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MW-motivic cohomology of Stiefel varieties and linear algebraic groups

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THÈSE

Pour obtenir le grade de

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École doctorale : MSTII - Mathématiques, Sciences et technologies de l'information, Informatique

Spécialité : Mathématiques

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Cohomologie MW-motivique des variétés de Stiefel et des groupes algébriques linéaires

MW-motivic cohomology of Stiefel varieties and linear algebraic groups

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*There is no Bodhi-tree,
Nor stand of a mirror bright.
Since all is void,
Where can the dust alight?*

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General introduction

0.1 Background

Motivic Homotopy theory (also known as \mathbb{A}^1 -Homotopy theory) is a unique approach that applies the principles of algebraic topology, particularly homotopy theory, to algebraic varieties and schemes. This theory was first introduced by Fabien Morel and Vladimir Voevodsky and was inspired by Voevodsky's explorations leading to Motivic Cohomology [Morel1999]. The fundamental concept of \mathbb{A}^1 -Homotopy is that it seeks to develop a purely algebraic approach to homotopy theory by substituting the unit interval $[0, 1]$ with the affine line \mathbb{A}^1 .

One guiding intuition behind this approach is that the relationship between the \mathbb{A}^1 -homotopy theory over S and the category Sm_S of smooth S -schemes is of the same nature as the relationship between the classical homotopy theory and the category of differentiable manifolds [Morel2012a].

Thus, for a smooth S -scheme X , we can determine its "homotopy type" through the \mathbb{A}^1 -homotopy category $X \in \mathcal{H}(S)$, and further establish its "stable homotopy type" $\Sigma_{\mathbb{P}^1}^\infty X_+$ by using the stable \mathbb{A}^1 -homotopy category $\mathcal{S}\mathcal{H}(S)$. In this framework, there are several possible replacements of singular cohomology in topology. One of the most meaningful such replacements is the so-called MW-motivic cohomology (here, MW stands for Milnor-Witt), which can be computed using the MW-motive $\tilde{M}(X) \in \widetilde{\text{DM}}(S)$ [Bachmann2020], an object that can be considered as an analogue of the singular chain complex $C_*(X) \in D(\text{Ab})$. As a result, MW-motivic cohomology is defined as:

$$H_{\text{MW}}^{p,q}(X) := \text{Hom}_{\widetilde{\text{DM}}(S)}(\tilde{M}(X), \tilde{\mathbb{Z}}(q)[p]).$$

It is important to note that there are two types of shifts (q) and $[p]$, where $[p]$ arises from simplicial spheres, while the Tate twist (q) comes from the pointed scheme $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$. For example, the \mathbb{A}^1 -sphere $\mathbb{A}^n \setminus \{0\}$ has the homotopy type $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$, and thus its motive is of the form $\tilde{M}(\mathbb{A}^n \setminus \{0\}) = \tilde{\mathbb{Z}}(n)[2n-1]$. Further, one can define η to be the "Hopf map" $\mathbb{A}^2 \setminus \{0\} \cong S^1 \wedge \mathbb{G}_m^{\wedge 2} \rightarrow \mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m^{\wedge 1}$, which translates into a nontrivial morphism of MW-motives

$$\eta : \tilde{\mathbb{Z}}(1)[1] \rightarrow \tilde{\mathbb{Z}}.$$

Setting $\eta = 0$, leads to the degeneration of the MW-motivic cohomology to the motivic cohomology H_M that Voevodsky initially defined. The latter forms a crucial aspect of various prominent areas of algebraic geometry, like algebraic cycles, Chow groups, algebraic K theory, and the Langlands program, among others. On the other hand, MW-motivic cohomology offers additional geometry information like orientation.

Inverting η is an alternative way of reduction. This process leads to the construction of

$$H_\eta^n(X) := \bigoplus_q H_{\text{MW}}^{n+q,q}(X)[\eta^{-1}].$$

This reduction approach aligns more closely with the nature of the manifold of real points associated to a smooth scheme over \mathbb{R} which we will discuss later, and can be seen in a precise sense as an analogue of singular cohomology in topology.

Conventions

The letter K will always denote a perfect field of characteristic different from 2, while S will denote a fixed Noetherian base scheme over K . Further, we will denote by Sm/S the category of smooth schemes of finite type over S .

0.2 Introduction to motivic homotopy theory and MW-motivic cohomology

\mathbb{A}^1 -homotopy and stable \mathbb{A}^1 -homotopy categories. We define the \mathbb{A}^1 -homotopy category, denoted by $\mathcal{H}(S)$, as

$$\mathcal{H}(S) := L_{\mathrm{Nis}, \mathbb{A}^1} \mathrm{PShv}(\mathrm{Sm}/S)$$

Here, $\mathrm{PShv}(\mathrm{Sm}/S)$ denotes the category of presheaves of simplicial sets on Sm/S , and $L_{\mathrm{Nis}, \mathbb{A}^1}$ is the localization functor with respect to the Nisnevich topology and \mathbb{A}^1 -invariance.

The stable \mathbb{A}^1 -homotopy category, denoted by $\mathcal{S}\mathcal{H}(S)$, is obtained by stabilizing the \mathbb{A}^1 -homotopy category with respect to $\Omega_{\mathbb{P}^1}$:

$$\Omega_{\mathbb{P}^1}^\infty : \mathcal{S}\mathcal{H}(S) \rightleftarrows \mathcal{H}_*(S) : \Sigma_{\mathbb{P}^1}^\infty$$

Cohomology theories. Given a spectrum $E \in \mathcal{S}\mathcal{H}(S)$, we can define a bigraded cohomology theory for smooth S -schemes, as follows:

$$E^{p,q}(X) := [\Sigma_{\mathbb{P}^1}^\infty X_+, \Sigma^{p,q} E]_{\mathcal{S}\mathcal{H}(S)}$$

where $\Sigma^{p,q} E := S^{p,q} \wedge E = S^{p-q} \wedge \mathbb{G}_m^q \wedge E$.

In our analysis, we focus on ring spectra, i.e. spectra endowed with a multiplication $m : E \wedge E \rightarrow E$ and unit map $\mathbf{1} \rightarrow E$ satisfying the usual axioms. This multiplication induces a cup product $\cup : E^{*,*}(X) \otimes E^{*,*}(X) \rightarrow E^{*,*}(X)$. It is worth noting that $E^{*,*}(X)$ is a bigraded $(-1, \langle -1 \rangle)$ -commutative ring, where $\langle -1 \rangle = 1 + \eta[-1] \in \mathbf{K}_0^{\mathrm{MW}}$. That is, for any $a \in E^{p,q}(X)$ and $b \in E^{s,t}(X)$, we have $a \cup b = (-1)^{p+s} \langle -1 \rangle^{q+t} b \cup a$. All the cohomology rings we compute here are bigraded $(-1, \langle -1 \rangle)$ -commutative rings.

Milnor-Witt motives. We define Milnor-Witt motivic cohomology and state some properties that will be used in the proof of Theorem 0.3.1. We start with the (big) triangulated category of motives $\widetilde{\mathrm{DM}}(S) := \widetilde{\mathrm{DM}}_{\mathrm{Nis}}(S, \mathbb{Z})$ defined in [Déglise2016, Definition 3.3.2]. We also just denote $\widetilde{\mathrm{DM}}(S)$ by $\widetilde{\mathrm{DM}}$ if S is specified. The category of Milnor-Witt motives $\widetilde{\mathrm{DM}}(S)$ is equipped with a pair of adjoint functors

$$\widetilde{\gamma}_* : \mathcal{S}\mathcal{H}(S) \rightleftarrows \widetilde{\mathrm{DM}} : \widetilde{\gamma}^*$$

We will make use of the following property of $\widetilde{\mathrm{DM}}$.

Proposition 0.2.1 (Gysin Triangle). *Let X be a smooth K -scheme, let $Z \subset X$ be a smooth closed subscheme of codimension c and let $U = X \setminus Z$. Suppose that the normal cone $N_X Z$ admits a trivialization $\phi : N_X Z \cong Z \times \mathbb{A}^c$. Then, there is a Gysin triangle*

$$\widetilde{\mathrm{M}}(U) \rightarrow \widetilde{\mathrm{M}}(X) \rightarrow \widetilde{\mathrm{M}}(Z)(c)[2c] \xrightarrow{+1}$$

where the last two arrows depend on the choice of ϕ .

Proof. We have an adjunction of triangulated categories

$$\tilde{\gamma}_* : \mathcal{S}\mathcal{H}(S) \rightleftarrows \widetilde{\mathbf{DM}} : \tilde{\gamma}^*$$

obtained by combining the adjunction of [Déglise2017, §4.1] and the classical Dold-Kan correspondence (e.g. [Cisinski2019, 5.3.35]). The functor $\mathcal{S}\mathcal{H}(S) \rightarrow \widetilde{\mathbf{DM}}$ being exact, the statement follows for instance from [Morel1999, Chapter 3, Theorem 2.23]. \square

We have a functor

$$\widetilde{\mathbf{M}} : \mathbf{Sm}/S \rightarrow \widetilde{\mathbf{DM}}$$

given by $\widetilde{\mathbf{M}}(X) := \tilde{\gamma}_*(\Sigma_{\mathbb{P}^1}^\infty X_+)$. The category $\widetilde{\mathbf{DM}}$ is symmetric monoidal [Déglise2016, Proposition 3.3.4] with unit $\widetilde{\mathbb{Z}} := \widetilde{\mathbf{M}}(S)$. As written above, the Milnor-Witt motivic spectrum $\mathbf{H}_{\mathbf{MW}}$ can be defined as $\mathbf{H}_{\mathbf{MW}} := \tilde{\gamma}^*(\widetilde{\mathbb{Z}})$. Let $\widetilde{\mathbb{Z}}(q)[p] := \tilde{\gamma}_*(S^{p,q})$ denote the Tate twists and shifts, we obtain Milnor-Witt motivic cohomology groups

$$\mathbf{H}_{\mathbf{MW}}^{p,q}(X) := [\Sigma_{\mathbb{P}^1}^\infty X_+, \Sigma^{p,q} E]_{\mathcal{S}\mathcal{H}(S)} \cong \mathrm{Hom}_{\widetilde{\mathbf{DM}}(S)}(\widetilde{\mathbf{M}}(X), \widetilde{\mathbb{Z}}(q)[p]).$$

By [Déglise2016, Proposition 4.1.2], motivic cohomology groups can be computed as the Zariski hypercohomology groups of explicit complexes of sheaves.

There are homomorphisms (functorial in X) for any $p, q \in \mathbb{Z}$

$$\mathbf{H}_{\mathbf{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow \mathbf{H}_{\mathbf{M}}^{p,q}(X, \mathbb{Z})$$

where the right-hand side denotes the ordinary motivic cohomology defined by Voevodsky.

As illustrated by the list of properties below, the Milnor-Witt motivic cohomology groups behave in a fashion similar to ordinary motivic cohomology groups, even though there are some central differences (for instance, there are no reasonable Chern classes).

In a tensor category \mathcal{C} , we have the flexibility of omitting the tensor unit when expressing tensor products, denoted as $A \otimes B := A \otimes_{\mathbb{1}} B$. In the context of $\widetilde{\mathbf{DM}}(S)$, the tensor unit takes the form of $\widetilde{\mathbf{M}}(S)$. Hence, we can simply write $A \otimes B$ for $A \otimes_{\widetilde{\mathbf{M}}(S)} B$.

Properties of Milnor-Witt motivic cohomology. The Milnor-Witt motivic cohomology groups satisfy most of the formal properties of ordinary motivic cohomology and were computed in a couple of situations:

1. If $q \leq 1$, there are canonical isomorphisms

$$\mathbf{H}_{\mathbf{MW}}^{p,q}(X, \mathbb{Z}) \cong \mathbf{H}_{\mathbf{Nis}}^{p-q}(X, \mathbf{K}_q^{\mathbf{MW}}) \cong \mathbf{H}_{\mathbf{Zar}}^{p-q}(X, \mathbf{K}_q^{\mathbf{MW}})$$

where $\mathbf{K}_q^{\mathbf{MW}}$ is the unramified Milnor-Witt K -theory sheaf (in weight q) introduced in [Morel2012a].

Besides, in the case $q < 0$, we obtain isomorphisms

$$\mathbf{H}_{\mathbf{MW}}^{p,q}(X, \mathbb{Z}) \cong \mathbf{H}^{p-q}(X, \mathbf{K}_q^{\mathbf{MW}}) = \mathbf{H}^{p-q}(X, \mathbf{W})$$

where \mathbf{W} represents the Witt sheaf.

2. If L/K is a finitely generated field extension there are isomorphisms $\mathbf{H}_{\mathbf{MW}}^{n,n}(L, \mathbb{Z}) \cong \mathbf{K}_n^{\mathbf{MW}}(L)$ fitting in a commutative diagram for any $n \in \mathbb{Z}$

$$\begin{array}{ccc} \mathbf{H}_{\mathbf{MW}}^{n,n}(L, \mathbb{Z}) & \xrightarrow{\sim} & \mathbf{K}_n^{\mathbf{MW}}(L) \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathbf{M}}^{n,n}(L, \mathbb{Z}) & \xrightarrow{\sim} & \mathbf{K}_n^{\mathbf{M}}(L) \end{array}$$

where $K_n^M(L)$ is the (n -th) Milnor K -theory group of L , the bottom horizontal map is the isomorphism of Suslin-Nesterenko-Totaro, and the right-hand vertical map is the natural homomorphism from Milnor-Witt K -theory to Milnor K -theory. This result has the following consequence: The Milnor-Witt motivic cohomology groups are computed via an explicit complex of Nisnevich sheaves $\widetilde{\mathbb{Z}}(q)$ for any integer $q \in \mathbb{Z}$. The above result shows that there is a morphism of complexes of sheaves

$$\widetilde{\mathbb{Z}}(q) \rightarrow \mathbf{K}_q^{\text{MW}}[-q]$$

where the right-hand side is the complex whose only non-trivial sheaf is \mathbf{K}_q^{MW} in degree $-q$. For any essentially smooth scheme X over K , this yields group homomorphisms

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H^{p-q}(X, \mathbf{K}_q^{\text{MW}})$$

which are compatible with the ring structure on both sides.

In the particular case $p = 2n, q = n$ for some $n \in \mathbb{Z}$, we obtain isomorphisms (functorial in X)

$$H_{\text{MW}}^{2n,n}(X, \mathbb{Z}) \xrightarrow{\sim} H^n(X, \mathbf{K}_n^{\text{MW}}) \xrightarrow{\sim} \widetilde{\text{CH}}^n(X)$$

where the right-hand term is the n -th Chow-Witt group of X (defined in [Barge2000] and [Fasel2008]). Again, these isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H_{\text{MW}}^{2n,n}(X, \mathbb{Z}) & \xrightarrow{\sim} & \widetilde{\text{CH}}^n(X) \\ \downarrow & & \downarrow \\ H_{\text{M}}^{2n,n}(X, \mathbb{Z}) & \xrightarrow{\sim} & \text{CH}^n(X) \end{array}$$

where the right-hand vertical homomorphism is the natural map from Chow-Witt groups to Chow groups.

3. The total Milnor-Witt motivic cohomology has Borel classes for symplectic bundles [Yang2019] but in general the projective bundle theorem fails [Yang2020].
4. If X is a smooth scheme over \mathbb{R} , there are two interesting realization maps. On the one hand, one may consider the composite

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H_{\text{M}}^{p,q}(X, \mathbb{Z}) \rightarrow H_{\text{sing}}^p(X(\mathbb{C}), \mathbb{Z})$$

where the right-hand map is the complex realization map. On the other hand, one may also consider the following composite

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H^{p-q}(X, \mathbf{K}_q^{\text{MW}}) \rightarrow H^{p-q}(X, \mathbf{I}^q) \rightarrow H_{\text{sing}}^{p-q}(X(\mathbb{R}), \mathbb{Z})$$

where \mathbf{I}^q is the unramified sheaf associated to the q -th power of the fundamental ideal in the Witt ring, $\mathbf{K}_q^{\text{MW}} \rightarrow \mathbf{I}^q$ is the canonical projection and $H^{p-q}(X, \mathbf{I}^q) \rightarrow H_{\text{sing}}^{p-q}(X(\mathbb{R}), \mathbb{Z})$ is Jacobson's signature map [Jacobson2017].

We note here that these two realization maps show that Milnor-Witt motivic cohomology is in some sense the analogue of both the singular cohomology of the complex and the real points of X .

η -inverted cohomology. Consider once again the map $\eta : S^{3,2} \cong \Sigma_{\mathbb{P}^1}^\infty \mathbb{A}^2 \setminus \{0\} \rightarrow \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1 \cong S^{2,1}$, or more precisely the induced map

$$\eta : \tilde{\mathbb{Z}}(1)[1] \rightarrow \tilde{\mathbb{Z}}$$

known as the \mathbb{A}^1 -Hopf map. By definition, η induces a morphism $\eta : E^{p,q}(X) \rightarrow E^{p-1,q-1}(X)$ for any (ring) spectrum E and we may localize with respect to η to obtain the η -inverted cohomology theory $E[\eta^{-1}]$, which has the property that the multiplication by

$$\eta : E[\eta^{-1}]^{p,q}(X) \xrightarrow{\sim} E[\eta^{-1}]^{p-1,q-1}(X)$$

an isomorphism.

In the case of H_{MW} , since $X \mapsto H_{\text{MW}}^{p,q}(X)[\eta^{-1}] \cong H^{p-q}(X, \mathbf{W})$ for $q < 0$, we see that $H_\eta := H_{\text{MW}}[\eta^{-1}]$ represents the cohomology with coefficients in the Witt sheaf, in the sense that

$$[\Sigma_{\mathbb{P}^1}^\infty X_+, \Sigma^{p,q} H_\eta]_{\mathcal{S}\mathcal{H}(S)} = H^{p-q}(X, \mathbf{W}).$$

More generally, for an η -inverted spectrum E_η , we obtain single graded cohomology groups $E_\eta^*(X) := \bigoplus_{n \in \mathbb{Z}} E_\eta^{n+*,n}(X) \cong E_\eta^{*,0}(X)[\eta, \eta^{-1}]$.

Real realization. There is a close relationship between the stable homotopy category over \mathbb{R} or \mathbb{C} and its classical analogue in topology, via the realization functors. First, consider the functor $\text{Sm}_{\mathbb{R}} \rightarrow \text{Top}$ (resp. $\text{Sm}_{\mathbb{C}} \rightarrow \text{Top}$) mapping a smooth scheme X to the set $X(\mathbb{R})$ endowed with the Euclidean topology (resp. the set $X(\mathbb{C})$ endowed with the analytic topology). These two functors can then be extended to the stable homotopy category $\mathcal{S}\mathcal{H}(K) \rightarrow \text{SH}$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. For the real realization, \mathbb{G}_m is sent to two points, specifically S^0 , which trivializes Tate twists. It is noteworthy that inverting the morphism $\rho : S^0 \rightarrow \mathbb{G}_m$ in $\mathcal{S}\mathcal{H}(\mathbb{R})$ retrieves the usual stable homotopy category in the sense that the realization functor discussed above induces an equivalence $\mathcal{S}\mathcal{H}(\mathbb{R})(\rho^{-1}) \rightarrow \text{SH}$ [Bachmann2018]. Furthermore, it is worth mentioning that this real realization bears close resemblance with the η inverted case, where inverting the morphism ρ after inverting 2, is equivalent to inverting η .

Milnor-Witt cycle module. We can construct a t-structure δ on $\mathcal{S}\mathcal{H}(S)$ [Bondarko2017], whose heart can be understood using either homotopy modules [Déglise2011] or equivalently (cohomological) Milnor-Witt (MW)-cycle modules ([Feld2020] or [Déglise2022]).

We denote by \mathcal{F}_S the category of spectra of fields essentially of finite type over S . A (cohomological) Milnor-Witt cycle module is a functor from $\mathcal{F}_S^{\text{op}}$ to the category of \mathbb{Z} -graded abelian groups $\text{Abel}^{\mathbb{Z}}$

$$M_* : \mathcal{F}_S \rightarrow \text{Abel}^{\mathbb{Z}}$$

satisfying a list of axioms. The axioms of M_* include an action by Milnor-Witt K -theory K_*^{MW} , which is the fundamental example of a MW-cycle module.

Let R be a commutative ring over S . The Milnor-Witt K -theory $K_*^{\text{MW}}(R)$ of R is defined to be the graded algebra freely generated by elements of degree 1 of the form $[a], a \in R^\times$ and an element η in degree -1 , subject to the relations:

1. $[a][1-a] = 0$ for any a such that $a, 1-a \in R^\times \setminus \{1\}$.
2. $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in R^\times$.
3. $\eta[a] = [a]\eta$ for any $a \in R^\times$.
4. $\eta(2 + \eta[-1]) = 0$.

It defines an MW-cycle module via $(x : \text{Spec } F \rightarrow S) \in \mathcal{F}_S \mapsto \mathbf{K}_*^{\text{MW}}(F)$ and a homotopy module via $X \mapsto \mathbf{K}_*^{\text{MW}}(\mathcal{O}(X))$, where the right-hand term denotes the unramified Milnor-Witt K -theory, or alternatively the sheaf associated to the above presheaf. On the other hand, one can also consider the Milnor-Witt motivic cohomology (bigraded) presheaf

$$X \mapsto \mathbf{H}_{\text{MW}}(X) := \bigoplus_{p,q} \mathbf{H}_{\text{MW}}^{p,q}(X, \mathbb{Z})$$

By [Déglise2016, Theorem 4.2.2], there is a morphism of presheaves

$$s : \bigoplus_{n \in \mathbb{Z}} \mathbf{K}_n^{\text{MW}}(-) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{n,n}(-, \mathbb{Z}) \subset \mathbf{H}_{\text{MW}}(X) \quad (1)$$

which specializes to the above isomorphism if $X = \text{Spec}(F)$ where F is a finitely generated field extension of K [Calmès2017].

Homotopy Leray spectral sequences. To compute cohomology using fiber sequences, we can use the Leray spectral sequence in the \mathbb{A}^1 -setting by making use of the t-structure δ [Asok2018].

Let H_δ^p be the cohomology of the t-structure δ , which takes value in the heart, i.e. MW-cycle modules. In general, for any spectrum $\mathbb{F} \in \mathcal{S}\mathcal{H}(S)$, a spectral sequence can be obtained using the theory of spectral diagrams:

$$E_2^{p,q}(\mathbb{E}) = {}^\delta A^p(S, H_\delta^q \mathbb{F}) \Rightarrow H^{p+q}(S, \mathbb{F})$$

When applied to the push-forward $f_* \mathbb{E}$, we obtain:

Theorem 0.2.2 ([Asok2018], Theorem 4.2.5). *Let $f : T \rightarrow B$ be a morphism of schemes, and let $\mathbb{E} \in \mathcal{S}\mathcal{H}(T)$ be a ring spectrum over T , we have a spectral sequence of the form:*

$$E_2^{p,q}(f, \mathbb{E}) = {}^\delta A^p(B, H_\delta^q f_* \mathbb{E}) \Rightarrow H^{p+q}(T, \mathbb{E})$$

where the differentials d_r satisfy the usual Leibniz rule.

0.3 Main Results

For a scheme X over S , we write $\mathbf{H}_{\text{MW}}(X)$ or $\mathbf{H}_{\text{MW}}^{*,*}(X)$ for the total MW-motivic cohomology ring $\bigoplus_{p,q \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{p,q}(X, \mathbb{Z})$.

Let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes. In Chapter 1, we compute the total Milnor-Witt cohomology ring

$$\mathbf{H}_{\text{MW}}(U) = \bigoplus_{p,q} \mathbf{H}_{\text{MW}}^{p,q}(U, \mathbb{Z})$$

using methods very similar to [Chatzistamatiou2007], with however some necessary modifications.

For all $f \in \mathbb{G}_m(U)$, we use (f) to indicate the corresponding generator in the corresponding free algebras (e.g. $\mathbf{K}_n^{\text{MW}}(K)\{\mathbb{G}_m(U)\}$) and $[f]$ to indicate the corresponding elements in the cohomology groups (e.g. $\mathbf{H}_{\text{MW}}^{1,1}(U, \mathbb{Z})$).

Theorem 0.3.1. *Let K be a perfect field of characteristic different from 2 and let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes. There is an isomorphism of $\mathbf{H}_{\text{MW}}(K)$ -algebras*

$$\mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \cong \mathbf{H}_{\text{MW}}(U)$$

defined by mapping $(f) \in \mathbb{G}_m(U)$ to the class $[f]$ in $\mathbf{H}_{\text{MW}}^{1,1}(U, \mathbb{Z})$ corresponding to f under the morphism s of (1). Here, $\mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}$ is the free (associative) graded $\mathbf{H}_{\text{MW}}(K)$ -algebra generated by $\mathbb{G}_m(U)$ in degree $(1, 1)$ and J_U is the ideal generated by the following elements:

- (1) $(f) - [f]$, if $f \in K^\times \subset \mathbb{G}_m(U)$,
- (2) $(f) + (g) + \eta(f)(g) - (fg)$, if $f, g \in \mathbb{G}_m(U)$,
- (3) $(f_1)(f_2) \cdots (f_t)$, for any $f_1, \dots, f_t \in \mathbb{G}_m(U)$ such that $\sum_{i=1}^t f_i = 1$,
- (4) $(f)^2 - [-1](f)$, if $f \in \mathbb{G}_m(U)$.

As indicated above, this theorem and its proof are inspired from the computation of the (ordinary) motivic cohomology of U in [Chatzistamatiou2007]. We can recover the main theorem [Chatzistamatiou2007, Theorem 3.5] of the motivic cohomology case by taking $\eta = 0$. As a corollary, we obtain the following result.

Corollary 0.3.2. *Let $U \subset \mathbb{A}^N$ be the complement of a finite union of hyperplanes. The isomorphism of Theorem 0.3.1 induces an isomorphism*

$$\bigoplus_{n \in \mathbb{Z}} K_n^{\text{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \rightarrow \bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(U, \mathbb{Z}).$$

We do not know if the left-hand side coincides with $K_*^{\text{MW}}(U)$. At the end of the chapter, we spend a few lines on the real realization homomorphism

$$H_{\text{MW}}(U, \mathbb{Z}) \rightarrow H_{\text{sing}}^*(U(\mathbb{R}), \mathbb{Z})$$

when U is over $K = \mathbb{R}$. We prove in particular that both sides have essentially the same generators, and that the map is surjective.

Chapter 2 and 4 aim to provide a comprehensive overview of the interplay between Milnor-Witt motivic cohomology and linear algebraic groups and to demonstrate the usefulness of Milnor-Witt motivic cohomology in understanding the properties and structure of these groups. Linear algebraic groups play a fundamental role in various mathematical disciplines. In order to study them, cohomology has emerged as a critical invariant. In this part, we compute the cohomology of an algebraic group G , seen as a variety, using the Milnor-Witt motivic cohomology H_{MW} instead of employing the more commonly used group cohomology of the classifying space BG , although the cohomology of G and BG are connected.

In the sequel, recall that $\tilde{M}(X)$ denotes the Milnor-Witt motive of a scheme X . If $x \in X(S)$, we denote by $\tilde{M}(X, x)$ the reduced Milnor-Witt motive of X . For instance, if $n \geq 1$, we obtain for $\mathbb{A}^n \setminus \{0\} := \mathbb{A}_S^n \setminus \{0\}$ the following motives

$$\tilde{M}(\mathbb{A}^n \setminus \{0\}) = \tilde{M}(S) \oplus \tilde{M}(S)(n)[2n-1],$$

while $\tilde{M}(\mathbb{A}^n \setminus \{0\}, x) = \tilde{M}(S)(n)[2n-1]$ for any S -rational point x .

For the original motivic cohomology, i.e. for the case $\eta = 0$, the computations and motivic decompositions of split reductive groups [Biglari2012] and Stiefel varieties [Williams2012] in motivic cohomology are well known. However, there are at present only limited results for Milnor-Witt motivic cohomology, or even for η -inverted Milnor-Witt motivic cohomology, and the main goal of this part is to fill this gap.

In Chapter 2, we compute the Milnor-Witt motivic decomposition of Sp_{2n} in $\widetilde{\text{DM}}(S)$, over any reasonable base S , by utilizing the Sp -orientation on Milnor-Witt motivic cohomology. The final result is the following.

Theorem 0.3.3 (Theorem 2.3.4). *For any $n \geq 1$, We have an isomorphism in $\widetilde{\text{DM}}(S)$*

$$\tilde{M}(\text{Sp}_{2n}) \cong \tilde{M}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \tilde{M}(\text{Sp}_{2n-2}),$$

yielding a decomposition of the form

$$\tilde{M}(\text{Sp}_{2n}) \cong \tilde{M}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \tilde{M}(\mathbb{A}^{2n-2} \setminus \{0\}) \otimes \dots \otimes \tilde{M}(\mathbb{A}^2 \setminus \{0\}) \cong \bigoplus_{1 \leq i_1 < \dots < i_j \leq n} \tilde{\mathbb{Z}}(d(i_1, \dots, i_j)).$$

where $\tilde{\mathbb{Z}}(d(i_1, \dots, i_j)) := \tilde{\mathbb{Z}}(\sum_{l=1}^j 2i_l)[4(\sum_{l=1}^j i_l) - j]$.

