Some minimiztion problems related to the Schrödinger equations

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Quelques problèmes de minimisation en lien avec les équations de Schrödinger non-linéaires

Anthony MUR

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Chapitre 1

Introduction

Dans l'étude des équations aux dérivées partielles issues de la physique, les solutions qui minimisent ou maximisent certaines quantités - comme l'énergie, le moment ou la masse - jouent un rôle primordial dans la dynamique des systèmes considérés.

Cette thèse est dédiée à l'étude de deux problèmes de minimisation / maximisation dans l'espace euclidien. Dans les deux cas, les suites minimisantes ou maximisantes manquent de compacité à cause de l'invariance des problèmes par des translations et/ou des dilatations. Pour pallier à ce défaut, la mise en oeuvre d'outils adaptés a été nécessaire (décomposition en profils dans le premier cas, concentration-compacité dans le second).

Dans une première partie on s'intéresse aux inégalités de Strichartz qui sont des outils essentiels dans l'étude des équations dispersives, comme l'équation de Schrödinger. Elles interviennent dans l'étude du problème de Cauchy, du scattering, du compotement qualitatif des solutions ou encore de l'explosion. On considère l'équation de Schrödinger avec Laplacien fractionnaire dans $\mathbf{R} \times \mathbf{R}^N$:

$$i\partial_t U + (-\Delta)^\sigma U = 0, \tag{1.0.1}$$

où $(-\Delta)^\sigma = \mathcal{F}^{-1}\left(|\xi|^{2\sigma}\mathcal{F}u\right)$ et $\mathcal{F}$ ou $\widehat{\ }$ désignent la transformée de Fourier dans $\mathbf{R}^N$. Pour toute donnée initiale u définie sur $\mathbf{R}^N$, on note par $U(t,x) = [S(t)u](x)$ la solution de (1.0.1) qui vérifie la condition initiale $U(0,x) = u(x)$ et par $(S(\cdot))_{t\in\mathbf{R}}$ le groupe de Schrödinger associé au Laplacien fractionnaire $(-\Delta)^\sigma$. Plus précisément, on a $S(t)u = \mathcal{F}^{-1}\left(e^{it|\xi|^{2\sigma}}\widehat{u}\right)$. Pour toute distribution tempérée u on définit $|\nabla|^\gamma u$ par $|\nabla|^\gamma u = \mathcal{F}^{-1}\left(|\cdot|^\gamma \widehat{u}\right)$. On considère les espaces fonctionnels suivants :

- $\dot{H}^\alpha_q(\mathbf{R}^N)$ est le complété de $\mathcal{C}^\infty_c(\mathbf{R}^N)$ pour la norme

$$\|u\|_{\dot{H}^\alpha_q} := \||\nabla|^\alpha u\|_{L^q(\mathbf{R}^N)}.$$

Bien évidemment, on a $\dot{H}^0_q = L^q(\mathbf{R}^N)$. Remarquons que $\dot{H}^\alpha_2 = \dot{H}^\alpha$ est un espace de Hilbert et la formule de Plancherel donne

$$\|u\|^2_{\dot{H}^\alpha} = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} |\xi|^{2\alpha}|\hat{u}(\xi)|^2 d\xi.$$

• $L_t^p \dot{H}_q^\alpha := L^p(\mathbf{R}, \dot{H}_q^\alpha(\mathbf{R}^N))$, muni de la *norme de Strichartz*

$$\|U\|_{L_t^p \dot{H}_q^\alpha} := \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} ||\nabla|^\alpha U(t,x)|^q \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \quad \text{lorsque} \quad p, q \in [1, \infty),$$

respectivement

$$\|U\|_{L_t^\infty \dot{H}_q^\alpha} := \sup_{t \in \mathbf{R}} \left(\int_{\mathbf{R}^N} ||\nabla|^\alpha U(t,x)|^q dx \right)^{\frac{1}{q}} \quad \text{pour} \quad q \in [1, \infty).$$

Remarquons le comportement des normes par rapport aux dilatations : en posant $u_\tau(x) = u\left(\frac{x}{\tau}\right)$, on a

$$\|u_\tau\|_{\dot{H}_q^\alpha} = \tau^{\frac{N}{q} - \alpha} \|u\|_{\dot{H}_q^\alpha} \qquad \text{et} \qquad S(t) u_\tau = \left[S\left(\frac{t}{\tau^{2\sigma}} \right) u \right]_\tau \qquad \text{pour tout } \tau > 0.$$

Lorsque U est une fonction definie sur $\mathbf{R} \times \mathbf{R}^N$ et $U_{a,b}(t,x) = U\left(\frac{t}{a}, \frac{x}{b}\right)$, on a

$$\|U_{a,b}(\cdot + t_0, \cdot + x_0)\|_{L_t^p \dot{H}_q^\alpha} = a^{\frac{1}{p}} b^{\frac{N}{q} - \alpha} \|U\|_{L_t^p \dot{H}_q^\alpha}.$$

En particulier, quelque soient $\lambda \in (0, \infty)$, $t_0 \in \mathbf{R}$ et $x_0 \in \mathbf{R}^N$ on a

$$\left\| \lambda^{\gamma - \frac{N}{2^*}} U\left(\frac{\cdot + t_0}{\lambda^{2\sigma}}, \frac{\cdot + x_0}{\lambda} \right) \right\|_{L_t^p \dot{H}_q^\gamma} = \|U\|_{L_t^p \dot{H}_q^\gamma} \qquad \text{si } \frac{2\sigma}{p} + \frac{N}{q} = \frac{N}{2} - s, \quad \text{et} \tag{1.0.2}$$

$$\left\| \lambda^{\gamma - \frac{N}{2^*}} u\left(\frac{\cdot + x}{\lambda} \right) \right\|_{\dot{H}^{s+\gamma}} = \|u\|_{\dot{H}^{s+\gamma}}. \tag{1.0.3}$$

Les inégalités de Strichartz donnent des estimations de la solution $S(\cdot)u$ de (1.0.1) avec donnée initiale u en termes de certaines normes de u :

$$\|S(\cdot)u\|_{L_t^p \dot{H}_q^\gamma} \leq C_S \|u\|_{\dot{H}^{s+\gamma}}. \tag{1.0.4}$$

Compte tenu du comportement des normes par changement d'échelle, l'inégalité (1.0.4) ne peut avoir lieu que si la relation $\frac{2\sigma}{p} + \frac{N}{q} = \frac{N}{2} - s$ est vérifiée. On suppose que $s \in (0, \frac{N}{2})$, de sorte que l'espace $\dot{H}^s(\mathbf{R}^N)$ s'injecte continûment dans $L^{2^*}(\mathbf{R}^N)$, où $2^* = \frac{2N}{N-2s}$. Pour simplifier la notation, on pose $\mathfrak{d}_\lambda u := \lambda^{\gamma - \frac{N}{2^*}} u(\frac{\cdot}{\lambda})$. On a alors

$$\mathfrak{d}_\lambda \left[S(t)u \right] = S(\lambda^{2\sigma} t) \mathfrak{d}_\lambda u. \tag{1.0.5}$$

Pour s fixé, les exposants (p, q) sont *admissibles* si $p \in [2, \infty]$, $q \in [2, \infty)$, et

$$\frac{2\sigma}{p} + \frac{N}{q} = \frac{N}{2} - s = \frac{N}{2^*}. \tag{1.0.6}$$

On a alors $s = \frac{N}{2} - \frac{N}{q} - \frac{2\sigma}{p}$. Si la paire (p,q) est admissible, on doit avoir $p \in [2,\infty]$ et $q \in \left[2^*, \frac{2N}{N-2(s+\sigma)}\right]$. Les paires admissibles *extremales* sont $(2, \frac{2N}{N-2(s+\sigma)})$ et $(\infty, 2^*)$. Toutes les autres sont *non-extremales*. Il existe une seule paire admissible avec $p = q$. On la note $(\overline{q}, \overline{q})$ et on voit facilement que $\overline{q} = \frac{2(N+2\sigma)}{N-2s} = 2^* + \frac{4\sigma}{N-2s}$. On observe que $L_t^{\overline{q}} L_x^{\overline{q}} = L^{\overline{q}}(\mathbf{R}^{N+1})$; cette caractérisation sera très utile par la suite. Dans une longue série de travaux par des nombreux auteurs (commencée par R. Strichartz en 1979, et jusqu'au résultats récents de V. D. Dinh en 2017), les inégalités de Strichartz (1.0.4) ont été demontrées pour toutes les paires admisibles.

L'application $u \longmapsto S(\cdot)u$ de $\dot{H}^{s+\gamma}$ dans $L_t^p \dot{H}_q^\gamma$ est continue, mais n'est pas compacte à cause des invariances par translations et par changement d'échelle. Nous montrons que ces invariances sont les seuls obstacles à la compacité. Plus précisément, nous démontrons le résultat suivant de *décomposition en profils :*

Theorem 1.0.1. *Soit $N \in \mathbf{N}^*$ et $0 \leqslant s + \gamma < \frac{N}{2}$. Soit $\mathbf{u} = (u_n)_{n\geq 1}$ une suite bornée de $\dot{H}^{s+\gamma}$. Il existe une application strictement croissante $j : \mathbf{N}^* \to \mathbf{N}^*$, des fonctions $V^i \in \dot{H}^{s+\gamma}$, et pour tout $i \in \mathbf{N}^*$ il existe des suites $\boldsymbol{\lambda}^i = (\lambda_n^i)_{n\in\mathbf{N}^*} \subset (0,\infty)$, $\mathbf{t}^i = (t_n^i)_{n\in\mathbf{N}^*} \subset \mathbf{R}$, $\mathbf{x}^i = (x_n^i)_{n\in\mathbf{N}^*} \subset \mathbf{R}^N$ telles que :*

a) Si $i \neq \ell$, les suites $\left(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i\right)$ et $\left(\boldsymbol{\lambda}^\ell, \mathbf{t}^\ell, \mathbf{x}^\ell\right)$ sont "orthogonales" dans le sens suivant :

$$\lim_{n\to\infty} \frac{\lambda_n^i}{\lambda_n^\ell} + \frac{\lambda_n^\ell}{\lambda_n^i} + \frac{|t_n^i - t_n^\ell|}{(\lambda_n^i)^{2\sigma}} + \frac{|x_n^i - x_n^\ell|}{\lambda_n^i} = +\infty.$$

b) Pour tout $n \in \mathbf{N}^*$ et $k \in \mathbf{N}^*$ on a

$$u_{j(n)} = \sum_{i=1}^{k} S(-t_n^i)\left[(\lambda_n^i)^{\gamma - \frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right] + w_n^k, \qquad \text{et} \tag{1.0.7}$$

$$\lim_{k\to\infty}\left(\limsup_{n\to\infty} \|S(\cdot)w_n^k\|_{L_t^\infty \dot{H}_{2^*}^\gamma}\right) = 0. \tag{1.0.8}$$

c) Les conditions de "presque orthogonalité" suivantes sont vérifiées pour tout $k \in \mathbf{N}^*$:

$$\lim_{n\to\infty} \|S(\cdot)u_{j(n)}\|_{L_t^{\overline{q}}\dot{H}_{\overline{q}}^\gamma}^{\overline{q}} = \sum_{i=1}^{\infty} \|S(\cdot)V^i\|_{L_t^{\overline{q}}\dot{H}_{\overline{q}}^\gamma}^{\overline{q}}, \tag{1.0.9}$$

$$\|u_{j(n)}\|_{\dot{H}^{s+\gamma}}^2 = \sum_{i=1}^{k} \|V^i\|_{\dot{H}^{s+\gamma}}^2 + \|w_n^k\|_{\dot{H}^{s+\gamma}}^2 + \underset{n\to\infty}{o(1)}, \qquad \text{et} \tag{1.0.10}$$

$$\limsup_{k\to\infty}\left(\limsup_{n\to\infty} \|w_n^k\|_{\dot{H}^{s+\gamma}}\right) \leqslant \limsup_{n\to\infty} \|u_{j(n)}\|_{\dot{H}^{s+\gamma}}. \tag{1.0.11}$$

Pour toute paire admissible (p,q) on a

$$\lim_{n\to\infty}\|S(\cdot)u_{j(n)}\|^q_{L^p_t\dot H^\gamma_q}\leqslant\sum_{i=1}^k\|S(\cdot)V^i\|^q_{L^p_t\dot H^\gamma_q}+\underset{k\to\infty}{o(1)}\qquad \textit{lorsque } p>q,\qquad \textit{respectivement}$$

$$\lim_{n\to\infty}\|S(\cdot)u_{j(n)}\|^p_{L^p_t\dot H^\gamma_q}\leqslant\sum_{i=1}^k\|S(\cdot)V^i\|^p_{L^p_t\dot H^\gamma_q}+\underset{k\to\infty}{o(1)}\qquad \textit{lorsque } p<q.$$

Il existe plusieurs résultats de décomposition en profils dans la littérature, à commencer par celui de Patrick Gérard en 1998 dans les espaces $\dot H^s$, suivi de S. Keraani (2001). Le Théorème 1.0.1 ci-dessus atteint une généralité jamais obtenue auparavant et fait appel à peu d'outils techniques. L'ingrédient principal de la démonstration est le procédé d'extraction diagonale de Cantor. On fait très peu appel à des inégalités de Sobolev améliorées.

En utilisant le Théorème 1.0.1 on obtient le résultat suivant.

Theorem 1.0.2. *On suppose que $0\leqslant s+\gamma<\frac{N}{2}$. Soit (p,q) une paire admissible non-extremale. Il existe des maximiseurs pour l'inégalité (1.0.4), c'est à dire*

$$\sup_{u\in\dot H^{s+\gamma},u\neq0}\frac{\|S(\cdot)u\|_{L^p_t\dot H^\gamma_q}}{\|u\|_{\dot H^{s+\gamma}}}$$

est atteint.

Dans la seconde partie de la thèse on s'intéresse aux ondes progressives pour l'équation de Schrödinger

$$i\frac{\partial\Phi}{\partial t}+\Delta\Phi+F(|\Phi|^2)\Phi=0\qquad\text{dans }\mathbf{R}^2\times\mathbf{R}. \tag{1.0.12}$$

Pour $\Lambda>0$ fixé, on cherche des solutions $\Phi:\mathbf{R}^2\times\mathbf{R}\longrightarrow\mathbf{C}$ de (3.1.1) $\Lambda-$périodiques par rapport à la seconde variable, $\Phi(x,y+\Lambda,t)=\Phi(x,y,t)$, qui satisfont la "condition aux limites" $|\Phi(x,y,t)|\longrightarrow1$ lorsque $x\longrightarrow\pm\infty$. On suppose que la nonlinéarité F satisfait $F(1)=0$ et $F'(1)=-1$. Le cas modèle est celui de l'équation de Gross-Pitaevskii, où $F(s)=1-s$.

On préfère normaliser la période Λ à 1 en posant $\tilde\Phi(x,y,t)=\Phi(x,\Lambda y,t)$. Il est évident que Φ est solution de (1.0.12) si et seulement si $\tilde\Phi$ satisfait l'équation

$$i\frac{\partial\tilde\Phi}{\partial t}+\frac{\partial^2\tilde\Phi}{\partial x^2}+\frac{1}{\Lambda^2}\frac{\partial^2\tilde\Phi}{\partial y^2}+F(|\tilde\Phi|^2)\tilde\Phi=0\qquad\text{dans }\mathbf{R}^2. \tag{1.0.13}$$

On note $\lambda=\frac{1}{\Lambda}$ et on considère "l'énergie renormalisée"

$$E_\lambda(\Psi)=\int_{\mathbf{R}\times[0,1]}\Big|\frac{\partial\Psi}{\partial x}\Big|^2+\lambda^2\Big|\frac{\partial\Psi}{\partial y}\Big|^2+V(|\Psi|^2)\,dx\,dy, \tag{1.0.14}$$

où $V(s) = \int_s^1 F(\tau)\,d\tau$. Les ondes progressives de (1.0.13) sont des solutions spéciales de la forme $\tilde{\Phi}(x,y,t) = \psi(x+ct,y)$. Le profil ψ est $1-$périodique par rapport à y et satisfait l'équation

$$ic\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2} + \frac{1}{\Lambda^2}\frac{\partial^2\psi}{\partial y^2} + F(|\psi|^2)\psi = 0 \qquad \text{dans } \mathbf{R}^2. \tag{1.0.15}$$

L'équation (1.0.13) admet deux quantités conservées : l'énergie E_λ ci-dessus, et le moment. Formellement le moment devrait être $Q(\psi) = \int_{\mathbf{R}\times(0,1)} \langle i\frac{\partial\psi}{\partial x}, \psi\rangle\,dx\,dy$. Si ψ est une fonction telle que $E_\lambda(\psi) < \infty$, la fonction $\langle i\frac{\partial\psi}{\partial x}, \psi\rangle$ n'est pas nécessairement intégrable et définir correctement le moment est une difficulté en soi. Nous proposons une définition adéquate dans la section 3.3. À cause des oscillations que les fonctions d'énergie finie peuvent avoir à $\pm\infty$, le moment est une fonctionnelle qui prend des valeurs dans $\mathbf{R}/2\pi\mathbf{Z}$.

On étudie d'abord les ondes progressives unidimensionnelles de (1.0.13). Ce sont les ondes progressives qui ne dépendent pas de la variable y. Leur profil ne dépend que de la variable x et satisfait une équation différentielle ordinaire qu'on peut résoudre. On trouve ainsi toutes les ondes progressives d'énergie finie. On résout ensuite le problème qui consiste à minimiser l'énergie à moment constant. Les minimiseurs sont des ondes progressives, et leurs vitesses c sont précisément les multiplicateurs de Lagrange associés au problème de minimisation. Dans le cas modèle de la non-linéarité de Gross-Pitaevskii $F(s) = 1 - s$, il était connu que toutes les ondes progressives d'énergie finie étaient aussi des minimiseurs. Nous construisons des exemples de non-linéarités lisses pour lesquelles il existe des ondes progressives qui ne sont pas des minimiseurs de l'énergie à moment constant.

En dimension deux, on considère l'espace

$$\begin{aligned}\mathcal{E} \;=\; & \{\psi \in L^1_{loc}(\mathbf{R}^2) \;\mid\; \psi \text{ est } 1-\text{périodique par rapport à la seconde variable et} \\ & \nabla\psi \in L^2(\mathbf{R}\times[0,1]) \text{ et } V(|\psi|^2) \in L^1(\mathbf{R}\times[0,1])\}.\end{aligned}$$

On note par $Q(\psi)$ le moment de la fonction ψ et on définit

$$E_{\lambda,min}(p) = \inf\{E_\lambda(\psi) \;\mid\; \psi \in \mathcal{E} \text{ et } Q(\psi) = p\}.$$

ainsi que

$$E^1_{min}(p) = \inf\{E_\lambda(\psi) \;\mid\; \psi \in \mathcal{E}, \psi \text{ ne dépend pas de } y \text{ et } P(\psi) = p\}.$$

On suppose que $V > 0$ sur $[0,\infty)\setminus\{1\}$. Si V prend des valeurs négatives, on peut montrer que $E_{\lambda,min}(p) = -\infty$ pour tout p.

Nos principaux résultats peuvent être résumés ainsi :

Theorem 1.0.1. *Sous des conditions générales sur la non-linéarité F, on a :*

i) La fonction E^1_{min} est non-négative, paire, $2\pi-$périodique et concave sur $[0,2\pi]$, $E^1_{min}(p) \leqslant \sqrt{2}p$ et la dérivée à droite de E^1_{min} à l'origine est $\sqrt{2}$. Si, de plus, F est C^2 dans un voisinage de 1 et $F''(1) < \frac{9}{4}$ on a $E^1_{min}(p) < \sqrt{2}|p|$ pour tout $p \neq 0$.

Pour tout $p \in (0,\pi]$ tel que $E^1_{min}(p) < \sqrt{2}p$ il existe des minimiseurs pour $E^1_{min}(p)$ dans $\mathcal{E}(\mathbf{R})$ et toutes les suites minimisantes sont pré-compactes.

ii) Pour tout $\lambda > 0$ fixé, la fonction $p \longmapsto E_{\lambda,min}(p)$ est $2\pi-$périodique, concave sur $[0,2\pi]$, $E_{\lambda,min}(p) \leqslant \sqrt{2}p$, la dérivée à droite de $E_{\lambda,min}$ à l'origine est $\sqrt{2}$, et $E_{\lambda,min}(p) \leqslant E^1_{min}(p)$.

iii) Pour tout p fixé il existe $\lambda_*(p) > 0$ tel que l'application $\lambda \longmapsto E_{\lambda,min}(p)$ est strictement croissante sur $(0,\lambda_*(p)]$ et $E_{\lambda,min}(p) = E^1_{min}(p)$ pour tout $\lambda \geqslant \lambda_*(p)$.

iv) Pour tout $\lambda > 0$ et tout $p \in (0,\pi]$ qui satisfait $E_{\lambda,min}(p) < \sqrt{2}p$ il existe des minimiseurs pour $E_{\lambda,min}(p)$ dans $\mathcal{E}$ et toutes les suites minimisantes sont pré-compactes modulo des translations.

Les minimiseurs donnés par le théorème ci-dessus sont des fonctions lisses et sont des ondes progressives ayant des vitesses c comprises entre les dérivées à droite et à gauche de la fonction $E_{\lambda,min}$.

Lorsque λ est grand (ce qui correspond à des périodes $\Lambda = \frac{1}{\lambda}$ petites), la dépendance d'une fonction ψ par rapport à la variable y est fortement pénalisée par le terme $\lambda^2\Big|\dfrac{\partial\psi}{\partial y}\Big|^2$ qui apparaît dans l'énergie $E_\lambda(\psi)$. C'est la raison pour laquelle les minimiseurs de $E_{\lambda,min}(p)$ ne dépendent que de x. En revanche, lorsque λ est petit on peut exploiter la concavité stricte de $E^1_{\lambda,min}$ dans un voisinage de p pour construire des fonctions $\psi \in \mathcal{E}$ qui ont un moment égal à p et une énergie strictement inférieure à $E^1_{\lambda,min}(p)$. Cela signifie que pour p fixé et λ suffisamment petit, les minimiseurs de $E_{\lambda,min}(p)$ dépendent effectivement des deux variables x et y.

Chapitre 2

Profile decomposition and maximizers for the Sobolev-Strichartz inequalities for Schrödinger's equation

2.1 Introduction

We consider the linear fractional Schrödinger equation

$$i\partial_t U + (-\Delta)^\sigma U = 0 \text{ in } \mathbf{R}^N, \tag{2.1.1}$$

where $(-\Delta)^\sigma = \mathcal{F}^{-1}\left(|\xi|^{2\sigma}\mathcal{F}u\right)$ and $\mathcal{F}$ (or $\widehat{\ }$) is the classical Fourier transform,

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbf{R}^N} e^{ix\cdot\xi}u(x)dx.$$

For a given complex-valued function u defined on $\mathbf{R}^N$, the solution U of (2.1.1) satisfying $U(0,x) = u(x)$ is denoted by

$$U(t,x) = \left[S(t)u\right](x).$$

Here $(S(\cdot))_{t\in\mathbf{R}}$ is the Schrödinger group associated to the fractionnal Laplacian $(-\Delta)^\sigma$. We recall that $S(t)u = \mathcal{F}^{-1}\left(e^{it|\xi|^{2\sigma}}\widehat{u}\right)$, and S verifies $S(0) = Id$ and $S(t_1+t_2) = S(t_1)S(t_2)$ for every $t_1, t_2 \in \mathbf{R}$.

We denote by $\dot{H}^\alpha_q(\mathbf{R}^N)$, or shortly $\dot{H}^\alpha_q$, the homogeneous Sobolev space which is the completion of $\mathcal{C}^\infty_c(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_{\dot{H}^\alpha_q} := \||\nabla|^\alpha u\|_{L^q(\mathbf{R}^N)},$$

where $|\nabla|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is the operator defined by $|\nabla|^\alpha u = \mathcal{F}^{-1}\left(|\cdot|^\alpha\widehat{u}\right)$. Obviously, we have $\dot{H}^0_q = L^q$. In the case $q = 2$, $\dot{H}^\alpha_2 = \dot{H}^\alpha$ is a Hilbert space and by Plancherel's formula we have

$$\|u\|_{\dot{H}^\alpha} = \frac{1}{(2\pi)^{\frac{N}{2}}} \left(\int_{\mathbf{R}^N} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Let us recall that the $\dot{H}_2^\alpha$-norm is invariant by the Schrödinger group associated to (2.1.1)

$$\|S(t)u\|_{\dot{H}_2^\alpha} = \|u\|_{\dot{H}_2^\alpha}, \ \forall t \in \mathbf{R}. \tag{2.1.2}$$

By the Sobolev embedding theorem (see, e.g., [19], Theorem 8.1 p. 301) for any $s \in \left(0, \frac{N}{2}\right)$ the space $\dot{H}^s(\mathbf{R}^N)$ is continuously embedded into $L^{2^*}(\mathbf{R}^N)$, where $2^* := \frac{2N}{N-2s}$.

We consider the space $L_t^p \dot{H}_q^\alpha := L^p(\mathbf{R}, \dot{H}_q^\alpha(\mathbf{R}^N))$, endowed with the *Strichartz norm*

$$\|U\|_{L_t^p \dot{H}_q^\alpha} := \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} ||\nabla|^\alpha U(t,x)|^q \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \text{ if } p, q \in [1, \infty),$$

respectively

$$\|U\|_{L_t^\infty \dot{H}_q^\alpha} := \sup_{t \in \mathbf{R}} \left(\int_{\mathbf{R}^N} ||\nabla|^\alpha U(t,x)|^q dx \right)^{\frac{1}{q}} \text{ if } q \in [1, \infty).$$

The following scaling properties hold. If u is a function defined on $\mathbf{R}^N$ and $u_\tau(x) = u\left(\frac{x}{\tau}\right)$, we have

$$\|u_\tau\|_{\dot{H}_q^\alpha} = \tau^{\frac{N}{q} - \alpha} \|u\|_{\dot{H}_q^\alpha} \qquad \text{and} \qquad S(t)u_\tau = \left[S\left(\frac{t}{\tau^{2\sigma}} \right) u \right]_\tau \qquad \text{for any } \tau > 0.$$

If U is a function defined on $\mathbf{R} \times \mathbf{R}^N$ and $U_{a,b}(t,x) = U\left(\frac{t}{a}, \frac{x}{b}\right)$, then for all a, $b > 0$ we have

$$\|U_{a,b}\|_{L_t^p \dot{H}_q^\alpha} = a^{\frac{1}{p}} b^{\frac{N}{q} - \alpha} \|U\|_{L_t^p \dot{H}_q^\alpha}.$$

The $L_t^p \dot{H}_q^\gamma$ and the $\dot{H}^{s+\gamma}$ norms are invariant under translation in time and in space and appropriate scaling : for all $\lambda \in (0, \infty)$, $t_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^N$ we have

$$\left\| \lambda^{\gamma - \frac{N}{2^*}} U\left(\frac{\cdot + t_0}{\lambda^{2\sigma}}, \frac{\cdot + x_0}{\lambda} \right) \right\|_{L_t^p \dot{H}_q^\gamma} = \|U\|_{L_t^p \dot{H}_q^\gamma} \qquad \text{provided that } \frac{2\sigma}{p} + \frac{N}{q} = \frac{N}{2} - s, \tag{2.1.3}$$

and

$$\left\| \lambda^{\gamma - \frac{N}{2^*}} u\left(\frac{\cdot + x}{\lambda} \right) \right\|_{\dot{H}^{s+\gamma}} = \|u\|_{\dot{H}^{s+\gamma}}. \tag{2.1.4}$$

We define $\mathfrak{d}_\lambda u := \lambda^{\gamma - \frac{N}{2^*}} u(\frac{\cdot}{\lambda})$. Because of the aforementioned invariances of the Schrödinger equation, we have the following relationship between scale change in space and scale change in time :

$$\mathfrak{d}_\lambda \left[S(t)u \right] = S(\lambda^{2\sigma} t) \mathfrak{d}_\lambda u. \tag{2.1.5}$$

A pair (p,q) is called *admissible* if $p \in [2,\infty]$, $q \in [2,\infty)$, and $\frac{2}{p} + \frac{N}{q} \leq \frac{N}{2}$. If (p,q) is an admissible pair and σ is as in (2.1.1), we define

$$s = \frac{N}{2} - \frac{N}{q} - \frac{2\sigma}{p}. \tag{2.1.6}$$

Notice that $s > 0$ whenever $0 < \sigma < 1$. If $0 < s < \frac{N}{2}$, (2.1.6) can be written as

$$\frac{2\sigma}{p} + \frac{N}{q} = \frac{N}{2} - s = \frac{N}{2^*}, \tag{2.1.7}$$

where $2^* = \frac{2N}{N-2s}$ is the *Sobolev exponent* so that we have the embedding $\dot{H}^s(\mathbf{R}^N) \subset L^{2^*}(\mathbf{R}^N)$.

The following *Strichartz inequalities* hold : if (p,q) is an admissible pair and s is given by (2.1.6), then for any $\gamma \in \mathbf{R}$, there exist $C_S > 0$ depending only on γ, N, p, q, σ such that

$$\|S(\cdot)u\|_{L^p_t \dot{H}^\gamma_q} \leq C_S \|u\|_{\dot{H}^{s+\gamma}}. \tag{2.1.8}$$

Using scaling properties it is easily seen that (p,q) must be an admissible pair for (2.1.8) to hold.

Establishing (2.1.8) for all Schrödinger admissible pairs (as in (2.1.7)) has been a long story. When $(p,q) = (\infty, 2)$, the estimate (2.1.8) follows immediately from (2.1.2). For the classical Schrödinger equation in $L^2(\mathbf{R}^N)$ (that is, in the case $\sigma = 1$ and $s = \gamma = 0$), the admissibility condition (2.1.7) becomes $\frac{2}{p} + \frac{N}{q} = \frac{N}{2}$. In this case the unique admissible pair with $p = q$ is $\left(2 + \frac{4}{N}, 2 + \frac{4}{N}\right)$, and inequality (2.1.8) with exponents $p = q = 2 + \frac{4}{N}$ was proved by R. Strichartz in [21]. Inequality (2.1.8) has been established by J. Ginibre and G. Velo in [10] for all admissible pairs (p,q) with $p > 2$. If $N \geq 3$, the pair $(2, \frac{2N}{N-2})$ is admissible ; the delicate *endpoint estimate* in the case $(p,q) = (2, \frac{2N}{N-2})$ has been proven by M. Keel and T. Tao in [12]. We refer to Theorem 2.3.3 p. 33 in [5] for an elementary proof (inspired from [10]) of non-endpoint Strichartz estimates and for further comments. The case of the fractional Laplacian (that is, $\sigma \neq 1$, $\sigma \neq \frac{1}{2}$) follows from the general Strichartz-type estimates obtained by Y. Cho, T. Ozawa and S. Xia in [6]. For the the inequality (2.1.8) above, see Corollary 14 p. 496 in the recent work [7] by Van Duong Dinh. We also refer to [6] and [7] for further references. In the case $\sigma = \frac{1}{2}$, equation (2.1.1) becomes the half-wave equation and the Strichartz estimates are somehow different (see [7]).

In the sequel we will always assume that $s \in (0, \frac{N}{2})$ and p, q, s satisfy (2.1.6). Then :

(i) For any $s \in (0, \frac{N}{2})$ there exists a unique admissible pair of the form $(\overline{q}, \overline{q})$ satisfying (2.1.6) and $\overline{q}$ is given by $\overline{q} = \frac{2(N+2\sigma)}{N-2s} = 2^* + \frac{4\sigma}{N-2s}$.

(ii) If (p,q) is admissible and (2.1.7) holds, then $p \in [2,\infty]$ and $q \in \left[2^*, \frac{2N}{N-2(s+\sigma)}\right]$.

(iii) If (p,q) is admissible and $q < \overline{q}$ we have necessarily $p > \overline{q} > q$; if (p,q) is admissible and $q > \overline{q}$, then $p < \overline{q} < q$.

Let $q_{max} = \frac{2N}{N-2(s+\sigma)}$, so that for any admissible pair (p,q) we have $q \in [2^*, q_{max}]$. If $q \in (2^*, q_{max})$ then (p,q) is called a *non-extremal* admissible pair. The pair $(p,q) = \left(2, \frac{2N}{N-2(s+\sigma)}\right)$ is called *endpoint* admissible pair. If $q \neq \frac{2N}{N-2(s+\sigma)}$ then we say that (p,q) is a *non-endpoint* admissible pair.

Assume that (p_1, q_1), (p_2, q_2) and (p, q) are admissible pairs and $\frac{1}{q} = \frac{\kappa}{q_1} + \frac{1-\kappa}{q_2}$ for some $a \in (0, 1)$. By (2.1.7) we have $\frac{1}{p} = \frac{\kappa}{p_1} + \frac{1-\kappa}{p_2}$ and then using Hölder's inequality we get

$$\|u\|_{\dot{H}^\gamma_q} \leq \|u\|^{\kappa}_{\dot{H}^\gamma_{q_1}} \|u\|^{1-\kappa}_{\dot{H}^\gamma_{q_2}} \qquad \text{for any } u \in \dot{H}^\gamma_{q_1} \cap \dot{H}^\gamma_{q_2}, \tag{2.1.9}$$

$$\|U\|_{L^p_t \dot{H}^\gamma_q} \leq \|U\|^{\kappa}_{L^{p_1}_t \dot{H}^\gamma_{q_1}} \|U\|^{1-\kappa}_{L^{p_2}_t \dot{H}^\gamma_{q_2}} \qquad \text{for any } U \in L^{p_1}_t \dot{H}^\gamma_{q_1} \cap L^{p_2}_t \dot{H}^\gamma_{q_2}. \tag{2.1.10}$$

Using (2.1.10) we obtain the following interpolation inequalities for non-extremal admissible pairs (p, q) :

$$\|S(\cdot)u\|_{L^{\overline{q}}_t \dot{H}^\gamma_{\overline{q}}} \leqslant \|S(\cdot)u\|^{1-p/\overline{q}}_{L^\infty_t \dot{H}^\gamma_{2^*}} \|S(\cdot)u\|^{p/\overline{q}}_{L^p_t \dot{H}^\gamma_q} \text{ if } p < q, \tag{2.1.11}$$

$$\|S(\cdot)u\|_{L^p_t \dot{H}^\gamma_q} \leqslant \|S(\cdot)u\|^{1-\overline{q}/p}_{L^\infty_t \dot{H}^\gamma_{2^*}} \|S(\cdot)u\|^{\overline{q}/p}_{L^{\overline{q}}_t \dot{H}^\gamma_{\overline{q}}} \text{ if } p > q, \tag{2.1.12}$$

$$\|S(\cdot)u\|_{L^p_t \dot{H}^\gamma_q} \leqslant \|S(\cdot)u\|^{\kappa}_{L^2_t \dot{H}^\gamma_{q_{max}}} \|S(\cdot)u\|^{1-\kappa}_{L^{\overline{q}}_t \dot{H}^\gamma_{\overline{q}}} \text{ if } p < q, \qquad \text{where } \kappa = \frac{\frac{1}{\overline{q}} - \frac{1}{q}}{\frac{1}{\overline{q}} - \frac{1}{q_{max}}} = \frac{\frac{1}{p} - \frac{1}{\overline{q}}}{\frac{1}{2} - \frac{1}{\overline{q}}}. \tag{2.1.13}$$

In this article we prove compactness of maximizing sequences for the Strichartz inequality (2.1.8) for fractional Schrödinger equations and for any non-extremal admissible pair (p, q). To do this we first establish a linear profile decomposition result (Theorem 3.1.1 below) in which "almost orthogonality" is characterized using the $L^{\overline{q}}_t \dot{H}^\gamma_{\overline{q}}$-norm. We believe that this new profile decomposition result is of independent interest.

We will prove that it is possible to extract a suitable profile decomposition from every bounded sequence $\mathbf{u} = (u_n)_{n\in\mathbf{N}} \subset \dot{H}^{s+\gamma}$. To do this, we need to take into account the weak limits of $\mathbf{u}$ up to translations and scaling invariances. We need the following

Definition 2.1.1. *We say that two sequences* $(\boldsymbol{\lambda}, \mathbf{t}, \mathbf{x}) = (\lambda_n, t_n, x_n)_{n\geqslant 1}$ *and* $(\tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{t}}, \tilde{\mathbf{x}}) = (\tilde{\lambda}_n, \tilde{t}_n, \tilde{x}_n)_{n\geqslant 1}$ *in* $(0, \infty) \times \mathbf{R} \times \mathbf{R}^N$ *are orthogonal if :*

$$\lim_{n\to\infty} \frac{\lambda_n}{\tilde{\lambda}_n} + \frac{\tilde{\lambda}_n}{\lambda_n} + \frac{|t_n - \tilde{t}_n|}{(\lambda_n)^{2\sigma}} + \frac{|x_n - \tilde{x}_n|}{\lambda_n} = +\infty. \tag{2.1.14}$$

He have the following profile decomposition result for bounded sequences in $\dot{H}^{s+\gamma}$.

Theorem 2.1.1. *Let* $N \in \mathbf{N}^*$ *and* $0 \leqslant s + \gamma < \frac{N}{2}$. *Let* $\mathbf{u} = (u_n)_{n\geqslant 1}$ *be a bounded sequence in* $\dot{H}^{s+\gamma}$. *Then there exists an increasing mapping* $j : \mathbf{N}^* \to \mathbf{N}^*$, *there exist functions* $V^i \in \dot{H}^{s+\gamma}$, *and for all* $i \in \mathbf{N}^*$ *there are sequences* $\boldsymbol{\lambda}^i = (\lambda^i_n)_{n\in\mathbf{N}^*} \subset (0, \infty)$, $\mathbf{t}^i = (t^i_n)_{n\in\mathbf{N}^*} \subset \mathbf{R}$, $\mathbf{x}^i = (x^i_n)_{n\in\mathbf{N}^*} \subset \mathbf{R}^N$ *such that :*

a) For $i \neq \ell$, the sequences $\left(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i\right)$ and $\left(\boldsymbol{\lambda}^\ell, \mathbf{t}^\ell, \mathbf{x}^\ell\right)$ are orthogonal.

b) For each $n \in \mathbf{N}^*$ and $k \in \mathbf{N}^*$ we have

$$u_{j(n)} = \sum_{i=1}^{k} S(-t_n^i)\left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right] + w_n^k, \qquad \textit{and} \tag{2.1.15}$$

$$\lim_{k\to\infty}\left(\limsup_{n\to\infty} \|S(\cdot)w_n^k\|_{L_t^\infty \dot{H}^\gamma_{2^*}}\right) = 0. \tag{2.1.16}$$

c) The following "almost orthogonality" conditions are satisfied for every $k \in \mathbf{N}^*$:

$$\lim_{n\to\infty} \|S(\cdot)u_{j(n)}\|^{\overline{q}}_{L_t^{\overline{q}}\dot{H}^\gamma_{\overline{q}}} = \sum_{i=1}^{\infty} \|S(\cdot)V^i\|^{\overline{q}}_{L_t^{\overline{q}}\dot{H}^\gamma_{\overline{q}}}, \tag{2.1.17}$$

$$\|u_{j(n)}\|^2_{\dot{H}^{s+\gamma}} = \sum_{i=1}^{k} \|V^i\|^2_{\dot{H}^{s+\gamma}} + \|w_n^k\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}, \qquad \textit{and} \tag{2.1.18}$$

$$\limsup_{k\to\infty}\left(\limsup_{n\to\infty} \|w_n^k\|_{\dot{H}^{s+\gamma}}\right) \leqslant \limsup_{n\to\infty} \|u_{j(n)}\|_{\dot{H}^{s+\gamma}}. \tag{2.1.19}$$

For any admissible pair (p,q) and $s \in (0, \frac{N}{2})$ satisfying (2.1.7), we have

$$\lim_{n\to\infty} \|S(\cdot)u_{j(n)}\|^q_{L_t^p\dot{H}^\gamma_q} \leqslant \sum_{i=1}^{k} \|S(\cdot)V^i\|^q_{L_t^p\dot{H}^\gamma_q} + \underset{k\to\infty}{o(1)} \qquad \textit{if } p > q, \qquad \textit{respectively} \tag{2.1.20}$$

$$\lim_{n\to\infty} \|S(\cdot)u_{j(n)}\|^p_{L_t^p\dot{H}^\gamma_q} \leqslant \sum_{i=1}^{k} \|S(\cdot)V^i\|^p_{L_t^p\dot{H}^\gamma_q} + \underset{k\to\infty}{o(1)} \qquad \textit{if } p < q. \tag{2.1.21}$$

Remark. Inequalities (2.1.11), (2.1.12), (2.1.13) together with (2.1.16) imply that

$$\lim_{k\to\infty}\left(\limsup_{n\to\infty} \|S(\cdot)w_n^k\|_{L_t^p\dot{H}^\gamma_q}\right) = 0$$

for all (p,q) non-endpoint admissible pairs satisfying (2.1.7).

The functions V^i are called *profiles* of $\mathbf{u}$. Theorem 3.1.1 will be crucial in the proof of the following theorem which gives the existence of maximizers for Sobolev-Strichartz inequality as well as the precompactness, modulo the symmetries of the problem, of any maximizing sequence in $\dot{H}^{s+\gamma}$.

Theorem 2.1.2. *Let $N \in \mathbf{N}^*$ and $0 \leqslant s+\gamma < \frac{N}{2}$. Let (p,q) be a non-extremal admissible pair and let $\mathbf{u}$ be an optimizing sequence for the Strichartz inequality (2.1.8) in the following sense :*

$$\|u_n\|_{\dot{H}^{s+\gamma}} = 1 \qquad \text{for all } n, \qquad \text{and} \tag{2.1.22}$$

$$\|S(\cdot)u_n\|_{L_t^p L_x^q} \underset{n\to\infty}{\longrightarrow} C_S(p,q,\gamma), \tag{2.1.23}$$

where $C_S(p,q,\gamma) := \sup_{\|u\|_{\dot{H}^{s+\gamma}}=1} \|S(\cdot)u\|_{L_t^p \dot{H}_q^\gamma}$ is the optimal Strichartz constant.

Then there exist $V \in \dot{H}^{s+\gamma}$, an increasing mapping $j : \mathbf{N}^* \to \mathbf{N}^*$, and a sequence $(\boldsymbol{\lambda},\mathbf{t},\mathbf{x})$ with $\boldsymbol{\lambda} = (\lambda_n)_{n\geqslant 1} \subset (0,\infty)$, $\mathbf{t} = (t_n)_{n\geqslant 1} \subset \mathbf{R}$, $\mathbf{x} = (x_n)_{n\geqslant 1} \subset \mathbf{R}^N$ such that

$$\mathfrak{d}_{\lambda_n^{-1}}\left[S(t_n)u_{j(n)}(\cdot + x_n)\right] \underset{n\to\infty}{\longrightarrow} V \text{ in } \dot{H}^{s+\gamma}. \tag{2.1.24}$$

In particular, V is a maximizer for (2.1.8), that is

$$\|V\|_{\dot{H}^{s+\gamma}} = 1 \tag{2.1.25}$$

and

$$\|S(\cdot)V\|_{L_t^p \dot{H}_q^\gamma} = C_S(p,q,\gamma). \tag{2.1.26}$$

There are several results about the existence of optimal functions for the Strichartz inequalities in the literature. We firstly mention M. Kunze [14], who proved the existence of maximizers of the $L^6_{t,x}$-Strichartz inequality for the 1-D Schrödinger equation. In [9], D. Foschi found the best constant of the inequality and also the shape of the maximizers for the 1D and 2D-Schrödinger propagators. L. Fanelli, L. Vega, and N. Visciglia proved the existence of maximizers in the case of a differential operator $h(D)$ more general than the classical Laplacian, for which Sobolev-Strichartz inequalities hold in the $L_t^{\overline{q}} L_x^{\overline{q}}$-space (identified with $L^{\overline{q}}(\mathbf{R}^{1+N})$). They used a result of H. Brezis about non-vanishing sequences to recover compactness and they used operator continuity properties to extend the properties of maximizing sequences to the limit.
S. Shao proved in [20] existence of maximizers in the case of classical Laplacian ($\sigma = 1$) by using a powerful profile decomposition result initiated by P. Gérard in [11] to describe the defect of compactness of bounded sequences in homogeneous Sobolev spaces, and then developed by S. Keraani ([13]), J. Bourgain ([2]), F. Merle and L. Vega ([18]), and P. Bégout and A. Vargas ([4]).

In the next section we give some elementary results which will be used in our proofs, then we prove Theorems 3.1.1 and 2.1.2 in section 3. The main observation in the proof of Theorem 2.1.2 is that there can be only one non-trivial profile V^i in the profile decomposition of an optimizing sequence, and all other profiles must be zero. This clearly implies the compactness of the optimizing sequence up to the invariances of the problem.

2.2 Preliminary results

An important technical tool that we use is Lieb's Lemma (see Lemma 6 p. 447 in [15] in the H^1 case or Lemma 6.1 and Lemma 6.2 in the appendix of [16] in the case of fractional derivatives).

Lemma 2.2.1. *(E. H. Lieb, [15]) (i) Let $N \in \mathbf{N}^*$ and $s \in (0,\infty)$. Assume that $u \in \dot{H}^s(\mathbf{R}^N)$ satisfies*

$$\|u\|_{\dot{H}^s} \leqslant M \text{ and } \mathcal{L}\left(\{x \in \mathbf{R}^N \mid |u(x)| \geqslant \varepsilon\}\right) \geqslant \alpha,$$

where M, ε and α are given positive constants and $\mathcal{L}$ is the Lebesgue measure in $\mathbf{R}^N$. Then for all $\delta \in (0,\varepsilon)$, there exists a constant $\beta = \beta(N,s,M,\alpha,\varepsilon,\delta) > 0$, independent of u, and there exists $x_0 \in \mathbf{R}^N$ such that

$$\mathcal{L}\left(\{x \in B(x_0,1) \mid |u(x)| \geqslant \delta\}\right) \geqslant \beta.$$

(ii) Let $N \in \mathbf{N}^*$, $s \in (0,\infty)$ and $\gamma \in [0,\infty)$. Assume that $u \in \dot{H}^{s+\gamma}$ satisfies

$$\|u\|_{\dot{H}^{s+\gamma}} \leqslant M \text{ and } \mathcal{L}\left(\{x \in \mathbf{R}^N \mid ||\nabla|^\gamma u(x)| \geqslant \varepsilon\}\right) \geqslant \alpha,$$

where M, ε and α are given positive constants. Then for all $\delta \in (0,\varepsilon)$, there exists a constant $\beta = \beta(N,s,M,\alpha,\varepsilon,\delta) > 0$, independent of u, and there exists $x_0 \in \mathbf{R}^N$ such that

$$\mathcal{L}\left(\{x \in B(x_0,1) \mid ||\nabla|^\gamma u(x)| \geqslant \delta\}\right) \geqslant \beta. \tag{2.2.1}$$

We use part (ii) of the above lemma in the sequel, but it is merely a rephrase of part (i) with $|\nabla|^\gamma u$ instead of u.

To prove Theorem 1.1, we will study weak limits after translation and rescaling of subsequences of the bounded sequence $\mathbf{u}$. We use the formalism in [17] (see the Proof of Theorem 1.9 there). For any sequence $\mathbf{u}$, we consider the set

$$\Gamma(\mathbf{u}) := \left\{ V \in \dot{H}^{s+\gamma} \;\middle|\; \begin{array}{l} \text{there exist a subsequence } (u_{n_k})_{k\in\mathbf{N}^*} \text{ and sequences} \\ (x_k)_{k\in\mathbf{N}^*} \subset \mathbf{R}^N,\ (t_k)_{k\in\mathbf{N}^*} \subset \mathbf{R} \text{ and } (\lambda_k)_{k\in\mathbf{N}^*} \subset (0,\infty) \text{ such that} \\ \mathfrak{d}_{\lambda_k^{-1}}\left[S(t_k)u_{n_k}(\cdot + x_k)\right] \underset{k\to\infty}{\rightharpoonup} V \text{ in } \dot{H}^{s+\gamma} \end{array} \right\}$$

and we denote

$$\Lambda(\mathbf{u}) := \sup_{V\in\Gamma(\mathbf{u})} \|S(\cdot)V\|^{2^*}_{L^\infty_t \dot{H}^\gamma_{2^*}}.$$

It is easy to see that if $\mathbf{u}$ is bounded in $\dot{H}^{s+\gamma}$, we have always $0 \in \Gamma(\mathbf{u})$. The Strichartz inequality (2.1.8) implies that for any admissible pair (p,q) satisfying (2.1.6), the mapping $u \longmapsto S(\cdot)u$ is linear continuous from $H^{s+\gamma}(\mathbf{R}^N)$ to $L^p_t \dot{H}^\gamma_q$. Therefore, if

$$\mathfrak{d}_{\lambda_k^{-1}}\left[S(t_k)u_{n_k}(\cdot + x_k)\right] \underset{k\to\infty}{\rightharpoonup} V \quad \text{weakly in } \dot{H}^{s+\gamma},$$

then we have

$$S(\cdot)\mathfrak{d}_{\lambda_k^{-1}}\left[S(t_k)u_{n_k}(\cdot+x_k)\right] \underset{k\to\infty}{\rightharpoonup} S(\cdot)V \quad \text{weakly in } L_t^p\dot{H}_q^\gamma$$

for any admissible pair (p,q), and using (2.1.5) and (2.1.3) we find

$$\begin{aligned}\|S(\cdot)V\|_{L_t^p\dot{H}_q^\gamma} &\leqslant \liminf_{k\to\infty}\|S(\cdot)\mathfrak{d}_{\lambda_k^{-1}}\left[S(t_k)u_{n_k}(\cdot+x_k)\right]\|_{L_t^p\dot{H}_q^\gamma} = \liminf_{k\to\infty}\|S(\cdot)u_{n_k}\|_{L_t^p\dot{H}_q^\gamma}\\ &\leqslant \limsup_{k\to\infty}\|S(\cdot)u_{n_k}\|_{L_t^p\dot{H}_q^\gamma} \leqslant C_S(p,q,\gamma)\limsup_{n\to\infty}\|u_n\|_{\dot{H}^{s+\gamma}}.\end{aligned}$$

Since the above inequality holds for any $V\in\Gamma(\mathbf{u})$ and any admissible pair (p,q), we get

$$\Lambda(\mathbf{u}) \leqslant C_S(\infty,2^*,\gamma)^{2^*}\limsup_{n\to\infty}\|u_n\|_{\dot{H}^{s+\gamma}}^{2^*}.$$

We recall some results about Lorentz spaces $L_x^{p,q}$ (see Chapter 1.3 in [1]). Given a function $f:\mathbf{R}^N\to\mathbf{C}$, the distribution function of f is the function $F:(0,\infty)\longrightarrow[0,\infty]$ defined by $F(s):=\mathcal{L}\left(\{x\in\mathbf{R}^N|\ |f(x)|>s\}\right)$. Let $\|f\|_{L^{p,q}}:=\left(\displaystyle\int_0^\infty s^{q-1}F(s)^{\frac{q}{p}}ds\right)^{\frac{1}{q}}$ if $1\leqslant p,q<\infty$ and $\|f\|_{L^{p,\infty}}:=\sup_{s>0}t\,(F(s))^{\frac{1}{p}}$. The Lorentz space $L^{p,q}$ is the space of measurable functions f such that $\|f\|_{L^{p,q}}$ is finite. It is well known that $\|f\|_{L^p}^p=p\|f\|_{L^{p,p}}^p$ and that $L^{p,q_1}\subset L^{p,q_2}$ if $q_1<q_2$. If $0<s<N/2$ we have the improved Sobolev embedding $\dot{H}_2^s\subset L^{2^*,2}$ (see Theorem 8.1 page 301 in [19]), and there exists a constant $C>0$ depending only on N and s such that

$$\|f\|_{L^{2^*,2}}\leqslant C\|f\|_{\dot{H}_2^s} \text{ for any } u\in\dot{H}_2^s. \tag{2.2.2}$$

On the other hand, Hölder's inequality gives

$$\|f\|_{L^{2^*}}\leqslant\|f\|_{L^{2^*,2}}^{\frac{2}{2^*}}\|f\|_{L^{2^*,\infty}}^{1-\frac{2}{2^*}}. \tag{2.2.3}$$

The next lemma is the "space-time" counterpart of Lemma 6.1 in [17].

