $i > m$ , il n'y a aucun chemin entre  $u$  et  $r(y(i))$ , car sinon on trouverait  $k$  assez grand pour que  $n_k > i$  et  $y$  a les mêmes  $i$  premiers digits que  $y_k$ , et il y aurait un chemin entre  $u$  et  $v_{n_k} = r(y_k(n_k))$ .

Soit  $x \in X_B$  tel que  $r(x(m)) = u$ . Supposons par l'absurde que  $\varphi_B$  est minimal. On peut alors trouver  $k \in \mathbb{Z}$  tel que  $\varphi_B^k(y)(i) = x(i)$  pour tout  $i \leq m$ . On peut supposer sans perte de généralité que  $\varphi_B^k(y)$  et  $y$  ont la même queue (c'est à dire que  $\varphi_B^k(y)(N) = y(N)$  pour tout  $N$  assez grand). En effet, le seul cas problématique est celui où  $y \in \text{Orb}(e_{max})$ , mais la Proposition 1.1 assure qu'on peut prendre  $k$  positif ou négatif et ainsi rester dans la demi-orbite dont la queue ne change pas. Or  $\varphi_B^k(y)$  exhibe donc un chemin entre  $u = r(x(m)) = r(\varphi_B^k(y)(m))$  et  $v_{n_k} = r(y(n_k)) = r(\varphi_B^k(y(n_k)))$  pour  $k$  assez grand. Ceci est absurde.  $\square$

Nous mentionnions au début de cette section que les diagrammes de Bratteli sont une façon alternative de voir les partitions K-R : voyons en détail comment cette correspondance fonctionne. Étant donné un diagramme de Bratteli simple  $B = (V, E, \leq)$ , on peut construire une suite de partitions K-R de la façon suivante : pour un chemin  $p$  de  $R$  jusqu'à  $u \in V_n$  on pose

$$C(p) = \{x \in X_B : \forall i \leq n \ x(i) = p(i)\}.$$

Chaque  $C(p)$  est bien clopen et on peut poser

$$\Xi_n = \{C(p) : p \text{ un chemin de } R \text{ à un certain } u \in V_n\}.$$

Il y aura alors une tour dans la partition  $\Xi_n$  pour chaque sommet de  $V_n$ , puisque  $\varphi_B$  agit transitivement sur les chemins entre  $R$  et un sommet donné (cf Proposition 1.7), et il y aura un atome dans la tour correspondant au sommet  $u \in V_n$  pour chaque chemin reliant  $R$  à  $u$ .

Le résultat suivant montre en fait que la correspondance fonctionne aussi dans l'autre sens : tout homéomorphisme peut être représenté comme une application de Vershik.

**Théorème 1.5** (Herman-Putnam-Skau [HPS], Théorème 4.6). *Soit  $\varphi \in \text{Homeo}(X)$  un homéomorphisme minimal et  $x_0 \in X$ . Alors il existe un diagramme de Bratteli simple  $B = (V, E, \leq)$  tel que  $(\varphi, X, x_0)$  et  $(\varphi_B, X_B, e_{min})$  sont conjugués.*

La figure 1.6, qui reprend la figure employée pour illustrer le procédé de découpage et empilement sera particulièrement utile pour ne pas se perdre dans les

notations. L'idée de la preuve est que chaque tour dans la partition  $\Xi_n$  correspondra à un sommet à l'étage  $n$  du diagramme correspondant, et les arêtes seront les "morceaux" de tours de l'étage précédent, ordonnés dans l'ordre dans lequel apparaissent ces "morceaux" dans la tour de l'étage  $n$ .

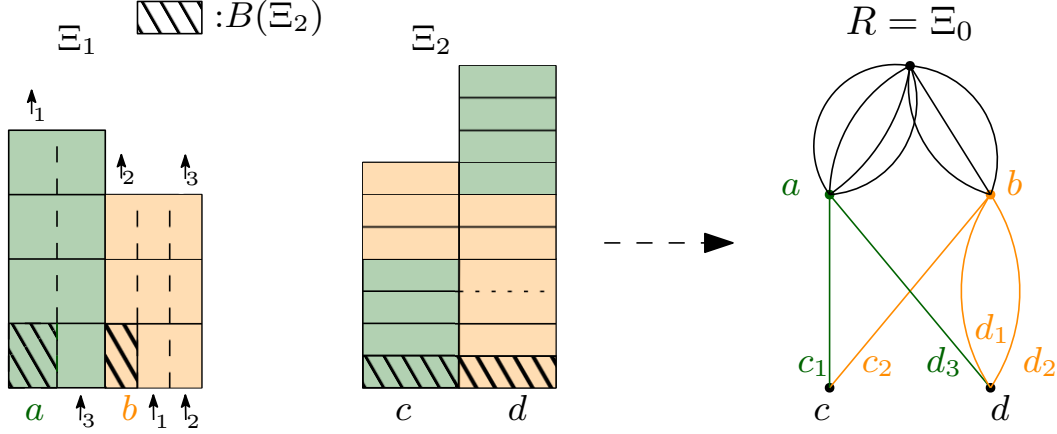


Figure 1.6: Construction du diagramme de Bratteli associé à une suite de partitions

*Preuve.*

Soit  $\Xi_n = \{D_{i,j}^n : 1 \leq i \leq K^n, 0 \leq j \leq H_i^n\}$  une suite de partitions K-R vérifiant les propriétés habituelles, avec  $x_0 = \bigcap_{n \in \mathbb{N}} B(\Xi_n)$  comme point-base. Pour aller avec notre convention selon laquelle  $R$  est l'unique racine d'un diagramme de Bratteli, demandons également  $\Xi_0 = \{X\}$ . Enfin, par souci de praticité, réordonnons les tours de sorte que le point-base  $x_0$  soit toujours dans la première tour. Construisons maintenant le diagramme approprié à partir de cette suite. Les sommets de  $V_n$  sont les tours de  $\Xi_n$  :  $V_n = \mathcal{T}(\Xi_n)$ . Pour savoir s'il y a une (ou plusieurs) arêtes entre deux sommets  $T_k^n$  et  $T_i^{n+1}$ , on regarde s'il y a des morceaux de  $T_k^n$  dans  $T_i^{n+1}$  (rappelons qu'une tour de  $\Xi_{n+1}$  est obtenue à partir de celles de  $\Xi_n$  par découpage et empilement, cf figure 1.6). Plus formellement, si  $D_{i,j}^{n+1} \subset D_{k,0}^n$  on met une arête entre  $T_k^n$  et  $T_i^{n+1}$ . Quant à la façon d'ordonner les arêtes, c'est assez naturel : elles sont ordonnées en fonction de l'ordre dans lequel les morceaux de tours apparaissent dans la tour qui est le sommet d'arrivée dans la procédure par découpage et empilement. En symboles : si une arête  $e_1$  correspond à l'inclusion  $D_{i,j_1}^{n+1} \subset D_{k_1,0}^n$  et une arête  $e_2$  correspond à l'inclusion  $D_{i,j_2}^{n+1} \subset D_{k_2,0}^n$ , alors on aura  $e_1 \leq e_2$  si et seulement si  $j_1 \leq j_2$ . Notons que  $B$  est essentiellement simple : à un chemin minimal on peut associer une suite décroissante d'atomes qui soient toujours dans la base de la suite de partitions, et la seule telle suite est celle des atomes contenant le point-base  $x_0$ . De même un chemin maximal correspond à une suite décroissante d'atomes qui soient toujours au sommet de la partition, et la seule telle suite est celle des atomes contenant le point-sommet  $\varphi^{-1}(x_0)$ .

Montrons maintenant que les deux systèmes pointés sont conjugués. Nous

allons construire l'homéomorphisme  $\xi: X \rightarrow X_B$  implémentant la conjugaison. Soit  $x \in X$  quelconque et  $n \geq 1$ . Soient respectivement  $D_{i_n, j_n}^{n-1}$  et  $D_{i_n, j_n}^n$  les atomes de  $\Xi_{n-1}$  et  $\Xi_n$  contenant  $x$ . On a alors  $j_{n-1} \leq j_n$ , et  $D_{i_n, j_n - j_{n-1}}^n \subset D_{i_{n-1}, 0}^{n-1}$  et on définit  $\xi(x)(n)$  comme étant l'arête représentée par cette inclusion. En résumé  $\xi(x)$  est le chemin entre les sommets qui correspondent à l'unique tour  $T_{i_n}^n$  contenant  $x$  dans chaque  $\Xi_n$ , et l'arête entre ces sommets correspond à l'inclusion du morceau de la tour  $T_{i_n}^n$  contenant  $x$  dans  $T_{i_{n+1}}^{n+1}$ .

Nous prétendons que l'atome de  $\Xi_n$  contenant  $x$  est déterminé par  $\xi(x)|_n$ , c'est à dire

$$\forall i \leq n, \xi(x)(i) = \xi(y)(i) \iff x \text{ et } y \text{ sont dans le même atome de } \Xi_n. \quad (1.1)$$

L'implication  $\Leftarrow$  est claire. Montrons l'implication réciproque par contraposée : supposons que  $x$  et  $y$  ne sont pas dans le même atome de  $\Xi_n$ , et notons  $m \in \llbracket 1, n \rrbracket$  le premier entier tel que  $x$  et  $y$  ne sont pas dans le même atome de  $\Xi_m$ . Comme ils étaient dans le même atome de  $\Xi_{m-1}$ , ils ont été séparés par le procédé de découpage, et n'appartiennent donc pas au même morceau de tour : il s'ensuit que les arêtes  $\xi(x)(m)$  et  $\xi(y)(m)$  diffèrent.

Il est maintenant facile de vérifier toutes les propriétés de  $\xi$ . La continuité est évidente par l'équivalence 1.1. L'injectivité l'est également, puisque les atomes des  $\Xi_n$  engendrent la topologie. La surjectivité est claire par construction. Donc  $\xi$  est un homéomorphisme (une application continue sur un compact est ouverte). Le fait que  $\xi \circ \varphi = \varphi_B \circ \xi$  est également une vérification élémentaire.  $\square$

### 1.2.4.3 Un homéomorphisme orbitalement équivalent à l'odomètre

Dans cette sous-section nous présentons un exemple qui provient d'un travail non publié de John D. Clemens dans [C].

Pour comprendre la démarche, donnons un peu de contexte. Une part importante de la difficulté dans la compréhension de l'équivalence orbitale, que ce soit dans notre travail ou dans celui de Giordano, Putnam et Skau, provient des théorèmes d'absorption (voir Théorème 3.5). Ceux-ci garantissent dans le cas présent que deux homéomorphismes minimaux  $\varphi$  et  $\psi$  dont les orbites seraient toutes les mêmes, sauf une orbite de  $\varphi$  qui serait "découpée" en deux et correspondrait à deux orbites pour  $\psi$  sont orbitalement équivalents (en fait, ceci est un cas particulier, le théorème s'énonce en termes de relations d'équivalence, et l'existence d'un homéomorphisme induisant la relation est déjà un résultat important du théorème). Il n'est pas difficile de se convaincre de ce résultat à posteriori, à l'aide du Théorème 1.1. En effet on voit aisément que deux tels homéomorphismes préservent les mêmes mesures<sup>4</sup>, car les mesures "ne voient pas" l'orbite problématique qui est dénombrable (toute mesure invariante pour un homéomorphisme apériodique -et à fortiori minimal- attribue une valeur nulle à chaque

<sup>4</sup>Le terme *mesure* désigne une mesure de probabilité, cf Section 1.2.5 suivante



singleton, donc à chaque orbite). Plus formellement, en prenant  $x_0$  et  $y_0$  tels que  $\text{Orb}_\psi(x_0) = \text{Orb}_\varphi(x_0) \sqcup \text{Orb}_\varphi(y_0)$  les mesures sur  $X$  correspondent aux mesures sur  $X' = X \setminus \text{Orb}_\psi(x_0)$ , et sur ce dernier espace il s'agit de montrer que deux systèmes ayant les mêmes orbites préservent les mêmes mesures, ce qui est très classique : pour toute mesure  $\mu \in M(X')$  invariante par  $\psi$  on a

$$\forall U \in \mathcal{B}(X'), \mu(U) = \sum_{k \in \mathbb{Z}} \mu(U_k) = \sum_{k \in \mathbb{Z}} \mu(\psi^k(U_k)) = \mu\left(\bigsqcup_{k \in \mathbb{Z}} \psi^k(U_k)\right) = \mu(\varphi^{-1}(U))$$

et donc  $\mu$  est  $\varphi$ -invariante, avec  $U_k = \{x \in U : \varphi^{-1}(x) = \psi^k(x)\}$ . Les rôles de  $\varphi$  et  $\psi$  étant symétriques, ceci est suffisant.

Toutefois, comprendre explicitement à quoi peut bien ressembler une telle équivalence orbitale reste un peu mystérieux, d'où l'idée d'essayer de trouver un exemple explicite dans l'espoir que cela éclaire la preuve de ces théorèmes d'absorption. L'un des homéomorphismes minimaux les plus simples à notre disposition est l'odomètre qui, rappelons-le, correspond à ajouter 1 avec retenue à droite (cf 1.2.2). Or la relation d'équivalence engendrée par ses orbites est presque la classique relation de queue  $E_0$ , définie par

$$x \underset{E_0}{\sim} y \iff \exists N \in \mathbb{N}, \forall i \geq N, x(i) = y(i)$$

à ceci près que les orbites de  $0^\infty$  et  $1^\infty$  sous  $E_0$  n'en forment qu'une sous l'action de  $\sigma_2$ , puisque  $\sigma_2(1^\infty) = 0^\infty$ .

Or Clemens a réussi à expliciter un homéomorphisme dont la relation induite par les orbites est exactement  $E_0$ , que l'on note  $\varphi_0$ , sous la forme de l'application de Vershik du diagramme de Bratteli de la figure 1.7, dont les arêtes arrivant à un sommet donné sont ordonnées de gauche (petite) à droite (grande).

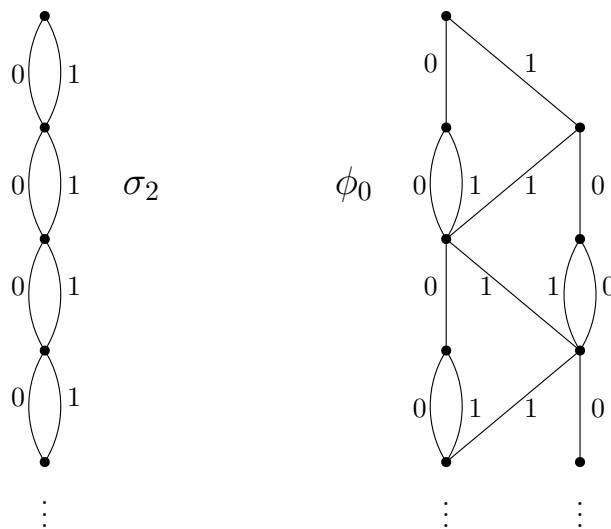


Figure 1.7: Diagrammes de Bratteli de l'odomètre  $\sigma_2$  et de  $\varphi_0$

Pour la lectrice qui lirait cet exemple avec une compréhension préalable de l'équivalence orbitale forte (par exemple après avoir lu le Chapitre 2), on peut préciser que l'équivalence orbitale entre  $\varphi_0$  et  $\sigma_2$  n'est pas une équivalence orbitale forte. En effet, comme tout clopen apparaît ultimement dans l'algèbre Booléenne engendrée par une partition K-R pour l'odomètre, et que chacune de ces partitions ne comporte qu'une seule tour, l'action d'un groupe ample associé à  $\sigma_2$ <sup>5</sup> est transitive sur les clopens. Or on peut remarquer que le diagramme définissant  $\varphi_0$  est construit de telle sorte qu'à partir du troisième étage, tout sommet est le but d'un nombre impair d'arêtes, et donc les tours des partitions K-R associées contiennent toutes un nombre impair d'atomes commençant par 1 et un nombre pair d'atomes commençant par 0, puisque c'est le cas au deuxième étage. Ainsi  $N_0$  et  $N_1$  sont dans deux orbites distinctes sous l'action d'un groupe ample associé à  $\varphi_0$ , ce qui montre que l'action n'est pas transitive sur les clopens, et par conséquent l'équivalence orbitale ne peut pas être une équivalence orbitale forte (voir Chapitre 2, Théorème 2.3).

### 1.2.5 Mesures invariantes

**Le terme *mesure* désignera toujours une mesure de probabilité Borélienne,  $\sigma$ -additive.**

Dans cette partie nous allons parler de l'ensemble  $M(X)$  des mesures sur  $X$ , ainsi que du rôle central que jouent les mesures invariantes d'un système dans la compréhension de celui-ci. Les techniques utilisées relèveront plus de la théorie ergodique que de la dynamique topologique, mais nous choisirons d'en inclure certaines en raison de l'importance cruciale qu'auront certains des résultats présentés ici dans la suite.

Tout d'abord  $M(X)$  est un espace topologique compact et métrisable. En effet, une mesure sur  $X$  peut être vue comme un élément de  $C(X)^*$ , le dual (topologique) des fonctions continues de  $X$  dans  $\mathbb{R}$ , via l'identification

$$\begin{array}{ccc} i: M(X) & \longrightarrow & C(X)^* \\ \mu & \longmapsto & \int d\mu: C(X) \longrightarrow \mathbb{R} \\ & & f \longmapsto \int f d\mu \end{array} .$$

On a bien  $\|\int d\mu\|_\infty = 1$  pour tout  $\mu \in M(X)$ , et  $i(M(X))$  est bien inclus dans la boule unité fermée de  $(C(X)^*, \|\cdot\|_\infty)$ , qui est compacte pour la topologie faible-\* d'après le Théorème de Banach-Alaoglu. Le fait que  $i(M(X))$  soit fermé découle du théorème de représentation de Riesz-Markov, qui garantit que toute forme linéaire positive valant 1 lorsqu'elle est évaluée en l'identité appartient à  $i(M(X))$ . Une preuve et un énoncé précis peuvent être trouvées dans n'importe quel livre de référence d'analyse réelle, par exemple [B1].

En pratique, rappelons que la topologie faible-\* est la topologie de la convergence simple, et qu'une base de voisinages ouverts d'une mesure  $\mu$  est donnée par

<sup>5</sup>Pour la définition d'un groupe ample associé à un homéomorphisme minimal, voir le Chapitre 2, section 2.2.2.

la famille

$$U(\mu, A_1, \dots, A_n, \varepsilon) = \{v \in M(X) : |\mu(A_i) - v(A_i)| < \varepsilon \text{ pour tout } i \leq n\}$$

où les  $A_i$  sont des clopens (ceci suffit par densité de l'algèbre engendrée par la famille  $(\mathbb{1}_A)_{A \in CO(X)}$  dans  $C(X)$ ).

*Définition 1.13.* Nous sommes particulièrement intéressés par les mesures invariantes par un système dynamique : à un homéomorphisme  $\varphi$ , on associe l'espace

$$M(\varphi) = \{\mu \in M(X) : \varphi_*\mu = \mu\}$$

où  $\varphi_*\mu$  est le poussé en avant de  $\mu$  par  $\varphi$ , défini par  $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$  pour tout Borélien  $A$ .

Notons que par définition de la convergence simple, et comme  $\varphi$  est une fonction continue sur  $X$ ,  $M(\varphi)$  est un fermé, donc un compact, de  $M(X)$ . De plus, le Théorème de Krylov-Bogoliubov assure que cet ensemble n'est jamais vide :

**Théorème 1.6** (Krylov-Bogoliubov). *Soit  $\varphi \in \text{Homeo}(X)$ . Alors l'ensemble  $M(\varphi)$  n'est pas vide.*

Nous ne donnerons qu'un aperçu de la preuve, qui est classique en théorie ergodique : il s'agit de partir d'un point  $x \in X$  quelconque, et de moyenner les mesures de Dirac en  $\varphi^k(x)$  pour  $k \in \llbracket 1; n \rrbracket$ , créant ainsi des mesures "de plus en plus  $\varphi$ -invariantes". La compacité de  $M(X)$  nous donne alors une valeur d'adhérence de la suite, dont on peut montrer qu'elle appartient à  $M(\varphi)$ .

La propriété suivante montre que, relativement aux mesures invariantes par un homéomorphisme minimal, chaque clopen non vide a une mesure minimale (non-nulle).

**Proposition 1.9.** *Soit  $\varphi$  un homéomorphisme minimal et  $A$  un clopen. Alors*

$$\inf \{\mu(A) : \mu \in M(\varphi)\} > 0.$$

*Preuve.*

Par la Propriété 1.1, il existe  $n \in \mathbb{N}^*$  tel que  $X = \bigcup_{i=1}^n \varphi^i(A)$ , et donc toute mesure invariante  $\mu$  vérifie  $\mu(A) \geq \frac{1}{n}$ . □

D'un autre côté, si  $d$  est une métrique compatible sur  $X$ , la métrique permet de contrôler les mesures invariantes, au sens où un clopen "suffisamment petit" au sens de la métrique sera aussi petit qu'on le souhaite au sens des mesures invariantes, comme précisé par la proposition suivante :

**Proposition 1.10.** *Soit  $\varphi$  un homéomorphisme minimal (ici apériodique suffirait). Si  $(V_n)_{n \in \mathbb{N}}$  est une suite de voisinages clopens d'un point  $x \in X$  dont le diamètre tend vers 0, alors  $\sup_{\mu \in M(\varphi)} \mu(V_n)$  tend également vers 0.*

*Preuve.*

Soit  $N \in \mathbb{N}$ . Nous allons montrer que  $\sup_{\mu \in M(\varphi)} \mu(V_n) < \frac{1}{N}$  à partir d'un certain rang. Pour cela considérons les points (distincts par apériodicité)  $x, \varphi(x), \dots, \varphi^N(x)$ , et choisissons des voisinages clopens  $U_i$  de  $\varphi^i(x)$  deux à deux disjoints. On obtient alors un voisinage clopen  $V = \bigcap_{i \leq N} \varphi^{-i}(U_i)$  de  $x$  qui vérifie

$$\forall i \neq j, \varphi^i(V) \cap \varphi^j(V) = \emptyset.$$

Comme  $\text{diam}_d(V_n) \xrightarrow{n \rightarrow +\infty} 0$ , on a  $V_n \subset V$  à partir d'un certain rang, et dès lors

$$1 = \mu(X) \geq \mu\left(\bigsqcup_{i \leq N} \varphi^i(V_n)\right) = (N+1)\mu(V_n) \text{ pour toute mesure } \mu \in M(\varphi)$$

d'où comme annoncé  $\sup_{\mu \in M(\varphi)} \mu(V_n) < \frac{1}{N}$ .  $\square$

Nous arrivons maintenant à deux résultats particulièrement importants pour notre travail, dûs à Glasner et Weiss dans [GW]. Nous les réutiliserons souvent, et leurs preuves sont de plus assez représentatives des manipulations que nous serons amenés à effectuer. Ils établissent que la capacité du groupe plein  $[\varphi]$  d'un homéomorphisme minimal à envoyer un clopen sur ou dans un autre se lit sur les mesures  $\varphi$ -invariantes de ces clopens. Pour comprendre la portée de ces résultats, remarquons déjà que les éléments de  $[\varphi]$  préservent toute mesure de  $M(\varphi)$ , i.e

$$\forall g \in [\varphi], \forall A \in CO(X), \forall \mu \in M(\varphi), \mu(g(A)) = \mu(A).$$

En effet, on peut partitionner  $A$  selon les valeurs que prend le cocycle  $n_g$  :

$$A = \bigsqcup_{k \in \mathbb{Z}} A_k, \text{ avec } A_k = \{x \in A : n_g(x) = k\}.$$

Les  $A_k$  sont des Boréliens, et on a bien, pour  $\mu \in M(\varphi)$ ,

$$\mu(g(A)) = \mu\left(\bigsqcup_{k \in \mathbb{Z}} g(A_k)\right) = \mu\left(\bigsqcup_{k \in \mathbb{Z}} \varphi^k(A_k)\right) = \mu\left(\bigsqcup_{k \in \mathbb{Z}} A_k\right) = \mu(A).$$

De plus, si  $g(A)$  est strictement inclus dans un clopen  $B$ , alors  $\mu(A) < \mu(B)$  pour toute mesure  $\mu \in M(\varphi)$ . En fait on a même  $\inf \{\mu(B) - \mu(A) : \mu \in M(\varphi)\} > 0$  par la Proposition 1.9, mais ce n'est pas important ici. Ce que Glasner et Weiss ont démontré, c'est que ces conditions nécessaires sont en fait suffisantes, et même un peu plus précis que cela, puisque dans le premier cas, on peut envoyer un clopen dans l'autre par un élément du groupe plein topologique :

**Théorème 1.7** ([GW], Lemme 2.5). *Soit  $\varphi$  un homéomorphisme minimal, et  $A, B$  des clopens tels que  $\mu(A) < \mu(B)$  pour toute mesure  $\mu \in M(\varphi)$ . Alors il existe un élément  $g \in [[\varphi]]$  tel que  $g(A) \subset B$ . De plus, on peut trouver un tel élément vérifiant également  $g^2 = \text{Id}$  et  $\text{supp}(g) \subset A \cup B$ .*

**Théorème 1.8** ([GW], Proposition 2.6). *Soit  $\varphi$  un homéomorphisme minimal, et  $A, B$  des clopens tels que  $\mu(A) = \mu(B)$  pour toute mesure  $\mu \in M(\varphi)$ . Alors il existe un élément  $g \in [\varphi]$  tel que  $g(A) = B$ , mais également  $g^2 = \text{Id}$ ,  $\text{supp}(g) \subset A \cup B$ , et le cocycle  $n_g$  ait au plus deux points de discontinuité.*

*Preuve du Théorème 1.7.*

On peut supposer que  $A \cap B = \emptyset$ . En effet, il suffit de définir  $g$  comme  $\text{Id}$  sur  $A \cap B$ , et de travailler avec  $A \setminus (A \cap B)$  et  $B \setminus (A \cap B)$ .

Posons  $f = \mathbb{1}_B - \mathbb{1}_A$ . Nous allons montrer qu'il existe un entier  $N_0 > 0$  tel que tout morceau d'orbite de taille supérieure à  $N_0$  contienne plus d'éléments dans  $B$  que dans  $A$ . En effet, si ce n'était pas le cas, alors il existerait une suite d'entiers strictement croissante  $(n_k)_{k \in \mathbb{N}}$  et des points  $(x_k)_{k \in \mathbb{N}}$  tels que pour tout  $k \in \mathbb{N}$  on ait

$$\sum_{i=0}^{n_k} f(\varphi^i(x_k)) < 0.$$

Mais alors on obtiendrait une suite de mesures donnant un plus fort poids à  $A$  qu'à  $B$  en posant

$$\mu_k = \frac{1}{n_k + 1} \sum_{i=0}^{n_k} \varphi^i \circ \delta_{x_k} = \frac{1}{n_k + 1} \sum_{i=0}^{n_k} \delta_{\varphi^i(x_k)}.$$

Ces mesures ne sont toutefois pas  $\varphi$ -invariantes, mais quitte à extraire une sous-suite on peut supposer que  $\mu_k \xrightarrow[k \rightarrow \infty]{} \nu$ , et nous allons montrer que cette limite est, elle,  $\varphi$ -invariante. En effet, pour toute fonction  $g \in C(X)$  on a d'une part

$$\int g \, d\mu_k = \frac{1}{n_k + 1} \sum_{i=0}^{n_k} g(\varphi^i(x_k))$$

et d'autre part

$$\int g \, d\varphi_*\mu_k = \frac{1}{n_k + 1} \sum_{i=0}^{n_k} g(\varphi^{i+1}(x_k)) = \int g \, d\mu_k + \frac{1}{n_k + 1} (g(\varphi^{n_k+1}(x_k)) - g(x_k))$$

d'où

$$\left| \int g \, d\varphi_*\mu_k - \int g \, d\mu_k \right| \leq \frac{2\|g\|_\infty}{n_k + 1}$$

Ceci implique que  $\varphi_*\nu = \nu$ , i.e  $\nu \in M(\varphi)$ . De plus, par définition de convergence sur  $M(\varphi)$ , on a en particulier

$$\left| \int f \, d\nu - \int f \, d\mu_k \right| \xrightarrow[k \rightarrow \infty]{} 0$$

d'où  $\nu(B) - \nu(A) \leq 0$ , ce qui contredit l'hypothèse. Ainsi, puisque tout morceau d'orbite de taille supérieure à  $N_0$  contient plus d'éléments dans  $B$  que dans  $A$ , on peut prendre un clopen  $D \subset B$  tel que  $\varphi^i(D) \cap \varphi^j(D) = \emptyset$  pour tout  $i, j \leq N_0$

différents, et construire une partition K-R ayant pour base  $D$ , puis appliquer la Proposition 1.4 pour la découper en une partition dont les atomes engendrent  $A$  et  $B$ . Cette partition n'a alors que des tours de hauteur supérieure à  $N_0$  et chacune contient donc plus d'atomes inclus dans  $B$  que d'atomes inclus dans  $A$ . On voit alors naturellement l'élément recherché du groupe plein  $g$  se dessiner : il suffit de le définir comme envoyant chaque atome inclus dans  $A$  dans un atome inclus dans  $B$  de la même tour par la puissance de  $\varphi$  adéquate.  $\square$

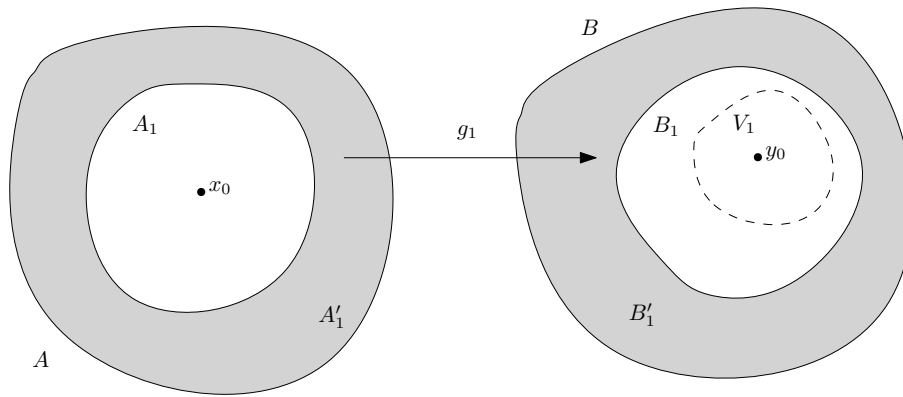


Figure 1.8: Première étape de la construction de  $g$  par va-et-vient

*Preuve du Théorème 1.8.*

Supposons  $A$  et  $B$  non vides, sinon il n'y a rien à prouver. On peut de plus supposer comme précédemment  $A$  et  $B$  disjoints. La suite du raisonnement est une sorte de va-et-vient qui ne sera pas étranger à la lectrice ayant fait un peu de logique, dont l'idée est de construire notre élément  $g$  sur des morceaux de plus en plus gros de  $A$ , comme le résume la figure 1.8. Prenons un point  $x_0 \in A$ . par minimalité de  $\varphi$ , il existe  $n_0$  tel que  $y_0 = \varphi^{n_0}(x_0) \in B$ . Fixons également une métrique compatible  $d$  sur  $X$ . On choisit un voisinage clopen  $A_1$  de  $x_0$  dans  $A$ , de diamètre inférieur à 1. Le complémentaire  $A'_1 = A \setminus A_1$  satisfera automatiquement

$$\mu(A'_1) < \mu(A) = \mu(B) \text{ pour toute mesure } \mu \in M(\varphi).$$

Toutefois on aimerait utiliser le Théorème 1.7 tout en laissant de la place autour de  $y_0$ , qui sera l'image de  $x_0$  par l'élément du groupe plein que nous sommes en train de construire. On trouve donc un voisinage clopen  $V_1$  de  $y_0$  dans  $B$  vérifiant

$$\mu(A'_1) < \mu(B \setminus V_1) \text{ pour toute mesure } \mu \in M(\varphi).$$

Ceci est possible par la Propriété 1.10. On peut maintenant utiliser le Théorème 1.7 dans de bonnes conditions, ce qui nous donne une involution  $g_1 \in [[\varphi]]$  telle que  $g_1(A'_1) = B'_1 \subset B \setminus V_1$ . On pose alors  $B_1 = B \setminus B'_1$ . Notons que par la remarque faite avant l'énoncé du théorème, on a  $\mu(B_1) = \mu(A_1)$  pour toute mesure  $\varphi$ -invariante. On réitère alors le processus dans l'autre sens, et on obtient ce faisant un voisinage

clopen  $B_2 \subset B_1$  de  $y_0$  de diamètre inférieur à  $\frac{1}{2}$ ,  $B'_2 = B_1 \setminus B_2$ , et une involution  $g_2 \in \llbracket \varphi \rrbracket$  telle que  $g_2(B'_2) = A'_2 \subset A_1 \setminus \{x_0\}$ . En itérant ce processus, on obtient une décomposition

$$A \cup B = \left( \bigsqcup_{n \in \mathbb{N}} A'_n \sqcup B'_n \right) \sqcup \{x_0, y_0\}$$

et on peut poser  $g \in [\varphi]$  défini par

$$g(x) = \begin{cases} x & \text{si } x \notin A \cup B \\ g_n(x) & \text{si } x \in A'_n \cup B'_n \\ y_0 & \text{si } x = x_0 \\ x_0 & \text{si } x = y_0 \end{cases}.$$

Les cocycles des  $g_n$  étant continus sur les clopens  $A'_n \cup B'_n$ , les seuls points de discontinuité éventuels du cocycle de  $g$  sont  $x_0$  et  $y_0$ .  $\square$

### 1.3 Résumé des chapitres

Les chapitres de ce manuscrit peuvent être lus indépendamment, quoiqu'ils fassent partie d'une même démarche. Certaines notions pourront être définies plusieurs fois pour assurer l'indépendance des chapitres. Chaque chapitre suit de près un article publié.

#### 1.3.1 Strong orbit equivalence and simple locally finite groups

Ce chapitre est une version légèrement modifiée de [R].

Nous commençons par y expliquer comment un certain groupe localement fini (et même *ample* d'après la terminologie de Krieger, que nous réutiliserons) peut être canoniquement associé à une suite de partitions de Kakutani-Rokhlin provenant d'un homéomorphisme minimal  $\varphi$  et à un point-base. L'étape suivante est de montrer que ce groupe  $\Gamma_x^\varphi$  ne dépend en fait pas de la suite de partitions choisie, mais seulement du point-base  $x$  de celle-ci, et qu'il s'agit en fait du sous-groupe de  $\llbracket \varphi \rrbracket$  constitué des éléments qui préservent l'orbite positive de  $x$ . De plus, nous montrons également que l'action de ces groupes sur l'algèbre Booléenne  $CO(X)$  ne dépend en fait même pas de ce point-base  $x$ , et est donc canoniquement associée au système. Ceci redémontre au passage que  $\overline{\Gamma_{x,\varphi}} = \overline{\llbracket \varphi \rrbracket}$ , résultat déjà présent dans [IM2] et [GM], mais nous permet surtout d'utiliser la machinerie développée par Krieger dans [K1] sur les groupes amples, à savoir le Théorème 1.3.3 développé dans l'Appendice A.

Grâce à cet outil, nous donnerons une preuve élémentaire (i.e n'utilisant que des manipulations sur des clopens) du résultat dû à Giordano, Putnam et Skau qui admet la reformulation suivante (cf la définition 1.3 de l'équivalence orbitale forte) :

**Théorème** ([GPS2], Corollaire 4.11). *Deux systèmes minimaux sur l'espace de Cantor  $(\varphi, X)$  et  $(\psi, X)$  sont fortement orbitalement équivalents si et seulement si  $\Gamma_x^\varphi$  et  $\Gamma_y^\psi$  sont isomorphes pour tout  $x, y \in X$  (ou de manière équivalente s'il existe de tels  $x, y$ ).*

Ensuite, nous montrons que le groupe dérivé  $D(\Gamma_x^\varphi)$  est toujours dense dans  $\Gamma_x^\varphi$  (ce qui se traduit en disant que leurs orbites sur  $CO(X)$  sont les mêmes), qu'il est toujours simple (en adaptant des outils tirés de [BM]), et qu'un isomorphisme entre deux tels groupes est toujours spatial (en mimant cette fois des arguments de [M4]). Tout ceci combiné donne la caractérisation suivante :

**Corollaire 1.9.** *Deux systèmes minimaux  $\varphi$  et  $\psi$  sont fortement orbitalement équivalents si et seulement si  $D(\Gamma_x^\varphi)$  et  $D(\Gamma_y^\psi)$  sont isomorphes en tant que groupe abstraits pour tous (ou de manière équivalente pour certains)  $x, y \in X$ .*

Celle-ci admet finalement une application dans un domaine tout à fait différent, celui de la réductibilité Borélienne, qui cherche à comparer la complexité des relations d'équivalences (pour les énoncés précis, se référer au Chapitre 2).