Next, we address the computation for Stiefel varieties and in particular for GL_n in Chapter 4. As already stated before, one of the fundamental difference between Milnor-Witt motivic cohomology and its more classical version is that the former is not GL -oriented (i.e. doesn't have a reasonable theory of Chern classes), thus making the computations more delicate. For this chapter, the η -inverted Milnor-Witt cohomology H_η provides the most effective mean of computations. This allows us to parallel the classical computations for both O_n and SO_n [Hatcher2002]. Using the Leray spectral sequence, we can inductively compute for all Stiefel varieties. In the next statement, we write $V_k(\mathbb{A}^n)$ for the Stiefel variety of full rank $k \times n$ matrices, where $k \leq n$, and we set

$$\begin{aligned}\mathbf{HS}_{2k} &:= \tilde{M}_\eta(S) \oplus \tilde{M}_\eta(S)(2k)[4k-1], \\ \mathbf{HS}_{2k+1} &:= \tilde{M}_\eta(S) \oplus \tilde{M}_\eta(S)(4k+1)[8k].\end{aligned}$$

Here, let us note once again that $\tilde{M}_\eta(S)(1)[1]$ is canonically isomorphic to $\tilde{M}_\eta(S)$, via multiplication by η , and we could have therefore set

$$\begin{aligned}\mathbf{HS}_{2k} &\cong \tilde{M}_\eta(S) \oplus \tilde{M}_\eta(S)[2k-1], \\ \mathbf{HS}_{2k+1} &\cong \tilde{M}_\eta(S) \oplus \tilde{M}_\eta(S)[4k-1].\end{aligned}$$

We however prefer to keep the previous notation, mainly since we plan to compute the integral motive of Stiefel varieties in $\widetilde{DM}(S)$ in the close future.

Theorem 0.3.4 (Theorem 3.0.1). *We have an isomorphism of graded algebras*

$$H_\eta^*(V_k(\mathbb{A}^n)) \cong H_\eta^*(S)[G]/(g^2 - \delta_{\gamma_0}(g)[-1]g, g \in G)$$

where the generators G are given by

$$G = \begin{cases} G_1 \sqcup G_2 \sqcup G_3 & n \text{ even, } n-k \text{ even,} \\ G_2 \sqcup G_3 & n \text{ odd, } n-k \text{ even,} \\ G_1 \sqcup G_2 & n \text{ even, } n-k \text{ odd,} \\ G_2 & n \text{ odd, } n-k \text{ odd,} \end{cases}$$

for

$$\begin{aligned}G_1 &= \{\alpha_{n-1} \in H_\eta^{n-1}(\mathbf{HS}_n)\} \\ G_2 &= \{\beta_{4j-1} \in H_\eta^{4j-1}(\mathbf{HS}_{2j+1}) \mid n-k < 2j < n\} \\ G_3 &= \{\gamma_{n-k} \in H_\eta^{n-k}(\mathbb{A}^{n-k+1} \setminus \{0\})\}\end{aligned}$$

And $\delta_{\gamma_0}(\gamma_0) = 1$, otherwise $\delta_{\gamma_0}(g) = 0$.

As a consequence, we obtain the following theorem.

Theorem 0.3.5 (Theorem 3.0.2). *We have a motivic decomposition of the following form for any $i, j \in \mathbb{N}$:*

$$\begin{aligned}\tilde{M}_\eta(V_{2j}(\mathbb{A}^{2i})) &\cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-2)} \otimes \tilde{M}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\}) \\ \tilde{M}_\eta(V_{2j+1}(\mathbb{A}^{2i})) &\cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-1)} \\ \tilde{M}_\eta(V_{2j}(\mathbb{A}^{2i+1})) &\cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)} \\ \tilde{M}_\eta(V_{2j+1}(\mathbb{A}^{2i+1})) &\cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)} \otimes \tilde{M}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\}).\end{aligned}$$

One easy observation is that all these generators coincide with classical generators of the cohomology groups with coefficients in $\mathbb{Z}[1/2]$ of the Stiefel manifolds, and that they are indeed sent to the latter under the real realization functor discussed above.

Introduction générale

0.1 Contexte

La théorie de l'homotopie motivique (également connue sous le nom de théorie \mathbb{A}^1 -homotopique) est une approche originale qui applique les principes de la topologie algébrique, en particulier de l'homotopie, aux variétés et aux schémas algébriques. Cette théorie a été introduite pour la première fois par Fabien Morel et Vladimir Voevodsky, et elle a été inspirée par le développement de la cohomologie motivique par Voevodsky [Morel1999]. Le concept fondamental de la théorie de l'homotopie motivique est qu'elle cherche à développer une approche purement algébrique de la théorie de l'homotopie en remplaçant l'intervalle unité $[0, 1]$ par la droite affine \mathbb{A}^1 .

Une intuition directrice derrière cette approche est que la relation entre la théorie de l'homotopie motivique sur une base S raisonnable et la catégorie Sm_S des schémas lisses S est de même nature que la relation entre la théorie classique de l'homotopie et la catégorie des variétés différentiables [Morel2012a].

Ainsi, pour un S -schéma X lisse, nous pouvons déterminer son "type d'homotopie" au travers de sa classe $X \in \mathcal{H}(S)$ dans la catégorie de l'homotopie motivique sur S , et établir en outre son "type d'homotopie stable" $\Sigma_{\mathbb{P}^1}^\infty X_+$ en utilisant la catégorie \mathbb{A}^1 -homotopique stable $\mathcal{S}\mathcal{H}(S)$. Dans ce cadre, il y a plusieurs manières possibles de remplacer la cohomologie singulière par une théorie purement motivique. L'un des approches les plus fertiles est donné par la cohomologie MW-motivique (où MW est une abréviation de Milnor-Witt), qui peut être calculée en utilisant le MW-motif $\widetilde{M}(X) \in \widetilde{DM}(S)$ [Bachmann2020], un objet qui est un remplacement du complexe de chaînes singulier $C_*(X) \in D(\mathrm{Ab})$ associé à X . Dans ce contexte, la cohomologie MW-motivique peut être définie par

$$H_{\mathrm{MW}}^{p,q}(X) := \mathrm{Hom}_{\widetilde{DM}(S)}(\widetilde{M}(X), \widetilde{Z}(q)[p]).$$

Il est important de noter qu'il existe deux types de suspensions (q) et $[p]$, où $[p]$ provient des sphères simpliciales, tandis que la torsion de Tate (q) provient du schéma pointé $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$. Par exemple, l' \mathbb{A}^1 -sphère $\mathbb{A}^n \setminus \{0\}$ a pour type d'homotopie $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$, et il s'ensuit que $\widetilde{M}(\mathbb{A}^n \setminus \{0\}) = \widetilde{Z}(n)[2n-1]$. De plus, on peut considérer l'application de Hopf $\mathbb{A}^2 \setminus \{0\} \cong S^1 \wedge \mathbb{G}_m^{\wedge 2} \rightarrow \mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m^{\wedge 1}$, qu'on peut voir comme un morphisme non trivial

$$\eta : \widetilde{Z}(1)[1] \rightarrow \widetilde{Z}.$$

Poser $\eta = 0$ mène à la dégénérescence de la MW-cohomologie motivique en la cohomologie motivique H_M que Voevodsky a initialement définie. Cette dernière forme un objet central dans divers domaines importants de la géométrie algébrique, comme les cycles algébriques, les groupes de Chow, la K-théorie algébrique et le programme de Langlands, entre autres. D'un autre côté, la cohomologie motivique MW offre des informations géométriques supplémentaires telles que l'orientation.

L'inversion de η est une autre méthode de réduction. Ce processus conduit à la construction de

$$H_\eta^n(X) := \bigoplus_q H_{\mathrm{MW}}^{n+q,q}(X)[\eta^{-1}].$$

Cette méthode de réduction est plus en accord avec la nature de la variété des points réels associée à un schéma réel que nous discuterons plus tard, et peut être considérée en un sens précis comme un analogue de la cohomologie singulière en topologie.

Conventions. La lettre K désignera toujours un corps parfait de caractéristique différente de 2, tandis que S désignera un schéma de base noethérien fixe sur K . De plus, nous noterons par Sm/S la catégorie des schémas lisses de type fini sur S .

0.2 Introduction à la théorie de l'homotopie motivique et à la cohomologie MW-motivique

Les catégories \mathbb{A}^1 -homotopique et \mathbb{A}^1 -homotopique stable. Nous définissons la catégorie d'homotopie motivique, notée $\mathcal{H}(S)$, de la manière suivante :

$$\mathcal{H}(S) := L_{\mathrm{Nis}, \mathbb{A}^1} \mathrm{PShv}(\mathrm{Sm}/S)$$

Ici, $\mathrm{PShv}(\mathrm{Sm}/S)$ désigne la catégorie des préfaisceaux en ensembles simpliciaux sur Sm/S et $L_{\mathrm{Nis}, \mathbb{A}^1}$ est le foncteur de localisation par rapport à la topologie de Nisnevich et l' \mathbb{A}^1 -invariance.

La catégorie \mathbb{A}^1 -homotopique stable, notée $\mathcal{S}\mathcal{H}(S)$, est obtenue en stabilisant la catégorie \mathbb{A}^1 -homotopique par rapport à $\Omega_{\mathbb{P}^1}$:

$$\Omega_{\mathbb{P}^1}^\infty : \mathcal{S}\mathcal{H}(S) \rightleftarrows \mathcal{H}_*(S) : \Sigma_{\mathbb{P}^1}^\infty$$

Théorie de cohomologie. Étant donné un spectre $E \in \mathcal{S}\mathcal{H}(S)$, nous pouvons définir une théorie cohomologique à deux variables pour les S -schémas lisses, comme suit :

$$E^{p,q}(X) = [\Sigma_{\mathbb{P}^1}^\infty X_+, \Sigma^{p,q} E]_{\mathcal{S}\mathcal{H}(S)}$$

avec $\Sigma^{p,q} E := S^{p,q} \wedge E = S^{p-q} \wedge \mathbb{G}_m^q \wedge E$.

Dans notre analyse, nous nous concentrons sur les spectres en anneaux, c'est-à-dire les spectres dotés d'une multiplication $m : E \wedge E \rightarrow E$ et d'une unité $\mathbf{1} \rightarrow E$ satisfaisant les axiomes habituels. Cette multiplication induit un produit $\cup : E^{*,*}(X) \otimes E^{*,*}(X) \rightarrow E^{*,*}(X)$. Il convient de noter que $E^{*,*}(X)$ est un anneau commutatif bigradué de degré $(-1, \langle -1 \rangle)$, où $\langle -1 \rangle = 1 + \eta[-1] \in \mathbb{K}_0^{\mathrm{MW}}$. Autrement dit, pour tout $a \in E^{p,q}(X)$ et $b \in E^{s,t}(X)$, on a $a \cup b = (-1)^{p+s} \langle -1 \rangle^{q+t} b \cup a$. Tous les anneaux de cohomologie que nous calculons ici sont des anneaux commutatifs bigradués de degré $(-1, \langle -1 \rangle)$.

Motifs de Milnor-Witt Nous définissons la cohomologie motivique de Milnor-Witt et énonçons certaines propriétés qui seront utilisées dans la démonstration du théorème 0.3.1. Nous commençons par la (grande) catégorie triangulée de motifs $\widetilde{\mathrm{DM}}(S) := \widetilde{\mathrm{DM}}_{\mathrm{Nis}}(S, \mathbb{Z})$ définie dans [Déglise2016, Définition 3.3.2]. Nous notons également $\widetilde{\mathrm{DM}}$ si S est spécifié. La catégorie des motifs de Milnor-Witt $\widetilde{\mathrm{DM}}(S)$ est équipée d'une paire de foncteurs adjoints

$$\widetilde{\gamma}_* : \mathcal{S}\mathcal{H}(S) \rightleftarrows \widetilde{\mathrm{DM}} : \widetilde{\gamma}^*$$

Nous utiliserons la propriété suivante de $\widetilde{\mathrm{DM}}$.

Proposition 0.2.1 (Triangle de Gysin). *Soit X un K -schéma régulier, soit $Z \subset X$ un sous-schéma fermé régulier de codimension c et soit $U = X \setminus Z$. Supposons que le cône normal $N_X Z$ admet une trivialisatation $\phi : N_X Z \cong Z \times \mathbb{A}^c$. Alors, il existe un triangle de Gysin*

$$\widetilde{M}(U) \rightarrow \widetilde{M}(X) \rightarrow \widetilde{M}(Z)(c)[2c] \xrightarrow{+1}$$

où les deux dernières flèches dépendent du choix de ϕ .

Proof. Nous avons une adjonction de catégories triangulées

$$\widetilde{\gamma}_* : \mathcal{S}\mathcal{H}(S) \rightleftarrows \widetilde{DM} : \widetilde{\gamma}^*$$

obtenue en combinant l'adjonction de [Déglise2017, §4.1] et la correspondance de Dold-Kan classique (par exemple [Cisinski2019, 5.3.35]). Le foncteur $\mathcal{S}\mathcal{H}(S) \rightarrow \widetilde{DM}$ étant exact, le résultat découle par exemple de [Morel1999, Chapitre 3, Théorème 2.23]. \square

Nous pouvons ensuite définir le foncteur

$$\widetilde{M} : \text{Sm}/S \rightarrow \widetilde{DM}$$

donné par $\widetilde{M}(X) := \widetilde{\gamma}_*(\Sigma_{\mathbb{P}^1}^\infty X_+)$. La catégorie \widetilde{DM} est symétrique monoidale [Déglise2016, Proposition 3.3.4] avec unité $\widetilde{\mathbb{Z}} := \widetilde{M}(S)$. Comme écrit ci-dessus, le spectre de la cohomologie Milnor-Witt -motivique H_{MW} peut être défini comme $H_{MW} := \widetilde{\gamma}^*(\widetilde{\mathbb{Z}})$. Posant $\widetilde{\mathbb{Z}}(q)[p] := \widetilde{\gamma}_*(S^{p,q})$, nous obtenons les groupes de cohomologie motivique de Milnor-Witt

$$H_{MW}^{p,q}(X) := [\Sigma_{\mathbb{P}^1}^\infty X_+, \Sigma^{p,q} E]_{\mathcal{S}\mathcal{H}(S)} \cong \text{Hom}_{\widetilde{DM}(S)}(\widetilde{M}(X), \widetilde{\mathbb{Z}}(q)[p]).$$

Par [Déglise2016, Proposition 4.1.2], les groupes de cohomologie motivique peuvent être calculés sous forme de groupes d'hypercohomologie de Zariski de complexes explicites de faisceaux.

Il existe des homomorphismes d'oubli (fonctoriels en X) pour tout $p, q \in \mathbb{Z}$

$$H_{MW}^{p,q}(X, \mathbb{Z}) \rightarrow H_M^{p,q}(X, \mathbb{Z})$$

où le côté droit indique la cohomologie motivique ordinaire définie par Voevodsky.

Comme indiqué par la liste de propriétés ci-dessous, les groupes de cohomologie motivique Milnor-Witt se comportent de manière similaire aux groupes de cohomologie motivique ordinaires, mais il existe des différences centrales (par exemple, il n'y a pas de classes de Chern raisonnables).

Propriétés de la cohomologie MW-motivique Comme indiqué ci-dessus, les groupes de cohomologie MW-motivique satisfont la plupart des propriétés formelles de la cohomologie motivique ordinaire et ont été calculés dans quelques situations :

1. Si $q \leq 1$, il existe des isomorphismes canoniques

$$H_{MW}^{p,q}(X, \mathbb{Z}) \cong H_{\text{Nis}}^{p-q}(X, \mathbf{K}_q^{\text{MW}}) \cong H_{\text{Zar}}^{p-q}(X, \mathbf{K}_q^{\text{MW}})$$

où \mathbf{K}_q^{MW} est le faisceau de la K -théorie de Milnor-Witt non ramifiée (en poids q) introduit dans [Morel2012a].

Par ailleurs, dans le cas où $q < 0$, nous obtenons les isomorphismes suivants

$$H_{MW}^{p,q}(X, \mathbb{Z}) \cong H^{p-q}(X, \mathbf{K}_q^{\text{MW}}) = H^{p-q}(X, \mathbf{W})$$

où \mathbf{W} est le faisceau associé au groupe de Witt.

2. Si L/K est une extension de corps de dimension finie, il existe des isomorphismes $H_{\text{MW}}^{n,n}(L, \mathbb{Z}) \cong K_n^{\text{MW}}(L)$ qui satisfont un diagramme commutatif pour tout entier $n \in \mathbb{Z}$

$$\begin{array}{ccc} H_{\text{MW}}^{n,n}(L, \mathbb{Z}) & \xrightarrow{\sim} & K_n^{\text{MW}}(L) \\ \downarrow & & \downarrow \\ H_{\text{M}}^{n,n}(L, \mathbb{Z}) & \xrightarrow{\sim} & K_n^{\text{M}}(L) \end{array}$$

où $K_n^{\text{M}}(L)$ est le groupe (n -ième) de K -théorie de Milnor de L , la flèche horizontale inférieure est l'isomorphisme de Suslin-Nesterenko-Totaro, et la flèche verticale de droite est l'homomorphisme naturel de la K -théorie de Milnor-Witt à la K -théorie de Milnor. Ce résultat a la conséquence suivante: les groupes de cohomologie MW-motivique sont calculés via un complexe explicite de faisceaux Nisnevich (ou Zariski) $\tilde{\mathbb{Z}}(q)$ pour tout entier $q \in \mathbb{Z}$. Le résultat précédent montre qu'il existe une flèche de morphismes de faisceaux

$$\tilde{\mathbb{Z}}(q) \rightarrow \mathbf{K}_q^{\text{MW}}[-q]$$

où le côté droit est le complexe dont le seul faisceau non trivial est \mathbf{K}_q^{MW} en degré $-q$. Pour tout schéma essentiellement lisse X sur K , cela produit des homomorphismes de groupes

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H^{p-q}(X, \mathbf{K}_q^{\text{MW}})$$

compatibles avec la structure d'anneau des deux côtés.

Dans le cas particulier où $p = 2n, q = n$ pour certains $n \in \mathbb{Z}$, nous obtenons des isomorphismes (fonctoriels en X)

$$H_{\text{MW}}^{2n,n}(X, \mathbb{Z}) \xrightarrow{\sim} H^{p-q}(X, \mathbf{K}_q^{\text{MW}}) \xrightarrow{\sim} \widetilde{\text{CH}}^n(X)$$

où le terme de droite est le n -ième groupe Chow-Witt de X (défini dans [Barge2000] and [Fasel2008]). À nouveau, ces isomorphismes s'inscrivent dans des diagrammes commutatifs

$$\begin{array}{ccc} H_{\text{MW}}^{2n,n}(X, \mathbb{Z}) & \xrightarrow{\sim} & \widetilde{\text{CH}}^n(X) \\ \downarrow & & \downarrow \\ H_{\text{M}}^{2n,n}(X, \mathbb{Z}) & \xrightarrow{\sim} & \text{CH}^n(X) \end{array}$$

où la flèche verticale de droite est la flèche naturelle des groupes de Chow-Witt vers les groupes de Chow.

3. Les classes de Borel pour les faisceaux symplectiques existent dans la MW-cohomologie motivique totale [Yang2019], mais en général, le théorème du fibré projectif échoue [Yang2020].
4. Pour un schéma lisse X sur \mathbb{R} , il existe deux applications de réalisation intéressantes. D'une part, on peut considérer la composition

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H_{\text{M}}^{p,q}(X, \mathbb{Z}) \rightarrow H_{\text{sing}}^p(X(\mathbb{C}), \mathbb{Z})$$

où la flèche de droite est l'application de réalisation complexe. D'autre part, on peut aussi considérer la composition suivante

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H^{p-q}(X, \mathbf{K}_q^{\text{MW}}) \rightarrow H^{p-q}(X, \mathbf{I}^q) \rightarrow H_{\text{sing}}^{p-q}(X(\mathbb{R}), \mathbb{Z})$$

où \mathbf{I}^q est le faisceau non ramifié associé à la q -ième puissance de l'idéal fondamental dans l'anneau de Witt, $\mathbf{K}_q^{\text{MW}} \rightarrow \mathbf{I}^q$ est la projection canonique et $H^{p-q}(X, \mathbf{I}^q) \rightarrow H_{\text{sing}}^{p-q}(X(\mathbb{R}), \mathbb{Z})$ est l'application de signature de Jacobson [Jacobson2017].

Remarquons ici que ces deux applications de réalisation montrent que la cohomologie motivique de Milnor-Witt est en quelque sorte l'analogue à la fois de la cohomologie singulière des points complexes et réels de X .

Cohomologie η -divisible Considérons une fois encore l'application de Hopf $\eta : S^{3,2} \cong \Sigma_{\mathbb{P}^1}^{\infty} \mathbb{A}^2 \setminus \{0\} \longrightarrow \Sigma_{\mathbb{P}^1}^{\infty} \mathbb{P}^1 \cong S^{2,1}$, ou plus précisément le morphisme induit

$$\eta : \widetilde{\mathbb{Z}}(1)[1] \rightarrow \widetilde{\mathbb{Z}}$$

encore appelé application de Hopf de manière un peu abusive. Par définition, η induit un morphisme $\eta : E^{p,q}(X) \rightarrow E^{p-1,q-1}(X)$. Localisant par rapport à η , on obtient la cohomologie η -divisible $E[\eta^{-1}]$, qui a la propriété que la multiplication par $\eta : E[\eta^{-1}]^{p,q}(X) \xrightarrow{\sim} E[\eta^{-1}]^{p-1,q-1}(X)$ est un isomorphisme.

Dans le cas de H_{MW} , puisque $X \mapsto H_{\text{MW}}^{p,q}(X)[\eta^{-1}] \cong H^{p-q}(X, \mathbf{W})$ pour $q < 0$, nous voyons que $H_{\eta} := H_{\text{MW}}[\eta^{-1}]$ représente la cohomologie dans le faisceau de Witt, au sens où

$$[\Sigma_{\mathbb{P}^1}^{\infty} X_+, \Sigma^{p,q} H_{\eta}]_{\mathcal{S}\mathcal{H}(S)} = H^{p-q}(X, \mathbf{W}).$$

Plus généralement, pour un spectre η -divisible E_{η} , on obtient des groupes de cohomologie gradués $E_{\eta}^*(X) := \bigoplus_{n \in \mathbb{Z}} E_{\eta}^{n+*,n}(X) \cong E_{\eta}^{*,0}(X)[\eta, \eta^{-1}]$.

Réalisation réelle Il y a une relation étroite entre les catégories homotopiques stables sur \mathbb{R} ou sur \mathbb{C} et leur analogue en topologie, obtenue au moyen des foncteurs de réalisation. Tout d'abord, considérons le foncteur $\text{Sm}_{\mathbb{R}} \rightarrow \text{Top}$ (resp. $\text{Sm}_{\mathbb{C}} \rightarrow \text{Top}$) associant à un schéma lisse X l'ensemble $X(\mathbb{R})$ muni de la topologie euclidienne (resp. l'ensemble $X(\mathbb{C})$ muni de la topologie analytique). Ces foncteurs peuvent être ensuite étendus à la catégorie \mathbb{A}^1 -homotopique stable $\mathcal{S}\mathcal{H}(K) \rightarrow \text{SH}$ pour $K = \mathbb{R}$ ou $K = \mathbb{C}$. Pour la réalisation réelle, on peut observer que \mathbb{G}_m est envoyé sur deux points, spécifiquement S^0 , trivialisant ainsi les torsions de Tate. Il est à noter qu'en inversant le morphisme $\rho : S^0 \rightarrow \mathbb{G}_m$ envoyant le point complémentaire du point de base sur -1 dans $\mathcal{S}\mathcal{H}(\mathbb{R})$, on retrouve la catégorie homotopique (topologique) stable. Plus précisément, le foncteur de réalisation $\mathcal{S}\mathcal{H}(\mathbb{R})(\rho^{-1}) \rightarrow \text{SH}$ est une équivalence [Bachmann2018]. En outre, il convient de mentionner que cette réalisation réelle présente une forte ressemblance avec l'inversion d' η considéré ci-dessus, puisque l'inversion du morphisme ρ après l'inversion de 2 est équivalent à l'inversion de η .

Les MW-modules de cycle Il est possible de construire une t-structure δ sur $\mathcal{S}\mathcal{H}(S)$ [Bondarko2017], dont le coeur peut être compris en utilisant soit les modules homotopiques [Déglise2011] soit les MW-modules de cycle (cohomologiques) ([Feld2020] ou [Déglise2022]).

Nous notons par \mathcal{F}_S la catégorie des spectres des corps essentiellement finis sur S . Un module de cycle de Milnor-Witt (cohomologique) est un foncteur de $\mathcal{F}_S^{\text{op}}$ dans la catégorie $\text{Abel}^{\mathbb{Z}}$ des groupes abéliens \mathbb{Z} -gradués

$$M_* : \mathcal{F}_S \rightarrow \text{Abel}^{\mathbb{Z}}$$

satisfaisant une liste d'axiomes. Les axiomes de M_* incluent l'action de la K -théorie de Milnor-Witt K_*^{MW} , qui est l'exemple fondamental d'un module de cycle de Milnor-Witt. De manière plus précise, on considère le préfaisceau de Milnor-Witt $K_*^{\text{MW}}(R)$ associé à toute \mathcal{O}_S -algèbre R défini comme étant l'algèbre graduée librement engendrée par des éléments de degré 1 de la forme $[a], a \in R^{\times}$ et un élément η de degré -1 , soumise aux relations suivantes:

1. $[a][1-a] = 0$ pour tout a tel que $a, 1-a \in R^{\times} \setminus \{1\}$.
2. $[ab] = [a] + [b] + \eta[a][b]$ pour tout $a, b \in R^{\times}$.
3. $\eta[a] = [a]\eta$ pour tout $a \in R^{\times}$.
4. $\eta(2 + \eta[-1]) = 0$.

Cela définit un module de cycle de Milnor-Witt via $(x : \text{Spec } F \rightarrow S) \in \mathcal{F}_S \mapsto \mathbf{K}_*^{\text{MW}}(F)$ et un module homotopique via $X \mapsto \mathbf{K}_*^{\text{MW}}(\mathcal{O}(X))$, où le terme de droite est le faisceau associé au préfaisceau ci-dessus. D'autre part, on peut aussi considérer le préfaisceau de cohomologie MW-motivique (bigradué)

$$X \mapsto \mathbf{H}_{\text{MW}}(X) := \bigoplus_{p,q} \mathbf{H}_{\text{MW}}^{p,q}(X, \mathbb{Z})$$

Par [Déglise2016, Theorem 4.2.2], il existe un morphisme de préfaisceaux

$$s : \bigoplus_{n \in \mathbb{Z}} \mathbf{K}_n^{\text{MW}}(-) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{n,n}(-, \mathbb{Z}) \subset \mathbf{H}_{\text{MW}}(X)$$

qui se spécialise à l'isomorphisme ci-dessus si $X = \text{Spec}(F)$ où F est une extension de corps finiment engendrée de K [Calmès2017].

Les suites spectrales de Leray en homotopie. Pour calculer la cohomologie en utilisant les suites de fibrations, on peut utiliser la suite spectrale de Leray dans le cadre de l'homotopie motivique en utilisant la t -structure δ [Asok2018].

Soit H_δ^p la cohomologie de la t -structure δ , qui prend ses valeurs dans le coeur de la catégorie homotopique stable, c'est-à-dire les MW-modules de cycle. En général, pour tout spectre $\mathbb{F} \in \mathcal{S}\mathcal{H}(S)$, on peut obtenir une suite spectrale en utilisant la théorie des diagrammes spectraux :

$$E_2^{p,q}(\mathbb{E}) = {}_\delta A^p(S, H_\delta^q \mathbb{F}) \Rightarrow H^{p+q}(S, \mathbb{F})$$

Lorsqu'on l'applique au poussé en avant $f_* \mathbb{E}$, on obtient :

Theorem 0.2.2 ([Asok2018], Théorème 4.2.5). *Soit $f : T \rightarrow B$ un morphisme de schémas, et soit $\mathbb{E} \in \mathcal{S}\mathcal{H}(T)$ un spectre en anneaux sur T . Il existe une suite spectrale de la forme :*

$$E_2^{p,q}(f, \mathbb{E}) = {}_\delta A^p(B, H_\delta^q f_* \mathbb{E}) \Rightarrow H^{p+q}(T, \mathbb{E})$$

où les différentielles d_r satisfont la règle usuelle de Leibniz.

0.3 Résultats Principaux

Rappelons que si X est un schéma lisse sur S , nous notons $\mathbf{H}_{\text{MW}}(X)$ ou $\mathbf{H}_{\text{MW}}^{*,*}(X)$ l'anneau de cohomologie MW-motivique total $\bigoplus_{p,q \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{p,q}(X, \mathbb{Z})$.

Dans le chapitre 1, nous calculons l'anneau de cohomologie MW-motivique total $\mathbf{H}_{\text{MW}}(U)$ du complémentaire $U \subset \mathbb{A}_K^N$ d'une union finie d'hyperplans en utilisant des méthodes très similaires à celles de [Chatzistamatiou2007], avec certaines modifications nécessaires.

Pour tout $f \in \mathbb{G}_m(U)$, nous utilisons (f) pour indiquer le générateur obtenu dans les algèbres libres correspondantes (par exemple, $\mathbf{K}_n^{\text{MW}}(K)\{\mathbb{G}_m(U)\}$) et $[f]$ pour indiquer les éléments correspondants dans les groupes de cohomologie (par exemple, $\mathbf{H}_{\text{MW}}^{1,1}(U, \mathbb{Z})$).

Theorem 0.3.1. *Soit K un corps parfait de caractéristique différente de 2 et soit $U \subset \mathbb{A}_K^N$ le complément d'une union finie d'hyperplans. Il existe un isomorphisme de $\mathbf{H}_{\text{MW}}(K)$ -algèbres*

$$\mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \cong \mathbf{H}_{\text{MW}}(U)$$

qui est défini en envoyant $(f) \in \mathbb{G}_m(U)$ sur la classe $[f]$ dans $\mathbf{H}_{\text{MW}}^{1,1}(U, \mathbb{Z})$ correspondant à f sous l'homomorphisme s . Ici, $\mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}$ est la $\mathbf{H}_{\text{MW}}(K)$ -algèbre libre (associative) graduée engendrée par $\mathbb{G}_m(U)$ en degré $(1, 1)$ et J_U est l'idéal engendré par les éléments suivants :

- (1) $(f) - [f]$, si $f \in K^\times \subset \mathbb{G}_m(U)$,
- (2) $(f) + (g) + \eta(f)(g) - (fg)$, si $f, g \in \mathbb{G}_m(U)$,
- (3) $(f_1)(f_2) \cdots (f_t)$, pour tout $f_1, \dots, f_t \in \mathbb{G}_m(U)$ tels que $\sum_{i=1}^t f_i = 1$,
- (4) $(f)^2 - [-1](f)$, si $f \in \mathbb{G}_m(U)$.