Lemma 2.2.2. *Let $0<s<N/2$, $M>0$, and $a>0$. Let $\mathbf{u}=(u_n)_{n\geqslant1}$ be a sequence in $\dot{H}^{s+\gamma}$ such that $\|u_n\|_{\dot{H}^{s+\gamma}}\leqslant M$ for all n. Assume that there is a sequence $(t_n)_{n\geqslant1}\subset\mathbf{R}$ satisfying*

$$\|S(t_n)\left(|\nabla|^\gamma u_n\right)\|_{L^{2^*,\infty}(\mathbf{R}^N)}\geqslant a \quad \textit{for all } n. \tag{2.2.4}$$

Then there exists a constant $\delta=\delta(a,\gamma,s,\sigma,N,M)>0$ such that

$$\Lambda(\mathbf{u})\geqslant\delta. \tag{2.2.5}$$

Démonstration. Assume that $\mathbf{u}=(u_n)_{n\geqslant1}$ satisfies (2.2.4). Passing to a subsequence of $\mathbf{u}$, stil denoted the same, there exists sequences $\mathbf{t}=(t_n)_{n\geqslant1}\subset\mathbf{R}$ and $\mathbf{s}=(s_n)_{n\geqslant1}\subset(0,\infty)$ such that

$$s_n\mathcal{L}\left(\{x\in\mathbf{R}^N|\ |S(t_n)|\nabla|^\gamma u_n(x)|>s_n\}\right)^{\frac{1}{2^*}}>\frac{a}{2}.$$

Let $\lambda_n^{-1} := s_n^{\frac{2^*}{N}}$ and $v_n := \mathfrak{d}_{\lambda_n^{-1}}[S(t_n)u_n]$. Then $v_n \in \dot{H}^{s+\gamma}$, $\|v_n\|_{\dot{H}^{s+\gamma}} = \|u_n\|_{\dot{H}^{s+\gamma}}$ and

$$\mathcal{L}\left(\{x \in \mathbf{R}^N |\ ||\nabla|^\gamma v_n(x)| > 1\}\right) = s_n^{2^*}\mathcal{L}\left(\{x \in \mathbf{R}^N |\ |S(t_n)|\nabla|^\gamma u_n(x)| > s_n\}\right) > \left(\frac{a}{2}\right)^{2^*}.$$

Using Lieb's lemma, there exist $b > 0$, depending only on N, s, γ, a and M, and a sequence $\mathbf{x} \subset \mathbf{R}^N$ such that

$$\mathcal{L}\left(\{x \in B(x_n, 1) \mid ||\nabla|^\gamma v_n(x)| > 0.5\}\right) > b,\ \forall n \in \mathbf{N}. \tag{2.2.6}$$

Since $\dot{H}^{s+\gamma}$ is a Hilbert space, there exists $V \in \dot{H}^{s+\gamma}$ such that, up to a subsequence, $v_n(\cdot + x_n) \rightharpoonup V$ weakly in $\dot{H}^{s+\gamma}$. Then $|\nabla|^\gamma v_n(\cdot + x_n) \rightharpoonup |\nabla|^\gamma V$ weakly in $\dot{H}^s$. By compact Sobolev embeddings, up to a subsequence we have $|\nabla|^\gamma v_n(\cdot + x_n) \underset{n\to\infty}{\longrightarrow} |\nabla|^\gamma V$ strongly in $L^q(B(0,R))$ for all $q \in [1, 2^*)$ and all $R > 0$. One can easily check that $V \in \Gamma(\mathbf{u})$ and then using (2.2.6), we infer that

$$\||\nabla|^\gamma v_n(\cdot + x_n)\|_{L_x^q(B(0,1))} \geqslant \frac{1}{2}b^{\frac{1}{q}} \quad \text{for all } n \in \mathbf{N}.$$

Passing to the limit we obtain $\||\nabla|^\gamma V\|_{L_x^q(B(0,1))} \geqslant \frac{1}{2}b^{\frac{1}{q}}$. Then by Hölder's inequality we get

$$\||\nabla|^\gamma V\|_{L_x^{2^*}(B(0,1))} \geqslant \mathcal{L}(B(0,1))^{\frac{1}{2^*}-\frac{1}{q}}\||\nabla|^\gamma V\|_{L^q(B(0,1))} \geqslant \frac{1}{2}\mathcal{L}(B(0,1))^{\frac{1}{2^*}-\frac{1}{q}}b^{\frac{1}{q}},$$

from which we deduce, by letting $q \nearrow 2^*$, that $\||\nabla|^\gamma V\|_{L_x^{2^*}(B(0,1))}^{2^*} \geqslant \frac{1}{2^{2^*}}b$. Thus we may take $\delta := \frac{1}{2^{2^*}}b$. So far we have proved that $V \in \Gamma(\mathbf{u})$ and $\|\ |\nabla|^\gamma V\|_{L_x^{2^*}} = \|S(0)V\|_{\dot{H}_{2^*}^\gamma}^{2^*} \geqslant \delta$.

On the other hand, we have $S(t)V \rightharpoonup V$ weakly in $\dot{H}^{s+\gamma}$. Indeed, for any $\varphi \in \dot{H}^{s+\gamma}$ using the dominated convergence theorem we find

$$\langle S(t)V, \varphi\rangle_{\dot{H}^{s+\gamma}} = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} |\xi|^{2(s+\gamma)}e^{it|\xi|^{2\sigma}}\widehat{V}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi$$

$$\underset{t\to 0}{\longrightarrow} \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} |\xi|^{2(s+\gamma)}\widehat{V}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi = \langle V, \varphi\rangle_{\dot{H}^{s+\gamma}}.$$

We infer that $S(t)\left(|\nabla|^\gamma V\right) \underset{t\to 0}{\rightharpoonup} |\nabla|^\gamma V$ in $\dot{H}^s$, from which we deduce that $S(t)\left(|\nabla|^\gamma V\right) \underset{t\to 0}{\rightharpoonup} |\nabla|^\gamma V$ in L^{2^*} and

$$\delta \leqslant \||\nabla|^\gamma V\|_{L^{2^*}}^{2^*} \leqslant \liminf_{t\to 0}\|S(t)|\nabla|^\gamma V\|_{L^{2^*}}^{2^*} \leqslant \|S(\cdot)|\nabla|^\gamma V\|_{L_t^\infty L^{2^*}}^{2^*} = \|S(\cdot)V\|_{L_t^\infty \dot{H}_{2^*}^\gamma}^{2^*} \leqslant \Lambda(\mathbf{u}).$$

□

Moreover, by the interpolation inequality (2.2.3) we have immediately the following

Corollary 2.2.1. *Let $s \in \left(0, \frac{N}{2}\right)$, $\gamma \in \mathbf{R}$ and let $\mathbf{u} = (u_n)_{n\geqslant 1}$ be a sequence in $\dot{H}^{s+\gamma}$ such that $\|u_n\|_{\dot{H}^{s+\gamma}} \leqslant M$ for all n. We have*

$$\Lambda(\mathbf{u}) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|S(\cdot)u_n\|_{L_t^\infty \dot{H}^\gamma_{2^*}} = 0. \tag{2.2.7}$$

Moreover, if $\mathbf{u}^i = (u_n^i)_{n\geqslant 1}$ are sequences in $\dot{H}^{s+\gamma}$ such that $\|u_n^i\|_{\dot{H}^{s+\gamma}} \leq M$ for all n and i, then

$$\lim_{i\to\infty} \Lambda(\mathbf{u}^i) = 0 \quad \Longrightarrow \quad \lim_{i\to\infty} \left(\limsup_{n\to\infty} \|S(\cdot)u_n^i\|_{L_t^\infty \dot{H}^\gamma_{2^*}} \right) = 0 \tag{2.2.8}$$

Proof. From Lemma 2.1 we deduce that if $\Lambda(\mathbf{u}) = 0$ then $\|S(\cdot)|\nabla|^\gamma u_n\|_{L_t^\infty L^{2^*,\infty}} \underset{n\to\infty}{\longrightarrow} 0$, and then we use the interpolation inequality (2.2.3) and the improved Sobolev embedding (2.2.2) to get the first part. The second part is similar. □

If a sequence $\mathbf{u} = (u_n)_{n\geqslant 1}$ is bounded in $\dot{H}^{s+\gamma}$ and $S(\cdot)u_n$ tends to zero in $L_t^\infty \dot{H}^\gamma_{2^*}$, then the interpolation inequality (2.1.11) implies that $S(\cdot)u_n$ tends to zero in $L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}$. The converse is also true, as it can be seen in the following corollary.

Corollary 2.2.2. *Let $\mathbf{u} = (u_n)_{n\geqslant 1}$ be a bounded sequence in $\dot{H}^{s+\gamma}$. Then*

$$\|S(\cdot)u_n\|_{L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}} \underset{n\to\infty}{\longrightarrow} 0 \quad \Longrightarrow \quad \|S(\cdot)u_n\|_{L_t^\infty \dot{H}^\gamma_{2^*}} \underset{n\to\infty}{\longrightarrow} 0. \tag{2.2.9}$$

Démonstration. Suppose that $\|S(\cdot)u_n\|_{L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}} \underset{n\to\infty}{\longrightarrow} 0$ and $\|S(\cdot)u_n\|_{L_t^\infty \dot{H}^\gamma_{2^*}} \underset{n\to\infty}{\nrightarrow} 0$. Then, by (2.2.3), there exists $a > 0$ such that

$$\limsup_{n\to\infty} \|S(\cdot)|\nabla|^\gamma u_n\|_{L_t^\infty L^{2^*,\infty}} \geqslant a.$$

By Lemma 2.1, there exists a non-trivial profile $V \neq 0 \in \Gamma(\mathbf{u})$ and a triplet of sequences $(\boldsymbol{\lambda}, \mathbf{t}, \mathbf{x}) \subset (0,\infty) \times \mathbf{R} \times \mathbf{R}^N$ such that up to a subsequence of $\mathbf{u}$, we have

$$v_n := \mathfrak{d}_{\lambda_n} \left[S(t_n)u_n(\cdot + x_n) \right] \underset{n\to\infty}{\rightharpoonup} V \ \text{ in } \ \dot{H}^{s+\gamma}.$$

Then $S(\cdot)v_n \underset{n\to\infty}{\rightharpoonup} S(\cdot)V$ in $L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}$ and using the scale invariances (2.1.3)-(2.1.5) we find

$$0 < \|S(\cdot)V\|_{L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}} \leqslant \liminf_{n\to\infty} \|S(\cdot)v_n\|_{L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}} = \liminf_{n\to\infty} \|S(\cdot)u_n\|_{L_t^{\overline{q}} \dot{H}^\gamma_{\overline{q}}} = 0,$$

which is a contradiction. □

2.3 Proof of Theorem 1.1, Profile Decomposition

We start with the following simple lemmas. Their proofs are straightforward and are left to the reader.

Lemma 2.3.1. *Let the sequences $\boldsymbol{\lambda} \subset (0,\infty)$, $\mathbf{t} \subset \mathbf{R}$ and $\mathbf{x} \subset \mathbf{R}^N$. Assume that one of the following conditions holds*

— *$\lambda_n \to 0$ or $\lambda_n \to \infty$ and $\mathbf{t}$, $\mathbf{x}$ are arbitrary, or*

— *$\boldsymbol{\lambda}$, $\mathbf{t}$ are arbitrary and $|x_n| \to \infty$, or*

— *$\boldsymbol{\lambda}$, $\mathbf{x}$ are arbitrary and $|t_n| \to \infty$.*

Then for every $u \in \dot{H}^{s+\gamma}$ we have $S(t_n)(\mathfrak{d}_{\lambda_n} u)(\cdot + x_n) \underset{n\to\infty}{\rightharpoonup} 0$ in $\dot{H}^{s+\gamma}$.

Lemma 2.3.2. *Assume that $u_n \underset{n\to\infty}{\rightharpoonup} u$ weakly in $\dot{H}^{s+\gamma}$, $\lambda_n \underset{n\to\infty}{\to} \lambda_* \in (0,\infty)$, $t_n \to t_* \in \mathbf{R}$ and $x_n \to x_* \in \mathbf{R}^N$. Then $S(t_n)(\mathfrak{d}_{\lambda_n} u_n)(\cdot + x_n) \underset{n\to\infty}{\rightharpoonup} S(t_*)(\mathfrak{d}_{\lambda_*} u)(\cdot + x_*)$ in $\dot{H}^{s+\gamma}$.*

Lemma 2.3.3. *Let (p,q) be a non-extremal admissible pair. Assume that the sequences $\boldsymbol{\lambda} = (\lambda_n)_{n\geqslant 1} \subset (0,\infty)$, $\mathbf{t} = (t_n)_{n\geqslant 1} \subset \mathbf{R}$ and $\mathbf{x} \subset \mathbf{R}^{N+1}$ satisfy one of the following conditions :*

— *either $\lambda_n \to 0$ or $\lambda_n \to \infty$, $\mathbf{t}$ and $\mathbf{x}$ are arbitrary, or*

— *$\boldsymbol{\lambda}$ is arbitrary and $|t_n| \to \infty$ or $|x_n| \to \infty$.*

Then for any u, $v \in \dot{H}^{s+\gamma}$ we have

$$I_n := \int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} \left| S\left(\cdot + t_n\right) |\nabla|^{\gamma} \left[\mathfrak{d}_{\lambda_n} u\right] \left(\cdot + x_n\right) \right| \times \left| S(\cdot) |\nabla|^{\gamma} v \right|^{q-1} dx \right)^{\frac{p}{q}} dt \underset{n\to\infty}{\longrightarrow} 0. \tag{2.3.1}$$

Démonstration. For any $U \in L_t^{\overline{q}}(\mathbf{R}\times\mathbf{R}^N) = L_t^{\overline{q}} L_x^{\overline{q}}$ and any $\lambda > 0$, $t_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^N$ we denote

$$U_{\lambda,t_0,x_0}(t,x) = \lambda^{-\frac{N}{2^*}} U\left(\frac{t+t_0}{\lambda^{2\sigma}}, \frac{x+x_0}{\lambda}\right).$$

For $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^N$ and $R > 0$ we consider the cylinder

$$\Omega_{t_0,x_0,R} = (t_0 - R^{2\sigma}, t_0 + R^{2\sigma}) \times B(x_0, R) \subset \mathbf{R}\times\mathbf{R}^N.$$

A simple change of variables gives

$$\int_{\Omega_{0,0,R}} \left|U_{\lambda,t_0,x_0}\right|^{\overline{q}} dx\, dt = \int_{\Omega_{\frac{t_0}{\lambda^{2\sigma}}, \frac{x_0}{\lambda}, \frac{R}{\lambda}}} \left|U(t,x)\right|^{\overline{q}} dx\, dt. \tag{2.3.2}$$

We claim that for any $U, V \in L_t^{\overline{q}} L_x^{\overline{q}}$ and any sequences $\boldsymbol{\lambda}$, $\mathbf{t}$ and $\mathbf{x}$ as in Lemma 2.3.3 we have

$$J_n := \int_{\mathbf{R}} \int_{\mathbf{R}^N} \left|U_{\lambda_n,t_n,x_n}\right|^{\frac{\overline{q}}{2}} \left|V\right|^{\frac{\overline{q}}{2}} dx\, dt \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{2.3.3}$$

To see this, fix $\varepsilon > 0$. Since $|U|^{\overline{q}}, |V|^{\overline{q}} \in L^1(\mathbf{R}^{N+1})$, there exists $R_\varepsilon > 0$ such that

$$\int_{\mathbf{R}^{N+1}\setminus\Omega_{0,0,R_\varepsilon}} \left|U\right|^{\overline{q}} dx\, dt < \varepsilon, \qquad \int_{\mathbf{R}^{N+1}\setminus\Omega_{0,0,R_\varepsilon}} \left|V\right|^{\overline{q}} dx\, dt < \varepsilon, \tag{2.3.4}$$

and and there is $r_\varepsilon > 0$ such that for any $t_0 \in \mathbf{R}$ and any $x_0 \in \mathbf{R}^N$ we have

$$\int_{\Omega_{t_0,x_0,r_\varepsilon}} |U|^{\overline{q}}\,dx\,dt < \varepsilon \quad \text{and} \quad \int_{\Omega_{t_0,x_0,r_\varepsilon}} |V|^{\overline{q}}\,dx\,dt < \varepsilon. \tag{2.3.5}$$

We have

$$\begin{aligned} J_n &= \int_{\Omega_{0,0,R_\varepsilon}} \left|U_{\lambda_n,t_n,x_n}\right|^{\frac{\overline{q}}{2}} |V|^{\frac{\overline{q}}{2}}\,dx\,dt + \int_{\Omega^c_{0,0,R_\varepsilon}} \left|U_{\lambda_n,t_n,x_n}\right|^{\frac{\overline{q}}{2}} |V|^{\frac{\overline{q}}{2}}\,dx\,dt \\ &\leqslant \left(\int_{\Omega_{0,0,R_\varepsilon}} \left|U_{\lambda_n,t_n,x_n}\right|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} \left(\int_{\Omega_{0,0,R_\varepsilon}} |V|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} \\ &\qquad + \left(\int_{\Omega^c_{0,0,R_\varepsilon}} \left|U_{\lambda_n,t_n,x_n}\right|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} \left(\int_{\Omega^c_{0,0,R_\varepsilon}} |V|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{\Omega_{\frac{t_n}{\lambda_n^{2\sigma}},\frac{x_n}{\lambda_n},\frac{R_\varepsilon}{\lambda_n}}} |U|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} \left(\int_{\Omega_{0,0,R_\varepsilon}} |V|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \left(\int_{\mathbf{R}^{N+1}} |U|^{\overline{q}}\,dx\,dt\right)^{\frac{1}{2}}. \end{aligned} \tag{2.3.6}$$

If $\lambda_n \longrightarrow \infty$, for all n sufficiently large we have $\frac{R_\varepsilon}{\lambda_n} < r_\varepsilon$ and using (2.3.5) we see that for all such n we have $J_n \leqslant \varepsilon^{\frac{1}{2}} \left(\|U\|^{\frac{\overline{q}}{2}}_{L^{\overline{q}}(\mathbf{R}^{N+1})} + \|V\|^{\frac{\overline{q}}{2}}_{L^{\overline{q}}(\mathbf{R}^{N+1})}\right)$. Since ε was arbitrary, (2.3.3) follows.

If $\lambda_n \longrightarrow 0$, a simple change of variables gives

$$J_n = \int_{\mathbf{R}\times\mathbf{R}^N} |U|^{\frac{\overline{q}}{2}} \left|V_{\frac{1}{\lambda_n},-\frac{t_n}{\lambda_n^{2\sigma}},-\frac{x_n}{\lambda_n}}\right|^{\frac{\overline{q}}{2}}\,dx\,dt$$

and (2.3.3) is proven exactly as above.

If there exist positive constants a, A such that $a \leqslant \lambda_n \leqslant A$ for all sufficiently large n and $|t_n| \longrightarrow \infty$ or $|x_n| \longrightarrow \infty$, for n large enough we have $\Omega_{\frac{t_n}{\lambda_n^{2\sigma}},\frac{x_n}{\lambda_n},\frac{R_\varepsilon}{\lambda_n}} \cap \Omega_{0,0,R_\varepsilon} = \emptyset$ and then using (2.3.4) and (2.3.6) we see that for all such n we have $J_n \leqslant \varepsilon^{\frac{1}{2}} \left(\|U\|^{\frac{\overline{q}}{2}}_{L^{\overline{q}}(\mathbf{R}^{N+1})} + \|V\|^{\frac{\overline{q}}{2}}_{L^{\overline{q}}(\mathbf{R}^{N+1})}\right)$. As previously, we infer that (2.3.3) holds.

The above situations cover all possible cases and (2.3.3) is proven.

Let $u, v \in \dot{H}^{s+\gamma}$. Using (2.3.3) with $U = S(\cdot)|\nabla|^\gamma u$ and $V = S(\cdot)|\nabla|^\gamma v$ we see that

$$\int_{\mathbf{R}} \int_{\mathbf{R}^N} \left|S\left(\cdot + t_n\right)|\nabla|^\gamma \left[\mathfrak{d}_{\lambda_n} u\right](\cdot + x_n)\, S(\cdot)|\nabla|^\gamma v\right|^{\frac{\overline{q}}{2}} dx\,dt \underset{n\to\infty}{\longrightarrow} 0.$$

or equivalently

$$\left\|S\left(\cdot + t_n\right)|\nabla|^\gamma \left[\mathfrak{d}_{\lambda_n} u\right](\cdot + x_n)\, S(\cdot)|\nabla|^\gamma v\right\|_{L_t^{\frac{\overline{q}}{2}} L_x^{\frac{\overline{q}}{2}}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.3.7}$$

Using Hölder's inequality in $\mathbf{R}^N$, then in $\mathbf{R}$ and (2.1.3), we have

$$I_n \leqslant \left\|S\left(\cdot+t_n\right)|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n}u\right]\left(\cdot+x_n\right)S(\cdot)|\nabla|^{\gamma}v\right\|^{\frac{p}{q}}_{L_t^{\frac{p}{2}}L_x^{\frac{q}{2}}} \times \left\|S(\cdot)v\right\|^{\frac{(q-2)p}{q}}_{L_t^p\dot{H}_q^{\gamma}}. \tag{2.3.8}$$

Since the pair (p,q) is non-extremal, there exists another admissible pair $(p^\sharp,q^\sharp)$ satisfying (2.1.7) and $\kappa\in[0,1)$ such that $\frac{1}{p}=\frac{1-\kappa}{\overline{q}}+\frac{\kappa}{p^\sharp}$ and $\frac{1}{q}=\frac{1-\kappa}{\overline{q}}+\frac{\kappa}{q^\sharp}$. Then using Hölder's inequality we get

$$\begin{aligned}
&\left\|S\left(\cdot+t_n\right)|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n}u\right]\left(\cdot+x_n\right)S(\cdot)|\nabla|^{\gamma}v\right\|_{L_t^{\frac{p}{2}}L_x^{\frac{q}{2}}}\\
&\leqslant \left\|S\left(\cdot+t_n\right)|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n}u\right]\left(\cdot+x_n\right)S(\cdot)|\nabla|^{\gamma}v\right\|^{1-\kappa}_{L_t^{\frac{\overline{q}}{2}}L_x^{\frac{\overline{q}}{2}}}\\
&\qquad\times\left\|S\left(\cdot+t_n\right)|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n}u\right]\left(\cdot+x_n\right)S(\cdot)|\nabla|^{\gamma}v\right\|^{\kappa}_{L_t^{\frac{p^\sharp}{2}}L_x^{\frac{q^\sharp}{2}}}\\
&\leqslant \left\|S\left(\cdot+t_n\right)|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n}u\right]\left(\cdot+x_n\right)S(\cdot)|\nabla|^{\gamma}v\right\|^{1-\kappa}_{L_t^{\frac{\overline{q}}{2}}L_x^{\frac{\overline{q}}{2}}}\left\|S(\cdot)u\right\|^{\kappa}_{L_t^{p^\sharp}\dot{H}^{\gamma}_{q^\sharp}}\left\|S(\cdot)v\right\|^{\kappa}_{L_t^{p^\sharp}\dot{H}^{\gamma}_{q^\sharp}}.
\end{aligned} \tag{2.3.9}$$

From (2.3.7) - (2.3.9) and Strichartz' inequality (2.1.8) we deduce that $I_n\longrightarrow 0$ as $n\longrightarrow\infty$ and Lemma 2.3.3 is proven. □

Lemma 2.3.4. *Let $\mathbf{u}=(u_n)_{n\geqslant 1}$ be a bounded sequence in $\dot{H}^{s+\gamma}$. Assume that there exist sequences $(\boldsymbol{\lambda},\mathbf{t},\mathbf{x})\subset(0,\infty)\times\mathbf{R}\times\mathbf{R}^N$ and $V\in\dot{H}^{s+\gamma}$ such that*

$$\mathfrak{d}_{\lambda_n^{-1}}\left[S(t_n)u_n(\cdot+x_n)\right]\underset{n\to\infty}{\rightharpoonup}V \qquad \text{in } \dot{H}^{s+\gamma}.$$

Denote $w_n:=u_n-S(-t_n)(\mathfrak{d}_{\lambda_n}V)(\cdot-x_n)$. Then

$$\Gamma(\mathbf{w})\subset\Gamma(\mathbf{u}).$$

Moreover, if there exist an increasing mapping $j:\mathbf{N}^*\to\mathbf{N}^*$ and a triplet of sequences $(\tilde{\boldsymbol{\lambda}},\tilde{\mathbf{t}},\tilde{\mathbf{x}})\subset(\mathbf{0},\infty)\times\mathbf{R}\times\mathbf{R}^{\mathbf{N}}$ such that

$$\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n)w_{j(n)}(\cdot+\tilde{x}_n)\right]\underset{n\to\infty}{\rightharpoonup}w\neq 0 \text{ in } \dot{H}^{s+\gamma},$$

then $(\boldsymbol{\lambda},\mathbf{t},\mathbf{x})_{j(\cdot)}$ and $(\tilde{\boldsymbol{\lambda}},\tilde{\mathbf{t}},\tilde{\mathbf{x}})$ are orthogonal in the sense of Definition 1.1.

Démonstration. Assume that $w\in\Gamma(\mathbf{w})$ and $w\neq 0$. Then there exists a subsequence $(w_{j(n)})_{n\in\mathbf{N}^*}$ and a triplet $(\tilde{\boldsymbol{\lambda}},\tilde{\mathbf{t}},\tilde{\mathbf{x}})$ such that $\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n)w_{j(n)}(\cdot+\tilde{x}_n)\right]\underset{n\to\infty}{\rightharpoonup}w$ in $\dot{H}^{s+\gamma}$, which we can rewrite by using (2.1.5) as

$$\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n)u_{j(n)}(\cdot+\tilde{x}_n)\right]-S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\left(\mathfrak{d}_{\tilde{\lambda}_n^{-1}\lambda_{j(n)}}V\right)\left(\cdot+\frac{\tilde{x}_n-x_{j(n)}}{\tilde{\lambda}_n}\right)\underset{n\to\infty}{\rightharpoonup}w. \tag{2.3.10}$$

We have to discuss two cases :

Case A. $(\boldsymbol{\lambda},\mathbf{t},\mathbf{x})_{j(\cdot)}$ and $(\tilde{\boldsymbol{\lambda}},\tilde{\mathbf{t}},\tilde{\mathbf{x}})$ are orthogonal. This means, by applying Lemma 3.1, that $S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\left(\mathfrak{d}_{\tilde{\lambda}_n^{-1}\lambda_{j(n)}}V\right)\left(\cdot+\frac{\tilde{x}_n-x_{j(n)}}{\tilde{\lambda}_n}\right)\underset{n\to\infty}{\rightharpoonup}0$. In particular

$$\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n)u_{j(n)}(\cdot+\tilde{x}_n)\right]\underset{n\to\infty}{\rightharpoonup} w \text{ in } \dot{H}^{s+\gamma} \text{ and consequently } w\in\Gamma(\mathbf{u}).$$

Case B. $(\boldsymbol{\lambda},\mathbf{t},\mathbf{x})_{j(\cdot)}$ and $(\tilde{\boldsymbol{\lambda}},\tilde{\mathbf{t}},\tilde{\mathbf{x}})$ are not orthogonal. We may assume, up to extracting a subsequence still denoted the same, that $\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\to t_*\in\mathbf{R}$, $\tilde{\lambda}_n^{-1}\lambda_{j(n)}\to\lambda_*\in(0,\infty)$ and $\frac{\tilde{x}_n-x_{j(n)}}{\tilde{\lambda}_n}\to x_*\in\mathbf{R}^N$ as $n\to\infty$. Then, by Lemma 3.2, we have

$$S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\left(\mathfrak{d}_{\tilde{\lambda}_n^{-1}\lambda_{j(n)}}V\right)\left(\cdot+\frac{\tilde{x}_n-x_{j(n)}}{\tilde{\lambda}_n}\right)\underset{n\to\infty}{\rightharpoonup}S(t_*)(\mathfrak{d}_{\lambda_*}V)\left(\cdot+x_*\right). \tag{2.3.11}$$

On the other hand, we can rewrite the first term in (2.3.10) by using again (2.1.5),

$$\begin{aligned}
\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n)u_{j(n)}(\cdot+\tilde{x}_n)\right] &= \mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(\tilde{t}_n-t_{j(n)})S(t_{j(n)})u_{j(n)}(\cdot+x_{j(n)}+\tilde{x}_n-x_{j(n)})\right]\\
&= S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\mathfrak{d}_{\tilde{\lambda}_n^{-1}}\left[S(t_{j(n)})u_{j(n)}(\cdot+x_{j(n)}+\tilde{x}_n-x_{j(n)})\right]\\
&= S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\left(\mathfrak{d}_{\tilde{\lambda}_n^{-1}\lambda_{j(n)}}\mathfrak{d}_{\lambda_{j(n)}^{-1}}\right)\left[S(t_{j(n)})u_{j(n)}(\cdot+x_{j(n)}+\tilde{x}_n-x_{j(n)})\right]\\
&= S\left(\frac{\tilde{t}_n-t_{j(n)}}{(\tilde{\lambda}_n)^{2\sigma}}\right)\left(\mathfrak{d}_{\tilde{\lambda}_n^{-1}\lambda_{j(n)}}\right)\left(\mathfrak{d}_{\lambda_{j(n)}^{-1}}\left[S(t_{j(n)})u_{j(n)}(\cdot+x_{j(n)})\right]\right)\left(\cdot+\frac{\tilde{x}_n-x_{j(n)}}{\tilde{\lambda}_n}\right),
\end{aligned}$$

which converges weakly to $S(t_*)(\mathfrak{d}_{\lambda_*}V)\left(\cdot+x_*\right)$ by Lemma 3.2. Together with (2.3.10) and (2.3.11) this implies that $w=0$, which is a contradiction. We conclude that the second case does never occur and that $\Gamma(\mathbf{w})\subset\Gamma(\mathbf{u})$. □

Proof of Theorem 1.1. Let $\mathbf{u}$ be a bounded sequence in $\dot{H}^{s+\gamma}$.

If $\Lambda(\mathbf{u})=0$, by Corollary 2.1 we have $\limsup_{n\to\infty}\|u_n\|_{L_t^\infty\dot{H}^\gamma_{2^*}}=0$. By interpolation (2.1.12) this implies that $\|u_n\|_{L_t^p\dot{H}^\gamma_q}\to0$ as $n\to\infty$ for any non-endpoint admissible pair (p,q) and we can take $w_n^k:=u_n$ and $V^i=0$ for all i and $k\in\mathbf{N}^*$.

If $\Lambda(\mathbf{u})>0$, there exist $V^1\in\dot{H}^{s+\gamma}$, a triplet of sequences $\left(\boldsymbol{\lambda}^1,\mathbf{t}^1,\mathbf{x}^1\right)\subset(0,\infty)\times\mathbf{R}\times\mathbf{R}^N$ and an increasing mapping $j^1:\mathbf{N}^*\to\mathbf{N}^*$ such that

$$\mathfrak{d}_{(\lambda_n^1)^{-1}}\left[S(t_n^1)u_{j^1(n)}(\cdot+x_n^1)\right]\underset{n\to\infty}{\rightharpoonup}V^1 \qquad \text{in } \dot{H}^{s+\gamma} \tag{2.3.12}$$

and

$$\|S(\cdot)V^1\|^{2^*}_{L^\infty_t \dot{H}^\gamma_{2^*}} \geqslant \frac{3}{4}\Lambda(\mathbf{u}). \tag{2.3.13}$$

We denote $w^1_n := u_{j^1(n)} - S(-t^1_n)(\mathfrak{d}_{\lambda^1_n} V^1)(\cdot - x^1_n)$. Then by Lemma 3.4 we have $\Gamma\left(\mathbf{w}^1\right) \subset \Gamma(\mathbf{u})$ and $\Lambda\left(\mathbf{w}^1\right) \leqslant \Lambda(\mathbf{u})$.
Since $\dot{H}^{s+\gamma}$ is a Hilbert space, the weak convergence (2.3.12), the scaling invariance (2.1.4) and the norm conservation (2.1.2) imply that

$$\left\|u_{j^1(n)}\right\|^2_{\dot{H}^{s+\gamma}} = \left\|V^1\right\|^2_{\dot{H}^{s+\gamma}} + \left\|w^1_n\right\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}. \tag{2.3.14}$$

Now we repeat the previous procedure, with $\mathbf{u}$ replaced by $\mathbf{w^1}$.
If $\Lambda(\mathbf{w^1}) = 0$, take $V^i = 0$ for $i \geqslant 2$ and $w^k_n = w^1_n$ for all $k \in \mathbf{N}^*$. We have

$$u_{j^1(n)} = S(-t^1_n)\left[(\lambda^1_n)^{\gamma - \frac{N}{2^*}} V^1\left(\frac{\cdot - x^1_n}{\lambda^1_n}\right)\right] + w^1_n,$$

with $\limsup_{n\to\infty} \|w^1_n\|_{L^\infty_t L^{2^*}_x} = 0$ by Corollary 2.1.
If $\Lambda\left(\mathbf{w}^1\right) > 0$, there exist $V^2 \in \dot{H}^{s+\gamma}$, a triplet of sequences $(\boldsymbol{\lambda}^2, \mathbf{t}^2, \mathbf{x}^2) \subset (0,\infty) \times \mathbf{R} \times \mathbf{R}^N$ and an increasing mapping $j^2 : \mathbf{N}^* \to \mathbf{N}^*$ such that

$$\mathfrak{d}_{(\lambda^2_n)^{-1}}\left[S(t^2_n) w^1_{j^2(n)}(\cdot + x^2_n)\right] \underset{n\to\infty}{\rightharpoonup} V^2 \quad \text{in } \dot{H}^{s+\gamma}, \qquad \text{and} \tag{2.3.15}$$

$$\|S(\cdot)V^2\|^{2^*}_{L^\infty_t \dot{H}^\gamma_{2^*}} \geqslant \frac{3}{4}\Lambda\left(\mathbf{w}^1\right). \tag{2.3.16}$$

We denote

$$\begin{aligned} w^2_n :&= w^1_{j^2(n)} - S(-t^2_n)(\mathfrak{d}_{\lambda^2_n} V^2)(\cdot - x^2_n) \\ &= u_{j^1\circ j^2(n)} - S(-t^1_{j^2(n)})(\mathfrak{d}_{\lambda^1_{j^2(n)}} V^1)(\cdot - x^1_{j^2(n)}) - S(-t^2_n)(\mathfrak{d}_{\lambda^2_n} V^2)(\cdot - x^2_n). \end{aligned}$$

By Lemma 3.4 we have $\Gamma\left(\mathbf{w}^2\right) \subset \Gamma\left(\mathbf{w}^1\right) \subset \Gamma(\mathbf{u})$ and consequently $\Lambda\left(\mathbf{w}^2\right) \leqslant \Lambda\left(\mathbf{w}^1\right) \leqslant \Lambda(\mathbf{u})$.
Using the fact that $\dot{H}^{s+\gamma}$ is a Hilbert space, the weak convergence (2.3.15), the scaling invariance (2.1.4) and the norm conservation (2.1.2) we get as above

$$\left\|w^1_{j^2(n)}\right\|^2_{\dot{H}^{s+\gamma}} = \left\|V^2\right\|^2_{\dot{H}^{s+\gamma}} + \left\|w^2_n\right\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}. \tag{2.3.17}$$

Combining (2.3.14) with (2.3.17), we get

$$\left\|u_{j^1\circ j^2(n)}\right\|^2_{\dot{H}^{s+\gamma}} = \left\|V^1\right\|^2_{\dot{H}^{s+\gamma}} + \left\|V^2\right\|^2_{\dot{H}^{s+\gamma}} + \left\|w^2_n\right\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}.$$

At this stage of the proof it is important to remark that thanks to Lemma 3.4, $(\boldsymbol{\lambda}^{\mathbf{1}}, \mathbf{t}^{\mathbf{1}}, \mathbf{x}^{\mathbf{1}})_{\mathbf{j^2}(\cdot)}$ and $(\boldsymbol{\lambda}^{\mathbf{2}}, \mathbf{t}^{\mathbf{2}}, \mathbf{x}^{\mathbf{2}})$ are orthogonal in the sense of Definition 1.1.

Next we repeat the process described above. If there is $K \in \mathbf{N}^*$ such that $\Lambda(\mathbf{w}^K) = 0$ we take $V^i = 0$ and $w_n^k := w_n^K$ for all $k > K$. Otherwise the inductive process continues forever and for all $i \in \mathbf{N}^*$ we obtain a nontrivial function $V^i \in \dot{H}^{s+\gamma}$, a sequence $\mathbf{w}^i \subset \dot{H}^{s+\gamma}$, an increasing mapping $j_i : \mathbf{N}^* \to \mathbf{N}^*$, and triplets of sequences $(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i)$, such that the following properties are satisfied :

I) $V^i \in \Gamma(\mathbf{w}^{i-1})$ and $\|S(\cdot)V^i\|^{2^*}_{L^\infty_t \dot{H}^\gamma_{2^*}} \geqslant \frac{3}{4}\Lambda(\mathbf{w}^{i-1})$.

II) $\mathfrak{d}_{(\lambda_n^i)^{-1}} \left[S(t_n^i) w^{i-1}_{j^i(n)}(\cdot + x_n^i) \right] \underset{n\to\infty}{\rightharpoonup} V^i$ in $\dot{H}^{s+\gamma}$,

III) $w_n^i = w^{i-1}_{j^i(n)} - S(-t_n^i)(\mathfrak{d}_{\lambda_n^i} V^i)(\cdot - x_n^i)$

IV) $\left(\boldsymbol{\lambda}^{i-1}, \mathbf{t}^{i-1}, \mathbf{x}^{i-1}\right)_{j^i(\cdot)}$ and $\left(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i\right)$ are orthogonal in the sense of Definition 1.1.

We will show that $\Lambda\left(\mathbf{w}^k\right) \to 0$ for $k \to \infty$. As above, we deduce from (II) and (III) that for all $i, n \in \mathbf{N}^*$

$$\left\| w^{i-1}_{j^i(n)} \right\|^2_{\dot{H}^{s+\gamma}} = \left\| V^i \right\|^2_{\dot{H}^{s+\gamma}} + \left\| w_n^i \right\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}, \tag{2.3.18}$$

and we have constructed a decomposition of $\mathbf{u}$ of the form

$$u_{j^1 \circ \ldots \circ j^k(n)} = \sum_{i=1}^{k} S(-t^i_{j^{i+1}\circ\ldots\circ j^k(n)}) \left[(\lambda^i_{j^{i+1}\circ\ldots\circ j^k(n)})^{\gamma - \frac{N}{2^*}} V^i \left(\frac{\cdot - x^i_{j^{i+1}\circ\ldots\circ j^k(n)}}{\lambda^i_{j^{i+1}\circ\ldots\circ j^k(n)}} \right) \right] + w_n^k, \tag{2.3.19}$$

with the abuse of notation $j^k \circ j^k = j^k$ and $j^{k+1} \circ j^k = Id$.

We claim that for all $i < \ell$, $\left(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i\right)_{j^{i+1}\circ\ldots\circ j^\ell(\cdot)}$ and $\left(\boldsymbol{\lambda}^\ell, \mathbf{t}^\ell, \mathbf{x}^\ell\right)$ are orthogonal in the sense of Definition 1.1. Indeed, we firstly remark that :

$$\begin{aligned}
&\mathfrak{d}_{(\lambda_n^{i+1})^{-1}} \left[S(t_n^{i+1}) w^{i-1}_{j^i \circ j^{i+1}(n)}(\cdot + x_n^{i+1}) \right] \\
&= S\left(\frac{t_n^{i+1} - t^i_{j^{i+1}(n)}}{(\lambda_n^{i+1})^{2\sigma}} \right) \left(\mathfrak{d}_{(\lambda_n^{i+1})^{-1}\lambda^i_{j^{i+1}(n)}} V^i \right) \left(\cdot + \frac{x_n^{i+1} - x^i_{j^{i+1}(n)}}{\lambda^i_{j^{i+1}(n)}} \right) \\
&\qquad + \mathfrak{d}_{(\lambda_n^{i+1})^{-1}} \left[S(t_n^{i+1}) w^i_{j^{i+1}(n)}(\cdot + x_n^{i+1}) \right].
\end{aligned}$$

By Lemma 3.1, the first term above tends weakly to 0 in $\dot{H}^{s+\gamma}$ as $n \to \infty$ because of the orthogonality of $(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i)_{j^{i+1}(\cdot)}$ and $(\boldsymbol{\lambda}^{i+1}, \mathbf{t}^{i+1}, \mathbf{x}^{i+1})$. By property (II), the second term tends weakly to V^{i+1} in $\dot{H}^{s+\gamma}$ and then Lemma 3.4 implies that $(\boldsymbol{\lambda}^{i-1}, \mathbf{t}^{i-1}, \mathbf{x}^{i-1})_{j^i \circ j^{i+1}(\cdot)}$ and $(\boldsymbol{\lambda}^{i+1}, \mathbf{t}^{i+1}, \mathbf{x}^{i+1})$ are orthogonal.

More generally, by the same argument we obtain that for all $i < \ell$,

$$\mathfrak{d}_{(\lambda_n^\ell)^{-1}} \left[S(t_n^\ell) w^i_{j^{i+1}\circ\ldots\circ j^\ell(n)}(\cdot + x_n^\ell) \right] \underset{n\to\infty}{\rightharpoonup} V^\ell \qquad \text{in } \dot{H}^{s+\gamma}.$$

Using again Lemma 3.4, our claim about orthogonality follows.

Now let $j(n) := j^1 \circ ... \circ j^n(n)$ be the diagonal extraction map. To simplify notation we relabel the subsequences and we denote $\lambda^i_{j^{i+1}\circ...\circ j^n(n)}$ simply by λ^i_n ; we do the same for $\mathbf{t}^i$ and for $\mathbf{x}^i$. Then for any $i \neq \ell$, $(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i)$ and $(\boldsymbol{\lambda}^\ell, \mathbf{t}^\ell, \mathbf{x}^\ell)$ are orthogonal. Given any $k \in \mathbf{N}^*$ and $n \geqslant k$, we have

$$u_{j(n)} = \sum_{i=1}^{k} S(-t^i_n) \left[(\lambda^i_n)^{\gamma - \frac{N}{2^*}} V^i \left(\frac{\cdot - x^i_n}{\lambda^i_n} \right) \right] + w^k_n,$$

thus (2.1.15) holds. By (2.3.18) we get

$$\|u_{j(n)}\|^2_{\dot{H}^{s+\gamma}} = \sum_{i=1}^{k} \|V^i\|^2_{\dot{H}^{s+\gamma}} + \|w^k_n\|^2_{\dot{H}^{s+\gamma}} + \underset{n\to\infty}{o(1)}.$$

This proves (2.1.18).

Then we have immediately $\limsup_{k\to\infty} \left(\limsup_{n\to\infty} \|w^k_n\|_{\dot{H}^{s+\gamma}} \right) \leqslant \limsup_{n\to\infty} \|u_{j(n)}\|_{\dot{H}^{s+\gamma}}$.

Next we prove (2.1.16) which means that the reminder term w^k_n must be small with respect to the Strichartz norm. We claim that for all $k \in \mathbf{N}^*$,

$$\sum_{i=1}^{k} \left\| S(\cdot)V^i \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}} \leqslant \liminf_{n\to\infty} \left\| S(\cdot)u_{j(n)} \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}}. \tag{2.3.20}$$

We fix k and we proceed by induction. As in the proof of Lemma 2.3.3, we denote $\Omega_{t,x,R} := (t - R^{2\sigma}, t + R^{2\sigma}) \times B(x, R) \subset \mathbf{R} \times \mathbf{R}^N$. Given $\varepsilon > 0$, there exists $R^1 > 0$ such that

$$\left\| S(\cdot)V^1 \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}} - \frac{\varepsilon}{2} < \left\| S(\cdot)V^1 \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}(\Omega_{0,0,R^1})},$$

where $\|S(\cdot)V\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}(\Omega)} = \int_\Omega \left| \mathcal{F}^{-1} \left(|\cdot|^\gamma e^{it|\cdot|^{2\sigma}} \widehat{V} \right) \right|^{\overline{q}}(x)\, dx\, dt.$

Since $\mathfrak{d}_{(\lambda^1_n)^{-1}} \left[S(t^1_n) u_{j(n)}(\cdot + x^1_n) \right] \underset{n\to\infty}{\rightharpoonup} V^1$ in $\dot{H}^{s+\gamma}$, we have

$$S(\cdot) \left(\mathfrak{d}_{(\lambda^1_n)^{-1}} \left[S(t^1_n) u_{j(n)}(\cdot + x^1_n) \right] \right) \underset{n\to\infty}{\rightharpoonup} S(\cdot)V^1 \text{ in } L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}$$

and consequently, after a straightforward change of variables we get

$$\begin{aligned} \left\| S(\cdot)V^1 \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}} - \frac{\varepsilon}{2} &< \liminf_{n\to\infty} \left\| S(\cdot)\mathfrak{d}_{(\lambda^1_n)^{-1}} \left[S(t^1_n) u_{j(n)}(\cdot + x^1_n) \right] \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}}(\Omega_{0,0,R^1})} \\ &= \liminf_{n\to\infty} \left\| S(\cdot)u_{j(n)} \right\|^{\overline{q}}_{L^{\overline{q}}_t \dot{H}^{\gamma}_{\overline{q}} \left(\Omega_{t^1_n, x^1_n, \lambda^1_n R^1} \right)}. \end{aligned}$$

Similarly, there exists $R^2 > 0$ such that

$$\left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}} - \frac{\varepsilon}{4} < \left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}(\Omega_{0,0,R^2})},$$

and we find

$$\begin{aligned}\left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}} - \frac{\varepsilon}{4} &< \liminf_{n\to\infty}\left\|S(\cdot)\mathfrak{d}_{(\lambda_n^2)^{-1}}\left[S(t_n^2)u_{j(n)}(\cdot+x_n^2)\right]\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}(\Omega_{0,0,R^2})}\\ &= \liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{t_n^2,x_n^2,\lambda_n^2R^2}\right)}.\end{aligned}$$

If, up to a subsequence, $\frac{\lambda_n^1}{\lambda_n^2} \longrightarrow \alpha \in (0,\infty)$, then we have by orthogonality that $\frac{|x_n^1-x_n^2|}{\lambda_n^2} \longrightarrow \infty$ or $\frac{|t_n^1-t_n^2|}{(\lambda_n^2)^{2\sigma}} \longrightarrow \infty$. Hence $\Omega_{t_n^1,x_n^1,\lambda_n^1R^1}$ and $\Omega_{t_n^2,x_n^2,\lambda_n^2R^2}$ are disjoint for n sufficiently large and we obtain

$$\begin{aligned}\left\|S(\cdot)V^1\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}} + \left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}} - \left(\frac{\varepsilon}{2}+\frac{\varepsilon}{4}\right) &< \liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{t_n^1,x_n^1,\lambda_n^1R^1}\bigcup\Omega_{t_n^2,x_n^2,\lambda_n^2R^2}\right)}\\ &\leqslant \liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}.\end{aligned}$$

Otherwise we may suppose that after permutation of the indices we have $\frac{\lambda_n^1}{\lambda_n^2} \longrightarrow 0$, and we have to treat two sub-cases :

— $\frac{|x_n^1-x_n^2|}{\lambda_n^2} \longrightarrow \infty$ or $\frac{|t_n^1-t_n^2|}{(\lambda_n^2)^{2\sigma}} \longrightarrow \infty$, and then $\Omega_{t_n^1,x_n^1,\lambda_n^1R^1}$ and $\Omega_{t_n^2,x_n^2,\lambda_n^2R^2}$ are disjoint for n sufficiently large. We argue exactly as above.