Un résultat classique de ce domaine est que pour tout groupe Polonais  $G$ , il existe une relation universelle  $E_G$  provenant d'une action Borélienne de  $G$  sur un espace de probabilité standard telle que toute autre telle action se Borel-réduise à  $E_G$ . Informellement parlant,  $E_G$  est la relation "la plus complexe possible" (au sens de la réductibilité Borélienne) qui provienne d'une telle action. Dans [M5], Melleray a prouvé que dans le cas  $G = S_\infty$  (le groupe de permutation des entiers, qui admet une panoplie impressionnante d'actions), la relation d'équivalence orbitale forte était une telle relation universelle. Cette dernière étant encodée par les groupes  $D(\Gamma_x^\varphi)$ , nous avons donc montré le théorème suivant :

**Théorème** (R. 2020). *La relation d'isomorphisme sur les groupes localement finis, simples et dénombrables est une relation universelle pour les actions Borélienne de  $S_\infty$ .*

### 1.3.2 From invariant measures to orbit equivalence, via locally finite groups

Ce chapitre est issu d'un travail en commun avec Julien Melleray et suit de près [MR].

La réalisation principale de ce chapitre est de donner une preuve élémentaire du Théorème 1.1 prouvé par Giordano, Putnam et Skau dans [GPS1], qui, rappelons-le, caractérise l'équivalence orbitale pour les actions de  $\mathbb{Z}$  sur  $X$  en termes de mesures invariantes.

Pour y parvenir, nous élargissons un peu la définition d'équivalence orbitale utilisée jusqu'ici pour inclure des actions d'autres groupes dénombrables que  $\mathbb{Z}$  en disant que deux actions de tels groupes  $\Gamma$  et  $\Lambda$  sont orbitalement équivalentes si

$$\forall x, x' \in X \quad (\exists \gamma \in \Gamma \gamma(x) = x') \Leftrightarrow (\exists \lambda \in \Lambda \lambda(g(x)) = g(x')).$$

Comme nos actions seront toutes par homéomorphismes, on se permettra de dire que les sous-groupes  $\Gamma$  et  $\Lambda$  de  $\text{Homeo}(X)$  sont orbitalement équivalents.



La démarche s'articule alors en deux étapes majeures :

- D'une part, montrer le théorème de classification pour les groupes amples (analogue du Théorème 1.1, et déjà présent dans [GPS1], cf Théorème 2.3) :

**Théorème 1.10.** [Giordano–Putnam–Skau] *Deux sous-groupes amples  $\Gamma$  et  $\Lambda$  de  $\text{Homeo}(X)$  agissant minimalement sont orbitalement équivalents si et seulement si il existe un homéomorphisme  $g$  de  $X$  tel que  $g_*M(\Gamma) = M(\Lambda)$ .*

- Relier un homéomorphisme minimal  $\varphi$  à un groupe ample de manière à pouvoir déduire le Théorème 1.1 de classification des actions de  $\mathbb{Z}$  de son analogue ci-dessus. Le groupe ample en question sera le groupe  $\Gamma_x^\varphi$  rencontré plus tôt, et on obtiendra

**Théorème 1.11.** *Soient  $\varphi$  un homéomorphisme minimal et  $x \in X$ . Alors les action de  $(\mathbb{Z}$  associée à)  $\varphi$  et de  $\Gamma_x^\varphi$  sont orbitalement équivalentes.*

L'essentiel de la difficulté réside dans la preuve du premier point, dans le sens où le Théorème d'absorption 1.12 ci-dessous (qui est vraiment le coeur de la difficulté, et dont l'enjeu à été présenté dans la section 1.2.4.3) utilisé pour prouver le second point aura déjà été utilisé dans le premier. Rappelons toutefois que c'est toujours une conséquence de [GPS3, Theorem 4.6].

**Théorème 1.12.** *Soit  $\Gamma$  un groupe ample agissant minimalement,  $K = K_1 \sqcup \sigma(K_1)$  un fermé de  $X$  ne rencontrant chaque orbite de  $\Gamma$  qu'en au plus un point, où  $\sigma \in \text{Homeo}(K)$  est une involution. Alors il existe un groupe ample  $\Sigma$  agissant minimalement dont les orbites sont exactement les orbites de  $\Gamma$ , hormis les orbites de  $k$  et de  $\sigma(k)$  qui sont "recollées" en une seul  $\Sigma$ -orbite. De plus  $\Gamma$  et  $\Sigma$  sont orbitalement équivalents.*

On dit d'un groupe ample  $\Gamma$  qu'il est saturé s'il est dense dans son groupe plein  $[\Gamma] = \{h \in \text{Homeo}(X) : \forall x \in X, \exists \gamma \in \Gamma, h(x) = \gamma(x)\}$ . Le Théorème de Krieger 1.3.3 garantit immédiatement que deux groupes amples saturés agissant minimalement vérifient le Théorème 1.10, il suffit en effet de remarquer que

$$\overline{[\Gamma]} = \{h \in \text{Homeo}(X) : \forall \mu \in M(\Gamma), h_*\mu = \mu\}.$$

La stratégie est donc, partant d'un groupe ample  $\Gamma$ , de réussir à le saturer en un nouveau groupe  $\tilde{\Gamma}$  qui lui reste orbitalement équivalent.

Cette étape est assez technique, et se base sur le renforcement du Théorème de Krieger suivant, qui nous permet d'obtenir le Théorème d'absorption 1.12, et nous paraît intéressant en soi.

**Théorème 1.13** ([MR]Théorème de Krieger avec contrôle sur un fermé). *Soient  $\Gamma, \Lambda$  des groupes amples minimaux de clôtures isomorphes (ce qui est une autre manière de dire que leurs orbites sur les clopens sont identifiables par un homéomorphisme, cf le Théorème 1.3.3 dans l'Appendice A). Soient  $K, L$  des fermés de  $X$  tels que  $K$  rencontre chaque orbite de  $\Gamma$  en au plus un point, et de même  $L$  rencontre chaque orbite de  $\Lambda$  en au plus un point. Soit encore  $h : K \rightarrow L$  un homéomorphisme.*

*Alors il existe  $g \in \text{Homeo}(X)$  tel que  $g\Lambda g^{-1} = \Gamma$ , et  $g|_K = h$ .*

L'Appendice A vise à rendre ce résultat plus accessible.

### 1.3.3 Appendice : Une preuve du Théorème de Krieger

Cet appendice est une réécriture agrémentée de détails et de figures du Théorème 1.3.3 qui est central dans notre approche. Il s'agit du résultat principal de l'article [K1], légèrement adapté à notre contexte.

**Théorème.** ([K1], Théorème 3.5) *Deux sous-groupes amples  $\Gamma$  et  $\Lambda$  de  $\text{Homeo}(X)$  sont spatialement conjugués (i.e il existe un homéomorphisme  $g$  tel que  $\Lambda = g\Gamma g^{-1}$ ) si et seulement si il existe un homéomorphisme  $g$  tel que*

$$\bar{g}: \begin{cases} CO(X)/\Gamma \longrightarrow CO(X)/\Lambda \\ Orb_{\Gamma}(A) \longmapsto Orb_{\Lambda}(gA) \end{cases} \text{ soit une bijection.}$$

Dans le Chapitre 3, nous avons amélioré ce résultat en montrant que l'on pouvait avoir du contrôle sur l'homéomorphisme qui conjugue les groupes lorsque l'on part de deux groupes ayant des orbites identifiables sur  $CO(X)$ . Néanmoins, la preuve du théorème initial est déjà relativement lourde en notations et en détails techniques, c'est pourquoi nous avons voulu en donner une preuve la plus accessible possible, qui aide à comprendre les idées en jeu avant de se lancer dans la compréhension du Théorème 3.2 (espérons que ce soit le cas).

En ce sens, nous avons également ajouté à la fin de l'annexe un exercice qui est une version intermédiaire entre les Théorèmes 1.3.3 et 3.2 et est une adaptation assez facile de la preuve du premier, en espérant que cela permette de mettre en lumière certains points de la preuve du second, qui ne sont ni plus ni moins que tenter d'étendre les ingrédients utilisés (en particulier les Théorèmes 1.7 et 1.8).



# Strong orbit equivalence and simple locally finite groups

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This chapter follows closely [R].

## 2.1 Introduction

Algebraic tools (notably, dimension groups) have long been known as a useful and effective way to study minimal homeomorphisms of the Cantor space; sometimes this leads to relatively complex, high-powered proofs of theorems that one would like to understand dynamically, so as to gain a different perspective or extend them to other contexts. Such is one of the purposes of this article. Using a different perspective can lead to new results: even though the core of this article is about topological dynamics, our main result is about Borel reducibility theory. We only briefly explain the general context here and refer the interested reader to [BK], section 3, [G], section 5 and references therein for more advanced results.

*Definition.* Let  $E, F$  be equivalence relations on standard Borel spaces. We say that  $E$  is *Borel reducible* to  $F$ , and write  $E \leq F$ , if there exists a Borel map  $f$  such that

$$\forall x, x' \quad xEx' \iff f(x)Ff(x').$$

We call such a map  $f$  a *Borel reduction* from  $E$  to  $F$ . If  $E \leq F$  and  $F \leq E$ , we say that  $E$  and  $F$  are *Borel bireducible*.

The idea behind this definition is that solving the classification problem associated to  $E$  (i.e. deciding when two elements are in the same  $E$ -class) is then simpler than solving the one associated to  $F$ , hence the relation  $F$  can be considered "more complex" than  $E$ . However, without a requirement on  $f$ , this notion would only detect the number of equivalence classes, and such a map could be very chaotic, this is why one wants to ensure that the correspondence is somehow computable, and in this paper, the notion of computable retained is Borel.

An important source of equivalence relations is given by actions of Polish groups. The following theorem shows that given a Polish group, there always exists an equivalence relation arising from it that is as complex as possible :

**Théorème** ([BK], Corollary 3.5.2). *Let  $G$  be a Polish group. There exists an equivalence relation  $E_G$  arising from a Borel  $G$ -action on a standard Borel space such that any other such relation Borel reduces to it.*

We will call such a relation  $G$ -universal. We will be interested in equivalence relations arising from actions of  $S_\infty$ , the permutation group of a countably infinite set, which is a well known Polish group (see [G], section 2.4), and we will denote  $E_\infty$  a (unique up to Borel bireducibility) universal relation arising from an action of it. Our main theorem is the following:

**Théorème.** *The relation of isomorphism of countable, locally finite, simple groups is a universal relation arising from a Borel action of  $S_\infty$  (i.e is Borel bireducible to  $E_\infty$ ).*

A proof of this result is given in section 2.4. We use a result of [M5], namely that strong orbit equivalence (a notion from topological dynamics that we will define straight after this) is Borel bireducible to  $E_\infty$ , and we find a Borel way to associate to minimal homeomorphisms some locally finite simple groups that are isomorphic exactly when the homeomorphisms are strong orbit equivalent.

In order to describe more precisely the groups at stake, let us move on to some notions of dynamics.

Any action of a group  $G$  on a set  $X$  induces an equivalence relation, whose equivalence classes are the  $G$ -orbits. Forgetting the action to focus only on this equivalence relation, two actions can be very different dynamically speaking, even coming from different groups, yet generate the same equivalence relation up to a bijection. This leads to the notion of orbit equivalence:

*Definition.* Let  $G, G'$  be groups and  $X, X'$  be sets. Two actions  $\alpha: G \curvearrowright X$  and  $\beta: G' \curvearrowright X'$  are *orbit equivalent* if there exists a bijection  $h: X \rightarrow X'$  (called *an orbit equivalence between  $\alpha$  and  $\beta$* ) that realizes a bijective correspondence between  $\alpha$ -orbits and  $\beta$ -orbits, i.e

$$\forall x \in X \quad h(\text{Orb}_\alpha(x)) = \text{Orb}_\beta(h(x))$$

If the sets  $X$  and  $X'$  are endowed with a structure, it is natural to require  $h$  to preserve this structure, i.e. to be an isomorphism from  $X$  to  $X'$ , and this will be a part of our conventions below. It is then tempting to try and classify actions up to

orbit equivalence. In a measure theoretical context, this is now a well-understood problem; A combination of works by Dye ([D2]) and later Ornstein–Weiss ([OW]) gives the following famous result: there is only one probability measure preserving, ergodic action of an amenable group on a standard probability space up to orbit equivalence.

However, in a topological context, when  $X$  is a Cantor space, the situation is more complicated. In [GPS1] and [GPS2], Giordano, Putnam and Skau managed, using  $C^*$ -algebra techniques, to obtain results about  $\mathbb{Z}$ -actions. Before quoting them, let us recall some basic objects from topological dynamics:

*Definition.* A homeomorphism  $\varphi$  on the Cantor space  $X$  is said to be *minimal* if every  $\varphi$ -orbit is dense in  $X$ :  $\forall x \in X \overline{Orb_\varphi(x)} = X$ .

*Definition.* The *full group* of  $\varphi$ , denoted by  $[\varphi]$ , consists of all homeomorphisms that preserve  $\varphi$ -orbits:

$$[\varphi] = \left\{ f \in \text{Homeo}(X) : \forall x \in X \exists n_f(x) \in \mathbb{Z} \text{ such that } f(x) = \varphi^{n_f(x)}(x) \right\}$$

For  $f \in [\varphi]$ , the application  $n_f : \begin{cases} X \longrightarrow \mathbb{Z} \\ x \longmapsto n_f(x) \end{cases}$  is called a *cocycle* of  $f$ .

*Remark.* Since  $\varphi$  is aperiodic (because it is minimal), cocycles are uniquely defined.

*Definition.* The *topological full group* of  $\varphi$ , denoted by  $\llbracket \varphi \rrbracket$ , is the subgroup of elements in  $[\varphi]$  which have a continuous cocycle.

Using this vocabulary, Giordano, Putnam and Skau's Theorem about orbit equivalence is the following:

**Théorème** ([GPS2], Corollary 4.6). *Let  $(\varphi_1, X_1)$  and  $(\varphi_2, X_2)$  be minimal Cantor systems. Then the following are equivalent:*

- (i) *The two systems  $(\varphi_1, X_1)$  and  $(\varphi_2, X_2)$  are orbit equivalent*
- (ii) *The full groups  $[\varphi_1]$  and  $[\varphi_2]$  are isomorphic as abstract groups*
- (iii) *There exists a homeomorphism  $g : X_1 \rightarrow X_2$  that pushes forward  $\varphi_1$ -invariant measures onto  $\varphi_2$ -invariant measures, i.e  $g_*M(\varphi_1) = M(\varphi_2)$ , where*

$$M(\varphi_i) = \{ \mu \text{ probability measure on } X_i : \varphi_{i*}\mu = \mu \}$$

In particular, point (iii) implies that there exist continuum many pairwise non orbit equivalent Cantor minimal systems, see for example [D1].

Giordano–Putnam–Skau also considered a notion which differs slightly from orbit equivalence and that we are concerned with in this article, which they call *strong orbit equivalence*. To explain it, let us introduce some terminology: an orbit equivalence  $h$  between two minimal systems  $(\varphi, X)$  and  $(\psi, X)$  gives rise to two applications  $n$  and  $m$  from  $X$  to  $\mathbb{Z}$  such that

$$\forall x \in X \ h(\varphi(x)) = \psi^{n(x)}(h(x)) \text{ and conversely } h^{-1}(\psi(x)) = \varphi^{m(x)}(h^{-1}(x)).$$

These two applications are called *cocycles associated to  $h$* . The orbit equivalence  $h$  is called a *strong orbit equivalence* if both cocycles  $n$  and  $m$  have at most one point of discontinuity. This notion is, as the next theorem highlights, closely related to the group  $\Gamma_x^\varphi$  (associated to a minimal system  $(\varphi, X)$  and a point  $x \in X$ ) consisting of all elements  $\gamma$  of  $[[\varphi]]$  such that  $\gamma(Orb_\varphi^+(x)) = Orb_\varphi^+(x)$ .

**Théorème** ([GPS2], Corollary 4.11). *Two Cantor minimal systems  $(\varphi, X)$  and  $(\psi, X)$  are strong orbit equivalent if and only if  $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$  are isomorphic as abstract groups for all  $x, y \in X$ .*

An elementary proof of this result, which justifies our original interest for the groups  $\Gamma_x^\varphi$ , is given in section 2.3. By *elementary*, we mean that the proof is only based on manipulations on closed-open sets. The objects we mainly use are sequences of Kakutani-Rokhlin partitions, the point of which is to describe in finite time the image of (almost) any closed-open set by a minimal homeomorphism (details in section 2.2.1). Our general strategy is based on a theorem of Krieger ([K1], Theorem 3.5). Recall that two subgroups  $\Gamma$  and  $\Lambda$  of  $\text{Homeo}(X)$  are *spatially isomorphic* if there exists a homeomorphism  $g$  of  $X$  such that

$$\Gamma = g\Lambda g^{-1}.$$

Krieger's theorem establishes that two countable, locally finite groups  $\Gamma$  and  $\Lambda$  satisfying some extra properties, called *ample groups*, are spatially isomorphic if and only if there exists a homeomorphism  $g$  such that

$$\overline{g}: \begin{cases} CO(X)/\Gamma \longrightarrow CO(X)/\Lambda \\ Orb_\Gamma(A) \longmapsto Orb_\Lambda(g(A)) \end{cases}$$

is a bijection.

To set the stage for the application of Krieger's theorem, we first recall the construction of the groups  $\Gamma_x^\varphi$  out of a sequence of Kakutani-Rokhlin partitions, which shows that they are locally finite (and actually even ample). Afterwards, an important step is to show that the relation induced by the  $\Gamma_x^\varphi$  on closed-open sets does not depend on the point  $x$ , i.e using Krieger's vocabulary that they have the same *dimension range*. We also notice that this approach recovers the result that  $\overline{\Gamma_x^\varphi} = \overline{[[\varphi]]}$  (see [IM1], Theorem 5.6, or [GM])

Finally, the groups  $\Gamma_x^\varphi$  are not necessarily simple, but in section 2.4, adapting arguments from [BM] on the one hand, and [M4] on the other hand, we show respectively that the groups  $D(\Gamma_x^\varphi)$  are always simple, and that any isomorphism between two such groups is spatial, which makes them locally finite, simple groups attesting to strong orbit equivalence of homeomorphisms they are attached to, as desired.

## 2.2 Preliminaries

For the remainder of this article,  $X$  denotes the Cantor space;  $\varphi$  denotes a minimal homeomorphism of  $X$  and  $x_0$  a point of  $X$ ; we call a closed-open set a clopen set,

and  $CO(X)$  stands for the Boolean algebra of clopen subsets of  $X$ . Moreover, we use the notation  $\llbracket i; j \rrbracket$  to denote the interval of integers  $\{i, i+1, \dots, j\}$ , which is by convention empty if  $i > j$ . Finally we draw the reader's attention to the following fact: for us,  $\mathbb{N}$  is the set of non-negative integers, and  $Orb_\varphi^+(x_0) = \{\varphi^n(x_0)\}_{n \in \mathbb{N}}$ .

Most of this section consists of well-known facts and objects, and the reader knowing what a Kakutani-Rokhlin partition is can skim through it until Theorem 2.2 and discussion below.

The following property is very useful to build and understand elements of a topological full group in practice:

**Proposition 2.1.** *Let  $f$  be a homeomorphism of  $X$ . Then  $f \in \llbracket \varphi \rrbracket$  if and only if there exist  $A_1, \dots, A_n \in CO(X)$  and  $k_1, \dots, k_n \in \mathbb{Z}$  such that  $X = \bigsqcup_{i=1}^n A_i$  and  $f|_{A_i} = \varphi|_{A_i}^{k_i}$ .*

The following theorem is well-known (and easy to check in this situation):

**Théorème 2.1.** *(Stone's representation theorem for Boolean algebras)*

*For every automorphism  $\alpha: CO(X) \rightarrow CO(X)$ , there exists a (unique) homeomorphism  $g$  on  $X$  that extends  $\alpha$  (i.e.  $g(C) = \alpha(C)$  for every  $C \in CO(X)$ )*

*Remark 2.1.*  $\text{Aut}(CO(X))$  is a closed subset of  $CO(X)^{CO(X)}$  equipped with the product topology (of the discrete topology on  $CO(X)$ ). Thus this theorem implies that  $\text{Homeo}(X)$  inherits a Polish group structure from that topology; a sub-basis is given by the sets

$$[A \rightarrow B] = \{f \in \text{Homeo}(X) : f(A) = B\}$$

where  $A$  and  $B$  are clopen sets. Actually, this is the only Polish topology on  $\text{Homeo}(X)$  (see [RS]). We always consider  $\text{Homeo}(X)$  as endowed with this topology, and think of it in this way, even though one can use a complete metric on  $X$  and think of  $\text{Homeo}(X)$  as being endowed with the topology of uniform convergence.

### 2.2.1 Kakutani-Rokhlin partitions

Kakutani-Rokhlin partitions (for which we will use the term "K-R partitions" for brevity) are an essential tool for an elementary study of the dynamics of minimal homeomorphisms.

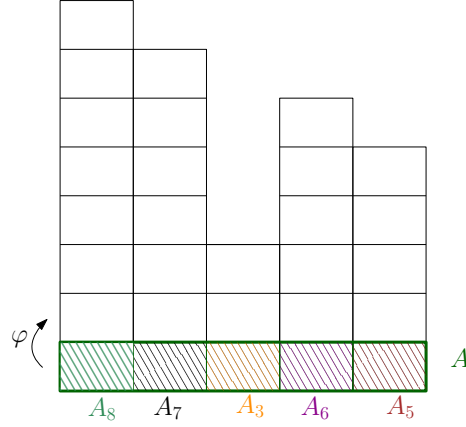
The idea is the following: given a nonempty clopen set  $A$ , we look at the first return map

$$\varphi_A: \begin{cases} A \longrightarrow A \\ x \longmapsto \varphi^{\tau_A(x)}(x) \end{cases}, \text{ where}$$

$$\tau_A: \begin{cases} A \longrightarrow \mathbb{N} \\ x \longmapsto \min\{k > 0 : \varphi^k(x) \in A\} \end{cases}$$

is well-defined because a homeomorphism is minimal if and only if every forward orbit is dense in  $X$ . We can also consider  $\varphi_A$  as a homeomorphism of  $X$  by requiring  $\varphi_A|_{A^c} = id|_{A^c}$ . Since  $A$  is compact and  $\tau_A$  is continuous,  $\tau_A(A)$  is finite,



Figure 2.1: K-R partition built from  $A$ 

and  $A$  admits a finite clopen partition consisting of points that return to  $A$  in a certain number of steps. More formally,  $A = \bigsqcup_{k \in \mathbb{N}} A_k$ , where  $A_k := \tau_A^{-1}(k)$  is clopen, and all but a finite number of them are empty. We then obtain a partition of  $X$  that can be represented as on figure 2.1.

More generally we define K-R partitions as follows:

*Definition 2.1.* A K-R partition associated to  $\varphi$  is a clopen partition

$$\Xi = (D_{i,j})_{i \in \llbracket 1, N \rrbracket, j \in \llbracket 0, H_i - 1 \rrbracket}$$

of  $X$  such that  $\varphi(D_{i,j-1}) = D_{i,j}$  for all  $i \in \llbracket 1, N \rrbracket$  and all  $j \in \llbracket 1, H_i - 1 \rrbracket$ .

We call, for  $i \in \llbracket 1, N \rrbracket$ , *tower  $i$  of  $\Xi$*  the set  $T_i = \bigsqcup_{j \in \llbracket 0, H_i - 1 \rrbracket} D_{i,j}$ , and *height* of this tower the number  $H_i$ . We also talk about the  $k$ -th floor of  $\Xi$  to designate  $\bigsqcup_{i \in \llbracket 1, N \rrbracket} \varphi^k(D_{i,0})$ . The 0-th floor is called *the base* of  $\Xi$ , and denoted by  $B(\Xi)$ , and the  $-1$ -th one is called *the top* of the partition.

In the case where the partition carries an index ( $\Xi_n$  instead of  $\Xi$ ), which will be the case very soon, we talk about the tower  $T_i^n$ , the atom  $D_{i,j}^n$ , etc.

*Remark 2.2.* Note that  $\bigsqcup_{i \in \llbracket 1, N \rrbracket} D_{i, H_i - 1}$  is mapped by  $\varphi$  onto  $B(\Xi)$ , and thus is the top of the partition. However,  $D_{i, H_i - 1}$  can be mapped anywhere in  $B(\Xi)$ .

**Notation.** We denote by  $\langle \Xi \rangle$  the Boolean algebra generated by the atoms  $(D_{i,j})_{i \in \llbracket 1, N \rrbracket, j \in \llbracket 0, H_i - 1 \rrbracket}$  of  $\Xi$ .

The idea behind K-R partitions is that they represent how  $\varphi$  acts on the atoms of  $\Xi$  which are not contained in the top of  $\Xi$ . We want to get better and better approximations of  $\varphi$  in terms of how it acts on clopen sets. In the same spirit as Theorem 2.1, one can check that knowing how  $\varphi$  acts on clopen sets that does not contain a particular point determines  $\varphi$ . In our case, we want to construct a sequence of K-R partitions in which every clopen set appears, and such that the intersection of their bases (and hence of their tops) is a single point. To do that, we have to be able to refine a K-R partition into another one, without losing information we have already obtained. This is the purpose of the following proposition:

**Proposition 2.2.** *Let  $A \in CO(X)$ , and  $\Xi$  be a K-R partition. Then there exists a K-R partition  $\Xi'$  finer than  $\Xi$  such that  $A \in \langle \Xi' \rangle$ .*

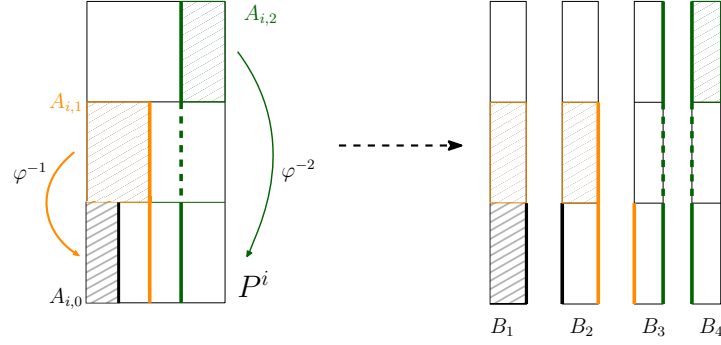


Figure 2.2: Cutting process in tower  $i$  of  $\Xi$ ,  $A_{i,j} = A \cap D_{i,j}$

*Proof.* For each tower  $T_i$  of  $\Xi$ , we define an equivalence relation  $\mathcal{R}_i$  on  $D_{i,0}$  by

$$x \mathcal{R}_i y \Leftrightarrow \forall j < H_i \left( (\varphi^j(x) \in A \wedge \varphi^j(y) \in A) \vee (\varphi^j(x) \in A^c \wedge \varphi^j(y) \in A^c) \right)$$

Denote by  $P^i$  the partition of  $D_{i,0}$  associated to  $\mathcal{R}_i$ . (cf figure 2.2)

Denoting by  $(B_l)_{l \in \llbracket 0, N_i \rrbracket}$  the atoms of  $P^i$ , we just "cut" the  $i$ -th tower into  $N_i$  towers whose bases are the  $B_l$ 's. Following this method for each tower, we obtain a K-R partition  $\Xi'$  finer than  $\Xi$  such that every  $A \cap D_{i,j}$  is in  $\langle \Xi' \rangle$ , and so  $A$  is also in  $\langle \Xi' \rangle$ .  $\square$

Let  $(U_n)_{n \in \mathbb{N}}$  be a clopen basis of the topology. Applying Proposition 2.2 inductively, we obtain the following:

**Corollary 2.3.** *There exists a sequence of K-R partitions  $(\Xi_n)_{n \in \mathbb{N}}$  such that the following conditions hold for all  $n \in \mathbb{N}$ :*

- (i)  $\Xi_{n+1}$  is finer than  $\Xi_n$
- (ii)  $B(\Xi_{n+1}) \subset B(\Xi_n)$  and  $\bigcap_i B(\Xi_i) = \{x_0\}$
- (iii)  $U_n \in \langle \Xi_n \rangle$
- (iv) The minimal height of  $\Xi_n$  is greater than  $n$ .

When we consider a sequence of K-R partitions, we assume from now on that it fulfills properties (i) to (iv).

*Remark 2.3.* Property (iv) is obtained thanks to aperiodicity of  $\varphi$ : the points  $x_0, \varphi(x_0), \dots, \varphi^n(x_0)$  are all distinct, thus for a sufficiently small neighbourhood  $U$  of  $x_0$ , we have that  $U, \varphi(U), \dots, \varphi^n(U)$  are pairwise disjoint.

*Definition 2.2.* We call *base point* of a sequence of K-R partitions  $(\Xi_n)$  the point  $x_0$  that appears in property (ii), and *top point* of the sequence the point  $\varphi^{-1}(x_0)$ .

*Remark 2.4.* An important thing to understand about this construction is that a tower of  $\Xi_{n+1}$  is obtained by cutting (vertically) the towers of  $\Xi_n$  and then stacking some of them on top of each other. This is what is called "cutting and stacking" (see figure 2.3; arrows tell us where a clopen set at the top of the partition is sent by  $\varphi$ . If there is none, it means that it is sent into the base of  $\Xi_{n+1}$ ). Indeed, for every  $i$ ,  $D_{i,0}^{n+1} \subset D_{i_0,0}^n \subset B(\Xi_n)$  for some  $i_0$ , so  $\bigsqcup_{k=0}^{H_{i_0}^n-1} D_{i,k}^{n+1}$  is exactly obtained by cutting the  $i_0$ -th tower of  $\Xi_n$ . If  $D_{i,H_{i_0}^n-1}^{n+1}$  is not on the top of  $\Xi_{n+1}$ , it suffices to look at the tower of  $\Xi_n$  in which it is mapped by  $\varphi$  to know which "tower" to stack over this one: if  $\varphi(D_{i,H_{i_0}^n-1}^{n+1}) \subset D_{i_1,0}^n$ , then  $\bigsqcup_{k=H_{i_0}^n}^{H_{i_0}^n+H_{i_1}^n-1} D_{i,k}^{n+1}$  is obtained by cutting the  $i_1$ -th tower of  $\Xi_n$ , and so on.

### 2.2.2 A locally finite group associated to a minimal homeomorphism

In this section we study the group of homeomorphisms in  $[[\varphi]]$  that preserve the non-negative semi-orbit of  $x_0$ . We denote this group

$$\Gamma_{x_0}^\varphi := \{f \in [[\varphi]] : f(Orb^+(x_0)) = Orb^+(x_0)\},$$

or just  $\Gamma_{x_0}$  if the homeomorphism involved is clear from the context. As recalled in the introduction, these groups are used in [GPS2] and are one of the main objects under consideration in our article. For the moment we focus on explaining how they are related to K-R partitions.

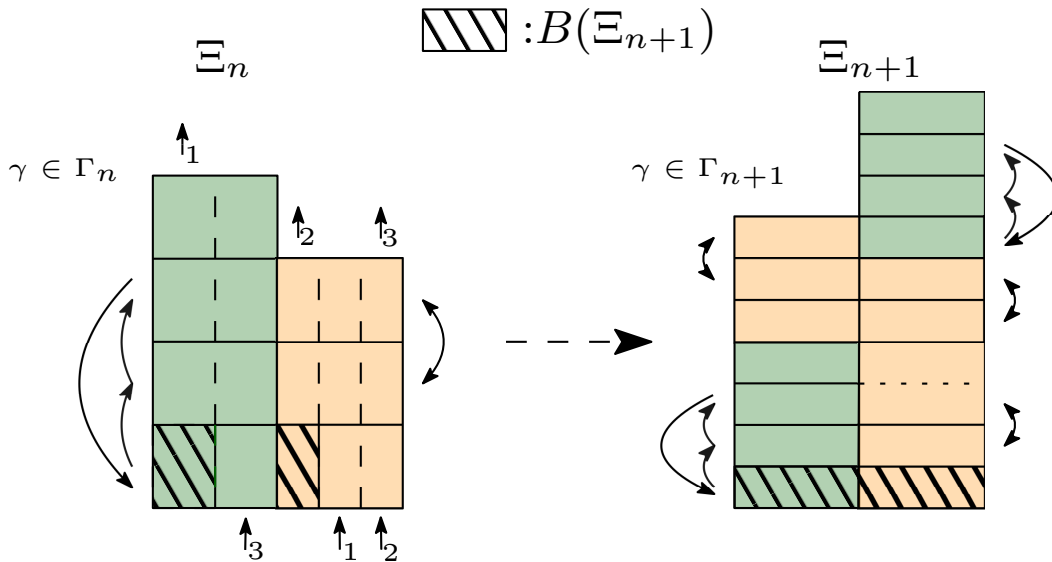


Figure 2.3: an element  $\gamma$  in  $\Gamma_n$  and in  $\Gamma_{n+1}$

Let  $(\Xi_n)_{n \in \mathbb{N}}$  be a sequence of K-R partitions (satisfying properties (i) to (iv)). For each  $n \in \mathbb{N}$ , we consider the group  $\Gamma_n$  of elements of  $[[\varphi]]$  that have a constant

cocycle on atoms of  $\Xi_n$  and that "stay" in each tower of the partition, meaning that an atom can not go through either the top of its tower nor its base under the action of a  $\gamma$  in  $\Gamma_n$ .

More formally, if  $(D_{i,j})_{i,j}$  are the atoms of  $\Xi_n$  and  $H_i$  is the height of the  $i$ -th tower, then

$$\Gamma_n = \{f \in \llbracket \varphi \rrbracket : \forall i, j \ f \upharpoonright_{D_{i,j}} = \varphi^k \upharpoonright_{D_{i,j}}, \ k \in \llbracket -j, H_i - j - 1 \rrbracket\}$$

An element of this group may be thought of as permuting atoms in each tower, but it is important to remember that it acts on points, in particular that helps to see the inclusion  $\Gamma_n \subset \Gamma_{n+1}$ .

Indeed, as  $\Xi_{n+1}$  is constructed from  $\Xi_n$  by cutting and stacking, each  $\gamma \in \Gamma_n$  also belongs to  $\Gamma_{n+1}$  (see figure 2.3)

*Definition 2.3.* We have obtained from a sequence of K-R partitions  $(\Xi_n)$  a locally finite group  $\Gamma_\Xi = \bigcup_{n \in \mathbb{N}} \Gamma_n$ . If there is risk of confusion, we write  $\Gamma_\Xi^\varphi$  to mention the homeomorphism involved.

For the moment, it seems that  $\Gamma_\Xi$  depends on the sequence of K-R partitions we have chosen. We note that it only depends on the choice of the base point. Recall that

$$\Gamma_{x_0}^\varphi = \{\gamma \in \llbracket \varphi \rrbracket \mid \gamma(\text{Orb}_\varphi^+(x_0)) = \text{Orb}_\varphi^+(x_0)\}$$

**Proposition 2.4.** *The groups  $\Gamma_\Xi$  and  $\Gamma_{x_0}$  are equal.*

*Proof.* Let  $\gamma \in \Gamma_\Xi$ , and  $y \in \text{Orb}^+(x_0)$ . Say  $y = \varphi^k(x_0)$  for a  $k \in \mathbb{N}$ . By property (iv) of Corollary 2.3, there exists  $N \in \mathbb{N}$  big enough that  $\gamma \in \Gamma_N$  and each tower of  $\Gamma_N$  has an height greater or equal to  $k + 1$ . Then  $x_0$  and  $y$  belong to the same tower of  $\Xi_N$ , and  $x_0 \in B(\Xi_N)$ . Since  $\gamma \in \Gamma_N$ ,  $\gamma(y) = \varphi^j(y)$  with  $j \geq -k$ , we have  $\gamma(y) \in \text{Orb}^+(x_0)$ . Since the same property holds for  $\gamma^{-1}$ , we obtain

$$\gamma(\text{Orb}^+(x_0)) = \text{Orb}^+(x_0).$$

Conversely, let  $h \in \Gamma_{x_0}$ . Let  $X = \bigsqcup_{i=1}^n A_i$  be a clopen partition such that  $h \upharpoonright_{A_i} = \varphi^{k_i} \upharpoonright_{A_i}$ . Define  $m_i = \min\{k \geq 0 : \varphi^k(x_0) \in A_i\}$ . Necessarily,  $k_i \geq -m_i$ . On the other hand, since  $\varphi^k(x_0) \notin A_i$  for all  $k \in \llbracket 0, m_i - 1 \rrbracket$ , there exists a clopen set  $U_k^i \ni \varphi^k(x_0)$  disjoint from  $A_i$ . Thus  $\bigcap_i \bigcap_{0 \leq k < m_i} \varphi^{-k}(U_k^i)$  is a neighbourhood of  $x_0$ , hence  $B(\Xi_N)$  is contained in it for  $N$  big enough. Moreover, we can take  $N$  big enough that every  $A_i \in \langle \Xi_N \rangle$ , and every tower in  $\Xi_N$  has an height greater than  $\max_i(m_i)$ . Then every atom of  $\Xi_N$  is included into one of the  $A_i$ 's, and an atom contained in  $A_i$  cannot appear before the  $m_i$ -th floor (because the  $k$ -th floor is a subset of  $U_k^i$  which is disjoint from  $A_i$ ).

