Hybrid optimal control : optimalityconditions and applications

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Contrôle optimal hybride: conditions d'optimalité et applications

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General introduction

General context. Optimal control theory was developed at the end of the fifties with two major mathematical theorems, namely, the *Hamilton-Jacobi-Bellman equation* [22] (in short, HJB equation) and the *Pontryagin Maximum Principle* [101] (in short, PMP). The HJB equation focuses on sufficient optimality conditions, while the PMP represents, in some way, its counterpart about necessary optimality conditions. The objective of this thesis is to extend the latter in a new framework related to *hybrid control systems* and address a new set of problems known as *optimal control problems with loss control regions*, which will be described later in parts of this general introduction.
Firstly, we provide an informal statement of a typical optimal control problem and its PMP. The goal is to establish the basic concepts and terminology without getting into technical details. The first fundamental component of an optimal control problem is the *control system* that is given by

$$\dot{x}(t) = f(x(t), u(t), t), \text{ for a.e. } t \in [0, T], \quad x(0) = x_0. \tag{1}$$

In the above control system, $x : [0, T] \to \mathbb{R}^n$ represents the *state* function (or *trajectory*), and $u : [0, T] \to \mathbb{R}^m$ represents the *control* function. The dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^n$ is assumed to be of sufficient regularity, with $x_0 \in \mathbb{R}^n$ denoting the initial condition and $T > 0$ denoting the final time. The values $n, m \in \mathbb{N}^*$ are the dimensions of the state and control spaces, respectively.
The second fundamental component is the *cost* function, which encompasses factors such as energy, time transfer, and more, given by

$$\overbrace{\underbrace{\phi(x(T))}_{\text{Mayer cost}} + \underbrace{\int_0^T L(x(t), u(t), t)dt}_{\text{Lagrange cost}},}^{\text{Bolza cost}} \tag{2}$$

where the function $\phi : \mathbb{R}^n \to \mathbb{R}$ represents a final cost (also known as a *Mayer* cost), and the function $L : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ represents a *running cost* (also known as a *Lagrange* cost). These costs together form what is commonly referred to as a *Bolza* cost (as shown in (2)). These functions are assumed to be of sufficient regularity.
In this general introduction, we have chosen to consider a basic framework. In fact, we consider the following optimal control problem involving (only) a *Mayer* cost:

$$\begin{array}{ll} \text{minimize} & \phi(x(T)), \\ \text{subject to} & x : [0,T] \to \mathbb{R}^n,\ u : [0,T] \to \mathbb{R}^m \text{ solution to (1)}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \end{array} \tag{3}$$

where U is a nonempty subset of $\mathbb{R}^m$. We say that a pair (x,u) is *admissible* if it satisfies all constraints of Problem (3). Therefore, the goal is to find an optimal pair (x^*,u^*) that minimizes the Mayer cost over all admissible pairs (x,u). It is usual to define the *Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ associated with Problem (3) by the formula:

$$H(x,u,p,t) := \langle p, f(x,u,t)\rangle_{\mathbb{R}^n},$$

for all $(x,u,p,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T]$. Secondly, concerning Problem (3), we present the statement of the (easy) PMP (that can be found in [28], [101]), which asserts the following: given a solution (x^*,u^*) to Problem (3), there exists an *adjoint vector* (also called *costate*) $p : [0,T] \to \mathbb{R}^n$ satisfying:

(i) the *adjoint equation*:

$$-\dot{p}(t) = \nabla_x H(x^*(t),u^*(t),p(t),t),$$

for almost every $t \in [0,T]$;

(ii) the *transversality condition*:

$$p(T) = -\nabla\phi(x^*(T));$$

(iii) the *Hamiltonian maximization condition*:

$$u^*(t) \in \arg\max_{\omega \in \mathrm{U}} H(x^*(t),\omega,p(t),t),$$

for almost every $t \in [0,T]$.

The proof of the (easy) PMP is based on the application of *needle-like perturbations* to the optimal control. Precisely, we perform a perturbation of the control with a constant value over a small time interval. Afterwards, through an examination of the associated perturbation of the cost function, we can establish the adjoint vector in a backward way, fulfilling both the transversality condition and the adjoint equation. Finally, this process allows us to derive the Hamiltonian maximization condition. As a result, the optimal control can be expressed as a *feedback* of the state and a costate function.
Furthermore, one can derive a PMP for optimal control problems that include, for instance, *terminal state constraints* (constraints on $x(0)$ and $x(T)$). These constraints are commonly encountered in various applications of optimal control theory. Several methods have been developed in the literature to take into account such terminal state constraints. One can invoke the *Ekeland variational principle* [61] or some *implicit function* arguments (see, e.g., [2], [110]) or the use of *Lagrange multiplier rules* [3].
In the literature, the PMP has been derived in various versions to address different types of optimal control

problems, including deterministic or stochastic settings that involve continuous or discrete systems, among others. The PMP has also been extended to remarkable fields like nondifferentiable optimization and infinite-dimensional optimization (partial differential equations), where necessary optimality conditions are presented in a PMP form.
From a numerical standpoint, there are different methods for solving optimal control problems in optimal control theory. Two important categories of these methods are *direct* and *indirect methods*. On one hand, direct methods involve discretizing the state and control variables, simplifying the problem into a nonlinear optimization problem. On the other hand, indirect methods (based on the PMP) solve the problem by considering a boundary value problem and using a *shooting method*.

Motivation of this thesis. In many fields of application, such as aerospace, epidemiology, biology, and others, it is often not possible to maintain a *permanent* control (in the sense that the control value is authorized to be modified at any instant of time). However, in practice, some constraints may prevent changing the control value freely at any time. In that case, we speak of a *nonpermanent control*. As an example, in automatic, when the control is digital, the control value can be modified only in a discrete way in time, resulting into a piecewise constant control (also called *sampled-data control*). A version of the PMP has been obtained for sampled-data controls, in which the Hamiltonian maximization condition is replaced by a weaker condition known as the *averaged Hamiltonian gradient condition* (see [1], [24], [34], [36], [41]). Also, in the aerospace domain, permanent controls cannot be maintained in the presence of an *eclipse constraint* [64], [69], [73]. This constraint applies to satellites that use solar power and cannot be actuated while they are in a *shadow region* (the area that is not directly exposed to sunlight). Consequently, the control input can only be set to zero due to the absence of power.
Furthermore, in viability theory, we frequently come across constraint sets associated with the controlled dynamics. These sets can be linked to thresholds that must not be exceeded, and they can, in general, be described by a set of inequalities involving state variables. For example, in prey-predator systems, we are interested in minimizing the time spent by the system in a non-desired set where the number of prey is smaller than a given threshold $\underline{x} > 0$ as depicted in Figure 1 (we refer to [20] for more details). In this context, to reduce operating costs, it can be advantageous to use constant controls (as depicted in Figures 3b) when the system evolves in a desired set. Allowing constant controls with *moderated values* (values within the interior of the control constraint set) or *saturated values* (values on the boundary of the control constraint set) leads to strategies that outperform those based on zero controls but are still less optimal than permanent controls.

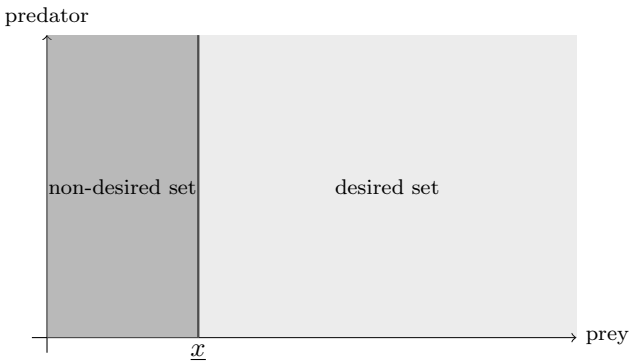


Figure 1: Partition of the the state space into two sets (desired and non-desired) based on a fixed threshold $\underline{x}$.

The main goal of this manuscript is to address optimal control problems *with loss control regions*. In this context, the state space is partitioned into disjoint sets referred to as regions, which are classified into two types: *control regions* and *loss control regions* (see Figure 2). When the state belongs to the first type of

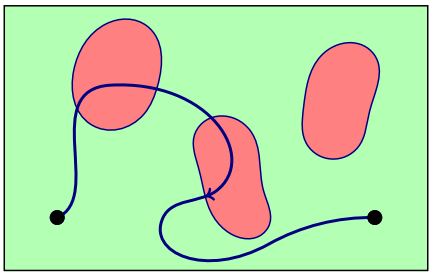
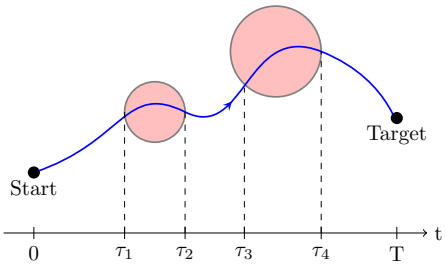


Figure 2: A state visits regions (in red) where permanent controls are not allowed, only constant ones.

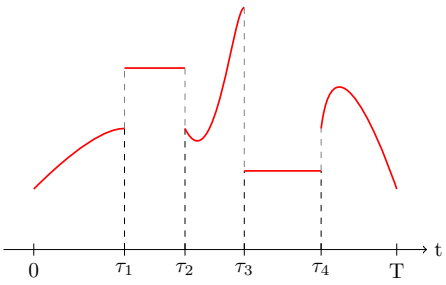
region, a permanent control can be applied (the control value can be modified at any time). However, when the state belongs to the second type of region, the control must remain constant, equal to the last assigned value before the state enters the loss control region. Roughly speaking, one has

$$\exists \lambda \in \mathrm{U},\ u(t) = \lambda, \quad \text{when } x \text{ is in a loss control region.}$$

This value $\lambda \in \mathrm{U}$ is maintained until the state exits the region, as depicted in Figure 3, where the time instants $\tau_1, \tau_3 \in (0,T)$ (respectively, $\tau_2, \tau_4 \in (0,T)$) represent the entrance instants (respectively, exit instants) to both loss control regions.



(a) The state x (in blue) traverses two loss control regions (in red) to reach the target.



(b) The control u (in red) is permanent on control regions and frozen in loss control regions.

Figure 3: behavior of the pair (x,u) in both control and loss control regions.

Furthermore, although using constant controls in loss control regions may appear restrictive, it is important to note that this approach can be extended to include a broader range of controls. For instance, we can consider *feedback* controls of the form $Ax(t) + b$ for some $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$. Such controls are commonly used in biology, as seen in the case of artificial micro-swimmers [106]. By using a change of variables, we can actually transform optimal control problems with feedback controls in loss control regions into optimal control problems with constant controls in loss control regions. Therefore, this thesis focuses (only) on constant controls in loss control regions.

Methodology. To elaborate on this concept, let us emphasize that when considering a partition of the state space $\mathbb{R}^n$ divided into control regions and loss control regions, we associate a control system

similar to the one described in equation (2). The novelty lies in the control function's ability to change its behavior (from permanent to constant and vice versa) based on the position of the state. Thus, we are dealing with discontinuous dynamics in relation to the position of the state. Roughly speaking, we consider the following partition[1] of the state space:

$$\mathbb{R}^n = \bigcup_{j\in\mathcal{J}} \overline{X_j},$$

where $\mathcal{J}$ is family of indexes and the sets X_j, called *regions*, are disjoint nonempty open subsets of $\mathbb{R}^n$. Moreover, when considering a region X_j, it can be of two types: either a control region or a loss control region. When the state belongs to a control region, permanent controls are permitted, while when the state belongs to a loss control region, only constant controls are permitted. To this state partition, we associate a *control system with loss control regions* that is given by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t), \quad \text{for a.e. } t \in [0, T], \\ u \text{ is constant when } x \text{ is in a loss control region.} \end{cases} \tag{4}$$

The challenge that arises when considering the above control system is that the change in behavior of the control function depends entirely on the state position. Therefore, our approach is to rewrite (4) as follows:

$$\dot{x}(t) = \begin{cases} f(x(t), u(t), t), & \text{if } x(t) \text{ is in a control region,} \\ f(x(t), \lambda(t), t), & \text{if } x(t) \text{ is in a loss control region,} \end{cases}$$

where λ (that is only active when the state belongs to a loss control region) is a piecewise constant function. It maintains a constant value within a loss control region and changes only when the state position x moves to another region of the same type or revisits the same loss control region. We refer to this particular type of function as a *regionally switching parameter*. Furthermore, it is important to note that the dynamics changes discontinuously (only) according to the state position. As a result, control systems with loss control regions fall within the domain of *spatially*[2] *hybrid control systems with a regionally switching parameter*. Consequently, we focus on this broader framework. Before considering a regionally switching parameter, we present the state of the art regarding the various hybrid settings, as well as the existing first-order necessary optimality conditions (if available).

State of the art. *Hybrid systems* are, in a broad sense, dynamical systems that exhibit both continuous and discrete behaviors. They are particularly used in automation and robotics to describe complex systems in which, for example, logic decisions are combined with physical processes. We refer to [71], [91], [115] for an elementary introduction to hybrid systems. This theory is very large and it is commonly accepted that it includes ODEs with *heterogeneous dynamics*, that is, ODEs involving a family of different dynamics (used for example to describe evolution in heterogeneous media) where the transitions from one dynamics to another are seen as discrete events. The PMP has been extended to hybrid control systems, especially in the context of ODEs with heterogeneous dynamics (see, e.g., [5], [25], [59], [68], [99], [108], [111]), resulting in theorems often referred to as *Hybrid Maximum Principle* (in short, HMP).
We emphasize that hybrid frameworks are very varied. Very often, the rule governing the *switching*

[1] In this general introduction, for simplicity, we do not consider time-dependent regions $X_j(t)$. However, we will examine them in Chapter 2.

[2] The term *spatially* is used to emphasize the spatial partition involved.

mode, which refers to the pattern of transitions between different dynamics, is already defined. In cases where the pattern is given, the values of the transition instances can be of different types: either fixed or treated as additional (discrete) control variables. We have chosen to call such a framework *temporally* hybrid. A fundamental difference that distinguishes a spatially hybrid setting from a temporal one is that in a spatially hybrid setting, the switching mode and the switching instants (often referred to as *crossing times*) depend entirely on the state position. Moreover, unlike the temporally hybrid setting, these crossing times cannot be fixed or treated as additional variables. This issue makes studying systems in this setting more challenging. Unfortunately, in the literature, we believe that there is no clear classification that separates each framework from the others. Therefore, we propose a suitable classification for each possible case (see Figure 4).

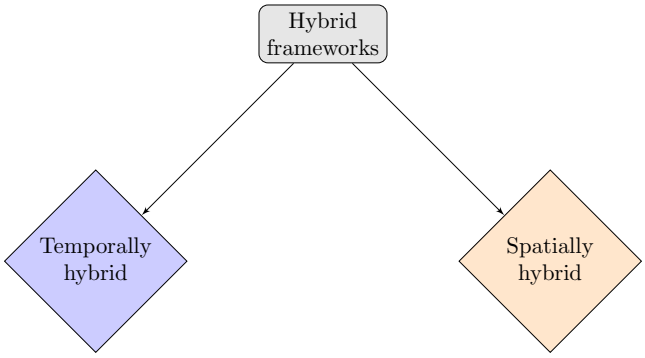


Figure 4: Diagram of the classification of hybrid systems with heterogeneous dynamics.

To delve deeper into all of these different frameworks, we provide an explanation of different situations that have been considered in the literature in each setting (temporal and spatial).

(i) **Temporally hybrid setting.** On one hand, the framework of temporally hybrid optimal control problems has been widely addressed in the literature in different type of problems.

(a) A first type of problems was considered in [68], [100], [111], [112]. In this situation, the authors consider a control system that is of the form:

$$\dot{x}(t) = f_{q(t)}(x(t), u(t)), \quad \text{for a.e. } t \in [0, T]. \tag{5}$$

A *hybrid trajectory* is defined as a pair $X(t) := (q(t), x(t))$ associated with a time partition $\{t_k\}_{k=0,\dots,N}$ of the time interval $[0, T]$. Here, $q : [0, T] \to \mathcal{Q}$ is a piecewise constant function with respect to $\{t_k\}_{k=0,\dots,N}$ (referred to as a *location*) where $\mathcal{Q}$ is a finite set and $N \in \mathbb{N}^*$. The function q provides the switching mode. Furthermore, the trajectory x is continuous within each interval of the form $[t_k, t_{k+1}]$, although it may exhibit discontinuities at switching times t_k. These switching times t_k are considered as additional variables. Furthermore, the hybrid optimal control problem considered in this situation contains a cost that involves all hybrid trajectories X with the same (fixed) switching mode. For the statement of the hybrid maximum principle as in [111], [112], or hybrid necessary conditions (in short, HNP) such as in [68], we must emphasize that the set of variations involves only trajectories of the same structure (or the same *history*) as the optimal trajectory. Let us also mention that following [68], the perturbations of the control affect only the trajectory x but not the switching times t_k.

(b) The inclusion of *intermediate constraints* (or *boundary condition*) led to a second type of

temporally hybrid optimal control problems in [50], [52], [57], [59], [117]. On one hand, intermediate constraints were considered in [57], [59]. In this situation, we assume that the switching mode is fixed, and the values of the switching instances are considered to be fixed or free, with the possibility of adding intermediate constraints given a vector p of the form:

$$p = ((t_0, x(t_0)), (t_1, x(t_1)), \ldots, (t_N, x(t_N))),$$

where $t_0 < t_1 < \ldots < t_N$. We associate intermediate constraints of the form $\eta(p) \leq 0$ and $\varphi(p) = 0$, where η and φ are smooth mappings. Furthermore, for the statement of the hybrid maximum principle, as in [59], we must emphasize that the set of variations involves only trajectories of the same structure as the optimal trajectory. In the papers [57], [59], the authors provide a practical technique known as the *augmentation technique*, which allows the reduction of the temporally hybrid optimal control problem to a classical one. This technique will be a key tool that we will rely on in this manuscript. In fact, it will be carefully adapted in Chapters 3, 4 and 5 within the spatially hybrid setting. It is important to note that this latter setting, it is more technical to manage the crossing of boundaries between regions and maintain the dynamics associated with each region. Hence, we will briefly demonstrate later in this manuscript (by means of a counterexample) that applying the augmentation technique directly in a spatially hybrid setting is not possible. On the other hand, boundary conditions were considered in a more general setting known as *optimal multiprocesses* (see [50], [52]) that covers optimal control problems of a very general nature. In fact, it allows nonsmooth controlled systems (and also differential inclusions), state constraints, variable dimensions of the state and the time intervals $[t_k, t_{k+1}]$ considered are disjoint. It should be noticed that our framework differs since we consider optimal control problems governed by hybrid control systems defined over a partition of the state space. Therefore, a change of dynamics occurs whenever the trajectory goes from one region to another, which, to our best knowledge, is not considered in these works.

(c) In a third type of temporally hybrid optimal control problems, a series of papers are found [97]–[99], [108], [109]. In this situation, the authors primarily focus on hybrid control systems that combine two types of switching: *controlled switching*, where switching times are considered as additional (discrete) control variables. More precisely, the control system is similar to the one given in (5). The change from the discrete state q_{k-1} to q_k (where $q_k = q(t)$ for all $t \in [t_{k-1}, t_k)$ and $q_{k+1} = q(t)$ for all $t \in [t_k, t_{k+1})$) at a controlled switching time t_k occurs through a discrete control function $\sigma : [0, T] \to \Sigma$, where Σ denotes the finite set of all possible switching modes. Therefore, the dynamics changes from f_{q_k} to $f_{q_{k+1}}$ at t_k by means of the discrete control σ, which satisfies:

$$q_{k+1} = A(q_k, \sigma(t_k)),$$

where $A : \mathcal{Q} \times \Sigma \to \mathcal{Q}$ is a smooth mapping. *Autonomous switching*, on the other hand occurs at t_k when the trajectory passes through a switching manifold, of the form $\{x \in \mathbb{R}^n \mid m(x) = 0\}$, as follows:

$$m(x(t_k)) = 0,$$

where $m : \mathbb{R}^n \to \mathbb{R}$ is a smooth mapping. It is important to emphasize that the set of

variations involved in the statement of the hybrid maximum principle in this setting only includes trajectories with the same structure as the optimal trajectory.

(ii) **Spatially setting.** On the other hand, the framework of spatially hybrid optimal control problems was considered in a series of papers [8], [10], [73], [95], [96]. However, to the best of our knowledge, the corresponding necessary optimality conditions are given in [10], [73]. In this hybrid setting, we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j\in\mathcal{J}} \overline{X}_j,$$

where $\mathcal{J}$ is a family of indices and the sets X_j are disjoint nonempty open subsets of $\mathbb{R}^n$. To this latter partition, we associate the following *spatially hybrid* control system:

$$\dot{x}(t) = h(x(t), u(t)), \quad \text{for a.e. } t \in [0,T],\ x(0) = x_0, \tag{6}$$

where the hybrid dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is defined *regionally* as follows:

$$h(x,u) := h_j(x,u), \quad \text{if } x \in X_j,$$

where $x_0 \in \mathbb{R}^n$ and mappings $h_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are assumed to be of sufficient regularity for all $j \in \mathcal{J}$. Hence, the statement of a spatially hybrid optimal control problem is given as follows: find a control $u : [0,T] \to \mathbb{R}^m$ that minimizes the Mayer cost $\phi(x(T))$ over all admissible pairs (x,u) that satisfy (6) and the control u verify $u(t) \in \mathrm{U}$ that is a nonempty subset of $\mathbb{R}^m$.
One can observe that the key difference is that the dynamics is entirely determined by the position of the state. This feature makes these problems more difficult and technical to handle. Unlike the temporally hybrid setting, where the time partition is predefined (with fixed or free switching times) and the switching mode is fixed. In this case, the time partition is not predefined, and the switching mode is determined by the position of the state. It is noteworthy that such difficulties have been raised in the framework of optimal control problems with discontinuous dynamics [4], [83], [85]. These works consider a very particular setting where the state space is separated into two disjoint sets by a *switching surface* of the form $S(x) = \langle d, x\rangle_{\mathbb{R}^n}$ for some $d \in \mathbb{R}^n$ dividing the state space into two disjoint sets $\{x \in \mathbb{R}^n \mid S(x) > 0\}$ and $\{x \in \mathbb{R}^n \mid S(x) < 0\}$. To each set, one associates a particular dynamics of the form $a_k(x) + b_k(x)u$, where $a_k : \mathbb{R}^n \to \mathbb{R}^n$, $b_k : \mathbb{R}^n \to \mathbb{R}^n$, and $u : [0,T] \to \mathbb{R}$ denote the control function for $k = 1, 2$. It is worth mentioning that these works rely on the augmentation technique. Another particular family of problems was considered in the work [25], referred to as *variable structure systems*. Furthermore, we believe that the general framework of a spatially hybrid optimal control problem was introduced in [73]. In this manuscript, we follow the framework presented in that work, while also considering a regionally switching parameter.

Actually, we shall next see that the derivation of necessary optimality conditions in a spatially hybrid setting requires a careful approach. In fact, in such a setting we identify two important challenges that have not been addressed in the literature. Firstly, the admissibility of needle-like perturbations in a spatially hybrid setting fails to hold true. Secondly, the use of the augmentation technique (which has proven effective for temporally optimal control problems as shown in [57], [59]) cannot be directly applied in a spatially hybrid setting or in optimal control problems with a discontinuous right-hand side, as described in [82], [83]. Hereafter, we provide a detailed discussion of these issues through the use of two counterexamples.

Challenges occurred in the spatially hybrid setting. Hereafter, we emphasize two main issues that may occur in a hybrid setting using counterexamples.

(i) **Nonadmissibility of needle-like perturbations.** In optimal control theory, the PMP is derived using needle-like perturbations. This involves considering a perturbed control defined by $u^\alpha(t) := v$ for $t \in (\sigma - \alpha, \sigma)$ and $u^\alpha(t) := u(t)$ elsewhere, where $\alpha > 0$, $v \in \mathbb{R}^m$, and $\sigma \in (0, T)$ are fixed (as illustrated in Figure 5a). Then, we examine the perturbed trajectory x^α (as illustrated in Figure 5b) and the corresponding variation vector is obtained as the limit of $\frac{x^\alpha - x}{\alpha}$ as $\alpha \to 0$. However, in a

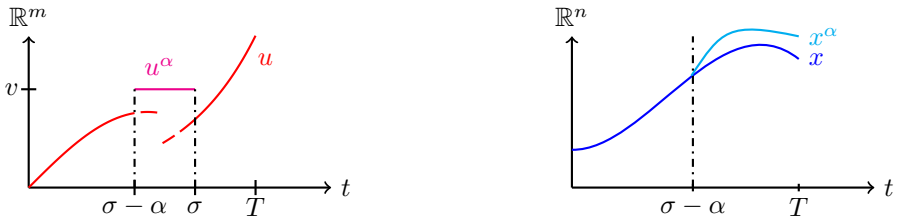


Figure 5: Illustrations of a needle-like perturbation (left) and the corresponding perturbed trajectory (right).

spatially hybrid setting, needle-like perturbations may not be admissible. For instance, the perturbed trajectory x^α may not converge uniformly to x over $[0, T]$ as $\alpha \to 0$, or it may not be globally defined over the entire interval $[0, T]$.

Let us provide a simple counterexample which highlights this issue which is not encountered in the classical (non-hybrid) setting. Consider $T = 2$, $n = m = 1$ and the space partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2}$, where $X_1 = \{y \in \mathbb{R} \mid y < 1\}$ and $X_2 = \{y \in \mathbb{R} \mid y > 1\}$. Now consider the spatially hybrid control system given by

$$\dot{x}(t) = \begin{cases} +u(t) & \text{if } x(t) \in X_1, \\ -u(t) & \text{if } x(t) \in X_2, \end{cases}$$

with the initial condition $x_0 = 0$. By taking the control $u(t) = +1$ over $[0, 1)$ and $u(t) = -1$ over $(1, 2]$, we get the corresponding trajectory x given by $x(t) = t$ over $[0, 2]$, with $\tau_1 = 1$ as unique crossing time. Now we apply needle-like perturbations of the control u at some $\sigma \in (0, 1)$ and we refer to Figure 6 for illustrations.

(i) If $v = -1$ we get a perturbed trajectory x^α satisfying $x^\alpha(t) \in X_1$ over the whole interval $[0, 2]$ and thus x^α does not uniformly converge to x over $[0, 2]$ when $\alpha \to 0$.

(ii) If $v = 2$ we get a perturbed trajectory x^α defined over $[0, \tilde{\tau}_1(\alpha))$ for some $\tilde{\tau}_1(\alpha) < 1$ and not $[0, T]$.

In a spatially hybrid setting, this issue arises because standard needle-like perturbations of the control u do not take into account the perturbation of the crossing time (we refer to Chapter 2 for more details).

(ii) **Adapting the augmentation technique.** Hereafter, our aim is twofold. In the first part, we present the mechanism of the augmentation technique in a simple temporally hybrid setting. In the second part, we develop a counterexample showing that this technique cannot be directly adapted to a spatially hybrid setting.

Part I. In optimal control theory, it is well known in the literature (see, e.g., [28], [44], [48], [54]) that the augmentation technique allows us to deal with more general Bolza costs, free final time,

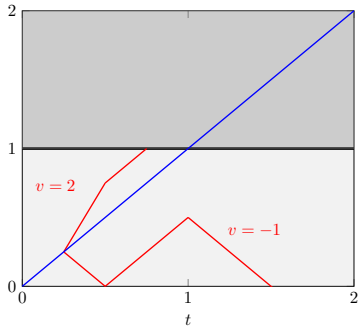


Figure 6: Representation of x (in blue) and x^α with $\sigma = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ (in red).

and time-dependent dynamics. Moreover, this technique remains excellent to use in a hybrid setting. Indeed, the augmentation technique has been used to cover the temporally hybrid case (which includes intermediate constraints [57], [59]). It is based on reducing the temporally hybrid problem to a classical one so that one can use the classical PMP. To illustrate this reduction, we consider a (simple) temporally hybrid optimal control problem given by

$$\begin{aligned}
&\text{minimize} && \phi(x(T)) \\
&\text{subject to} && x : [0,T] \to \mathbb{R}^n,\ u : [0,T] \to \mathbb{R}^m,\ \tau \in [0,T], \\
&&& \dot{x}(t) = f(x(t),u(t)), \quad \text{for a.e. } t \in [0,\tau], \\
&&& \dot{x}(t) = g(x(t),u(t)), \quad \text{for a.e. } t \in [\tau,T], \\
&&& x(0) = x_0,\ F(x(\tau)) = 0, \\
&&& u(t) \in \mathrm{U}, \quad \text{for a.e. } t \in [0,T],
\end{aligned} \tag{7}$$

where the Mayer cost $\phi : \mathbb{R}^n \to \mathbb{R}$, both dynamics $f, g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the mapping $F : \mathbb{R}^n \to \mathbb{R}$ that describes the intermediate constraint are assumed to have sufficient regularity, $x_0 \in \mathbb{R}^n$, $T > 0$ and U is a nonempty subset of $\mathbb{R}^m$. Given a global solution (x^*, u^*, τ^*) to Problem (7), the first step is to apply an affine variable change (to reduce $[0,\tau^*]$ and $[\tau^*,T]$ to $[0,1]$). This is done as follows:

$$y_1^*(s) := x^*(\tau^* s) \quad \text{and} \quad y_2^*(s) := x^*(\tau^* + (T-\tau^*)s),$$

and

$$v_1^*(s) := u^*(\tau^* s) \quad \text{and} \quad v_2^*(s) := u^*(\tau^* + (T-\tau^*)s),$$

for all $s \in [0,1]$ which allows us to generate an augmented state $y^* : [0,T] \to \mathbb{R}^{2n}$ and an augmented control $v^* : [0,T] \to \mathbb{R}^{2m}$. The second step involves defining an augmented classical optimal control

problem given by

$$\begin{aligned}
&\text{minimize} && \phi(y_2(1)) \\
&\text{subject to} && y:[0,1]\to\mathbb{R}^{2n},\ v:[0,1]\to\mathbb{R}^{2m},\ \tau\in[0,T], \\
&&& \dot{y}_1(s)=\tau f(y_1(s),v_1(s)), \quad \text{for a.e. } s\in[0,1], \\
&&& \dot{y}_2(s)=(T-\tau)g(y_2(s),v_2(s)), \quad \text{for a.e. } s\in[0,1], \\
&&& y_1(0)=x_0,\ F(y_1(1))=0,\ y_1(1)-y_2(0)=0, \\
&&& (v_1(s),v_2(s))\in \mathrm{U}^2, \quad \text{for a.e. } s\in[0,1].
\end{aligned}$$

and verifying that the admissible triplet (y^*,v^*,τ^*) is a *local*[3] solution. Lastly, the third step involves applying the classical PMP to (y^*,v^*,τ^*). Consequently, the derivation of a temporally HMP for Problem (7) is a result of inverting the augmentation procedure.
Unfortunately, such a technique could not be directly employed in a spatially hybrid setting like [10], [82]. The reason is that we cannot recover a solution in a (standard) local sense mentioned earlier. Therefore, we cannot apply the classical PMP to obtain a HMP by inverting the augmentation procedure. Hereafter, we dedicate the rest of this section to presenting a counterexample, which shows that we do not recover a (standard) local solution for the classical problem in general. We emphasize that a more detailed presentation is provided in Chapter 4, including more precise hypotheses and notations.

Part II. Let us consider the two-dimensional case $n=2$, the state space partition $\mathbb{R}^2=\overline{X_1}\cup\overline{X_2}$ where $X_1:=(-\infty,1)\times\mathbb{R}$ and $X_2:=(1,+\infty)\times\mathbb{R}$ (see Figure 7), and the spatially hybrid optimal control problem given by

$$\begin{aligned}
&\text{minimize} && -(x_1(2)-2)^3-\rho x_2(2), \\
&\text{subject to} && x:[0,2]\to\mathbb{R}^2,\ u:[0,2]\to\mathbb{R}, \\
&&& \dot{x}(t)=h(x(t),u(t)), \quad \text{a.e. } t\in[0,2], \\
&&& x(0)=0_{\mathbb{R}^2}, \\
&&& u(t)\in[-1,1], \quad \text{a.e. } t\in[0,2],
\end{aligned} \tag{8}$$

where the spatially heterogeneous dynamics $h:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ is defined by

$$h(x,u):=\begin{cases}\Big(1,((1-x_1)^+)^2\Big), & \text{if } x\in X_1,\\ \Big(u,((1-x_1)^+)^2\Big), & \text{if } x\in X_2,\end{cases}$$

for all $x=(x_1,x_2)\in X_1\cup X_2$ and all $u\in\mathbb{R}$, and where $\rho>96$ and $y^+:=\max(y,0)$ for all $y\in\mathbb{R}$.
- A global solution (x^*,u^*) (with one crossing time $\tau_1^*=1$) to Problem (8) is given by

$$x_1^*(t):=t, \quad x_2^*(t):=\begin{cases}\frac{1}{3}((t-1)^3+1) & \text{if } t\in[0,1],\\ \frac{1}{3} & \text{if } t\in[1,2],\end{cases} \qquad u^*(t):=1,$$

for all $t\in[0,2]$, and the corresponding optimal cost is given by $\mathcal{C}^*:=-\frac{\rho}{3}$.

[3] In uniform norm for the state / in L^1-norm for the control.

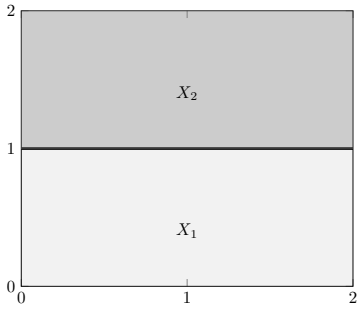


Figure 7: Partition of the state space $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2}$.

Now, after providing a global solution to Problem (8), we can introduce the augmented state $y^* : [0,1] \to \mathbb{R}^4$ and control $v^* : [0,1] \to \mathbb{R}^2$ based on a variable change that transforms both time intervals $[0, \tau_1^*]$ and $[\tau_1^*, 2]$ into the common time interval $[0,1]$ as follows:

$$(y_1^1)^*(s) = s, \quad (y_2^1)^*(s) = s + 1, \quad (y_1^2)^*(s) = \frac{1}{3}((s-1)^3 + 1), \quad (y_2^2)^*(s) = \frac{1}{3},$$

and $v_1^*(s) = v_2^*(s) = 1$ for all $s \in [0,1]$, and $\mathbb{T}^* = \{0,1,2\}$. As expected the triplet $(y^*, v^*, \mathbb{T}^*)$ is admissible for the classical optimal control problem with *mixed terminal state constraints* and a parameter

$$\begin{aligned}
\text{minimize} \quad & -(y_2^1(1) - 2)^3 - \rho y_2^2(1), \\
\text{subject to} \quad & y : [0,1] \to \mathbb{R}^4,\ v : [0,1] \to \mathbb{R}^2,\ \mathbb{T} \in \mathbb{R}^3, \\
& \dot{y}_1^1(s) = \tau_1, \quad \text{a.e. } s \in [0,1], \\
& \dot{y}_1^2(s) = \tau_1((1 - y_1^1(s))^+)^2, \quad \text{a.e. } s \in [0,1], \\
& \dot{y}_2^1(s) = (2 - \tau_1)v_2(s), \quad \text{a.e. } s \in [0,1], \\
& \dot{y}_2^2(s) = (2 - \tau_1)((1 - y_2^1(s))^+)^2, \quad \text{a.e. } s \in [0,1], \\
& y_1^1(0) = 0, \quad y_1^2(0) = 0, \quad y_1^1(1) - 1 = 0, \\
& y_2^1(0) - y_1^1(1) = 0, \quad y_2^2(0) - y_1^2(1) = 0, \\
& \tau_0 = 0, \quad \tau_1 \in [0,2], \quad \tau_2 = 2, \\
& v_1(s),\ v_2(s) \in [-1,1], \quad \text{a.e. } s \in [0,1],
\end{aligned} \tag{9}$$

with the (same) cost $\mathcal{C}^* = -\frac{\rho}{3}$.

- The triplet $(y^*, v^*, \mathbb{T}^*)$ is not a (standard) local solution to Problem (9). Indeed, one can find controls that are close in L^1–norm, that provide a better cost: for any $\varepsilon > 0$ small enough, we introduce the triplet $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$ defined by $(y_1^1)^\varepsilon := (y_1^1)^*$, $(y_1^2)^\varepsilon := (y_1^2)^*$, $v_1^\varepsilon = v_1^*$, $\mathbb{T}^\varepsilon = \mathbb{T}^*$, and by

$$(y_2^1)^\varepsilon(s) := \begin{cases} s + 1, & \text{if } s \in [0, \varepsilon], \\ 2\varepsilon - s + 1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ s - 4\varepsilon + 1, & \text{if } s \in [3\varepsilon, 1], \end{cases} \qquad v_2^\varepsilon(s) := \begin{cases} 1, & \text{if } s \in [0, \varepsilon], \\ -1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ 1, & \text{if } s \in [3\varepsilon, 1], \end{cases}$$

and

$$(y_2^2)^\varepsilon(s) := \begin{cases} \frac{1}{3}, & \text{if } s \in [0, 2\varepsilon], \\ \frac{1}{3}((s-2\varepsilon)^3 + 1), & \text{if } s \in [2\varepsilon, 3\varepsilon], \\ \frac{1}{3}((s-4\varepsilon)^3 + 2\varepsilon^3 + 1), & \text{if } s \in [3\varepsilon, 4\varepsilon], \\ \frac{1}{3}(2\varepsilon^3 + 1), & \text{if } s \in [4\varepsilon, 1], \end{cases}$$

for all $s \in [0,1]$. One can easily conclude that the triplet (y^*, v^*, τ^*) is not a local solution to Problem (9) since:

(i) The triplet $(y^\varepsilon, v^\varepsilon, \tau^\varepsilon)$ is admissible for Problem (9) for any $\varepsilon > 0$.

(ii) For any $\varepsilon > 0$, the cost $\mathcal{C}^\varepsilon$ associated with the triplet $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$ is given by

$$\mathcal{C}^\varepsilon = -\frac{\rho}{3} - \left(\frac{2\rho}{3} - 64\right)\varepsilon^3 < -\frac{\rho}{3} = \mathcal{C}^*.$$

Hence, the above counterexample highlights the impossibility of directly using the augmentation technique. In Chapter 4, we overcome this issue by introducing an appropriate notion of a local minimum, and secondly, we derive the corresponding PMP to obtain a HMP in a spatially hybrid setting.

Hereafter, we describe the main contribution of Chapters 2, 3, 4 and 5 of this thesis.

Contribution of Chapter 2. As we mentioned earlier, in a spatially hybrid framework, in particular the derivation of necessary optimality conditions has not been visited so often in the literature. To the best of our knowledge, we could find only in [10], [73] a general presentation of spatially hybrid optimal control problems together with a presentation of the corresponding necessary optimality conditions. Precisely in [73], the authors provide a sketch of the proof of a HMP similar to the proof of the PMP. In fact, it is based on needle-like perturbations and implicit function arguments. Hence, in a first attempt, we have considered spatially hybrid optimal control problem with a regionally switching parameter of the form

$$\begin{aligned} &\text{minimize} && \phi(x(T)), \\ &\text{subject to} && x : [0,T] \to \mathbb{R}^n,\ \lambda : [0,T] \to \mathbb{R}^d,\ u : [0,T] \to \mathbb{R}^m, \\ &&& \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0,T], \\ &&& x(0) = x_0, \\ &&& (\lambda(t), u(t)) \in \Lambda \times \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\ &&& \lambda \text{ is a regionally switching parameter associated with } x. \end{aligned} \tag{10}$$

Here, the control system involves a hybrid dynamics $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ that is defined similarly to (6) and includes a regionally switching parameter λ (that is a piecewise constant function), which satisfies the constraint $\lambda(t) \in \Lambda$ that is a nonempty convex subset of $\mathbb{R}^d$ with $d \in \mathrm{N}^*$. Our objective is to follow this well-known proof method, providing a complete and rigorous proof of the HMP with the novelty of considering a regionally switching parameter. This latter is here to model loss control regions.

However, we encountered an important issue, that is, the nonadmissibility of needle-like perturbations. This latter can occur in a spatially hybrid setting (we refer to the discussion provided previously in the first counterexample). In fact, the reason behind this issue is that we do not take into account the perturbed crossing time of the trajectory after performing a needle-like perturbation. This is why we

may obtain perturbed trajectories that are not defined over the entire time frame or do not converge uniformly towards the nominal trajectory (as highlighted in the first counterexample). To overcome this challenge, we introduce a new tool called *auxiliary controls*. These controls coincide with the nominal one over the interval between two consecutive crossing times and are continuously extended as constant functions outside of this interval. It is worth noting that similar tools were introduced in other hybrid settings, as discussed in [68], [108].

After performing a needle-like perturbation on an auxiliary control, we obtain perturbed auxiliary trajectory that admit a perturbed crossing time. This is a consequence of applying an implicit function theorem. Hence, to obtain a perturbed trajectory that overcomes the difficulties raised in the first challenge, we need to create the following pieces of the perturbed trajectory based on the perturbed crossing times that follow the needle-like perturbation. Similarly, we perform convex perturbations (as in [34], [41]) to *auxiliary parameters*. Let us emphasize that this construction is made possible due to the regularity assumptions made on the nominal trajectory and control. Specifically, the existence of left and right limits of the nominal control ensures a continuous extension in the definition of an auxiliary control. The application of the implicit function theorem relies on a *transverse condition* made on the nominal trajectory which means that this latter crosses the boundary between regions transversally and not tangentially. Such a hypothesis is quite common in a spatially hybrid setting.

The proof of the HMP with a regionally switching parameter follows a similar approach to the proof of the PMP (we refer to the paper [13]). After conducting the sensitivity analysis described above within the hybrid framework, we obtain a variation vector with discontinuous jumps at each crossing time. Therefore, we construct an adjoint vector with discontinuous jumps at each crossing time to ensure the constancy of the inner product of all variation vectors. This enables us to obtain the Hamiltonian maximization condition for the permanent control and the averaged Hamiltonian gradient condition for the regionally switching parameter that is given by

$$\int_{\tau_{k-1}}^{\tau_k} \nabla_\lambda H(x(s), \lambda_k, u(s), p(s))\, \mathrm{d}s \in \mathrm{N}_\Lambda[\lambda_k],$$

where $H : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian function associated with Problem (10) and $\mathrm{N}_\Lambda[y]$ denotes the *normal cone*[4] of Λ at $y \in \mathbb{R}^d$ and λ_k denotes the constant value of λ over $[\tau_{k-1}, \tau_k)$.
Lastly, it is worth mentioning that our primary motivation is to derive necessary optimality conditions for optimal control problems with loss control regions. Since such problems can be formulated as spatially hybrid optimal control problems with a regionally switching parameter, we directly obtain a PMP with loss control regions (we refer to the conference proceedings [14]). In this case, the adjoint vector exhibits discontinuous jumps at each crossing time due to the change in behavior of the control (permanent in control regions and constant in loss control regions). Additionally, the optimal control satisfies the Hamiltonian maximization condition in control regions and the averaged Hamiltonian gradient condition in loss control regions.

Contribution of Chapter 3. In optimal control theory, terminal constraints are commonly encountered in various applications. Therefore, it seems natural to extend the contributions of Chapter 2 to include such constraints. Following the method of proof based on needle-like perturbations, the proof would become more cumbersome and technical, involving heavy notation due to the use of more sophisticated

[4] the precise definition will be provided in Section 1.1 of Chapter 1.

tools such as *multiple* needle-like perturbations and *Pontryagin cone* (we refer to [3], [38], [58], [81], [101]). To avoid having a more complicated and longer proof, we rely on the *augmentation technique*. This technique was first introduced in [54] to handle minor difficulties, such as reducing Lagrange costs to Mayer costs, converting free final time to fixed final time, and transforming time-dependent dynamics to autonomous dynamics. Furthermore, this technique proved extremely useful in tackling major difficulties, including reducing temporally hybrid optimal control problems to classical ones (we refer to [57], [59]). In fact, applying such a technique allows to derive a temporally HMP as a consequence of the classical PMP, highlighting the strength of this approach.

In this chapter, and in a second attempt, we aim to take advantage of this technique to derive a PMP with loss control regions for a particular type of problem often encountered in many domains of applications. It is about the class of *minimum time problems*, but with the novelty of considering loss control regions. The goal is to study and provide (similarly to the classical case) an optimal synthesis for any given initial condition for the minimum time problem to reach the origin for the *double integrator*. However, in contrast with the classical version of this problem, we consider a loss control region. Roughly speaking, we consider the following optimal control problem

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & x : [0,T] \to \mathbb{R}^2,\ u : [0,T] \to \mathbb{R},\ T \in (0,+\infty), \\ & \dot{x}_1(t) = x_2(t) \quad \text{for a.e. } t \in [0,T], \\ & \dot{x}_2(t) = u(t) \quad \text{for a.e. } t \in [0,T], \\ & x(0) = x_0,\ x(T) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1,1], \quad \text{a.e. } t \in [0,T], \\ & u \text{ is constant when } x \text{ belongs to } \{x \in \mathbb{R}^2 \mid x_1 < 0\}. \end{array} \tag{11}$$

To solve Problem (11), we first derive a PMP with loss control regions, adapted for a general minimum time problems under a strong transverse condition. Roughly speaking, we consider a partition of the state space $\mathbb{R}^n = \overline{X_1} \bigcup \overline{X_2}$ and impose a strong transverse condition at the boundary given by $\partial X := \{x \in \mathbb{R}^n \mid F(x) = 0\}$, which remains valid for all controls:

$$\forall (x,u) \in \partial X \times \mathrm{U}, \quad \langle \nabla F(x), f(x,u) \rangle_{\mathbb{R}^n} \neq 0,$$

where both mappings $F : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (representing the dynamics) are smooth. It is noteworthy that the above strong condition is verified in the case of the minimum time problem for the double integrator with a loss control region, which is the focus of this chapter and the paper [15].

Let us emphasize that the derivation of a PMP with loss control regions, adapted for minimum time problems and under the above strong transverse condition, becomes a consequence of the classical PMP. In fact, after verifying that the augmented solution is a standard local solution, we apply the classical PMP. Regarding the optimal synthesis of the double integrator problem with a loss control region, when the optimal trajectory crosses the loss control region, the constant value of the control is determined using the averaged Hamiltonian gradient condition. We notice that for a given initial condition such that the trajectory visits the loss control region, the corresponding constant value of the control (in this region) is nontrivial and it belongs to the interior of the control constraint set. These new phenomena raise questions and challenges in optimal control problems with loss control regions, both in theory and practice.

Contribution of Chapter 4. In this chapter, we address more into details the augmentation technique since we have seen that it is very useful for handling minimum time problems including a loss control region. However, here we consider a more general setting: spatially hybrid optimal control problems with general terminal mixed state constraints of the form

$$\begin{array}{ll} \text{minimize} & \phi(x(T)), \\ \text{subject to} & x:[0,T]\to\mathbb{R}^n,\ u:[0,T]\to\mathbb{R}^m, \\ & \dot{x}(t)=h(x(t),u(t)), \quad \text{a.e. } t\in[0,T], \\ & g(x(0),x(T))\in \mathrm{S}, \\ & u(t)\in \mathrm{U}, \quad \text{a.e. } t\in[0,T]. \end{array} \tag{12}$$

Here, the hybrid control system similar to (6) where the mapping $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^\ell$ is sufficiently regular and the constraint set S is a nonempty closed convex subset of $\mathbb{R}^\ell$ with $\ell\in\mathbb{N}^*$. Moreover, we consider a weaker transverse condition, which only involves the optimal path. This is in contrast to the strong transverse condition made in Chapter 3, which is a stronger hypothesis than the one only on the nominal path.

It turns out that considering a weaker transverse condition (which is only satisfied by the optimal solution) raises an important issue: impossibility of adapting the augmentation technique. In fact, after applying the augmentation procedure, we obtain that the global solution to the spatially hybrid optimal control problem does not yield a standard local solution to the augmented problem. This is due to the fact that nearby trajectories (close in L^1–norm for controls) does not generate admissible trajectories for the spatially hybrid optimal control problem. This issue was previously discussed through the use of a counterexample that emphasizes the second challenge encountered in a spatially hybrid setting.
To overcome this issue, we first fix an optimal solution (x^*,u^*) to the spatially hybrid optimal control problem, with $\mathbb{T}^*$ denoting the vector of crossing times. Since we known that the augmentation technique generates an admissible triplet $(y^*,v^*,\mathbb{T}^*)$ to the augmented problem, we will focus on addressing the following question:

Is the triplet $(y^*,v^*,\mathbb{T}^*)$ solution to the augmented problem? If yes, in which sense?

Answering the above question led to the introduction of a novel notion of local solutions, namely, the $\mathrm{L}^1_\square$–*local* solution (we refer to the work [16]). The main feature of this notion is that it forbids variations of the control in neighborhoods of each crossing time. This solves the issue highlighted in the second counterexample. Indeed, by not allowing the control to vary in neighborhoods of each crossing, then we can prove that nearby trajectories (in this new local sense) generates admissible trajectories for the spatially hybrid problem.
Lastly, in order to derive a spatially HMP, we first need to derive a PMP that corresponds to the new notion of local solutions. We refer to this as the PMP for $\mathrm{L}^1_\square$-local solutions. Consequently, the spatially HMP is obtained as a consequence of the latter by inverting the augmentation procedure. Also, it is worth noting that in a spatially hybrid setting, the choice of the transverse conditions affects the quality of local solutions to the augmented problem.

Contribution of Chapter 5. In this chapter, our goal is twofold. First, we aim to cover a general

class of optimal control problems with loss control regions of the form

$$\begin{array}{ll} \text{minimize} & \phi(x(T)), \\ \text{subject to} & x : [0,T] \to \mathbb{R}^n,\ u : [0,T] \to \mathbb{R}^m, \\ & \dot{x}(t) = f(x(t),u(t)), \quad \text{a.e. } t \in [0,T], \\ & g(x(0),x(T)) \in \mathrm{S}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\ & u \text{ is constant when } x \text{ is in a loss control region,} \end{array} \tag{13}$$

and establish the corresponding PMP. Second, we provide a numerical approach based on both direct and indirect methods adapted for these types of problems, as outlined in the work [17].

To address our first objective, we initially derive a spatially HMP with a regionally switching parameter. This latter presents the most general result of this manuscript since it includes general mixed terminal state constraints and a regionally switching parameter. As explained earlier, this framework allows us to address optimal control problems with loss control regions. The methodology remains the same. In fact, we rely on the augmentation technique, which has proven to be a useful tool in this framework, even under weak transverse conditions that only involve the optimal solution. However, we emphasize the importance of simultaneously using the novel concept of $\mathrm{L}^1_{\square}$–local solution (that is developed in Chapter 4). This concept ensures the correct correspondence between a global solution of the spatially hybrid optimal control problem and a local solution (in this novel sense) of the augmented problem.

Let us emphasize that the augmentation procedure remains the same even when considering a regionally switching parameter. In fact, since this latter remains constant within each region, we can treat it as a state variable satisfying the differential equation $\dot{\lambda} = 0$. Moreover, we consider additional terminal constraints to ensure that the constraint made on the regionally switching parameter is not violated. Similar to Chapter 3, we obtain the spatially HMP with a regionally switching parameter as a consequence of the PMP for $\mathrm{L}^1_{\square}$–local solutions.

To address our second objective, we introduce a two-step numerical approach to solve optimal control problems with loss control regions. First, we use a direct numerical method applied to a *regularized* problem. The regularization is necessary to overcome the discontinuities that arise when transforming the optimal control problem with loss control regions into a hybrid optimal control problem with a regionally switching parameter. This initial step is crucial for determining the *structure* of the optimal trajectory, i.e., the ordered sequence of regions that the optimal trajectory visits. Secondly, we initialize an indirect numerical method applied to the original problem, which is based on the PMP with loss control regions. The novelty of this approach is the incorporation of the averaged Hamiltonian gradient condition, as well as the discontinuity jumps of the adjoint vector, to define an appropriate shooting function. This addition supplements the classical terms that define the shooting function for non-hybrid control problems (see [45], [53]). Finally, this approach is applied to numerically solve some illustrative examples, precisely a Zermelo-type problem [6] and a version of the minimum time problem for the harmonic oscillator [107] both including loss control regions.

Organization of the manuscript. This manuscript is composed of 6 chapters. Chapter 1 is devoted to the preliminary notions required throughout this thesis in order to describe spatially hybrid optimal control problems and optimal control problems with loss control regions. Precisely, we define the functional

spaces encountered in these problems. We also provide a quick recall of the general theory of ordinary differential equations and present some standard sensitivity analysis results. Additionally, we introduce standard notions of minima of optimal control problems and provide necessary optimality conditions, in a PMP form, associated with each notion of local minimum. Finally, we provide a quick recap overview of the notion of Filippov's solutions. These preliminaries are useful to adequately present our results in chapters 2, 3, 4 and 5.

Chapter 2 is devoted to the derivation of a spatially HMP with a regionally switching parameter (see Theorem 2.2.1).We also give an orevieviw of the proof Theorem 2.2.1 in Section 2.2.3. An application example is then provided in Section 2.3. In Section 2.4, we develop a general result of thorough sensitivity analysis in the case of non-hybrid control systems. This allows us to provide a thorough sensitivity analysis in the hybrid case by introducing the notions of auxiliary control, auxiliary parameters and auxiliary trajectories in Section 2.5. Then, based on these technical results, Section 2.6 is devoted to the complete proof of Theorem 2.2.1. Finally, Section 2.7 deals with the derivation of a PMP with loss control regions.

Chapter 3 is devoted to the study of a variant of the minimum time problem for the double integrator with a loss control region. In Section 3.2, we recall the well known solution to the classical (without loss control region) minimum time problem for the double integrator. Next we state a version of the PMP adapted to a minimum time problem with a loss control region (see Proposition 3.2.2). In Section 3.3, our main result (Theorem 3.3.1) is stated, providing an exact analytical solution to the minimum time problem for the double integrator with a loss control region. Its proof is given immediately after, being divided into several cases arising in the application of Proposition 3.2.2. Section 3.4 gives a list of additional comments on Theorem 3.3.1 and its proof. We conclude with open questions and perspectives about optimal control problems with loss control regions. Finally Section 3.5 contains the proof of Proposition 3.2.2.

Chapter 4 is devoted to the derivation of a spatially HMP based on a careful use of the augmentation technique. In Section 4.2, a classical optimal control problem is considered, the new notion of $L^1_\square$–local solution is introduced and a corresponding PMP is established (see Theorem 4.2.1). In Section 4.3, a hybrid optimal control problem with spatially heterogeneous dynamics is introduced, applying the augmentation procedure, Proposition 4.3.1 states that an augmented solution to a hybrid optimal control problem is a $L^1_\square$–local solution to the corresponding classical augmented problem. Hence, applying the above new PMP and inverting the affine changes of time variable, a HMP for Problem (4.2) is obtained (see Theorem 4.3.1). An explicit counterexample showing that an augmented solution to Problem (4.2) is not a local solution (in the usual sense) to the corresponding classical augmented problem is provided in Section 4.3.4. Finally the technical proofs of Proposition 4.3.1 and Theorem 4.3.1 are provided in Sections 4.4 and 4.5 respectively.

Chapter 5 is devoted to the derivation a PMP with loss control regions, but also a two-step numerical approeach to solve this type of problems. In Section 5.2, a general hybrid optimal control problem with regionally switching parameter is introduced. Then Proposition 5.2.1 asserts that the augmentation of a global solution to Problem (5.1) leads to a $L^1_\square$–local solution to a classical augmented optimal control problem. Hence, applying the PMP for $L^1_\square$–local solutions (Theorem 4.2.1 provided in Chapter 4) and reversing the augmentation procedure, a HMP for spatially hybrid optimal control problems with a regionally switching parameter is obtained (see Theorem 5.2.1). In Section 5.3, we deal with a general optimal control problem with loss control regions. By rewriting this problem as a spatially hybrid optimal

control problem with a regionally switching parameter and applying the previous Theorem 5.2.1, a PMP with loss control regions is obtained. In Section 5.4, a two-steps numerical scheme is proposed to solve optimal control problems with loss control regions. Afterwards it is applied to numerically solve some illustrative examples, precisely a Zermelo-type problem [6] and a version of the minimum time problem for the harmonic oscillator [107] both including loss control regions. Finally the technical proofs of Proposition 5.2.1 and Theorem 5.2.1 are provided in Sections 5.5 and 5.6 respectively.

Finally, in the general conclusion of this manuscript presented in Chapter 6, we review the outcome of the investigations undertaken during this PhD thesis. We also provide several possible perspectives, including further personal research projects to be undertaken in both frameworks:

(A) Spatially hybrid optimal control problems with a regionally switching parameter.

(B) Optimal control problems with loss control regions.

1

Preliminaries and notations

Contents

This chapter introduces the necessary preliminary notions for describing the two main frameworks of this manuscript: spatially hybrid optimal control problems with a regionally switching parameter and optimal control problems with loss control regions. In Section 1.1, we define the functional spaces that describe the state functions, the regionally switching parameter, and control functions encountered these frameworks in Chapters 2, 3, 4, and 5. In Section 1.2, we give recalls on the general theory of ordinary differential equations. This section covers key concepts such as solutions, some regularity assumptions

and fundamental results, which are required for providing existence and uniqueness results. We also cover some fundamental results on the linear Cauchy problem theory that will be used in the proof of the HMP with a regionally switching parameter stated in Chapter 2, such as the *transition matrix* and *Duhamel formula*. Moreover, in order to provide a complete proof of the HMP with a regionally switching parameter, we rely on thorough sensitivity analysis within a hybrid setting. Therefore, in Section 1.3, we review the standard results in sensitivity analysis. This includes needle-like perturbations of the control, L^∞-convex perturbations of the parameter, and perturbations of the initial condition. Lastly, in Section 1.4, we focus on various notions of minima and afterwards we give the statement of the classical PMP, accompanied by a list of remarks and comments. This is important because in Chapter 4, we introduce a new notion of a local minimum (which is also used in Chapter 5). Also in Chapters 4 and 5, we provide an extension of the classical PMP in the spatially hybrid setting and the setting for optimal control problems with loss control regions.