— $\frac{|x_n^1-x_n^2|}{\lambda_n^2} \longrightarrow x^{1,2}$ or $\frac{|t_n^1-t_n^2|}{(\lambda_n^2)^{2\sigma}} \longrightarrow t^{1,2}$, up to a subsequence still denoted the same. Then there exists $\delta > 0$ such that

$$\begin{aligned}&\left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}} - \frac{\varepsilon}{4} < \left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^2}\setminus\Omega_{t^{1,2},x^{1,2},\delta}\right)}\\ &\leqslant \liminf_{n\to\infty}\left\|S(\cdot)\mathfrak{d}_{(\lambda_n^2)^{-1}}\left[S(t_n^2)u_{j(n)}(\cdot+x_n^2)\right]\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^2}\setminus\Omega_{t^{1,2},x^{1,2},\delta}\right)}\\ &= \liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{t_n^2,x_n^2,\lambda_n^2R^2}\setminus\Omega_{t_n^2+(\lambda_n^2)^{2\sigma}t^{1,2},x_n^2+\lambda_n^2x^{1,2},\lambda_n^2\delta}\right)},\end{aligned}$$

and it is easy to see that for n sufficiently large we have

$$\Omega_{t_n^1,x_n^1,\lambda_n^1R^1} \subset \Omega_{t_n^2+(\lambda_n^2)^{2\sigma}t^{1,2},x_n^2+\lambda_n^2x^{1,2},\lambda_n^2\delta}.$$

As previously, we get the desired inequality :

$$\begin{aligned}
&\left\|S(\cdot)V^1\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}+\left\|S(\cdot)V^2\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\left(\frac{\varepsilon}{2}+\frac{\varepsilon}{4}\right)\\
&<\liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{t^1_n,x^1_n,\lambda^1_nR^1}\bigcup\Omega_{t^2_n,x^2_n,\lambda^2_nR^2}\setminus\Omega_{t^2_n+(\lambda^2_n)^{2\sigma}t^{1,2},x^2_n+\lambda^2_nx^{1,2},\lambda^2_n\delta}\right)}\\
&\leqslant\liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}.
\end{aligned}$$

Continuing this process, we may assume that after a permutation of indices $1,\dots,k$ and extraction of a subsequence we have either $\frac{\lambda^i_n}{\lambda^\ell_n}\longrightarrow\alpha\in(0,\infty)$, or $\frac{\lambda^i_n}{\lambda^\ell_n}\longrightarrow 0$ for $i\leqslant\ell$.
Suppose that for $\ell\in\{1,...,k-1\}$ we have proved that there exists $R^1,...,R^\ell>0$ such that

$$\|S(\cdot)V^\ell\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\frac{\varepsilon}{2^\ell}\leqslant\|S(\cdot)V^\ell\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^\ell}\right)},$$

and

$$\sum_{i=1}^{\ell}\|S(\cdot)V^i\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\sum_{i=1}^{\ell}\frac{\varepsilon}{2^i}\leqslant\liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}.$$

Then we choose $R^{\ell+1}>0$ such that

$$\|S(\cdot)V^{\ell+1}\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\frac{\varepsilon}{2^{\ell+1}}\leqslant\|S(\cdot)V^{\ell+1}\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^{\ell+1}}\right)},$$

and we define $I_1:=\left\{i\in\{1,..,\ell\},\ \frac{|t^{\ell+1}_n-t^i_n|}{(\lambda^{\ell+1}_n)^{2\sigma}}\longrightarrow\infty\text{ or }\frac{|x^{\ell+1}_n-x^i_n|}{\lambda^{\ell+1}_n}\longrightarrow\infty\right\}$
and $I_2:=\left\{i\in\{1,..,\ell\},\ \frac{|t^{\ell+1}_n-t^i_n|}{(\lambda^{\ell+1}_n)^{2\sigma}}\longrightarrow t^{i,\ell+1}\text{ and }\frac{|x^{\ell+1}_n-x^i_n|}{\lambda^{\ell+1}_n}\longrightarrow x^{i,\ell+1}\text{ up to a subsequence}\right\}$.
If $i\in I_1$, the cylinders $\Omega_{t^i_n,x^i_n,\lambda^i_nR^i}$ and $\Omega_{t^{\ell+1}_n,x^{\ell+1}_n,\lambda^{\ell+1}_nR^{\ell+1}}$ are disjoint for n sufficiently large.
For $i\in I_2$, there exists $\delta>0$ such that

$$\begin{aligned}
&\left\|S(\cdot)V^{\ell+1}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\frac{\varepsilon}{2^{\ell+1}}<\left\|S(\cdot)V^{\ell+1}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^{\ell+1}}\setminus\bigcup_{i\in I_2}\Omega_{t^{i,\ell+1},x^{i,\ell+1},\delta}\right)}\\
&\leqslant\liminf_{n\to\infty}\left\|S(\cdot)\mathfrak{d}_{(\lambda^{\ell+1}_n)^{-1}}\left[S(t^{\ell+1}_n)u_{j(n)}(\cdot+x^{\ell+1}_n)\right]\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{0,0,R^{\ell+1}}\setminus\bigcup_{i\in I_2}\Omega_{t^{i,\ell+1},x^{i,\ell+1},\delta}\right)}\\
&=\liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}\left(\Omega_{t^{\ell+1}_n,x^{\ell+1}_n,\lambda^{\ell+1}_nR^{\ell+1}}\setminus\bigcup_{i\in I_2}\Omega_{t^{\ell+1}_n+(\lambda^{\ell+1}_n)^{2\sigma}t^{i,\ell+1},x^{\ell+1}_n+\lambda^{\ell+1}_nx^{i,\ell+1},\lambda^{\ell+1}_n\delta}\right)}.
\end{aligned}$$

As previously, it is easy to see that there exists $n_{\ell+1}$ such that for all $n\geqslant n_{\ell+1}$ and for all $i\in I_2$ we have $\Omega_{t^i_n,x^i_n,\lambda^i_nR^i}\subset\Omega_{t^{\ell+1}_n+(\lambda^{\ell+1}_n)^{2\sigma}t^{i,\ell+1},x^{\ell+1}_n+\lambda^{\ell+1}_nx^{i,\ell+1},\lambda^{\ell+1}_n\delta}$. Hence $\Omega_{t^i_n,x^i_n,\lambda^i_nR^i}$ and
$\Omega_{t^{\ell+1}_n,x^{\ell+1}_n,\lambda^{\ell+1}_nR^{\ell+1}}\setminus\bigcup_{i\in I_2}\Omega_{t^{\ell+1}_n+(\lambda^{\ell+1}_n)^{2\sigma}t^{i,\ell+1},x^{\ell+1}_n+\lambda^{\ell+1}_nx^{i,\ell+1},\lambda^{\ell+1}_n\delta}$ are disjoint and we get the desired inequality :

$$\sum_{i=1}^{\ell+1}\|S(\cdot)V^i\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}-\sum_{i=1}^{\ell+1}\frac{\varepsilon}{2^i}\leqslant\liminf_{n\to\infty}\left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L^{\overline{q}}_t\dot{H}^{\gamma}_{\overline{q}}}.$$

Then we have for all $k \in \mathbf{N}^*$ and for all $\varepsilon > 0$,

$$\sum_{i=1}^{k} \|S(\cdot)V^i\|^{\overline{q}}_{L_t^{\overline{q}}\dot{H}^{\gamma}_{\overline{q}}} - \varepsilon\left(1 - \frac{1}{2^k}\right) \leqslant \liminf_{n\to\infty} \left\|S(\cdot)u_{j(n)}\right\|^{\overline{q}}_{L_t^{\overline{q}}\dot{H}^{\gamma}_{\overline{q}}},$$

and this yields (2.3.20) because ε is arbitrary.

It follows from (2.3.20) that the series $\left(\sum \|S(\cdot)V^i\|^{\overline{q}}_{L_t^{\overline{q}}\dot{H}^{\gamma}_{\overline{q}}}\right)_{i\geqslant 1}$ converges and in particular $\|S(\cdot)V^i\|_{L_t^{\overline{q}}\dot{H}^{\gamma}_{\overline{q}}} \underset{i\to\infty}{\longrightarrow} 0$. By Corollary 2.2.2 we have $\|S(\cdot)V^i\|_{L_t^{\infty}\dot{H}^{\gamma}_{2^*}} \underset{i\to\infty}{\longrightarrow} 0$ and property (I) above, which derives from the construction of $(V^i)_{i\in\mathbf{N}^*}$, enables us to affirm that

$$\Lambda(\mathbf{w}^k) \underset{k\to\infty}{\longrightarrow} 0.$$

By Corollary 2.2.1 we have

$$\lim_{k\to\infty}\left(\limsup_{n\to\infty} \|S(\cdot)w_n^k\|_{L_t^{\infty}\dot{H}^{\gamma}_{2^*}}\right) = 0,$$

and we get (2.1.16). By the interpolation inequalities (2.1.11)-(2.1.13) it is easy to see that

$$\lim_{k\to\infty}\left(\limsup_{n\to\infty} \|S(\cdot)w_n^k\|_{L_t^{p}\dot{H}^{\gamma}_{q}}\right) = 0$$

for any admissible pair (p, q). Now, we claim that for all $k \in \mathbf{N}^*$,

$$\left\|\sum_{i=1}^{k} S(\cdot - t_n^i)\left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right]\right\|^{q_*}_{L_t^{p}\dot{H}^{\gamma}_{q}} \leqslant \sum_{i=0}^{k} \left\|S(\cdot)V^i\right\|^{q_*}_{L_t^{p}\dot{H}^{\gamma}_{q}} + \underset{n\to\infty}{o(1)}, \tag{2.3.21}$$

where $q_* := \min(p, q)$. We will use the following elementary inequality : for any $k \in \mathbf{N}^*$ and any $q \in (1, \infty)$, there exists $C_k > 0$, depending only on k and on q such that

$$\left|\left|\sum_{i=1}^{k} a_i\right|^q - \sum_{i=1}^{k} |a_i|^q\right| \leqslant C_k \sum_{\substack{i,l=1\\ i\neq l}}^{k} |a_i|^{q-1}|a_l|, \ \forall a_1, ..., a_k \in \mathbf{C}.$$

We denote

$$f_n^i := S(t - t_n^i)|\nabla|^{\gamma}\left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right] = S(t - t_n^i)\left(|\nabla|^{\gamma}\left[\mathfrak{d}_{\lambda_n^i} V^i\right](\cdot - x_n^i)\right).$$

We fix $t \in \mathbf{R}$, and we denote the reminder of the sum

$$R_n^k(t) := \int_{\mathbf{R}^N} \left|\sum_{i=1}^{k} f_n^i\right|^q - \sum_{i=1}^{k} \left|f_n^i\right|^q dx.$$

By using the elementary inequality above and the subadditivity of the norm in $L_t^{\frac{p}{q}}$ in the case $p \geqslant q$, respectively the subadditivity of the mapping $\tau \mapsto \tau^{\frac{p}{q}}$ on $[0,\infty)$ and the fact that $\frac{q_*}{p} = 1$ in the case $p < q$ we get

$$\begin{aligned}\left(\int_{\mathbf{R}} \left|R_n^k(t)\right|^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}} &\leqslant C_k^{\frac{q_*}{q}} \left(\int_{\mathbf{R}} \left|\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{k} \int_{\mathbf{R}^N} |f_n^i||f_n^\ell|^{q-1} dx\right|^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}} \\ &\leqslant C_k^{\frac{q_*}{q}} \sum_{\substack{i,\ell=1\\ i\neq\ell}}^{k} \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} |f_n^i||f_n^\ell|^{q-1} dx\right)^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}}.\end{aligned}$$

By a change of variables we obtain the following identity :

$$\begin{aligned}&\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} |f_n^i||f_n^\ell|^{q-1} dx\right)^{\frac{p}{q}} dt \\ &\quad = \int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} \left|S\left(\cdot + \frac{t_n^\ell - t_n^i}{(\lambda_n^\ell)^{2\sigma}}\right) |\nabla|^\gamma \left[\mathfrak{d}_{\lambda_n^i(\lambda_n^\ell)^{-1}} V^i\right]\left(\cdot + \frac{x_n^\ell - x_n^i}{\lambda_n^\ell}\right)\right| \left|S(\cdot)|\nabla|^\gamma V^\ell\right|^{q-1} dx\right)^{\frac{p}{q}} dt,\end{aligned}$$

The last expression tends to zero as $n \to \infty$ by Lemma 2.3.3 because $(\boldsymbol{\lambda}^i, \mathbf{t}^i, \mathbf{x}^i)$ and $(\boldsymbol{\lambda}^\ell, \mathbf{t}^\ell, \mathbf{x}^\ell)$ are orthogonal for $i \neq \ell$. Then $\displaystyle\int_{\mathbf{R}} \left|R_n^k(t)\right|^{\frac{p}{q}} dt = \underset{n\to\infty}{o(1)}$ and using again the subadditivity of the norm in $L_t^{\frac{p}{q}}$ in the case $p \geqslant q$, respectively the concavity of the mapping $\tau \mapsto \tau^{\frac{p}{q}}$ on $[0,\infty)$ if $p < q$, as well as (2.1.3), it is easy to see that

$$\begin{aligned}\left\|\sum_{i=1}^k f_n^i\right\|_{L_t^p L_x^q}^{q_*} &= \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} \sum_{i=1}^k \left|f_n^i\right|^q dx + R_n^k(t)\right)^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}} \\ &\leqslant \sum_{i=1}^k \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^N} \left|f_n^i\right|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}} + \left(\int_{\mathbf{R}} \left|R_n^k(t)\right|^{\frac{p}{q}} dt\right)^{\frac{q_*}{p}} \\ &= \sum_{i=1}^k \left\|S(\cdot)V^i\right\|_{L_t^p \dot{H}_q^\gamma}^{q_*} + \underset{n\to\infty}{o(1)},\end{aligned}$$

and this yields (2.3.21). It remains to prove (2.1.17), (2.1.20) and (2.1.21). We have :

$$\begin{aligned}\|S(\cdot)u_{j(n)}\|_{L_t^p \dot{H}_q^\gamma}^{q_*} &= \left\|\sum_{i=1}^k S(\cdot - t_n^i)\left[(\lambda_n^i)^{\gamma - \frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right] + S(\cdot)w_n^k\right\|_{L_t^p \dot{H}_q^\gamma}^{q_*} \\ &\leqslant \left(\left\|\sum_{i=1}^k S(\cdot - t_n^i)\left[(\lambda_n^i)^{\gamma - \frac{N}{2^*}} V^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right)\right]\right\|_{L_t^p \dot{H}_q^\gamma} + \left\|S(\cdot)w_n^k\right\|_{L_t^p \dot{H}_q^\gamma}\right)^{q_*},\end{aligned}$$

and by using (2.3.21) and by passing to the limit,

$$
\begin{aligned}
&\limsup_{n\to\infty} \|S(\cdot)u_{j(n)}\|^{q_*}_{L^p_t\dot H^\gamma_q} \\
&\leqslant \left(\limsup_{n\to\infty} \left\| \sum_{i=1}^k S(\cdot - t_n^i) \left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i \left(\frac{\cdot - x_n^i}{\lambda_n^i} \right) \right] \right\|_{L^p_t\dot H^\gamma_q} + \limsup_{n\to\infty} \left\| S(\cdot) w_n^k \right\|_{L^p_t\dot H^\gamma_q} \right)^{q_*} \\
&= \left(\limsup_{n\to\infty} \left\| \sum_{i=1}^k S(\cdot - t_n^i) \left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i \left(\frac{\cdot - x_n^i}{\lambda_n^i} \right) \right] \right\|_{L^p_t\dot H^\gamma_q} + \underset{k\to\infty}{o(1)} \right)^{q_*} \\
&= \limsup_{n\to\infty} \left\| \sum_{i=1}^k S(\cdot - t_n^i) \left[(\lambda_n^i)^{\gamma-\frac{N}{2^*}} V^i \left(\frac{\cdot - x_n^i}{\lambda_n^i} \right) \right] \right\|^{q_*}_{L^p_t\dot H^\gamma_q} + \underset{k\to\infty}{o(1)} \\
&\leqslant \sum_{i=1}^k \left\| S(\cdot) V^i \right\|^{q_*}_{L^p_t\dot H^\gamma_q} + \underset{k\to\infty}{o(1)}.
\end{aligned}
$$

and this yields (2.1.20) and (2.1.21). The above inequality in the case $p = q = \overline{q}$ and (2.3.20) give (2.1.17). □

To prove Theorem 1.2, we will show that if we have a profile decomposition of an optimizing sequence $\mathbf{u}$ as in Theorem 1.1, then there is only one non-trivial profile $V^{i_0} \neq 0$ and we must have $V^i = 0$ for all $i \neq i_0$.

Proof of Theorem 1.2. Let $\mathbf{u}$ be a maximizing sequence. We use the same notation as in Theorem 1.1.
Since $\|S(\cdot)V^i\|_{L^p_t\dot H^\gamma_q}$ goes to zero as $i \to \infty$, there exists $i_0 \in \mathbf{N}^*$ such that $\|S(\cdot)V^{i_0}\|_{L^p_t\dot H^\gamma_q} = \sup_{i\in\mathbf{N}^*} \|S(\cdot)V^i\|_{L^p_t\dot H^\gamma_q}$.
Moreover, by (2.1.18), $\sum_{i=1}^\infty \|V^i\|^2_{\dot H^{s+\gamma}} \leqslant \lim_{n\to\infty} \|u_{j(n)}\|^2_{\dot H^{s+\gamma}} = 1$.
By Theorem 1.1, we get

$$
\begin{aligned}
(C_S)^{q_*} = \lim_{n\to\infty} \|S(\cdot)u_{j(n)}\|^{q_*}_{L^p_t\dot H^\gamma_q} &\leqslant \sum_{i=1}^\infty \|S(\cdot)V^i\|^{q_*}_{L^p_t\dot H^\gamma_q} \\
&\leqslant \|S(\cdot)V^{i_0}\|^{q_*-2}_{L^p_t\dot H^\gamma_q} \sum_{i=1}^\infty \|S(\cdot)V^i\|^2_{L^p_t\dot H^\gamma_q} \\
&\leqslant (C_S)^{q_*-2} \|V^{i_0}\|^{q_*-2}_{\dot H^{s+\gamma}} (C_S)^2 \sum_{i=1}^\infty \|V^i\|^2_{\dot H^{s+\gamma}} \\
&\leqslant (C_S)^{q_*} \|V^{i_0}\|^{q_*-2}_{\dot H^{s+\gamma}}.
\end{aligned}
$$

Then $1 \leqslant \|V^{i_0}\|_{\dot{H}^{s+\gamma}}$ and this implies that $\|V^{i_0}\|_{\dot{H}^{s+\gamma}} = 1$ and $\|V^i\|_{\dot{H}^{s+\gamma}} = 0$ for all $i \neq i_0$. Recall that by Theorem 1.1, $\mathfrak{d}_{(\lambda_n^{i_0})^{-1}} \left[S(t_n^{i_0})u_{j(n)}(\cdot + x_n^{i_0})\right] \underset{n\to\infty}{\rightharpoonup} V^{i_0}$ in $\dot{H}^{s+\gamma}$.
Since we have weak convergence and convergence of norms in the Hilbert space $\dot{H}^{s+\gamma}$, we infer that

$$\mathfrak{d}_{(\lambda_n^{i_0})^{-1}} \left[S(t_n^{i_0})u_{j(n)}(\cdot + x_n^{i_0})\right] \underset{n\to\infty}{\to} V^{i_0} \qquad \text{strongly in } \dot{H}^{s+\gamma},$$

and this is the desired conclusion. □

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Chapitre 3

Periodic traveling waves for nonlinear Schrödinger equations with non-zero conditions at infinity in $\mathbf{R}^2$

3.1 Introduction

We consider the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta \Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^2 \times \mathbf{R}. \tag{3.1.1}$$

For a given $\Lambda > 0$, we are interested in solutions $\Phi : \mathbf{R}^2 \times \mathbf{R} \longrightarrow \mathbf{C}$ of (3.1.1) that are $\Lambda-$periodic with respect to the second variable, namely $\Phi(x, y + \Lambda, t) = \Phi(x, y, t)$ for all (x, y, t) and satisfy the "boundary condition" $|\Phi(x, y, t)| \longrightarrow r_0$ as $x \longrightarrow \pm\infty$, where $r_0 > 0$ and F is a real-valued function on $[0, \infty)$ such that $F(r_0^2) = 0$.

If $F'(r_0^2) < 0$ (which means that (3.1.1) is defocusing), a simple scaling enables us to assume that $r_0 = 1$ and $F'(r_0^2) = -1$ (see [11], p. 108) ; we will do so throughout this paper. The sound velocity at infinity associated to (3.1.1) is then $v_s = r_0\sqrt{-2F'(r_0^2)} = \sqrt{2}$.

Equation (3.1.1) has a Hamiltonian structure. Indeed, let $V(s) = \int_s^1 F(\tau)\, d\tau$. It is then easy to see that if Φ is a solution to (3.1.1) and Φ is $\Lambda-$periodic with respect to the second variable, then at least formally, the "energy"

$$E(\Phi) = \int_{\mathbf{R}\times[0,\Lambda]} |\frac{\partial \Phi}{\partial x}|^2 + |\frac{\partial \Phi}{\partial y}|^2 + V(|\Phi|^2)\, dx\, dy \tag{3.1.2}$$

does not depend on t.

In the sequel we prefer to normalize the period to 1 : instead of working with functions Φ that are $\Lambda-$periodic with respect to the second variable, we will consider the function $\tilde{\Phi}(x, y, t) = \Phi(x, \Lambda y, t)$

which is is 1−periodic with respect to the second variable. It is clear that Φ is a solution of (3.1.1) if an only if $\tilde{\Phi}$ satisfies the equation

$$i\frac{\partial\tilde{\Phi}}{\partial t}+\frac{\partial^2\tilde{\Phi}}{\partial x^2}+\frac{1}{\Lambda^2}\frac{\partial^2\tilde{\Phi}}{\partial y^2}+F(|\tilde{\Phi}|^2)\tilde{\Phi}=0 \qquad \text{in } \mathbf{R}^2. \tag{3.1.3}$$

We denote $\lambda=\frac{1}{\Lambda}$ and we consider the renormalized energy

$$E_\lambda(\Psi)=\int_{\mathbf{R}\times[0,1]}\Big|\frac{\partial\Psi}{\partial x}\Big|^2+\lambda^2\Big|\frac{\partial\Psi}{\partial y}\Big|^2+V(|\Psi|^2)\,dx\,dy. \tag{3.1.4}$$

If Φ and $\tilde{\Phi}$ are as above, we have $E(\Phi)=\frac{1}{\lambda}E_\lambda(\tilde{\Phi})$.

We are interested in traveling waves for (3.1.1), which are solutions of the form $\Phi(x,y,t)=\psi(x+ct,y)$. If ψ is a traveling wave and is $\Lambda-$periodic with respect to the variable y, then $\tilde{\psi}(x,y)=\psi(x,\Lambda y)=\psi(x,\frac{y}{\lambda})$ is 1−periodic with respect to y and satisfies the equation

$$ic\frac{\partial\tilde{\psi}}{\partial x}+\frac{\partial^2\tilde{\psi}}{\partial x^2}+\frac{1}{\Lambda^2}\frac{\partial^2\tilde{\psi}}{\partial y^2}+F(|\tilde{\psi}|^2)\tilde{\psi}=0 \qquad \text{in } \mathbf{R}^2. \tag{3.1.5}$$

Assumptions and some comments on the assumptions. We will work with general nonlinearities F. We will consider the set of assumptions **(A1), (A2), (B1), (B2)** that we describe below. We will assume throughout the paper that **(A1)** holds. The other assumptions are not needed all the time and sometimes they can be slightly relaxed. For each result we will indicate the precise conditions that we use. For instance, assumption **(A2)** is not necessary when we deal with one-dimensional traveling waves to (3.1.1). We aimed to consider sufficiently general nonlinearities and in the meantime to focus on ideas, avoiding irrelevant technicalities.

(A1) The function F is continuous on $[0,\infty)$, C^1 in a neighborhood of 1, $F(1)=0$ and $F'(1)<0$.

(A2) There exist $C>0$ and $1<p_0<\infty$ such that $|F(s)|\le C(1+s^{p_0})$ for any $s\ge 0$.

We denote

$$V(s)=\int_s^1 F(\tau)\,d\tau,$$

so that $V(1)=0$ and $V'(s)=-F(s)$. If assumption **(A1)** holds, we have

$$V(s)=\frac{1}{2}(s-1)^2+o\Big((s-1)^2\Big) \qquad \text{as } s\longrightarrow 1. \tag{3.1.6}$$

If **(A2)** holds, there is $C'>0$ such that $|V(s)|\leqslant C's^{p_0+1}$ for all $s\geqslant 2$.

The natural function space associated to (3.1.3) is

$$\mathcal{E}=\{\psi\in L^1_{loc}(\mathbf{R}^2)\ \mid\ \psi \text{ is } 1-\text{periodic with respect to the second variable,}\\ \nabla\psi\in L^2(\mathbf{R}\times[0,1]) \text{ and } V(|\psi|^2)\in L^1(\mathbf{R}\times[0,1])\}.$$

We will also consider the one-dimensional variant of $\mathcal{E}$, namely

$$\mathcal{E}(\mathbf{R}) = \{\psi \in L^1_{loc}(\mathbf{R}^2) \ \mid \ \psi' \in L^2(\mathbf{R}) \text{ and } V(|\psi|^2) \in L^1(\mathbf{R})\}$$

and the associated $1-$dimensional energy

$$E^1(\psi) = \int_{\mathbf{R}} |\psi'(x)|^2 + V(|\psi|^2)(x)\, dx.$$

As we can see, assumption **(A1)** determines the behaviour of the nonlinear potential V in a neighbourhood of 1, and **(A2)** gives upper bounds on $V(s)$ for large s. In view of (3.1.6), the renormalized Ginzburg-Landau energy

$$E_{GL,\lambda}(\psi) = \int_{\mathbf{R}\times[0,1]} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2 \Big|\frac{\partial\psi}{\partial y}\Big|^2 + \frac{1}{2}\left(1 - |\psi|^2\right)^2 dx\, dy,$$

together with its $1-$dimensional variant

$$E^1_{GL}(\psi) = \int_{\mathbf{R}} |\psi'(x)|^2 + \frac{1}{2}\left(1 - |\psi(x)|^2\right)^2 dx \qquad \text{for } \psi \in \mathcal{E}(\mathbf{R})$$

will be relevant throughout the article. Notice that the renormalized Ginzburg-Landau energy is simply the energy associated to the Gross-Pitaevskii nonlinear potential $V(s) = \frac{1}{2}(1-s)^2$ corresponding to the nonlinearity $F(s) = 1 - s$.

Whenever minimization of energy at fixed momentum is considered, we need to assume that $V \geqslant 0$ on $[0,\infty)$ (for otherwise, the infimum is $-\infty$). For simplicity, we will assume that $V(s) > 0$ for $s \neq 1$. The next assumptions give lower bounds on V and will be useful to estimate the Ginzburg-Landau energy $E_{GL}(\psi)$ in terms of $E(\psi)$; see Lemmas 3.2.1 and 3.2.3 in the next section.

(B1) We have $V > 0$ on $[0,\infty) \setminus \{1\}$ and, denoting $H(s) = \displaystyle\int_1^s |V(\tau^2)|^{\frac{1}{2}} d\tau$, we have

$$H(s) \longrightarrow \infty \qquad \text{as } s \longrightarrow \infty.$$

(B2) $V > 0$ on $[0,\infty) \setminus \{1\}$ and there exists $\gamma > 0$ and $s_0 \geqslant 1$ such that $V(s) \geqslant s^\gamma$ for all $s \geqslant s_0$.

Clearly, **(B2)** is stronger than **(B1)**. If $V > 0$ on $[0,\infty) \setminus \{1\}$, then the function H is strictly increasing on $[0,\infty)$ and $H(1) = 0$. If we do not assume that $H(s) \longrightarrow \infty$ as $s \longrightarrow \infty$, it is possible to construct sequences of functions $\psi_n : \mathbf{R} \longrightarrow \mathbf{C}$ such that $E^1(\psi_n)$ is bounded, but the Ginzburg-Landau energy $E^1_{GL}(\psi_n)$ is unbounded, a situation that we would like to avoid. Assumption **(A1)** is enough to study $1-$dimensional traveling-waves of (3.1.1) by using ODE arguments, as we do in section 4. Assumptions **(A1)** and **(B1)** are sufficient to obtain $1-$dimensional traveling-waves by minimizing the energy at fixed momentum, as in section 5.

When we consider functions defined on the whole space $\mathbf{R}^N$, $N \geqslant 2$, assumptions **(A1)** and **(A2)** (with some p_0 smaller than the crtical Sobolev exponent) and the fact that $V \geqslant 0$ are sufficient

to prove that $E_{GL}(\psi)$ is bounded whenever $E(\psi)$ is bounded, and vice-versa ; see Lemma 4.8 p. 177 in [6]. This is no longer true in the one-dimensional space $\mathbf{R}$ or in a strip $\mathbf{R} \times (0,1)$.

Momentum. There is another important quantity conserved by equations (3.1.1) and by (3.1.3), namely *the momentum.* It carries some topological information and defining rigorously the momentum is a difficulty in itself. We will address this issue in Section 3. Roughly speaking, the momentum is a functional Q whose Gâteaux differential is $Q'(\psi) = 2i\frac{\partial \psi}{\partial x}$. If $\psi \in \mathcal{E}$ is a function such that there exists $A \geqslant 0$ such that $\psi(x,y) = 1$ if $x \leqslant -A$ and if $x \geqslant A$, we have $Q(\psi) = \int_{\mathbf{R}\times[0,1]} \langle i\frac{\partial \psi}{\partial x}, \psi\rangle \, dx\, dy$. Given an arbitrary function $\psi \in \mathcal{E}$, the function $\langle i\frac{\partial \psi}{\partial x}, \psi\rangle$ does not necessarily belong to $L^1(\mathbf{R}\times[0,1])$ and giving a meaning to the above integral is not obvious. For functions $\psi \in \mathcal{E}(\mathbf{R})$ there is a one-dimensional variant of the momentum that we will denote $P(\psi)$. We will see that the momentum can be defined only modulo 2π.

Brief description of the results. In this article we will focus on traveling waves for (3.1.1) that minimize the energy when the momentum is kept fixed. In view of a celebrated result by T. Cazenave and P.-L. Lions [4], such solutions are expected to be orbitally stable by the flow associated to (3.1.1).

We will denote

$$E_{\lambda,min}(p) = \inf\{E_\lambda(\psi) \ \mid \ \psi \in \mathcal{E} \text{ and } Q(\psi) = p\}.$$

and

$$E^1_{min}(p) = \inf\{E_\lambda(\psi) \ \mid \ \psi \in \mathcal{E}(\mathbf{R}) \text{ and } P(\psi) = p\}.$$

The main results of this article can be summarized as follows.

Theorem 3.1.1. *Assume that the conditions* **(A1), (A2), (B2)** *above are satisfied. Then :*

i) The function E^1_{min} is nonnegative, $2\pi-$periodic, concave on $[0,2\pi]$, $E^1_{min}(-p) = E^1_{min}(p)$, $E^1_{min}(p) \leqslant \sqrt{2}p$ and the right-derivative of E^1_{min} at the origin is $\sqrt{2}$. If F is C^2 near 1 and $F''(1) < \frac{9}{4}$ we have $E^1_{min}(p) < \sqrt{2}|p|$ for any $p \neq 0$.

For any $p \in (0,\pi]$ satisfying $E^1_{min}(p) < \sqrt{2}p$ there exist minimizers for $E^1_{min}(p)$ in $\mathcal{E}(\mathbf{R})$ and all minimizing sequences are pre-compact[1] *modulo translations.*

ii) For any fixed $\lambda > 0$ the function $p \longmapsto E_{\lambda,min}(p)$ is $2\pi-$periodic, concave on $[0,2\pi]$, $E_{\lambda,min}(p) \leqslant \sqrt{2}p$, the right-derivative of $E_{\lambda,min}$ at the origin is $\sqrt{2}$, and $E_{\lambda,min}(p) \leqslant E^1_{min}(p)$.

iii) For any fixed p there exists $\lambda_*(p) > 0$ such that the mapping $\lambda \longmapsto E_{\lambda,min}(p)$ is strictly increasing on $(0,\lambda_*(p)]$, and $E_{\lambda,min}(p) = E^1_{min}(p)$ for all $\lambda \geqslant \lambda_*(p)$.

iv) For any $\lambda > 0$ and any $p \in (0,\pi]$ satisfying $E_{\lambda,min}(p) < \sqrt{2}p$ there exist minimizers for $E_{\lambda,min}(p)$ in $\mathcal{E}$ and all minimizing sequences are pre-compact[2] *modulo translations.*

1. This statement is vague because we did not introduce a distance on $\mathcal{E}(\mathbf{R})$. Please see Theorem 3.5.2 for a precise statement.

2. See Theorem 3.6.6 for a precise statement.

Outline of the paper. In the next section we present the functional setting and we show that one can bound the energies E^1 and E_λ in terms of the associated Ginzburg-Landau energies, and vice-versa. In Section 3 we give a rigorous definition of the momentum, firstly for functions in $\mathcal{E}(\mathbf{R})$ and then for functions in $\mathcal{E}$. Section 4 is devoted to the study of one-dimensional traveling waves by using ODE arguments. In the case of the Gross-Pitaevskii equation ($F(s) = 1-s$) those results were already known, see [1] and references therein. Different behaviour may occur when we consider other nonlinearities. In section 5 we study the $1-$dimensional minimization problem associated to E^1_{min} and we prove part (i) in Theorem 3.1.1. Section 6 is devoted to the $2-$dimensional minimization problem for $E_{\lambda,min}$, and to the rest of the proof of Theorem 3.1.1.

Notation. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbf{C} \simeq \mathbf{R}^2$, namely $\langle a+ib, c+id\rangle = ac+bd$, and by $\mathcal{L}^N$ the Lebesgue measure in $\mathbf{R}^N$. We denote by $C_{a_1,\dots,a_\ell}$ or by $C(a_1,\dots,a_\ell)$ a positive constant that may change from line to line, but depends only on the parameters $a_1,\dots,a_\ell$.

3.2 Energy and function spaces

The precise representative of a given function $f \in L^1_{loc}(\mathbf{R}^N)$ is the function f^* defined on $\mathbf{R}^N$ by $f^*(x) = \lim\limits_{r\to 0} \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} f(y)\,dy$ if the limit exists, and 0 otherwise. It is well-known that for any $f \in L^1_{loc}(\mathbf{R}^N)$ we have $f = f^*$ almost everywhere (see, e.g., Corollary 1 p. 44 in [7]). In the sequel we will always replace functions in $L^1_{loc}(\mathbf{R}^N)$ by their precise representatives. We denote

$$\mathcal{E}(\mathbf{R}) = \{\psi \in L^1_{loc}(\mathbf{R}) \ \mid \ \psi' \in L^2(\mathbf{R}),\, V(|\psi|^2) \in L^1(\mathbf{R})\}$$

For any given $\psi \in \mathcal{E}(\mathbf{R})$ and any interval $I \subset \mathbf{R}$ we denote

$$E^{1,I}(\psi) = \int_I |\psi'|^2 + V(|\psi|^2)\,dx \qquad \text{and} \qquad E^{1,I}_{GL}(\psi) = \int_I |\psi'|^2 + \frac{1}{2}(|\psi|^2-1)^2\,dx.$$

When $I = \mathbf{R}$ we write simply $E^1(\psi)$ and $E^1_{GL}(\psi)$.

Lemma 3.2.1. *Assume that* **(A1)** *and* **(B1)** *hold. Then :*

i) Any function $\psi \in \mathcal{E}(\mathbf{R})$ is bounded and $\frac{1}{2}-$Hölder continuous on $\mathbf{R}$. There exists a function $b : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\lim\limits_{\tau\to 0} b(\tau) = 0$, $\lim\limits_{\tau\to\infty} b(\tau) = \infty$ such that

$$\| \, |\psi| - 1\|_{L^\infty(\mathbf{R})} \leqslant b(E^1(\psi)) \qquad \textit{for any } \psi \in \mathcal{E}(\mathbf{R}).$$

ii) For any $\psi \in \mathcal{E}(\mathbf{R})$ we have $|\psi| - 1 \in H^1(\mathbf{R})$ and

$$\begin{aligned} \mathcal{E}(\mathbf{R}) &= \{\psi \in H^1_{loc}(\mathbf{R}) \ \mid \ \psi' \in L^2(\mathbf{R}) \textit{ and } |\psi|^2 - 1 \in L^2(\mathbf{R})\} \\ &= \{\psi \in H^1_{loc}(\mathbf{R}) \ \mid \ \psi' \in L^2(\mathbf{R}) \textit{ and } |\psi| - 1 \in L^2(\mathbf{R})\}. \end{aligned}$$

Furthermore, there exist functions $b_1, b_2 : [0,\infty) \longrightarrow [0,\infty)$ *such that* $\lim_{\tau\to 0} b_i(\tau) = 0$, $\lim_{\tau\to\infty} b_i(\tau) = \infty$ *for* $i = 1, 2$, *and we have*

$$E^1(\psi) \leqslant b_1(E^1_{GL}(\psi)) \quad \text{and} \quad E^1_{GL}(\psi) \leqslant b_2(E^1(\psi)) \quad \text{for any } \psi \in \mathcal{E}(\mathbf{R}).$$

For any $\psi \in \mathcal{E}(\mathbf{R})$ *and any* $v \in H^1(\mathbf{R})$ *we have* $\psi + v \in \mathcal{E}(\mathbf{R})$.

Proof. (i) Let $\psi \in \mathcal{E}(\mathbf{R})$. Since $\psi \in L^1_{loc}(\mathbf{R})$ and $\psi' \in L^1_{loc}(\mathbf{R})$, it follows from Theorem 8.2 p. 204 in [3] that ψ is equal almost everywhere to a continuous function and

$$\psi(b) - \psi(a) = \int_a^b \psi'(s)\, ds.$$

The Cauchy-Schwarz inequality gives

$$|\psi(b) - \psi(a)| \leqslant |b-a|^{\frac{1}{2}} \|\psi'\|_{L^2([a,b])} \qquad \text{for any } a, b \in \mathbf{R}. \tag{3.2.1}$$

It is well-known that $|\psi| \in L^1_{loc}(\mathbf{R})$ and $\big|\,|\psi|'\,\big| \leqslant |\psi'|$ almost everywhere. Using the Cauchy-Schwarz inequality and denoting $h(s) = \sqrt{V(s^2)}$ we get for any $a, b \in \mathbf{R}$, $a < b$,

$$\begin{aligned} \int_a^b |\psi'|^2 + V(|\psi|^2)\, dx &\geqslant \int_a^b \big|\,|\psi|'\,\big|^2 + h^2(|\psi|)\, dx \\ &\geqslant 2\Big|\int_a^b h(|\psi|)\cdot|\psi|'\, dx\Big| = 2\Big|H(|\psi(b)|) - H(|\psi(a)|)\Big|. \end{aligned} \tag{3.2.2}$$

Using (3.2.2) and **(B1)** it is easily seen that any function $\psi \in \mathcal{E}(\mathbf{R})$ is bounded. Since $V(|\psi|^2) \in L^1(\mathbf{R})$, there are sequences $a_n \longrightarrow -\infty$ and $b_n \longrightarrow \infty$ such that $|\psi(a_n)| \longrightarrow 1$ and $|\psi(b_n)| \longrightarrow 1$. For any $x \in \mathbf{R}$ we use (3.2.2) on $[a_n, x]$ and on $[x, b_n]$ and we let $n \longrightarrow \infty$ to get $4\big|H(|\psi(x)|)\big| \leqslant E^1(\psi)$. We infer that

$$\big|\,|\psi(x)| - 1\big| \leqslant \max\left(H^{-1}\left(\frac{1}{4}E^1(\psi)\right) - 1, 1 - H^{-1}\left(-\frac{1}{4}E^1(\psi)\right)\right) \qquad \text{if } 0 \leqslant |\psi(x)| \leqslant 2,$$

respectively $|\psi(x)| - 1 \leqslant H^{-1}\left(\frac{1}{4}E^1(\psi)\right) - 1$ if $|\psi(x)| > 2$, and (i) is proven.

(ii) Assume that $\psi' \in L^2(\mathbf{R})$ and $V(|\psi|^2) \in L^1(\mathbf{R})$. By **(A1)** and **(B1)** there exists $C > 0$ such that $(1-s)^2 \leqslant CV(s)$ for all $s \in [0,4]$. Then $\big(1 - |\psi|^2\big)^2 \mathbb{1}_{\{|\psi|\leqslant 2\}} \leqslant CV(|\psi|^2) \in L^1(\mathbf{R})$.

The set $A = \{x \in \mathbf{R} \;\mid\; |\psi(x)| > 2\}$ has finite Lebesgue measure in $\mathbf{R}$. Indeed, by (i) we know that ψ is bounded. If $m := \sup|\psi| > 2$, let $i(m) = \inf_{s\in[4,m^2]} V(s)$. By **(B1)** we know that $i(m) > 0$ and we have the rough estimate $\int_{\mathbf{R}} V(|\psi|^2)\, dx \geqslant \int_A V(|\psi|^2)\, dx \geqslant i(m)\mathcal{L}^1(A)$, hence $\mathcal{L}^1(A) \leqslant \frac{1}{i(m)}\|V(|\psi|^2)\|_{L^1(\mathbf{R})}$ and

$$\int_A \Big(1 - |\psi|^2\Big)^2\, dx \leqslant (m^2-1)^2\mathcal{L}^1(A) \leqslant \frac{(m^2-1)^2}{i(m)}\|V(|\psi|^2)\|_{L^1(\mathbf{R})}.$$

It follows from the above estimates that $1-|\psi|^2 \in L^2(\mathbf{R})$ and there exists a function b_2 with the desired properties.

Conversely, if $1-|\psi|^2 \in L^2(\mathbf{R})$ a similar argument shows that ψ is bounded, $V(|\psi|^2) \in L^1(\mathbf{R})$ and $E^1(\psi)$ can be estimated in terms of $E^1_{GL}(\psi)$.

We have $\big|\,|\psi|-1\big| = \frac{\big||\psi|^2-1\big|}{|\psi|+1} \leqslant \big||\psi|^2-1\big|$, hence $|\psi|-1 \in L^2(\mathbf{R})$ whenever $|\psi|^2-1 \in L^2(\mathbf{R})$. Conversely, if $\psi' \in L^2(\mathbf{R})$ and $|\psi|-1 \in L^2(\mathbf{R})$ we see as above that ψ is bounded and therefore $|\psi|^2-1 \in L^2(\mathbf{R})$.

For the last statement proceed as in the proof of Lemma 3.2.4 (i) below. □

The following result is contained in Theorem 1.8 p. 134 in [8] :

Lemma 3.2.2. *([8]) Let $\psi \in \mathcal{E}(\mathbf{R})$. Then :*

i) There exist a real-valued function $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in H^1(\mathbf{R})$ such that $\psi = e^{i\varphi}+w$.

ii) If (φ_1, w_1) and (φ_2, w_2) are as above, there exist $k_-, k_+ \in \mathbf{Z}$ such that $\varphi_1 - \varphi_2 - 2\pi k_\pm \in L^2(\mathbf{R}_\pm)$.

iii) Moreover, the function φ can be chosen such that $\varphi \in C^\infty(\mathbf{R})$ and $\varphi^{(k)} \in L^2(\mathbf{R})$ for any $k \in \mathbf{N}^*$.

As already mentioned in the introduction, the natural "energy space" for the study of (3.1.3) is

$$\begin{aligned}\mathcal{E} \;=\; & \{\psi \in L^1_{loc}(\mathbf{R}^2) \;\mid\; \psi \text{ is } 1-\text{periodic with respect to the second variable and} \\ & \nabla\psi \in L^2(\mathbf{R}\times[0,1]) \text{ and } V(|\psi|^2) \in L^1(\mathbf{R}\times[0,1])\}.\end{aligned} \tag{3.2.3}$$

Obviously, for any $\psi \in \mathcal{E}(\mathbf{R})$ the function $\psi^\sharp$ defined by $\psi^\sharp(x,y) = \psi(x)$ belongs to $\mathcal{E}$. We will also denote

$$H^1_{per} = \{v \in H^1_{loc}(\mathbf{R}^2) \;\mid\; v \text{ is } 1-\text{periodic in the second variable and } v \in H^1(\mathbf{R}\times(0,1))\}.$$

It is clear that $E_\lambda(\psi)$ is well-defined for any $\psi \in \mathcal{E}$ and for any $\lambda > 0$, where E_λ is as in (3.1.4). We will show in Lemma 3.2.3 below that for any $\psi \in \mathcal{E}$ we have $\big(|\psi|^2-1\big)^2 \in L^1(\mathbf{R}\times[0,1])$. For any $\psi \in \mathcal{E}$, $\lambda > 0$ and for any interval $I \subset \mathbf{R}$ we denote

$$E^I_\lambda(\psi) = \int_{I\times[0,1]} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial x}\Big|^2 + V(|\psi|^2)\,dx\,dy \quad \text{ and}$$

$$E^I_{GL,\lambda}(\psi) = \int_{I\times[0,1]} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial x}\Big|^2 + \frac{1}{2}\Big(|\psi|^2-1\Big)^2\,dx\,dy.$$

We will simply write $E_\lambda(\psi)$, respectively $E_{GL,\lambda}(\psi)$ when $I = \mathbf{R}$. We will write $E(\psi)$ and $E_{GL}(\psi)$ when $\lambda = 1$.

Lemma 3.2.3. *Assume that the conditions* **(A1), (A2)** *and* **(B2)** *in the introduction are satisfied. Let $\mathcal{E}$ be as in (3.2.3). Then we have*

$$\begin{aligned}\mathcal{E} &= \{\psi \in H^1_{loc}(\mathbf{R}^2) \mid \psi \text{ is } 1\text{−periodic with respect to the second variable and } \nabla\psi \in L^2(\mathbf{R}\times[0,1]) \text{ and } |\psi|-1 \in L^2(\mathbf{R}\times[0,1])\}\\ &= \{\psi \in H^1_{loc}(\mathbf{R}^2) \mid \psi \text{ is } 1\text{−periodic with respect to the second variable and } \nabla\psi \in L^2(\mathbf{R}\times[0,1]) \text{ and } |\psi|^2-1 \in L^2(\mathbf{R}\times[0,1])\}.\end{aligned}$$

In particular, for any $\psi \in \mathcal{E}$ we have $|\psi|-1 \in H^1_{per}$.

Moreover, for any $\lambda > 0$ there exist $a, b, c, d > 0$ such that for all $\psi \in \mathcal{E}$ and for any interval $I \subset \mathbf{R}$ of length at least 1 we have

$$E^I_\lambda(\psi) \leqslant aE^I_{GL,\lambda}(\psi) + bE^I_{GL,\lambda}(\psi)^{p_0+1} \qquad \text{and} \qquad E^I_{GL,\lambda}(\psi) \leqslant cE^I_\lambda(\psi) + dE^I_\lambda(\psi)^{\frac{2}{\gamma}}, \tag{3.2.4}$$

where $\gamma \in (0,1]$ is the exponent appearing in **(B2)**.

Proof. We have $\big|\,|\psi|^2-1\big| = \big|\,|\psi|-1\big| \cdot \big|\,|\psi|+1\big| \geqslant \big|\,|\psi|-1\big|$. If $|\psi|^2-1 \in L^2(\mathbf{R}\times[0,1])$, it is obvious that $|\psi|-1 \in L^2(\mathbf{R}\times[0,1])$.

Conversely, assume that $|\psi|-1 \in L^2(\mathbf{R}\times[0,1])$ and $\nabla\psi \in L^2(\mathbf{R}\times[0,1])$. Since $\big|\nabla|\psi|\,\big| \leqslant |\nabla\psi|$ almost everywhere, we infer that $|\psi|-1 \in H^1(\mathbf{R}\times[0,1])$ and the Sobolev embedding implies that $|\psi|-1 \in L^p(\mathbf{R}\times[0,1])$ for any $p \in [2,\infty)$. We have

$$\big|\,|\psi|^2-1\big| \leqslant 5\big|\,|\psi|-1\big|\mathbb{1}_{\{|\psi|\leqslant 4\}} + \frac{5}{3}\big|\,|\psi|-1\big|^2\mathbb{1}_{\{|\psi|>4\}}$$

and we infer that $|\psi|^2-1 \in L^2(\mathbf{R}\times[0,1])$.

We will repeatedly use the following simple observation. Let $I \subset \mathbf{R}$ be an interval of length greater than or equal to 1. Proceeding as in [3], Section 9.2 we may use four successive "mirror symmetries" to extend any function $u \in H^1(I\times(0,1))$ to a function $\tilde{u} \in H^1(\Omega_I)$, where Ω_I is a domain containing $(I+(-1,1))\times(-1,2))$. Then we choose a cut-off function $\chi \in C^\infty_c(\Omega_I)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $I\times[0,1]$ and $\nabla\chi$ is bounded independently of I. Denoting $P(u) = \chi\tilde{u}$, we see that $P(u) \in H^1(\mathbf{R}^2)$ and $\|Pu\|_{H^1(\mathbf{R}^2)} \leqslant C\|u\|_{H^1(I\times(0,1))}$, where C is independent of I. Using the Sobolev embedding in $\mathbf{R}^2$ we see that for any $p \in [2,\infty)$ there exists $C_p > 0$ depending only on p such that

$$\|u\|_{L^p(I\times(0,1))} \leqslant \|P(u)\|_{L^p(\mathbf{R}^2)} \leqslant C_p\|P(u)\|_{H^1(\mathbf{R}^2)} \leqslant CC_p\|u\|_{H^1(I\times(0,1))}.$$

Assume that $\psi \in H^1_{loc}(\mathbf{R}\times(0,1))$ and $E_{GL}(\psi) < \infty$. From the above arguments it follows that $|\psi|-1 \in H^1(\mathbf{R}\times(0,1))$ and $\||\psi|-1\|^2_{H^1(I\times(0,1))} \leqslant CE^I_{GL}(\psi)$ for any interval I of length at least 1. By **(A1)** and **(A2)** there exist $C_1, C_2 > 0$ such that

$$V(s^2) \leqslant C_1\left(s^2-1\right)^2 \quad \text{if } 0 \leqslant s \leqslant 2, \qquad \text{and} \qquad V(s^2) \leqslant C_2(s-1)^{2p_0+2} \quad \text{if } s > 2.$$

Using the Sobolev embedding we have

$$\int_{I\times(0,1)} V(|\psi|^2)\,dx \leqslant \int_{I\times(0,1)} C_1\left(|\psi|^2-1\right)^2 + C_2\big|\,|\psi|-1\big|^{2p_0+2}\,dx$$

$$\leqslant C_1E^I_{GL}(\psi) + C_2C_{2p_0+2}\|\,|\psi|-1\|^{2p_0+2}_{H^1(I\times(0,1))} \leqslant C_1E^I_{GL}(\psi) + C_2'E^I_{GL}(\psi)^{p_0+1}.$$

The first estimate in (3.2.4) is thus proven. If $I=\mathbf{R}$ we see that $V(|\psi|^2)\in L^1(\mathbf{R}\times(0,1))$.

Conversely, assume that **(B2)** holds, $\nabla\psi\in L^2(I\times[0,1])$ and $V(|\psi|^2)\in L^1(I\times[0,1])$. Without loss of generality we may assume that $0<\gamma<1$. Take an increasing concave function $G:\mathbf{R}\longrightarrow\mathbf{R}$ such that $G(s)=s$ if $s\leqslant 4$ and $G(s)=s^\gamma$ for $s\geqslant s_1$, where $s_1>4$, and $0<G'\leqslant 1$. By assumptions **(A1)** and **(B2)** there is $C>0$ such that $G(|\psi|-1)^2\leqslant CV(|\psi|^2)$, hence $G(|\psi|-1)\in L^2(I\times[0,1])$.