Since  $h \in \llbracket \varphi \rrbracket$ , we also have

$$h(\text{Orb}_\varphi^{<0}(x_0)) = \text{Orb}_\varphi^{<0}(x_0)$$

(where  $Orb_\varphi^{\leq 0}(x_0)$  stands for  $\{\varphi^k(x_0), k < 0\}$ ). Thus, defining

$$m'_i = \max\{k < 0: \varphi^k(x_0) \in A_i\}$$

and using the same argument, we get that  $k_i < -m'_i$  and an atom contained in  $A_i$  cannot appear higher than the  $m'_i$ -th floor (i.e there is at least  $\text{abs } m'_i - 1$  atoms above it before reaching the top of the tower).

This shows that  $h$  cannot send an atom of  $\Xi_N$  through the top or the base of the partition, and so  $h \in \Gamma_\Xi$ .  $\square$

In the next section, we use as a key ingredient the following theorem which is a combination of two results: the equivalence between Conditions 2 and 3 is due to Krieger ([K1], Theorem 3.5), and its proof is elementary (actually, it is based on a back-and-forth argument that is easy to set up directly in the case of the groups  $\Gamma_x^\varphi$ ; this proof is developed in Appendix A). The equivalence between Conditions 1 and 2 is known since Giordano, Putnam and Skau (see [GPS2], Theorem 4.2) in the case  $(\Gamma, \Lambda) = (\Gamma_{x_1}^\varphi, \Gamma_{x_2}^\psi)$ . New proofs based on reconstruction theorems have been developed later, and we give a proof in the case  $(\Gamma, \Lambda) = (D(\Gamma_{x_1}^\varphi), D(\Gamma_{x_2}^\psi))$  using this approach in Proposition 2.8.

**Théorème 2.2.** *Let  $\varphi, \psi$  be minimal homeomorphisms,  $x_1, x_2$  be two points in  $X$ , and  $(\Gamma, \Lambda)$  denote either  $(\Gamma_{x_1}^\varphi, \Gamma_{x_2}^\psi)$  or their commutator subgroups  $(D(\Gamma_{x_1}^\varphi), D(\Gamma_{x_2}^\psi))$ . Then the following are equivalent:*

1. *The abstract groups  $\Gamma$  and  $\Lambda$  are isomorphic*
2. *There exists a homeomorphism  $g$  such that  $\Gamma = g\Lambda g^{-1}$*
3. *There exists a homeomorphism  $g$  such that*

$$\bar{g}: \begin{cases} CO(X)/\Gamma \longrightarrow CO(X)/\Lambda \\ Orb_\Gamma(A) \longmapsto Orb_\Lambda(gA) \end{cases} \text{ is a bijection.}$$

*Remark 2.5.* Moreover, a bit more is true: if Condition 3 is satisfied, and  $(x, y), (x', y')$  are two pairs of point belonging respectively to different  $\Lambda$  and  $\Gamma$  orbits, then the homeomorphism  $g$  of Condition 2 can be chosen such that  $g(x) = x'$  and  $g(y) = y'$ . This is not part of the original theorem in [K1] but can easily be seen following the proof (See the Exercise at the end of Appendix A). A proof of a much stronger generalization is given in Chapter 3 (Theorem 3.2).

*Definition 2.4* (Krieger's vocabulary). The relation induced by a group  $\Gamma_x^\varphi$  on  $CO(X)$  is called *dimension range of  $\Gamma_x^\varphi$* . A homeomorphism  $g$  such that  $\bar{g}$  (defined as in 3) is a bijection is an *isomorphism between the dimension ranges of  $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$* , and those two groups are said to have *isomorphic dimension ranges*.

## 2.3 An elementary proof of Giordano, Putnam and Skau's characterization of strong orbit equivalence

We are now ready to prove Giordano, Putnam and Skau's characterization of strong orbit equivalence. Let us recall the definition of strong orbit equivalence :

*Definition 2.5.* Two minimal homeomorphisms  $\varphi$  and  $\psi$  are called *strong orbit equivalent* if there exists a homeomorphism  $g$  such that

$$\forall x \in X \quad g(\text{Orb}_\varphi(x)) = \text{Orb}_\psi(g(x))$$

and such that the two associated cocycles  $n$  and  $m$ , defined by

$$\forall x \in X \quad g(\varphi(x)) = \psi^{n(x)}(g(x)) \text{ and } g^{-1}(\psi(x)) = \varphi^{m(x)}(g^{-1}(x))$$

have at most one point of discontinuity each.

First of all we need to gain a better understanding of the relation induced by  $\Gamma_x$  on clopen sets (the dimension range of  $\Gamma_x$ , cf definition 2.4). We have seen that the group does not depend on the sequence of partitions out of which it is constructed. Now we go further, showing that the dimension range of  $\Gamma_x$  does not depend on the base point  $x$  either.

**Lemma 2.5.** *For every  $x, x' \in X$ ,  $\Gamma_x$  and  $\Gamma_{x'}$  have the same dimension range.*

*Proof.* Let  $(\Xi_n)$  and  $(\Xi'_n)$  be two sequences of K-R partitions, with bases  $(D^n)$  and  $(F^n)$  respectively, such that  $\bigcap D^n = \{x\}$  and  $\bigcap F^n = \{x'\}$ . Let also  $A$  and  $B$  be two clopen sets such that  $B \in \text{Orb}_{\Gamma_\Xi}(A)$ . There exists  $n \in \mathbb{N}$  and  $\gamma \in \Gamma_{n,\Xi}$  such that  $\gamma(A) = B$ , and  $A, B$  are in  $\langle \Xi_n \rangle$ . Then in each tower of  $\Xi_n$  there are as many atoms which are contained in  $A$  as in  $B$ . Suppose  $x' \in D^n_{i,j}$ , and choose  $m \in \mathbb{N}$  such that  $\Xi'_m$  refines  $\Xi_n$ , and  $F^m \subset D^n_{i,j}$ . Let  $T'_k$  be a tower of  $\Xi'_m$  with base  $F^m_{k,0}$ . Then

$$H_k^m = H_i^n - j + H_{i_1}^n + \dots + H_{i_t}^n + j \text{ for some indices } i_1, \dots, i_t$$

and  $T'_k$  is composed of a stacking of  $\bigcup_{p \geq j} D^n_{i,p} \cap T'_{k'} \cap T'_{i_1} \cap T'_{i_2} \dots \cap T'_{i_t} \cap T'_{k'}$ , and eventually  $\bigcup_{p < j} D^n_{i,p} \cap T'_k$  (see figure 2.4). So there are as many atoms of  $T'_k$  that are included in  $A$  as in  $B$ . We can then naturally define an involution  $g \in \Gamma_{\Xi'_m}$  such that  $g(A) = B$  (see figure 2.4); whence  $B \in \text{Orb}_{\Gamma_{\Xi'}}(A)$ , which concludes the proof.  $\square$

*Remark 2.6.* This lemma could also be seen as an easy consequence of the fact that  $\overline{\Gamma_\varphi} = \overline{[\varphi]}$  (see [GM] or [IM1], Theorem 5.6). Conversely, Lemmas 2.5 and 2.6 give an elementary proof of that fact, using very similar arguments to those in the proof of Theorem 2.3.

**Théorème 2.3** ([GPS2], Corollary 4.11). *Let  $\varphi$  and  $\psi$  be two minimal homeomorphisms. The following are equivalent:*

1.  $\varphi$  is strong orbit equivalent to  $\psi$

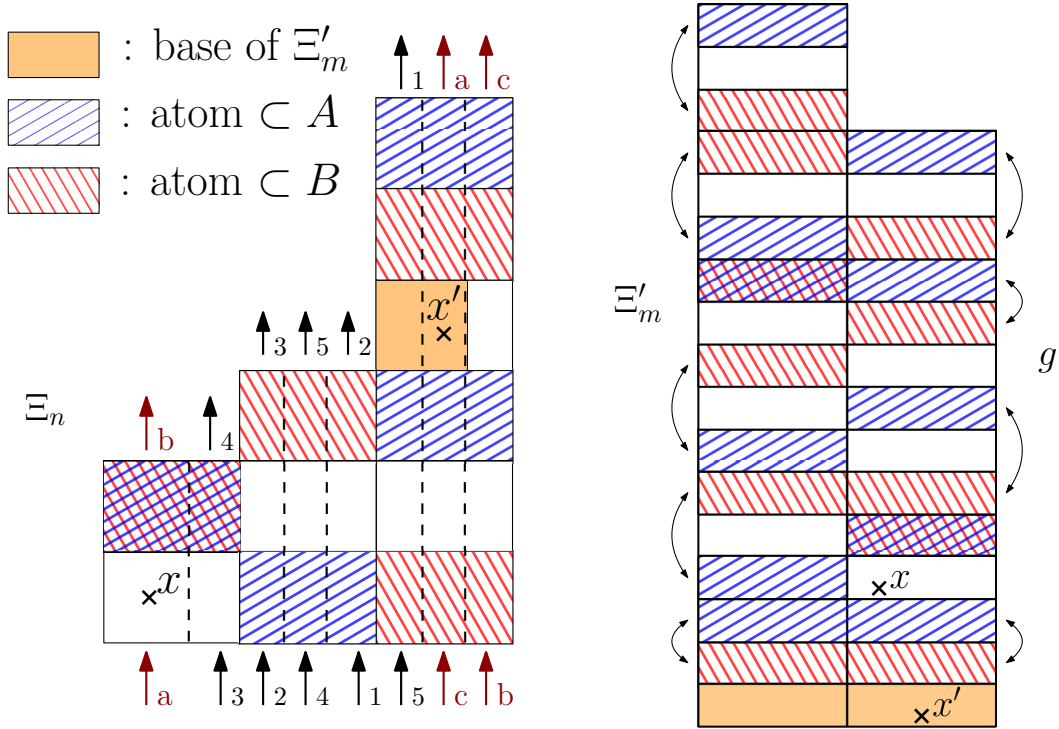


Figure 2.4: Key figure for understanding of Lemma 2.5

2. There exist  $x, y \in X$  such that  $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$  are isomorphic as abstract groups
3. For all  $x, y \in X$ ,  $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$  are isomorphic as abstract groups.

*Remark 2.7.* Thanks to Theorem 2.2, for  $x, y \in X$  the condition “ $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$  are isomorphic as abstract groups” is equivalent to “ $\Gamma_x^\varphi$  and  $\Gamma_y^\psi$  have isomorphic dimension ranges”.

**Notation.** For simplicity, in the following proof we write  $\Lambda_x$  instead of  $\Gamma_x^\psi$ , and  $\Gamma_x$  instead of  $\Gamma_x^\varphi$  (for  $x \in X$ ).

*Proof.* First of all, note that conditions 2 and 3 are equivalent by Lemma 2.5 and Remark 2.7.

Let  $x$  be in  $X$  and suppose that  $\Gamma_x$  and  $\Lambda_x$  have isomorphic dimension ranges. By continuity of  $\varphi$ , the top point of a sequence of partitions  $(\Xi_N)_N \in \mathbb{N}$  associated to  $\Gamma_x$  is  $\varphi^{-1}(x) = y_1$ . We also denote  $y_2 = \psi^{-1}(x)$ . By Krieger’s theorem (Theorem 2.2) and Remark 2.5, there exists a homeomorphism  $g$  such that  $\Lambda_x = g\Gamma_x g^{-1}$ ,  $g(x) = x$  and  $g(y_1) = y_2$ . This control on these two pairs of points ensures that  $g$  is indeed an orbit equivalence. Let  $z \neq y_1$ . Then for a large enough  $N$ ,  $z$  does not belong to the top of  $\Xi_N$ , hence there exists  $h \in \Gamma_x$  and a neighbourhood  $W$  of  $z$  such that  $h|_W = \varphi|_W$ . On the other hand  $ghg^{-1} \in \Lambda_x \subset \llbracket \psi \rrbracket$ , so there exists a neighbourhood

$V$  of  $g(z)$  and an integer  $n_0$  such that  $ghg^{-1}|_V = \psi|_V^{n_0}$ . Finally we get:

$$\forall z' \in g^{-1}(V) \cap W \quad g(\varphi(z')) = g(h(z')) = ghg^{-1}(g(z')) = \psi^{n_0}(g(z')),$$

and we deduce that the cocycle  $n$  is continuous at the point  $z$ , hence at every point except  $y_1$ . The situation being symmetric, we also get that the other cocycle is continuous everywhere, except maybe in one point, and by definition  $g$  is then a strong orbit equivalence between  $\varphi$  and  $\psi$ .

Conversely, suppose that  $g$  realizes a strong orbit equivalence between  $\varphi$  and  $\psi$ . By replacing  $\psi$  by  $g^{-1}\psi g$ , we can assume that  $g = id$ . Then

$$\forall x \in X \quad \varphi(x) = \psi^{n(x)}(x) \text{ and } \psi(x) = \varphi^{m(x)}(x)$$

with  $n$  and  $m$  being continuous except maybe in  $y$  and  $y'$  respectively. We set  $x = \varphi(y)$  and  $x' = \psi(y')$ , and construct  $\Gamma_x$  associated to a sequence of K-R partitions  $(\Xi_N)_{N \in \mathbb{N}}$  with bases  $(B^N)_{N \in \mathbb{N}}$  and  $\Lambda_{x'}$  associated to a sequence of K-R partitions

$$(\Xi'_N)_{N \in \mathbb{N}} \text{ with bases } (D^N)_{N \in \mathbb{N}}. \text{ We want to show that } \overline{id}: \begin{cases} CO(X) / \Gamma_x \longrightarrow CO(X) / \Lambda_{x'} \\ Orb_{\Gamma_x}(A) \longmapsto Orb_{\Lambda_{x'}}(A) \end{cases}$$

is a bijection.

By symmetry, if we prove that it is well defined, the assertion follows. We use the following lemma, that we will prove right after the end of the current proof:

**Lemma 2.6.** *Let  $f$  be a homeomorphism and  $A = \bigsqcup_{i=1}^r A_i$  a clopen partition of  $A$ . Assume that for all  $i$ ,  $f(A_i) \in Orb_{\Lambda_{x'}}(A_i)$ . Then  $f(A) \in Orb_{\Lambda_{x'}}(A)$ .*

Let  $A$  be a clopen set, and  $\gamma \in \Gamma_{N,x}$ , where  $N$  is sufficiently large that  $A$  is in  $\langle \Xi_N \rangle$ . We have to show that  $\gamma(A) \in Orb_{\Lambda_{x'}}(A)$ . Thanks to Lemma 2.6, it is sufficient to show that two atoms of the same tower of  $\Xi_N$  are in the same  $\Lambda_{x'}$ -orbit, or, told differently, that  $\varphi(A) \in Orb_{\Lambda_{x'}}(A)$  for every atom  $A$  of  $\Xi_N$  which is not on the top of the partition. For such an atom  $A$ , one has  $y \notin A$ , and so the cocycle  $n$  is continuous on  $A$  which is compact, hence  $A$  can be partitioned into pieces on which  $n$  is constant. Thus using Lemma 2.6 once more, we can assume that  $n$  is constant on  $A$ , equal to a certain  $n_0$ . We set  $M = \text{abs } n_0$ . Now, for all  $x$  in  $X$  there exists a clopen neighbourhood  $V_x$  of  $x$  such that  $\psi^{-M}(V_x), \dots, \psi^{M+1}(V_x)$  are pairwise disjoint (see Remark 2.3). Then by compactness of  $A$  we get a partition  $A = \bigsqcup_{i=1}^r A_i$ , with each  $A_i$  such that  $\psi^{-M}(A_i), \dots, \psi^{M+1}(A_i)$  are pairwise disjoint. Let  $i \in \llbracket 1, r \rrbracket$  be fixed, and choose  $x_i \notin \bigcup_{j=-M}^M \psi^j(A_i)$  (for example  $x_i \in \psi^{M+1}(A_i)$ ), so that none of the  $\psi^j(x_i)$  is in  $A_i$  for  $j \in \llbracket -M, M \rrbracket$ . That means that no atom contained in  $A_i$  belongs to  $\psi^j(D_k^i)$  (where the  $D_k^i$ 's are the bases of a sequence of K-R partitions  $(\Xi_k^i)_{k \in \mathbb{N}}$  associated to  $\Lambda_{x_i}$ ) for  $k$  large enough and  $j \in \llbracket -M, M \rrbracket$ , i.e every atom contained in  $A_i$  is at distance at least  $M$  from the top and the bottom of its tower, which means that  $\varphi(A_i) = \psi^{n_0}(A_i) \in Orb_{\Lambda_{x_i}}(A_i) = Orb_{\Lambda_{x'}}(A_i)$  (this last equality being due to Lemma 2.5). Using Lemma 2.6 one last time, we conclude that  $\varphi(A) \in Orb_{\Lambda_{x'}}(A)$ , which concludes the proof.  $\square$



*Proof of Lemma 2.6.* Let  $N$  be large enough that for all  $i$ ,  $A_i$  and  $f(A_i)$  are in  $\langle \Xi'_N \rangle$  and there exists  $h_i \in \Lambda_{N,x'}$  such that  $f(A_i) = h_i(A_i)$ . For all  $k$ , and for all  $j \in \llbracket 0, H_k^N - 1 \rrbracket$ , we define  $h \in \Lambda_{N,x'}$  on an atom  $D_{k,j}^N$  of  $\Xi'_N$  as follows (see figure 2.5):

- if  $D_{k,j}^N \not\subseteq A \cup f(A)$ , we set  $h = id$  on  $D_{k,j}^N$  ;
- if  $D_{k,j}^N \subset A_i$  for some  $i$ , we set  $h = h_i$  on  $D_{k,j}^N$  ;
- if  $D_{k,j}^N \subset f(A) \setminus A$ , we find the least positive integer  $m$  and an integer  $k_m$  such that

$$f^{-m}(D_{k,j}^N) \subset A \setminus f(A) \text{ and } \psi^{k_m}(D_{k,j}^N) = f^{-m}(D_{k,j}^N)$$

and we set  $h = \psi^{k_m}$  on  $D_{k,j}^N$ .

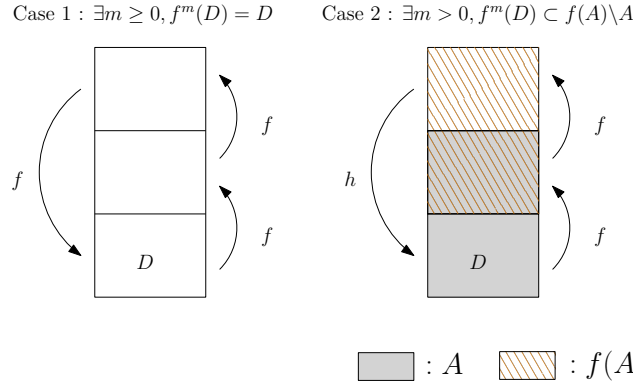


Figure 2.5: Construction of  $h$ , example with  $m = 2$

It is clear that  $h \in \Lambda_{N,x'}$  and  $h(A) = \bigsqcup_{i=1}^r h_i(A_i) = \bigsqcup_{i=1}^r f(A_i) = f(A)$  □

## 2.4 Borel complexity of isomorphism of countable, locally finite simple groups

In this section we study algebraically in some detail the countable, locally finite groups  $\Gamma$  that we have used above. Since for  $x, y \in X$ ,  $\Gamma_x$  and  $\Gamma_y$  are (spatially) isomorphic by a combination of Lemma 2.5 and Theorem 2.2, we denote them  $\Gamma$  and do not specify which semi-orbit it preserves. Observing that those groups are not always simple, we study their commutator subgroups and show that they are simple; further, we point out that if the commutator subgroups of two of these groups are isomorphic, then the groups themselves are isomorphic. It enables us to show that the strong orbit equivalence relation is, in a way we are going to recall precisely (Borel reducibility theory, cf section 2.4.2), “simpler” than the isomorphism relation between countable, locally finite simple groups.

## 2.4. Borel complexity of isomorphism of countable, locally finite simple groups 47

### 2.4.1 Study of $D(\Gamma)$

We have claimed that  $\Gamma^\varphi$  is not always simple, so let us give an explicit example. Let  $\sigma_3$  denote the 3-odometer on the Cantor space  $X = \{0, 1, 2\}^{\mathbb{N}}$ , and  $N_\alpha$  be the clopen subset consisting of all sequences starting by the finite sequence  $\alpha$ . Then we have a sequence of K-R partitions, each consisting of a single tower of basis  $N_{0^n}$  for the  $n$ -th partition. The cutting and stacking process is then very simple: the  $n + 1$ -th tower is a stacking of three copies of the  $n$ -th tower. The element  $\gamma$  of  $\Gamma^{\sigma_3}$  defined by

$$\gamma \upharpoonright_{N_0} = \sigma_3 \upharpoonright_{N_0} \text{ and } \gamma \upharpoonright_{N_1} = \sigma_3^{-1} \upharpoonright_{N_1}$$

is clearly not in the commutator subgroup, since on  $\Xi_n$  it decomposes as a product of  $3^{n-1}$  transpositions, which belongs to  $\mathfrak{S}_{3^n} \setminus \mathcal{A}_{3^n}$ .

We however have the following:

**Proposition 2.7.** *For every minimal homeomorphism  $\varphi$ ,  $D(\Gamma^\varphi)$  is simple.*

To see that, it suffices to read carefully section 3 in [BM], where a similar result is established for commutator subgroups of topological full groups; and to observe that the argument given there works exactly the same way, replacing  $\llbracket \varphi \rrbracket$  by  $\Gamma^\varphi$ .

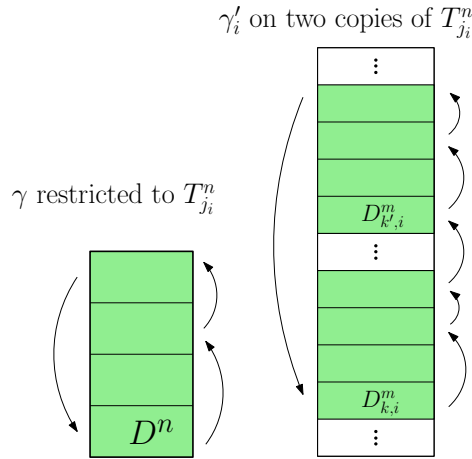


Figure 2.6: Construction of  $\gamma'_i$  in Remark 2.8

*Remark 2.8.* The only remark we may need to make, in order to follow the argument of [BM], is that  $D(\Gamma)$  is dense in  $\Gamma$ . Indeed, let  $\gamma \in \Gamma$ , and  $A_1, \dots, A_r$  be disjoint clopen sets. There are some integers  $n$  and  $m$  such that  $A_i$  belongs to  $\langle \Xi_n \rangle$  for all  $i$  and towers of  $\Xi_m$  are high enough that they must contain at least two copies of some tower of  $\Xi_n$ . Let us assume that the tower  $T_i^m$  contains two copies of  $T_{j_i}^n$ . Let  $D_{i,k}^m$  and  $D_{i,k'}^m$  be distinct atoms of  $T_i^m$  such that  $D_{i,k}^m \cup D_{i,k'}^m \subset D^n \subset T_{j_i}^n$  (where  $D^n$  is an atom of  $T_{j_i}^n$ , no matter which one). Let  $N = \min\{l > 0: \gamma^l(D^n) = D^n\}$ . We

$$\text{define } \gamma'_i \in \Gamma_m \text{ by } \gamma'_i(D) = \begin{cases} D_{i,k'}^m & \text{if } D = \gamma^{N-1}(D_{i,k}^m) \\ D_{i,k}^m & \text{if } D = \gamma^{N-1}(D_{i,k'}^m) \\ \gamma(D) & \text{elsewhere} \end{cases}$$

Figure 2.6 shows how it works with  $N = 3$ . The element  $\gamma'_i$  is constructed in such a way that, viewing  $\gamma_{\uparrow T_i^m}$  and  $\gamma'_i_{\uparrow T_i^m}$  as permutations of atoms of the tower  $T_i^m$ , exactly one of them belongs to the alternating group. Denote this restriction by  $\tilde{\gamma}_i$ . Then  $\tilde{\gamma} = \bigsqcup_i \tilde{\gamma}_i \in D(\Gamma)$  and acts the same way as  $\gamma$  on  $\langle \Xi_n \rangle$ , in particular on every  $A_i$ .

We now need the following:

**Proposition 2.8.** *Let  $\varphi$  and  $\psi$  be two minimal homeomorphisms. Then any group isomorphism  $\alpha: D(\Gamma^\varphi) \rightarrow D(\Gamma^\psi)$  is spatial.*

The proof is very much inspired by [M4], though less technical. It uses as its principal ingredient the reconstruction Theorem 384D of [F]. In order to use it we need the following definitions:

- Definition 2.6.*
- An open set  $A$  is called *regular* if  $A = \text{int}(\overline{A})$ . The set of all regular open sets forms a Boolean algebra we denote by  $RO(X)$ . Obviously clopen sets are regular, and so  $CO(X) \subset RO(X)$
  - A group  $G \leq \text{Homeo}(X)$  is said to *have many involutions* if for any regular open set  $A$ , there exists an involution  $g \in G$  whose support is included in  $A$ .

*Remark 2.9.* The "support of  $g$ " is here defined by  $\text{int}(\{x \in X: g(x) \neq x\})$ , but since in our case  $g$  always belongs to the topological full group it makes no difference with the usual definition.

Note that for every minimal homeomorphism  $\varphi$ ,  $D(\Gamma^\varphi)$  has many involutions. Indeed, if  $A$  is a regular open set, there exists a nonempty clopen set  $C \subset A$ . Think of  $\Gamma^\varphi$  as being constructed from a sequence  $\Xi$  of K-R partitions whose base point is  $x_0 \in X$ . By minimality of  $\varphi$ , for  $n$  large enough there are at least 4 atoms in the tower  $T_{i_0}^n$  of  $\Xi_n$  containing  $x_0$  which are contained in  $C$ , say  $D_{i_0, k_0}^n, D_{i_0, k_1}^n, D_{i_0, k_2}^n, D_{i_0, k_3}^n$ , with  $k_j < k_{j+1}$ . Then we can easily define an involution  $g \in D(\Gamma_n)$  whose support is contained in  $C$ , and thus in  $A$  (for example the double transposition  $(D_{i_0, k_0}^n D_{i_0, k_1}^n)(D_{i_0, k_2}^n D_{i_0, k_3}^n)$ ).

*Proof.* Let  $\alpha: D(\Gamma^\varphi) \rightarrow D(\Gamma^\psi)$  be an isomorphism. Thanks to remark 2.9, we now know that  $D(\Gamma^\varphi)$  and  $D(\Gamma^\psi)$  both have many involutions. Theorem 384D applies and gives us an automorphism of Boolean algebras

$$\Lambda: RO(X) \rightarrow RO(X)$$

such that  $\alpha(g)(V) = \Lambda g \Lambda^{-1}(V)$  for all  $g \in D(\Gamma^\varphi)$  and  $V \in RO(X)$ . If we can show that  $\Lambda(CO(X)) = CO(X)$ , then  $\Lambda$  is induced by a homeomorphism of  $X$  and the proof is over. We proceed to explain why this is true.

We note that  $CO(X)$  is generated by the supports of the involutions in  $D(\Gamma)$ , where  $\Gamma$  stands for either  $\Gamma^\varphi$  or  $\Gamma^\psi$  (or more generally for any  $\Gamma$  associated to a minimal homeomorphism). Indeed, take a clopen set  $C$ , and look at  $\Gamma$  as constructed out of a sequence of K-R partitions  $\Xi$ . For  $n$  large enough,  $C$  belongs to  $\langle \Xi_n \rangle$  and every tower of  $\Xi_n$  has an height greater than 7. Now, given an atom  $A$  of  $\Xi_n$ , it is easy to construct two permutations  $\gamma_1, \gamma_2 \in D(\Gamma)$  such that  $\text{supp}(\gamma_1) \cap \text{supp}(\gamma_2) = A$ ,

## 2.4. Borel complexity of isomorphism of countable, locally finite simple groups 49

indeed it suffices to take two double transpositions defined on  $A$  and three other atoms each, pairwise disjoint, which is possible because of the height of the tower being greater than 7. The proof is now over by applying the following lemma, that follows from the proof of the Theorem 384D of [F]:

**Lemma 2.9.** *If  $g \in D(\Gamma^\varphi)$  is an involution, then  $\text{supp}(\alpha(g)) = \Lambda(\text{supp}(g))$ .*

Indeed, this shows that  $\Lambda(\text{CO}(X)) \subset \text{CO}(X)$ , and since the situation is symmetric we obtain as desired the equality  $\Lambda(\text{CO}(X)) = \text{CO}(X)$ .  $\square$

Let us sum up what we know: given two minimal homeomorphisms  $\varphi$  and  $\psi$ , we have

$$\begin{aligned} & D(\Gamma^\varphi), D(\Gamma^\psi) \text{ are isomorphic as abstract groups} \\ \iff & D(\Gamma^\varphi), D(\Gamma^\psi) \text{ are spatially isomorphic} \\ \iff & D(\Gamma^\varphi), D(\Gamma^\psi) \text{ have isomorphic dimension ranges, (Theorem 2.2)} \\ \iff & \Gamma^\varphi, \Gamma^\psi \text{ have isomorphic dimension ranges, (Remark 2.8)} \\ \iff & \varphi, \psi \text{ are strong orbit equivalent, (Theorem 2.3)} \end{aligned}$$

### 2.4.2 Isomorphism on the space of countable, locally finite simple groups is a universal relation

Let us start by recalling the definition of Borel reducibility from the introduction :

*Definition 2.7.* Let  $E, F$  be equivalence relations on standard Borel spaces  $X, Y$  respectively.  $E$  is said to be *Borel reducible to  $F$* , and we write  $E \leq F$  if there exists a Borel map  $f: X \rightarrow Y$  such that

$$\forall x, x' \in X \quad xEx' \iff f(x)Ff(x').$$

We call such a map  $f$  a *Borel reduction from  $E$  to  $F$* . If  $E \leq F$  and  $F \leq E$ , we say that  $E$  and  $F$  are *Borel bireducible*.

We also recall that there exists a universal equivalence  $E_\infty$  arising from an action of  $S_\infty$  on a standard Borel space, that we denote  $E_\infty$ . A theorem of Melleray gives us a particular realization of  $E_\infty$  :

**Théorème 2.4 ([M5]).** *The equivalence relation of strong orbit equivalence of minimal homeomorphisms on the Cantor space is Borel bireducible to  $E_\infty$ .*

We have obtained in the previous subsection (see the series of equivalences at the end of subsection 2.4.1) that strong orbit equivalence between two minimal homeomorphisms  $\varphi$  and  $\psi$  is characterized by the isomorphism of  $D(\Gamma^\varphi)$  and  $D(\Gamma^\psi)$ , which are countable, simple and locally finite. We need to exhibit a Borel way to associate to a minimal homeomorphism  $\varphi$  its countable, locally finite simple group  $D(\Gamma_{x_0}^\varphi)$ . There are plenty of ways to do it and the one we have chosen requires a lot of steps; but it is quite easy to check that each of those steps is

Borel. Let  $\text{Homeomin}(X)$  be the Borel subset of  $\text{Homeo}(X)$  that consists of minimal homeomorphisms of  $X$ .

First of all we define

$$\text{EnumTfg}: \text{Homeomin}(X) \rightarrow \text{Homeo}(X)^\omega$$

such that  $\text{EnumTfg}(\varphi)$  is an enumeration of  $\llbracket \varphi \rrbracket$ .

Let  $(n_1^i, \dots, n_{k_i}^i, A_1^i, \dots, A_{k_i}^i)_{i \in \mathbb{N}}$  be an enumeration of  $\bigsqcup_{k \in \mathbb{N}} \mathbb{N}^k \times \text{CO}(X)^k$ . We define  $\text{EnumTfg}$  as follows: for all  $i \in \mathbb{N}$ , if  $\bigsqcup_{j=1}^{k_i} A_j^i = X$  and  $\bigsqcup_{j=1}^{k_i} \varphi_{|A_j^i}^{n_j^i} \in \llbracket \varphi \rrbracket$ , then  $\text{EnumTfg}(\varphi)(i) = \bigsqcup_{j=1}^{k_i} \varphi_{|A_j^i}^{n_j^i}$ , else, put it equal to  $id$ . This defines an enumeration of  $\llbracket \varphi \rrbracket$  with repetitions. Note that one can always make a sequence injective in a Borel way by removing duplicates.

Now that we have encoded  $\llbracket \varphi \rrbracket$ , we define

$$\text{Enum}\Gamma_{x_0}: \text{Homeomin}(X) \rightarrow \text{Homeo}(X)^\omega$$

such that  $\text{Enum}\Gamma_{x_0}(\varphi)$  enumerates  $\Gamma_{x_0}^\varphi$ . To do this, we introduce the function

$$\text{IsIn}\Gamma: \text{Homeo}(X) \times \text{Homeomin}(X) \rightarrow \text{Homeo}(X)$$

that associates  $\alpha$  to  $(\alpha, \varphi)$  if it belongs to  $\Gamma_{x_0}^\varphi$ , and  $id$  otherwise. The condition “ $\alpha$  belongs to  $\Gamma_{x_0}^\varphi$ ” can be written

$$\begin{aligned} \alpha \in \llbracket \varphi \rrbracket \quad & \text{and } \forall n \in \mathbb{N} \exists m \in \mathbb{N} \alpha(\varphi^n(x_0)) = \varphi^m(x_0) \\ & \text{and } \forall n \in \mathbb{N} \exists m \in \mathbb{N} \alpha(\varphi^m(x_0)) = \varphi^n(x_0) \end{aligned}$$

whence  $\text{IsIn}\Gamma$  is a Borel function. Then it is clear that  $\text{Enum}\Gamma_{x_0}$  defined by

$$\text{Enum}\Gamma_{x_0}(\varphi)_i = \text{IsIn}\Gamma((\text{EnumTfg}(\varphi)_i, \varphi))$$

is a Borel function.

It remains to encode  $D(\Gamma)$ . Since one can define Borel functions  $\text{Commu}$  and  $\text{GenBy}$  which enumerate respectively all commutators of a given sequence of homeomorphisms and the group generated by it, we can define

$$\text{Enum}D\Gamma = \text{GenBy} \circ \text{Commu} \circ \text{Enum}\Gamma_{x_0}$$

It is a Borel reduction of the strong orbit equivalence relation to the isomorphism relation on countable, locally finite simple groups, and we have proved the following :

**Théorème 2.5.** *The relation of isomorphism of countable, locally finite, simple groups is a universal relation arising from a Borel action of  $S_\infty$ .*

So in particular, in the case of Borel reducibility it is as complicated to classify simple, locally finite groups as it is to classify countable groups.

# From invariant measures to orbit equivalence, via locally finite groups

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This is joint work with Julien Melleray. It follows closely [MR].

## 3.1 Introduction

This paper is concerned with continuous actions of countable groups by homeomorphisms on the Cantor space  $X$ ; we say that such an action is *minimal* if all of its orbits are dense. Two actions of countable groups  $\Gamma, \Lambda$  on  $X$  are *orbit equivalent* if there exists a homeomorphism  $g$  of  $X$  such that

$$\forall x, x' \in X \quad (\exists \gamma \in \Gamma \ \gamma(x) = x') \Leftrightarrow (\exists \lambda \in \Lambda \ \lambda(g(x)) = g(x'))$$

In words, the actions of  $\Gamma$  and  $\Lambda$  are orbit equivalent when the equivalence relations that they induce are isomorphic. In what follows, we will often see  $\Gamma, \Lambda$  as subgroups of  $\text{Homeo}(X)$ , and then we simply say that  $\Gamma, \Lambda$  are orbit equivalent.