Comme indiqué ci-dessus, ce théorème et sa preuve sont inspirés par le calcul de la cohomologie motivique (ordinaire) de U dans [Chatzistamatiou2007]. Nous pouvons récupérer le théorème principal [Chatzistamatiou2007, Théorème 3.5] en prenant $\eta = 0$. En conséquence, nous obtenons le résultat suivant.

Corollary 0.3.2. *Soit $U \subset \mathbb{A}^N$ le complément d'une union finie d'hyperplans. L'isomorphisme du Théorème 0.3.1 induit un isomorphisme*

$$\bigoplus_{n \in \mathbb{Z}} K_n^{\text{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \rightarrow \bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(U, \mathbb{Z}).$$

Nous ne savons pas si le membre de gauche coïncide avec $K_*^{\text{MW}}(U)$, mais cela semble probable. À la fin de le chapitre, nous consacrons quelques lignes à l'homomorphisme de réalisation réelle

$$H_{\text{MW}}(U, \mathbb{Z}) \rightarrow H_{\text{sing}}^*(U(\mathbb{R}), \mathbb{Z})$$

lorsque U est défini sur $K = \mathbb{R}$. Nous prouvons en particulier que les deux côtés ont essentiellement les mêmes générateurs, et que l'application est surjective.

Les chapitres 2 et 4 visent à fournir un aperçu complet de l'interaction entre la cohomologie MW-motivique et les groupes linéaires algébriques, ainsi qu'à démontrer l'utilité de la cohomologie MW-motivique pour comprendre les propriétés et la structure de ces groupes. Les groupes linéaires algébriques jouent un rôle fondamental dans diverses disciplines mathématiques. Dans cette partie, nous calculons la cohomologie d'un groupe algébrique G , vu comme une variété, en utilisant la cohomologie motivique de Milnor-Witt H_{MW} au lieu d'utiliser la cohomologie de groupe plus couramment utilisée de l'espace classifiant BG , bien que la cohomologie de G et BG soient liées.

Dans la suite, rappelons que $\tilde{M}(X)$ dénote le MW-motif d'un schéma X . Si $x \in X(S)$, nous notons par $\tilde{M}(X, x)$ le motif de Milnor-Witt réduit de X . Par exemple, si $n \geq 1$, nous obtenons pour $\mathbb{A}^n \setminus \{0\} := \mathbb{A}_S^n \setminus \{0\}$ les motifs suivants

$$\tilde{M}(\mathbb{A}^n \setminus \{0\}) = \tilde{M}(S) \oplus \tilde{M}(S)(n)[2n - 1],$$

alors que $\tilde{M}(\mathbb{A}^n \setminus \{0\}, x) = \tilde{M}(S)(n)[2n - 1]$ pour tout point S -rationnel x .

Pour la cohomologie motivique originale, c'est-à-dire dans le cas $\eta = 0$, les calculs et les décompositions motiviques des groupes réductifs déployés [Biglari2012] et des variétés de Stiefel [Williams2012] en cohomologie motivique sont bien connus.

Cependant, à l'heure actuelle, il n'y a que des résultats très partiels pour la cohomologie MW-motivique, même dans le cas où η est inversé. Le but principal de cette partie est de combler cette lacune.

Dans le chapitre 2, nous calculons la décomposition MW-motivique de Sp_{2n} dans $\widetilde{\text{DM}}(S)$, sur n'importe quelle base raisonnable S , en utilisant la Sp -orientation de la cohomologie MW-motivique, et plus précisément les classes de Borel associées à cette orientation. Le résultat final est le suivant.

Theorem 0.3.3 (Théorème 2.3.4). *Pour tout $n \geq 1$, nous avons un isomorphisme dans $\widetilde{\text{DM}}(S)$*

$$\tilde{M}(\text{Sp}_{2n}) \cong \tilde{M}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \tilde{M}(\text{Sp}_{2n-2}),$$

conduisant à une décomposition de la forme

$$\widetilde{\mathbf{M}}(\mathrm{Sp}_{2n}) \cong \widetilde{\mathbf{M}}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \widetilde{\mathbf{M}}(\mathbb{A}^{2n-2} \setminus \{0\}) \otimes \dots \otimes \widetilde{\mathbf{M}}(\mathbb{A}^2 \setminus \{0\}) \cong \bigoplus_{1 \leq i_1 < \dots < i_j \leq n} \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j)).$$

où $\widetilde{\mathbb{Z}}(d(i_1, \dots, i_j))$ dénote $\widetilde{\mathbb{Z}}(\sum_{l=1}^j 2i_l)[4(\sum_{l=1}^j i_l) - j]$.

Ensuite, nous abordons le calcul pour les variétés de Stiefel et en particulier pour GL_n au chapitre 4. Comme énoncé ci-dessus, une des différences fondamentales entre la cohomologie motivique de Milnor-Witt et sa version plus classique est que la première n'est pas GL -orientée (c'est-à-dire qu'elle n'a pas une théorie raisonnable de classes de Chern), rendant ainsi les calculs plus délicats. Pour ce chapitre, la cohomologie MW-motivique η -inversée H_η fournit le moyen de calcul le plus efficace. Cela nous permet de transposer au cadre motivique les calculs classiques pour O_n et SO_n [Hatcher2002]. En utilisant la suite spectrale de Leray, nous pouvons calculer par induction pour toutes les variétés de Stiefel. Dans l'énoncé suivant, nous écrivons $V_k(\mathbb{A}^n)$ pour la variété de Stiefel de rang complet des matrices $k \times n$, où $k \leq n$, et nous posons

$$\begin{aligned} \mathbf{HS}_{2k} &:= \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)(2k)[4k-1], \\ \mathbf{HS}_{2k+1} &:= \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)(4k+1)[8k]. \end{aligned}$$

Notons encore une fois que $\widetilde{\mathbf{M}}_\eta(S)(1)[1]$ est canoniquement isomorphe à $\widetilde{\mathbf{M}}_\eta(S)$, via la multiplication par η , et nous aurions donc pu poser

$$\begin{aligned} \mathbf{HS}_{2k} &\cong \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)[2k-1], \\ \mathbf{HS}_{2k+1} &\cong \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)[4k-1]. \end{aligned}$$

Nous préférons cependant garder la notation précédente, principalement parce que nous prévoyons de calculer le MW-motif intégral des variétés de Stiefel dans $\widetilde{\mathrm{DM}}(S)$ dans un proche avenir.

Theorem 0.3.4 (Théorème 3.0.1). *Nous avons un isomorphisme d'algèbres graduées:*

$$H_\eta^*(V_k(\mathbb{A}^n)) \cong H_\eta^*(S)[G]/(g^2 - \delta_{\gamma_0}(g)[-1]g, g \in G)$$

où les générateurs G sont donnés par

$$G = \begin{cases} G_1 \sqcup G_2 \sqcup G_3 & n \text{ est pair, } n-k \text{ est pair,} \\ G_2 \sqcup G_3 & n \text{ est impair, } n-k \text{ est pair,} \\ G_1 \sqcup G_2 & n \text{ est pair, } n-k \text{ est impair,} \\ G_2 & n \text{ est impair, } n-k \text{ est impair,} \end{cases}$$

avec

$$\begin{aligned} G_1 &= \{\alpha_{n-1} \in H_\eta^{n-1}(\mathbf{HS}_n)\} \\ G_2 &= \{\beta_{4j-1} \in H_\eta^{4j-1}(\mathbf{HS}_{2j+1}) \mid n-k < 2j < n\} \\ G_3 &= \{\gamma_{n-k} \in H_\eta^{n-k}(\mathbb{A}^{n-k+1} \setminus \{0\})\} \end{aligned}$$

et $\delta_{\gamma_0}(\gamma_0) = 1$, sinon $\delta_{\gamma_0}(g) = 0$.

En conséquence, nous obtenons le théorème suivant.

Theorem 0.3.5 (Théorème 3.0.2). *Nous avons une décomposition motivique de la forme suivante pour tout $i, j \in \mathbb{N}$:*

$$\tilde{\mathbf{M}}_\eta(V_{2j}(\mathbb{A}^{2i})) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-2)} \otimes \tilde{\mathbf{M}}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\})$$

$$\tilde{\mathbf{M}}_\eta(V_{2j+1}(\mathbb{A}^{2i})) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-1)}$$

$$\tilde{\mathbf{M}}_\eta(V_{2j}(\mathbb{A}^{2i+1})) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)}$$

$$\tilde{\mathbf{M}}_\eta(V_{2j+1}(\mathbb{A}^{2i+1})) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)} \otimes \tilde{\mathbf{M}}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\})$$

Une observation facile est que tous ces générateurs coïncident avec les générateurs des groupes de cohomologie singulière à coefficient dans $\mathbb{Z}[1/2]$ des variétés de Stiefel topologiques, et que la réalisation réelle discutée ci-dessus envoie les uns sur les autres.

1

Computations on the complement of hyperplane arrangements

In this chapter, we compute the (total) Milnor-Witt motivic cohomology of the complement of a hyperplane arrangement in an affine space as an algebra with given generators and relations. We also obtain some corollaries by realization to classical cohomology.

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1.1 Basic structure of the cohomology ring

Let V be an affine space, i.e. $V \cong \mathbb{A}_K^N$ for some $N \in \mathbb{N}$. We consider finite families I of hyperplanes in V (that we suppose to be distinct). We denote by $|I|$ the cardinality of I and set $U_I^V := V \setminus (\cup_{Y \in I} Y)$ and simply write U_I^N when a specified isomorphism $V = \mathbb{A}_K^N$ is understood. For any hyperplane Y , we put $I_Y := \{Y_i \cap Y \mid Y_i \in I, Y_i \neq Y\}$.

Proposition 1.1.1. *Let V and I be as above. We have*

$$\tilde{M}(U_I^V) \cong \bigoplus_{j \in J} \tilde{\mathbb{Z}}(n_j)[n_j]$$

for some set J and integers $n_j \geq 0$.

Proof. We proceed by induction on the dimension N of V and $|I|$. If $|I| = 0$, then $\tilde{M}(U_I^V) = \tilde{M}(V) \cong \tilde{\mathbb{Z}}$ and we are done. Let then $|I| \geq 1$ and $Y \in I$. The Gysin triangle reads as

$$\tilde{M}(U_I^V) \rightarrow \tilde{M}(U_{I-\{Y\}}^V) \xrightarrow{\phi} \tilde{M}(U_Y^V)(1)[2] \xrightarrow{+1}.$$
 (1.1)

If $\phi = 0$, then the triangle is split and consequently we obtain an isomorphism

$$\tilde{M}(U_I^V) \cong \tilde{M}(U_{I-\{Y\}}^V) \oplus \tilde{M}(U_Y^V)(1)[1].$$
 (1.2)

Since $|I - \{Y\}| < |I|$ and $\dim(Y) = \dim(V) - 1$ we conclude by induction that the right-hand side has the correct form. We are then reduced to show that $\phi = 0$. By induction, we have

$$\begin{aligned} \phi &\in \text{Hom}_{\widetilde{\text{DM}}}(\tilde{M}(U_{I-\{Y\}}^V), \tilde{M}(U_Y^V)(1)[2]) \\ &\cong \bigoplus_{j,k} \text{Hom}_{\widetilde{\text{DM}}}(\tilde{\mathbb{Z}}(n_j)[n_j], \tilde{\mathbb{Z}}(m_k)[m_k + 1]) \end{aligned}$$

for some integers $n_j, m_k \geq 0$, and it suffices to prove that $\text{Hom}_{\widetilde{\text{DM}}}(\tilde{\mathbb{Z}}, \tilde{\mathbb{Z}}(m)[m + 1]) = 0$ for any $m \in \mathbb{Z}$ to conclude. Now,

$$\text{Hom}_{\widetilde{\text{DM}}}(\tilde{\mathbb{Z}}, \tilde{\mathbb{Z}}(m)[m + 1]) = \text{H}_{\text{MW}}^{m+1, m}(K, \mathbb{Z})$$

and the latter is trivial by [Dégli2016, Proposition 4.1.2] and [Dégli2016, proof of Theorem 4.2.4]. \square

As an immediate corollary, we obtain the following result.

Corollary 1.1.2. *The total MW-motivic cohomology $\text{H}_{\text{MW}}(U_I^V)$ is a finitely generated free $\text{H}_{\text{MW}}(K)$ -module.*

To obtain more precise results, we now study the Gysin (split) triangle (1.1) in more detail. We can rewrite it as

$$\tilde{M}(U_Y^V)(1)[1] \xrightarrow{\beta^Y} \tilde{M}(U_I^V) \xrightarrow{\alpha^Y} \tilde{M}(U_{I-\{Y\}}^V) \xrightarrow{0}$$

and therefore we obtain the following short (split) exact sequence in which the morphisms are induced by the first two morphisms in the triangle

$$0 \rightarrow \bigoplus_{p,q} \text{H}_{\text{MW}}^{p,q}(U_{I-\{Y\}}^V, \mathbb{Z}) \xrightarrow{\alpha_*^Y} \bigoplus_{p,q} \text{H}_{\text{MW}}^{p,q}(U_I^V, \mathbb{Z}) \xrightarrow{\beta_*^Y} \bigoplus_{p,q} \text{H}_{\text{MW}}^{p-1, q-1}(U_Y^V, \mathbb{Z}) \rightarrow 0. \quad (1.3)$$

The inclusion $Y \subset V$ yields a morphism $U_Y^V \rightarrow U_{I-\{Y\}}^V$ and therefore a morphism $\iota : \tilde{M}(U_Y^V) \rightarrow \tilde{M}(U_{I-\{Y\}}^V)$. The global section f of V corresponding to the equation of Y becomes invertible in U_I^V and therefore yields a morphism $[f] : \tilde{M}(U_I^V) \rightarrow \tilde{\mathbb{Z}}(1)[1]$ corresponding to the class $[f] \in \text{H}_{\text{MW}}^{1,1}(U_I^V, \mathbb{Z})$ given by the morphism

$$s : \bigoplus_{n \in \mathbb{Z}} \text{K}_n^{\text{MW}}(-) \rightarrow \bigoplus_{n \in \mathbb{Z}} \text{H}_{\text{MW}}^{n,n}(-, \mathbb{Z}).$$

Lemma 1.1.3. *The following diagram commutes*

$$\begin{array}{ccc} \tilde{\mathbf{M}}(U_{I_Y}^Y)(1)[1] & \xrightarrow{\iota^{(1)}[1]} & \tilde{\mathbf{M}}(U_{I-\{Y\}}^V)(1)[1] \\ \beta^Y \downarrow & & \uparrow \alpha^Y \otimes [f] \\ \tilde{\mathbf{M}}(U_I^V) & \xrightarrow{\Delta} & \tilde{\mathbf{M}}(U_I^V) \otimes \tilde{\mathbf{M}}(U_I^V). \end{array}$$

Proof. The commutative diagram of schemes

$$\begin{array}{ccc} U_I^V & \longrightarrow & U_{I-\{Y\}}^V \\ \downarrow (\text{Id}, f) & & \downarrow (\text{Id}, f) \\ U_{I-\{Y\}}^V \times \mathbb{G}_m & \longrightarrow & U_{I-\{Y\}}^V \times \mathbb{A}^1 \end{array}$$

yields a morphism of Gysin triangles and thus a commutative diagram

$$\begin{array}{ccccccc} \tilde{\mathbf{M}}(U_{I_Y}^Y)(1)[1] & \xrightarrow{\beta^Y} & \tilde{\mathbf{M}}(U_I^V) & \xrightarrow{\alpha^Y} & \tilde{\mathbf{M}}(U_{I-\{Y\}}^V) & \longrightarrow & \dots \\ \downarrow \iota^{(1)}[1] & & \downarrow (\text{Id}, f) & & \downarrow & & \\ \tilde{\mathbf{M}}(U_{I-\{Y\}}^V)(1)[1] & \longrightarrow & \tilde{\mathbf{M}}(U_{I-\{Y\}}^V \times \mathbb{G}_m) & \longrightarrow & \tilde{\mathbf{M}}(U_{I-\{Y\}}^V \times \mathbb{A}^1) & \longrightarrow & \dots \\ & & \downarrow & & & & \\ & & \tilde{\mathbf{M}}(U_{I-\{Y\}}^V)(1)[1] & & & & \end{array}$$

in which the map $\tilde{\mathbf{M}}(U_{I-\{Y\}}^V \times \mathbb{G}_m) \rightarrow \tilde{\mathbf{M}}(U_{I-\{Y\}}^V)(1)[1]$ is just the projection. We conclude by observing that the middle vertical composite is just $(\alpha^Y \otimes [f]) \circ \Delta$. \square

We may now prove the main result of this section.

Proposition 1.1.4. *The cohomology ring $\mathbf{H}_{\text{MW}}(U)$ is generated by the classes of units in U as an $\mathbf{H}_{\text{MW}}(K)$ -algebra. In particular, the homomorphism*

$$s : \bigoplus_{n \in \mathbb{Z}} \mathbf{K}_n^{\text{MW}}(U) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{n,n}(U, \mathbb{Z})$$

is surjective.

Proof. We still prove the result by induction on $|I|$ and the dimension of V , the case $|I| = 0$ being obvious. Suppose then that the result holds for $U_{I_Y}^Y$ and $U_{I-\{Y\}}^V$ and consider the split sequence (1.3). For any $x \in \mathbf{H}_{\text{MW}}(U) = \mathbf{H}_{\text{MW}}(U_I^V)$, we have that $\beta_*^Y(x) \in \mathbf{H}_{\text{MW}}(U_{I_Y}^Y)$ is in the subalgebra generated by $\{[f] \mid f \in \mathbb{G}_m(U_{I_Y}^V)\}$ and η . For any $f_1, \dots, f_n \in \mathbb{G}_m(U_{I-\{Y\}}^V)$, Lemma 1.1.3 yields

$$\beta_*^Y([(f_1)_{|U_I^V}] \cdots [(f_n)_{|U_I^V}] \cdot [t]) = [(f_1)_{|U_{I_Y}^V}] \cdots [(f_n)_{|U_{I_Y}^V}].$$

The map $\mathbb{G}_m(U_{I-\{Y\}}^V) \rightarrow \mathbb{G}_m(U_{I_Y}^V)$ being surjective, it follows that there exists $x' \in \mathbf{H}_{\text{MW}}(U_I^V)$ in the subalgebra generated by units such that $\beta_*^Y(x - x') = 0$. Thus $x - x' = \alpha_*(y)$ for some $y \in \mathbf{H}_{\text{MW}}(U_{I-\{Y\}}^V)$ and the result now follows from the fact that α_* is just induced by the inclusion $U_I^V \subset U_{I-\{Y\}}^V$. \square

1.2 Relations in the cohomology ring

The purpose of this section is to prove that the relations of Theorem 0.3.1 hold in $H_{\text{MW}}(U)$. The first two relations are obviously satisfied since the homomorphism is induced by the ring homomorphism

$$s: \bigoplus_{n \in \mathbb{Z}} \mathbf{K}_n^{\text{MW}}(U) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{H}_{\text{MW}}^{n,n}(U, \mathbb{Z}).$$

Recall now that the last two relations are

3. $[f_1][f_2] \cdots [f_t]$, if $f_i \in \mathbb{G}_m(U)$ for any i and $\sum_{i=1}^t f_i = 1$,
4. $[f]^2 - [-1][f]$, if $f \in \mathbb{G}_m(U)$.

We will prove that they are equal to 0 in $H_{\text{MW}}(U)$. Actually, it will be more convenient to work with the following relations:

- 3'. $R(f_0, \dots, f_t)$ defined by

$$\sum_{i=0}^t \varepsilon^{t+i} [f_0] \cdots [\widehat{f_i}] \cdots [f_t] + \sum_{0 \leq i_0 < \dots < i_k \leq t} (-1)^k [-1]^k [f_0] \cdots [\widehat{f_{i_0}}] \cdots [\widehat{f_{i_k}}] \cdots [f_t]$$

for $f_i \in \mathbb{G}_m(U)$ such that $\sum_{i=0}^t f_i = 0$,

- 4'. (anti-commutativity) $[f][g] - \varepsilon[g][f]$,

where $\varepsilon := -\langle -1 \rangle = -1 - \eta[-1]$.

Lemma 1.2.1. *The two sets of relations are equivalent in $H_{\text{MW}}(U)$.*

Proof. We first assume that 3. and 4. are satisfied. Since 1. and 2. are satisfied, we have $[-f] = [-1] + \langle -1 \rangle [f]$. As 4. is satisfied and $[-1] = \varepsilon[-1]$ in $\mathbf{K}_*^{\text{MW}}(K)$,

$$[-f][f] = [-1][f] + \langle -1 \rangle [f]^2 = \varepsilon([-1][f] - [f]^2) = 0$$

and then $[fg][-fg] = [f][g] + \varepsilon[g][f]$ for any $g, f \in \mathbb{G}_m(U)$ by [Morel2012a, proof of Lemma 3.7]. Suppose next that $\sum_{i=0}^t f_i = 0$, so that $\sum_{i=1}^t \frac{f_i}{-f_0} = 1$. Combining 3. and the anti-commutativity law, we obtain

$$0 = [1] = [f_j^{-1}] + \langle f_j^{-1} \rangle [f_j] \quad (\text{by 2.}) \quad (1.4)$$

$$\left[\frac{-f_i}{f_j} \right] = \langle f_j^{-1} \rangle [-f_i] + [f_j^{-1}] \quad (1.5)$$

$$= \langle f_j^{-1} \rangle ([-f_i] - [f_j]) \quad (\text{by 4.})$$

$$= \langle f_j^{-1} \rangle (\langle -1 \rangle [f_i] + [-1] - [f_j])$$

$$([f_0] - [-1])^k = \sum_{i=0}^k \binom{k}{i} [-1]^{k-i} [f_0]^i \quad (1.6)$$

$$= \left(\sum_{i=0}^{k-1} \binom{k}{i} \right) [-1]^{k-1} [f_0] + (-1)^k [-1]^k \quad (\text{by 4.})$$

$$= (-1)^{k-1} [-1]^{k-1} [f_0] + (-1)^k [-1]^k$$

$$\begin{aligned}
 0 &= (-\langle f_0 \rangle)^t \left[\frac{-f_1}{f_0} \right] \left[\frac{-f_2}{f_0} \right] \cdots \left[\frac{-f_t}{f_0} \right] && \text{(by 3.)} \\
 &= ([f_0] - [-1] - \langle -1 \rangle [f_1]) \cdots ([f_0] - [-1] - \langle -1 \rangle [f_t]) && \text{(by 5)} \\
 &= \varepsilon^t \widehat{[f_0]} [f_1] \cdots [f_t] + \sum_{i=1}^t \varepsilon^{t-1} [f_1] \cdots \widehat{[f_i]} ([f_0] - [-1]) \cdots [f_t] + \\
 &\quad \sum_{i < j} ([f_0] - [-1])^2 [f_1] \cdots \widehat{[f_i]} \cdots \widehat{[f_j]} \cdots [f_t] + \cdots \\
 &= \sum_{i=0}^t \varepsilon^{t+i} [f_0] \cdots \widehat{[f_i]} \cdots [f_t] + \\
 &\quad \sum_{0 \leq i_0 < \cdots < i_k \leq t} (-1)^k [-1]^k [f_0] \cdots \widehat{[f_{i_0}]} \cdots \widehat{[f_{i_k}]} \cdots [f_t] && \text{(by 6)} \\
 &= R(f_0, \dots, f_t).
 \end{aligned}$$

Conversely, suppose that 3' and 4' hold. A direct calculation shows that we have $R(-1, f_1, \dots, f_t) = (-\langle -1 \rangle)^t [f_1] \cdots [f_t] = \varepsilon^t [f_1] \cdots [f_t]$, and consequently that 3. also holds. For every field $K \neq \mathbb{F}_2$, we have $1 + a + b = 0$ for some $a, b \neq 0$ and it follows from $[-a][-b] = 0$ in $K_*^{\text{MW}}(K)$ that

$$\begin{aligned}
 R(f, af, bf) &= R(f, af, bf) - [-a][-b] = R(f, af, bf) - \left[-\frac{af}{f} \right] \left[-\frac{bf}{f} \right] \\
 &= R(f, af, bf) - (\langle -1 \rangle [af] + [-1] - [f]) (\langle -1 \rangle [bf] + [-1] - [f]) \\
 &= -[-1][f] + [-1]^2 - ([f] - [-1])^2 = [-1][f] - [f]^2.
 \end{aligned}$$

□

Remark 1. Observe that the following properties of the relations R and anti-commutativity hold:

1. For any $a, b \in \mathbb{G}_m(U)$, we have $[a/b] = -\langle b^{-1} \rangle R(b, -a)$.
2. For any $f_0, \dots, f_t \in \mathbb{G}_m(U)$, by direct computation, we have

$$R(f_0, \dots, f_t) - \varepsilon^i [f_i] R(f_0, \dots, \widehat{f_i}, \dots, f_t) = P(f_0, \dots, \widehat{f_i}, \dots, f_t)$$

for some polynomial P . We shall use the anti-commutativity and the fact that $[-1] = \varepsilon^j [-1]$ for any $j \geq 0$ in the computation.

3. For any $f_0, \dots, f_t \in K^\times$ such that $\sum_{i=0}^t f_i = 0$, we have $R(f_0, \dots, f_t) = 0$ in $K_*^{\text{MW}}(K)$.

The following lemma will prove useful in the proof of the main theorem.

Lemma 1.2.2. Any morphism $\phi : \widetilde{\mathbb{M}}(U_I^Y) \rightarrow T$ in $\widetilde{\text{DM}}(K)$ such that

$$\widetilde{\mathbb{M}}(U_{I'}^Y)(1)[1] \xrightarrow{\beta^Y} \widetilde{\mathbb{M}}(U_I^Y) \xrightarrow{\phi} T$$

is trivial for every $Y \in I$ factors through $\widetilde{\mathbb{Z}}$, i.e. there is a morphism $\psi : \widetilde{\mathbb{Z}} \rightarrow T$ such that the following diagram

$$\begin{array}{ccc}
 \widetilde{\mathbb{M}}(U_I^Y) & \xrightarrow{\phi} & T \\
 \downarrow & \nearrow \psi & \\
 \widetilde{\mathbb{Z}} & &
 \end{array}$$

is commutative.

Proof. We prove as usual the result by induction on $|I|$, the result being trivial if $|I| = 0$, i.e. if $U_I^V \cong \mathbb{A}_K^N$. By assumption, ϕ factors through $\tilde{\mathbb{M}}(U_{I-\{Y\}}^V)$, i.e. we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{M}}(U_I^V) & \xrightarrow{\alpha^Y} & \tilde{\mathbb{M}}(U_{I-\{Y\}}^V) \\ & \searrow \phi & \downarrow \phi_0 \\ & & T. \end{array}$$

For $H \in I' = I - \{Y\}$, we have an associated Gysin morphism $\beta^H : \tilde{\mathbb{M}}(U_H^H)(1)[1] \rightarrow \tilde{\mathbb{M}}(U_I^V)$ which induces a commutative diagram

$$\begin{array}{ccccc} \tilde{\mathbb{M}}(U_H^H)(1)[1] & \xrightarrow{\beta^H} & \tilde{\mathbb{M}}(U_I^V) & \xrightarrow{\phi} & T \\ \alpha^Y(1)[1] \downarrow & & \downarrow \alpha^Y & \nearrow \phi_0 & \\ \tilde{\mathbb{M}}(U_H^H)(1)[1] & \xrightarrow{\beta^H} & \tilde{\mathbb{M}}(U_{I'}^V) & & \end{array}$$

in which the morphism $\alpha^Y(1)[1]$ on the left is split surjective. It follows that $\phi_0 \circ \beta^H \circ \alpha^Y(1)[1] = \phi \circ \beta^H = 0$ implies $\phi_0 \circ \beta^H = 0$. We conclude by induction. \square

Proposition 1.2.3. *Let S be an essentially smooth K -scheme and let $f_i \in \mathbb{G}_m(S)$ be such that $\sum_{i=0}^t f_i = 0$. Then*

$$R(f_0, \dots, f_t) = 0 \text{ in } \mathbf{H}_{\text{MW}}(S).$$

Proof. The global sections f_0, \dots, f_t yield a morphism $j = (f_0, \dots, f_t) : S \rightarrow \mathbb{A}_K^{t+1}$ which restrict to a morphism $j : S \rightarrow U_I^H$, where $H \subset \mathbb{A}_K^{t+1}$ is given by $\sum_{i=0}^t x_i = 0$ and $I = \{\{x_1 = 0\}, \dots, \{x_t = 0\}\}$. Since $R(f_0, \dots, f_t) = j^*(R(x_0, \dots, x_t))$, we can reduce the proposition to $S = U_I^H$.