We claim that $G(|\psi|-1)\in H^1(I\times(0,1))$ and $\nabla\left(G(|\psi|-1)\right)=G'(|\psi|-1)\nabla(|\psi|)$ almost everywhere. Indeed, let $u_n=\min(|\psi|-1,n)$. We have $|\nabla u_n|\leqslant|\nabla\psi|$ a.e., and there is $C_n>0$ such that $|u_n|^2\leqslant C_nV(|\psi|^2)$, hence $u_n\in H^1(I\times(0,1))$. By Proposition 9.5 p. 270 in [3] we have $G(u_n)\in H^1(I\times(0,1))$ and $\nabla(G(u_n))=G'(u_n)\nabla u_n=G'(u_n)\nabla(|\psi|)\mathbb{1}_{\{|\psi|\leqslant n+1\}}$ a.e. The claim follows by letting $n\longrightarrow\infty$ and using the Dominated Convergence Theorem.

It is obvious that $\big|\nabla(G(|\psi|-1))\big|\leqslant|\nabla\psi|$ a.e. and we conclude that $\|G(|\psi|-1)\|^2_{H^1(I\times(0,1))}\leqslant C_\lambda E^I_\lambda(\psi)$. By the Sobolev embedding we have $\|G(|\psi|-1)\|_{L^p(I\times(0,1))}\leqslant C(p,\lambda)E^I_\lambda(\psi)^{\frac{p}{2}}$ for any $p\in[2,\infty)$. We have

$$(|\psi|^2-1)^2\leqslant 25G(|\psi|-1)^2\mathbb{1}_{\{|\psi|\leqslant 4\}} + CG(|\psi|-1)^{\frac{4}{\gamma}}\mathbb{1}_{\{|\psi|>4\}}$$

and we infer that

$$\int_{I\times(0,1)}(|\psi|^2-1)^2\leqslant 25\|G(|\psi|-1)\|^2_{L^2(I\times(0,1))} + C\|G(|\psi|-1)\|^{\frac{4}{\gamma}}_{L^{\frac{4}{\gamma}}(I\times(0,1))}\leqslant C_1E^I_\lambda(\psi)+C_2E^I_\lambda(\psi)^{\frac{2}{\gamma}}.$$

This gives the second estimate in (3.2.4). □

Lemma 3.2.4. *Assume that* **(A1), (A2)** *and* **(B2)** *hold. Then :*

i) For any $\psi\in\mathcal{E}$ and any $v\in H^1_{per}$ we have $\psi+v\in\mathcal{E}$.

ii) Let $\psi\in\mathcal{E}$. Then for almost all $y\in\mathbf{R}$ the mapping $\psi(\cdot,y)$ belongs to $\mathcal{E}(\mathbf{R})$ and for almost all $x\in\mathbf{R}$ the mapping $\psi(x,\cdot)$ belongs to $H^1((0,1))$.

iii) For a given $\psi\in\mathcal{E}$, define $\breve{\psi}(x)=\int_0^1\psi(x,y)\,dy$ and $v_\psi(x,y)=\psi(x,y)-\breve{\psi}(x)$. Then we have $\breve{\psi}\in\mathcal{E}(\mathbf{R})$, $v_\psi\in H^1_{per}$ and

$$(\breve{\psi})'(x)=\int_0^1\frac{\partial\psi}{\partial x}(x,y)\,dy,\qquad \frac{\partial v_\psi}{\partial y}=\frac{\partial\psi}{\partial y}\quad \textit{almost everywhere.}$$

For any interval $I\subset\mathbf{R}$ of length greater than or equal to 1 we have

$$\|v_\psi\|^2_{H^1(I\times(0,1))}\leqslant 2\|\nabla\psi\|^2_{L^2(I\times(0,1))}\quad\textit{and}\quad E^{1,I}_{GL}(\breve{\psi})\leqslant C\Big(E^I_{GL}(\psi)+E^I_{GL}(\psi)^2\Big). \tag{3.2.5}$$

iv) For any $\psi \in \mathcal{E}$ there exist a real-valued function $\varphi \in C^\infty(\mathbf{R})$ satisfying $\varphi^{(k)} \in L^2(\mathbf{R})$ for any $k \in \mathbf{N}^*$ and $w \in H^1_{per}$ such that $\psi(x,y) = e^{i\varphi(x)} + w(x,y)$.

Proof. (i) It is clear that $\psi + v \in H^1_{loc}(\mathbf{R}^2)$ and $\nabla(\psi + v) \in L^2(\mathbf{R} \times [0,1])$. We only need to show that $|\psi + v|^2 - 1 \in L^2(\mathbf{R} \times [0,1])$. Recall that $|\psi| - 1, v \in H^1_{per} \subset L^p(\mathbf{R} \times [0,1])$ for any $p \in [2, \infty)$ by the Sobolev embedding. We have

$$|\psi + v|^2 - 1 = \Big(|\psi|^2 - 1\Big) + 2\langle \psi, v\rangle + |v|^2. \tag{3.2.6}$$

It is clear that $|\psi|^2 - 1 \in L^2(\mathbf{R} \times [0,1])$ because $\psi \in \mathcal{E}$ and $|v|^2 \in L^2(\mathbf{R} \times [0,1])$ because $v \in L^4(\mathbf{R} \times [0,1])$. We have also

$$|\langle \psi, v\rangle| \leqslant |\psi| \cdot |v| \leqslant \big|\,|\psi| - 1\big| \cdot |v| + |v|. \tag{3.2.7}$$

The last function is in $L^2(\mathbf{R} \times [0,1])$ because $|\psi| - 1, v \in L^4(\mathbf{R} \times [0,1])$ and $v \in L^2(\mathbf{R} \times [0,1])$.

(ii) is a consequence of Theorem 2 p. 164 in [7] and of Fubini's Theorem.

(iii) By Fubini's Theorem, $\breve{\psi}$ is measurable. By the Cauchy-Schwarz inequality we have $\big|\breve{\psi}(x)\big|^2 \leqslant \int_0^1 |\psi(x,y)|^2\, dy$ and we infer that $\breve{\psi} \in L^2_{loc}(\mathbf{R})$.

Let $g(x) = \int_0^1 \frac{\partial \psi}{\partial x}(x,y)\, dy$. As above, using the Cauchy-Schwarz inequality we find $|g(x)|^2 \leqslant \int_0^1 \big|\frac{\partial \psi}{\partial x}(x,y)\big|^2\, dy$ and we infer that $g \in L^2(\mathbf{R})$ and $\|g\|_{L^2(I)} \leqslant \big\|\frac{\partial \psi}{\partial x}\big\|_{L^2(I \times (0,1))}$ for any interval $I \subset \mathbf{R}$. For any $\phi \in C^\infty_c(\mathbf{R})$ we have

$$\int_{\mathbf{R}} \breve{\psi}(x)\phi'(x)\, dx = \int_{\mathbf{R}} \left(\int_0^1 \psi(x,y)\phi'(x)\, dy\right) dx$$

$$= \int_0^1 \left(\int_{\mathbf{R}} \psi(x,y)\phi'(x)\, dx\right) dy \quad \text{by Fubini because } \psi(x,y)\phi'(x) \in L^1(\mathbf{R} \times [0,1])$$

$$= -\int_0^1 \left(\int_{\mathbf{R}} \frac{\partial \psi}{\partial x}(x,y)\phi(x)\, dx\right) dy \quad \text{because } \psi(\cdot, y) \in H^1_{loc}(\mathbf{R}) \text{ for a.e. } y \in [0,1]$$

$$= -\int_{\mathbf{R}} \left(\int_0^1 \frac{\partial \psi}{\partial x}(x,y)\, dy\right) \cdot \varphi(x)\, dx \quad \text{by Fubini again because } \frac{\partial \psi}{\partial x}\phi \in L^1(\mathbf{R} \times [0,1])$$

$$= -\int_{\mathbf{R}} g(x)\phi(x)\, dx.$$

We conclude that $\breve{\psi} \in H^1_{loc}(\mathbf{R})$ and $\big(\breve{\psi}\big)' = g$.

It is clear that v_ψ is $1-$ periodic with respect to the second variable and $\frac{\partial v_\psi}{\partial y} = \frac{\partial \psi}{\partial y}$. For almost every $x \in \mathbf{R}$ we have $v_\psi(x,\cdot) = \psi(x,\cdot) - \breve{\psi}(x) \in H^1((0,1))$ and $\int_0^1 v_\psi(x,y)\, dy = 0$. For any such x, using the Poincaré-Wirtinger inequality (see [3] p. 233) we get

$$\int_0^1 |v_\psi(x,y)|^2\, dy \leqslant \|v_\psi(x,\cdot)\|^2_{L^\infty(0,1)} \leqslant \Big\|\frac{\partial v_\psi}{\partial y}(x,\cdot)\Big\|^2_{L^1((0,1))} \leqslant \int_0^1 \Big|\frac{\partial \psi}{\partial y}(x,y)\Big|^2\, dy.$$

Integrating with respect to x we infer that $v_\psi \in L^2(\mathbf{R}\times[0,1])$ and we have $\|v_\psi\|_{L^2(I\times[0,1])} \leqslant \|\frac{\partial\psi}{\partial y}\|_{L^2(I\times[0,1])}$ for any interval $I \subset \mathbf{R}$.

We have

$$\frac{\partial v_\psi}{\partial x}(x,y) = \frac{\partial\psi}{\partial x}(x,y) - \left(\breve{\psi}\right)'(x) = \frac{\partial\psi}{\partial x}(x,y) - \int_0^1 \frac{\partial\psi}{\partial x}(x,y)\,dy$$

and we see that for almost every $x \in \mathbf{R}$ there holds

$$\int_0^1 \left|\frac{\partial v_\psi}{\partial x}(x,y)\right|^2 dy = \int_0^1 \left|\frac{\partial\psi}{\partial x}(x,y)\right|^2 dy - \left(\int_0^1 \frac{\partial\psi}{\partial x}(x,y)\,dy\right)^2 \leqslant \int_0^1 \left|\frac{\partial\psi}{\partial x}(x,y)\right|^2 dy.$$

The above estimates imply that $v_\psi \in H^1_{per}$ and the first estimate in (3.2.5) holds.

We have $\breve{\psi} = \psi - v_\psi$. Then using (3.2.6), (3.2.7) and the Sobolev inequality

$$\|\,|\psi|-1\|_{L^4(I\times[0,1])} \leqslant C\|\,|\psi|-1\|_{H^1(I\times[0,1])} \leqslant C E^I_{GL}(\psi)^{\frac{1}{2}}$$

as well as the similar estimate for v_ψ we get the second estimate in (3.2.5).

(iv) By Lemma 3.2.2 there exist $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi^{(k)} \in L^2(\mathbf{R})$ for any $k \geqslant 1$ and $w_1 \in H^1(\mathbf{R})$ such that $\breve{\psi} = e^{i\varphi} + w_1$. Letting $w(x,y) = v_\psi(x,y) + w_1(x)$ we see that $w \in H^1_{per}$ and (iv) holds. □

3.3 The momentum

3.3.1 Definition of the momentum on $\mathcal{E}(\mathbf{R})$

From a mathematical point of view, the momentum should be a functional defined on $\mathcal{E}(\mathbf{R})$ such that for any $\psi \in \mathcal{E}(\mathbf{R})$ and for any $v \in H^1(\mathbf{R})$,

$$\lim_{t\to 0} \frac{P(\psi+tv)-P(\psi)}{t} = 2\int_{\mathbf{R}} \langle i\psi', v\rangle\,dx. \tag{3.3.1}$$

.

Notice that functions in $\mathcal{E}(\mathbf{R})$ may oscillate at infinity. One can introduce a distance and define a manifold structure on $\mathcal{E}(\mathbf{R})$, see [8]. The tangent space of $\mathcal{E}(\mathbf{R})$ at ψ contains $H^1(\mathbf{R})$, but is larger than $H^1(\mathbf{R})$ (see [8] p. 140). We require (3.3.1) to hold only for $v \in H^1(\mathbf{R})$, hence condition (3.3.1) is weaker than Gâteaux differentiability and this allows some flexibility in the choice of the definition of the momentum.

We were inspired by the definitions of the momentum in higher space dimensions given in [11, 6]. The energy space associated to eq. (3.1.1) in $\mathbf{R}^N$ with "boundary condition" $|\psi| \longrightarrow 1$ as $|x| \longrightarrow \infty$ is

$$\mathcal{E}(\mathbf{R}^N) = \{\psi \in H^1_{loc}(\mathbf{R}^N) \;|\; \nabla\psi \in L^2(\mathbf{R}^N) \text{ and } |\psi|-1 \in L^2(\mathbf{R}^N)\}$$

(see (1.11) p. 154 in [6]). The momentum with respect to the x_1-direction is a functional $P : \mathcal{E}(\mathbf{R}^N) \longrightarrow \mathbf{R}$ satisfying

$$\lim_{t\to 0} \frac{P(\psi+tv)-P(\psi)}{t} = 2\int_{\mathbf{R}^N} \langle i\frac{\partial \psi}{\partial x_1}, v\rangle \, dx \quad \text{ for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.}$$

Formally we should take $P(\psi) = \displaystyle\int_{\mathbf{R}^N} \langle i\frac{\partial \psi}{\partial x_1}, \psi\rangle \, dx$, except that for an arbitrary function $\psi \in \mathcal{E}(\mathbf{R}^N)$, the function $\langle i\frac{\partial \psi}{\partial x_1}, \psi\rangle$ is not necessarily in $L^1(\mathbf{R}^N)$. However, it has been shown in [11, 6] that for any $\psi \in \mathcal{E}(\mathbf{R}^N)$ there exist $f \in L^1(\mathbf{R}^N)$ and $g \in \dot{H}^1(\mathbf{R}^N)$ such that

$$\langle i\frac{\partial \psi}{\partial x_1}, \psi\rangle = f + \partial_{x_1} g.$$

Obviously, f and g are not unique. However, if $h \in \dot{H}^1(\mathbf{R}^N)$ and $\partial_{x_1} h \in L^1(\mathbf{R}^N)$, then Lemma 2.3 p. 122 in [11] implies that necessarily $\int_{\mathbf{R}^N} \partial_{x_1} h \, dx = 0$. This allows to define unambiguously the momentum by $P(\psi) = \int_{\mathbf{R}^N} f(x)\, dx$.

The situation is different in space dimension one. If $h \in \dot{H}^1(\mathbf{R})$ and $h' \in L^1(\mathbf{R})$, the integral $\int_{-\infty}^{\infty} h'(x)\, dx$ can take any value. This is due to the fact that functions in $\dot{H}^1(\mathbf{R})$ may have different limits or may oscillate at $\pm\infty$. To give an example, let $\alpha \in (\frac{1}{2}, 1)$ and consider $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\chi'(x) = \frac{1}{|x|^\alpha}$ on $(-\infty, -1] \cup [1, \infty)$. Then $\chi(x) = \frac{1}{1-\alpha} x^{1-\alpha} + C_1$ on $[1, \infty)$ and a similar formula holds on $(-\infty, -1]$. We have $\chi \in \dot{H}^1(\mathbf{R})$, $e^{i\chi} \in \mathcal{E}(\mathbf{R})$ and $\chi' \notin L^1(\mathbf{R})$. For $a, b \in \mathbf{R}$, $a < b$, let

$$\chi_{a,b}(x) = \begin{cases} \chi(a) & \text{if } x < a, \\ \chi(x) & \text{if } x \in [a, b], \\ \chi(b) & \text{if } x > b. \end{cases}$$

We have $\chi_{a,b} \in \dot{H}^1(\mathbf{R})$, $e^{i\chi_{a,b}} \in \mathcal{E}(\mathbf{R})$, $\chi'_{a,b} \in L^1(\mathbf{R})$ and $\int_{-\infty}^{\infty} \chi'_{a,b}(x)\, dx$ may take any value in $\mathbf{R}$ as a and b vary.

Assume that $\psi \in H^1_{loc}(\mathbf{R}^N)$ can be written as $\psi = e^{i\varphi} + w$, where φ is real-valued and $\varphi, w \in H^1_{loc}(\mathbf{R}^N)$. A simple computation gives

$$\langle i\frac{\partial \psi}{\partial x_1}, \psi\rangle = -\frac{\partial \varphi}{\partial x_1} + \frac{\partial}{\partial x_1}\left(\langle iw, e^{i\varphi}\rangle\right) - 2\langle \frac{\partial \varphi}{\partial x_1} e^{i\varphi}, w\rangle + \langle i\frac{\partial w}{\partial x_1}, w\rangle. \tag{3.3.2}$$

In space dimension $N = 2$, a variant of Lemma 3.2.2 asserts that for any $\psi \in \mathcal{E}(\mathbf{R}^2)$ there exist a real-valued function $\varphi \in \dot{H}^1(\mathbf{R}^2)$ and $w \in H^1(\mathbf{R}^2)$ such that $\psi = e^{i\varphi} + w$ (see Theorem 1.8 p. 134 in [8]). Then we have $\langle iw, e^{i\varphi}\rangle \in H^1(\mathbf{R}^2)$ (see the proof of Lemma 2.1 p. 158 in [6]). According to Lemma 2.3 p. 122 in [11] we must have $\int_{\mathbf{R}^2} -\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x}\left(\langle iw, e^{i\varphi}\rangle\right) dx\, dy = 0$ whenever $-\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x}\left(\langle iw, e^{i\varphi}\rangle\right) \in L^1(\mathbf{R}^2)$. This observation enables to define unambiguously the momentum on $\mathcal{E}(\mathbf{R}^2)$ by

$$Q(\psi) = \int_{\mathbf{R}^2} -2\langle \frac{\partial \varphi}{\partial x} e^{i\varphi}, w\rangle + \langle i\frac{\partial w}{\partial x}, w\rangle \, dx\, dy$$

for any function $\psi = e^{i\varphi} + w$, where φ and w are as above. The integrand belongs to $L^1(\mathbf{R}^2)$ by the Cauchy-Schwarz inequality, and the value of the integral does not depend on the choice of the functions φ and w satisfying the above properties.

The situation is more complicated in space dimension $N = 1$ because the integral of a derivative of a function in $\dot{H}^1(\mathbf{R})$, when it exists, does not necessarily vanish. Nevertheless, we will use an analogy to the two-dimensional case. More precisely, for any real-valued function $\varphi \in \dot{H}^1(\mathbf{R})$ and for any $w \in H^1(\mathbf{R})$ we define

$$p(\varphi, w) = \int_{\mathbf{R}} -2\langle \varphi' e^{i\varphi}, w\rangle + \langle iw', w\rangle \, dx. \tag{3.3.3}$$

Notice that the integrand is in $L^1(\mathbf{R})$ by the Cauchy-Schwarz inequality and consequently $p(\varphi, w)$ is well-defined.

Assume that $\psi \in \mathcal{E}(\mathbf{R})$ can be written as $\psi = e^{i\varphi_1} + w_1 = e^{i\varphi_2} + w_2$, where (φ_1, w_1) and (φ_2, w_2) are as in Lemma 3.2.2. By (3.3.2) we have

$$-2\langle \varphi_j' e^{i\varphi_j}, w_j\rangle + \langle iw_j', w_j\rangle = \langle i\psi', \psi\rangle + \varphi_j' - \Big(\langle iw_j, e^{i\varphi_j}\rangle\Big)'$$

for $j = 1, 2$, therefore

$$\begin{aligned} &(-2\langle \varphi_2' e^{i\varphi_2}, w_2\rangle + \langle iw_2', w_2\rangle) - (-2\langle \varphi_1' e^{i\varphi_1}, w_1\rangle + \langle iw_1', w_1\rangle) \\ &= \varphi_2' - \varphi_1' - \big(\langle iw_2, e^{i\varphi_2}\rangle\big)' + \big(\langle iw_1, e^{i\varphi_1}\rangle\big)'. \end{aligned} \tag{3.3.4}$$

By Lemma 3.2.2 there exist $k_+, k_- \in \mathbf{Z}$ such that $\varphi_2 - \varphi_1 - 2k_-\pi \in L^2((-\infty, 0])$ and $\varphi_2 - \varphi_1 - 2k_+\pi \in L^2([0, \infty))$. Then we have $\varphi_2 - \varphi_1 - 2k_-\pi \in H^1((-\infty, 0))$ and $\varphi_2 - \varphi_1 - 2k_+\pi \in H^1((0, \infty))$, hence $\varphi_2 - \varphi_1 \longrightarrow 2k_-\pi$ as $x \longrightarrow -\infty$ and $\varphi_2 - \varphi_1 \longrightarrow 2k_+\pi$ as $x \longrightarrow \infty$.

It is easy to see that $\langle iw_j, e^{i\varphi_j}\rangle \in H^1(\mathbf{R})$ for $j = 1, 2$. For any function $f \in H^1(\mathbf{R})$ we have $f(s) \longrightarrow 0$ as $s \longrightarrow \pm\infty$ and $\int_{-R}^{R} f'(t)\, dt = f(R) - f(-R) \longrightarrow 0$ as $R \longrightarrow \infty$. Using (3.3.3), Lebesgue's dominated convergence theorem, then (3.3.4) we get

$$\begin{aligned} &p(\varphi_2, w_2) - p(\varphi_1, w_1) \\ &= \lim_{R\to\infty} \int_{-R}^{R} (-2\langle \varphi_2' e^{i\varphi_2}, w_2\rangle + \langle iw_2', w_2\rangle) - (-2\langle \varphi_1' e^{i\varphi_1}, w_1\rangle + \langle iw_1', w_1\rangle)\, dx \\ &= \lim_{R\to\infty} \int_{-R}^{R} \varphi_2' - \varphi_1' - \Big(\langle iw_2, e^{i\varphi_2}\rangle\Big)' + \Big(\langle iw_1, e^{i\varphi_1}\rangle\Big)'\, dx = 2\pi(k_+ - k_-). \end{aligned} \tag{3.3.5}$$

Let $\psi = e^{i\varphi} + w \in \mathcal{E}(\mathbf{R})$, where φ and w are as in Lemma 3.2.2. Let $k \in \mathbf{Z}$. Consider a real-valued function $\chi \in C^\infty(\mathbf{R})$ such that $\chi = 0$ on $(-\infty, 0]$ and $\chi = 1$ on $[1, \infty)$. Define $\tilde{\varphi} = \varphi + 2k\chi$ and $\tilde{w} = w + e^{i\varphi} - e^{i\tilde{\varphi}}$. It is easily seen that $\psi = e^{i\tilde{\varphi}} + \tilde{w}$, where $\tilde{\varphi}$ and $\tilde{w}$ also satisfy the conclusion of Lemma 3.2.2, and the above computation shows that $p(\tilde{\varphi}, \tilde{w}) - p(\varphi, w) = 2k\pi$.

We conclude given any $\psi \in \mathcal{E}(\mathbf{R})$, it can be written as $\psi = e^{i\varphi} + w \in \mathcal{E}(\mathbf{R})$, where φ and w are as in Lemma 3.2.2, but the quantity $p(\varphi, w)$ is well-defined only modulo $2\pi\mathbf{Z}$. We denote by $\lfloor \cdot \rfloor$ the projection of $\mathbf{R}$ onto $\mathbf{R}/2\pi\mathbf{Z}$, namely $\lfloor x \rfloor = \{x + 2k\pi \mid k \in \mathbf{Z}\}$.

Definition 3.3.1. *Given any $\psi \in \mathcal{E}(\mathbf{R})$, the momentum of ψ is*

$$\lfloor P \rfloor(\psi) = \lfloor p(\varphi, w) \rfloor,$$

where $\psi = e^{i\varphi} + w$ and φ and w are as in Lemma 3.2.2 (i). We call a valuation *of the momentum of ψ any number in the set $\lfloor P \rfloor(\psi)$, and the* canonical valuation *the only number in the set $[0, 2\pi) \cap \lfloor P \rfloor(\psi)$.*

It follows from the above discussion that a number $p \in \mathbf{R}$ is a valuation of the momentum of a mapping $\psi \in \mathcal{E}(\mathbf{R})$ if and only if there exist $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$ and $w \in H^1(\mathbf{R}, \mathbf{C})$ such that $\psi = e^{i\varphi} + w$ and $p(\varphi, w) = p$.

For any $\psi = e^{\varphi} + w \in \mathcal{E}(\mathbf{R})$ and any $\alpha \in \mathbf{R}$ we have $e^{i\alpha}\psi = e^{i(\varphi+\alpha)} + e^{i\alpha}w$. It is clear that $p(\varphi, w) = p(\varphi + \alpha, e^{i\alpha}w)$, and consequently we have $\lfloor P \rfloor(e^{i\alpha}\psi) = \lfloor P \rfloor(\psi)$.

For all $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$ we have $e^{i\varphi} \in \mathcal{E}(\mathbf{R})$ and $E^1(e^{i\varphi}) = \|\varphi'\|^2_{L^2(\mathbf{R})}$. Definition 3.3.1 gives $\lfloor P \rfloor(e^{i\varphi}) = \lfloor p(\varphi, 0) \rfloor = \lfloor 0 \rfloor$. However, we have $\langle i\left(e^{i\varphi}\right)', e^{i\varphi}\rangle = -\varphi'$ and φ' does not necessarily belong to $L^1(\mathbf{R})$; when it does, $\int_{\mathbf{R}} \varphi'(x)\, dx$ can take any value.

Assume that $\psi \in \mathcal{E}(\mathbf{R})$ is constant outside a bounded interval $[a, b]$, say $\psi(x) = e^{i\alpha_1}$ on $(-\infty, a]$ and $\psi(x) = e^{i\alpha_1}$ on $[b, \infty)$, where $\alpha_1, \alpha_2 \in \mathbf{R}$. Consider any function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi = \alpha_1$ on $(-\infty, a]$ and $\varphi = \alpha_2$ on $[b, \infty)$. Let $w = \psi - e^{i\varphi}$. Then $w \in H^1(\mathbf{R})$, $\mathrm{supp}(w) \subset [a, b]$ and using (3.3.2) we get

$$p(\varphi, w) = \alpha_2 - \alpha_1 + \int_a^b \langle i\psi', \psi\rangle\, dx.$$

In particular, if $\alpha_1 = \alpha_2$ we see that a valuation of the momentum of ψ is $\int_{\mathbf{R}} \langle i\psi', \psi\rangle\, dx$.

Given any $\psi = e^{i\varphi} + w \in \mathcal{E}(\mathbf{R})$ and any $v \in H^1(\mathbf{R})$, we have $\psi + v = e^{i\varphi} + (w + v)$, hence a valuation of the momentum of $\psi + v$ is $p(\varphi, w + v)$. It is obvious that

$$p(\varphi, w + v) - p(\varphi, w) = \int_{\mathbf{R}} -2\langle \varphi' e^{i\varphi}, v\rangle + 2\langle iw', v\rangle + \langle iv', v\rangle\, dx = \int_{\mathbf{R}} \langle i\left(2\psi' + v'\right), v\rangle\, dx$$

and

$$\lim_{t\to 0} \frac{p(\varphi, w + tv) - p(\varphi, w)}{t} = 2\int_{\mathbf{R}} \langle -\varphi' e^{i\varphi}, v\rangle + \langle iw', v\rangle\, dx = 2\int_{\mathbf{R}} \langle i\psi', v\rangle\, dx.$$

Remark 3.3.2. Assume that $\psi \in \mathcal{E}(\mathbf{R})$ admits a lifting $\psi(x) = \rho(x)e^{i\theta(x)}$ where $\rho = |\psi|$ and $\theta \in \dot{H}^1(\mathbf{R})$. By Lemma 3.2.3 we have $1 - \rho \in H^1(\mathbf{R})$. Let $w = (\rho - 1)e^{i\theta}$. It is easy to see that $w \in H^1(\mathbf{R})$, $\psi = e^{i\theta} + w$ and

$$-2\langle \theta' e^{i\theta}, w\rangle + \langle iw', w\rangle = (1 - \rho^2)\theta'.$$

Therefore a valuation of the momentum of ψ is $p(\theta, w) = \int_{\mathbf{R}} (1-\rho^2)\theta'\, dx$. This is in perfect agreement with formula (2.12) p. 123 in [11] and with formula (2.7) p. 159 in [6].

3.3.2 Definition of the momentum on $\mathcal{E}$

For any given $\varphi \in \dot{H}^1(\mathbf{R})$ and any $w \in H^1(\mathbf{R} \times (0,1))$ we define

$$d[\varphi, w](x,y) = -2\langle \varphi'(x) e^{i\varphi(x)}, w(x,y)\rangle + \langle i\frac{\partial w}{\partial x}(x,y), w(x,y)\rangle.$$

The Cauchy-Schwarz inequality implies that $d[\varphi, w] \in L^1(\mathbf{R} \times (0,1))$, thus we may define

$$q(\varphi, w) = \int_{\mathbf{R}\times(0,1)} d[\varphi, w]\, dx\, dy.$$

Assume that $\varphi_1, \varphi_2 \in \dot{H}^1(\mathbf{R})$, $w_1, w_2 \in H^1(\mathbf{R}\times(0,1))$ and $e^{i\varphi_1} + w_1 = e^{i\varphi_2} + w_2$ a.e. on $\mathbf{R}\times(0,1)$. Let $h(x) = e^{i\varphi_2(x)} - e^{i\varphi_1(x)}$. Then we have $h \in \dot{H}^1(\mathbf{R})$ and $h = w_1 - w_2 \in H^1(\mathbf{R}\times(0,1))$. By Fubini's theorem it follows that $h \in L^2(\mathbf{R})$, hence $h \in H^1(\mathbf{R})$.

By Theorem 2 p. 164 in [7], there is a set $A \subset (0,1)$ such that $(0,1) \setminus A$ has zero Lebesgue measure and for any $y \in A$ the mappings $w_j(\cdot, y)$ belong to $H^1(\mathbf{R})$ for $j = 1, 2$. For all $y \in A$ we have $\psi(\cdot, y) = e^{i\varphi_j} + w_j(\cdot, y) \in \mathcal{E}(\mathbf{R})$. Using Lemma 3.2.2 (ii), there exist $k_+, k_- \in \mathbf{Z}$ such that $\lim\limits_{x\to\pm\infty} \varphi_2(x) - \varphi_1(x) = 2\pi k_\pm$. Using (3.3.2) we get

$$d[\varphi_2, w_2] - d[\varphi_1, w_1] = \varphi_2' - \varphi_1' - \frac{\partial}{\partial x}\left(\langle iw_2, e^{i\varphi_2}\rangle - \langle iw_1, e^{i\varphi_1}\rangle\right) = \varphi_2' - \varphi_1' + \frac{\partial}{\partial x}\left(\langle ih, e^{i\varphi_1} + w_2\rangle\right).$$

Proceeding exactly as in (3.3.5) we see that for any $y \in A$ there holds

$$\int_{\mathbf{R}} d[\varphi_2, w_2](x,y) - d[\varphi_1, w_1](x,y)\, dx = 2\pi(k_+ - k_-)$$

and then integrating with respect to y and using Fubini's theorem we get

$$q(\varphi_2, w_2) - q(\varphi_1, w_1) = 2\pi(k_+ - k_-).$$

Let $\varphi \in \dot{H}^1(\mathbf{R})$, $w \in H^1(\mathbf{R}\times(0,1))$ and $k \in \mathbf{Z}$ be arbitrary. Take a real-valued function $\chi \in C^\infty(\mathbf{R})$ such that $\chi = 0$ on $(-\infty, 0]$ and $\chi = 1$ on $[1, \infty)$ and define $\tilde{\varphi} = \varphi + 2k\chi$ and $\tilde{w} = w + e^{i\varphi} - e^{i\tilde{\varphi}}$. It is easily seen that $e^{i\varphi} + w = e^{i\tilde{\varphi}} + \tilde{w}$ and $q(\tilde{\varphi}, \tilde{w}) - q(\varphi, w) = 2k\pi$.

Given any $\psi \in \mathcal{E}$, by Lemma 3.2.4 (iv) there exist $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in H^1(\mathbf{R}\times(0,1))$ such that $\psi(x,y) = e^{i\varphi(x)} + w(x,y)$. The previous discussion shows that the quantity $q(\varphi, w)$ is well-defined modulo $2\pi\mathbf{Z}$. This enables us to give the following

Definition 3.3.3. *Given any $\psi \in \mathcal{E}$, the momentum of ψ is*

$$\lfloor Q \rfloor(\psi) = \lfloor q(\varphi, w) \rfloor,$$

where $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$ and $w \in H^1(\mathbf{R}\times(0,1), \mathbf{C})$ are such that $\psi = e^{i\varphi} + w$. A valuation *of the momentum of ψ is any number in the set $\lfloor Q \rfloor(\psi)$, and the* canonical valuation *is the only number in the set $\lfloor Q \rfloor(\psi) \cap [0, 2\pi)$.*

Remark 3.3.4. i) As in the one-dimensional case, a number $q \in \mathbf{R}$ is a valuation of the momentum of a mapping $\psi \in \mathcal{E}$ if and only if there exist $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$ and $w \in H^1_{per}$ such that $\psi = e^{i\varphi} + w$ and $q(\varphi, w) = q$. If ψ, φ and w are as above, then for almost any $y \in \mathbf{R}$ we have $\psi(\cdot, y) \in \mathcal{E}(\mathbf{R})$ and $w(\cdot, y) \in H^1(\mathbf{R})$, and then $p(\varphi, w(\cdot, y))$ is a valuation of the momentum of $\psi(\cdot, y) \in \mathcal{E}(\mathbf{R})$. By Fubini's Theorem we have

$$q(\varphi, w) = \int_0^1 p(\varphi, w(\cdot, y))\, dy.$$

ii) **Warning !** If $Q_0(\psi)$ and $P_0(\psi(\cdot, y))$ are the canonical valuations of the momenta of $\psi \in \mathcal{E}$ and of $\psi(\cdot, y) \in \mathcal{E}(\mathbf{R})$, respectively, we may have

$$Q_0(\psi) \neq \int_0^1 P_0(\psi(\cdot, y))\, dy.$$

iii) For $\psi = e^{i\varphi} + w \in \mathcal{E}$ and $v \in H^1(\mathbf{R} \times (0,1))$, a valuation of the momentum of $\psi + v$ is $q(\varphi, w + v)$ and we have

$$\begin{aligned} q(\varphi, w+v) - q(\varphi, w) &= \int_{\mathbf{R}\times[0,1]} -2\langle \varphi' e^{i\varphi}, v\rangle + 2\langle i\frac{\partial w}{\partial x}, v\rangle + \langle i\frac{\partial v}{\partial x}, v\rangle \, dx\, dy \\ &= \int_{\mathbf{R}\times[0,1]} 2\langle i\frac{\partial \psi}{\partial x}, v\rangle + \langle i\frac{\partial v}{\partial x}, v\rangle \, dx\, dy = \int_{\mathbf{R}\times[0,1]} \langle i\frac{\partial \psi}{\partial x} + i\frac{\partial(\psi+v)}{\partial x}, v\rangle \, dx\, dy. \end{aligned} \tag{3.3.6}$$

Using the Cauchy-Schwarz inequality we get

$$|q(\varphi, w+v) - q(\varphi, w)| \leqslant \|v\|_{L^2(\mathbf{R}\times[0,1])} \left(\left\| \frac{\partial \psi}{\partial x} \right\|_{L^2(\mathbf{R}\times[0,1])} + \left\| \frac{\partial(\psi+v)}{\partial x} \right\|_{L^2(\mathbf{R}\times[0,1])} \right). \tag{3.3.7}$$

From (3.3.6) we obtain

$$\lim_{t\to 0} \frac{q(\varphi, w+tv) - q(\varphi, w)}{t} = 2\int_{\mathbf{R}\times(0,1)} \langle i\frac{\partial \psi}{\partial x}, v\rangle \, dx\, dy. \tag{3.3.8}$$

Notice that (3.3.6)-(3.3.8) are analogous to Lemma 2.5 and Corollary 2.6 p. 123-124 in [11] and to Lemma 2.3 and Corollary 2.4 p. 159 in [6].

Lemma 3.3.5. *Assume that $\psi \in \mathcal{E}$ satisfies $\rho_0 := \inf_{(x,y)\in\mathbf{R}^2} |\psi(x,y)| > 0$. Let $\rho = |\psi|$. There exists a real-valued function $\theta \in H^1_{per}$ such that $\psi = \rho e^{i\theta}$ and a valuation of the momentum of ψ is*

$$\int_{\mathbf{R}\times(0,1)} (1-\rho^2)\frac{\partial \theta}{\partial x}\, dx\, dy.$$

Proof. It is well-known that $\rho \in H^1_{loc}(\mathbf{R}^2)$ and $|\nabla \rho| \leqslant |\nabla \psi|$ almost everywhere. The mapping $\frac{\psi}{\rho}$ belongs to $H^1_{loc}(\mathbf{R}^2, \mathbb{S}^1)$, hence it admits a lifting, in other words there exists $\theta \in H^1_{loc}(\mathbf{R}^2, \mathbf{R})$ such that $\frac{\psi}{\rho} = e^{i\theta}$, or equivalently $\psi = \rho e^{i\theta}$. We have

$$|\nabla \psi|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2 \quad \text{almost everywhere}$$

and we infer that $\nabla\theta \in L^2(\mathbf{R}\times[0,1])$.

We claim that θ is $1-$periodic with respect to the second variable y. Indeed, since ψ and ρ are $1-$periodic with respect to y, we infer that for any $(x,y)\in\mathbf{R}^2$ there holds $1=\frac{\psi(x,y+1)}{\psi(x,y)}=e^{i(\theta(x,y+1)-\theta(x,y))}$. The mapping $(x,y)\longmapsto\theta(x,y+1)-\theta(x,y)$ belongs to $H^1_{loc}(\mathbf{R}^2,\mathbf{R})$ and takes values in $2\pi\mathbf{Z}$, hence it must be constant. We infer that there exists $k_\theta\in\mathbf{Z}$ such that $\theta(x,y+1)=\theta(x,y)+2\pi k_\theta$ for any $(x,y)\in\mathbf{R}^2$. For almost any $x\in\mathbf{R}$ we have $\theta(x,\cdot)\in H^1((-1,2))$ and then using the Cauchy-Schwarz inequality we get

$$2\pi|k_\theta|=|\theta(x,1)-\theta(x,0)|=\left|\int_0^1\frac{\partial\theta}{\partial y}(x,y)\,dy\right|\leqslant\int_0^1\left|\frac{\partial\theta}{\partial y}(x,y)\right|^2dy.$$

Integrating the above inequality with respect to x and using the fact that $\frac{\partial\theta}{\partial y}\in L^2(\mathbf{R}\times(0,1))$ we see that necessarily $k_\theta=0$ and the claim is proven.

Denote $\breve{\theta}(x)=\int_0^1\theta(x,y)\,dy$ and $\theta^\sharp(x,y)=\theta(x,y)-\breve{\theta}(x)$. Proceeding as in the proof of Lemma 3.2.4 (iii) we see that $\breve{\theta}'\in L^2(\mathbf{R})$ and $\theta^\sharp\in H^1(\mathbf{R}\times(0,1))$. Let $w(x,y)=\psi(x,y)-e^{i\breve{\theta}(x)}$. We have

$$w=(\rho-1)e^{i\theta}+e^{i\breve{\theta}}\left(e^{i\theta^\sharp}-1\right).$$

It is easy to see that $w\in H^1(\mathbf{R}\times(0,1))$. Using (3.3.2) we have

$$\begin{aligned}
d[\breve{\theta},w]&=\langle i\frac{\partial\psi}{\partial x},\psi\rangle+\breve{\theta}'-\frac{\partial}{\partial x}\left(\langle iw,e^{i\breve{\theta}}\rangle\right)\\
&=-\rho^2\frac{\partial\theta}{\partial x}+\breve{\theta}'-\frac{\partial}{\partial x}\left(\langle i\rho e^{i\theta}-ie^{i\breve{\theta}},e^{i\breve{\theta}}\rangle\right)=-\rho^2\frac{\partial\theta}{\partial x}+\breve{\theta}'+\frac{\partial}{\partial x}\left(\rho\sin(\theta-\breve{\theta})\right)\\
&=(1-\rho^2)\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial x}\left(-\theta^\sharp+\rho\sin(\theta^\sharp)\right).
\end{aligned}\tag{3.3.9}$$

By the Cauchy-Schwarz inequality we have $d[\breve{\theta},w]\in L^1(\mathbf{R}\times(0,1))$ and $(1-\rho^2)\frac{\partial\theta}{\partial x}\in L^1(\mathbf{R}\times(0,1))$, and then (3.3.9) gives $\frac{\partial}{\partial x}\left(-\theta^\sharp+\rho\sin(\theta^\sharp)\right)\in L^1(\mathbf{R}\times(0,1))$.

Next we use the following simple observation : whenever $f\in H^1(\mathbf{R}\times(0,1))$ satisfies $\partial_1 f\in L^1(\mathbf{R}\times(0,1))$ we must have $\int_{\mathbf{R}\times(0,1)}\partial_1 f(x,y)\,dx\,dy=0$. Indeed, using Theorem 2 p. 164 in [7] and Fubini's Theorem we infer that for almost every $y\in(0,1)$ we have $f(\cdot,y)\in H^1(\mathbf{R})$ and $\frac{d}{dx}[f(\cdot,y)]=\partial_1 f(\cdot,y)\in L^1\cap L^2(\mathbf{R})$. For any such y we have

$$\int_{\mathbf{R}}\partial_1 f(x,y)\,dx=\lim_{R\to\infty}\int_{-R}^R\partial_1 f(s,y)\,ds=\lim_{R\to\infty}\left(f(R,y)-f(-R,y)\right)=0.$$

Integrating with respect to y we get the desired result.

Since $\theta^\sharp\in H^1(\mathbf{R}\times(0,1))$ and $\rho-1\in H^1(\mathbf{R}\times(0,1))$ it is easily seen that $-\theta^\sharp+\rho\sin(\theta^\sharp)=-\theta^\sharp+\sin(\theta^\sharp)+(\rho-1)\sin(\theta^\sharp)\in H^1(\mathbf{R}\times(0,1))$ and then using (3.3.9) and the above observation we infer that

$$q(\breve{\theta},w)=\int_{\mathbf{R}\times(0,1))}d[\breve{\theta},w]\,dx\,dy=\int_{\mathbf{R}\times(0,1))}(1-\rho^2)\frac{\partial\theta}{\partial x}\,dx\,dy.$$

This completes the proof of Lemma 3.3.5. Notice that this is in agreement with formula (2.12) p. 123 in [11] and with formula (2.7) p. 159 in [6]. □

Lemma 3.3.6. *i) For any $\psi \in \mathcal{E}(\mathbf{R})$ there exist sequences $(\varphi_n)_{n\geqslant 1} \subset C^\infty(\mathbf{R},\mathbf{R})$ and $(w_n)_{n\geqslant 1} \subset C_c^\infty(\mathbf{R})$ such that*

$$\lfloor P \rfloor \left(e^{i\varphi_n} + w_n\right) = \lfloor P \rfloor(\psi) \qquad \text{for all } n,$$

$$\left(e^{i\varphi_n} + w_n\right)' \longrightarrow \psi' \qquad \text{in } L^2(\mathbf{R}),$$

$$e^{i\varphi_n} + w_n \longrightarrow \psi \quad \text{in } H^1_{loc}(\mathbf{R}),$$

$$V\left(\left|e^{i\varphi_n} + w_n\right|^2\right) \longrightarrow V(|\psi|^2) \ \text{ and } \ \left(1 - \left|e^{i\varphi_n} + w_n\right|^2\right)^2 \longrightarrow \left(1 - |\psi|^2\right)^2 \qquad \text{in } L^1(\mathbf{R}),$$

and there exist sequences of real numbers $(A_n)_{n\geqslant 1}$, $(\alpha_n)_{n\geqslant 1}$, $(\beta_n)_{n\geqslant 1}$ such that $A_n \longrightarrow \infty$, $supp(w_n) \subset [-A_n, A_n]$, $\varphi_n(x) = \alpha_n$ for $x \in (-\infty, -A_n]$, and $\varphi_n(x) = \beta_n$ for $x \in [A_n, \infty)$.

ii) For any $\psi \in \mathcal{E}$ there exist sequences $(\varphi_n)_{n\geqslant 1} \subset C^\infty(\mathbf{R},\mathbf{R})$ and $(w_n)_{n\geqslant 1} \subset C_c^\infty(\mathbf{R}\times(0,1))$ satisfying

$$\lfloor Q \rfloor \left(e^{i\varphi_n} + w_n\right) = \lfloor Q \rfloor(\psi) \qquad \text{for all } n,$$

$$\nabla\left(e^{i\varphi_n} + w_n\right) \longrightarrow \nabla\psi \qquad \text{in } L^2(\mathbf{R}\times(0,1)),$$

$$e^{i\varphi_n} + w_n \longrightarrow \psi \quad \text{in } H^1_{loc}(\mathbf{R}\times(0,1)),$$

$$V\left(\left|e^{i\varphi_n} + w_n\right|^2\right) \longrightarrow V(|\psi|^2) \ \text{ and } \ \left(1 - \left|e^{i\varphi_n} + w_n\right|^2\right)^2 \longrightarrow \left(1 - |\psi|^2\right)^2 \qquad \text{in } L^1(\mathbf{R}\times(0,1)),$$

and there exist sequences $(A_n)_{n\geqslant 1}$, $(\alpha_n)_{n\geqslant 1}$, $(\beta_n)_{n\geqslant 1} \subset \mathbf{R}$ such that $A_n \longrightarrow \infty$, $supp(w_n) \subset [-A_n, A_n]\times(0,1)$, $\varphi_n(x) = \alpha_n$ for $x \in (-\infty, -A_n]$, and $\varphi_n(x) = \beta_n$ for $x \in [A_n, \infty)$.

Proof. We only prove (ii). The proof of (i) is similar.

If $\frac{\partial\psi}{\partial x} = 0$ in $L^2(\mathbf{R}\times(0,1))$, then for almost all $y \in (0,1)$ the mapping $\psi(\cdot, y)$ belongs to $H^1_{loc}(\mathbf{R})$ and $\psi(\cdot, y)$ is constant. We infer that there exists $h \in H^1(0,1)$ such that $\psi(x,y) = h(y)$ almost everywhere in $\mathbf{R}\times(0,1)$, and then $\frac{\partial\psi}{\partial y} = h'(y)$. Since $\frac{\partial\psi}{\partial y} \in L^2(\mathbf{R}\times(0,1))$, using Fubini's theorem we see that $h' = 0$ in $L^2(0,1)$ and consequently h is constant, hence ψ is constant. In this case it suffices to take φ_n a constant function such that $e^{i\varphi_n} = \psi$, and $w_n = 0$.

If $\frac{\partial\psi}{\partial x} \neq 0$ in $L^2(\mathbf{R}\times(0,1))$, there exists $v \in C_c^\infty(\mathbf{R}\times(0,1))$ such that

$$\int_{\mathbf{R}\times(0,1)} \langle i\frac{\partial\psi}{\partial x}, v\rangle \, dx\, dy \neq 0.$$

By Lemma 3.2.4 (iv) there exist $\varphi \in C^\infty(\mathbf{R},\mathbf{R})$ such that $\varphi^{(k)} \in L^2(\mathbf{R})$ for any $k \in \mathbf{N}^*$ and $w \in H^1_{per}$ such that $\psi = e^{i\varphi} + w$. There exists a sequence $(\tilde{w}_n)_{n\geqslant 1} \subset C_c^\infty(\mathbf{R}\times(0,1))$ such that $\tilde{w}_n \longrightarrow w$ in $H^1(\mathbf{R}\times(0,1))$. We have

$$q(\varphi, \tilde{w}_n + tv) = q(\varphi, \tilde{w}_n) + 2t\int_{\mathbf{R}\times(0,1)} \langle i\frac{\partial}{\partial x}\left(e^{i\varphi} + \tilde{w}_n\right), v\rangle \, dx\, dy + t^2 \int_{\mathbf{R}\times(0,1)} \langle i\frac{\partial v}{\partial x}, v\rangle \, dx\, dy.$$

Since $q(\varphi,\tilde{w}_n)\longrightarrow q(\varphi,w)$ and $\int_{\mathbf{R}\times(0,1)}\langle i\frac{\partial}{\partial x}\left(e^{i\varphi}+\tilde{w}_n\right),v\rangle\,dx\,dy\longrightarrow\int_{\mathbf{R}\times(0,1)}\langle i\frac{\partial\psi}{\partial x},v\rangle\,dx\,dy\neq 0$, we infer that there is a sequence $t_n\longrightarrow 0$ such that $q(\varphi,\tilde{w}_n+t_nv)=q(\varphi,w)$ for all n sufficiently large. Then we take $w_n=\tilde{w}_n+t_nv$. If $\mathrm{supp}(w_n)\subset[a_n,b_n]\times(0,1)$, we choose $A_n>2\max(|a_n|,|b_n|)+1$ such that $A_n\longrightarrow\infty$. Take $\chi\in C^\infty(\mathbf{R},\mathbf{R})$ such that $\mathrm{supp}(\chi)\subset[-2,2]$ and $\chi=1$ on $[-1,1]$. Take $\varphi_n(x)=\int_0^x\chi\left(\frac{2s}{A_n}\right)\varphi'(s)$, so that $\varphi_n=\varphi$ on $[-A_n/2,A_n/2]$, and φ_n is constant on $(-\infty,-A_n]$ and on $[A_n,\infty)$. It is easy to see that $(\varphi_n,w_n)_{n\geqslant 1}$ satisfy all desired properties. □

Lemma 3.3.7. *Let $\varphi\in\dot{H}^1(\mathbf{R})$ and let $w\in C^2(\mathbf{R}^2)$ such that there exists $a>0$ satisfying $supp(w)\subset[-a,a]\times\mathbf{R}$. Let $\psi(x,y)=e^{i\varphi(x)}+w(x,y)$. Then for any $y_1,y_2\in\mathbf{R}$, $y_1<y_2$ we have*

$$p(\varphi,w(\cdot,y_2))-p(\varphi,w(\cdot,y_1))=2\int_{\mathbf{R}\times[y_1,y_2]}\langle i\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y}\rangle\,dx\,dy$$

and consequently

$$|p(\varphi,w(\cdot,y_2))-p(\varphi,w(\cdot,y_1))|\leqslant 2\Big\|\frac{\partial\psi}{\partial x}\Big\|_{L^2(\mathbf{R}\times[y_1,y_2])}\Big\|\frac{\partial\psi}{\partial y}\Big\|_{L^2(\mathbf{R}\times[y_1,y_2])}$$

Proof. We have $d[\varphi,w],\langle i\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y}\rangle\in L^1(\mathbf{R}\times[y_1,y_2])$ and a standard computation gives

$$p(\varphi,w(\cdot,y_2))-p(\varphi,w(\cdot,y_1))=\int_{\mathbf{R}}d[\varphi,w](x,y_2)-d[\varphi,w](x,y_1)\,dx$$

$$=\int_{\mathbf{R}}\int_{y_1}^{y_2}\frac{\partial}{\partial y}\left(d[\varphi,w](x,y)\right)\,dy\,dx$$

$$=\int_{\mathbf{R}}\int_{y_1}^{y_2}\langle -2\varphi'(x)e^{i\varphi(x)},\frac{\partial w}{\partial y}(x,y)\rangle+\langle i\frac{\partial^2w}{\partial y\partial x},w\rangle+\langle i\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\rangle\,dy\,dx$$

$$=\int_{y_1}^{y_2}\int_{\mathbf{R}}\langle -2\varphi'(x)e^{i\varphi(x)},\frac{\partial w}{\partial y}(x,y)\rangle+\langle i\frac{\partial^2w}{\partial y\partial x},w\rangle+\langle i\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\rangle\,dx\,dy\quad\text{(Fubini)}$$

$$=2\int_{y_1}^{y_2}\int_{\mathbf{R}}\langle -\varphi'(x)e^{i\varphi(x)},\frac{\partial w}{\partial y}(x,y)\rangle+\langle i\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\rangle\,dx\,dy\quad\text{(integration by parts)}$$

$$=2\int_{y_1}^{y_2}\int_{\mathbf{R}}\langle i\frac{\partial}{\partial x}(e^{i\varphi}+w),\frac{\partial w}{\partial y}\rangle\,dx\,dy=2\int_{y_1}^{y_2}\int_{\mathbf{R}}\langle i\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y}\rangle\,dx\,dy.$$

The second statement follows from the first one and the Cauchy-Schwarz inequality. □

3.4 One-dimensional traveling waves for (3.1.1)

We consider (3.1.1) in $\mathbf{R}\times\mathbf{R}$ and we look for traveling waves, namely solutions of the form $\Phi(x,t)=\psi(x+ct)$. The traveling wave profile ψ satisfies the ordinary differential equation

$$ic\psi'+\psi''+F(|\psi|^2)\psi=0\qquad\text{in }\mathbf{R}. \tag{3.4.1}$$

We will only consider solutions of (3.4.1) in $\mathcal{E}(\mathbf{R})$. In the case of the Gross-Pitaevskii equation ($F(s) = 1 - s$), an extensive study of solutions to (3.4.1) has been carried out in [1], Section 2. For more general nolinearities we refer to [5] and to [10] (the latter focuses mainly on the non-existence of supersonic traveling waves).