For any minimal homeomorphism  $\varphi$  of  $X$  (i.e. a homeomorphism such that the associated action of  $\mathbb{Z}$  on  $X$  is minimal), denote by  $M(\varphi)$  the set of all  $\varphi$ -invariant Borel probability measures. Our motivation here is to give a proof of the following famous theorem of Giordano, Putnam and Skau, which we call throughout the paper the “classification theorem for minimal homeomorphisms”.

**Théorème** (Giordano–Putnam–Skau [GPS1, Theorem 2.2]). *Assume that  $\varphi, \psi$  are minimal homeomorphisms of the Cantor space  $X$ . The following conditions are equivalent.*

- *The  $\mathbb{Z}$ -actions induced by  $\varphi$  and  $\psi$  are orbit equivalent.*
- *There exists a homeomorphism  $g$  of  $X$  such that  $g_*M(\varphi) = M(\psi)$ . (where  $g_*M(\varphi) = \{g_*\mu : \mu \in M(\varphi)\}$ , with  $g_*\mu(A) = \mu(g^{-1}(A))$  for any Borel subset  $A$  of  $X$ )*

The implication from top to bottom above is easy to see, and any  $g$  witnessing that  $\varphi$  and  $\psi$  are orbit equivalent is such that  $g_*M(\varphi) = M(\psi)$ . The converse is much more surprising, and somewhat mysterious; in particular, two minimal homeomorphisms may have the same invariant Borel probability measures but induce different equivalence relations on  $X$ . The original proof in [GPS1] uses homological algebra in an essential way; different arguments have been proposed since then (see for instance [P1] or [P2], as well as the “elementary” proof given in [HKY]) but it remains difficult to “understand the dynamics that lie beneath”, to quote Glasner and Weiss [GW].

Our aim in this paper is to present a self-contained proof of the classification theorem for minimal  $\mathbb{Z}$ -actions. We exploit a connection with minimal actions of certain locally finite groups, which was already noticed by Giordano, Putnam and Skau. Recall that a subgroup  $\Gamma$  of  $\text{Homeo}(X)$  is a *full group* if

$$\Gamma = \{g \in \text{Homeo}(X) : \exists A \text{ finite } \subset \Gamma \forall x \in X \exists \gamma \in A \gamma(x) = x\}$$

The condition above says that  $\Gamma$  coincides with the group of homeomorphisms which are obtained by gluing together finitely many elements of  $\Gamma$  (see section 3.2.1 for some more details on full groups, as well as more background on other notions that we use in this introduction. In the context of minimal  $\mathbb{Z}$ -actions on the Cantor space, full groups were first investigated in [GPS2]). Following Krieger [K1], we say that a subgroup  $\Gamma$  of  $\text{Homeo}(X)$  is *ample* if it is a countable, locally finite full group such that  $\{x : \gamma(x) = x\}$  is clopen for every  $\gamma \in \Gamma$ . Equivalence relations induced by minimal ample groups correspond to the “affable” equivalence relations introduced by Giordano, Putnam and Skau, which play an important part in their theory (see [GPS3] and [P2]). For any ample group  $\Gamma$ , the set  $M(\Gamma)$  of all  $\Gamma$ -invariant Borel probability measures on  $X$  is nonempty, and Giordano, Putnam and Skau established the following result, which we call the “classification theorem for minimal ample groups” (see also Putnam [P1] for an interesting take on this result).

**Theorem** (Giordano–Putnam–Skau [GPS1, Theorem 2.3]). *Let  $X$  be the Cantor space. Given two ample subgroups of  $\text{Homeo}(X)$  acting minimally, the following conditions are equivalent.*

- *The actions of  $\Gamma$  and  $\Lambda$  are orbit equivalent.*
- *There exists a homeomorphism  $g$  of  $X$  such that  $g_*M(\Gamma) = M(\Lambda)$ .*

Giordano, Putnam and Skau observed that every minimal  $\mathbb{Z}$ -action is orbit equivalent to an action of an ample group (the group in question is naturally associated to the  $\mathbb{Z}$ -action, see section 3.2.3) and then pointed out that one can deduce the classification theorem for minimal  $\mathbb{Z}$ -actions from the corresponding classification theorem for actions of ample groups. Our strategy is the same, although our proof of this fact (Theorem 3.7 here) is new. Like Giordano, Putnam and Skau we use an “absorption theorem” (Theorem 3.5) to establish the classification theorem for minimal ample groups.

As an example of such an absorption theorem (indeed, a fundamental particular case), consider the relation  $E_0$  on  $\{0, 1\}^{\mathbb{N}}$ , where for all  $x, y \in \{0, 1\}^{\mathbb{N}}$  one has  $x E_0 y$  iff  $x(n) = y(n)$  for all large enough  $n$ . This relation is induced by the countable, locally finite group  $\Gamma$  of all dyadic permutations; these are the homeomorphisms of  $\{0, 1\}^{\mathbb{N}}$  obtained by first choosing a bijection  $\sigma$  of some  $\{0, 1\}^n$  to itself, then setting  $\tilde{\sigma}(u \frown x) = \sigma(u) \frown x$ . Denote by  $\varphi$  the dyadic odometer (which corresponds to “adding 1 with right-carry”), which is a minimal homeomorphism; the relation  $R_\varphi$  is obtained from  $E_0$  by gluing the classes of  $0^\infty$  and  $1^\infty$  together (indeed,  $\varphi$  can only change finitely many coordinates at a time, except for the special case  $\varphi(1^\infty) = 0^\infty$ ). Giordano, Putnam and Skau’s absorption theorem implies in particular that  $R_\Gamma$  and  $R_\varphi$  are isomorphic; this means that there exists a homeomorphism  $g$  of  $\{0, 1\}^{\mathbb{N}}$  which turns  $E_0$  into the relation obtained from  $E_0$  by gluing two orbits together. This extraordinary fact is quite hard to visualize, and we do not know of any “concrete” construction of such a  $g$ . Clemens, in unpublished work [C], gave a nice explicit construction of a minimal homeomorphism inducing  $E_0$  (via a Bratteli diagram) and asked whether  $E_0$  can be generated by a Lipschitz automorphism of  $\{0, 1\}^{\mathbb{N}}$ .

While our absorption theorem (and its strengthening, Theorem 3.9) follows from known results ([GPS3, Lemma 4.15], itself generalized in [GMPS1, Theorem 4.6] and [M2, Theorem 3.2]), its proof is relatively elementary, and based on the following strengthening of a theorem of Krieger which we find interesting in its own right.

**Théorème 3.1.** *[see Krieger [K1]Theorem 3.5] Let  $\Gamma, \Lambda$  be two ample subgroups of  $\text{Homeo}(X)$ ; assume that for any  $A, B \in CO(X)$  there exists  $\gamma \in \Gamma$  such that  $\gamma(A) = B$  iff there exists  $\lambda \in \Lambda$  such that  $\lambda(A) = B$ .*

*Assume additionally that  $K$  is a closed subset of  $X$  which intersects each  $\Gamma$ -orbit in at most one point;  $L$  is a closed subset of  $X$  which intersects each  $\Lambda$ -orbit in at most one point; and  $h: K \rightarrow L$  is a homeomorphism.*

*Then there exists  $g \in \text{Homeo}(X)$  such that  $g\Gamma g^{-1} = \Lambda$ , and  $g|_K = h$ .*

In terms of the Polish topology of  $\text{Homeo}(X)$ , the original statement of Krieger’s theorem (i.e. with  $K = L = \emptyset$  above) can be seen as saying that two ample groups



$\Gamma, \Lambda$  are conjugate in  $\text{Homeo}(X)$  as soon as their closures are. A lemma due to Glasner and Weiss (Lemma 3.8 below), which is essential to our approach, implies that, whenever  $\varphi$  is a minimal homeomorphism, the closure of its full group is equal to the group of all homeomorphisms which preserve every measure in  $M(\varphi)$ . Also, two homeomorphisms  $\varphi$  and  $\psi$  are orbit equivalent if and only if their full groups are conjugate in  $\text{Homeo}(X)$ . Thus one can see the classification theorem for minimal  $\mathbb{Z}$ -actions as the statement that the full groups of two minimal homeomorphisms are conjugate as soon as their closures coincide, which suggests a relationship with Krieger's theorem. That is the motivation for our approach.

Say that an ample subgroup  $\Gamma$  of  $\text{Homeo}(X)$  acting minimally is *saturated* if

$$\bar{\Gamma} = \{g \in \text{Homeo}(X) : \forall \mu \in M(\Gamma) \ g_*\mu = \mu\}$$

It follows from Krieger's theorem and a variation on the aforementioned result of Glasner and Weiss (Lemma 3.7 below) that two saturated ample groups  $\Gamma, \Lambda$  are conjugate iff there exists  $g \in \text{Homeo}(X)$  such that  $g_*M(\Gamma) = M(\Lambda)$ . In particular, a strong form of the classification theorem, in the special case of saturated ample groups, follows directly from a combination of Glasner–Weiss's compactness argument and Krieger's theorem: two saturated minimal ample groups which preserve the same Borel probability measures are conjugate.

It is then natural to try and prove the classification theorem by establishing that any minimal action of an ample subgroup of  $\text{Homeo}(X)$  is orbit equivalent to a minimal action of a saturated ample subgroup (a closely related strategy was suggested in [IM2]). This is the approach that we follow here.

We need to control some tension between two equivalence relations, one on clopen subsets of  $X$  and the other on points. Starting from an ample group  $\Gamma$ , we want to coarsen the relation induced by the action of  $\Gamma$  on  $CO(X)$  so as to make  $\Gamma$  saturated; but this changes the relation induced by the action of  $\Gamma$  on  $X$ , and the only technique we have at our disposal to control this relation is via our strengthening of Krieger's theorem. Given a minimal ample  $\Gamma$ , we manipulate Cantor sets  $K \sqcup \sigma(K)$ , where  $\sigma$  is a homeomorphic involution, which intersect each  $\Gamma$ -orbit in at most one point, and consider the relation  $R_{\Gamma, K}$  which is obtained by joining the  $\Gamma$ -orbit of  $x$  and  $\sigma(x)$  together for all  $x \in K$ , and leaving the other orbits unchanged. By Theorem 3.2, for any two such  $K, L$  the relations  $R_{\Gamma, K}$  and  $R_{\Gamma, L}$  are isomorphic. Then we prove that there exists such a  $K$  for which  $R_{\Gamma, K}$  is induced by an ample group with the same orbits as  $\Gamma$  on  $CO(X)$  (hence this ample group is conjugate to  $\Gamma$ ); and another such  $K$  for which  $R_{\Gamma, K}$  is induced by a saturated minimal ample group. It follows that  $\Gamma$  is orbit equivalent to a saturated minimal ample group, thereby establishing the classification theorem for minimal ample groups. This is where we need, and prove, an absorption theorem: the argument above establishes that  $R_\Gamma$  is isomorphic to the relation obtained by gluing together the orbits of  $x$  and  $\sigma(x)$  for each  $x \in K$ .

We conclude this introduction by a few words about the structure of this article, which we tried to make as self-contained as possible, in the hope of making the proof accessible to a broad mathematical audience. Sections 3.2 and 3.3 mostly

consist of background material, the exception being Theorem 3.2, which is the main driving force in our proof of the classification theorems. Section 3.4 develops an auxiliary combinatorial tool which we need to extend an ample group to a saturated ample group while having some control on the equivalence relation induced by the bigger group (the underlying idea is related to work from [IM2]). Then we use our tools to prove the classification theorem for minimal ample groups. In the last section, we explain how to deduce the classification theorem for minimal  $\mathbb{Z}$ -actions.

**Acknowledgements.** The first-named author spent an embarrassing amount of time, spread over several years, trying to understand how to prove the classification theorem for minimal  $\mathbb{Z}$ -actions via elementary methods. He wishes to thank his co-author for helping to put him out of his misery, as well as present his sincere apologies to the many colleagues whom he has bored with various complaints related to this problem, false proofs, and assorted existential crises (some of which occurred uncomfortably late in this project).

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## 3.2 Background and terminology

### 3.2.1 Full groups.

*Definition 3.1.* Given a subgroup  $\Gamma$  of  $\text{Homeo}(X)$ , we set

$$F(\Gamma) = \{g \in \text{Homeo}(X) : \exists A \text{ finite } \subset X \forall x \in X \exists \gamma \in \Gamma g(x) = \gamma(x)\}$$

We say that  $F(\Gamma)$  is the topological full group associated to the action of  $\Gamma$  on  $X$ ; and that  $\Gamma$  is a *full group* if  $\Gamma = F(\Gamma)$ .

**Notation.** Below, whenever  $\Gamma$  is a subgroup of  $\text{Homeo}(X)$ , we denote by  $R_\Gamma$  the equivalence relation induced by the action of  $\Gamma$ , i.e.

$$\forall x, y \in X \quad (xR_\Gamma y) \Leftrightarrow (\exists \gamma \in \Gamma \gamma(x) = y)$$

For  $\varphi \in \text{Homeo}(X)$ , we simply denote by  $R_\varphi$  the equivalence relation induced by the action of  $\{\varphi^n : n \in \mathbb{Z}\}$ .

*Definition 3.2.* Let  $R$  be an equivalence relation on  $X$ . The *full group* of  $R$  is

$$[R] = \{g \in \text{Homeo}(X) : \forall x g(x)Rx\}$$

Note that  $[R]$  is indeed a full group as defined above; also, two actions of groups  $\Gamma, \Lambda$  on  $X$  are orbit equivalent if and only if there exists  $g \in \text{Homeo}(X)$  such that  $g[R_\Gamma]g^{-1} = [R_\Lambda]$ .

When  $\varphi$  is a homeomorphism of  $X$ , we can consider the topological full group of the associated action of  $\mathbb{Z}$ , which we denote  $F(\varphi)$ ; or the full group of the equivalence relation  $R_\varphi$  induced by the action, which we denote  $[R_\varphi]$ . The standard notations for these objects are  $[[\varphi]]$  for what we denote  $F(\varphi)$ , and  $[\varphi]$  for  $[R_\varphi]$ ;  $F(\varphi)$  is called the topological full group of  $\varphi$ , and  $[\varphi]$  the full group of  $\varphi$ . We find these notations potentially confusing, especially in this paper where it will be important to keep in mind the difference between the full group associated to an action of a countable group, which is a countable group of  $\text{Homeo}(X)$ , and the full group associated to an equivalence relation, a typically much bigger group. We refer the reader to [GPS2], where topological full groups of minimal homeomorphisms are investigated in detail.

### 3.2.2 Invariant measures for minimal actions

**Notation.** Whenever  $\Gamma$  is a subgroup of  $\text{Homeo}(X)$ , we denote by  $M(\Gamma)$  the set of  $\Gamma$ -invariant Borel probability measures. For any  $\varphi \in \text{Homeo}(X)$ , we simply denote  $M(\varphi)$  for  $M(\{\varphi^n : n \in \mathbb{Z}\})$ .

The set  $M(\Gamma)$  is nonempty as soon as  $\Gamma$  is amenable, which is the case for the groups we are concerned with, namely  $\mathbb{Z}$ , some locally finite groups (the *ample* groups considered in Section 3.3), and full groups associated to these groups and the equivalence relations they induce.

**Lemma 3.1.** *Let  $\Gamma$  be a countable subgroup of  $\text{Homeo}(X)$ . Then  $M(\Gamma) = M([R_\Gamma])$ .*

*Proof.* Since  $\Gamma$  is contained in  $[R_\Gamma]$ , we have  $M([R_\Gamma]) \subseteq M(\Gamma)$  by definition.

Conversely, let  $\mu \in M(\Gamma)$  and  $g \in [R_\Gamma]$ . There exists a Borel partition  $(B_n)$  of  $X$  and elements  $\gamma_n \in \Gamma$  such that  $g|_{B_n} = \gamma_n|_{B_n}$  for all  $n$ . For any Borel  $A$ , we have

$$\begin{aligned} \mu(g(A)) &= \sum_{n \in \mathbb{N}} \mu(g(A \cap B_n)) \\ &= \sum_{n \in \mathbb{N}} \mu(\gamma_n(A \cap B_n)) \\ &= \sum_{n \in \mathbb{N}} \mu(A \cap B_n) \\ &= \mu(A) \end{aligned}$$

□

Given two homeomorphisms  $\varphi, \psi$ , an orbit equivalence from  $R_\varphi$  to  $R_\psi$  is the same as a homeomorphism  $g$  of  $X$  such that  $g[R_\varphi]g^{-1} = [R_\psi]$ . Hence whenever  $g$  is an orbit equivalence from  $R_\varphi$  to  $R_\psi$ , we have  $g_*M([\varphi]) = M([\psi])$ , that is,  $g_*M(\varphi) = M(\psi)$ . This establishes the easy direction of the classification theorem for minimal  $\mathbb{Z}$ -actions.

We collect some well-known facts about invariant measures for minimal actions.

**Lemma 3.2.** *Let  $\Gamma$  be a countable subgroup of  $\text{Homeo}(X)$  acting minimally; assume that  $M(\Gamma) \neq \emptyset$ . Then:*

- *Any  $\mu \in M(\Gamma)$  is atomless.*
- *For any nonempty clopen  $U$ , we have  $\inf_{\mu \in M(\Gamma)} \mu(U) > 0$ .*
- *Fix a compatible distance on  $X$ . For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any clopen  $A$  of diameter less than  $\delta$ , one has  $\sup_{\mu \in M(\Gamma)} \mu(A) < \varepsilon$ .*

In particular, any  $\Gamma$ -invariant measure has full support.

*Proof.* Fix  $x \in X$  and  $\mu \in M(\Gamma)$ . The set  $\{\gamma(x) : \gamma \in \Gamma\}$  is infinite, so  $\mu(\{x\}) = 0$ . This proves the first property.

To see why the second property holds, fix a nonempty clopen  $U$ ; since  $\Gamma$  acts minimally, we have  $\bigcup_{\gamma \in \Gamma} \gamma(U) = X$ , so by compactness there exist  $\gamma_1, \dots, \gamma_n \in \Gamma$  with  $X = \bigcup_{i=1}^n \gamma_i(U)$ , whence  $\mu(U) \geq \frac{1}{n}$  for any  $\mu \in M(\Gamma)$ .

For the third point, we use the same argument as in ([BM, Proposition 2.3]). Arguing by contradiction, we assume that there exists a sequence of clopens  $(A_n)$  of vanishing diameter and  $\mu_n \in M(\Gamma)$  such that  $\mu_n(A_n) \geq \varepsilon$  for all  $n$ . Up to some extraction, we may assume that  $(A_n)$  converges to  $x \in X$  for the Vietoris topology on the space of compact subsets of  $X$ , and  $(\mu_n)$  converges to  $\mu \in M(\Gamma)$ . If  $O$  is a clopen neighborhood of  $x$ , we have that  $A_n \subseteq O$  for large enough  $n$  so that  $\mu_n(O) \geq \mu_n(A_n) \geq \varepsilon$ , whence  $\mu(O) \geq \varepsilon$  for all  $n$ . Hence  $\mu(\{x\}) \geq \varepsilon$ , contradicting the fact that  $\mu$  is atomless.  $\square$

The following observation will also play a part later on.

**Lemma 3.3.** *Let  $\Gamma$  be a subgroup of  $\text{Homeo}(X)$ . Denote*

$$G_\Gamma = \{g \in \text{Homeo}(X) : \forall \mu \in M(\Gamma) \ g_*\mu = \mu\}$$

*Then  $G_\Gamma$  is a full group and  $M(G_\Gamma) = M(\Gamma)$ .*

*Proof.* Clearly  $G_\Gamma$  is a full group; since  $\Gamma \subseteq G_\Gamma$  we have  $M(G_\Gamma) \subseteq M(\Gamma)$ . Conversely, for any  $g \in G_\Gamma$  and any  $\mu \in M(\Gamma)$  we have  $g_*\mu = \mu$ , so  $M(\Gamma) \subseteq M(G_\Gamma)$ .  $\square$

### 3.2.3 Kakutani–Rokhlin partitions

In this subsection, we fix a minimal homeomorphism  $\varphi$  of  $X$ . For any nonempty  $U \in CO(X)$ , we have  $X = \bigcup_{n \in \mathbb{Z}} \varphi^n(U)$ , thus by compactness of  $X$  there exists  $N$  such that

$$X = \bigcup_{n=-N}^N \varphi^n(U) = \bigcup_{n=-2N-1}^{-1} \varphi^n(U) = \bigcup_{n=0}^{2N} \varphi^n(U)$$

This proves that the forward and backward orbit of any  $x \in X$  are both dense. For any  $x$  there exists some  $n \geq 1$  such that  $\varphi^n(x) \in U$ , and we define

$$n_U(x) = \min \{n \geq 1 : \varphi^n(x) \in U\}$$

Since  $U$  is clopen, the map  $n_U$  is continuous, so it takes finitely many values on  $U$  since  $U$  is compact. Let  $I = n_U(X)$  and  $U_i = \{x \in U : n_U(x) = i\}$ . Each  $U_i$  is clopen and for any  $i, j \in I$  and any  $n \leq i - 1, m \leq j - 1$  we have  $\varphi^n(U_i) \cap \varphi^m(U_j) = \emptyset$  as soon as  $(i, n) \neq (j, m)$ . This leads us to the following definition.

*Definition 3.3.* A Kakutani-Rokhlin-partition of  $X$  is a clopen partition  $(U_{i,j})_{i \in I, j < n_i}$  such that for any  $i$  and any  $j < n_i - 1$  one has  $\varphi(U_{i,j}) = U_{i,j+1}$ .

The *base* of the partition is  $U = \bigcup_{i \in I} U_{i,0}$ , and its *top* is  $\varphi^{-1}(U) = \bigcup_{i \in I} U_{i,n_i-1}$ .

We say that  $(U_{i,j})_{0 \leq j < n_i}$  is a *column* of the partition, and that  $n_i$  is the *height* of this column.

The construction outlined before the definition of Kakutani–Rokhlin partitions shows that for any nonempty clopen  $U$  there exists a Kakutani–Rokhlin partition whose base is equal to  $U$ .

*Definition 3.4.* Let  $\mathcal{A} = (U_{i,j})_{i \in I, 0 \leq j < n_i}$  and  $\mathcal{B} = (B_{k,l})_{k \in K, 0 \leq l < m_k}$  be two Kakutani–Rokhlin partitions.

We say that  $\mathcal{B}$  refines  $\mathcal{A}$  if every  $B_{k,l}$  is contained in some  $A_{i,j}$  and the base of  $\mathcal{B}$  is contained in the base of  $\mathcal{A}$  (then the top of  $\mathcal{B}$  is also contained in the top of  $\mathcal{A}$ ).

Note that if  $\mathcal{B}$  refines  $\mathcal{A}$  and  $B_{k,l}$  is contained in some atom  $A_{i,j}$  with  $j < n_i - 1$ , then  $l < m_k - 1$  and  $B_{k,l+1} = \varphi(B_{k,l}) \subseteq A_{i,j+1}$ ; one often says that the columns of  $\mathcal{B}$  have been obtained by *cutting and stacking* from the columns of  $\mathcal{A}$ . Going back to the example of a Kakutani–Rokhlin partition defined from the first return map to some clopen  $U$ , the intuition is that if we shrink the base to some  $V \subset U$  then  $\varphi^k(y)$  can only belong to  $V$  if it belongs to  $U$ ; and if  $\varphi^k(x) \in U \setminus V$ , then before coming back to  $U$  one will have to go through the whole column containing  $x$  for the Kakutani–Rokhlin partition based on  $U$  (see figure 3.1 below).

*Definition 3.5.* A Kakutani–Rokhlin partition  $\mathcal{A}$  is *compatible* with  $U \in CO(X)$  if  $U$  belongs to the Boolean algebra generated by  $\mathcal{A}$ .

**Lemma 3.4.** *Let  $\mathcal{A}$  be a Kakutani–Rokhlin partition, and  $U \in CO(X)$ . There exists a Kakutani–Rokhlin partition  $\mathcal{B}$  which refines  $\mathcal{A}$  and is compatible with  $U$ .*

*Proof.* The proof only involves cutting, and no stacking. Let  $\mathcal{A} = (U_{i,j})_{i \in I, 0 \leq j < n_i}$ . For any  $i \in I$ , consider the equivalence relation  $R_i$  on  $U_{i,0}$  defined by

$$(xR_i y) \Leftrightarrow \left( \forall j < n_i \left( \varphi^j(x) \in U \Leftrightarrow \varphi^j(y) \in U \right) \right)$$

The equivalence classes of  $R_i$  are clopen, call them  $D_k^i$ , for  $k \in K_i$ .

Then the Kakutani–Rokhlin partition  $\mathcal{B}$  whose columns are  $(\varphi^j(D_k^i))_{0 \leq j < n_i}$  refines  $\mathcal{A}$  and is compatible with  $U$ .  $\square$

Note that if  $\mathcal{A}$  is compatible with some  $U \in CO(X)$ , and  $\mathcal{B}$  refines  $\mathcal{A}$ , then  $\mathcal{B}$  is also compatible with  $U$ .

Since there are countably many clopen sets, it follows from the previous discussion that, given any  $x \in X$ , we may build a refining sequence of Kakutani–Rokhlin partitions  $(\mathcal{A}_n)$  with the following properties:

1. For any  $U \in CO(X)$ , there exists  $n$  such that  $\mathcal{A}_m$  is compatible with  $U$  for all  $m \geq n$ .
2. The intersection of the bases of  $\mathcal{A}_n$  is equal to  $\{x\}$  (and then the intersection of the tops is equal to  $\varphi^{-1}(\{x\})$ ).

We fix such a sequence of partitions  $(\mathcal{A}_n)$  for the remainder of this section.

*Remark 3.1.* Let  $k$  be a given natural integer. The fact that  $\varphi$  acts minimally (aperiodicity of the action would suffice) and condition (2) ensure that for any big enough  $n$ , the base  $B$  of  $\mathcal{A}_n$  is such that  $B, \varphi(B), \dots, \varphi^{k-1}(B)$  are pairwise disjoint, and thus every column of  $\mathcal{A}_n$  is of height larger than  $k$ .

*Definition 3.6.* For any  $n \in \mathbb{N}$ , we let  $\Gamma_n$  consist of all  $g \in \text{Homeo}(X)$  with the following property: for every atom  $U_{i,j}$  of  $\mathcal{A}_n$  there exists an integer  $k_{i,j}$  with  $0 \leq j + k_{i,j} < n_i$  and such that  $g(y) = \varphi^{k_{i,j}}(y)$  for all  $y \in U_{i,j}$ .

Set  $\Gamma_x(\varphi) = \bigcup_n \Gamma_n$  (note that  $\Gamma_n$  is a subgroup of  $\Gamma_{n+1}$  for all  $n$ , see figure 3.1 below).

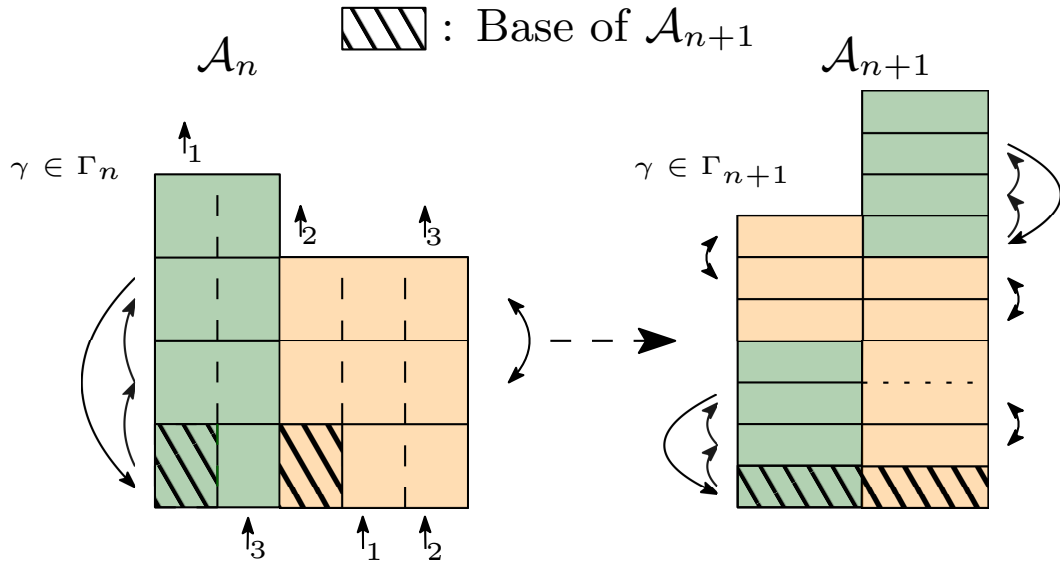


Figure 3.1: Cutting-and-stacking procedure, and  $\Gamma_n < \Gamma_{n+1}$

By definition,  $\Gamma_x(\varphi)$  is a subgroup of the topological full group  $F(\varphi)$ .

Each  $\gamma \in \Gamma_n$  induces a permutation of the atoms of each column of  $\mathcal{A}_n$ ; and if we know how  $\gamma$  permutes the atoms within each column then we have completely determined  $\gamma$  (this will lead us to the concept of *unit system* in the next section). In particular, each  $\Gamma_n$  is finite (and is isomorphic to a direct product of finite symmetric groups). So  $\Gamma_x(\varphi)$  is locally finite.

Since each clopen is eventually a union of atoms of  $\mathcal{A}_n$ ,  $\Gamma_x(\varphi)$  is a full group.

It seems from the definition that the group  $\Gamma_x(\varphi)$  depends on the choice of sequence of Kakutani–Rokhlin partitions, but it only depends on the choice of  $x$ ;

and even then, we will later see as a consequence of Krieger's theorem that  $\Gamma_x(\varphi)$  and  $\Gamma_y(\varphi)$  are conjugate in  $\text{Homeo}(X)$  for any  $x, y \in X$ .

**Lemma 3.5.** Denote  $O^+(x) = \{\varphi^n(x) : n \geq 0\}$ ,  $O^-(x) = \{\varphi^n(x) : n < 0\}$ .

Then  $\Gamma_x(\varphi)x = O^+(x)$ ,  $\Gamma_x(\varphi)\varphi^{-1}(x) = O^-(x)$ , and for any  $y$  which is not in the  $\varphi$ -orbit of  $x$  we have  $\Gamma_x(\varphi)y = \{\varphi^n(y) : n \in \mathbb{Z}\}$ .

Further,

$$\Gamma_x(\varphi) = \{g \in F(\varphi) : g(O^+(x)) = O^+(x)\}$$

*Proof.* Fix  $k > 0$  and let  $n$  be such that each column of  $\mathcal{A}_n$  has height bigger than  $k$ . Letting  $U_{i,0}$  be the atom of  $\mathcal{A}_n$  containing  $x$ , we then have  $\varphi^k(x) \in U_{i,k}$ , and there is an element of  $\Gamma_n$  which is equal to  $\varphi^k$  on  $U_{i,0}$ , so  $\varphi^k(x) \in \Gamma_x(\varphi)x$ . Hence  $O^+(x) \subseteq \Gamma_x(\varphi)x$  and a similar argument (or this argument applied to  $\varphi^{-1}$ ) shows that  $O^-(x) \subseteq \Gamma_x(\varphi)\varphi^{-1}(x)$ .

The converse inclusions are immediate from the definition of  $\Gamma_x(\varphi)$ : for  $k \geq 0$ , and any  $n$  such that the height of each column of  $\mathcal{A}_n$  is bigger than  $k$ , the  $\Gamma_n$ -orbit of  $\varphi^k(x)$  consists of  $x, \dots, \varphi^n(x)$  (and similarly for negative  $k$ ).

Next, denote by  $U_n$  the base of  $\mathcal{A}_n$ . If  $y$  does not belong to the  $\varphi$ -orbit of  $x$ , then for any sufficiently large  $n$   $y$  does not belong to  $U_n \cup \varphi^{-1}(U_n)$ . If we denote by  $V_n$  the atom of  $\mathcal{A}_n$  which contains  $y$ , this means that the map which is equal to  $\varphi$  on  $V_n$  and  $\varphi^{-1}$  on  $\varphi(V_n)$ , as well as the map which is equal to  $\varphi^{-1}$  on  $V_n$  and  $\varphi$  on  $\varphi^{-1}(V_n)$  both belong to  $\Gamma_n$ . Thus  $y$  belongs to the same  $\Gamma_x(\varphi)$ -orbit as  $\varphi^{\pm 1}(y)$ , so  $\Gamma_x(\varphi)y = \{\varphi^n(y) : n \in \mathbb{Z}\}$ . This completes the description of the  $\Gamma_x(\varphi)$ -orbits.

We still have to prove the second assertion. One inclusion comes from what we just established. Let  $g \in F(\varphi)$  and set

$$F = \{k \in \mathbb{Z} : \exists x \ g(x) = \varphi^k(x)\}$$

$F$  is a finite set; for any sufficiently large  $n$ , on any atom  $U_{i,j}$  of  $\mathcal{A}_n$  there exists  $k \in F$  such that  $g(y) = \varphi^k(y)$  for all  $y \in U_{i,j}$ . If  $g$  does not belong to  $\Gamma_x(\varphi)$ , there must exist some  $k \in F$  such that for infinitely many  $n$  there exists some atom  $U_{i,j}$  of  $\mathcal{A}_n$  such that either  $k+j \geq n_i$  or  $k+j < 0$ . Considering only a subsequence of  $(\mathcal{A}_n)$ , we may assume that we are always in the first case and  $n_i - j$  is constant equal to some integer  $m$  (since  $0 \leq n_i - j \leq k$ , only finitely many values are possible); or always in the second case and  $j$  is constant. In the first case,  $g$  maps  $\varphi^{-m-1}(x) \in O^-(x)$  to  $\varphi^{k-m}(x) \in O^+(x)$ ; in the second case  $g$  maps  $\varphi^j(x) \in O^+(x)$  to  $\varphi^{k+j}(x) \in O^-(x)$ .

We just proved that if  $g \in F(\varphi) \setminus \Gamma_x(\varphi)$  then  $g(O^+(x)) \neq O^+(x)$ , which is the contrapositive of the implication we were aiming for.  $\square$

### 3.3 Ample groups and a pointed version of Krieger's theorem

Now we go over some notions from Krieger [K1]. We will in particular establish a strengthening of the main result of [K1] (theorem 3.2 below).

### 3.3.1 Ample groups

*Definition 3.7* (Krieger [K1]). A subgroup  $\Gamma$  of  $\text{Homeo}(X)$  is an *ample group* if

- $\Gamma$  is a locally finite, countable, full group.
- For all  $\gamma \in \Gamma$ ,  $\{x: \gamma(x) = x\}$  is clopen.

Our main example comes from the objects introduced in the previous section: given a minimal homeomorphism  $\varphi$  and  $x \in X$ , the group  $\Gamma_x(\varphi)$  is an ample group (the fact that for each  $\gamma$  the set  $\{x: \gamma(x) = x\}$  is clopen comes from the fact that  $\varphi$  has no periodic points, since it acts minimally). Actually, all ample groups are of this form, see Theorem 3.4.

*Definition 3.8* (Krieger [K1]). Let  $\Gamma$  be a subgroup of  $\text{Homeo}(X)$ , and  $\mathcal{A}$  be a Boolean subalgebra of  $CO(X)$ . We say that  $(\mathcal{A}, \Gamma)$  is a *unit system* if:

- Every  $\gamma \in \Gamma$  induces an automorphism of  $\mathcal{A}$ .
- If  $\gamma \in \Gamma$  is such that  $\gamma(A) = A$  for all  $A \in \mathcal{A}$ , then  $\gamma = 1$ .
- If  $g \in \text{Homeo}(X)$  induces an automorphism of  $\mathcal{A}$ , and for any atom of  $\mathcal{A}$  there exists  $\gamma_A \in \Gamma$  such that  $\gamma_{A|A} = g|_A$ , then  $g \in \Gamma$ .

We say that  $(\mathcal{A}, \Gamma)$  is a *finite unit system* if  $\mathcal{A}$  is finite (in which case  $\Gamma$  is finite also).

By definition, if we know how  $\gamma \in \Gamma$  acts on atoms of  $\mathcal{A}$  then  $\gamma$  is uniquely determined. We say that a unit system  $(\mathcal{B}, \Delta)$  *refines* another unit system  $(\Gamma, \mathcal{A})$  if  $\Gamma \subseteq \Delta$  and  $\mathcal{A} \subseteq \mathcal{B}$ .