For any x_j , we set $Y_j := \{x_j = 0\} \subset H$ and we obtain a Gysin morphism $\beta_j : \tilde{\mathbb{M}}(U_{Y_j}^{Y_j})(1)[1] \rightarrow \tilde{\mathbb{M}}(U_I^H)$ and a composite

$$\tilde{\mathbb{M}}(U_{Y_j}^{Y_j})(1)[1] \xrightarrow{\beta_j} \tilde{\mathbb{M}}(U_I^H) \xrightarrow{R(x_0, \dots, x_t)} \tilde{\mathbb{Z}}(t)[t].$$

By Remark 1 and Lemma 1.1.3,

$$\begin{aligned} R(x_0, \dots, x_t) \circ \beta_j &= (\varepsilon^j[x_j]R(x_0, \dots, \hat{x}_j, \dots, x_t) + P(x_0, \dots, \hat{x}_j, \dots, x_t)) \circ \beta_j \\ &= \varepsilon^j[x_j]R(x_0, \dots, \hat{x}_j, \dots, x_t) \circ \beta_j + P(x_0, \dots, \hat{x}_j, \dots, x_t) \circ \alpha_j \circ \beta_j \\ &= \varepsilon^j R(x_0|_{U_{Y_j}^{Y_j}}, \dots, \hat{x}_j, \dots, x_t|_{U_{Y_j}^{Y_j}}). \end{aligned}$$

As $R(f, -f) = 0$ for $f \in \mathbb{G}_m(S)$ by Remark 1, we obtain by induction that $R(x_0, \dots, x_t) \circ \beta_j = 0$ for any $j = 0, \dots, t$. Applying Lemma 1.2.2, we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{M}}(U_I^H) & \xrightarrow{R(x_0, \dots, x_t)} & \tilde{\mathbb{Z}}(t)[t] \\ \downarrow & \nearrow \psi & \\ \tilde{\mathbb{M}}(K) & & \end{array}$$

As $\text{char}(K) \neq 2$, U_I^H has a K -rational point $(\lambda_0, \dots, \lambda_t) \in \mathbb{A}_K^{t+1}$, and we obtain a diagram

$$\begin{array}{ccc}
 \tilde{M}(K) & & \\
 \downarrow u & \searrow R(\lambda_0, \dots, \lambda_t) & \\
 \tilde{M}(U_I^H) & \xrightarrow{R(x_0, \dots, x_t)} & \tilde{\mathbb{Z}}(t)[t] \\
 \downarrow & \nearrow \psi & \\
 \tilde{M}(K) & &
 \end{array}$$

The vertical composite being the identity, $\psi = R(\lambda_0, \dots, \lambda_t)$ and the latter is trivial by the relations in Milnor-Witt K -theory. \square

Applying lemma 1.2.1, we obtain the following corollary.

Corollary 1.2.4. *Let S be an essentially smooth smooth K -scheme.*

1. For any $f_1, \dots, f_t \in \mathbb{G}_m(S)$ such that $\sum_{i=1}^t f_i = 1$, we have

$$[f_1][f_2] \cdots [f_t] = 0 \in \mathbf{H}_{\text{MW}}(S).$$

2. For any $f \in \mathbb{G}_m(S)$, we have $[f]^2 - [-1][f] = 0$ in $\mathbf{H}_{\text{MW}}(S)$.

1.3 Proof of the main theorem

In this section, we prove Theorem 0.3.1. We denote by $J_U \subset \mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}$ the ideal generated by the relations

- (1) $(f) - [f]$, for $f \in K^\times \subset \mathbb{G}_m(U)$.
- (2) $(f) + (g) + \eta(f)(g) - (fg)$, for $f, g \in \mathbb{G}_m(U)$.
- (3) $(f_1)(f_2) \cdots (f_t)$, for any $f_1, \dots, f_t \in \mathbb{G}_m(U)$ such that $\sum_{i=1}^t f_i = 1$.
- (4) $(f)^2 - [-1](f)$, for $f \in \mathbb{G}_m(U)$.

By Lemma 1.2.1, $J_U \subset \mathbf{H}_{\text{MW}}(K)\{\mathbb{G}_m(U)\}$ is in fact generated by

- (1) $(f) - [f]$, for $f \in K^\times \subset \mathbb{G}_m(U)$.
- (2) $(f) + (g) + \eta(f)(g) - (fg)$, for $f, g \in \mathbb{G}_m(U)$.
- (3') (anti-commutativity) $(f)(g) - \varepsilon(g)(f)$ for any $f, g \in \mathbb{G}_m(U)$.
- (4') $R(f_0, \dots, f_t)$ given by

$$\sum_{i=0}^t \varepsilon^{t+i}(f_0) \cdots \widehat{(f_i)} \cdots (f_t) + \sum_{0 \leq i_0 < \dots < i_k \leq t} (-1)^k [-1]^k (f_0) \cdots \widehat{(f_{i_0})} \cdots \widehat{(f_{i_k})} \cdots (f_t),$$

for any $f_0, \dots, f_t \in \mathbb{G}_m(U)$ such that $\sum_{i=0}^t f_i = 0$.

In view of Corollary 1.2.4, the morphism $\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\} \rightarrow \mathrm{H}_{\mathrm{MW}}(U)$ defined by $(f) \mapsto [f]$ induces a morphism of $\mathrm{H}_{\mathrm{MW}}(K)$ -algebras

$$\rho : \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \rightarrow \mathrm{H}_{\mathrm{MW}}(U).$$

Now, choose linear polynomials ϕ_1, \dots, ϕ_s that define the hyperplanes $Y_i \in I$ and let $J'_U \subset \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ be the ideal generated by the relations (1), (2), (3') and (4') for elements of the form $f_j = \lambda_j \phi_{i_j}$ or $f_j = \lambda_j$ for $\lambda_j \in K^\times$ and $\phi_{i_j} \in \{\phi_1, \dots, \phi_s\}$. We have a string of surjective morphisms of $\mathrm{H}_{\mathrm{MW}}(K)$ -algebras

$$\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J'_U \rightarrow \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \xrightarrow{\rho} \mathrm{H}_{\mathrm{MW}}(U)$$

whose composite we denote by ρ' .

Theorem 1.3.1. *The morphism of $\mathrm{H}_{\mathrm{MW}}(K)$ algebra*

$$\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \xrightarrow{\rho} \mathrm{H}_{\mathrm{MW}}(U)$$

is an isomorphism.

Proof. It suffices to prove that ρ' is an isomorphism. To see this, we work again by induction on $|I|$. If $|I| = 0$, we have $U \cong \mathbb{A}_K^N$ for some $N \in \mathbb{N}$. By homotopy invariance, we have to prove that the map

$$\rho' : \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(K)\}/J'_K \rightarrow \mathrm{H}_{\mathrm{MW}}(K)$$

is an isomorphism. Now, the morphism of $\mathrm{H}_{\mathrm{MW}}(K)$ -algebras

$$\mathrm{H}_{\mathrm{MW}}(K) \rightarrow \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(K)\}/J'_K$$

is surjective by Relation 1. Its composite with ρ' is the identity and we conclude in that case.

Assume now that $Y \in I$ is defined by $\phi_1 = 0$ and that we have isomorphisms

$$\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I'}^Y)\}/J'_{U_{I'}^Y} \xrightarrow{\sim} \mathrm{H}_{\mathrm{MW}}(U_{I'}^Y),$$

$$\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I_Y}^Y)\}/J'_{U_{I_Y}^Y} \xrightarrow{\sim} \mathrm{H}_{\mathrm{MW}}(U_{I_Y}^Y).$$

The morphism $U_I^Y \rightarrow U_{I'}^Y$ induces a morphism $\mathbb{G}_m(U_{I'}^Y) \rightarrow \mathbb{G}_m(U_I^Y)$ and then a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I'}^Y)\}/J'_{U_{I'}^Y} & \xrightarrow{\tilde{\alpha}} & \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^Y)\}/J'_{U_I^Y} & \xrightarrow{\tilde{\beta}} & \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I_Y}^Y)\}/J'_{U_{I_Y}^Y} & \longrightarrow & 0 \\ \cong \downarrow & & \downarrow \rho' & & \downarrow \cong & & \\ 0 \longrightarrow & \mathrm{H}_{\mathrm{MW}}(U_{I'}^Y) & \xrightarrow{\alpha_*^Y} & \mathrm{H}_{\mathrm{MW}}(U_I^Y) & \xrightarrow{\beta_*^Y} & \mathrm{H}_{\mathrm{MW}}(U_{I_Y}^Y) & \longrightarrow 0 \end{array}$$

in which $\tilde{\beta}$ is the unique lift of $\beta_*^Y \circ \rho$ and the bottom row is exact. We are thus reduced to prove that the top sequence is short exact to conclude. It is straightforward to check that $\tilde{\alpha}$ is injective and $\tilde{\beta}$ is surjective. Moreover, the commutativity of the diagram and the fact that $\beta_*^Y \circ \alpha_*^Y = 0$ imply that $\tilde{\beta} \circ \tilde{\alpha} = 0$, so we are left to prove exactness in the middle.

Let $x \in \mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^Y)\}/J'_{U_I^Y}$. The group $\mathbb{G}_m(U_I^Y)$ being generated by $\mathbb{G}_m(U_{I'}^Y)$ and ϕ_1 , we may use relations (2) and (4) to see that $x = (\phi_1)\tilde{\alpha}(x_1) + \tilde{\alpha}(x_0)$ in $\mathrm{H}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^Y)\}/J'_{U_I^Y}$. By Lemma 1.1.3, we get $\tilde{\beta}(x) = \tilde{i}(x_1)$, where \tilde{i} is induced by the restriction $\mathbb{G}_m(U_{I'}^Y) \rightarrow \mathbb{G}_m(U_{I_Y}^Y)$. Consequently, we need to prove that if $\tilde{i}(x_1) = 0$ then $(\phi_1)\tilde{\alpha}(x_1)$ is in the image of $\tilde{\alpha}$. With this in mind, we now prove that the kernel of \tilde{i} is generated by elements of the form

$$R(f_0, \dots, f_t)$$

where $f_j = \lambda \phi_j$ with $i_j > 1$ or $f_j = \lambda$ and $\sum_{i=0}^t f_i|_{U_Y^Y} = 0$. Denote by L' the ideal of $\mathbb{H}_{\text{MW}}(K)\{\mathbb{G}_m(U_{I'}^V)\}$ generated by such elements. By construction, the restriction induces a homomorphism

$$L' + J'_{U_{I'}^V} \rightarrow J'_{U_{I'}^Y}$$

which is surjective. Indeed, relations (1), (2) and (3') can be lifted using the fact that the map $\mathbb{G}_m(U_{I'}^V) \rightarrow \mathbb{G}_m(U_{I'}^Y)$ is surjective, while an element satisfying relation (4) with every f_j of the form $f_j = \lambda_j \phi_j$ or $f_j = \lambda_j$ for $\lambda_j \in K^\times$ (with $i_j \neq 1$) lifts to an element in L' . As in [Chatzistamatiou2007, proof of Theorem 3.5], we see that the kernel of the group homomorphism $\mathbb{G}_m(U_{I'}^V) \rightarrow \mathbb{G}_m(U_{I'}^Y)$ is generated by elements of the form

1. $\lambda \frac{\phi_i}{\phi_j}$ with i, j such that $Y_1 \cap Y_i = Y_1 \cap Y_j$ and $\lambda = \frac{(\phi_j)|_{Y_1}}{(\phi_i)|_{Y_1}}$.
2. $\lambda \phi_i$ where i is such that $Y_1 \cap Y_i = \emptyset$ and $\lambda = \frac{1}{(\phi_i)|_{Y_1}}$.

Remark 1 yields $[\frac{\lambda \cdot \phi_i}{\phi_j}] = -\langle \phi_j^{-1} \rangle R(\phi_j, -\lambda \cdot \phi_i) \subset L'$, while $[\lambda \phi_i] = \varepsilon R(-1, \lambda \cdot \phi_i) \subset L'$ showing that $\ker(\mathbb{G}_m(U_{I'}^V) \rightarrow \mathbb{G}_m(U_{I'}^Y)) \subset L' + J'_{U_{I'}^V}$. We deduce that $\ker(\tilde{\iota}) = L'$.

We now conclude. If $\tilde{\iota}(x_1) = 0$, then $x_1 \in L'$ and we may suppose that $x_1 = R(f_0, \dots, f_t)$ for f_0, \dots, f_t such that $\sum_{i=0}^t f_i|_{U_Y^Y} = 0$. It follows that $\sum_{i=0}^t f_i = -\mu \phi_1$ for $\mu \in K$. If $\mu = 0$ there is nothing to do. Else, use $R(\mu \phi_1, f_0, \dots, f_t) = 0$ and Remark 1 to get

$$\begin{aligned} (\phi_1) \tilde{\alpha}(x_1) &= (\mu \phi_1) \tilde{\alpha}(x_1) - \langle \phi_1 \rangle (\mu) \tilde{\alpha}(x_1) \\ &= (\mu \phi_1) \tilde{\alpha}(x_1) + R(\mu \phi_1, f_0, \dots, f_t) - \langle \phi_1 \rangle (\mu) \tilde{\alpha}(x_1) \\ &= \tilde{\alpha}(P(f_0, \dots, f_t)) - \tilde{\alpha}(\langle \phi_1 \rangle (\mu) x_1) \in \text{image}(\tilde{\alpha}). \end{aligned}$$

□

Corollary 1.3.2. *The graded ring isomorphism of Theorem 1.3.1 induces an isomorphism*

$$\bigoplus_{n \in \mathbb{Z}} K_n^{\text{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \rightarrow \bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(U, \mathbb{Z}).$$

Proof. Notice that the ideal J_U of Theorem 1.3.1 is homogeneous, and it follows that $\bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(U, \mathbb{Z})$ can be computed as $H_{\text{MW}}^{*,*}(K)\{\mathbb{G}_m(U)\}/J_U$, where $H_{\text{MW}}^{*,*}(K)$ is the diagonal of $H_{\text{MW}}(K)$.

□

1.4 Combinatorial description

In this section, we fix an affine space $V = \mathbb{A}_K^N$, a family of hyperplanes I and we set $U := U_I^N$. We let $Q(U)$ be the cokernel of the group homomorphism $\mathbb{G}_m(K) \rightarrow \mathbb{G}_m(U)$, and we observe that the divisor map

$$\mathbb{G}_m(U) \xrightarrow{\text{div}} \bigoplus_{Y_i \in I} \mathbb{Z} \cdot Y_i$$

in \mathbb{A}_K^N induces an isomorphism $Q(U) \cong \bigoplus_{Y_i \in I} \mathbb{Z} \cdot Y_i$. We consider the exterior algebra $\Lambda_{\mathbb{Z}} Q(U)$ and write $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U) := \mathbb{Z}[\eta]/2\eta \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}} Q(U)$. The abelian group $Q(U)$ being free, the $\mathbb{Z}[\eta]/2\eta$ -module $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$ is also free, with the same basis. To provide a combinatorial description of $H_{\text{MW}}(U)$, we will have to slightly modify the definition of the divisor map above, in order to incorporate the action of η . We then define a map

$$\mathbb{G}_m(U) \xrightarrow{\tilde{\text{div}}} \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$$

as follows:

1. If $f = \lambda\phi$ or $f = \lambda$ where $\lambda \in \mathbb{G}_m(K)$ and ϕ is a linear polynomial as above, then $\widetilde{\text{div}}(f) = \text{div}(f)$.
2. If $f, g \in \mathbb{G}_m(U)$ then $\widetilde{\text{div}}(fg) = \widetilde{\text{div}}(f) + \widetilde{\text{div}}(g) + \eta \cdot \widetilde{\text{div}}(f) \wedge \widetilde{\text{div}}(g)$.

Lemma 1.4.1. *The map $\widetilde{\text{div}}$ is well-defined.*

Proof. We first notice that $\widetilde{\text{div}}(fg) = \widetilde{\text{div}}(gf)$, since

$$\widetilde{\text{div}}(fg) - \widetilde{\text{div}}(gf) = \eta \cdot \widetilde{\text{div}}(f) \wedge \widetilde{\text{div}}(g) - \eta \cdot \widetilde{\text{div}}(g) \wedge \widetilde{\text{div}}(f) = 2\eta \cdot \widetilde{\text{div}}(f) \wedge \widetilde{\text{div}}(g) = 0.$$

Let $f_1, f_2, g_1, g_2 \in \mathbb{G}_m(U)$ be such that $f_1g_1 = f_2g_2$. Let $Y \in I$ be such that $f_i = Y^{n_i} \cdot f'_i$ with $\text{div}_Y(f'_i) = 0$ and $g_i = Y^{m_i} \cdot g'_i$ with $\text{div}_Y(g'_i) = 0$ for $i = 1, 2$ and $m_i, n_i \in \mathbb{Z}$. We get

$$\widetilde{\text{div}}(f_1g_1) = (m_1 + n_1) \cdot Y + \widetilde{\text{div}}(f'_1g'_1) + (m_1 + n_1)\eta(Y \wedge \widetilde{\text{div}}(f'_1g'_1))$$

and

$$\widetilde{\text{div}}(f_2g_2) = (m_2 + n_2) \cdot Y + \widetilde{\text{div}}(f'_2g'_2) + (m_2 + n_2)\eta(Y \wedge \widetilde{\text{div}}(f'_2g'_2)).$$

As $\Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)$ is free with usual basis, we deduce that $\widetilde{\text{div}}(f'_2g'_2) = \widetilde{\text{div}}(f'_1g'_1)$ which allows to conclude by induction on the number of non-trivial factors in the decomposition of f_1g_1 . \square

Let now $L_U \subset \Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)$ be the ideal generated by the following elements:

1. $Y_1 \wedge \dots \wedge Y_s$, for $Y_i \in I$ such that $Y_1 \cap \dots \cap Y_s = \emptyset$;
2. $\sum_{j=1}^s (-1)^k Y_1 \wedge \dots \wedge \widehat{Y}_j \wedge \dots \wedge Y_s$, for $Y_i \in I$ such that $Y_1 \cap \dots \cap Y_s \neq \emptyset$ and $\text{codim}(Y_1 \cap \dots \cap Y_s) < s$.

As a consequence of Lemma 1.4.1, the map $\widetilde{\text{div}}$ induces a morphism of $\mathbb{Z}[\eta]/2\eta$ -algebras

$$\psi : (\mathbb{Z}[\eta]/2\eta)\{\mathbb{G}_m(U)\} \rightarrow \Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)/L_U$$

It is now time to introduce the ring

$$A_0(U) := \mathbf{K}_*^{\text{MW}}(K)\{\mathbb{G}_m(U)\} / (J_U + K^\times \cdot \mathbf{K}_*^{\text{MW}}(K)\{\mathbb{G}_m(U)\}).$$

As $\varepsilon = -1 - [-1]\eta \sim -1$ in $A_0(U)$, it follows that $A_0(U)$ is an exterior algebra. Moreover, the coefficient ring $\mathbf{K}_*^{\text{MW}}(K)$ can be reduced to $\mathbf{K}_*^{\text{MW}}(K)/(K^\times \cdot \mathbf{K}_*^{\text{MW}}(K)) \cong \mathbb{Z}[\eta]/2\eta$.

Proposition 1.4.2. *The morphism of $\mathbb{Z}[\eta]/2\eta$ -algebras*

$$\psi : \mathbb{Z}[\eta]/2\eta\{\mathbb{G}_m(U)\} \rightarrow \Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)/L_U$$

induces an isomorphism

$$\Psi : A_0(U) \rightarrow \Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)/L_U.$$

Proof. We first prove that Ψ is well-defined, which amounts to show that the image of J_U is contained in L_U . For $f \in K^\times$, we have $[f] \in K^\times \cdot \mathbf{K}_*^{\text{MW}}(K)\{\mathbb{G}_m(U)\}$ and $\widetilde{\text{div}}(f) = 0$, showing that the first relation is satisfied. The second relation is satisfied by definition of $\widetilde{\text{div}}$, while relation (3') is satisfied as $\Lambda_{\mathbb{Z}[\eta]/2\eta}\mathcal{Q}(U)/L_U$ is an exterior algebra. As in the proof of Theorem 1.3.1, we are then left with elements of J'_U , i.e. elements of the form $R(f_0, \dots, f_t)$ for $\sum_{i=0}^t f_i = 0$, where $f_j = \lambda_j \phi_j$ or $f_j = \lambda_j$. Modulo $K^\times \cdot \mathbf{K}_*^{\text{MW}}(K)\{\mathbb{G}_m(U)\}$, we have $R(f_0, \dots, f_t) \sim \sum_{i=0}^t (-1)^{t+i} [f_0] \dots \widehat{[f_i]} \dots [f_t]$ and we just need to prove that

$$\alpha := (-1)^t \psi(R(f_0, \dots, f_t)) = \sum_{i=0}^t (-1)^i \widetilde{\text{div}}(f_0) \wedge \dots \wedge \widehat{\widetilde{\text{div}}(f_i)} \wedge \dots \wedge \widetilde{\text{div}}(f_t)$$

is an element of L_U . Note that if there are more than two constant functions among f_j , α would be trivial. Suppose that $f_0 = \lambda_0$ is the only constant, and let $f_j = \lambda_j \phi_j$ with kernel $Y_j \in I$, so that $\alpha = Y_1 \wedge \dots \wedge Y_t$. Since $\sum_{j=1}^t \lambda_j \phi_j = -\lambda_0 \neq 0$, we can easily get that $Y_1 \cap \dots \cap Y_t = \emptyset$ and $\alpha = Y_1 \wedge \dots \wedge Y_t \in L_U$. In the case where none of the f_j are constant, $\alpha = \sum_{i=0}^t (-1)^i Y_0 \wedge \dots \wedge \widehat{Y}_i \wedge \dots \wedge Y_t$. And for every i , we have $\sum_{j=0, j \neq i}^t \lambda_j \phi_j = -\lambda_i \phi_i$, which means $Y_i \subseteq Y_0 \cap \dots \cap \widehat{Y}_i \cap \dots \cap Y_t = Y_0 \cap \dots \cap Y_t$. If $Y_0 \cap \dots \cap Y_t = \emptyset$, so is $Y_0 \cap \dots \cap \widehat{Y}_i \cap \dots \cap Y_t$, thus $Y_0 \wedge \dots \wedge \widehat{Y}_i \wedge \dots \wedge Y_t \in L_U$; else, $\text{codim}(Y_0 \cap \dots \cap Y_t) = \text{codim}(Y_0 \cap \dots \cap \widehat{Y}_i \cap \dots \cap Y_t) \leq t < t + 1$, which just fits the condition (2) of L_U . This proves that Ψ is well-defined.

To prove that Ψ is an isomorphism, we construct the inverse map by

$$\Phi : \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U \rightarrow A_0(U), Y_i \mapsto (\phi_i)$$

and prove that it is well-defined. As above, we just need to discuss elements of L_U . If $Y_1 \cap \dots \cap Y_s = \emptyset$, then we can find $\lambda_i \in K^\times$ such that $\sum_i \lambda_i \phi_i = 1$, and thus $(\phi_1) \cdots (\phi_s) \sim (\lambda_1 \phi_1) \cdots (\lambda_s \phi_s) = 0$ in $A_0(U)$. In the case $\text{codim}(Y_1 \cap \dots \cap Y_s) < s$, we have $\sum_i \lambda_i \phi_i = 0$ for some $\lambda_i \in K^\times$. Then $\sum_{i=1}^s (-1)^i (\phi_1) \cdots (\widehat{\phi_i}) \cdots (\phi_s) \sim (-1)^{s-1} R(\lambda_1 \phi_1, \dots, \lambda_s \phi_s) = 0$ in $A_0(U)$. This shows that the inverse map is well-defined. \square

The following corollary shows that the rank of the free $H_{\text{MW}}(K)$ -module $H_{\text{MW}}(U)$ is exactly the same as the rank of the free $H_{\text{M}}(K)$ -module $H_{\text{M}}(U)$ [Chatzistamatiou2007, Proposition 3.11].

Corollary 1.4.3. *The rank of the free $H_{\text{MW}}(K)$ -module $H_{\text{MW}}(U)$ is equal to the rank of the free module $\Lambda_{\mathbb{Z}} Q(U)/L_U$.*

Proof. It is clear that $\text{rk}_{\mathbb{Z}[\eta]/2\eta}(\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U) = \text{rk}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}} Q(U)/L_U)$. As all generators in $H_{\text{MW}}(U)$ are from $H_{\text{MW}}^{p,p}(U, \mathbb{Z})$, we have

$$\text{rk}_{H_{\text{MW}}(K)}(H_{\text{MW}}(U)) = \text{rk}_{\mathbf{K}_*^{\text{MW}}(K)}\left(\bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(U, \mathbb{Z})\right) = \text{rk}_{\mathbb{Z}[\eta]/2\eta}(A_0(U))$$

\square

1.5 I-cohomology and singular cohomology

As already introduced above, there is a realization functor from ordinary motivic cohomology and the singular cohomology of complex points. This yields the following comparative result.

Proposition 1.5.1. [Chatzistamatiou2007, Proposition 3.9] *In the case $K = \mathbb{C}$, there is an isomorphism of rings:*

$$\bigoplus_n H_{\text{M}}^{n,n}(U, \mathbb{Q}) \otimes_{H_{\text{M}}(K)} \mathbf{K}_*^{\text{M}}(K)/K^\times \cdot \mathbf{K}_*^{\text{M}}(K) \xrightarrow{\cong} \bigoplus_n H_{\text{sing}}^n(U(\mathbb{C}), \mathbb{Q}).$$

In this section, we provide an analogue for the singular cohomology of the real points of the complement of a hyperplane arrangement defined over \mathbb{R} . We start with some results about the I-cohomology [Fasel2008].

As recalled in Section 0.2, we have natural homomorphisms from Milnor-Witt motivic cohomology to \mathbf{I}^* -cohomology

$$H_{\text{MW}}^{p,q}(X, \mathbb{Z}) \rightarrow H^{p-q}(X, \mathbf{K}_q^{\text{MW}}) \rightarrow H^{p-q}(X, \mathbf{I}^q)$$

which induce a ring homomorphism $H_{\text{MW}}(X) \rightarrow \bigoplus_{r,q} H^r(X, \mathbf{I}^q)$ (where $\mathbf{I}^q = \mathbf{K}_q^{\text{MW}} = \mathbf{W}$ for $q < 0$). In case $X = \text{Spec}(K)$, we obtain in particular a ring homomorphism $H_{\text{MW}}(K) \rightarrow \bigoplus_{r,q} H^r(K, \mathbf{I}^q) = \bigoplus_{q \in \mathbb{Z}} \mathbf{I}^q(K)$

Proposition 1.5.2. *The morphism of $\bigoplus_{q \in \mathbb{Z}} \mathbf{I}^q(K)$ -algebras*

$$j : \mathbf{H}_{\text{MW}}(U) \otimes_{\mathbf{H}_{\text{MW}}(K)} \left(\bigoplus_{q \in \mathbb{Z}} \mathbf{I}^q(K) \right) \rightarrow \bigoplus_{r,q} \mathbf{H}^r(U, \mathbf{I}^q)$$

is an isomorphism. Moreover $\mathbf{H}^r(U, \mathbf{I}^q) = 0$, for $r \neq 0$.

Proof. Throughout the proof, we write $\mathbf{H}_{\text{MW}}(U) \otimes \mathbf{I}$ for the graded ring $\mathbf{H}_{\text{MW}}(U) \otimes_{\mathbf{H}_{\text{MW}}(K)} \left(\bigoplus_q \mathbf{I}^q(K) \right)$. We follow the same induction process as in the proof of the main theorem. When $|I| = 0$, we only need to consider $\text{Spec}(K)$ by homotopy invariance, and the result is trivial.

Assume now that $Y \in I$ and that we have isomorphisms for $U_{I'}^Y$ and $U_{I_Y}^Y$. Notice that for \mathbf{I} -cohomology we still have a Gysin long exact sequence [Fasel2008, Remarque 9.3.5]. The proof of the main theorem yields the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{H}_{\text{MW}}(U_{I'}^Y) \otimes \mathbf{I} & \rightarrow & \mathbf{H}_{\text{MW}}(U_I^Y) \otimes \mathbf{I} & \rightarrow & \mathbf{H}_{\text{MW}}(U_{I_Y}^Y) \otimes \mathbf{I} \longrightarrow 0 \\ & & \cong \downarrow & & j \downarrow & & \cong \downarrow \\ \bigoplus_q \mathbf{H}^{-1}(U_{I_Y}^Y, \mathbf{I}^{q-1}) & \rightarrow & \bigoplus_q \mathbf{H}^0(U_{I'}^Y, \mathbf{I}^q) & \rightarrow & \bigoplus_q \mathbf{H}^0(U_I^Y, \mathbf{I}^q) & \rightarrow & \bigoplus_q \mathbf{H}^0(U_{I_Y}^Y, \mathbf{I}^{q-1}) \rightarrow \bigoplus_q \mathbf{H}^1(U_{I'}^Y, \mathbf{I}^q) \end{array}$$

By our assumption, $\mathbf{H}^{-1}(U_{I_Y}^Y, \mathbf{I}^{q-1})$ and $\mathbf{H}^1(U_{I'}^Y, \mathbf{I}^q)$ are both 0, so the second line is also short exact. We conclude that j is an isomorphism as well. The same argument implies that $\mathbf{H}^r(U_I^Y, \mathbf{I}^q) = 0$ for $r \neq 0$. \square

The analogue of Corollary 1.1.2 in this setting then reads as follows.

Corollary 1.5.3. *There is a finite set J and integers $n_j \geq 0$ for any $j \in J$ such that*

$$\mathbf{H}^0(U_I^Y, \mathbf{I}^q) \cong \bigoplus_{j \in J} \mathbf{I}^{q-n_j}(K) b_j$$

as a free $\bigoplus_q \mathbf{I}^q(K)$ -module with basis $b_j \in \mathbf{H}^0(U_I^Y, \mathbf{I}^{n_j})$.

Proof. Every step is the same as in Proposition 1.1.1, except the splitting, which comes from the identification with $\mathbf{H}_{\text{MW}}(U_I^Y) \otimes \mathbf{I}$. \square

As in [Jacobson2017] and [Hornbostel2019], we can compute the cohomology of the real spectrum using \mathbf{I} -cohomology.