1.1 Notations and functional framework

In this section, we provide a review of the basic functional spaces to be used throughout this manuscript. For spatially hybrid optimal control problems with a regionally switching parameter (as well as optimal control problems with loss control regions), the state function (or trajectory) is absolutely continuous, the regionally switching parameter is only piecewise constant, and the control function is essentially bounded.

For any positive integer $d \in \mathbb{N}^*$, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ (resp. $\| \cdot \|_{\mathbb{R}^d}$) the standard inner product (resp. Euclidean norm) of $\mathbb{R}^d$. For any subset $S \subset \mathbb{R}^d$, we denote by ∂S the boundary of S defined by $\partial S := \overline{S} \backslash \mathrm{Int}(S)$, where $\overline{S}$ and $\mathrm{Int}(S)$ stand for the closure and the interior of S respectively. Given a (Lebesgue) measurable subset $A \subset \mathbb{R}$, we denote by $\mu(A)$ its (Lebesgue) measure. Furthermore, given a closed convex set $Y \subset \mathbb{R}^d$, the normal cone to Y at some point $y \in Y$ is defined by

$$\mathrm{N}_Y[y] := \{y'' \in \mathbb{R}^d \mid \forall y' \in Y,\ \langle y'' - y, y \rangle_{\mathbb{R}^d} \leq 0\}.$$

Now, for any extended-real number $r \in [1, +\infty]$ and any nonempty real interval $I \subset \mathbb{R}$, we denote by:

- $\mathrm{L}^r(I, \mathbb{R}^d)$ the standard Lebesgue space of r-integrable functions defined on I with values in $\mathbb{R}^d$, endowed with its usual norm $\| \cdot \|_{\mathrm{L}^r}$.
- $\mathrm{C}(I, \mathbb{R}^d)$ the standard space of continuous functions defined on I with values in $\mathbb{R}^d$, endowed with its standard uniform norm $\| \cdot \|_{\mathrm{C}}$.

As usual in the literature, when $(\mathcal{Z}, \mathrm{d}_{\mathcal{Z}})$ is a metric set, we denote by $\mathrm{B}_{\mathcal{Z}}(z, \varepsilon)$ (resp. $\overline{\mathrm{B}}_{\mathcal{Z}}(z, \varepsilon)$) the standard open (resp. closed) ball of $\mathcal{Z}$ centered at $z \in \mathcal{Z}$ and of radius $\varepsilon > 0$. Finally, For a differentiable map $\psi : \mathbb{R}^d \to \mathbb{R}^{d'}$, with $d' \in \mathbb{N}^*$, we denote by $\nabla\psi(x) := \big(\nabla\psi_1(x) \dots \nabla\psi_{d'}(x)\big) \in \mathbb{R}^{d \times d'}$ the gradient of ψ at some $x \in \mathbb{R}^d$. We say that ψ is *submersive* at $x \in \mathbb{R}^d$ if the differential $D\psi(x) = \nabla\psi(x)^\top \in \mathbb{R}^{d' \times d}$ is surjective.

1.1.1 Recap on absolutely continuous functions

In optimal control theory, it is standard to assume that the state function is absolutely continuous. Let us fix two real numbers a and b such that $a < b$ and a positive integer $n \in \mathbb{N}^*$. Hereafter, we will provide the definition of an absolutely continuous function.

Definition 1.1.1 (Absolutely continuous function)**.** *A function $x : [a, b] \to \mathbb{R}^n$ is said to be absolutely continuous over $[a, b]$ if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall r \in \mathbb{N},\ \forall \{(a_k, b_k)\}_{k=1,\dots,r} \text{ pairwise disjoint open sub-intervals of } [a, b],$$
$$\sum_{k=0}^{r} (b_k - a_k) < \delta \implies \sum_{k=0}^{r} \|x(b_k) - x(a_k)\|_{\mathbb{R}^n} < \varepsilon.$$

We denote by $\mathrm{AC}(I, \mathbb{R}^n)$ the space of absolutely continuous functions.

Now, we provide a fundamental characterization of absolutely continuous functions.

Proposition 1.1.1. *Let $x : [a, b] \to \mathbb{R}^n$ be a given function. The following conditions are equivalent:*

(i) x is absolutely continuous over $[a, b]$.

(ii) x is differentiable almost everywhere over $[a, b]$ with $\dot{x} \in \mathrm{L}^1([a, b], \mathbb{R}^n)$ and

$$x(t) = x(a) + \int_a^b \dot{x}(s) ds,$$

for all $t \in [a, b]$.

(iii) there exists $c \in \mathbb{R}^n$ and $y \in \mathrm{L}^1([a, b], \mathbb{R}^n)$ such that

$$x(t) = c + \int_a^t y(s) ds,$$

for all $t \in [a, b]$.

in that case it holds that $c = x(a)$ and $y = \dot{x}$.

1.1.2 Recap on piecewise constant functions and piecewise absolutely continuous functions

In this manuscript, our focus is on investigating spatially hybrid control systems with a regionally switching parameter. Within these systems, and given a partition of the state space, the value of this (new) parameter remains constant as long as the state belongs to a specific region. However, it can change once the state transitions into a different region. This characteristic leads to the use of piecewise constant functions to describe the behavior of this (new) parameter. Therefore, this section provides a recap on this type of functions.

If a function $\gamma : I \to \mathbb{R}^d$ admits left and right limits at some $\tau \in \mathrm{Int}(I)$, we denote by

$$\gamma^-(\tau) := \lim_{\substack{t \to \tau \\ t < \tau}} \gamma(t) \quad \text{and} \quad \gamma^+(\tau) := \lim_{\substack{t \to \tau \\ t > \tau}} \gamma(t).$$

Now take $I = [0, T]$ for some $T > 0$. Recall that a partition of the interval $[0, T]$ is a finite set $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$, for some positive integer $N \in \mathbb{N}^*$, such that $0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < \tau_N = T$. In this manuscript:

- A function $\gamma \in \mathrm{L}^\infty([0, T], \mathbb{R}^d)$ is said to be *piecewise constant*, with respect to a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0, T]$, if the restriction of γ over each open interval (τ_{k-1}, τ_k) is almost everywhere equal to a constant denoted by $\gamma_k \in \mathbb{R}^d$. If so, γ is identified to the function $\gamma : [0, T] \to \mathbb{R}^d$

given by

$$\gamma(t) := \begin{cases} \gamma_k & \text{if } t \in [\tau_{k-1}, \tau_k) \text{ for all } k \in \{1, \dots, N-1\}, \\ \gamma_N & \text{if } t \in [\tau_{N-1}, \tau_N] \text{ for } k = N, \end{cases}$$

for all $t \in [0, T]$.

- A function $\gamma : [0, T] \to \mathbb{R}^d$ is said to be *piecewise absolutely continuous*, with respect to a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0, T]$, if γ is continuous at 0 and T and the restriction of γ over each open interval (τ_{k-1}, τ_k) admits an extension over $[\tau_{k-1}, \tau_k]$ that is absolutely continuous. If so, γ admits left and right limits at each $\tau_k \in (0, T)$, denoted by $\gamma^-(\tau_k)$ and $\gamma^+(\tau_k)$ respectively.

In what follows we denote by $\mathrm{PC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$ (resp. $\mathrm{PAC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$) the space of all piecewise constant functions (resp. piecewise absolutely continuous functions) respecting a given partition $\mathbb{T}$ of $[0, T]$. Finally we denote by $\mathrm{PC}([0, T], \mathbb{R}^d)$ (resp. $\mathrm{PAC}([0, T], \mathbb{R}^d)$) the set of all piecewise constant functions (resp. piecewise absolutely continuous functions), independently of the partition considered.

1.2 General theory of ordinary differential equations

In this section, we fix $n \in \mathbb{N}^*$ as a positive integer and a and b as two real numbers such that $a < b$. Let $g : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ be a mapping and $x_a \in \mathbb{R}^n$ be fixed. In this section we focus on the *Cauchy problem* given by

$$\begin{cases} \dot{x}(t) = g(x(t), t), & \text{for a.e. } t \in [a, b], \\ x(a) = x_a, \end{cases} \tag{1.1}$$

where the *dynamics* g is assumed to be continuous in its first variable, but only measurable in its second variable. We say that g is a *Carathéodory function*. In the following section, we provide different notions of solutions and some fundamental results necessary to state the existence and uniqueness results. These concepts are also required for performing sensitivity analysis.

1.2.1 Notions of solutions and fundamental results

Definition 1.2.1 (Local solution)**.** *We say that the couple (x, I) is a local solution to* (1.1) *if the following conditions are satisfied:*

(i) *I is an interval such that $\{a\} \subseteq I \subsetneq [a, b]$.*

(ii) *$x : [a, c] \to \mathbb{R}^n$ in an absolutely continuous function such that $\dot{x}(t) = g(x(t), t)$ for almost every $t \in [a, c]$ for all $c \in I$.*

(iii) *$x(a) = x_a$.*

Definition 1.2.2 (Extension of a solution)**.** *Let (x_1, I_1), (x_2, I_2) be two local solutions to* (1.1)*. We say that (x_2, I_2) is an extension of (x_1, I_1) if*

(i) *$I_1 \subseteq I_2$.*

(ii) *$x_2(t) = x_1(t)$ for all $t \in I_1$.*

We say that the extension is strict if $I_1 \subsetneq I_2$.

Definition 1.2.3 (Maximal and global solutions)**.** *Let (x, I) be a local solution to* (1.1)*. We say that*

(i) *(x, I) is a maximal solution to* (1.1) *if it does not admit any strict extension.*

(ii) *(x, I) is a global solution to* (1.1) *if $I = [a, b]$.*

Note that a global solution is necessarily a maximal solution.

Before revisiting the standard existence and uniqueness results for solutions to (1.1), it is necessary to establish specific assumptions regarding the regularity and integrability of the dynamics g.

Definition 1.2.4 (Locally integrable)**.** *We say that g is locally integrable if for all $R \geq 0$, there exists $\ell_R \in \mathrm{L}^1([a,b],\mathbb{R})$ such that*

$$\|g(x,t)\|_{\mathbb{R}^n} \leq \ell_R(t),$$

for all $x \in \overline{\mathrm{B}}_{\mathbb{R}^n}(0_{\mathbb{R}^n}, R)$ for almost every $t \in [a,b]$.

Definition 1.2.5 (Locally Lipschitz in its first variable)**.** *We say that g is locally Lipschitz in its first variable if for all $(x^*,t^*) \in \mathbb{R}^n \times [a,b]$, there exists $\varepsilon > 0$ and $L \geq 0$ such that*

$$\|g(x_2,t) - g(x_1,t)\|_{\mathbb{R}^n} \leq L\|x_2 - x_1\|_{\mathbb{R}^n},$$

for all $x_1, x_2 \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*,\varepsilon)$ for almost every $t \in [t^*-\varepsilon, t^*+\varepsilon] \cap [a,b]$.

Definition 1.2.6 (Globally Lipschitz in its first variable)**.** *We say that g is globally Lipschitz in its first variable if there exists $L \geq 0$ such that*

$$\|g(x_2,t) - g(x_1,t)\|_{\mathbb{R}^n} \leq L\|x_2 - x_1\|_{\mathbb{R}^n},$$

for all $x_1, x_2 \in \mathbb{R}^n$ for almost every $t \in [a,b]$.

Let us now turn to *Grönwall's lemma*, which will be useful in providing a continuous dependence result in Section 1.3.

Lemma 1.2.1 (Grönwall's lemma)**.** *Let $\alpha, \beta : [a,b] \to \mathbb{R}$ be two Lebesgue integrable functions and $z : [a,b] \to \mathbb{R}$ an absolutely continuous function satisfying the inequality*

$$\dot{z}(t) \leq \alpha(t)z(t) + \beta(t), \quad \text{for a.e. } t \in [a,b].$$

Then, for any $t \in [a,b]$, one has

$$z(t) \leq z(a)e^{\int_a^t \alpha(s)ds} + \int_a^t \beta(s)e^{\int_s^t \alpha(\tau)d\tau}ds.$$

Now, we are in a position to state the classical results of existence and uniqueness of solutions to ordinary differential equations.

1.2.2 Existence and uniqueness results

In this section, we present the classical results of existence and uniqueness of solutions to an ordinary differential equation without control, considering smooth dynamics. In fact, we consider dynamics that are time-dependent and measurable with respect to the time variable, which is more suitable for the control setting later on. The proofs are omitted and can be found [77].

Theorem 1.2.1 (Maximal Cauchy-Lipschitz theorem)**.** *If g is locally integrable and locally Lipschitz in its first variable. Then, the Cauchy problem* (1.1) *admits a unique maximal solution.*

Theorem 1.2.2 (Alternative theorem)**.** *If g is locally integrable and locally Lipschitz in its first variable, and let (x,I) be the unique maximal solution to* (1.1)*. We consider two cases:*

(i) either $I = [a,b]$, that is, the maximal solution (x,I) is global.

(ii) either $I \subsetneq [a,b]$, and in that case, $I = [a,c)$ for some $a < c \leq b$, and x is unbounded over I.

In the second case, we say that the solution x "explodes" in the neighborhood of c.

1.2.3 Linear Cauchy problems

In this section, we focus on the (forward) linear Cauchy problem given by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t), & \text{for a.e. } t \in [a,b], \\ x(a) = x_a, \end{cases} \tag{1.2}$$

where $x_a \in \mathbb{R}^n$, $A \in \mathrm{L}^\infty([a,b],\mathbb{R}^{n\times n})$, and $B \in \mathrm{L}^1([a,b],\mathbb{R}^n)$. The next result can be easily derived from Theorem 1.2.1.

Theorem 1.2.3. *The linear Cauchy problem* (1.2) *admits a unique maximal solution. Moreover, this maximal solution is global.*

Now, we are in position to define the state transition matrix. For all $s \in [a,b]$, we denote by $\Phi(\cdot,s) : [a,b] \to \mathbb{R}^{n\times n}$ the unique maximal solution (which is global) of the forward/backward linear matrix Cauchy problem given by

$$\begin{cases} \dot{\Phi}(t) = A(t)\Phi(t), & \text{for a.e. } t \in [a,b], \\ \Phi(s) = \mathrm{Id}_{\mathbb{R}^n}. \end{cases} \tag{1.3}$$

The matrix function $\Phi : [a,b]^2 \to \mathbb{R}^{n\times n}$ has two variables and is called the state transition matrix associated with A.

Remark 1.2.1. In this remark, we provide a *duality result*. For all $t \in [a,b]$, the matrix function $\Phi(t,\cdot) : [a,b] \to \mathbb{R}^{n\times n}$ is the unique maximal solution (which is global) of the forward/backward linear Cauchy problem given by

$$\begin{cases} \dot{\Phi}(s) = -\Phi(s)A(s), & \text{for a.e. } s \in [a,b], \\ \Phi(t) = \mathrm{Id}_{\mathbb{R}^n}. \end{cases} \tag{1.4}$$

Theorem 1.2.4 (Duhamel formula)**.** *The unique maximal solution to* (1.2) *(which is global) is given by the explicit Duhamel formula:*

$$x(t) = \Phi(t,a)x(a) + \int_a^t \Phi(t,s)B(s)\,ds,$$

for all $t \in [a,b]$.

Remark 1.2.2. In this remark, we comment on the Duhamel formula.

(i) If $A(\cdot) = A \in \mathbb{R}^{n\times n}$ is constant, the unique maximal solution to (1.2) (which is global) is given by the explicit Duhamel formula:

$$x(t) = e^{(t-a)A}x(a) + \int_a^t e^{(t-s)A}B(s)\,ds,$$

for all $t \in [a,b]$.

(ii) A Duhamel formula for the unique maximal solution (which is global) of the backward linear Cauchy

problem given by

$$\begin{cases} \dot{y}(s) = -A(s)^\top y(s) - B(s), \quad \text{for a.e. } s \in [a,b], \\ y(b) = y_b, \end{cases} \tag{1.5}$$

for some fixed $y_b \in \mathbb{R}^n$, is given by the explicit Duhamel formula:

$$y(s) = \Phi(b,s)^\top y_b + \int_s^b \Phi(b,t)^\top B(t)\, dt,$$

for all $s \in [a,b]$.

(iii) We emphasize that Duhamel formula is used in the proof of the HMP with a regionally switching parameter in Chapter 2 (similarly to the proof of the Pontryagin maximum principle [39], [40]).

1.3 Sensitivity analysis of the state equation

In order to introduce Chapter 2, we would like here to remind classical properties (continuous dependence and differentiability results) of solutions to differential equations that depends on an additional (constant) parameter.

Let n, m and d be three positive integers in $\mathbb{N}^*$, and let $T > 0$ be a fixed positive real number. In this section, we will study general nonlinear control systems with (constant) parameter of the form:

$$\dot{x}(t) = f(x(t), \lambda, u(t), t), \quad \text{for a.e. } t \in [0,T],$$

where the dynamics $f : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$ is of class C^1. In that situation, we will generally deal with bounded controls $u \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$. The next result can be easily derived from Theorem 1.2.1.

Theorem 1.3.1. *Let us fix $\lambda \in \mathbb{R}^d$, $u \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ and $x_0 \in \mathbb{R}^n$. The forward Cauchy problem given by*

$$\begin{cases} \dot{x}(t) = f(x(t), \lambda, u(t), t), \quad \textit{for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1.6}$$

admits a unique maximal solution denoted by $x(\cdot, \lambda, u, x_0)$ and defined on the maximal interval denoted by $I(\lambda, u, x_0)$.

Definition 1.3.1 (Admissibility for globality)**.** *A triplet $(\lambda, u, x_0) \in \mathbb{R}^d \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times \mathbb{R}^n$ is said to be admissible for globality if $I(\lambda, u, x_0) = [0,T]$, that is, if the corresponding maximal solution $x(\cdot, \lambda, u, x_0)$ is global. In what follows,* $\mathrm{Glob}(f)$ *will stand for the set of all triplets $(\lambda, u, x_0) \in \mathbb{R}^d \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times \mathbb{R}^n$ that are admissible for globality.*

Now, we are in position to state a continuous dependence result.

Lemma 1.3.1. *Let $(\lambda, u, x_0) \in \mathrm{Glob}(f)$. For all $R > \|u\|_{\mathrm{L}^\infty}$, there exists $\varepsilon_R > 0$ such that the neighborhood*

$$\mathcal{N}_R(\lambda, u, x_0) := \overline{\mathrm{B}}_{\mathbb{R}^d}(\lambda, \varepsilon_R) \times \overline{\mathrm{B}}_{\mathrm{L}^1}(u, \varepsilon_R) \cap \overline{\mathrm{B}}_{\mathrm{L}^\infty}(0, R) \times \overline{\mathrm{B}}_{\mathbb{R}^n}(x_0, \varepsilon_R),$$

is included in $\mathrm{Glob}(f)$*. Roughly speaking,* $\mathrm{Glob}(f)$ *is open with respect to the $\mathbb{R}^d \times \mathrm{L}^1 \times \mathbb{R}^n$-distance.*

Moreover, the map

$$\begin{array}{rcl} \mathcal{F} : \mathcal{N}_R(\lambda, u, x_0) & \to & \mathrm{C}([0,T], \mathbb{R}^n), \\ (\lambda', u', x_0') & \mapsto & x(\cdot, \lambda', u', x_0'), \end{array}$$

is Lipschitz continuous with respect to the $\mathbb{R}^d \times \mathrm{L}^1 \times \mathbb{R}^n$*-distance.*

Proof. **First part.** Let $(\lambda, u, x_0) \in \mathrm{Glob}(f)$ and $R > \|u\|_{\mathrm{L}^\infty}$. From continuity of $x(\cdot, \lambda, u, x_0)$ over $[0,T]$, we deduce that the set

$$K_R := \{(\mu, y, v, t) \in \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \mid \|\mu - \lambda\|_{\mathbb{R}^d} \leq 1 \text{ and } \|y - x(t, \lambda, u, x_0)\|_{\mathbb{R}^n} \leq 1 \text{ and } \|v\|_{\mathbb{R}^m} \leq R\},$$

is a compact subset of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \times [0,T]$. Since f is of class C^1, one can easily deduce that $\|\nabla_1 f\|_{\mathbb{R}^{n\times n}}$, $\|\nabla_2 f\|_{\mathbb{R}^{n\times d}}$ and $\|\nabla_3 f\|_{\mathbb{R}^{n\times m}}$ are bounded over K_R by some $L_R > 0$ and that

$$\|f(\mu_2, y_2, v_2, t) - f(\mu_1, y_1, v_1, t)\|_{\mathbb{R}^n} \leq L_R(\|\mu_2 - \mu_1\|_{\mathbb{R}^d} + \|y_2 - y_1\|_{\mathbb{R}^n} + \|v_2 - v_1\|_{\mathbb{R}^m}),$$

for all $(\mu_1, y_1, v_1, t), (\mu_2, y_2, v_2, t) \in K_R$. We fix some $0 < \varepsilon_R < 1$ such that $\varepsilon_R(1 + 2L_R)e^{L_R T} < 1$. Let us consider $(\lambda', u', x_0') \in \mathcal{N}_R(\lambda, u, x_0)$. Our objective is to prove that $T \in I(\lambda', u', x_0')$. To this aim, we introduce the set

$$\mathcal{T} := \{t \in I(\lambda', u', x_0') \mid \|x(t, \lambda', u', x_0') - x(t, \lambda, u, x_0)\|_{\mathbb{R}^n} > 1\}.$$

If $\mathcal{T} = \emptyset$, then the solution $x(\cdot, \lambda', u', x_0')$ is bounded. In that case, we deduce from Theorem 1.2.2 that $x(\cdot, \lambda', u', x_0')$ is global, that is, $I(\lambda', u', x_0') = [0,T]$. Hence, let us prove that $\mathcal{T} = \emptyset$.
By contradiction, let us assume that $\mathcal{T} \neq \emptyset$ and let us denote $t_1 := \inf \mathcal{T}$. From continuity and the definition of t_1, we know that $\|x(t_1, \lambda', u', x_0') - x(t_1, \lambda, u, x_0)\|_{\mathbb{R}^n} > 1$. Note that $t_1 > 0$ since $\|x(0, \lambda', u', x_0') - x(0, \lambda, u, x_0)\|_{\mathbb{R}^n} = \|x_0' - x_0\|_{\mathbb{R}^n} \leq \varepsilon_R < 1$. From continuity and the definition of t_1, we deduce that $\|x(t, \lambda', u', x_0') - x(t, \lambda, u, x_0)\|_{\mathbb{R}^n} \leq 1$ for all $t \in [0, t_1]$. We conclude that $(\lambda', x(t, \lambda', u', x_0'), u'(t), t)$ and $(\lambda, x(t, \lambda, u, x_0), u(t), t)$ are elements of K_R for almost every $t \in [0, t_1]$. Since one has

$$x(t, \lambda', u', x_0') - x(t, \lambda, u, x_0) = x_0' - x_0 + \int_0^t \left(f(x(s, \lambda', u', x_0'), u'(s), s) - f(x(s, \lambda, u, x_0), u(s), s)\right) ds,$$

for all $t \in [0, t_1]$, we get that

$$\begin{aligned} &\|x(t, \lambda', u', x_0') - x(t, \lambda, u, x_0)\|_{\mathbb{R}^n} \\ &\quad \leq \|x_0' - x_0\|_{\mathbb{R}^n} + L_R \int_0^t \|\lambda - \lambda'\|_{\mathbb{R}^d} + \|u(s) - u'(s)\|_{\mathbb{R}^m} + \|x(s, \lambda', u', x_0') - x(s, \lambda, u, x_0)\|_{\mathbb{R}^n} ds \\ &\quad \leq \varepsilon_R(1 + 2L_R) + L_R \int_0^t \|x(s, \lambda', u', x_0') - x(s, \lambda, u, x_0)\| ds, \end{aligned}$$

and from Grönwall's lemma, we get

$$\|x(t, \lambda', u', x_0') - x(t, \lambda, u, x_0)\|_{\mathbb{R}^n} \leq \varepsilon_R(1 + 2L_R)e^{L_R t} \leq \varepsilon_R(1 + 2L_R)e^{L_R T} < 1,$$

for all $t \in [0, t_1]$, which raises a contradiction at $t = t_1$ and completes the proof of the first part.
Second part. In the first part, we have proved that $\mathcal{T} = \emptyset$, and thus $(x(t, \lambda', u', x_0'), u'(t), t) \in K_R$ for

almost every $t \in [0,T]$, for all $(\lambda', u', x_0') \in \mathcal{N}_R(\lambda, u, x_0)$. Now let $(\lambda', u', x_0'), (\lambda'', u'', x_0'') \in \mathcal{N}_R(\lambda, u, x_0) \subset \mathrm{Glob}(f)$. It holds that

$$x(t,\lambda'',u'',x_0'') - x(t,\lambda',u',x_0') = x_0'' - x_0' + \int_0^t \big(f(x(s,\lambda'',u'',x_0''),u''(s),s) - f(x(s,\lambda',u',x_0'),u'(s),s)\big)ds,$$

for all $t \in [0,T]$, we obtain that

$$\begin{aligned}\|x(t,\lambda'',u'',x_0'') - x(t,\lambda',u',x_0')\|_{\mathbb{R}^n} &\leq \|x_0'' - x_0'\|_{\mathbb{R}^n} \\ &\quad + L_R \int_0^t \|u''(s) - u'(s)\|_{\mathbb{R}^m} + \|x(s,\lambda'',u'',x_0'') - x(s,\lambda',u',x_0')\|_{\mathbb{R}^n} ds,\end{aligned}$$

and, from Grönwall's lemma, that

$$\|x(t,\lambda'',u'',x_0'') - x(t,\lambda',u',x_0')\|_{\mathbb{R}^n} \leq \left(\|x_0'' - x_0'\|_{\mathbb{R}^n} + L_R\|u'' - u'\|_{\mathrm{L}^1}\right) e^{L_R T},$$

for all $t \in [0,T]$. This concludes the proof of the second part. □

Remark 1.3.1. It is noteworthy that in Chapter 2, we expand on the previous result of continuous dependence with respect to the triplet (λ, u, x_0). Specifically, we consider the case where we establish a continuous dependence with respect to the quadruplet (r, λ, u, x_0), where r represents the initial time. Such a result has a similar proof to Lemma 1.3.1.

1.3.1 Needle-like perturbation of the control

For this section, we fix a triplet $(\lambda, u, x_0) \in \mathrm{Glob}(f)$ and we denote by $x := x(\cdot, \lambda, u, x_0)$ the corresponding maximal solution which is global. Our aim now is to state a differentiability result for the trajectory x with respect to small L^1–perturbations of the of the control u.

Definition 1.3.2 (Lebesgue point)**.** *Let us fix $f \in \mathrm{L}^1([0,T],\mathbb{R}^n)$. A point $\tau \in [0,T]$ is called a Lebesgue point of f if the following condition holds:*

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{\tau-\delta}^{\tau+\delta} \|f(t) - f(\tau)\|_{\mathbb{R}^n}\, dt = 0.$$

Proposition 1.3.1. *Let us fix $(\lambda, u, x_0) \in \mathrm{Glob}(f)$, $v \in \mathbb{R}^m$ and $\tau \in (0,T]$ being a Lebesgue point of the map $t \mapsto f(x(t), \lambda, u(t), t)$. We consider the needle-like perturbation of u given by*

$$u^\alpha(t) := \begin{cases} v & \text{if } t \in (\tau - \alpha, \tau], \\ u(t) & \text{if } t \notin (\tau - \alpha, \tau], \end{cases}$$

for all $0 \leq \alpha \leq 1$. Then:

(i) There exists $\overline{\alpha} \in (0,1]$ such that $x(\cdot, \lambda, u^\alpha, x_0) \in \mathrm{Glob}(f)$ for all $\alpha \in [0, \overline{\alpha}]$.

(ii) The map

$$\alpha \in [0, \overline{\alpha}] \mapsto x(\cdot, \lambda, u^\alpha, x_0) \in \mathrm{C}([0,T], \mathbb{R}^n)$$

is differentiable at $\alpha = 0$, and its derivative is equal to w_u, which is the unique solution (that is

global) to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_x f(x(t), \lambda, u(t), t)w(t), & \text{for a.e. } t \in [\tau, T], \\ w(\tau) = f(x(\tau), \lambda, v, \tau) - f(x(\tau), \lambda, u(\tau), \tau). \end{cases}$$

Proof. Let us fix $R = \|v\|_{\mathbb{R}^m} + \|u\|_{\mathrm{L}^\infty}$. Consider $\varepsilon_R > 0$ provided in Lemma 1.3.1. It is clear that $(\lambda, u^\alpha, x_0) \in \mathcal{N}_R(\lambda, u, x_0)$ for sufficiently small $\alpha > 0$. As a consequence, from Lemma 1.3.1, there exists $0 < \overline{\alpha} \leq 1$ such that $(\lambda, u^\alpha, x_0) \in \mathrm{Glob}(f)$ for all $\alpha \in [0, \overline{\alpha}]$ which concludes the proof of the first item. Now our aim is to prove the second item. Let us introduce the notation

$$z^\alpha(t) := \frac{x^\alpha(t) - x(t)}{\alpha} - w_u(t),$$

with $x^\alpha(\cdot) := x(\cdot, \lambda, u^\alpha, x_0)$ and $x(\cdot) := x(\cdot, \lambda, u, x_0)$ for all $t \in [0, T]$ and for all $\alpha \in (0, \overline{\alpha}]$. Our aim is to prove that z^α converges uniformly to zero over $[\tau, T]$ when α goes to zero. It holds true that

$$z^\alpha(t) = z^\alpha(\tau) + \int_\tau^t \left(\frac{f(x^\alpha(s), \lambda, u(s), s) - f(x(s), \lambda, u(s), s)}{\alpha} - \nabla_x f(x(s), \lambda, u(s), s) w_u(s) \right) ds,$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. From the Taylor expansion with integral rest, we have

$$f(x^\alpha(s), \lambda, u(s), s) - f(x(s), \lambda, u(s), s) = \Big(\int_0^1 \nabla_x f(x(s) + \theta(x^\alpha(s) - x(s)), \lambda, u(s), s) d\theta \Big)(x^\alpha(s) - x(s)),$$

for almost every $s \in [\tau, T]$, which implies that

$$\begin{aligned} z^\alpha(t) = z^\alpha(\tau) &+ \int_\tau^t \left(\int_0^1 \nabla_x f(x(s) + \theta(x^\alpha(s) - x(s)), \lambda, u(s), s) d\theta \right) z^\alpha(s) ds \\ &+ \int_\tau^t \left(\int_0^1 \nabla_x f(x(s) + \theta(x^\alpha(s) - x(s)), \lambda, u(s), s) - \nabla_x f(x(s), \lambda, u(s), s) d\theta \right) w_u(s) ds, \end{aligned}$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. We get that

$$\|z^\alpha(t)\|_{\mathbb{R}^n} \leq \|z^\alpha(\tau)\|_{\mathbb{R}^n} + \Gamma(\alpha) + L_R \int_0^t \|z^\alpha(s)\|_{\mathbb{R}^n} ds,$$

where

$$\Gamma(\alpha) := \int_\tau^T \left(\int_0^1 \|\nabla_x f(x(s) + \theta(x^\alpha(s) - x(s)), \lambda, u(s), s) - \nabla_x f(x(s), \lambda, u(s), s)\|_{\mathbb{R}^{n \times n}} d\theta \right) \|w_u(s)\|_{\mathbb{R}^n} ds,$$

and then, from Grönwall's lemma we get that

$$\|z^\alpha(t)\|_{\mathbb{R}^n} \leq (\|z^\alpha(\tau)\|_{\mathbb{R}^n} + \Gamma(\alpha)) e^{L_R T},$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. It can be easily proved from the dominated convergence theorem that $\Gamma(\alpha)$ goes to zero when α goes to zero. Now let us prove that $\|z^\alpha(\tau)\|_{\mathbb{R}^n}$ tends to zero when α goes to

zero. Since $x^\alpha(\tau) = x(\tau)$ for all $\alpha \in (0, \overline{\alpha}]$, we know that

$$z^\alpha(\tau) = \int_{\tau-\alpha}^{\tau} \left(f(x^\alpha(s), \lambda, v, s) - f(x(s), \lambda, u(s), s)\right) ds - w_u(\tau - \alpha),$$

that is,

$$z^\alpha(\tau) = \int_{\tau-\alpha}^{\tau} \left(\frac{f(x(s), \lambda, v, s) - f(x(s), \lambda, u(s), s)}{\alpha}\right) ds \\ - w_u(\tau - \delta) + \int_{\tau-\delta}^{\tau} \left(\frac{f(x^\alpha(s), \lambda, v, s) - f(x(s), \lambda, v, s)}{\alpha}\right) ds.$$

The sum of the first two terms tends to zero since τ is a Lebesgue point of the map $t \mapsto f(x(t), \lambda, u(t), t)$ and since $w_u(\tau - \delta)$ tends to $w_u(\tau)$. The last term tends to zero from a Taylor expansion with integral rest and since x^α converges uniformly on $[0, T]$ to x. The proof is complete. □

Remark 1.3.2. Let us emphasize that in Chapter 2 we extend the result given in Proposition 1.3.1 to a spatially hybrid setting, where needle-like perturbations are no longer admissible. To do so, we perform needle-like perturbations on *auxiliary controls*, this process generates (modified) needle-like perturbations that are admissible (we refer to Chapter 2 for details).

1.3.2 L^∞–convex perturbation of the parameter

For this section, we fix a triplet $(\lambda, u, x_0) \in \mathrm{Glob}(f)$ and we denote by $x := x(\cdot, \lambda, u, x_0)$ the corresponding maximal solution which is global. Our aim now is to state a differentiability result for the trajectory x with respect to L^∞–convex perturbations of the parameter λ.

Proposition 1.3.2. *Let us fix $(\lambda, u, x_0) \in \mathrm{Glob}(f)$ and $\lambda' \in \mathbb{R}^d$. We consider the L^∞–convex perturbation of λ given by $\lambda + \alpha(\lambda' - \lambda)$ for all $\alpha \in [0, 1]$. Then:*

(i) There exists $\overline{\alpha} \in (0, 1]$ such that $x(\cdot, \lambda + \alpha(\lambda' - \lambda), u, x_0) \in \mathrm{Glob}(f)$ for all $\alpha \in [0, \overline{\alpha}]$.

(ii) The map

$$\alpha \in [0, \overline{\alpha}] \mapsto x(\cdot, \lambda + \alpha(\lambda' - \lambda), u, x_0) \in \mathrm{C}([0, T], \mathbb{R}^n)$$

is differentiable at $\alpha = 0$, and its derivative is equal to w_λ, which is the unique solution (that is global) to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_x f(x(t), \lambda, u(t), t) w(t) + \nabla_\lambda f(x(t), \lambda, u(t), t)(\lambda' - \lambda), & \text{for a.e. } t \in [0, T], \\ w(0) = 0_{\mathbb{R}^n}. \end{cases}$$

Remark 1.3.3. It is noteworthy that the differentiability result provided in Proposition 1.3.2 will be extended in two areas: first, to a larger spatially hybrid setting, and second, to cover regionally switching parameters. Also similarly to Remark 1.3.2, we rely on the notion of *auxiliary parameters* (we refer to Chapter 2 for details).

1.3.3 Perturbation of the initial condition

For this section, we fix a triplet $(\lambda, u, x_0) \in \mathrm{Glob}(g)$ and we denote by $x := x(\cdot, \lambda, u, x_0)$ the corresponding maximal solution which is global. Our aim now is to state a differentiability result for the trajectory x

with respect to L^∞–perturbations of the initial condition x_0.

Proposition 1.3.3. *Let us fix $(\lambda, u, x_0) \in \mathrm{Glob}(f)$ and $y \in \mathbb{R}^n$. Then:*

(i) There exists $\overline{\alpha} \in (0,1]$ such that $x(\cdot, u, x_0 + \alpha y) \in \mathrm{Glob}(f)$ for all $\alpha \in [0, \overline{\alpha}]$.

(ii) The map

$$\alpha \in [0, \overline{\alpha}] \mapsto x(\cdot, u, x_0 + \alpha y) \in \mathrm{C}([0,T], \mathbb{R}^n)$$

is differentiable at $\alpha = 0$, and its derivative is equal to w_{x_0}, which is the unique solution (that is global) to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_x f(x(t), \lambda, u(t), t) w(t), & \text{for a.e. } t \in [0,T], \\ w(0) = y. \end{cases}$$

Remark 1.3.4. It is well-known that the results given in Propositions 1.3.1, 1.3.2, and 1.3.3 are presented separately. However, in Chapter 2, we provide a differentiability result that takes into account perturbations of the quadruplet (r, λ, u, x_0) simultaneously (r denotes the initial time). Such a result has a similar proof to Proposition 1.3.1.

1.4 Standard notions of minima and statement of the PMP

We fix n, m, d and ℓ four positive integers in $\mathbb{N}^*$ and a fixed positive real number $T > 0$. In this section, our aim is twofold. First, we aim to provide various standard notions of global and local minimum for classical optimal control problems (we refer to [93]). Second, we present the statement of the classical PMP along with a series of comments. Let us consider a classical Mayer optimal control problem with parameter and mixed terminal state constraints given by

$$\begin{array}{ll} \text{minimize} & \phi(x(T)), \\ \text{subject to} & (x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathbb{R}^d \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \\ & \dot{x}(t) = f(x(t), \lambda, u(t)), \quad \text{a.e. } t \in [0,T], \\ & g(x(0), x(T), \lambda) \in \mathrm{S}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \end{array} \tag{1.7}$$

where the Mayer cost function $\phi : \mathbb{R}^n \to \mathbb{R}$, the dynamics $f : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^\ell$, defined by $g(x_0, x_1, \lambda)$ for all $(x_0, x_1, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ are of class C^1, and where $\mathrm{S} \subset \mathbb{R}^\ell$ is a nonempty closed convex subset and $\mathrm{U} \subset \mathbb{R}^m$ is a nonempty subset. As usual in the literature, $x \in \mathrm{AC}([0,T], \mathbb{R}^n)$ is called the *state* (or the *trajectory*), $u \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is called the *control* and $\lambda \in \mathbb{R}^d$ is called the *parameter*. A triplet $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathbb{R}^d \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is said to be *admissible* for Problem (1.7) if it satisfies all the constraints of Problem (1.7).

Remark 1.4.1. In this remark we comment on the setting of Problem (1.7).

(i) Let us emphasize that we have chosen to deal with optimal control problems with (only) Mayer cost, fixed final time and autonomous dynamics. It is well known in the literature (see, *e.g.*, [28], [44], [48]) that standard techniques (such as augmentation or changes of variables) allow to deal with more general Bolza cost, free final time and time-dependent dynamics.

(ii) Note that we consider a (constant) parameter $\lambda \in \mathbb{R}^d$, which can be handled with using an augmentation technique, for instance. It is noteworthy that one of our aims in this thesis is to

extend this notion to the concept of a regionally switching parameter (we refer to Chapters 2 and 5 for more details).

The classical PMP [102] has originally been developed for global solutions but, as usual in optimization, it remains valid for *local solutions*. As a consequence, several notions of local solutions to classical optimal control problems as well as the corresponding versions of the PMP have been developed in the literature (see, *e.g.*, [29], [93]). Since in Chapter 4 we introduce the notion of $\mathrm{L}^1_{\square}$*–local solution*, we recall various (standard) notions of minima following [93]:

Definition 1.4.1. *Let us consider* $(x^*, \lambda^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathbb{R}^d \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$. *The triplet* (x^*, λ^*, u^*) *is said to be:*

(i) a global minimum of Problem (1.7) *if*

$$\phi(x^*(T)) \leq \phi(x(T)) \text{ for all } (x, \lambda, u) \text{ admissible for Problem } (1.7),$$

(ii) a strong minimum of Problem (1.7) *if there exists* $\varepsilon > 0$ *such that*

$$\phi(x^*(T)) \leq \phi(x(T)) \text{ for all } (x, \lambda, u) \text{ admissible for Problem } (1.7) \text{ such that}$$

$$\|x^* - x\|_{\mathrm{C}} \leq \varepsilon,$$

(iii) a Pontryagin minimum of Problem (1.7) *if for any* $R > \|u^*\|_{\mathrm{L}^\infty}$, *there exists* $\varepsilon > 0$ *such that*

$$\phi(x^*(T)) \leq \phi(x(T)) \text{ for all } (x, \lambda, u) \text{ admissible for Problem } (1.7) \text{ such that}$$

$$\|x^* - x\|_{\mathrm{C}} + \|\lambda^* - \lambda\|_{\mathbb{R}^d} + \|u^* - u\|_{\mathrm{L}^1} \leq \varepsilon \text{ and } \|u\|_{\mathrm{L}^\infty} \leq R,$$

(iv) a weak minimum of Problem (1.7) *if there exists* $\varepsilon > 0$ *such that*

$$\phi(x^*(T)) \leq \phi(x(T)) \text{ for all } (x, \lambda, u) \text{ admissible for Problem } (1.7) \text{ such that}$$

$$\|x^* - x\|_{\mathrm{C}} + \|\lambda^* - \lambda\|_{\mathbb{R}^d} + \|u^* - u\|_{\mathrm{L}^\infty} \leq \varepsilon.$$

Remark 1.4.2. In this remark, we comment some aspects related to Definition 1.4.1.

(i) It is worth mentioning that for the terminology, a Pontryagin minimum is also referred to as a L^1–local minimum, also a weak minimum is also referred to as a L^∞–local minimum. Moreover, we have the following implications:

$$\text{global minimum} \implies \text{strong minimum} \implies \text{Pontryagin minimum} \implies \text{weak minimum}$$

(ii) We highlight that the presence of $R > 0$ in the definition fo the notion of a Pontryagin minimum (or L^1–local minimum) is due to the Ekeland approach that is used to prove the PMP. Indeed, as one can see in detail in [39], since the set U is not bounded a priori, the Ekeland approach requires at some step to intersect U with the ball $\overline{\mathrm{B}}_{\mathbb{R}^m}(0_{\mathbb{R}^m}, R)$ with $R > \|u^*\|_{\mathrm{L}^\infty}$ (to guarantee the validity of some useful estimations in the sensitivity analysis of the control sytem), and then, at the end of the proof, to make tend $R \to +\infty$.

(iii) It is noteworthy that in a spatially hybrid setting and under certain assumptions, the standard notions of minima given in Definition 1.4.1 cannot be used. Therefore, we will introduce a new

notion of local minimum referred to as $\mathrm{L}^1_\square$–*local* minimum (we refer to Chapter 4 for more details). Before stating the PMP associated with Problem (1.7), we just need to recall the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (1.7), that is given by

$$H(x, \lambda, u, p) := \langle p, f(x, \lambda, u)\rangle_{\mathbb{R}^n},$$

for all $(x, \lambda, u, p) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n$.

Theorem 1.4.1 (PMP). *If (x^*, u^*, λ^*) is a global solution to Problem (4.1), for a measurable subset $A \subset [0, T]$, such that g is submersive at $(x^*(0), x^*(T), \lambda^*)$, then there exists a nontrivial pair $(p, p^0) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:*

(i) the Hamiltonian system

$$\dot{x^*}(t) = \nabla_p H(x^*(t), \lambda^*, u^*(t), p(t)), \text{ and } -\dot{p}(t) = \nabla_x H(x^*(t), \lambda^*, u^*(t), p(t)),$$

for almost every $t \in [0, T]$;

(ii) the transversality condition

$$p(0) = \nabla_{x_0} g(x^*(0), x^*(T), \lambda^*)\xi,$$

and

$$p(T) = p^0 \nabla\phi(x^*(T)) + \nabla_{x_1} g(x^*(0), x^*(T), \lambda^*)\xi,$$

for some $\xi \in \mathrm{N_S}[g(x^*(0), x^*(T), \lambda^*)]$;

(iii) the averaged Hamiltonian gradient condition

$$\int_0^T \nabla_\lambda H(x^*(s), u^*(s), \lambda^*, p(s))\, ds = \nabla_\lambda g(x^*(0), x^*(T), \lambda^*)\xi;$$

(iv) the Hamiltonian maximization condition

$$u^*(t) \in \arg\max_{\omega \in \mathrm{U}} H(x^*(t), \lambda^*, \omega, p(t)),$$

for almost every $t \in [0, T]$.

Remark 1.4.3. In this remark we comment on some aspects related to Theorem 1.4.1.

(i) As usual in optimal control theory, the nontrivial pair (p, p^0) provided in Theorem 1.4.1 is defined up to a positive multiplicative constant. It is said to be *normal* whenever $p^0 > 0$, and *abnormal* whenever $p^0 = 0$. In the normal case $p^0 > 0$, it is usual to renormalize it so that $p^0 = 1$.

(ii) As explained in [24], [35], the submersiveness hypothesis made in Theorem 1.4.1 can be removed. In that case, all items of Theorem 1.4.1 remain valid except Item (ii).

(iii) Using the Hamiltonian system and the Hamiltonian maximization condition over $[0, T]$ and applying [63, Theorem 2.6.1], we obtain the *Hamiltonian constancy condition*

$$H(x^*(t), \lambda^*, u^*(t), p(t)) = c,$$

for almost every $t \in [0, T]$, for some $c \in \mathbb{R}$.

(iv) Theorem 1.4.1 is stated for a solution to (1.7) in a global sense. However, this result remain valid for a solution to (1.7) only in a local sense, take for instance L^1–local solution (also called a Pontryagin

minimum, see Definition 1.4.1).

(v) It is noteworthy that under certain assumptions (assuming U is closed and convex), one can obtain a *weak version* of the classical PMP. This version is adapted to L^∞–local solutions (we refer to [42] and discussion therein). Precisely, we obtain a weaker version of Theorem 1.4.1, wherein the Hamiltonian constancy condition is omitted, and more importantly, the Hamiltonian maximization condition is replaced by the weaker *Hamiltonian gradient condition*.

$$\nabla_u H(x^*(t), \lambda^*, u^*(t), p(t)) \in \mathrm{N}_{\mathrm{U}}[u^*(t)],$$

for almost every $t \in [0, T]$.

We shall see in this thesis the extension of Theorem 1.4.1 to two frameworks:

(A) Spatially hybrid optimal control problems with a regionally switching parameter.

(B) Optimal control problems with loss control regions.

As explained in the General Introduction 0, for problems of type (A), the corresponding first-order necessary optimality conditions are referred to as the spatially hybrid maximum principle (HMP, in short) with a regionally switching parameter. For problems of type (B), we refer to them as the PMP with loss control regions. Moreover, the most general statement for a spatially HMP with a regionally switching parameter and a PMP with loss control regions are provided in Chapter 5.

1.5 Filippov solutions for hybrid systems

In this section, we provide a quick recap on the existence of solutions to discontinuous differential equations. Indeed, in this thesis, we mainly deal with spatially hybrid control systems and such a dynamics exhibit discontinuities (with respect to the state) at the boundaries.

1.5.1 Notion of Filippov solutions for discontinuous differential equations

Let us consider a time-dependent Cauchy problem:

$$\begin{cases} \dot{x}(t) = f(x(t), t), \quad \text{for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.8}$$

where $x_0 \in \mathbb{R}^n$ and $T > 0$ denote the initial condition and the final time respectively. It is well-known in the literature that if the dynamics $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is smooth (at least continuous) then the existence of a solution is guaranteed by *Peano's existence theorem*. However, in the case where the dynamics f is not continuous with respect to x the classical theory of ODEs can no longer be applied. Hence, several notions of solutions have been introduced in the literature, take for instance *Filippov* and *Krasovskij* solutions (we refer to [65], [66]).

Hereafter, we focus only on a quick overview of Filippov solutions. Before providing the definition of solution along with corresponding existence result. We assume that dynamics f is measurable and locally bounded[1].

[1] for any $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ there exists a neighborhood K of (x_0, t_0) such that $f(K)$ is a bounded set.

The Filippov regularization consists in replacing the system (1.8) by the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in \mathcal{F}(x(t), t), \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.9}$$

where

$$\mathcal{F}(y, s) := \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{\text{co}} f(s, \mathrm{B}_{\mathbb{R}^n}(y, \delta) \backslash N),$$

for all $(y, s) \in [0, T] \times \mathbb{R}^n$ where $\overline{\text{co}} A$ represents the closure of convex hull of the set $A \subset \mathbb{R}^n$. Following [7], and under the assumptions made on f we get that the set-valued map $\mathcal{F}$ is upper semicontinuous with compact and convex values. This implies the existence of a *Filippov solution* which refers to an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ that is a solution to (1.9). So, in this setting, this is a way to give a sense to a solution of (1.8).

1.5.2 Existence results for spatially hybrid systems

In this section, we highlight the preceding concept of Filippov solution in a particular setting. Doing so, consider (for simplicity) the following partition of the state space $\mathbb{R}^n = \overline{X_1} \cup \overline{X_2}$, where X_1 and X_2 are nonempty open subsets of $\mathbb{R}^n$. To this space partition, we associate a *spatially hybrid dynamics* $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is defined regionally as follows:

$$h(x, u) := \begin{cases} h_1(x, u), \text{ if } x \in X_1, \\ h_2(x, u), \text{ if } x \in X_2, \end{cases}$$

for all $x \in X_1 \cup X_2$ and all $u \in \mathbb{R}^m$ where $h_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 for $j = 1, 2$. Now, we are in position to consider a spatially hybrid control system given by

$$\begin{cases} \dot{x}(t) = h(x(t), u(t)), \quad \text{for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.10}$$

where $x_0 \in \mathbb{R}^n$ is a fixed initial condition and $T > 0$ is a fixed final time. For a fixed control $u \in \mathrm{L}^\infty([0, T], \mathbb{R}^m)$, the control system given in (1.10) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = g(x(t), t), \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

with $g : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is given by $g(y, s) := h(y, u(s))$ for all $(y, s) \in\in X_1 \cup X_2 \times [0, T]$. Note that the above differential equation is a particular instance of *piecewise smooth systems* (in short, PWS systems); for more details we refer to [7], [55], [66] and references therein. This type of system deals with differential equations that admit discontinuities on a boundary, in this case, the boundary $\overline{X_1} \cap \overline{X_2}$. Based on Section 1.5.1, we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in \mathcal{G}(x(t), t), \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where the set-valued map $\mathcal{G}$ is given by

$$\mathcal{G}(y,s) := \begin{cases} h_1(y,s), \text{ if } y \in X_1, \\ \{\alpha h_1(y,s) + (1-\alpha)h_2(y,s) \mid \alpha \in [0,1]\}, \text{ if } y \in \overline{X_1} \cap \overline{X_2}, \\ h_2(y,s), \text{ if } y \in X_2, \end{cases}$$

for all $(y,s) \in \mathbb{R}^n \times [0,T]$. Let us emphasize that in this case, we may have solution that cross the boundary *transversally* or have a *sliding mode* as depicted in Figure 1.1 (we refer to [55]).

Figure 1.1: Transverse crossing to the left and sliding mode to the right.

In this thesis, we considered in every chapter nominal trajectories that cross transversally the boundary between two regions. So, actually, we shall not need to deal with Filippov solution (since the velocity is defined almost everywhere). Considering trajectories that could enter into a region tangentially or considering nominal trajectories having a sliding mode [10], [82], [84] could be addressed in future works (see our perspectives in Chapter 6). Such cases would require dealing with Filippov's solutions. Our methodology in this thesis also mainly relies on perturbation of the nominal trajectory. So, it is of utmost importance to have a transverse hypotheses on the nominal paths (see Chapters 2, 3, 4 and 5). If this would not be the case, perturbations may be more delicate to introduce and to handle, that is why, we first studied the case of transversal crossings.

2

Hybrid maximum principle with regionally switching parameter

Contents

This chapter is based on the article "Hybrid maximum principle with regionally switching parameter" by T. Bayen, A. Bouali, and L. Bourdin (see [13]), which covers spatially hybrid optimal control problems with a regionally switching parameter. Moreover, we present the application of this framework to optimal control problems with loss control regions, based on the conference proceeding entitled "Optimal control problems with non-control regions: necessary optimality conditions" by T. Bayen, A. Bouali, and L. Bourdin (see [14]).

2.1 Introduction

In this chapter we consider a general spatially hybrid control system, involving both a permanent control and a regionally switching parameter, given by

$$\dot{x}(t) = h(x(t), \lambda(t), u(t), t), \qquad \text{a.e. } t \in [0, T],$$

where λ is a regionally switching parameter (as presented in General introduction 0) and where $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^n$ is a spatially hybrid dynamics, in the sense that it is defined regionally by

$$h(x, \lambda, u, t) := h_j(x, \lambda, u, t), \quad \text{when } x \in X_j,$$

where the $h_j : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^n$ are C^1 functions. Let us insist here on the fact that control systems with loss control regions (which constitutes our initial motivation as has been mentioned in General introduction 0) are just a particular case of the above spatially hybrid setting (indeed, one has just to take $h_j(x, \lambda, u, t) = f(x, u, t)$ when X_j is a control region, and $h_j(x, \lambda, u, t) = f(x, \lambda, t)$ when X_j is a loss control region, see [14] which is presented in Section 2.7 for details).
The main objective of this chapter is to provide first-order necessary optimality conditions in a PMP form for a Mayer optimal control problem

$$\text{minimize } \phi(x(T)),$$

among solutions to the above spatially hybrid control system. Therefore, our main result (Theorem 2.2.1) is called *hybrid maximum principle with regionally switching parameter*. As one can expect, given an optimal triplet (x, λ, u), Theorem 2.2.1 asserts that u satisfies the classical Hamiltonian maximization condition, while λ satisfies the averaged Hamiltonian gradient condition over each region visited. Furthermore, as usual in hybrid settings, a discontinuity jump of the costate function is observed at each interface crossing.
Now let us discuss briefly our proof of Theorem 2.2.1 and the main difficulties encountered. First of all, as explained in General introduction 0, our spatially hybrid setting cannot be easily rewritten as a classical optimal control problem and therefore, in contrast to sampled-data controls, the regionally switching parameter cannot be easily treated thanks to an augmentation technique (see Remark 2.2.1 for details). As a consequence, and as in abstract optimization, to derive necessary optimality conditions, we

have to perform a sensitivity analysis of the constraints. In our setting, this translates into a sensitivity analysis of the hybrid control system. To this aim, we consider a perturbation of the control (needle-like perturbation as in [10], [63], [102]) and of the regionally switching parameter (convex perturbation as in [34], [41]). Under such local perturbations, we obtain a perturbed trajectory, but also a perturbed crossing time. We stress that the major difficulty of this work lies in handling this perturbed crossing time. To prove its existence, we rely on implicit function arguments which require two regularity assumptions: left and right continuity of the nominal control at the crossing times, and a transverse crossing condition on the nominal trajectory. Such hypotheses are commonly used in spatially hybrid settings (see, *e.g.*, [18], [73], [99]). In addition, since the perturbed crossing time does perturb the next one, and so on, and so on, a rigorous inductive reasoning is required to prove the existence of the remaining perturbed crossing times. Once the sensitivity analysis is complete, our proof follows similar steps to the PMP's proof which is based on the construction of an adequate adjoint vector to maintain the constancy of the inner product with all variation vectors. Let us note that, as usual in hybrid settings, since the variation vectors admit discontinuity jumps at each crossing time (due to the perturbed ones, as explained above), the adjoint vector also admits discontinuity jumps at each crossing time. Finally, the main novelties of the present chapter are the variation vectors obtained under convex perturbations of the regionally switching parameter (which lead to the averaged Hamiltonian gradient condition) and the applicability of our main result to control systems with loss control regions (which is developed in Section 2.7 based on a companion proceeding [14]). Furthermore we emphasize that our goal was also to provide a very complete and rigorous proof of the HMP. Therefore the proof is quite long and technical and it is postponed to Section 2.6. Nevertheless, for pedagogical reasons and for the reader's convenience, we provide in Section 2.2.3 a short overview of the proof of Theorem 2.2.1.

Some remarks. Hereafter we provide a short list of comments before starting the chapter:

(i) In contrast to what is claimed above (for simplicity), we actually consider in the present work the possibility of a state partition that can be infinite, and also that can be not static (in other words, that can be time dependent).

(ii) In this chapter we give a simple counterexample showing, as noticed in [68], that a standard needle-like perturbation of the control (as used in the literature for non-hybrid control systems) can produce a non-admissible trajectory in the spatially hybrid setting (see Item 2 in Section 2.2.3). This important subtlety, which seems to be ignored in some recent works, leads us to consider the construction of *auxiliary controls* on which we perform needle-like perturbations to obtain admissible trajectories (see Section 2.2.3 for details).

(iii) The present chapter does not cover *terminal state constraints* (that is, constraints on $x(0)$ and $x(T)$). In the classical non-hybrid setting, several methods have been developed in the literature to take into account such constraints. One can invoke the use of Ekeland's principle [61], of some implicit function arguments [2], [86], [110], of Lagrange multiplier rules [3], etc. To the best of our knowledge, the Ekeland approach does not apply in the present spatially hybrid setting for several reasons, while the methods based on implicit function arguments and Lagrange multiplier rules should be adaptable but at the price of a heavy formalism (see Item (i) of Remark 2.2.7 for details). Since our main objective in this chapter was to focus on the new concept of regionally switching parameter and on the corresponding averaged Hamiltonian gradient condition, we decided to avoid the technicalities related to the presence of terminal state constraints which are already well known in the literature.

(iv) Section 2.2.4 is dedicated to a list of comments on our framework and assumptions, on our main result and its consequences, and also on possible relaxations and extensions. For instance we discuss the behavior of the Hamiltonian function, and the possible extension to a general Bolza cost (instead of a Mayer cost) or to a free final time $T > 0$.

(v) By means of a simple academic example, we show in Section 2.3 how to use Theorem 2.2.1 and how our new framework, in some way, fills a gap in the literature. Indeed, in that example, and as one can expect, the optimal solution associated with a regionally switching parameter has a better cost than the one associated with a constant parameter, but has a worse cost than the one considering a permanent control instead of the regionally switching parameter. We highlight that this example remains simple and academic. More concrete and complex examples (such as double integrators, harmonic oscillators, Zermelo navigation problems, all of them including loss control regions) will be treated in Chapters 3, 4 and 5. Let us refer to Section 2.7 for the specification of our main result to (non-hybrid) optimal control problems with loss control regions.