**Lemma 3.6** (Krieger). *Let  $\Gamma$  be an ample group. There exists a refining sequence  $(\mathcal{A}_n, \Gamma_n)$  of finite unit systems such that*

$$CO(X) = \bigcup_n \mathcal{A}_n \quad ; \quad \Gamma = \bigcup_n \Gamma_n$$

*We say that such a sequence of unit systems is exhaustive.*

*Proof.* Fix enumerations  $(\gamma_n)_{n \in \mathbb{N}}$  of  $\Gamma$  and  $(U_n)_{n \in \mathbb{N}}$  of  $CO(X)$ . We construct inductively a refining sequence of finite unit systems  $(\mathcal{A}_n, \Gamma_n)_{n \in \mathbb{N}}$  such that

$$\forall n \in \mathbb{N}, U_n \in \mathcal{A}_n \text{ and } \gamma_n \in \Gamma_n.$$

Assume  $\gamma_0 = id$  and  $U_0 = X$ , so that  $(\{X, \emptyset\}, \{\gamma_0\})$  is already a unit system, and set  $\mathcal{A}_0 = \{X, \emptyset\}, \Gamma_0 = \{\gamma_0\}$ .

Then assume  $(\mathcal{A}_n, \Gamma_n)$  is constructed for some  $n \geq 0$ , and let  $\Gamma'_n$  denote the (finite) group generated by  $\Gamma_n$  and  $\gamma_{n+1}$ . Fix  $\gamma \in \Gamma'_n$ . For any integer  $p$ , the set  $U_p(\gamma)$  of points which have period exactly  $p$  for  $\gamma$  is clopen. Let  $I(\gamma)$  denote the (finite) set of all  $p$  such that  $U_p(\gamma) \neq \emptyset$ .

Whenever  $p \geq 2$  and  $x \in U_p(\gamma)$ , we can find some clopen neighborhood  $V_p$  of  $x$  such that  $V_p, \gamma(V_p), \dots, \gamma^{p-1}(V_p)$  are pairwise disjoint. Since  $U_p(\gamma)$  is covered



by finitely many such  $V_p$ , we can then produce a clopen  $W_p(\gamma)$  such that for any  $p \in I(\gamma)$  one has

$$U_p(\gamma) = \bigsqcup_{k=0}^{p-1} \gamma^k(W_p(\gamma))$$

Then the family  $(\gamma^k(W_p(\gamma)))_{p \in I, 0 \leq k \leq p-1}$  forms a clopen partition, which generates a finite Boolean subalgebra  $\mathcal{B}_\gamma$  of  $CO(X)$ . For any atom  $A$  of this partition, either  $\gamma$  is equal to the identity on  $A$ , or  $\gamma(A) \cap A = \emptyset$ .

Let  $\mathcal{A}_{n+1}$  denote the coarsest  $\Gamma'_n$ -invariant Boolean subalgebra of  $CO(X)$  which refines each  $\mathcal{B}_\gamma$  as well as  $\mathcal{A}_n$  and the subalgebra  $\{\emptyset, U_{n+1}, X \setminus U_{n+1}, X\}$ . It is a finite subalgebra of  $CO(X)$ .

Finally, let  $\Gamma_{n+1}$  denote all  $\gamma \in \Gamma$  such that for any atom  $U$  of  $\mathcal{A}_{n+1}$  there exists  $\delta \in \Gamma'_n$  which coincides with  $\gamma$  on  $U$ .

By construction,  $\Gamma'_n$  is contained in  $\Gamma_{n+1}$ , and  $U_{n+1} \in \mathcal{A}_{n+1}$ . It remains to point out that  $(\mathcal{A}_{n+1}, \Gamma_{n+1})$  is a unit system; to see this, choose  $\gamma \in \Gamma'_n$  and  $U$  an atom of  $\mathcal{A}_{n+1}$  such that  $\gamma(U) = U$ . Then  $U$  is contained in some atom  $A$  of  $\mathcal{B}_\gamma$ , and either  $\gamma$  coincides with the identity on  $A$  or  $\gamma(A) \cap A = \emptyset$ . Since  $U \subseteq A$  and  $\gamma(U) = U$  we must be in the first situation, which concludes the proof.  $\square$

The following result is an analogue of a lemma due to Glasner and Weiss [GW, Lemma 2.5 and Proposition 2.6]; the argument we use in the proof is essentially the same as in [GW].

**Lemma 3.7.** *Let  $\Gamma$  be an ample group; recall that  $M(\Gamma)$  is the set of all  $\Gamma$ -invariant Borel probability measures on  $X$ . Let  $A, B$  be two clopen subsets of  $X$ .*

1. *Assume that  $\mu(A) < \mu(B)$  for all  $\mu \in M(\Gamma)$ . Then there exists  $\gamma \in \Gamma$  such that  $\gamma(A) \subset B$ .*
2. *Assume that  $\mu(A) = \mu(B)$  for all  $\mu \in M(\Gamma)$  and that  $\Gamma$  acts topologically transitively. Then there exists  $g \in [R_\Gamma]$  such that  $g(A) = B$ .*

We only assume that  $\Gamma$  acts topologically transitively above because that is the natural hypothesis to make the argument work. Since our concern is with minimal actions, we mostly stick with the minimality assumption throughout the paper but make an exception here.

*Proof.* (1) Find an exhaustive sequence  $(\mathcal{A}_n, \Gamma_n)$  of finite unit systems. There exists  $m \in \mathbb{N}$  such that for all  $n \geq m$  both  $A$  and  $B$  are unions of atoms of  $\mathcal{A}_n$ . For  $U$  an atom of  $\mathcal{A}_n$ , we may thus consider

$$a_n(U) = |\{V \in \Gamma_n U : V \subset A\}|$$

$$b_n(U) = |\{V \in \Gamma_n U : V \subset B\}|$$

(where  $|F|$  stands for the cardinality of a finite set  $F$ )

Assume that for any  $n \geq m$  there exists  $p \geq n$  and an atom  $U_p \in \mathcal{A}_p$  such that  $a_p(U_p) \geq b_p(U_p)$ ; pick  $x_p \in U_p$  and set

$$\mu_p = \frac{1}{|\Gamma_p|} \sum_{\gamma \in \Gamma_p} \delta_{\gamma(x_p)}$$

where  $\delta_y$  stands for the Dirac measure at  $y \in X$ . Then  $\mu_p$  is a  $\Gamma_p$ -invariant Borel probability measure for all  $p$ , and  $\mu_p(A) \geq \mu_p(B)$ . Using compactness of the space of all Borel probability measures on  $X$ , we may take a cluster point  $\mu$  of  $(\mu_p)$ , and  $\mu$  is a  $\Gamma$ -invariant measure such that  $\mu(A) \geq \mu(B)$ , contradicting our assumption.

Hence we see that, for any sufficiently large  $n \geq m$ , any atom  $U$  of  $\mathcal{A}_n$  is such that  $a_n(U) < b_n(U)$ . From this we obtain the existence of  $\gamma \in \Gamma_n$  such that  $\gamma(A) \subset B$  (any permutation of the atoms of a column of  $\mathcal{A}_n$  is induced by an element of  $\Gamma_n$ ).

(2) We may (and do) assume that  $A, B$  are nonempty and  $A \cap B = \emptyset$ ; we fix  $a \in A$  and  $b \in B \cap \Gamma a$  (here we are using topological transitivity).

Fix a compatible ultrametric  $d$  on  $X$ ; we use a back-and-forth argument to build sequences of clopen subsets  $(U_n), (V_n)$  of  $X$ , and a sequence  $(\gamma_n)$  of elements of  $\Gamma$  such that for all  $n$ :

- $U_n \subseteq A, a \in A \setminus U_n$ , and  $U_n \cap \bigcup_{i=0}^{n-1} U_i = \emptyset$ .
- $V_n \subseteq B, b \in A \setminus V_n$ , and  $V_n \cap \bigcup_{i=0}^{n-1} V_i = \emptyset$ .
- The diameters of  $A \setminus \bigcup_{i=0}^n U_i$  and  $B \setminus \bigcup_{i=0}^n V_i$  converge to 0.
- For all  $n, \gamma_n(U_n) = V_n$ .

Assuming that this is indeed possible, we obtain the desired  $g$  by setting  $g = \gamma_n$  on  $U_n, g = \gamma_n^{-1}$  on  $V_n, g(a) = b, g(b) = a$  (and  $g$  is the identity outside  $A \cup B$ ).

We use even steps of the process to make the diameter of  $X \setminus \bigcup_{i=0}^n U_i$  decrease, and odd steps to control  $X \setminus \bigcup_{i=0}^n V_i$ ; since the conditions are symmetric, let us explain what we do when  $U_0, V_0, \dots, U_{n-1}, V_{n-1}$  have been defined and  $n$  is even. Set

$$\tilde{A} = A \setminus \bigcup_{i=0}^{n-1} U_i, \quad \tilde{B} = B \setminus \bigcup_{i=0}^{n-1} V_i$$

Then  $a \in \tilde{A}, b \in \tilde{B}$ , and  $\mu(\tilde{A}) = \mu(\tilde{B})$  for all  $\mu \in M(\Gamma)$ .

Pick  $\varepsilon > 0$  such that  $B(a, \varepsilon) \subset \tilde{A}$ . It follows from Lemma 3.2 that for small enough  $\delta > 0$ , we have

$$\sup_{\mu \in M(\Gamma)} \mu(\tilde{A} \setminus B(a, \varepsilon)) < \inf_{\mu \in M(\Gamma)} \mu(\tilde{B} \setminus B(b, \delta))$$

Set  $U_{n+1} = \tilde{A} \setminus B(a, \varepsilon)$ ; by (1) we can find  $\gamma_{n+1} \in \Gamma$  such that  $\gamma_{n+1}(U_{n+1}) \subset \tilde{B} \setminus B(b, \delta)$ . We set  $V_{n+1} = \gamma(U_{n+1})$  and move on to the next step.  $\square$

For future reference, we state the original lemma of Glasner and Weiss, whose proof is very similar to the proof of Lemma 3.7.

**Lemma 3.8** (Glasner–Weiss [GW, Lemma 2.5]). *Let  $\varphi$  be a minimal homeomorphism, and let  $A, B$  be two clopen sets such that  $\mu(A) < \mu(B)$  for any  $\mu \in M(\varphi)$ .*

*Then there exists an integer  $N$  such that, whenever  $\mathcal{A}$  is a Kakutani–Rokhlin partition compatible with  $A, B$  and such that all columns of  $\mathcal{A}$  have height  $\geq N$ , for any column  $C$  of  $\mathcal{A}$  one has*

$$|\{\alpha \in C: \alpha \subseteq A\}| < |\{\alpha \in C: \alpha \subseteq B\}|$$

We reformulate Lemma 3.7 using the Polish group topology of  $\text{Homeo}(X)$ ; we recall that this topology can be viewed either as the topology of compact-open convergence on  $X$ , or as the permutation group topology induced by the action of  $\text{Homeo}(X)$  on  $CO(X)$ . Using this last point of view, a neighborhood basis of the identity is given by the subgroups

$$G_{\mathcal{A}} = \{g \in \text{Homeo}(X): \forall A \in \mathcal{A} \ g(A) = A\}$$

where  $\mathcal{A}$  ranges over all clopen partitions of  $X$ .

**Lemma 3.9.** *Assume that  $\Gamma$  is a minimal ample group. Then*

$$\overline{[R_{\Gamma}]} = \{g \in \text{Homeo}(X): \forall \mu \in M(\Gamma) \ g_*\mu = \mu\}$$

*Proof.* Any element  $g$  of  $[R_{\Gamma}]$  must be such that  $g_*\mu = \mu$  for all  $\mu \in M(\Gamma)$  (see Lemma 3.1) and  $\{g: g_*\mu = \mu\}$  is a closed subset of  $\text{Homeo}(X)$  for all  $\mu$ . This proves the inclusion from left to right.

To see the converse inclusion, take  $g$  such that  $g_*\mu = \mu$  for all  $\mu \in M(\Gamma)$ . Let  $U$  be a neighborhood of  $g$  in  $\text{Homeo}(X)$ ; by definition of the topology of  $\text{Homeo}(X)$ , there exists a clopen partition  $\mathcal{A}$  of  $X$  such that

$$\{h \in \text{Homeo}(X): \forall A \in \mathcal{A} \ h(A) = g(A)\} \subseteq U$$

Lemma 3.7 shows that for any  $A \in \mathcal{A}$  there exists  $h_A \in [R_{\Gamma}]$  such that  $h_A(A) = g(A)$ , whence there exists  $h \in [R_{\Gamma}] \cap U$  obtained by setting  $h(x) = h_A(x)$  for all  $x \in A$  and all  $A \in \mathcal{A}$ .  $\square$

The heart of the above argument is the fact that, for a full group  $G$  contained in  $\text{Homeo}(X)$ , the closure  $\overline{G}$  consists of all homeomorphisms  $h$  such that for any  $A \in CO(X)$  there exists  $g \in G$  such that  $h(A) = g(A)$ .

The following lemma will help us deduce the classification theorem for minimal homeomorphisms from the classification theorem for minimal ample groups.

**Lemma 3.10.** *Let  $\varphi$  be a minimal homeomorphism, and  $x \in X$ .*

*Then  $M(\Gamma_x(\varphi)) = M(\varphi)$ , and*

$$\overline{[R_{\Gamma_x(\varphi)}]} = \overline{[R_{\varphi}]} = \{g \in \text{Homeo}(X): \forall \mu \in M(\varphi) \ g_*\mu = \mu\}$$

*Proof.* To simplify notation, denote  $\Gamma = \Gamma_x(\varphi)$ .

Since  $\Gamma \subset [R_{\varphi}]$ , we have  $[R_{\Gamma}] \subseteq [R_{\varphi}]$  and  $M(\varphi) = M([R_{\varphi}]) \subseteq M(\Gamma)$ .

Pick  $\mu \in M(\Gamma)$ , and let  $U \in CO(X)$ ; assume first that  $\varphi(U) \cap U = \emptyset$ , and  $U$  does not contain  $\varphi^{-1}(x)$ . Then the involution  $\gamma$  equal to  $\varphi$  on  $U$ ,  $\varphi^{-1}$  on  $\varphi(U)$  and the identity elsewhere belongs to  $\Gamma$ . Thus

$$\mu(\varphi(U)) = \mu(\gamma(U)) = \mu(U)$$

If  $U \in CO(X)$  is any clopen not containing  $\varphi^{-1}(x)$ , we can write it (by exhaustion) as a disjoint union of clopens  $U_i$  such that  $\varphi(U_i) \cap U_i = \emptyset$ , so that

$$\mu(\varphi(U)) = \sum_{i=1}^n \mu(\varphi(U_i)) = \sum_{i=1}^n \mu(U_i) = \mu(U)$$

Finally, if  $\varphi^{-1}(x) \in U$ , we can find a clopen  $V$  containing  $x$  and such that  $V$  and  $\varphi(V)$  both have arbitrarily small diameter (for some compatible distance), thus if we fix  $\varepsilon > 0$  we can find a clopen  $V \subset U$  such that  $\varphi^{-1}(x) \in V$  and  $\mu(V), \mu(\varphi(V))$  are both  $< \varepsilon$ . Since  $\mu(U \setminus V) = \mu(\varphi(U) \setminus \varphi(V))$ , we have

$$|\mu(U) - \mu(\varphi(U))| = |\mu(V) - \mu(\varphi(V))| \leq 2\varepsilon$$

This is true for any  $\varepsilon > 0$ , so  $\mu(\varphi(U)) = \mu(U)$  for any clopen  $U$ : in other words,  $\mu \in M(\varphi)$ .

This establishes the first assertion; the second one is an immediate consequence of this, since then we have by Lemma 3.9 the equality

$$\overline{[R_{\Gamma_x(\varphi)}]} = \{g \in \text{Homeo}(X) : \forall \mu \in M(\varphi) \ g_*\mu = \mu\}$$

and the right-hand side of this equality is closed and contains  $[R_\varphi]$ .  $\square$

### 3.3.2 A strengthening of Krieger's theorem

We turn to the proof of a version of Krieger's theorem that is instrumental to our approach. The proof is based on a back-and-forth argument already present in Krieger's proof.

*Definition 3.9.* Let  $\Gamma$  be a subgroup of  $\text{Homeo}(X)$ . Given  $U, V \in CO(X)$  we write  $U \sim_\Gamma V$  if there exists  $\gamma \in \Gamma$  such that  $\gamma(U) = V$ .

*Definition 3.10.* Let  $\Gamma, \Lambda$  be two ample subgroups of  $\text{Homeo}(X)$ . We say that  $\Gamma, \Lambda$  have *isomorphic closures* if there exists  $g \in \text{Homeo}(X)$  such that  $g\overline{\Gamma}g^{-1} = \overline{\Lambda}$  or, equivalently,

$$\forall U, V \in CO(X) \quad (U \sim_\Gamma V) \Leftrightarrow (g(U) \sim_\Lambda g(V))$$

The fact that both conditions in the previous definition are equivalent follows from the remark immediately following Lemma 3.9; they are also equivalent (in our context) to what Krieger calls "isomorphism of dimension ranges", though we formulate it in the way which we find most suitable for our purposes in this article.

*Definition 3.11.* Let  $\Gamma$  be a minimal ample group. We say that a closed subset  $K$  of  $X$  is  $\Gamma$ -sparse if each  $\Gamma$ -orbit intersects  $K$  in at most one point. For such a  $K$ , we say that a finite unit system  $(\mathcal{A}, \Sigma)$  with  $\Sigma \leq \Gamma$  is  $K$ -compatible if any  $\Sigma$ -orbit (for the action of  $\Sigma$  on the atoms of  $\mathcal{A}$ ) has at most one element which intersects  $K$ .

Our aim in this subsection is to prove the following strengthening of Krieger's theorem [K1, Theorem 3.5]. This result plays a crucial role in our proof of the classification theorem for minimal ample groups.

**Théorème 3.2.** *Let  $\Gamma, \Lambda$  be minimal ample groups with isomorphic closures. Let  $K, L$  be closed subsets of  $X$  such that  $K$  is  $\Gamma$ -sparse, and  $L$  is  $\Lambda$ -sparse. Assume that  $h: K \rightarrow L$  is a homeomorphism.*

*Then there exists  $g \in \text{Homeo}(X)$  such that  $g\Lambda g^{-1} = \Gamma$ , and  $g|_K = h$ .*

*Remark 3.2.* Although we do not need this result here, we note that whenever  $\varphi$  is a minimal homeomorphism of  $X$ ,  $\Gamma_x(\varphi)$  and  $\Gamma_y(\varphi)$  induce the same relation on  $\text{CO}(X)$  for any  $x, y \in X$  (see [R] for an elementary proof), and it then follows from Theorem 3.2 (with  $K = L = \emptyset$ ) that they are conjugate.

We begin working towards the proof of Theorem 3.2.

**Lemma 3.11.** *Let  $\Gamma$  be a minimal ample group, and  $K$  a  $\Gamma$ -sparse closed subset of  $X$ .*

- *For any finite unit system  $(\mathcal{A}, \Sigma)$  with  $\Sigma \leq \Gamma$ , there exists a  $K$ -compatible finite unit system  $(\mathcal{B}, \Sigma)$  with  $\mathcal{B}$  refining  $\mathcal{A}$ .*
- *There exists a refining sequence  $(\mathcal{B}_n, \Gamma_n)$  of  $K$ -compatible finite unit systems such that  $\text{CO}(X) = \bigcup \mathcal{B}_n$  and  $\Gamma = \bigcup \Gamma_n$ .*

*Proof.* Fix a finite unit system  $(\mathcal{A}, \Sigma)$  with  $\Sigma \leq \Gamma$ .

Choose  $U_1, \dots, U_k$  representatives of the  $\mathcal{A}$ -orbits. For each  $i$ , the  $\mathcal{A}$ -orbit of  $U_i$  is of the form  $U_i \sqcup \gamma_{i,1}U_i \sqcup \gamma_{i,k_i}U_i$ . For all  $x$  in  $U_i$ , at most one element of  $\{x, \gamma_{1,1}(x), \dots, \gamma_{i,k_i}(x)\}$  can belong to  $K$ .

So we can write  $U_i = \bigsqcup_{j=1}^{n_i} U_{i,j}$ , where, for every  $j$ ,  $U_{i,j} \sqcup \gamma_{i,1}U_{i,j} \dots \sqcup \gamma_{i,k_i}U_{i,j}$  intersects  $K$  in at most one point.

Finally, let  $\mathcal{B}$  be the algebra whose atoms are the  $\gamma(U_{i,j})$ ,  $\gamma \in \Sigma$ .

This proves the first part of the lemma's statement; the second part immediately follows from the first part and Lemma 3.6.  $\square$

**Lemma 3.12.** *Let  $\Gamma$  be a minimal ample group, and  $K$  be a  $\Gamma$ -sparse closed subset of  $X$ .*

*Let  $U$  be a nontrivial clopen subset of  $X$ , and  $A$  be a clopen subset of  $K$ . Let  $V \in \text{CO}(X)$  be such that  $A \subset V \cap K$  and  $\mu(U) < \mu(V)$  for all  $\mu \in M(\Gamma)$ .*

*There exists  $U' \in \text{CO}(X)$  such that  $U' \cap K = A$ ,  $U' \subset V$ , and  $U' \sim_{\Gamma} U$ .*

*Proof.* Fix some integer  $N$ . Let  $(\Gamma_n)$  be an increasing sequence of finite groups such that  $\bigcup_n \Gamma_n = \Gamma$ . Assume first, for a contradiction, that for all  $n$  there exists  $x_n \in X$  such that  $\Gamma_n x_n$  has cardinality  $< N$ . By compactness, we may assume that  $(x_n)$  converges to  $x$ ; since  $\Gamma$  acts aperiodically, we can find  $\gamma_1, \dots, \gamma_N$  such that

$\gamma_i(x) \neq \gamma_j(x)$  for all  $i \neq j$ . Then we also have  $\gamma_i(x_n) \neq \gamma_j(x_n)$  for  $n$  large enough and  $i \neq j$ , which is the desired contradiction.

Thus, for any  $n$  large enough, every  $\Gamma_n$ -orbit has cardinality  $\geq N$ . Thus there exists a finite unit system  $(\mathcal{A}, \Sigma)$  with  $\Sigma \leq \Gamma$  such that every orbit for the action of  $\Sigma$  on the set of atoms of  $\mathcal{A}$  has cardinality  $\geq N$ . This allows us to find  $\gamma \in \Gamma$  such that every  $x \in X$  has a  $\gamma$ -orbit of cardinality  $\geq N$ .

Since  $K, \dots, \gamma^{N-1}(K)$  are closed and pairwise disjoint, we can find some disjoint clopens  $B_i$  such that  $\gamma^i(K) \subseteq B_i$  for all  $i \in \{0, \dots, N-1\}$ . Then  $B = B_0 \cap \bigcup_{i=1}^{N-1} \gamma^{-i}(B_i)$  is clopen, contains  $K$ , and  $B, \dots, \gamma^{N-1}(B)$  are pairwise disjoint.

So there exists  $B \in CO(X)$  containing  $K$  and such that  $\mu(B) \leq \frac{1}{N}$  for all  $\mu \in K$ . Hence we can find  $B$  clopen, containing  $K$ , and such that  $\mu(B) < \mu(U)$  for all  $\mu \in M(\Gamma)$ .

There exists  $C \in CO(X)$  such that  $A = K \cap C$ , so  $D = B \cap C$  is clopen,  $D \cap K = A$ , and  $\mu(D) < \mu(U)$  for all  $\mu \in M(\Gamma)$ . Similarly we can find  $E \in CO(X)$  such that  $E \cap K = K \setminus A$ ,  $\mu(E) < \mu(X \setminus U)$  for all  $\mu \in M(\Gamma)$ , and  $E \cap D = \emptyset$ .

There exists  $\gamma \in \Gamma$  such that  $\gamma(D) \subset U$ ,  $\gamma(E) \subset X \setminus U$ ; set  $W = \gamma^{-1}(U)$ . We have  $W \cap K = A$  and  $W \sim_{\Gamma} U$ .

Set  $V' = V \cap W$ ; it is clopen, contained in  $V$ , and  $V' \cap K = A$ . Since there are clopen subsets of arbitrarily small measures containing  $K$ , there exists a clopen  $V''$  contained in  $V$ , disjoint from  $K$ , and such that  $\mu(U) < \mu(V'')$  for all  $\mu \in M(\Gamma)$ . So  $\mu(W \setminus V') < \mu(V'' \setminus V')$  for all  $\mu \in M(\Gamma)$ . Hence there exists  $Y \sim_{\Gamma} W \setminus V'$  such that  $Y \subset V'' \setminus V'$ . We may finally set  $U' = V' \sqcup Y$ .  $\square$

*Notation.* For the remainder of this section, we fix two minimal ample groups  $\Gamma, \Lambda$ , and assume that  $\sim_{\Gamma}$  and  $\sim_{\Lambda}$  coincide (we reduce to this situation by conjugating  $\Lambda$  if necessary). We denote this equivalence relation on  $CO(X)$  by  $\sim$ . We also fix closed subsets  $K, L$  such that  $K$  is  $\Gamma$ -sparse and  $L$  is  $\Lambda$ -sparse, and a homeomorphism  $h: K \rightarrow L$ .

*Definition 3.12.* Let  $\Delta$  be a finite subgroup of  $\Gamma$ ,  $\Sigma$  a finite subgroup of  $\Lambda$ , and assume that  $(\mathcal{A}, \Delta), (\mathcal{B}, \Sigma)$  are finite unit systems.

A Boolean algebra isomorphism  $\Phi: \mathcal{A} \rightarrow \mathcal{B}$  respects  $\sim$  if for any  $A \in \mathcal{A}$  one has  $\Phi(A) \sim A$ . We say that  $\Phi$  conjugates  $(\mathcal{A}, \Delta)$  on  $(\mathcal{B}, \Sigma)$  if  $\Sigma|_{\mathcal{B}} = \Phi\Delta|_{\mathcal{A}}\Phi^{-1}$ .

*Definition 3.13.* Let  $\Delta$  be a finite subgroup of  $\Gamma$ ,  $\Sigma$  a finite subgroup of  $\Lambda$ , and assume that  $(\mathcal{A}, \Delta)$  is a  $K$ -compatible finite unit system, and  $(\mathcal{B}, \Sigma)$  is a  $L$ -compatible finite unit system.

We say that a Boolean algebra isomorphism  $\Phi: \mathcal{A} \rightarrow \mathcal{B}$  is  $h$ -compatible if:

1.  $\Phi$  respects  $\sim$ .
2.  $\Phi$  conjugates  $(\mathcal{A}, \Delta)$  on  $(\mathcal{B}, \Sigma)$ .
3. For every atom  $\alpha \in \mathcal{A}$ ,  $\Phi(\alpha) \cap L = h(\alpha \cap K)$ .

**Lemma 3.13.** *Assume that  $(\mathcal{A}, \Delta), (\mathcal{B}, \Sigma)$  are respectively  $K$ - and  $L$ -compatible finite unit systems with  $\Delta \leq \Gamma, \Sigma \leq \Lambda$ , and  $\Phi: \mathcal{A} \rightarrow \mathcal{B}$  is a  $h$ -compatible Boolean algebra isomorphism.*

Let  $(\mathcal{A}', \Delta')$  be a  $K$ -compatible finite unit system refining  $(\mathcal{A}, \Delta)$  with  $\Delta' \leq \Gamma$ .

Then one can find a  $L$ -compatible finite unit system  $(\mathcal{B}', \Sigma')$  refining  $(\mathcal{B}, \Sigma)$ , with  $\Sigma' \leq \Lambda$  and a  $h$ -compatible isomorphism  $\Phi': \mathcal{A}' \rightarrow \mathcal{B}'$  which extends  $\Phi$ .

*Proof.* For every orbit  $\rho$  of the action of  $\Delta$  on the atoms of  $\mathcal{A}$ , we choose a representative  $A_\rho$ . If  $\rho$  intersects  $K$ , we choose  $A_\rho$  so that  $A_\rho \cap K \neq \emptyset$  (and it is the unique such atom in  $\rho$ , because  $\mathcal{A}$  is  $K$ -compatible).

For every  $A \in \rho$ , we denote by  $\delta(\rho, A)$  the element of  $\Delta$  which maps  $A$  to  $A_\rho$ ,  $A_\rho$  to  $A$ , and is the identity everywhere else. This is an involution (and it is uniquely defined by definition of a unit system); in the particular case where  $A = A_\rho$  we have  $\delta(\rho, A_\rho) = id$ . Similarly, we denote  $\sigma(\rho, A)$  the involution of  $\Sigma$  exchanging  $\Phi(A)$  and  $\Phi(A_\rho)$  and which is the identity everywhere else.

For every  $\rho$  we have

$$A_\rho = \bigsqcup_{C \in \text{atoms}(\mathcal{A}'): C \subseteq A_\rho} C$$

Let  $C_1, \dots, C_p$  denote the atoms of  $\mathcal{A}'$  contained in  $A_\rho$ . Applying Lemma 3.12, find a clopen  $U(C_1) \sim_\Gamma C_1$  contained in  $\Phi(A_\rho)$  and such that  $U(C_1) \cap L = h(C_1 \cap K)$ ; then a clopen  $U(C_2) \sim_\Gamma C_2$  contained in  $\Phi(A_\rho)$  disjoint from  $C_1$  and such that  $U(C_2) \cap L = h(C_2 \cap K)$ ; and so on.

We now have

$$\Phi(A_\rho) = \bigsqcup_{C \in \text{atoms}(\mathcal{A}'): C \subseteq A_\rho} U(C)$$

where  $U(C) \sim C$ , and  $U(C) \cap L = h(C \cap K)$  for all  $C$ .

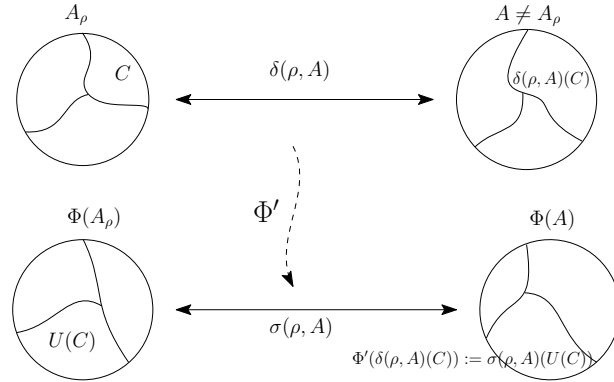


Figure 3.2: Construction of  $\mathcal{B}'$  and  $\Phi'$

We define the algebra  $\mathcal{B}'$  by setting as its atoms all  $U(C)$ , for  $C$  an atom of  $\mathcal{A}'$  contained in some  $A_\rho$ , as well as all  $\sigma(\rho, A)(U(C))$  for  $A \in \rho$  and  $C$  contained in  $A_\rho$  (see figure 3.2). We obtain an isomorphism  $\Phi': \mathcal{A}' \rightarrow \mathcal{B}'$  by setting  $\Phi(C) = U(C)$  for every atom of  $\mathcal{A}'$  contained in some  $A_\rho$ ; and then for any atom  $C$  of  $\mathcal{A}'$  contained in some  $A \in \mathcal{A}$  whose  $\Delta$ -orbit is  $\rho$ ,

$$\Phi'(C) = \sigma(\rho, A)(U(\delta(\rho, A)(C)))$$

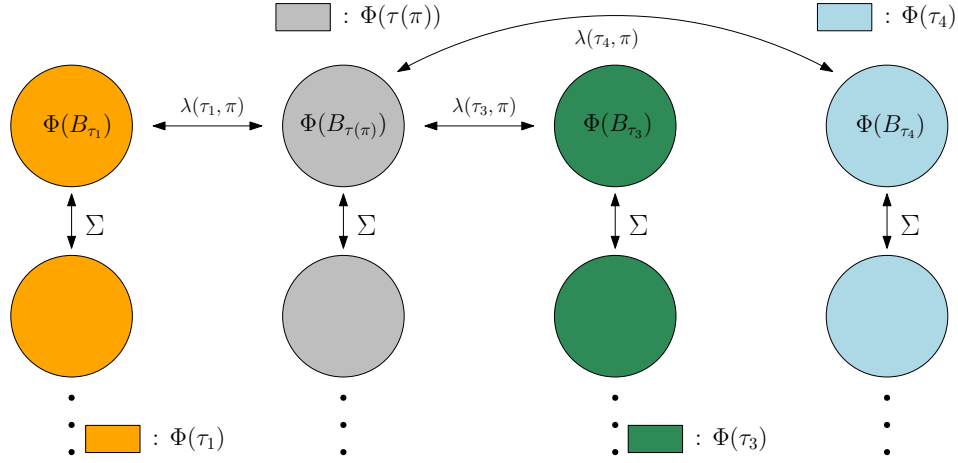


Figure 3.3: Construction of  $\Sigma'$  imitating the behavior of  $\Delta'$  on the image of a  $\Delta'$ -orbit  $\pi$  containing four  $\Delta$ -orbits

We now need to construct the group  $\Sigma'$ . In the remainder of the proof, the letter  $\tau$  always stands for an orbit of the action of  $\Delta$  on the atoms of  $\mathcal{A}'$ , and the letter  $\pi$  for an orbit of the action of  $\Delta'$  on the atoms of  $\mathcal{A}'$ . For any  $\tau$  there exists a unique  $\pi$  which contains  $\tau$ .

For any  $\tau$  we choose a representative  $B_\tau$ , and among all  $B_\tau$  contained in a given  $\pi$  we choose one  $B_{\tau(\pi)}$ . For every  $\tau$  contained in  $\pi$ , we choose an involution  $\lambda(\tau, \pi) \in \Lambda$  mapping  $\Phi'(B_{\tau(\pi)})$  to  $\Phi'(B_\tau)$ , and equal to the identity elsewhere. Let  $\Sigma'$  be the group generated by  $\Sigma$  and  $\{\lambda(\tau, \pi) : \tau \subset \pi\}$ . Then  $(\mathcal{B}', \Sigma')$  is a finite unit system (because we have added at most one link between any two  $\Sigma$ -orbits) and  $\Phi'$  conjugates  $(\mathcal{A}', \Delta')$  to  $(\mathcal{B}', \Sigma')$ .

For every atom  $A$  of  $\mathcal{A}'$  we have  $\Phi(A) \cap L = h(A \cap K)$  by choice of  $U(A)$ . Since  $\Phi'$  conjugates  $(\mathcal{A}', \Delta')$  to  $(\mathcal{B}', \Sigma')$ , for any two atoms  $C, D$  of  $\mathcal{A}'$  which intersect  $K$ ,  $\Phi'(C)$  and  $\Phi'(D)$  belong to different  $\Sigma'$ -orbits. This proves that  $(\mathcal{B}', \Sigma')$  is  $L$ -compatible, and completes the proof.  $\square$

*End of the proof of Theorem 3.2.* Fix sequences  $(\mathcal{A}_n, \Gamma_n)$  and  $(\mathcal{B}_n, \Lambda_n)$  of respectively  $K$  and  $L$ -compatible finite unit systems as in Lemma 3.11, with  $\mathcal{A}_0 = \mathcal{B}_0 = \{\emptyset, X\}$  and  $\Gamma_0 = \Lambda_0 = \{\text{id}\}$ .

Then  $(\mathcal{A}_0, \Gamma_0), (\mathcal{B}_0, \Lambda_0)$  are respectively  $K$  and  $L$ -compatible finite unit systems, and  $\Phi_0 = \text{id}$  is  $h$ -compatible.

Applying Lemma 3.13 we build inductively refining sequences  $(\mathcal{A}'_n, \Gamma'_n)$ , (resp.  $(\mathcal{B}'_n, \Lambda'_n)$ ) of  $K$ -compatible (resp.  $L$ -compatible) finite unit systems contained in  $(CO(X), \Gamma)$  and  $(CO(X), \Lambda)$  respectively, as well as  $h$ -compatible isomorphisms  $\Phi_n$  conjugating  $(\mathcal{A}'_n, \Gamma'_n)$  to  $(\mathcal{B}'_n, \Lambda'_n)$  such that for odd  $n$   $(\mathcal{A}'_n, \Gamma'_n)$  refines  $(\mathcal{A}_n, \Gamma_n)$ , and for even  $n$   $(\mathcal{B}'_n, \Lambda'_n)$  refines  $(\mathcal{B}_n, \Lambda_n)$ .