Proposition 1.5.4. [Hornbostel2019, Proposition 3.6] *The signature map induces an isomorphism*

$$\mathbf{H}^r(X, \text{Colim}_{q \geq 0} \mathbf{I}^q) \xrightarrow{\text{sign}_\infty} \mathbf{H}_{\text{sing}}^r(\text{Sper}(X), \mathbb{Z})$$

where $\text{Sper}(X)$ is the real spectrum of X . In particular,

$$\text{Colim}_{q \geq 0} \mathbf{I}^q(K) \cong \mathbf{H}_{\text{sing}}^0(\text{Sper}(K), \mathbb{Z}).$$

In our case, since U is always noetherian and $\text{Colim}_{q \geq 0}$ is filtered, we have a canonical isomorphism

$$\mathbf{H}^r(U, \text{Colim}_{q \geq 0} \mathbf{I}^q) \cong \text{Colim}_{q \geq 0} \mathbf{H}^r(U, \mathbf{I}^q).$$

Combining with Corollary 1.5.3, we obtain following proposition.

Proposition 1.5.5. *There exists an integer $N > 0$ such that*

$$\mathbf{H}^0(U_I^Y, \mathbf{I}^N) \otimes_{\bigoplus_{q \geq 0} \mathbf{I}^q(K)} \mathbf{H}_{\text{sing}}^0(\text{Sper}(K), \mathbb{Z}) \xrightarrow{2^{-N} \text{sign}} \mathbf{H}_{\text{sing}}^0(\text{Sper}(U_I^Y), \mathbb{Z})$$

is an isomorphism. Moreover $\mathbf{H}_{\text{sing}}^r(\text{Sper}(U_I^Y), \mathbb{Z}) = 0$ for $r \neq 0$.

Proof. As discussed above, we can rewrite the right-hand side as $\text{Colim}_{q \geq 0} \mathbf{H}^0(U_I^V, \mathbf{I}^q)$. Applying Corollary 1.5.3, we get

$$\begin{aligned} \text{Colim}_{q \geq 0} (\oplus_{j \in J} \mathbf{I}^{q-n_j}(K) b_j) &\cong \oplus_{j \in J} (\text{Colim}_{q \geq 0} \mathbf{I}^{q-n_j}(K) b_j) \\ &\cong \oplus_{j \in J} \mathbf{H}_{\text{sing}}^0(\text{Sper}(K), \mathbb{Z}) b_j. \end{aligned}$$

Let $N \in \mathbb{N}$ be such that $\forall j \in J, N \geq n_j$. Using again Corollary 1.5.3,

$$\mathbf{H}^0(U_I^V, \mathbf{I}^N) \cong \oplus_{j \in J} \mathbf{I}^{N-n_j}(K) b_j$$

which implies

$$\oplus_{j \in J} \mathbf{I}^{N-n_j}(K) b_j \otimes_{(\oplus_{q \geq 0} \mathbf{I}^q(K))} \mathbf{H}_{\text{sing}}^0(\text{Sper}(K), \mathbb{Z}) \cong \oplus_{j \in J} \mathbf{H}_{\text{sing}}^0(\text{Sper}(K), \mathbb{Z}) b_j$$

since for every j we have $N - n_j \geq 0$. That proves the first part, while the second part is trivial. \square

Taking $K = \mathbb{R}$, we have $\mathbf{H}_{\text{sing}}^0(\mathbb{R}, \mathbb{Z}) = \mathbb{Z}$ and we recover the classical result for complements of hyperplane arrangements:

$$\mathbf{H}^0(U_I^V, \mathbf{I}^N) \xrightarrow[\cong]{2^{-N} \text{sign}} \mathbf{H}_{\text{sing}}^0(U_I^V(\mathbb{R}), \mathbb{Z}) \cong \bigoplus_{R_i \in \text{connected components}} \mathbb{Z}\{R_i\}.$$

2

Computations on symplectic groups

In this chapter, we compute the MW-motivic cohomology of the symplectic groups Sp_{2n} for any $n \in \mathbb{N}$ using the Sp-orientation and the associated Borel classes.

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In this section, our focus is on the additive structure of the cohomology groups of general Sp-oriented theories. It is important to note that all the isomorphisms discussed in this section are up to modules. In section 4, we will study the multiplicative structure, specifically for H_{MW} .

2.1 Recollections on oriented cohomology theories.

In our context, a G -orientation of a cohomology theory (i.e. a spectrum) E is simply a morphism $\theta : MG \rightarrow E$, where MG is the Thom spectrum associated with the classifying space BG of the group G . If such an orientation exists, we say that E is G -oriented. One of the main cases of interest is the Sp-oriented theory [Panin2023].

The canonical morphism

$$B\text{Sp}_n \rightarrow M\text{Sp}_n \rightarrow \Sigma^{2n,n}M\text{Sp}$$

gives rise to a universal Euler class $\theta \in M\text{Sp}^{2n,n}(B\text{Sp}_n)$, which can also be defined for Sp-oriented theories as $\theta \in E^{2n,n}(B\text{Sp}_n)$ through the orientation.

In fact, for Sp-oriented theories, we have the splitting principle, which allows us to define Borel classes (e.g. [Panin2022])

$$b_j \in E^{2j,j}(B\text{Sp}_{2n})$$

for $j \leq n$, satisfying $E^{*,*}(B\text{Sp}_{2n}) \cong E^{*,*}(K)[[b_1, \dots, b_n]]$ with $b_n = \theta$.

In particular, the Milnor-Witt motivic cohomology H_{MW} is Sp-oriented but not GL-oriented.

2.2 Suslin's theorem

Let $E \in \mathcal{SH}(S)$ be a ring spectrum, and let G be a smooth scheme over S . We aim to compute the cohomology ring $E^{*,*}(G) = \bigoplus E^{p,q}(G)$ as an $E^{*,*}(S)$ -algebra.

Example 1. Consider $G = \mathbb{A}_S^n \setminus \{0\}$. Since $E^{p,q}(\mathbb{A}_S^n \setminus \{0\}) \cong E^{p,q}(S) \oplus E^{p-(2n-1),q-n}(S)$, we obtain

$$E^{*,*}(\mathbb{A}_S^n \setminus \{0\}) \cong E^{*,*}(S) \cdot 1 \oplus E^{*,*}(S) \cdot \theta_n \cong E^{*,*}(S)[\theta_n]/(\theta_n^2), n > 1$$

and

$$E^{*,*}(\mathbb{A}_S^1 \setminus \{0\}) \cong E^{*,*}(S)[\theta_1]/(\theta_1^2 - [-1]\theta_1),$$

where $\theta_n \in E^{2n-1,n}(\mathbb{A}_S^n \setminus \{0\})$ is the canonical element induced by the unit of the spectrum.

To compute the cohomology ring $E^{*,*}(G_n)$ for algebraic groups, we define a family $\{G_n\}, n \in \mathbb{N}$ of S -schemes to be a k -**fiber family** over S for some $k \in \mathbb{N}^*$ if there exists for each G_n a Zariski locally trivial fibration

$$G_{n-1} \hookrightarrow G_n \xrightarrow{f_n} \mathbb{A}_S^{kn} \setminus \{0\},$$

with $G_0 = S$ and a convenient choice of base point for $\mathbb{A}_S^{kn} \setminus \{0\}$. For any morphism $T \rightarrow S$, the base-changed family $G_{n,T} = G_n \times_S T$ is also a k -fiber family over T .

Example 2. $\{\text{Sp}_{2n} \times S | n \in \mathbb{N}\}$ is a 2-fiber family over S , and $\{\text{GL}_n \times S | n \in \mathbb{N}\}$ is a 1-fiber family over S .

We can now state our theorem for computing the cohomology ring $E^{*,*}(G_n)$ of a k -fiber family, which is based on Suslin's result [Suslin1991].

Theorem 2.2.1. For $k > 1$, consider a k -fiber family over S and a ring spectrum $E \in \mathcal{S}\mathcal{H}(S)$. Assume that for all n , G_n has series of classes $c(n)_j \in E^{2kj-1, kj}(G_n)$, where $1 \leq j \leq n$, such that:

1. $c(n)_n = uf_n^*(\theta_{kn})$, where u is a unit;
2. for any fiber embedding $i: G_{n-1} \hookrightarrow G_n$, the generators $(i^*(c(n)_1), \dots, i^*(c(n)_{n-1}))$ are isomorphic to $(c(n-1)_1, \dots, c(n-1)_{n-1})$ up to an isomorphism in $GL_{n-1}(\mathbb{Z})$.

Then the cohomology ring $E^{*,*}(G_n)$ is a freely generated square-zero algebra.

$$E^{*,*}(G_n) \cong E^{*,*}(S)[c(n)_i] / (c(n)_i^2, 1 \leq i \leq n)$$

and hence, it is free as $E^{*,*}(S)$ -module with basis $c(n)_{i_1} \cup \dots \cup c(n)_{i_j}, 1 \leq i_1 < \dots < i_j \leq n$ and $0 \leq j \leq n$.

Remark 2. The assumption $k > 1$ was only made to avoid the exceptional case for $E^{*,*}(\mathbb{A}_S^1 \setminus \{0\})$. For the module structure, the theorem holds for all $k > 0$.

Since the cohomology ring is free as a module, we have the Kunnetth formula and the following corollary holds.

Corollary 2.2.2. The base changed k -fiber family $G_{n,T} = G_n \times_S T$ and the classes $p_2^*(c(n)_i)$ also satisfy the conditions of theorem 2.2.1, and we have

$$E^{*,*}(G_{n,T}) \cong E^{*,*}(T) \otimes_{E^{*,*}(S)} E^{*,*}(G_n)$$

Giving a morphism of ring spectra $\sigma: E \rightarrow F$, if E satisfies the conditions of theorem 2.2.1, then so do F : The classes are given by $\sigma(c(n)_j)$. This leads to another corollary.

Corollary 2.2.3. With the setting above, we have an isomorphism

$$F^{*,*}(G_n) \cong F^{*,*}(S) \otimes_{E^{*,*}(S)} E^{*,*}(G_n)$$

To prove Theorem 2.2.1, we need a simple lemma.

Lemma 2.2.4. Let $f: X \rightarrow Y$ be a morphism of smooth schemes and let g_1, \dots, g_n be some homogeneous elements of $E^{*,*}(X)$. Suppose that there exist a finite open covering $Y = \bigcup U_i$ such that for any i and any open $V \subset U_i$, the ring $E^{*,*}(f^{-1}(V))$ is a free $E^{*,*}(V)$ -module with basis $g_k|_{f^{-1}(V)}$. Then $E^{*,*}(X)$ is a free $E^{*,*}(Y)$ -module with basis $\{g_k\}$

Proof. Applying the Mayer-Vietoris long exact sequence, we obtain a commutative diagram

$$\begin{array}{ccccc} E^{i,*}(V_1 \cup V_2) \otimes \mathbb{Z}^{n_i} & \longrightarrow & (E^{i,*}(V_1) \oplus E^{i,*}(V_2)) \otimes \mathbb{Z}^{n_i} & \longrightarrow & E^{i,*}(V_1 \cap V_2) \otimes \mathbb{Z}^{n_i} \\ \alpha \downarrow & & \cong \downarrow & & \downarrow \cong \\ E^{i,*}(f^{-1}(V_1 \cup V_2)) & \longrightarrow & E^{i,*}(f^{-1}(V_1)) \oplus E^{i,*}(f^{-1}(V_2)) & \longrightarrow & E^{i,*}(f^{-1}(V_1 \cap V_2)) \end{array}$$

where the vertical arrows are obtained by making use of the basis g_k . As both rows are exact, α is an isomorphism by the 5-lemma. Note that the g_k gives the sections of $R^i f_* E$. \square

Remark 3. This lemma actually gives a condition for the Leray spectral sequence to collapse at the E_2 -page. We will see more of this perspective in the next part of the article.

Proof of Theorem 2.2.1. We prove it by induction on n . For $n = 1$, this can be concluded by Example 1 since $G_1 \cong \mathbb{A}_S^k \setminus \{0\}$.

For general n we apply the lemma to f_n , then by our assumption for any open $V \subset U_i$, $f_n^{-1}(V) = G_{n-1} \times_S V$ and $c(n)_j|_{f_n^{-1}(V)}$ coincides with $p_2^* i^*(c(n)_j)$ in $E^{*,*}(f_n^{-1}(V))$. Using Assumption 2 and the induction hypothesis, we get that $(p_2^* i^*(c(n)_1), \dots, p_2^* i^*(c(n)_{n-1})) \sim (p_2^*(c(n-1)_1), \dots, p_2^*(c(n-1)_{n-1}))$ forms generators of $E^{*,*}(f_n^{-1}(V)) \cong E^{*,*}(V) \otimes_{E^{*,*}(S)} E^{*,*}(G_{n-1})$. This implies that $E^{*,*}(G_n)$ is a free $E^{*,*}(\mathbb{A}_S^{kn} \setminus \{0\})$ square-zero algebra with generators $(c(n)_1, \dots, c(n)_{n-1})$, i.e. the basis is given by $c(n)_{i_1} \cup \dots \cup c(n)_{i_j}, 1 \leq i_1 < \dots < i_j \leq n-1$.

Then combining with Example 1, we can conclude that $E^{*,*}(G_n)$ is a free $E^{*,*}(S)$ square-zero algebra with generators $(c(n)_1, \dots, c(n)_{n-1}, f_n^*(\theta_{kn})) \sim (c(n)_1, \dots, c(n)_n)$. \square

2.3 Application to Sp-oriented cohomology

Thanks to the corollary on the base change formula, we omit from now on the subscript S , at least when it is clear from the context. For instance, we write $\mathbb{A}^n \setminus \{0\} := \mathbb{A}_S^n \setminus \{0\}$.

Recall that for any Sp-oriented ring spectrum $E \in \mathcal{S}\mathcal{H}(S)$, we can define Borel classes $b_j \in E^{4j,2j}(B\mathrm{Sp}_{2n}), j \leq n$. Using loop spaces, we can define a new series of classes $\beta(n)_j \in E^{4j-1,2j}(\mathrm{Sp}_{2n})$:

$$\beta(n)_j : \mathrm{Sp}_{2n} \cong \Omega B\mathrm{Sp}_{2n} \xrightarrow{\Omega b_j} \Sigma^{4j-1,2j} E.$$

Note further that Sp_{2n} is a 2-fiber family since we have (locally split) fiber sequences for any $n \in \mathbb{N}$:

$$\mathrm{Sp}_{2n-2} \hookrightarrow \mathrm{Sp}_{2n} \xrightarrow{p_n} \mathbb{A}^{2n} \setminus \{0\}.$$

Proposition 2.3.1. *For any Sp-oriented ring spectrum E , Sp_{2n} and $\beta(n)_j$ satisfy the conditions of Theorem 2.2.1.*

Proof. We just need to prove the result for the universal Sp-oriented spectrum $\mathrm{M}\mathrm{Sp}$. Condition 2 is automatically satisfied by the properties of Borel classes. We only need to verify that $\beta(n)_n$ satisfy Condition 1. For this, we consider the morphism g defined as the looping of the canonical map

$$B\mathrm{Sp}_{2n} \rightarrow \mathrm{Th}(\mathcal{U}_{2n})$$

where $\mathrm{Th}(\mathcal{U}_{2n})$ is the Thom space of the tautological symplectic bundle \mathcal{U}_{2n} over $B\mathrm{Sp}_{2n}$, and the morphism f induced by the following diagram, where $\mathcal{U}_{2n}^* = \mathcal{U}_{2n} \setminus B\mathrm{Sp}_{2n}$ is the complement of zero section and i is the fiber embedding along $S = \mathrm{Sp}_0 \rightarrow \mathrm{Sp}_{2n}$

$$\begin{array}{ccc} \mathbb{A}^{2n} \setminus \{0\} & \longrightarrow & \mathbb{A}^{2n} \\ \downarrow & \lrcorner & \downarrow i \\ \mathcal{U}_{2n}^* & \longrightarrow & \mathcal{U}_{2n} \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{Th}(\mathcal{U}_{2n}) \end{array}$$

To prove the statement, we need to show that the diagram

$$\begin{array}{ccc} \mathrm{Sp}_{2n} & \xrightarrow{p_n} & \mathbb{A}^{2n} \setminus \{0\} \\ & \searrow g & \downarrow f \\ & & \Omega \mathrm{Th}(\mathcal{U}_{2n}) \theta_{2n} \\ & \searrow \beta(n)_n & \downarrow \\ & & \Sigma^{4n-1,2n} \mathrm{M}\mathrm{Sp} \end{array}$$

is commutative, which amounts to show that $g = f p_n$.

This comes from the following commutative diagram, where z is the zero section, and the fact

that rh coincides with i in the diagram above.

$$\begin{array}{ccccc}
 \mathrm{Sp}_{2n} & \xrightarrow{P_n} & \mathbb{A}^{2n} \setminus \{0\} & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow h & \lrcorner & \downarrow \\
 * & \longrightarrow & \mathrm{BSp}_{2n-2} & \longrightarrow & \mathrm{BSp}_{2n} \\
 & & \downarrow r \cong & & \downarrow z \cong \\
 & & \mathcal{U}_{2n}^* & \longrightarrow & \mathcal{U}_{2n} \\
 & & \downarrow & \lrcorner & \downarrow \\
 & & * & \longrightarrow & \mathrm{Th}(\mathcal{U}_{2n})
 \end{array}$$

□

Theorem 2.3.2. *The graded ring $H_{\mathrm{MW}}^{*,*}(\mathrm{Sp}_{2n})$ is a free $H_{\mathrm{MW}}^{*,*}(S)$ -square-zero algebra generated by the classes $\beta(n)_j \in E^{4j-1,2j}(\mathrm{Sp}_{2n})$ for $j = 1, \dots, n$.*

Proof. We note that $H_{\mathrm{MW}}^{*,*}(\mathrm{Sp}_{2n})$ is Sp -oriented, and combining this with our previous results yields the theorem. □

We now establish the cofiber sequences required to obtain a motivic decomposition of Sp_{2n} .

Lemma 2.3.3. *For any $n \geq 2$, there exists a cofiber sequence in $\mathcal{H}(S)$*

$$(\mathrm{Sp}_{2n-2})_+ \rightarrow (\mathrm{Sp}_{2n})_+ \rightarrow (\mathrm{Sp}_{2n-2})_+ \wedge (\mathbb{A}^{2n} \setminus \{0\})$$

Proof. We provide a proof of this lemma in Appendix A using matrix computations. We also refer to Section 3.2, where we will see similar computations. □

With this sequence in place, we can now derive the required decomposition result.

Theorem 2.3.4. *The cofiber sequence of Lemma 2.3.3 splits in $\widetilde{\mathrm{DM}}(S)$. In particular, it induces an isomorphism*

$$\widetilde{\mathrm{M}}(\mathrm{Sp}_{2n}) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \widetilde{\mathrm{M}}(\mathrm{Sp}_{2n-2})$$

Moreover, the following decomposition holds in $\widetilde{\mathrm{DM}}(S)$:

$$\widetilde{\mathrm{M}}(\mathrm{Sp}_{2n}) \cong \widetilde{\mathrm{M}}(\mathbb{A}^{2n} \setminus \{0\}) \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2n-2} \setminus \{0\}) \otimes \dots \otimes \widetilde{\mathrm{M}}(\mathbb{A}^2 \setminus \{0\}) \cong \bigoplus_{1 \leq i_1 < \dots < i_j \leq n} \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j)).$$

where $\widetilde{\mathbb{Z}}(d(i_1, \dots, i_j))$ stands for $\widetilde{\mathbb{Z}}(\sum_{l=1}^j 2i_l)[4(\sum_{l=1}^j i_l) - j]$.

Proof. We will prove this theorem by induction on n . For the base case $n = 1$, we note that $\mathrm{Sp}_2 \cong \mathbb{A}^2 \setminus \{0\}$, and that $\widetilde{\mathrm{M}}(\mathbb{A}^2 \setminus \{0\}) \cong \widetilde{\mathbb{Z}} \oplus \widetilde{\mathbb{Z}}(2)[3]$.

Now, suppose the theorem holds for some $n - 1 \geq 1$. If the sequence in Lemma 2.3.3 splits after taking $\widetilde{\mathrm{M}}$, we then have

$$\begin{aligned}
 \widetilde{\mathrm{M}}(\mathrm{Sp}_{2n}) &\cong \widetilde{\mathrm{M}}(\mathrm{Sp}_{2n-2}) \oplus (\widetilde{\mathrm{M}}(\mathrm{Sp}_{2n-2}) \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2n} \setminus \{0\}) / \widetilde{\mathbb{Z}}) \\
 &\cong \bigoplus_{1 \leq i_1 < \dots < i_j \leq n-1} \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j)) \oplus \bigoplus_{1 \leq i_1 < \dots < i_j \leq n-1} \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j, n))
 \end{aligned}$$

To complete the proof, we need to show that the triangle

$$\widetilde{\mathrm{M}}(\mathrm{Sp}_{2n-2}) \xrightarrow{i} \widetilde{\mathrm{M}}(\mathrm{Sp}_{2n}) \rightarrow \widetilde{\mathrm{M}}(\mathrm{Sp}_{2n-2}) \otimes \widetilde{\mathrm{M}}(\mathbb{A}^{2n} \setminus \{0\}) / \widetilde{\mathbb{Z}} \xrightarrow{+1}$$

splits via a section $s : \widetilde{M}(\mathrm{Sp}_{2n}) \rightarrow \widetilde{M}(\mathrm{Sp}_{2n-2})$ such that $si = \mathrm{id}$.

As $\widetilde{M}(\mathrm{Sp}_{2n-2}) \cong \bigoplus_{1 \leq i_1 < \dots < i_j \leq n-1} \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j))$, we get the following diagram:

$$\begin{array}{ccc}
 \mathrm{Hom}_{\widetilde{\mathrm{DM}}}(\widetilde{M}(\mathrm{Sp}_{2n}), \widetilde{M}(\mathrm{Sp}_{2n-2})) & \xrightarrow{i^*} & \mathrm{Hom}_{\widetilde{\mathrm{DM}}}(\widetilde{M}(\mathrm{Sp}_{2n-2}), \widetilde{M}(\mathrm{Sp}_{2n-2})) \\
 \downarrow p_{i_1, \dots, i_j} & & \downarrow p_{i_1, \dots, i_j} \\
 \mathrm{Hom}_{\widetilde{\mathrm{DM}}}(\widetilde{M}(\mathrm{Sp}_{2n}), \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j))) & \xrightarrow{i^*} & \mathrm{Hom}_{\widetilde{\mathrm{DM}}}(\widetilde{M}(\mathrm{Sp}_{2n-2}), \widetilde{\mathbb{Z}}(d(i_1, \dots, i_j))) \\
 \downarrow & & \downarrow \\
 \mathrm{H}_{\mathrm{MW}}^{d(i_1, \dots, i_j)}(\mathrm{Sp}_{2n}, \mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}_{\mathrm{MW}}^{d(i_1, \dots, i_j)}(\mathrm{Sp}_{2n-2}, \mathbb{Z})
 \end{array}$$

where p_{i_1, \dots, i_j} is the projector onto the direct summand of $\widetilde{M}(\mathrm{Sp}_{2n-2})$ corresponding to (i_1, \dots, i_j) . Since we have $i^*(\beta(n)_j) = \beta(n-1)_j$ for $1 \leq j \leq n-1$, we can deduce that i^* is surjective using Corollary 2.3.2 on the bottom row. Thus the first line of the diagram is also surjective. Therefore we can define such $s : \widetilde{M}(\mathrm{Sp}_{2n}) \rightarrow \widetilde{M}(\mathrm{Sp}_{2n-2})$ by $s = \bigoplus_{1 \leq j_1 \leq \dots \leq j_k \leq n-1} \beta(n)_{j_1} \cup \dots \cup \beta(n)_{j_k}$ and the theorem is proven. \square

Remark 4. Since $\mathrm{Hom}_{\mathrm{Mod}_E(\mathcal{S}\mathcal{H}(S))}(\Sigma_+^\infty X, \Sigma^{p,q} E) \cong E^{p,q}(X)$, the same proof also works for the decomposition of $E \wedge \Sigma_+^\infty \mathrm{Sp}_{2n} \in \mathrm{Mod}_E(\mathcal{S}\mathcal{H}(S))$ for any Sp-oriented spectrum E .

Remark 5. The results of this section also apply to other oriented theory. For example, for GL-oriented theories, in particular H_M , we can also get similar results for $\mathrm{H}_M(\mathrm{GL}_n)$ and $M(\mathrm{GL}_n) \in \mathrm{DM}(S)$ with Chern classes, etc.

3

Computations on Stiefel varieties

In this chapter, following the classical computations and using the analogue in motivic homotopy theory of the Leray spectral sequence, we compute the η -inverted MW-motivic cohomology of general Stiefel varieties, obtaining in particular the computation of the η -inverted MW-motivic cohomology of the general linear groups GL_n and special linear groups SL_n for any $n \in \mathbb{N}$.

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As we mentioned before, H_{MW} is not a GL-oriented cohomology theory, making the methods used for computing the ordinary motives of algebraic groups and Stiefel varieties impossible to apply. Instead, we will follow the classic computations of the singular cohomology groups of O_n , which make use of the Leray spectral sequences and inductive computation involving all Stiefel manifolds.

For $k \leq n$, let then $V_k(\mathbb{A}^n)$ be the Stiefel variety, which is the variety of k -linear independent vectors in \mathbb{A}_K^n . In particular, $V_n(\mathbb{A}_K^n) \cong \text{GL}_n$ and $V_1(\mathbb{A}_K^n) \cong \mathbb{A}_K^n \setminus \{0\}$. We can also consider these varieties over more a general base scheme S over K , in which case the relevant varieties are just pulled-back from K . In the sequel, we omit the subscript with the convention that we always work over a base scheme S . There is a canonical identification

$$V_k(\mathbb{A}^n) \simeq \text{GL}_n / \text{GL}_{n-k}$$

which allows to see Stiefel varieties as homogeneous spaces.

We first notice that for $l < k \leq n$ there are fiber sequences for Stiefel varieties of the form

$$V_{k-l}(\mathbb{A}^{n-l}) \rightarrow V_k(\mathbb{A}^n) \rightarrow V_l(\mathbb{A}^n) \quad (3.1)$$

as one can see from [Asok2014, Theorem 2.1]. In fact, these fiber sequences are Zariski locally split sequences.

Let $\mathbf{HS}_{2k} := \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)(2k)[4k-1]$ and $\mathbf{HS}_{2k+1} := \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)(4k+1)[8k]$. As explained in the introduction, we might as well have set $\mathbf{HS}_{2k} \cong \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)[2k-1]$ and $\mathbf{HS}_{2k+1} \cong \widetilde{\mathbf{M}}_\eta(S) \oplus \widetilde{\mathbf{M}}_\eta(S)[4k-1]$. Let H_η be the η -inverted cohomology theory induced by the η -inverted cycle module \mathbf{W} (see more on section 3.3) over K . Then H_η can be treated as a one-graded theory: $H_\eta^* := \bigoplus_n H_\eta^{n+*,n} \cong H_\eta^{*,0}[\eta, \eta^{-1}]$.

Theorem 3.0.1. *The following computations hold:*

$$H_\eta^*(V_k(\mathbb{A}^n)) \cong \begin{cases} H_\eta^*(V_{k-1}(\mathbb{A}^n)) \otimes_{H_\eta^*(S)} H_\eta^*(\mathbb{A}^{n-k+1} \setminus \{0\}) & \text{if } n-k \text{ is even.} \\ H_\eta^*(V_k(\mathbb{A}^n)) \cong H_\eta^*(V_{k-2}(\mathbb{A}^n)) \otimes_{H_\eta^*(S)} H_\eta^*(\mathbf{HS}_{n-k+2}) & \text{if } n-k \text{ is odd.} \end{cases}$$

Consequently, we inductively obtain an isomorphism of graded algebras

$$H_\eta^*(V_k(\mathbb{A}^n)) \cong H_\eta^*(S)[G]/(g^2 - \delta_{\gamma_0}(g)[-1]g, g \in G)$$

where the generators G are given by

$$G = \begin{cases} G_1 \sqcup G_2 \sqcup G_3 & n \text{ even, } n-k \text{ even,} \\ G_2 \sqcup G_3 & n \text{ odd, } n-k \text{ even,} \\ G_1 \sqcup G_2 & n \text{ even, } n-k \text{ odd,} \\ G_2 & n \text{ odd, } n-k \text{ odd,} \end{cases}$$

with

$$\begin{aligned} G_1 &= \{\alpha_{n-1} \in H_\eta^{n-1}(\mathbf{HS}_n)\}, \\ G_2 &= \{\beta_{4j-1} \in H_\eta^{4j-1}(\mathbf{HS}_{2j+1}) \mid n-k < 2j < n\}, \\ G_3 &= \{\gamma_{n-k} \in H_\eta^{n-k}(\mathbb{A}^{n-k+1} \setminus \{0\})\}, \end{aligned}$$

and $\delta_{\gamma_0}(\gamma_0) = 1$, otherwise $\delta_{\gamma_0}(g) = 0$.