Organization of this chapter. This chapter is structured as follows. Section 2.2 starts with notation and functional framework. Then we introduce a general Mayer optimal control problem governed by a hybrid control system involving a regionally switching parameter. Our main result (Theorem 2.2.1) about the corresponding first-order necessary optimality conditions in a PMP form is stated right after. Next we give an overview of the proof of Theorem 2.2.1, as well as a list of comments and perspectives. Section 2.3 is dedicated to a simple academic example. Sections 2.4, 2.5 and 2.6 are dedicated to the quite long and technical proof of Theorem 2.2.1. Precisely, in the preliminary Sections 2.4 and 2.5, we provide a thorough sensitivity analysis of non-hybrid and hybrid control systems respectively. Based on these technical results, Section 2.6 is devoted to the complete proof of Theorem 2.2.1. Finally, Section 2.7 is devoted to the study of optimal control problems including loss control regions.

2.2 Main result

This section is dedicated to state our main result of the paper [13]. In Section 2.2.1, the hybrid optimal control problem with regionally switching parameter considered in this chapter is presented, with terminology and assumptions. In Section 2.2.2, the corresponding hybrid maximum principle, which constitutes our main result, is provided (see Theorem 2.2.1). The proof of Theorem 2.2.1 is quite long and technical. Therefore it is postponed to Sections 2.4, 2.5 and 2.6. Nonetheless, for the reader's convenience, an overview of the proof of Theorem 2.2.1 is proposed in Section 2.2.3. Finally a list of general comments on our framework and assumptions, on Theorem 2.2.1 and its consequences, and also on possible relaxations and extensions, is provided in Section 2.2.4. Finally, the last section is devoted to the study of optimal control problems including loss control regions.

2.2.1 A hybrid optimal control problem with regionally switching parameter

Let n, d, $m \in \mathbb{N}^*$ be three fixed positive integers and $T > 0$ be a fixed positive real number. In this chapter, in the spirit of [73], we consider a *time dependent partition* of $\mathbb{R}^n$ given by

$$\forall t \in [0,T], \qquad \mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j(t)},$$

where $\mathcal{J}$ is a (possibly infinite) family of indexes and where, for all $t \in [0,T]$, the nonempty connected open subsets $X_j(t)$, called *regions*, are disjoint. This time dependent partition is furthermore assumed to satisfy two basic *continuity conditions* given by:

($\mathcal{C}1$) for all $j \in \mathcal{J}$ and all $x \in \mathrm{C}([a,b],\mathbb{R}^n)$ satisfying $x(t) \in X_j(t)$ over $[a,b]$, for some $0 \leq a \leq b \leq T$, there exists a uniform $\sigma > 0$ such that $\overline{\mathrm{B}}_{\mathbb{R}^n}(x(t),\sigma) \subset X_j(t)$ for all $t \in [a,b]$.

($\mathcal{C}2$) for all $t^c \in (0,T)$ and all $x \in \mathrm{C}([t^c-\delta, t^c+\delta],\mathbb{R}^n)$ satisfying $x(t) \in X_j(t)$ over $[t^c-\delta,t^c)$ and $x(t) \in X_{j'}(t)$ over $(t^c,t^c+\delta]$, for some j, $j' \in \mathcal{J}$ with $j \neq j'$ and some small $\delta > 0$, it holds that $x(t^c) \in \partial X_j(t^c) \cap \partial X_{j'}(t^c)$.

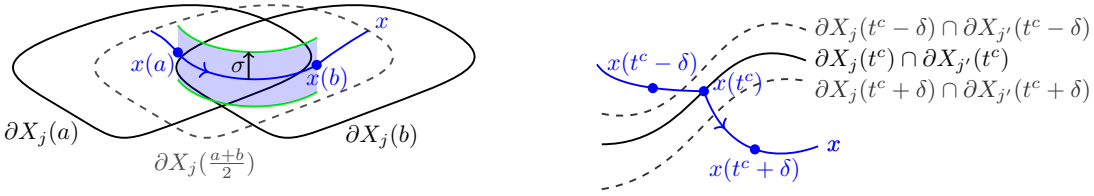


Figure 2.1: Illustrations of Condition ($\mathcal{C}1$) on the left, and of Condition ($\mathcal{C}2$) on the right.

We refer to Item (i) of Remark 2.2.1 for comments on Conditions ($\mathcal{C}1$) and ($\mathcal{C}2$), and to Section 2.2.3 for details on how they are used in our approach. Additionally to the above time dependent partition of $\mathbb{R}^n$, we consider a *hybrid dynamics* $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$ defined *regionally* by

$$\forall (x,\lambda,u,t) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T], \quad h(x,\lambda,u,t) := h_j(x,\lambda,u,t) \text{ when } x \in X_j(t),$$

where the maps $h_j : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$ are of class C^1. Note that $h(x,\lambda,u,t)$ is not defined when $x \notin \cup_{j\in\mathcal{J}} X_j(t)$ but this fact will have no impact on the rest of this work. In this chapter we focus on the *hybrid control system with regionally switching parameter* given by

$$\begin{cases} (x,\lambda,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{PC}([0,T],\mathbb{R}^d) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m), \\ \dot{x}(t) = h(x(t),\lambda(t),u(t),t), \quad \text{a.e. } t \in [0,T], \\ x(0) = x_{\mathrm{init}}, \\ \lambda \text{ is a regionally switching parameter associated with } x, \end{cases} \tag{2.1}$$

where the fixed initial condition x_{init} belongs to $X_{j_1}(0)$ for some $j_1 \in \mathcal{J}$. In the control system (2.1), as usual in the literature, $x \in \mathrm{AC}([0,T],\mathbb{R}^n)$ is called the *state* (or the *trajectory*) and $u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is called the *control*.

The novelty of the present work lies in the consideration of a *regionally switching parameter* $\lambda \in \mathrm{PC}([0,T],\mathbb{R}^d)$ meaning, roughly speaking, that the parameter λ remains constant while the trajectory x stays inside a region, but is authorized to *switch* (that is, to change its value) when the trajectory x crosses a boundary, going from one region to another. The precise definition of a solution to (2.1) is given below.

Definition 2.2.1 (Solution to (2.1))**.** *A triple* $(x,\lambda,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{PC}([0,T],\mathbb{R}^d) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *is said to be a solution to* (2.1) *if the following conditions are satisfied:*

(i) *There exists a partition* $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ *of the interval* $[0,T]$ *such that*

$$\forall k \in \{1,\dots,N\}, \quad \exists j(k) \in \mathcal{J}, \quad \forall t \in (t_{k-1}^c, t_k^c), \quad x(t) \in X_{j(k)}(t),$$

with $j(k) \neq j(k-1)$ *for all* $k \in \{2, \dots, N\}$, *with* $x(0) \in X_{j(1)}(0)$ *and* $x(T) \in X_{j(N)}(T)$.

(ii) λ *is a regionally switching parameter associated with* x, *that is,* $\lambda \in \mathrm{PC}_{\mathbb{T}}([0,T], \mathbb{R}^d)$.

(iii) *The state equation* $\dot{x}(t) = h_{j(k)}(x(t), \lambda_k, u(t), t)$ *is satisfied for almost every* $t \in (t_{k-1}^c, t_k^c)$ *and all* $k \in \{1, \dots, N\}$.

(iv) *The initial condition* $x(0) = x_{\mathrm{init}}$ *is satisfied (and thus* $j(1) = j_1$*).*

In that case, for the ease of notations, we simply denote by $f_k := h_{j(k)}$ *and* $E_k := X_{j(k)}$ *for all* $k \in \{1, \dots, N\}$. *With this system of notations we get that*

$$\forall k \in \{1, \dots, N\}, \qquad \begin{cases} x(t) \in E_k(t), & \forall t \in (t_{k-1}^c, t_k^c), \\ \dot{x}(t) = f_k(x(t), \lambda_k, u(t), t), & \text{a.e. } t \in (t_{k-1}^c, t_k^c), \end{cases}$$

and $x(0) \in E_1(0)$, $x(T) \in E_N(T)$. *Furthermore the times* t_k^c, *for* $k \in \{1, \dots, N-1\}$, *are called the crossing times, corresponding to the times at which the trajectory* x *goes from the region* E_k *to the region* E_{k+1}, *and thus* $x(t_k^c) \in \partial E_k(t_k^c) \cap \partial E_{k+1}(t_k^c)$ *from the continuity condition* $(\mathcal{C}2)$. *We refer to Figure 2.2 for an illustration.*

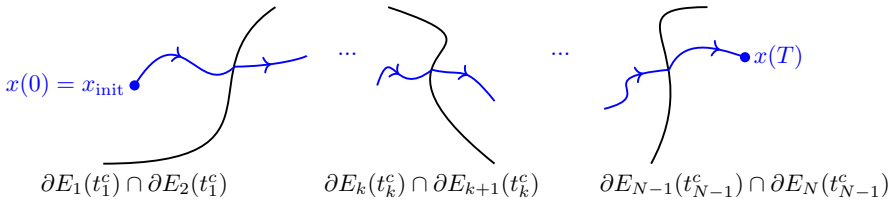


Figure 2.2: Illustration of Definition 2.2.1.

Our objective in the present work is to derive first-order necessary optimality conditions in a PMP form for the *hybrid optimal control problem with regionally switching parameter* given by

$$\begin{aligned} &\text{minimize} && \phi(x(T)), \\ &\text{subject to} && (x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \text{ solution to (2.1)}, \\ &&& (\lambda(t), u(t)) \in \Lambda \times \mathrm{U}, \quad \text{a.e. } t \in [0,T], \end{aligned} \tag{2.2}$$

where the Mayer cost function $\phi : \mathbb{R}^n \to \mathbb{R}$ is of class C^1, the parameter constraint set Λ is a nonempty convex subset of $\mathbb{R}^d$ and the control constraint set U is a nonempty closed subset of $\mathbb{R}^m$. We refer to Item 11 of Section 2.2.3 for details on how the hypotheses made on Λ and U are used in our approach, and to Items (i) and (ii) of Remark 2.2.6 for possible relaxations.

2.2.2 Hybrid maximum principle with regionally switching parameter

Our main result (Theorem 2.2.1) is based on some regularity assumptions made on the behavior of the optimal triple (x, λ, u) at the crossing times t_k^c. These hypotheses are precised in the next definition. We refer to Item (ii) of Remark 2.2.1 for comments on these hypotheses, and to Section 2.2.3 for details on how they are used in our approach.

Definition 2.2.2 (Regular solution to (2.1))**.** *Following the notations introduced in Definition 2.2.1, a solution* $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ *to* (2.1) *is said to be regular if there*

exist $0 < \delta \leq \frac{1}{3}\min_{k=1,\dots,N}|t_k - t_{k-1}|$ *and* $\nu > 0$ *such that:*

($\mathcal{A}1$) *At each crossing time* t_k^c*, the control* u *is continuous over* $[t_k^c - \delta, t_k^c)$ *and over* $(t_k^c, t_k^c + \delta]$*, and admits left and right limits at* t_k^c*, denoted by* $u^-(t_k^c)$ *and* $u^+(t_k^c)$ *respectively.*

($\mathcal{A}2$) *At each crossing time* t_k^c*, there exists a* C^1 *function* $F_k : \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \nu) \times [t_k^c - \delta, t_k^c + \delta] \to \mathbb{R}$ *such that*

$$\forall (y,t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c),\nu) \times [t_k^c - \delta, t_k^c + \delta], \quad \begin{cases} y \in E_k(t) & \Leftrightarrow \quad F_k(y,t) < 0, \\ y \in \partial E_k(t) \cap \partial E_{k+1}(t) & \Leftrightarrow \quad F_k(y,t) = 0, \\ y \in E_{k+1}(t) & \Leftrightarrow \quad F_k(y,t) > 0. \end{cases}$$

In particular it holds that $F_k(x(t_k^c), t_k^c) = 0$.

($\mathcal{A}3$) *At each crossing time* t_k^c*, the transverse conditions given by*

$$\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c) > 0,$$

$$\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_{k+1})^+(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c) > 0,$$

where $(f_k)^-(t_k^c) := f_k(x(t_k^c), \lambda_k, u^-(t_k^c), t_k^c)$ *and* $(f_{k+1})^+(t_k^c) := f_{k+1}(x(t_k^c), \lambda_{k+1}, u^+(t_k^c), t_k^c)$*, are both satisfied. We refer to Figure 2.3 for a geometrical illustration.*

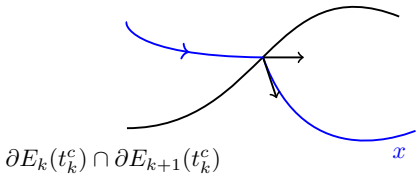


Figure 2.3: Geometrical illustration of a transversal boundary crossing (Assumption ($\mathcal{A}3$)).

Before stating the main result of this chapter we just need to recall the usual definition of the *Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ associated with the optimal control problem (2.2) given by

$$\forall (x,\lambda,u,p,t) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T], \quad H(x,\lambda,u,p,t) := \langle p, h(x,\lambda,u,t) \rangle_{\mathbb{R}^n}.$$

We are now in a position to state the main result established in the paper [13].

Theorem 2.2.1 (Hybrid maximum principle with regionally switching parameter)**.** *If* $(x,\lambda,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{PC}([0,T],\mathbb{R}^d) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *is a solution to* (2.2)*, that is moreover a regular solution to* (2.1)*, then, following the notations introduced in Definitions 2.2.1 and 2.2.2, there exists an adjoint vector* $p \in \mathrm{PAC}_{\mathbb{T}}([0,T],\mathbb{R}^n)$ *(also called costate) such that:*

(i) *The adjoint equation* $\dot{p}(t) = -\nabla_x f_k(x(t),\lambda_k,u(t),t)^\top p(t)$ *is satisfied for almost every* $t \in (t_{k-1}^c, t_k^c)$ *and all* $k \in \{1,\dots,N\}$.

(ii) *The final condition* $p(T) = -\nabla\phi(x(T))$ *is satisfied.*

(iii) *At each crossing time* t_k^c*, the adjoint discontinuity condition*

$$p^+(t_k^c) - p^-(t_k^c) = -\frac{\langle p^+(t_k^c), (f_{k+1})^+(t_k^c) - (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} \nabla_x F_k(x(t_k^c),t_k^c), \tag{2.3}$$

is satisfied.

(iv) *The Hamiltonian maximization condition*

$$u(t) \in \arg\max_{v \in \mathrm{U}} H(x(t), \lambda_k, v, p(t), t), \tag{2.4}$$

holds true for almost every $t \in (t_{k-1}^c, t_k^c)$ *and all* $k \in \{1, \ldots, N\}$.

(v) *The averaged Hamiltonian gradient condition*

$$\int_{t_{k-1}^c}^{t_k^c} \nabla_\lambda H(x(s), \lambda_k, u(s), p(s), s)\, \mathrm{d}s \in \mathrm{N}_\Lambda[\lambda_k], \tag{2.5}$$

holds true for all $k \in \{1, ..., N\}$.

The proof of Theorem 2.2.1 is quite long and technical. Therefore it is postponed to Section 2.6, after the two preliminary Sections 2.4 and 2.5 that are dedicated to sensitivity analyses of non-hybrid and hybrid control systems respectively. Nonetheless, for the reader's convenience, an overview of the proof of Theorem 2.2.1 is proposed in the next Section 2.2.3.

2.2.3 Overview of the proof of Theorem 2.2.1

This section is dedicated to an overview of the proof of Theorem 2.2.1. For the reader's convenience, our presentation is divided into twelve major items in which we take care to highlight at which point of the proof the continuity conditions ($\mathcal{C}1$) and ($\mathcal{C}2$) and the regularity assumptions ($\mathcal{A}1$), ($\mathcal{A}2$) and ($\mathcal{A}3$) are used. Before, we would like to emphasize a crucial subtlety: Item 2 provides a simple example showing that a standard needle-like perturbation of a control may be not admissible in a spatially hybrid setting, in the sense that the corresponding perturbed trajectory may not uniformly converge to the nominal one, or may not be a global solution to the control system. This counterexample reveals an erroneous assertion in [73, beginning of Section 2.1.1] and highlights interesting comments given in [68, pp. 1872]. As a conclusion, handling needle-like perturbations of a control in a spatially hybrid setting requires a careful attention.

1. From the point of view of abstract optimization, sensitivity analysis of constraints (with respect to given parameters) plays a fundamental role in order to derive necessary optimality conditions. In optimal control theory, this translates into a sensitivity analysis of the control system with respect to perturbations of the control u. To derive the classical PMP, the standard method is to consider a *needle-like perturbation* defined by $u^\alpha(t) := v$ for all $t \in (\tau - \alpha, \tau)$ and $u^\alpha(t) := u(t)$ elsewhere, for all $\alpha > 0$ and where $v \in \mathbb{R}^m$ and $\tau \in (0, T)$ are fixed. Then one has to identify the corresponding variation vector, that is the uniform limit of $\frac{x^\alpha - x}{\alpha}$ when $\alpha \to 0$, where x^α stands for the perturbed trajectory associated with the perturbed control u^α (see Figure 2.4), as the solution to a linearized control system.
2. However a needle-like perturbation may be not admissible in a spatially hybrid setting, in the sense that the corresponding perturbed trajectory x^α does not necessarily converge uniformly to x over $[0, T]$ when $\alpha \to 0$, or even may be not defined globally over the whole interval $[0, T]$. Let us provide a simple counterexample which highlights this issue which is not encountered in the classical non-hybrid setting. Consider $T = 2$, $n = m = 1$ and the static partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2}$, where $X_1 = \{y \in \mathbb{R} \mid y < 1\}$ and $X_2 = \{y \in \mathbb{R} \mid y > 1\}$. Now consider the spatially hybrid control

Figure 2.4: Illustrations of a needle-like perturbation (left) and the corresponding perturbed trajectory (right).

system given by

$$\dot{x}(t) = \begin{cases} +u(t) & \text{if } x(t) \in X_1, \\ -u(t) & \text{if } x(t) \in X_2, \end{cases}$$

with the initial condition $x_{\text{init}} = 0$. By taking the control $u(t) = +1$ over $[0, 1)$ and $u(t) = -1$ over $(1, 2]$, we get the corresponding trajectory x given by $x(t) = t$ over $[0, 2]$, with $t_1^c = 1$ as unique crossing time. Note that all conditions considered in this chapter are satisfied, including the regularity assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and $(\mathcal{A}3)$. Now we apply needle-like perturbations of the control u at some $\tau \in (0, 1)$ and we refer to Figure 2.5 for illustrations.

(i) If $v = -1$ we get a perturbed trajectory x^α satisfying $x^\alpha(t) \in X_1$ over the whole interval $[0, 2]$ and thus x^α does not uniformly converge to x over $[0, 2]$ when $\alpha \to 0$.

(ii) If $v = 2$ we get a perturbed trajectory x^α defined over $[0, \tilde{t}(\alpha))$ for some $\tilde{t}(\alpha) < 1$. Note that x^α is not defined over $[\tilde{t}(\alpha), 2]$ since, by contradiction, one would obtain $\dot{x}^\alpha(\tilde{t}(\alpha)^-) = +1$ and $\dot{x}^\alpha(\tilde{t}(\alpha)^+) = -1$ implying that x^α does not enter into the open region X_2 over $(\tilde{t}(\alpha), 1)$. In that context, note that different approaches can be explored, such as differential inclusions (see, *e.g.*, [7]) and sliding modes (see, *e.g.*, [114]), to consider a generalized notion of solution to the hybrid control system. However these approaches would not solve the issue presented in Item (i) anyway, and thus we will not go any further in that direction.

The reason of this feature in a spatially hybrid setting lies in the fact that standard needle-like perturbations of the control u do not take into account the perturbation of the next crossing time.

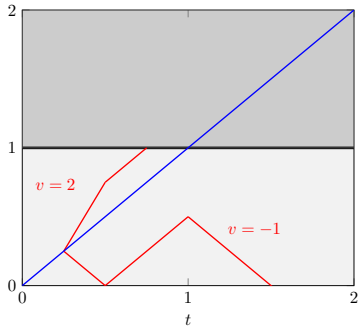


Figure 2.5: Illustration of the counterexample given in Item 2 of Section 2.2.3.

3. We are now in a position to provide an overview of the proof of Theorem 2.2.1. Let (x, λ, u) be a solution to (2.2), that is moreover a regular solution to (2.1). To overcome the difficulty of handling needle-like perturbations in the spatially hybrid setting (as discussed in Item 2 above), we shall introduce, for all $k \in \{1, \dots, N\}$, an *auxiliary control*, denoted by $\tilde{u}_k$, that coincides with the

control u over (t^c_{k-1}, t^c_k) and that is continuously extended to a constant function outside (t^c_{k-1}, t^c_k) thanks to Assumption $(\mathcal{A}1)$ (see Figure 2.6 and the exact definition of $\tilde{u}_k$ in Section 2.5.1). In the sequel we will apply needle-like perturbations only to auxiliary controls $\tilde{u}_k$ (and not to the nominal control u).

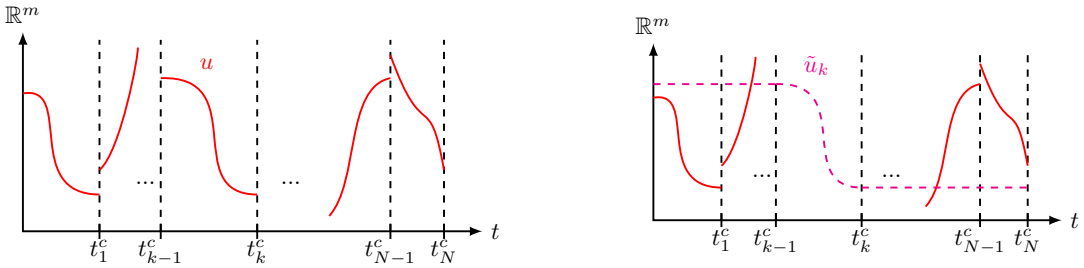


Figure 2.6: Illustration of an auxiliary control $\tilde{u}_k$. In this illustration, for simplicity, we have chosen a control u that is continuous over each (t^c_{k-1}, t^c_k) but it is not necessary. We only know that u satisfies the continuity properties given in Assumption $(\mathcal{A}1)$.

4. Now let us fix $k \in \{1, ..., N\}$ (from now and until Item 11). The pair $(\lambda_k, \tilde{u}_k)$ allows us to define the *auxiliary non-hybrid trajectory*, denoted by $\tilde{z}_k$, as the unique solution to the non-hybrid state equation defined with the dynamics f_k only (that is, with the dynamics f_k all over $\mathbb{R}^n$, even outside E_k) and with the constant parameter λ_k only (that is, with the constant parameter λ_k all over $[0, T]$, even outside (t^c_{k-1}, t^c_k)), together with the initial condition $\tilde{z}_k(t^c_{k-1}) = x(t^c_{k-1})$. Observe that $\tilde{z}_k$ represents an extension of the nominal trajectory x as illustrated in Figure 2.7.

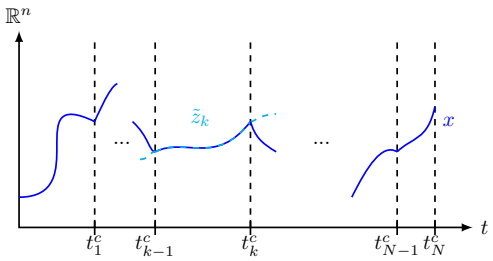


Figure 2.7: Illustration of the auxiliary non-hybrid trajectory $\tilde{z}_k$.

5. Now we will consider either a basic convex perturbation of λ_k given by $\lambda_k + \alpha(\overline{\lambda}_k - \lambda_k)$ for some $\overline{\lambda}_k \in \Lambda$, either a classical needle-like perturbation of the auxiliary control $\tilde{u}_k$ associated with some $\tau \in (t^c_{k-1}, t^c_k)$ and some $v \in \mathrm{U}$ (see Figure 2.8). In both cases, this gives us a *perturbed auxiliary non-hybrid trajectory* denoted by $\tilde{z}^\alpha_k$. Since we deal here with a classical non-hybrid setting (with the dynamics f_k only), we can use standard results from the literature such as the uniform convergence of $\tilde{z}^\alpha_k$ to $\tilde{z}_k$ when $\alpha \to 0$, and the existence of the corresponding variation vector, denoted by w_k, solution to a linearized control system with an initial condition at t^c_{k-1} reduced to $0_{\mathbb{R}^n}$.
6. The next step is to prove that the trajectory $\tilde{z}^\alpha_k$ crosses the boundary $\partial E_k \cap \partial E_{k+1}$ at a perturbed crossing time $\tilde{t}_k(\alpha)$ (see Figure 2.9). To this aim we invoke an implicit function theorem (Lemma 2.5.1) to the map $G_k : (\alpha, t) \mapsto F_k(\tilde{z}^\alpha_k(t), t)$ that can be applied thanks to the regularity assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and $(\mathcal{A}3)$ and the construction of $\tilde{u}_k$. In particular note that $\nabla_t G_k$ is invertible at $(0, t^c_k)$ thanks to the first transverse condition in Assumption $(\mathcal{A}3)$.

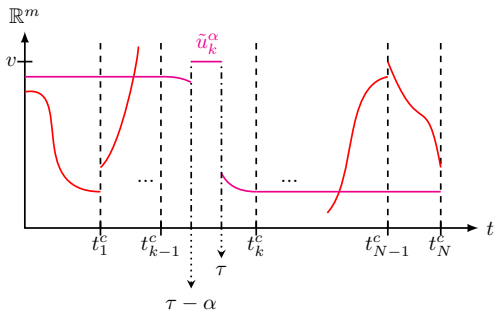


Figure 2.8: Illustration of a needle-like perturbation of $\tilde{u}_k$ (recall Figure 2.6).

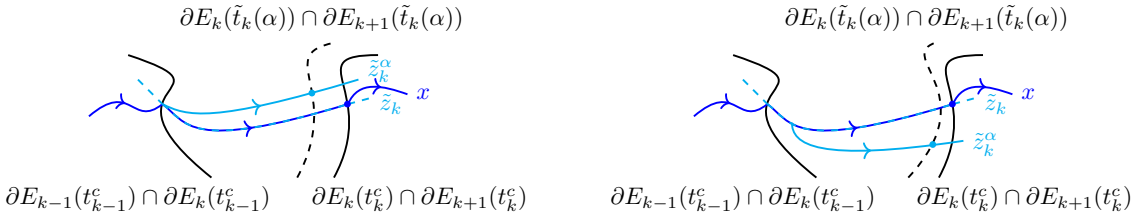


Figure 2.9: Plot of $\tilde{z}_k^\alpha$ under a convex perturbation of λ_k (left). Plot of $\tilde{z}_k^\alpha$ under a needle-like perturbation of $\tilde{u}_k$ (right). In both cases $\tilde{z}_k^\alpha$ crosses the boundary $\partial E_k \cap E_{k+1}$ at some time $\tilde{t}_k(\alpha)$.

7. From the construction of the trajectory $\tilde{z}_k^\alpha$, it can be proved that $\tilde{z}_k^\alpha$ stays inside E_k over $(t_{k-1}^c, \tilde{t}_k(\alpha))$. Indeed, thanks to Assumption $(\mathcal{A}3)$, one can prove by contradiction that there exist $t_{k-1}^c < s_k' < s_k < \min\{t_k^c, \tilde{t}_k(\alpha)\}$, uniformly with respect to α, such that $\tilde{z}_k^\alpha$ has values in E_k over (t_{k-1}^c, s_k') and over $(s_k, \tilde{t}_k(\alpha))$ (see Lemmas 2.5.3 and 2.5.6 for technical details). Then, from Condition $(\mathcal{C}1)$ and the uniform convergence of $\tilde{z}_k^\alpha$ to $\tilde{z}_k = x$ over $[s_k', s_k]$, we obtain that $\tilde{z}_k^\alpha$ has values in E_k over $[s_k', s_k]$ also.
8. After having considered perturbations in the region E_k (see Items 4 and 5) and the consequences in the region E_k only (see Items 6 and 7), our aim now is to analyze the resulting perturbations in the next regions $E_{k+1}, \ldots, E_N$. For the reader's convenience, we will detail here only the passage to the region E_{k+1} (the other regions are treated with a basic induction, see Item 10). Similarly to Item 4, the pair $(\lambda_{k+1}, \tilde{u}_{k+1})$ allows us to define the auxiliary non-hybrid trajectory, denoted by $\tilde{z}_{k+1}$, as the unique solution to the non-hybrid state equation considered with the dynamics f_{k+1} only and with the constant parameter λ_{k+1} only, together with the initial condition $\tilde{z}_{k+1}(t_k^c) = x(t_k^c)$. Now, in contrast to Item 5 (in which we have proceeded either to a perturbation of the parameter, either to a perturbation of the control), we will consider here the perturbation of the initial time t_k^c by $\tilde{t}_k(\alpha)$ (constructed in Item 6) and the perturbation of the initial condition $x(t_k^c)$ by $\tilde{z}_k^\alpha(\tilde{t}_k(\alpha))$. This gives us the perturbed auxiliary non-hybrid trajectory $\tilde{z}_{k+1}^\alpha$. This construction will allow us to proceed to a concatenation of the perturbed auxiliary non-hybrid trajectories $\tilde{z}_k^\alpha$ and $\tilde{z}_{k+1}^\alpha$ (see Figure 2.10). Since we deal here with a classical non-hybrid setting (with the dynamics f_{k+1} only), we can use standard results from the literature such as the uniform convergence of $\tilde{z}_{k+1}^\alpha$ to $\tilde{z}_{k+1}$ when $\alpha \to 0$, and the existence of the corresponding variation vector, denoted by w_{k+1}, solution to a linearized control system with an initial condition at t_k^c given by $w_k(t_{k-1}^c)$ plus an additional term due to the perturbations of the initial time and of the initial condition. Finally, similarly to Item 6, we prove

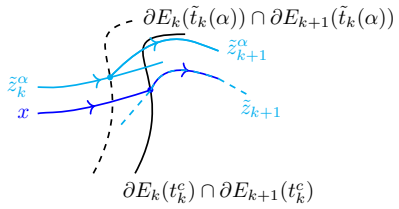


Figure 2.10: Perturbed auxiliary non-hybrid trajectory $\tilde{z}^\alpha_{k+1}$ under perturbations of the initial time and of the initial condition.

that $\tilde{z}^\alpha_{k+1}$ crosses the boundary $\partial E_{k+1} \cap \partial E_{k+2}$ at a perturbed crossing time $\tilde{t}_{k+1}(\alpha)$.

9. Using similar arguments to Item 7, it can be proved that the trajectory $\tilde{z}^\alpha_{k+1}$ stays inside E_{k+1} over $(\tilde{t}_k(\alpha), \tilde{t}_{k+1}(\alpha))$.
10. Finally we proceed by induction, region after region, in order to construct the perturbed auxiliary non-hybrid trajectories $\tilde{z}^\alpha_q$ and the corresponding variation vectors w_q for all $q \in \{k, ..., N\}$. Then we construct a "global" perturbed trajectory x^α of x over the whole time interval $[0, T]$ (resp. a "global" variation vector w) by concatenation of the perturbed auxiliary non-hybrid trajectories $\tilde{z}^\alpha_q$ over $[\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)]$ (resp. of the variation vectors w_q over $[t^c_{q-1}, t^c_q)$). This construction allows to guarantee several properties. First x^α visits exactly (and in the same order) the same regions that the nominal trajectory x. Second x^α converges uniformly to x over $[0, T]$ when $\alpha \to 0$. Third the "global" variation vector w corresponds to the variation vector associated with the "global" perturbed trajectory x^α of x. It is worth mentioning that, as reported in Item 8, the "global" variation vector w has a discontinuity jump at each crossing time t^c_q.
11. From convexity of Λ, note that the convex perturbation of λ_k belongs to Λ. Similarly, since $v \in \mathrm{U}$ and from the construction of $\tilde{u}_k$ and the closedness of U, note that the needle-like perturbation of $\tilde{u}_k$ has values in U. Therefore the constraints of Problem (2.2) are satisfied and thus, from optimality of the triple (x, λ, u), it is clear that $\frac{\phi(x^\alpha(T)) - \phi(x(T))}{\alpha} \geq 0$ which leads to $\langle \nabla\phi(x(T)), w(T)\rangle_{\mathbb{R}^n} \geq 0$ when $\alpha \to 0$. One has to note that this last inequality is satisfied for any variation vector w constructed as in the previous items, and thus is satisfied for any $\overline{\lambda}_k \in \Lambda$, any $v \in \mathrm{U}$, any $\tau \in (0, T)$ and for any $k \in \{1, \ldots, N\}$.
12. To conclude the proof, the method is now very similar to the standard non-hybrid setting found in the literature. The idea is to construct an adjoint vector p which guarantees the constancy of the inner product between the adjoint vector p and any variation vector w constructed as in the previous items. To this aim we define p as solution to the opposite of the transpose of the linearized control system satisfied by the variation vectors w (which corresponds exactly to the adjoint equation in Theorem 2.2.1). On the other hand, to handle the discontinuity jumps of the variation vectors w at each crossing time, we impose appropriate discontinuity jumps on p (which correspond exactly to the adjoint discontinuity jumps in Theorem 2.2.1). We refer to Remark 2.2.4 for details on the expression of the discontinuity jumps of p. Finally, imposing the final condition $p(T) = -\nabla\phi(x(T))$, we obtain that $\langle p(T), w(T)\rangle_{\mathbb{R}^n} \leq 0$ for any variation vector w. Using the classical Duhamel formula and thanks to the constancy of the inner product between the adjoint vector p and any variation vector w, this last inequality can be rewritten as the averaged Hamiltonian gradient condition in Theorem 2.2.1 (if we have considered a variation vector w associated with a convex perturbation of the parameter) or as the Hamiltonian maximization condition in Theorem 2.2.1 (if we have

considered a variation vector w associated with a needle-like perturbation of an auxiliary control). The proof is complete.

2.2.4 A list of general comments

This section is dedicated to a list of general comments on our framework and assumptions, on Theorem 2.2.1 and its consequences, but also on possible relaxations and extensions.

Remark 2.2.1. Comments on our framework and assumptions. Recall that several versions of necessary optimality conditions for spatially hybrid optimal control problems (without regionally switching parameter) are already available in the literature (see references in General introduction 0). Note that the following two extensions could be done with no major difficulty:

- First, considering an additional constant parameter $\lambda \in \Lambda$ could be easily treated thanks to the classical technique of augmenting the control system with the equation $\dot{\lambda}(t) = 0_{\mathbb{R}^d}$ and by considering the final state constraint $\lambda(T) \in \Lambda$ (see, *e.g.*, [28]). One would obtain a necessary optimality condition written as an averaged Hamiltonian gradient condition over the whole interval $[0, T]$.
- Second, in the case where $\lambda \in \mathrm{PC}_{\mathbb{T}'}([0, T], \mathbb{R}^d)$ is a piecewise constant control (also known as *sampled-data control*) with values in Λ, associated with a fixed partition $\mathbb{T}' = \{t'_k\}_{k=0,\dots,N'}$ (independent of the state position and of the interface crossings), one could easily deduce necessary optimality conditions from the previous item. Indeed, using an adequate change of time variable (transforming all intervals $[t'_{k-1}, t'_k]$ into the common interval $[0, T]$), all the values $\lambda_k \in \Lambda$ would become constant parameters and one could deduce an averaged Hamiltonian gradient condition over each interval $[t'_{k-1}, t'_k]$ (see, *e.g.*, [35], [41]). Note that, in case of a free partition $\mathbb{T}'$ (but with a fixed positive integer $N' \in \mathbb{N}^*$), one could consider each t'_k as a parameter and derive a corresponding necessary optimality condition (see, *e.g.*, [34]).

On the other hand, to the best of our knowledge, regionally switching parameters in spatially hybrid control systems have never been discussed yet, and our aim in the present work is to fill this gap in the literature. We insist on the fact that, even if considering a regionally switching parameter might seem as easy as dealing with sampled-data controls, it is not. The two main technical issues are the fact that the possibility (or not) of changing the parameter value depends on the state position $x(t)$ (and not on the time variable t) and the fact that spatially hybrid optimal control problems cannot be rewritten easily as classical optimal control problems. This is the reason why, to derive necessary optimality conditions, our new framework requires a thorough sensitivity analysis (presented in Section 2.5) of the spatially hybrid control system under convex perturbations of the regionally switching parameter and under needle-like perturbations of the control. Finally note that this technical sensitivity analysis is allowed thanks to the various assumptions introduced all along Sections 2.2.1 and 2.2.2 that we comment now:

(i) Let us comment the continuity conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$. First, note that they are automatically satisfied whenever the partition is *static* (that is, independent of the time variable t). In contrast, when the partition is not static, the continuity conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$ guarantee, as one might expect, a kind of smooth and reasonable time evolution of the regions composing the partition (see Figure 2.1). In the sensitivity analysis developed in Section 2.5, they are used to construct perturbed trajectories which visit exactly (and in the same order) the same regions than the nominal one. We refer to Section 2.2.3 for a brief overview and some illustrations.

(ii) Let us comment the Assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and $(\mathcal{A}3)$. In the sensitivity analysis developed in Section 2.5, when considering (local) perturbations of the regionally switching parameter and of the

control in a given region E_k, we obtain a perturbed trajectory, but also a perturbed crossing time between the two consecutive regions E_k and E_{k+1}. To guarantee the existence of this perturbed crossing time, we rely on the application of an implicit function theorem (Lemma 2.5.1) which requires Assumption $(\mathcal{A}2)$ to benefit a local description (in space and time) of the boundary between E_k and E_{k+1}. This gives us an explicit function whose regularity is guaranteed by Assumption $(\mathcal{A}1)$ and whose the invertibility of the partial derivative (with respect to time) is guaranteed by the first transverse condition in Assumption $(\mathcal{A}3)$. Finally, the second transverse condition in Assumption $(\mathcal{A}3)$ is used to guarantee that the perturbed trajectory enters in the next open region E_{k+1}. We then proceed by induction, region after region. We refer to Section 2.2.3 for a brief overview and some illustrations.

(iii) Finally note that, in this chapter, the fixed initial condition x_{init} belongs to a region (and not to a boundary) and, according to Definition 2.2.1, we deal (only) with trajectories x whose final condition $x(T)$ also belongs to a region (and not to a boundary). These restrictions allow us, similarly to the continuity conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$, to avoid situations in which the sensitivity analysis of the hybrid control system would involve perturbed trajectories that would visit more regions than the nominal one. However we are confident that, at the price of a slightly more cumbersome analysis, the methodology developed in this chapter (in particular the assumptions and techniques used to deal with the boundary crossings) could be adapted to deal with terminal conditions that might belong to boundaries.

Remark 2.2.2. Comments on the novelty of Theorem 2.2.1. We would like to emphasize that Items (i), (ii), (iii) and (iv) of Theorem 2.2.1 already appear in the literature on spatially hybrid optimal control problems (see references in General introduction 0), but without the consideration of a regionally switching parameter. Let us also mention that the last Item (v) is also well known in the literature on optimal sampled-data control problems (see references in General introduction 0) which is a simpler and very different setting since the partition of the piecewise constant control is temporal, and not spatial. As a consequence, all items of Theorem 2.2.1 have already appeared in the literature, but in different contexts. The novelty of Theorem 2.2.1 is to derive all these items together in order to deal with our new framework given by spatially hybrid optimal control problems with regionally switching parameter.

Remark 2.2.3. Comments on the averaged Hamiltonian gradient condition. Consider the framework of Theorem 2.2.1. Note that the averaged Hamiltonian gradient condition (2.5) is implicit in general since λ_k intervenes, not only in both sides of the equation, but also in the values of x and p along the interval (t_{k-1}^c, t_k^c). Furthermore we do not know in advance the values of t_{k-1}^c and t_k^c. Nevertheless, as already seen in [34]–[36], [41] and as we will see in Section 2.3 on a simple example, the averaged Hamiltonian gradient condition (2.5) can be useful to determine the optimal values of regionally switching parameters. From a numerical point of view, when Λ is closed, note that λ_k can be expressed as the fixed point

$$\lambda_k = \mathrm{proj}_\Lambda\Big(\lambda_k + \int_{t_{k-1}^c}^{t_k^c} \nabla_\lambda H(x(s), \lambda_k, u(s), p(s), s)\,\mathrm{d}s\Big),$$

where $\mathrm{proj}_\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ stands for the standard projection operator onto Λ. Finally, let us mention that, under some advanced convexity assumptions developed by Halkin and Holtzmann [79] (which are more general than just assuming a global convexity of the Hamiltonian H with respect to its second variable),

it should be possible to derive an *averaged Hamiltonian maximization condition* of the form

$$\lambda_k \in \arg\max_{\mu\in\Lambda} \int_{t_{k-1}^c}^{t_k^c} H(x(s),\mu,u(s),p(s),s)\,\mathrm{d}s.$$

However this point is out of the scope of the present work.

Remark 2.2.4. Comments on the adjoint vector and its discontinuity jumps. Consider the framework of Theorem 2.2.1. In the first item below, in order to avoid any confusion, we give details on the discontinuous structure of the adjoint vector p and then, in the second item, we discuss the origin of its discontinuity jumps.

(i) The adjoint vector $p \in \mathrm{PAC}_{\mathbb{T}}([0,T],\mathbb{R}^n)$ is piecewise absolutely continuous, respecting the same partition $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ associated with the solution (x,λ,u) (see Definition 2.2.1). Hence the restriction of p over each open interval (t_{k-1}^c, t_k^c) admits an extension over $[t_{k-1}^c, t_k^c]$ that is absolutely continuous, satisfying the adjoint equation provided in Theorem 2.2.1. Furthermore, at each crossing time t_k^c, the adjoint vector p admits a discontinuity jump satisfying the equality (2.3). Note that (2.3) is written in a backward way, in the sense that $p^-(t_k^c)$ is expressed explicitly in terms of $p^+(t_k^c)$. Nevertheless we emphasize that (2.3) can also be written in a forward way as

$$p^+(t_k^c) - p^-(t_k^c) = -\frac{\langle p^-(t_k^c), (f_{k+1})^+(t_k^c) - (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^+(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} \nabla_x F_k(x(t_k^c),t_k^c).$$

(ii) To explain the origin of the discontinuity jumps of the adjoint vector p, we need to point out that, in spatially hybrid settings, a variation vector w (obtained from a perturbation of the regionally switching parameter or of the control in our setting) has a discontinuity jump at each crossing time t_k^c given by

$$\xi_k := w^+(t_k^c) - w^-(t_k^c) = \frac{\langle \nabla_x F_k(x(t_k^c),t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} ((f_{q+1})^+(t_k^c) - (f_q)^-(t_k^c)).$$

We refer to Propositions 2.5.1 and 2.5.2 for details. To derive our main result (Theorem 2.1), just as in classical optimal control theory, we need to construct an adjoint vector p which guarantees the constancy of the inner product between p and any variation vector w. To this aim, in Section 2.6, we define p as solution to the opposite of the transpose of the linearized control system satisfied by the variation vectors w. On the other hand, to handle the discontinuity jumps of the variation vectors w at each crossing time, we also need to ensure that $\langle p^+(t_k^c), w^+(t_k^c)\rangle_{\mathbb{R}^n} = \langle p^-(t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}$. To this aim, we introduce discontinuity jumps $\chi_k := p^+(t_k^c) - p^-(t_k^c)$ for the adjoint vector p. Hence we look for a value of χ_k such that $\langle p^-(t_k^c), \xi_k\rangle_{\mathbb{R}^n} + \langle \chi_k, w^-(t_k^c) + \xi_k\rangle_{\mathbb{R}^n} = 0$. Replacing the value of ξ_k recalled above, we look for a value of χ_k such that

$$\begin{aligned}\frac{\langle \nabla_x F_k(x(t_k^c),t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} \langle p^-(t_k^c), ((f_{q+1})^+(t_k^c) - (f_q)^-(t_k^c))\rangle_{\mathbb{R}^n} + \langle \chi_k, w^-(t_k^c)\rangle_{\mathbb{R}^n} \\ + \frac{\langle \nabla_x F_k(x(t_k^c),t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} \langle \chi_k, ((f_{q+1})^+(t_k^c) - (f_q)^-(t_k^c))\rangle_{\mathbb{R}^n} = 0.\end{aligned}$$

Thus, in view of factorizing the above equality by $\langle \nabla_x F_k(x(t_k^c),t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}$, we make the Ansatz that χ_k is proportionate to $\nabla_x F_k(x(t_k^c),t_k^c)$, that is, there exists $\sigma_k \in \mathbb{R}$ such that

$\chi_k = \sigma_k \nabla_x F_k(x(t_k^c), t_k^c)$. Finally, we look for a value of σ_k such that

$$\frac{\langle \nabla_x F_k(x(t_k^c), t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)} \Big(\langle p^-(t_k^c), ((f_{q+1})^+(t_k^c) - (f_q)^-(t_k^c))\rangle_{\mathbb{R}^n} + \sigma_k(\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_{k+1})^+(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)) \Big) = 0,$$

which gives

$$\sigma_k = -\frac{\langle p^-(t_k^c), ((f_{q+1})^+(t_k^c) - (f_q)^-(t_k^c))\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_{k+1})^+(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)}.$$

Finally we define the adjoint vector p backward in time, using the discontinuity jumps given by $\chi_k = \sigma_k \nabla_x F_k(x(t_k^c), t_k^c)$.

To conclude this remark, we precise, as in Remark 2.2.2, that the expression of the discontinuity jumps of the adjoint vector p can be found in several references in the literature such as [25], [73], [98], [99], [108] but without the consideration of a regionally switching parameter. As far as we know, it is an open question to know if this choice is unique, that is, if every adjoint vector p that fulfills Theorem 2.2.1 (except Item (iii)) necessarily has discontinuity jumps χ_k that are proportional to $\nabla_x F_k(x(t_k^c), t_k^c)$ at each crossing time t_k^c.

Remark 2.2.5. Comments on the Hamiltonian function. In this remark we would like to recall some standard properties satisfied by the Hamiltonian function in spatially hybrid settings (see references in Introduction) that are preserved in our setting with regionally switching parameter. Consider the framework of Theorem 2.2.1. Note that the Hamiltonian system

$$\dot{x}(t) = \nabla_p H(x(t), \lambda_k, u(t), p(t), t), \qquad \dot{p}(t) = -\nabla_x H(x(t), \lambda_k, u(t), p(t), t),$$

is satisfied for almost every $t \in (t_{k-1}^c, t_k^c)$ and all $k \in \{1, \dots, N\}$. As usual in the literature let us introduce the *Hamiltonian function* $\mathcal{H} : [0, T] \to \mathbb{R}$ defined by

$$\mathcal{H}(t) := H(x(t), \lambda(t), u(t), p(t), t),$$

for almost every $t \in [0, T]$. Using similar arguments as in [63, Theorem 2.6.1 pp. 71], one can prove from the above Hamiltonian system, from (2.4) and from the piecewise constancy of the parameter λ, that $\mathcal{H}$ is equal almost everywhere over each interval (t_{k-1}^c, t_k^c) to an absolutely continuous function which satisfies

$$\dot{\mathcal{H}}(t) = \nabla_t H(x(t), \lambda_k, u(t), p(t), t),$$

for almost every $t \in (t_{k-1}^c, t_k^c)$ and all $k \in \{1, \dots, N\}$. Therefore we write $\mathcal{H} \in \mathrm{PAC}_{\mathbb{T}}([0, T], \mathbb{R})$ and one can easily obtain from simple computations that the discontinuity jumps of $\mathcal{H}$ are given by

$$\mathcal{H}^+(t_k^c) - \mathcal{H}^-(t_k^c) = \frac{\langle p^-(t_k^c), (f_{k+1})^+(t_k^c) - (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c)), (f_{k+1})^+(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)} \nabla_t F_k(x(t_k^c), t_k^c).$$

As in Remark 2.2.4, we emphasize that the above formula can be rewritten in terms of $p^+(t_k^c)$ (instead of $p^-(t_k^c)$). Finally, from the results presented in this remark, we deduce that:

(i) If the partition is static, then the discontinuity jumps of $\mathcal{H}$ are reduced to zero and thus $\mathcal{H} \in$

$\mathrm{C}([0,T],\mathbb{R})$.[1]

(ii) If the hybrid dynamics is *autonomous* (that is, h does not depend on the time variable t), then $\mathcal{H}$ is constant over each interval (t_{k-1}^c, t_k) and thus $\mathcal{H} \in \mathrm{PC}_{\mathbb{T}}([0,T],\mathbb{R})$.

(iii) In the joint case where the partition is static and the hybrid dynamics is autonomous, then $\mathcal{H}$ is constant over $[0,T]$.

Remark 2.2.6. Some possible relaxations and easy extensions of Theorem 2.2.1. In this chapter we have considered a certain framework which is, of course, not the most general possible. In fact we have made some choices to make the presentation and the notations as simple and pleasant to read as possible, while keeping the essence of our work. In this remark our aim is to gather a number of possible relaxations and easy extensions of Theorem 2.2.1. One can easily be convinced by the validity of these generalizations by reading the proof of Theorem 2.2.1 in Sections 2.4, 2.5 and 2.6 (or the brief overview of the proof provided in Section 2.2.3).

(i) The convexity hypothesis made on Λ can be removed by using a generalized version of the normal cone. Precisely, instead of using basic convex perturbations of the form $\lambda_k + \alpha(\overline{\lambda}_k - \lambda_k)$, one can invoke a general perturbation $\tilde{\lambda}_k(\alpha)$ where $\tilde{\lambda}_k : [0,1] \to \Lambda$ is a continuous function satisfying $\tilde{\lambda}_k(0) = \lambda_k$ and that is differentiable at 0 with derivative denoted by $\tilde{\lambda}_k'(0)$. Therefore Theorem 2.2.1 remains valid by considering the generalized notion of normal cone to Λ at some $\lambda \in \Lambda$ given by

$$\mathrm{N}_{\Lambda}^{gen}[\lambda] := \{\lambda'' \in \mathbb{R}^d \mid \langle \lambda'', \tilde{\lambda}'(0)\rangle_{\mathbb{R}^d} \leq 0 \text{ for all continuous functions } \tilde{\lambda} : [0,1] \to \Lambda \text{ with } \tilde{\lambda}(0) = \lambda \text{ and differentiable at 0 with derivative denoted by } \tilde{\lambda}'(0)\}.$$

(ii) The closedness hypothesis made on U can be removed by assuming in Theorem 2.2.1 that all the limits $u^-(t_k^c)$ and $u^+(t_k^c)$ belong to U. Indeed, in our proof of Theorem 2.2.1, we only need that the auxiliary controls $\tilde{u}_k$ are with values in U.

(iii) The right continuity after each crossing time t_k^c, and the left continuity before the last crossing time t_{N-1}^c, of the control u (see Definition 2.2.2) are useless in our proof of Theorem 2.2.1. We have adopted these hypotheses for the sake of simplicity of the presentation. However they can be removed.

(iv) Theorem 2.2.1 is stated for a solution to (2.2) in a global sense. However Theorem 2.2.1 remains valid for a solution to (2.2) in (only) a local sense to be precised (for example, in L^∞-norm for the trajectory and the regionally switching parameter, and in L^1-norm for the control).

(v) The C^1-regularity of the map ϕ can be relaxed. Indeed only the differentiability of ϕ is required for our proof of Theorem 2.2.1. Similarly the C^1-regularity of the maps h_j can be relaxed. Indeed our proof of Theorem 2.2.1 (precisely the sensitivity analyses developed in the preliminary Sections 2.4 and 2.5) requires (only) that, for all $j \in \mathcal{J}$, the map h_j is continuous, is differentiable with respect to its two first variables with $\nabla_x h_j$ and $\nabla_\lambda h_j$ continuous, and is Lipschitz continuous with respect to its three first variables on any compact subset of $\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T]$. We refer to [23] where similar relaxed regularity assumptions have been considered.

(vi) A possible extension of our work is to consider, for each region X_j, a hybrid dynamics $h_j : \mathbb{R}^n \times \mathbb{R}^{d_j} \times \mathbb{R}^{m_j} \times [0,T] \to \mathbb{R}^n$ with possibly different dimensions d_j and m_j. Indeed, one can rigorously see that, in our proof, the construction of auxiliary controls has no impact constraining

[1] We just note that this result is kind of in accordance with the main result obtained in [34] stating that, when the partition associated with sampled-data controls is free, then the corresponding necessary optimality condition coincides with the continuity of the Hamiltonian function $\mathcal{H}$.

the controls to be with values in the same space $\mathbb{R}^m$. Accordingly, another possible extension of our work is to consider possibly different parameter constraint sets $\Lambda_j \subset \mathbb{R}^{d_j}$ and possibly different control constraint sets $\mathrm{U}_j \subset \mathbb{R}^{m_j}$. This generalized context is interesting to impose the values of the regionally switching parameter and/or of the control in some regions (for example, by taking $\Lambda_j = \{0_{\mathbb{R}^{d_j}}\}$ and/or $\mathrm{U}_j = \{0_{\mathbb{R}^{m_j}}\}$ for some $j \in \mathcal{J}$).

(vii) A possible generalization of our work is to extend the space partition of $\mathbb{R}^n$ to a space-time partition of the form $\mathbb{R}^n \times [0,T] = \cup_{j\in\mathcal{J}} Y_j$. Such an extension would cover, in particular, the framework of *optimal sampled-data control problems* developed in [34]–[36], [41]. Note that, to obtain this extension, one can simply use the classical augmentation technique of considering the time variable t as an additional state variable x_{n+1} satisfying $\frac{dx_{n+1}}{dt} = 1$ (see, *e.g.*, [28, Section 1.3.3]). Actually, from a more general point of view, this technique could have been used all along the present chapter. However, to allow the reader who may face a time-dependent problem to apply directly our main result (Theorem 2.2.1), without having to proceed himself to the augmentation technique, we decided to keep the time variable t in our setting.

(viii) Using the classical technique of augmenting the state of the control system (see, *e.g.*, [28]), one can easily extend Theorem 2.2.1 to deal with Bolza costs, that is, when the cost of (2.2) is replaced by a cost of the form

$$\phi(x(T)) + \int_0^T L(x(s),\lambda(s),u(s),s)\,\mathrm{d}s,$$

where the hybrid Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ is defined regionally by

$$\forall (x,\lambda,u,t) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T], \quad L(x,\lambda,u,t) := L_j(x,\lambda,u,t) \text{ when } x \in X_j(t),$$

where the maps $L_j : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ are of class C^1. In that context Theorem 2.2.1 remains valid by replacing the definition of the Hamiltonian H by

$$H(x,\lambda,u,p,t) := \langle p, h(x,\lambda,u,t)\rangle_{\mathbb{R}^n} - L(x,\lambda,u,t),$$

for all $(x,\lambda,u,p,t) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T]$, and by replacing (2.3) by

$$\begin{aligned} &p^+(t_k^c) - p^-(t_k^c) \\ &\qquad = -\frac{\langle p^+(t_k^c), (f_{k+1})^+(t_k^c) - (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} - ((L_{k+1})^+(t_k^c) - (L_k)^-(t_k^c))}{\langle \nabla_x F_k(x(t_k^c),t_k^c), (f_k)^-(t_k^c)\rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c),t_k^c)} \nabla_x F_k(x(t_k^c),t_k^c). \end{aligned}$$

(ix) One can also consider a hybrid control system of the form $\dot{x}(t) = h(x(t),\mu,\lambda(t),u(t),t)$ involving an additional constant parameter $\mu \in \mathbb{R}^{d'}$ for some $d' \in \mathbb{N}^*$. In that context we emphasize that μ is not a regionally switching parameter: it is constant over the whole interval $[0,T]$. Then one can consider the additional parameter constraint $\mu \in \mathrm{M}$ in Problem (2.2), where M is a nonempty convex subset of $\mathbb{R}^{d'}$. By adapting the proof of Theorem 2.2.1, one can easily see that Theorem 2.2.1 remains valid by replacing the definition of the Hamiltonian H by

$$H(x,\mu,\lambda,u,p,t) := \langle p, h(x,\mu,\lambda,u,t)\rangle_{\mathbb{R}^n},$$

for all $(x,\mu,\lambda,u,p,t) \in \mathbb{R}^n \times \mathbb{R}^{d'} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T]$, and by adding the necessary optimality

condition given by

$$\int_0^T \nabla_\mu H(x(s), \mu, \lambda(s), u(s), p(s), s)\, \mathrm{d}s \in \mathrm{N}_{\mathrm{M}}[\mu].$$

One may think that adding a constant parameter μ to our framework can be treated as easy as explained at the beginning of Remark 2.2.1, but unfortunately it cannot. Indeed, to follow the idea of Remark 2.2.1, one would need a version of Theorem 2.2.1 that can handle final state constraints. Note that such an extension is one of our perspectives as explained in Remark 2.2.7. Hence, (only) when such an extension will be available in the literature, adding a constant parameter μ to our framework will be very easy.

(x) One can also consider a version of Problem (2.2) with a *free final time* $T > 0$ and in which the Mayer cost is of the form $\phi(T, x(T))$. Such a framework is important to deal with *minimal time problems* (see, *e.g.*, [46, Section 3]). By using the classical technique of change of time variable $t = Ts$, one can transform the variable T to optimize as a constant parameter. Thanks to the previous item and the results presented in Remark 2.2.5, one can prove that Theorem 2.2.1 remains valid with the additional necessary optimality condition given by

$$\mathcal{H}(T) = \nabla_T \phi(T, x(T)),$$

where $\mathcal{H} : [0, T] \to \mathbb{R}$ is the Hamiltonian function introduced in Remark 2.2.5.

Remark 2.2.7. Some possible perspectives. This last remark is dedicated to a nonexhaustive list of possible perspectives:

(i) The present chapter does not cover *terminal state constraints* (that is, constraints on $x(0)$ and $x(T)$) which are common in most applications of optimal control theory. This is clearly a criticism that can be made to the present work. In the classical non-hybrid setting, several methods have been developed in the literature to take into account such terminal state constraints. One can invoke the Ekeland variational principle [61] or some implicit function arguments (see, *e.g.*, [2], [110]) or the use of Lagrange multiplier rules [3]. In one hand, it is worth mentioning that, to the best of our knowledge, the Ekeland approach does not apply in the present spatially hybrid setting for two main reasons. First, this approach requires to define a continuous penalized functional on a L^1-neighborhood of the optimal control u. However we have seen in Item 2 of Section 2.2.3 that such a construction is obstructed in the present spatially hybrid setting. Second, the control sequence produced by the Ekeland variational principle (which converges in L^1-norm to the optimal control u) would have no reason to satisfy the regularity assumption ($\mathcal{A}1$) and therefore the sensitivity analysis developed in the preliminary Section 2.5 may be not valid on the control sequence. On the other hand, we are confident that a method based on an implicit function argument or on a Lagrange multiplier rule could be adapted to the present spatially hybrid setting. However these approaches are based on the so-called *Pontryagin convex cone* constructed thanks to the consideration of *multiple* needle-like perturbations of the control (see, *e.g.*, [1], [3], [38], [56], [58], [81], [101] and references therein). In the present spatially hybrid setting, this would have required the consideration of multiple needle-like perturbations of the control in each region simultaneously. This would have significantly increased the complexity of the analysis and the notations. Since our main objective in this work was to focus on the concept of *regionally switching parameter* and on the corresponding averaged Hamiltonian gradient condition (2.5), we decided to avoid the technicalities related to the presence of terminal state constraints which are already well known in the literature and to keep the

reading of the technical proof of Theorem 2.2.1 as pleasant as possible.