To see why this is possible, assume that we have carried out this construction up to some even  $n$  (the odd case is symmetric). We pick  $k \geq n + 1$  such that  $(\mathcal{A}_k, \Gamma_k)$  refines  $(\mathcal{A}'_n, \Gamma'_n)$ . The unit system  $(\mathcal{A}_k, \Gamma_k)$  (resp.  $(\mathcal{B}'_n, \Lambda'_n)$ ) is  $K$ -compatible



(resp.  $L$ -compatible), so applying Lemma 3.13 gives us some  $L$ -compatible finite unit system  $(\mathcal{B}'_{n+1}, \Lambda'_{n+1})$  contained in  $(CO(X), \Lambda)$  which refines  $(\mathcal{B}'_n, \Lambda_n)$ , and a  $h$ -compatible isomorphism  $\Phi_{n+1}: \mathcal{A}_k \rightarrow \mathcal{B}'_{n+1}$ . Setting  $\mathcal{A}'_{n+1} = \mathcal{A}_k$ ,  $\Gamma'_{n+1} = \Gamma_k$ , we are done.

This construction produces an isomorphism  $\Phi$  of  $CO(X)$  (the union of the sequence  $\Phi_n$ ) such that  $\Phi\Gamma\Phi^{-1} = \Lambda$  (here,  $\Gamma$  and  $\Lambda$  are viewed as subgroups of the automorphism group of the Boolean algebra  $CO(X)$ ). By Stone duality, there exists a unique  $g \in \text{Homeo}(X)$  such that  $g(U) = \Phi(U)$  for any  $U \in CO(X)$ , and we have  $g\Gamma g^{-1} = \Lambda$ .

Clopen subsets of the form  $\alpha \cap K$ , where  $\alpha$  is an atom of some  $\mathcal{A}'_n$ , generate the topology of  $K$ . For each such clopen we have  $\Phi(\alpha) \cap L = h(\alpha \cap K)$ , in other words  $g(\alpha) \cap L = h(\alpha \cap K)$ . It follows that  $g|_K = h$ , and we are done.  $\square$

*Definition 3.14.* We say that an ample group  $\Gamma$  is *saturated* if  $\bar{\Gamma} = \overline{[R_\Gamma]}$ .

The classification theorem for minimal actions of saturated ample groups follows immediately from Krieger's theorem.

**Théorème 3.3.** *Let  $\Gamma, \Lambda$  be two saturated ample subgroups of  $\text{Homeo}(X)$  acting minimally. Then the following conditions are equivalent:*

1.  $\Gamma$  and  $\Lambda$  are conjugated in  $\text{Homeo}(X)$ .
2.  $\Gamma$  and  $\Lambda$  are orbit equivalent.
3. There exists  $g \in \text{Homeo}(X)$  such that  $g_*M(\Gamma) = M(\Lambda)$ .

*Proof.* Clearly the first condition implies the second, and we already know that the second implies the third. Now, assume that  $\Gamma, \Lambda$  are saturated ample groups acting minimally, and  $M(\Gamma) = M(\Lambda)$  (as usual, we reduce to this situation by conjugating  $\Lambda$  if necessary).

Then we have  $\overline{[R_\Gamma]} = \overline{[R_\Lambda]}$  (see Lemma 3.9), hence also  $\bar{\Gamma} = \bar{\Lambda}$ . So  $\Gamma, \Lambda$  are conjugated by Theorem 3.2.  $\square$

### 3.4 Balanced partitions

In this section, we fix an ample group  $\Gamma$  acting minimally, and an exhaustive sequence  $(\mathcal{A}_n, \Gamma_n)$  of finite unit systems.

*Definition 3.15.* We consider two equivalence relations on  $CO(X)$ , defined as follows:

- $U \sim_\Gamma V$  iff there exists  $g \in \Gamma$  such that  $g(U) = V$ .
- $U \sim_\Gamma^* V$  iff  $\mu(U) = \mu(V)$  for any  $\mu \in M(\Gamma)$  (equivalently, there exists  $g \in [R_\Gamma]$  such that  $g(U) = V$ ).

The main difficulty in our proof of the classification theorem comes from the fact that  $\sim_\Gamma^*$  may be strictly coarser than  $\sim_\Gamma$ .

*Definition 3.16.* An equivalence relation  $\simeq$  on  $CO(X)$  is *full* if for any clopens  $A, B$ , whenever  $A = \bigsqcup A_i, B = \bigsqcup B_i$  and  $A_i \simeq B_i$  for all  $i$ , we have  $A \simeq B$ .

The relation  $\sim_\Gamma$  is full: indeed, consider  $(A_i), (B_i)$  as above; applying Lemma 3.6 we find a finite unit system  $(\mathcal{A}, \Delta)$  with  $\Delta$  a finite subgroup of  $\Gamma$ , such that for all  $i$   $A_i$  and  $B_i$  are unions of atoms of  $\mathcal{A}$ , and there exists  $\delta_i \in \Delta$  such that  $\delta_i A_i = B_i$ . We then see that there exists  $\delta \in \Delta$  such that  $\delta(\bigsqcup A_i) = \bigsqcup B_i$ .

It is immediate that  $\sim_\Gamma^*$  is full. We note the following question: is  $\sim_\Lambda$  full whenever  $\Lambda$  is the full group associated to an action of a countable group?

*Definition 3.17.* Let  $\simeq$  be a full equivalence relation on  $CO(X)$ .

A  $\simeq$ -partition of  $X$  is a clopen partition  $\mathcal{A} = (A_{i,j})_{(i,j) \in I}$  such that

$$\forall i, j, k \quad ((i, j) \in I \text{ and } (i, k) \in I) \Rightarrow A_{i,j} \simeq A_{i,k}$$

We denote  $I_i$  the set  $\{j : (i, j) \in I\}$ . For any element  $\alpha = A_{i,j}$  of  $\mathcal{A}$ , the set  $\{A_{i,k} : k \in I_i\}$  is called the  $\mathcal{A}$ -orbit of  $\alpha$  and denoted  $O(\alpha)$ .

Given an orbit  $O = (A_{i,j})_{j \in I_i}$ , a family of clopens  $(B_{i,j})_{j \in I_i}$  such that

$$\forall j \in I_i \quad B_{i,j} \subset A_{i,j} \text{ and } \forall j, k \in I_i \quad B_{i,j} \simeq B_{i,k}$$

is called a *fragment* of  $O$ .

Fragments of orbits are the abstract counterpart to the copies of columns that appear during the cutting procedure for Kakutani–Rokhlin partitions. For  $A \subseteq \alpha$ , with  $\alpha$  an atom of  $\mathcal{A}$ , we may abuse notation and talk about  $O(A)$  to designate a fragment of  $O(\alpha)$  containing  $A$  (such a fragment is obviously not unique, but we usually do not care about the particular choices we are making when selecting the atoms of our fragment).

*Definition 3.18.* Let  $\simeq$  be a full equivalence relation on  $CO(X)$ .

A  $\simeq$ -partition  $\mathcal{A}$  is *compatible* with a given clopen set  $U$  if  $U$  is a union of elements of  $\mathcal{A}$  (equivalently, if  $U$  belongs to the Boolean algebra generated by  $\mathcal{A}$ ).

If  $\mathcal{A}$  is compatible with  $U$ , and  $O$  is a  $\mathcal{A}$ -orbit, we let  $n_O(U)$  be the number of elements of  $O$  which are contained in  $U$ .

*Definition 3.19.* Let  $\simeq$  be a full equivalence relation on  $CO(X)$ , and  $\mathcal{A}, \mathcal{B}$  be two  $\simeq$ -partitions.

We say that  $\mathcal{B}$  *refines*  $\mathcal{A}$  if:

- $\mathcal{B}$  refines  $\mathcal{A}$  as a clopen partition;
- Whenever  $\alpha, \beta$  belong to the same  $\mathcal{A}$ -orbit, we have  $n_O(\alpha) = n_O(\beta)$  for any  $\mathcal{B}$ -orbit  $O$ . By analogy with Kakutani–Rokhlin partitions, we say that there are  $n_O(\alpha)$  copies of the  $\mathcal{A}$ -orbit of  $\alpha$  contained in  $O$ .

**Lemma 3.14.** Any two  $\sim_\Gamma$ -partitions  $\mathcal{A}, \mathcal{B}$  admit a common refinement.

*Proof.* For  $n$  big enough,  $\mathcal{A}_n$  is compatible with any atom of  $\mathcal{A}$ , and for any  $\alpha, \beta$  belonging to the same  $\mathcal{A}$ -orbit, there exists  $\gamma \in \Gamma_n$  such that  $\gamma(\alpha) = \beta$ . Hence in any  $\mathcal{A}_n$ -orbit  $O$  we have  $n_O(\alpha) = n_O(\beta)$  for large  $n$ . So any  $\mathcal{A}_n$  refines  $\mathcal{A}$  as long as  $n$  is large enough, and it follows that  $\mathcal{A}, \mathcal{B}$  have a common refinement.  $\square$

This implies in particular that, for any  $\sim_\Gamma$ -partition  $\mathcal{A}$  and any clopen  $U$ , there exists a refinement of  $\mathcal{A}$  which is compatible with  $U$ .

We seize the opportunity to note the following fact, closely related to [GPS1, Lemma 6.1] and [GPS3, Theorem 4.8]. This result will not be needed in our proof of the classification theorems (though we use it at the end of the paper to prove Theorem 3.9).

**Théorème 3.4.** *There exists a minimal homeomorphism  $\varphi$  and  $x \in X$  such that  $\Gamma = \Gamma_x(\varphi)$ .*

*Proof.* We consider ordered  $\sim_\Gamma$ -partitions, i.e.  $\Gamma$ -partitions  $\mathcal{A}$  where each orbit is totally ordered. Given two ordered  $\Gamma$ -partitions  $\mathcal{A}, \mathcal{B}$ , we say that  $\mathcal{B}$  refines  $\mathcal{A}$  if:

- $\mathcal{B}$  refines  $\mathcal{A}$  as a  $\sim_\Gamma$ -partition.
- For any  $\mathcal{B}$ -orbit  $O$ , each fragment  $O_{\mathcal{A}}(\alpha)$  of  $\mathcal{A}$ -orbit contained in  $O$  is an interval for the ordering of  $O$ , and the ordering on each  $O_{\mathcal{A}}(\alpha)$  induced by  $\mathcal{B}$  coincides with the ordering induced by  $\mathcal{A}$ .

Given an ordered  $\sim_\Gamma$ -partition and a  $\mathcal{A}$ -orbit  $O$ , we call  $\text{base}(O)$  the minimal element of  $O$ , and  $\text{top}(O)$  its maximal element; the base of  $\mathcal{A}$  is the union of the bases of all  $\mathcal{A}$ -orbits, and we similarly define the top of  $\mathcal{A}$ .

If  $\mathcal{A}$  is an ordered  $\Gamma$ -partition, and  $\mathcal{B}$  is a  $\Gamma$ -partition which refines  $\mathcal{A}$ , then we may turn  $\mathcal{B}$  into an ordered  $\Gamma$ -partition refining  $\mathcal{A}$ .

We fix a compatible metric on  $X$ . We build a sequence  $\mathcal{B}_n$  of ordered  $\sim_\Gamma$ -partitions, and a sequence  $\Delta_n$  of finite subgroups of  $\Gamma$ , such that (when we forget the ordering on  $\mathcal{B}_n$ ) the sequence  $(\mathcal{B}_n, \Delta_n)$  is an exhaustive sequence of finite unit systems and the diameters of the top and base of  $\mathcal{B}_n$  both converge to 0. Assume for the moment that this is possible. Then there exists a unique homeomorphism  $\varphi$  of  $X$  such that, for every  $n$  and every atom  $\beta$  of  $\mathcal{B}_n$  which does not belong to  $\text{top}(\mathcal{B}_n)$ ,  $\beta$  is mapped by  $\varphi$  to its successor in  $O(\beta)$ ; and  $\varphi$  maps the top of  $\mathcal{B}_n$  to the base of  $\mathcal{B}_n$ . Let  $\{x\} = \bigcap_n \text{base}(\mathcal{B}_n)$ . Then  $\mathcal{B}_n$  is a sequence of Kakutani-Rokhlin partitions for the minimal homeomorphism  $\varphi$ , and  $\Gamma_x(\varphi)$  has the same orbits as  $\Gamma$  on  $CO(X)$ . We deduce from Krieger's theorem that  $\Gamma$  is conjugate to  $\Gamma_x(\varphi)$  in  $\text{Homeo}(X)$ , which gives the desired result.

It remains to explain how to build  $(\mathcal{B}_n, \Delta_n)$ . We let  $\mathcal{B}_0$  be the trivial partition and  $\Delta_0 = \{\text{id}\}$  to initialize the construction. Assume that  $\mathcal{B}_n$  has been constructed. By cutting an orbit if necessary, we may assume that  $\mathcal{B}_n$  has two orbits  $O, O'$  such that  $\alpha = \text{base}(O)$  and  $\beta = \text{top}(O')$  have diameter less than  $2^{-n}$ .

Next, we find  $N$  big enough that  $\Gamma_N$  contains both  $\Delta_n$  and  $\Gamma_n$ ;  $\mathcal{A}_N$  refines both  $\mathcal{A}_n$  and  $\mathcal{B}_n$  (as unordered  $\Gamma$ -partitions); and each  $\mathcal{A}_N$ -orbit contains a fragment of  $O$  as well as a fragment of  $O'$  (this last property holds for any large enough  $N$  by minimality of  $\Gamma$ ). Then we may turn  $\mathcal{A}_N$  into an ordered  $\Gamma$ -partition refining  $\mathcal{B}_n$  and such that each  $\mathcal{B}_n$ -orbit has its base contained in  $\alpha$  and its top contained in  $\beta$ . We let  $\mathcal{B}_{n+1} = \mathcal{A}_N$  (endowed with such an ordering),  $\Delta_{n+1} = \Gamma_N$ . This concludes the proof.  $\square$

We turn to a construction that is more specifically needed in our proof of the classification theorem for minimal ample groups.

*Definition 3.20.* Let  $U, V$  be two clopen sets, and  $\mathcal{A}$  be a  $\sim_\Gamma$ -partition compatible with  $U$  and  $V$ .

- We say that a  $\mathcal{A}$ -orbit  $O$  is  $(U, V)$ -balanced if  $n_O(U) = n_O(V)$ .
- We say that a pair of  $\mathcal{A}$  orbits  $(O(\alpha), O(\beta))$  is  $(U, V)$ -balanced if

$$n_{O(\alpha)}(U) - n_{O(\alpha)}(V) = n_{O(\beta)}(V) - n_{O(\beta)}(U)$$

and  $\mu(\alpha) = \mu(\beta)$  for all  $\mu \in M(\Gamma)$ .

- We say that  $U, V$  are  $\mathcal{A}$ -equivalent if  $n_O(U) = n_O(V)$  for any  $\mathcal{A}$ -orbit  $O$ .

Note that two clopens  $U, V$  are  $\mathcal{A}$ -equivalent for some  $\sim_\Gamma$ -partition  $\mathcal{A}$  if and only if  $U \sim_\Gamma V$ . If  $U \sim_\Gamma^* V$  but  $U \not\sim_\Gamma V$ , there cannot exist a  $\sim_\Gamma$  partition  $\mathcal{A}$  such that  $U, V$  are  $\mathcal{A}$ -equivalent; it is however convenient to manipulate  $\sim_\Gamma$ -partitions which are as close as possible to making  $U, V$   $\mathcal{A}$ -equivalent. This is what leads us to consider balanced pairs of orbits, and motivates the next definition and lemma.

*Definition 3.21.* Let  $\bar{U} = (U_1, \dots, U_n), \bar{V} = (V_1, \dots, V_n)$  be tuples of clopen sets. Let  $\mathcal{A}$  be a  $\sim_\Gamma$ -partition. We say that  $\bar{U}$  and  $\bar{V}$  are *almost  $\mathcal{A}$ -equivalent* if  $\mathcal{A}$  is compatible with each  $U_i, V_i$  and there exists  $k \leq n$  and  $\mathcal{A}$ -orbits  $C_1, \dots, C_k, D_1, \dots, D_k$ , such that:

- Any  $\mathcal{A}$ -orbit which does not belong to  $\{C_1, \dots, C_k, D_1, \dots, D_k\}$  is  $(U_j, V_j)$ -balanced for all  $j$ .
- For any  $i \in \{1, \dots, k\}$  and any  $j \in \{1, \dots, n\}$ ,  $(C_i, D_i)$  is a  $(U_j, V_j)$ -balanced pair of orbits.

We call  $C_1, \dots, C_k, D_1, \dots, D_k$  the *exceptional  $\mathcal{A}$ -orbits*.

In the above definition, we allow  $k = 0$ , in which case  $\bar{U}, \bar{V}$  are  $\mathcal{A}$ -equivalent.

**Proposition 3.15.** *Let  $\mathcal{A}$  be a  $\sim_\Gamma$ -partition, and  $\bar{U} = (U_1, \dots, U_n), \bar{V} = (V_1, \dots, V_n)$  be two tuples of clopen sets such that  $U_i \sim_\Gamma^* V_i$  for all  $i \in \{1, \dots, n\}$ .*

*Then there exists a  $\sim_\Gamma$ -partition  $\mathcal{B}$  which refines  $\mathcal{A}$  and is such that  $(\bar{U}, \bar{V})$  are almost  $\mathcal{B}$ -equivalent.*

*Additionally, one can ensure that the following condition holds: denote*

$$N_i(\mathcal{B}) = \max\{|n_O(U_i) - n_O(V_i)| : O \text{ is a } \mathcal{B}\text{-orbit}\}$$

*Let  $N(\mathcal{B}) = \sum_{i=1}^n N_i(\mathcal{B})$  and denote by  $h$  be the number of atoms of  $\mathcal{A}$ . Then any exceptional  $\mathcal{B}$ -orbit contains more than  $3hN(\mathcal{B})$  copies of every  $\mathcal{A}$ -orbit.*

The proof of the proposition is based on an argument initially used in the proof of [IM2, Proposition 3.5]; it is the key combinatorial step of our argument. The proof is elementary and based on repeated applications of Lemma 3.7, but somewhat tedious. The last part of the statement plays a technical part in the proof of the classification theorem for minimal ample groups (in the language used in the proof of that theorem, it is used to ensure that the  $\tilde{\Gamma}$ -orbits of singular points are distinct) and can safely be ignored on first reading.

We prove the first part of the statement (before “additionally...”) by induction on  $n$ . We treat the case  $n = 1$  separately.

*The case  $n = 1$ .* Fix  $\mathcal{A}$  and  $U, V$  such that  $U \sim_{\Gamma}^* V$ .

There exists a  $\sim_{\Gamma}$  partition  $\mathcal{B}$  which refines  $\mathcal{A}$  and is compatible with  $U, V$ ; for such partitions we simply denote

$$N(\mathcal{B}) = \max\{|n_O(U) - n_O(V)| : O \text{ is a } \mathcal{B}\text{-orbit}\}$$

Consider a partition  $\mathcal{B}$  refining  $\mathcal{A}$  and for which  $N(\mathcal{B})$  is minimal (denote it  $N$  from now on); we assume that  $N \geq 1$ , otherwise  $(U, V)$  are  $\mathcal{B}$ -equivalent and we have nothing to do.

Let  $O(\alpha_1), \dots, O(\alpha_p)$  enumerate the  $\mathcal{B}$ -orbits for which  $n_O(U) - n_O(V) = N$  (if any such orbit exists). Let also  $O(\beta_1), \dots, O(\beta_q)$  enumerate the orbits such that  $n_O(V) > n_O(U)$ .

We have, for all  $\mu \in M(\Gamma)$ ,

$$\begin{aligned} \mu(U) - \mu(V) &\geq N\mu\left(\bigsqcup_{i=1}^p \alpha_i\right) - \sum_{j=1}^q \left(n_{O(\beta_j)}(V) - n_{O(\beta_j)}(U)\right) \mu(\beta_j) \\ &\geq N\mu\left(\bigsqcup_{i=1}^p \alpha_i\right) - N\mu\left(\bigsqcup_{j=1}^q \beta_j\right) \end{aligned}$$

Since  $\mu(U) = \mu(V)$  for all  $\mu \in M(\Gamma)$ , we must have that

$$\forall \mu \in M(\Gamma) \quad \mu\left(\bigsqcup_{i=1}^p \alpha_i\right) \leq \mu\left(\bigsqcup_{j=1}^q \beta_j\right)$$

and there can be equality only if  $|n_O(V) - n_O(U)| = N$  for any orbit such that  $n_O(V) \neq n_O(U)$ . If we are not in this situation, we may apply Lemma 3.7 to find an element  $\gamma$  of  $\Gamma$  mapping  $\bigsqcup_{i=1}^p \alpha_i$  into  $\bigsqcup_{j=1}^q \beta_j$ ; from this, we can produce a  $\sim_{\Gamma}$ -partition  $\mathcal{C}$  refining  $\mathcal{B}$ , and such that there does not exist any  $\mathcal{C}$ -orbit for which  $n_O(U) - n_O(V) = N$ . Let us detail a bit here how  $\mathcal{C}$  is produced; this explanation is intended for readers who are not familiar with the cutting-and-stacking procedure and every partition produced in the remainder of this proof will be obtained similarly. Take the  $\sim_{\Gamma}$ -partition  $\mathcal{B}'$  obtained by cutting  $\mathcal{B}$  (as in Lemma 3.4) to refine the  $\gamma(\alpha_i)$ 's and the  $\gamma^{-1}(\beta_j)$ 's. Call, for all  $i \leq p, j \leq q$ ,

$$\alpha_{i,j} = \alpha_i \cap \gamma^{-1}(\beta_j) \text{ and } \beta_{i,j} = \beta_j \cap \gamma(\alpha_i)$$

Clearly, for all  $i \leq p, j \leq q$ , one has that  $\beta_{i,j} = \gamma(\alpha_{i,j})$ . We now form a new  $\sim_\Gamma$ -partition by joining the  $\mathcal{B}'$ -orbit of  $\alpha_{i,j}$  and  $\beta_{i,j}$ , for each  $i, j$ , and leaving the other orbits unchanged (this is the analogue of what we call stacking for Kakutani–Rokhlin partitions). Denote by  $\mathcal{C}$  this new  $\sim_\Gamma$ -partition.

Using the same argument with the roles of  $U$  and  $V$  reversed, we see that we can build a  $\sim_\Gamma$ -partition  $\mathcal{D}$  refining  $\mathcal{C}$  and such that  $N(\mathcal{D}) < N$ , unless any orbit  $O$  of  $\mathcal{C}$  which is not  $(U, V)$ -balanced is such that  $|n_O(U) - n_O(V)| = N$ . By definition of  $N$ , we must thus be in that particular case.

Let again  $O(\alpha_1), \dots, O(\alpha_p)$  enumerate all  $\mathcal{C}$ -orbits  $O$  with  $n_O(U) > n_O(V)$ , and  $O(\beta_1), \dots, O(\beta_q)$  enumerate those with  $n_O(V) > n_O(U)$ . Assume that  $p \geq 2$ . There exists  $\gamma \in \Gamma$  mapping  $\bigsqcup_{i=2}^p \alpha_i$  into  $\bigsqcup_{j=1}^q \beta_j$ . By forming appropriate clopen partitions of  $\bigsqcup_{i=2}^p \alpha_i$ ,  $\bigsqcup_{j=1}^q \beta_j$  and matching them together (similarly to how we defined  $\mathcal{C}$  above), we produce a  $\sim_\Gamma$ -partition with only one orbit  $O$  for which  $n_O(U) > n_O(V)$ . Applying the same argument to this new partition, with the roles of  $U$  and  $V$  reversed, we obtain a  $\sim_\Gamma$ -partition with exactly one orbit  $O$  for which  $n_O(U) - n_O(V) > 0$ , and one orbit  $P$  for which  $n_P(V) - n_P(U) > 0$ . If  $O = O(\alpha)$ ,  $P = O(\beta)$ , we must have  $\mu(\alpha) = \mu(\beta)$  for all  $\mu \in M(\Gamma)$ , since

$$\forall \mu \in M(\Gamma) \quad 0 = \mu(U) - \mu(V) = N\mu(\alpha) - N\mu(\beta)$$

This proves the first part of the Proposition's statement in case  $n = 1$ .  $\square$

Now we assume that the first part of the statement of Proposition 3.15 has been established for some  $n$ , and need to explain why it is true for  $n + 1$ .

We fix  $\mathcal{A}$ ,  $(U_1, \dots, U_{n+1})$  and  $(V_1, \dots, V_{n+1})$  such that  $U_i \sim_\Gamma^* V_i$  for all  $i \in \{1, \dots, n + 1\}$ . To simplify notation below we let  $U = U_{n+1}$  and  $V = V_{n+1}$ . We consider partitions  $\mathcal{B}$  refining  $\mathcal{A}$  and such that  $(U_1, \dots, U_n), (V_1, \dots, V_n)$  are almost  $\mathcal{B}$ -equivalent (these exist by our induction hypothesis), as witnessed by exceptional orbits  $O(\alpha_1^{\mathcal{B}}), \dots, O(\alpha_k^{\mathcal{B}}), O(\beta_1^{\mathcal{B}}), \dots, O(\beta_k^{\mathcal{B}})$  for some  $k \leq n$ .

For such a  $\mathcal{B}$ , We denote

$$Y^{\mathcal{B}} = X \setminus \bigsqcup_{i=1}^k (O(\alpha_i^{\mathcal{B}}) \sqcup O(\beta_i^{\mathcal{B}}))$$

**Lemma 3.16.** *Assume that there exists  $\mathcal{B}$  refining  $\mathcal{A}$ , such that  $(U_1, \dots, U_n)$  and  $(V_1, \dots, V_n)$  are almost  $\mathcal{B}$ -equivalent and each pair  $(O(\alpha_i^{\mathcal{B}}), O(\beta_i^{\mathcal{B}}))$  is  $(U, V)$ -balanced. Then there exists  $\mathcal{C}$  refining  $\mathcal{A}$  and such that  $(U_1, \dots, U_{n+1})$  and  $(V_1, \dots, V_{n+1})$  are almost  $\mathcal{C}$ -equivalent.*

*Proof.* We may assume that  $Y^{\mathcal{B}}$  is nonempty: if it is empty, take  $\tau \subseteq \alpha_1^{\mathcal{B}}$ , then find  $\pi \subseteq \beta_1^{\mathcal{B}}$  such that  $\tau \sim_\Gamma \pi$  (Lemma 3.7 shows that this is possible). Then form a partition  $\mathcal{C}$  refining  $\mathcal{B}$  by stacking together the  $\mathcal{B}$ -orbits of  $\tau$  and  $\pi$ , and having  $O(\alpha_1^{\mathcal{B}} \setminus \tau), O(\beta_1^{\mathcal{B}} \setminus \pi)$  form an exceptional column pair. This partition is still such that  $(U_1, \dots, V_n)$  and  $(V_1, \dots, V_n)$  are  $\mathcal{C}$ -equivalent, and  $Y^{\mathcal{C}}$  is nonempty since it contains  $\tau$ .

Assuming that  $Y^{\mathcal{B}}$  is nonempty, note that  $U \cap Y^{\mathcal{B}} \sim_{\Gamma}^* V \cap Y^{\mathcal{B}}$ ; we simply repeat the argument of the proof of the case  $n = 1$  of Proposition 3.15, working inside  $Y^{\mathcal{B}}$ . The point is that for each  $i$  the construction joins together fragments of orbits which are  $(U_i, V_i)$ -balanced for all  $i \leq n$ , and these new orbits are still  $(U_i, V_i)$ -balanced for each  $i \leq n$ .

This produces the desired  $C$ ; every  $C$ -orbit outside  $Y^{\mathcal{B}}$  coincides with a  $\mathcal{B}$ -orbit. This adds at most one exceptional pair of orbits to the exceptional pairs of orbits of  $\mathcal{B}$  (which remain exceptional for  $C$ ).  $\square$

*Inductive step of the proof of Proposition 3.15.* For a  $\sim_{\Gamma}$ -partition  $\mathcal{B}$  refining  $\mathcal{A}$  and for which  $(U_1, \dots, U_n), (V_1, \dots, V_n)$  are almost  $\mathcal{B}$ -equivalent, we denote

$$M(\mathcal{B}) = \max_{1 \leq i \leq k} |n_{O(\alpha_i^{\mathcal{B}})}(U) + n_{O(\beta_i^{\mathcal{B}})}(U) - n_{O(\alpha_i^{\mathcal{B}})}(V) - n_{O(\beta_i^{\mathcal{B}})}(V)|$$

Take  $\mathcal{B}$  so that  $M(\mathcal{B})$  is minimal (denote it  $M$  from now on). Our goal is to prove that  $M = 0$ , for then we obtain the desired result by applying Lemma 3.16.

A key point to note here is that, if we join together a fragment of some  $O(\alpha_i^{\mathcal{B}})$  with a fragment of  $O(\beta_i^{\mathcal{B}})$  (using Lemma 3.7 as we did earlier) to form a finer partition  $C$ , this finer partition satisfies the same conditions as  $\mathcal{B}$ , and the new orbit obtained after this joining is contained in  $Y^C$ .

Now, assume for a contradiction that  $M \geq 1$ . Up to reordering (and exchanging the roles of  $U$  and  $V$ ), we suppose that

$$n_{O(\alpha_1^{\mathcal{B}})}(U) + n_{O(\beta_1^{\mathcal{B}})}(U) - n_{O(\alpha_1^{\mathcal{B}})}(V) - n_{O(\beta_1^{\mathcal{B}})}(V) = M$$

If there is no orbit  $O$  in  $Y_{\mathcal{B}}$  such that  $n_O(V) - n_O(U) = K > 0$ , there must exist  $1 \leq j \leq k$  such that

$$n_{O(\alpha_j^{\mathcal{B}})}(V) + n_{O(\beta_j^{\mathcal{B}})}(V) - n_{O(\alpha_j^{\mathcal{B}})}(U) - n_{O(\beta_j^{\mathcal{B}})}(U) > 0$$

As we already mentioned above, if we join together (using  $\Gamma$ ) a fragment of  $O(\alpha_j^{\mathcal{B}})$  with a fragment of  $O(\beta_j^{\mathcal{B}})$ , we create a partition  $C$  refining  $\mathcal{B}$ , satisfying all the relevant conditions, and such that an orbit  $O$  with  $n_O(V) - n_O(U) = K$  exists in  $Y^C$ ; so we may as well assume that such an orbit exists in  $Y^{\mathcal{B}}$ .

Let  $K = qM + r$  be the euclidean division of  $K$  by  $M$ . If  $q > 0$ , we can join together a small fragment of  $O(\alpha_1^{\mathcal{B}})$ , one of  $O(\beta_1^{\mathcal{B}})$  and one of  $O$  to create a finer partition  $\mathcal{B}'$  along with an orbit  $O'$  contained in  $Y_{\mathcal{B}'}$  and such that  $n_{O'}(V) - n_{O'}(U) = K - M$ . Repeating this  $q - 1$  times, we obtain a finer partition  $\mathcal{B}''$  along with an orbit  $O(\delta)$  inside  $Y^{\mathcal{B}''}$  such that

$$0 < n_{O(\delta)}(V) - n_{O(\delta)}(U) = M + r < 2M$$

If  $q = 0$  then choose  $\delta$  such that  $O = O(\delta)$ ; the above inequality is also satisfied. Again to simplify notation, we assume that such an orbit already exists in  $\mathcal{B}$  (since we just reduced to that case).

Choose  $\alpha'_1$  inside  $\alpha_1^{\mathcal{B}}$  such that  $0 < \mu(\alpha'_1) < \mu(\delta)$  for all  $\mu \in M(\Gamma)$ .

Then find  $\gamma \in \Gamma$  such that  $\gamma(\alpha'_1) \subset \beta_1^{\mathcal{B}}$ , and set  $\beta'_1 = \gamma(\alpha'_1)$ . Cut the orbit of  $\alpha_1^{\mathcal{B}}$  to form two orbits  $O(\alpha'_1)$  and  $O(\alpha_1^{\mathcal{B}} \setminus \alpha'_1)$ ; do the same to the orbit of  $\beta_1^{\mathcal{B}}$ . Then join together the orbit of  $\alpha_1^{\mathcal{B}} \setminus \alpha'_1$  and the orbit of  $\beta_1^{\mathcal{B}} \setminus \beta'_1$  (thus producing an orbit which is  $(U_i, V_i)$ -balanced for all  $i \in \{1, \dots, n\}$ ); and the orbit of  $\alpha'_1$  with a fragment of  $O(\delta)$ , which is possible because  $\mu(\alpha'_1) < \mu(\delta)$  for all  $\mu \in M(\Gamma)$ .

We obtain a  $\sim_\Gamma$ -partition  $\mathcal{C}$  which refines  $\mathcal{B}$ , and such that  $(U_1, \dots, U_n)$  and  $(V_1, \dots, V_n)$  are almost  $\mathcal{C}$ -equivalent with exceptional  $\mathcal{C}$ -orbits of the form

$$O(\alpha'_1), O(\alpha_2^{\mathcal{B}}), \dots, O(\alpha_k^{\mathcal{B}}), O(\beta'_1), \dots, O(\beta_k^{\mathcal{B}})$$

Further,

$$|n_{O(\alpha'_1)}(U) + n_{O(\beta'_1)}(U) - n_{O(\alpha'_1)}(V) - n_{O(\beta'_1)}(V)| < M$$

Applying this argument repeatedly, we conclude that there exists a  $\sim_\Gamma$ -partition  $\mathcal{D}$  refining  $\mathcal{B}$  and such that  $M(\mathcal{D}) < M(\mathcal{B})$ . This is a contradiction, so we conclude that  $M = 0$  as promised.  $\square$

*End of the proof of Proposition 3.15.* Let us now see why the “additionally” part holds true. We first pick a  $\sim_\Gamma$ -partition  $\mathcal{B}$  satisfying the conditions in the first part of the lemma, and denote as before  $O(\alpha_1^{\mathcal{B}}), \dots, O(\alpha_k^{\mathcal{B}}), O(\beta_1^{\mathcal{B}}), \dots, O(\beta_k^{\mathcal{B}})$  the exceptional  $\mathcal{B}$ -orbits and

$$Y^{\mathcal{B}} = X \setminus \bigsqcup_{i=1}^k (O(\alpha_i^{\mathcal{B}}) \cup O(\beta_i^{\mathcal{B}}))$$

By joining together fragments of some  $O(\alpha_i^{\mathcal{B}})$  and  $O(\beta_i^{\mathcal{B}})$  if necessary, we can guarantee that for every  $\mathcal{A}$ -orbit  $O$  there is a fragment of  $O$  contained in  $Y^{\mathcal{B}}$ . Then, by joining together some sufficiently small fragments of orbits contained in  $Y^{\mathcal{B}}$ , we ensure that in  $Y^{\mathcal{B}}$  there exists an orbit  $O(\delta)$  which contains more than  $3hN(\mathcal{B})$  copies of each  $\mathcal{A}$ -orbit.

Again by joining together if necessary some fragments of  $O(\alpha_i^{\mathcal{B}})$  and  $O(\beta_i^{\mathcal{B}})$ , we can assume that  $2n\mu(\alpha_i^{\mathcal{B}}) < \mu(\delta)$  for all  $\mu \in M(\Gamma)$  and all  $i$ . Then we may join for each  $i$  a fragment of  $O(\delta)$  and a fragment of  $O(\alpha_i^{\mathcal{B}})$ , as well as a fragment of  $O(\delta)$  and a fragment of  $O(\beta_i^{\mathcal{B}})$ . Each orbit in these new exceptional orbit pairs contains many copies of each  $\mathcal{A}$ -orbit.  $\square$

The fact that  $k \leq n$  in the lemma above does not play a part in our arguments; but it is important to have some control over the exceptional orbits.

In the remainder of the paper, we often identify a clopen partition  $\mathcal{A}$  with the Boolean algebra it generates (and start doing this in the lemma below).

**Lemma 3.17.** *Let  $\simeq$  be a full equivalence relation on  $CO(X)$ , and  $(\mathcal{B}_n)$  be a sequence of  $\simeq$ -partitions such that*

- For all  $n$ ,  $\mathcal{B}_{n+1}$  refines  $\mathcal{B}_n$ .
- For each clopen  $U$ , there exists  $n$  such that  $\mathcal{B}_n$  is compatible with  $U$ .



Then there exists an ample group  $\Lambda = \bigcup_n \Lambda_n$  such that

- For all  $n$ ,  $(\mathcal{B}_n, \Lambda_n)$  is a unit system.
- For all  $n$ , all  $\alpha \in \mathcal{B}_n$ , the  $\mathcal{B}_n$ -orbit of  $\alpha$  coincides with  $\Lambda_n \alpha$ .
- For any clopen  $A$  and any  $\lambda \in \Lambda$  one has  $A \simeq \lambda(A)$ .