We can also give the motivic decomposition of $\widetilde{\mathbf{M}}_\eta(V_k(\mathbb{A}^n))$ in $\widetilde{\text{DM}}(S)[\eta^{-1}]$ for any $k \leq n$. We note that in this category $\widetilde{\mathbb{Z}}[\eta^{-1}] \cong \mathbf{W}_*$, the Milnor-Witt cycle module $\mathbf{W}_* := W[\eta, \eta^{-1}]$ where η is of degree -1 , i.e. $\mathbf{W}_i := \eta^{-i}\mathbf{W}_0$. Therefore, we have

$$\mathbf{HS}_{2k} = \mathbf{W}_0 \oplus \mathbf{W}_{2k}[2k-1], \quad \mathbf{HS}_{2k+1} = \mathbf{W}_0 \oplus \mathbf{W}_{4k+1}[4k-1].$$

Theorem 3.0.2. In $\widetilde{\text{DM}}(S)[\eta^{-1}]$, we have the following motivic decompositions

$$\widetilde{\mathbf{M}}_\eta(V_k(\mathbb{A}^{2i})) \cong \mathbf{HS}_{2i} \otimes \widetilde{\mathbf{M}}_\eta(V_{k-1}(\mathbb{A}^{2i-1}))$$

and

$$\widetilde{\mathbf{M}}_\eta(V_k(\mathbb{A}^{2i+1})) \cong \mathbf{HS}_{2i+1} \otimes \widetilde{\mathbf{M}}_\eta(V_{k-2}(\mathbb{A}^{2i-1})).$$

Consequently, we obtain the following full decompositions by induction:

$$\widetilde{\mathbf{M}}_\eta(V_{2j}(\mathbb{A}^{2i})) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-2)} \otimes \widetilde{\mathbf{M}}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\})$$

$$\widetilde{\mathbf{M}}_\eta(V_{2j+1}(\mathbb{A}^{2i})) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_{2i-1-2(j-1)}$$

$$\widetilde{\mathbf{M}}_\eta(V_{2j}(\mathbb{A}^{2i+1})) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)}$$

$$\widetilde{\mathbf{M}}_\eta(V_{2j+1}(\mathbb{A}^{2i+1})) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_{2i+1-2(j-1)} \otimes \widetilde{\mathbf{M}}_\eta(\mathbb{A}^{2i+1-2j} \setminus \{0\})$$

It is important to note that the generators in Theorem 3.0.1 correspond to the decompositions in Theorem 3.0.2.

Corollary 3.0.3. In $\widetilde{\text{DM}}(S)[\eta^{-1}]$, we have

$$\widetilde{\mathbf{M}}_\eta(\text{GL}_{2i}) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_3 \otimes \widetilde{\mathbf{M}}_\eta(\mathbb{G}_m)$$

$$\widetilde{\mathbf{M}}_\eta(\text{GL}_{2i+1}) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_3 \otimes \widetilde{\mathbf{M}}_\eta(\mathbb{G}_m)$$

and

$$\widetilde{\mathbf{M}}_\eta(\text{SL}_{2i}) \cong \mathbf{HS}_{2i} \otimes \mathbf{HS}_{2i-1} \otimes \mathbf{HS}_{2i-3} \otimes \dots \otimes \mathbf{HS}_3$$

$$\widetilde{\mathbf{M}}_\eta(\text{SL}_{2i+1}) \cong \mathbf{HS}_{2i+1} \otimes \mathbf{HS}_{2i-1} \otimes \dots \otimes \mathbf{HS}_3.$$

Proof. For the case of SL_n , we just need to show that $\text{SL}_n \cong V_{n-1}(\mathbb{A}^n)$ which follows from the fact that $V_{n-1}(\mathbb{A}^n) \simeq \text{GL}_n/\text{GL}_1$. \square

To prove the main theorems, we will use induction. Let $\text{PH}(k, n)$ (resp. $\text{P}(k, n)$) denote the property that Theorem 3.0.1 (resp. Theorem 3.0.2) holds for $V_k(\mathbb{A}^n)$. Our strategy is to use induction to prove that $\text{PH}(k, n)$ holds, and deduce that $\text{P}(k, n)$ also holds. We notice that for $k = 1$ everything is clear from the results on motivic spheres, and we proceed with the case $k = 2$ in the next section.

3.1 A cofiber sequence for $V_2(\mathbb{A}^n)$.

The points of $V_2(\mathbb{A}^n)$ can be expressed as $\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$ (with the condition on the rank of the matrix). For such a matrix, we write a_i^* or $(a_i \cdots a_j)^*$ if $(a_i, \dots, a_j) \in \mathbb{A}^{j-i+1} \setminus \{0\}$ and a_i° or $(a_i \cdots a_j)^\circ$ if $(a_i, \dots, a_j) \in \mathbb{A}^{j-i+1}$ without extra condition.

Next we consider the open subvariety $U \subset V_2(\mathbb{A}^n)$, which is defined by $(a_1, \dots, a_{n-1}) \in \mathbb{A}^{n-1} \setminus \{0\}$, that is

$$\begin{bmatrix} (a_1 \cdots a_{n-1})^* & a_n^\circ \\ b_1 & \cdots & \cdots & b_n \end{bmatrix} \in U.$$

Lemma 3.1.1. *There is a retraction $r : U \rightarrow \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\}$ given by elementary transformations of the form*

$$\begin{bmatrix} (a_1 & \cdots & a_{n-1})^* & a_n^\circ \\ b_1 & \cdots & \cdots & b_n \end{bmatrix} \mapsto \begin{bmatrix} (a_1 & \cdots & a_{n-1})^* & 1 \\ (b'_1 & \cdots & b'_{n-1})^* & 0 \end{bmatrix}.$$

In particular, r induces an isomorphism $U \cong \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\}$ up to \mathbb{A}^1 -homotopy.

Proof. It is convenient to replace $\mathbb{A}^{n-1} \setminus \{0\}$ with \mathcal{Q}_{2n-3} , whose points consist of pairs (\mathbf{a}, \mathbf{x}) such that $\mathbf{a} \cdot \mathbf{x}^T = 1$, while the projection $\mathcal{Q}_{2n-3} \rightarrow \mathbb{A}^{n-1} \setminus \{0\}$ is given by $(\mathbf{a}, \mathbf{x}) \mapsto \mathbf{a}$. Note that the projection is an \mathbb{A}^1 -equivalence, since it is locally trivial with affine fibers. The pull-back U' of \mathcal{Q}_{2n-3} along the projection $U \rightarrow \mathbb{A}^{n-1} \setminus \{0\}$ is also \mathbb{A}^1 -weak equivalent to U and we may then consider the elements in U' as pairs $(\mathbf{x}, \begin{bmatrix} \mathbf{a}^* & a_n^\circ \\ \mathbf{b} & b_n \end{bmatrix})$ with $\mathbf{a} \cdot \mathbf{x}^T = 1$.

Using the following elementary transformations (which are \mathbb{A}^1 -weak equivalences), we can change a_n to 1:

$$\left(\mathbf{x}, \begin{bmatrix} \mathbf{a}^* & a_n^\circ \\ \mathbf{b} & b_n \end{bmatrix} \right) \xrightarrow{\begin{bmatrix} \mathbf{I} & -(a_n - 1)\mathbf{x}^T \\ 0 & 1 \end{bmatrix}} \left(\mathbf{x}, \begin{bmatrix} \mathbf{a}^* & 1 \\ \mathbf{b} & b_n - (a_n - 1)\mathbf{b} \cdot \mathbf{x} \end{bmatrix} \right).$$

Now, we may consider the following transformation

$$\left(\mathbf{x}, \begin{bmatrix} \mathbf{a}^* & 1 \\ \mathbf{b} & b_n - (a_n - 1)\mathbf{b} \cdot \mathbf{x} \end{bmatrix} \right) \xrightarrow{\begin{bmatrix} 1 & 0 \\ -b_n - (a_n - 1)\mathbf{b} \cdot \mathbf{x} & 1 \end{bmatrix}} \left(\mathbf{x}, \begin{bmatrix} \mathbf{a}^* & 1 \\ \mathbf{b} - (b_n - (a_n - 1)\mathbf{b} \cdot \mathbf{x})\mathbf{a} & 0 \end{bmatrix} \right).$$

Note that indeed $\mathbf{b} - (b_n - (a_n - 1)\mathbf{b} \cdot \mathbf{x})\mathbf{a} \in \mathbb{A}^{n-1} \setminus \{0\}$ as the matrix is of rank 2. Finally, we conclude by applying Lemma A.0.1. \square

The elements in the complement of U are simply given by $\begin{bmatrix} 0 & a_n^* \\ (b_1 & \cdots & b_{n-1})^* & b_n^\circ \end{bmatrix}$. As a result, the complement is isomorphic to $\mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\}$ up to homotopy, and its normal bundle is trivial, with an obvious choice of trivialization. Thus, we obtain the following cofiber sequence:

$$(\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\})_+ \rightarrow V_2(\mathbb{A}^n)_+ \rightarrow (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ \xrightarrow{\alpha} (3.2)$$

We now consider the following diagram

$$\begin{array}{ccccc} (\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & (\mathbb{A}^{n-1} \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \\ \parallel & & \downarrow & & \downarrow \\ (\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & V_2(\mathbb{A}^n)_+ & \longrightarrow & (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ \\ \downarrow & & \parallel & & \downarrow \\ (\mathbb{A}^{n-1} \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & V_2(\mathbb{A}^n)_+ & \longrightarrow & C \end{array}$$

in which the top cofiber sequence is given by the obvious morphism

$$\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\} \rightarrow \mathbb{A}^{n-1} \times \mathbb{A}^{n-1} \setminus \{0\}.$$

The octahedron axiom shows that the right-hand column is a cofiber sequence, in which the top vertical map is induced by the map $\mathbb{A}^{n-1} \setminus \{0\} \rightarrow \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\}$ given by $\mathbf{a} \mapsto (1, \mathbf{a})$. Since $(\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_m \cong \mathbb{A}^n \setminus \{0\}$, we obtain that $C \simeq \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}_+$ and the bottom cofiber sequence becomes:

$$\mathbb{A}^{n-1} \setminus \{0\}_+ \rightarrow V_2(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \xrightarrow{\beta}$$

Our goal is to comprehend the connecting map β . To achieve our goal, we go back to the cofiber sequence (3.2), which shows that β originates from the relevant direct summand of α . To make the latter explicit, we refer to the diagram below:

$$\begin{array}{ccccc} (\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & (\mathbb{A}^{n-1} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ & \longrightarrow & (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ \xrightarrow{i} \\ \downarrow j_1 & & \downarrow j_2 & & \parallel \\ \gamma_+ \left(U_+ \right. & \longrightarrow & V_2(\mathbb{A}^n)_+ & \longrightarrow & (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\})_+ \xrightarrow{\alpha} \\ \left. \cong \downarrow r \right) & & & & \\ (\mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^{n-1} \setminus \{0\})_+ & & & & \end{array}$$

in which the morphisms j_1 and j_2 are defined by the formulas

$$j_1 : (\mathbf{a}, a_n, \mathbf{b}) \mapsto \begin{bmatrix} (a_1 & \cdots & a_{n-1})^* & a_n^* \\ (b_1 & \cdots & b_{n-1})^* & 0 \end{bmatrix}, \quad j_2 : (\mathbf{a}, a_n, \mathbf{b}) \mapsto \begin{bmatrix} (a_1 & \cdots & a_{n-1})^\circ & a_n^* \\ (b_1 & \cdots & b_{n-1})^* & 0 \end{bmatrix}.$$

Therefore, we have $\alpha = \Sigma\gamma_+ \circ i$, where i denotes the canonical Gysin map induced by the connecting map $(\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \rightarrow \Sigma\mathbb{A}^{n-1} \setminus \{0\}$ and $\gamma = r \circ j_1$. Indeed, we have the following lemma.

Lemma 3.1.2. *Let*

$$\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_2 \xrightarrow{g} \mathcal{L}_3 \xrightarrow{i} \Sigma\mathcal{L}_1$$

be a cofiber sequence, and let \mathcal{X} be a space. Then, the following sequence

$$(\mathcal{L}_1 \times \mathcal{X})_+ \xrightarrow{f \times 1} (\mathcal{L}_2 \times \mathcal{X})_+ \xrightarrow{p_{\mathcal{L}_3}(g \times 1)} \mathcal{L}_3 \wedge \mathcal{X}_+ \xrightarrow{i \wedge 1} \Sigma\mathcal{L}_1 \wedge \mathcal{X}_+,$$

in which $p_{\mathcal{L}_3} : (\mathcal{L}_3 \times \mathcal{X})_+ \rightarrow \mathcal{L}_3 \wedge \mathcal{X}_+$ is the canonical projection, is a cofiber sequence.

Proof. We have a cofiber sequence

$$\mathcal{L}_1 \wedge \mathcal{X}_+ \xrightarrow{f \wedge 1} \mathcal{L}_2 \wedge \mathcal{X}_+ \xrightarrow{g \wedge 1} \mathcal{L}_3 \wedge \mathcal{X}_+ \xrightarrow{i \wedge 1} \Sigma\mathcal{L}_1 \wedge \mathcal{X}_+$$

and a commutative diagram in which the rows and columns are cofiber sequences

$$\begin{array}{ccccccc} \mathcal{X}_+ & \xlongequal{\quad} & \mathcal{X}_+ & \longrightarrow & \star & \longrightarrow & \Sigma(\mathcal{X}_+) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathcal{L}_1 \times \mathcal{X})_+ & \xrightarrow{f \times 1} & (\mathcal{L}_2 \times \mathcal{X})_+ & \longrightarrow & C & \longrightarrow & \Sigma(\mathcal{L}_1 \times \mathcal{X})_+ \\ p_{\mathcal{L}_1} \downarrow & & p_{\mathcal{L}_2} \downarrow & & \vdots & & \Sigma p_{\mathcal{L}_1} \downarrow \\ \mathcal{L}_1 \wedge \mathcal{X}_+ & \xrightarrow{f \wedge 1} & \mathcal{L}_2 \wedge \mathcal{X}_+ & \xrightarrow{g \wedge 1} & \mathcal{L}_3 \wedge \mathcal{X}_+ & \xrightarrow{i \wedge 1} & \Sigma\mathcal{L}_1 \wedge \mathcal{X}_+ \end{array}$$

The claim now follows from the fact that the dotted arrow is an equivalence, and from the equality $(g \wedge 1)p_{\mathcal{L}_2} = p_{\mathcal{L}_3}(g \times 1)$. \square

We now provide the explicit expression of γ after substituting $\mathbb{A}^{n-1} \setminus \{0\}$ with Q_{2n-3} .

Lemma 3.1.3. *The morphism $\gamma = r \circ j_1 : Q_{2n-3} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \rightarrow Q_{2n-3} \times \mathbb{A}^{n-1} \setminus \{0\}$ is defined by*

$$(\mathbf{x}, \mathbf{a}, a_n, \mathbf{b}) \mapsto (\mathbf{x}, \mathbf{a}, \mathbf{bM}(\mathbf{x}, \mathbf{a}, a_n)^T)$$

where $\mathbf{M}(\mathbf{x}, \mathbf{a}, a_n) = \mathbf{I} + (a_n - 1)\mathbf{a}^T \mathbf{x}$. Moreover, $\mathbf{M}(\mathbf{x}, \mathbf{a}, a_n)$ is invertible, and \mathbf{a} is an eigenvector of the eigenvalue a_n .

Proof. This first part follows immediately from the definition of the retraction r in Lemma 3.1.1.

Now, let us consider the matrix $\begin{bmatrix} 1 & -\mathbf{x} \\ (a_n - 1)\mathbf{a}^T & \mathbf{I} \end{bmatrix}$. Applying the elementary transformations

given by the matrices $\begin{bmatrix} 1 & \mathbf{x} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ and $\begin{bmatrix} 1 & \mathbf{0} \\ -(a_n - 1)\mathbf{a}^T & \mathbf{I} \end{bmatrix}$, it becomes $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + (a_n - 1)\mathbf{a}^T \mathbf{x} \end{bmatrix}$ and

$\begin{bmatrix} 1 + (a_n - 1)\mathbf{a}\mathbf{x}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ respectively. Thus $\det(\mathbf{I} + (a_n - 1)\mathbf{a}^T \mathbf{x}) = a_n$, which is invertible.

Finally, we have $(\mathbf{I} + (a_n - 1)\mathbf{a}^T \mathbf{x})\mathbf{a}^T = \mathbf{a}^T + (a_n - 1)\mathbf{a}^T = a_n \mathbf{a}^T$. \square

Corollary 3.1.4. *The Gysin map β can be expressed as*

$$\beta = \Sigma \gamma'_+ \circ i',$$

where $i' : (\mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_m \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \rightarrow \Sigma \mathbb{A}^{n-1} \setminus \{0\} \wedge (\mathbb{G}_m)_+ \wedge \mathbb{A}^{n-1} \setminus \{0\}_+$ is the canonical split injection, and

$$\gamma' : Q_{2n-3} \times \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \rightarrow \mathbb{A}^{n-1} \setminus \{0\}$$

is defined by $(\mathbf{x}, \mathbf{a}, a_n, \mathbf{b}) \mapsto \mathbf{bM}(\mathbf{x}, \mathbf{a}, a_n)^T$.

Remark 6. *The morphism γ' can be seen as the analogue of the holonomy data in the classical case.*

Now consider the morphism

$$d : (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_m \rightarrow (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_m \times (\mathbb{A}^{n-1} \setminus \{0\})$$

defined by $d(\mathbf{a}, a_n) = (\mathbf{a}, a_n, \mathbf{a})$, which induces a split injection

$$d : (\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m)_+ \rightarrow (\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m)_+ \wedge \mathbb{A}^{n-1} \setminus \{0\}_+.$$

From above, it is easy to see that $\gamma' \circ d = \theta : (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{A}^{n-1} \setminus \{0\}$, $(\mathbf{a}, a_n) \mapsto a_n \mathbf{a}$, and γ' is trivial on the summand $(\mathbb{A}^{n-1} \setminus \{0\}) \wedge (\mathbb{G}_m)_+ \wedge \mathbb{A}^{n-1} \setminus \{0\}$ for degree reasons. Combining with ([Morel2020, Lemma 2.48]), we obtain the following result.

Proposition 3.1.5. *In $D_{\mathbb{A}^1}(S)$, the composite morphism*

$$\mathbb{A}^n \setminus \{0\} \cong \Sigma \mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \xrightarrow{\Sigma d} \mathbb{A}^n \setminus \{0\} \wedge (\mathbb{A}^{n-1} \setminus \{0\})_+ \xrightarrow{\beta} \Sigma \mathbb{A}^{n-1} \setminus \{0\}$$

is given by $n_\varepsilon \eta$, i.e. it is trivial if n is even, and equal to η if n is odd. Furthermore, β is trivial on the factor $\mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}$.

Proof. As discussed above, we need to understand θ which can be done passing to the first non-trivial homotopy sheaf. We then obtain a map $\Sigma d \circ \beta = \Sigma \theta : \mathbf{K}_{n-1}^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}[n] \rightarrow \mathbf{K}_{n-1}^{\text{MW}}[n]$ and the result follows from [Morel2020, Lemma 2.48]. \square

Remark 7. For n odd, after inverting η , this gives a non-canonical splitting: the composition with the projection to the summand

$$\mathbb{A}^n \setminus \{0\} \xrightarrow{\Sigma d} \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \cong \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\} \vee \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{A}^n \setminus \{0\}$$

is the identity. Thus the complement of Σd is given by

$$C(\Sigma d) \cong \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\} \sqcup_{\Sigma d, \mathbb{A}^n \setminus \{0\}} \mathbb{A}^n \setminus \{0\} \cong \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}.$$

Corollary 3.1.6. In the η -inverted category $D_{\mathbb{A}^1}(S)[\eta^{-1}]$, we have the following split cofiber sequences:

$$\mathbb{A}^{n-1} \setminus \{0\}_+ \rightarrow V_2(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge (\mathbb{A}^{n-1} \setminus \{0\}_+) \xrightarrow{0}$$

for n even, and

$$S^0 \rightarrow V_2(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\} \xrightarrow{0}$$

for n odd.

Proof. The result is obvious if n is even. In case n is odd, we consider the following diagram:

$$\begin{array}{ccccc} S^0 & \longrightarrow & \mathbb{A}^{n-1} \setminus \{0\}_+ & \longrightarrow & \mathbb{A}^{n-1} \setminus \{0\} \\ \parallel & & \downarrow & & \downarrow \text{---} \curvearrowright \text{---} \\ S^0 & \longrightarrow & V_2(\mathbb{A}^n)_+ & \longrightarrow & C(\Sigma^{-1}\beta) \quad +1 \beta \\ \downarrow & & \parallel & & \downarrow \text{---} \\ \mathbb{A}^{n-1} \setminus \{0\}_+ & \longrightarrow & V_2(\mathbb{A}^n)_+ & \longrightarrow & \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \end{array}$$

We can deduce that $C(\Sigma^{-1}\beta) \cong C(d) \cong \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}$. □

The same outcome holds in $\widetilde{DM}(S)[\eta^{-1}]$.

Corollary 3.1.7. The following decompositions hold in $\widetilde{DM}(S)[\eta^{-1}]$

$$\widetilde{M}_\eta(V_2(\mathbb{A}^n)) \cong \widetilde{M}_\eta(\mathbb{A}^n \setminus \{0\}) \otimes \widetilde{M}_\eta(\mathbb{A}^{n-1} \setminus \{0\}) \cong \mathbf{HS}_n \otimes \widetilde{M}_\eta(\mathbb{A}^{n-1} \setminus \{0\})$$

for n even, and

$$\widetilde{M}_\eta(V_2(\mathbb{A}^n)) \cong \mathbf{HS}_n$$

for n odd.

If we apply these results to the cohomology theory H_η , this implies the decomposition of $H_\eta(V_2(\mathbb{A}^n))$ in terms of the cohomology of $\mathbb{A}^n \setminus \{0\}$ and $\mathbb{A}^{n-1} \setminus \{0\}$.

Corollary 3.1.8. We have an isomorphism

$$H_\eta^*(V_2(\mathbb{A}^n)) \cong \begin{cases} H_\eta^*(\mathbb{A}^n \setminus \{0\}) \otimes_{H_\eta^*(S)} H_\eta^*(\mathbb{A}^{n-1} \setminus \{0\}) & \text{for } n \text{ even.} \\ H_\eta^*(\mathbf{HS}_n) & \text{for } n \text{ odd.} \end{cases}$$

Note that the two corollaries show that both $P(2, n)$ and $PH(2, n)$ hold true for all n .

3.2 Cofiber sequences for $V_k(\mathbb{A}^n)$

In this section, we show that computations analogous to the ones in Section 3.1 can be applied to $V_k(\mathbb{A}^n)$ for $n \geq k \geq 2$.

Applying the same transformations

$$\begin{bmatrix} (a_1 & \cdots & a_{n-1})^* & a_n \circ \\ \vdots & & & \\ b_{j1} & \cdots & \cdots & b_{jn} \\ \vdots & & & \end{bmatrix} \mapsto \begin{bmatrix} (a_1 & \cdots & a_{n-1})^* & 1 \\ \vdots & & & \\ (b'_{j1} & \cdots & b'_{j,n-1})^* & 0 \\ \vdots & & & \end{bmatrix}$$

and lemma A.0.1, we obtain the following cofiber sequences for $k \geq 2$:

$$V_{k-1}(\mathbb{A}^{n-1})_+ \rightarrow V_k(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{\beta}$$

and the corresponding diagrams:

$$\begin{array}{ccccc} (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1})_+ & \rightarrow & (\mathbb{A}^{n-1} \times \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1})_+) & \rightarrow & \mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\} \wedge (\mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1})_+) \rightarrow \\ \downarrow j_1 & & \downarrow j_2 & & \parallel \\ \gamma \left(\begin{array}{c} U \\ \cong \\ \downarrow \\ (\mathbb{A}^{n-1} \setminus \{0\}) \times V_{k-1}(\mathbb{A}^{n-1})_+ \end{array} \right) & \longrightarrow & V_k(\mathbb{A}^n)_+ & \longrightarrow & \mathbb{A}^{n-1}/\mathbb{A}^{n-1} \setminus \{0\} \wedge (\mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1})_+) \xrightarrow{\alpha} \end{array}$$

We also define a morphism $\gamma: \mathcal{Q}_{2n-3} \times \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1}) \rightarrow \mathcal{Q}_{2n-3} \times V_{k-1}(\mathbb{A}^{n-1})$ as

$$(\mathbf{x}, \mathbf{a}, a_n, \mathbf{B}) \mapsto (\mathbf{x}, \mathbf{a}, \mathbf{B}\mathbf{M}(\mathbf{x}, \mathbf{a}, a_n)^T)$$

where $\mathbf{M}(\mathbf{x}, \mathbf{a}, a_n) = \mathbf{I} + (a_n - 1)\mathbf{a}^T \mathbf{x}$.

In analogy with the previous section, we expect to get split decompositions.

Proposition 3.2.1. *Using our inductive hypothesis and PH(k, n), we conclude that we have the following split cofiber sequences in $D_{\mathbb{A}^1}(S)[\eta^{-1}]$*

$$V_{k-1}(\mathbb{A}^{n-1})_+ \rightarrow V_k(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{0}$$

for n even, and

$$V_{k-2}(\mathbb{A}^{n-2})_+ \rightarrow V_k(\mathbb{A}^n)_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{0},$$

for n odd. In particular, P(k, n) holds.

Proof. As we have assumed PH(k, n) and already constructed the split surjection using the same method as in the proof of Theorem 2.3.4, we just need to show that the cofiber sequences actually exist. For n even, there is nothing to do. For $n = 2i + 1$ odd, consider the following diagram:

$$\begin{array}{ccccc} V_{k-2}(\mathbb{A}^{n-2})_+ & \xrightarrow{i} & V_{k-1}(\mathbb{A}^{n-1})_+ & \longrightarrow & \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \\ \parallel & & \downarrow & \dashleftarrow j & \downarrow \\ V_{k-2}(\mathbb{A}^{n-2})_+ & \longrightarrow & V_k(\mathbb{A}^n)_+ & \longrightarrow & C(\Sigma^{-1}\tau) +_1 \tau \\ \downarrow & & \parallel & & \downarrow \\ V_{k-1}(\mathbb{A}^{n-1})_+ & \longrightarrow & V_k(\mathbb{A}^n)_+ & \longrightarrow & \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \end{array}$$

The morphism τ is simply the composite

$$\tau : \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{\beta} \Sigma V_{k-1}(\mathbb{A}^{n-1})_+ \rightarrow \Sigma \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+$$

and we now prove that it is split in $D_{\mathbb{A}^1}(S)[\eta^{-1}]$, by constructing a morphism

$$\mu : \mathbb{A}^n \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-1})_+ \rightarrow \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+$$

such that $\tau\mu$ is an isomorphism.

Remark 8. Notice that μ shall NOT be $1 \wedge i$. Instead, we will define a morphism analogous to $\Sigma d : \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}_+^{n-1}$ in the previous section.

Since $n - 1$ is even, we have the split cofiber sequence

$$V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{i} V_{k-1}(\mathbb{A}^{n-1})_+ \rightarrow \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+$$

for which we choose a splitting

$$j : \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \rightarrow V_{k-1}(\mathbb{A}^{n-1})_+.$$

Denoting by $f : V_{k-1}(\mathbb{A}^{n-1}) \rightarrow \mathbb{A}^{n-1} \setminus \{0\}$ the first row map, we define a morphism of schemes

$$d' : \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1}) \rightarrow \mathbb{G}_m \times \mathbb{A}^{n-1} \setminus \{0\} \times V_{k-1}(\mathbb{A}^{n-1})$$

by $d'(x, y) = (x, f(y), y)$, and we obtain a morphism in $D_{\mathbb{A}^1}(S)[\eta^{-1}]$ of the form

$$d : \mathbb{G}_m \wedge \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{1 \wedge j} \mathbb{G}_m \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{d'} \mathbb{G}_m \wedge \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+.$$

Let $\mu = \Sigma d$, we can now prove that the composition

$$\mathbb{A}^n \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{\Sigma d} \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{\tau} \Sigma \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+$$

is simply $\eta \wedge id$.

Lemma 3.2.2. *the composition*

$$\mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1}) \xrightarrow{d'} \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1}) \xrightarrow{pr_2 \circ \gamma} V_{k-1}(\mathbb{A}^{n-1})$$

is homotopic to the morphism

$$(c, \begin{bmatrix} a_1 & \cdots & a_{n-1} \\ \vdots & & \\ b_{j1} & \cdots & b_{j,n-1} \\ \vdots & & \end{bmatrix}) \mapsto \begin{bmatrix} ca_1 & \cdots & ca_{n-1} \\ \vdots & & \\ b_{j1} & \cdots & b_{j,n-1} \\ \vdots & & \end{bmatrix}$$

Thus this morphism induces

$$\eta \wedge id : \mathbb{A}^n \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{\Sigma d} \mathbb{A}^n \setminus \{0\} \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \xrightarrow{\tau} \Sigma \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+$$

Proof. First we define $QV_k(\mathbb{A}^n) : \{(\mathbf{B}, \mathbf{x}) \in V_k(\mathbb{A}^n) \times \mathbb{A}^n, s.t. \mathbf{B}\mathbf{x}^T = [1, 0, \dots, 0]^T\}$. This gives a \mathbb{A}^{n-k} bundle over $V_k(\mathbb{A}^n)$.

Now we replace $V_{k-1}(\mathbb{A}^{n-1})$ with $QV_{k-1}(\mathbb{A}^{n-1})$ and we find that d' is equivalent to the morphism

$$\mathbb{G}_m \times QV_{k-1}(\mathbb{A}^{n-1}) \rightarrow \mathcal{Q}_{2n-3} \times \mathbb{G}_m \times V_{k-1}(\mathbb{A}^{n-1}), (c, \mathbf{B}, \mathbf{x}) \mapsto ((\mathbf{x}, f(\mathbf{B})), c, \mathbf{B})$$

Following the definition of γ , we have

$$pr_2 \circ \gamma((\mathbf{x}, f(\mathbf{B})), c, \mathbf{B}) = B + (c-1)[1, 0, \dots, 0]^T f(\mathbf{B})$$

which is exactly the matrix

$$\begin{bmatrix} ca_1 & \cdots & ca_{n-1} \\ & \ddots & \\ b_{j1} & \cdots & b_{j,n-1} \\ & & \ddots \end{bmatrix}$$

□

We now give a slightly different presentation of d which will be more convenient for our purpose. As $D_{\mathbb{A}^1}(S)[\eta^{-1}]$ is an additive category, we may define the partial diagonal embedding

$$d_0 : \mathbb{A}^{n-1} \setminus \{0\} \rightarrow \mathbb{A}^{n-1} \setminus \{0\} \wedge (\mathbb{A}^{n-1} \setminus \{0\})_+$$

by $(a) \mapsto (a) \wedge 1 + (a) \wedge (a)$. We now see that d can be identified (up to permuting the factors) with the composite

$$\mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{d_0 \wedge id} \mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\}_+ \wedge \mathbb{G}_m \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{id \wedge (i \vee j)} \mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \wedge V_{k-1}(\mathbb{A}^{n-1})_+$$

Therefore, it is evident that the composite

$$\mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \wedge V_{k-2}(\mathbb{A}^{n-2})_+ \xrightarrow{d} \mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \wedge V_{k-1}(\mathbb{A}^{n-1})_+ \rightarrow \mathbb{A}^{n-1} \setminus \{0\} \wedge \mathbb{G}_m \wedge V_{k-2}(\mathbb{A}^{n-2})_+$$

is the identity.