If assumption **(A1)** is satisfied, it has been shown in Theorem 5.1 p. 1099 in [10] that the only solutions of (3.1.1) with $c^2 > v_s^2 = 2$ are constants. It follows from the proof of Theorem 5.1 in [10] that all traveling waves in $\mathcal{E}(\mathbf{R})$ are C^2 functions on $\mathbf{R}$. Let $\psi \in \mathcal{E}(\mathbf{R})$ be a solution of (3.4.1) and let $\varrho = |\psi|^2$. Then $\varrho - 1 \in H^1(\mathbf{R})$ by Lemma 3.2.1 and it can be shown that ϱ satisfies the equation

$$\varrho'' + c^2(\varrho - 1) - 2V(\varrho) + 2\varrho F(\varrho) = 0 \qquad \text{in } \mathbf{R} \tag{3.4.2}$$

(for the proof see (5.10) p. 1100 in [10]). Multiplying (3.4.2) by $2\varrho'$ and integrating we get

$$(\varrho')^2 + c^2(\varrho - 1)^2 - 4\varrho V(\varrho) = 0. \tag{3.4.3}$$

Denote

$$g(s, c) = 4sV(s) - c^2(s - 1)^2.$$

By (3.4.3), for any $x \in \mathbf{R}$ we must have $g(\varrho(x), c) \geqslant 0$ and $\varrho'(x) = \pm\sqrt{g(\varrho(x), c)}$. Since $g(0, c) = -c^2$, we see that for any $c \neq 0$, solutions of (3.4.3) must stay away from zero. This implies that for $c \neq 0$, any solution $\psi \in \mathcal{E}(\mathbf{R})$ of (3.4.1) does not vanish, and therefore has a lifting $\psi(x) = \sqrt{\varrho(x)}e^{i\theta(x)}$, where the function θ is C^2 on $\mathbf{R}$. Taking the scalar product of (3.4.1) with $i\psi$ we get

$$\frac{c}{2}\varrho' + (\varrho\theta')' = 0.$$

We infer that there is a constant $k_1 \in \mathbf{R}$ such that $\frac{c}{2}\varrho + \varrho\theta' = k_1$ in $\mathbf{R}$. We have $\varrho(x) \longrightarrow 1$ as $x \longrightarrow \pm\infty$ because $\varrho - 1 \in H^1(\mathbf{R})$. Since $|\psi'|^2 = \frac{|\varrho'|^2}{4\varrho} + \varrho|\theta'|^2 \in L^1(\mathbf{R})$, we deduce that $\theta' \in L^2(\mathbf{R})$, therefore we must have $k_1 = \frac{c}{2}$ and consequently

$$\theta' = \frac{c}{2}\frac{1 - \varrho}{\varrho}. \tag{3.4.4}$$

Let $c \neq 0$. If we are able to solve (3.4.2) and we get a solution ϱ such that $\varrho - 1 \in H^1(\mathbf{R})$, it follows from (3.4.3) that $\inf_{x \in \mathbf{R}} \varrho(x) > 0$, and then from (3.4.4) we obtain θ up to a constant. Then by (3.4.4) we have $\theta' \in H^1(\mathbf{R})$, $\sqrt{\varrho}e^{i\theta} \in \mathcal{E}(\mathbf{R})$ and it is straightforward to see that $\sqrt{\varrho}e^{i\theta}$ is a solution of (3.4.1). Moreover, all solutions of (3.4.1) in $\mathcal{E}(\mathbf{R})$ are obtained in this way. Notice that (3.4.1) is invariant by translations and by multiplication by complex numbers of modulus 1, so the phase θ can be determined only up to a constant.

If assumption **(A1)** is satisfied, we have $V(s) = \frac{1}{2}(s-1)^2 + o((s-1)^2)$ and $g(s, c) = (2 - c^2)(s - 1)^2 + o((s - 1)^2)$ as $s \longrightarrow 1$. If $c^2 < v_s^2 = 2$ we have $g(s, c) > 0$ whenever s is sufficiently close to 1 and $s \neq 1$. Since $g(0, c) = -c^2 \leqslant 0$, we infer that there exists $\zeta \in [0, 1)$ such that $g(\zeta, c) = 0$ and we denote $\zeta(c) = \sup\{\zeta \in [0, 1) \mid g(\zeta, c) = 0\}$. It is clear that $g(\zeta(c), c) = 0$, $g > 0$ on $(\zeta(c), 1)$

and the mapping $c \longmapsto \zeta(c)$ is even on $(-\sqrt{2},\sqrt{2})$ and is strictly increasing on $(0,\sqrt{2})$. We denote $D = \{(s,c) \in (0,1)\times(-\sqrt{2},\sqrt{2}) \mid \zeta(c) < s < 1\}$. The set D is connected, but not necessarily open. We consider a continuous function $G : D \longrightarrow \mathbf{R}$ such that $\frac{\partial G}{\partial s}$ exists and $\frac{\partial G}{\partial s}(s,c) = \frac{1}{\sqrt{g(s,c)}}$ for any $(s,c) \in D$, and G is C^2 in $\mathring{D}$. (Such a function exists : it suffices to take a smooth curve $c \longmapsto a(c)$ defined on $(-\sqrt{2},\sqrt{2})$ such that $\zeta(c) < a(c) < 1$ for all c, then put $G(s,c) = \int_{a(c)}^{s} \frac{1}{\sqrt{g(\tau,c)}}\, d\tau$.) For any fixed $c \in (-\sqrt{2},\sqrt{2})$, the mapping $s \longmapsto G(s,c)$ is strictly increasing on $(\zeta(c),1)$ and tends to ∞ as $s \longrightarrow 1$ because $\frac{1-s}{\sqrt{g(s,c)}} \longrightarrow \frac{1}{\sqrt{2-c^2}}$. Let $L(c) = \lim\limits_{s\downarrow\zeta(c)} G(s,c)$. Then $G(\cdot,c)$ is a C^2-diffeomorphism between $(\zeta(c),1)$ and $(L(c),\infty)$.

Proposition 3.4.1. *Assume that assumption* **(A1)** *is satisfied and let $c \in (-\sqrt{2},\sqrt{2})$. Then :*

i) Equation (3.4.1) admits a solution $\psi \in \mathcal{E}(\mathbf{R})$ satisfying $\inf\limits_{x\in\mathbf{R}} |\psi(x)| < 1$ if and only if $L(c) := \lim\limits_{s\downarrow\zeta(c)} G(s,c)$ is finite.

Whenever $L(c)$ is finite, let

$$\varrho_c(x) = \begin{cases} G(\cdot,c)^{-1}(L(c)-x) & \text{if } x < 0 \\ \zeta(c) & \text{if } x = 0, \\ G(\cdot,c)^{-1}(L(c)+x) & \text{if } x > 0 \end{cases} \qquad \text{and} \quad \theta_c(x) = \frac{c}{2}\int_0^x \frac{1-\varrho_c(s)}{\varrho_c(s)}\, ds \quad \text{if } c \neq 0.$$

If $c \neq 0$ we define $\psi_c(x) = \sqrt{\varrho(x)}e^{i\theta_c(x)}$.
If $c = 0$ there are three subcases :

- either there exists $s \in (0,1)$ such that $V(s) = 0$, then we have $\zeta(0) > 0$ and we put $\psi_0(x) = \sqrt{\varrho_0(x)}$;

- or $V > 0$ on $[0,1)$, so that $\zeta(0) = 0$ and we put $\psi_0(x) = \mathrm{sgn}(x)\sqrt{\varrho_0(x)}$;

- or $V > 0$ on $(0,1)$ and $V(0) = 0$, and then $V(s) \leqslant Cs$ for s sufficiently small and consequently $L(0) = -\infty$; in this subcase we do not define ψ_0.

Whenever ψ_c is defined as above, we have $\psi_c \in \mathcal{E}(\mathbf{R})$ and ψ_c is a solution of (3.4.1).

ii) Equation (3.4.1) admits a solution $\psi \in \mathcal{E}(\mathbf{R})$ satisfying $\sup\limits_{x\in\mathbf{R}} |\psi(x)| > 1$ if and only if

- *the mapping $g(\cdot,c)$ admits zeroes in $(1,\infty)$, and*
- *denoting $\tilde{\zeta}(c) = \inf\{\zeta > 1 \mid g(\zeta,c) = 0\}$ and by $\tilde{G}(\cdot,c)$ a primitive of $\frac{1}{\sqrt{g(\cdot,c)}}$ on the interval $(1,\tilde{\zeta}_c)$, the limit $\tilde{L}(c) := \lim\limits_{s\uparrow\tilde{\zeta}(c)} \tilde{G}(s,c)$ is finite.*

In this case, define

$$\tilde{\varrho}_c(x) = \begin{cases} \tilde{G}(\cdot,c)^{-1}(\tilde{L}(c)+x) & \text{if } x < 0, \\ \tilde{\zeta}(c) & \text{if } x = 0, \\ \tilde{G}(\cdot,c)^{-1}(\tilde{L}(c)-x) & \text{if } x > 0, \end{cases} \qquad \text{and} \quad \tilde{\theta}_c(x) = \frac{c}{2}\int_0^x \frac{1-\tilde{\varrho}_c(s)}{\tilde{\varrho}_c(s)}\, ds \quad \text{if } c \neq 0.$$

Let $\tilde{\psi}_c(x) = \sqrt{\tilde{\varrho}_c(x)}e^{i\tilde{\theta}_c(x)}$ if $c \neq 0$, respectively $\tilde{\psi}_0(x) = \sqrt{\tilde{\varrho}_0(x)}$ if $c = 0$.

Then $\tilde{\psi}_c \in \mathcal{E}(\mathbf{R})$ and $\tilde{\psi}_c$ is a solution of (3.4.1).

iii) Let $\psi \in \mathcal{E}(\mathbf{R})$ be a nonconstant solution of (3.4.1). We have either $|\psi(x)| < 1$ for any $x \in \mathbf{R}$, or $|\psi| > 1$ for any $x \in \mathbf{R}$. For any $\varepsilon \in (0, \sqrt{2-c^2})$ there exist $A_\varepsilon, B_\varepsilon$ (depending on ε and on ψ) such that

$$e^{-|x|\sqrt{2-c^2+\varepsilon^2}} \leqslant \big||\psi|^2 - 1\big| \leqslant e^{-|x|\sqrt{2-c^2-\varepsilon^2}} \qquad \text{on } (-\infty, A_\varepsilon] \cup [B_\varepsilon, \infty). \tag{3.4.5}$$

iv) Any solution $\psi \in \mathcal{E}(\mathbf{R})$ of (3.4.1) satisfies $|\psi'(x)|^2 = V(|\psi(x)|^2)$ for any $x \in \mathbf{R}$. In particular, we have $E(\psi) = 2\int_{\mathbf{R}} |\psi'|^2\, dx = 2\int_{\mathbf{R}} V(|\psi|^2)\, dx$.

v) If the nonlinearity F is locally Lipschitz, then any solution $\psi \in \mathcal{E}(\mathbf{R})$ of (3.4.1) is either a constant of modulus one, or is of the form $e^{i\alpha}\psi_c(\cdot - x_0)$ or $e^{i\alpha}\tilde{\psi}_c(\cdot - x_0)$ for some $\alpha, x_0 \in \mathbf{R}$, where ψ_c and $\tilde{\psi}_c$ are as in (i) and (ii), respectively.

vi) If $c_0 \neq 0$ and the mapping $c \longmapsto \zeta(c)$ is differentiable at c_0, then $L(c_0)$ is finite. Consequently, equation (3.4.1) admits solutions in $\mathcal{E}(\mathbf{R})$ for almost every $c \in (-\sqrt{2}, \sqrt{2})$.

Proof. (i) "$\Longrightarrow$" Let $\psi \in \mathcal{E}(\mathbf{R})$ be a solution of (3.4.1) and let $\varrho = |\psi|^2$. Then $\varrho \longrightarrow 1$ at $\pm\infty$, ϱ is a C^2 function on $\mathbf{R}$ and satisfies (3.4.2) and (3.4.3). Assume that $\inf \varrho < 1$. Take $x_1 \in \mathbf{R}$ such that $\varrho(x_1) \in (\zeta(c), 1)$. Let $I = (a, b)$ be the maximal interval containing x_1 such that $\varrho(x) \in (\zeta(c), 1)$ for any $x \in I$. It follows from (3.4.3) that $\varrho' \neq 0$ on I, hence ϱ' has constant sign and ϱ is strictly monotonic on I. If $\varrho' > 0$ on I, from (3.4.3) we get $\dfrac{\varrho'(x)}{g(\varrho(x), c)} = 1$ on I, and integrating we obtain $G(\varrho(x), c) = x + k_2$ on I, where k_2 is a constant, hence $\varrho(x) = G(\cdot, c)^{-1}(x + k_2)$ on I. Then necessarily $b = \infty$. Indeed, if b is finite we have $\zeta(c) < \varrho(x) < G(\cdot, c)^{-1}(b + k_2) < 1$ on I, hence there is some $\varepsilon > 0$ such that $\zeta(c) < \varrho(x) < 1$ on $[b, b + \varepsilon)$, contradicting the maximality of I. If $L(c) = -\infty$ we must have $a = -\infty$ (for otherwise, $\zeta(c) < G(\cdot, c)^{-1}(a + k_2) = \varrho(a) < \varrho(x) < 1$ for all $x \in I$, and we would have $\zeta(c) < \varrho(x)$ for $x \in (a - \varepsilon, a]$ for some positive ε, contradicting again the maximality of I). But if $a = -\infty$ we have $\lim\limits_{x \to -\infty} \varrho(x) = \lim\limits_{x \to -\infty} G(\cdot, c)^{-1}(x + k_2) = \zeta(c)$, impossible because $\varrho(x) \longrightarrow 1$ as $x \longrightarrow -\infty$. Thus necessarily $L(c)$ is finite. This implies that a is finite and $\varrho(a) = \zeta(c)$, and we find $a + k_2 = L(c)$. In conclusion, if I is a maximal interval such that $\zeta(c) < \varrho < 1$ and $\varrho' > 0$ on I then necessarily I is of the form (a, ∞) for some $a \in \mathbf{R}$ and we have $\varrho(x) = G(\cdot, c)^{-1}(x - a + L(c))$ on I. Similarly, if I is a maximal interval such that $\zeta(c) < \varrho < 1$ and $\varrho' < 0$ on I we show that $L(c)$ must be finite (for otherwise $I = \mathbf{R}$ and ϱ would not tend to 1 at ∞) and I is of the form $(-\infty, b)$ for some $b \in \mathbf{R}$ and $\varrho(x) = G(\cdot, c)^{-1}(-x + b + L(c))$ on I.

"$\Longleftarrow$" One easily proves that ϱ_c satisfies (3.4.3) and (3.4.2), and is C^2 in $\mathbf{R}$. It is obvious that θ_c satisfies (3.4.4) and then en easy computation shows that ψ_c solves (3.4.1). By (3.4.5) proven below we have $\varrho_c - 1 \in L^2(\mathbf{R})$, and then (3.4.3), the boundedness of ϱ_c and **(A1)** imply that $\varrho' \in L^2(\mathbf{R})$. By (3.4.4) we get $\theta' \in L^2(\mathbf{R})$ and then we infer that $\psi_c \in \mathcal{E}(\mathbf{R})$. Notice that ϱ may vanish only if $c = 0$, but in this case we have $\theta_0' = 0$ in $\mathbf{R}^*$ by (3.4.4).

(ii) The proof of (ii) is similar to the proof of part (i), so we omit it.

(iii) Let ψ be a solution of (3.4.1) and let $\varrho = |\psi|^2$. If $\varrho = 1$, we may write $\psi = e^{i\theta}$ and (3.4.4) implies that $\theta' = 0$, hence ψ is constant. Assume that there exist $x_1, x_2 \in \mathbf{R}$ such that $\varrho(x_1) < 1$ and $\varrho(x_2) \geqslant 1$. If $x_1 < x_2$, there exists $x_3 \in (x_1, x_2)$ such that $\zeta(c) < \varrho(x_3) < 1$ and $\varrho'(x_3) > 0$. The argument in the proof of part (i) "$\Longrightarrow$" shows that the maximal interval I containing x_3 such that $\zeta(c) < \varrho < 1$ and $\varrho' > 0$ on I is of the form (b, ∞), contradicting the fact that $\varrho(x_2) \geqslant 1$. If $x_1 > x_2$, there exists $x_3 \in (x_2, x_1)$ such that $\zeta(c) < \varrho(x_3) < 1$ and $\varrho'(x_3) < 0$. As above, we have then $\zeta(c) < \varrho(x_3) < 1$ and $\varrho'(x_3) < 0$ on $(-\infty, x_3]$, contradicting the fact that $\varrho(x_2) \geqslant 1$.

A similar argument leads to a contradiction if we assume that there exist $x_1, x_2 \in \mathbf{R}$ such that $\varrho(x_1) > 1$ and $\varrho(x_2) \leqslant 1$.

Fix $\varepsilon \in (0, \sqrt{2 - c^2})$. There is $\delta_\varepsilon > 0$ such that

$$(2 - c^2 - \varepsilon^2)(s-1)^2 \leqslant g(s,c) \leqslant (2 - c^2 + \varepsilon^2) \qquad \text{for any } s \in (1 - \delta_\varepsilon, 1 + \delta_\varepsilon).$$

and integrating we see that there exist constants $C_1, C_2 \in \mathbf{R}$ such that

$$C_1 - \frac{\ln|1-s|}{\sqrt{2 - c^2 + \varepsilon^2}} \leqslant G(s,c) \leqslant C_2 - \frac{\ln|1-s|}{\sqrt{2 - c^2 - \varepsilon^2}} \qquad \text{for } s \in [1 - \delta_\varepsilon, 1) \cup (1, 1 + \delta_\varepsilon). \tag{3.4.6}$$

If $\varrho < 1$, there exist $a, b, k_1, k_2 \in \mathbf{R}$ such that $a < b$, $\varrho(x) \in (1 - \delta_\varepsilon, 1)$ for all $x \in (-\infty, a) \cup (b, \infty)$, and $G(\varrho(x), c) = -x + k_1$ on $(-\infty, a)$, respectively $G(\varrho(x), c) = x + k_2$ on (b, ∞). Then using (3.4.6) we see that there are constants $C_3, C_4 \in \mathbf{R}$ such that

$$C_3 e^{-|x|\sqrt{2-c^2+\varepsilon^2}} \leqslant \big|1 - \varrho(x)\big| \leqslant C_4 e^{-|x|\sqrt{2-c^2-\varepsilon^2}} \qquad \text{on } (-\infty, a) \cup (b, \infty).$$

We obtain (3.4.5) for any $0 < \varepsilon' < \varepsilon$ by choosing conveniently $A_{\varepsilon'} < a$ and $B_{\varepsilon'} > b$. The proof of (3.4.5) is similar if $\varrho > 1$.

(iv) Taking the scalar product of (3.4.1) with $2\psi'$ we get $(|\psi'|^2)' - (V(|\psi|^2))' = 0$, hence $|\psi'|^2 - V(|\psi|^2)$ is constant. Since $|\psi'|^2$ and $V(|\psi|^2)$ belong to $L^1(\mathbf{R})$, the constant must be zero.

(v) Let $\psi \in \mathcal{E}(\mathbf{R})$ be a traveling wave of speed c. Let $\varrho = |\psi|^2$. Assume that $\varrho(x) < 1$ for some $x \in \mathbf{R}$. By (i) we know that $L(c)$ is finite. There is some $x_1 \in \mathbf{R}$ such that $\zeta(c) < \varrho(x_1) < 1$ and $\varrho'(x_1) < 0$. As in the proof of part (i), there exists $x_0 > x_1$ such that $\varrho(x) = G(\cdot, c)^{-1}(L(c) + x_0 - x)$ for all $x \in (-\infty, x_0)$ and $\varrho(x_0) = \zeta(c)$. Using (3.4.3) and the continuity of ϱ' we get $\varrho'(x_0) = \lim_{x \uparrow x_0} \varrho'(x) = \lim_{x \uparrow x_0} (-\sqrt{g(\varrho(x), c)}) = 0$. Let ϱ_c be as in (i). Then (ϱ, ϱ') and $(\varrho_c(\cdot - x_0), \varrho_c'(\cdot - x_0))$ are both solutions of the Cauchy problem

$$\begin{cases} y'(x) = z(x) \\ z'(x) = -c^2(y(x) - 1) + 2V(y(x)) - 2y(x)F(y(x)) \\ (y(x_0), z(x_0)) = (\zeta(c), 0) \end{cases}$$

in $[x_0, \infty)$. Since F is locally Lipschitz, the solution of the above Cauchy problem is unique and we infer that $\varrho = \varrho_c(\cdot - x_0)$ on $[x_0, \infty)$, thus on $\mathbf{R}$. Then using (3.4.4) we see that the difference between the phase of ψ and $\theta_c(\cdot - x_0)$ is a constant, say α, and therefore $\psi = e^{i\alpha}\psi_c(\cdot - x_0)$.

The proof is analogous if there exists $x \in \mathbf{R}$ such that $\varrho(x) > 1$.

(vi) We have $g(\zeta(c), c) = 0$ for any $c \in (-\sqrt{2}, \sqrt{2})$. If ζ is differentiable at c_0, differentiating this equality we get

$$\partial_1 g(\zeta(c_0), c_0) \cdot \zeta'(c_0) + \partial_2 g(\zeta(c_0), c_0) = 0,$$

that is $\partial_1 g(\zeta(c_0), c_0) \cdot \zeta'(c_0) = 2c_0(\zeta(c_0)-1)^2 > 0$. We infer that $\partial_1 g(\zeta(c_0), c_0) \neq 0$. Since $g(\zeta(c_0), c_0) = 0$ and $g(s, c_0) > 0$ on $(\zeta(c_0), 1)$, we have $\partial_1 g(\zeta(c_0), c_0) > 0$. Then $\frac{\sqrt{s-\zeta(c_0)}}{\sqrt{g(s,c_0)}} \longrightarrow \frac{1}{\sqrt{\partial_1 g(\zeta(c_0),c_0)}} > 0$ as $s \searrow \zeta(c_0)$, hence $s \longmapsto \frac{1}{\sqrt{g(s,c_0)}}$ is integrable on an interval $(\zeta(c_0), \zeta(c_0) + \varepsilon)$ and $L(c_0)$ is finite.

It is well-known that a monotonic function is differentiable almost everywhere. □

The following question arises naturally : is it true that (3.4.1) admits solutions in $\mathcal{E}(\mathbf{R})$ for all but countably many c's ?

Example 3.4.2. Consider the particular case of the Gross-Pitaevskii equation, namely $F(s) = 1-s$. We have $V(s) = \frac{1}{2}(1-s)^2$, $g(s,c) = (2s-c^2)(1-s)^2$, $\zeta(c) = \frac{c^2}{2}$ and $g(s,c) > 0$ if $s > 1$. One can use the change of variable $t = \sqrt{2s-c^2}$ to compute a primitive of $\frac{1}{\sqrt{g(\cdot,c)}}$ and it is easily seen that we may take

$$G(s,c) = \frac{1}{\sqrt{2-c^2}} \ln \Big| \frac{\sqrt{2-c^2} + \sqrt{2s-c^2}}{\sqrt{2-c^2} - \sqrt{2s-c^2}} \Big|. \tag{3.4.7}$$

With this choice of G we have $L(c) = \lim\limits_{s \searrow \zeta(c)} G(s,c) = \lim\limits_{s \searrow \frac{c^2}{2}} G(s,c) = 0$ for all $c \in (-\sqrt{2}, \sqrt{2})$. For any fixed c, the function $G(\cdot, c)$ is an increasing diffeomorphism between $(\frac{c^2}{2}, 1)$ and $(0, \infty)$. We find $G(\cdot,c)^{-1}(x) = \frac{c^2}{2} + \frac{2-c^2}{2} \left[\tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)\right]^2$. Proceeding as in Proposition 3.4.1 (i) we get

$$\varrho_c(x) = \frac{c^2}{2} + \frac{2-c^2}{2} \left[\tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)\right]^2 \qquad \text{for any } x \in \mathbf{R}. \tag{3.4.8}$$

Using (3.4.4) and the change of variable $t = \tanh\left(\frac{\sqrt{2-c^2}}{2}s\right)$ we compute θ_c and we find

$$\theta_c(x) = \frac{c}{2} \int_0^x \frac{1-\varrho_c(s)}{\varrho_c(s)}\, ds = \arctan\left[\frac{\sqrt{2-c^2}}{c} \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)\right]. \tag{3.4.9}$$

Since $\cos(\arctan(z)) = \frac{1}{\sqrt{1+z^2}}$ and $\sin(\arctan(z)) = \frac{z}{\sqrt{1+z^2}}$, we finally obtain

$$\begin{aligned} \psi_c(x) = \sqrt{\varrho(x)} e^{i\theta(x)} &= \tfrac{c}{\sqrt{2}} + i\sqrt{1-\tfrac{c^2}{2}} \tanh\left(\tfrac{\sqrt{2-c^2}}{2}x\right) \\ &= i\left[\sqrt{1-\tfrac{c^2}{2}} \tanh\left(\tfrac{\sqrt{2-c^2}}{2}x\right) - i\tfrac{c}{\sqrt{2}}\right]. \end{aligned} \tag{3.4.10}$$

It follows from Proposition 3.4.1 (i), (ii), (v) that all traveling waves for the Gross-Pitaevskii equation in $\mathcal{E}(\mathbf{R})$ are either constants of modulus one, or are of the form $e^{i\alpha}\psi_c(\cdot - x_0)$ for some $\alpha, x_0 \in \mathbf{R}$.

Using Proposition 3.4.1 (iv), then the change of variable $t = \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)$ we get

$$\begin{aligned} E^1(\psi_c) = E^1_{GL}(\psi_c) &= 2\int_{\mathbf{R}} |\psi_c'|^2\,dx = 2\int_{\mathbf{R}} V(|\psi_c|^2)\,dx = \int_{\mathbf{R}} (1-\varrho_c(x))^2\,dx \\ &= \left(\frac{2-c^2}{2}\right)^2 \frac{2}{\sqrt{2-c^2}}\int_{-1}^{1} 1-t^2\,dt = \frac{2}{3}(2-c^2)^{\frac{3}{2}}. \end{aligned} \tag{3.4.11}$$

If $c \neq 0$, it follows from Remark 3.3.2, identity (3.4.4) and the change of variable $t = \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)$ that a valuation of the momentum of ψ_c is

$$\begin{aligned} \int_{\mathbf{R}} (1-\varrho_c)\theta_c'\,dx &= \frac{c}{2}\int_{\mathbf{R}} \frac{(1-\varrho_c)^2}{\varrho_c}\,dx = \frac{c}{4}(2-c^2)^{\frac{3}{2}}\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{c^2}{2}+\frac{2-c^2}{2}t^2}\frac{1}{1-t^2}\,dt \\ &= \frac{c}{4}(2-c^2)^{\frac{3}{2}}\int_{-1}^{1} \frac{1-t^2}{\frac{c^2}{2}+\frac{2-c^2}{2}t^2}\,dt = \frac{c}{2}(2-c^2)^{\frac{1}{2}}\int_{-1}^{1} \frac{1-t^2}{\frac{c^2}{2-c^2}+t^2}\,dt \\ &= 2\arctan\left(\frac{\sqrt{2-c^2}}{c}\right) - c\sqrt{2-c^2}. \end{aligned} \tag{3.4.12}$$

For more information on traveling waves for the Gross-Pitaevskii equation and for further references we refer to [1], Section 2. It has been shown in [1] that the functions ψ_c minimize the energy when the momentum is kept fixed. Formulas (3.4.11) and (3.4.12) here above correspond to formulas (2.23) and (2.24) p. 63 in [1], respectively.

Example 3.4.3. Consider a function $V \in C^\infty(\mathbf{R}, \mathbf{R})$ having the following properties :

- There exists $\delta_1 > 0$ such that $V(s) = \frac{1}{2}(1-s)^2$ for all $s \in [1-\delta_1, 1+\delta_1]$,
- There exist $c_0 \in (0, \sqrt{2})$, $s_0 \in (0,1)$ and $a > 0$, $\delta_2 > 0$ such that

$$V(s) = \frac{1}{4s}\left(c_0^2(1-s)^2 + a^2(s-s_0)^3\right) \text{ on } (s_0-\delta_2, s_0+\delta_2], \quad \text{and } V(s) > \frac{c_0^2(1-s)^2}{4s} \text{ on } (s_0, 1),$$

- $V(s) \geqslant \frac{(s-1)^2}{2s}$ on $(1, \infty)$.

Let $F(s) = -V'(s)$. It is obvious that F satisfies the assumption **(A1)**. We consider equation (3.1.1) with nonlinearity F. Using Proposition 3.4.1 (i), (ii) and (v) it is easily seen that for c sufficiently close to $\sqrt{2}$ (more precisely, for $c \in [\sqrt{2(1-\delta_1)}, \sqrt{2})$), traveling waves of speed c for this equation in $\mathcal{E}(\mathbf{R})$ are the same as traveling waves for the Gross-Pitaevskii equation (and they are either constant, or are equal to functions ψ_c in Example 3.4.2 up to a translation and a phase shift). Letting $g(s,c) = 4sV(s) - c^2(s-1)^2$, we have $g(s, c_0) = a^2(s-c_0)^3$ on $(s_0, s_0+\delta_2)$ and $g(s, c_0) > 0$ on $(s_0, 1) \cup (1, \infty)$, hence $\zeta(c_0) = s_0$ and $L(c_0) = -\infty$. Then Proposition 3.4.1 (i) and (ii) implies that all traveling waves of (3.1.1) in $\mathcal{E}(\mathbf{R})$ with speed c_0 must be constant.

Lemma 3.4.4. *Assume that $V > 0$ on $[0,1)$. Then we have*

$$\inf\left\{E^1(\psi) \mid \psi \in \mathcal{E}(\mathbf{R}) \text{ and } \inf_{x\in\mathbf{R}} |\psi(x)| = 0\right\} = 4\int_0^1 \sqrt{V(s^2)}\, ds.$$

If the infimum is achieved by a function ψ, then there exist $x_0, \alpha_-, \alpha_+ \in \mathbf{R}$ such that

$$\psi(x) = \begin{cases} \sqrt{\varrho_0(x-x_0)}e^{i\alpha_-} & \text{if } x < x_0, \\ \sqrt{\varrho_0(x+x_0)}e^{i\alpha_+} & \text{if } x \geqslant x_0, \end{cases} \tag{3.4.13}$$

where ϱ_0 is as in Proposition 3.4.1 (i).

Proof. For any $\psi \in \mathcal{E}(\mathbf{R})$ we have $|\psi| - 1 \in H^1(\mathbf{R})$, hence $|\psi|$ is continuous and tends to 1 at $\pm\infty$. If $\inf\limits_{x\in\mathbf{R}} |\psi(x)| < 1$, then the infimum is achieved at some $x_0 \in \mathbf{R}$.

Consider any $\psi \in \mathcal{E}(\mathbf{R})$ such that $\psi(x_0) = 0$ for some $x_0 \in \mathbf{R}$. Take two sequences $(x_n^\pm)_{n\geqslant 1} \subset \mathbf{R}$ such that $x_n^\pm \longrightarrow \pm\infty$. For n sufficiently large we have $x_n^- < x_0 < x_n^+$ and using (3.2.2) we get

$$\int_{x_n^-}^{x_0} |\psi'|^2 + V(|\psi|^2)\, dx \geqslant 2\big|H(|\psi(x_0)|) - H(|\psi(x_n^-)|)\big|$$

and

$$\int_{x_0}^{x_n^+} |\psi'|^2 + V(|\psi|^2)\, dx \geqslant 2\big|H(|\psi(x_n^-)|) - H(|\psi(x_0)|)\big|.$$

Summing up and letting $n \longrightarrow \infty$ we get $E^1(\psi) \geqslant 4|H(0)| = 4\displaystyle\int_0^1 \sqrt{V(s^2)}\, ds$.

If $V(0) > 0$ it follows that any primitive of the function $s \longmapsto \frac{1}{\sqrt{g(s,0)}} = \frac{1}{2\sqrt{sV(s)}}$ has finite limit at $0+$ (in other words, $L(0)$ is finite), and the function ϱ_0 in Proposition 3.4.1 (i) is well-defined and satisfies

$$\varrho_0' = -2\sqrt{\varrho_0 V(\varrho_0)} \text{ on } (-\infty, 0), \quad \text{respectively} \quad \varrho_0' = 2\sqrt{\varrho_0 V(\varrho_0)} \text{ on } (0,\infty).$$

Let $\rho_0 = \sqrt{\varrho_0}$. It is easily seen that

$$\rho_0' = -\sqrt{V(\rho_0^2)} \text{ on } (-\infty, 0), \qquad \text{respectively} \qquad \rho_0' = \sqrt{V(\rho_0^2)} \text{ on } (0,\infty). \tag{3.4.14}$$

Using (3.4.14) and the change of variable $\tau = \rho_0(x)$ we obtain

$$\int_{-\infty}^0 |\rho_0'|^2 + V(\rho_0^2)\, dx = -2\int_{-\infty}^0 \sqrt{V(\rho_0^2(x))}\rho_0'(x)\, dx = 2\int_0^1 \sqrt{V(\tau^2)}\, d\tau$$

and a similar computation holds on $[0,\infty)$, hence $E^1(\rho_0) = 4\displaystyle\int_0^1 \sqrt{V(\tau^2)}\, d\tau$.

Consider any $\psi \in \mathcal{E}(\mathbf{R})$ such that $E^1(\psi) = 4\int_0^1 \sqrt{V(\tau^2)}\,d\tau$ and there exists $x_0 \in \mathbf{R}$ such that $\psi(x_0) = 0$. As above, using (3.2.2) we see that

$$\int_{-\infty}^{x_0} |\psi'|^2 + V(|\psi|^2)\,dx \geqslant 2\int_0^1 \sqrt{V(\tau^2)}\,d\tau \text{ and } \int_{x_0}^{-\infty} |\psi'|^2 + V(|\psi|^2)\,dx \geqslant 2\int_0^1 \sqrt{V(\tau^2)}\,d\tau. \tag{3.4.15}$$

Hence we must have equality in both inequalities in (3.4.15) and we infer that the point x_0 must be unique, and $\rho = |\psi|$ must satisfy $\rho' = \pm\sqrt{V(\rho^2)}$ on each of the intervals $(-\infty, x_0)$ and (x_0, ∞). Then it follows easily that $\rho(\cdot + x_0)$ satisfies (3.4.14), and finally that $\rho(\cdot + x_0) = \rho_0$. The function $\frac{\psi}{\rho}$ must be constant on each of the intervals $(-\infty, x_0)$ and (x_0, ∞) (for otherwise, we would have $\psi = \rho e^{i\theta}$ on those intervals for some $\theta \in H^1_{loc}$, and then $|\psi'|^2 = |\rho'|^2 + \rho^2|\theta'|^2 \geqslant |\rho'|^2$. If $\theta' \not\equiv 0$ we would get $\int_{\mathbf{R}} |\psi'|^2\,dx > \int_{\mathbf{R}} |\rho'|^2\,dx$, hence $E^1(\psi) > E^1(\rho)$, a contradiction). □

Corollary 3.4.5. *For any $\psi \in \mathcal{E}(\mathbf{R})$ satisfying $E^1(\psi) < 4\int_0^1 \sqrt{V(\tau^2)}\,d\tau$ we have $\inf_{x\in\mathbf{R}} |\psi(x)| > 0$ and there exists a lifting $\psi = \rho e^{i\theta}$, where $1 - \rho \in H^1(\mathbf{R})$ and $\theta \in \dot{H}^1(\mathbf{R})$.*

The same conclusion holds if $E^1(\psi) = 4\int_0^1 \sqrt{V(\tau^2)}\,d\tau$ and ψ is not one of the functions in (3.4.13).

Remark 3.4.6. It can be shown that for any $a \in (0, 1)$ we have

$$\inf\left\{E^1(\psi) \mid \psi \in \mathcal{E}(\mathbf{R}) \text{ and } \inf_{x\in\mathbf{R}} |\psi(x)| = a\right\} = 4\int_a^1 \sqrt{V(s^2)}\,ds.$$

Moreover, the only minimizers, up to translations in $\mathbf{R}$ and multiplication by complex numbers of modulus 1, are the functions $\rho_a(x) = \begin{cases} \rho_0(\cdot - b) & \text{if } x < 0, \\ \rho_0(\cdot + b) & \text{if } x \geqslant 0, \end{cases}$ where ρ_0 is as in (3.4.14) and $b > 0$ is chosen so that $\rho_0(b) = a$.

3.5 Minimizing the energy at fixed momentum in $\mathcal{E}(\mathbf{R})$

We define

$$E^1_{min}(p) = \inf\{E^1(\psi) \mid \psi \in \mathcal{E}(\mathbf{R}) \text{ and } \lfloor P \rfloor(\psi) = \lfloor p \rfloor\}.$$

We collect in the next Lemma the main properties of the function E^1_{min}.

Lemma 3.5.1. *Assume that V satisfies* **(A1)**. *The function E^1_{min} has the following properties :*

i) E^1_{min} is non-negative, $2\pi-$periodic, $E^1_{min}(-p) = E^1_{min}(p)$ for all $p \in \mathbf{R}$ and

$$E^1_{min}(p) = \inf\{E^1(e^{i\varphi} + w) \mid \varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R}), w \in H^1(\mathbf{R}, \mathbf{C}) \text{ and } p(\varphi, w) = p\}.$$

ii) $E^1_{min}(p) \leqslant \sqrt{2}p$.

iii) For any $\varepsilon > 0$ there exists $p_\varepsilon > 0$ such that $E^1_{min}(p) \geqslant (1-\varepsilon)\sqrt{2}p$ for any $p \in (0, p_\varepsilon)$.

iv) E^1_{min} is sub-additive : for any $p_1, p_2 \in \mathbf{R}$ there holds

$$E_{min}(p_1 + p_2) \leqslant E^1_{min}(p_1) + E^1_{min}(p_2).$$

v) E^1_{min} is Lipschitz on $\mathbf{R}$ and its best Lipschitz constant is $v_s = \sqrt{2}$.

vi) Assume that there exists $\delta > 0$ such that $V(s) \leqslant \frac{1}{2}(1-s)^2 + \frac{3}{8}(1-s)^3$ for any $s \in [1-\delta, 1)$ [3]. Then we have $E^1_{min}(p) < \sqrt{2}p$ for any $p > 0$.

Assume, in addition, that $V > 0$ on $[0, 1)$. Then :

vii) For any $p \in \mathbf{R}$ we have $E^1_{min}(p) \leqslant 4\displaystyle\int_0^1 \sqrt{V(\tau^2)}\, d\tau$.

viii) E^1_{min} is nondecreasing on $[0, \pi]$ and is concave on $[0, 2\pi]$.

Proof. (i) For any given $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$ and $w \in H^1(\mathbf{R})$, let $\tilde{\varphi}(x) = \varphi(-x)$ and $\tilde{w}(x) = w(-x)$. Then we have $\tilde{\varphi} \in \dot{H}^1(\mathbf{R}, \mathbf{R})$, $\tilde{w} \in H^1(\mathbf{R})$, $p(\tilde{\varphi}, \tilde{w}) = -p(\varphi, w)$ and $E^1(e^{i\tilde{\varphi}} + \tilde{w}) = E^1(e^{i\varphi} + w)$. This implies that $E^1_{min}(-p) = E^1_{min}(p)$ for any $p \in \mathbf{R}$.

Let $p \in \mathbf{R}$. Let $k \in \mathbf{Z}$. Consider $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in H^1_{per}$ satisfying $p(\varphi, w) = p$. Let $\chi \in C^\infty(\mathbf{R})$ such that $\chi = 0$ on $(-\infty, 0]$ and $\chi = 1$ on $[1, \infty)$. Let $\tilde{\varphi}(x) = \varphi(x) + 2k\pi\chi(x)$ and $\tilde{w}(x, y) = e^{i\varphi(x)} - e^{i\tilde{\varphi}(x)} + w(x, y)$. Then $p(\tilde{\varphi}, \tilde{w}) = p(\varphi, w) + 2k\pi = p + 2k\pi$ and $e^{i\varphi} + w = e^{i\tilde{\varphi}} + \tilde{w}$. Since for any $(\varphi, w) \in \dot{H}^1(\mathbf{R}, \mathbf{R}) \times H^1_{per}$ satisfying $p(\varphi, w) = p$ we may construct $(\tilde{\varphi}, \tilde{w})$ as previously, we conclude that $E^1_{min}(p + 2k\pi) = E^1_{min}(p)$. The rest of part (i) is obvious.

(ii) The proof is very similar to the proof of Lemma 3.3 p. 604 in [2] and of Lemma 4.5 p. 173 in [6]. Take $\chi \in C^\infty_c(\mathbf{R})$ such that $\int_{\mathbf{R}} |\chi'(x)|^2\, dx = 1$ and $\int_{\mathbf{R}} \chi'(x)^3\, dx = 0$ (for instance, we may take χ an even function). Let $A = \int_{\mathbf{R}} |\chi''(t)|^2\, dt$ and $B = \int_{\mathbf{R}} |\chi(t)|^4\, dt$. For $\varepsilon, \lambda, \sigma > 0$ (to be chosen later), let $\rho_{\varepsilon,\lambda}(x) = 1 - \frac{\varepsilon}{\lambda}\chi'\left(\frac{x}{\lambda}\right)$, $\theta_{\lambda,\sigma}(x) = \sigma\chi\left(\frac{x}{\lambda}\right)$ and $\psi_{\varepsilon,\lambda,\sigma}(x) = \rho_{\varepsilon,\lambda}(x)e^{i\theta_{\lambda,\sigma}(x)}$. It is clear that $\psi_{\varepsilon,\lambda,\sigma} \in \mathcal{E}(\mathbf{R})$ and a simple computation gives

$$\int_{\mathbf{R}} |\psi'_{\varepsilon,\lambda,\sigma}(x)|^2\, dx = \int_{\mathbf{R}} |\rho'_{\varepsilon,\lambda}(x)|^2 + |\rho_{\varepsilon,\lambda}|^2|\theta'_{\lambda,\sigma}(x)|^2\, dx$$

$$= \int_{\mathbf{R}} \frac{\varepsilon^2}{\lambda^3}|\chi''(t)|^2 + \frac{\sigma^2}{\lambda}|\chi'(t)|^2\left(1 - \frac{2\varepsilon}{\lambda}\chi'(t) + \frac{\varepsilon^2}{\lambda^2}|\chi'(t)|^2\right) dt = \frac{\sigma^2}{\lambda} + A\frac{\varepsilon^2}{\lambda^3} + B\frac{\varepsilon^2\sigma^2}{\lambda^3},$$

$$\int_{\mathbf{R}} (1 - \rho^2_{\varepsilon,\lambda})\theta'_{\lambda,\sigma}(x)\, dx = \int_{\mathbf{R}} \frac{2\sigma\varepsilon}{\lambda}|\chi'(t)|^2 - \frac{\sigma\varepsilon^2}{\lambda^2}\chi'(t)^3\, dt = \frac{2\sigma\varepsilon}{\lambda},$$

$$\int_{\mathbf{R}} (1 - \rho^2_{\varepsilon,\lambda})^2\, dx = \int_{\mathbf{R}} \frac{4\varepsilon^2}{\lambda}|\chi'(t)|^2 - \frac{4\varepsilon^3}{\lambda^2}\chi'(t)^3 + \frac{\varepsilon^4}{\lambda^3}\chi'(t)^4\, dt = \frac{4\varepsilon^2}{\lambda} + B\frac{\varepsilon^4}{\lambda^3}.$$

3. This condition is fulfilled, for instance, if F is C^2 near 1 and $F''(1) < \frac{9}{4}$.

Fix $p > 0$. For $\lambda > 0$ we choose $\varepsilon = \varepsilon(\lambda) = 2^{-\frac{3}{4}}\sqrt{p\lambda}$ and $\sigma = \sigma(\lambda) = 2^{-\frac{1}{4}}\sqrt{p\lambda}$. Let $\psi_\lambda = \psi_{\varepsilon(\lambda),\lambda,\sigma(\lambda)}$. As $\lambda \longrightarrow \infty$ we have $\frac{\varepsilon(\lambda)}{\lambda} \longrightarrow 0$, $\frac{\sigma(\lambda)}{\lambda} \longrightarrow 0$, and $\big|1-|\psi_\lambda|\big| = \big|1-\rho_{\varepsilon(\lambda),\lambda}\big| \leqslant \frac{\varepsilon(\lambda)}{\lambda}\|\chi'\|_{L^\infty} \longrightarrow 0$. For all λ sufficiently large we have $\psi_\lambda \in \mathcal{E}(\mathbf{R})$ and the above computations show that a valuation of the momentum of ψ_λ is p, and

$$\int_{\mathbf{R}} |\psi'_{\lambda,}(x)|^2\,dx = 2^{-\frac{1}{2}}p + 2^{-\frac{3}{2}}\frac{p}{\lambda^2}A + \frac{p^2}{4\lambda}B \longrightarrow \frac{p}{\sqrt{2}} \quad \text{ as } \lambda \longrightarrow \infty.$$

By assumption **(A1)** we have $V(s) = \Big(\frac{1}{2} + o(|s-1|)\Big)\,(s-1)^2$ as $s \longrightarrow 1$, hence

$$\int_{\mathbf{R}} V(|\psi_\lambda|^2)\,dx = \left(\frac{1}{2} + o\left(\frac{\varepsilon(\lambda)}{\lambda}\|\chi'\|_{L^\infty}\right)\right)\int_{\mathbf{R}}(1-\rho^2_{\varepsilon(\lambda),\lambda})^2\,dx \longrightarrow \frac{p}{\sqrt{2}} \quad \text{ as } \lambda \longrightarrow \infty.$$

Since $E^1_{min}(p) \leqslant E^1(\psi_\lambda)$ for all sufficiently large λ and $E^1(\psi(\lambda)) \longrightarrow \sqrt{2}p$ as $\lambda \longrightarrow \infty$, (ii) follows.

(iii) Fix $\varepsilon > 0$. We may assume that $\varepsilon \leqslant \frac{1}{2}$. By assumption **(A1)** there exists $\delta = \delta(\varepsilon) > 0$ such that $(1-\delta)^2 \geqslant 1-\varepsilon$ and

$$V(\rho^2) > \frac{1}{2}(1-\varepsilon)(1-\rho^2)^2 \qquad \text{for any } \rho \in [1-\delta, 1+\delta].$$

By Lemma 3.2.1 (i) there exists $\kappa > 0$ such that for any $\psi \in \mathcal{E}(\mathbf{R})$ satisfying $E^1(\psi) \leqslant \kappa$ we have $\|1-|\psi|\|_{L^\infty(\mathbf{R})} \leqslant \delta$. Let $p_\varepsilon = \min\left(\frac{\kappa}{2\sqrt{2}}, \frac{\pi}{4}\right)$ and let $p \in (0, p_\varepsilon]$. Consider any $\varphi \in \dot{H}^1(\mathbf{R})$ and any $w \in H^1(\mathbf{R})$ such that $p(\varphi, w) = p$ and $E^1(e^{i\varphi} + w) \leqslant 2\sqrt{2}p$. Denoting $\psi = e^{i\varphi} + w$, we have $\psi \in \mathcal{E}(\mathbf{R})$ and $E^1(\psi) \leqslant \kappa$, and then Lemma 3.2.1 (i) implies that $\|1-|\psi|\|_{L^\infty(\mathbf{R})} \leqslant \delta$. We infer that ψ admits a lifting $\psi = \rho e^{i\theta}$ where $\theta \in \dot{H}^1(\mathbf{R})$ and $1-\delta \leqslant \rho(x) \leqslant 1+\delta$ on $\mathbf{R}$. Then we have

$$\begin{aligned} E^1(\psi) &= \int_{\mathbf{R}} |\rho'|^2(x) + \rho^2(x)|\theta'|^2(x) + V(\rho^2(x))\,dx \\ &\geqslant \int_{\mathbf{R}} (1-\delta)^2|\theta'|^2(x) + \frac{1}{2}(1-\varepsilon)(1-\rho^2(x))^2\,dx \geqslant (1-\varepsilon)\sqrt{2}\Big|\int_{\mathbf{R}}(1-\rho^2(x))\theta'(x)\,dx\Big|. \end{aligned} \tag{3.5.1}$$

By Definition 3.3.1 and Remark 3.3.2, $p = p(\varphi, w)$ and $\displaystyle\int_{\mathbf{R}}(1-\rho^2(x))\theta'(x)\,dx$ are both valuations of the momentum of ψ, hence $\displaystyle\int_{\mathbf{R}}(1-\rho^2(x))\theta'(x)\,dx = p + 2k\pi$ for some $k \in \mathbf{Z}$. We have $0 < p \leqslant p_\varepsilon \leqslant \frac{\pi}{4}$ and by (3.5.1) we get $\Big|\displaystyle\int_{\mathbf{R}}(1-\rho^2(x))\theta'(x)\,dx\Big| \leqslant \frac{E^1(\psi)}{(1-\varepsilon)\sqrt{2}} \leqslant 4p_\varepsilon < \pi$, and we conclude that necessarily $\displaystyle\int_{\mathbf{R}}(1-\rho^2(x))\theta'(x)\,dx = p$. Then (3.5.1) gives $E^1(\psi) = E^1(e^{i\varphi}+w) \geqslant (1-\varepsilon)\sqrt{2}p$. Since this inequality holds for any φ and w as above, (iii) follows.