*Proof.* Denote by  $O_n$  the set of  $\mathcal{B}_n$ -orbits, enumerate it as  $(O_1^n, \dots, O_{k_n}^n)$ , and for  $i \in \{1, \dots, k_n\}$  denote by  $h_i^n$  the cardinality of  $O_i^n$ . Each group  $\Lambda_n$  will be isomorphic, as an abstract group, to  $\prod_{i=1}^{k_n} \mathfrak{S}_{O_i^n}$ .

Let  $\mathcal{B}_{-1}$  be the trivial partition, and  $\Lambda_{-1} = \{\text{id}\}$  to initialize the process; then assume that  $\Lambda_n$  has been constructed. For every  $i$ , denote by  $(A_{i,j})_{j < h_i^n}$  an enumeration of  $O_i^n$ , and by  $\tau_j^i$  the transposition  $(A_{i,j}, A_{i,j+1})$  of  $\mathfrak{S}_{O_i^n}$  for  $j < h_i^n - 1$ . We first use the fact that  $\mathcal{B}_{n+1}$  refines  $\mathcal{B}_n$  to extend each  $\tau_j^i$  to a permutation of the atoms of each  $O_k^{n+1}$ ; to see how this is done, fix some  $\tau = \tau_j^i$ , and let  $\alpha, \beta \in O_i^n$  be the two only atoms that are not fixed by  $\tau$ . For any  $\mathcal{B}_{n+1}$ -orbit  $O_k^{n+1}$ , there are as many atoms  $\alpha_1^k, \dots, \alpha_l^k$  contained in  $\alpha$  as there are atoms  $\beta_1^k, \dots, \beta_l^k$  contained in  $\beta$ , because  $\mathcal{B}_{n+1}$  refines  $\mathcal{B}_n$ . So we can extend  $\tau$  to a permutation of the atoms of  $\mathcal{B}_{n+1}$ , by setting  $\tau(\alpha_m^k) = \beta_m^k$ ,  $\tau(\beta_m^k) = \alpha_m^k$  for every  $k, m$  (we note here that our extension depends on the choice of enumerations of the atoms  $\alpha_l^k, \beta_l^k$ ; this freedom will be used during the proof of the classification theorem for minimal ample groups).

Once this is done for every  $\tau_j^i$ , we see  $\Lambda_n$  as a subgroup of the permutation group  $\Lambda_{n+1} = \prod_{i=1}^{k_{n+1}} \mathfrak{S}_{O_i^{n+1}}$  and can move on to the next step.

Since any clopen set eventually belongs to some  $\mathcal{B}_n$ , we can view each  $\lambda \in \Lambda$  as an automorphism of  $CO(X)$ , that is, a homeomorphism of  $X$ .

By construction,  $\Lambda$  is then an ample subgroup of  $\text{Homeo}(X)$ , and the  $\mathcal{B}_n$ -orbit of any  $\alpha \in \mathcal{B}_n$  coincides with  $\Lambda_n \alpha$ .

To check the last point, fix a clopen  $A$  and  $\lambda \in \Lambda$ . There exists  $n$  such that  $A$  is a union of atoms  $\alpha_1, \dots, \alpha_n$  of  $\mathcal{B}_n$ , and  $\lambda$  belongs to  $\Lambda_n$ . Then

$$\lambda(A) = \bigsqcup \lambda(\alpha_n)$$

and for all  $n$ , we have that  $\lambda(\alpha_n) \simeq \alpha_n$  since  $\mathcal{B}_n$  is a  $\simeq$ -partition. Since  $\simeq$  is full, this establishes the final condition of the lemma. □

## 3.5 The classification theorem for minimal ample groups

### 3.5.1 An absorption theorem

We now prove an “absorption theorem,” which is a particular case of [GPS3, Theorem 4.6]; see also the further generalizations in [GMPS1, Lemma 4.15] and [M2, Theorem 3.2].

**Théorème 3.5.** *Let  $\Gamma$  be an ample subgroup acting minimally. Let  $K = K_1 \sqcup \sigma(K_1)$  be a closed subset of  $X$  without isolated points, where  $K_1$  is closed and  $\sigma \in \text{Homeo}(K)$  is an involution. Assume that  $K$  is  $\Gamma$ -sparse.*

*Denote by  $R_{\Gamma,K}$  the finest equivalence relation coarser than  $R_\Gamma$  and for which  $k, \sigma(k)$  are equivalent for all  $k \in K$ . Then there exists an ample group  $\Sigma$  whose action induces  $R_{\Gamma,K}$ ; furthermore  $\Sigma$  and  $\Gamma$  are orbit equivalent.*

At the end of the paper, we explain how one can remove the assumption that  $K$  has no isolated points from the above statement.

*Proof.* The strategy of proof is as follows: we show first that there exists a sparse set  $S_1 \sqcup \pi(S_1)$  without isolated points (with  $\pi$  a homeomorphic involution), such that  $R_{\Gamma,S}$  is induced by an ample group which is orbit equivalent to  $\Gamma$ . Then we use Theorem 3.2 to conclude that this result holds for  $R_{\Gamma,K}$ .

First, we fix a refining sequence  $(\mathcal{A}_n, \Gamma_n)$  of finite unit systems such that  $\Gamma = \bigcup_n \Gamma_n$ ,  $CO(X) = \bigcup_n \mathcal{A}_n$ , with the additional property that for all  $n$  there exists two disjoint  $\mathcal{A}_n$ -orbits  $O_n(\alpha_n), O_n(\beta_n)$  (which we call the *exceptional orbits*) such that:

- For all  $n$ ,  $\alpha_n$  and  $\beta_n$  are  $\mathcal{A}_{n+1}$ -equivalent.
- For all  $n$ , denote by  $h_n$  the cardinality of  $\mathcal{A}_n$ . Then  $O_{n+1}(\alpha_{n+1})$  contains at least  $2h_n$  copies of  $O_n(\alpha_n)$ , and  $O_{n+1}(\beta_{n+1})$  contains at least  $2h_n$  copies of  $O_n(\beta_n)$ .

It is straightforward to build such a sequence using the same techniques (cutting and stacking, and Lemma 3.7) as in the previous section, so we do not give details here. We denote by  $\tilde{\mathcal{A}}_n$  the refinement of  $\mathcal{A}_n$  obtained by joining together  $O(\alpha_n)$  and  $O(\beta_n)$  (the corresponding orbit is called the *exceptional orbit* of  $\tilde{\mathcal{A}}_n$ ). Since each orbit of  $\mathcal{A}_{n+1}$  is  $(\alpha_n, \beta_n)$ -balanced,  $\tilde{\mathcal{A}}_{n+1}$  refines  $\tilde{\mathcal{A}}_n$ .

We define inductively a homeomorphic involution  $\pi$  with the following properties:

- For all  $n$ ,  $\pi$  induces an automorphism of  $\tilde{\mathcal{A}}_n$ , which maps each non-exceptional orbit to itself.
- Denoting by  $\Lambda_n$  the subgroup of  $\text{Aut}(\tilde{\mathcal{A}}_n)$  generated by  $\Gamma_n$  and  $\pi$ ,  $(\tilde{\mathcal{A}}_n, \Lambda_n)$  is a unit system and the  $\Lambda_n$ -orbit of any atom of  $\tilde{\mathcal{A}}_n$  coincides with its  $\tilde{\mathcal{A}}_n$ -orbit.
- Say that an atom  $\alpha$  of  $\mathcal{A}_n$  is *singular* if  $\pi$  does not coincide with an element of  $\Gamma_n$  on  $\alpha$  ( $\alpha$  must belong to one of the two exceptional orbits, and be mapped by  $\pi$  to the other exceptional  $\mathcal{A}_n$ -orbit). Then for every atom  $\alpha$  of  $\mathcal{A}_{n+1}$  there exists at most one singular atom in  $\Gamma_n \alpha$ ; and for every singular atom  $\alpha$  of  $\mathcal{A}_n$  there exist at least two singular atoms of  $\mathcal{A}_{n+1}$  contained in  $\alpha$ .

Initialize the construction by setting  $\pi(\alpha_0) = \beta_0$ ,  $\sigma(\beta_0) = \alpha_0$ , and  $\pi$  is the identity on the other atoms of  $\mathcal{A}_0$ .

Next, assume that we have extended  $\pi$  to an automorphism of  $\tilde{\mathcal{A}}_n$  satisfying the various conditions above. In particular, on all nonsingular atoms of  $\mathcal{A}_n$ , we

have already declared  $\pi$  to be equal to some element of  $\Gamma_n$ , hence the extension of  $\pi$  to atoms of  $\tilde{\mathcal{A}}_{n+1}$  contained in a nonsingular atom is already defined. Since each  $\mathcal{A}_n$ -orbit is  $(\alpha, \pi(\alpha))$ -balanced for every singular atom of  $\mathcal{A}_n$ , we can also extend  $\pi$  to all nonexceptional  $\mathcal{A}_{n+1}$ -orbits so that it coincides with an element of  $\Gamma_{n+1}$  on these orbits.

It remains to explain how to extend  $\pi$  to the two exceptional orbits  $O_{n+1}(\alpha_{n+1})$ ,  $O_{n+1}(\beta_{n+1})$ . Let  $\tau_1, \dots, \tau_p$  be singular atoms of  $\mathcal{A}_n$  which belong to distinct  $\pi$ -orbits and such that for any singular atom  $\alpha$  of  $\mathcal{A}_n$  there exists  $i$  such that  $\alpha = \tau_i$  or  $\alpha = \pi(\tau_i)$ . For all  $i$ , we set aside two distinct atoms  $\theta_1^i, \theta_2^i$  of  $O_{n+1}(\alpha_{n+1})$  contained in  $\tau_i$ , and two distinct atoms  $\delta_1^i, \delta_2^i$  of  $O_{n+1}(\beta_{n+1})$  contained in  $\pi(\tau_i)$ . We do this while ensuring that no two of these atoms belong to the same  $\Gamma_n$ -orbit (which is possible since there are many copies of  $O_n(\alpha_n)$  in  $O_{n+1}(\alpha_{n+1})$ , and many copies of  $O_n(\beta_n)$  in  $O_{n+1}(\beta_{n+1})$ ). Next we set  $\pi(\theta_j^i) = \delta_j^i, \pi(\delta_j^i) = \theta_j^i$ .

For any  $i \in \{1, \dots, p\}$ , there remain as many atoms of  $O_{n+1}(\alpha_{n+1})$  contained in  $\tau_i$  on which  $\pi$  is yet to be defined as there are such atoms in  $\pi(\tau_i)$ ; so we can extend  $\pi$  so that it coincides with an element of  $\Gamma_{n+1}$  on those atoms. We do the same in  $O_{n+1}(\beta_{n+1})$  to finish extending  $\pi$  to an involutive automorphism of  $\tilde{\mathcal{A}}_{n+1}$ .

Say that  $x \in X$  is *singular* if the atom  $\alpha_n(x)$  of  $\mathcal{A}_n$  which contains  $x$  is singular for all  $n$ . Since any singular atom of  $\mathcal{A}_n$  contains at least two singular atoms of  $\mathcal{A}_{n+1}$ , we see that the set  $S$  of singular points does not have any isolated point. Also, for every singular point  $x$ ,  $\alpha_{n+1}(x)$  is the unique singular atom contained in  $\Gamma_n \alpha_{n+1}(x)$ , which means that  $x$  is the unique singular point in  $\Gamma_n x$ . Thus  $S$  is  $\Gamma$ -sparse (clearly  $S$  is closed since it is an intersection of clopen sets).

Let  $\Lambda$  be the full group generated by  $\Gamma$  and  $\pi$ . It is an ample group since  $(\tilde{\mathcal{A}}_n, \Lambda_n)$  is a sequence of finite unit systems and  $\Lambda = \bigcup_n \Lambda_n$ .  $R_\Lambda$  is obtained from  $R_\Gamma$  by gluing together the  $\Gamma$ -orbits of  $x$  and  $\pi(x)$  for all  $x \in S$ ; in other words, it is the finest equivalence relation coarser than  $R_\Gamma$  and such that  $x, \pi(x)$  are equivalent for all  $x \in S$ .

By construction, the actions of  $\Lambda$  and  $\Gamma$  have the same orbits when they act on  $CO(X)$ . Hence  $\Lambda$  and  $\Gamma$  are orbit equivalent by Krieger's theorem.

We can find a closed subset  $S_1$  of  $S$  such that  $S = S_1 \sqcup \pi(S_1)$ . Let  $h: S_1 \rightarrow K_1$  be any homeomorphism (both  $K_1$  and  $S_1$  are Cantor sets), and extend  $h$  to a homeomorphism from  $S$  to  $K$  by setting  $h(\pi(x)) = \sigma(h(x))$  for all  $x \in S_1$ .

By Theorem 3.2, there exists  $g \in \text{Homeo}(X)$  such that  $g\Gamma g^{-1} = \Gamma$ , and  $g|_S = h$ . Then  $\Sigma = g\Lambda g^{-1}$  is an ample group which induces  $R_{\Gamma, K}$ . This ample group is conjugate to  $\Lambda$ , hence orbit equivalent to  $\Gamma$ .  $\square$

### 3.5.2 Proof of the classification theorem for minimal ample groups

We are now ready to prove the key result leading to the classification of minimal  $\mathbb{Z}$ -actions on the Cantor space.

**Théorème 3.6** (The classification theorem for minimal ample groups). *Let  $X$  be the Cantor space. Given two ample subgroups of  $\text{Homeo}(X)$  acting minimally, the following*

conditions are equivalent.

- The actions of  $\Gamma$  and  $\Lambda$  are orbit equivalent.
- There exists a homeomorphism  $g$  of  $X$  such that  $g_*M(\Gamma) = M(\Lambda)$ .

We fix an ample group  $\Gamma$  acting minimally on  $X$ , and reuse the same notations as in the previous section.

We saw earlier (cf. Theorem 3.3) that Krieger's theorem gives a proof of the classification theorem for saturated minimal ample groups. Thus our work consists in proving that  $\Gamma$  is orbit equivalent to a saturated, ample subgroup  $\Lambda$  of  $\text{Homeo}(X)$ .

The idea of the proof is as follows: using ideas similar to those of the previous section (and Proposition 3.15) we build a Cantor set  $K$ , and a homeomorphic involution  $\pi$  such that  $K \cap \pi(K) = \emptyset$ ,  $K \cup \pi(K)$  is  $\Gamma$ -sparse, and the equivalence relation induced by gluing together the  $\Gamma$ -orbits of  $x$  and  $\pi(x)$  for every  $x \in K$  is induced by an action of a saturated minimal ample group  $\Lambda$ . The absorption theorem 3.5 yields that  $\Gamma$  is orbit equivalent to  $\Lambda$ , from which we obtain as desired that  $\Gamma$  is orbit equivalent to a saturated minimal ample group.

We now begin the proof. Let  $(U_n, V_n)$  be an enumeration of all pairs of  $\sim_{\Gamma}^*$ -equivalent clopens, and assume for notational simplicity that  $U_0, V_0$  are disjoint and  $U_0 \not\sim_{\Gamma} V_0$ .

**Lemma 3.18.** *We may build a sequence of  $\sim_{\Gamma}$ -partitions  $(\mathcal{A}_n)$ , with distinguished orbit pairs  $O(\alpha_1^n), \dots, O(\alpha_{k_n}^n), O(\beta_1^n), \dots, O(\beta_{k_n}^n)$  ( $k_n \geq 1$  for all  $n$ ) satisfying the following conditions.*

1.  $k_0 = 1$ ,  $\alpha_1^0 = U_0$ ,  $\beta_1^0 = V_0$  and  $\mathcal{A}_0 = \{\alpha_1^0, \beta_1^0, X \setminus (\alpha_1^0 \cup \beta_1^0)\}$  (three orbits of cardinality 1).

For all  $n$  one has:

2.  $\mathcal{A}_{n+1}$  refines  $\mathcal{A}_n$ .
3. If  $U_n \sim_{\Gamma} V_n$  then  $U_n$  and  $V_n$  are  $\mathcal{A}_n$ -equivalent.
4. The tuples  $(\alpha_1^n, \dots, \alpha_{k_n}^n, U_{n+1})$  and  $(\beta_1^n, \dots, \beta_{k_n}^n, V_{n+1})$  are almost  $\mathcal{A}_{n+1}$ -equivalent, as witnessed by the exceptional orbits

$$O(\alpha_1^{n+1}), \dots, O(\alpha_{k_{n+1}}^{n+1}), O(\beta_1^{n+1}), \dots, O(\beta_{k_{n+1}}^{n+1})$$

5. For all  $i, j$   $\alpha_i^j \not\sim_{\Gamma} \beta_i^j$ .

6. Let  $h_n$  be the number of atoms of  $\mathcal{A}_n$ ; denote

$$N_i^n = \max\{|n_O(\alpha_i^n) - n_O(\beta_i^n)| : O \text{ is a } \mathcal{A}_{n+1} \text{-orbit}\} \quad (i \leq k_n)$$

$$N^{(n)} = \sum_{i=1}^{k_n} N_i^n$$

Then every exceptional  $\mathcal{A}_{n+1}$ -orbit contains more than  $3h_n N^{(n)}$  copies of every  $\mathcal{A}_n$ -orbit.

*Proof.* Assume the construction has been carried out up to some  $n$  (the case  $n = 0$  being dealt with by the first condition above).

If  $U_{n+1} \sim_{\Gamma} V_{n+1}$ , find a  $\sim_{\Gamma}$ -partition  $\mathcal{B}$  refining  $\tilde{\mathcal{A}}_n$  and such that  $U_{n+1}$  and  $V_{n+1}$  are  $\mathcal{B}$ -equivalent; such a partition exists because there exists a  $\sim_{\Gamma}$ -partition for which  $U_{n+1}$  and  $V_{n+1}$  are  $\mathcal{B}$ -equivalent, and any two  $\sim_{\Gamma}$ -partitions have a common refinement. Then apply Proposition 3.15 to this partition and  $(\alpha_1^n, \dots, \alpha_{k_n}^n)$  and  $(\beta_1^n, \dots, \beta_{k_n}^n)$ . Choose  $\mathcal{A}_{n+1}$  with a minimal number of exceptional columns, which ensures that no  $\alpha_j^{n+1}$  and  $\beta_j^{n+1}$  belong to the same  $\Gamma$ -orbit.

If  $U_{n+1} \not\sim_{\Gamma} V_{n+1}$ , apply Proposition 3.15 (again, with a minimal number of exceptional columns) to  $\mathcal{A}_n, (\alpha_1^n, \dots, \alpha_{k_n}^n, U_{n+1})$  and  $(\beta_1^n, \dots, \beta_{k_n}^n, V_{n+1})$ .  $\square$

We obtain a sequence of  $\sim_{\Gamma}^*$ -partitions  $(\mathcal{B}_n)$  by joining together the  $\mathcal{A}_n$ -orbits of each  $\alpha_i^n$  and  $\beta_i^n$  (i.e. every pair of exceptional orbits of  $\mathcal{A}_n$  are joined together so as to form a single  $\mathcal{B}_n$ -orbit consisting of two  $\mathcal{A}_n$ -orbits; the other orbits are unchanged). The construction ensures that, for all  $n$ :

- $\mathcal{B}_{n+1}$  refines  $\mathcal{B}_n$ . Indeed, for all  $n$  and  $i \leq k_n$ ,  $\alpha_i^n$  and  $\beta_i^n$  are  $\mathcal{B}_{n+1}$ -equivalent (since they are almost  $\mathcal{A}_{n+1}$ -equivalent, and we have joined together each exceptional pair of orbits of  $\mathcal{A}_{n+1}$  to form a single  $\mathcal{B}_{n+1}$ -orbit). And any two atoms belonging to the same non-exceptional  $\mathcal{A}_n$ -orbit are  $\mathcal{A}_{n+1}$ -equivalent, hence  $\mathcal{B}_{n+1}$ -equivalent.
- For any  $n$   $U_n$  and  $V_n$  are  $\mathcal{B}_n$ -equivalent: If  $U_n \sim_{\Gamma} V_n$  then  $U_n$  and  $V_n$  are already  $\mathcal{A}_n$ -equivalent and  $\mathcal{B}_n$  refines  $\mathcal{A}_n$ ; if  $U_n \not\sim_{\Gamma} V_n$  then stacking the exceptional  $\mathcal{A}_n$ -orbits together makes them  $\mathcal{B}_n$ -equivalent. It follows that  $U_n, V_n$  are  $\mathcal{B}_m$ -equivalent for all  $m \geq n$ .

**Lemma 3.19.** *We may build:*

- An ample group  $\tilde{\Gamma} = \bigcup_n \tilde{\Gamma}_n$  such that  $(\mathcal{A}_n, \tilde{\Gamma}_n)$  is a unit system and the  $\tilde{\Gamma}_n$ -orbit of any atom  $\alpha$  of  $\mathcal{A}_n$  is equal to  $\tilde{\Gamma}_n \alpha$ .
- An involution  $\pi$  such that for all  $n$   $\pi$  induces an automorphism of  $\mathcal{B}_n$ ,  $\pi(\alpha_i^n) = \beta_i^n$  for all  $n$  and all  $i \in \{1, \dots, k_n\}$ , and  $\pi$  is trivial outside of  $\alpha_1^0 \sqcup \beta_1^0$ .

Denoting by  $\Lambda_n$  the subgroup of  $\text{Aut}(\mathcal{B}_n)$  generated by  $\tilde{\Gamma}_n$  and  $\pi$ , we ensure that  $(\mathcal{B}_n, \Lambda_n)$  is a unit system and the  $\Lambda_n$ -orbit of any atom of  $\mathcal{B}_n$  coincides with its  $\mathcal{B}_n$ -orbit.

We say that atoms of  $\tilde{\Gamma}_n$  on which  $\pi$  does not coincide with an element of  $\tilde{\Gamma}_n$  are singular; for every such atom  $\alpha$   $\pi(\alpha)$  does not belong to  $\tilde{\Gamma}_n \alpha$ .

We also ensure that the following conditions hold:

- (\*) For every atom  $\alpha$  of  $\mathcal{A}_{n+1}$  there exists at most one singular atom of  $\mathcal{A}_{n+1}$  contained in  $\tilde{\Gamma}_n \alpha$ .
- (\*\*) For every singular atom  $\alpha$  of  $\mathcal{A}_n$ , there are at least two singular atoms of  $\mathcal{A}_{n+1}$  contained in  $\alpha$ .

*Proof.* The construction proceeds as follows: first, we define  $\tilde{\Gamma}_0$ , which is the trivial group. Then we define an involution  $\pi \in \text{Aut}(\mathcal{B}_0)$  by setting  $\pi(\alpha_0^1) = \beta_0^1$ ,  $\pi(\beta_0^1) = \alpha_0^1$ , and  $\pi$  is the identity on the other atom. Our desired conditions are satisfied for  $n = 0$ .

Now assume that we are at step  $n$  of our construction, i.e. we have built  $\tilde{\Gamma}_n$  and an automorphism  $\pi$  of  $\mathcal{B}_n$  satisfying our conditions. First, we extend  $\tilde{\Gamma}_n$  to  $\tilde{\Gamma}_{n+1}$  as in the proof of Lemma 3.17, which is possible since  $\mathcal{A}_{n+1}$  refines  $\mathcal{A}_n$ . On atoms of  $\mathcal{B}_{n+1}$  contained in an atom of  $\mathcal{A}_n$  which is not singular, we have no choice for the extension of  $\pi$ : it must coincide with the extension of an element of  $\tilde{\Gamma}_n$  to an automorphism of  $\text{Aut}(\mathcal{B}_{n+1})$ , and that extension has already been defined.

Let  $\alpha$  be a singular atom of  $\mathcal{A}_n$  and  $\beta$  be an atom of  $\mathcal{A}_{n+1}$  contained in  $\alpha$ . If the  $\mathcal{A}_{n+1}$ -orbit  $O(\beta)$  is not exceptional, then it is  $(\alpha, \pi(\alpha))$ -balanced, so we may find an involution  $\gamma \in \tilde{\Gamma}_{n+1}$  such that  $\gamma(\delta) \subset \pi(\alpha)$  for all atoms  $\delta$  of  $O(\beta)$  contained in  $\alpha$ . Declare  $\pi$  to be equal to  $\gamma$  on those atoms.

So the real work consists of extending  $\pi$  to the atoms of the exceptional  $\mathcal{A}_{n+1}$ -orbits contained in some singular atom of  $\mathcal{A}_n$ . Let  $\tau_1, \dots, \tau_p$  be singular atoms of  $\mathcal{A}_n$  belonging to distinct  $\pi$ -orbits and such that for every singular atom  $\alpha$  of  $\mathcal{A}_n$  there exists  $i$  such that  $\alpha = \tau_i$  or  $\alpha = \pi(\tau_i)$  (we use below the fact that  $p \leq h_n$ , since there are fewer singular atoms than there are atoms in  $\mathcal{A}_n$ ). Let  $(O, O')$  be an exceptional pair of  $\mathcal{A}_{n+1}$ -orbits. Note that  $(O, O')$  is a  $(\tau_i, \pi(\tau_i))$ -balanced pair of orbits for all  $i$ .

Denote for all  $i$   $m_O(i) = n_O(\tau_i) - n_O(\pi(\tau_i))$ . We need to do something to balance the columns for which  $|m_O(i)| \geq 1$ ; we distinguish two cases.

- If  $m_O(i) = 1$ , choose two atoms  $\theta_1(i), \theta_2(i)$  of  $O$  contained in  $\tau_i$ , and one atom  $\theta_3(i)$  of  $O$  contained in  $\pi(\tau_i)$ ; as well as two atoms  $\delta_1(i), \delta_2(i)$  of  $O'$  contained in  $\pi(\tau_i)$ , and an atom  $\delta_3(i)$  of  $O'$  contained in  $\tau_i$ . If  $m_O(i) = -1$ , do the same thing but with  $\theta_1(i), \theta_2(i)$  in  $\pi(\tau_i)$  and  $\theta_3(i)$  in  $\tau_i$ ; and  $\delta_1(i), \delta_2(i)$  in  $\tau_i$  while  $\delta_3(i)$  belongs to  $\pi(\tau_i)$ .
- If  $m_O(i) \geq 2$ , we set aside atoms  $\theta_1(i), \dots, \theta_{m_O(i)}(i)$  of  $O$  contained in  $\tau_i$ , and atoms  $\delta_1(i), \dots, \delta_{m_O(i)}(i)$  of  $O'$  contained in  $\pi(\tau_i)$ ; similarly, if  $m_O(i) \leq -2$  we set aside atoms  $\theta_1(i), \dots, \theta_{m_O(i)}(i)$  of  $O$  contained in  $\pi(\tau_i)$ , and atoms  $\delta_1(i), \dots, \delta_{m_O(i)}(i)$  of  $O'$  contained in  $\tau_i$ .

Overall, this involves choosing fewer than  $3h_n N$  atoms in each of  $O, O'$ , so we can additionally ensure that no two of these atoms belong to the same  $\tilde{\Gamma}_n$ -orbit. We then set  $\pi(\delta_j(i)) = \theta_j(i)$ ,  $\pi(\theta_j(i)) = \delta_j(i)$  for all  $i, j$ .

We do this for all exceptional orbit pairs  $(O, O')$  and all  $i$ . Now, in each exceptional orbit  $O$  of  $\mathcal{A}_{n+1}$ , and for any  $i$ , there remain as many atoms of  $O$  contained in  $\tau_i$  on which  $\pi$  is yet to be extended as there are such atoms contained in  $\pi(\tau_i)$ . This means that we can extend  $\pi$  so that it coincides with an element of  $\tilde{\Gamma}_{n+1}$  on those atoms.

This enables us to move on to the next step. Condition (\*) has been guaranteed when we chose our new singular atoms in distinct  $\tilde{\Gamma}_n$ -orbits. To see why condition

(\*\*) holds, let  $\alpha$  be a singular atom of  $\mathcal{A}_n$ . Since  $\pi(\alpha) \not\sim_{\Gamma} \alpha$ , there must be some exceptional column of  $\mathcal{A}_{n+1}$  which is not  $(\alpha, \pi(\alpha))$ -balanced; and we took care to include more than two singular atoms contained in  $\alpha$  in such a column (this is why we singled out the case  $|m_O(i)| = 1$  above).  $\square$

We let  $\tilde{\Gamma} = \bigcup_n \tilde{\Gamma}_n$ ,  $\Lambda = \bigcup_n \Lambda_n$ . They are both ample groups, and  $\Lambda$  is the full group generated by  $\tilde{\Gamma}$  and  $\pi$ .

**Lemma 3.20.** *For any  $U, V \in CO(X)$  such that  $U \sim_{\Gamma}^* V$ , there exists  $\lambda \in \Lambda$  such that  $\lambda(U) = V$ .*

*Proof.* First, note that for any clopen  $U, V$  such that  $U \sim_{\Gamma} V$ , there exists some  $n$  such that  $(U, V) = (U_n, V_n)$ , so that  $(U, V)$  are  $\mathcal{A}_n$ -equivalent, hence also  $\mathcal{B}_n$ -equivalent, so there exists  $\lambda \in \Lambda_n$  such that  $\lambda(U) = V$ . So  $U \sim_{\Lambda} V$ .

If  $U \not\sim_{\Gamma} V$  but  $U \sim_{\Gamma}^* V$ , there exists  $n$  such that  $(U, V) = (U_n, V_n)$ , so  $(U, V)$  are  $\mathcal{B}_n$ -equivalent and  $U \sim_{\Lambda} V$ .  $\square$

**Lemma 3.21.**  *$\Lambda$  acts minimally on  $X$ ;  $M(\Lambda) = M(\Gamma)$ ; and  $\Lambda$  is a saturated ample group.*

*Proof.* Given any nonempty clopen set  $U$ , there exists  $\gamma_1, \dots, \gamma_n \in \Gamma$  such that  $X = \bigcup_{i=1}^n \gamma_i(U)$ , since  $\Gamma$  acts minimally. By Lemma 3.20, for all  $i$  there exists some  $\lambda_i \in \Lambda$  such that  $\lambda_i(U) = \gamma_i(U)$ , whence  $\Lambda$  also acts minimally.

Let  $G_{\Gamma} = \{g \in \text{Homeo}(X) : \forall \mu \in M(\Gamma) \ g_*\mu = \mu\}$ . By construction,  $\Lambda \subset G_{\Gamma}$  since each  $\mathcal{B}_n$  is a  $\sim_{\Gamma}^*$ -partition (see Lemma 3.17).

If  $U, V \in CO(X)$  are such that  $\mu(U) = \mu(V)$  for all  $\mu \in M(\Gamma)$ , we know by the previous lemma that there exists  $\lambda \in \Lambda$  such that  $\lambda(U) = V$ , so  $\overline{\Lambda}$  contains  $G_{\Gamma}$ . Hence  $\overline{\Lambda} = G_{\Gamma}$ , thus for any Borel probability measure  $\mu$  on  $X$  we have

$$\begin{aligned} \mu \in M(\Lambda) &\Leftrightarrow \mu \in M(\overline{\Lambda}) \\ &\Leftrightarrow \mu \in M(G_{\Gamma}) \\ &\Leftrightarrow \mu \in M(\Gamma) \quad (\text{recall that } M(\Gamma) = M(G_{\Gamma}), \text{ see Lemma 3.3}) \end{aligned}$$

Thus  $\overline{\Lambda} = G_{\Gamma} = \overline{[R_{\Gamma}]}$ , and  $\overline{[R_{\Gamma}]} = \overline{[R_{\Lambda}]}$  since  $M(\Lambda) = M(\Gamma)$  (see Lemma 3.9). Hence  $\overline{\Lambda} = [R_{\Lambda}]$ , i.e.  $\Lambda$  is saturated.  $\square$

**Lemma 3.22.** *The orbits of the action of  $\tilde{\Gamma}$  on  $CO(X)$  coincide with the orbits of the action of  $\Gamma$  on  $CO(X)$ .*

*Proof.* One inclusion is immediate from the definition of  $\tilde{\Gamma}$ .

If  $U, V$  are two clopens such that  $U \sim_{\Gamma} V$ , then there exists  $n$  such that  $(U, V)$  are  $\mathcal{A}_n$ -equivalent, so there exists  $\gamma \in \tilde{\Gamma}_n$  such that  $\gamma(U) = V$  (recall that the  $\mathcal{A}_n$ -orbit of any atom  $\alpha$  of  $\mathcal{A}_n$  coincides with  $\tilde{\Gamma}_n \alpha$ ).  $\square$

Using Krieger's theorem, we conclude that  $\Gamma$  and  $\tilde{\Gamma}$  are conjugate.

*Definition 3.22.* We say that  $x \in X$  is *singular* if for any  $n$  the atom  $\alpha_n(x)$  of  $\mathcal{B}_n$  which contains  $x$  is singular (recall that this means that  $\pi$  does not coincide on  $\alpha_n(x)$  with an element of  $\tilde{\Gamma}_n$ ; and then  $\pi(\alpha_n(x))$  does not belong to  $\tilde{\Gamma}_n \alpha_n(x)$ ).

If  $x \in X$  is not singular, then  $\pi$  coincides on a neighborhood of  $x$  with an element of  $\tilde{\Gamma}$ . Since  $\Lambda$  is generated, as a full group, by  $\tilde{\Gamma}$  and  $\pi$ , this means that  $R_\Lambda$  is the finest equivalence relation coarser than  $R_{\tilde{\Gamma}}$  and such that  $x$  is equivalent to  $\pi(x)$  for each singular point  $x$ . Denote by  $S$  the set of singular points. It is closed since it is an intersection of clopen sets.

**Lemma 3.23.**  *$S$  is  $\tilde{\Gamma}$ -sparse and has no isolated points.*

*Proof.* Condition  $(*)$  from the construction of  $\pi$  implies that, for any  $n$ , there is no singular point besides  $x$  in  $\tilde{\Gamma}_n \alpha_{n+1}(x)$ . In particular,  $x$  is the unique singular point in  $\tilde{\Gamma}_n x \cap S$  for any  $n$ . This proves that  $S$  is  $\tilde{\Gamma}$ -sparse.

Let  $x$  be singular, and  $U$  a clopen subset containing  $x$ . There exists  $n$  such that  $\alpha_n(x) \subset U$ ; condition  $(**)$  implies that there exists a singular atom of  $\mathcal{A}_{n+1}$  contained in  $\alpha_n(x)$  and disjoint from  $\alpha_{n+1}(x)$ . Since each singular atom contains a singular point, there is a singular point distinct from  $x$  and belonging to  $U$ . So  $S$  has no isolated points.  $\square$

Let us recap what we have done. Starting from a minimal ample group  $\Gamma$ , we built two new minimal ample groups  $\tilde{\Gamma}$ ,  $\Lambda$ . The group  $\tilde{\Gamma}$  is conjugated to  $\Gamma$  since their actions on  $CO(X)$  have the same orbits;  $\Lambda$  is saturated. The orbits of the action of  $\Lambda$  on  $X$  are obtained by gluing together pairs of orbits for the action of  $\tilde{\Gamma}$  along a  $\tilde{\Gamma}$ -sparse closed subset which has no isolated points, so Theorem 3.5 tells us that  $\tilde{\Gamma}$  is orbit equivalent to  $\Lambda$ .

Hence  $\Gamma$  is orbit equivalent to  $\Lambda$ ; we have finally proved that every minimal ample group is orbit equivalent to a saturated minimal ample group, and this concludes the proof of the classification theorem for minimal ample groups.

### 3.6 The classification theorem for minimal homeomorphisms

The following theorem is a consequence of [GPS1, Theorem 2.3], as well as a particular case of [GPS3, Theorem 4.16]; it can be seen as an absorption theorem where one only needs to glue two orbits together.

**Théorème 3.7.** *Let  $\varphi$  be a minimal homeomorphism, and  $x \in X$ . Then the relations induced by  $\varphi$  and by  $\Gamma_x(\varphi)$  are isomorphic.*

*Proof.* Fix an element  $y \in X$  which does not belong to the  $\varphi$ -orbit of  $x$ , and consider the ample group  $\Delta = \Gamma_x(\varphi) \cap \Gamma_y(\varphi)$ , which acts minimally (we skip the proof of this fact since we prove a more general statement below when establishing Theorem 3.8).

For any integer  $N$ , we can find a clopen  $U$  such that  $x, y \in U$  and  $U \cap \varphi^i(U) = \emptyset$  for any  $i \in \{1, \dots, N\}$ . By considering a Kakutani–Rokhlin partition with basis such a set  $U$ , we see thanks to Lemma 3.8 that for any clopen  $A, B$  such that  $\mu(A) < \mu(B)$  for any  $\mu \in M(\Gamma)$ , there exists  $\delta \in \Delta$  such that  $\delta(A) \subset B$ .