(ii) The regularity assumptions introduced in Definition 2.2.2 are crucial to develop the sensitivity analysis of the spatially hybrid control system in the preliminary Section 2.5, precisely to construct perturbed trajectories which visit exactly (and in the same order) the same regions than the nominal trajectory. To the best of our knowledge, an open question is how to obtain a hybrid maximum principle without these regularity assumptions (for example, without assuming that the optimal control is left-continuous and right-continuous at each crossing time). In that direction, note that a similar sensitivity analysis can be developed in a spatially hybrid framework where Assumption ($\mathcal{A}3$) is removed, as it has been studied in the case of two static regions from a dynamic programming standpoint in [10].

(iii) As mentioned in Introduction, the original motivation of the present work was to deal with (non-hybrid) optimal control problems involving *non-control regions* (in which the control must remain constant). We insist here on the fact that this framework is just a particular case of the present work (see [14] for details) and it is motivated by applications related to aerospace, for example thrust problems with shadow zones causing inability to thrust while the spacecraft is passing through an eclipse, due to the low power generated by the solar panels (see [64], [72], [80], [118]). One could also consider a slightly different setting where, in non-control regions, the control is an affine feedback of the state (and thus is not necessarily constant). Again, this framework can be seen as a particular case of the present work and will be developed in details in a forthcoming research work.

(iv) In the field of mathematical epidemiology, hybrid frameworks provide an accurate description of some infectious diseases and their spread. We refer for instance to [88] where the authors take into account that the contact rate between members of the population changes throughout each season, or to [26] in which the authors provide a version of the SIR model that takes into consideration different control strategies (vaccination, isolation, culling, etc.). An interesting research perspective would be to consider a time crisis problem (such as in [18], [19]) related to a COVID-19 model, in order to provide better control strategies. To this aim, using the approach of optimal control problems with non-control regions presented in [14] (which is a particular case of the present work) is privileged. Moreover, since time crisis problems deal with Bolza costs with a discontinuous Lagrangian function, one can note that our main result (Theorem 2.2.1) tackles perfectly this discontinuity (see Item (viii) of Remark 2.2.6).

(v) In this work we have investigated the necessary optimality conditions for spatially hybrid optimal control problems with regionally switching parameter. However note that many other standard investigations from optimal control theory can be developed for that framework. First, one may develop existence results, by extending for example the classical Filippov theorem [65]. This would certainly require to introduce adequate differential inclusions (see, *e.g.*, [7]), in particular at the interfaces where the dynamics is not defined. Sufficient optimality conditions could also be investigated, at least in the case of LQ-problems (see related studies in [103], [104], [119] for switched systems). Also a complete extension of the Riccati theory in the present spatially hybrid setting with regionally switching parameter constitutes an attractive perspective for future works (see [37] for a related study with sampled-data controls). Finally, from a numerical point of view, another perspective could be the formulation of a multiple shooting method as in [73] taking into account the averaged Hamiltonian gradient condition (2.5).

2.3 Application to a simple academic example

The objective of this section is to show that, in some way, our framework fills a gap in the literature. Precisely, based on the necessary optimality conditions derived in Theorem 2.2.1, we solve a hybrid optimal control problem involving a regionally switching parameter and we show (see Figure 2.12) that the corresponding optimal cost is strictly between, in one hand, the best cost that can be obtained when replacing the regionally switching parameter by a classical permanent control and, in the other hand, the best cost that can be obtained when replacing the regionally switching parameter by a classical constant parameter. The example studied in this section is a simple academic example whose only purpose is to fulfill the objective of this section. In particular we emphasize that this example can probably be solved, not only from Theorem 2.2.1, but also from other results derived in the literature, or even by direct computations. The application of Theorem 2.2.1 to concrete and sophisticated application models as evoked in Items (iii) and (iv) of Remark 2.2.7, in particular to optimal control problems with loss control regions (as specified in the paper [14] which is presented in Section 2.7), will be the focus of our forthcoming research works.

2.3.1 Presentation of the example

Take $T = 8$, $n = d = m = 1$, $x_{\text{init}} = -1$ and the static partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3}$ where

$$X_1 = \left\{ y \in \mathbb{R} \mid y < -\frac{1}{2} \right\}, \quad X_2 = \left\{ y \in \mathbb{R} \mid -\frac{1}{2} < y < \frac{1}{4} \right\}, \quad X_3 = \left\{ y \in \mathbb{R} \mid y > \frac{1}{4} \right\}.$$

In this section we consider the hybrid optimal control problem with regionally switching parameter given by

$$\begin{aligned}
&\text{minimize} && -x(8), \\
&\text{subject to} && (x, \lambda, u) \in \mathrm{AC}([0,8], \mathbb{R}) \times \mathrm{PC}([0,8], \mathbb{R}) \times \mathrm{L}^\infty([0,8], \mathbb{R}), \\
& && \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0,8], \\
& && x(0) = -1, \\
& && \lambda \text{ is a regionally switching parameter associated with } x, \\
& && \lambda(t) \in [-\tfrac{3}{2}, \tfrac{3}{4}], \quad \text{a.e. } t \in [0,8], \\
& && u(t) \in [-1,1], \quad \text{a.e. } t \in [0,8],
\end{aligned} \tag{2.6}$$

where the (autonomous) hybrid dynamics $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$h(x, \lambda, u) = \begin{cases} u(x-1) + \lambda & \text{if} \quad x \in X_1, \\ \lambda x + \frac{1}{2} u & \text{if} \quad x \in X_2, \\ u(x-1) + \lambda & \text{if} \quad x \in X_3, \end{cases}$$

for all $(x, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. We refer to Figure 2.11 for an illustration of the setting of Problem (2.6) in which the objective is to maximize the final value $x(8)$ starting from the initial condition $x_{\text{init}} = -1$.

Then let us recall that the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ associated with Problem (2.6) is given by

$$H(x, \lambda, u, p) = \begin{cases} p(u(x-1) + \lambda) & \text{if} \quad x \in X_1, \\ p(\lambda x + \frac{1}{2}u) & \text{if} \quad x \in X_2, \\ p(u(x-1) + \lambda) & \text{if} \quad x \in X_3, \end{cases}$$

for all $(x, \lambda, u, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

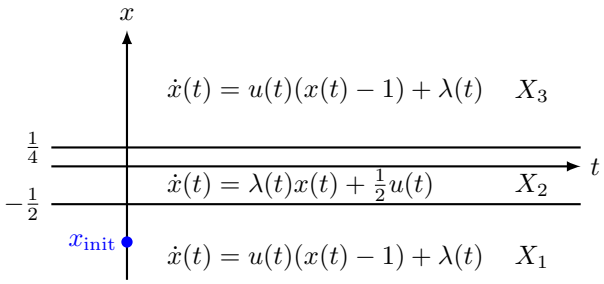


Figure 2.11: Illustration of the setting of Problem (2.6). Here the objective is to maximize the final value $x(8)$ starting from the initial condition $x_{\text{init}} = -1$.

Since existence results are out of the scope of the present work (see Item (v) of Remark 2.2.7), we assume here that (2.6) has a solution (x, λ, u) and we denote by $\mathbb{T}$ the corresponding partition. Our aim in the next section is to identify such a triple thanks to the necessary optimality conditions stated in Theorem 2.2.1. Therefore we assume furthermore that the regularity assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and $(\mathcal{A}3)$ are fulfilled. Finally, to simplify the analysis and according to the nature of the objective of (2.6), we assume that $x(8) > 1$, that x is increasing over $[0, 8]$, with exactly two crossing times $0 < t_1^c < t_2^c < 8$, and that

$$\begin{aligned} &\forall t \in (0, t_1^c), \quad (x(t), \lambda(t)) \in X_1 \times \{\lambda_1\}, \\ &\forall t \in (t_1^c, t_2^c), \quad (x(t), \lambda(t)) \in X_2 \times \{\lambda_2\}, \\ &\forall t \in (t_2^c, 8), \quad (x(t), \lambda(t)) \in X_3 \times \{\lambda_3\}, \end{aligned}$$

for some $\lambda_1, \ \lambda_2, \ \lambda_3 \in [-\frac{3}{2}, \frac{3}{4}]$.

2.3.2 Application of Theorem 2.2.1

Let us denote by $p \in \text{PAC}_{\mathbb{T}}([0, T], \mathbb{R})$ the adjoint vector provided in Theorem 2.2.1. Our aim in this section is to identify the triple (x, λ, u) thanks to the necessary optimality conditions stated in Theorem 2.2.1. To this aim we reason backward in time.

Analysis over the interval $(t_2^c, 8)$. Since $x(t) \in X_3$ over $(t_2^c, 8)$, the adjoint equation and the final condition give

$$\begin{cases} \dot{p}(t) = -u(t)p(t), & \text{a.e. } t \in (t_2^c, 8), \\ p(8) = 1, \end{cases}$$

which implies that $p(t) > 0$ over $(t_2^c, 8]$. Since the averaged Hamiltonian gradient condition writes

$$\int_{t_2^c}^{8} p(s)\,\mathrm{d}s \in \mathrm{N}_{[-\frac{3}{2},\frac{3}{4}]}(\lambda_3),$$

we deduce that $\lambda_3 = \frac{3}{4}$. Since $p(t) > 0$ over $(t_2^c, 8]$, one can easily derive from the Hamiltonian maximization condition that

$$u(t) \in \underset{v\in[-1,1]}{\arg\max}\ v(x(t)-1),$$

for almost every $t \in (t_2^c, 8)$. Since $x(8) > 1$, $x(t_2^c) = \frac{1}{4}$ and x is increasing over $[t_2^c, 8]$, there exists a unique $t^* \in (t_2^c, 8)$ such that $x(t^*) = 1$ and we obtain that

$$u(t) = \begin{cases} +1, & \text{a.e. } t \in (t^*, 8), \\ -1, & \text{a.e. } t \in (t_2^c, t^*). \end{cases}$$

We deduce that

$$p(t) = \begin{cases} e^{8-t}, & \text{for all } t \in [t^*, 8], \\ e^{8-t^*}e^{t-t^*}, & \text{for all } t \in (t_2^c, t^*], \end{cases} \quad \text{and} \quad x(t) = \begin{cases} \frac{3}{4}e^{t-t^*} + \frac{1}{4}, & \text{for all } t \in [t^*, 8], \\ -\frac{6}{4}e^{t_2^c - t} + \frac{7}{4}, & \text{for all } t \in [t_2^c, t^*]. \end{cases}$$

Finally, from the continuity of x at t^*, we get that $t^* = t_2^c + \ln(2)$.

Analysis over the interval (t_1^c, t_2^c). The adjoint discontinuity condition at t_2^c writes $p^-(t_2^c) = p^+(t_2^c)\frac{(f_3)^+(t_2^c)}{(f_2)^-(t_2^c)}$ which implies, from Assumption $(\mathcal{A}3)$, that $p^-(t_2^c) > 0$ and, from $t^* = t_2^c + \ln(2)$, that $p^-(t_2^c) = \frac{3e^{8-t_2^c}}{2\lambda_2 + 4u^-(t_2^c)}$. Since $x(t) \in X_2$ over (t_1^c, t_2^c), when adding the adjoint equation, we obtain that

$$\begin{cases} \dot{p}(t) = -\lambda_2 p(t), & \text{a.e. } t \in (t_1^c, t_2^c), \\ p^-(t_2^c) = \frac{3e^{8-t_2^c}}{2\lambda_2 + 4u^-(t_2^c)}, \end{cases}$$

which implies that $p(t) > 0$ over (t_1^c, t_2^c). The Hamiltonian maximization condition leads to

$$u(t) \in \underset{v\in[-1,1]}{\arg\max}\ vp(t),$$

and thus to $u(t) = 1$ for almost every $t \in (t_1^c, t_2^c)$. We deduce that

$$p(t) = \frac{3e^{8-t_2^c}}{2\lambda_2 + 4}e^{\lambda_2(t_2^c - t)} \quad \text{and} \quad x(t) = \begin{cases} \frac{1}{2\lambda_2}\left(e^{\lambda_2(t-t_2^c)} - 1\right) + \frac{1}{4}e^{\lambda_2(t-t_2^c)}, & \text{if } \lambda_2 \neq 0, \\ \frac{t-t_2^c}{2} + \frac{1}{4}, & \text{if } \lambda_2 = 0, \end{cases}$$

for all $t \in (t_1^c, t_2^c)$. Since $x(t_1^c) = -\frac{1}{2}$ and $\lambda_2 \in [-\frac{3}{2}, \frac{3}{4}]$, we get that

$$t_2^c - t_1^c = \begin{cases} \frac{1}{\lambda_2}\ln\left(\frac{1+\frac{\lambda_2}{2}}{1-\lambda_2}\right), & \text{if } \lambda_2 \neq 0, \\ \frac{3}{2}, & \text{if } \lambda_2 = 0. \end{cases}$$

Since $\frac{3e^{8-t_2^c}}{2\lambda_2+4} > 0$, the averaged Hamiltonian gradient condition is equivalent to $\upsilon(\lambda_2) \in \mathrm{N}_{[-\frac{3}{2},\frac{3}{4}]}(\lambda_2)$ where

$$\upsilon(\lambda_2) := \begin{cases} \displaystyle\int_{t_1^c}^{t_2^c} \frac{1}{2\lambda_2} + \frac{1}{4} - \frac{1}{2\lambda_2} e^{\lambda_2(t_2^c - s)} \,\mathrm{d}s = \frac{1}{2\lambda_2^2}\left(1 + \frac{\lambda_2}{2}\right) \ln\left(\frac{1+\frac{\lambda_2}{2}}{1-\lambda_2}\right) - \frac{3}{4\lambda_2(1-\lambda_2)}, & \text{if } \lambda_2 \neq 0, \\ \displaystyle\int_{t_1^c}^{t_2^c} \frac{s - t_2^c}{2} + \frac{1}{4} \,\mathrm{d}s = -\frac{3}{16}, & \text{if } \lambda_2 = 0. \end{cases}$$

We find that:

- if $\lambda_2 = \frac{3}{4}$, then $\upsilon(\lambda_2) \simeq -1.916 < 0$, while $\mathrm{N}_{[-\frac{3}{2},\frac{3}{4}]}(\lambda_2) = \mathbb{R}_+$, which is a contradiction.
- if $\lambda_2 = -\frac{3}{2}$, then $\upsilon(\lambda_2) \simeq 0.072 > 0$, while $\mathrm{N}_{[-\frac{3}{2},\frac{3}{4}]}(\lambda_2) = \mathbb{R}_-$, which is a contradiction.

We deduce that $\lambda_2 \in (-\frac{3}{2}, \frac{3}{4})$ and thus $\mathrm{N}_{[-\frac{3}{2},\frac{3}{4}]}(\lambda_2) = \{0\}$. Solving the equation $\upsilon(\lambda_2) = 0$ over $(-\frac{3}{2}, \frac{3}{4})$, we find that $\lambda_2 \simeq -0.754$.

Analysis over the interval $(0, t_1^c)$. The adjoint discontinuity condition at t_1^c writes $p^-(t_1^c) = p^+(t_1^c)\frac{(f_2)^+(t_1^c)}{(f_1)^-(t_1^c)}$ which implies, from Assumption $(\mathcal{A}3)$, that $p^-(t_1^c) > 0$ and, from $t_2^c - t_1^c = \frac{1}{\lambda_2}\ln(\frac{1+\frac{\lambda_2}{2}}{1-\lambda_2})$, one can obtain that $p^-(t_1^c) = \frac{3e^{8-t_2^c}}{8(\lambda_1 - \frac{3}{2}u^-(t_1^c))}$. Since $x(t) \in X_1$ over $(0, t_1^c)$, when adding the adjoint equation, we obtain that

$$\begin{cases} \dot{p}(t) = -u(t)p(t), & \text{a.e. } t \in (0, t_1^c), \\ p^-(t_1^c) = \frac{3e^{8-t_2^c}}{8(\lambda_1 - \frac{3}{2}u^-(t_1^c))}. \end{cases}$$

Following similar arguments as in the analysis over the interval $(t_2^c, 8)$, one can prove that $\lambda_1 = \frac{3}{4}$ and $u(t) = -1$ for almost every $t \in (0, t_1^c)$.

Conclusion. From the above analysis we obtain that

$$x(t) = \begin{cases} -\frac{11}{4}e^{-t} + \frac{7}{4}, & \text{for all } t \in [0, t_1^c], \\ -\frac{1}{2\lambda_2}((\lambda_2 - 1)e^{\lambda_2(t - t_1^c)} + 1), & \text{for all } t \in [t_1^c, t_2^c], \\ -\frac{6}{4}e^{-(t - t_2^c)} + \frac{7}{4}, & \text{for all } t \in [t_2^c, t^*], \\ +\frac{3}{4}e^{t - t^*} + \frac{1}{4}, & \text{for all } t \in [t^*, 8], \end{cases}$$

and

$$\lambda(t) = \begin{cases} \frac{3}{4}, & \text{for a.e. } t \in (0, t_1^c), \\ \lambda_2, & \text{for a.e. } t \in (t_1^c, t_2^c), \\ \frac{3}{4}, & \text{for a.e. } t \in (t_2^c, 8), \end{cases} \qquad u(t) = \begin{cases} -1, & \text{for a.e. } t \in (0, t_1^c), \\ +1, & \text{for a.e. } t \in (t_1^c, t_2^c), \\ -1, & \text{for a.e. } t \in (t_2^c, t^*), \\ +1, & \text{for a.e. } t \in (t^*, 8). \end{cases}$$

with $\lambda_2 \simeq -0.754$, $t_1^c = \ln(\frac{11}{9}) \simeq 0.2$, $t_2^c = t_1^c + \frac{1}{\lambda_2}\ln(\frac{1+\frac{\lambda_2}{2}}{1-\lambda_2}) \simeq 1.57$ and $t^* = t_2^c + \ln(2) \simeq 2.26$.

2.3.3 Comparisons with standard settings found in the literature

Our objective in this section is to emphasize that, in some way, our work fills a gap in the literature. To this aim we will show on the present academic example that the optimal trajectory x computed in the previous section (associated with a regionally switching parameter λ) is exactly between the optimal trajectory $x^\dagger$ when λ is considered as a classical permanent control (that is, when $\lambda \in \mathrm{L}^\infty([0,8], \mathbb{R})$), and the optimal

trajectory $\hat{x}$ when λ is considered as a classical constant parameter (that is, when $\lambda \in \mathbb{R}$). Precisely:

- First, let us consider the case where $\lambda \in \mathrm{L}^\infty([0,8],\mathbb{R})$ is a classical permanent control in Problem (2.6) (and not a regionally switching parameter that has to remain constant in each region) constrained to be with values in $[-\frac{3}{2},\frac{3}{4}]$. By developing a similar analysis to the previous section, but by using a Hamiltonian maximization condition to determine the values $\lambda(t)$ over the whole interval $[0,T]$, we get that the optimal triple $(x^\dagger,\lambda^\dagger,u^\dagger)$ is given by

$$x^\dagger(t) = \begin{cases} -\frac{11}{4}e^{-t} + \frac{7}{4}, & \text{for all } t \in [0,(t_1^c)^\dagger], \\ -\frac{5}{6}e^{-\frac{3}{2}(t-(t_1^c)^\dagger)} + \frac{1}{3}, & \text{for all } t \in [(t_1^c)^\dagger,(t_1^*)^\dagger], \\ +\frac{2}{3}e^{\frac{3}{4}(t-(t_1^*)^\dagger)} - \frac{2}{3}, & \text{for all } t \in [(t_1^*)^\dagger,(t_2^c)^\dagger], \\ -\frac{3}{2}e^{-(t-(t_2^c)^\dagger)} + \frac{7}{4}, & \text{for all } t \in [(t_2^c)^\dagger,(t_2^*)^\dagger], \\ +\frac{3}{4}e^{t-(t_2^*)^\dagger} + \frac{1}{4}, & \text{for all } t \in [(t_2^*)^\dagger,8], \end{cases}$$

  and

$$\lambda^\dagger(t) = \begin{cases} +\frac{3}{4}, & \text{for a.e. } t \in (0,(t_1^c)^\dagger), \\ -\frac{3}{2}, & \text{for a.e. } t \in ((t_1^c)^\dagger,(t_1^*)^\dagger), \\ +\frac{3}{4}, & \text{for a.e. } t \in ((t_1^*)^\dagger,(t_2^c)^\dagger), \\ -\frac{3}{2}, & \text{for a.e. } t \in ((t_2^c)^\dagger,(t_2^*)^\dagger), \\ +\frac{3}{4}, & \text{for a.e. } t \in ((t_2^*)^\dagger,8), \end{cases} \qquad u^\dagger(t) = \begin{cases} -1, & \text{for a.e. } t \in (0,(t_1^c)^\dagger), \\ +1, & \text{for a.e. } t \in ((t_1^c)^\dagger,(t_2^c)^\dagger), \\ -1, & \text{for a.e. } t \in ((t_2^c)^\dagger,(t_2^*)^\dagger), \\ +1, & \text{for a.e. } t \in ((t_2^*)^\dagger,8). \end{cases}$$

  with $(t_1^c)^\dagger = t_1^c \simeq 0.2$, $(t_1^*)^\dagger \simeq 0.81$, $(t_2^c)^\dagger \simeq 1.23$, $(t_2^*)^\dagger \simeq 2.07$. The detailed computations are left to the reader.
- Second, let us consider the case where $\lambda \in \mathbb{R}$ is a classical constant parameter in Problem (2.6) (that cannot switch at boundary crossings) constrained to belong to $[-\frac{3}{2},\frac{3}{4}]$. By developing a similar analysis to the previous section, but by using the averaged Hamiltonian gradient condition given in Item (ix) of Remark 2.2.6, we get that the optimal triple $(\hat{x},\hat{\lambda},\hat{u})$ is given by

$$\hat{x}(t) = \begin{cases} -\frac{11}{4}e^{-t} + \frac{7}{4}, & \text{for all } t \in [0,\hat{t_1^c}], \\ +\frac{5}{3}e^{\frac{3}{4}(t-\hat{t_1^c})} - \frac{2}{3}, & \text{for all } t \in [\hat{t_1^c},\hat{t_2^c}], \\ -\frac{3}{2}e^{-(t-\hat{t_2^c})} + \frac{7}{4}, & \text{for all } t \in [\hat{t_2^c},\hat{t_2^*}], \\ -\frac{3}{4}e^{-(t-\hat{t^*})} + 7/4, & \text{for all } t \in [\hat{t_2^*},8], \end{cases}$$

  and

$$\hat{\lambda} = \frac{3}{4}, \qquad \hat{u}(t) = \begin{cases} -1, & \text{for a.e. } t \in (0,\hat{t_1^c}), \\ +1, & \text{for a.e. } t \in (\hat{t_1^c},\hat{t_2^c}), \\ -1, & \text{for a.e. } t \in (\hat{t_2^c},\hat{t^*}), \\ +1, & \text{for a.e. } t \in (\hat{t^*},8). \end{cases}$$

  with $\hat{t_1^c} = t_1^c \simeq 0.2$, $\hat{t_2^c} \simeq 2.47$, and $\hat{t^*} = 3.16$. The detailed computations are left to the reader.

We refer to Figure 2.12 for the plots of the three trajectories $x^\dagger$, x and $\hat{x}$. As expected, the trajectory $x^\dagger$ (associated with a classical permanent control) provides a better cost than the trajectory x (associated with a regionally switching parameter) which provides a better cost than the trajectory $\hat{x}$ (associated with a classical constant parameter). This figure emphasizes that, in some way, our

framework fills a gap in the literature.

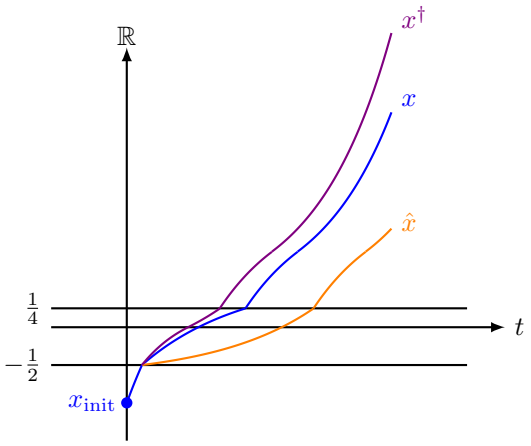


Figure 2.12: Trajectories $x^\dagger$, x and $\hat{x}$ (zoom on the time interval $[0, \frac{7}{2}]$).

2.4 Preliminaries: sensitivity analysis in the non-hybrid context

As explained in the overview of the proof of Theorem 2.2.1 developed in Section 2.2.3, for the needs of the sensitivity analysis in the hybrid context performed in the next Section 2.5, we need to provide a complete sensitivity analysis of a general non-hybrid parameterized control system with respect to perturbations of the parameter, the control, the initial time and the initial condition. This is precisely the content of the present section. We will work on the time interval $[0,T]$. Nevertheless our results can be trivially extended to any compact time interval $[a,b]$ with $a<b$. In fact note that we will use these results in the next Section 2.5 on compact subintervals of $[0,T]$.
Let $g:\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^m\times[0,T]\to\mathbb{R}^n$ be a general (non-hybrid) dynamics of class C^1. For any quadruplet $\theta=(\lambda,u,r,y_r)\in\mathbb{R}^d\times\mathrm{L}^\infty([0,T],\mathbb{R}^m)\times[0,T]\times\mathbb{R}^n$, the Cauchy-Lipschitz theorem ensures the existence and the uniqueness of the maximal solution to the Cauchy problem

$$\begin{cases} \dot{y}(t)=g(y(t),\lambda,u(t),t), & \text{a.e. } t\in[0,T],\\ y(r)=y_r. \end{cases}$$

This maximal solution is denoted by $y(\cdot,g,\theta)$ and is defined over the maximal interval denoted by $I(g,\theta)\subset[0,T]$. Recall that the blow-up theorem ensures that, either $I(g,\theta)=[0,T]$ (in that case we speak of a global solution), either $y(\cdot,g,\theta)$ is unbounded over $I(g,\theta)$. In the sequel we denote by $\mathrm{Glob}(g)$ the set of all quadruplets θ such that $I(g,\theta)=[0,T]$.
For the technical needs of this section, for any quadruplet $\theta=(\lambda,u,r,y_r)\in\mathrm{Glob}(g)$ and any $R\geq\|u\|_{\mathrm{L}^\infty}$, we denote by $M(g,\theta,R)\geq 0$ a common bound of $\|g\|_{\mathbb{R}^n}$, $\|\nabla_x g\|_{\mathbb{R}^{n\times n}}$, $\|\nabla_\lambda g\|_{\mathbb{R}^{n\times d}}$ and $\|\nabla_u g\|_{\mathbb{R}^{n\times m}}$ over the compact set

$$K(g,\theta,R):=\Big\{(x,\mu,v,t)\in\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^m\times[0,T]\;|\;\|x-y(t,g,\theta)\|_{\mathbb{R}^n}\leq 1,\ \|\mu-\lambda\|_{\mathbb{R}^d}\leq 1,\ \|v\|_{\mathbb{R}^m}\leq R\Big\}.$$

Note that $(y(t,g,\theta),\lambda,u(t),t)\in K(g,\theta,R)$ for almost every $t\in[0,T]$. Since $K(g,\theta,R)$ is convex with

respect to its first three components, one can easily get that

$$\|g(y_2,\mu_2,v_2,t)-g(y_1,\mu_1,v_1,t)\|_{\mathbb{R}^n}\leq M(g,\theta,R)(\|y_2-y_1\|_{\mathbb{R}^n}+\|\mu_2-\mu_1\|_{\mathbb{R}^d}+\|v_2-v_1\|_{\mathbb{R}^m}),$$

for all $(y_2,\mu_2,v_2,t),(y_1,\mu_1,v_1,t)\in K(g,\theta,R)$.
We are now in a position to state and prove the next continuous dependence result for the trajectory $y(\cdot,g,\theta)$ with respect to the quadruplet θ.

Lemma 2.4.1. *For any quadruplet $\theta=(\lambda,u,r,y_r)\in \mathrm{Glob}(g)$ and any $R\geq\|u\|_{\mathrm{L}^\infty}$, there exists $\varepsilon>0$ such that the neighborhood of θ given by*

$$\mathcal{N}(g,\theta,R,\varepsilon):=\overline{\mathrm{B}}_{\mathbb{R}^d}(\lambda,\varepsilon)\times\Big(\overline{\mathrm{B}}_{\mathrm{L}^1}(u,\varepsilon)\cap\overline{\mathrm{B}}_{\mathrm{L}^\infty}(0_{\mathrm{L}^\infty},R)\Big)\times\Big([r-\varepsilon,r+\varepsilon]\cap[0,T]\Big)\times\overline{\mathrm{B}}_{\mathbb{R}^n}(y_r,\varepsilon),$$

is included in $\mathrm{Glob}(g)$. *Furthermore, for all quadruplets $\theta'=(\lambda',u',r',y'_r)\in\mathcal{N}(g,\theta,R,\varepsilon)$, it holds that $(y(t,g,\theta'),\lambda',u'(t),t)\in K(g,\theta,R)$ for almost every $t\in[0,T]$. Finally the map*

$$\begin{array}{rccc}\mathcal{F}:&\mathcal{N}(g,\theta,R,\varepsilon)&\to&\mathrm{C}([0,T],\mathbb{R}^n)\\&\theta'&\mapsto&y(\cdot,g,\theta'),\end{array}$$

is Lipschitz continuous, in the sense that there exists $L(g,\theta,R)\geq 0$ such that

$$\|y(\cdot,g,\theta'')-y(\cdot,g,\theta')\|_{\mathrm{C}}\leq L(g,\theta,R)(\|\lambda''-\lambda'\|_{\mathbb{R}^d}+\|u''-u'\|_{\mathrm{L}^1}+|r''-r'|+\|y''_r-y'_r\|_{\mathbb{R}^n}),$$

for all $\theta'=(\lambda',u',r',y'_r)$, $\theta''=(\lambda'',u'',r'',y''_r)\in\mathcal{N}(g,\theta,R,\varepsilon)$.

Proof. Let $\theta=(\lambda,u,r,y_r)\in\mathrm{Glob}(g)$ and $R\geq\|u\|_{\mathrm{L}^\infty}$. In this proof, for the ease of notations, we denote by $M:=M(g,\theta,R)$. Let us fix $\varepsilon>0$ such that $\varepsilon(1+M(2+T))e^{MT}<1$ and let us prove that $\mathcal{N}(g,\theta,R,\varepsilon)\subset\mathrm{Glob}(g)$. To this aim let $\theta'=(\lambda',u',r',y'_r)\in\mathcal{N}(g,\theta,R,\varepsilon)$ and introduce the sets

$$\mathcal{I}_1:=\{t\in I(g,\theta')\cap[0,r']\mid\|y(t,g,\theta')-y(t,g,\theta)\|_{\mathbb{R}^n}>1\}$$
$$\text{and}\quad\mathcal{I}_2:=\{t\in I(g,\theta')\cap[r',T]\mid\|y(t,g,\theta')-y(t,g,\theta)\|_{\mathbb{R}^n}>1\}.$$

If $\mathcal{I}_1\cup\mathcal{I}_2=\emptyset$, then the solution $y(\cdot,g,\theta')$ is bounded over $I(g,\theta')$, and thus $\theta'\in\mathrm{Glob}(g)$ from the blow-up theorem. Therefore, by contradiction, let us assume that $\mathcal{I}_1\cup\mathcal{I}_2\neq\emptyset$. In the sequel we only deal with the case $\mathcal{I}_2\neq\emptyset$ (the case where $\mathcal{I}_2=\emptyset$, and thus $\mathcal{I}_1\neq\emptyset$, is similar). From integral representations it holds that

$$y(t,g,\theta')-y(t,g,\theta)=(y'_r-y_r)+\int_{r'}^{t}g(y(s,g,\theta'),\lambda',u'(s),s)-g(y(s,g,\theta),\lambda,u(s),s)\,\mathrm{d}s$$
$$-\int_{r}^{r'}g(y(s,g,\theta),\lambda,u(s),s)\,\mathrm{d}s,$$

for all $t\in I(g,\theta')$. Now let $t_2:=\inf\mathcal{I}_2\geq r'$. From continuity and definition of t_2 we know that

$\|y(t_2,g,\theta')-y(t_2,g,\theta)\|_{\mathbb{R}^n}\geq 1$ and thus $r'<t_2$ since

$$\|y(r',g,\theta')-y(r',g,\theta)\|_{\mathbb{R}^n}\leq \|y'_r-y_r\|_{\mathbb{R}^n}+\left|\int_r^{r'}\|g(y(s,g,\theta),\lambda,u(s),s)\|_{\mathbb{R}^n}\,\mathrm{d}s\right|$$
$$\leq \|y'_r-y_r\|_{\mathbb{R}^n}+M|r'-r|\leq \varepsilon(1+M)<1.$$

From definition of t_2 we deduce that $\|y(t,g,\theta')-y(t,g,\theta)\|_{\mathbb{R}^n}\leq 1$ for all $t\in[r',t_2]$. Therefore, since moreover $\|\lambda'-\lambda\|_{\mathbb{R}^d}\leq\varepsilon<1$ and $\|u'\|_{\mathrm{L}^\infty}\leq R$, we deduce that $(y(t,g,\theta'),\lambda',u'(t),t)\in K(g,\theta,R)$ for almost every $t\in[r',t_2]$. Hence, from integral representations, we get that

$$\begin{aligned}&\|y(t,g,\theta')-y(t,g,\theta)\|_{\mathbb{R}^n}\\&\quad\leq \|y'_r-y_r\|_{\mathbb{R}^n}+M|r'-r|+M\int_{r'}^t\|y(s,g,\theta')-y(s,g,\theta)\|_{\mathbb{R}^n}+\|\lambda'-\lambda\|_{\mathbb{R}^d}+\|u'(s)-u(s)\|_{\mathbb{R}^m}\,\mathrm{d}s\\&\quad\leq \|y'_r-y_r\|_{\mathbb{R}^n}+M|r'-r|+M\int_{r'}^t\|y(s,g,\theta')-y(s,g,\theta)\|_{\mathbb{R}^n}\,\mathrm{d}s+MT\|\lambda'-\lambda\|_{\mathbb{R}^d}+M\|u'-u\|_{\mathrm{L}^1},\end{aligned}$$

for all $t\in[r',t_2]$. From the Grönwall lemma we obtain that

$$\|y(t,g,\theta')-y(t,g,\theta)\|_{\mathbb{R}^n}\leq(\|y'_r-y_r\|_{\mathbb{R}^n}+M|r'-r|+MT\|\lambda'-\lambda\|_{\mathbb{R}^d}+M\|u'-u\|_{\mathrm{L}^1})e^{MT}$$
$$\leq\varepsilon(1+M(2+T))e^{MT}<1,$$

for all $t\in[r',t_2]$, which raises a contradiction at $t=t_2$. Thus we have proved that $\mathcal{I}_1\cup\mathcal{I}_2=\emptyset$ which gives $\theta'\in\mathrm{Glob}(g)$ but also $(y(t,g,\theta'),\lambda',u'(t),t)\in K(g,\theta,R)$ for almost every $t\in[0,T]$. Hence the proofs of the first two parts of Lemma 2.4.1 are complete. Now let us prove the last part. To this aim let $\theta'=(\lambda',u',r',y'_r)$, $\theta''=(\lambda'',u'',r'',y''_r)\in\mathcal{N}(g,\theta,R,\varepsilon)$. From integral representations it holds that

$$y(t,g,\theta'')-y(t,g,\theta')=(y''_r-y'_r)+\int_{r''}^t g(y(s,g,\theta''),\lambda'',u''(s),s)-g(y(s,g,\theta'),\lambda',u'(s),s)\,\mathrm{d}s$$
$$-\int_{r'}^{r''}g(y(s,g,\theta'),\lambda',u'(s),s)\,\mathrm{d}s,$$

for all $t\in[0,T]$. Using similar arguments than before (in particular using the Grönwall lemma), we get that

$$\|y(t,g,\theta'')-y(t,g,\theta')\|_{\mathbb{R}^n}\leq(\|y_{r''}-y_{r'}\|_{\mathbb{R}^n}+M|r''-r'|+MT\|\lambda''-\lambda'\|_{\mathbb{R}^d}+M\|u''-u'\|_{\mathrm{L}^1})e^{MT},$$

for all $t\in[0,T]$, which concludes the proof of the last part of Lemma 2.4.1. □

In the next proposition we state a differentiability result for the trajectory $y(\cdot,g,\theta)$ with respect to perturbations of the quadruplet $\theta\in\mathrm{Glob}(g)$. As explained in the overview of the proof of Theorem 2.2.1 developed in Section 2.2.3, this proposition will be useful in the next Section 2.5 to construct perturbed trajectories of the hybrid control system (2.1) which visit exactly (and in the same order) the same regions than a given nominal trajectory.

Proposition 2.4.1. *Consider the perturbation of a quadruplet $\theta=(\lambda,u,r,y_r)\in\mathrm{Glob}(g)$ given by $\tilde\theta(\alpha):=(\tilde\lambda(\alpha),\tilde u(\alpha),\tilde r(\alpha),\tilde y_r(\alpha))$ for all $\alpha\in[0,1]$ where:*

- $\tilde{\lambda} : [0,1] \to \mathbb{R}^d$ *satisfies* $\tilde{\lambda}(0) = \lambda$ *and is differentiable at* 0 *with derivative denoted by* $\tilde{\lambda}'(0)$.
- *either* $\tilde{u} : [0,1] \to \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *is given by* $\tilde{u}(\alpha) := u$ *for all* $\alpha \in [0,1]$ *(no perturbation of the control), either* $\tilde{u} : [0,1] \to \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *is the needle-like perturbation of* u *given by*

$$\tilde{u}(\alpha)(t) := \begin{cases} v & \text{if } t \in [\tau-\alpha,\tau), \\ u(t) & \text{if } t \notin [\tau-\alpha,\tau), \end{cases} \tag{2.7}$$

  for almost every $t \in [0,T]$ *and all* $\alpha \in [0,1]$*, where* $v \in \mathbb{R}^m$ *and* $\tau \in (0,T]$ *is a Lebesgue point of the map* $g(y(\cdot,g,\theta),\lambda,u(\cdot),\cdot)$.
- *either* $\tilde{r} : [0,1] \to [0,T]$ *is constantly equal to* r *(no perturbation of the initial time), either* $\tilde{r} : [0,1] \to [0,T]$ *satisfies* $\tilde{r}(0) = r$ *and is differentiable at* 0 *with derivative denoted by* $\tilde{r}'(0)$ *(in that second context, assume that* $r \in [0,T)$ *is a Lebesgue point of the map* $g(y(\cdot,g,\theta),\lambda,u(\cdot),\cdot)$ *and, in case of needle-like perturbation of the control, assume furthermore that* $r \neq \tau$*).*
- $\tilde{y}_r : [0,1] \to \mathbb{R}^n$ *satisfies* $\tilde{y}_r(0) = y_r$ *and is differentiable at* 0 *with derivative denoted by* $\tilde{y}_r'(0)$.

Then:

(i) *There exists* $0 < \overline{\alpha} \leq 1$ *such that* $\tilde{\theta}(\alpha) \in \mathrm{Glob}(g)$ *for all* $\alpha \in [0,\overline{\alpha}]$.

(ii) *The perturbed trajectory* $y(\cdot,g,\tilde{\theta}(\alpha))$ *uniformly converges to* $y(\cdot,g,\theta)$ *over* $[0,T]$ *when* $\alpha \to 0$.

(iii) *The map*

$$\begin{array}{rccc} \mathcal{P} : & [0,\overline{\alpha}] & \to & \mathrm{C}([\varsigma,T],\mathbb{R}^n) \\ & \alpha & \mapsto & y(\cdot,g,\tilde{\theta}(\alpha)), \end{array}$$

with $\varsigma := \tau$ *in case of needle-like perturbation of the control and* $\varsigma := 0$ *otherwise, is differentiable at* 0 *and its derivative is equal to* $w_{\tilde{\lambda}} + w_{\tilde{u}} + w_{(\tilde{r},\tilde{y_r})}$*, where* $w_{\tilde{\lambda}}$*,* $w_{\tilde{u}}$ *and* $w_{(\tilde{r},\tilde{y_r})}$ *are the three variation vectors respectively defined as the unique maximal solutions (which are global) to the three linearized Cauchy problems given by*

$$\begin{cases} \dot{w}(t) = \nabla_x g(y(t,g,\theta),\lambda,u(t),t)w(t) + \nabla_\lambda g(y(t,g,\theta),\lambda,u(t),t)\tilde{\lambda}'(0), & \text{a.e. } t \in [0,T], \\ w(r) = 0_{\mathbb{R}^n}, \end{cases}$$

$$\begin{cases} \dot{w}(t) = \nabla_x g(y(t,g,\theta),\lambda,u(t),t)w(t), & \text{a.e. } t \in [0,T], \\ w(\tau) = g(y(\tau,g,\theta),\lambda,v,\tau) - g(y(\tau,g,\theta),\lambda,u(\tau),\tau), \end{cases}$$

$$\begin{cases} \dot{w}(t) = \nabla_x g(y(t,g,\theta),\lambda,u(t),t)w(t), & \text{a.e. } t \in [0,T], \\ w(r) = \tilde{y}_r'(0) - \tilde{r}'(0)g(y(r,g,\theta),\lambda,u(r),r). \end{cases}$$

(iv) *If furthermore the three functions* $\tilde{\lambda}$*,* $\tilde{y}_r$ *and* $\tilde{r}$ *are assumed to be continuous over* $[0,1]$*, then the map* $(\alpha,t) \in [0,\overline{\alpha}] \times [0,T] \mapsto y(t,g,\tilde{\theta}(\alpha)) \in \mathbb{R}^n$ *is continuous.*

Proof. This proof is dedicated to the case of a needle-like perturbation of the control and of a perturbation of the initial time (the other cases are similar and simpler). Let $R \geq \|u\|_{\mathrm{L}^\infty} + \|v\|_{\mathbb{R}^m}$. As in the proof of Lemma 2.4.1, we denote by $M := M(g,\theta,R)$. Consider $\varepsilon > 0$ provided in Lemma 2.4.1. It is clear that $\tilde{\theta}(\alpha) \in \mathcal{N}(g,\theta,R,\varepsilon)$ for sufficiently small $\alpha > 0$. As a consequence, from Lemma 2.4.1, there exists $0 < \overline{\alpha} \leq 1$ such that $\tilde{\theta}(\alpha) \in \mathrm{Glob}(g)$ for all $\alpha \in [0,\overline{\alpha}]$ which concludes the proof of the first item. The second and fourth items are trivial consequences of the Lipschitz continuity provided in Lemma 2.4.1.

Now our aim is to prove the third item. To this aim let us introduce

$$\chi^\alpha(t) := \frac{y(t,g,\tilde{\theta}(\alpha)) - y(t,g,\theta)}{\alpha} - w_{\tilde{\lambda}}(t) - w_{\tilde{u}}(t) - w_{(\tilde{r},\tilde{y_r})}(t),$$

for all $t \in [0,T]$ and all $\alpha \in (0,\overline{\alpha}]$. Our aim is to prove that χ^α uniformly converges to zero over $[\tau,T]$ when $\alpha \to 0$. To this aim we write $\chi^\alpha = \chi_1^\alpha + \chi_2^\alpha + \chi_3^\alpha$ where

$$\chi_1^\alpha(t) := \frac{\tilde{y}^\alpha(t) - \tilde{y}_1^\alpha(t)}{\alpha} - w_{\tilde{\lambda}}(t), \quad \chi_2^\alpha(t) := \frac{\tilde{y}_2^\alpha(t) - y(t)}{\alpha} - w_{\tilde{u}}(t), \quad \chi_3^\alpha(t) := \frac{\tilde{y}_1^\alpha(t) - \tilde{y}_2^\alpha(t)}{\alpha} - w_{(\tilde{r},\tilde{y_r})}(t),$$

for all $t \in [0,T]$ and all $\alpha \in (0,\overline{\alpha}]$, where we use the notations

$$\tilde{y}^\alpha(t) := y(t,g,\tilde{\theta}(\alpha)), \quad \tilde{y}_1^\alpha(t) := y(t,g,\tilde{\theta}_1(\alpha)), \quad \tilde{y}_2^\alpha(t) := y(t,g,\tilde{\theta}_2(\alpha)), \quad y(t) := y(t,g,\theta),$$

and

$$\tilde{\theta}_1(\alpha) := (\lambda, \tilde{u}(\alpha), \tilde{r}(\alpha), \tilde{y}_r(\alpha)) \in \mathcal{N}(g,\theta,R,\varepsilon) \quad \text{and} \quad \tilde{\theta}_2(\alpha) := (\lambda, \tilde{u}(\alpha), r, y_r) \in \mathcal{N}(g,\theta,R,\varepsilon),$$

for all $t \in [0,T]$ and all $\alpha \in [0,\overline{\alpha}]$. From Lemma 2.4.1, for almost every $t \in [0,T]$ and all $\alpha \in [0,\overline{\alpha}]$, the five elements

$$(y(t),\lambda,u(t),t), \quad (\tilde{y}^\alpha(t),\tilde{\lambda}(\alpha),u(\alpha)(t),t), \quad (\tilde{y}^\alpha(t),\lambda,u(\alpha)(t),t), \quad (\tilde{y}_1^\alpha(t),\lambda,\tilde{u}(\alpha)(t),t), \quad (\tilde{y}_2^\alpha(t),\lambda,\tilde{u}(\alpha)(t),t),$$

belong to $K(g,\theta,R)$, but also their convex combinations. Also note that $\tilde{y}^\alpha$, $\tilde{y}_1^\alpha$ and $\tilde{y}_2^\alpha$ uniformly converge to y over $[0,T]$ when $\alpha \to 0$ from the Lipschitz continuity provided in Lemma 2.4.1.
In what follows, as in the proof of Lemma 2.4.1, we will use integral representations and the Grönwall lemma to prove that χ_1^α, χ_2^α and χ_3^α uniformly converge to zero over $[\tau,T]$ when $\alpha \to 0$. To reduce the notation in integrands, we will use the notation $\rho(\cdot) := (y(\cdot,g,\theta),\lambda,u(\cdot),\cdot)$.

Step 1: Let us prove that χ_1^α uniformly converges to zero over $[0,T]$ when $\alpha \to 0$. From integral representations it holds that

$$\begin{aligned}\chi_1^\alpha(t) = \chi_1^\alpha(r) &+ \int_r^t \frac{g(\tilde{y}^\alpha(s),\tilde{\lambda}(\alpha),\tilde{u}(\alpha)(s),s) - g(\tilde{y}^\alpha(s),\lambda,\tilde{u}(\alpha)(s),s)}{\alpha} - \nabla_\lambda g(\rho(s))\tilde{\lambda}'(0)\,\mathrm{d}s \\ &+ \int_r^t \frac{g(\tilde{y}^\alpha(s),\lambda,\tilde{u}(\alpha)(s),s) - g(\tilde{y}_1^\alpha(s),\lambda,\tilde{u}(\alpha)(s),s)}{\alpha} - \nabla_x g(\rho(s))w_{\tilde{\lambda}}(s)\,\mathrm{d}s,\end{aligned}$$

for all $t \in [0,T]$ and all $\alpha \in (0,\overline{\alpha}]$. Using Taylor expansions with integral rest, we obtain that

$$\|\chi_1^\alpha(t)\|_{\mathbb{R}^n} \leq \|\chi_1^\alpha(r)\|_{\mathbb{R}^n} + \underbrace{\int_0^T \int_0^1 \left\|\nabla_\lambda g(\tilde{y}^\alpha(s), \lambda + \eta(\tilde{\lambda}(\alpha) - \lambda), \tilde{u}(\alpha)(s), s)\right\|_{\mathbb{R}^{n\times d}} \left|\frac{\tilde{\lambda}(\alpha) - \lambda}{\alpha} - \tilde{\lambda}'(0)\right| \mathrm{d}\eta\, \mathrm{d}s}_{\Gamma_1(\alpha)}$$
$$+ \underbrace{\int_0^T \int_0^1 \left\|\nabla_\lambda g(\tilde{y}^\alpha(s), \lambda + \eta(\tilde{\lambda}(\alpha) - \lambda), \tilde{u}(\alpha)(s), s) - \nabla_\lambda g(\rho(s))\right\|_{\mathbb{R}^{n\times d}} |\tilde{\lambda}'(0)|\, \mathrm{d}\eta\, \mathrm{d}s}_{\Gamma_2(\alpha)}$$
$$+ \left| \int_r^t \int_0^1 \|\nabla_x g(\tilde{y}_1^\alpha(s) + \eta(\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)), \lambda, \tilde{u}(\alpha)(s), s)\|_{\mathbb{R}^{n\times n}} \left\| \underbrace{\frac{\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)}{\alpha} - w_{\tilde{\lambda}}(s)}_{\chi_1^\alpha(s)} \right\|_{\mathbb{R}^n} \mathrm{d}\eta\, \mathrm{d}s \right|$$
$$+ \underbrace{\int_0^T \int_0^1 \|\nabla_x g(\tilde{y}_1^\alpha(s) + \eta(\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)), \lambda, \tilde{u}(\alpha)(s), s) - \nabla_x g(\rho(s))\|_{\mathbb{R}^{n\times n}} \|w_{\tilde{\lambda}}(s)\|_{\mathbb{R}^n}\, \mathrm{d}\eta\, \mathrm{d}s}_{\Gamma_3(\alpha)},$$

for all $t \in [0, T]$ and all $\alpha \in (0, \overline{\alpha}]$. The Grönwall lemma leads to

$$\|\chi_1^\alpha(t)\|_{\mathbb{R}^n} \leq \Big(\|\chi_1^\alpha(r)\|_{\mathbb{R}^n} + \Gamma_1(\alpha) + \Gamma_2(\alpha) + \Gamma_3(\alpha)\Big) e^{MT},$$

for all $t \in [0, T]$ and all $\alpha \in (0, \overline{\alpha}]$. From the differentiability of $\tilde{\lambda}(\cdot)$ at 0, the boundedness of $\|\nabla_x g\|_{\mathbb{R}^{n\times n}}$ and $\|\nabla_\lambda g\|_{\mathbb{R}^{n\times d}}$ over $K(g, \theta, R)$, the uniform convergences of $\tilde{y}^\alpha$ and $\tilde{y}_1^\alpha$ to y over $[0, T]$ when $\alpha \to 0$ and from the dominated convergence theorem, we prove that $\Gamma_1(\alpha), \Gamma_2(\alpha)$ and $\Gamma_3(\alpha)$ converge to zero when $\alpha \to 0$. It remains to prove that $\|\chi_1^\alpha(r)\|_{\mathbb{R}^n}$ converges to zero when $\alpha \to 0$. From integral representations it holds that

$$\chi_1^\alpha(r) = \frac{1}{\alpha} \int_{\tilde{r}(\alpha)}^r g(\tilde{y}^\alpha(s), \tilde{\lambda}(\alpha), \tilde{u}(\alpha)(s), s) - g(\tilde{y}_1^\alpha(s), \lambda, \tilde{u}(\alpha)(s), s)\, \mathrm{d}s,$$

for all $\alpha \in (0, \overline{\alpha}]$, and, using similar arguments than before, we obtain that

$$\|\chi_1^\alpha(r)\|_{\mathbb{R}^n} \leq \frac{M}{\alpha} \left| \int_{\tilde{r}(\alpha)}^r \|\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)\|_{\mathbb{R}^n} + \|\tilde{\lambda}(\alpha) - \lambda\|_{\mathbb{R}^d}\, \mathrm{d}s \right|$$
$$\leq M \left| \frac{\tilde{r}(\alpha) - r}{\alpha} \right| (\|\tilde{y}^\alpha - \tilde{y}_1^\alpha\|_{\mathrm{C}} + \|\tilde{\lambda}(\alpha) - \lambda\|_{\mathbb{R}^d}),$$

for all $\alpha \in (0, \overline{\alpha}]$, which concludes the proof of Step 1 from the differentiability of $\tilde{r}(\cdot)$ and the continuity of $\tilde{\lambda}(\cdot)$ at 0 and from the uniform convergences of $\tilde{y}^\alpha$ and $\tilde{y}_1^\alpha$ to y over $[0, T]$ when $\alpha \to 0$. The proof of Step 1 is complete.

Step 2: Let us prove that χ_2^α uniformly converges to zero over $[\tau, T]$ when $\alpha \to 0$. From integral representations it holds that

$$\chi_2^\alpha(t) = \chi_2^\alpha(\tau) + \int_\tau^t \frac{g(\tilde{y}_2^\alpha(s), \lambda, u(s), s) - g(y(s), \lambda, u(s), s)}{\alpha} - \nabla_x g(\rho(s)) w_{\tilde{u}}(s)\, \mathrm{d}s,$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. Using a Taylor expansion with integral rest, we obtain that

$$\begin{aligned}\|\chi_2^\alpha(t)\|_{\mathbb{R}^n} \leq{}& \|\chi_2^\alpha(\tau)\|_{\mathbb{R}^n} \\ &+ \int_\tau^t \int_0^1 \|\nabla_x g(y(s) + \eta(\tilde{y}_2^\alpha(s) - y(s)), \lambda, u(s), s)\|_{\mathbb{R}^{n\times n}} \left\| \underbrace{\frac{\tilde{y}_2^\alpha(s) - y(s)}{\alpha} - w_{\tilde{u}}(s)}_{\chi_2^\alpha(s)} \right\|_{\mathbb{R}^n} \mathrm{d}\eta\, \mathrm{d}s \\ &+ \underbrace{\int_\tau^T \int_0^1 \|\nabla_x g(y(s) + \eta(\tilde{y}_2^\alpha(s) - y(s)), \lambda, u(s), s) - \nabla_x g(\rho(s))\|_{\mathbb{R}^{n\times n}} \|w_{\tilde{u}}(s)\|_{\mathbb{R}^n}\, \mathrm{d}\eta\, \mathrm{d}s}_{\Gamma_4(\alpha)},\end{aligned}$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. The Grönwall lemma leads to

$$\|\chi_2^\alpha(t)\|_{\mathbb{R}^n} \leq (\|\chi_2^\alpha(\tau)\|_{\mathbb{R}^n} + \Gamma_4(\alpha))\, e^{MT},$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \overline{\alpha}]$. From the uniform convergence of $\tilde{y}_2^\alpha$ to y over $[0, T]$ when $\alpha \to 0$ and from the dominated convergence theorem, we prove that $\Gamma_4(\alpha)$ converges to zero when $\alpha \to 0$. It remains to prove that $\|\chi_2^\alpha(\tau)\|_{\mathbb{R}^n}$ converges to zero when $\alpha \to 0$. From integral representations it holds that

$$\chi_2^\alpha(\tau) = \int_{\tau-\alpha}^{\tau} \frac{g(\tilde{y}_2^\alpha(s), \lambda, v, s) - g(y(s), \lambda, v, s)}{\alpha}\, \mathrm{d}s + \int_{\tau-\alpha}^{\tau} \frac{g(y(s), \lambda, v, s) - g(y(s), \lambda, u(s), s)}{\alpha}\, \mathrm{d}s - w_{\tilde{u}}(\tau).$$

for all $\alpha \in (0, \overline{\alpha}]$. From the uniform convergence of $\tilde{y}_2^\alpha$ to y over $[0, T]$ when $\alpha \to 0$, one can easily prove that the first term tends to $0_{\mathbb{R}^n}$ when $\alpha \to 0$. Finally, since τ is a Lebesgue point of the map $g(y(\cdot), \lambda, u(\cdot), \cdot)$ and from the value of $w_{\tilde{u}}(\tau)$, the second term tends to $0_{\mathbb{R}^n}$ when $\alpha \to 0$. The proof of Step 2 is complete.