(iv) Let $p_1, p_2 \in \mathbf{R}$. Fix $\varepsilon > 0$. By Lemma 3.3.6 (i) there exist $\varphi_1, \varphi_2 \in C^\infty(\mathbf{R})$, $w_1, w_2 \in C^\infty_c(\mathbf{R})$ and $A > 0$ such that $\operatorname{supp}(w_j) \subset (-A, A)$, $\varphi_j = \alpha_i$ on $(-\infty, A]$, $\varphi_j = \beta_j$ on $[A, \infty)$,

$$p(\varphi_j, w_j) \in \lfloor p_j \rfloor \qquad \text{and} \qquad E^1(e^{i\varphi_j} + w_j) < E^1_{min}(p_j) + \frac{\varepsilon}{2} \qquad \text{for } j = 1, 2.$$

Let

$$\psi(x) = \begin{cases} e^{i\varphi_1(x)} + w_1(x) \text{ if } x \leqslant 2A, \\ \\ e^{i(\beta_1-\alpha_2)} \left(e^{i\varphi_2(x-3A)} + w_2(x-3A) \right) \text{ if } x > 2A. \end{cases}$$

Then we have $\psi \in \mathcal{E}(\mathbf{R}) \cap C^\infty(\mathbf{R})$ and

$$\lfloor P \rfloor(\psi) = \lfloor P \rfloor(e^{i\varphi_1} + w_1) + \lfloor P \rfloor \left(e^{i(\beta_1-\alpha_2)} \left(e^{i(\varphi_2(\cdot-3A))} + w_2(\cdot - 3A) \right) \right) = \lfloor p_1 \rfloor + \lfloor p_2 \rfloor.$$

We infer that

$$\begin{aligned} E^1_{min}(p_1+p_2) &\leqslant E^1(\psi) = E^1\left(e^{i\varphi_1} + w_1\right) + E^1\left(e^{i(\beta_1-\alpha_2)} \left(e^{i(\varphi_2(\cdot-3A))} + w_2(\cdot - 3A) \right) \right) \\ &= E^1\left(e^{i\varphi_1} + w_1\right) + E^1\left(e^{i\varphi_2} + w_2\right) < E^1_{min}(p_1) + E^1_{min}(p_2) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, (iv) follows.

(v) The sub-additivity of E^1_{min} and part (ii) imply that

$$|E^1_{min}(p_2) - E^1_{min}(p_1)| \leqslant E^1_{min}(p_2 - p_1) \leqslant \sqrt{2}|p_2 - p_1| \qquad \text{for any } p_1, p_2 \in \mathbf{R}.$$

Part (iii) implies that $\sqrt{2}$ is the best Lipschitz constant of E^1_{min}.

(vi) The sub-additivity of E^1_{min} gives $E^1_{min}(np) \leqslant nE^1_{min}(p)$ for any $p > 0$ and any $n \in \mathbf{N}^*$. Hence it suffices to show that $E^1_{min}(p) < \sqrt{2}p$ for sufficiently small p.

We use as "test functions" the traveling-waves for the Gross-Pitaevskii equation in Example 3.4.2. Proceeding as in (3.4.11) and using the change of variable $t = \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right)$ we get

$$\int_{\mathbf{R}} \left(1 - |\psi_c|^2\right)^3 dx = \int_{\mathbf{R}} (1 - \varrho_c(x))^3 \, dx = \left(\frac{2-c^2}{2}\right)^3 \frac{2}{\sqrt{2-c^2}} \int_{-1}^{1} (1-t^2)^2 \, dt = \frac{4}{15}(2-c^2)^{\frac{5}{2}}. \tag{3.5.2}$$

There is some $c_\delta \in (0, \sqrt{2})$ such that for all $c \in (c_\delta, \sqrt{2})$ we have $1 - \delta < |\psi_c| < 1$. Using the fact that $V(s) \leqslant \frac{1}{2}(1-s)^2 + \frac{3}{8}(1-s)^3$ for $s \in [1-\delta, 1)$, (3.4.11) and (3.5.2) we get

$$E^1(\psi_c) \leqslant E^1_{GL}(\psi_c) + \frac{3}{8}\int_{\mathbf{R}} \left(1 - |\psi_c|^2\right)^3 dx \leqslant \frac{2}{3}(2-c^2)^{\frac{3}{2}} + \frac{1}{10}(2-c^2)^{\frac{5}{2}}.$$

It is useful to denote $\varepsilon(c) = \sqrt{2-c^2}$, so that $\varepsilon(c) \longrightarrow 0$ as $c \nearrow \sqrt{2}$. The above inequality can be written as

$$E^1(\psi_c) \leqslant f(\varepsilon(c)), \qquad \text{where } f(\varepsilon) = \frac{2}{3}e^3 + \frac{1}{10}\varepsilon^5.$$

Recall that by (3.4.12), a valuation of the momentum of ψ_c is

$$m(c) := 2\arctan\left(\frac{\sqrt{2-c^2}}{c}\right) - c\sqrt{2-c^2} = 2\arctan\left(\frac{\varepsilon(c)}{\sqrt{2-\varepsilon^2(c)}}\right) - \varepsilon(c)\sqrt{2-\varepsilon^2(c)} =: g(\varepsilon(c)).$$

We have $g(0) = 0$ and an elementary computation gives $g'(\varepsilon) = \frac{2\varepsilon^2}{\sqrt{2-\varepsilon^2}}$ for $\varepsilon \in (0, \sqrt{2})$. The function g is increasing and continuous on $[0, \sqrt{2})$, and therefore $c \longmapsto m(c)$ is decreasing, positive on $(c_\delta, \sqrt{2})$, and tends to 0 as $c \nearrow \sqrt{2}$. We have

$$E^1_{min}(m(c)) \leqslant E^1(\psi_c) \leqslant f(\varepsilon(c))$$

and it suffices to show that $f(\varepsilon) < \sqrt{2}g(\varepsilon)$ for all sufficiently small ε. A straightforward computation gives

$$g'(\varepsilon) = \sqrt{2}\left(\varepsilon^2 + \frac{1}{4}\varepsilon^4 + \frac{3}{32}\varepsilon^6 + o(\varepsilon^6)\right) \quad \text{and}$$

$$g(\varepsilon) = \sqrt{2}\left(\frac{1}{3}\varepsilon^3 + \frac{1}{20}\varepsilon^5 + \frac{3}{224}\varepsilon^7 + o(\varepsilon^7)\right) \quad \text{as } \varepsilon \longrightarrow 0.$$

Hence there is $\varepsilon_0 > 0$ such that $f(\varepsilon) < \sqrt{2}g(\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_0)$, as desired.

(vii) Let $p \in \mathbf{R}$. Fix $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta V(0) < \varepsilon$. Let ϱ_0 be as in Proposition 3.4.1 (i). Define

$$\rho(x) = \begin{cases} \sqrt{\varrho_0(x)} & \text{if } x < 0, \\ 0 & \text{if } 0 \leqslant x \leqslant \delta, \\ \sqrt{\varrho_0(x-\delta)} & \text{if } x \geqslant x_0, \end{cases}$$

It is easily seen that $1 - \rho \in H^1(\mathbf{R})$. Choose $\theta \in C^\infty(\mathbf{R})$ such that θ is constant on $(-\infty, 0]$ and on $[\delta, \infty)$, and $\int_0^\delta \theta'\, dx = \theta(\delta) - \theta(0) = p$. Let $w = (\rho - 1)e^{i\theta}$ and $\psi = \rho e^{i\theta} = e^{i\theta} + w$. We have $w \in H^1(\mathbf{R})$, $\psi \in \mathcal{E}(\mathbf{R})$ and using Remark 3.3.2 we get

$$p(\theta, w) = \int_{\mathbf{R}} (1 - \rho^2)\theta'\, dx = p.$$

We have $|\rho'|^2 + V(\rho^2) = V(0)$ on $(0, \delta)$ and consequently

$$E^1_{min}(p) \leqslant E^1(\psi) = \int_{(-\infty,0]\cup[\delta,\infty)} \left(|\rho'|^2 + V(\rho^2)\right) dx + \delta V(0) \leqslant 4\int_0^1 \sqrt{V(\tau^2)}\, d\tau + \varepsilon.$$

Since ε was arbitrary, the conclusion follows.

(viii) We proceed in several steps.

Step 1. "Reflection" of functions in $\mathcal{E}(\mathbf{R})$ that have a lifting. Assume that $\psi \in \mathcal{E}(\mathbf{R})$ can be written in the form $\psi = \rho e^{i\theta}$, where ρ and θ are real-valued functions, $1 - \rho \in H^1(\mathbf{R})$ and $\theta \in \dot{H}^1(\mathbf{R})$. For any $t \in \mathbf{R}$ we define $\psi_t(x) = e^{-i\theta(t)}\psi(x) = \rho(x)e^{i(\theta(x)-\theta(t))}$. It is obvious that

$$\psi_t \in \mathcal{E}(\mathbf{R}), \quad \psi_t(t) = \rho(t) \in [0, \infty), \quad E^1(\psi_t) = E^1(\psi) \quad \text{and} \quad \lfloor P \rfloor(\psi_t) = \lfloor P \rfloor(\psi).$$

We define

$$\psi_{t,1}(x) = \begin{cases} \psi_t(x) \text{ if } x \leqslant t, \\ \overline{\psi_t}(2t - x) \text{ if } x > t, \end{cases} \qquad \psi_{t,2}(x) = \begin{cases} \overline{\psi_t}(2t - x) \text{ if } x < t, \\ \psi_t(x) \text{ if } x \geqslant t, \end{cases} \tag{3.5.3}$$

$$\theta_{t,1}(x) = \begin{cases} \theta(x) - \theta(t) \text{ if } x \leqslant t, \\ -\theta(2t-x) + \theta(t) \text{ if } x > t, \end{cases} \qquad \theta_{t,2}(x) = \begin{cases} -\theta(2t-x) + \theta(t) \text{ if } x < t, \\ \theta(x) - \theta(t) \text{ if } x \geqslant t, \end{cases} \tag{3.5.4}$$

$$\rho_{t,1}(x) = \begin{cases} \rho(x) \text{ if } x \leqslant t, \\ \rho(2t-x) \text{ if } x > t, \end{cases} \qquad \rho_{t,2}(x) = \begin{cases} \rho(2t-x) \text{ if } x < t, \\ \rho(x) \text{ if } x \geqslant t. \end{cases} \tag{3.5.5}$$

It is easy to check that $\rho_{t,j} \in H^1(\mathbf{R})$, $\theta_{t,j} \in \dot{H}^1(\mathbf{R})$, $\psi_{t,j} \in \mathcal{E}(\mathbf{R})$ and $\psi_{t,j} = \rho_{t,j} e^{i\theta_{t,j}} = e^{i\theta_{t,j}} + w_{t,j}$ for $j = 1, 2$, where $w_{t,j} = (\rho_{t,j} - 1)e^{i\theta_{t,j}} \in H^1(\mathbf{R})$. An immediate computation gives

$$E^1(\psi_{t,1}) = 2\int_{-\infty}^{t} |\psi'|^2 + V(|\psi|^2)\, dx \quad \text{and} \quad E^1(\psi_{t,2}) = 2\int_{t}^{\infty} |\psi'|^2 + V(|\psi|^2)\, dx, \tag{3.5.6}$$

$$\begin{aligned} p(\theta_{t,1}, w_{t,1}) &= \int_{\mathbf{R}} (1 - \rho_{t,1}^2)\theta'_{t,1}\, dx = 2\int_{-\infty}^{t} (1-\rho^2)\theta'\, dx, \\ p(\theta_{t,2}, w_{t,2}) &= \int_{\mathbf{R}} (1 - \rho_{t,2}^2)\theta'_{t,2}\, dx = 2\int_{t}^{\infty} (1-\rho^2)\theta'\, dx, \end{aligned} \tag{3.5.7}$$

so that

$$E^1(\psi_{t,1}) + E^1(\psi_{t,2}) = 2E^1(\psi) \quad \text{and} \quad p(\theta_{t,1}, w_{t,1}) + p(\theta_{t,2}, w_{t,2}) = 2p\left(\theta, (\rho-1)e^{i\theta}\right). \tag{3.5.8}$$

Step 2. For any $p \in (0, \pi]$ satisfying $E^1_{min}(p) < 4\int_0^1 \sqrt{V(s^2)}\, ds$ and for any $\varepsilon > 0$ there exists $\psi \in \mathcal{E}(\mathbf{R})$ such that $E(\psi) < E^1_{min}(p) + \varepsilon$, $\psi = \rho e^{i\theta}$ with $\rho \in H^1(\mathbf{R})$, $\theta \in \dot{H}^1(\mathbf{R})$ and $p\left(\theta, (\rho-1)e^{i\theta}\right) = p$, and ψ is constant on some intervals $(-\infty, -A]$ and $[A, \infty)$.

Let p be as above. Let $0 < \varepsilon < 4\int_0^1 \sqrt{V(s^2)}\, ds - p$. By the definition of E^1_{min} and by Lemma 3.3.6 (i), there exists $\psi \in C^\infty(\mathbf{R})$ such that $\lfloor P \rfloor(\psi) = \lfloor p \rfloor$, $E^1(\psi) < E^1_{min} + \varepsilon < 4\int_0^1 \sqrt{V(s^2)}\, ds$ and ψ is constant on $(-\infty, -A]$ and on $[A, \infty)$ for some $A > 0$. By Corollary 3.4.5 we have $|\psi| > 0$ on $\mathbf{R}$, hence there exist $\theta, w \in C^\infty$ such that $\psi = \rho e^{i\theta}$, and $\operatorname{supp}(w), \operatorname{supp}(\theta') \subset [-A, A]$. We have $p(\theta, w) = p + 2k\pi$, where $k \in \mathbf{Z}$. If $k = 0$, the functions ψ, θ and ρ satisfy all requirements of Step 2.

Otherwise we construct $\psi_{t,1} = \rho_{t,1} e^{i\theta_{t,1}}$ and $\psi_{t,2} = \rho_{t,1} e^{i\theta_{t,1}}$ as in (3.5.3) - (3.5.5). Let $w_{t,j} = (\rho_{t,j} - 1)e^{i\theta_{t,j}}$ for $j = 1, 2$. By (3.5.7) we see that the mappings $m_j(t) := p(\theta_{t,j}, w_{t,j})$ are continuous, $m_1 = 0$ on $(-\infty, -A]$ and $m_1 = 2p + 4k\pi$ on $[A, \infty)$, $m_2 = 2p + 4k\pi$ on $(-\infty, -A]$ and $m_2 = 0$ on $[A, \infty)$.

If $k \geqslant 1$ we may choose $t_1 < t_2$ such that $m_1(t_1) = m_2(t_2) = p$. We have

$$E(\psi_{t_1,1}) + E(\psi_{t_2,2}) = 2\int_{(-\infty,t_1]\cup[t_2,\infty)} |\psi'|^2 + V(|\psi|^2)\, dx \leqslant 2E(\psi),$$

hence $E(\psi_{t_1,1}) \leqslant E(\psi)$ or $E(\psi_{t_2,2}) \leqslant E(\psi)$. In the former case we replace ψ by $\psi_{t_1,1}$ and in the latter case we replace ψ by $\psi_{t_2,2}$.

If $k \leqslant -1$ we have $2k\pi + p \leqslant -\pi \leqslant -p$ because $p \in (0, \pi]$. In this case by (3.5.7) there exist $t_1 \leqslant t_2$ such that $m_1(t_1) = -p$ and $m_2(t_2) = -p$. As above, we have $E(\psi_{t_1,1}) + E(\psi_{t_2,2}) \leqslant 2E(\psi)$ and we replace ψ by $\overline{\psi_{t_1,1}}$ if $E(\psi_{t_1,1}) \leqslant E(\psi)$, respectively by $\overline{\psi_{t_2,2}}$ if $E(\psi_{t_2,2}) \leqslant E(\psi)$.

Step 3. E^1_{min} is nondecreasing on $[0, \pi]$.

Let $0 \leqslant p_1 < p_2 \leqslant \pi$. If $E_{min}(p_2) = 4\displaystyle\int_0^1 \sqrt{V(s^2)}\, ds$ we have $E^1_{min}(p_1) \leqslant E^1_{min}(p_2)$ by (vii). Otherwise, consider any ε such that $0 < \varepsilon < 4\displaystyle\int_0^1 \sqrt{V(s^2)} - E^1_{min}(p_2)$, then choose $\psi = \rho e^{i\theta} = e^{i\theta} + w$ as in Step 2. Define $\psi_{t,j} = \rho_{t,j} e^{i\theta_{t,j}} = e^{i\theta_{t,j}} + w_{t,j}$ for $j = 1, 2$ as in (3.5.3) - (3.5.5). Using (3.5.7) we see that there exist $t_1 < t_2$ such that $p(\theta_{t_1,1}, w_{t_1,1}) = p(\theta_{t_2,2}, w_{t_2,2}) = p_1$. By (3.5.6) we have $E^1(\psi_{t_1,1}) + E^1(\psi_{t_2,2}) \leqslant 2E^1(\psi)$. Then we have

$$E^1_{min}(p_1) \leqslant \min(E^1(\psi_{t_1,1}),\, E^1(\psi_{t_2,2})) \leqslant E^1(\psi) \leqslant E^1_{min}(p_2) + \varepsilon.$$

Since ε was arbitrary we get $E^1_{min}(p_1) \leqslant E^1_{min}(p_2)$, as desired.

Step 4. E^1_{min} is concave.

Let $0 < p_1 < p_2 \leqslant \pi$ and let $p = \frac{p_1+p_2}{2}$. We will prove that

$$E^1_{min}\left(\frac{p_1 + p_2}{2}\right) \geqslant \frac{1}{2} E^1_{min}(p_1) + \frac{1}{2} E^1_{min}(p_2). \tag{3.5.9}$$

If $E^1_{min}\left(\frac{p_1+p_2}{2}\right) = 4\displaystyle\int_0^1 \sqrt{V(s^2)}\, ds$, (3.5.9) obviously holds. Otherwise, take any ε such that $0 < \varepsilon < 4\displaystyle\int_0^1 \sqrt{V(s^2)}\, ds - E^1_{min}(p)$, then choose $\psi = \rho e^{i\theta} = e^{i\theta} + w$ as in Step 2. There exists $t_0 \in \mathbf{R}$ such that $\displaystyle\int_{-\infty}^{t_0} (1 - \rho^2)\theta'\, dx = \frac{p_1}{2}$, and then we have necessarily $\displaystyle\int_{t_0}^{\infty} (1 - \rho^2)\theta'\, dx = \frac{p_2}{2}$. Let $\psi_{t_0,j} = \rho_{t_0,j} e^{i\theta_{t_0,j}} = e^{i\theta_{t_0,j}} + w_{t_0,j}$ $j = 1, 2$, be as in Step 1. It is clear that $p(\theta_{t_0,j}, w_{t_0,j}) = p_j$ for $j = 1, 2$ and using (3.5.6) we get

$$E^1_{min}(p_1) + E^1_{min}(p_2) \leqslant E^1(\psi_{t_0,1}) + E^1(\psi_{t_0,2}) = 2E(\psi) \leqslant 2E^1_{min}(p) + 2\varepsilon.$$

Since ε was arbitrary, we infer that (3.5.9) holds.

We have shown that E^1_{min} is continuous and satisfies (3.5.9) for any $0 < p_1 < p_2 \leqslant \pi$. It is then standard to prove that E^1_{min} is concave on $[0, \pi]$. We have $E^1_{min}(2\pi - p) = E^1_{min}(-p) = E^1_{min}(p)$ for any p and we infer that E^1_{min} is non-decreasing and concave on $[\pi, 2\pi]$, and then it follows that it is concave on $[0, 2\pi]$. □

Theorem 3.5.2. *Assume that conditions* **(A1)** *and* **(B1)** *in the introduction hold and $p \in (0, \pi]$ satisfies $E^1_{min}(p) < \sqrt{2}p$. Let $(\psi_n)_{n\geqslant 1} \subset \mathcal{E}(\mathbf{R})$ be a sequence satisfying*

$$\lfloor P \rfloor(\psi_n) \longrightarrow \lfloor p \rfloor \qquad \text{and} \qquad E(\psi_n) \longrightarrow E^1_{min}(p) \qquad \text{as } n \longrightarrow \infty. \tag{3.5.10}$$

Then there exist a subsequence $(\psi_{n_k})_{k\geqslant 1}$*, a sequence* $(x_k)_{k\geqslant 1}\subset\mathbf{R}$ *and* $\psi\in\mathcal{E}(\mathbf{R})$ *satisfying* $\lfloor P\rfloor(\psi)=\lfloor p\rfloor$*,* $E^1(\psi)=E_{min}(p)$*, and*

$$\begin{aligned}&\psi_{n_k}(\cdot+x_k)\longrightarrow\psi \quad \text{uniformly on } [-R,R] \text{ for any } R>0,\\ &\psi'_{n_k}(\cdot+x_k)\longrightarrow\psi' \quad \text{in } L^2(\mathbf{R}),\\ &|\psi_{n_k}(\cdot+x_k)|-|\psi|\longrightarrow 0 \quad \text{in } L^p(\mathbf{R}) \text{ for } 2\leqslant p<\infty.\end{aligned}$$

Proof. Let $p\in(0,\pi]$ such that $E^1_{min}(p)<\sqrt{2}p$ and let $(\psi_n)_{n\geqslant 1}\subset\mathcal{E}(\mathbf{R})$ be a sequence satisfying (3.6.36). We denote by $p_n\in[0,2\pi)$ the canonical valuation of the momentum of ψ_n. Then we have $p_n\longrightarrow p$ as $n\longrightarrow\infty$.

Denoting $M:=\sup_{n\geqslant 1}\|\psi'_n\|_{L^2(\mathbf{R})}<\infty$, by (3.2.1) we have

$$|\psi_n(x)-\psi_n(y)|\leqslant M|x-y|^{\frac{1}{2}} \qquad \text{for all } x,y\in\mathbf{R} \text{ and all } n\in\mathbf{N}^*. \tag{3.5.11}$$

We will use the celebrated concentration-compactness principle introduced in [9]. The sequence $f_n:=|\psi'_n|^2+V(|\psi_n|^2)$ is bounded in $L^1(\mathbf{R})$. We denote by Λ_n the concentration function of f_n, namely

$$\Lambda_n(t)=\sup_{x\in\mathbf{R}}\int_{x-t}^{x+t}|\psi'_n(y)|^2+V(|\psi_n(y)|^2)\,dy.$$

Obviously, Λ_n is a non-decreasing function on $[0,\infty)$, $\Lambda_n(0)=0$ and $\Lambda_n(t)\longrightarrow E^1(\psi_n)$ as $t\longrightarrow\infty$. Proceeding as in [9] we see that there exists a subsequence of $(\psi_n,\Lambda_n)_{n\geqslant 1}$, still denoted $(\psi_n,\Lambda_n)_{n\geqslant 1}$, and there is a non-decreasing function $\Lambda:[0,\infty)\longrightarrow\infty$ satisfying

$$\Lambda_n(t)\longrightarrow\Lambda(t) \text{ a.e on } [0,\infty) \text{ as } n\longrightarrow\infty. \tag{3.5.12}$$

Let $\alpha=\lim\limits_{t\to\infty}\Lambda(t)$. It is clear that $0\leqslant\alpha\leqslant\lim\limits_{n\to\infty}\left(\lim\limits_{t\to\infty}\Lambda_n(t)\right)=E^1_{min}(p)$. We will show that $\alpha=E^1_{min}(p)$.

We prove first that $\alpha>0$. We argue by contradiction and we assume that $\alpha=0$. This implies that $\Lambda(t)=0$ for any $t>0$, which means that for any fixed $t>0$ we have

$$\sup_{x\in\mathbf{R}}\int_{x-t}^{x+t}|\psi'_n(y)|^2+V(|\psi_n(y)|^2)\,dy\longrightarrow 0 \qquad \text{as } n\longrightarrow\infty. \tag{3.5.13}$$

We claim that if (3.5.13) occurs we have $|\psi_n|\longrightarrow 1$ uniformly on $\mathbf{R}$. By **(A1)** there is $\eta_0>0$ such that

$$\frac{1}{4}(1-s^2)^2<V(s^2)<(1-s^2)^2 \qquad \text{for all } s\in[1-\eta_0,1+\eta_0].$$

Fix $\eta\in(0,\frac{\eta_0}{2}]$. Assume that there exists $x_n\in\mathbf{R}$ such that $\big|\,|\psi_n(x_n)|-1\big|=\eta$. From (3.5.11) it follows that there exists $r>0$, independent of n, such that $\big|\,|\psi_n(x_n)|-1\big|\in[\frac{\eta}{2},\frac{3\eta}{2}]$ for $y\in[x_n-r,x_n+r]$ and therefore

$$\Lambda_n(r)\geqslant\int_{x_n-r}^{x_n+r}V(|\psi_n(y)|^2)\,dy\geqslant\int_{x_n-r}^{x_n+r}\frac{1}{4}\big|\,|\psi_n(x_n)|-1\big|^2\,dy\geqslant\frac{1}{8}\eta^2 r.$$

By (3.5.13) there exists $n_\eta \in \mathbf{N}$ such that for all $n \geqslant n_\eta$ we have $\Lambda_n(r) < \frac{1}{8}\eta^2 r$, and the above inequalities imply that for any $n \geqslant n_\eta$ we must have $1-\eta < |\psi_n| < 1+\eta$. Thus $|\psi_n| \longrightarrow 1$ uniformly on $\mathbf{R}$.

Choose $\delta \in (0,1)$ such that $E^1_{min}(p) < (1-\delta)^2\sqrt{2}p$ (this is possible because $E^1_{min}(p) < \sqrt{2}p$). By **(A1)** there is $\eta_\delta > 0$ such that $V(s^2) \geqslant \frac{1}{2}(1-\delta)^2(1-s^2)^2$ for any $s \in [1-\eta_\delta, 1+\eta_\delta]$. For all sufficiently large n we have $\big|\,|\psi_n| - 1\big| < \min(\delta, \eta_\delta)$ on $\mathbf{R}$. For any such n we may write $\psi_n = \rho_n e^{i\theta_n}$ where $1-\rho_n \in H^1(\mathbf{R})$ and $\theta_n \in \dot{H}^1(\mathbf{R},\mathbf{R})$, and we have

$$
\begin{aligned}
E^1(\psi_n) &\geqslant \int_{\mathbf{R}} \rho_n^2|\theta_n'|^2 + V(\rho_n^2)\,dx \geqslant (1-\delta)^2 \int_{\mathbf{R}} |\theta_n'|^2 + \frac{1}{2}(1-\rho_n^2)^2\,dx \\
&\geqslant (1-\delta)^2\sqrt{2}\left|\int_{\mathbf{R}} (1-\rho_n^2)\theta_n'\,dx\right|.
\end{aligned}
\tag{3.5.14}
$$

Recall that by Remark 3.3.2, $\displaystyle\int_{\mathbf{R}} (1-\rho_n^2)\theta_n'\,dx$ is a valuation of the momentum of ψ_n, hence there exists $\ell_n \in \mathbf{Z}$ such that $\displaystyle\int_{\mathbf{R}} (1-\rho_n^2)\theta_n'\,dx = p_n + 2\ell_n\pi$ and consequently

$$
\left|\int_{\mathbf{R}} (1-\rho_n^2)\theta_n'\,dx\right| \geqslant \operatorname{dist}(p_n, 2\pi\mathbf{Z}).
$$

Letting $n \longrightarrow \infty$ in (3.5.14) we get

$$
E^1_{min}(p) \geqslant (1-\delta)^2\sqrt{2}\lim_{n\to\infty} \operatorname{dist}(p_n, 2\pi\mathbf{Z}) = (1-\delta)^2\sqrt{2}p,
$$

contradicting the choice of δ. We have thus proved that $\alpha \neq 0$.

Assume that $0 < \alpha < E^1_{min}(p)$. Arguing as in the proof of Theorem 5.3 in [11] (see (5.12) p. 156 there) we infer that there is a nondecreasing sequence $R_n \longrightarrow \infty$ such that

$$
\lim_{n\to\infty} \Lambda_n(2R_n) = \lim_{n\to\infty} \Lambda_n(R_n) = \alpha.
$$

For each n choose $x_n \in \mathbf{R}$ such that

$$
\int_{x_n-R_n}^{x_n+R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy > \Lambda_n(R_n) - \frac{1}{n}.
$$

Then we have

$$
\Lambda_n(R_n) - \frac{1}{n} < \int_{x_n-2R_n}^{x_n+2R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \leqslant \Lambda_n(2R_n),
$$

and we infer that

$$
\int_{x_n-R_n}^{x_n+R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \longrightarrow \alpha, \tag{3.5.15}
$$

(3.5.16) $$\int_{x_n-2R_n}^{x_n-R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy + \int_{x_n+R_n}^{x_n+2R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \longrightarrow 0, \qquad \text{and}$$

(3.5.17) $$\int_{-\infty}^{x_n-2R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy + \int_{x_n+2R_n}^{\infty} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \longrightarrow E^1_{min}(p) - \alpha.$$

Assume that for infinitely many n's we have

$$\int_{-\infty}^{x_n-2R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \geqslant \int_{x_n+2R_n}^{\infty} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy.$$

(A similar argument will work if the opposite inequality holds true for infinitely many n's.) Passing to a subsequence, still denoted the same, we may assume that

(3.5.18) $$\int_{-\infty}^{x_n-2R_n} |\psi_n'(y)|^2 + V(|\psi_n(y)|^2)\,dy \longrightarrow \beta > 0 \quad \text{as } n \longrightarrow \infty, \quad \text{where } 0 < \beta < E^1_{min}(p).$$

Let $z_n = x_n - \frac{3}{2}R_n$. For n sufficiently large we have $[z_n - 2, z_n + 2] \subset [x_n - 2R_n, x_n - R_n]$. Then using (3.5.11) and (3.5.16) and arguing as in the proof of the fact that $\alpha > 0$ we infer that $\sup_{y\in[z_n-1,z_n+1]} |\,|\psi_n(y)| - 1| \longrightarrow 0$ as $n \longrightarrow \infty$. For n large enough we have $\frac{1}{2} < |\psi_n| < \frac{3}{2}$ on $[z_n-1, z_n+1]$, thus we have a lifting $\psi_n = \rho_n e^{i\theta_n}$ on that interval. Let $r_n = \rho_n(z_n)$ and $\alpha_n = \theta_n(z_n)$. It is clear that $r_n \longrightarrow 1$. Define

$$\psi_{n,1}(x) = \begin{cases} \psi_n(x) \text{ if } x \leqslant z_n, \\ ((1-r_n)(x-z_n) + r_n)e^{i\alpha_n} \text{ if } x \in (z_n, z_n+1), \\ e^{i\alpha_n} \text{ if } x \geqslant z_n + 1, \end{cases}$$

$$\psi_{n,2}(x) = \begin{cases} e^{i\alpha_n} \text{ if } x \leqslant z_n - 1, \\ ((r_n-1)(x-z_n) + r_n)e^{i\alpha_n} \text{ if } x \in (z_n-1, z_n), \\ \psi_n(x) \text{ if } x \geqslant z_n. \end{cases}$$

It is easy to see that $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}(\mathbf{R})$. It follows form (3.5.16) and (3.5.18) that $E^1(\psi_{n,1}) \longrightarrow \beta$ and $E^1(\psi_{n,2}) \longrightarrow E^1_{min}(p) - \beta$ as $n \longrightarrow \infty$. Since $0 < \beta < E^1_{min}(p) \leqslant 4\int_0^1 \sqrt{V(s^2)}\,ds$, for all n sufficiently large we have $E^1(\psi_{n,j}) < 4\int_0^1 \sqrt{V(s^2)}\,ds$, $j = 1, 2$, and then using Corollary 3.4.5 we infer that $\psi_{n,1}$ and $\psi_{n,2}$ do not vanish and consequently these functions admit liftings, thus we may write $\psi_{n,j} = \rho_{n,j}e^{i\theta_{n,j}}$ on $\mathbf{R}$. Replacing $\theta_{n,j}$ by $\theta_{n,j} + 2k\pi$ for some $k \in \mathbf{Z}$, we may assume that $\theta_{n,1} = \alpha_n$ on $[z_n, \infty)$ and that $\theta_{n,2} = \alpha_n$ on $(-\infty, z_n]$. Let

$$\rho_n(x) = \begin{cases} \rho_{n,1}(x) & \text{if } x \leqslant z_n, \\ \rho_{n,2}(x) & \text{if } x > z_n, \end{cases} \qquad \theta_n(x) = \begin{cases} \theta_{n,1}(x) & \text{if } x \leqslant z_n, \\ \theta_{n,2}(x) & \text{if } x > z_n. \end{cases}$$

Then $1-\rho_n \in H^1(\mathbf{R})$, $\theta \in \dot{H}^1(\mathbf{R})$ and $\psi_n = \rho_n e^{i\theta_n}$ on $\mathbf{R}$. By Remark 3.3.2 and the fact that $\theta'_{n,1} = 0$ on (z_n, ∞), $\theta'_{n,2} = 0$ on $(-\infty, z_n)$, valuations of momenta of ψ_n, $\psi_{n,1}$ and $\psi_{n,2}$ are, respectively,

$$\tilde{p}_n = \int_{\mathbf{R}} (1-\rho_n^2)\theta'_n \, dx, \quad \tilde{p}_{n,1} = \int_{-\infty}^{z_n} (1-\rho_n^2)\theta'_n \, dx, \quad \tilde{p}_{n,2} = \int_{z_n}^{\infty} (1-\rho_n^2)\theta'_n \, dx. \tag{3.5.19}$$

Let $p_n,\ p_{n,1},\ p_{n,2} \in [0, 2\pi)$ be the canonical valuations of momenta of these functions, respectively. From (3.5.19) it is obvious that $\tilde{p}_n = \tilde{p}_{n,1} + \tilde{p}_{n,2}$, and this implies $p_n = p_{n,1} + p_{n,2} \pmod{2\pi}$. Since $p_{n,1} + p_{n,2} \in [0, 4\pi)$, we have

$$p_{n,1} + p_{n,2} = p_n \qquad \text{or} \qquad p_{n,1} + p_{n,2} = p_n + 2\pi. \tag{3.5.20}$$

Choose $p_0 \in (0, p)$ such that

$$\max(\beta, E^1_{min}(p) - \beta) < E^1_{min}(p_0) < E^1_{min}(p).$$

For all n sufficiently large and for $j = 1, 2$ we have

$$E^1_{min}(p_{n,j}) = E^1_{min}(\tilde{p}_{n,j}) \leqslant E^1(\psi_{n,j}) < E^1_{min}(p_0).$$

Since E^1_{min} is non-decreasing on $[0, \pi]$ and non-increasing on $[\pi, 2\pi]$, we infer that for all large n we have either $p_{n,j} \in [0, p_0)$, or $p_{n,j} \in (2\pi - p_0, 2\pi]$. We cannot have $p_{n,1} \in [0, p_0)$ and $p_{n,2} \in (2\pi - p_0, 2\pi]$ because this would give

$$p < 2\pi - p_0 < p_{n,2} \leqslant p_{n,1} + p_{n,2} < p_0 + 2\pi < p + 2\pi,$$

contradicting (3.5.20). We deduce that one of the following situations occurs :

Case A : $p_{n,1}, p_{n,2} \in [0, p_0)$, or

Case B : $p_{n,1}, p_{n,2} \in (2\pi - p_0, 2\pi]$.

Assume that we are in case A for infinitely many n's. Passing to a further subsequence we may assume that $p_{n,j} \longrightarrow p^*_j \in [0, p_0]$ for $j = 1, 2$, and $p^*_1 + p^*_2 = p$, hence $p^*_j \geqslant p - p_0$. We have

$$E^1_{min}(p_{n,1}) + E^1_{min}(p_{n,2}) \leqslant E^1(\psi_{n,1}) + E^1(\psi_{n,2}) = E^1(\psi_n) + o(1)$$

and letting $n \longrightarrow \infty$ we discover $E^1_{min}(p^*_1) + E^1_{min}(p^*_2) \leqslant E^1_{min}(p)$. The concavity of E^1_{min} on $[0, 2\pi]$ implies that $E^1_{min}(p^*_j) \geqslant \frac{p^*_j}{p} E^1_{min}(p)$, and equality may occur if and only if E^1_{min} is linear on $[0, p]$. Summing up these inequalities for $j = 1, 2$ and comparing to the previous inequality we infer that we must have equality, and consequently E^1_{min} must be linear on $[0, p]$. Taking into account the behaviour of E^1_{min} at the origin (see Lemma 3.5.1 (ii) and (iii)) we infer that $E^1_{min}(s) = \sqrt{2}s$ for all $s \in [0, p]$, contradicting the fact that $E^1_{min}(p) < \sqrt{2}p$.

Assume that we are in case B for infinitely many n's. From (3.5.20) we see that necessarily $p_{n,1} + p_{n,2} = p_n + 2\pi$, hence $(2\pi - p_{n,1}) + (2\pi - p_{n,2}) = 2\pi - p_n$. Passing again to a subsequence we

may assume that $(2\pi - p_{n,j}) \longrightarrow p_j^\sharp \in [0, p_0]$ for $j = 1, 2$. Then we have $p_1^\sharp + p_2^\sharp = 2\pi - p \geqslant p$, thus necessarily $p_j^\sharp \geqslant p - p_0$. We have

$$E^1_{min}(2\pi - p_{n,j}) = E^1_{min}(p_{n,j}) \leqslant E^1(\psi_{n,j}).$$

Summing up for $j = 1, 2$ and letting $n \longrightarrow \infty$ we get

$$E^1_{min}(p_1^\sharp) + E^1_{min}(p_2^\sharp) \leqslant E^1_{min}(p) = E^1_{min}(2\pi - p).$$

The last inequality and the concavity of E^1_{min} on $[0, 2\pi]$ imply that E^1_{min} must be linear on $[0, 2\pi - p]$. Since $2\pi - p \geqslant \pi$, E^1_{min} is nondecreasing on $[0, \pi]$ and nonincreasing on $[\pi, 2\pi]$ and is not constant, we must have $2\pi - p = \pi$, hence $p = \pi$. Then by Lemma 3.5.1 (ii) and (iii) we get $E^1_{min}(s) = \sqrt{2}s$ for all $s \in [0, p]$, hence $E^1_{min}(p) = \sqrt{2}p$, a contradiction.

We conclude that we cannot have $0 < \alpha < E^1_{min}(p)$, thus necessarily $\alpha = E^1_{min}(p)$.

It is then standard to prove that there exists a sequence $(x_n)_{n\geqslant 1} \subset \mathbf{R}^N$ such that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ satisfying

$$\int_{-\infty}^{x_n - R_\varepsilon} |\psi_n'|^2 + V(|\psi_n|^2)\, dx + \int_{x_n + R_\varepsilon}^{\infty} |\psi_n'|^2 + V(|\psi_n|^2)\, dx < \varepsilon$$

for all sufficiently large n. Let $\tilde{\psi}_n = \psi_n(\cdot + x_n)$. Then for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that

$$\int_{-\infty}^{-R_\varepsilon} |\tilde{\psi}_n'|^2 + V(|\tilde{\psi}_n|^2)\, dx + \int_{R_\varepsilon}^{\infty} |\tilde{\psi}_n'|^2 + V(|\tilde{\psi}_n|^2)\, dx < \varepsilon \qquad \text{for all } n \geqslant n_\varepsilon. \tag{3.5.21}$$

Take $\eta_0 > 0$ such that $\frac{1}{4}(1 - s^2)^2 < V(s^2) < (1 - s^2)^2$ for any $s \in [1 - \eta_0, 1 + \eta_0]$. Let H be as in assumption **(B1)**. Let $\varepsilon_1 = 2\min(|H(\frac{1}{2})|, H(\frac{3}{2}), |H(1 - \eta_0)|, H(1 + \eta_0))$. Take $R_1 = R_{\varepsilon_1} > 0$ such that (3.5.21) holds with ε_1 instead of ε. Using (3.2.2) we see that for all n sufficiently large and for all $x \in (-\infty, -R_1] \cup [R_1, \infty)$ we have

$$|\tilde{\psi}_n(x)| \in \left[\frac{1}{2}, \frac{3}{2}\right] \cap [1 - \eta_0, 1 + \eta_0] \ \text{ and } \ \frac{1}{4}(1 - |\tilde{\psi}_n(x)|^2)^2 < V(|\tilde{\psi}_n(x)|^2) < (1 - |\tilde{\psi}_n(x)|^2)^2. \tag{3.5.22}$$

Then using (3.5.11) we see that $\tilde{\psi}_n$ are uniformly bounded on $[-R_1, R_1]$, hence on $\mathbf{R}$. Since $(\tilde{\psi}_n')_{n\geqslant 1}$ is bounded in $L^2(\mathbf{R})$, it is standard to prove that there exists a function $\psi \in H^1_{loc}(\mathbf{R})$ such that $\psi' \in L^2(\mathbf{R})$ and there is a subsequence $(\tilde{\psi}_{n_k})_{k\geqslant 1}$ of $(\tilde{\psi}_n)_{n\geqslant 1}$ satisfying

$$\begin{aligned} &\tilde{\psi}_{n_k}' \rightharpoonup \psi' && \text{weakly in } L^2(\mathbf{R}),\\ &\tilde{\psi}_{n_k} \longrightarrow \psi && \text{strongly in } L^p([-R, R]) \text{ for any } R > 0 \text{ and any } 1 \leqslant p \leqslant \infty,\\ &\tilde{\psi}_{n_k} \longrightarrow \psi && \text{uniformly on } [-R, R] \text{ for any } R > 0. \end{aligned} \tag{3.5.23}$$

We may use (3.5.11) and the Arzelà-Ascoli Theorem to get uniform convergence on compact intervals. The weak convergence $\tilde{\psi}_{n_k}' \rightharpoonup \psi'$ in $L^2(\mathbf{R})$ implies that for any interval $I \subset \mathbf{R}$ we have

$$\int_I |\psi'|^2\, dx \leqslant \liminf_{n\to\infty} \int_I |\tilde{\psi}_{n_k}'|^2\, dx. \tag{3.5.24}$$

Fatou's Lemma gives

$$\int_I V(|\psi|^2)\,dx \leqslant \liminf_{n\to\infty} \int_I V\left(|\tilde\psi_{n_k}|^2\right)\,dx. \tag{3.5.25}$$

Fix $\varepsilon > 0$. Take R_ε as in (3.5.21). Since $(\tilde\psi_{n_k})_{k\geqslant 1}$ is bounded in $L^\infty([-R_\varepsilon, R_\varepsilon]$, converges to ψ almost everywhere and V is continuous, the dominated convergence theorem gives $V(|\tilde\psi_{n_k}|^2) \longrightarrow V(|\psi|^2)$ in $L^1([-R_\varepsilon, R_\varepsilon])$, hence we may choose $n'_\varepsilon \geqslant n_\varepsilon$ such that

$$\int_{-R_\varepsilon}^{R_\varepsilon} |V(|\tilde\psi_{n_k}|^2) - V(|\psi|^2)|\,dx < \varepsilon \qquad \text{for all } n \geqslant n'_\varepsilon.$$

Then using (3.5.21) we have

$$\begin{aligned}
&\int_{\mathbf{R}} |V(|\tilde\psi_{n_k}|^2) - V(|\psi|^2)|\,dx \\
&\leqslant \int_{-R_\varepsilon}^{R_\varepsilon} |V(|\tilde\psi_{n_k}|^2) - V(|\psi|^2)|\,dx + \int_{(-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty)} V(|\tilde\psi_{n_k}|^2) + V(|\psi|^2)\,dx \leqslant 3\varepsilon.
\end{aligned}$$

Since ε was arbitrary we get $V(|\tilde\psi_{n_k}|^2) \longrightarrow V(|\psi|^2)$ in $L^1(\mathbf{R})$, and in particular

$$\int_{\mathbf{R}} V(|\tilde\psi_{n_k}|^2)\,dx \longrightarrow \int_{\mathbf{R}} V(|\psi|^2)\,dx \qquad \text{as } k \longrightarrow \infty. \tag{3.5.26}$$

Similarly we show that $\left(1 - |\tilde\psi_{n_k}|^2\right) \longrightarrow (1 - |\psi|^2)$ strongly in $L^p(\mathbf{R})$ for any $p \in [2, \infty)$.

On $(-\infty, -R_1]$ and on $[R_1, \infty)$ we have liftings, that is we may write $\tilde\psi_{n_k} = \rho_k e^{i\theta_k}$ and $\psi = \rho e^{i\theta}$. Given $\varepsilon > 0$, take $R_\varepsilon > R_1$ such that (3.5.21) holds. Since $\frac{\tilde\psi_{n_k}}{\psi}(\pm R_\varepsilon) \longrightarrow 1$ as $k \longrightarrow \infty$, we may replace if necessary θ_k by $\theta_k + 2\ell_k\pi$ for some $\ell_k \in \mathbf{Z}$ on $(-\infty, -R_1]$ or on $[R_1, \infty)$ in such a way that $\theta_k(-R_\varepsilon) = \theta(-R_\varepsilon) + \alpha_k^-$ and $\theta_k(R_\varepsilon) = \theta(R_\varepsilon) + \alpha_k^+$, where $\alpha_k^\pm \longrightarrow 0$ as $k \longrightarrow \infty$. Then we extend θ and θ_k as affine functions on $[-R_\varepsilon, R_\varepsilon]$. It is easily seen that $e^{i\theta_k} \longrightarrow e^{i\theta}$ uniformly on $[-R_\varepsilon, R_\varepsilon]$ and in $H^1([-R_\varepsilon, R_\varepsilon])$, and that $\theta_k, \theta \in \dot H^1(\mathbf{R})$. Denote $w_k = \tilde\psi_{n_k} - e^{i\theta_k}$ and $w = \psi - e^{i\theta}$. From (3.5.23) it follows that $w_k, w \in H^1(\mathbf{R})$, $w'_k \rightharpoonup w'$ weakly in $L^2([-R_\varepsilon, R_\varepsilon])$, and $w_k \longrightarrow w$ strongly in $L^2([-R_\varepsilon, R_\varepsilon])$. Then we infer that

$$\int_{-R_\varepsilon}^{R_\varepsilon} -2\langle \theta'_k e^{i\theta_k}, w_k\rangle + \langle iw'_k, w_k\rangle\,dx \longrightarrow \int_{-R_\varepsilon}^{R_\varepsilon} -2\langle \theta' e^{i\theta}, w\rangle + \langle iw', w\rangle\,dx \quad \text{as } n \longrightarrow \infty. \tag{3.5.27}$$

On the other hand, on $(-\infty, -R_1] \cup [R_1, \infty)$ we have $-2\langle \theta'_k e^{i\theta_k}, w_k\rangle + \langle iw'_k, w_k\rangle = (1 - \rho_k^2)\theta'_k$. Using (3.5.22), proceeding as in (3.5.14), then using (3.5.21) we get

$$\left|\int_{-\infty}^{-R_\varepsilon} (1-\rho_k^2)\theta'_k\,dx\right| \leqslant 2\int_{-\infty}^{-R_\varepsilon} \frac{1}{4}|\theta'_k|^2 + \frac{1}{4}(1-\rho_k^2)\,dx \leqslant 2\int_{-\infty}^{-R_\varepsilon} \rho_k^2|\theta'_k|^2 + V(\rho_k^2)\,dx \leqslant 2\varepsilon. \tag{3.5.28}$$

A similar estimate holds true on $[R_\varepsilon, \infty)$, as well as for the function ψ. Thus we get

$$\begin{aligned} &\Big| \int_{\mathbf{R}} -2\langle \theta_k' e^{i\theta_k}, w_k\rangle + \langle iw_k', w_k\rangle \, dx - \int_{\mathbf{R}} -2\langle \theta' e^{i\theta}, w\rangle + \langle iw', w\rangle \, dx \Big| \\ &\leqslant 8\varepsilon + \Big| \int_{-R_\varepsilon}^{R_\varepsilon} -2\langle \theta_k' e^{i\theta_k}, w_k\rangle + \langle iw_k', w_k\rangle \, dx - \int_{\mathbf{R}} -2\langle \theta' e^{i\theta}, w\rangle + \langle iw', w\rangle \, dx \Big| \end{aligned} \tag{3.5.29}$$

for all k sufficiently large. Then using (3.5.27) we see that the right-hand side of (3.5.29) is smaller than 9ε if k is large enough. Since ε was arbitrary, we have proved that

$$p(\theta_k, w_k) \longrightarrow p(\theta, w) \qquad \text{as } k \longrightarrow \infty. \tag{3.5.30}$$

We infer that $p(\theta, w) = p + 2\ell\pi$ for some $\ell \in \mathbf{Z}$, and consequently we have

$$E^1(\psi) \geqslant E^1_{min}(p) = \lim_{k\to\infty} E^1(\tilde{\psi}_{n_k}).$$

On the other hand, we have $\displaystyle\int_{\mathbf{R}} |\psi'|^2 \, dx \leqslant \liminf_{k\to\infty} \int_{\mathbf{R}} |\tilde{\psi}_{n_k}'|^2 \, dx$ because $\tilde{\psi}_{n_k}' \rightharpoonup \psi'$ in $L^2(\mathbf{R})$. Taking into account (3.5.26), we infer that necessarily $\|\tilde{\psi}_{n_k}'\|^2_{L^2(\mathbf{R})} \longrightarrow \|\psi'\|^2_{L^2}$ as $k \longrightarrow \infty$. The weak convergence and the convergence of norms imply that $\tilde{\psi}_{n_k}' \longrightarrow \psi'$ strongly in $L^2(\mathbf{R})$. □

Proposition 3.5.3. *Let $p \in (0, \pi]$ and let $\psi \in \mathcal{E}(\mathbf{R})$ be a solution of the minimization problem considered above, that is $\lfloor P \rfloor(\psi) = \lfloor p \rfloor$ and $E^1(\psi) = E^1_{min}(p)$. Then there exists $c \in [(E^1_{min})'_r(p), (E^1_{min})'_\ell(p)]$, where $(E^1_{min})'_\ell(p)$ and $(E^1_{min})'_r(p)$ are the left and right derivatives of E^1_{min} at p, respectively, such that*

$$ic\psi' + \psi'' + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}.$$

In other words, ψ is a one-dimensional traveling wave of speed c for (3.1.1), and $\psi \in C^2(\mathbf{R})$.