Finally, with the same argument as in Remark 7, we can conclude that $C(\Sigma^{-1}\tau) \cong C(\Sigma d) \cong \mathbb{A}^n \setminus \{0\} \wedge \mathbb{A}^{n-1} \setminus \{0\} \wedge V_{k-2}(\mathbb{A}^{n-2})_+$. □

Now we are only left to prove $\text{PH}(k, n)$.

3.3 Recollections on Milnor-Witt cycle module

We can construct a t-structure δ on $\mathcal{S}\mathcal{H}(S)$ [Bondarko2017], whose heart can be understood using either homotopy modules [Déglise2011] or equivalently (cohomological) Milnor-Witt (MW)-cycle modules ([Feld2020] or [Déglise2022]).

We denote by \mathcal{F}_S the category of spectra of fields essentially of finite type over S . A (cohomological) Milnor-Witt cycle module is a functor from $\mathcal{F}_S^{\text{op}}$ to the category of \mathbb{Z} -graded abelian groups $\text{Abel}^{\mathbb{Z}}$

$$M_* : \mathcal{F}_S \rightarrow \text{Abel}^{\mathbb{Z}}$$

satisfying the axioms listed in [Déglise2022, §5.1]. The axioms that M_* should satisfy include an action by Milnor-Witt K -theory K_*^{MW} , which is the fundamental example of a MW-cycle module. In the sequel, we will denote by $\text{MWmod}^{\text{coh}}(S)$ the category of cohomological MW-cycle modules over S . We note that the dual notion of homological MW-cycle module is developed in [Déglise2022], but we choose to the former notion to fit our computations.

For any $p : X \rightarrow S$ and a given Milnor-Witt cycle module $M_* \in \text{MWmod}^{\text{coh}}(S)$, there is an associated Gersten complex ${}^\delta C^*(X, M_*)$, which actually defines a spectrum ${}^\delta \mathbf{HM} \in \mathcal{S}\mathcal{H}(S)$ satisfying the following property

$${}^\delta A^i(X, M_j) := H^i({}^\delta C^*(X, M_j)) \cong {}^\delta \mathbf{HM}^{i+j, j}(X).$$

The Gersten complex gives the heart of the t-structure δ . Correspondingly, we can also get a homotopy module, i.e. a \mathbb{Z} -graded strictly \mathbb{A}^1 -invariant sheaf $\mathbb{M}_* : U \mapsto {}^\delta C^0(U, M_*)$.

The above cohomology theory H_η gives an example, corresponding to the MW-cycle module \mathbf{W}_* associated to the Witt sheaf. Here, $\mathbf{W}_i(k)$ is just the equivalence classes of non-degenerate quadratic forms up to hyperbolic planes. Sometimes, we omit the ${}^\delta \mathbf{H}$ while identifying the MW-cycle module and its corresponding spectrum.

We can also pull back MW-cycle modules to schemes $p : X \rightarrow S$, obtaining an object $p^*(M)_* \in \text{MWmod}^{\text{coh}}(X)$ given by

$$p^*(M)_*((v \rightarrow X) \in \mathcal{F}_X) := M_*(v \rightarrow X \xrightarrow{p} S).$$

We say that such a MW-cycle module $p^*(M)_*$ over X is (cohomologically) S -simple.

Next, we would like to define twists of a given MW-cycle module with respect to a principal bundle. If G is an algebraic group (or more generally a sheaf of groups over S) $M_* \in \text{MWmod}^{\text{coh}}(S)$, we want to consider actions of G on M_* which preserve all the constraints satisfied by M_* . In the framework of this thesis, we will only consider the following notion. A *compatible action of G on M_** is a linear action $A(F) : G(F) \times M_*(F) \rightarrow M_*(F)$ for any $F \in \mathcal{F}_S$, subject to the condition that for any discrete valuation ring \mathcal{O}_v over S (i.e. endowed with a morphism $\mathcal{O}(S) \rightarrow \mathcal{O}_v$) with fraction field $K(v)$ and residue field $\kappa(v)$

$$K(v) \xleftarrow{\xi} \mathcal{O}_v \xrightarrow{s} \kappa(v)$$

the following diagram commutes

$$\begin{array}{ccc} G(\mathcal{O}_v) \times M_*(K(v)) & \xrightarrow{\xi_* \times \text{id}} & G(K(v)) \times M_*(K(v)) & \xrightarrow{A(K(v))} & M_*(K(v)) \\ & \searrow_{s_* \times \partial} & & & \downarrow \partial \\ & & G(\kappa(v)) \times M_{*-1}(\kappa(v), N_v^\vee) & \xrightarrow{A(\kappa(v))} & M_{*-1}(\kappa(v), N_v^\vee) \end{array} \quad (3.3)$$

where N_v^\vee is the dual of the normal bundle to $\kappa(v)$ in \mathcal{O}_v , ∂ is the residue map of Axiom (D4) in [Déglise2022, §5.1], and the action $A(\kappa(v))$ of $G(\kappa(v))$ on $M_{*-1}(\kappa(v), N_v^\vee)$ is the natural one induced by the action on $M_{*-1}(\kappa(v))$ (the action on the line bundle is trivial).

Given a principal G -bundle P over X , we set

$$(M \otimes_A P)_*((x : v \rightarrow X) \in \mathcal{F}_X) := P_x \otimes_{G_x}^A p^*(M)_*(x).$$

Notice that P_x is non-canonically isomorphic to G_x , and thus $(M \otimes_A P)_*(x)$ is also non-canonically isomorphic to $p^*(M)_*(x)$. A direct verification shows that the axioms of a cohomological MW-cycle module are verified (use (3.3) above for (D4)) and we therefore obtain a well-defined object $(M \otimes_A P)_*$ in the category of cohomological MW-cycle modules.

In the particular example where $G = \mathbb{G}_m$, we can make G act on M_* via the obvious map $\mathbb{G}_m \rightarrow \mathbb{K}_0^{\text{MW}}$. If L is a line bundle over X , it is an easy exercise to check that $M \otimes_A L$ is the usual cycle module twisted by L , with Gersten complex

$$\delta C_*(X, M \otimes_A L) = \delta C_*(X, M, L).$$

A compatible action of G on M_* also yields an action on the Gersten complex associated to M_* . Indeed, for any $p, q \in \mathbb{Z}$ and any dimension function δ , we obtain a string of maps for any open subscheme $U \subset X$

$$G(U) \times \bigoplus_{\delta(x)=p} M_{p+q}(\kappa(x), \lambda_x) \xrightarrow{\oplus s_x \times \text{id}} \bigoplus_{\delta(x)=p} G(\kappa(x)) \times M_{p+q}(\kappa(x), \lambda_x) \xrightarrow{\oplus A(\kappa(x))} \bigoplus_{\delta(x)=p} M_{p+q}(\kappa(x), \lambda_x).$$

The fact that Diagram (3.3) commutes precisely shows that this action commutes with the differentials of the complex, and we thus obtain an induced action

$$G(U) \times \delta C_*(U, M_q) \rightarrow \delta C_*(U, M_q)$$

for any $q \in \mathbb{Z}$. In particular, G acts on the spectrum $\delta \mathbf{HM}$ associated to M_* .

Remark 9. *We could also define the twist of a homotopy module \mathbb{M} in a similar way.*

An η -inverted cycle module M_η is a MW-cycle module for which $\eta \in \mathbb{K}_{-1}^{\text{MW}}(K)$ acts as an isomorphism. The MW-cycle module \mathbf{W} is such an example and one can check that it is the tensor unit in the category of η -inverted cycle modules. More generally, we can define the η -localization of any given MW-cycle module M as

$$M[\eta^{-1}]_* := \mathbf{W}_* \otimes_{\mathbb{K}_*^{\text{MW}}} M_*.$$

This is obviously an η -inverted cycle module, and one can check that this functor is adjoint with the inclusion of the category of η -inverted MW-cycles modules into the category of MW-cycle modules.

Finally, let us observe that given an η -inverted cycle module M_* , we only need to consider $(M_\eta)_0$. From now on, we only consider η -inverted MW-cycle modules.

3.4 Recollections on the motivic Leray spectral sequences

To compute cohomology using fiber sequences, we can use the motivic analogue of the Leray spectral sequence by making use of the t-structure δ [Asok2018].

Let H_δ^p be the cohomology of the t-structure δ , which takes value in the heart, i.e. MW-cycle modules. In general, for any spectrum $\mathbb{F} \in \mathcal{S}\mathcal{H}(S)$, a spectral sequence can be obtained using the theory of spectral diagrams:

$$E_2^{p,q}(\mathbb{F}) = \delta A^p(S, H_\delta^q \mathbb{F}) \Rightarrow H^{p+q}(S, \mathbb{F})$$

Applying to the push-forward $f_* \mathbb{E}$ along a morphism $f : T \rightarrow B$, we obtain:

Theorem 3.4.1 ([Asok2018], Theorem 4.2.5). *Let $f : T \rightarrow B$ be a morphism of schemes, and let $\mathbb{E} \in \mathcal{S}\mathcal{H}(T)$ be a ring spectrum over T , we have a spectral sequence of the form:*

$$E_2^{p,q}(f, \mathbb{E}) = \delta A^p(B, H_\delta^q f_* \mathbb{E}) \Rightarrow H^{p+q}(T, \mathbb{E})$$

where the differentials d_r satisfy the usual Leibniz rule.

When $\mathbb{E} = f^*\mathbb{F}$ for some $\mathbb{F} \in \mathcal{S}\mathcal{H}(B)$, we have the non-canonical isomorphism

$$(H_\delta^p f_* \mathbb{E})_q(b : v \rightarrow B) \cong H^{p+q}(T_b, b^! \mathbb{F}),$$

where $T_b := v \times_B T$.

However, computing $H_\delta^p f_* \mathbb{E}$ as a MW-cycle module is challenging in general. In our scenario, we focus on the fiber sequence

$$F \rightarrow T \xrightarrow{f} B$$

over S using and η -inverted cycle module $M \in \text{MWmod}^{\text{coh}}(S)$ as our coefficients.

In classical computations, simple connectedness is used to further reduce $\mathbf{R}^q f_* \mathbb{Z}$ into the constant sheaf associated to $H^q(F, \mathbb{Z})$ over B . However, additional care is necessary in the motivic setting. Instead, we show that $H_\delta^q f_* \mathbb{E}$ is (cohomologically) S -simple, i.e. $H_\delta^q f_* \mathbb{E} \cong p_B^*(N)$, for $N \in \text{MWmod}^{\text{coh}}(S)$

We begin by considering a covering $U \rightarrow B$ satisfying

$$U \times_B T \xrightarrow[\cong]{\chi} U \times F.$$

We further assume that $H^*(F, \mathbf{W})$ is a free $H^*(S, \mathbf{W})$ -module, giving

$$H^*(V \times_S F, M) = H^*(V, M) \otimes_{H^*(S, \mathbf{W})} H^*(F, \mathbf{W})$$

for any V over S . In order to comprehend the MW-cycle module $H_\delta^p f_* M$, we use the trivialization $\chi : U \times_B T \rightarrow U \times F$ and the trivial F -bundle $f_U : U \times_B T \rightarrow U$. When we compare the trivializations induced on the two projections, namely $\text{pr}_1, \text{pr}_2 : U \times_B U \rightarrow U$, we obtain the transformation morphism

$$g : U \times_B U \times F \xrightarrow{\text{pr}_1^* \chi^{-1}} U \times_B U \times_B T \xrightarrow{\text{pr}_2^* \chi} U \times_B U \times F.$$

We observe that the η -inverted cycle module $H_\delta^p f_{U*} M$ over U is simple, as $H^q(F_{(-)}, M) \cong H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M$. Therefore, the η -inverted cycle module $H_\delta^p f_* M$ over B is locally simple. It follows from our assumption that, for a (semi-)local $V \subset U$, $H^q(V \times F, M) = H^0(V, M) \otimes H^q(F, \mathbf{W})$. Actually $H_\delta^p f_M$ is the twist $(H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M) \otimes_g P_T$ for some $\text{Aut}(H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M)$ principal bundle P_T over B . Moreover, g induces an automorphism

$$g^* : H^*(U \times_B U, M) \otimes H^*(F, \mathbf{W}) \rightarrow H^*(U \times_B U, M) \otimes H^*(F, \mathbf{W})$$

that provides the $\text{Aut}(H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M)$ principal bundle P_T gluing the cycle module $H_\delta^p f_* M$.

Finally, if we assume that another automorphism exists, $h : H^q(U \times F, M) \rightarrow H^q(U \times F, M)$ such that $g^* = \text{pr}_1^* h^{-1} \circ \text{pr}_2^* h$, we may conclude that P_T is indeed a trivial $\text{Aut}(H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M)$ principle bundle, and thus $H_\delta^p f_* M$ is simple as the $H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M$.

Remark 10. *Although Higher-order coherence data exists, in this instance, the $\text{Aut}(H^q(F, \mathbf{W}) \otimes_{\mathbf{W}} M)$ -principal bundle is the only variable necessary for consideration due to the cohomology groups landing in Set . Later in our scenario, we will even reduce to the case of a \mathbb{G}_m -principal bundle, i.e. a line bundle.*

3.5 Leray spectral sequence and Stiefel varieties

In the above discussion, we only need to consider the case $M = \mathbf{W}$, so we may omit the coefficient \mathbf{W} for simplicity.

3.5.1 First case of Stiefel varieties

We begin with the special cases of the fiber sequences (3.1) for Stiefel varieties

$$\begin{aligned}
 V_{k-1}(\mathbb{A}^{n-1}) &\rightarrow V_k(\mathbb{A}^n) \xrightarrow{f} \mathbb{A}^n \setminus \{0\} \\
 f : \begin{bmatrix} a_1 & \cdots & a_n \\ & \ddots & \\ b_{j1} & \cdots & b_{jn} \\ & & \ddots \end{bmatrix} &\mapsto [a_1, \dots, a_n] = \mathbf{a}.
 \end{aligned}$$

We follow the processes above for these fiber sequences. Note that

$$H^*(S \times V_{k-1}(\mathbb{A}^{n-1}), \mathbf{W}) = H^*(S, \mathbf{W}) \otimes H^*(V_{k-1}(\mathbb{A}^{n-1}), \mathbf{W})$$

is already given by $\text{PH}(k-1, n-1)$

First we consider the covering $U = \sqcup U_i \rightarrow \mathbb{A}^n \setminus \{0\}$ where $U_i = \{a_i \neq 0\} \cong \mathbb{A}^{i-1} \times \mathbb{G}_m \times \mathbb{A}^{n-i}$. The trivializations $\chi_i : U_i \times_{\mathbb{A}^n \setminus \{0\}} V_k(\mathbb{A}^n) \rightarrow U_i \times V_{k-1}(\mathbb{A}^{n-1})$ are given by the following matrix representations

$$\begin{aligned}
 \begin{bmatrix} a_{1\circ} & \cdots & a_i^* & \cdots & a_{n\circ} \\ & \ddots & & & \\ b_{j1} & \cdots & b_{ji} & \cdots & b_{jn} \\ & & & \ddots & \end{bmatrix} &\mapsto \begin{bmatrix} a_{1\circ} & \cdots & a_i^* & \cdots & a_{n\circ} \\ & \ddots & & & \\ b_{j1} - a_1 b_{ji}/a_i & \cdots & 0 & \cdots & b_{jn} - a_n b_{ji}/a_i \\ & & & \ddots & \end{bmatrix} \\
 &= \left([a_{1\circ}, \dots, a_i^*, \dots, a_{n\circ}], \begin{bmatrix} \ddots & & & \\ b_{j1} - a_1 b_{ji}/a_i & \cdots & b_{jn} - a_n b_{ji}/a_i \\ & \ddots & \end{bmatrix} \right).
 \end{aligned}$$

Let $C(i)$ be the matrix $\begin{bmatrix} \ddots & & \\ b_{j1} - a_1 b_{ji}/a_i & \cdots & b_{jn} - a_n b_{ji}/a_i \\ & \ddots & \end{bmatrix}$. We aim to understand the variation

of $C(i)$ between two open subschemes, that is, between U_i and U_k . We use the notation $U_{ik} = U_i \cap U_k$ with $i < k$. We compare the formulas for $C(i)$ and $C(k)$ to obtain the relation $C(i)G_{ik}(\mathbf{a})E_{ik} = C(k)$ where E_{ik} is the permutation matrix of columns given by $(i \ i+1 \ \cdots \ k-1)$ and

$$G_{ik}(\mathbf{a}) = \begin{bmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ -a_1/a_k & \cdots & -a_{i-1}/a_k & -a_{i+1}/a_k & \cdots & -a_{k-1}/a_k & -a_i/a_k & \cdots & -a_n/a_k \\ & & & & & & & \ddots & \end{bmatrix}.$$

Here, $G_{ik}(\mathbf{a})_{k-1,k-1} = -a_i/a_k$; for $j < i$, $G_{ik}(\mathbf{a})_{k-1,j} = -a_j/a_k$; for other j , $G_{ik}(\mathbf{a})_{k-1,j} = -a_{j+1}/a_k$, and otherwise $G_{ik}(\mathbf{a})_{pq} = \delta_{pq}$. From this, we obtain the transformation morphisms $g_{ik} : U_{ik} \times V_{k-1}(\mathbb{A}^{n-1}) \rightarrow U_{ik} \times V_{k-1}(\mathbb{A}^{n-1})$

Up to \mathbb{A}_1 -homotopy, $G_{ik}(\mathbf{a}) \sim \text{diag}(-a_i/a_k, 1, \dots, 1)$, and $\det(E_{ik}) = (-1)^{k-i-1}$, hence $G_{ik}(\mathbf{a})E_{ik} \sim D((-1)^{k-i}a_i/a_k) := \text{diag}((-1)^{k-i}a_i/a_k, 1, \dots, 1)$.

Remark 11. We observe that only the columns i and k are relevant, hence we only need to consider the transformation restricted to those columns. This crucial observation will be helpful in the sequel, when we will have to understand the general case.

As we mentioned above, the transformation morphisms induce morphisms of cohomology groups, which are defined up to homotopy. Therefore, we obtain

$$g_{ik}^* = D((-1)^{k-i}a_i/a_k)^* : H^q(U_{ik} \times V_{k-1}(\mathbb{A}^{n-1})) \rightarrow H^q(U_{ik} \times V_{k-1}(\mathbb{A}^{n-1})).$$

To show that the torsor is trivial, define $h_i = D((-1)^i a_i^{-1}) : U_i \times V_{k-1}(\mathbb{A}^{n-1}) \rightarrow U_i \times V_{k-1}(\mathbb{A}^{n-1})$, and obviously, $g_{ik}^* = h_i^{-1*} h_k^*$.

In conclusion, we have shown that $H_{\delta}^q f_* \mathbf{W}$ is the simple cycle module $H^q(V_{k-1}(\mathbb{A}^{n-1})) \otimes \mathbf{W}$ on $\mathbb{A}^n \setminus \{0\}$.

3.5.2 More general cases

We can perform similar computations for all fiber sequences of the form

$$V_{k-1}(\mathbb{A}^{n-1}) \rightarrow V_k(\mathbb{A}^n) \xrightarrow{f_l} V_l(\mathbb{A}^n),$$

$$f_l : \begin{bmatrix} \mathbf{a}_{L1} & \cdots & \mathbf{a}_{Ln} \\ & \ddots & \\ b_{j1} & \cdots & b_{jn} \\ & & \ddots \end{bmatrix} := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{l1} & \cdots & a_{ln} \\ & \ddots & \\ b_{j1} & \cdots & b_{jn} \\ & & \ddots \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{a}_{L1} & \cdots & \mathbf{a}_{Ln} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{l1} & \cdots & a_{ln} \end{bmatrix}.$$

We write respectively \mathbf{a}_{Si} and \mathbf{b}_{jS} for the column vector $\begin{bmatrix} \vdots \\ a_{ti} \\ \vdots \end{bmatrix}$ and row vector $[\cdots b_{ti} \cdots]$,

where $t \in S$ denotes a subset of index.

Let I be an l -subset of $\{1, \dots, n\}$. Given a $l \times l$ matrix

$$\mathbf{A}_I = \begin{bmatrix} \cdots & \mathbf{a}_{Lt} & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & a_{1t} & \cdots \\ & \ddots & \\ \cdots & a_{lt} & \cdots \end{bmatrix}, t \in I,$$

we have the open cover of $V_l(\mathbb{A}^n)$ by $U = \sqcup U_I$ where $U_I = \{\det(\mathbf{A}_I) \neq 0\} \cong \text{GL}_l$. We could refine the cover U further to make it cohomologically trivial, but this does not affect the computations here.

Now, let us consider the trivializations $\chi_I : f_I^{-1}(U_I) \rightarrow U_I \times V_{k-I}(\mathbb{A}^{n-l})$, which are given by the following expression:

$$\begin{aligned} \begin{bmatrix} \mathbf{a}_{L1^\circ} & \cdots & \mathbf{A}_I^* & \cdots & \mathbf{a}_{Ln^\circ} \\ & \ddots & & & \\ b_{j1} & \cdots & \mathbf{b}_{jI} & \cdots & b_{jn} \\ & & & \ddots & \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{a}_{L1^\circ} & \cdots & \mathbf{A}_I^* & \cdots & \mathbf{a}_{Ln^\circ} \\ & \ddots & & & \\ b_{j1} - \mathbf{b}_{jI} \mathbf{A}_I^{-1} \cdot \mathbf{a}_{L1} & \cdots & 0 & \cdots & b_{jn} - \mathbf{b}_{jI} \mathbf{A}_I^{-1} \cdot \mathbf{a}_{Ln} \\ & & & \ddots & \end{bmatrix} \\ = \left(\begin{bmatrix} \mathbf{a}_{L1^\circ} & \cdots & \mathbf{A}_I^* & \cdots & \mathbf{a}_{Ln^\circ} \end{bmatrix}, \begin{bmatrix} \cdots & & & & \\ \cdots & b_{js} - \mathbf{b}_{jI} \mathbf{A}_I^{-1} \cdot \mathbf{a}_{Ls} & \cdots & & \\ & & & \ddots & \end{bmatrix} \right), s \notin I \end{aligned}$$

To compare two trivializations (up to homotopy) over two open subschemes U_I and U_K , we only need to focus on the columns in the union $I \cup K$. Given the symmetry of the rows, we pick a row j to construct the transformation from \mathbf{c}_{jK} to \mathbf{c}_{jI} . Here $\mathbf{c}_{jK} = [\dots b_{jk} - \mathbf{b}_{jI} \mathbf{A}_I^{-1} \mathbf{a}_{Lk} \dots]_{k \in K, k \notin I} = P_{K \setminus I}(\mathbf{b}_{jK} - \mathbf{b}_{jI} \mathbf{A}_I^{-1} \mathbf{A}_K)$ and $\mathbf{c}_{jI} = P_{I \setminus K}(\mathbf{b}_{jI} - \mathbf{b}_{jK} \mathbf{A}_K^{-1} \mathbf{A}_I)$, where P_S is the first projection to the subspace spanned by S before embedding it into the entire row vector space \mathbb{A}^{n-l} .

Hence, the transformation from \mathbf{c}_{jK} to \mathbf{c}_{jI} is given by ${}_{K \setminus I} \mathbf{A}_K^{-1} \mathbf{A}_I |_{I \setminus K} E_{IK}$, where ${}_S \mathbf{A} |_T$ denotes the operation of selecting S -rows and T -columns of \mathbf{A} before embedding it into GL_{n-l} . Finally, E_{IK} is the permutation that maps K -columns to I -columns.

Although we restrict the matrix to a portion, one simple observation is that \mathbf{A}_I and \mathbf{A}_K share some columns. By using a permutation ε , we have

$$\mathbf{A}_I - \varepsilon \mathbf{A}_K = \begin{bmatrix} \cdots & \mathbf{a}_{Ll} & \cdots & 0 & \cdots \end{bmatrix}$$

and therefore

$$\mathbf{A}_K^{-1} \mathbf{A}_I = \varepsilon + \begin{bmatrix} \cdots & * & \cdots & 0 & \cdots \end{bmatrix}.$$

Consequently, we obtain $\det({}_{K \setminus I} \mathbf{A}_K^{-1} \mathbf{A}_I |_{I \setminus K}) = \det(\mathbf{A}_K^{-1} \mathbf{A}_I)$.

Therefore the transformation \mathbf{g}_{IK} can be expressed as $G_{IK}(\mathbf{A}) E_{IK} \sim \mathrm{diag}(\det(-\mathbf{A}_K^{-1} \mathbf{A}_I), 1, \dots, 1) E_{IK}$. After some calculations, we obtain

$$\det(E_{IK}) = \prod_{k_1 < \dots < k_l \in K, i_1 < \dots < i_l \in I} \det(E_{i_r k_r}) = (-1)^{\sum_{k_1 < \dots < k_l \in K, i_1 < \dots < i_l \in I} k_r - i_r - 1} = (-1)^{(\sum_{k \in K} k - \sum_{i \in I} i) - l},$$

and finally, the transformation matrix up to homotopy is nothing but

$$\mathbf{g}_{IK} \sim G_{IK}(\mathbf{A}) E_{IK} \sim \mathbf{D}((-1)^{\sum_{k \in K} k - \sum_{i \in I} i} \det(\mathbf{A}_K)^{-1} \det(\mathbf{A}_I)).$$

Using the same method as above, we can define $\mathbf{h}_I = \mathbf{D}((-1)^{\sum_{i \in I} i} \det(\mathbf{A}_I)^{-1})$ and $\mathbf{g}_{IK}^* = \mathbf{h}_I^{-1} \mathbf{h}_K^*$ and we obtain the desired outcome.

Finally, we have shown as in the previous section that $H_\delta^q f_{I*} \mathbf{W}$ is the simple cycle module $H^q(V_{k-1}(\mathbb{A}^{n-1})) \otimes \mathbf{W}$ on $V_I(\mathbb{A}^n)$.

3.5.3 Convergence of the relevant spectral sequences

We now examine the following special cases of the fiber sequences (3.1):

$$\mathbb{A}^{n-k+1} \setminus \{0\} \rightarrow V_k(\mathbb{A}^n) \rightarrow V_{k-1}(\mathbb{A}^n)$$

for $n - k$ even, and

$$V_2(\mathbb{A}^{n-k+2}) \rightarrow V_k(\mathbb{A}^n) \rightarrow V_{k-2}(\mathbb{A}^n)$$

for $n - k$ odd.

For $H_\eta = \delta \mathbf{HW}$ we obtain when is $n - k$ even,

$$E_2^{p,q} = H^p(V_{k-1}(\mathbb{A}^n), H^q(\mathbb{A}^{n-k+1} \setminus \{0\}, \mathbf{W})) \cong H^p(V_{k-1}(\mathbb{A}^n), \mathbf{W}) \otimes_{H^*(S, \mathbf{W})} H^q(\mathbb{A}^{n-k+1} \setminus \{0\}, \mathbf{W}) \Rightarrow H^{p+q}(V_k(\mathbb{A}^n), \mathbf{W}).$$

On the other hand, when $n - k$ is odd,

$$E_2^{p,q} = H^p(V_{k-2}(\mathbb{A}^n), H^q(V_2(\mathbb{A}^{n-k+2}), \mathbf{W})) \cong H^p(V_{k-2}(\mathbb{A}^n), \mathbf{W}) \otimes_{H^*(S, \mathbf{W})} H^q(V_2(\mathbb{A}^{n-k+2}), \mathbf{W}) \Rightarrow H^{p+q}(V_k(\mathbb{A}^n), \mathbf{W}).$$

It is important to note that the spectral sequences in the two cases collapse at the E_2 -page.

Proposition 3.5.1. *If we assume that the conditions $\text{PH}(k-1, n)$ and $\text{PH}(k-2, n)$ hold, then $\text{PH}(k, n)$ also holds.*

Proof. While we showed this for the case $k = 2$, we assume $k > 2$ here.

With the statement of $\text{PH}(k, n)$, we know that H_η^* forms a free algebra. Thus, using the Kuneth formula, we only need to demonstrate when $S = \text{Spec } K$. In order to prove that these spectral sequences collapse at the E_2 -page, it is enough to examine the generators, and show that all differentials degenerate. For this, recall that the differentials in the spectral sequence satisfy Leibnitz rule. If $n - k$ is even, we obtain

$$\begin{array}{cccccccc}
 n-k & \gamma_{n-k} & & 0 & & \alpha_{n-1}\gamma_{n-k} & & \beta_{4j-1}\gamma_{n-k} \\
 & \searrow^{d_2} & & & & & & \\
 \vdots & & & & & & & \\
 0 & 1 & & 0 & & \alpha_{n-1} & & \beta_{4j-1} \\
 & & 0 & \cdots & n-k+1 & \cdots & n-1(\text{for } n \text{ even}) & \cdots & 4j-1(n-k+1 < 2j < n) & \cdots
 \end{array}$$

while for $n - k$ odd, we have

$$\begin{array}{cccccccc}
 2(n-k)+1 & \beta_{2(n-k)+1} & & 0 & & \alpha_{n-1}\beta_{2(n-k)+1} & & \beta_{4j-1}\beta_{2(n-k)+1} \\
 & \searrow^{d_2} & & & & & & \\
 \vdots & & & & & & & \\
 0 & 1 & & 0 & & \alpha_{n-1} & & \beta_{4j-1} \\
 & & 0 & \cdots & 2(n-k)+2 & \cdots & n-1(\text{for } n \text{ even}) & \cdots & 4j-1(n-k+2 < 2j < n) & \cdots
 \end{array}$$

The claim follows. \square

4

Computations on Multiplicative Structures

In this chapter, we will complete the proof for multiplicative structures of cohomology rings. And we will also introduce symplectic Stiefel varieties.