Step 3: Let us prove that χ_3^α uniformly converges to zero over $[0, T]$ when $\alpha \to 0$. From integral representations it holds that

$$\chi_3^\alpha(t) = \chi_3^\alpha(r) + \int_r^t \frac{g(\tilde{y}_1^\alpha(s), \lambda, \tilde{u}(\alpha)(s), s) - g(\tilde{y}_2^\alpha(s), \lambda, \tilde{u}(\alpha)(s), s)}{\alpha} - \nabla_x g(\rho(s)) w_{(\tilde{r}, \tilde{y}_r)}(s)\, \mathrm{d}s,$$

for all $t \in [0, T]$ and all $\alpha \in (0, \overline{\alpha}]$. Using a Taylor expansion with integral rest, we obtain that

$$\begin{aligned}\|\chi_3^\alpha(t)\|_{\mathbb{R}^n} \leq{}& \|\chi_3^\alpha(r)\|_{\mathbb{R}^n} \\ &+ \left| \int_r^t \int_0^1 \|\nabla_x g(\tilde{y}_2^\alpha(s) + \eta(\tilde{y}_1^\alpha(s) - \tilde{y}_2^\alpha(s)), \lambda, \tilde{u}(\alpha)(s), s)\|_{\mathbb{R}^{n\times n}} \left\| \underbrace{\frac{\tilde{y}_1^\alpha(s) - \tilde{y}_2^\alpha(s)}{\alpha} - w_{(\tilde{r}, \tilde{y}_r)}(s)}_{\chi_3^\alpha(s)} \right\|_{\mathbb{R}^n} \mathrm{d}\eta\, \mathrm{d}s \right| \\ &+ \underbrace{\int_0^T \int_0^1 \|\nabla_x g(\tilde{y}_2^\alpha(s) + \eta(\tilde{y}_1^\alpha(s) - \tilde{y}_2^\alpha(s)), \lambda, \tilde{u}(\alpha)(s), s) - \nabla_x g(\rho(s))\|_{\mathbb{R}^{n\times n}} \|w_{(\tilde{r}, \tilde{y}_r)}(s)\|_{\mathbb{R}^n}\, \mathrm{d}\eta\, \mathrm{d}s}_{\Gamma_5(\alpha)},\end{aligned}$$

for all $t \in [0,T]$ and all $\alpha \in (0,\overline{\alpha}]$. The Grönwall lemma leads to

$$\|\chi_3^\alpha(t)\|_{\mathbb{R}^n} \leq (\|\chi_3^\alpha(r)\|_{\mathbb{R}^n} + \Gamma_5(\alpha))\, e^{MT},$$

for all $t \in [0,T]$ and all $\alpha \in (0,\overline{\alpha}]$. From the uniform convergences of $\tilde{y}_1^\alpha$ and $\tilde{y}_2^\alpha$ to y over $[0,T]$ when $\alpha \to 0$ and from the dominated convergence theorem, we prove that $\Gamma_5(\alpha)$ converges to zero when $\alpha \to 0$. It remains to prove that $\|\chi_3^\alpha(r)\|_{\mathbb{R}^n}$ converges to zero when $\alpha \to 0$. From integral representations it holds that

$$\chi_3^\alpha(r) = \left(\frac{\tilde{y}_r(\alpha) - y_r}{\alpha} - \tilde{y}_r'(0)\right) + \left(\tilde{r}'(0) g(y(r),\lambda,u(r),r) + \frac{1}{\alpha}\int_{\tilde{r}(\alpha)}^{r} g(\tilde{y}_1^\alpha(s),\tilde{\lambda}(\alpha),\tilde{u}(\alpha)(s),s)\,\mathrm{d}s\right),$$

for all $\alpha \in (0,\overline{\alpha}]$. From differentiability of $\tilde{y}_r(\cdot)$ at 0, the first term converges to $0_{\mathbb{R}^n}$ when $\alpha \to 0$. Since $r \neq \tau$ and from the continuity of $\tilde{r}(\cdot)$ at 0, we know that the second term can be rewritten as

$$\tilde{r}'(0) g(y(r),\lambda,u(r),r) + \frac{1}{\alpha}\int_{\tilde{r}(\alpha)}^{r} g(y(s),\lambda,u(s),s)\,\mathrm{d}s - \frac{1}{\alpha}\int_{\tilde{r}(\alpha)}^{r} g(y(s),\lambda,u(s),s) - g(\tilde{y}_1^\alpha(s),\tilde{\lambda}(\alpha),u(s),s)\,\mathrm{d}s,$$

for sufficiently small $\alpha > 0$. Since r is a Lebesgue point of the map $g(y(\cdot,g,\theta),\lambda,u(\cdot),\cdot)$ and from the differentiability of $\tilde{r}(\cdot)$ at 0, the sum of the two first terms in the above equation converges to $0_{\mathbb{R}^n}$ when $\alpha \to 0$. Finally the norm of the last term in the above equation can be bounded by

$$\left|\frac{1}{\alpha}\int_{\tilde{r}(\alpha)}^{r} \|g(y(s),\lambda,u(s),s) - g(\tilde{y}_1^\alpha(s),\tilde{\lambda}(\alpha),u(s),s)\|_{\mathbb{R}^n}\,\mathrm{d}s\right| \leq M\left|\frac{\tilde{r}(\alpha)-r}{\alpha}\right|(\|y-\tilde{y}_1^\alpha\|_{\mathrm{C}} + \|\lambda - \tilde{\lambda}(\alpha)\|_{\mathbb{R}^d}),$$

which tends to zero when $\alpha \to 0$, thanks to the differentiability of $\tilde{r}(\cdot)$ at 0, to the continuity of $\tilde{\lambda}(\alpha)$ at 0 and from the uniform convergence of $\tilde{y}_1^\alpha$ to y over $[0,T]$ when $\alpha \to 0$. The proof of Step 3 is complete. This completes the proof of Proposition 2.4.1. □

2.5 Preliminaries: sensitivity analysis in the hybrid context

As explained in the overview of the proof of Theorem 2.2.1 developed in Section 2.2.3, a sensitivity analysis of the hybrid control system (2.1) has to be performed to construct perturbed trajectories which visit exactly (and in the same order) the same regions than a given nominal trajectory. This is exactly the content of the present section. To this aim we will use the results stated in the previous Section 2.4, but we will also invoke at several occasions the following conic implicit function theorem to prove the existence of perturbed crossing times (see Section 2.2.3 for details).

Lemma 2.5.1 (A conic implicit function theorem)**.** *Let $\overline{\alpha} > 0$, $t^c \in (0,T)$ and $\delta > 0$. Consider a continuous map*

$$\begin{array}{rccc} \mathcal{G}: & [0,\overline{\alpha}] \times [t^c-\delta, t^c+\delta] & \to & \mathbb{R} \\ & (\alpha,t) & \mapsto & \mathcal{G}(\alpha,t), \end{array}$$

satisfying $\mathcal{G}(0,t^c) = 0$, such that $\nabla_\alpha \mathcal{G}(0,t^c)$ exists and such that $\nabla_t \mathcal{G}$ exists and is continuous over $[0,\overline{\alpha}] \times [t^c-\delta, t^c+\delta]$ with $\nabla_t \mathcal{G}(0,t^c) \neq 0$. Then there exist $0 < \overline{\beta} \leq \overline{\alpha}$ and an implicit function $\tilde{t} \in \mathrm{C}([0,\overline{\beta}],[t^c-\delta,t^c+\delta])$, satisfying $\tilde{t}(0) = t^c$ and $\mathcal{G}(\alpha,\tilde{t}(\alpha)) = 0$ for all $\alpha \in [0,\overline{\beta}]$, that is differentiable at 0 with derivative $\tilde{t}'(0) = -\frac{\nabla_\alpha \mathcal{G}(0,t^c)}{\nabla_t \mathcal{G}(0,t^c)}$.

Proof. Consider the extension $\mathcal{G}_0 : [-\overline{\alpha}, \overline{\alpha}] \times [t^c - \delta, t^c + \delta] \to \mathbb{R}$ defined by

$$\forall (\alpha, t) \in [-\overline{\alpha}, \overline{\alpha}] \times [t^c - \delta, t^c + \delta], \quad \mathcal{G}_0(\alpha, t) := \begin{cases} \mathcal{G}(\alpha, t) & \text{if} \quad \alpha \in [0, \overline{\alpha}], \\ 2\mathcal{G}(0, t) - \mathcal{G}(-\alpha, t) & \text{if} \quad \alpha \in [-\overline{\alpha}, 0]. \end{cases}$$

From the assumptions of Lemma 2.5.1, one can easily derive that $\mathcal{G}_0(0, t^c) = 0$, $\nabla_\alpha \mathcal{G}_0(0, t^c)$ exists and $\nabla_t \mathcal{G}_0$ exists and is continuous over $[-\overline{\alpha}, \overline{\alpha}] \times [t^c - \delta, t^c + \delta]$ with $\nabla_t \mathcal{G}_0(0, t^c) \neq 0$. Using a classical version of the implicit function theorem (see [89, Theorem 9.3] and [75, Theorem E]), there exist $0 < \overline{\beta} \leq \overline{\alpha}$ and an implicit function $\tilde{t} \in \mathrm{C}([-\overline{\beta}, \overline{\beta}], [t^c - \delta, t^c + \delta])$, satisfying $\tilde{t}(0) = t^c$ and $\mathcal{G}_0(\alpha, \tilde{t}(\alpha)) = 0$ for all $\alpha \in [-\overline{\beta}, \overline{\beta}]$, differentiable at 0 with derivative $\tilde{t}'(0) = -\frac{\nabla_\alpha \mathcal{G}_0(0, t^c)}{\nabla_t \mathcal{G}_0(0, t^c)}$. To conclude the proof, one has just to consider the restriction of the function $\tilde{t}$ to the interval $[0, \overline{\beta}]$ and to use the facts that $\nabla_\alpha \mathcal{G}_0(0, t^c) = \nabla_\alpha \mathcal{G}(0, t^c)$ and $\nabla_t \mathcal{G}_0(0, t^c) = \nabla_t \mathcal{G}(0, t^c)$. □

2.5.1 A regular solution to (2.1) and auxiliary non-hybrid trajectories

Throughout Section 2.5 we fix $(x, \lambda, u) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{PC}([0, T], \mathbb{R}^d) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m)$ being a regular solution to (2.1) and we will use the notations introduced in Definitions 2.2.1 and 2.2.2. For all $k \in \{1, ..., N\}$, we introduce, following the notations from Section 2.4, the *auxiliary non-hybrid trajectory* $\tilde{z}_k := y(\cdot, f_k, \theta_k)$ associated with the quadruplet $\theta_k := (\lambda_k, \tilde{u}_k, t^c_{k-1}, x(t^c_{k-1}))$, where the *auxiliary control* $\tilde{u}_k \in \mathrm{L}^\infty([0, T], \mathbb{R}^m)$ is defined by

$$\tilde{u}_k(t) := \begin{cases} u^+(t^c_{k-1}), & \text{for a.e. } t \in (t^c_0, t^c_{k-1}), \\ u(t), & \text{for a.e. } t \in (t^c_{k-1}, t^c_k), \\ u^-(t^c_k), & \text{for a.e. } t \in (t^c_k, t^c_N). \end{cases}$$

We refer to Figure 2.6 in Section 2.2.3. Note that $\tilde{z}_k = x$ over $[t^c_{k-1}, t^c_k]$ for all $k \in \{1, ..., N\}$ (see Figure 2.7 in Section 2.2.3). As a consequence, from Cauchy-Lipschitz theorem and up to reducing $\delta > 0$ provided in Definition 2.2.2, we will consider in the sequel that $[t^c_{k-1} - \delta, t^c_k + \delta] \cap [0, T] \subset I(f_k, \theta_k)$ for all $k \in \{1, ..., N\}$. Furthermore, up to reducing $\delta > 0$ again, we will consider that $\tilde{z}_k(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t^c_{k-1}), \frac{\nu}{2})$ for all $t \in [t^c_{k-1} - \delta, t^c_{k-1} + \delta]$ and all $k \in \{2, \ldots, N\}$, and that $\tilde{z}_k(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t^c_k), \frac{\nu}{2})$ for all $t \in [t^c_k - \delta, t^c_k + \delta]$ and all $k \in \{1, \ldots, N-1\}$.

Furthermore, from $(\mathcal{A}1)$ and for any $k \in \{1, \ldots, N-1\}$, note that $\tilde{u}_k$ is continuous over $[t^c_k - \delta, T]$ and thus $\tilde{z}_k$ is of class C^1 over $[t^c_k - \delta, t^c_k + \delta]$ with $\dot{z}_k(t) = f_k(\tilde{z}_k(t), \lambda_k, \tilde{u}_k(t), t)$ for all $t \in [t^c_k - \delta, t^c_k + \delta]$. In particular t^c_k is a Lebesgue point of the map $f_k(\tilde{z}_k(\cdot), \lambda_k, \tilde{u}_k(\cdot), \cdot)$ and it holds that $\dot{z}_k(t^c_k) = (f_k)^-(t^c_k)$. Similarly, from $(\mathcal{A}1)$ and for any $k \in \{1, \ldots, N-1\}$, note that $\tilde{u}_{k+1}$ is continuous over $[0, t^c_k + \delta]$ and thus $\tilde{z}_{k+1}$ is of class C^1 over $[t^c_k - \delta, t^c_k + \delta]$ with $\dot{z}_{k+1}(t) = f_{k+1}(\tilde{z}_{k+1}(t), \lambda_{k+1}, \tilde{u}_{k+1}(t), t)$ for all $t \in [t^c_k - \delta, t^c_k + \delta]$. In particular t^c_k is a Lebesgue point of the map $f_{k+1}(\tilde{z}_{k+1}(\cdot), \lambda_{k+1}, \tilde{u}_{k+1}(\cdot), \cdot)$ and it holds that $\dot{z}_{k+1}(t^c_k) = (f_{k+1})^+(t^c_k)$.

2.5.2 Convex perturbation of the regionally switching parameter

Consider the framework of Section 2.5.1. This entire Section 2.5.2 is dedicated to the proof of the next proposition which states a differentiability result at time $t = T$ for the trajectory x with respect to a convex perturbation of the regionally switching parameter λ.

Proposition 2.5.1. *Consider the framework of Section 2.5.1. Let $k \in \{1, ..., N\}$ and let $\overline{\lambda}_k \in \mathbb{R}^d$. Then there exists $0 < \overline{\alpha} \leq 1$ such that, for all $\alpha \in (0, \overline{\alpha}]$, there exists a perturbed solution $(x^\alpha, \lambda^\alpha, u^\alpha) \in$*

$\mathrm{AC}([0,T],\mathbb{R}^n)\times\mathrm{PC}([0,T],\mathbb{R}^d)\times\mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *to* (2.1) *such that:*

(i) *The corresponding perturbed partition of* $[0,T]$*, denoted by* $\{\tilde{t}_q(\alpha)\}_{q=0,\dots,N(\alpha)}$*, satisfies* $N(\alpha)=N$*, with* $\tilde{t}_q(\alpha)=t_q^c$ *for all* $q\in\{1,...,k-1\}$*, and* $\tilde{t}_q(\alpha)$ *tends to* t_q^c *when* $\alpha\to 0$ *for all* $q\in\{k,...,N-1\}$*.*

(ii) *The perturbed trajectory* x^α *follows the same regions than* x*, that is,* x^α *satisfies*

$$x^\alpha(t)\in E_q(t) \text{ for all } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\},$$

with $x^\alpha(0)=x_{\mathrm{init}}\in E_1(0)$ *and* $x^\alpha(T)\in E_N(T)$*. Moreover* x^α *uniformly converges to* x *over* $[0,T]$ *when* $\alpha\to 0$*.*

(iii) *The perturbed regionally switching parameter* λ^α *is given by the convex perturbation*

$$\lambda^\alpha(t)=\begin{cases}\lambda_k+\alpha(\overline{\lambda}_k-\lambda_k) & \text{for a.e. } t\in(\tilde{t}_{k-1}(\alpha),\tilde{t}_k(\alpha)),\\ \lambda_q & \text{for a.e. } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\}\backslash\{k\}.\end{cases}$$

(iv) *The perturbed control* u^α *is given by*

$$u^\alpha(t)=\tilde{u}_q(t) \text{ for a.e. } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\},$$

where $\tilde{u}_q$ *stands for the auxiliary control defined in Section 2.5.1 for all* $q\in\{1,\dots,N\}$*.*

(v) *The limit*

$$\lim_{\alpha\to 0}\frac{x^\alpha(T)-x(T)}{\alpha}=w(T),$$

holds true, where

$$w(t):=\begin{cases}w_q(t) & \text{for all } t\in[t_{q-1}^c,t_q^c) \text{ and all } q\in\{k,...,N-1\},\\ w_N(t) & \text{for all } t\in[t_{N-1}^c,t_N^c],\end{cases}$$

where w_k *is the variation vector defined as the unique maximal solution (which is global) to the linearized Cauchy problem given by*

$$\begin{cases}\dot{w}(t)=\nabla_x f_k(\tilde{z}_k(t),\lambda_k,\tilde{u}_k(t),t)w(t)+\nabla_\lambda f_k(\tilde{z}_k(t),\lambda_k,\tilde{u}_k(t),t)(\overline{\lambda}_k-\lambda_k),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t\in[t_{k-1}^c-\delta,t_k^c+\delta]\cap[0,T],\\ w(t_{k-1}^c)=0_{\mathbb{R}^n},\end{cases}$$

and w_q *is the variation vector defined by induction as the unique maximal solution (which is global) to the linearized Cauchy problem given by*

$$\begin{cases}\dot{w}(t)=\nabla_x f_q(\tilde{z}_q(t),\lambda_q,\tilde{u}_q(t),t)w(t), \quad \text{a.e. } t\in[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T],\\ w(t_{q-1}^c)=w_{q-1}(t_{q-1}^c)+\xi_{q-1},\end{cases}$$

for all $q\in\{k+1,...,N\}$*, where* $\xi_q\in\mathbb{R}^n$ *stands for the jump vector defined by*

$$\xi_q:=\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)}((f_{q+1})^+(t_q^c)-(f_q)^-(t_q^c)),$$

for all $q\in\{k,...,N-1\}$*.*

(vi) *The limit*

$$\lim_{\alpha\to 0}\frac{\tilde{t}_q(\alpha)-t_q^c}{\alpha}=-\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)},$$

holds true for all $q\in\{k,\dots,N-1\}$.

Remark 2.5.1. Consider the framework of Proposition 2.5.1. It is worth noticing that the variation vector w, defined in Item (v), satisfies the discontinuity jump

$$w^+(t_q^c)-w^-(t_q^c)=\xi_q=\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w^-(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)}((f_{q+1})^+(t_q^c)-(f_q)^-(t_q^c)),$$

at each crossing time t_q^c for $q\in\{k,\dots,N-1\}$.

Construction of perturbed auxiliary non-hybrid trajectories

Lemma 2.5.2 (Construction of perturbed auxiliary non-hybrid trajectories)**.** *Consider the frameworks of Section 2.5.1 and Proposition 2.5.1. Let* $k\in\{1,...,N\}$ *and let* $\overline{\lambda}_k\in\mathbb{R}^d$. *Then there exists* $0<\overline{\alpha}\leq 1$ *and, for all* $q\in\{k,...,N-1\}$, *there exists a function* $\tilde{t}_q\in \mathrm{C}([0,\overline{\alpha}],[t_q^c-\delta,t_q^c+\delta])$ *differentiable at* 0 *with* $\tilde{t}_q(0)=t_q^c$ *and*

$$\tilde{t}_q'(0)=-\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)},$$

such that the perturbed auxiliary non-hybrid trajectories $\tilde{z}_q^\alpha:=y(\cdot,f_q,\theta_q^\alpha)$ *associated with the perturbed quadruplets* θ_q^α *defined by the induction*

$$\theta_q^\alpha:=\begin{cases}(\lambda_k+\alpha(\overline{\lambda}_k-\lambda_k),\tilde{u}_k,t_{k-1}^c,x(t_{k-1}^c)) & \text{if}\quad q=k,\\ (\lambda_q,\tilde{u}_q,\tilde{t}_{q-1}(\alpha),\tilde{z}_{q-1}^\alpha(\tilde{t}_{q-1}(\alpha))) & \text{if}\quad q\in\{k+1,\dots,N\},\end{cases}$$

for all $\alpha\in[0,\overline{\alpha}]$ *and all* $q\in\{k,\dots,N\}$, *satisfy:*

- *for all* $q\in\{k,\dots,N\}$, *it holds that* $[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T]\subset I(f_q,\theta_q^\alpha)$ *for all* $\alpha\in[0,\overline{\alpha}]$, *that* $\tilde{z}_q^\alpha$ *uniformly converges to* $\tilde{z}_q$ *over* $[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T]$ *when* $\alpha\to 0$, *and*

$$\lim_{\alpha\to 0}\frac{\tilde{z}_q^\alpha(t_q^c)-\tilde{z}_q(t_q^c)}{\alpha}=w_q(t_q^c).$$

- *for all* $q\in\{k,\dots,N-1\}$, *it holds that* $\tilde{z}_q^\alpha(t)\in\overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_q^c),\nu)$ *for all* $(\alpha,t)\in[0,\overline{\alpha}]\times[t_q^c-\delta,t_q^c+\delta]$, *that* $F_q(\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)),\tilde{t}_q(\alpha))=0$ *for all* $\alpha\in[0,\overline{\alpha}]$, *and that the map* $\alpha\in[0,\overline{\alpha}]\mapsto\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))\in\mathbb{R}^n$ *is continuous over* $[0,\overline{\alpha}]$ *and differentiable at* 0 *with*

$$\lim_{\alpha\to 0}\frac{\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))-\tilde{z}_q(t_q^c)}{\alpha}=w_q(t_q^c)+\tilde{t}_q'(0)(f_q)^-(t_q^c).$$

Proof. Let us fix $k\in\{1,...,N\}$ and $\overline{\lambda}_k\in\mathbb{R}^d$. The case $k=N$ follows directly from Proposition 2.4.1. In the sequel we deal with the case $k\in\{1,\dots,N-1\}$ and we will proceed by induction over $q\in\{k,\dots,N\}$. Note that we will construct $0<\overline{\alpha}\leq 1$ in the base case and that it will be reduced a finite number of times at each step of the induction.

Base case $q=k$. We deduce from Proposition 2.4.1 that there exists $0<\overline{\alpha}\leq 1$ such that $[t_{k-1}^c-\delta,t_k^c+$

$\delta] \cap [0,T] \subset I(f_k, \theta_k^\alpha)$ for all $\alpha \in [0,\overline{\alpha}]$, that $\tilde{z}_k^\alpha$ uniformly converges to $\tilde{z}_k$ over $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0,T]$ when $\alpha \to 0$ (as illustrated in Figure 2.9(a) in Section 2.2.3), and that the map

$$(\alpha, t) \in [0,\overline{\alpha}] \times ([t_{k-1}^c - \delta, t_k^c + \delta] \cap [0,T]) \mapsto \tilde{z}_k^\alpha(t) \in \mathbb{R}^n, \tag{2.8}$$

is continuous. Since moreover $\tilde{z}_k(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \frac{\nu}{2})$ for all $t \in [t_k^c - \delta, t_k^c + \delta]$, up to reducing $\overline{\alpha} > 0$, we have $\tilde{z}_k^\alpha(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \nu)$ for all $(\alpha, t) \in [0,\overline{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$. We are now in a position to define the map

$$\begin{array}{rccl} G_k: & [0,\overline{\alpha}] \times [t_k^c - \delta, t_k^c + \delta] & \to & \mathbb{R} \\ & (\alpha, t) & \mapsto & F_k(\tilde{z}_k^\alpha(t), t), \end{array}$$

where $F_k : \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \nu) \times [t_k^c - \delta, t_k^c + \delta] \to \mathbb{R}$ is the C^1 function provided in Definition 2.2.2.
Let us check that G_k satisfies all the assumptions of the conic implicit function theorem (Lemma 2.5.1). First, G_k is continuous from the continuity of the map (2.8) and $G_k(0, t_k^c) = F_k(x(t_k^c), t_k^c) = 0$. Second, for any $\alpha \in [0,\overline{\alpha}]$, since $\tilde{u}_k$ is continuous over $[t_k^c - \delta, t_k^c + \delta]$ (see Figure 2.6 in Section 2.2.3), we know that $\tilde{z}_k^\alpha$ is of class C^1 over $[t_k^c - \delta, t_k^c + \delta]$. This implies that $\nabla_t G_k(\alpha, t)$ exists with

$$\nabla_t G_k(\alpha, t) = \langle \nabla_x F_k(\tilde{z}_k^\alpha(t), t), f_k(\tilde{z}_k^\alpha(t), \lambda_k + \alpha(\overline{\lambda}_k - \lambda_k), \tilde{u}_k(t), t) \rangle_{\mathbb{R}^n} + \nabla_t F_k(\tilde{z}_k^\alpha(t), t),$$

for all $(\alpha, t) \in [0,\overline{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$. Furthermore, from the continuity of the map (2.8), one can see that $\nabla_t G_k$ is continuous over $[0,\overline{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$ and, from $(\mathcal{A}3)$, it holds that

$$\nabla_t G_k(0, t_k^c) = \langle \nabla_x F_k(x(t_k^c), t_k^c), (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c) \neq 0.$$

Finally, from the third item of Proposition 2.4.1, we get that

$$\lim_{\alpha \to 0} \frac{\tilde{z}_k^\alpha(t_k^c) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c),$$

which implies that $\nabla_\alpha G_k(0, t_k^c)$ exists with $\nabla_\alpha G_k(0, t_k^c) = \langle \nabla_x F_k(x(t_k^c), t_k^c), w_k(t_k^c) \rangle_{\mathbb{R}^n}$.
We deduce from the conic implicit function theorem (Lemma 2.5.1) that, up to reducing $\overline{\alpha} > 0$ (precisely, by taking $\overline{\alpha} = \overline{\beta}$), there exists a function $\tilde{t}_k \in \mathrm{C}([0,\overline{\alpha}], [t_k^c - \delta, t_k^c + \delta])$, such that $\tilde{t}_k(0) = t_k^c$ and $F_k(\tilde{z}_k^\alpha(\tilde{t}_k(\alpha)), \tilde{t}_k(\alpha)) = 0$ for all $\alpha \in [0,\overline{\alpha}]$ (see Figure 2.9(a) in Section 2.2.3), that is differentiable at 0 with

$$\tilde{t}_k'(0) = -\frac{\langle \nabla_x F_k(x(t_k^c), t_k^c), w_k(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)}.$$

From the continuities of the function $\tilde{t}_k$ and of the map (2.8), we deduce that the map $\alpha \in [0,\overline{\alpha}] \mapsto \tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) \in \mathbb{R}^n$ is continuous over $[0,\overline{\alpha}]$. It remains to prove that

$$\lim_{\alpha \to 0} \frac{\tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c) + \tilde{t}_k'(0)(f_k)^-(t_k^c).$$

To this aim, using integral representations, one can write

$$\frac{\tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) - \tilde{z}_k(t_k^c)}{\alpha} = \frac{\tilde{z}_k^\alpha(t_k^c) - \tilde{z}_k(t_k^c)}{\alpha} + \frac{\tilde{t}_k(\alpha) - t_k^c}{\alpha} \frac{1}{\tilde{t}_k(\alpha) - t_k^c} \int_{t_k^c}^{\tilde{t}_k(\alpha)} f_k(\tilde{z}_k(s), \lambda_k, \tilde{u}_k(s), s)\, \mathrm{d}s$$
$$+ \frac{1}{\alpha} \int_{t_k^c}^{\tilde{t}_k(\alpha)} f_k(\tilde{z}_k^\alpha(s), \lambda_k + \alpha(\overline{\lambda}_k - \lambda_k), \tilde{u}_k(s), s) - f_k(\tilde{z}_k(s), \lambda_k, \tilde{u}_k(s), s)\, \mathrm{d}s,$$

for all $\alpha \in (0, \overline{\alpha}]$. We already proved that the first term tends to $w_k(t_k^c)$ when $\alpha \to 0$. Since t_k^c is a Lebesgue point of the map $f_k(\tilde{z}_k(\cdot), \lambda_k, \tilde{u}_k(\cdot), \cdot)$ and since $\tilde{t}_k$ is differentiable at 0, the second term tends to $\tilde{t}_k'(0)(f_k)^-(t_k^c)$, finally the third term tends to zero when $\alpha \to 0$, since $\tilde{z}_k^\alpha$ uniformly converges to $\tilde{z}_k$ over $[t_k^c - \delta, t_k^c + \delta]$, f_k is of class C^1 and $\tilde{t}_k$ is differentiable at 0. Hence the proof for the base case is complete.

Inductive step. Let $q \in \{k+1, ..., N\}$ and assume that the induction hypothesis holds true for all $\ell \in \{k, ..., q-1\}$. The case $q = N$ follows directly from Proposition 2.4.1 and from the induction hypothesis (in particular from the differentiabilities at 0 of the function $\tilde{t}_{N-1}$ and of the map $\alpha \in [0, \overline{\alpha}] \mapsto \tilde{z}_{N-1}^\alpha(\tilde{t}_{N-1}(\alpha)) \in \mathbb{R}^n$). Therefore, in the sequel, we deal with the case $q \in \{k+1, \ldots, N-1\}$ and we will proceed similarly to the base case. Therefore some details will be omitted.
Thanks to the induction hypothesis ensuring the continuities of the function $\tilde{t}_{q-1}$ and of the map $\alpha \in [0, \overline{\alpha}] \mapsto \tilde{z}_{q-1}^\alpha(\tilde{t}_{q-1}(\alpha))$, we deduce from Proposition 2.4.1 that, up to reducing $\overline{\alpha}$, it holds that $[t_{q-1}^c - \delta, t_q^c + \delta] \subset I(f_q, \theta_q^\alpha)$ for all $\alpha \in [0, \overline{\alpha}]$, that $\tilde{z}_q^\alpha$ uniformly converges to $\tilde{z}_q$ over $[t_{q-1}^c - \delta, t_q^c + \delta]$ when $\alpha \to 0$ (see Figure 2.10 in Section 2.2.3 where $q = k+1$), and that the map

$$(\alpha, t) \in [0, \overline{\alpha}] \times [t_{q-1}^c - \delta, t_q^c + \delta] \mapsto \tilde{z}_q^\alpha(t) \in \mathbb{R}^n, \tag{2.9}$$

is continuous. Similarly to the base case, up to reducing $\overline{\alpha} > 0$, we get that $\tilde{z}_q^\alpha(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_q^c), \nu)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t_q^c - \delta, t_q^c + \delta]$ and thus we are in a position to define the map

$$\begin{array}{rccl} G_q : & [0, \overline{\alpha}] \times [t_q^c - \delta, t_q^c + \delta] & \to & \mathbb{R} \\ & (\alpha, t) & \mapsto & F_q(\tilde{z}_q^\alpha(t), t). \end{array}$$

Similarly to the base case, G_q is continuous, $G_q(0, t_q^c) = F_q(x(t_q^c), t_q^c) = 0$ and $\nabla_t G_q(\alpha, t)$ exists and is continuous over $[0, \overline{\alpha}] \times [t_q^c - \delta, t_q^c + \delta]$ and

$$\nabla_t G_q(0, t_q^c) = \langle \nabla_x F_q(x(t_q^c), t_q^c), (f_q)^-(t_q^c) \rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c), t_q^c) \neq 0.$$

Finally, from the third item of Proposition 2.4.1 and from the induction hypothesis (in particular from the differentiabilities at 0 of the function $\tilde{t}_{q-1}$ and of the map $\alpha \in [0, \overline{\alpha}] \mapsto \tilde{z}_{q-1}^\alpha(\tilde{t}_{q-1}(\alpha)) \in \mathbb{R}^n$), we get that

$$\lim_{\alpha \to 0} \frac{\tilde{z}_q^\alpha(t_q^c) - \tilde{z}_q(t_q^c)}{\alpha} = w_q(t_q^c),$$

which implies that $\nabla_\alpha G_q(0, t_q^c)$ exists with $\nabla_\alpha G_q(0, t_q^c) = \langle \nabla_x F_q(x(t_q^c), t_q^c), w_q(t_q^c) \rangle_{\mathbb{R}^n}$.
From the conic implicit function theorem (Lemma 2.5.1), up to reducing $\overline{\alpha} > 0$, there exists a function $\tilde{t}_q \in \mathrm{C}([0, \overline{\alpha}], [t_q^c - \delta, t_q^c + \delta])$, such that $\tilde{t}_q(0) = t_q^c$ and $F_q(\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)), \tilde{t}_q(\alpha)) = 0$ for all $\alpha \in [0, \overline{\alpha}]$, that is

differentiable at 0 with

$$\tilde{t}_q'(0) = -\frac{\langle \nabla_x F_q(x(t_q^c), t_q^c), w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_q(x(t_q^c), t_q^c), (f_q)^-(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c), t_q^c)}.$$

From the continuities of the function $\tilde{t}_q$ and of the map (2.9), we deduce that the map $\alpha \in [0, \overline{\alpha}] \mapsto \tilde{z}_q^\alpha(\tilde{t}_q(\alpha)) \in \mathbb{R}^n$ is continuous over $[0, \overline{\alpha}]$. Similarly to the base case, one can easily prove that

$$\lim_{\alpha \to 0} \frac{\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)) - \tilde{z}_q(t_q^c)}{\alpha} = w_q(t_q^c) + \tilde{t}_q'(0)(f_q)^-(t_q^c),$$

which completes the proof for the inductive step. □

Admissibility of the perturbed auxiliary non-hybrid trajectories

Lemma 2.5.3. *Consider the framework of Lemma 2.5.2. Then, up to reducing $\overline{\alpha}$, the following properties are satisfied:*

1. *There exists $s_{k-1}' \in (t_{k-1}^c, t_{k-1}^c + \delta]$ such that $\tilde{z}_k^\alpha(t) \in E_k(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (t_{k-1}^c, s_{k-1}']$ (and for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t_0^c, s_{k-1}']$ if $k = 1$).*
2. *For all $q \in \{k, ..., N-1\}$, there exists $s_q \in [t_q^c - \delta, t_q^c)$ such that $\tilde{z}_q^\alpha(t) \in E_q(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [s_q, \tilde{t}_q(\alpha))$.*
3. *For all $q \in \{k, ..., N-1\}$, there exists $s_q' \in (t_q^c, t_q^c + \delta]$ such that $\tilde{z}_{q+1}^\alpha(t) \in E_{q+1}(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (\tilde{t}_q(\alpha), s_q']$.*
4. *There exists $s_N \in [t_N^c - \delta, t_N^c)$ such that $\tilde{z}_N^\alpha(t) \in E_N(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [s_N, t_N^c]$.*

Proof. This proof does not require induction. We will prove each item separately. Note that we will reduce $\overline{\alpha}$ in each item.

Proof of the fourth item. Recall that $\tilde{z}_N = x$ over $[t_{N-1}^c, t_N^c]$ and that $x(t) \in E_N(t)$ for all $t \in [t_N^c - \delta, t_N^c]$. From $(\mathcal{C}1)$ and since $\tilde{z}_N^\alpha$ converges uniformly to $\tilde{z}_N$ over $[t_{N-1}^c - \delta, t_N^c] \cap [0, T]$ when $\alpha \to 0$, one can easily conclude the proof of the fourth item by reducing $\overline{\alpha} > 0$ and by taking $s_N = t_N^c - \delta$.

Proof of the first item. If $k = 1$, then the proof is similar to the above fourth item. Therefore let us deal with the case $k \in \{2, \dots, N\}$. Recall that $\tilde{z}_k(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_{k-1}^c), \frac{\nu}{2})$ for all $t \in [t_{k-1}^c - \delta, t_{k-1}^c + \delta]$. Since $\tilde{z}_k^\alpha$ uniformly converges to $\tilde{z}_k$ over $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T]$ when $\alpha \to 0$, up to reducing $\overline{\alpha} > 0$, we get that $\tilde{z}_k^\alpha(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_{k-1}^c), \nu)$, and therefore $\tilde{z}_k^\alpha(t) \in E_k(t)$ if and only if $F_{k-1}(\tilde{z}_k^\alpha(t), t) > 0$, for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t_{k-1}^c - \delta, t_{k-1}^c + \delta]$. By contradiction let us assume that

$$\forall s_{k-1}' \in (t_{k-1}^c, t_{k-1}^c + \delta], \quad \forall 0 < \beta \leq \overline{\alpha}, \quad \exists \alpha \in [0, \beta], \quad \exists t \in (t_{k-1}^c, s_{k-1}'], \quad F_{k-1}(\tilde{z}_k^\alpha(t), t) \leq 0. \quad (2.10)$$

Let $s_{k-1}' \in (t_{k-1}^c, t_{k-1}^c + \delta]$ and $0 < \beta \leq \overline{\alpha}$ and consider (α, t) given in (2.10). Since $F_{k-1}(\tilde{z}_k^\alpha(t_{k-1}^c), t_{k-1}^c) = F_{k-1}(x(t_{k-1}^c), t_{k-1}^c) = 0$, we obtain that

$$F_{k-1}(\tilde{z}_k^\alpha(t), t) - F_{k-1}(\tilde{z}_k^\alpha(t_{k-1}^c), t_{k-1}^c) \leq 0.$$

Since $\tilde{z}_k^\alpha$ is of class C^1 over $[t_{k-1}^c - \delta, t_{k-1}^c + \delta]$, note that the above inequality can be rewritten as

$$\frac{1}{t - t_{k-1}^c} \int_{t_{k-1}^c}^t \Psi_{k-1}(s)\, \mathrm{d}s \leq \frac{1}{t - t_{k-1}^c} \int_{t_{k-1}^c}^t \Psi_{k-1}(s) - \Psi_{k-1}^\alpha(s)\, \mathrm{d}s, \quad (2.11)$$

where

$$\Psi_{k-1}(s) := \langle \nabla_x F_{k-1}(\tilde{z}_k(s), s), f_k(\tilde{z}_k(s), \lambda_k, \tilde{u}_k(s), s)\rangle_{\mathbb{R}^n} + \nabla_t F_{k-1}(\tilde{z}_k(s), s),$$

and

$$\Psi^\alpha_{k-1}(s) := \langle \nabla_x F_{k-1}(\tilde{z}^\alpha_k(s), s), f_k(\tilde{z}^\alpha_k(s), \lambda_k + \alpha(\overline{\lambda}_k - \lambda_k), \tilde{u}_k(s), s)\rangle_{\mathbb{R}^n} + \nabla_t F_{k-1}(\tilde{z}^\alpha_k(s), s),$$

for all $s \in [t^c_{k-1} - \delta, t^c_{k-1} + \delta]$. Since $\tilde{u}_k$ is continuous at t^c_{k-1}, note that t^c_{k-1} is a Lebesgue point of Ψ_{k-1}. Therefore, when making tend $s'_{k-1} \to t^c_{k-1}$ and $\beta \to 0$, we make tend $\alpha \to 0$ and $t \to t^c_{k-1}$ and thus the left term of (2.11) tends to

$$\langle \nabla_x F_{k-1}(x(t^c_{k-1}), t^c_{k-1}), (f_k)^+(t^c_{k-1})\rangle_{\mathbb{R}^n} + \nabla_t F_{k-1}(x(t^c_{k-1}), t^c_{k-1}).$$

It remains to prove that the right term of (2.11) converges to zero when $\alpha \to 0$ and $t \to t^c_k$. To this aim recall that $\tilde{z}^\alpha_k(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t^c_{k-1}), \nu)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t^c_{k-1} - \delta, t^c_{k-1} + \delta]$ and that $\nabla_x F_{k-1}$ and $\nabla_t F_{k-1}$ are uniformly continuous over the compact set $\overline{\mathrm{B}}_{\mathbb{R}^n}(x(t^c_{k-1}), \nu) \times [t^c_{k-1} - \delta, t^c_{k-1} + \delta]$ (since F_{k-1} is of class C^1). Therefore, since $\tilde{z}^\alpha_k$ uniformly converges to $\tilde{z}_k$ over $[t^c_{k-1} - \delta, t^c_{k-1} + \delta]$ when $\alpha \to 0$, one can easily prove that the right term of (2.11) tends to zero when $\alpha \to 0$ and $t \to t^c_k$.
Hence we have obtained that

$$\langle \nabla_x F_{k-1}(x(t^c_{k-1}), t^c_{k-1}), (f_k)^+(t^c_{k-1})\rangle_{\mathbb{R}^n} + \nabla_t F_{k-1}(x(t^c_{k-1}), t^c_{k-1}) \leq 0,$$

which raises a contradiction with $(\mathcal{A}3)$. Therefore we have proved the negation of (2.10) which is given by

$$\exists s'_{k-1} \in (t^c_{k-1}, t^c_{k-1} + \delta], \quad \exists 0 < \beta \leq \overline{\alpha}, \quad \forall \alpha \in [0, \beta], \quad \forall t \in (t^c_{k-1}, s'_{k-1}], \quad F_{k-1}(\tilde{z}^\alpha_k(t), t) > 0,$$

which concludes the proof of the first item by reducing $\overline{\alpha} > 0$ to β.

Proof of the second item. Let $q \in \{k, ..., N-1\}$ be fixed. This proof is similar to the above one, with an additional difficulty due to the presence of the implicit function $\tilde{t}_q$. Recall that $\tilde{z}^\alpha_q(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t^c_q), \nu)$, and therefore $\tilde{z}^\alpha_q(t) \in E_q(t)$ if and only if $F_q(\tilde{z}^\alpha_q(t), t) < 0$, for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t^c_q - \delta, t^c_q + \delta]$. Also recall that $\tilde{t}_q(\alpha)$ tends to t^c_q when $\alpha \to 0$. Therefore, for any $s_q \in [t^c_q - \delta, t^c_q)$, there exists $0 < \overline{\beta}(s_q) \leq \overline{\alpha}$ such that $s_q < \tilde{t}_q(\alpha) \leq t^c_q + \delta$ for all $\alpha \in [0, \overline{\beta}(s_q)]$. By contradiction let us assume that

$$\forall s_q \in [t^c_q - \delta, t^c_q), \quad \forall 0 < \beta \leq \overline{\beta}(s_q), \quad \exists \alpha \in [0, \beta], \quad \exists t \in [s_q, \tilde{t}_q(\alpha)), \quad F_q(\tilde{z}^\alpha_q(t), t) \geq 0. \tag{2.12}$$

Let $s_q \in [t^c_q - \delta, t^c_q)$ and $0 < \beta \leq \overline{\beta}(s_q)$ and consider (α, t) given in (2.12). Since $F_q(\tilde{z}^\alpha_q(\tilde{t}_q(\alpha)), \tilde{t}_q(\alpha)) = 0$ (see Lemma 2.5.2), we obtain that

$$F_q(\tilde{z}^\alpha_q(\tilde{t}_q(\alpha)), \tilde{t}_q(\alpha)) - F_q(\tilde{z}^\alpha_q(t), t) \leq 0.$$

Since $\tilde{z}^\alpha_q$ is of class C^1 over $[t^c_q - \delta, t^c_q + \delta]$, note that the above inequality can be rewritten as

$$\frac{1}{\tilde{t}_q(\alpha) - t} \int_t^{\tilde{t}_q(\alpha)} \Psi_q(s)\, \mathrm{d}s \leq \frac{1}{\tilde{t}_q(\alpha) - t} \int_t^{\tilde{t}_q(\alpha)} \Psi_q(s) - \Psi^\alpha_q(s)\, \mathrm{d}s, \tag{2.13}$$

where

$$\Psi_q(s) := \begin{cases} \langle \nabla_x F_k(\tilde{z}_k(s),s), f_k(\tilde{z}_k(s),\lambda_k,\tilde{u}_k(s),s)\rangle_{\mathbb{R}^n} + \nabla_t F_k(\tilde{z}_k(s),s), & \text{if } q=k, \\ \langle \nabla_x F_q(\tilde{z}_q(s),s), f_q(\tilde{z}_q(s),\lambda_q,\tilde{u}_q(s),s)\rangle_{\mathbb{R}^n} + \nabla_t F_q(\tilde{z}_q(s),s), & \text{if } q\in\{k+1,...,N-1\}, \end{cases}$$

and

$$\Psi_q^\alpha(s) := \begin{cases} \langle \nabla_x F_k(\tilde{z}_k^\alpha(s),s), f_k(\tilde{z}_k^\alpha(s),\lambda_k+\alpha(\overline{\lambda}_k-\lambda_k),\tilde{u}_k(s),s)\rangle_{\mathbb{R}^n} + \nabla_t F_k(\tilde{z}_k^\alpha(s),s), & \text{if } q=k, \\ \langle \nabla_x F_q(\tilde{z}_q^\alpha(s),s), f_q(\tilde{z}_q^\alpha(s),\lambda_q,\tilde{u}_q(s),s)\rangle_{\mathbb{R}^n} + \nabla_t F_q(\tilde{z}_q^\alpha(s),s), & \text{if } q\in\{k+1,...,N-1\}, \end{cases}$$

for all $s\in[t_q^c-\delta,t_q^c+\delta]$. Since $\tilde{u}_q$ is continuous at t_q^c, note that t_q^c is a Lebesgue point of Ψ_q. Therefore, when making tend $s_q\to t_q^c$ and $\beta\to 0$, we make tend $\alpha\to 0$, $\tilde{t}_q(\alpha)\to t_q^c$ and $t\to t_q^c$ and thus the left term of (2.13) tends to

$$\langle \nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c),t_q^c),$$

and, using similar arguments as in the proof of the first item, we obtain that the right term of (2.13) tends to zero when $\alpha\to 0$, $\tilde{t}_q(\alpha)\to t_q^c$ and $t\to t_q^c$.
Hence we have obtained that

$$\langle \nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c),t_q^c)\leq 0,$$

which raises a contradiction with $(\mathcal{A}3)$. Therefore we have proved the negation of (2.12) which is given by

$$\exists s_q\in[t_q^c-\delta,t_q^c),\quad \exists 0<\beta\leq\overline{\beta}(s_q),\quad \forall\alpha\in[0,\beta],\quad \forall t\in[s_q,\tilde{t}_q(\alpha)),\quad F_q(\tilde{z}_q^\alpha(t),t)>0.$$

which concludes the proof of the second item by reducing $\overline{\alpha}>0$ to β.

Proof of the third item. The proof is similar to the above one. □

Lemma 2.5.4 (Admissibility of the perturbed auxiliary non-hybrid trajectories)**.** *Consider the framework of Lemma 2.5.2. Then, up to reducing $\overline{\alpha}>0$, it holds that:*

1. *$\tilde{z}_k^\alpha(t)\in E_k(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times(t_{k-1}^c,\tilde{t}_k(\alpha))$ (and for all $(\alpha,t)\in[0,\overline{\alpha}]\times[t_0^c,\tilde{t}_k(\alpha))$ if $k=1$).*
2. *$\tilde{z}_q^\alpha(t)\in E_q(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha))$ and all $q\in\{k+1,...,N-1\}$.*
3. *$\tilde{z}_N^\alpha(t)\in E_N(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times(\tilde{t}_{N-1}(\alpha),t_N^c]$.*

Proof. This proof does not require induction. Let us prove the second item only. The other items can be proved similarly (and note that $\overline{\alpha}>0$ is reduced in each item). Let $q\in\{k+1,\dots,N-1\}$. From Lemma 2.5.3, we know that:

- there exists $s'_{q-1}\in(t_{q-1}^c,t_{q-1}^c+\delta]$ such that $\tilde{z}_q^\alpha(t)\in E_q(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times(\tilde{t}_{q-1}(\alpha),s'_{q-1}]$.
- there exists $s_q\in[t_q^c-\delta,t_q^c)$ such that $\tilde{z}_q^\alpha(t)\in E_q(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times[s_q,\tilde{t}_q(\alpha))$.

Now recall that $\tilde{z}_q=x$ over $[t_{q-1}^c,t_q^c]$ and that $x(t)\in E_q(t)$ for all $t\in(t_{q-1}^c,t_q^c)$ and thus for all $t\in[s'_{q-1},s_q]$. From $(\mathcal{C}1)$ and since $\tilde{z}_q^\alpha$ converges uniformly to $\tilde{z}_q$ over $[t_{q-1}^c-\delta,t_q^c+\delta]$ when $\alpha\to 0$, one can easily see that, up to reducing $\overline{\alpha}>0$, one has $\tilde{z}_q^\alpha(t)\in E_q(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times[s'_{q-1},s_q]$. We finally deduce that $\tilde{z}_q^\alpha(t)\in E_q(t)$ for all $(\alpha,t)\in[0,\overline{\alpha}]\times(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha))$. The proof of the second item is complete. □

Proof of Proposition 2.5.1

Let us fix $k \in \{1, ..., N\}$ and $\overline{\lambda}_k \in \mathbb{R}^d$. Consider the perturbed auxiliary non-hybrid trajectories $\tilde{z}_q^\alpha = y(\cdot, f_q, \theta_q^\alpha)$ over $[t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T]$ for all $q \in \{k, \dots, N\}$ and all $\alpha \in (0, \overline{\alpha}]$ constructed in Lemma 2.5.2, together with the corresponding implicit functions $\tilde{t}_q$ for all $q \in \{k, \dots, N-1\}$. As explained in Section 2.2.3, we define by concatenation

$$x^\alpha(t) := \begin{cases} x(t) & \text{for all } t \in [t_0^c, t_{k-1}^c], \\ \tilde{z}_k^\alpha(t) & \text{for all } t \in [t_{k-1}^c, \tilde{t}_k(\alpha)], \\ \tilde{z}_q^\alpha(t) & \text{for all } t \in [\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)] \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \tilde{z}_N^\alpha(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c], \end{cases}$$

and

$$\lambda^\alpha(t) := \begin{cases} \lambda(t) & \text{for a.e. } t \in (t_0^c, t_{k-1}^c), \\ \lambda_k + \alpha(\overline{\lambda}_k - \lambda_k) & \text{for a.e. } t \in (t_{k-1}^c, \tilde{t}_k(\alpha)), \\ \lambda_q & \text{for a.e. } t \in (\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)) \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \lambda_N & \text{for a.e. } t \in (\tilde{t}_{N-1}(\alpha), t_N^c), \end{cases}$$

and

$$u^\alpha(t) := \begin{cases} u(t) & \text{for a.e. } t \in (t_0^c, t_{k-1}^c), \\ \tilde{u}_k(t) & \text{for a.e. } t \in (t_{k-1}^c, \tilde{t}_k(\alpha)), \\ \tilde{u}_q(t) & \text{for a.e. } t \in (\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)) \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \tilde{u}_N(t) & \text{for a.e. } t \in (\tilde{t}_{N-1}(\alpha), t_N^c), \end{cases}$$

for all $\alpha \in (0, \overline{\alpha}]$. From the construction and the results developed in Lemmas 2.5.2 and 2.5.4, one can easily see that $(x^\alpha, \lambda^\alpha, u^\alpha) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is a (perturbed) solution to (2.1), admitting the $\tilde{t}_q(\alpha)$ as crossing times, where we have introduced $\tilde{t}_q(\alpha) := t_q^c$ for all $q \in \{1, \dots, k-1\}$ and all $\alpha \in (0, \overline{\alpha}]$. The first, third, fourth and sixth items of Proposition 2.5.1 also directly follow, as well as the first assertion of the second item. The second assertion of the second item follows from the uniform convergence of $\tilde{z}_q^\alpha$ to $\tilde{z}_q$ over $[t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T]$ for all $q \in \{k, \dots, N\}$ when $\alpha \to 0$, from the convergence of $\tilde{t}_q(\alpha)$ to t_q^c for all $q \in \{k, \dots, N-1\}$ when $\alpha \to 0$, and from the equality $\tilde{z}_q = x$ over $[t_{q-1}^c, t_q^c]$ for all $q \in \{k, \dots, N\}$. Finally the fifth item follows from Lemma 2.5.2 since it holds that

$$\lim_{\alpha \to 0} \frac{x^\alpha(T) - x(T)}{\alpha} = \lim_{\alpha \to 0} \frac{\tilde{z}_N^\alpha(t_N^c) - \tilde{z}_N(t_N^c)}{\alpha} = w_N(t_N^c) = w(T),$$

which concludes the proof of Proposition 2.5.1.

2.5.3 Needle-like perturbation of the control

Consider the framework of Section 2.5.1. This entire Section 2.5.3 is dedicated to the proof of the next proposition which states a differentiability result at time $t = T$ for the trajectory x with respect to a needle-like perturbation of the control u. Since the proofs of this section are very similar to the ones of the previous Section 2.5.2, they are omitted.

Proposition 2.5.2. *Consider the framework of Section 2.5.1. Let $k \in \{1, ..., N\}$, let $v \in \mathbb{R}^m$ and let $\tau \in (t_{k-1}^c, t_k^c)$ be a Lebesgue point of the map $h(x(\cdot), \lambda(\cdot), u(\cdot), \cdot)$. Then there exists $0 < \overline{\alpha} < \min\{1, \tau - t_{k-1}^c\}$ such that, for all $\alpha \in (0, \overline{\alpha}]$, there exists a perturbed solution $(x^\alpha, \lambda^\alpha, u^\alpha) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times$*

$\mathrm{L}^\infty([0,T],\mathbb{R}^m)$ *to* (2.1) *such that:*

(i) *The corresponding perturbed partition of* $[0,T]$*, denoted by* $\{\tilde{t}_q(\alpha)\}_{q=0,\dots,N(\alpha)}$*, satisfies* $N(\alpha)=N$*, with* $\tilde{t}_q(\alpha)=t_q^c$ *for all* $q\in\{1,...,k-1\}$*, and* $\tilde{t}_q(\alpha)$ *tends to* t_q^c *when* $\alpha\to 0$ *for all* $q\in\{k,...,N-1\}$*.*

(ii) *The perturbed trajectory* x^α *follows the same regions than* x*, that is,* x^α *satisfies*

$$x^\alpha(t)\in E_q(t) \text{ for all } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\},$$

with $x^\alpha(0)=x_{\text{init}}\in E_1(0)$ *and* $x^\alpha(T)\in E_N(T)$*. Moreover* x^α *uniformly converges to* x *over* $[0,T]$ *when* $\alpha\to 0$*.*

(iii) *The perturbed regionally switching parameter* λ^α *is given by*

$$\lambda^\alpha(t)=\lambda_q \text{ for a.e. } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\}.$$

(iv) *The perturbed control* u^α *is given by*

$$u^\alpha(t)=\begin{cases} v & \text{for a.e. } t\in(\tau-\alpha,\tau),\\ \tilde{u}_k(t) & \text{for a.e. } t\in(t_{k-1}^c,\tau-\alpha)\cup(\tau,\tilde{t}_k(\alpha)),\\ \tilde{u}_q(t) & \text{for a.e. } t\in(\tilde{t}_{q-1}(\alpha),\tilde{t}_q(\alpha)) \text{ and all } q\in\{1,\dots,N\}\backslash\{k\},\end{cases}$$

where $\tilde{u}_q$ *stands for the auxiliary control defined in Section 2.5.1 for all* $q\in\{1,\dots,N\}$*.*

(v) *The limit*

$$\lim_{\alpha\to 0}\frac{x^\alpha(T)-x(T)}{\alpha}=w(T),$$

holds true, where

$$w(t):=\begin{cases} w_q(t), & \text{for all } t\in[t_{q-1}^c,t_q^c) \text{ and all } q\in\{k,...,N-1\},\\ w_N(t), & \text{for all } t\in[t_{N-1}^c,t_N^c],\end{cases}$$

where w_k *is the variation vector defined as the unique maximal solution (which is global) to the linearized Cauchy problem given by*

$$\begin{cases}\dot{w}(t)=\nabla_x f_k(\tilde{z}_k(t),\lambda_k,\tilde{u}_k(t),t)w(t), & \text{a.e. } t\in[t_{k-1}^c-\delta,t_k^c+\delta]\cap[0,T],\\ w(\tau)=f_k(\tilde{z}_k(\tau),\lambda_k,v,\tau)-f_k(\tilde{z}_k(\tau),\lambda_k,\tilde{u}_k(\tau),\tau),\end{cases}$$

and w_q *is the variation vector defined by induction as the unique maximal solution (which is global) to the linearized Cauchy problem given by*

$$\begin{cases}\dot{w}(t)=\nabla_x f_q(\tilde{z}_q(t),\lambda_q,\tilde{u}_q(t),t)w(t), & \text{a.e. } t\in[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T],\\ w(t_{q-1}^c)=w_{q-1}(t_{q-1}^c)+\xi_{q-1},\end{cases}$$

for all $q\in\{k+1,...,N\}$*, where* $\xi_q\in\mathbb{R}^n$ *stands for the jump vector defined by*

$$\xi_q:=\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)}((f_{q+1})^+(t_q^c)-(f_q)^-(t_q^c)),$$

for all $q\in\{k,...,N-1\}$*.*

(vi) *The limit*

$$\lim_{\alpha\to 0}\frac{\tilde{t}_q(\alpha)-t_q^c}{\alpha}=-\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)},$$

holds true for all $q\in\{k,\dots,N-1\}$.

Remark 2.5.2. Consider the framework of Proposition 2.5.2. It is worth noticing that the variation vector w, defined in Item (v), satisfies the discontinuity jump

$$w^+(t_q^c)-w^-(t_q^c)=\xi_q=\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w^-(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)}((f_{q+1})^+(t_q^c)-(f_q)^-(t_q^c)),$$

at each crossing time t_q^c for $q\in\{k,\dots,N-1\}$.