Moreover, for any $p \in (0, \pi]$ such that $E^1_{min}(p) < \sqrt{2}p$ and $(E^1_{min})'_\ell(p) > (E^1_{min})'_r(p)$, there exist two minimizers $\psi_1, \psi_2 \in \mathcal{E}(\mathbf{R})$ for $E^1_{min}(p)$ that solve (3.4.1) with speeds $c_1 = (E^1_{min})'_\ell(p)$ and $c_2 = (E^1_{min})'_r(p)$, respectively.

The proof of Proposition 3.5.3 is standard and is similar to the proof of Proposition 4.14 in [6], so we omit it.

Example 3.5.4. Consider $V \in C^\infty([0, \infty)$ such that $V(s) = \frac{1}{2}(1-s)^2$ on $[1-\delta, \infty)$ for some $\delta > 0$, V is decreasing on $[0, 1)$ and $4\int_0^1 \sqrt{V(s^2)}\, ds > \sqrt{2}\pi$. Then all solutions of (3.4.1) are given by Proposition 3.4.1 (i). Let $\zeta(c)$ be as in Proposition 3.4.1. If ψ_c is a solution of (3.4.1), the infimum of $|\psi|^2$ is $\zeta(c)$. We have $\zeta(c) \longrightarrow 0$ as $c \longrightarrow 0$, and using (3.2.2) we see that

$$E^1(\psi_c) \geqslant 4|H(\sqrt{\zeta(c)})| \longrightarrow 4\int_0^1 \sqrt{V(s^2)}\, ds > \sqrt{2}\pi \qquad \text{as } c \longrightarrow 0.$$

By Lemma 3.5.1 (vi) we have $E^1_{min}(p) < \sqrt{2}p$ for any $p > 0$, thus $E^1_{min}(p) \leqslant E^1_{min}(\pi) < \sqrt{2}\pi$ for any $p \in (0, \pi]$.

There is $c_0 > 0$ such that for $c \in [0, c_0)$ we have $E^1(\psi_c) > E^1_{min}(\pi) \geqslant E^1_{min}(p)$ for any $p \in (0, \pi]$. Thus ψ_c cannot be a minimizer for $E^1_{min}(p)$ if $0 \leqslant c < c_0$.

On the other hand,Theorem 3.5.2 implies that there exist minimizers for $E^1_{min}(p)$ for all $p \in (0, \pi]$. By Proposition 3.5.3, the minimizers are necessarily solutions of (3.4.1), but they must have speeds $c \geqslant c_0$. We infer that $\sup_{p \leqslant \pi} \left(E^1_{min}\right)'_\ell(p) \geqslant c_0$, and E^1_{min} has a cusp at $p = \pi$.

3.6 Minimizing the energy at fixed momentum in $\mathcal{E}$

Throughout this section we suppose that the assumptions **(A1), (A2), (B2)** in the introduction hold. We define

$$E_{\lambda,min}(q) = \inf\{E_\lambda(\psi) \mid \psi \in \mathcal{E} \text{ and } \lfloor Q \rfloor(\psi) = \lfloor q \rfloor\}.$$

The next lemma collects the main properties of the function E_{min}.

Lemma 3.6.1. *Assume that V satisfies the assumptions* **(A1)**, **(A2)** *and* **(B2)** *in the introduction. The function $E_{\lambda,min}$ has the following properties :*

i) $E_{\lambda,min}$ is non-negative, $2\pi-$periodic, $E_{\lambda,min}(-p) = E_{\lambda,min}(p)$ for all $p \in \mathbf{R}$ and

$$E_{\lambda,min}(p) = \inf\{E_\lambda(e^{i\varphi} + w) \mid \varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R}), w \in H^1_{per}(\mathbf{R}, \mathbf{C}) \text{ and } q(\varphi, w) = p\}.$$

ii) For any $p \in \mathbf{R}$ and any $\lambda > 0$ we have $E_{\lambda,min}(p) \leqslant E^1_{min}(p)$. Consequently we have $E_{\lambda,min}(p) \leqslant \sqrt{2}|p|$ for all p.

iii) $E_{\lambda,min}$ is sub-additive : for any $p_1, p_2 \in \mathbf{R}$ there holds

$$E_{\lambda,min}(p_1 + p_2) \leqslant E_{\lambda,min}(p_1) + E_{\lambda,min}(p_2).$$

iv) $E_{\lambda,min}$ is $\sqrt{2}-$Lipschitz on $\mathbf{R}$.

v) For any $\delta > 0$ there exists $p_\delta > 0$ such that $E_{\lambda,min}(p) \geqslant (1 - \delta)\sqrt{2}p$ for any $p \in (0, p_\delta)$.

vi) For any fixed p, the mapping $\lambda \longmapsto E_{\lambda,min}(p)$ is non-decreasing. Assume that $p \in (0, 2\pi)$ and there exist $p_1, p_2 \in (0, 2\pi)$ satisfying $p = \frac{p_1+p_2}{2}$ and $E^1_{min}(p) > \frac{1}{2}\left(E^1_{min}(p_1) + E^1_{min}(p_2)\right)$. Then there exists $\lambda_*(p) > 0$ such that $E_{\lambda,min}(p) < E^1_{min}(p)$ for any $\lambda \in (0, \lambda_*(p))$.

vii) Assume that $p_0 \in (0, 2\pi)$ satisfies

$$\liminf_{h \to 0} \frac{E^1_{min}(p_0 + h) + E^1_{min}(p_0 - h) - 2E^1_{min}(p_0)}{h^2} > -\infty.$$

Then there exists $\lambda^*(p_0) > 0$ such that $E_{\lambda,min}(p_0) = E^1_{min}(p_0)$ for any $\lambda \geqslant \lambda^*(p_0)$.

viii) E^1_{min} is concave on $[0, 2\pi]$.

Remark 3.6.2. i) If the assumption in Lemma 3.6.1 (vi) does not hold for some $p \in (0, \pi]$, the concavity of E^1_{min} implies that $E^1_{min}(p) = \frac{1}{2}\left(E^1_{min}(p-\delta) + E^1_{min}(p+\delta)\right)$ for any $\delta \in (0, p)$, and then we infer that E^1_{min} must be affine on $[0, 2p]$. This is impossible if $p \in (\frac{\pi}{2}, \pi]$ because $E^1_{min}(0) = 0$ and E^1_{min} achieves its positive maximum at π. If $V > 0$ on $[0, 1)$ and $V(s) \leqslant \frac{1}{2}(1-s)^2 + \frac{3}{8}(1-s)^3$ on some interval $(1-\eta, 1]$, Lemma 3.5.1 (ii) and (vi) implies that E^1_{min} cannot be linear on $[0, p]$ for any $p \in (0, \pi]$. Remember that $E_{\lambda,\min}(p) = E_{\lambda,\min}(2\pi - p)$. Therefore the conclusion of Lemma 3.6.1 (vi) holds without any additional assumption if $p \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and it holds for any $p \in (0, 2\pi)$ under the assumption of Lemma 3.5.1 (vi).

ii) It is well-known that a concave function is twice differentiable almost everywhere. The limit in Lemma 3.6.1 (vii) exists and is equal to $(E^1_{min})''(p)$ for almost every $p \in (0, 2\pi)$, and for any such p the conclusion of Lemma 3.6.1 (vii) holds true. Then parts (vi) and (vii) of the above lemma show that there is some critical value $\lambda^*(p)$ such that $E_{\lambda,\min}(p) < E^1_{min}(p)$ for $\lambda < \lambda^*(p)$, and $E_{\lambda,\min}(p) = E^1_{min}(p)$ for $\lambda \geqslant \lambda^*(p)$. The proof shows that one can give upper bounds for the critical value $\lambda^*(p)$ if a lower bound for $(E^1_{min})''(p)$ is known (see (3.6.3) below).

Using an argument in the proof of part (viii) one can show that $\lambda \longmapsto E_{\lambda,min}(p)$ is strictly increasing on $(0, \lambda^*(p)]$.

The results in Lemma 3.6.1 (vi) and (vii) are not surprising. Recall that E_λ is a rescaled energy that comes from minimizing the energy E on $\mathbf{R} \times \mathbf{T}_{\frac{1}{\lambda}}$ where $\mathbf{T}_\tau$ is the $1-$dimensional torus of length τ. When λ is large the torus $\mathbf{T}_{\frac{1}{\lambda}}$ is too narrow and variations with respect to the variable y would be energetically too costly. On the contrary, on large tori one can find better competitors than the $1-$dimensional minimizers of E^1_{min}.

In the case of the Gross-Pitaevskii nonlinearity $F(s) = 1 - s$, the function E^1_{min} is known explicitly (see [1] or Example 3.4.2) and it turns out that it is C^2 on $(0, 2\pi)$, hence in this particular case the conclusion of Lemma 3.6.1 (vii) holds for any value of p.

Proof. The proof of (i) is similar to the proof of Lemma 3.5.1 (i).

(ii) Consider any $\psi \in \mathcal{E}(\mathbf{R})$ satisfying $\lfloor P \rfloor(\psi) = p$. Let $\psi^\sharp(x, y) = \psi(x)$. It is obvious that $\psi^\sharp \in \mathcal{E}$ and $E_\lambda(\psi^\sharp) = E^1(\psi)$ for any $\lambda > 0$. If $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in H^1(\mathbf{R})$ are such that $\psi = e^{i\varphi} + w$ in $\mathbf{R}$, we have $\psi^\sharp = e^{i\varphi(x)} + w^\sharp$, where $w^\sharp(x, y) = w(x)$. It is obvious that $q(\varphi, w^\sharp) = p(\varphi, w)$, thus $\lfloor Q \rfloor(\psi^\sharp) = \lfloor P \rfloor(\psi) = \lfloor p \rfloor$. We infer that $E_{\lambda,min}(p) \leqslant E_\lambda(\psi^\sharp) = E^1(\psi)$. Since the last inequality holds for all $\psi \in \mathcal{E}(\mathbf{R})$ such that $\lfloor P \rfloor(\psi) = p$, the conclusion follows.

The proofs of (iii) and (iv) are very similar to the proofs of Lemma 3.5.1 (iv) and (v), respectively, and we omit them. For the proof of part (v) we need some results from [6] and [11]. These results are stated in Lemma 3.6.4 below. We postpone the proof of part (v) after the proof of Lemma 3.6.5.

vi) If $0 < \lambda_1 < \lambda_2$ it is obvious that $E_{\lambda_1}(\psi) \leqslant E_{\lambda_2}(\psi)$ for any $\psi \in \mathcal{E}$, and this trivially implies that $E_{\lambda_1,min}(p) \leqslant E_{\lambda_2,min}(p)$ for all p.

It suffices to consider the case $p \in (0, \pi]$. Fix $p_1, p_2 \in (0, 2\pi)$ and $\varepsilon > 0$ such that $p = \frac{p_1+p_2}{2}$ and

$$10\varepsilon < E^1_{min}(p) - \frac{1}{2}\left(E^1_{min}(p_1) + E^1_{min}(p_2)\right).$$

By Lemma 3.3.6 (i), there exist functions $\psi_j = e^{i\varphi_j} + w_j \in \mathcal{E}(\mathbf{R})$ such that $\varphi_j \in \dot{H}^1 \cap C^\infty(\mathbf{R},\mathbf{R})$, $w_j \in C_c^\infty(\mathbf{R},\mathbf{C})$ and $\tilde{A} > 0$ such that $\mathrm{supp}(w_j) \subset [-\tilde{A},\tilde{A}]$, φ_j are constant on $(-\infty,-\tilde{A}]$ and on $[\tilde{A},\infty)$ and

$$p(\varphi_j, w_j) = p_j \qquad \text{and} \qquad E^1(\psi_j) < E^1_{min}(p_j) + \varepsilon \qquad \text{for } j = 1,2.$$

If φ_1 and φ_2 take different values near $\pm\infty$, we modify φ_j on $(-\infty,-\tilde{A}]$ and on $[\tilde{A},\infty)$ in such a way that $\varphi_1 = \varphi_2 = \textit{constant}$ on $(-\infty,-A]\cup[A,\infty)$ (where A may eventually be much larger than $\tilde{A}$) and $\int_{-A}^{-\tilde{A}} |\varphi_j'|^2\,dx + \int_{\tilde{A}}^{A} |\varphi_j'|^2\,dx < \varepsilon$ for $j = 1,2$. We still denote φ_j the modified functions. After this modification we have $p(\varphi_j, w_j) = p_j$ and $E^1(\psi_j) < E^1_{min}(p_j) + 2\varepsilon$ for $j = 1,2$.

Let $0 < \eta < \frac{1}{8}$ (the value of η will be chosen later). Take $\chi \in C^\infty(\mathbf{R})$ such that χ is $1-$periodic, $0 \leqslant \chi \leqslant 1$,

$$\chi = 1 \text{ on } \left[0, \frac{1}{4} - \eta\right] \cup \left[\frac{3}{4} + \eta, 1\right], \quad \chi = 0 \text{ on } \left[\frac{1}{4} + \eta, \frac{3}{4} - \eta\right], \quad \text{and } \int_0^1 \chi(y)\,dy = \frac{1}{2}.$$

Let

$$\varphi(x) = \tfrac{1}{2}(\varphi_1(x) + \varphi_2(x)),$$

$$w(x,y) = \chi(y)\left[e^{i\varphi_1(x)} + w_1(x)\right] + (1-\chi(y))\left[e^{i\varphi_2(x)} + w_2(x)\right] - e^{i\varphi(x)}, \quad \text{and}$$

$$\psi(x,y) = e^{i\varphi(x)} + w(x,y) = \chi(y)\psi_1(x) + (1-\chi(y))\psi_2(x).$$

Obviously, $\varphi \in \dot{H}^1 \cap C^\infty(\mathbf{R},\mathbf{R})$, $w \in C^\infty(\mathbf{R}^2)$ and w is $1-$periodic with respect to the second variable, and $w = 0$ on $((-\infty,-A]\cup[A,\infty)) \times \mathbf{R}$. Let $p(\varphi,w)$ be as in (3.3.3) and let $d[\varphi,w]$ be as at the beginning of section 3.3.2. Since $\varphi = \varphi_1 = \varphi_2 = \textit{constant}$ on $(-\infty,-A]\cup[A,\infty)$, using (3.3.5) we get

$$p(\varphi_j, w_j) = p(\varphi, w_j + e^{i\varphi_j} - e^{i\varphi}) \qquad \text{for } j = 1,2.$$

Using this simple observation, after a straightforward computation we obtain

$$\int_{\mathbf{R}} d[\varphi,w](x,y)\,dx = \chi(y)p(\varphi_1,w_1) + (1-\chi(y))p(\varphi_2,w_2)$$

$$-\chi(y)(1-\chi(y))\int_{\mathbf{R}} \langle i\left(e^{i\varphi_1} + w_1\right)' - i\left(e^{i\varphi_2} + w_2\right)', e^{i\varphi_1} + w_1 - (e^{i\varphi_2} + w_2)\rangle\,dx.$$

Integrating on $[0,1]$ we find

$$q(\varphi,w) = \frac{1}{2}p(\varphi_1,w_1) + \frac{1}{2}p(\varphi_2,w_2) - \int_0^1 \chi(y)(1-\chi(y))\,dy \cdot \int_{\mathbf{R}} \langle i\psi_1' - i\psi_2', \psi_1 - \psi_2\rangle\,dx.$$

Notice that $0 \leqslant \int_0^1 \chi(y)(1-\chi(y))\,dy < \eta$ because $0 \leqslant \chi(y)(1-\chi(y)) \leqslant \frac{1}{4}$ and $\chi(y)(1-\chi(y)) = 0$ on $[0,\frac{1}{4}-\eta]\cup[\frac{1}{4}+\eta,\frac{3}{4}-\eta]\cup[\frac{3}{4}+\eta,1]$. Denoting $K = \left|\int_{\mathbf{R}} \langle i\psi_1' - i\psi_2', \psi_1 - \psi_2\rangle\,dx\right|$, we have shown

that

$$|q(\varphi, w) - \frac{p_1 + p_2}{2}| < \eta K.$$

We know that for any $\lambda > 0$, the function $E_{\lambda,min}$ is $\sqrt{2}-$Lipschitz. If $\eta K < \frac{\varepsilon}{\sqrt{2}}$ we have

$$\left| E_{\lambda,min}(q(\varphi, w)) - E_{\lambda,min}\left(\frac{p_1 + p_2}{2}\right) \right| < \varepsilon \qquad \text{for any } \lambda > 0.$$

Let $M = \sup_{t\in[0,1]} E^1\left(t\psi_1 + (1-t)\psi_2 - e^{i\varphi}\right)$. It is straightforward to see that M is finite. Since $\psi(\cdot, y) = \psi_1$ for $y \in [0, \frac{1}{4} - \eta] \cup [\frac{3}{4} + \eta, 1]$ and $\psi(\cdot, y) = \psi_2$ for $y \in [\frac{1}{4} + \eta, \frac{3}{4} - \eta]$, we infer that

$$\int_{\mathbf{R}\times[0,1]} \left|\frac{\partial\psi}{\partial x}\right|^2 + V(|\psi|^2)\, dx\, dy \leqslant \left(\frac{1}{2} - 2\eta\right) E^1(\psi_1) + \left(\frac{1}{2} - 2\eta\right) E^1(\psi_2) + 4\eta M.$$

Now choose η such that $0 < \eta < \min\left(\frac{\varepsilon}{\sqrt{2}K}, \frac{\varepsilon}{4M}\right)$. Then for any $\lambda > 0$ we have

$$\begin{aligned}
&E_{\lambda,\min}(p) = E_{\lambda,min}\left(\tfrac{p_1+p_2}{2}\right) \leqslant E_{\lambda,min}(q(\varphi, w)) + \varepsilon \leqslant E_\lambda(\psi) + \varepsilon \\
&= \int_{\mathbf{R}\times[0,1]} \left|\frac{\partial\psi}{\partial x}\right|^2 + V(|\psi|^2)\, dx\, dy + \lambda^2 \left\|\frac{\partial\psi}{\partial y}\right\|^2_{L^2(\mathbf{R}\times[0,1])} + \varepsilon \\
&\leqslant \frac{1}{2}E^1(\psi_1) + \frac{1}{2}E^1(\psi_2) + 2\varepsilon + \lambda^2 \left\|\frac{\partial\psi}{\partial y}\right\|^2_{L^2(\mathbf{R}\times[0,1])} \\
&\leqslant \frac{1}{2}E^1_{min}(p_1) + \frac{1}{2}E^1_{min}(p_2) + 4\varepsilon + \lambda^2 \left\|\frac{\partial\psi}{\partial y}\right\|^2_{L^2(\mathbf{R}\times[0,1])}.
\end{aligned}$$

For λ sufficiently small, so that $\lambda^2 \left\|\frac{\partial\psi}{\partial y}\right\|^2_{L^2(\mathbf{R}\times[0,1])} < \varepsilon$, we get

$$E_{\lambda,\min}(p) < \frac{1}{2}E^1_{min}(p_1) + \frac{1}{2}E^1_{min}(p_2) + 5\varepsilon < E^1_{min}(p),$$

as desired.

vii) Choose $\delta > 0$ sufficiently small and $L > -\infty$ such that $(p_0 - \delta, p_0 + \delta) \subset (0, 2\pi)$ and

$$\frac{E^1_{min}(p_0 + h) + E^1_{min}(p_0 - h) - 2E^1_{min}(p_0)}{h^2} \geqslant L > -\infty \qquad \text{for all } h \in (0, \delta]. \tag{3.6.1}$$

Let $M = E^1_{min}(p_0) + 1$. Consider any $\psi = e^{i\varphi(x)} + w(x, y) \in \mathcal{E}$, where $\varphi \in \dot{H}^1(\mathbf{R}, \mathbf{R})$, $\varphi \in C^2(\mathbf{R} \times [0, 1])$ and $\mathrm{supp}(w) \subset [-a, a] \times [0, 1]$ for some $a > 0$ such that $q(\varphi, w) = p_0$ and

$$E_\lambda(\psi) = \int_{\mathbf{R}\times[0,1]} \left|\frac{\partial\psi}{\partial x}\right|^2 + \lambda^2\left|\frac{\partial\psi}{\partial y}\right|^2 + V(|\psi|^2)\, dx\, dy \leqslant M.$$

In particular, we have $\left\|\frac{\partial\psi}{\partial x}\right\|_{L^2(\mathbf{R}\times[0,1])} \leqslant \sqrt{M}$ and $\left\|\frac{\partial\psi}{\partial y}\right\|_{L^2(\mathbf{R}\times[0,1])} \leqslant \frac{\sqrt{M}}{\lambda}$. By Lemma 3.3.7, for any $y_1, y_2 \in [0,1]$ we have

$$|p(\varphi, w(\cdot, y_2)) - p(\varphi, w(\cdot, y_1))| \leqslant 2\left\|\frac{\partial\psi}{\partial x}\right\|_{L^2((\mathbf{R}\times[y_1,y_2])}\left\|\frac{\partial\psi}{\partial y}\right\|_{L^2((\mathbf{R}\times[y_1,y_2])} \leqslant \frac{2M}{\lambda}. \tag{3.6.2}$$

Choose $\lambda^*(p_0)$ sufficiently large, so that

$$\frac{2M}{\lambda^*(p_0)} < \frac{\delta}{2} \qquad \text{and} \qquad \frac{L}{2} + \frac{(\lambda^*(p_0))^2}{4M} > 1. \tag{3.6.3}$$

From now on we will assume that $\lambda > \lambda^*(p_0)$. For ψ as above, let $p_\psi = p(\varphi, w(\cdot, \frac{1}{2}))$. By (3.6.2) we have $|p(\varphi, w(\cdot, y) - p_\psi| \in [p_\psi - \frac{\delta}{2}, p_\psi + \frac{\delta}{2}]$ and then using Remark 3.3.4 (i) we see that a valuation of the momentum of ψ is

$$q(\varphi, w) = \int_0^1 p(\varphi, w(\cdot, y))\, dy \in \left[p_\psi - \frac{\delta}{2}, p_\psi + \frac{\delta}{2}\right]. \tag{3.6.4}$$

By Lemma 3.3.6 (ii) and the discussion preceding Definition 3.3.3, there exist maps $\psi = e^{i\varphi} + w \in \mathcal{E}$ such that $\varphi \in \dot{H}^1(\mathbf{R}) \cap C^\infty(\mathbf{R})$, w is the $1-$periodic extension with respect to the second variable of a function in $C_c^\infty(\mathbf{R}\times(0,1))$, $q(\varphi, w) = p_0$ and $E_\lambda(\psi)$ is arbitrarily close to $E_{\lambda,min}(p_0)$, in particular $E_{\lambda,min}(\psi) < M$. If $\lambda \geqslant \lambda^*(p_0)$, using (3.6.2) and (3.6.4) we see that for any such map and for any $y \in [0,1]$ we have

$$|p(\varphi, w(\cdot, y)) - p_0| < \delta.$$

Denote $p(y) = p(\varphi, w(\cdot, y))$ and let $\delta_\psi = \sup_{y\in[0,1]} |p(y) - p_0|$. Obviously, for any $y \in [0,1]$ we have $\psi(\cdot, y)e^{i\varphi} + w(\cdot, y) \in \mathcal{E}(\mathbf{R})$ and consequently

$$E^1(\psi(\cdot, y)) \geqslant E^1_{min}(p(\varphi, w(\cdot, y))) = E^1_{min}(p(y)). \tag{3.6.5}$$

If $f : [a,b] \longrightarrow \mathbf{R}$ is concave and continuous, for any $t \in [a,b]$ we have

$$f(t) \geqslant f(a) + \frac{t-a}{b-a}(f(b) - f(a)).$$

Since E^1_{min} is concave on $(0, 2\pi)$ and $p(y) \in [p_0 - \delta_\psi, p_0 + \delta_\psi]$, we get

$$E^1_{min}(p(y)) \geqslant E^1_{min}(p_0 - \delta_\psi) + \frac{p(y) - (p_0 - \delta_\psi)}{2\delta_\psi}\left[E^1_{min}(p_0 + \delta_\psi) - E^1_{min}(p_0 - \delta_\psi)\right]. \tag{3.6.6}$$

Using Fubini's Theorem, (3.6.5), (3.6.6) and the fact that $\int_0^1 p(y)\, dy = p_0$ (see (3.6.4)) we obtain

$$\begin{aligned} &\int_{\mathbf{R}\times[0,1]} \left|\frac{\partial\psi}{\partial x}\right|^2 + V(|\psi|^2)\, dx = \int_0^1 E^1(\psi(\cdot, y))\, dy \geqslant \int_0^1 E^1_{min}(p(y))\, dy \\ &\geqslant \frac{1}{2}\left[E^1_{min}(p_0 + \delta_\psi) + E^1_{min}(p_0 - \delta_\psi)\right]. \end{aligned} \tag{3.6.7}$$

By (3.6.2) the mapping $y \longmapsto p(y)$ is continuous, hence there exists $y_0 \in [0,1]$ such that $p(y_0) = p_0$. From (3.6.2) we get

$$\Big\|\frac{\partial \psi}{\partial y}\Big\|_{L^2((\mathbf{R}\times[0,1])} \geqslant \frac{\delta_\psi}{2\Big\|\frac{\partial \psi}{\partial x}\Big\|_{L^2((\mathbf{R}\times[0,1])}} \geqslant \frac{\delta_\psi}{2\sqrt{M}}. \tag{3.6.8}$$

From (3.6.7) and (3.6.8) we obtain

$$E_\lambda(\psi) - E^1_{min}(p_0) \geqslant \frac{1}{2}\left[E^1_{min}(p_0+\delta_\psi) + E^1_{min}(p_0-\delta_\psi)\right] - E^1_{min}(p_0) + \frac{\lambda^2\delta_\psi^2}{4M}$$

$$= \delta_\psi^2\left(\frac{1}{2}\frac{E^1_{min}(p_0+\delta_\psi) + E^1_{min}(p_0-\delta_\psi) - 2E^1_{min}(p_0)}{\delta_\psi^2} + \frac{\lambda^2}{4M}\right) > \delta_\psi^2$$

because $\delta_\psi \leqslant \delta$ and $\lambda \geqslant \lambda^*(p_0)$ (see (3.6.3). Since the above estimate holds for any ψ as considered above, the conclusion follows.

(viii) We proceed in several steps to prove the concavity of $E_{\lambda,min}$.

Step 0. Functional setting. We consider the space

$$\begin{aligned}\mathcal{X} \;=\; \{&w : \mathbf{R}^2 \longrightarrow \mathbf{C} \;\mid\; w \text{ is } 1-\text{periodic with respect to the second variable,}\\ &\text{is piecewise } C^2 \text{ and there exists } a>0 \text{ such that } \mathrm{supp}(\mathbf{w}) \subset [-a,a]\times\mathbf{R}\}.\end{aligned}$$

By "piecewise C^2" we mean that w is continuous and there exist finitely many points $0 = y_0 < y_1 < \cdots < y_n = 1$ such that for each $j \in \{1,\dots,n\}$ there exists a mapping $\tilde{w}^j$ that is C^2 on some larger strip $\mathbf{R}\times(y_{j-1}-\delta, y_j+\delta)$, and $w_{|\mathbf{R}\times[y_{j-1},y_j]} = \tilde{w}^j_{|\mathbf{R}\times[y_{j-1},y_j]}$. We consider the space $\mathcal{X}$ for the following reasons : we need a function space $\mathcal{X} \subset H^1_{per}$ such that any function $\psi \in \mathcal{E}$ can be approximated by functions of the form $e^{i\varphi(x)} + w(x,y)$, where $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in \mathcal{X}$ (this can be done in view of Lemma 3.3.6 (ii)), we need to use Lemma 3.3.7 (which obviously extends to functions in $\mathcal{X}$), and we need $\mathcal{X}$ to be stable by a reflection procedure that we will describe below.

Given any $w \in \mathcal{X}$ and $y_0 \in \mathbf{R}$, we define the functions $T_{1,y_0}w$ and $T_{2,y_0}w$ on $\mathbf{R}\times[0,1]$ as follows, then we extend them to $\mathbf{R}^2$ as $1-$periodic functions with respect to the second variable :

$$T_{1,y_0}w(x,y) = \begin{cases} w(x,y_0+y) & \text{if } y \in [0,\frac{1}{2}],\\ w(x,y_0+1-y) & \text{if } y \in [\frac{1}{2},1],\end{cases}$$

$$T_{2,y_0}w(x,y) = \begin{cases} w(x,y_0+1-y) & \text{if } y \in [0,\frac{1}{2}],\\ w(x,y_0+y) & \text{if } y \in [\frac{1}{2},1].\end{cases}$$

It is obvious that $T_{1,y_0}w$ and $T_{2,y_0}w$ belong to $\mathcal{X}$ for any $w \in \mathcal{X}$ and any $y_0 \in \mathbf{R}$, and we have $T_{j,y_0}w(x,y) = T_{j,y_0}w(x,1-y)$ for all $y \in [0,1]$ and $j = 1,2$.

For $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in \mathcal{X}$, let $\psi(x,y) = e^{i\varphi(x)} + w(x,y)$ and let $d[\varphi,w]$ and $q(\varphi,w)$ be as in subsection 3.3.2. It is easily seen that

$$q(\varphi, T_{1,y_0}w) = \int_0^1 \int_{\mathbf{R}} d[\varphi, T_{1,y_0}w](x,y)\,dx\,dy = 2\int_{y_0}^{y_0+\frac{1}{2}} d[\varphi,w](x,y)\,dx\,dy,$$

$$q(\varphi, T_{2,y_0}w) = \int_0^1 \int_{\mathbf{R}} d[\varphi, T_{2,y_0}w](x,y)\,dx\,dy = 2\int_{y_0-\frac{1}{2}}^{y_0} d[\varphi,w](x,y)\,dx\,dy,$$

(3.6.9)

$$E_\lambda(e^{i\varphi} + T_{1,y_0}w) = 2\int_{y_0}^{y_0+\frac{1}{2}} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial y}\Big|^2 + V(|\psi|^2)\,dx\,dy,$$

$$E_\lambda(e^{i\varphi} + T_{2,y_0}w) = 2\int_{y_0-\frac{1}{2}}^{y_0} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial y}\Big|^2 + V(|\psi|^2)\,dx\,dy, \qquad \text{and}$$

$$E_\lambda(e^{i\varphi} + T_{1,y_0}w) + E_\lambda(e^{i\varphi} + T_{1,y_0}w) = 2E_\lambda(\psi).$$

For any φ and w as above, the function

$$\Upsilon_{\varphi,w}(t) = \int_t^{t+\frac{1}{2}} d[\varphi,w](x,y)\,dx\,dy - \int_{t-\frac{1}{2}}^{t} d[\varphi,w](x,y)\,dx\,dy$$

is continuous, 1−periodic on $\mathbf{R}$ and $\Upsilon_{\varphi,w}(t+\frac{1}{2}) = -\Upsilon_{\varphi,w}(t)$ for any t. We denote

$$\omega(\varphi,w) = \sup_{t\in[0,1]} |\,\Upsilon_{\varphi,w}(t)\,|\,.$$

For any $q \in (0,2\pi)$ and any $\lambda > 0$, we denote

$$\delta_\lambda(q) = \inf_{\varepsilon>0}\Big(\sup\Big\{\omega(\varphi,w) \;\mid\; \varphi\in\dot{H}^1(\mathbf{R}), w\in\mathcal{X}, q(\varphi,w)=q \text{ and } E_\lambda(e^{i\varphi}+w) < E_{\lambda,min}(q)+\varepsilon\Big\}\Big).$$

Step 1. Assume that $\delta_\lambda(q) > 0$. Then for any $\eta \in (0,\delta_\lambda(q))$ we have

$$E_{\lambda,min}(q-\eta) + E_{\lambda,min}(q+\eta) \leqslant 2E_{\lambda,min}(q). \tag{3.6.10}$$

To see this, fix $\eta \in (0,\delta_\lambda(q))$. Fix $\varepsilon > 0$. By the definition of $\delta_\lambda(q)$, there exist $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in \mathcal{X}$ such that $q(\varphi,w) = q$, $E(e^{i\varphi}+w) < E_{\lambda,min}(q)+\varepsilon$ and $\omega(\varphi,w) > \eta$. Since $\Upsilon_{\varphi,w}$ is continuous, 1−periodic and $\Upsilon_{\varphi,w}(t+\frac{1}{2}) = -\Upsilon_{\varphi,w}(t)$ for any t, there exists $t_0 \in [0,1]$ such that $\Upsilon_{\varphi,w}(t_0) = \eta$. Let $w_1 = T_{1,t_0}w$ and $w_2 = T_{2,t_0}w$. From the first two equalities in (3.6.9) we get $q(\varphi,w_1) = q+\eta$ and $q(\varphi,w_2) = q-\eta$, and we infer that $E_\lambda(e^{i\varphi}+w_1) \geqslant E_{\lambda,min}(q+\eta)$ and $E_\lambda(e^{i\varphi}+w_2) \geqslant E_{\lambda,min}(q-\eta)$. Then using the last equality in (3.6.9) we find

$$E_{\lambda,min}(q-\eta) + E_{\lambda,min}(q+\eta) \leqslant E_\lambda(e^{i\varphi}+w_2) + E_\lambda(e^{i\varphi}+w_1) = 2E_\lambda(e^{i\varphi}+w) < 2E_{\lambda,min}(q) + 2\varepsilon.$$

Since the above inequality holds for any $\varepsilon > 0$, (3.6.10) is proven.

Step 2. If $q \in (0, 2\pi)$ is such that $\delta_\lambda(q) = 0$, then $E_{\lambda,min}(q) = E^1_{min}(q)$.

Let $M = E_{\lambda,min}(q) + 1$. Choose $r > 0$ such that $(q - r, q + r) \subset (0, 2\pi)$. Then choose $\varepsilon_0 \in (0, 1)$ such that $\varepsilon_0 < r$ and $\frac{4\sqrt{M\varepsilon_0}}{\lambda} < r$.

Fix $\varepsilon \in (0, \varepsilon_0)$. By the definition of $\delta_\lambda(q)$, there exist $\eta_\varepsilon > 0$ such that for any $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in \mathcal{X}$ satisfying $q(\varphi, w) = q$ and $E_\lambda(e^{i\varphi} + w) < E_{\lambda,min}(q) + 2\eta_\varepsilon$ we have $\omega(\varphi, w) < \varepsilon$. We may assume that $\eta_\varepsilon \leqslant \varepsilon$.

Consider $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in \mathcal{X}$ such that $q(\varphi, w) = q$ and $E(e^{i\varphi} + w) < E_{\lambda,min}(q) + \eta_\varepsilon$. Proceeding as in step 1, we see that there exists $y_0 \in [0, 1]$ such that $\Upsilon_{\varphi,w}(y_0) = 0$. Denote $w_1 = T_{1,y_0}w$, $w_2 = T_{2,y_0}w$, $\psi = e^{i\varphi} + w$, and $\psi_j(x, y) = e^{i\varphi(x)} + w_j(x, y)$ for $j = 1, 2$. By (3.6.9) we have

$$q(\varphi, w_1) = q(\varphi, w_2) = q(\varphi, w) = q \quad \text{and} \quad E_\lambda(\psi_1) + E_\lambda(\psi_2) = 2E_\lambda(\psi) < 2E_{\lambda,min}(q) + 2\eta_\varepsilon$$

and we infer that

$$E_{\lambda,min}(q) \leqslant E_\lambda(\psi_j) < E_{\lambda,min}(q) + 2\eta_\varepsilon \qquad \text{for } j = 1, 2.$$

Taking into account how η_ε was chosen, we infer that $\omega(\varphi, w_j) < \varepsilon$ for $j = 1, 2$.

Let $w_{j,1} = T_{1,\frac{1}{4}}w_j$ and $w_{j,2} = T_{2,\frac{1}{4}}w_j$ for $j = 1, 2$, then let $\psi_{j,\ell}(x, y) = e^{i\varphi(x)} + w_{j,\ell}(x, y)$ for $j, \ell \in \{1, 2\}$. Using the first equality in (3.6.9) we find

$$|q(\varphi, w_{1,1}) - q(\varphi, w_1)| = |\Upsilon_{\varphi,w_1}\left(\frac{1}{4}\right)| \leqslant \omega(\varphi, w_j) < \varepsilon.$$

Similarly we get $|q(\varphi, w_{j,\ell}) - q(\varphi, w_j)| < \varepsilon$ for $j, \ell \in \{1, 2\}$, and this gives $q(\varphi, w_{j,\ell}) \in (q - \varepsilon, q + \varepsilon)$. Since $E_{\lambda,min}$ is $\sqrt{2}-$Lipschitz, we get

$$E_{\lambda,min}(q(\varphi, w_{j,\ell})) \geqslant E_{\lambda,min}(q) - \sqrt{2}\varepsilon \qquad \text{for } j, \ell \in \{1, 2\}. \tag{3.6.11}$$

Now we observe that by construction, the functions $w_{j,\ell}$ are $\frac{1}{2}-$periodic with respect to the second variable y. Let $\tilde{w}_{j,\ell}(x, y) = w_{j,\ell}(x, \frac{y}{2})$. Then we have $\tilde{w}_{j,\ell} \in \mathcal{X}$ and

$$q(\varphi, \tilde{w}_{j,\ell}) = \int_0^1 \int_{\mathbf{R}} d[\varphi, \tilde{w}_{j,\ell}](x, y)\, dx\, dy = 2\int_0^{\frac{1}{2}} \int_{\mathbf{R}} d[\varphi, w_{j,\ell}](x, y)\, dx\, dy = q(\varphi, w_{j,\ell})$$

because $\int_0^{\frac{1}{2}} \int_{\mathbf{R}} d[\varphi, w_{j,\ell}](x, y)\, dx\, dy = \int_{\frac{1}{2}}^1 \int_{\mathbf{R}} d[\varphi, w_{j,\ell}](x, y)\, dx\, dy$ (the last equality is a consequence of the fact that $w_{j,\ell}$ is $\frac{1}{2}-$periodic with respect to y). Denoting $\tilde{\psi}_{j,\ell}(x, y) = e^{i\varphi(x)} + \tilde{w}_{j,\ell}(x, y) = \psi_{j,\ell}(x, \frac{y}{2})$ we have $\tilde{\psi}_{j,\ell} \in \mathcal{E}$ and using (3.6.11) we get

$$E_\lambda(\tilde{\psi}_{j,\ell}) \geqslant E_{\lambda,min}(q(\varphi, \tilde{w}_{j,\ell})) = E_{\lambda,min}(q(\varphi, w_{j,\ell})) \geqslant E_{\lambda,min}(q) - \sqrt{2}\varepsilon.$$

A simple computation gives

$$E_\lambda(\tilde{\psi}_{j,\ell}) = E_\lambda(\psi_{j,\ell}) - \frac{3}{4}\lambda^2 \Big\| \frac{\partial \psi_{j,\ell}}{\partial y} \Big\|^2_{L^2(\mathbf{R}\times[0,1])}.$$

From the last sequence of inequalities we obtain

$$\frac{3}{4}\lambda^2 \Big\| \frac{\partial \psi_{j,\ell}}{\partial y} \Big\|^2_{L^2(\mathbf{R}\times[0,1])} \leqslant E_\lambda(\psi_{j,\ell}) - E_{\lambda,min}(q) + \sqrt{2}\varepsilon. \tag{3.6.12}$$

Summing up the inequalities (3.6.12) for $j, \ell \in \{1, 2\}$ we get

$$3\lambda^2 \Big\| \frac{\partial \psi}{\partial y} \Big\|^2_{L^2(\mathbf{R}\times[0,1])} \leqslant 4E_\lambda(\psi) - 4E_{\lambda,min}(q) + 4\sqrt{2}\varepsilon < 4\eta_\varepsilon + 4\sqrt{2}\varepsilon < 12\varepsilon. \tag{3.6.13}$$

Using (3.6.13), the fact that $\Big\| \frac{\partial \psi}{\partial x} \Big\|^2_{L^2(\mathbf{R}\times[0,1])} \leqslant E_\lambda(\psi) \leqslant M$ and Lemma 3.3.7 we infer that

$$|p(\varphi, w(\cdot, y_2)) - p(\varphi, w(\cdot, y_1))| \leqslant 2 \Big\| \frac{\partial \psi}{\partial x} \Big\|_{L^2(\mathbf{R}\times[0,1])} \Big\| \frac{\partial \psi}{\partial y} \Big\|_{L^2(\mathbf{R}\times[0,1])} \leqslant \frac{4\sqrt{\varepsilon M}}{\lambda}$$

for any $y_1, y_2 \in [0, 1]$. Since $y \longmapsto p(\varphi, w(\cdot, y))$ is continuous and $\displaystyle\int_0^1 p(\varphi, w(\cdot, y))\, dy = q$, there exists $y_* \in [0, 1]$ such that $p(\varphi, w(\cdot, y_*)) = q$ and consequently

$$p(\varphi, w(\cdot, y)) \subset \left(q - \frac{4\sqrt{\varepsilon M}}{\lambda}, q + \frac{4\sqrt{\varepsilon M}}{\lambda} \right) \qquad \text{for any } y \in [0, 1]. \tag{3.6.14}$$

Using the fact that E^1_{min} is $\sqrt{2}-$Lipschitz, we deduce that

$$E^1(\psi(\cdot, y)) = \int_{\mathbf{R}} \Big| \frac{\partial \psi}{\partial x}(x, y) \Big|^2 + V(|\psi|^2)(x, y)\, dx \geqslant E^1_{min}(q) - \frac{4\sqrt{2\varepsilon M}}{\lambda}. \tag{3.6.15}$$

Integrating (3.6.15) over $[0, 1]$ we discover

$$E_{\lambda,min}(q) + \varepsilon > E_\lambda(\psi) \geqslant \int_0^1 E^1(\psi(\cdot, y))\, dy \geqslant E^1_{min}(q) - \frac{4\sqrt{2\varepsilon M}}{\lambda}. \tag{3.6.16}$$

Since (3.6.16) holds for any $\varepsilon \in (0, \varepsilon_0)$, we infer that $E_{\lambda,min}(q) \geqslant E^1_{min}(q)$. Thus necessarily $E_{\lambda,min}(q) = E^1_{min}(q)$ (see part (ii)) and the proof of step 2 is completed.

Step 3. Conclusion. The concavity of $E_{\lambda,min}$ on $[0, 2\pi]$ follows from steps 1 and 2 and from Lemma 3.6.3 below with $[a, b] = [0, 2\pi]$, $f = E_{\lambda,min}$ and $g = E^1_{min}$.

Lemma 3.6.3. *Let $f, g : [a,b] \longrightarrow \mathbf{R}$ be two continuous functions. Assume that*
a) g is concave and $f \leqslant g$ on $[a,b]$, and
b) for any $x \in (a,b)$ we have either $f(x) = g(x)$, or there exists $\delta_x > 0$ such that

$$f(x) \geqslant \frac{1}{2}(f(x-\eta) + f(x+\eta)) \qquad \text{for any } 0 < \eta < \delta_x. \tag{3.6.17}$$

Then f is concave on $[a,b]$.

Proof. For any $x \in (a,b)$ there exists $\delta_x > 0$ such that (3.6.17) holds. If $f(x) < g(x)$, this follows from assumption (b). If $f(x) = g(x)$ we may take $\delta_x = \min(x-a, b-x)$. Indeed, if $0 < \eta < \min(x-a, b-x)$ we have $x-\eta, x+\eta \in [a,b]$. By assumption (a) we get

$$f(x) = g(x) \geqslant \frac{1}{2}(g(x-\eta) + g(x+\eta)) \geqslant \frac{1}{2}(f(x-\eta) + f(x+\eta)).$$

Next we see that for any $\alpha, \beta \in \mathbf{R}$, the function $f_{\alpha,\beta}(t) = f(t) - \alpha t - \beta$ cannot achieve a minimum on an interval $(x_1, x_2) \subset (a,b)$ unless it is constant on $[x_1, x_2]$. Indeed, assume that $f_{\alpha,\beta}$ reaches a minimum on (x_1, x_2) at some point $y \in (x_1, x_2)$. It is obvious that $f_{\alpha,\beta}$ also satisfies (3.6.17). Let $S = \{z \in (x_1, x_2) \mid f_{\alpha,\beta}(z) = f_{\alpha,\beta}(y)\}$. By (3.6.17) we see that $z-\eta, z+\eta \in S$ for any $0 \leqslant \eta < \min(\delta_y, y - x_1, x_2 - y)$ and we infer that S is open in (x_1, x_2). By the continuity of f, the set S is also relatively closed in (x_1, x_2). Hence $S = (x_1, x_2)$.

Let $x_1, x_2 \in [a,b]$, $x_1 < x_2$. The function $t \longmapsto f(t) - \frac{x_2 - t}{x_2 - x_1} f(x_1) - \frac{t - x_1}{x_2 - x_1} f(x_2)$ takes the value 0 at $t = x_1$ and at $t = x_2$, hence it must be nonnegative on $[x_1, x_2]$. This means that

$$f(t) \geqslant \frac{x_2 - t}{x_2 - x_1} f(x_1) + \frac{t - x_1}{x_2 - x_1} f(x_2) \qquad \text{for any } t \in [x_1, x_2].$$

Since x_1 and x_2 were arbitrary, f is concave on $[a,b]$. □

To perform minimization of the energy at fixed momentum in $\mathcal{E}$ we will use a "regularization by minimization procedure" that has been developed in [11] and [6]. It will enable us to get rid of small defects of Sobolev functions and to approximate functions in $\mathcal{E}$ that have very small energy on every ball of fixed radius by functions whose modulus is close to 1.

We consider a function $\nu \in C^\infty(\mathbf{R})$ such that ν is odd, $\nu(s) = s$ for $s \in [0,2]$, $0 \leq \nu' \leq 1$ on $\mathbf{R}$ and $\nu(s) = 3$ for $s \geq 4$. Given $\psi \in \mathcal{E}$ and $\lambda > 0$, the modified Ginzburg-Landau energy of ψ in Ω is

$$E_{GLm,\lambda}(\psi) = \int_{\mathbf{R}\times[0,1]} \Big|\frac{\partial \psi}{\partial x}\Big|^2 + \lambda^2 \Big|\frac{\partial \psi}{\partial y}\Big|^2 + \frac{1}{2}\Big(\nu^2(|\psi|) - 1\Big)^2 \, dx\, dy. \tag{3.6.18}$$

For any given $\psi \in \mathcal{E}$, and $h, \lambda > 0$ we consider the functional

$$\begin{aligned} G^\psi_{h,\lambda}(\zeta) &= E_{GLm,\lambda}(\zeta) + \tfrac{1}{h^2}\|\zeta - \psi\|^2_{L^2(\mathbf{R}\times[0,1])} \\ &= \int_{\mathbf{R}\times[0,1]} \Big|\frac{\partial \zeta}{\partial x}\Big|^2 + \lambda^2 \Big|\frac{\partial \zeta}{\partial y}\Big|^2 + \frac{1}{2}\Big(\nu^2(|\zeta|) - 1\Big)^2 + \frac{1}{h^2}|\zeta - \psi|^2 \, dx\, dy. \end{aligned} \tag{3.6.19}$$

Notice that $G^{\psi}_{h,\lambda}(\zeta) < \infty$ for any $\zeta \in \mathcal{E}$ satisfying $\zeta - \psi \in H^1_{per}$. We will consider the problem of minimizing $G^{\psi}_{h,\lambda}$ in the set $\{\zeta \in \mathcal{E} \mid \zeta - \psi \in H^1_{per}\}$.

Let $\Lambda = \frac{1}{\lambda}$. Denoting $\tilde{\psi}(x,y) = \psi(x,\lambda y)$ and $\tilde{\zeta}(x,y) = \zeta(x,\lambda y)$ we see that $\tilde{\psi}$ and $\tilde{\zeta}$ are $\Lambda-$periodic with respect to the second variable and

$$G^{\psi}_{h,\lambda}(\zeta) = \lambda \int_{\mathbf{R}\times[0,\Lambda]} \Big|\frac{\partial\tilde{\zeta}}{\partial x}\Big|^2 + \Big|\frac{\partial\tilde{\zeta}}{\partial y}\Big|^2 + \frac{1}{2}\left(\nu^2(|\tilde{\zeta}|)-1\right)^2 + \frac{1}{h^2}|\tilde{\zeta}-\psi|^2\,dx\,dy = \lambda \tilde{G}^{\tilde{\psi}}_h(\tilde{\zeta}).$$

Therefore ζ is a minimizer for $G^{\psi}_{h,\lambda}$ among $1-$periodic functions with respect to the second variable if and only if $\tilde{\zeta}$ is a minimizer for $\tilde{G}^{\tilde{\psi}}_h$ among functions that are $\Lambda-$periodic with respect to the second variable. This observation enables us to use directly the results established in [6, 11].