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In this section, we will study the multiplicative structures and complete the proof of the computations of the cohomology rings $H_{\text{MW}}^{*,*}$ and H_η^* . As mentioned earlier, we only need to focus on the cases where the base is over the field K . For our purposes, let us introduce another index of degree: $(p, \{q\}) := (p + q, q)$.

Remark 12. We recall from [Bachmann2020, §1.3] that for H_{MW} , we have:

$$H_{\text{MW}}^{p, \{q\}}(K) = \begin{cases} 0 & \text{if } p > 0 \\ K_q^{\text{MW}}(K) & \text{if } p = 0 \\ 0 & \text{if } p < 0 \text{ and } q \leq 0 \end{cases}$$

Furthermore, for H_η is rather clear that:

$$H_\eta^{p, \{q\}}(K) = H_\eta^p(K) = \begin{cases} 0 & \text{if } p \neq 0 \\ W(K) & \text{if } p = 0 \end{cases}$$

More generally, we say a cohomology theory E is **coconnected** if $E^{p, \{q\}}(K) = 0$ for $p > 0$.

4.0.1 Basic examples

Let us now consider the basic cases of $\mathbb{A}^n \setminus \{0\}$.

Lemma 4.0.1. Following the same notation of Example 1, for a the coconnected cohomology theory E we have the isomorphism of rings

$$E^{*,*}(\mathbb{G}_m) \cong E^{*,*}(K)[\theta_1]/(\theta_1^2 - [-1]\theta_1)$$

and for $n > 1$

$$E^{*,*}(\mathbb{A}^n \setminus \{0\}) \cong E^{*,*}(K)[\theta_n]/(\theta_n^2)$$

Proof. From Example 1, we already know that this is an isomorphism of modules. For \mathbb{G}_m , this is well-known [Morel2012b]. For $n > 1$, we only need to show $\theta_n^2 = 0$. Let $\theta_n^2 = a + b\theta_n$, where $a \in E^{2n-2, \{2n\}}(K)$ and $b \in E^{n-1, \{n\}}(K)$. By the definition of coconnectedness, we have $a = 0$ and $b = 0$. \square

Thus this result can be applied to both H_{MW} and H_η .

With the same argument, we can show that $H_\eta^*(V_2(\mathbb{A}^{2k+1})) \cong H_\eta^*(K)[\beta_{4k-1}]/(\beta_{4k-1}^2)$, where $\beta_{4k-1} \in H_\eta^{4k-1}$.

4.0.2 Multiplicative Structures of $H_\eta^*(V_k(\mathbb{A}^n))$

We now aim to show that the multiplicative structures of $H_\eta^*(V_k(\mathbb{A}^n))$ are as simple as they can be. As mentioned earlier, over K , the coefficient ring $H_\eta^*(K)$ is concentrated on $H_\eta^0(K) = W(K)$.

Proof of multiplicative structure in Theorem 3.0.1. It is clear for all n , when $k = 1$ and for n is odd, when $k = 2$. We follow the notations in the proof of Proposition 3.5.1. We just need to show inductively that for the new generators $\beta_{2(n-k)+1}^2 = 0$ or $\gamma_{n-k}^2 = 0$ (except $\gamma_0^2 = [-1]\gamma_0$). The degrees of $\beta_{2(n-k)+1}^2$ and γ_{n-k}^2 are too small to be products of two or more other generators. In other words, $\beta_{2(n-k)+1}^2$ and γ_{n-k}^2 must be of the form $c\alpha_{n-1}$ (for n even) + $\sum c_j \beta_{4j-1}$. These must be equal to 0 (or $\gamma_0^2 = [-1]\gamma_0$) since the degree of the former is even while the degree of the latter is odd and we apply Remark 12. \square

4.0.3 Symplectic Stiefel Varieties

Following the same idea, we can define the symplectic Stiefel varieties $\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \subset V_{2k}(\mathbb{A}^{2n})$ as the varieties of symplectic morphisms $\mathrm{Hom}_{\mathrm{Sym}}(\mathbb{A}^{2k}, \mathbb{A}^{2n})$, where we give \mathbb{A}^{2n} the canonical symplectic structure ω . Similar to the case of Stiefel varieties, we can also identify symplectic Stiefel varieties as homogeneous spaces:

$$\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \cong \mathrm{Sp}_{2n}/\mathrm{Sp}_{2(n-k)}.$$

We will show that from $\mathrm{Sp}V_{2k}(\mathbb{A}^{2n})$, it is enough to "add one row" to get $\mathrm{Sp}V_{2k+2}(\mathbb{A}^{2n})$.

Lemma 4.0.2. *Let $f: V_{2k+1}(\mathbb{A}^{2n}) \rightarrow V_{2k}(\mathbb{A}^{2n})$ take the first $2k$ rows, and let $Y := f^{-1}(\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}))$. Then $g: \mathrm{Sp}V_{2k+2}(\mathbb{A}^{2n}) \rightarrow Y$ is an affine bundle over Y . In particular, we have the homotopy equivalence $\mathrm{Sp}V_{2k+2}(\mathbb{A}^{2n}) \cong Y$.*

Proof. First, we replace Y with X , such that in X , the $2k+1$ -th row is symplectic orthogonal to the other rows. This gives an affine bundle and a retraction, thus $Y \cong X$.

Then we can check it locally. Let $U \subset X$ be an open subvariety such that for $\mathbf{A} \in U$ and $\mathbf{A}_{2k+1} = \mathbf{x}$, where \mathbf{A}_{2k+1} is the $2k+1$ -th row of \mathbf{A} , we have $x_j \neq 0$.

Next, for the matrix in the fiber $g^{-1}(\mathbf{A}) \ni \mathbf{B}$, the $2k+2$ -nd row of \mathbf{B} should be $\mathbf{y} + c_{2k+1}\mathbf{x} + \mathbf{z}$, where $y_{n+j} = x_j^{-1}$, otherwise $y_i = 0$, and $\mathbf{z} \perp_{\omega} \{\mathbf{x}, \mathbf{y}\}$. This shows that the fiber should be also an affine space. \square

Notice that when we take $k=0$ this shows that $\mathrm{Sp}V_2(\mathbb{A}^{2n}) \cong \mathbb{A}^{2n} \setminus \{0\}$.

In the meantime, Y can also be understood as the complement of a subbundle in a vector bundle over $\mathrm{Sp}V_{2k}(\mathbb{A}^{2n})$: Let $\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \times \mathbb{A}^{2n}$ denote the free addition of the $2k+1$ -th row, and we define the embedding

$$i: \mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \times \mathbb{A}^{2k} \rightarrow \mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \times \mathbb{A}^{2n}$$

by simply

$$(\mathbf{A}, \mathbf{x}) \mapsto \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\mathbf{x} \end{bmatrix}.$$

Clearly, Y is the complement of i . Moreover, we have the following localization cofiber sequence:

$$\mathrm{Sp}V_{2k+2}(\mathbb{A}^{2n}) \cong Y \rightarrow \mathrm{Sp}V_{2k}(\mathbb{A}^{2n}) \rightarrow \mathrm{Th}(N_i)$$

where N_i is the normal bundle of i . We claim that N_i is SL-oriented: in fact, $\det(N_i)$ is exactly the torsor we computed in Section 3.5.2, which is trivial. Since H_{MW} is SL-oriented, in the computation of H_{MW} we can choose an orientation

$$\tau \in H_{\mathrm{MW}}^{2n-2k, \{2n-2k\}}(\Sigma \mathbb{A}^{2n-2k} \setminus \{0\}) \hookrightarrow H_{\mathrm{MW}}^{2n-2k, \{2n-2k\}}(\mathrm{Th}(N_i)),$$

and we can identify $H_{\mathrm{MW}}^{p, \{q\}}(\mathrm{Th}(N_i))$ with $H_{\mathrm{MW}}^{p-2(n-k), \{q-2(n-k)\}}(\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}))\tau$.

We now follow the same idea of [Williams2012, Proposition 5] to show the following theorem:

Theorem 4.0.3. *The graded ring $H_{\mathrm{MW}}^{*,*}(\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}))$ is a free $H_{\mathrm{MW}}^{*,*}(S)$ -square-zero algebra generated by the classes $\beta_j \in H_{\mathrm{MW}}^{2j-1, \{2j\}}(\mathrm{Sp}V_{2k}(\mathbb{A}^{2n}))$ for $n-k < \dots \leq n$. In other words, we have an isomorphism of rings:*

$$H_{\mathrm{MW}}^{*,*}(\mathrm{Sp}V_{2k}(\mathbb{A}^{2n})) \cong H_{\mathrm{MW}}^{*,*}(K)[\beta_{n-k+1}, \dots, \beta_n]/(\beta_{n-k+1}^2, \dots, \beta_n^2).$$

Proof. We will prove this result inductively. For the base case $k = 1$, the result is already known. Using the localization cofiber sequence, we obtain the long exact sequence

$$\rightarrow H_{\text{MW}}^{p-2(n-k), \{q-2(n-k)\}}(\text{Sp}V_{2k}(\mathbb{A}^{2n}))\tau \xrightarrow{i_*} H_{\text{MW}}^{p, \{q\}}(\text{Sp}V_{2k}(\mathbb{A}^{2n})) \rightarrow H_{\text{MW}}^{p, \{q\}}(\text{Sp}V_{2k+2}(\mathbb{A}^{2n})) \rightarrow$$

First, we will show that $i_* = 0$, which is equivalent to showing that $i_*(\tau) = 0 \in H_{\text{MW}}^{2(n-k), \{2(n-k)\}}(\text{Sp}V_{2k}(\mathbb{A}^{2n}))$. Instead, we will demonstrate that $H_{\text{MW}}^{2(n-k), \{2(n-k)\}}(\text{Sp}V_{2k}(\mathbb{A}^{2n})) = 0$.

By our inductive hypothesis, all the generators of $H_{\text{MW}}^{*,*}(\text{Sp}V_{2k}(\mathbb{A}^{2n}))$ have degree at least $(2(n-k) + 1, \{2(n-k) + 2\})$. Therefore, for any $c \in H_{\text{MW}}^{2(n-k), \{2(n-k)\}}(\text{Sp}V_{2k}(\mathbb{A}^{2n}))$, we can write $c = \sum c_l \beta_l$. Moreover, by Remark 12, all the coefficients $c_l \in H_{\text{MW}}^{p, \{q\}}(K) = 0$ since $p, q < 0$.

Remark 13. *This argument cannot be applied to H_{MW} of Stiefel varieties, as in that case, it is possible for c to be of the form $c = c_0 \gamma_{n-k}$, where $c_0 \neq 0 \in H_{\text{MW}}^{0, \{-1\}}(K)$.*

This gives us the following module isomorphism:

$$H_{\text{MW}}^{*,*}(\text{Sp}V_{2k+2}(\mathbb{A}^{2n})) \cong H_{\text{MW}}^{*,*}(\text{Sp}V_{2k}(\mathbb{A}^{2n})) \oplus H_{\text{MW}}^{*,*}(\text{Sp}V_{2k}(\mathbb{A}^{2n}))\beta_{n-k}$$

where $\beta_{n-k} = \Omega\tau \in H_{\text{MW}}^{2(n-k)-1, \{2(n-k)\}}(\text{Sp}V_{2k+2}(\mathbb{A}^{2n}))$. This shows the additive structure of the theorem.

To determine the multiplicative structure, we only need to show that $\beta_{n-k}^2 = 0 \in H_{\text{MW}}^{4(n-k)-2, \{4(n-k)\}}(\text{Sp}V_{2k+2}(\mathbb{A}^{2n}))$. By considering the degrees as before, we see that the degree of β_{n-k}^2 is too small to be the product of two or more generators. Therefore, $\beta_{n-k}^2 = \sum c_j \beta_j$, where $c_j \in H_{\text{MW}}^{4(n-k)-2j-1, \{4(n-k)-2j\}}$. Applying Remark 12, if $4(n-k) - 2j > 0$, then $4(n-k) - 2j - 1 > 0$, and we have $c_j = 0$. Similarly, if $4(n-k) - 2j \leq 0$, then $4(n-k) - 2j - 1 < 0$, and again we have $c_j = 0$. \square

A

Appendix

We summarize some detail of computations here.

A crucial lemma for matrix computations is that the elementary matrices induce \mathbb{A}^1 -homotopies that all to construct retractions. Our aim in this appendix is to give a general unified framework for the constructions we need in the main body of this thesis. For any $n \in \mathbb{N}$, we set $M_n := \mathbb{A}^{n \times n}$ and we consider the matrix multiplication

$$M_n \times M_n \rightarrow M_n.$$

We say that a subvariety $C \subset M_n$ is *conic* if it is a submonoid for the matrix multiplication and there exists an integer $m \in \mathbb{N}$ and an embedding $f_C : \mathbb{A}^m \rightarrow M_n$ such that $f_C(0) = \text{Id}$, and $C = f_C(\mathbb{A}^m)$. In other words, C is a linear subspace of M_n containing the identity. In particular, we are interested in the conic subvariety consisting of elementary matrices.

Lemma A.0.1 (Matrix Retraction). *Let $n \in \mathbb{N}$ and let X be a variety and let $A \subset X$ be a subvariety. Suppose that there exists a conic subvariety $C \subset M_n$ acting on X , and that there exists a morphism $T : X \rightarrow C$ such that $xT(x) \in A$ and $T(a) = \text{Id}$ for all $a \in A$. Then, the morphism $r : X \rightarrow A$ defined by $x \mapsto xT(x)$ satisfies $r \circ i = \text{Id}_A$ and $i \circ r$ is homotopic to Id_X . Consequently, $A \cong X$ in $\mathcal{H}(S)$.*

Proof. We only need to show that the morphism $r' : X \rightarrow X$ defined by $x \mapsto xT(x)$ is \mathbb{A}^1 -homotopic to the identity. We just set $T(x) = f_C(\mathbf{a}_x)$ and define a homotopy $F : X \times \mathbb{A}^1 \rightarrow X$ by $(x, t) \mapsto xf_C(t\mathbf{a}_x)$. A direct computation shows that F does the trick. \square

To fix the ideas, recall that the symplectic group of size $2n$ is defined as the subgroup of invertible matrices satisfying

$$A^T J_{2n} A = J_{2n}$$

where $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $J_{2n} := \text{diag}(J_2, J_{2n-2})$ is defined by induction. Now we consider the morphism $p : \text{Sp}_{2n} \rightarrow \mathbb{A}^{2n-1}$ defined by $p(A) = (A_{12}, \dots, A_{1,2n})$.

Lemma A.0.2. *We have an isomorphism $p^{-1}(0) \xrightarrow{\sim} \mathbb{G}_m \times \text{Sp}_{2n-2} \times \mathbb{A}^{2n-1}$, yielding a weak equivalence $p^{-1}(0) \cong \mathbb{G}_m \times \text{Sp}_{2n-2}$.*

Proof. Let $A \in p^{-1}(0)$ be of the form

$$\begin{bmatrix} \begin{bmatrix} \lambda & 0 \\ a & b \end{bmatrix} & \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} \\ \begin{bmatrix} \mathbf{w}_1^T & \mathbf{w}_2^T \end{bmatrix} & M \end{bmatrix}$$

Using the fact that $A^T J_{2n} A = J_{2n}$, we obtain that $M \in \text{Sp}_{2n-2}$ and an equations

$$\lambda b J_2 + \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} J_{2n-2} \begin{pmatrix} \mathbf{w}_1^T & \mathbf{w}_2^T \end{pmatrix} = J_2$$

and

$$\begin{pmatrix} \lambda \mathbf{v} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} J_{2n-2} M = 0.$$

By the second equality, we can conclude that

$$\mathbf{w}_1 = -\lambda \mathbf{v} (J_{2n-2} M)^{-1}, \mathbf{w}_2 = 0$$

Therefore

$$\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} J_{2n-2} \begin{pmatrix} \mathbf{w}_1^T & \mathbf{w}_2^T \end{pmatrix} = 0$$

and $b = \lambda^{-1}$. Hence A is uniquely determined by λ, a, v and M . \square

Since $p^{-1}(0)$ is a complete intersection in Sp_{2n} , its normal bundle admits an obvious trivialization (i.e. $A_{12} = \dots = A_{1,2n} = 0$). We then obtain

$$\mathrm{Sp}_{2n}/p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\}) \cong p^{-1}(0)_+ \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\}.$$

and a cofiber sequence

$$p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})_+ \rightarrow \mathrm{Sp}_{2n+} \rightarrow \mathrm{Sp}_{2n}/p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\}) \cong (\mathbb{G}_m \times \mathrm{Sp}_{2n-2})_+ \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\} \quad (\text{A.1})$$

To decide the homotopy type of $p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$, we make use of lemma A.0.1. We consider the conic subvariety C of Sp_{2n} given by $f : \mathbb{A}^{2n-1} \rightarrow \mathrm{Sp}_{2n}$ defined by

$$(a, \mathbf{w}) \mapsto \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -\mathbf{w}J_{2n-2} \end{bmatrix} \\ \begin{bmatrix} \mathbf{w}^T & 0 \end{bmatrix} & \mathbf{I} \end{bmatrix}$$

Lemma A.0.3. *There exists an \mathbb{A}^1 -homotopy retraction*

$$r : p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\}) \rightarrow \mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\}$$

of the inclusion $\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\} \subset p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$, which is consequently a weak equivalence.

Proof. Let $Q_{2m-1} \rightarrow \mathbb{A}^m \setminus \{0\}$ be the affine bundle

$$\mathrm{Spec} K[a_1, \dots, a_m, x_1, \dots, x_m] / (\mathbf{a} \cdot \mathbf{x}^T - 1) \xrightarrow{(a_1, \dots, a_m)} \mathbb{A}^m \setminus \{0\}.$$

For any variety V over $\mathbb{A}^m \setminus \{0\}$, we can consider $QV := Q_{2m-1} \times_{\mathbb{A}^m \setminus \{0\}} V$. Since the projection $Q_{2m-1} \rightarrow \mathbb{A}^m \setminus \{0\}$ is an affine bundle, the same is true for the projection $QV \rightarrow V$ and it follows that the latter is a weak equivalence. This applies in particular to $V = p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$ and we denote by W_{2n} the scheme $Qp^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$.

Let $\tilde{M} = (\mathbf{x}, M) \in W_{2n}$ be such that $M \in p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$ is of the form

$$M = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w}^T & M_1 \end{pmatrix}$$

with $\mathbf{x} \cdot \mathbf{v}^T = 1$. We define the transformation matrix $T_1(\tilde{M}) \in C$ to be

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ (a-1)x_1 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -(a-1)\mathbf{x}_2 J_{2n-2} \end{bmatrix} \\ \begin{bmatrix} (a-1)\mathbf{x}_2^T & 0 \end{bmatrix} & \mathbf{I} \end{bmatrix}$$

where $(x_1, \mathbf{x}_2) = \mathbf{x}$. We obtain

$$M' := MT_1(\tilde{M}) = \begin{pmatrix} 1 & \mathbf{v}' \\ \mathbf{w}'^T & M'_1 \end{pmatrix}$$

Since $\mathbf{v}' = (v_1, \mathbf{v}_2 - v_1(a-1)\mathbf{x}_2 J_{2n-2})$ is still in $\mathbb{A}^{2n-1} \setminus \{0\}$, it follows that $M' \in p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$. For the same reason, $p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$ is stable under multiplication by C .

Similarly, we can define

$$T_2(M') = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -w'_1 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ \mathbf{w}'_2 J_{2n-2} \end{bmatrix} \\ \begin{bmatrix} -\mathbf{w}'_2{}^T & 0 \end{bmatrix} & \mathbf{I} \end{bmatrix}$$

and after multiplying, we get

$$M'' = T_2(M')M' = \begin{pmatrix} 1 & \mathbf{v}'' \\ 0 & M''_1 \end{pmatrix}$$

As in the previous lemma, we find that M'' must have the form

$$\begin{bmatrix} \begin{bmatrix} 1 & \mathbf{v}''_1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \mathbf{v}'' \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \mathbf{u} \end{bmatrix} & M_2 \end{bmatrix}$$

where $\mathbf{u} = \mathbf{v}''(J_{2n-2}M_2)^{-1}$ and $M_2 \in \mathrm{Sp}_{2n-2}$. This also yields an embedding $i : \mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\} \hookrightarrow p^{-1}(\mathbb{A}^{2n-1} \setminus \{0\})$. And by our construction, using lemma A.0.1, we obtain a \mathbb{A}^1 -homotopy retraction $r : W_{2n} \rightarrow \mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\}$. \square

Therefore the cofiber sequence (A.1) now becomes

$$(\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\})_+ \rightarrow \mathrm{Sp}_{2n+} \rightarrow (\mathbb{G}_m \times \mathrm{Sp}_{2n-2})_+ \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\}$$

Finally, we consider the following diagram

$$\begin{array}{ccccc} (\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\})_+ & \rightarrow & (\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1})_+ & \longrightarrow & \mathrm{Sp}_{2n-2+} \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\} \\ \parallel & & \downarrow & & \vdots \\ (\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1} \setminus \{0\})_+ & \longrightarrow & \mathrm{Sp}_{2n+} & \longrightarrow & (\mathbb{G}_m \times \mathrm{Sp}_{2n-2})_+ \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\} \\ \downarrow & & \parallel & & \vdots \\ (\mathrm{Sp}_{2n-2} \times \mathbb{A}^{2n-1})_+ & \longrightarrow & \mathrm{Sp}_{2n+} & \longrightarrow & \mathbb{G}_m \wedge \mathrm{Sp}_{2n-2+} \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\} \end{array}$$

Using the bottom line, the octahedral axiom, and $\mathbb{G}_m \wedge \Sigma \mathbb{A}^{2n-1} \setminus \{0\} \cong \mathbb{A}^{2n} \setminus \{0\}$ we obtain a cofiber sequence

$$\mathrm{Sp}_{2n-2+} \rightarrow \mathrm{Sp}_{2n+} \rightarrow \mathrm{Sp}_{2n-2+} \wedge \mathbb{A}^{2n} \setminus \{0\}.$$

Bibliography

- [Asok2014] A. Asok et J. Fasel. *A cohomological classification of vector bundles on smooth affine threefolds*. Duke Math. J., vol. 163, no. 14, pages 2561–2601, 2014.
- [Asok2018] Aravind Asok, Frédéric Déglise et Jan Nagel. *The homotopy Leray spectral sequence*. arXiv preprint arXiv:1812.09574, 2018.
- [Bachmann2018] Tom Bachmann. *Motivic and real étale stable homotopy theory*. Compositio Mathematica, vol. 154, no. 5, pages 883–917, 2018.
- [Bachmann2020] Tom Bachmann, Baptiste Calmès, Frédéric Déglise, Jean Fasel et Paul Arne Østvær. *Milnor-Witt motives*. arXiv preprint arXiv:2004.06634, 2020.
- [Barge2000] Jean Barge et Fabien Morel. *Groupe de Chow des cycles orientés et classe d’Euler des fibrés vectoriels*. C. R. Acad. Sci. Paris Sér. I Math., vol. 330, no. 4, pages 287–290, 2000.
- [Biglari2012] Shahram Biglari. *Motives of reductive groups*. American Journal of Mathematics, vol. 134, no. 1, pages 235–257, 2012.
- [Bondarko2017] Mikhail Bondarko et Frédéric Déglise. *Dimensional homotopy t-structures in motivic homotopy theory*. Advances in Mathematics, vol. 311, pages 91–189, 2017.
- [Calmès2017] B. Calmès et J. Fasel. A comparison theorem for MW-motivic cohomology. arXiv:1708.06100, 2017.
- [Chatzistamatiou2007] A. Chatzistamatiou. *Motivic cohomology of the complement of hyperplane arrangements*. Duke Math. J., vol. 138, no. 3, pages 375–389, 2007.
- [Cisinski2019] D.-C. Cisinski et F. Déglise. *Triangulated categories of mixed motives*. Springer Monographs in Mathematics. Springer International Publishing, 2019.
- [Déglise2011] Frédéric Déglise. *Modules homotopiques*. Documenta Mathematica, vol. 16, pages 411–455, 2011.
- [Déglise2016] F. Déglise et J. Fasel. MW-motivic complexes. arXiv:1708.06095, 2016.
- [Déglise2017] F. Déglise et J. Fasel. The Milnor-Witt motivic ring spectrum and its associated theories. arXiv:1708.06102, 2017.
- [Déglise2022] Frédéric Déglise, Niels Feld et Fangzhou Jin. *Perverse homotopy heart and MW-modules*. arXiv preprint arXiv:2210.14832, 2022.
- [Fasel2008] Jean Fasel. *Groupes de Chow-Witt*. Mém. Soc. Math. Fr. (N.S.), vol. 113, pages viii+197, 2008.

- [Feld2020] Niels Feld. *Milnor-Witt cycle modules*. Journal of Pure and Applied Algebra, vol. 224, no. 7, page 106298, 2020.
- [Hatcher2002] A. Hatcher. Algebraic topology. Algebraic Topology. Cambridge University Press, 2002.
- [Hornbostel2019] Jens Hornbostel, Matthias Wendt, Heng Xie et Marcus Zibrowius. *The real cycle class map*. arXiv preprint arXiv:1911.04150, 2019.
- [Jacobson2017] J. A. Jacobson. *Real cohomology and the powers of the fundamental ideal in the Witt ring*. Annals of K-theory, vol. 2, no. 3, pages 357–385, 2017.
- [Morel1999] F. Morel et V. Voevodsky. \mathbb{A}^1 -homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math., vol. 90, pages 45–143 (2001), 1999.
- [Morel2012a] F. Morel. \mathbb{A}^1 -Algebraic Topology over a Field, volume 2052 of *Lecture Notes in Math*. Springer, New York, 2012.
- [Morel2012b] Fabien Morel. \mathbb{A}^1 -algebraic topology over a field, volume 2052. Springer, 2012.
- [Morel2020] Fabien Morel et Anand Sawant. *Cellular \mathbb{A}^1 -homology and the motivic version of Matsumoto’s theorem*. arXiv preprint arXiv:2007.14770, 2020.
- [Panin2022] Ivan Panin et Charles Walter. *Quaternionic Grassmannians and Borel classes in algebraic geometry*. St. Petersburg Mathematical Journal, vol. 33, no. 1, pages 97–140, 2022.
- [Panin2023] I Panin et Charles Walter. *On the algebraic cobordism spectra MSL and MSp*. St. Petersburg Mathematical Journal, vol. 34, no. 1, pages 109–141, 2023.
- [Suslin1991] Andrei Suslin. *K-theory and K-cohomology of certain group varieties*. Advances in Soviet Mathematics, vol. 4, pages 53–74, 1991.
- [Williams2012] Ben Williams. *The motivic cohomology of Stiefel varieties*. Journal of K-theory, vol. 10, no. 1, pages 141–163, 2012.
- [Yang2019] N. Yang. *Quaternionic Projective Bundle Theorem and Gysin Triangle in MW-Motivic Cohomology*. Manuscripta Math., 2019.
- [Yang2020] N. Yang. *Projective Bundle Theorem in MW-Motivic Cohomology*. arXiv:2006.11774, 2020.

Cohomologie MW-motivique des variétés de Stiefel et des groupes algébriques linéaires

MW-motivic cohomology of Stiefel varieties and linear algebraic groups

Résumé

Dans ce mémoire, nous présentons plusieurs calculs autour de la cohomologie motivique de Milnor-Witt .

Nous calculons d'abord la cohomologie motivique de Milnor-Witt (totale) du complément d'un arrangement d'hyperplans dans un espace affine en tant qu'algèbre avec des générateurs et des relations donnés. Nous obtenons également des corollaires par réalisation à la cohomologie classique.

Deuxièmement, nous calculons la cohomologie motivique de Milnor-Witt des groupes symplectiques Sp_{2n} pour tout $n \in \mathbb{N}$ en utilisant l'orientation Sp et les classes de Borel associées. Ensuite, en suivant les calculs classiques et en utilisant l'analogie de la suite spectrale de Leray en \mathbb{A}^1 -homotopie, nous calculons la cohomologie motivique de Milnor-Witt inversée par η des variétés de Stiefel générales, obtenant en particulier le calcul de la cohomologie motivique de Milnor-Witt inversée par η des groupes linéaires généraux GL_n et des groupes linéaires spéciaux SL_n pour tout $n \in \mathbb{N}$.

Mots-clés : cohomologie MW-motivique, group algébrique, théorie de l'homotopie motivique

Abstract

In this thesis, we present several computations around Milnor-Witt motivic cohomology.

We first compute the (total) Milnor-Witt motivic cohomology of the complement of a hyperplane arrangement in an affine space as an algebra with given generators and relations. We also obtain some corollaries by realization to classical cohomology.

Secondly, we compute the Milnor-Witt motivic cohomology of the symplectic groups Sp_{2n} for any $n \in \mathbb{N}$ using the Sp -orientation and the associated Borel classes. Then, following the classical computations and using the analogue in \mathbb{A}^1 -homotopy of the Leray spectral sequence, we compute the η -inverted Milnor-Witt motivic cohomology of general Stiefel varieties, obtaining in particular the computation of the η -inverted Milnor-Witt motivic cohomology of the general linear groups GL_n and special linear groups SL_n for any $n \in \mathbb{N}$.

Keywords : MW-motivic cohomology, algebraic group, motivic homotopy theory