Construction of perturbed auxiliary non-hybrid trajectories

Lemma 2.5.5 (Construction of perturbed auxiliary non-hybrid trajectories)**.** *Consider the frameworks of Section 2.5.1 and Proposition 2.5.2. Let* $k\in\{1,...,N\}$*, let* $v\in\mathbb{R}^m$ *and let* $\tau\in(t_{k-1}^c,t_k^c)$ *be a Lebesgue point of the map* $h(x(\cdot),\lambda(\cdot),u(\cdot),\cdot)$*. Then there exists* $0<\overline{\alpha}<\min\{1,\tau-t_{k-1}^c\}$ *and, for all* $q\in\{k,...,N-1\}$*, there exists a function* $\tilde{t}_q\in\mathrm{C}([0,\overline{\alpha}],[t_q^c-\delta,t_q^c+\delta])$ *differentiable at* 0 *with* $\tilde{t}_q(0)=t_q^c$ *and*

$$\tilde{t}_q'(0)=-\frac{\langle\nabla_x F_q(x(t_q^c),t_q^c),w_q(t_q^c)\rangle_{\mathbb{R}^n}}{\langle\nabla_x F_q(x(t_q^c),t_q^c),(f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}+\nabla_t F_q(x(t_q^c),t_q^c)},$$

such that the perturbed auxiliary non-hybrid trajectories $\tilde{z}_q^\alpha:=y(\cdot,f_q,\theta_q^\alpha)$ *associated with the perturbed quadruplets* θ_q^α *defined by the induction*

$$\theta_q^\alpha:=\begin{cases}(\lambda_k,\tilde{u}_k^\alpha,t_{k-1}^c,x(t_{k-1}^c)) & \text{if}\quad q=k,\\ (\lambda_q,\tilde{u}_q,\tilde{t}_{q-1}(\alpha),\tilde{z}_{q-1}^\alpha(\tilde{t}_{q-1}(\alpha))) & \text{if}\quad q\in\{k+1,\dots,N\},\end{cases}$$

for all $\alpha\in[0,\overline{\alpha}]$ *and all* $q\in\{k,\dots,N\}$*, where* $\tilde{u}_k^\alpha$ *is the needle-like perturbation of* $\tilde{u}_k$ *(see Figure 2.8 in Section 2.2.3) given by*

$$\tilde{u}_k^\alpha(t):=\begin{cases}v & \text{if } t\in[\tau-\alpha,\tau),\\ \tilde{u}_k(t) & \text{if } t\notin[\tau-\alpha,\tau),\end{cases}$$

for almost every $t\in[0,T]$*, satisfy:*

- *for all* $q\in\{k,\dots,N\}$*, it holds that* $[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T]\subset I(f_q,\theta_q^\alpha)$ *for all* $\alpha\in[0,\overline{\alpha}]$*, that* $\tilde{z}_q^\alpha$ *uniformly converges to* $\tilde{z}_q$ *over* $[t_{q-1}^c-\delta,t_q^c+\delta]\cap[0,T]$ *when* $\alpha\to 0$*, and*

$$\lim_{\alpha\to 0}\frac{\tilde{z}_q^\alpha(t_q^c)-\tilde{z}_q(t_q^c)}{\alpha}=w_q(t_q^c).$$

- *for all* $q\in\{k,\dots,N-1\}$*, it holds that* $\tilde{z}_q^\alpha(t)\in\overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_q^c),\nu)$ *for all* $(\alpha,t)\in[0,\overline{\alpha}]\times[t_q^c-\delta,t_q^c+\delta]$*, that* $F_q(\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)),\tilde{t}_q(\alpha))=0$ *for all* $\alpha\in[0,\overline{\alpha}]$*, and that the map* $\alpha\in[0,\overline{\alpha}]\mapsto\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))\in\mathbb{R}^n$ *is continuous over* $[0,\overline{\alpha}]$ *and differentiable at* 0 *with*

$$\lim_{\alpha\to 0}\frac{\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))-\tilde{z}_q(t_q^c)}{\alpha}=w_q(t_q^c)+\tilde{t}_q'(0)(f_q)^-(t_q^c).$$

Proof. The proof is very similar to the one of Lemma 2.5.2 and thus is omitted. The only difference is that,

for the base case, one must note that we fix $\delta_0 \in [0, \delta]$ such that $\tau < t_k^c - \delta_0$ in order to have $\tilde{u}_k^\alpha = \tilde{u}_k$ for almost every $t \in [t_k^c - \delta_0, t_k^c + \delta_0]$ where $\tau \in (t_{k-1}^c, t_k^c)$ stands for a Lebesgue point of $h(x(\cdot), \lambda(\cdot), u(\cdot), \cdot)$. □

Admissibility of the perturbed auxiliary non-hybrid trajectories

Lemma 2.5.6. *Consider the framework of Lemma 2.5.5. Then, up to reducing $\overline{\alpha} > 0$, the following properties are satisfied:*

1. *There exists $s'_{k-1} \in (t_{k-1}^c, t_{k-1}^c + \delta]$ such that $\tilde{z}_k^\alpha(t) \in E_k(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (t_{k-1}^c, s'_{k-1}]$ (and for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t_0^c, s'_{k-1}]$ if $k = 1$).*
2. *For all $q \in \{k, ..., N-1\}$, there exists $s_q \in [t_q^c - \delta, t_q^c)$ such that $\tilde{z}_q^\alpha(t) \in E_q(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [s_q, \tilde{t}_q(\alpha))$.*
3. *For all $q \in \{k, ..., N-1\}$, there exists $s'_q \in (t_q^c, t_q^c + \delta]$ such that $\tilde{z}_{q+1}^\alpha(t) \in E_{q+1}(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (\tilde{t}_q(\alpha), s'_q]$.*
4. *There exists $s_N \in [t_N^c - \delta, t_N^c)$ such that $\tilde{z}_N^\alpha(t) \in E_N(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times [s_N, t_N^c]$.*

Proof. The proof is very similar to the one of Lemma 2.5.3 and thus is omitted. □

Lemma 2.5.7 (Admissibility of the perturbed auxiliary non-hybrid trajectories)**.** *Consider the framework of Lemma 2.5.5. Then, up to reducing $\overline{\alpha} > 0$, it holds that:*

1. *$\tilde{z}_k^\alpha(t) \in E_k(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (t_{k-1}^c, \tilde{t}_k(\alpha))$ (and for all $(\alpha, t) \in [0, \overline{\alpha}] \times [t_0^c, \tilde{t}_k(\alpha))$ if $k = 1$).*
2. *$\tilde{z}_q^\alpha(t) \in E_q(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha))$ and all $q \in \{k+1, ..., N-1\}$.*
3. *$\tilde{z}_N^\alpha(t) \in E_N(t)$ for all $(\alpha, t) \in [0, \overline{\alpha}] \times (\tilde{t}_{N-1}(\alpha), t_N^c]$.*

Proof. The proof is very similar to the one of Lemma 2.5.4 and thus is omitted. □

Proof of Proposition 2.5.2

Let us fix $k \in \{1, ..., N\}$, $v \in \mathbb{R}^m$ and $\tau \in (t_{k-1}^c, t_k^c)$ being a Lebesgue point of the map $h(x(\cdot), \lambda(\cdot), u(\cdot), \cdot)$. Consider the perturbed auxiliary non-hybrid trajectories $\tilde{z}_q^\alpha = y(\cdot, f_q, \theta_q^\alpha)$ over $[t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T]$ for all $q \in \{k, \dots, N\}$ and all $\alpha \in (0, \overline{\alpha}]$ constructed in Lemma 2.5.5, together with the corresponding implicit functions $\tilde{t}_q$ for all $q \in \{k, \dots, N-1\}$. As explained in Section 2.2.3, we define by concatenation

$$x^\alpha(t) := \begin{cases} x(t) & \text{for all } t \in [t_0^c, t_{k-1}^c], \\ \tilde{z}_k^\alpha(t) & \text{for all } t \in [t_{k-1}^c, \tilde{t}_k(\alpha)], \\ \tilde{z}_q^\alpha(t) & \text{for all } t \in [\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)] \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \tilde{z}_N^\alpha(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c], \end{cases}$$

and

$$\lambda^\alpha(t) := \begin{cases} \lambda(t) & \text{for a.e. } t \in (t_0^c, t_{k-1}^c), \\ \lambda_k & \text{for a.e. } t \in (t_{k-1}^c, \tilde{t}_k(\alpha)), \\ \lambda_q & \text{for a.e. } t \in (\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)) \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \lambda_N & \text{for a.e. } t \in (\tilde{t}_{N-1}(\alpha), t_N^c), \end{cases}$$

and

$$u^\alpha(t) = \begin{cases} u(t) & \text{for a.e. } t \in (t_0^c, t_{k-1}^c), \\ v & \text{for a.e. } t \in (\tau-\alpha, \tau), \\ \tilde{u}_k(t) & \text{for a.e. } t \in (t_{k-1}^c, \tau-\alpha) \cup (\tau, \tilde{t}_k(\alpha)), \\ \tilde{u}_q(t) & \text{for a.e. } t \in (\tilde{t}_{q-1}(\alpha), \tilde{t}_q(\alpha)) \text{ and all } q \in \{k+1, \dots, N-1\}, \\ \tilde{u}_N(t) & \text{for a.e. } t \in (\tilde{t}_{N-1}(\alpha), t_N^c), \end{cases}$$

for all $\alpha \in (0, \overline{\alpha}]$. The remaining details of the proof are omitted, since we use similar arguments than the proof of Proposition 2.5.1.

2.6 Proof of Theorem 2.2.1

Let $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ be a solution to (2.2), that is moreover a regular solution to (2.1). In the sequel we will use the notations introduced in Definitions 2.2.1 and 2.2.2 and the results obtained in the previous Section 2.5.

Definition of an adjoint vector. We define an adjoint vector $p \in \mathrm{PAC}_{\mathbb{T}}([0,T], \mathbb{R}^n)$ as

$$p(t) := \begin{cases} p_1(t) & \text{for all } t \in [t_0^c, t_1^c), \\ p_k(t) & \text{for all } t \in (t_{k-1}^c, t_k^c) \text{ and all } k \in \{2, ..., N-1\}, \\ p_N(t) & \text{for all } t \in (t_{N-1}^c, t_N^c], \end{cases}$$

where p_N is defined as the unique maximal solution (which is global) to the linear Cauchy problem given by

$$\begin{cases} \dot{p}(t) = -\nabla_x f_N(\tilde{z}_N(t), \lambda_N, \tilde{u}_N(t), t)^\top p(t), & \text{a.e. } t \in [t_{N-1}^c - \delta, T] \cap [0,T], \\ p(T) = -\nabla \phi(x(T)), \end{cases}$$

and p_k is defined by backward induction as the unique maximal solution (which is global) to the linear Cauchy problem given by

$$\begin{cases} \dot{p}(t) = -\nabla_x f_k(\tilde{z}_k(t), \lambda_k, \tilde{u}_k(t), t)^\top p(t), & \text{a.e. } t \in [t_{k-1}^c - \delta, t_k^c + \delta] \cap [0,T], \\ p^-(t_k^c) = p_{k+1}^+(t_k^c) - \chi_k, \end{cases}$$

for all $k \in \{1, ..., N-1\}$, where $\chi_k \in \mathbb{R}^n$ stands for the jump vector defined by

$$\chi_k := -\frac{\langle p_{k+1}^+(t_k^c), (f_{k+1})^+(t_k^c) - (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla_x F_k(x(t_k^c), t_k^c), (f_k)^-(t_k^c) \rangle_{\mathbb{R}^n} + \nabla_t F_k(x(t_k^c), t_k^c)} \nabla_x F_k(x(t_k^c), t_k^c),$$

for all $k \in \{1, ..., N-1\}$. With the above construction, observe that $\chi_k = p^+(t_k^c) - p^-(t_k^c)$ corresponds to the discontinuity jump of p at each crossing time t_k^c. We refer to Remark 2.2.4 for details on the choice of such an expression for the discontinuity jumps of p.
Recall that $\tilde{z}_k(t) = x(t)$ for all $t \in [t_{k-1}^c, t_k^c]$ and all $k \in \{1, ..., N\}$. Through concatenation of the above linear Cauchy problems, one can easily see that the first item of Theorem 2.2.1 is fulfilled. Furthermore, from the above Cauchy conditions, the second and third items of Theorem 2.2.1 also trivially follow.

The Hamiltonian maximization condition. Let us fix $k \in \{1, ..., N\}$, $v \in \mathrm{U}$ and $\tau \in (t_{k-1}^c, t_k^c)$ being a Lebesgue point of $h(x(\cdot), \lambda(\cdot), u(\cdot), \cdot)$. Consider $0 < \overline{\alpha} < \min\{1, \tau - t_{k-1}^c\}$ given in Proposition 2.5.2.

From the construction detailed in Proposition 2.5.2 and explained in details in Section 2.2.3, and since U is assumed to be closed (which guarantees that the limits $u^-(t_k^c)$ and $u^+(t_k^c)$ belongs to U), one can easily see that the perturbed solution $(x^\alpha, \lambda^\alpha, u^\alpha)$ to (2.1) satisfies all the constraints of Problem (2.2) for all $\alpha \in (0, \overline{\alpha}]$. Thus, from optimality of the triplet (x, λ, u), we get that

$$\frac{\phi(x^\alpha(T)) - \phi(x(T))}{\alpha} \geq 0,$$

for all $\alpha \in (0, \overline{\alpha}]$ and, taking the limit $\alpha \to 0$, we get from Proposition 2.5.2 that $\langle \nabla\phi(x(T)), w(T)\rangle_{\mathbb{R}^n} \geq 0$ which can be rewritten as $\langle p(T), w(T)\rangle_{\mathbb{R}^n} \leq 0$.
From the linear Cauchy problems satisfied by p and w over each open interval (t_{q-1}^c, t_q^c) for $q \in \{k, \dots, N-1\}$ (and over $(t_{N-1}^c, t_N^c]$ for $q = N$), one can easily see that the scalar product $\langle p(\cdot), w(\cdot)\rangle_{\mathbb{R}^n}$ is constant over each of these intervals.
Now let us prove that $\langle p^+(t_q^c), w^+(t_q^c)\rangle_{\mathbb{R}^n} = \langle p^-(t_q^c), w^-(t_q^c)\rangle_{\mathbb{R}^n}$ at each crossing time t_q for $q \in \{k, \dots, N-1\}$. To this aim note that the definition of χ_q has been selected to get that

$$\frac{\langle p^+(t_q^c), (f_{q+1})^+(t_q^c) - (f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_q(x(t_q^c), t_q^c), (f_q)^-(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c), t_q^c)} = \frac{\langle p^-(t_q^c), (f_{q+1})^+(t_q^c) - (f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_q(x(t_q^c), t_q^c), (f_{q+1})^+(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c), t_q^c)},$$

by replacing $p^-(t_q^c)$ in the above right-hand term by $p^-(t_q^c) = p^+(t_q^c) - \chi_q$. In particular χ_q can thus be rewritten as

$$\chi_q = -\frac{\langle p^-(t_q^c), (f_{q+1})^+(t_q^c) - (f_q)^-(t_q^c)\rangle_{\mathbb{R}^n}}{\langle \nabla_x F_q(x(t_q^c), t_q^c), (f_{q+1})^+(t_q^c)\rangle_{\mathbb{R}^n} + \nabla_t F_q(x(t_q^c), t_q^c)} \nabla_x F_q(x(t_q^c), t_q^c),$$

which leads to $\langle p^-(t_q^c), \xi_q\rangle_{\mathbb{R}^n} + \langle \chi_q, w^-(t_q^c)\rangle_{\mathbb{R}^n} + \langle \chi_q, \xi_q\rangle_{\mathbb{R}^n} = 0$. Therefore, from the equality $\langle p^+(t_q^c), w^+(t_q^c)\rangle_{\mathbb{R}^n} = \langle p^-(t_q^c) + \chi_q, w^-(t_q^c) + \xi_q\rangle_{\mathbb{R}^n}$, we get that $\langle p^+(t_q^c), w^+(t_q^c)\rangle_{\mathbb{R}^n} = \langle p^-(t_q^c), w^-(t_q^c)\rangle_{\mathbb{R}^n}$.
Finally, by simple backward induction, we have obtained that $\langle p(\tau), w(\tau)\rangle_{\mathbb{R}^n} \leq 0$. From the value of $w(\tau)$ given in Proposition 2.5.2, this inequality gives

$$H(\tilde{z}_k(\tau), \lambda_k, v, p_k(\tau), \tau) \leq H(\tilde{z}_k(\tau), \lambda_k, \tilde{u}_k(\tau), p_k(\tau), \tau),$$

which can be rewritten as

$$H(x(\tau), \lambda(\tau), v, p(\tau), \tau) \leq H(x(\tau), \lambda(\tau), u(\tau), p(\tau), \tau),$$

which concludes this paragraph.

The averaged Hamiltonian gradient condition. Let us fix $k \in \{1, ..., N\}$. Consider some $\overline{\lambda}_k \in \Lambda$ and $0 < \overline{\alpha} \leq 1$ given in Proposition 2.5.1. From the convexity of Λ and the construction detailed in Proposition 2.5.1 and explained in details in Section 2.2.3, one can easily see that the perturbed solution $(x^\alpha, \lambda^\alpha, u^\alpha)$ to (2.1) satisfies all the constraints of Problem (2.2) for all $\alpha \in (0, \overline{\alpha}]$. Thus, from optimality of the triplet (x, λ, u), we get that

$$\frac{\phi(x^\alpha(T)) - \phi(x(T))}{\alpha} \geq 0,$$

for all $\alpha \in (0, \overline{\alpha}]$ and, taking the limit $\alpha \to 0$, we get from Proposition 2.5.1 that $\langle \nabla\phi(x(T)), w(T)\rangle_{\mathbb{R}^n} \geq 0$ which can be rewritten as $\langle p(T), w(T)\rangle_{\mathbb{R}^n} \leq 0$.
Using similar arguments than in the previous paragraph, one can derive that $\langle p^-(t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n} \leq 0$. Now recall that the classical Duhamel formula leads to $p(s) = \Phi(t_k^c, s)^\top p^-(t_k^c)$ for all $s \in (t_{k-1}^c, t_k^c)$ and

$$w^-(t_k^c) = \int_{t_{k-1}^c}^{t_k^c} \Phi(t_k^c, s)\nabla_\lambda f_k(\tilde{z}_k(s), \lambda_k, \tilde{u}_k(s), s)(\overline{\lambda}_k - \lambda_k)\, \mathrm{d}s,$$

where Φ stands for the state transition matrix associated with the matrix function $\nabla_x f_k(\tilde{z}_k(\cdot), \lambda_k, \tilde{u}_k(\cdot), \cdot)$. Therefore the inequality $\langle p^-(t_k^c), w^-(t_k^c)\rangle_{\mathbb{R}^n} \leq 0$ gives

$$\left\langle \int_{t_{k-1}^c}^{t_k^c} \nabla_\lambda f_k(\tilde{z}_k(s), \lambda_k, \tilde{u}_k(s), s)^\top p(s)\, \mathrm{d}s, \overline{\lambda}_k - \lambda_k \right\rangle_{\mathbb{R}^d} \leq 0,$$

which can be rewritten as

$$\left\langle \int_{t_{k-1}^c}^{t_k^c} \nabla_\lambda H(x(s), \lambda_k, u(s), p(s), s)\, \mathrm{d}s, \overline{\lambda}_k - \lambda_k \right\rangle_{\mathbb{R}^d} \leq 0.$$

Since the above inequality is satisfied for any $\overline{\lambda}_k \in \Lambda$, this paragraph is complete, and so is the proof of Theorem 2.2.1.

2.7 Application to optimal control problems with loss control regions

This section is organized as follows. In Section 2.7.1, we introduce an optimal control problem with loss control regions, along with terminology and assumptions. In Section 2.7.2, we discuss the notion of regular solution to the corresponding control system and we state the main theoretical result of this section (Theorem 2.7.1) which is a PMP with loss control regions. Its proof is based on Theorem 2.2.1 that was established previously.

2.7.1 Mayer optimal control problem with loss control regions

Throughout this section we consider a partition of $\mathbb{R}^n$ given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where $\mathcal{J}$ is a (possibly infinite) family of indexes and the nonempty connected open subsets X_j (called *regions*) are disjoint. We assume that each region is either a control region, either a loss control region (see General introduction 0 for details), and thus, for all $j \in \mathcal{J}$, we introduce

$$q_j := \begin{cases} 1 & \text{if } X_j \text{ is a control region,} \\ 0 & \text{if } X_j \text{ is a loss control region.} \end{cases}$$

Now we introduce the control system

$$\begin{cases} (x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m), \\ \dot{x}(t) = f(x(t),u(t)) \text{ a.e. } t \in [0,T], \\ x(0) = x_{\mathrm{init}}, \\ u \text{ is constant in loss control regions}, \end{cases} \tag{2.14}$$

where the initial condition x_{init} is fixed with $x_{\mathrm{init}} \in X_{j_1}$ for some $j_1 \in \mathcal{J}$, and the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1. As usual in the literature, x is called *state* (or *trajectory*) and u is called *control*. We now give a precise definition of a solution to (2.14).

Definition 2.7.1. *We say that a pair $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is a solution to (2.14) if the four following conditions are satisfied:*

1. *There exists a partition $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ of $[0,T]$, with $N \in \mathbb{N}^*$, such that, for all $k \in \{1,\dots,N\}$, there exists $j(k) \in \mathcal{J}$ such that*

$$\forall t \in (t_{k-1}^c, t_k^c), \quad x(t) \in X_{j(k)},$$

   with $j(k) \neq j(k-1)$ for all $k \in \{2,\dots,N\}$. In the sequel, the times t_k^c, for $k \in \{1,\dots,N-1\}$, are called crossing times and correspond to the instants at which the trajectory x goes from one region to another (in particular $x(t_k^c)$ belongs to the interface $\partial X_{j(k)} \cap \partial X_{j(k+1)}$).
2. *It holds that $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.*
3. *The state equation $\dot{x}(t) = f(x(t),u(t))$ is satisfied for almost every $t \in [0,T]$ and $x(0) = x_{\mathrm{init}}$ (and thus $j(1) = j_1$).*
4. *For all $k \in \{1,\dots,N\}$ such that $q_{j(k)} = 0$, the control u is constant over (t_{k-1}^c, t_k^c) (the constant value being denoted by u_k in the sequel).*

Our objective in the present work is to derive first-order necessary optimality conditions (in a PMP form) for the Mayer optimal control problem with loss control regions given by

$$\begin{array}{ll} \text{minimize} & \phi(x(T)), \\ \text{subject to} & (x,u) \text{ solution to (2.14)}, \\ & u(t) \in \mathrm{U} \text{ a.e. } t \in [0,T], \end{array} \tag{2.15}$$

where the Mayer cost function $\phi : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and the control constraint set U is a nonempty closed convex subset of $\mathbb{R}^m$.

2.7.2 Regular solution and necessary optimality conditions

The main result of the paper [14] that we are presenting in this section is based on some regularity assumptions. It concerns the transverse behavior of the optimal trajectory at the interfaces between regions. The precise hypotheses are provided in the next definition and are standard (see, *e.g.*, [19], [73]).

Definition 2.7.2. *Consider a solution (x,u) to (2.14) and the notations introduced in Definition 2.7.1. Set $\alpha := \frac{1}{3}\min_{k=1,\dots,N} |t_k - t_{k-1}| > 0$. We say that (x,u) is regular if there exist $\delta \in (0,\alpha)$ and $\nu > 0$ such that:*

1. *At each crossing time t_k^c, the control u is continuous over $[t_k^c - \delta, t_k^c)$ and over $(t_k^c, t_k^c + \delta]$, and admits left and right limits denoted by $u^-(t_k^c)$ and $u^+(t_k^c)$.*

2. *At each crossing time* t_k^c*, there exists a* C^1 *function* $F_k : \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \nu) \to \mathbb{R}$ *such that*

$$\begin{cases} y \in X_{j(k)} & \Leftrightarrow \quad F_k(y) < 0, \\ y \in \partial X_{j(k)} \cap \partial X_{j(k+1)} & \Leftrightarrow \quad F_k(y) = 0, \\ y \in X_{j(k+1)} & \Leftrightarrow \quad F_k(y) > 0, \end{cases}$$

for all $y \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(t_k^c), \nu)$.

3. *At each crossing time* t_k^c*, the transverse conditions depicted in Figure 2.13 and given by*

$$\langle \nabla F_k(x(t_k^c)), (f)^-(t_k^c) \rangle > 0,$$

$$\langle \nabla F_k(x(t_k^c)), (f)^+(t_k^c) \rangle > 0,$$

are fulfilled, where $(f)^\pm(t_k^c) := f(x(t_k^c), u^\pm(t_k^c))$.

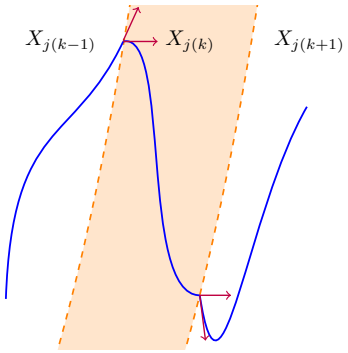


Figure 2.13: Illustration of a regular trajectory (in blue) crossing transversally the interfaces between regions.

Let $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ stand for the Hamiltonian function associated with the Mayer optimal control problem (2.15) defined by

$$H(x, u, p) := \langle p, f(x, u) \rangle,$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. Our main result is as follows.

Theorem 2.7.1. *Let* (x, u) *be a solution to* (2.15)*, which is moreover a regular solution to* (2.14)*, and consider the notations introduced in Definitions 2.7.1 and 2.7.2. Then there exists a piecewise absolutely continuous costate* $p : [0, T] \to \mathbb{R}^n$*, respecting the partition* $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ *of* $[0, T]$*, such that:*

1. *The adjoint equation* $\dot{p}(t) = -\nabla_x f(x(t), u(t))^\top p(t)$ *is satisfied for almost every* $t \in [0, T]$.
2. *The final condition* $p(T) = -\nabla \phi(x(T))$ *is satisfied.*
3. *At each crossing time* t_k^c*, the discontinuity condition*

$$p^+(t_k^c) - p^-(t_k^c) = \beta_k \nabla F_k(x(t_k^c)),$$

with

$$\beta_k := -\frac{\langle p^+(t_k^c), (f)^+(t_k^c) - (f)^-(t_k^c) \rangle}{\langle \nabla F_k(x(t_k^c)), (f)^-(t_k^c) \rangle},$$

is fulfilled.

4. *For all* $k \in \{1, \dots, N\}$ *such that* $q_{j(k)} = 1$*, the Hamiltonian maximization condition*

$$u(t) \in \arg\max_{v \in \mathrm{U}} H(x(t), v, p(t)),$$

is fulfilled for almost every $t \in (t_{k-1}^c, t_k^c)$.

5. *For all* $k \in \{1, ..., N\}$ *such that* $q_{j(k)} = 0$, *the averaged Hamiltonian gradient condition*

$$\int_{t_{k-1}^c}^{t_k^c} \nabla_u H(x(s), u_k, p(s))\, \mathrm{d}s \in \mathrm{N}_{\mathrm{U}}(u_k).$$

holds true, where $\mathrm{N}_{\mathrm{U}}(u_k)$ *stands for the normal cone to* U *at* u_k.

The proof of Theorem 2.7.1 is a direct application of the hybrid maximum principle developed in the paper [13] and in Section 2.2 that takes into account a regionally switching parameter. Indeed, one has just to see that the control system (2.14) can be rewritten as the hybrid control system given by

$$\begin{cases} (x, u) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m), \\ \lambda : [0, T] \to \mathbb{R}^m \text{ is a regionally switching parameter associated with } x, \\ \dot{x}(t) = h(x(t), \lambda(t), u(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_{\mathrm{init}}, \end{cases}$$

where the hybrid dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by

$$h(x, \lambda, u) := h_j(x, \lambda, u) \text{ if } x \in X_j,$$

where

$$h_j(x, \lambda, u) := \begin{cases} f(x, u) & \text{if} \quad q_j = 1, \\ f(x, \lambda) & \text{if} \quad q_j = 0, \end{cases}$$

for all $(x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ and all $j \in \mathcal{J}$. Indeed, let us recall that a regionally switching parameter is a function that remains constant while the state position x stays inside a region, and can switch (that is, can change its value) only when the state position x goes from one region to another. To keep this work concise, we do not include the detailed proof of Theorem 2.7.1, but we refer to [13] and Section 2.2 for details.

2.7.3 Comments

• To summarize, Theorem 2.7.1 shows that, in each control region, the usual Hamiltonian maximization condition holds true, whereas, in each loss control region, a so-called averaged Hamiltonian gradient condition (in the spirit of the one obtained for optimal sampled-data control problems, see [34], [35], [41]) holds true. It is worth mentioning that the latter is implicit in general since u_k intervenes, not only in both sides of the equation, but moreover in the values of x and p along the interval (t_{k-1}^c, t_k^c). Furthermore we do not know in advance the values of t_{k-1}^c and t_k^c. However, as we will see in Section 2.8, the averaged Hamiltonian gradient condition can be useful to determine the optimal values of the control in loss control regions.

• In Theorem 2.7.1, and as usual in the literature, the Hamiltonian system $(\dot{x}, \dot{p}) = (\nabla_p H, -\nabla_x H)$ is satisfied which implies (together with the other necessary optimality conditions) that the Hamiltonian function

$$t \mapsto H(x(t), u(t), p(t)),$$

is constant almost everywhere over $[0, T]$. Indeed, in a control region (with the Hamiltonian maximization condition), one has just to use the standard argumentation (see [102]). In a loss control region, the result is straightforward since the control is constant. Finally the discontinuity conditions ensure the constancy at each crossing time t_k^c (see [73]).

• To benefit the most of Theorem 2.7.1 (and avoid unnecessary hypotheses), the partition of $\mathbb{R}^n$ must be written so that the number of regions involved is as small as possible. The idea is to avoid, for example, trajectories that would go from a control region to another one (which would be redundant from a model point of view).

• In Theorem 2.7.1, the discontinuity condition at each crossing time t_k^c is written backward in time. Nonetheless, it can also be written forward in time by noting that

$$\beta_k = -\frac{\langle p^-(t_k^c), (f)^+(t_k^c) - (f)^-(t_k^c)\rangle}{\langle \nabla F_k(x(t_k^c)), (f)^+(t_k^c)\rangle}.$$

• In Definition 2.7.2, note that the continuity and limit conditions on the control are superfluous in loss control regions (since u is constant in such a region).

• Several extensions of Theorem 2.7.1 could be of interest and can be easily derived. For the following possible extensions, we refer to [13] for details:

- Theorem 2.7.1 can be extended to a non-autonomous setting, as well on the dynamics as on the partition of the state space.
- The convexity (resp. closedness) hypothesis on U can be removed by using a generalized version of the normal cone (resp. by assuming that all the limits $u^-(t_k^c)$ and $u^+(t_k^c)$ belong to U).
- One can consider a control constraint set U_j in each region X_j. This would allow to impose the control value in loss control regions. For example, to deal with the case where no control input is allowed in loss control regions, take $\mathrm{U}_j = \{0_{\mathbb{R}^m}\}$ for all $j \in \mathcal{J}$ such that $q_j = 0$.
- One can consider a Bolza cost, involving a Lagrange cost associated with a hybrid Lagrangian function adapted to the partition. This setting would allow to deal with time crisis problems for which the constraint set K is a control region (or a loss control region).

2.8 Example

In this section, we highlight the use of Theorem 2.7.1 on a simple one-dimensional Mayer optimal control problem with one loss control region. Here $n = m = 1$ and $T = 8$.

2.8.1 Presentation of the example

Consider the partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3}$ with

$$\begin{aligned} X_1 &:= \{y \in \mathbb{R} \mid y < -1\}, \\ X_2 &:= \{y \in \mathbb{R} \mid -1 < y < \tfrac{1}{2}\}, \\ X_3 &:= \{y \in \mathbb{R} \mid y > \tfrac{1}{2}\}. \end{aligned}$$

In what follows, we suppose that X_1 and X_3 are control regions (that is $q_1 = q_3 = 1$) and X_2 is a loss control region (that is $q_2 = 0$). Now consider the Mayer optimal control problem given by

$$
\begin{aligned}
\text{minimize} \quad & -x(8), \\
\text{subject to} \quad & (x,u) \in \mathrm{AC}([0,8],\mathbb{R}) \times \mathrm{L}^\infty([0,8],\mathbb{R}), \\
& \dot{x}(t) = u(t)x(t) + 1 \text{ a.e. } t \in [0,8], \\
& x(0) = -2, \\
& u \text{ is constant in the loss control region } X_2, \\
& u(t) \in [-\tfrac{3}{2}, \tfrac{1}{2}] \text{ a.e. } t \in [0,8].
\end{aligned}
$$

The situation is depicted in Figure 2.14 and the corresponding Hamiltonian is given by

$$H(x,u,p) := p(ux+1),$$

for all $(x,u,p) \in \mathbb{R}^3$.

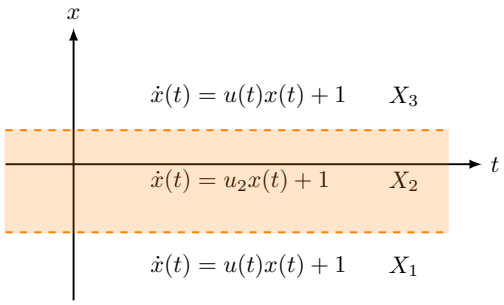


Figure 2.14: Illustration of the framework of Section 2.8.

2.8.2 Synthesis of an optimal control

In this section, we assume that there exists a solution (x,u), that is regular, and we suppose that it admits exactly two (unknown) crossing times $0 < t_1^c < t_2^c < 8$ and satisfies the following structure:

$$
\begin{aligned}
t \in [0, t_1^c) \quad &\Rightarrow \quad (x(t),u(t)) \in X_1 \times [-\tfrac{3}{2}, \tfrac{1}{2}], \\
t \in (t_1^c, t_2^c) \quad &\Rightarrow \quad (x(t),u(t)) \in X_2 \times \{u_2\}, \\
t \in (t_2^c, 8] \quad &\Rightarrow \quad (x(t),u(t)) \in X_3 \times [-\tfrac{3}{2}, \tfrac{1}{2}],
\end{aligned}
$$

where $u_2 \in [-\frac{3}{2}, \frac{1}{2}]$ is unknown and assumed to satisfy $u_2 \neq 0$. Now let us denote by $p : [0,8] \to \mathbb{R}$ the costate provided by Theorem 2.7.1. We proceed to the analysis backward in time.

• *Step 1: analysis in the region X_3.* The adjoint equation and final condition give

$$
\begin{cases}
\dot{p}(t) = -u(t)p(t) \text{ a.e. } t \in [t_2^c, 8], \\
p(8) = 1,
\end{cases}
$$

and the Hamiltonian maximization condition writes

$$u(t) \in \underset{v \in [-\frac{3}{2}, \frac{1}{2}]}{\arg\max}\, x(t)p(t)v \text{ a.e. } t \in (t_2^c, 8).$$

Since $p(t) > 0$ and $x(t) > 0$ over $(t_2^c, 8]$, we get that $u(t) = 1/2$ for almost every $t \in (t_2^c, 8)$ and thus, since $x(t_c^2) = 1/2$, we get that

$$p(t) = e^{4-(t/2)} \quad \text{and} \quad x(t) = \tfrac{5}{2}e^{(t-t_2^c)/2} - 2,$$

for all $t \in (t_2^c, 8]$.

• *Step 2: analysis in the region X_2.* From the discontinuity condition at t_2^c and the adjoint equation, the costate p satisfies

$$\begin{cases} \dot{p}(t) = -u_2 p(t) \text{ a.e. } t \in [t_1^c, t_2^c], \\ p^-(t_2^c) = \frac{5}{2u_2+4} e^{4-(t_2^c/2)}. \end{cases}$$

We get that

$$\begin{array}{rcl} p(t) & = & \frac{5}{2u_2+4} e^{4-(t_2^c/2)} e^{u_2(t_2^c - t)}, \\ x(t) & = & \frac{1}{2u_2}((u_2+2)e^{u_2(t-t_2^c)} - 2), \end{array} \tag{2.16}$$

for all $t \in (t_1^c, t_2^c)$. Since $x(t_1^c) = -1$, one deduces that

$$\tfrac{1}{2u_2}((u_2+2)e^{u_2(t_1^c - t_2^c)} - 2) = -1$$

and the relation between t_1^c and t_2^c given by

$$t_2^c = t_1^c + \frac{1}{u_2} \ln\left(\frac{u_2+2}{2(1-u_2)}\right). \tag{2.17}$$

• *Step 3: analysis in the region X_1.* From the discontinuity condition at t_1^c and the adjoint equation, the costate p satisfies

$$\begin{cases} \dot{p}(t) = -u(t)p(t) \text{ a.e. } t \in [0, t_1^c], \\ p^-(t_1^c) = \frac{5(1-u_2)}{(2u_2+4)(1-u^-(t_1^c))} e^{4-(t_2^c/2)} e^{u_2(t_2^c - t_1^c)}, \end{cases}$$

and the Hamiltonian maximization condition writes

$$u(t) \in \underset{v \in [-\frac{3}{2}, \frac{1}{2}]}{\arg\max} \; x(t)p(t)v \quad \text{a.e. } t \in (0, t_1^c).$$

Since $p(t) > 0$ and $x(t) < 0$ over $[0, t_1^c]$, we deduce that $u(t) = -3/2$ for almost every $t \in [0, t_1^c]$ and thus, since $x(0) = -2$, we get that

$$\begin{array}{rcl} p(t) & = & \frac{1}{2} e^{u_2(4-(t_2^c/2))} e^{(3/2)(t-t_1^c)}, \\ x(t) & = & \frac{2}{3} - \frac{8}{3} e^{-(3/2)t}, \end{array}$$

for all $t \in [0, t_1^c)$.

• *Step 4: global analysis.* From $x(t_1^c) = -1$, one can easily obtain that $t_1^c = \frac{2}{3}\ln(\frac{8}{5})$. Furthermore, we can now determine the value $u_2 \in [-\frac{3}{2}, \frac{1}{2}]$ thanks to the averaged Hamiltonian gradient condition which writes

$$\gamma(u_2) := \int_{t_1^c}^{t_2^c} x(s)p(s) \, \mathrm{d}s \in \mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2). \tag{2.18}$$

Using (2.16) and (2.17), we find that:

- if $u_2 = 1/2$, then $\mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2) = \mathbb{R}_+$ and $\gamma(u_2) \simeq -26.48 < 0$ which contradicts (2.18);
- if $u_2 = -3/2$, then $\mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2) = \mathbb{R}_-$ and $\gamma(u_2) \simeq 15.61 > 0$ which contradicts (2.18).

It follows that $u_2 \in (-\frac{3}{2}, \frac{1}{2})$ and thus (2.18) implies that $\gamma(u_2) = 0$ which amounts to solving the equation

$$(u_2 - 1)(u_2 + 2) \ln\left(\frac{1 + \frac{u_2}{2}}{1 - u_2}\right) + 3u_2 = 0.$$

This gives us a unique value for $u_2 \in (-\frac{3}{2}, \frac{1}{2})$ given approximately by $u_2 \simeq -0.75$. Finally the optimal control is given by

$$u(t) = \begin{cases} -\frac{3}{2} & \text{a.e. } t \in (0, t_1^c), \\ u_2 & \text{a.e. } t \in (t_1^c, t_2^c), \\ \frac{1}{2} & \text{a.e. } t \in (t_2^c, 8), \end{cases}$$

with $t_1^c = \frac{2}{3}\ln(\frac{8}{5}) \simeq 0.31$ and $t_2^c \simeq 1.68$.

2.8.3 Comparisons with other control strategies

We end-up this case study with comparisons of the optimal control u obtained in the previous section with different control strategies. To keep this work concise, the computations of this section are omitted.

- First, note that, if the region X_2 was a control region, then the classical PMP would imply that the optimal (permanent) control $\hat{u}$ (associated with the trajectory $\hat{x}$) satisfies

$$\hat{u}(t) = \begin{cases} -3/2 & \text{if} \quad \hat{x}(t) < 0, \\ 1/2 & \text{if} \quad \hat{x}(t) > 0, \end{cases}$$

  for almost every $t \in [0, 8]$. However, since X_2 (a loss control region) is a strip containing 0, the control $\hat{u}$ is not admissible (since it requires to change its value in the loss control region X_2, see Figure 2.15).
- Second, from the (nonadmissible) control $\hat{u}$, one might consider the admissible control $u^\perp$ (resp. $u^\dagger$) given by $u^\perp = -3/2$ in both regions X_1 and X_2 (resp. $u^\dagger = -3/2$ in region X_1) and by $u^\perp = 1/2$ in region X_3 (resp. $u^\dagger = 1/2$ in regions X_2 and X_3). The associated trajectory is denoted by $x^\perp$ (resp. $x^\dagger$).

On Figure 2.15, we depict the trajectories x, $\hat{x}$, $x^\perp$ and $x^\dagger$. As expected, the cost associated with $\hat{x}$ is the best, but is not admissible, while the cost associated with x is admissible and better than the other admissible costs associated with $x^\perp$ and $x^\dagger$. This example shows the relevancy of establishing a PMP in the present context of loss of control since, in general, the optimal constant values in loss control regions do not follow the values of the optimal permanent control obtained with the classical PMP. Furthermore, note that, in contrary to what is usually observed in the classical literature (with permanent controls) when the Hamiltonian is linear with respect to the control, the loss of control can induce optimal constant values in loss control regions that do not saturate the control constraint set U. With this example, we also emphasize that the avegared Hamiltonian gradient condition derived in Theorem 2.7.1 allows to determine such optimal values.

2.9 Conclusion and perspectives

In this work, we have introduced a new framework in optimal control theory letting the possibility for a control system to be subject to loss of control depending on its position in a partition of the state space. In our approach, the control value has to be fixed to an admissible value as long as the system belongs to a loss control region but we do not know in advance how long the system stays in such a region. The corresponding optimal constant value satisfies (and possibly is determined by) the averaged

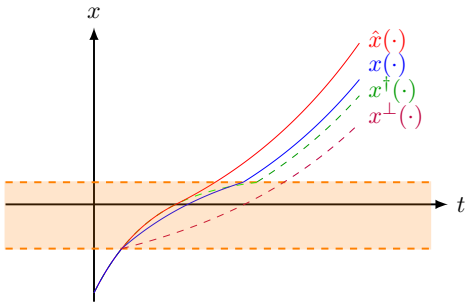


Figure 2.15: Trajectories x, $\hat{x}$, $x^{\perp}$ and $x^{\dagger}$ from Section 2.8.3 (zoom on the time interval $[0, 3]$).

Hamiltonian gradient condition. We believe that this setting differs from other frameworks covered by hybrid optimal control problems or state constrained optimal control problems (see, *e.g.*, [67]) and that it could be employed in various practical situations such as in aerospace (in particular for the determination of an optimal control strategy when a spacecraft enters into a shadow zone). Future works could focus on the determination of optimal control policies in this framework for SIR models or in population models in the context of time crisis problems when one is unable to control in the non-constraint set (see, *e.g.*, [20]). We are also interested in extending the necessary optimality conditions obtained in this section to the case of feedback controls in loss control regions (instead of frozen controls) and to the context of final state constraints, and in developing Riccati theory for linear control systems subject to loss of control. Also note that, in this section, we did not discuss the existence of a solution to (2.15) which may be a difficult question due to the presence of loss control regions. So, in Theorem 2.7.1, we have assumed that there exists a solution to (2.15), moreover with a finite number of crossing times, excluding that way other possible solutions with more complicated structures such as chattering, boundary arcs, tangential crossing, etc. On the other hand, note that considering loss control regions may impact the controllability of (2.14) but we did not discuss controllability issues here. All these subjects constitute interesting perspectives for further research works.

3

Minimum time problem for the double integrator with a loss control region

Contents

This chapter is based on the article "Minimum time problem for the double integrator with a loss control region" by T. Bayen, A. Bouali, and L. Bourdin (see [15]). Here, we address the minimum time problem for the double integrator system. Unlike the classical example, we consider a loss control region where the control remains constant. This unique constraint introduces new challenges and requires a thorough

analysis. As in [15], we derive a Pontryagin maximum principle adapted to this setting, which involves discontinuous jumps in the costate and an averaged Hamiltonian gradient condition. Hence, this chapter aims to highlight the novel behaviors observed, such as the absence of dynamical programming principles, feedback expressions, and the saturation of the control constraint set.

3.1 Introduction

Geometric control theory is developed since the sixties and it now plays a central role in optimal control theory. Based on the Pontryagin maximum principle [102] and differential geometry, it gathers mathematical tools and methods to determine optimal controls and to synthesize feedback expressions [2], [31], [33], [107]. Several well known examples of optimal control problem illustrate various phenomena observed in that field. We can cite for instance the minimum time problem for the double integrator for which every optimal control is bang-bang with zero or one switching time (depending on the initial condition), or for the harmonic oscillator for which every optimal control is bang-bang with a finite (but possibly large) number of switching times. We should also mention the classical Fuller's problem [120] for which every optimal control is bang-bang with an infinite number of switching times on a finite time interval.

This chapter is concerned with the study a variant of the minimum time problem for the double integrator in which the control is constrained to be constant as long as the corresponding state belongs to a loss control region. The consideration of optimal control problems involving loss control regions is motivated by various applications. For instance, in the context of aerospace, this question arises in order to take into account the shadow effect in the low-thrust transfer problem [69], [73]. Our choice, in the paper [15], is to focus on the double integrator is twofold. First, as far as we know, optimal control problems including loss control regions have not been treated in the literature yet. Therefore, the adaptation of an academic problem to this new setting could serve the community to highlight the construction of optimal paths in that context (see, *e.g.*, a related study [60] in which the double integrator is investigated under a linear pathwise constraint). Second, we shall see that the analysis of optimal trajectories for the double integrator in this new setting is more involved than in the usual case and it requires the use of an adapted methodology.

Our methodology is to follow the approach of our previous works [13], [14]. Precisely, first-order necessary optimality conditions (in a Pontryagin form) for hybrid optimal control problems involving regionally switching parameters are obtained in [13]. As a particular case, the paper [14] provides a Pontryagin maximum principle for optimal control problems involving loss control regions. This principle provides a so-called averaged Hamiltonian gradient condition to determine the optimal constant value of the control whenever the state belongs to a loss control region, as well as the usual Hamiltonian maximization condition whenever the state belongs to the other regions. Since our framework is related to hybrid optimal control problems, we recall that the costate obtained in the principle admits discontinuity jumps at the interfaces.

It is worth mentioning that the framework in [13], [14] does not allow terminal state constraints. Since the minimal time problem for the double integrator involves an endpoint constraint, we cannot resort to the results of [13], [14]. Therefore, in this chapter, we first prove an adapted version of the Pontryagin maximum principle for a general minimum time problem involving an arbitrary loss control region and endpoint constraints. We refer to Proposition 3.2.2 whose proof is based on an augmentation procedure in the spirit of [59]. Note that our framework involves a partition of the state space and thus the use of such an augmentation technique requires a careful study to relate a solution to the original problem to

a (local) solution to the augmented problem. This is made possible thanks to an hypothesis made on the velocity set at the boundary of the loss control region (in line with the usual transverse assumptions found in hybrid settings [13], [18], [59], [73]). We emphasize that Proposition 3.2.2 is established under quite strong hypotheses (see Remark 3.2.3 for details). However these hypotheses are all satisfied in the context of the double integrator with a loss control region which constitutes the major focus of the present work. Therefore Proposition 3.2.2 is sufficient for our purposes in this paper. The extension of Proposition 3.2.2 to more general settings should be the subject of further research papers.

Main result (Theorem 3.3.1) and new observations. Applying Proposition 3.2.2 to the minimum time problem for the double integrator with a loss control region, we prove that every optimal trajectory visits at most once the loss control region and then, thanks to the averaged Hamiltonian gradient condition, we are able to determine the corresponding optimal constant value of the control. The synthesis for each initial condition is given in Theorem 3.3.1 and the corresponding optimal trajectories are depicted in Figure 3.5. At this occasion, we observe new behaviors with respect to the classical setting (that is, without loss control region). For example, some optimal trajectories (for different initial conditions) cross each other, which implies that the classical *dynamical programming principle* does not hold true and that the optimal control cannot be expressed as a feedback. Furthermore, in contrary again to the classical setting, the optimal control takes moderated values, that is, values in the interior of the control constraint set (which is thus unsaturated). We refer to Remarks 3.2.2 and 3.4.1 for details. These new phenomena raise many questions and open new challenges to address (theoretically and/or numerically) optimal control problems with loss control regions in view of applications.

Organization of this chapter. This chapter is organized as follows. In the preliminary Section 3.2, we recall the well known solution to the classical (that is, without loss control region) minimum time problem for the double integrator. Next we state a version of the Pontryagin maximum principle adapted to a minimum time problem with a loss control region (see Proposition 3.2.2 whose proof is postponed in Section 3.5). In Section 3.3, our main result (Theorem 3.3.1) is stated, providing an exact analytical solution to the minimum time problem for the double integrator with a loss control region. Its proof (based on Proposition 3.2.2) is given immediately after, being divided into several cases arising in the application of Proposition 3.2.2. Section 3.4 gives a list of additional comments on Theorem 3.3.1 and its proof. We conclude with open questions and perspectives about optimal control problems with loss control regions (such as controllability/reachability issues, existence results, Hamilton-Jacobi-Bellman equation, etc.). Finally Section 3.5 contains the proof of Proposition 3.2.2.

3.2 Preliminaries

Let us start with some basic reminders on the classical minimum time problem for the double integrator.

3.2.1 Reminders on the classical minimum time problem for the double integrator

Recall that the classical minimum time problem for the double integrator [107] is given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^2) \times \mathrm{L}^\infty([0,T],\mathbb{R}) \times (0,+\infty), \\ & \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0,T], \\ & \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x^0, \quad x(T) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1,1], \quad \text{a.e. } t \in [0,T], \end{array} \tag{3.1}$$

where $x^0 \in \mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$. As usual in the literature $x = (x_1, x_2) \in \mathrm{AC}([0,T],\mathbb{R}^2)$ is called the *state* (or the *trajectory*), $u \in \mathrm{L}^\infty([0,T],\mathbb{R})$ is called the *control* and $T > 0$ is called the *final time*. Using the classical Filippov approach [65], it can be proved that Problem (3.1) admits (at least) one solution. Then, from the classical Pontryagin maximum principle [102], it can be proved that Problem (3.1) admits exactly one solution and its description can be separated into four cases according to the position of the initial condition x^0 in the partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \Gamma_0 \cup \Omega_1 \cup \Gamma_1 \cup \Omega_0$ (see Figure 3.1) where

$$\Gamma_0 := \left\{ \left(\frac{1}{2} x_2^2, x_2 \right) \mid x_2 < 0 \right\} \quad \text{and} \quad \Gamma_1 := \left\{ \left(-\frac{1}{2} x_2^2, x_2 \right) \mid x_2 > 0 \right\},$$

and where Ω_1 (resp. Ω_0) stands for the strict epigraph (resp. strict hypograph) of $\Gamma_0 \cup \Gamma_1 \cup \{0_{\mathbb{R}^2}\}$.[1]

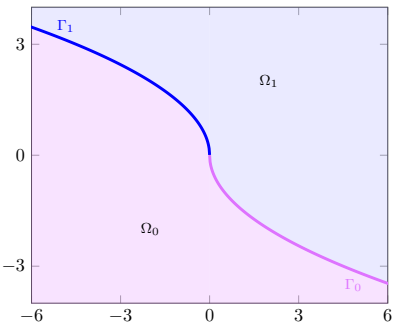


Figure 3.1: Partition of $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$ arising from the analysis of Problem (3.1) (see Proposition 3.2.1).

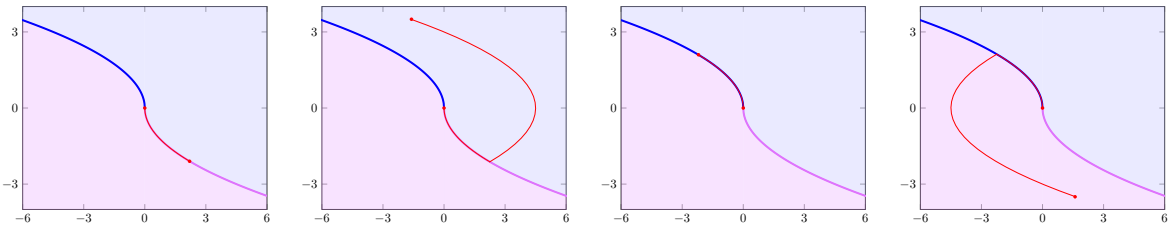
Precisely the following well known proposition is established [107].

Proposition 3.2.1. *If $(x^\dagger, u^\dagger, T^\dagger)$ is the unique solution to Problem (3.1), then an overview description of $(x^\dagger, u^\dagger)$ over the interval $[0, T^\dagger]$, according to the position of the initial condition x^0 in the partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \Gamma_0 \cup \Omega_1 \cup \Gamma_1 \cup \Omega_0$, can be summarized as follows:*

[1] The notation of the sets Γ_0, Ω_1, Γ_1 and Ω_0 may be non-intuitive with regards to other notations found in the literature. However they will be convenient and consistent with the setting developed in the next Section 3.3 (see Figure 3.4).

Position of x^0	*Overview description of* $x^\dagger(t)$ / $u^\dagger(t)$	*Figure*
Γ_0	Γ_0 1	*3.2a*
Ω_1	$\Omega_1 \rightsquigarrow \Gamma_0$ $-1 \rightsquigarrow 1$	*3.2b*
Γ_1	Γ_1 -1	*3.2c*
Ω_0	$\Omega_0 \rightsquigarrow \Gamma_1$ $1 \rightsquigarrow -1$	*3.2d*

For example, the second case of the above table can be read as follows: if $x^0 \in \Omega_1$*, then there exists a switching time* $\sigma^\dagger \in (0, T^\dagger)$ *such that* $x^\dagger(t) \in \Omega_1$ *and* $u^\dagger(t) = -1$ *over* $(0, \sigma^\dagger)$*, and* $x^\dagger(t) \in \Gamma_0$ *and* $u^\dagger(t) = 1$ *over* $(\sigma^\dagger, T^\dagger)$*.*



(a) **(b)** **(c)** **(d)**

Figure 3.2: Optimal trajectories (in red) in the four cases of Proposition 3.2.1.

Our objective in the present work is to state and prove a similar result to Proposition 3.2.1, but when adding a so-called loss control region in the control system. We refer to the next Section 3.2.2 for a general presentation of this new concept and to Section 3.3 for a specification to the double integrator.

Remark 3.2.1. Note that Proposition 3.2.1 is not as complete as it could be. Indeed the expressions of the final time $T^\dagger$ and of the (possible) switching time $\sigma^\dagger$ and switching state $x^\dagger(\sigma^\dagger)$, in function of the initial condition x^0, are not explicitly provided. Nevertheless these expressions can be easily obtained. To this aim define $\chi(\cdot, x^0, \mu) : \mathbb{R} \to \mathbb{R}^2$ as the unique solution to the control system, associated with the initial condition $x^0 \in \mathbb{R}^2$ and with the control constantly equal to $\mu \in \mathbb{R}$, whose explicit expression is given by

$$\chi(t, x^0, \mu) = \left(x_1^0 + x_2^0 t + \frac{\mu}{2} t^2, x_2^0 + \mu t\right), \tag{3.2}$$

for all $t \in \mathbb{R}$. For example, if $x^0 \in \Omega_1$, it holds from Proposition 3.2.1 that $x^\dagger(t) = \chi(t, x^0, -1)$ over $[0, \sigma^\dagger]$ and $x^\dagger(t) = \chi(t - \sigma^\dagger, x^\dagger(\sigma^\dagger), 1)$ over $[\sigma^\dagger, T^\dagger]$. Hence, in the case $x^0 \in \Omega_1$, one can easily deduce from (3.2) and simple computations that

$$\sigma^\dagger = x_2^0 + \sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}, \quad x^\dagger(\sigma^\dagger) = \left(\frac{1}{2}\left(\frac{1}{2}(x_2^0)^2 + x_1^0\right), -\sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}\right), \quad T^\dagger = x_2^0 + 2\sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}.$$

Therefore a complete and detailed description of the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (3.1), in function of the initial condition x^0, can be easily derived from Proposition 3.2.1, (3.2) and simple

computations.

Remark 3.2.2. Consider the framework of Proposition 3.2.1. In the present classical setting (that is, without loss control region), it is well known that:

(i) As usual with a classical minimum time problem, the *dynamical programming principle* holds true, in the sense that $(x^\dagger, u^\dagger)$ is not only the fastest way to reach the origin $0_{\mathbb{R}^2}$ from x^0, but also the fastest way to reach the origin $0_{\mathbb{R}^2}$ from any intermediate point $x^\dagger(s)$ with $s \in (0, T^\dagger)$, and also the fastest way to reach $x^\dagger(s)$ from x^0.

(ii) The optimal control $u^\dagger$ can be expressed as a *feedback control* (that is, as a function of the instantaneous position) given by

$$u^\dagger(t) = \begin{cases} -1 & \text{if } x^\dagger(t) \in \Gamma_1 \cup \Omega_1, \\ +1 & \text{if } x^\dagger(t) \in \Gamma_0 \cup \Omega_0, \end{cases}$$

over $(0, T^\dagger)$.

(iii) Furthermore, as is often the case with a classical optimal control problem for which the Hamiltonian is affine with respect to the control and without singular arc, the optimal control $u^\dagger$ *saturates* the control constraint set $[-1, 1]$, in the sense that it does not take any moderated value in the interior $(-1, 1)$.

As we will see in the next Section 3.3, these three well known properties are broken when considering a loss control region in the control system (see Remark 3.4.1).

3.2.2 Pontryagin maximum principle for a general minimum time problem with a loss control region

Let n, $m \in \mathbb{N}^\star$ be two positive integers and consider a state space partition $\mathbb{R}^n = \overline{\mathrm{X}}_1 \cup \overline{\mathrm{X}}_2$ where X_1, X_2 are two disjoint nonempty open subsets of $\mathbb{R}^n$ called *regions*. In the sequel we denote by $\partial\mathrm{X} := \overline{\mathrm{X}}_1 \cap \overline{\mathrm{X}}_2$ and we assume that there exists a C^1 description map $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathrm{X}_1 = \{x \in \mathbb{R}^n \mid F(x) > 0\}, \quad \partial\mathrm{X} = \{x \in \mathbb{R}^n \mid F(x) = 0\}, \quad \mathrm{X}_2 = \{x \in \mathbb{R}^n \mid F(x) < 0\}.$$

Consider the control system given by

$$\begin{cases} (x, u, T) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m) \times (0, +\infty), \\ \dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ \mathrm{X}_2 \text{ is a loss control region}, \end{cases} \tag{3.3}$$

where the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1. The novelty in the control system (3.3) is that X_2 is a *loss control region*, in the sense that the control value $u(t)$ is *frozen* (that is, cannot be modified) in the region X_2. In other words, the control value $u(t)$ remains constant on the intervals for which the state position $x(t)$ belongs to X_2. The precise definition of a solution to (3.3) is given as follows.

Definition 3.2.1 (Solution to (3.3))**.** *A triplet $(x, u, T) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m) \times (0, +\infty)$ is said to be a solution to* (3.3) *if the following conditions are satisfied:*

(i) *It holds that $\dot{x}(t) = f(x(t), u(t))$ for almost every $t \in [0, T]$.*

(ii) *There exists a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0, T]$ such that x is alternatively, over the open intervals (τ_{k-1}, τ_k), with values in X_1 and then with values in X_2. We denote by $\mathcal{I}_1$ (resp. $\mathcal{I}_2$)*

the set of indexes $k \in \{1, \dots, N\}$ such that x is with values in X_1 (resp. in X_2) over (τ_{k-1}, τ_k).

(iii) *For all $k \in \mathcal{I}_2$, there exists $\mu_k \in \mathbb{R}^m$ such that $u(t) = \mu_k$ for almost every $t \in (\tau_{k-1}, \tau_k)$.*

Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the general minimum time problem with a loss control region given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x, u, T) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times (0, +\infty), \\ & \dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x^0, \quad x(T) = x^{\mathrm{targ}}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\ & \mathrm{X}_2 \text{ is a loss control region}, \end{array} \tag{3.4}$$

where the initial condition $x^0 \in \mathbb{R}^n$ and the target $x^{\mathrm{targ}} \in \mathbb{R}^n$ are distinct and U is a nonempty compact convex subset of $\mathbb{R}^m$. A triplet $(x, u, T) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times (0, +\infty)$ is said to be *admissible* for Problem (3.4) if it satisfies all the constraints of Problem (3.4) (in particular it has to be a solution to (3.3) in the sense of Definition 3.2.1). Finally such an admissible triplet is said to be a *solution* to Problem (3.4) if it minimizes the final time among all admissible triplets.