As in the proof of Lemma 3.9, it follows that

$$\overline{[R_\Delta]} = \{g \in G : \forall \mu \in M(\varphi) \ g_*\mu = \mu\} = \overline{[R_{\Gamma_x(\varphi)}]}$$

Then the classification theorem for minimal ample groups implies that  $\Delta$  is orbit equivalent to  $\Gamma_x(\varphi)$ .

Since  $R_{\Gamma_x(\varphi)}$  is obtained from  $R_\Delta$  by joining the  $\Delta$ -orbits of  $y$  and  $\varphi^{-1}(y)$  together, and  $\Gamma_x(\varphi)$  is orbit equivalent to  $\Delta$ , we see that  $R_\Delta$  is isomorphic to the relation obtained from  $R_\Delta$  by joining the  $R_\Delta$ -orbits of  $y, \varphi^{-1}(y)$  together.

Hence, there exist  $z, t$  with distinct  $\Gamma_x(\varphi)$ -orbits such that  $R_{\Gamma_x(\varphi)}$  is isomorphic to the relation obtained by joining the orbits of  $z$  and  $t$  together; and by Theorem 3.2, there exists  $g$  which realizes an isomorphism from  $R_{\Gamma_x(\varphi)}$  to itself, with  $g(z) = x$  and  $g(t) = \varphi^{-1}(x)$ .

Finally,  $R_{\Gamma_x(\varphi)}$  is isomorphic to the relation induced from  $R_{\Gamma_x(\varphi)}$  by joining the orbits of  $x$  and  $\varphi^{-1}(x)$  together, which is equal to  $R_\varphi$ .  $\square$

The classification theorem for minimal homeomorphisms follows immediately from this and the classification theorem for minimal ample groups.

*Proof of the classification theorem for minimal homeomorphisms.* Assume that  $\varphi, \psi$  are two minimal homeomorphisms of  $X$  such that  $M(\varphi) = M(\psi)$ . Recall that for any  $x$  we have  $M(\Gamma_x(\varphi)) = M(\varphi)$  and  $M(\Gamma_x(\psi)) = M(\psi)$ .

Thus if  $M(\varphi) = M(\psi)$  then  $\Gamma_x(\varphi)$  and  $\Gamma_x(\psi)$  are orbit equivalent by the classification theorem for minimal ample groups, and then by the previous theorem  $\varphi$  and  $\psi$  are orbit equivalent.  $\square$

Using the same approach, we can also recover Theorem 4.16 of [GPS3]. We first note the following easy fact (we have used earlier in this paper an analogue of this for minimal ample groups).

**Lemma 3.24.** *Assume that  $\varphi$  is a minimal homeomorphism of  $X$ , and that  $Y$  is a closed subset of  $X$  which meets every  $\varphi$ -orbit in at most one point. Then for any  $N$  one can find a clopen subset  $U$  containing  $Y$  and such that  $U \cap \varphi^i(U) = \emptyset$  for all  $i \in \{1, \dots, N\}$ .*

*Proof.* Fix  $N$ . Since  $Y, \varphi(Y), \dots, \varphi^N(Y)$  are closed and pairwise disjoint, there exist disjoint clopen sets  $U_i$  such that  $\varphi^i(Y) \subset U_i$  for all  $i \in \{0, \dots, N\}$ . Set

$$U = \bigcap_{i=0}^N \varphi^{-i}(U_i)$$

Then  $U$  is clopen, contains  $Y$ , and  $U \cap \varphi^i(U) \subseteq U_0 \cap U_i = \emptyset$  for all  $i \in \{1, \dots, N\}$ .  $\square$

**Notation** (see [GPS3, Theorem 4.6]). Let  $\varphi$  be a minimal homeomorphism, and  $Y$  be a closed set meeting each  $\varphi$ -orbit in at most one point. For  $y \in Y$ , recall that

$$O^+(y) = \{\varphi^n(y) : n \geq 0\} \quad \text{and} \quad O^-(y) = \{\varphi^n(y) : n < 0\}$$

We denote by  $R_Y$  the equivalence relation obtained from  $R_\varphi$  by splitting the  $\varphi$ -orbit of each  $y \in Y$  into  $O^+(y)$  and  $O^-(y)$ , and leaving the other  $R$ -classes unchanged.

**Théorème 3.8** ([GPS3, Theorem 4.16]). *Let  $\varphi$  be a minimal homeomorphism, and  $Y$  be a closed set which meets every  $\varphi$ -orbit in at most one point. Then  $R_Y$  and  $R_\varphi$  are isomorphic.*

*Proof.* Consider the group  $\Gamma_Y = \bigcap_{y \in Y} \Gamma_y(\varphi)$ . It is ample since it is an intersection of ample groups.

We saw that for any given  $N$  there exists a Kakutani–Rokhlin partition for  $\varphi$  with base a clopen set  $U$  containing  $Y$  and such that  $U \cap \varphi^i(U) = \emptyset$  for all  $i \in \{1, \dots, N\}$ . Fix  $x$  whose orbit does not intersect  $Y$ , and find such a partition with  $x$  not belonging to  $U$  or  $\varphi^{-1}(U)$  (just shrink  $U$  if necessary) and say  $N = 2$ . Looking at the atom which contains  $x$ , we see that  $\varphi(x)$  and  $\varphi^{-1}(x)$  are both  $\Gamma_Y$ -equivalent to  $x$  (since the restriction of  $\varphi^{\pm 1}$  to some neighborhood of  $x$  belongs to  $\Gamma_Y$ ). This proves that the  $\Gamma_Y$ -orbit of any element whose  $\varphi$ -orbit does not intersect  $Y$  coincides with its  $\varphi$ -orbit: we can always move via  $\varphi^{\pm 1}$  along this orbit.

By definition of  $\Gamma_Y$ , the orbit of each  $y \in Y$  splits in at least two  $\Gamma_Y$ -orbits; and for each  $n \geq 0$  and  $y \in Y$ , the restriction of  $\varphi^n$  to some neighborhood of  $y$  belongs to  $\Gamma_Y$  (take a partition as above with  $N = n$ ); same argument for negative semi-orbits. Thus  $R_Y$  is the relation induced by  $\Gamma_Y$ , and in particular  $\Gamma_Y$  acts minimally.

As in the proof of Theorem 3.7, we deduce from the existence of Kakutani–Rokhlin partitions with arbitrarily large height and basis containing  $Y$  that  $\overline{[R_Y]} = \overline{[R_\varphi]}$ , so by the classification theorem  $R_Y$  is isomorphic to  $R_\varphi$ .  $\square$

We conclude this paper by improving our absorption theorem 3.5 (the result below is still a particular case of the absorption theorems in [GPS3], [GMPS1], [M2]).

**Théorème 3.9.** *Let  $\Gamma$  be an ample subgroup acting minimally. Let  $K = K_1 \sqcup \sigma(K_1)$ , where  $K_1$  is closed and  $\sigma \in \text{Homeo}(K)$  is an involution. Assume that  $K$  is  $\Gamma$ -sparse.*

*Denote by  $R_{\Gamma, K}$  the finest equivalence relation coarser than  $R_\Gamma$  and for which  $k, \sigma(k)$  are equivalent for all  $k \in K$ . Then there exists  $f \in \text{Homeo}(X)$  such that  $f\Gamma f^{-1}$  induces  $R_{\Gamma, K}$  (hence  $R_{\Gamma, K}$  and  $R_\Gamma$  are isomorphic).*

*Proof.* By the constructions used in the previous section, we know that we can find a nonempty closed subset  $F$  of  $X$  without isolated points, and a homeomorphism  $\pi$  of  $F$ , such that  $x \neq \pi(x)$  for all  $x \in F$  and the intersection of any  $\Gamma$ -orbit with  $F$  is either empty or equal to  $\{x, \pi(x)\}$  for some  $x \in F$ . Note that there exists a homeomorphic embedding  $g: K \rightarrow F$  such that  $g(\sigma(x)) = \pi(g(x))$  for all  $x \in K$ . Recall that  $K = K_1 \sqcup \sigma(K_1)$ ; let  $Y = g(K_1)$ .

Let  $\varphi$  be a minimal homeomorphism of  $X$  and  $x \in X$  such that  $\Gamma = \Gamma_x(\varphi)$  (see Theorem 3.4). We can assume that  $x$  is not in the  $\varphi$ -orbit of any element of  $Y$ . As in the proof of Theorem 3.8, denote  $\Gamma_Y = \bigcap_{y \in Y} \Gamma_y(\varphi)$ . We know that  $\Gamma_Y$  is orbit equivalent to  $\Gamma_x(\varphi) = \Gamma$ , so there exists  $\Lambda$  conjugate to  $\Gamma$  in  $\text{Homeo}(X)$  and which induces  $R_{\Gamma_Y}$ .

Further,  $R_\Gamma$  is obtained from  $R_{\Gamma_Y} = R_\Lambda$  by joining together the  $\Lambda$ -orbits of  $y$  and  $\varphi^{-1}(y)$  for all  $y \in Y$ .

Consider the homeomorphism  $h: Y \sqcup \varphi^{-1}(Y) \rightarrow K$  defined by  $h(y) = g^{-1}(x)$ ,  $h(\varphi^{-1}(y)) = \sigma(g^{-1}(x))$  for all  $y \in Y$ . Applying theorem 3.2 to  $\Lambda, \Gamma$  and  $h$ , we obtain a homeomorphism  $f$  of  $X$  such that  $f\Lambda f^{-1} = \Gamma$  and  $f|_{Y \sqcup \varphi^{-1}(Y)} = h$ .

Then  $f\Gamma f^{-1}$  induces the relation obtained from  $R_\Gamma$  by gluing together the  $\Gamma$ -orbit of  $h(y)$  and  $h(\varphi^{-1}(y)) = \sigma(h(y))$  for all  $y \in Y$ . This relation is exactly  $R_{\Gamma, K}$ .  $\square$

# Appendices



# Krieger's Theorem

We present in this appendix the main result of [K1], slightly adapting the proof to our less general context, and trying to highlight important steps. Also the original paper is quite old, and we hope that this refreshed version would be easier to read, especially thanks to the addition of some pictures. Also, we encourage readers to try to understand this version and to practice with the exercise at the end before trying to understand the Section 3.3 of Chapter 3, which contains a stronger version of this result.

Groups we are interested in here are countable locally finite subgroups of  $\text{Homeo}(X)$ , that leaves invariant a sub-Boolean algebra  $\mathcal{A}$  of  $\text{CO}(X)$ .

Given such a group  $G$  and a  $G$ -invariant sub-Boolean algebra  $\mathcal{A}$  of  $\text{CO}(X)$ , we can define new homeomorphisms from  $G$  and  $\mathcal{A}$  thanks to the following method : Let  $A_1, \dots, A_n \in \mathcal{A}$  and  $g_1, \dots, g_n \in G$  be such that  $\bigsqcup_{i=1}^n A_i = X = \bigsqcup_{i=1}^n g_i(A_i)$ . We can then define a homeomorphism  $h$  by "gluing" the  $g_i$ 's :  $h \upharpoonright_{A_i} = g_i \upharpoonright_{A_i}$ . We say that  $h$  is associated to  $G$  and  $\mathcal{A}$ .

**Notation.** Following Krieger's notations, we denote by  $[G, \mathcal{A}]$  the group of homeomorphisms associated to  $G$  and  $\mathcal{A}$ . We will say that  $G$  is full if  $G = [G, \text{CO}(X)]$ . Note that the full group and the topological full group of  $\phi$  are full groups.

We call  $g_{\mathcal{A}} : \begin{cases} \mathcal{A} \longrightarrow \mathcal{A} \\ A \longmapsto g(A) \end{cases}$ , for  $g \in G$ , the restriction of  $g$  to  $\mathcal{A}$ . We also write  $G_{\mathcal{A}} := \{g_{\mathcal{A}} : g \in G\}$ .  $G_{\mathcal{A}}$  is a group.

We are now ready to define ample groups and unit systems.

**Definition A.1** ([K1], section 2)). Let  $G$  be a countable locally finite subgroup of  $\text{Homeo}(X)$ , and  $\mathcal{A}$  a sub-Boolean algebra of  $\text{CO}(X)$ . We say that  $(G, \mathcal{A})$  is a *unit system* if it satisfies the following conditions :

1.  $G$  leaves  $\mathcal{A}$  invariant.
2.  $G = [G, \mathcal{A}]$ .
3.  $\text{restr}_{\mathcal{A}} : \begin{cases} G \longrightarrow G_{\mathcal{A}} \\ g \longmapsto g_{\mathcal{A}} \end{cases}$  is a group isomorphism.
4. For every  $g \in G$ , the set of fix points of  $g$  is in  $\mathcal{A}$ .

A unit system is said to be *finite* if  $\mathcal{A}$  is finite. In that case we denote by  $\text{at}(\mathcal{A})$  the set of atoms of  $\mathcal{A}$ .

We call  $G$  an *ample group* if  $(G, CO(X))$  is a unit system.

We say that a unit system  $(\mathcal{A}, G)$  is finer than another unit system  $(\mathcal{B}, H)$ , and we note  $(\mathcal{B}, H) \leq (\mathcal{A}, G)$ , if  $\mathcal{B} \subset \mathcal{A}$  and  $H \subset G$ .

*Remarque A.1.* For every  $x_0 \in X$ ,  $\Gamma_{x_0}$  is an ample group, thanks to Proposition 2.4 of Chapter 2.

This definition may seem not very clear, the idea is that if  $(\mathcal{A}, G)$  is a finite unit system, the group  $G$  is easy to understand, it essentially consists of some permutations of  $\mathcal{A}$ . Krieger's approach consists of approximating ample groups by finite unit systems, and then using a back-and-forth argument to construct a spatial isomorphism between the groups. In our case, it is easy to see that  $(\Xi_n, \Gamma_n)_n$  is an increasing sequence of finite unit systems such that  $\Gamma_\Xi = \bigcup_n \Gamma_n$  and  $CO(X) = \bigcup_n \Xi_n$ . This is true in general, but we will not give any proof of this :

**Lemma A.1.** *If  $G$  is an ample group, there exists an increasing sequence of finite unit systems  $(\mathcal{A}_n, G_n)_{n \in \mathbb{N}}$  such that  $CO(X) = \bigcup_n \mathcal{A}_n$  and  $G = \bigcup_n G_n$ .*

*Definition A.2* ([K1], section 3). Let  $G$  be an ample group. We will call *dimension range of  $G$*  the quotient  $CO(X)/G$  with the algebraic structure given by  $\gamma = \nu + \zeta$  if there exists  $E \in \nu$  and  $F \in \zeta$  disjoint such that  $E \cup F \in \gamma$ , where  $CO(X)/G$  stands for the orbit space of  $CO(X)$  under  $G$ . We also introduce the projection  $\delta_G: CO(X) \rightarrow CO(X)/G$ , and for a clopen  $A$ , we denote  $[A]_G$ , or simply  $[A]$ , the orbit  $\delta_G(A)$ .

Two ample groups are said to have *isomorphic dimension ranges* if there exists a bijection between their dimension ranges that preserves "+".

*Remarque A.2.* It is not hard to prove that the operation "+" defined above is well defined.

*Definition A.3.* Let  $f$  be an isomorphism between dimension ranges of two ample groups  $G$  and  $\overline{G}$ , and  $(\mathcal{D}, \mathcal{U}) \leq (CO(X), G)$ ,  $(\overline{\mathcal{D}}, \overline{\mathcal{U}}) \leq (CO(X), \overline{G})$  two finite unit systems. We say that a Boolean algebra isomorphism  $g: \mathcal{D} \rightarrow \overline{\mathcal{D}}$  respects  $f$  if  $\forall D \in \mathcal{D}, [g(D)] = f([D])$ . We also say that it *conjugates*  $(\mathcal{D}, \mathcal{U})$  on  $(\overline{\mathcal{D}}, \overline{\mathcal{U}})$  if  $\overline{\mathcal{U}}|_{\overline{\mathcal{D}}} = g\mathcal{U}|_{\mathcal{D}}g^{-1}$ .

The theorem we want to prove is the following :

**Theorem A.2.** ([K1], Theorem 3.5) *For two ample groups  $\Gamma$  and  $\Lambda$ , there exists a homeomorphism  $g$  such that  $\Lambda = g\Gamma g^{-1}$  if and only if  $\Gamma$  and  $\Lambda$  have isomorphic dimension ranges.*

*Remarque A.3.* In order to not to use lemma A.1 we have not proven, we will prove this theorem only for  $\Gamma = \Gamma^\phi$  and  $\Lambda = \Gamma^\psi$ ,  $\phi$  and  $\psi$  being minimal homeomorphisms.

The process of back-and-forth which is the heart of the proof is summed up on figure A.3. The main tool is the following lemma, that allows to construct inductively all of the arrows on this figure :

**Lemma A.3.** ([K1], Lemma 3.4) Let  $G$  and  $\bar{G}$  two ample groups with isomorphic dimension ranges, and let  $f: CO(X)/_G \rightarrow CO(X)/_{\bar{G}}$  be an isomorphism between them. Let also  $(\mathcal{D}, \mathcal{U}) \leq (CO(X), G)$  and  $(\bar{\mathcal{D}}, \bar{\mathcal{U}}) \leq (CO(X), \bar{G})$  be finite unit systems, and  $g: \mathcal{D} \rightarrow \bar{\mathcal{D}}$  be a Boolean algebra isomorphism.

Assume that the following conditions are satisfied :

1.  $g$  respects  $f$
2.  $g$  conjugates  $(\mathcal{D}, \mathcal{U})$  on  $(\bar{\mathcal{D}}, \bar{\mathcal{U}})$

Then, for every finite unit system  $(\mathcal{E}, \mathcal{V})$  such that  $(\mathcal{D}, \mathcal{U}) \leq (\mathcal{E}, \mathcal{V}) \leq (CO(X), G)$ , there exists a finite unit system  $(\bar{\mathcal{E}}, \bar{\mathcal{V}})$  such that  $(\bar{\mathcal{D}}, \bar{\mathcal{U}}) \leq (\bar{\mathcal{E}}, \bar{\mathcal{V}}) \leq (CO(X), \bar{G})$  and a isomorphism  $h: \mathcal{E} \rightarrow \bar{\mathcal{E}}$  extending  $g$  verifying the following conditions :

1.  $h$  respects  $f$
2.  $h$  conjugates  $(\mathcal{E}, \mathcal{V})$  on  $(\bar{\mathcal{E}}, \bar{\mathcal{V}})$

*Proof.* Fix an orbit  $\sigma \in at(\mathcal{D})/_\mathcal{U}$ , and choose a representative  $D_\sigma \in \sigma$ . Moreover, for  $D \in \sigma$ ,  $D \neq D_\sigma$ , we denote by  $u(\sigma, D)$  the unique element in  $\mathcal{U}$  that induces on  $at(\mathcal{D})$  the transposition  $(D_\sigma, D)$  (uniqueness comes from injectivity of  $restr_{\mathcal{D}}$ ). Equation (2) ensures that there exists  $\bar{u}(\sigma, D)$  in  $\bar{\mathcal{U}}$  that induces on  $at(\bar{\mathcal{D}})$  the transposition  $(g(D), g(D_\sigma))$ .

In addition,  $\mathcal{D} \subset \mathcal{E}$  implies that  $D_\sigma$  is partitioned by atoms of  $\mathcal{E}$  :  $D_\sigma = \bigsqcup_i E_i$ . Since  $f$  is a morphism, and thanks to Condition 1, we get

$$[g(D_\sigma)]_{\bar{G}} = f([D_\sigma]_G) = \bigoplus_i f([E_i]_G)$$

Then we can choose some  $\bar{E}_i \in CO(X)$  partitioning  $g(D_\sigma)$  such that  $[\bar{E}_i]_{\bar{G}} = f([E_i]_G)$ . Roughly speaking, we use  $f$  as an oracle to "copy" the algebra  $\mathcal{E}$  into  $\bar{\mathcal{D}}$ , the  $\bar{E}_i$ 's being "copies" of the  $E_i$ 's. (cf figure A.1)

We now construct the algebra  $\bar{\mathcal{E}}$  by putting as its atoms elements of

$$\bigcup_{\sigma \in at(\mathcal{D})/_\mathcal{U}} \{\bar{E} : E \subset D_\sigma\} \cup \{\bar{u}(\sigma, D)(\bar{E}) : D \in \sigma, D \neq D_\sigma\}.$$

We also define an isomorphism  $h: \mathcal{E} \rightarrow \bar{\mathcal{E}}$  extending  $g$  by  $h(E) = \bar{E}$ , and  $h(F) = \bar{u}(\sigma, D)(\bar{E})$  if  $F = u(\sigma, D)(E)$ ,  $E \subset D_\sigma$  and  $E, F \in at(\mathcal{E})$ . (cf figure A.1) Condition (1) is then satisfied by the choice of the  $\bar{E}_i$ 's.

It remains to construct a suitable group  $\bar{\mathcal{V}}$ . The idea is to realise  $\bar{\mathcal{V}}$  as "permutations" of  $at(\bar{\mathcal{E}})$  that left  $h(\zeta)$  invariant for  $\zeta \in at(\mathcal{E})/_\mathcal{V}$ .

To do this, we choose a representative  $E_\sigma$  in each orbit  $\sigma \in at(\mathcal{E})/_\mathcal{U}$ , and among them we choose a representative  $E_{\sigma(\zeta)}$  in each orbit  $\zeta \in at(\mathcal{E})/_\mathcal{V}$  (or equivalently



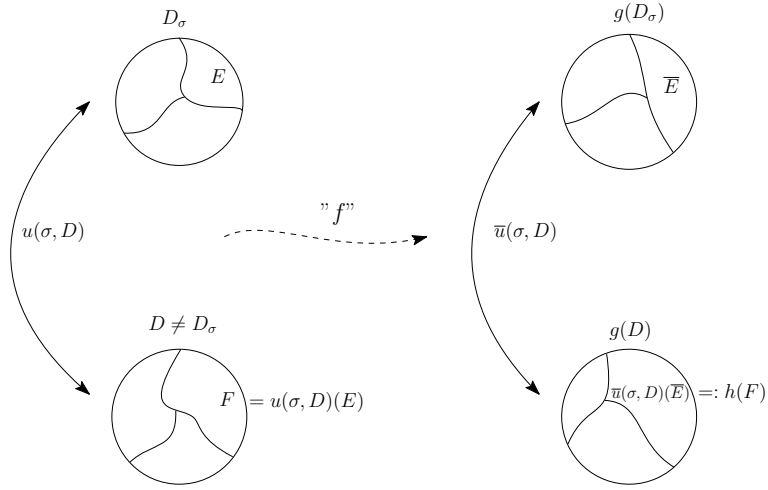


Figure A.1: Construction of the algebra  $\overline{\mathcal{E}}$  and its isomorphism  $h$

we choose a  $\mathcal{U}$ -orbit  $\sigma(\zeta) \subset \zeta$ . Heuristically, elements in  $\overline{\mathcal{U}}$  imitate those in  $\mathcal{U}$  but do not permit to cross from a  $\overline{\mathcal{U}}$ -orbit into another one, unlike what happens in  $\mathcal{V}$ . We will then add some links between  $h(E_{\sigma(\zeta)})$  and  $h(E_\sigma)$  for every  $\mathcal{U}$ -orbit  $\sigma$  enclosed in the  $\mathcal{V}$ -orbit  $\zeta$ . This is possible since according to (1)

$$[h(E_\sigma)]_{\overline{\mathcal{G}}} = f([E_\sigma]_{\mathcal{G}}) = f([E_{\sigma(\zeta)}]_{\mathcal{G}}) = [h(E_{\sigma(\zeta)})]_{\overline{\mathcal{G}}}$$

so there exists  $\bar{v}(\zeta, \sigma) \in [\overline{\mathcal{G}}, \overline{\mathcal{E}}]$  that induces on  $at(\overline{\mathcal{E}})$  the transposition  $(h(E_\sigma), h(E_{\sigma(\zeta)}))$ . We also have  $\bar{u}(\sigma, E) \in \overline{\mathcal{U}}$  that induces the transposition  $(h(E_\sigma), h(E))$  for any  $E \in \sigma$  verifying  $E \neq E_\sigma$ .

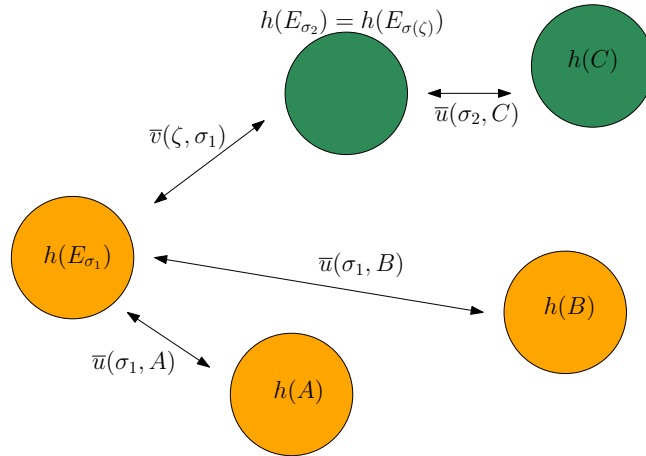


Figure A.2: A  $\overline{\mathcal{V}}$ -orbit  $h(\zeta)$  and two  $\overline{\mathcal{U}}$ -orbits  $h(\sigma_1)$  and  $h(\sigma_2)$  enclosed in it

We can now define a suitable group  $\overline{\mathcal{V}}$  as generated by the  $\{\bar{v}(\zeta, \sigma): \zeta \in at(\overline{\mathcal{E}}) / \overline{\mathcal{V}}, \sigma \subset \zeta\}$  and the  $\{\bar{u}(\sigma, E): \sigma \in at(\overline{\mathcal{E}}) / \overline{\mathcal{U}}, E \neq E_\sigma \in \sigma\}$  (cf figure A.2).

One can check that  $(\overline{\mathcal{E}}, \overline{\mathcal{V}})$  is a finite unit system finer than  $(\overline{\mathcal{D}}, \overline{\mathcal{U}})$  and that also satisfies (2). □

Everything is now ready to prove theorem A.2. The proof, as announced, will be a simple back-and-forth :

*Proof.* One implication is immediate, if  $g$  is a homeomorphism such that  $\Gamma = g\Lambda g^{-1}$ , then  $f: CO(X)/\Lambda \rightarrow CO(X)/\Gamma$  is well defined and is an isomorphism.

$$[A_\Lambda \mapsto [g(A)]_\Gamma$$

Conversely, let's suppose that  $\Gamma$  and  $\Lambda$  have isomorphic dimension ranges, i.e there exists an isomorphism  $f: CO(X)/\Gamma \rightarrow CO(X)/\Lambda$ . As noticed before Lemma A.1,  $(\Xi_n, \Gamma_n)_{n \in \mathbb{N}}$  and  $(\Xi'_n, \Lambda_n)_{n \in \mathbb{N}}$  are sequences of refining finite unit systems such that  $\bigcup \Xi_n = CO(X) = \bigcup \Xi'_n$ ,  $\Gamma = \bigcup \Gamma_n$ ,  $\Lambda = \bigcup \Lambda_n$  (where  $\Gamma$  is built from the sequence of K-R partitions  $\Xi$  and  $\Lambda$  from  $\Xi'$ ).

Let's show by a back-and-forth argument that there exists an isomorphism  $g$  of  $CO(X)$  such that  $\Lambda_{CO(X)} = g\Gamma_{CO(X)}g^{-1}$ . To do this we remark that the lemma A.3 shows that if  $g_k: \mathcal{C}_k \rightarrow \overline{\mathcal{C}}_k$  is a isomorphism such that  $[g_k(C)]_{\overline{\mathcal{C}}} = f([C]_{\mathcal{C}})$  for  $C \in \mathcal{C}_k$  and  $\overline{\mathcal{H}}_{k\overline{\mathcal{C}}_k} = g_k \mathcal{H}_{k\mathcal{C}_k} g_k^{-1}$ , with  $(\mathcal{C}_k, \mathcal{H}_k)$ ,  $(\overline{\mathcal{C}}_k, \overline{\mathcal{H}}_k)$  finite unit systems, and if  $(\mathcal{C}_k, \mathcal{H}_k) \leq (\Xi_{n_{k+1}}, \Gamma_{n_{k+1}})$ , then there exists a finite unit system  $(\overline{\mathcal{C}}_{k+1}, \overline{\mathcal{H}}_{k+1})$  finer than  $(\overline{\mathcal{C}}_k, \overline{\mathcal{H}}_k)$  and a isomorphism  $g_{k+1}: \Xi_{n_{k+1}} \rightarrow \overline{\mathcal{C}}_{k+1}$  extending  $g_k$  such that  $[g_{k+1}(A)]_{\overline{\mathcal{C}}} = f([A]_{\mathcal{C}})$  for  $A \in \Xi_{n_{k+1}}$  and  $\overline{\mathcal{H}}_{k+1\overline{\mathcal{C}}_{k+1}} = g_{k+1}\Gamma_{n_{k+1}\Xi_{n_{k+1}}}g_{k+1}^{-1}$ . The other way being strictly identical, we get two sequences of isomorphisms that extend each other as represented on figure A.3.

$$\begin{array}{ccccccc} (\{\emptyset, X\}, \{id\}) & \leq & (\Xi_{n_1}, \Gamma_{n_1}) & \leq & (C_1, H_1) & \leq & (\Xi_{n_2}, \Gamma_{n_2}) \leq \dots \\ & & \uparrow id & & \downarrow g_1 & & \uparrow h_1 & & \downarrow g_2 \\ (\{\emptyset, X\}, \{id\}) & \leq & (\overline{C}_1, \overline{H}_1) & \leq & (\Xi'_{\overline{n}_1}, \Lambda_{\overline{n}_1}) & \leq & (\overline{C}_2, \overline{H}_2) \leq \dots \end{array}$$

Figure A.3: Building an isomorphism of  $CO(X)$  by a back-and-forth argument

We define  $g$  as the increasing union of the  $g_k$ 's. It is an isomorphism by definition (surjective since  $h_k$  is defined on  $\Xi'_{\overline{n}_k}$  and  $(\overline{n}_k)_{k \in \mathbb{N}}$  is an increasing sequence);  $g$  also verifies  $\Lambda_{CO(X)} = g\Gamma_{CO(X)}g^{-1}$ . In addition, by the Stone's representation theorem for Boolean algebra, the result becomes : there exists a homeomorphism  $g$  such that  $\Lambda = g\Gamma g^{-1}$ . □

*Remarque A.4.* The proof shows in particular that an isomorphism between dimension ranges of ample groups is always implemented by an homeomorphism :  $\Gamma$  and  $\Lambda$  have isomorphic dimension ranges if and only if there exists a homeomorphism

$$g \text{ such that } \overline{g}: \begin{cases} CO(X)/\Gamma \rightarrow CO(X)/\Lambda \\ Orb_\Gamma(A) \mapsto Orb_\Lambda(gA) \end{cases} \text{ is a bijection.}$$

This theorem (and some improvements we make) plays an important role in our work. It is the center of our approach in Chapter 3, and using its vocabulary, Giordano, Putnam and Skau's theorem 2.3 states that two minimal homeomorphisms are strongly orbit equivalent if and only if their locally finite groups  $\Gamma^\phi$  and  $\Gamma^\psi$  have isomorphic dimension ranges (see Chapter 2).

*Exercice.* Using Lemma 1.7, prove that the construction of Lemma A.3 can be run while controlling the image of  $n$  points belonging to different orbits to obtain the following result when the ample groups act minimally :

*Lemma A.4.* Let  $G$  and  $\overline{G}$  two ample groups with isomorphic dimension ranges acting minimally on  $X$ , and let  $f: CO(X)/G \rightarrow CO(X)/\overline{G}$  be an isomorphism between them. Let also  $(\mathcal{D}, \mathcal{U}) \leq (CO(X), G)$  and  $(\overline{\mathcal{D}}, \overline{\mathcal{U}}) \leq (CO(X), \overline{G})$  be finite unit systems, and  $g: \mathcal{D} \rightarrow \overline{\mathcal{D}}$  be a Boolean algebra isomorphism. Finally, let  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  be two given tuples.

Assume that the following conditions are satisfied :

1.  $g$  respects  $f$
2.  $g$  conjugates  $(\mathcal{D}, \mathcal{U})$  on  $(\overline{\mathcal{D}}, \overline{\mathcal{U}})$
3.  $(\mathcal{D}, \mathcal{U})$  separates the  $x_i$ 's, meaning that the atoms  $A_i$ 's of  $\mathcal{D}$  containing the  $x_i$ 's have pairwise different orbits under  $\mathcal{U}$
4.  $g$  sends  $x_i$  compatibly with  $y_i$  for every  $i \leq n$ , meaning that we have  $y_i \in g(A_i)$  (with  $x_i \in A_i$ ).

Then, for every finite unit system  $(\mathcal{E}, \mathcal{V})$  satisfying Condition 3 (separation of the  $x_i$ 's) such that  $(\mathcal{D}, \mathcal{U}) \leq (\mathcal{E}, \mathcal{V}) \leq (CO(X), G)$ , there exists a finite unit system  $(\overline{\mathcal{E}}, \overline{\mathcal{V}})$  such that  $(\overline{\mathcal{D}}, \overline{\mathcal{U}}) \leq (\overline{\mathcal{E}}, \overline{\mathcal{V}}) \leq (CO(X), \overline{G})$  and a isomorphism  $h: \mathcal{E} \rightarrow \overline{\mathcal{E}}$  extending  $g$  satisfying the following conditions :

1.  $h$  respects  $f$
2.  $h$  conjugates  $(\mathcal{E}, \mathcal{V})$  on  $(\overline{\mathcal{E}}, \overline{\mathcal{V}})$
3.  $h$  sends  $x_i$  compatibly with  $y_i$  for every  $i \leq n$ .

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# Une approche par les groupes amples pour l'équivalence orbitale des actions de $\mathbb{Z}$ sur l'espace de Cantor

**Résumé.** Cette thèse s'inscrit dans le cadre de la dynamique topologique, branche des systèmes dynamiques s'intéressant aux comportements qualitatifs asymptotiques de transformations continues provenant d'une action de groupe ou de semigroupe sur un espace métrique usuellement compact. Souvent, de par leur caractère qualitatif et asymptotique, ces comportements ne dépendent pas précisément du système mais plutôt des *orbites* des points. D'où la notion d'*équivalence orbitale* au cœur de cette thèse, qui consiste à considérer que, après identification des espaces sous-jacents, deux systèmes dont tous les points auraient les mêmes orbites seraient "qualitativement les mêmes".

Ma principale contribution à travers le présent manuscrit consiste à apporter un éclairage et des preuves dynamiques élémentaires aux classifications obtenues par Giordano, Putnam et Skau des systèmes dynamiques minimaux provenant d'actions de  $\mathbb{Z}$  sur l'espace de Cantor à équivalence orbitale forte près (Chapitre 2) et à équivalence orbitale près (Chapitre 3). Chemin faisant, je démontrerai également un résultat de complexité Borélienne, à savoir que la relation d'isomorphisme de groupes dénombrables, localement finis et simples et une relation universelle provenant d'une action Borélienne de  $S_\infty$ , et nous améliorerons un résultat de Krieger sur la conjugaison des groupes amples.

**Mots-clés :** Équivalence Orbitale, Espace de Cantor, Groupes amples, Partitions de Kakutani-Rokhlin, Complexité Borélienne de la relation d'isomorphisme de groupes localement finis.

**Abstract.** This thesis takes place in the context of topological dynamics, a branch of dynamical systems concerned with the asymptotic qualitative behavior of continuous transformations arising from a group or semigroup action on a usually compact metric space. Often, because of their qualitative and asymptotic nature, these behaviors do not depend precisely on the system but rather on the *orbits* of the points. Hence the notion of orbit equivalence at the heart of this thesis, which consists in considering that, after identification of the underlying spaces, two systems whose points all have the same orbits would be "qualitatively the same".

My main contribution in the present manuscript is to bring an elementary perspective and dynamical proofs to the classifications obtained by Giordano, Putnam and Skau of dynamical systems arising from  $\mathbb{Z}$  actions on Cantor space up to strong orbit equivalence (Chapter 2) and orbit equivalence (Chapter 3). Along the way, I will also prove a result of Borelian complexity, namely that the isomorphism relation of countable, locally finite and simple groups and a universal relation arising from a Borelian action of  $S_\infty$ , and improve a result of Krieger about the conjugation of ample groups.

**Keywords:** Orbit equivalence, Cantor space, ample groups, Kakutani-Rokhlin partitions, Borelian complexity of the isomorphism relation between locally finite groups.



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