Proceeding exactly as in Lemma 3.1 p. 160 and Lemma 3.2 p. 164 in [6] (see also Lemma 3.1 p. 126 and Lemma 3.2 p. 132 in [11]), we get :

Lemma 3.6.4. *i) The functional $G^{\psi}_{h,\lambda}$ has a minimizer in the set $\mathcal{E}_\psi = \{\zeta \in \mathcal{E} \mid \zeta - \psi \in H^1_{per}\}$.*

ii) Any minimizer ζ_h satisfies

$$E_{GLm,\lambda}(\zeta_h) \leqslant E_{GLm,\lambda}(\psi); \tag{3.6.20}$$

$$\|\zeta_h - \psi\|^2_{L^2(\mathbf{R}\times[0,1])} \leqslant h^2 E_{GLm,\lambda}(\psi); \tag{3.6.21}$$

$$\int_{\mathbf{R}\times[0,1]} \Big|\left(\nu^2(|\zeta_h|)-1\right)^2 - \left(\nu^2(|\psi|)-1\right)^2\Big|\,dx\,dy \leqslant Ch E_{GLm,\lambda}(\psi). \tag{3.6.22}$$

If $\psi = e^{i\varphi} + w$ with $\varphi \in \dot{H}^1(\mathbf{R})$ and $w \in H^1_{per}$, then we have

$$|q(\varphi, w + (\zeta_h - \psi)) - q(\varphi, w)| \leqslant 2h E_{GLm,\lambda}(\psi). \tag{3.6.23}$$

iii) Let $L(z) = \left(\nu^2(|z|)-1\right)\nu(|z|)\nu'(|z|)\frac{z}{|z|}$ if $z \in \mathbf{C}^*$ and $L(0) = 0$. Then any minimizer ζ_h of $G^{\psi}_{h,\lambda}$ in $\mathcal{E}_\psi$ satisfies the equation

$$-\Delta\zeta_h + L(\zeta_h) + \frac{1}{h^2}(\zeta_h - \psi) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^2). \tag{3.6.24}$$

Moreover, we have $\zeta_h \in W^{2,p}_{loc}(\mathbf{R}^2)$ for $1 \leqslant p < \infty$, and consequently $\zeta_h \in C^{1,\alpha}_{loc}(\mathbf{R}^2)$ for any $\alpha \in [0,1)$.

iv) For any $h > 0$ and any $\delta > 0$ there exists a constant $K = K(h,\delta) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_{GLm,\lambda}(\psi) \leq K$ and for any minimizer ζ_h of $G^{\psi}_{h,\lambda}$ in $\mathcal{E}_\psi$ we have

$$1 - \delta < |\zeta_h(x)| < 1 + \delta \qquad \text{in } \mathbf{R}^2. \tag{3.6.25}$$

v) Let $(\psi_n)_{n\geqslant 1} \subset \mathcal{E}$ be a sequence of functions such that $(E_{GL,\lambda}(\psi_n))_{n\geqslant 1}$ is bounded and

$$\lim_{n\to\infty} \Big(\sup_{y\in\mathbf{R}} E_{GL,\lambda}^{([y-1,y+1])}(\psi_n) \Big) = 0. \tag{3.6.26}$$

There exists a sequence $h_n \longrightarrow 0$ such that for any minimizer ζ_n of $G_{h_n,\lambda}^{\psi_n}$ in $\mathcal{E}_{\psi_n}$ we have $\| \, |\zeta_n| - 1\|_{L^\infty(\mathbf{R}^2)} \longrightarrow 0$ as $n \longrightarrow \infty$.

The proof of Lemma 3.6.4 is the same as the poof of Lemmas 3.1 and 3.2 in [6], and we refer the interested reader to that article. See also Lemmas 3.1 and 3.2 in [11] for higher-dimensional variants. The existence of a minimizer is straightforward using the direct method in calculus of variations. Estimates (3.6.20) and (3.6.21) folow immediately from the fact that $G_{h,\lambda}^{\psi}(\zeta_h) \leqslant G_{h,\lambda}^{\psi}(\psi) = E_{GLm,\lambda}(\psi)$, and (3.6.23) comes from (3.3.7), (3.6.20) and (3.6.21). Part (v) is a version of Lemma 3.2 p. 164 in [6] in the periodic setting ; see also Lemma 3.2 p. 132 in [11].

For $\lfloor p \rfloor \in \mathbf{R}/2\pi\mathbf{Z}$ we denote $|\lfloor p \rfloor| = \inf\{|p'| \ \ | \ \ p' \in \lfloor p \rfloor\}$. Notice that $|\lfloor p \rfloor| \in [0,\pi]$ and the infimum is achieved. The distance between the classes $\lfloor p_1 \rfloor$ and $\lfloor p_2 \rfloor$ is $|\lfloor p_1 \rfloor - \lfloor p_2 \rfloor| = |\lfloor p_1 - p_2 \rfloor|$. It follows from Lemma 3.6.1 (i) that for any $p \in \mathbf{R}$ we have

$$E_{\lambda,min}(p) = E_{\lambda,min}(|\lfloor p \rfloor|).$$

Lemma 3.6.5. *Assume that $(\psi_n)_{n\geqslant 1} \subset \mathcal{E}$ is a sequence of functions satisfying :*

(a) There exists $M > 0$ such that $E_\lambda(\psi_n) \leqslant M$ for all n,

(b) $\lim_{n\to\infty} \Big(\sup_{y\in\mathbf{R}} E_{\lambda}^{([y-1,y+1])}(\psi_n) \Big) = 0$,

(c) $|\lfloor Q \rfloor(\psi_n)| \longrightarrow q \in [0,\pi]$ as $n \longrightarrow \infty$.

Then we have $\liminf_{n\to\infty} E_\lambda(\psi_n) \geqslant \sqrt{2}q$.

Proof. Using Lemma 3.2.3 we see that $E_{GL,\lambda}(\psi_n)$ is bounded and (3.6.26) holds. We denote

$$\tilde{M} = \sup_{n\geqslant 1} E_{GL,\lambda}(\psi_n) \qquad \text{and} \qquad \varepsilon_n = \sup_{x\in\mathbf{R}} E_{GL,\lambda}^{([x-1,x+1])}(\psi_n).$$

Let $u_n = |\psi_n| - 1$. Then we have $|\nabla u_n| \leqslant |\nabla \psi_n|$ almost everywhere on $\mathbf{R}^2$. We also have $|u_n| \leqslant |\,|\psi_n| - 1| \cdot |\,|\psi_n| + 1| = |\,|\psi_n|^2 - 1|$. We infer that $u_n \in H^1(\mathbf{R} \times (0,1))$ and

$$\|u_n\|^2_{H^1(\mathbf{R}\times(0,1))} \leqslant \max\left(1, \frac{1}{\lambda^2}\right) E_{GL,\lambda}(\psi_n) \leqslant \max\left(1, \frac{1}{\lambda^2}\right) \tilde{M}.$$

Similarly we get $\|u_n\|^2_{H^1([x-1,x+1]\times(0,1))} \leqslant \max\left(1, \frac{1}{\lambda^2}\right) \varepsilon_n$ for any $x \in \mathbf{R}$. Let $p \in (2,\infty)$. By the Sobolev embedding, there is $C_p > 0$ such that $\|u\|_{L^p([a-1,a+1]\times[0,1])} \leqslant C_p \|u\|_{H^1((a-1,a+1)\times(0,1))}$ for any $u \in H^1((a-1,a+1)\times(0,1))$ and we infer that

$$\int_{[a-1,a+1]\times[0,1]} |u_n|^p \, dx\, dy \leqslant C_p^p \|u\|^p_{H^1((a-1,a+1)\times(0,1))} \leqslant C(p,\lambda) \varepsilon_n^{\frac{p-2}{2}} \|u_n\|^2_{H^1((a-1,a+1)\times(0,1))}.$$

Take a sequence $(a_k)_{k\geqslant 1}$ such that $\mathbf{R} = \cup_{k\geqslant 1}[a_k - 1, a_k + 1]$ and each point $x \in \mathbf{R}$ belongs to at most two of the intervals $[a_k - 1, a_k + 1]$. We write the above inequality for each k and we sum over k to get

$$\|u_n\|^p_{L^p(\mathbf{R}\times[0,1])} \leqslant C(p,\lambda)\varepsilon_n^{\frac{p-2}{2}}\|u_n\|^2_{H^1(\mathbf{R}\times(0,1))} \leqslant C(p,\lambda)\varepsilon_n^{\frac{p-2}{2}}E_{GL,\lambda}(\psi_n) \leqslant C(p,\lambda)\varepsilon_n^{\frac{p-2}{2}}\tilde{M}.$$

We have thus proved that $\|u_n\|_{L^p(\mathbf{R}\times[0,1])} \longrightarrow 0$ as $n \longrightarrow \infty$ for any $p \in (2,\infty)$.

Next we show that

$$\int_{\mathbf{R}\times[0,1]} |V(|\psi_n|^2) - \frac{1}{2}\Big(|\psi_n|^2 - 1\Big)^2| \, dx\, dy \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.6.27}$$

Fix $\varepsilon > 0$. By assumption **(A1)** there exists $\eta(\varepsilon) > 0$ such that

$$|V(s^2) - \frac{1}{2}\Big(s^2 - 1\Big)^2| \leqslant \frac{\varepsilon}{2}\Big(s^2 - 1\Big)^2 \qquad \text{for any } s \in [1-\eta(\varepsilon), 1+\eta(\varepsilon)].$$

Choose $p \geqslant \min(4, 2p_0 + 2)$, where p_0 is as in **(A2)**. By assumption **(A2)** there is $C(\varepsilon, p) > 0$ such that

$$|V(s^2) - \frac{1}{2}\Big(s^2 - 1\Big)^2| \leqslant C(\varepsilon,p)|\,|s| - 1|^p \qquad \text{for any } s \in [0, 1-\eta(\varepsilon)] \cup [1+\eta(\varepsilon), \infty).$$

We find

$$\Big|V(|\psi_n|^2) - \frac{1}{2}\Big(|\psi_n|^2 - 1\Big)^2\Big| \leqslant \frac{\varepsilon}{2}\Big(|\psi_n|^2 - 1\Big)^2 + C(\varepsilon,p)|\,|\psi_n| - 1|^p$$

and integrating we obtain

$$\begin{aligned} &\int_{\mathbf{R}\times[0,1]} |V(|\psi_n|^2) - \frac{1}{2}\Big(|\psi_n|^2 - 1\Big)^2| \, dx\, dy \\ &\leqslant \frac{\varepsilon}{2}\int_{\mathbf{R}\times[0,1]} \Big(|\psi_n|^2 - 1\Big)^2 \, dx\, dy + C(\varepsilon,p)\|u_n\|^p_{L^p(\mathbf{R}\times[0,1])} \leqslant \varepsilon\tilde{M} + C(\varepsilon,p,\lambda)\varepsilon_n^{\frac{p-2}{2}}\tilde{M}. \end{aligned} \tag{3.6.28}$$

Since $\varepsilon_n \longrightarrow 0$, there exists $n(\varepsilon) \in \mathbf{N}^*$ such that $C(\varepsilon,p,\lambda)\varepsilon_n^{\frac{p-2}{2}} < \varepsilon$ for all $n \geqslant n(\varepsilon)$. Then the right-hand side in the above inequality is smaller than $2\varepsilon\tilde{M}$ for all $n \geqslant n(\varepsilon)$. Since ε was arbitrary, (3.6.27) is proven.

Assume that $\zeta \in \mathcal{E}$ satisfies $1 - \delta \leqslant |\zeta| \leqslant 1 + \delta$ for some $\delta \in (0,1)$. According to Lemma 3.3.5, ζ admits a lifting $\rho e^{i\theta}$, a valuation of the momentum of ζ is $\int_{\mathbf{R}\times[0,1]}(1-\rho^2)\frac{\partial\theta}{\partial x}\,dxdy$, and we have

$$\begin{aligned} &\sqrt{2}(1-\delta)\Big|\int_{\mathbf{R}\times[0,1]}(1-\rho^2)\frac{\partial\theta}{\partial x}\,dx\,dy\Big| \leqslant \int_{\mathbf{R}\times[0,1]}(1-\delta)^2\Big|\frac{\partial\theta}{\partial x}\Big|^2 + \frac{1}{2}(1-\rho^2)^2\,dx\,dy \\ &\leqslant \int_{\mathbf{R}\times[0,1]}\rho^2\Big|\frac{\partial\theta}{\partial x}\Big|^2 + \frac{(1-\rho^2)^2}{2}\,dx\,dy = \int_{\mathbf{R}\times[0,1]}\Big|\frac{\partial\zeta}{\partial x}\Big|^2 + \frac{1}{2}(1-|\zeta|^2)^2\,dx\,dy \leqslant E_{GLm,\lambda}(\zeta). \end{aligned} \tag{3.6.29}$$

We may use Lemma 3.6.4 (v) for the sequence $(\psi_n)_{n\geqslant 1}$. We infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there is a minimizer ζ_n of $G^{\psi_n}_{h_n,\lambda}$ in $\mathcal{E}_{\psi_n}$ such that $\| \, |\zeta_n| - 1\|_{L^\infty(\mathbf{R}^2)} \longrightarrow 0$ as $n \longrightarrow \infty$. Denote $\delta_n = \| \, |\zeta_n| - 1\|_{L^\infty(\mathbf{R}^2)}$, so that $1 - \delta_n \leqslant |\zeta_n| \leqslant 1 + \delta_n$. For all n sufficiently large we have $\delta_n < 1$, and then ζ_n admits a lifting $\zeta_n = \rho_n e^{i\theta_n}$ and (3.6.29) holds for ζ_n.

From (3.6.23) we have

$$|\lfloor Q \rfloor(\zeta_n) - \lfloor Q \rfloor(\psi_n)| \leqslant 2h_n E_{GLm,\lambda}(\psi_n) \leqslant 2h_n \tilde{M} \longrightarrow 0 \quad \text{ as } n \longrightarrow \infty.$$

Since $|\lfloor Q \rfloor(\psi_n)| \longrightarrow q$, we infer that $|\lfloor Q \rfloor(\zeta_n)| \longrightarrow q$ as $n \longrightarrow \infty$.

We have $E_{GL,\lambda}(\psi_n) \geqslant E_{GLm,\lambda}(\psi_n) \geqslant E_{GLm,\lambda}(\zeta_n)$, and using (3.6.29) we obtain

$$\begin{aligned} E_\lambda(\psi_n) &= E_{GL,\lambda}(\psi_n) + \int_{\mathbf{R}\times[0,1]} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2 \, dx \, dy \\ &\geq E_{GLm,\lambda}(\zeta_n) + \int_{\mathbf{R}\times[0,1]} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2 \, dx \, dy \\ &\geq \sqrt{2}(1-\delta_n)|\lfloor Q \rfloor(\zeta_n)| + \int_{\mathbf{R}\times[0,1]} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2 \, dx. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using (3.6.27) we get $\liminf\limits_{n\to\infty} E_\lambda(\psi_n) \geqslant \sqrt{2}q$ and Lemma 3.6.5 is proven. □

Proof of Lemma 3.6.1 (v). Fix $p \geqslant \max(4, 2p_0 + 2)$, where p_0 is as in **(A2)**. Coming back to (3.6.28) we see that for any $\varepsilon > 0$ there exists $C(\varepsilon, p) > 0$ such that for any $\psi \in \mathcal{E}$ we have

$$\begin{aligned} |E_\lambda(\psi) - E_{GL,\lambda}(\psi)| &\leqslant \int_{\mathbf{R}\times[0,1]} |V(|\psi_n|^2) - \frac{1}{2}\left(|\psi_n|^2 - 1\right)^2| \, dx \, dy \\ &\leqslant \frac{\varepsilon}{2} \int_{\mathbf{R}\times[0,1]} \left(|\psi_n|^2 - 1\right)^2 \, dx \, dy + C(\varepsilon, p) \| \, |\psi_n| - 1\|^p_{L^p(\mathbf{R}\times[0,1])} \\ &\leqslant \varepsilon E_{GL,\lambda}(\psi) + C(\varepsilon, p, \lambda) E_{GL,\lambda}(\psi)^{\frac{p}{2}}. \end{aligned} \tag{3.6.30}$$

Using (3.6.30) we infer that for any $\varepsilon > 0$ and for any $\lambda > 0$ there exists $\tilde{M}(\varepsilon, \lambda) > 0$ such that

$$|E_\lambda(\psi) - E_{GL,\lambda}(\psi)| \leqslant 2\varepsilon E_{GL,\lambda}(\psi) \qquad \text{for any } \psi \in \mathcal{E} \text{ satisfying } E_{GL,\lambda}(\psi) \leqslant \tilde{M}(\varepsilon, \lambda). \tag{3.6.31}$$

Fix $\delta \in (0, 1)$. Given any $\psi \in \mathcal{E}$ and any $h > 0$, denoting by ζ_h a minimizer of $G^\psi_{h,\lambda}$ in $\mathcal{E}_\psi$ we have $E_{GL,\lambda}(\psi) \geqslant E_{GLm,\lambda}(\psi) \geqslant E_{GLm,\lambda}(\zeta_h)$ and therefore

$$\begin{aligned} E_\lambda(\psi) - \sqrt{2}(1-\delta)|\lfloor Q \rfloor(\psi)| &\geqslant (E_\lambda(\psi) - E_{GL,\lambda}(\psi)) + \frac{\delta}{2} E_{GL,\lambda}(\psi) \\ &+ \left(1 - \frac{\delta}{2}\right)\left(E_{GLm,\lambda}(\zeta_h) - \frac{1-\delta}{1-\frac{\delta}{2}}\sqrt{2}|\lfloor Q \rfloor(\zeta_h)|\right) + (1-\delta)\sqrt{2}\left(|\lfloor Q \rfloor(\zeta_h)| - |\lfloor Q \rfloor(\psi)|\right). \end{aligned} \tag{3.6.32}$$

Choose $h = \frac{\delta}{16\sqrt{2}(1-\delta)}$. Using (3.6.23) we have

$$(1-\delta)\sqrt{2}\Big||\lfloor Q\rfloor(\zeta_h)| - |\lfloor Q\rfloor(\psi)|\Big| \leqslant (1-\delta)\sqrt{2}\cdot(2hE_{GLm,\lambda}(\psi)) \leqslant \frac{\delta}{8}E_{GL,\lambda}(\psi). \tag{3.6.33}$$

For h as above, by Lemma 3.6.4 (iv) there exists a constant $K(\delta) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_{GL,\lambda}(\psi) \leqslant K(\delta)$ and for any minimizer ζ_h of $G^{\psi}_{h,\lambda}$ in $\mathcal{E}_\psi$ we have $1 - \frac{\delta}{2} \leqslant |\zeta_h| \leqslant 1 + \frac{\delta}{2}$. Then using (3.6.29) for ζ_h we see that

$$\frac{1-\delta}{1-\frac{\delta}{2}}\sqrt{2}|\lfloor Q\rfloor(\zeta_h)| \leqslant \left(1-\frac{\delta}{2}\right)\sqrt{2}|\lfloor Q\rfloor(\zeta_h)| \leqslant E_{GLm,\lambda}(\zeta_h). \tag{3.6.34}$$

If $E_{GL,\lambda}(\psi) \leqslant \tilde{M}\left(\frac{\delta}{16},\lambda\right)$, using (3.6.31) we get

$$|E_\lambda(\psi) - E_{GL,\lambda}(\psi)| \leqslant \frac{\delta}{8}E_{GL,\lambda}(\psi). \tag{3.6.35}$$

By Lemma 3.2.3 there exists $m > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_\lambda(\psi) \leqslant m$ we have $E_{GL,\lambda}(\psi) \leqslant \min\left(K(\delta), \tilde{M}\left(\frac{\varepsilon}{16},\lambda\right)\right)$. Then using (3.6.32)-(3.6.35) we infer that

$$E_\lambda(\psi) - \sqrt{2}(1-\delta)|\lfloor Q\rfloor(\psi)| \geqslant \frac{\delta}{4}E_{GL,\lambda}(\psi) \geqslant 0 \qquad \text{for all } \psi \in \mathcal{E} \text{ satisfying } E_\lambda(\psi) \leqslant m.$$

The above inequality and the fact that $E_{\lambda,min}(p) \longrightarrow 0$ as $p \longrightarrow 0$ imply that there exists $p_\delta > 0$ such that $E_{\lambda,min}(p) \geqslant \sqrt{2}(1-\delta)p$ for all $p \in [0, p_\delta]$. □

Theorem 3.6.6. *Assume that V satisfies* **(A1)**, **(A2)**, *and* **(B2)**. *Let $p \in (0,\pi]$ and $\lambda > 0$ such that $E_{\lambda,min}(p) < \sqrt{2}p$. Let $(\psi_n)_{n\geqslant 1} \subset \mathcal{E}$ be a sequence satisfying*

$$\lfloor Q\rfloor(\psi_n) \longrightarrow \lfloor p\rfloor \qquad \text{and} \qquad E_\lambda(\psi_n) \longrightarrow E_{\lambda,min}(p) \qquad \text{as } n \longrightarrow \infty. \tag{3.6.36}$$

Then there exist a subsequence $(\psi_{n_k})_{k\geqslant 1}$, a sequence $(x_k)_{k\geqslant 1} \subset \mathbf{R}$ and $\psi \in \mathcal{E}$ satisfying $\lfloor Q\rfloor(\psi) = \lfloor p\rfloor$, $E_\lambda(\psi) = E_{\lambda,min}(p)$, and

$$\begin{aligned}
&|\psi_{n_k}|(\cdot + x_k) - |\psi| \longrightarrow 0 \quad \text{in } L^p(\mathbf{R}) \text{ for } 2 \leqslant p < \infty,\\
&V(|\psi_{n_k}(\cdot + x_k)|^2) \longrightarrow V(|\psi|^2) \quad \text{in } L^1(\mathbf{R}\times[0,1]),\\
&\nabla\psi_{n_k}(\cdot + x_k) \longrightarrow \nabla\psi \quad \text{in } L^2(\mathbf{R}\times[0,1]).
\end{aligned}$$

Remark 3.6.7. If $\lambda \geqslant \lambda^*(q)$ we have $E_{\lambda,\min}(q) = E^1_{min}(q)$. In this case the existence of minimizers follows from Theorem 3.5.2 in the previous section. However, even in this case Theorem 3.6.6 above is interesting because it gives the stability of the minimizers under two-dimensional periodic perturbations, and this stability does not follow directly from Theorem 3.5.2.

Proof of Theorem 3.6.6. Let $(\psi_n)_{n\geqslant 1} \subset \mathcal{E}$ be a sequence satisfying (3.6.36). Then $E_\lambda(\psi_n)$ is bounded.

As in the proof of Theorem 3.5.2, we use the concentration-compactness principle ([9]). We denote by Λ_n the concentration function of $f_n := \Big|\frac{\partial \psi_n}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial \psi_n}{\partial y}\Big|^2 + V(|\psi_n|^2)$, that is

$$\Lambda_n(t) = \sup_{a\in\mathbf{R}} \int_{[a-t,a+t]\times[0,1]} \Big|\frac{\partial \psi_n}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial \psi_n}{\partial y}\Big|^2 + V(|\psi_n|^2)\, dx\, dy.$$

It is clear that Λ_n is a non-decreasing function on $[0,\infty)$, $\Lambda_n(0) = 0$ and $\lim\limits_{t\to\infty} \Lambda_n(t) = E_\lambda(\psi_n)$. Then there exists a subsequence of $(\psi_n, \Lambda_n)_{n\geqslant 1}$, still denoted $(\psi_n, \Lambda_n)_{n\geqslant 1}$, and there is a non-decreasing function $\Lambda : [0,\infty) \longrightarrow [0,\infty)$ satisfying

$$\Lambda_n(t) \longrightarrow \Lambda(t) \text{ a.e on } [0,\infty) \text{ as } n \longrightarrow \infty. \tag{3.6.37}$$

Let $\alpha = \lim\limits_{t\to\infty} \Lambda(t)$. We have $0 \leqslant \alpha \leqslant \lim_{n\to\infty} E_\lambda(\psi_n) = E_{\lambda,min}(p)$.

We will show that $\alpha = E_{\lambda,min}(p)$.

We prove first that $\alpha > 0$. We argue by contradiction and we assume that $\alpha = 0$. Then we have $\Lambda(t) = 0$ for all $t \geqslant 0$, and in particular $\Lambda(1) = 0$. Then Lemma 3.6.5 implies that $\liminf\limits_{n\to\infty} E_\lambda(\psi_n) \geqslant \sqrt{2} \lim\limits_{n\to\infty} \big|\lfloor Q \rfloor(\psi_n)\big|$, that is $E_{\lambda,min}(p) \geqslant \sqrt{2}p$, contradictiong the assumption of Theorem 3.6.6.

Assume that $0 < \alpha < E_{\lambda,min}(p)$. Proceeding as in the proof of Theorem 3.5.2 (see (3.5.15) - (3.5.18) there) we see that there exist a sequence $R_n \longrightarrow \infty$, a sequence $(a_n)_{n\geqslant 1} \subset \mathbf{R}$, and $\beta \in (0, E_{\lambda,min}(p))$ such that, after possibly extracting a further subsequence, we have

$$E_\lambda^{(-\infty, a_n - R_n)}(\psi_n) \longrightarrow \beta, \qquad E_\lambda^{(a_n+R_n,\infty)}(\psi_n) \longrightarrow E_{\lambda,min}(p) - \beta, \quad \text{and} \tag{3.6.38}$$

$$E_\lambda^{[a_n-R_n, a_n+R_n]}(\psi_n) \longrightarrow 0. \tag{3.6.39}$$

Let $\breve{\psi}_n(x) = \int_0^1 \psi_n(x,y)\, dy$ and $v_{\psi_n}(x,y) = \psi_n(x,y) - \breve{\psi}_n(x)$. From (3.6.39) and Lemma 3.2.4 we infer that

$$E^{1,\,[a_n-R_n,a_n+R_n]}(\breve{\psi}_n) \longrightarrow 0 \quad \text{and} \quad \|v_{\psi_n}\|_{H^1(([a_n-R_n,a_n+R_n])} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.6.40}$$

Proceeding as in the proof of Theorem 3.5.2 we see that $\big\||\breve{\psi}_n| - 1\big\|_{L^\infty([a_n-R_n+1,a_n+R_n-1]} \longrightarrow 0$. For n sufficiently large, $\breve{\psi}_n$ admits a lifting $\breve{\psi}_n(x) = \rho_n(x)e^{i\theta_n(x)}$ on $[a_n - R_n + 1, a_n + R_n - 1]$, where $\rho_n = |\breve{\psi}_n|$. Since $E^{1,\,[a_n-R_n,a_n+R_n]}(\breve{\psi}_n) \longrightarrow 0$ we have $\|\rho_n - 1\|_{H^1([a_n-R_n+1,a_n+R_n-1])} \longrightarrow 0$.

By Lemma 3.2.2 there exist $\varphi_n \in \dot{H}^1(\mathbf{R})$ and $v_n \in H^1(\mathbf{R})$ such that $\breve{\psi}_n = e^{i\varphi_n} + v_n$. We may assume that $\varphi_n = \theta_n$ and $v_n = (\rho_n - 1)e^{i\theta_n}$ on $[a_n - R_n + 2, a_n + R_n - 2]$ (to see this, we take

$\chi_n \in C_c^\infty(\mathbf{R})$ such that $\mathrm{supp}(\chi_n) \subset [a_n - R_n + 1, a_n + R_n - 1]$ and $\chi_n = 1$ on $[a_n - R_n + 2, a_n + R_n - 2]$ and we replace φ_n by $\tilde{\varphi}_n = (1 - \chi_n)\varphi_n + \chi_n \theta_n$ and v_n by $\tilde{v}_n = \psi_n - e^{i\tilde{\varphi}_n})$. Then we may write

$$\psi_n = e^{i\varphi_n(x)} + w_n(x, y) \qquad \text{on } \mathbf{R}^2,$$

where $\varphi_n(x) = \theta_n(x)$ and $w_n(x, y) = (\rho_n(x) - 1)e^{i\theta_n(x)} + v_{\psi_n}(x, y)$ on $[a_n - R_n + 2, a_n + R_n - 2]$ and satisfy

$$\|\varphi_n'\|_{L^2([a_n - R_n + 2, a_n + R_n - 2])} \longrightarrow 0 \quad \text{and} \quad \|w_n\|_{H^1([a_n - R_n + 2, a_n + R_n - 2] \times [0,1])} \longrightarrow 0 \tag{3.6.41}$$

as $n \longrightarrow \infty$. Consider a nondecreasing function $\chi \in C^\infty(\mathbf{R})$ such that $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[0, \infty)$. Define

$$\varphi_{n,1}(x) = \begin{cases} \varphi_n(x) & \text{if } x \leqslant a_n, \\ \varphi_n(a_n) & \text{if } x > a_n, \end{cases} \qquad \varphi_{n,2}(x) = \begin{cases} \varphi_n(a_n) & \text{if } x < a_n, \\ \varphi_n(a_n) & \text{if } x \geqslant a_n, \end{cases}$$

$$w_{n,1}(x, y) = \chi(a_n - x)w_n(x, y), \qquad w_{n,2}(x, y) = \chi(x - a_n)w_n(x, y),$$

and $\psi_{n,j}(x, y) = e^{i\varphi_{n,j}(x)} + w_{n,j}(x, y)$ for $j = 1, 2$. It is then clear that $\psi_{n,j} \in \mathcal{E}$ and we have

$$\begin{array}{ll} \psi_{n,1} = \psi_n \text{ on } (-\infty, a_n], & \psi_{n,1} = e^{i\varphi_n(a_n)} = \textit{constant} \text{ on } [a_n + 1, \infty), \\ \psi_{n,2} = \psi_n \text{ on } [a_n, \infty), & \psi_{n,2} = e^{i\varphi_n(a_n)} = \textit{constant} \text{ on } (-\infty, a_n - 1]. \end{array} \tag{3.6.42}$$

Using (3.6.38), (3.6.39) and (3.6.41) it is easily seen that

$$E_\lambda(\psi_{n,1}) \longrightarrow \beta \qquad \text{and} \qquad E_\lambda(\psi_{n,2}) \longrightarrow E_{\lambda,\min}(p) - \beta \qquad \text{as } n \longrightarrow \infty. \tag{3.6.43}$$

Taking into account (3.6.42), valuations of the momenta of $\psi_{n,1}$ and of $\psi_{n,2}$ are, respectively,

$$\begin{aligned} q(\varphi_{n,1}, w_{n,1}) &= \int_{(-\infty, a_n] \times [0,1]} -2\langle \varphi_n' e^{i\varphi_n}, w_n \rangle + \langle i\frac{\partial w_n}{\partial x}, w_n \rangle \, dx\, dy \\ &\quad + \int_{[a_n, a_n + 1] \times [0,1]} \langle i\frac{\partial w_{n,1}}{\partial x}, w_{n,1} \rangle \, dx\, dy, \quad \text{and} \\ q(\varphi_{n,2}, w_{n,2}) &= \int_{[a_n, \infty) \times [0,1]} -2\langle \varphi_n' e^{i\varphi_n}, w_n \rangle + \langle i\frac{\partial w_n}{\partial x}, w_n \rangle \, dx\, dy \\ &\quad + \int_{[a_n - 1, a_n] \times [0,1]} \langle i\frac{\partial w_{n,2}}{\partial x}, w_{n,2} \rangle \, dx\, dy. \end{aligned}$$

Since $R_n \longrightarrow \infty$ and $\|w_{n,j}\|_{H^1([a_n - R_n + 2, a_n + R_n - 2] \times [0,1])} \longrightarrow 0$, it follows from the above that

$$q(\varphi_{n,1}, w_{n,1}) + q(\varphi_{n,2}, w_{n,2}) = q(\varphi_n, w_n) + o(1) \qquad \text{as } n \longrightarrow \infty. \tag{3.6.44}$$

For any $a, b \in \mathbf{R}$ we have $\big|\lfloor a+b \rfloor\big| \leqslant \big|\lfloor a \rfloor\big| + \big|\lfloor b \rfloor\big|$. Passing to a further subsequence we may assume that $\big|\lfloor Q \rfloor(\psi_{n,1})\big| \longrightarrow p_1 \in [0,\pi]$ and $\big|\lfloor Q \rfloor(\psi_{n,2})\big| \longrightarrow p_2 \in [0,\pi]$. Then using (3.6.44) we deduce that

$$p_1 + p_2 \geqslant p. \tag{3.6.45}$$

We have $E_\lambda(\psi_{n,j}) \geqslant E_{\lambda,min}\left(\big|\lfloor Q \rfloor(\psi_{n,j})\big|\right)$. Passing to the limit and using (3.6.43) we see that

$$\beta \geqslant E_{\lambda,min}(p_1) \qquad \text{and} \qquad E_{\lambda,min}(p) - \beta \geqslant E_{\lambda,min}(p_2). \tag{3.6.46}$$

Since $E_{\lambda,min}$ is nondecreasing on $[0,\pi]$, (3.6.46) implies that $p_1 < p$ and $p_2 < p$ and then from (3.6.45) we infer that $0 < p_j < p$ for $j = 1, 2$.

The concavity of $E_{\lambda,min}$ implies that $E_{\lambda,min}(p_j) \leqslant \frac{p_j}{p} E_{\lambda,min}(p)$ for $j = 1, 2$, and equality may occur if and only if $E_{\lambda,min}$ is linear on $[0, p]$. Summing up these two inequalities we find $E_{\lambda,min}(p_1) + E_{\lambda,min}(p_2) \geqslant E_{\lambda,min}(p)$, and equality implies that $E_{\lambda,min}$ must be linear on $[0, p]$. Comparing this to (3.6.46) we infer that necessarily $p_1 + p_2 = p$ and $E_{\lambda,min}(p_1) + E_{\lambda,min}(p_2) = E_{\lambda,min}(p)$, thus $E_{\lambda,min}$ is linear on $[0, p]$. Then using Lemma 3.6.1 (v) we deduce that $E_{\lambda,min}(p) = \sqrt{2}p$, contradicting the assumption that $E_{\lambda,min}(p) < \sqrt{2}p$. We conclude that we cannot have $0 < \alpha < E_{\lambda,min}(p)$.

So far we have proved that $\alpha = E_{\lambda,min}(p)$. Then it is standard to prove that there is a sequence $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ satisfying $E_\lambda^{(-\infty, x_n - R_\varepsilon)}(\psi_n) + E_\lambda^{(x_n + R_\varepsilon, \infty)}(\psi_n) < \varepsilon$ for all sufficiently large n. Denoting $\tilde{\psi}_n = \psi_n(\cdot + x_n)$, we see that for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that

$$E_\lambda^{(-\infty, -R_\varepsilon)}(\tilde{\psi}_n) + E_\lambda^{(R_\varepsilon, \infty)}(\tilde{\psi}_n) < \varepsilon \qquad \text{for all } n \geq n_\varepsilon. \tag{3.6.47}$$

Obviously, $(\nabla\tilde{\psi}_n)_{n\geqslant 1}$ is bounded in $L^2(\mathbf{R} \times (0,1))$ and using the boundedness of $E_{GL,\lambda}(\tilde{\psi}_n)$ it is easy to see that $(\tilde{\psi}_n)_{n\geqslant 1}$ is bounded in $L^2((-R,R)\times(0,1))$ for any $R > 0$. By a standard argument, there exist a function $\psi \in H^1_{loc}(\mathbf{R}^2)$ which is $1-$periodic with respect to the second variable, $\nabla\psi \in L^2(\mathbf{R} \times (0,1))$ and there is a subsequence $(\tilde{\psi}_{n_k})_{k\geq 1}$ satisfying

$$\begin{array}{ll}
\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi & \text{weakly in } L^2(\mathbf{R}\times(0,1)),\\
\tilde{\psi}_{n_k} \rightharpoonup \psi & \text{weakly in } H^1((-R,R)\times(0,1)) \text{ for any } R > 0,\\
\tilde{\psi}_{n_k} \longrightarrow \psi & \text{strongly in } L^p((-R,R)\times(0,1)) \text{ for any } R > 0 \text{ and } p \in [1,\infty),\\
\tilde{\psi}_{n_k} \longrightarrow \psi & \text{almost everywhere on } \mathbf{R}^2.
\end{array} \tag{3.6.48}$$

The weak convergence implies that for any interval $I \subset \mathbf{R}$ we have

$$\int_{I\times(0,1)} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial y}\Big|^2 \, dx\, dy \leqslant \liminf_{k\to\infty} \int_{I\times(0,1)} \Big|\frac{\partial\tilde{\psi}_{n_k}}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\tilde{\psi}_{n_k}}{\partial y}\Big|^2 \, dx\, dy. \tag{3.6.49}$$

The almost everywhere convergence and Fatou's Lemma give

$$\int_{I\times[0,1]} \Big(|\psi|^2 - 1\Big)^2 \, dx\, dy \leq \liminf_{k\to\infty} \int_{I\times[0,1]} \Big(|\tilde{\psi}_{n_k}|^2 - 1\Big)^2 \, dx\, dy \quad \text{and} \tag{3.6.50}$$

$$\int_{I\times[0,1]} V(|\psi|^2)\,dx\,dy \leq \liminf_{k\to\infty} \int_{I\times[0,1]} V(|\tilde\psi_{n_k}|^2)\,dx\,dy. \tag{3.6.51}$$

From (3.6.49) - (3.6.51) we get

$$E^I_{GL}(\psi) \leq \liminf_{k\to\infty} E^I_{GL}(\tilde\psi_{n_k}) \quad \text{and} \quad E^I_\lambda(\psi) \leq \liminf_{k\to\infty} E^I_\lambda(\tilde\psi_{n_k}). \tag{3.6.52}$$

We deduce in particular that $\psi \in \mathcal{E}$, $E_\lambda(\psi) \leqslant E_{\lambda,min}(p)$ and $E_\lambda^{(-\infty,-R_\varepsilon)}(\psi) + E_\lambda^{(R_\varepsilon,\infty)}(\psi) \leqslant \varepsilon$.

Using Lemma 3.2.4 (and taking eventually a larger R_ε) we see that the sequence $\breve{\tilde\psi}_{n_k}$ satisfies (3.5.21). As in the proof of Theorem 3.5.2 we infer that there is a function $\zeta \in H^1_{loc}(\mathbf{R})$ such that $\zeta' \in L^2(\mathbf{R})$ and we may extract a subsequence, still denoted the same, such that (3.5.23) holds for $\breve{\tilde\psi}_{n_k}$ and ζ. It is then clear that (3.5.24) and (3.5.25) hold for $\breve{\tilde\psi}_{n_k}$ and ζ, too, and consequently $\zeta \in \mathcal{E}(\mathbf{R})$.

Fix $R_1 > 0$ such that $\breve{\tilde\psi}_{n_k}$ and ζ satisfy (3.5.21) and (3.5.22) on $(-\infty,-R_1]\cup[R_1,\infty)$. Then the functions $\breve{\tilde\psi}_{n_k}$ and ζ admit liftings on $(-\infty,-R_1]\cup[R_1,\infty)$, say $\breve{\tilde\psi}_{n_k} = \rho_k e^{i\theta_k}$ and $\zeta = \rho e^{i\theta}$, where $\rho, \rho_k \in H^1((-\infty,-R_1]\cup[R_1,\infty))$, and θ, θ_k are continuous and $\theta', \theta_k' \in L^2((-\infty,-R_1]\cup[R_1,\infty))$.

Fix $\varepsilon > 0$. Then choose $R_\varepsilon > 0$ and $k_\varepsilon \in \mathbf{N}$ such that $\tilde\psi_{n_k}$ and ψ satisfy (3.6.47) and $\breve{\tilde\psi}_{n_k}$ and ζ satisfy (3.5.21) and (3.5.22) on $(-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty)$ for all $k \geqslant k_\varepsilon$. We proceed as in the proof of Theorem 3.5.2. We have $\breve{\tilde\psi}_{n_k}(\pm R_\varepsilon)/\zeta(\pm R_\varepsilon) \longrightarrow 1$ as $k \longrightarrow \infty$, thus we may replace θ_k by $\theta_k + 2\ell_k^\pm\pi$ on $(-\infty,-R_\varepsilon]$ and on $[R_\varepsilon,\infty)$, where $\ell_k^\pm \in \mathbf{Z}$, so that $\alpha_k^\pm := \theta_k(\pm R_\varepsilon) - \theta(\pm R_\varepsilon) \longrightarrow 0$ as $k \longrightarrow \infty$. We extend θ_k and θ as affine functions on $[-R_\varepsilon, R_\varepsilon]$. Then we have $\theta_k, \theta \in \dot H^1(\mathbf{R})$, and $\theta_k \longrightarrow \theta$, $e^{i\theta_k} \longrightarrow e^{i\theta}$ uniformly on $[-R_\varepsilon, R_\varepsilon]$ as well as in $H^1([-R_\varepsilon, R_\varepsilon])$.

We write

$$\tilde\psi_{n_k} = e^{i\theta_k} + w_k \qquad \text{and} \qquad \psi = e^{i\theta} + w,$$

where

$$w_k = \left(\breve{\tilde\psi}_{n_k} - e^{i\theta_k}\right) + \left(\tilde\psi_{n_k} - \breve{\tilde\psi}_{n_k}\right), \qquad \text{and} \qquad w = (\zeta - e^{i\theta}) + (\psi - \zeta).$$

It is clear that $w_k \in H^1(\mathbf{R}\times(0,1))$ and $\nabla w \in L^2(\mathbf{R}\times(0,1))$. On $(-\infty,-R_\varepsilon]$ and on $[R_\varepsilon,\infty)$ we have $\zeta - e^{i\theta} = (\rho-1)e^{i\theta} \in L^2((-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty))$ because $\zeta \in \mathcal{E}(\mathbf{R})$, and clearly $\zeta - e^{i\theta} \in L^\infty([-R_\varepsilon,R_\varepsilon]) \subset L^2([-R_\varepsilon,R_\varepsilon])$, hence $\zeta - e^{i\theta} \in L^2(\mathbf{R})$. We have $\tilde\psi_{n_k} - \breve{\tilde\psi}_{n_k} \longrightarrow \psi - \zeta$ almost everywhere on $\mathbf{R}^2$, and Lemma 3.2.4 gives

$$\|\tilde\psi_{n_k} - \breve{\tilde\psi}_{n_k}\|^2_{L^2(\mathbf{R}\times(0,1))} = \|v_{\tilde\psi_{n_k}}\|^2_{L^2(\mathbf{R}\times(0,1))} \leqslant C\Big\|\frac{\partial\tilde\psi_{n_k}}{\partial y}\Big\|^2_{L^2(\mathbf{R}\times(0,1))} \leqslant CM.$$

Then Fatou's Lemma implies that $\psi - \zeta \in L^2(\mathbf{R}\times(0,1))$, hence $w \in H^1(\mathbf{R}\times(0,1))$.

Denoting $v = \psi - \zeta$, an easy computation shows that on $((-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty))\times(0,1)$ we have

$$d(\theta,w) = d(\theta,(\rho-1)e^{i\theta}+v) = (1-\rho^2)\theta' + \Big\langle i\frac{\partial}{\partial x}(\psi + e^{i\theta}), v\Big\rangle + \langle i\frac{\partial v}{\partial x}, (\rho-1)e^{i\theta}\rangle.$$

We may estimate each term :

$$|(1-\rho^2)\theta'| \leqslant \frac{1}{2}|\theta'|^2 + \frac{1}{2}\left(1-\rho^2\right)^2,$$

$$\left|\left\langle i\frac{\partial}{\partial x}(\psi + e^{i\theta}), v\right\rangle\right| \leqslant \left(\left|\frac{\partial\psi}{\partial x}\right| + |\theta'|\right)|v| \leqslant \frac{1}{2}\left|\frac{\partial\psi}{\partial x}\right|^2 + \frac{1}{2}|\theta'|^2 + |v|^2,$$

$$|\langle i\frac{\partial v}{\partial x}, (\rho-1)e^{i\theta}\rangle| \leqslant \frac{1}{2}\left|\frac{\partial v}{\partial x}\right|^2 + \frac{1}{2}(\rho-1)^2 \leqslant \left|\frac{\partial\psi}{\partial x}\right|^2 + |\theta'|^2 + \frac{1}{2}(\rho^2-1)^2.$$

Since ψ and ζ satisfy (3.6.47) and (3.5.21), respectively, we find

$$\int_{((-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty))\times(0,1)} \left|d(\theta,w)\right| dx\, dy \leqslant C\varepsilon, \tag{3.6.53}$$

where C does not depend on ε.

Using the fact that $\tilde{\psi}_{n_k}$ and $\breve{\tilde{\psi}}_{n_k}$ satisfy (3.6.47) and (3.5.21), respectively, and proceeding similarly we find that θ_k and w_k also satisfy (3.6.53), with C independent of ε and of k.

On the other hand, we have

$$\begin{aligned}\int_{[-R_\varepsilon,R_\varepsilon]\times(0,1)} d(\theta_k,w_k)\, dx\, dy &= \int_{[-R_\varepsilon,R_\varepsilon]\times(0,1)} -2\langle\theta_k' e^{i\theta_k}, w_k\rangle + \langle i\frac{\partial w_k}{\partial x}, w_k\rangle\, dx\, dy \\ \longrightarrow \int_{[-R_\varepsilon,R_\varepsilon]\times(0,1)} -2\langle\theta' e^{i\theta}, w\rangle + \langle i\frac{\partial w}{\partial x}, w\rangle\, dx\, dy &= \int_{[-R_\varepsilon,R_\varepsilon]\times(0,1)} d(\theta,w)\, dx\, dy\end{aligned} \tag{3.6.54}$$

because $\theta_k' e^{i\theta_k} \rightharpoonup \theta' e^{i\theta}$ and $\frac{\partial w_k}{\partial x} \rightharpoonup \frac{\partial w}{\partial x}$ weakly in $L^2([-R_\varepsilon, R_\varepsilon]\times[0,1]$, while $w_k \longrightarrow w$ strongly in $L^2([-R_\varepsilon, R_\varepsilon]\times[0,1])$.

From (3.6.53) and (3.6.54) we deduce that for all sufficiently large k we have

$$|q(\theta_k,w_k) - q(\theta,w)| = \Big|\int_{\mathbf{R}\times(0,1)} d(\theta_k,w_k) - d(\theta,w)\, dx\, dy\Big| \leqslant (2C+1)\varepsilon.$$

Hence there exists $k(\varepsilon)\in\mathbf{N}$ such that for all $k \geqslant k(\varepsilon)$,

$$\left|\lfloor Q\rfloor(\tilde{\psi}_{n_k}) - \lfloor Q\rfloor(\psi)\right| \leqslant |q(\theta_k,w_k) - q(\theta,w)| \leqslant (2C+1)\varepsilon. \tag{3.6.55}$$

We have thus shown that $\lfloor Q\rfloor(\psi) = \lim\limits_{k\to\infty} \lfloor Q\rfloor(\tilde{\psi}_{n_k}) = \lfloor p\rfloor$. Therefore we must have

$$E_\lambda(\psi) \geqslant E_{\lambda,min}(p) = \lim_{k\to\infty} E_\lambda\left(\tilde{\psi}_{n_k}\right).$$

Comparing the above inequality to (3.6.49), (3.6.51) and (3.6.52) (with $I=\mathbf{R}$), we infer that necessarily

$$E_\lambda(\psi) = E_{\lambda,min}(p) = \lim_{k\to\infty} E_\lambda\left(\tilde{\psi}_{n_k}\right)$$

and

$$\int_{\mathbf{R}\times(0,1)} \Big|\frac{\partial\psi}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\psi}{\partial y}\Big|^2\,dx\,dy = \lim_{k\to\infty}\int_{\mathbf{R}\times(0,1)} \Big|\frac{\partial\tilde\psi_{n_k}}{\partial x}\Big|^2 + \lambda^2\Big|\frac{\partial\tilde\psi_{n_k}}{\partial y}\Big|^2\,dx\,dy, \tag{3.6.56}$$

$$\int_{\mathbf{R}\times(0,1)} V(|\psi|^2)\,dx\,dy = \lim_{k\to\infty}\int_{R\times(0,1)} V(|\tilde\psi_{n_k}|^2)\,dx\,dy. \tag{3.6.57}$$

The weak convergence $\nabla\tilde\psi_{n_k} \rightharpoonup \nabla\psi$ in $L^2(\mathbf{R}\times(0,1))$ and (3.6.56) give the strong convergence $\nabla\tilde\psi_{n_k} \longrightarrow \nabla\psi$ in $L^2(\mathbf{R}\times(0,1))$. The fact that $V\geqslant 0$, $V(|\tilde\psi_{n_k}|^2) \longrightarrow V(|\psi|^2)$ almost everywhere on $\mathbf{R}^2$ and (3.6.57) imply that $V(|\tilde\psi_{n_k}|^2) \longrightarrow V(|\psi|^2)$ in $L^1(\mathbf{R}\times(0,1))$.

Fix $\varepsilon>0$. Let R_ε be as in (3.6.47). Using (3.6.47) and Lemma 3.2.3 we find

$$\int_{((-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty))\times[0,1]} \big||\tilde\psi_{n_k}|-1\big|^2\,dx\,dy \leqslant CE_{GL,\lambda}^{(-\infty,-R_\varepsilon]\cup[R_\varepsilon,\infty)}(\tilde\psi_{n_k}) \leqslant \delta(\varepsilon), \tag{3.6.58}$$

where $\delta(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. It is obvious that a similar estimate holds for ψ. Since $\tilde\psi_{n_k}\longrightarrow\psi$ in $L^2([-R_\varepsilon,R_\varepsilon]\times[0,1])$, we have $\|\,|\tilde\psi_{n_k}|-|\psi|\|^2_{L^2(([-R_\varepsilon,R_\varepsilon]\times[0,1])} \leqslant \varepsilon$ for all sufficiently large k. It is obvious that $\Big(|\tilde\psi_{n_k}|-|\psi|\Big)^2 \leqslant \frac12\big|\tilde\psi_{n_k}|-1\big|^2 + \frac12\big|\,|\psi|-1\big|^2$ and using (3.6.58) we get

$$\|\,|\tilde\psi_{n_k}|-|\psi|\|^2_{L^2(\mathbf{R}\times[0,1])} \leqslant 2\delta(\varepsilon)+\varepsilon \tag{3.6.59}$$

for all sufficiently large k. We have thus shown that $|\tilde\psi_{n_k}|-|\psi|\longrightarrow 0$ in $L^2(\mathbf{R}\times[0,1])$. Since $\nabla\Big(|\tilde\psi_{n_k}|-|\psi|\Big)$ is bounded in $L^2(\mathbf{R}\times[0,1])$, using the Sobolev embedding and interpolation we infer that $|\tilde\psi_{n_k}|-|\psi|\longrightarrow 0$ in $L^p(\mathbf{R}\times[0,1])$ for any $p\in[2,\infty)$. □

Proposition 3.6.8. *Let $\lambda>0$ and $p\in(0,\pi]$. Assume that $\psi\in\mathcal{E}$ is a minimizer of E_λ in the set $\{\phi\in\mathcal{E}\mid |\lfloor Q\rfloor(\phi)|=p\}$. Then :*

(i) There is $c\in[(E_{\lambda,min})'_r(p),(E_{\lambda,min})'_\ell(p)]$ such that ψ satisfies

$$ic\psi_{x_1}+\Delta\psi+F(|\psi|^2)\psi=0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^2). \tag{3.6.60}$$

(ii) Any solution $\psi\in\mathcal{E}$ of (3.6.60) satisfies $\psi\in W^{2,p}_{loc}(\mathbf{R}^2)$ for any $p\in[2,\infty)$, ψ and $\nabla\psi$ are bounded and $\psi\in C^{1,\alpha}(\mathbf{R}^N)$ for any $\alpha\in[0,1)$.

(iii) If $c_1=(E_{\lambda,min})'_r(p)<(E_{\lambda,min})'_\ell(p)=c_2$, there exist $\psi_1,\psi_2\in\mathcal{E}$ such that each ψ_j is a minimizer of E_λ in the set $\{\phi\in\mathcal{E}\mid|\lfloor Q\rfloor(\phi)|=p\}$ and solves (3.6.60) with $c=c_j$ for $j=1,2$.

The proof of Proposition 3.6.8 is similar to the proof of Proposition 4.14 p. 187 in [6], so we omit it.

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