Recall that that the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (3.4) is defined by

$$H(x, u, p) := \langle p, f(x, u) \rangle_{\mathbb{R}^n},$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to provide a Pontryagin maximum principle for Problem (3.4) under the transverse assumption given by

$$\forall (x, u) \in (\partial \mathrm{X} \backslash \{x^{\mathrm{targ}}\}) \times \mathrm{U}, \quad \langle \nabla F(x), f(x, u) \rangle_{\mathbb{R}^n} \neq 0. \tag{3.5}$$

Proposition 3.2.2 (Pontryagin maximum principle for Problem (3.4))**.** *Under the transverse assumption (3.5), if $(x^\star, u^\star, T^\star)$ is a solution to Problem (3.4), associated with a partition $\mathbb{T}^\star = \{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0, T^\star]$, then there exists a nontrivial pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^\star}([0,T^\star], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:*

(i) The Hamiltonian system $\dot{x}^\star(t) = \nabla_p H(x^\star(t), u^\star(t), p(t))$ and $-\dot{p}(t) = \nabla_x H(x^\star(t), u^\star(t), p(t))$ for almost every $t \in [0, T^\star]$.

(ii) The Hamiltonian maximization condition $u^\star(t) \in \arg\max_{\omega \in \mathrm{U}} H(x^\star(t), \omega, p(t))$ for almost every $t \in (\tau_{k-1}^\star, \tau_k^\star)$, for all $k \in \mathcal{I}_1^\star$.

(iii) The averaged Hamiltonian gradient condition $\int_{\tau_{k-1}^\star}^{\tau_k^\star} \nabla_u H(x^\star(t), \mu_k^\star, p(t))\, dt \in \mathrm{N}_{\mathrm{U}}[\mu_k^\star]$ for all $k \in \mathcal{I}_2^\star$ where $\mathrm{N}_{\mathrm{U}}[\mu_k^\star]$ is the normal cone to U at $\mu_k^\star$.

(iv) The discontinuity jump condition $p^+(\tau_k^\star) - p^-(\tau_k^\star) = \nu_k \nabla F(x^\star(\tau_k^\star))$ for some $\nu_k \in \mathbb{R}$, for all $k \in \{1, \dots, N-1\}$.

(v) The constancy Hamiltonian condition $H(x^\star(t), u^\star(t), p(t)) = p^0$ for almost every $t \in [0, T^\star]$.

The proof of Proposition 3.2.2 is postponed in Section 3.5. It is based on an augmentation technique and the application of the classical Pontryagin maximum principle for local solutions to a classical (that is, without loss control region) augmented optimal control problem involving parameters and endpoint constraints.

Remark 3.2.3. Hereafter we provide a list of comments on Proposition 3.2.2 and its proof.

(i) First of all, we emphasize that Proposition 3.2.2 is established under strong hypotheses such as the

transverse assumption (3.5), the topological assumptions made on the control constraint set U or the global descriptions of the regions X_1 and X_2. However these hypotheses are all satisfied in the context of the double integrator with a loss control region considered in the next Section 3.3 which constitutes the central part of the present work. Therefore Proposition 3.2.2 is sufficient for our purposes in this chapter. We also emphasize that, in this paper, we do not consider in Definition 3.2.1 the possibility of an infinite number of instants τ_k (in the spirit of a chattering phenomenon [120]). The extension of Proposition 3.2.2 to more general contexts (including also general Bolza costs, not only minimum time problems) should be the subject of future research works.

(ii) The transverse assumption (3.5) has a geometrical interpretation. It implies that, for any admissible triplet (x, u, T) for Problem (3.4), if the trajectory x crosses the boundary $\partial\mathrm{X}$, then it does not cross it tangentially. This assumption plays a central role in the proof of Proposition 3.2.2 in order to guarantee that the reverse procedure of the augmentation technique produces (at least locally) admissible triplets for the original Problem (3.4). We refer to Section 3.5 for details. We also emphasize that, in the next Section 3.3, the non-equality in the transverse assumption (3.5) is not satisfied at x^{targ}. Fortunately, since we consider here a minimum time problem (and not a general Bolza cost), the non-equality in the transverse assumption (3.5) is not mandatory at x^{targ} thanks to a basic dynamical programming argument. We refer to Section 3.5 for details. To conclude on the transverse assumption (3.5), we mention that weaker assumptions could be considered. For example, one could consider a transverse assumption on the solution $(x^\star, u^\star, T^\star)$ only (and not everywhere). However, as explained in the previous item, it is not our objective here to provide a Pontryagin maximum principle for very general optimal control problems with loss control regions. Proposition 3.2.2 is sufficient for our purposes in this chapter.

(iii) From linearity, the nontrivial pair (p, p^0) in Proposition 3.2.2 is defined up to a positive multiplicative constant. When the pair is *normal* (that is, when $p^0 \neq 0$), we renormalize it so that $p^0 = 1$.

(iv) The averaged Hamiltonian gradient condition is well known in the context of sampled-data controls (that is, piecewise constant controls). We refer to [34], [36], [41] and references therein. In the present context, the control is imposed to be constant on intervals for which the state position lies in the loss control region. Therefore it is not surprising that the averaged Hamiltonian gradient condition appears in Proposition 3.2.2 as first-order necessary optimality condition on these *constancy intervals*. However note that our setting here is more involved than the framework of sampled-data controls since the constancy intervals of the control are determined by the state position $x(t)$, and not by the (independent) time variable t.

(v) The discontinuity jump condition on the costate p is well known in the literature on hybrid maximum principles (in which, for example, authors consider control systems with spatially heterogeneous dynamics). We refer to [10], [73] and references therein. As also well known, when the control $u^\star$ admits left and right limits at $\tau_k^\star$ for all $k \in \{1, \dots, N-1\}$, denoted respectively by $(u^\star)^-(\tau_k^\star)$ and $(u^\star)^+(\tau_k^\star)$, the constancy Hamiltonian condition allows to obtain (forward and backward) expressions for ν_k given by

$$\nu_k = -\frac{\langle p^{\pm}(\tau_k^\star), f(x^\star(\tau_k^\star), (u^\star)^+(\tau_k^\star)) - f(x^\star(\tau_k^\star), (u^\star)^-(\tau_k^\star))\rangle_{\mathbb{R}^n}}{\langle \nabla F(x^\star(\tau_k^\star)), f(x^\star(\tau_k^\star), (u^\star)^{\pm}(\tau_k^\star))\rangle_{\mathbb{R}^n}},$$

for all $k \in \{1, \dots, N-1\}$.

One can conclude from Items (iv) and (v) that the present framework of loss control region can be seen,

in some sense, as a mix of two well known topics in the literature, namely the sampled-data controls and the hybrid control systems.

3.3 Main result and its proof

In this section we focus on the minimum time problem for the double integrator with a loss control region given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^2) \times \mathrm{L}^\infty([0,T],\mathbb{R}) \times (0,+\infty), \\ & \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0,T], \\ & \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x^0, \quad x(T) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1,1], \quad \text{a.e. } t \in [0,T], \\ & \mathrm{X}_2 \text{ is a loss control region}, \end{array} \tag{3.6}$$

where $x^0 \in \mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$ and where the state space $\mathbb{R}^2 = \overline{\mathrm{X}}_1 \cup \overline{\mathrm{X}}_2$ has been partitioned (see Figure 3.3) with

$$\mathrm{X}_1 := \{x \in \mathbb{R}^2 \mid x_1 > 0\}, \quad \partial\mathrm{X} = \{x \in \mathbb{R}^2 \mid x_1 = 0\}, \quad \mathrm{X}_2 := \{x \in \mathbb{R}^2 \mid x_1 < 0\}.$$

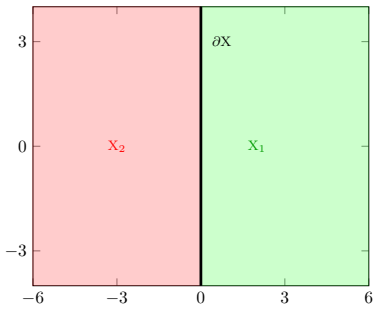


Figure 3.3: Partition of $\mathbb{R}^2$ into a loss control region (in red) and an "usual" region (in green).

In the case where $x^0 \in \Gamma_0 \cup \Omega_1 \cup \Gamma_1$, the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (3.1) is admissible for Problem (3.6) (since the control $u^\dagger$ remains frozen in the region X_2, see Proposition 3.2.1 and Figure 3.2) and therefore it is clear that $(x^\dagger, u^\dagger, T^\dagger)$ is the unique solution to Problem (3.6). On the contrary, when $x^0 \in \Omega_0$, the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (3.1) is not admissible for Problem (3.6) (since the control $u^\dagger$ requires a switch from $+1$ to -1 on the curve $\Gamma_1 \subset \mathrm{X}_2$, see Proposition 3.2.1 and Figure 3.2). Hence a rigorous analysis has to be performed in order to determine the candidate solution to Problem (3.6) in the case $x^0 \in \Omega_0$. This is the objective of the present section. To state and prove our main result (Theorem 3.3.1 below), we need to introduce several elements:

- The positive real number $\theta := \frac{1}{1+\sqrt{2}} > 0$ introduced to simplify notations.
- The partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \cup_{i=1}^{6} (\Gamma_{i-1} \cup \Omega_i)$ (see Figure 3.4) where Γ_0, Ω_1 and Γ_1 have already been

defined in Section 3.2.1, where $\Gamma_3 := \{(x_1, 0) \mid x_1 < 0\}$ and $\Gamma_5 := \{(0, x_2) \mid x_2 < 0\}$, where

$$\Gamma_2 := \left\{ \left(-\frac{1}{2\theta} x_2^2, x_2 \right) \mid x_2 > 0 \right\}, \quad \Gamma_4 := \left\{ \left(-\frac{1}{\theta} x_2^2, x_2 \right) \mid x_2 < 0 \right\},$$

and where Ω_i stands for the open region delimited by Γ_{i-1} and Γ_i for all $i \in \{1, \dots, 6\}$ (with $\Gamma_6 := \Gamma_0$ by convention).

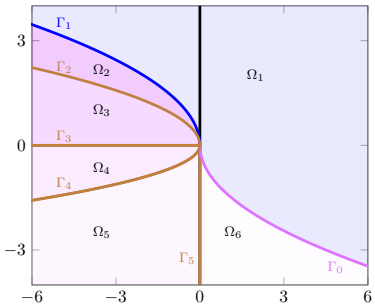


Figure 3.4: Partition of $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$ arising from the analysis of Problem (3.6) (see Theorem 3.3.1).

- The three real numbers

$$\lambda(x^0) := \frac{(x_2^0)^2}{2x_1^0} \quad \text{and} \quad \lambda^{\pm}(x^0) := \sqrt{\theta} \left(\sqrt{\theta} \pm 2\sqrt{-\lambda(x^0)} \right),$$

introduced for any initial condition $x^0 \in \Omega_0 \cap \mathrm{X}_2$, for which $x_1^0 < 0$ and $\lambda(x^0) \leq 0$. We refer to Remark 3.3.1 for additional comments on these numbers.

Theorem 3.3.1. *If $(x^\star, u^\star, T^\star)$ is a solution to Problem (3.6), then an overview description of $(x^\star, u^\star)$ over the interval $[0, T^\star]$, according to the position of the initial condition x^0 in the partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \cup_{i=1}^{6} (\Gamma_{i-1} \cup \Omega_i)$, can be summarized as follows: The column N allows to know how many crossing times (from X_1 to X_2, or from X_2 to X_1) are observed for the trajectory $x^\star$ (that is, $N - 1$). For example, if $x^0 \in \Omega_3$, then the trajectory $x^\star$ has only one crossing time $\tau_1^\star$ from X_2 to X_1. Precisely, in the case $x^0 \in \Omega_3$, there exist $0 < \tau_1^\star < \sigma^\star < T^\star$ such that $x^\star(t) \in \mathrm{X}_2$ and $u^\star(t) = \mu^\star$ over $(0, \tau_1^\star)$, and $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_1^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$.*

The results of Theorem 3.3.1 will be commented in Section 3.4.1. The rest of this section is dedicated to its proof which is based on the Pontryagin maximum principle stated in Proposition 3.2.2. To this aim let us fix a solution $(x^\star, u^\star, T^\star)$ to Problem (3.6), associated with a partition $\mathbb{T}^\star = \{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0, T^\star]$, and let us denote by $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^\star}([0, T^\star], \mathbb{R}^n) \times \mathbb{R}_+$ the nontrivial pair provided by Proposition 3.2.2 (whose hypotheses are all satisfied).

Remark 3.3.1. Before going any further in the proof of Theorem 3.3.1, we need to emphasize several facts.

(i) Note that $\Gamma_0 \subset \mathrm{X}_1$ and $\Gamma_1 \subset \mathrm{X}_2$, and that Ω_1 intersects both X_1 and X_2. Also note that $\Omega_0 \cap \mathrm{X}_2 = \Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5$, that $\Omega_0 \cap \partial \mathrm{X} = \Gamma_5$ and that $\Omega_0 \cap \mathrm{X}_1 = \Omega_6$.

(ii) For any initial condition $x^0 \in \Omega_0 \cap \mathrm{X}_2$, it holds that $\lambda(x^0) \leq 0$ (with equality if and only if $x^0 \in \Gamma_3$) and $\lambda^+(x^0) > 0$. Note that, if $x^0 \in \Gamma_1$ (resp. $x^0 \in \Gamma_2$, $x^0 \in \Gamma_3$, $x^0 \in \Gamma_4$), then $\lambda(x^0) = -1$ (resp. $\lambda^-(x^0) = \lambda(x^0)$, $\lambda^+(x^0) = \lambda^-(x^0)$, $\lambda^+(x^0) = 1$). Also note that, if $x^0 \in \Omega_2 \cup \Gamma_2$ (resp. $x^0 \in \Gamma_2 \cup \Omega_3 \cup \Gamma_3$, $x^0 \in \Gamma_3 \cup \Omega_4$), then $\lambda(x^0) \in (-1, 1)$ (resp. $\lambda^-(x^0) \in (-1, 1)$, $\lambda^+(x^0) \in (-1, 1)$).

Position of x^0	Overview description of $\begin{matrix} x^\star(t) \\ u^\star(t) \end{matrix}$	$\mu^\star \in (-1,1)$	N	Figure
Γ_0	$\begin{matrix} \Gamma_0 \\ 1 \end{matrix}$		1	3.5a
Ω_1	$\begin{matrix} \Omega_1 & \rightsquigarrow & \Gamma_0 \\ -1 & \rightsquigarrow & 1 \end{matrix}$		1 or 2	3.5b
Γ_1	$\begin{matrix} \Gamma_1 \\ -1 \end{matrix}$		1	3.5c
Ω_2	$\begin{matrix} \Omega_2 \\ \mu^\star \end{matrix}$	$\lambda(x^0)$	1	3.5d
Γ_2	$\begin{matrix} \Gamma_2 \\ \mu^\star \end{matrix}$	$\lambda(x^0) = \lambda^-(x^0)$	1	3.5e
Ω_3	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ \mu^\star & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$	$\lambda^-(x^0)$	2	3.5f
Γ_3	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ \mu^\star & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$	$\lambda^-(x^0) = \lambda^+(x^0)$	2	3.5g
Ω_4	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ \mu^\star & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$	$\lambda^+(x^0)$	2	3.5h
Γ_4	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ 1 & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$		2	3.5i
Ω_5	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ 1 & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$		2	3.5j
Γ_5	$\begin{matrix} \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ 1 & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$		2	3.5k
Ω_6	$\begin{matrix} \Omega_6 & \rightsquigarrow & \mathrm{X}_2 & \rightsquigarrow & \Omega_1 \cap \mathrm{X}_1 & \rightsquigarrow & \Gamma_0 \\ 1 & \rightsquigarrow & 1 & \rightsquigarrow & -1 & \rightsquigarrow & 1 \end{matrix}$		3	3.5l

(iii) Consider the framework of Proposition 3.2.1 in the case where $x^0 \in \Omega_1 \cap \partial\mathrm{X}$. In that context, from Remark 3.2.1, it holds that

$$\sigma^\dagger = \left(1 + \frac{\sqrt{2}}{2}\right) x_2^0, \quad x_2^\dagger(\sigma^\dagger) = -\frac{\sqrt{2}}{2} x_2^0, \quad T^\dagger = \frac{x_2^0}{\theta}.$$

In the sequel we denote by $(x^\dagger(\cdot, x^0), u^\dagger(\cdot, x^0), T^\dagger(x^0))$ the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (3.1) corresponding to such an initial condition $x^0 \in \Omega_1 \cap \partial\mathrm{X}$.

(iv) From the analysis of Problem (3.1) (see Proposition 3.2.1) and a basic dynamical programming argument, if $(x^\star, u^\star, T^\star)$ exits the loss control region X_2 at some time $\tau^\star \in (0, T^\star)$ with $x^\star(\tau^\star) \in \Omega_1 \cap \partial\mathrm{X}$, then it necessarily holds that $x^\star(t) = x^\dagger(t - \tau^\star, x^\star(\tau^\star))$ and $u^\star(t) = u^\dagger(t - \tau^\star, x^\star(\tau^\star))$ over $(\tau^\star, T^\star)$.

We are now in a position to pursue the proof of Theorem 3.3.1 by separating the cases according to the position of the initial condition x^0 in the partition of $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$ depicted in Figure 3.4. First, recall that the first three cases of Theorem 3.3.1 (that is, when $x^0 \in \Gamma_0 \cup \Omega_1 \cup \Gamma_1$) are trivial since, in these cases, the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (3.1) is admissible for Problem (3.6) and thus $(x^\star, u^\star, T^\star) = (x^\dagger, u^\dagger, T^\dagger)$ and we refer to Proposition 3.2.1 for the corresponding overview description. Note that, in the case $x^0 \in \Omega_1$ (since Ω_1 intersects both X_1 and X_2), we have $N = 2$ (resp. $N = 1$) if $x_1^0 < 0$ (resp. $x_1^0 \geq 0$).

In the sequel we will focus only on the case $x^0 \in \Omega_0$ and we separate it into three subcases given by $x^0 \in \Omega_0 \cap \mathrm{X}_2$ (see Section 3.3.1), $x^0 \in \Omega_0 \cap \partial\mathrm{X}$ (see Section 3.3.2) and $x^0 \in \Omega_0 \cap \mathrm{X}_1$ (see Section 3.3.3).

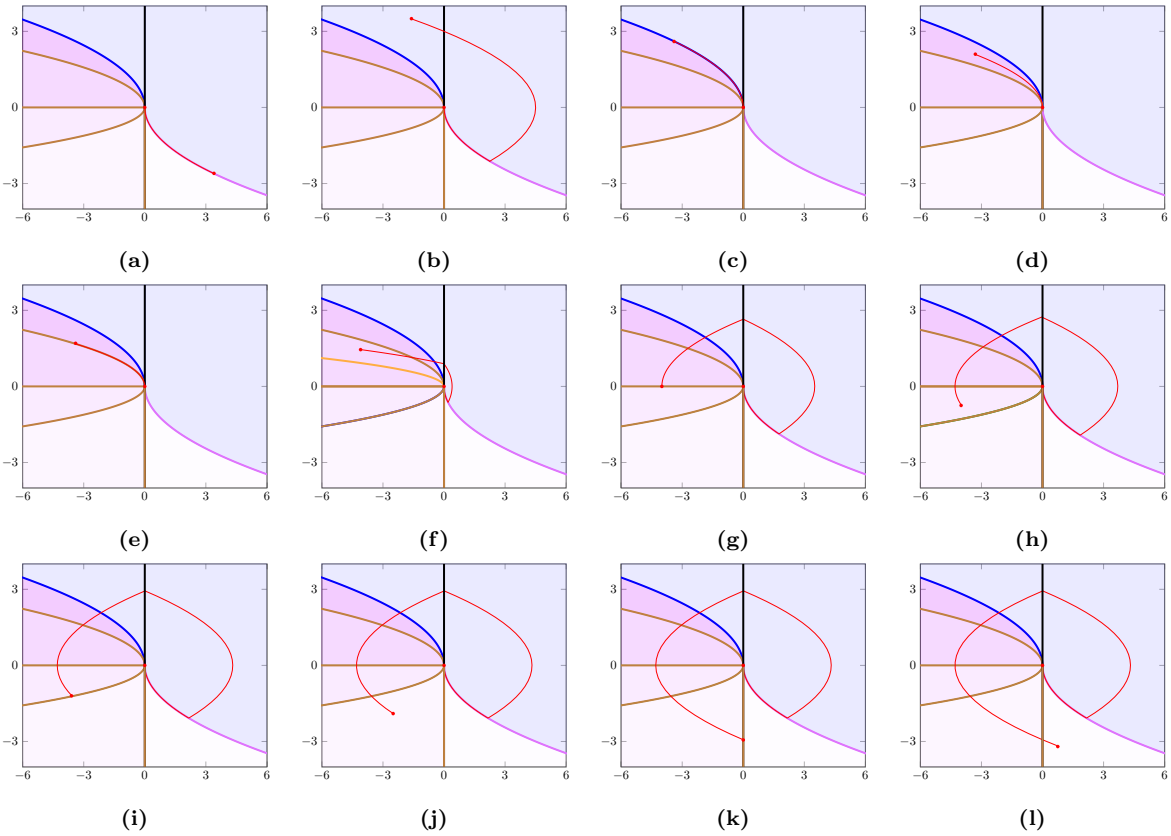


Figure 3.5: Optimal trajectories (in red) in the twelve cases of Theorem 3.3.1.

3.3.1 The case $x^0 \in \Omega_0 \cap \mathrm{X}_2$

Here we focus on the case $x^0 \in \Omega_0 \cap \mathrm{X}_2$. Since $x^\star(0) = x^0 \in \mathrm{X}_2$, we get that $x^\star(t) \in \mathrm{X}_2$ over $[0, \tau_1^\star)$. Moreover, since $x^\star(T^\star) = 0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star) = 0$ (independently of $N = 1$ or $N \geq 2$). Since X_2 is a loss control region, let us denote by $\mu^\star \in [-1, 1]$ the constant value of $u^\star$ over $(0, \tau_1^\star)$. Therefore it holds that $x^\star(t) = \chi(t, x^0, \mu^\star)$ over $[0, \tau_1^\star]$ (see Remark 3.2.1). From (3.2) and simple computations, one can easily derive the following lemma.

Lemma 3.3.1 (Case $x^0 \in \Omega_0 \cap \mathrm{X}_2$)**.** *The following five properties are satisfied:*

$$\text{(i)} \quad (x_2^0)^2 - 2\mu^\star x_1^0 \geq 0, \qquad \text{(ii)} \quad (x_2^0, \mu^\star) \notin \mathbb{R}^2_-, \qquad \text{(iii)} \quad \mu^\star \neq -1,$$

$$\text{(iv)} \quad \tau_1^\star = \begin{cases} \dfrac{\sqrt{(x_2^0)^2 - 2\mu^\star x_1^0} - x_2^0}{\mu^\star} & \text{if } \mu^\star \neq 0, \\ -\dfrac{x_1^0}{x_2^0} & \text{if } \mu^\star = 0. \end{cases}, \qquad \text{(v)} \quad x_2^\star(\tau_1^\star) = \sqrt{(x_2^0)^2 - 2\mu^\star x_1^0} \geq 0.$$

Proof. (i) Since $x_1^\star(\tau_1^\star) = 0$, the discriminant of $x_1^\star(t)$ is nonnegative. (ii) By contradiction, if $(x_2^0, \mu^\star) \notin \mathbb{R}^2_-$, then one would obtain that $x_1^\star(\tau_1^\star) \leq x_1^0 < 0$ which is absurd. (iii) By contradiction, if $\mu^\star = -1$, then, from the previous two items, one would obtain $(x_2^0)^2 + 2x_1^0 \geq 0$ and $x_2^0 > 0$, which contradicts $x^0 \in \Omega_0 \cap \mathrm{X}_2$. (iv)(v) Separating the cases $\mu^\star > 0$, $\mu^\star = 0$ and $\mu^\star < 0$ (note that $x_2^0 > 0$ in the last two cases), one can easily derive from (3.2) and simple computations the above expressions of $\tau_1^\star$ and $x_2^\star(\tau_1^\star)$. □

From Lemma 3.3.1 we deduce that $x^\star(\tau_1^\star) \in (\Omega_1 \cap \partial \mathrm{X}) \cup \{0_{\mathbb{R}^2}\}$. Precisely we obtain two cases:

1. Either $x_2^\star(\tau_1^\star) = 0$. In that case $x^\star(\tau_1^\star) = 0_{\mathbb{R}^2}$ and thus $T^\star = \tau_1^\star$ and $N = 1$. Furthermore, from Lemma 3.3.1, it holds that $\mu^\star = \lambda(x^0) \leq 0$ and thus this situation is possible only for $x_2^0 > 0$.
2. Either $x_2^\star(\tau_1^\star) > 0$. In that case $x^\star(\tau_1^\star) \in \Omega_1 \cap \partial \mathrm{X}$ and, from a basic dynamical programming argument, it holds that $x^\star = x^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ and $u^\star = u^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ over $(\tau_1^\star, T^\star)$ (see Remark 3.3.1). From Proposition 3.2.1 and Remark 3.3.1, we deduce that $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_1^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$, where $\sigma^\star = \tau_1^\star + (1 + \frac{\sqrt{2}}{2}) x_2^\star(\tau_1^\star)$, $x_2^\star(\sigma^\star) = -\frac{\sqrt{2}}{2} x_2^\star(\tau_1^\star) < 0$ and $T^\star = \tau_1^\star + \frac{x_2^\star(\tau_1^\star)}{\theta}$. In particular, in that case, we have $T^\star = \tau_2^\star$ and $N = 2$.

In the second case above, we already have a quite complete description of $(x^\star, u^\star)$ over $(\tau_1^\star, T^\star)$. Therefore we only need to determine the constant value $\mu^\star \in [-1, 1]$ of the optimal control $u^\star$ over $(0, \tau_1^\star)$. Our aim in the next lemma is to reduce the possibilities of values for $\mu^\star$ in that case. This lemma, whose proof is based on the application of the Pontryagin maximum principle stated in Proposition 3.2.2, allows to discriminate four values.

Lemma 3.3.2 (Case $x^0 \in \Omega_0 \cap \mathrm{X}_2$)**.** *If $x_2^\star(\tau_1^\star) > 0$, then $\mu^\star \in \{0, \lambda^-(x^0), \lambda^+(x^0), 1\}$.*

Proof. We only deal with the case $x_2^0 > 0$ (the other cases $x_2^0 = 0$ and $x_2^0 < 0$ are similar). Since $N = 2$ and from the Pontryagin maximum principle stated in Proposition 3.2.2 (precisely from the Hamiltonian system and the discontinuity jump condition), we get that

$$p_1(t) = \begin{cases} p_{11} & \text{if} \quad t \in [0, \tau_1^\star), \\ p_{12} & \text{if} \quad t \in (\tau_1^\star, T^\star], \end{cases} \quad \text{and} \quad p_2(t) = \begin{cases} -p_{11} t + p_2(0) & \text{if} \quad t \in [0, \tau_1^\star], \\ -p_{12}(t - \tau_1^\star) - p_{11}\tau_1^\star + p_2(0) & \text{if} \quad t \in [\tau_1^\star, T^\star], \end{cases}$$

with p_{11}, $p_{12} \in \mathbb{R}$. From the Hamiltonian maximization condition, since $x^\star(t) \in \mathrm{X}_1$ over $(\tau_1^\star, T^\star)$ and $u^\star$ changes its value at $\sigma^\star$, we deduce that $p_2(\sigma^\star) = 0$. From the Hamiltonian constancy (considered at 0, $\tau_1^\star$ and $\sigma^\star$), we obtain that

$$p_{11} x_2^0 + p_2(0)\mu^\star = p^0, \quad p_{11} x_2^\star(\tau_1^\star) + p_2(\tau_1^\star)\mu^\star = p^0, \quad p_{12} x_2^\star(\tau_1^\star) - p_2(\tau_1^\star) = p^0, \quad p_{12} x_2^\star(\sigma^\star) = p^0.$$

From these four equalities, one can easily prove in the one hand that

$$p_{12} = p_{11} + \frac{p_2(\tau_1^\star)}{x_2^\star(\tau_1^\star)}(1 + \mu^\star) = p_{11} + \frac{p_2(0) - p_{11}\tau_1^\star}{x_2^\star(\tau_1^\star)}(1 + \mu^\star), \tag{3.7}$$

and, in the other hand, using the nontriviality of the pair (p, p^0), that $p^0 \neq 0$ (by contradiction). In the sequel we take $p^0 = 1$ (see Remark 3.2.3) and we assume that $\mu^\star \notin \{0, 1\}$. Therefore it only remains to prove that $\mu^\star \in \{\lambda^-(x^0), \lambda^+(x^0)\}$. Since $\mu^\star \notin \{-1, 1\}$ (see Lemma 3.3.1), the averaged Hamiltonian gradient condition gives $\int_0^{\tau_1^\star} p_2(t)dt = 0$ and thus $p_2(0) = \frac{p_{11}}{2}$ (and thus $p_{11} \neq 0$ by contradiction). Using it in the equality $p_2(\sigma^\star) = 0$, we obtain that $-p_{12}(\sigma^\star - \tau_1^\star) - \frac{p_{11}}{2}\tau_1^\star = 0$. Replacing the value p_{12} from (4.13) and the value $\sigma^\star = \tau_1^\star + (1 + \frac{\sqrt{2}}{2}) x_2^\star(\tau_1^\star)$ (and dividing by $p_{11} \neq 0$ and by $x_2^\star(\tau_1^\star) \neq 0$), one can obtain that

$$\frac{2}{\theta} x_2^\star(\tau_1^\star) - \tau_1^\star \left(1 + \frac{\mu^\star}{\theta}\right) = 0.$$

Replacing the values $x_2^\star(\tau_1^\star)$ and $\tau_1^\star$ obtained in Lemma 3.3.1 (and dividing by $x_2^0 > 0$), we obtain that

$$\left(\frac{\mu^\star}{\theta} - 1\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} = -\left(1 + \frac{\mu^\star}{\theta}\right).$$

Squaring this last equality (and dividing by $\mu^\star \neq 0$), we obtain that

$$\left(\frac{\mu^\star}{\theta}\right)^2 - 2\left(\frac{\mu^\star}{\theta}\right) + \left(1 + \frac{4\lambda(x^0)}{\theta}\right) = 0.$$

which admits two solutions given by $\lambda^-(x^0)$ and $\lambda^+(x^0)$. The proof is complete. □

Finally, according to the previous analysis and using the equality $T^\star = \tau_1^\star + \frac{x_2^\star(\tau_1^\star)}{\theta}$, we can summarize the situation as follows:

(i) If $x_2^0 < 0$, then $\mu^\star \geq 0$ and $\mu^\star \in \{\lambda^-(x^0), \lambda^+(x^0), 1\}$ and

$$T^\star = -x_2^0\left(\left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} + \frac{1}{\mu^\star}\right). \tag{3.8}$$

(ii) If $x_2^0 = 0$, then $\mu^\star \in \{\theta, 1\}$ and

$$T^\star = \left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{-2\mu^\star x_1^0}. \tag{3.9}$$

(iii) If $x_2^0 > 0$, then $\mu^\star \in \{0, \lambda(x^0), \lambda^-(x^0), \lambda^+(x^0), 1\}$ and

$$T^\star = \begin{cases} x_2^0\left(\left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} - \frac{1}{\mu^\star}\right) & \text{if} \quad \mu^\star \neq 0, \\ x_2^0\left(\frac{1}{\theta} - \frac{1}{2\lambda(x^0)}\right) & \text{if} \quad \mu^\star = 0. \end{cases} \tag{3.10}$$

By comparing the value of $T^\star$ in function of the possibilities of value of $\mu^\star$, we get the following proposition which concludes the proof in the case $x^0 \in \Omega_0 \cap \mathrm{X}_2 = \Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5$.

Proposition 3.3.1. *It holds that:*

If $x^0 \in$	Ω_2	Γ_2	Ω_3	Γ_3	Ω_4	Γ_4	Ω_5
Then $\mu^\star =$	$\lambda(x^0)$	$\lambda(x^0) = \lambda^-(x^0)$	$\lambda^-(x^0)$	$\lambda^-(x^0) = \lambda^+(x^0)$	$\lambda^+(x^0)$	$\lambda^+(x^0) = 1$	1

Proof. In this proof we denote by $\mathcal{T}(\alpha)$ the value of $T^\star$ given in (3.8), (3.9) and (3.10) if $\mu^\star = \alpha$.

- Take $x^0 \in \Omega_2$. It holds that $-\frac{1}{2\theta}(x_2^0)^2 < x_1^0 < -\frac{1}{2}(x_2^0)^2$ and thus $-1 < \lambda(x^0) < -\theta$. In the one hand we deduce that $\mathcal{T}(0) > x_2^0(\frac{3}{2} + \sqrt{2})$, $\mathcal{T}(1) > x_2^0(1 + 2\sqrt{2})$ and $\mathcal{T}(\lambda(x^0)) < x_2^0(1 + \sqrt{2})$ and thus $\mu^\star \neq 0$ and $\mu^\star \neq 1$. In the other hand we deduce $\lambda^+(x^0) > 3\theta > 1$, and thus $\mu^\star \neq \lambda^+(x^0)$. By studying the quotient

$$\frac{\lambda^-(x^0)}{\lambda(x^0)} = \sqrt{\frac{\theta}{-\lambda(x^0)}}\left(2 - \sqrt{\frac{\theta}{-\lambda(x^0)}}\right),$$

  one can also obtain that $-1 < \lambda(x^0) < \lambda^-(x^0) < -\theta$ and thus $\mathcal{T}(\lambda^-(x^0)) > \mathcal{T}(\lambda(x^0))$. We conclude that $\mu^\star = \lambda(x^0)$.
- Take $x^0 \in \Gamma_2$. Similar to the first item.

- Take $x^0 \in \Gamma_3$. In that case it holds that $x_2^0 = 0$ and thus $\mu^\star \in \{\theta, 1\}$. Since $(\sqrt{\theta} - 1)^2 > 0$, one can easily obtain that $\mathcal{T}(\theta) < \mathcal{T}(1)$ and thus $\mu^\star = \theta = \lambda^-(x^0) = \lambda^+(x^0)$.
- Take $x^0 \in \Gamma_4 \cup \Omega_5$. In that case it holds that $-\frac{1}{\theta}(x_2^0)^2 \leq x_1^0 < 0$ and thus $\lambda(x^0) \leq -\frac{\theta}{2}$. One can deduce that $\lambda^+(x^0) \geq 1$ and $\lambda^-(x^0) < 0$. Therefore $\mu^\star = 1$ (and recall that $\lambda^+(x^0) = 1$ in the case $x^0 \in \Gamma_4$).

The cases $x^0 \in \Omega_3$ and $x^0 \in \Omega_4$ can be treated similarly but with more involved computations. For the sake of conciseness, these cases are omitted. □

3.3.2 The case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$

Here we focus on the case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$. This section is very similar (and even simpler) to the previous one, except that some minor adjustments have to be performed since $x_1^0 = 0$ and thus $\lambda(x^0)$ is not defined. Therefore, in this section, the proof is sketched.
From continuity of $\dot{x}_1^\star = x_2^\star$ and since $x_1^\star(0) = 0$ and $x_2^\star(0) = x_2^0 < 0$, we deduce that $x^\star(t) \in \mathrm{X}_2$ over $(0, \tau_1^\star)$. Since $x^\star(T^\star) = 0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star) = 0$ (independently of $N = 1$ or $N \geq 2$). Since X_2 is a loss control region, let us denote by $\mu^\star \in [-1, 1]$ the constant value of $u^\star$ over $(0, \tau_1^\star)$. Therefore it holds that $x^\star(t) = \chi(t, x^0, \mu^\star)$ over $[0, \tau_1^\star]$ (see Remark 3.2.1). From (3.2) and simple computations, one can easily derive the following lemma.

Lemma 3.3.3 (Case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$)**.** *The following three properties are satisfied:*

$$\text{(i)} \quad \mu^\star > 0, \qquad \text{(ii)} \quad \tau_1^\star = \frac{-2x_2^0}{\mu^\star}, \qquad \text{(iii)} \quad x_2^\star(\tau_1^\star) = -x_2^0 > 0.$$

In particular it holds that $x^\star(\tau_1^\star) = -x^0 \in \Omega_1 \cap \partial\mathrm{X}$.
Since $x^\star(\tau_1^\star) = -x^0 \in \Omega_1 \cap \partial\mathrm{X}$, it holds from a basic dynamical programming argument that $x^\star = x^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ and $u^\star = u^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ over $(\tau_1^\star, T^\star)$ (see Remark 3.3.1). From Proposition 3.2.1 and Remark 3.3.1, we deduce that $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_1^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$, where $\sigma^\star = \tau_1^\star + (1 + \frac{\sqrt{2}}{2})x_2^\star(\tau_1^\star)$, $x_2^\star(\sigma^\star) = -\frac{\sqrt{2}}{2}x_2^\star(\tau_1^\star) < 0$ and $T^\star = \tau_1^\star + \frac{x_2^\star(\tau_1^\star)}{\theta}$. In particular it holds that $T^\star = \tau_2^\star$ and $N = 2$. Hence we already have a quite complete description of $(x^\star, u^\star)$ over $(\tau_1^\star, T^\star)$. Therefore we only need to determine the constant value $\mu^\star \in [-1, 1]$ of the optimal control $u^\star$ over $(0, \tau_1^\star)$. To this aim one can follow the same steps than the proof of Lemma 3.3.2, except that one should assume by contradiction that $\mu^\star \neq 1$ (recall that $\mu^\star > 0$ from Lemma 3.3.3). At the step of replacing the values $x_2^\star(\tau_1^*)$ and $\tau_1^\star$ from Lemma 3.3.3, one obtains $\frac{1}{\mu^\star} = 0$ which is absurd. We get the following proposition which concludes the proof in the case $x^0 \in \Omega_0 \cap \partial\mathrm{X} = \Gamma_5$.

Proposition 3.3.2 (Case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$)**.** *It holds that* $\mu^\star = 1$ *and* $T^\star = -x_2^0(2 + \frac{1}{\theta})$.

3.3.3 The case $x^0 \in \Omega_0 \cap \mathrm{X}_1$

Here we focus on the case $x^0 \in \Omega_0 \cap \mathrm{X}_1$. This section is different from the previous two sections since our proof here is based, not only on a basic dynamical programming argument and the results of the previous section, but also on the application of the classical Pontryagin maximum principle on a classical (that is, without loss control region) optimal control problem. To this aim we first establish the next lemma.

Lemma 3.3.4 (Case $x^0 \in \Omega_0 \cap \mathrm{X}_1$)**.** *It holds that* $x^\star(t) \in \Omega_0 \cap \mathrm{X}_1$ *over* $[0, \tau_1^\star)$ *and* $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$.

Proof. In the one hand, since $x^\star(0) = x^0 \in \mathrm{X}_1$, we get that $x^\star(t) \in \mathrm{X}_1$ over $[0, \tau_1^\star)$. Moreover, since $x^\star(T^\star) = 0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star) = 0$ (independently of $N = 1$ or $N \geq 2$). On the other hand, from

the control system and since the control $u^\star$ is with values in $[-1,1]$, one has $x_1^\star(t) \leq \chi_1(t)$ and $x_2^\star(t) \leq \chi_2(t)$ over $[0,T^\star]$, where $\chi := \chi(\cdot, x^0, 1)$ (see Remark 3.2.1). Defining $r := -x_2^0 - \sqrt{(x_2^0)^2 - 2x_1^0} > 0$, from (3.2) and simple computations, one can easily obtain that $\chi_2(t) < 0 < \chi_1(t) < \frac{1}{2}\chi_2(t)^2$ (that is $\chi(t) \in \Omega_0 \cap \mathrm{X}_1$) over $[0,r)$ and that $\chi_2(r) < 0 = \chi_1(r)$ (that is $\chi(r) \in \Omega_0 \cap \partial\mathrm{X}$). Firstly, one can deduce that $\tau_1^\star \leq r$ (indeed, if $r < \tau_1^\star$, then $0 < x_1^\star(r) \leq \chi_1(r) = 0$ which is absurd). Secondly, one obtains that $x_2^\star(t) \leq \chi_2(t) < 0 < x_1^\star(t) \leq \chi_1(t) < \frac{1}{2}\chi_2(t)^2 \leq \frac{1}{2}x_2^\star(t)^2$ (and thus $x^\star(t) \in \Omega_0 \cap \mathrm{X}_1$) over $[0,\tau_1^\star)$ and $x_2^\star(\tau_1^\star) \leq \chi_2(\tau_1^\star) < 0 = x_1^\star(\tau_1^\star)$ (and thus $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$). □

From Lemma 3.3.4, it holds that $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$. From a basic dynamical programming argument, it holds from the previous section that $x^\star = \chi(\cdot - \tau_1^\star, x^\star(\tau_1^\star), 1)$ and $u^\star = 1$ over $(\tau_1^\star, \tau_2^\star)$, that $x^\star(\tau_2^\star) \in \Omega_1 \cap \partial\mathrm{X}$ and that $x^\star = x^\dagger(\cdot - \tau_2^\star, x^\star(\tau_2^\star))$ and $u^\star = u^\dagger(\cdot - \tau_2^\star, x^\star(\tau_2^\star))$ over $(\tau_2^\star, T^\star)$ (see Remark 3.3.1). From Proposition 3.2.1 and Remark 3.3.1, we deduce that $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_2^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$, where $\sigma^\star = \tau_2^\star + (1 + \frac{\sqrt{2}}{2})x_2^\star(\tau_2^\star)$, $x_2^\star(\sigma^\star) = -\frac{\sqrt{2}}{2}x_2^\star(\tau_2^\star) < 0$ and $T^\star = \tau_2^\star + \frac{x_2^\star(\tau_2^\star)}{\theta}$. In particular, in that case, we have $T^\star = \tau_3^\star$ and $N = 3$.
From the previous section, it also holds that $\tau_2^\star = \tau_1^\star - 2x_2^\star(\tau_1^\star)$ and $x_2^\star(\tau_2^\star) = -x_2^\star(\tau_1^\star)$. As a consequence we obtain that $T^\star = \tau_1^\star - (2 + \frac{1}{\theta})x_2^\star(\tau_1^\star)$. We deduce that the triplet $(x^\star, u^\star, \tau_1^\star)$ is a solution to the classical (that is, without loss control region) optimal control problem given by

$$
\begin{array}{ll}
\text{minimize} & \tau_1 - (3+\sqrt{2})x_2(\tau_1), \\
\text{subject to} & (x,u,\tau_1) \in \mathrm{AC}([0,\tau_1],\mathbb{R}^2) \times \mathrm{L}^\infty([0,\tau_1],\mathbb{R}) \times (0,+\infty), \\
& \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0,\tau_1], \\
& \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0,\tau_1], \\
& x(0) = x^0, \quad x_1(\tau_1) = 0, \\
& u(t) \in [-1,1], \quad \text{a.e. } t \in [0,\tau_1].
\end{array}
$$

Applying the classical Pontryagin maximum principle, there exists a nontrivial pair $(q, q^0) \in \mathrm{AC}([0,\tau_1^\star],\mathbb{R}^2) \times \mathbb{R}_+$ such that $-\dot{q}_2 = q_1$ is constant (and thus $q_2(t) = q_2(\tau_1^\star) + q_1(\tau_1^\star - t)$ is affine) over $[0,\tau_1^\star]$ and $q_1 x_2^\star(t) + q_2(t)u^\star(t) = q^0$ over $[0,\tau_1^\star]$ (and thus q_2 vanishes at most one time over $[0,\tau_1^\star]$ by contradiction), but also $u^\star(t) = \mathrm{sign}(q_2(t))$ over $[0,\tau_1^\star]$ and $q_2(\tau_1^\star) = q^0(3+\sqrt{2})$. Since $x_2^\star(\tau_1^\star) < 0$, we deduce that

$$
q_1 = \frac{q^0}{x_2^\star(\tau_1^\star)}(1 - (3+\sqrt{2})u_-^\star(t_1^\star)),
$$

and thus $q^0 \neq 0$ (by contradiction) that we renormalize so that $q^0 = 1$. We obtain that $q_2(\tau_1^\star) > 0$, $u_-^\star(\tau_1^\star) = 1$ and thus $q_1 > 0$. Finally we get that $q_2(t) > 0$ and thus $u^\star(t) = 1$ over $[0,\tau_1^\star]$, which concludes the proof in the case $x^0 \in \Omega_0 \cap \mathrm{X}_1 = \Omega_6$.

3.4 Comments and perspectives

This section is dedicated to comments on Theorem 3.3.1 and its proof (Section 3.4.1) and to several perspectives about the concept of loss control region for further research works (Section 3.4.2).

3.4.1 Comments on Theorem 3.3.1 and its proof

Remark 3.4.1. In connection with Remark 3.2.2, we emphasize that several well known properties observed in the classical (that is, without loss control region) minimal time problem for the double integrator are broken when considering a loss control region in the control system. First of all, we observe that some optimal trajectories obtained in Theorem 3.3.1 (from different initial conditions) intersect each other (see Figure 3.6). We deduce that, in the presence of a loss control region in the control system, the dynamical programming principle does not hold true and that the optimal control $u^\star$ cannot be expressed as a feedback. Furthermore we observe that, for initial conditions in $\Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4$, the optimal

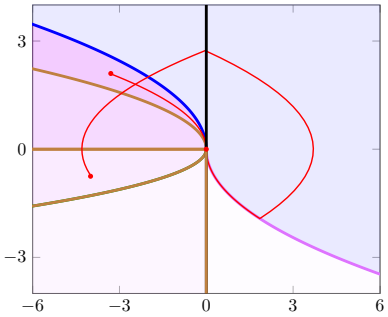


Figure 3.6: Illustration of intersecting optimal trajectories in Theorem 3.3.1.

control $u^\star$ takes a moderated value $\mu^\star$ in the interior $(-1, 1)$ of the control constraint set, and therefore does not saturate it.

Remark 3.4.2. In this remark we comment on the different behaviors observed in Theorem 3.3.1.

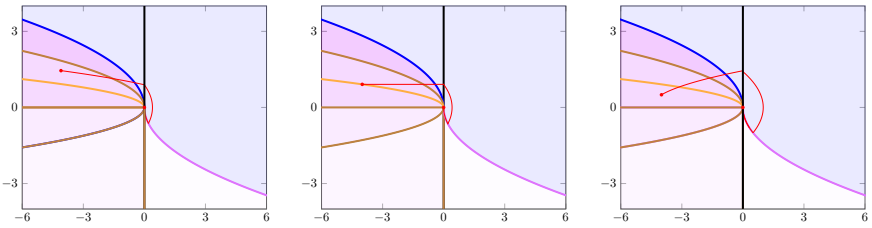
(i) For initial conditions in $\Omega_2 \cup \Gamma_2$, the optimal control $u^\star$ consists in taking a moderated value $\mu^\star \in (-1, 1)$ until reaching the origin $0_{\mathbb{R}^2}$. This behavior differs from the optimal strategies observed in classical (that is, without loss control region) minimum time problems (such as double integrator or harmonic oscillator). Indeed, for classical minimum time problems governed by affine systems with respect to the control, the target is usually reached by a so-called *bang-bang control* (apart singular arc and Fuller's phenomenon).

(ii) For initial conditions in $\Omega_3 \cup \Gamma_3 \cup \Omega_4$, the optimal control $u^\star$ takes a moderated value $\mu^\star \in (-1, 1)$ until reaching $\Omega_1 \cap \partial \mathrm{X}$ and then is bang-bang until reaching the origin $0_{\mathbb{R}^2}$. This analysis reveals that a moderated value can be associated with a bang-bang policy. Again, this feature differs from what is observed in classical settings.

(iii) For initial conditions in Ω_3, let us introduce the set Σ defined by

$$\Sigma := \left\{ \left(-\frac{2}{\theta} x_2^2, x_2 \right) \mid x_2 > 0 \right\},$$

which corresponds to the set of points $x^0 \in \Omega_3$ such that $\lambda^-(x^0) = 0$. Therefore, for initial conditions in Ω_3, we observe the three situations illustrated in Figure 3.7 in which the curve Σ is depicted in orange. In Figure 3.7(b), we observe that the part of the trajectory $x^\star$ in the region X_2 is an horizontal segment. This is due to the fact that, when $x^0 \in \Sigma$, it holds that $u^\star(t) = \mu^\star = \lambda^-(x^0) = 0$ in the region X_2. Finally, contrary to what Figures 3.7(a) and 3.7(c) above might suggest, the part of the trajectory $x^\star$ in the region X_2 is a not a segment, but a parabolic curve.



(a) **(b)** **(c)**

Figure 3.7: Three situations for initial conditions in Ω_3.

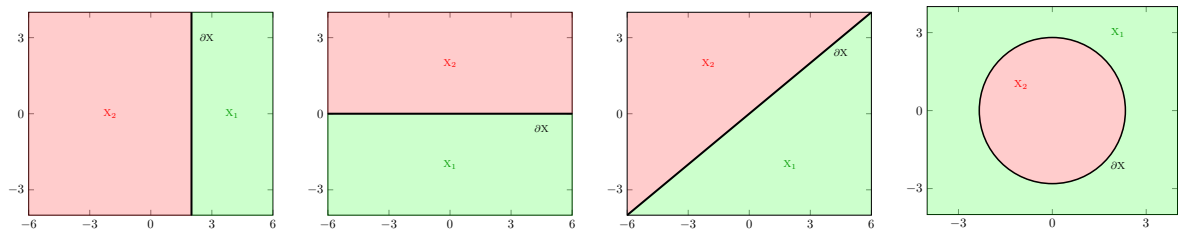
(iv) For initial conditions in $\Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5 \cup \Gamma_5 \cup \Omega_6$, we observe that the optimal control $u^\star$ admits two switching times. The structure of the optimal control in the presence of a loss control region is thus more complex than in the classical setting (for which every optimal control has at most one switching time).

(v) For initial conditions in Ω_6, we point out a non-intuitive property. Indeed it can be proved from the classical Pontryagin maximum principle that the fastest way to reach Γ_5 from Ω_6 consists in taking $u(t) = -1$. However, from Theorem 3.3.1, the optimal control $u^\star$ from an initial condition in Ω_6 consists in taking $u^\star(t) = +1$ until reaching Γ_5. We deduce that the optimal strategy in Theorem 3.3.1 from an initial condition in Ω_6 does not consist in reaching Γ_5 in minimal time.

Remark 3.4.3. As it is shown in the proof of Theorem 3.3.1, every optimal trajectory visits the loss control region X_2 at most one time. In view of this behavior, a direct analysis (that is, without using the Pontryagin maximum principle stated in Proposition 3.2.2) may lead to the same Theorem 3.3.1. Nevertheless our approach should also apply to more complicate situations in which the optimal trajectory would visit a loss control region more than one time. In particular it could be used to tackle a loss control region in a minimal time problem associated with the harmonic oscillator, or in an optimal control problem associated with an oscillatory controlled system (such as the Lotka-Volterra system [94]).

3.4.2 Several perspectives

In this chapter we have investigated the minimal time problem for the double integrator with a loss control region given by $\mathrm{X}_2 := \{x \in \mathbb{R}^2 \mid x_1 < 0\}$. Of course this study could be extended to many different loss control regions, such as the ones depicted in Figure 3.8.



(a) **(b)** **(c)** **(d)**

Figure 3.8: Illustration of other possible loss control regions.

As mentioned in Remark 3.4.3, this study could be extended to other control systems than the double integrator, such as the harmonic oscillator, Zermelo-type models (which will be covered in Chapter 5),

and controlled Lotka-Volterra system. This latter will be the subject of future research work.

Controllability/reachability. When adding a loss control region in the control system, it is clear that the set of admissible controls is reduced. As a consequence, controllability issues may appear. Typically, for a minimum time problem, depending on the choice of the loss control region, the target may not be reachable. Therefore a natural question concerns the robustness of the reachability of a target under the presence of a loss control region. We refer to [38] for a similar study in the context of control sampling. From a more general point of view, one could be interested in finding sufficient conditions on the control system, the target and the loss control region to ensure the reachability of the target. As a natural first step, one may look for including loss control regions in the classical Kalman theory about controllability of linear control systems.

Existence of an optimal control. In this chapter note that the existence of a solution to Problem (3.6) has not been investigated. From a general point of view, one may be interested in extending the classical Filippov's existence theorem [65] to the context of loss control regions (for minimal time problems or more general Bolza optimal control problems). We believe that, if one is able to give an upper bound on the number of times the state visits the loss control region, then existence of an optimal control could be ensured under standard hypotheses (such as compactness of the set of admissible triplets trajectory/control/final time and convexity of the so-called *augmented velocities set*).

Pontryagin maximum principle. Proposition 3.2.2 provides first-order necessary optimality conditions in a Pontryagin form for a general minimum time problem including a loss control region, but under strong hypotheses (see Remark 3.2.3). This result was sufficient to investigate Problem (3.6) in Section 3.3. In Chapters 4 and 5, we shall extend Proposition 3.2.2 to more general settings. First we want to cover the case of a general Bolza optimal control problem including mixed initial-final state constraints. Second, the transverse assumption (3.5) does not hold in general. Therefore we want to extend Proposition 3.2.2 under a weaker transverse assumption (involving only the optimal pair $(x^\star, u^\star)$ for example). This could be done by using an augmentation technique as in the proof of Proposition 3.2.2. It would serve to solve more involved application problems involving loss control regions, from a theoretical point of view as well as by using numerical tools as explained below.

Numerical methods. There are two predominant kinds of numerical methods in classical optimal control theory. In one hand, *direct numerical methods* consist in a full discretization of the optimal control problem which results into a constrained finite-dimensional optimization problem that can be solved using standard numerical optimization algorithms. On the other hand, *indirect numerical methods* consist in the numerical solving by a shooting method of the boundary value problem satisfied by the pair state/costate given by the Pontryagin maximum principle. We emphasize that neither method is inherently better to the other. For a detailed discussion on the advantages and drawbacks of each method, we refer to [113, pp. 178-179]. A challenge to solve application problems involving loss control regions would be to extend direct/indirect numerical methods to that context (which will be covered in Chapter 5). The main focus would be the possibility to constrain the control to be constant without knowing in advance when and how many times the corresponding state visits the loss control region. Furthermore note that the extension of indirect numerical methods is anyway conditioned in a first place by the extension of the Pontryagin maximum principle mentioned above.

Some insights into the HJB equation. In the literature, it is well known [9], [44], [107] how to define the Hamilton-Jacobi-Bellman (HJB) equation associated with the classical Problem (3.1). As well, the characterization of its value function V as the unique solution (in a certain sense) to the HJB equation is also well known. In contrast, when considering a loss control region, it is not clear how to define a HJB equation associated with Problem (3.6) and also if the corresponding value function W is a solution (in a certain sense) to this extended HJB equation. The aim of this paragraph is to give an insight into this question. In the sequel, in order to ease the notations, we write x in place of x^0 for the initial condition. Recall that the value function V associated with Problem (3.1) is given by

$$V(x) = \begin{cases} 2\sqrt{\frac{x_2^2}{2} + x_1} + x_2 & \text{if } x \in \Omega_1, \\ 2\sqrt{\frac{x_2^2}{2} - x_1} - x_2 & \text{if } x \in \Omega_0, \end{cases}$$

and that it is continuous and C^1-piecewise. Moreover, setting $H : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ the corresponding Hamiltonian defined by $H(x, u, p) := p_1 x_2 + p_2 u$ for all $(x, u, p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, the value function V can be characterized as the unique continuous and C^1-piecewise solution to the HJB equation

$$1 + \min_{u \in [-1,1]} H(x, u, \nabla V(x)) = 0, \quad x \in \Omega_1 \cup \Omega_0,$$

that can be rewritten as

$$1 + \partial_1 V(x_1, x_2) x_2 - |\partial_2 V(x_1, x_2)| = 0, \quad x \in \Omega_1 \cup \Omega_0. \tag{3.11}$$

Going back to our setting, one can show (from simple computations and from the results obtained in the proof of Theorem 3.3.1) that the value function W associated with Problem (3.6) is continuous, C^1-piecewise and that it fulfills the equalities

$$\begin{aligned} 1 + \partial_1 W(x_1, x_2) x_2 - |\partial_2 W(x_1, x_2)| = 0, &\quad \text{if } x \in \Omega_1, \\ 1 + \partial_1 W(x_1, x_2) x_2 + \partial_2 W(x_1, x_2) \mu^\star(x) = 0, &\quad \text{if } x \in \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6, \end{aligned} \tag{3.12}$$

where $\mu^\star(x)$ is given in Theorem 3.3.1 for $x \in \Omega_2 \cup \Omega_3 \cup \Omega_4$ and $\mu^\star(x) = 1$ for $x \in \Omega_5 \cup \Omega_6$. Note that both HJB equations (3.11) and (3.12) are the same in Ω_1 (since Problems (3.1) and (3.6) coincide for initial conditions in Ω_1). On the contrary, when $x \notin \Omega_1$, note that the term $\min_{u \in [-1,1]} H(x, u, \nabla V(x))$ in (3.11) is replaced by $H(x, \mu^\star(x), \nabla V(x))$ in (3.12).
Future works should investigate how to properly define a HJB equation when considering an optimal control problem involving a loss control region, as well as a characterization of the value function as the unique solution (in a certain sense) to this extended HJB equation. To this aim, a possible way could be to consider an augmentation technique (as in the proof of Proposition 3.2.2), to apply the classical methodology [11], [70], [78] to the augmented problem and try to reverse the augmentation procedure.

3.5 Proof of Proposition 3.2.2

In this section, we prove Proposition 3.2.2 by separating the two cases $x^{\mathrm{targ}} \notin \partial \mathrm{X}$ and $x^{\mathrm{targ}} \in \partial \mathrm{X}$.

3.5.1 The case $x^{\mathrm{targ}} \notin \partial \mathrm{X}$.

Assume that $x^{\mathrm{targ}} \notin \partial \mathrm{X}$ and let $(x^\star, u^\star, T^\star)$ be a solution to Problem (3.4), associated with a partition $\mathbb{T}^\star = \{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0, T^\star]$.

Step 1: augmentation procedure. Define $(y^\star, v^\star, \lambda^\star) \in \mathrm{AC}([0,1], \mathbb{R}^{nN}) \times \mathrm{L}^\infty([0,1], \mathbb{R}^{mN_1}) \times \mathbb{R}^{mN_2}$ by

$$\begin{cases} y_k^\star(s) := x^\star(\tau_{k-1}^\star + (\tau_k^\star - \tau_{k-1}^\star)s) & \text{for all } s \in [0,1] \text{ and all } k \in \{1,\dots,N\}, \\ v_k^\star(s) := u^\star(\tau_{k-1}^\star + (\tau_k^\star - \tau_{k-1}^\star)s) & \text{for all } s \in [0,1] \text{ and all } k \in \mathcal{I}_1^\star, \\ \lambda_k^\star := \overline{u}_k^\star & \text{for all } k \in \mathcal{I}_2^\star, \end{cases}$$

where $N_1 := \mathrm{card}(\mathcal{I}_1^\star)$ and $N_2 := \mathrm{card}(\mathcal{I}_2^\star)$. It is clear that the quadruplet $(y^\star, v^\star, \lambda^\star, \mathbb{T}^\star)$ is admissible for the classical (that is, without loss control region) *augmented* optimal control problem involving parameters and endpoint constraints given by

$$\begin{array}{ll} \text{minimize} & \tau_N, \\ \text{subject to} & (y, v, \lambda, \mathbb{T}) \in \mathrm{AC}([0,1], \mathbb{R}^{nN}) \times \mathrm{L}^\infty([0,1], \mathbb{R}^{mN_1}) \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1}, \\ & \dot{y}(s) = g(y(s), v(s), \lambda, \mathbb{T}), \quad \text{a.e. } s \in [0,1], \\ & v(s) \in \mathrm{U}^{N_1}, \quad \text{a.e. } s \in [0,1], \\ & (\lambda, \mathbb{T}) \in \mathrm{U}^{N_2} \times \Delta, \\ & y_1(0) = x^0, \quad y_N(1) = x^{\mathrm{targ}}, \\ & y_k(0) = y_{k-1}(1), \quad \text{for all } k \in \{2,\dots,N\}, \\ & F(y_k(1)) = 0, \quad \text{for all } k \in \{1,\dots,N-1\}, \end{array} \tag{3.13}$$

where $g = (g_k)_{k=1,\dots,N} : \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1} \to \mathbb{R}^{nN}$ is defined by

$$g_k(y, v, \lambda, \mathbb{T}) := \begin{cases} (\tau_k - \tau_{k-1}) f(y_k, v_k) & \text{if } k \in \mathcal{I}_1^\star, \\ (\tau_k - \tau_{k-1}) f(y_k, \lambda_k) & \text{if } k \in \mathcal{I}_2^\star, \end{cases}$$

for all $(y, v, \lambda, \mathbb{T}) \in \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1}$ and all $k \in \{1,\dots,N\}$, and where $\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0,\dots,N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{N-1} \leq \tau_N\}$ is a nonempty closed convex subset of $\mathbb{R}^{N+1}$.

Step 2: the quadruplet $(y^\star, v^\star, \lambda^\star, \mathbb{T}^\star)$ is a local solution to Problem (3.13). Let us prove that there exists $\eta > 0$ such that $\tau_N^\star \leq \tau_N$ for any quadruplet $(y, v, \lambda, \mathbb{T})$ admissible for Problem (3.13) satisfying

$$\|y - y^\star\|_{\mathrm{C}} + \|v - v^\star\|_{\mathrm{L}^1} + \|\lambda - \lambda^\star\|_{\mathbb{R}^{nN_1}} + \|\mathbb{T} - \mathbb{T}^\star\|_{\mathbb{R}^{N+1}} \leq \eta.$$

To this aim let $\eta > 0$ and $(y, v, \lambda, \mathbb{T})$ be an admissible triplet admissible for Problem (3.13) satisfying the above inequality. In the sequel we explain how to reduce $\eta > 0$ (step by step, and independently of the quadruplet $(y, v, \lambda, \mathbb{T})$) to obtain that $\tau_N^\star \leq \tau_N$.

(i) First one has to reduce $\eta > 0$ so that $0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < \tau_N$ and then one can correctly define $(x, u, T) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times (0, +\infty)$ by

$$x(t) := y_k\left(\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}\right) \text{ for all } t \in [\tau_{k-1}, \tau_k] \text{ and all } k \in \{1,\dots,N\},$$

and

$$u(t) := \begin{cases} v_k\left(\dfrac{t-\tau_{k-1}}{\tau_k-\tau_{k-1}}\right) & \text{for a.e. } t \in (\tau_{k-1},\tau_k) \text{ if } k \in \mathcal{I}_1^\star, \\ \lambda_k & \text{for a.e. } t \in (\tau_{k-1},\tau_k) \text{ if } k \in \mathcal{I}_2^\star, \end{cases} \qquad \text{for all } k \in \{1,\dots,N\},$$

and $T := \tau_N$. To obtain that $\tau_N^\star \leq \tau_N$, which is equivalent to $T^\star \leq T$, it is sufficient to prove that the triplet (x,u,T) is admissible for Problem (3.4). This is our aim in the next step.

(ii) It is clear that the triplet (x,u,T) satisfies $\dot{x}(t) = f(x(t),u(t))$ and $u(t) \in \mathrm{U}$ for almost every $t \in [0,T]$, and $x(0) = x^0$ and $x(T) = x^{\mathrm{targ}}$. Therefore, since u is constant over the intervals (τ_{k-1},τ_k) when $k \in \mathcal{I}_2^\star$, it only remains to prove that x is with values in X_1 (resp. in X_2) over the intervals (τ_{k-1},τ_k) when $k \in \mathcal{I}_1^\star$ (resp. when $k \in \mathcal{I}_2^\star$). This is possible by reducing $\eta > 0$ and by using the transverse assumption (3.5), the compactness of U, the fact that $x^{\mathrm{targ}} \notin \partial\mathrm{X}$ and the openness of the regions X_1 and X_2.

Step 3: application of the classical Pontryagin maximum principle. Consider the Hamiltonian $\widetilde{H} : \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN} \to \mathbb{R}$ associated with Problem (3.13) defined by

$$\begin{aligned} \widetilde{H}(y,v,\lambda,\mathbb{T},q) &:= \langle q, g(y,v,\lambda,\mathbb{T})\rangle_{\mathbb{R}^{nN}} \\ &= \sum_{k\in\mathcal{I}_1^\star} (\tau_k-\tau_{k-1})\langle q_k, f(y_k,v_k)\rangle_{\mathbb{R}^n} + \sum_{k\in\mathcal{I}_2^\star} (\tau_k-\tau_{k-1})\langle q_k, f(y_k,\lambda_k)\rangle_{\mathbb{R}^n}, \end{aligned}$$

for all $(y,v,\lambda,\mathbb{T},q) \in \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN}$. From the classical Pontryagin maximum principle applied to the quadruplet $(y^\star,v^\star,\lambda^\star,\mathbb{T}^\star)$, there exists a nontrivial pair $(q,q^0) \in \mathrm{AC}([0,1],\mathbb{R}^{nN}) \times \mathbb{R}_+$ such that:

(i) It holds that $-\dot{q}(s) = \nabla_y \widetilde{H}(y^\star(s),v^\star(s),\lambda^\star,\mathbb{T}^\star,q(s))$ for almost every $s \in [0,1]$.

(ii) It holds that $v^\star(s) \in \arg\max_{\omega\in\mathrm{U}^{N_1}} \widetilde{H}(y^\star(s),\omega,\lambda^\star,\mathbb{T}^\star,q(s))$ for almost every $s \in [0,1]$.

(iii) It holds that

$$\int_0^1 \nabla_\lambda \widetilde{H}(y^\star(s),v^\star(s),\lambda^\star,\mathbb{T}^\star,q(s))\,ds \in \mathrm{N}_{\mathrm{U}^{N_2}}[\lambda^\star].$$

(iv) It holds that $q_{k+1}(0) - q_k(1) = \nu_k \nabla F(y_k^\star(1))$ for some $\nu_k \in \mathbb{R}$ for all $k \in \{1,\dots,N-1\}$.

(v) It holds that

$$\int_0^1 \nabla_{\mathbb{T}} \widetilde{H}(y^\star(s),v^\star(s),\lambda^\star,\mathbb{T}^\star,q(s))\,ds \in q^0 e + \mathrm{N}_\Delta[\mathbb{T}^\star],$$

where $e = (0,\dots,0,1)^\top \in \mathbb{R}^{N+1}$.

Step 4: construction of the nontrivial pair (p,p^0). Define $p^0 := q^0 \in \mathbb{R}_+$ and $p \in \mathrm{PAC}_{\mathbb{T}^\star}([0,T^\star],\mathbb{R}^n)$ by $p(0) := q_1(0)$, $p(T^\star) := q_N(1)$ and by

$$p(t) := q_k\left(\frac{t-\tau_{k-1}^\star}{\tau_k^\star-\tau_{k-1}^\star}\right) \text{ for all } t \in (\tau_{k-1}^\star,\tau_k^\star) \text{ and all } k \in \{1,\dots,N\}.$$

From nontriviality of the pair (q,q^0), it is clear that the pair (p,p^0) is also nontrivial. Then the first four above items allows to obtain the first four items of Proposition 3.2.2. At this step, one can obtain that, for all $k \in \{1,\dots,N\}$, there exists $c_k \in \mathbb{R}$ such that $H(x^\star(t),u^\star(t),p(t)) = c_k$ for almost every $t \in (\tau_{k-1}^\star,\tau_k^\star)$. Indeed, the case $k \in \mathcal{I}_1^\star$ is obtained from the Hamiltonian maximization condition

over $(\tau^\star_{k-1}, \tau^\star_k)$ and [63, Theorem 2.6.3], and the case $k \in \mathcal{I}^\star_2$ is easily obtained from the constancy of $u^\star$ and the Hamiltonian system over $(\tau^\star_{k-1}, \tau^\star_k)$.
Finally the above fifth item allows to obtain, on the one hand, that $c_k = c_{k-1}$ for all $k \in \{2, \dots, N\}$ and, on the other hand, that $c_N = q^0$, which concludes the proof of Proposition 3.2.2 in the case $x^{\text{targ}} \notin \partial\text{X}$.

3.5.2 The case $x^{\text{targ}} \in \partial\text{X}$

Assume that $x^{\text{targ}} \in \partial\text{X}$ and let $(x^\star, u^\star, T^\star)$ be a solution to Problem (3.4), associated with a partition $\mathbb{T}^\star = \{\tau^\star_k\}_{k=0,\dots,N}$ of the interval $[0, T^\star]$. For any $\varepsilon > 0$ small enough (precisely $0 < \varepsilon < T^\star - \tau^\star_{N-1}$), we denote by $T_\varepsilon := T^\star - \varepsilon$ and, from a basic dynamical programming argument, one can easily see that $(x^\star, u^\star, T_\varepsilon)$ is a solution to Problem (3.4) when replacing x^{targ} by $x^{\text{targ}}_\varepsilon := x^\star(T_\varepsilon) \notin \partial\text{X}$. Therefore one can follow exactly the proof of Proposition 3.2.2 detailed in the previous subsection. For any $\varepsilon > 0$ small enough, it provides the existence of a nontrivial pair $(q_\varepsilon, q^0_\varepsilon) \in \text{AC}([0,1], \mathbb{R}^{nN}) \times \mathbb{R}_+$ satisfying the above five items (just replacing $T^\star$ by T_ε everywhere). Using the fact that the nontrivial pair $(q_\varepsilon, q^0_\varepsilon)$ can be renormalized (since it is defined up to a positive multiplicative constant), compactness arguments and the fact that $T_\varepsilon \to T^\star$ when $\varepsilon \to 0$, one can obtain the existence of a nontrivial pair $(q, q^0) \in \text{AC}([0,1], \mathbb{R}^{nN}) \times \mathbb{R}_+$ satisfying the above five items (with $T^\star$, and not with T_ε). Finally the proof of Proposition 3.2.2 is concluded in a similar way than Step 4 of the previous subsection.

4

The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for $L^1_\square$–local solutions

Contents

This chapter is based on the work "The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for $L^1_\square$–local solutions" by T. Bayen, A. Bouali and L. Bourdin (see [16]). Here, we present a novel adaptation of the well-known *augmentation technique* in a spatially hybrid setting. We accomplish this by introducing a

new notion of local solution ($L^1_\square$–*local solution*) for classical optimal control problems and establishing a corresponding Pontryagin maximum principle. Using this approach, we successfully derive a hybrid maximum principle specifically adapted to our setting.

4.1 Introduction

General context of the present chapter. In this chapter, we derive a *spatially hybrid maximum principle* with general mixed terminal state constraints based on a careful use of the *augmentation technique* as presented in the work [16]. Recall that the *Pontryagin Maximum Principle* (in short, PMP), established at the end of the 1950s (see [102]), has originally been developed for optimal control problems where the control system is described by an ordinary differential equation (in short, ODE). It states the corresponding first-order necessary optimality conditions, in terms of an (absolutely continuous) costate function. As usual in optimization, the PMP remains valid for *local solutions* only (typically in uniform norm for the state and in L^1-norm for the control). Since then, the PMP has been adapted to many situations, in particular for control systems of different natures.
On the other hand, *hybrid systems* are, in a broad sense, dynamical systems that exhibit both continuous and discrete behaviors. They are particularly used in automation and robotics to describe complex systems in which, for example, logic decisions are combined with physical processes. We refer to [115] for an elementary introduction to hybrid systems. This theory is very large and it is commonly accepted that it includes ODEs with *heterogeneous dynamics*, that is, ODEs involving a family of different dynamics (used for example to describe evolutions in heterogeneous media) where the transitions from one dynamics to another are seen as discrete events.

The PMP has been extended to hybrid control systems, especially in the context of ODEs with heterogeneous dynamics (see, *e.g.*, [59], [68], [99], [108], [109], [111]), resulting in theorems often referred to as *Hybrid Maximum Principle* (in short, HMP). We emphasize that the frameworks are very varied. Indeed the rule that supervises the transitions between the different dynamics is usually described by additional variables that can be free or constrained and, in that second case, the constraints can be of different natures. For example the *switching times* (*i.e.* the instants at which the control system moves from one dynamics to another) can be the resultant of a control decision or can be (fully or partially) determined by the time variable, the state position or both of them. Hence different versions of the HMP can be found in the literature, corresponding to different hybrid control systems that are presented under various names according to their nature (such as *multi-processes* [52], *switched systems* [105], *regional systems* [10], *systems on stratified domains* [43], *variable structure systems* [25]). In contrary to the classical PMP, the HMP is usually expressed in terms of an (only) piecewise absolutely continuous costate function that admits discontinuity jumps at the switching times. A common feature of most of the above references is that the mathematical framework somehow guarantees that local perturbations (typically in uniform norm for the state and in L^1-norm for the control) preserve the same hybrid structure (that is, the same sequence of dynamics) as the nominal one.

The augmentation technique of Dmitruk and Kaganovich. In the context of ODEs with heterogeneous dynamics, the difficult part of deriving a HMP lies in handling the dynamical discontinuities. To this end, an excellent strategy has been proposed in [59], in which the switching times are additional

variables satisfying equality/inequality constraints involving the corresponding intermediate state values. Roughly speaking, considering an optimal solution (associated with switching times denoted by τ_k^*), this technique consists in affine changes of time variable, mapping the intervals (τ_{k-1}^*, τ_k^*) into a common interval $(0, 1)$. This procedure augments the dimensions of the variables and thus is categorized in the set of *augmentation techniques*. The authors prove that the augmented solution is a local solution to the augmented problem which is classical (that is, non-hybrid) by construction (since the discontinuities have been positioned at the endpoints of the interval $[0, 1]$). Therefore the classical PMP can be applied to the augmented solution (expressed in terms of an augmented absolutely continuous costate function satisfying endpoint transversality conditions). Hence, by inverting the above affine changes of time variable, first-order necessary optimality conditions are derived for the original nonaugmented solution, expressed in terms of a nonaugmented (piecewise absolutely continuous) costate function satisfying discontinuity jumps at the switching times τ_k^* (whose expressions follow from the endpoint transversality conditions at 0 and 1 of the augmented costate function).
Hence Dmitruk and Kaganovich have entitled their paper [59] as *The hybrid maximum principle is a consequence of Pontryagin maximum principle*. The augmentation technique is particularly satisfactory because it allows to reduce the hybrid problem into a classical (non-hybrid) augmented problem, avoiding the use of technical arguments (such as implicit function theorems) to handle the dynamical discontinuities.

Framework and contributions of the present work. In the spirit of [10], [73], we consider a control system described by an ODE with *spatially* heterogeneous dynamics, in the sense that the state space is partitioned into several disjoint regions and each region has its own dynamics. In that context the sequence of dynamics followed by the trajectory and the corresponding switching times (called *crossing times* since they correspond to the instants at which the state goes from one region to another) are fully constrained by the state position.
A HMP corresponding to this setting has already been announced in [73] but with a sketch of proof which is, to our best knowledge, erroneous. Indeed the author invoke *needle-like perturbations* of the control, while they are not admissible in the present setting (see General introduction 0 for a counterexample). This issue has been corrected in our previous paper [13] by applying needle-like perturbations on *auxiliary controls*. Then, to handle the resulting perturbed crossing times, we used an inductive application of the implicit function theorem, which results into a technical and extended analysis. An attempt to derive a HMP corresponding to our setting, with the simpler approach of Dmitruk and Kaganovich, was also presented in [10]. Unfortunately, to our best knowledge, this proof is also incorrect. Indeed, in contrary to the framework of Dmitruk and Kaganovich in [59], our setting fails to guarantee that the augmented solution is a local solution to the classical augmented problem (see Section 4.3.4 for a counterexample) and, therefore, the classical PMP cannot be applied. We emphasize that our counterexample shows that, in our setting, a local perturbation (in uniform norm for the state and in L^1-norm for the control) does not preserve the hybrid structure of the nominal one in general.
Hence the main objective of the paper [16] is to derive a HMP for our setting, with a correct proof that adapts the augmentation procedure of Dmitruk and Kaganovich. To this aim a new notion of local solution to classical optimal control problems (see the definition of $L^1_\square$–local solution in Definition 4.2.2) and a corresponding version of the PMP (see Theorem 4.2.1) are required. Indeed we prove in Proposition 4.3.1 that, under appropriate assumptions (such as *transverse conditions* at the crossing times), the augmented solution is a $L^1_\square$–local solution to the classical augmented problem and therefore the above new PMP

can be applied. Finally, similarly to [59], by inverting the affine changes of time variable, a HMP for our setting is obtained (see Theorem 4.3.1).

Organization of the chapter. In Section 4.2, a classical optimal control problem is considered (see Problem (4.1)), the new notion of $L^1_\square$–local solution is introduced (see Definition 4.2.2) and a corresponding PMP is established (see Theorem 4.2.1). In Section 4.3, a hybrid optimal control problem with spatially heterogeneous dynamics is introduced (see Problem (4.2)). Applying the augmentation procedure, Proposition 4.3.1 states that an augmented solution to Problem (4.2) is a $L^1_\square$–local solution to the corresponding classical augmented problem of the form of Problem (4.1). Hence, applying the above new PMP and inverting the affine changes of time variable, a HMP for Problem (4.2) is obtained (see Theorem 4.3.1). An explicit counterexample showing that an augmented solution to Problem (4.2) is not a local solution (in the usual sense) to the corresponding classical augmented problem is provided in Section 4.3.4. Finally the technical proofs of Proposition 4.3.1 and Theorem 4.3.1 are provided in Sections 4.4 and 4.5 respectively.

4.2 Preliminaries and PMP for the new notion of $L^1_\square$–local solution

The section is devoted to one of the main contributions of the paper [16] to providing the new notion of $L^1_\square$–local solution and its corresponding PMP. In Section 4.2.1, we present a general optimal control problem with parameters, as well as the new notion of $L^1_\square$–local solution. In Section 4.2.2, we provide the corresponding PMP.

4.2.1 A classical optimal control problem and $L^1_\square$–local solution

Let n, m, d and $\ell \in \mathbb{N}^*$ be four fixed positive integers and $T > 0$ be a fixed positive real number. In the present section we consider a classical Mayer optimal control problem with parameter and mixed terminal state constraints given by

$$\begin{array}{ll} \text{minimize} & \phi(x(0), x(T), \lambda), \\ \text{subject to} & (x, u, \lambda) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times \mathbb{R}^d, \\ & \dot{x}(t) = f(x(t), u(t), \lambda), \quad \text{a.e. } t \in [0,T], \\ & g(x(0), x(T), \lambda) \in \mathrm{S}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \end{array} \tag{4.1}$$

where the Mayer cost function $\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^n$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^\ell$ are of class C^1, and where $\mathrm{S} \subset \mathbb{R}^\ell$ is a nonempty closed convex subset and $\mathrm{U} \subset \mathbb{R}^m$ is a nonempty subset. As usual in the literature, $x \in \mathrm{AC}([0,T], \mathbb{R}^n)$ is called the *state* (or the *trajectory*), $u \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is called the *control* and $\lambda \in \mathbb{R}^d$ is called the *parameter*. A triplet $(x, u, \lambda) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times \mathbb{R}^d$ is said to be *admissible* for Problem (4.1) if it satisfies all the constraints of Problem (4.1). Finally, such an admissible triplet is said to be a *global solution* to Problem (4.1) if it minimizes the Mayer cost $\phi(x(0), x(T), \lambda)$ among all admissible triplets.

Remark 4.2.1. (i) All along this paper (not only for Problem (4.1)), we have chosen to deal with optimal

control problems with (only) Mayer cost, fixed final time and autonomous dynamics. It is well known in the literature (see, *e.g.*, [28], [44], [48]) that standard techniques (such as augmentation or changes of variables) allow to deal with more general Bolza cost, free final time and time-dependent dynamics. Similarly, in Problem (4.1), we assume for simplicity that ϕ, f and g are of class C^1 and also some topological properties for S. However the results that we will present in this section can be extended to weaker assumptions (see, *e.g.*, [49], [116]). Overall, our aim in this paper is not to address the most general framework possible. We keep our setting as simple as possible to stay focused on the novel aspects of our work.

(ii) The presence of a parameter $\lambda \in \mathbb{R}^d$ in Problem (4.1) can also be treated thanks to an augmentation (see, *e.g.*, [28]). It is noteworthy that the main problem considered in the present work (see Problem (4.2) in the next Section 4.3) is a hybrid optimal control problem which does not involve any parameter. However the proof of our main result (Theorem 4.3.1) is based on a reduction of Problem (4.2) into a classical optimal control problem of the form of Problem (4.1) that involves parameters. This is the only reason why we need to consider the presence of a parameter $\lambda \in \mathbb{R}^d$ in Problem (4.1).

The classical PMP [102] has originally been developed for global solutions but, as usual in optimization, it remains valid for *local solutions*. As a consequence, several notions of local solution to classical optimal control problems, and the corresponding versions of the PMP, have been developed in the literature (see, *e.g.*, [29], [93]). Let us introduce two new notions of local solution which will play central roles in our work.

Definition 4.2.1 (L^1_A–local solution)**.** *An admissible triplet (x^*, u^*, λ^*) is said to be a L^1_A–local solution to Problem (4.1), for a measurable subset $A \subset [0,T]$, if, for all $R \geq \|u^*\|_{L^\infty}$, there exists $\eta > 0$ such that $\phi(x^*(0), x^*(T), \lambda^*) \leq \phi(x(0), x(T), \lambda)$ for all admissible triplets (x, u, λ) satisfying*

$$\begin{cases} \|x - x^*\|_C + \|u - u^*\|_{L^1} + \|\lambda - \lambda^*\|_{\mathbb{R}^d} \leq \eta, \\ \|u\|_{L^\infty} \leq R, \\ u(t) = u^*(t) \text{ a.e. } t \in [0,T] \backslash A. \end{cases}$$

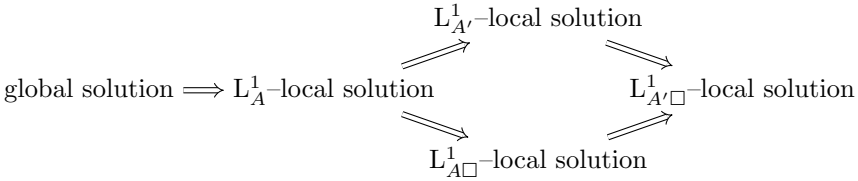
Definition 4.2.2 ($L^1_{A\square}$–local solution)**.** *An admissible triplet (x^*, u^*, λ^*) is said to be a $L^1_{A\square}$–local solution to Problem (4.1), for a measurable subset $A \subset [0,T]$, if there exists an increasing family $(A_\varepsilon)_{\varepsilon>0}$ of measurable subsets of A, satisfying $\mu(A_\varepsilon) \to \mu(A)$ as $\varepsilon \to 0$, such that (x^*, u^*, λ^*) is a $L^1_{A_\varepsilon}$–local solution to Problem (4.1) for all $\varepsilon > 0$.*

Remark 4.2.2. (i) The notations L^1_A and $L^1_{A\square}$ are very close, while the corresponding definitions are (slightly) different. Therefore the reader needs to be careful with these two different concepts, for which we will give each one a version of the PMP (see Lemma 4.2.1 for L^1_A–local solutions and Theorem 4.2.1 for $L^1_{A\square}$–local solutions).

(ii) The concept of $L^1_{[0,T]}$–local solution coincides with the classical notion of L^1–local solution well established in the literature (see, *e.g.*, [29], [93]). Therefore, in the sequel, we simply write L^1–local solution instead of $L^1_{[0,T]}$–local solution. To be consistent we simply write $L^1_\square$–local solution instead of $L^1_{[0,T]\square}$–local solution.

(iii) With respect to the classical concept of L^1–local solution, the refined notion of L^1_A–local solution imposes on admissible controls to match the nominal one almost everywhere outside the measurable subset $A \subset [0,T]$. This feature is crucial to reduce the hybrid optimal control problem considered in the next Section 4.3 into a classical optimal control problem. This is not possible with the classical concept of L^1–local solution, as shown by a counterexample in Section 4.3.4.

(iv) For a measurable subset $A \subset [0,T]$, it is clear that a L^1_A–local solution is automatically a $\mathrm{L}^1_{A\square}$–local solution. However the converse is not true in general (see the counterexample in Section 4.3.4). From a general point of view, the implications hold true for any measurable subsets $A' \subset A \subset [0,T]$, but not the



converses in general.

4.2.2 PMP for $\mathrm{L}^1_\square$–local solutions and comments

Recall first that the *normal cone* to S at some point $z \in \mathrm{S}$ is defined by

$$\mathrm{N}_\mathrm{S}[z] := \{z'' \in \mathbb{R}^\ell \mid \forall z' \in \mathrm{S},\ \langle z'', z'-z\rangle_{\mathbb{R}^\ell} \leq 0\},$$

and that g is said to be *submersive* at a point of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ if the differential of g at this point is surjective. Finally recall that the *Hamiltonian* $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (4.1) is defined by $\mathcal{H}(x,u,\lambda,p) := \langle p, f(x,u,\lambda)\rangle_{\mathbb{R}^n}$ for all $(x,u,\lambda,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n$. We are now in a position to establish a new version of the PMP that is dedicated to $\mathrm{L}^1_{A\square}$–local solutions to Problem (4.1).

Theorem 4.2.1 (PMP for $\mathrm{L}^1_{A\square}$–local solutions)**.** *If (x^*,u^*,λ^*) is a $\mathrm{L}^1_{A\square}$–local solution to Problem (4.1), for a measurable subset $A \subset [0,T]$, such that g is submersive at $(x^*(0),x^*(T),\lambda^*)$, then there exists a nontrivial pair $(p,p^0) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathbb{R}_+$ satisfying:*

(i) the Hamiltonian system $\dot{x^*}(t) = \nabla_p \mathcal{H}(x^*(t),u^*(t),\lambda^*,p(t))$ and $-\dot{p}(t) = \nabla_x \mathcal{H}(x^*(t),u^*(t),\lambda^*,p(t))$ for almost every $t \in [0,T]$;

(ii) the endpoint transversality condition

$$\begin{pmatrix} p(0) \\ -p(T) \\ \displaystyle\int_0^T \nabla_\lambda \mathcal{H}(x^*(s),u^*(s),\lambda^*,p(s))\,ds \end{pmatrix} = p^0 \nabla\phi(x^*(0),x^*(T),\lambda^*) + \nabla g(x^*(0),x^*(T),\lambda^*)\xi,$$

for some $\xi \in \mathrm{N}_\mathrm{S}[g(x^*(0),x^*(T),\lambda^*)]$;

(iii) the Hamiltonian maximization condition $u^*(t) \in \arg\max_{\omega\in\mathrm{U}} \mathcal{H}(x^*(t),\omega,\lambda^*,p(t))$ for almost every $t \in A$.

The proof of Theorem 4.2.1 is quite simple and will be developed in a few lines. It is based on the next preliminary PMP for L^1_A–local solutions to Problem (4.1).

Lemma 4.2.1 (PMP for L^1_A–local solutions)**.** *If (x^*,u^*,λ^*) is a L^1_A–local solution to Problem (4.1), for a measurable subset $A \subset [0,T]$, such that g is submersive at $(x^*(0),x^*(T),\lambda^*)$, then the conclusion of Theorem 4.2.1 holds true.*

About the proof of Lemma 4.2.1. A PMP for L^1_A–local solutions to classical optimal control problems can be established via many different methods known in the literature. In our context, since the measurable subset A can be of complex nature (such as a Cantor set of positive measure), the classical needle-like

perturbations of the control (see, *e.g.*, [44], [102]) may not be suitable for the sensitivity analysis of the control system and, therefore, one may prefer to use *implicit spike variations* (see, *e.g.*, [30], [39], [87]). To deal with the parameter $\lambda \in \mathbb{R}^d$ in Problem (4.1), one can simply augment the state variable from x to (x, λ) by adding the state equation $\dot{\lambda}(t) = 0_{\mathbb{R}^d}$ (see, *e.g.*, [28]). Finally, to deal with the general mixed terminal state constraints $g(x(0), x(T), \lambda) \in \mathrm{S}$ in Problem (4.1), one may use the Ekeland variational principle on a penalized functional involving the square of the distance function to S (see, *e.g.*, [39], [62]). Since all these techniques are very well known in the literature, the proof of Lemma 4.2.1 is omitted. □

Proof of Theorem 4.2.1. Consider an increasing family $(A_\varepsilon)_{\varepsilon>0}$ of measurable subsets of A associated with (x^*, u^*, λ^*) and a decreasing positive sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$. In the sequel we denote by $A_k := A_{\varepsilon_k}$ and by $(p_k, p_k^0) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathbb{R}_+$ the nontrivial pair provided by Lemma 4.2.1 (with $\xi_k \in \mathrm{N_S}[g(x^*(0), x^*(T), \lambda^*)]$) for all $k \in \mathbb{N}$. From linearity and submersiveness, the pair (ξ_k, p_k^0) is nontrivial and can be renormalized so that $\|(\xi_k, p_k^0)\|_{\mathbb{R}^\ell \times \mathbb{R}} = 1$ for all $k \in \mathbb{N}$. Therefore, up to a subsequence that we do not relabel, the sequence $(\xi_k, p_k^0)_{k\in\mathbb{N}}$ converges to some nontrivial pair (ξ, p^0) satisfying $(\xi, p^0) \in \mathrm{N_S}[g(x^*(0), x^*(T), \lambda^*)] \times \mathbb{R}_+$ from closure of the normal cone. Define $p \in \mathrm{AC}([0,T], \mathbb{R}^n)$ as the unique global solution to

$$\begin{cases} \dot{p}(t) = -\nabla_x f(x^*(t), u^*(t), \lambda^*)^\top p(t), \quad \text{a.e. } t \in [0,T], \\ p(T) = -p^0 \nabla_2 \phi(x^*(0), x^*(T), \lambda^*) - \nabla_2 g(x^*(0), x^*(T), \lambda^*)\xi. \end{cases}$$

The Hamiltonian system and the second component of the endpoint transversality condition are satisfied. Since p and p_k satisfy the same linear differential equation and $p_k(T) \to p(T)$, the sequence $(p_k)_{k\in\mathbb{N}}$ uniformly converges to p over $[0,T]$. We deduce the first and third components of the endpoint transversality condition and, from submersiveness, that the pair (p, p^0) is nontrivial. Still from Lemma 4.2.1, there exists a null set $N_k \subset A_k$ such that $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p_k(t)) \geq \mathcal{H}(x^*(t), \omega, \lambda^*, p_k(t))$ for all $\omega \in \mathrm{U}$ and all $t \in A_k \backslash N_k$, for all $k \in \mathbb{N}$. Now let us prove that the Hamiltonian maximization condition holds true at any $t \in \tilde{A} := (\cup_{k\in\mathbb{N}} A_k) \backslash (\cup_{k\in\mathbb{N}} N_k)$ which is a measurable subset of A with full measure. Let $t \in \tilde{A}$ and take $k_0 \in \mathbb{N}$ such that $t \in A_k \backslash N_k$ for all $k \geq k_0$. Therefore the previous inequality holds true at t for all $\omega \in \mathrm{U}$ and all $k \geq k_0$. From convergence of $p_k(t)$ to $p(t)$, we get that $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)) \geq \mathcal{H}(x^*(t), \omega, \lambda^*, p(t))$ for all $\omega \in \mathrm{U}$, which ends the proof. □

Remark 4.2.3. (i) First of all we bring the reader's attention to the fact that the Hamiltonian maximization condition in Lemma 4.2.1 and Theorem 4.2.1 holds true only almost everywhere over A (and not over the entire interval $[0,T]$). This is the only difference with the classical PMP and this is due, of course, to the restrictions to L^1_A– and $L^1_{A\square}$–local solutions (see Definitions 4.2.1 and 4.2.2 and Item (iii) of Remark 4.2.2).
(ii) Even if the conclusions of Lemma 4.2.1 and Theorem 4.2.1 are exactly the same, we recall that a $L^1_{A\square}$–local solution is not a L^1_A–local solution in general (see Item (iv) of Remark 4.2.2). Therefore Theorem 4.2.1 is not only a consequence of Lemma 4.2.1 but also a strict extension. From the diagram in Remark 4.2.2, it is also clear that the classical PMP (for global solutions or for L^1–local solutions) is a particular case of both Lemma 4.2.1 and Theorem 4.2.1 (by taking $A = [0,T]$).
(iii) As explained in [24], [35], the submersiveness hypothesis can be removed but, in that case, all items of Lemma 4.2.1 and Theorem 4.2.1 remain valid, except Item (ii).
(iv) Consider the framework of Theorem 4.2.1 for a $L^1_{\square}$–local solution (x^*, u^*, λ^*). Using the Hamiltonian system and the Hamiltonian maximization condition over $[0,T]$ and applying [63, Theorem 2.6.1], we obtain the *Hamiltonian constancy condition* $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)) = c$ for almost every $t \in [0,T]$, for

some $c \in \mathbb{R}$.

4.3 Derivation of a HMP for spatially heterogeneous dynamics

In this section we consider a partition of the state space $\mathbb{R}^n = \cup_{j\in\mathcal{J}} \overline{X_j}$, where $\mathcal{J}$ is a family of indexes and the nonempty open subsets $X_j \subset \mathbb{R}^n$, called *regions*, are disjoint. Our aim is to derive first-order necessary optimality conditions in a Pontryagin form for the *hybrid* optimal control problem with mixed terminal state constraints given by

$$\begin{array}{ll} \text{minimize} & \phi(x(0), x(T)), \\ \text{subject to} & (x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m), \\ & \dot{x}(t) = h(x(t),u(t)), \quad \text{a.e. } t \in [0,T], \\ & g(x(0),x(T)) \in \mathrm{S}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \end{array} \tag{4.2}$$

where the data assumptions and the terminology for Problem (4.2) are the same as those for Problem (4.1), except that the dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is *spatially heterogeneous*, in the sense that it is defined *regionally* by

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m, \quad h(x,u) := h_j(x,u) \quad \text{when } x \in X_j,$$

where the *subdynamics* $h_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are of class C^1. Note that $h(x,u)$ is not defined when $x \notin \cup_{j\in\mathcal{J}} X_j$ but this fact will have no impact on the rest of this work (see Item (i) in Remark 4.3.1). Finally, in contrary to Problem (4.1) and as explained in Item (ii) of Remark 4.2.1, note that Problem (4.2) does not involve any parameter.

4.3.1 Regular solutions to the hybrid control system

Due to the discontinuities of the spatially heterogeneous dynamics h, we need to precise the definition of a solution to the hybrid control system

$$\dot{x}(t) = h(x(t),u(t)), \quad \text{for a.e. } t \in [0,T], \tag{4.3}$$

associated with Problem (4.2).

Definition 4.3.1 (Solution to (4.3))**.** *A pair $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is said to be a solution to (4.3) if there exists a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ such that:*

(i) For all $k \in \{1,\dots,N\}$, there exists $j(k) \in \mathcal{J}$ (with $j(k) \neq j(k-1)$) such that $x(t) \in X_{j(k)}$ for almost every $t \in (\tau_{k-1},\tau_k)$.

(ii) $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.

(iii) $\dot{x}(t) = h_{j(k)}(x(t),u(t))$ for almost every $t \in (\tau_{k-1},\tau_k)$ and all $k \in \{1,\dots,N\}$.

In that case, to ease notation, we set $f_k := h_{j(k)}$ and $E_k := X_{j(k)}$ for all $k \in \{1,\dots,N\}$. With this system

of notations, we have

$$\begin{cases} x(t) \in E_1, \quad \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k, \quad \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N, \quad \forall t \in (\tau_{N-1}, \tau_N], \\ \dot{x}(t) = f_k(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{1, \dots, N\}. \end{cases}$$

Finally the times τ_k, for $k \in \{1, \dots, N-1\}$, are called crossing times since they correspond to the instants at which the trajectory x goes from the region E_k to the region E_{k+1}, and thus $x(\tau_k) \in \partial E_k \cap \partial E_{k+1}$.

Our main result (Theorem 4.3.1 stated in Section 4.3.3) is based on some regularity assumptions made on the behavior of the optimal pair of Problem (4.2) at each crossing time. These hypotheses are precised in the next definition.

Definition 4.3.2 (Regular solution to (4.3))**.** *Following the notations introduced in Definition 4.3.1, a solution $(x, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ to (4.3) is said to be regular if the following conditions are both satisfied:*

(i) At each crossing time τ_k, there exists a C^1 function $F_k : \mathbb{R}^n \to \mathbb{R}$ such that

$$\exists \nu_k > 0, \quad \forall z \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in E_k & \Leftrightarrow & F_k(z) < 0, \\ z \in \partial E_k \cap \partial E_{k+1} & \Leftrightarrow & F_k(z) = 0, \\ z \in E_{k+1} & \Leftrightarrow & F_k(z) > 0. \end{cases}$$

In particular it holds that $F_k(x(\tau_k)) = 0$.

(ii) At each crossing time τ_k, there exists $\alpha_k > 0$ and $\beta_k > 0$ such that the transverse conditions

$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k, \tau_k + \alpha_k], \end{cases} \tag{4.4}$$

are both satisfied.

Remark 4.3.1. (i) Definition 4.3.1 does not include the possibility of an infinite number of crossing times (excluding the Zeno phenomenon [120]). Also it does not allow trajectories bouncing against a boundary of a region, or moving along a boundary (excluding situations as described in [10]). This last restriction is the reason why the fact that $h(x, u)$ is not defined when $x \notin \cup_{j \in \mathcal{J}} X_j$ has no impact on the present work. Finally Definition 4.3.1 allows terminal states $x(0)$ and $x(T)$ that belong to regions only (and not to their boundaries). Possible relaxations are presented in Remark 4.3.4.

(ii) The transverse conditions (4.4) have a geometrical interpretation, meaning that x does not cross the boundary $\partial E_k \cap \partial E_{k+1}$ tangentially. At a crossing time τ_k, the transverse conditions

$$\begin{cases} u \text{ admits left and right limits at } \tau_k \text{ denoted by } u^-(\tau_k) \text{ and } u^+(\tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u^-(\tau_k)) \rangle_{\mathbb{R}^n} > 0 \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u^+(\tau_k)) \rangle_{\mathbb{R}^n} > 0, \end{cases} \tag{4.5}$$

considered in the papers [10], [73], are (slightly) stronger than (4.4).

4.3.2 Reduction into a classical optimal control problem with parameter

To establish a correspondence from the hybrid optimal control problem (4.2) to a classical optimal control problem with parameter of the form of Problem (4.1), we will proceed as in [59] to affine changes of time variable. Precisely let $(x^*, u^*) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ be a solution to (4.3), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, and let E_k^* and f_k^* stand for the corresponding regions and functions (see Definition 4.3.1). We introduce $(y^*, v^*) \in \mathrm{AC}([0,1],\mathbb{R}^{nN}) \times \mathrm{L}^\infty([0,1],\mathbb{R}^{mN})$ defined by

$$y_k^*(s) := x^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s) \quad \text{and} \quad v_k^*(s) := u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s), \tag{4.6}$$

for all $s \in [0,1]$ and all $k \in \{1,\dots,N\}$. To invert the changes of time variable, it holds

$$x^*(t) = y_k^*\left(\frac{t-\tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right) \quad \text{and} \quad u^*(t) = v_k^*\left(\frac{t-\tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right), \tag{4.7}$$

for all $t \in [\tau_{k-1}^*, \tau_k^*]$ and all $k \in \{1,\dots,N\}$. In particular note that $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$. From a more general point of view, it holds that $x^*(\tau_k^*) = y_{k+1}^*(0)$ for all $k \in \{0,\dots,N-1\}$ and $x^*(\tau_k^*) = y_k^*(1)$ for all $k \in \{1,\dots,N\}$. Note that the triplet $(y^*, v^*, \mathbb{T}^*)$ satisfies

$$\dot{y}^*(s) = f^*(y^*(s), v^*(s), \mathbb{T}^*), \quad \text{a.e. } s \in [0,1],$$

where $f^* : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \to \mathbb{R}^{nN}$ is the C^1 function defined by

$$f^*(y, v, \mathbb{T}) := \Big((\tau_1 - \tau_0) f_1^*(y_1, v_1), \dots, (\tau_N - \tau_{N-1}) f_N^*(y_N, v_N)\Big),$$

for all $y = (y_1,\dots,y_N) \in \mathbb{R}^{nN}$, $v = (v_1,\dots,v_N) \in \mathbb{R}^{mN}$ and $\mathbb{T} = \{\tau_0,\dots,\tau_N\} \in \mathbb{R}^{N+1}$. Furthermore it holds that

$$\begin{cases} y_1^*(s) \in E_1, & \forall s \in [0,1), \\ y_k^*(s) \in E_k, & \forall s \in (0,1), \quad \forall k \in \{2,\dots,N-1\}, \\ y_N^*(s) \in E_N, & \forall s \in (0,1], \end{cases} \tag{4.8}$$

and $y_{k+1}^*(0) = y_k^*(1) \in \partial E_k^* \cap \partial E_{k+1}^*$ for all $k \in \{1,\dots,N-1\}$. Also note that $\mathbb{T}^* \in \Delta$ where $\Delta \subset \mathbb{R}^{N+1}$ is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0,\dots,N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{N-1} \leq \tau_N = T\}.$$

Now assume that the pair (x^*, u^*) is moreover a regular solution to (4.3) and denote by F_k^* and $\nu_k^* > 0$ the corresponding functions and positive radii (see Definition 4.3.2). In that context note that $F_k^*(x(\tau_k^*)) = F_k^*(y_k^*(1)) = 0$ for all $k \in \{1,\dots,N-1\}$. Finally it is clear that, if the pair (x^*, u^*) is furthermore admissible for Problem (4.2), then the triplet $(y^*, v^*, \mathbb{T}^*)$ is admissible for the classical optimal control

problem with parameter given by

$$\begin{array}{ll} \text{minimize} & \phi^*(y(0),y(1),\mathbb{T}),\\ \text{subject to} & (y,v,\mathbb{T})\in \mathrm{AC}([0,1],\mathbb{R}^{nN})\times \mathrm{L}^\infty([0,1],\mathbb{R}^{mN})\times\mathbb{R}^{N+1},\\ & \dot y(s)=f^*(y(s),v(s),\mathbb{T}),\quad \text{a.e. } s\in[0,1],\\ & g^*(y(0),y(1),\mathbb{T})\in \mathrm{S}^*,\\ & v(s)\in \mathrm{U}^N,\quad \text{a.e. } s\in[0,1], \end{array} \tag{4.9}$$

where $\phi^*:\mathbb{R}^{nN}\times\mathbb{R}^{nN}\times\mathbb{R}^{N+1}\to\mathbb{R}$ and $g^*:\mathbb{R}^{nN}\times\mathbb{R}^{nN}\times\mathbb{R}^{N+1}\to\mathbb{R}^{\ell^*}$ are the C^1 functions defined by $\phi^*(y^0,y^1,\mathbb{T}):=\phi(y_1^0,y_N^1)$ and

$$g^*(y^0,y^1,\mathbb{T}):=(g(y_1^0,y_N^1),y_2^0-y_1^1,\dots,y_N^0-y_{N-1}^1,F_1^*(y_1^1),\dots,F_{N-1}^*(y_{N-1}^1),\mathbb{T}),$$

for all $y^0=(y_1^0,\dots,y_N^0)$, $y^1=(y_1^1,\dots,y_N^1)\in\mathbb{R}^{nN}$ and $\mathbb{T}=\{\tau_0,\dots,\tau_N\}\in\mathbb{R}^{N+1}$, where $\ell^*:=\ell+n(N-1)+(N-1)+(N+1)$, and where $\mathrm{S}^*\subset\mathbb{R}^{\ell^*}$ stands for the nonempty closed convex set defined by

$$\mathrm{S}^*:=\mathrm{S}\times\{0_{\mathbb{R}^n}\}^{N-1}\times\{0\}^{N-1}\times\Delta.$$

Proposition 4.3.1. *If $(x^*,u^*)\in \mathrm{AC}([0,T],\mathbb{R}^n)\times\mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is a global solution to Problem (4.2), that is moreover a regular solution to (4.3), associated with a partition $\mathbb{T}^*=\{\tau_k^*\}_{k=0,\dots,N}$, then the triplet $(y^*,v^*,\mathbb{T}^*)$ constructed above is a $\mathrm{L}^1_\square$–local solution to Problem (4.9).*

Proof. The proof of Proposition 4.3.1 is postponed to Appendix 4.4. We prove that the triplet $(y^*,v^*,\mathbb{T}^*)$ is a $\mathrm{L}^1_{[\varepsilon,1-\varepsilon]}$–local solution to Problem (4.9) for any $0<\varepsilon<1/2$. □

Remark 4.3.2. (i) Consider the framework of Proposition 4.3.1. In Section 4.3.4 we will provide a counterexample showing that the triplet $(y^*,v^*,\mathbb{T}^*)$ is not a L^1–local solution to Problem (4.9) in general. This highlights the fact that the classical PMP cannot be applied to the triplet $(y^*,v^*,\mathbb{T}^*)$. However, thanks to Proposition 4.3.1, we can apply the new PMP for $\mathrm{L}^1_\square$–local solution obtained in Theorem 4.2.1. This allows us to derive a HMP for Problem (4.2) in the next Section 4.3.3.

(ii) Consider the framework of Proposition 4.3.1. Given an admissible triplet $(y,v,\mathbb{T})$ for Problem (4.9), one can easily invert the augmentation procedure and obtain a pair (x,u) which satisfies all the constraints of Problem (4.2), except one. Precisely, even if (x,u) follows the same sequence $(f_k^*)_{k=1,\dots,N}$ of dynamics than the pair (x^*,u^*), it does not necessarily follow the same sequence of regions $(E_k^*)_{k=1,\dots,N}$ (and thus it is not necessarily admissible for Problem (4.2)). This is the major difficulty of the proof of Proposition 4.3.1 and, as we will see with a counterexample in Section 4.3.4, the notion of L^1–local solution (which consists in considering the triplet $(y,v,\mathbb{T})$ in a standard neighborhood of $(y^*,v^*,\mathbb{T}^*)$) fails to guarantee this property. This is because, even if transverse conditions are satisfied by the pair (x^*,u^*), allowing L^1-perturbations of u^* (with possibly far values in U from the ones of u^*) in the neighborhoods of the crossing times τ_k^* may lead to a perturbed pair (x,u) that does not satisfy the transverse conditions, and thus to a perturbed trajectory x that may visit a different sequence of regions than x^*. On the contrary, the new notion of $\mathrm{L}^1_{[\varepsilon,1-\varepsilon]}$–local solution, for $0<\varepsilon<1/2$, addresses this issue by allowing L^1-perturbations of u^* only outside neighborhoods of the crossing times τ_k^*.

4.3.3 HMP and comments

The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (4.2) is defined by $H(x,u,p) := \langle p, h(x,u)\rangle$ for all $(x,u,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to state the main result of the paper [16].

Theorem 4.3.1 (HMP). *If $(x^*,u^*) \in \mathrm{AC}([0,T],\mathbb{R}^n)\times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is a global solution to Problem (4.2), that is moreover a regular solution to (4.3), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, such that g is submersive at $(x^*(0),x^*(T))$, then there exists a nontrivial pair $(p,p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0,T],\mathbb{R}^n)\times\mathbb{R}_+$ satisfying:*

(i) the Hamiltonian system $\dot{x^*}(t) = \nabla_p H(x^*(t),u^*(t),p(t))$ and $-\dot{p}(t) = \nabla_x H(x^*(t),u^*(t),p(t))$ for almost every $t \in [0,T]$;

(ii) the endpoint transversality condition

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla\phi(x^*(0),x^*(T)) + \nabla g(x^*(0),x^*(T))\xi,$$

for some $\xi \in \mathrm{N_S}[g(x^*(0),x^*(T))]$;

(iii) the discontinuity condition $p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*))$ for some $\sigma_k \in \mathbb{R}$, for all $k \in \{1,\dots,N-1\}$;

(iv) the Hamiltonian maximization condition $u^*(t) \in \arg\max_{\omega\in \mathrm{U}} H(x^*(t),\omega,p(t))$ for almost every $t \in [0,T]$;

(v) the Hamiltonian constancy condition $H(x^*(t),u^*(t),p(t)) = c$ for almost every $t \in [0,T]$, for some $c \in \mathbb{R}$.

Proof. The proof of Theorem 4.3.1 is postponed to Section 4.5. It is based on Proposition 4.3.1 and on the application of Theorem 4.2.1 to the triplet $(y^*,v^*,\mathbb{T}^*)$. □

Remark 4.3.3. (i) In the classical PMP (that is, when the dynamics is not heterogeneous), the costate p is absolutely continuous over the entire interval $[0,T]$ and satisfies Items (i), (ii), (iv) and (v) of Theorem 4.3.1 (see, *e.g.*, [102]). In the present setting of heterogeneous dynamics, the costate p is (only) piecewise absolutely continuous over $[0,T]$, admitting at each crossing time τ_k^* a discontinuity jump satisfying Item (iii) of Theorem 4.3.1. Under the (slightly) stronger transverse conditions (4.5), the Hamiltonian constancy condition allows to obtain

$$\begin{aligned}\sigma_k &= -\frac{\Big\langle p^-(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*),(u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*),(u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}}{\Big\langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*),(u^*)^+(\tau_k^*))\Big\rangle_{\mathbb{R}^n}} \\ &= -\frac{\Big\langle p^+(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*),(u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*),(u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}}{\Big\langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*),(u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}},\end{aligned}$$

for all $k \in \{1,\dots,N-1\}$, and thus the discontinuity conditions can be expressed as forward (or backard) discontinuity jumps. Such discontinuity jumps are very standard in the literature on hybrid optimal control problems (see, *e.g.*, [25], [99]) and the discontinuity conditions have even been announced in our setting of spatially heterogeneous dynamics in the papers [10], [73]. However, as explained in Introduction, we recall that the proofs in [10], [73] are not satisfactory for several and different reasons.

(ii) Similarly to Item (iii) of Remark 4.2.3, and as explained in [24], [35], the submersiveness hypothesis made in Theorem 4.3.1 can be removed but, in that case, all items of Theorem 4.3.1 remain valid, except Item (ii).

Remark 4.3.4. (i) Consider the framework of Proposition 4.3.1. From Item (i) of Remark 4.3.2, we know that $(y^*, v^*, \mathbb{T}^*)$ is not a L^1–local solution to Problem (4.9) in general. Nevertheless, according to the ideas presented in Item (ii) of Remark 4.3.2, it may be possible to avoid the use of the notion of $L^1_\square$–local solution introduced in the paper [16]. However, to our best knowledge, this would not be possible without obtaining a weaker result and/or without restricting the framework. Let us develop two options in that direction:

- First, under the (slightly) stronger transverse conditions (4.5), it can be proved that $(y^*, v^*, \mathbb{T}^*)$ is a L^∞–local solution to Problem (4.9), in the sense that there exists $\eta > 0$ such that

$$\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T}),$$

  for all admissible triplets $(y, v, \mathbb{T})$ satisfying $\|y - y^*\|_C + \|v - v^*\|_{L^\infty} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta$. This idea is in-line with the approach developed in [19]. In that context, assuming for simplicity that U is closed and convex and applying a *weak version* of the classical PMP (that is, a version adapted to L^∞–local solutions, see [42] and discussion therein), one can derive a weaker version of Theorem 4.3.1, that is, without the Hamiltonian constancy condition and, above all, where the Hamiltonian maximization condition is replaced by the weaker *Hamiltonian gradient condition* $\nabla_u H(x^*(t), u^*(t), p(t)) \in N_U[u^*(t)]$ for a.e. $t \in [0, T]$.
- Second, under the (very) stronger transverse conditions given by

$$\forall \omega \in U, \quad \begin{cases} \langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \\ \langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \end{cases} \tag{4.10}$$

  for some $\beta_k > 0$ at each crossing time τ_k^*, it can be proved that $(y^*, v^*, \mathbb{T}^*)$ is a L^1–local solution to Problem (4.9). In that context one can derive Theorem 4.3.1 from the application of the classical PMP. However the strong transverse conditions (4.10) are quite restrictive and are not satisfied in practice (see the counterexample presented in the next Section 4.3.4).

From a general point of view, it can be observed that the choice of the transverse conditions (more or less strong) influences the *local quality* (L^1, L^∞ or $L^1_\square$) of the solution $(y^*, v^*, \mathbb{T}^*)$ to Problem (4.9) and thus the version of the PMP that can be applied to it, and finally the version of the HMP obtained on the original pair (x^*, u^*).

(ii) For simplicity, Definition 4.2.1 allows trajectories x such that $x(0)$ and $x(T)$ belong to regions only (and not to their boundaries). This restriction may limit the scope of our results. To overcome this restriction, some adjustments have to be performed. For instance, consider the framework of Theorem 4.3.1 with $x^*(0) \in E_1$ and $x^*(T) \in \partial E_N$ (other cases can be handled similarly). To deal with this situation, one has to add in Definition 4.3.2 the existence of a local C^1 description F_N of ∂E_N in a neighborhood of $x^*(T)$ and an adapted transverse condition of the form

$$\langle \nabla F_N^*(x^*(T)), f_N^*(x^*(T), u^*(t)) \rangle_{\mathbb{R}^n} \geq \beta_N, \quad \text{a.e. } t \in [T - \alpha_N, T),$$

with $\alpha_N > 0$ and $\beta_N > 0$. Then the augmented problem (4.9) must be adjusted carefully by adding the inequality constraint $F_N^*(y_N(1)) \leq 0$ to keep the validity of Proposition 4.3.1. Finally, adapting

the submersiveness hypothesis (involving both g and F_N^*), applying Theorem 4.2.1 and inverting the augmentation procedure, the conclusion of Theorem 4.3.1 remains valid, but with an additional term of the form $\zeta \nabla F_N^*(x^*(T))$ with $\zeta \geq 0$ in the expression of $-p(T)$.

(iii) In addition to the comments made in the previous Item (ii), we would like to emphasize that certain cases where $x^*(0)$ and $x^*(T)$ belong to boundaries of the regions can be treated without the adjusted procedure discussed above. For instance, if the initial condition is fixed on a boundary, then no information is expected for $p(0)$ and, furthermore, with the approach developed in this chapter, only perturbations of the control over $[\varepsilon, T]$ for small $\varepsilon > 0$ are considered. Hence the corresponding perturbed trajectories coincide with the nominal trajectory over $[0, \varepsilon]$ and thus satisfy the initial condition. Another example is provided with minimum time problems where the target belongs to a boundary of a region. In that context, a simple dynamical programming argument can eliminate the need of a transverse condition at T (see [15]).

(iv) Here we focus on possible extensions and perspectives of our work.

- First, one may consider a setting where the subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^{m_j} \to \mathbb{R}^n$ have possibly different control dimensions $m_j \in \mathbb{N}^*$ and with possibly different control constraint sets $\mathrm{U}_j \subset \mathbb{R}^{m_j}$. This generalized context is interesting to impose specific values for the control in certain regions (for example, by taking $\mathrm{U}_j = \{0_{\mathbb{R}^{m_j}}\}$ for some $j \in \mathcal{J}$). We believe that our methodology can be adapted to this framework without any major difficulty.
- Second, one may consider an extended setting that includes a *regionally switching parameter* (we refer to Chapter 5), meaning that the control system depends on a parameter that remains constant in each region but can change its value when the state crosses a boundary. This framework enables us to handle, as a specific case, control systems with *loss control regions* (see [14], [15], [17]).

4.3.4 A counterexample

Consider the framework of Proposition 4.3.1. This section is dedicated to an explicit counterexample showing that the triplet $(y^*, v^*, \mathbb{T}^*)$ is not a L^1–local solution to Problem (4.9) in general. To this aim consider the two-dimensional case $n = 2$, the state space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2}$ where $X_1 := (-\infty, 1) \times \mathbb{R}$ and $X_2 := (1, +\infty) \times \mathbb{R}$, and the hybrid optimal control problem given by

$$\begin{aligned} \text{minimize} \quad & -(x_1(2) - 2)^3 - \rho x_2(2), \\ \text{subject to} \quad & (x, u) \in \mathrm{AC}([0,2], \mathbb{R}^2) \times \mathrm{L}^\infty([0,2], \mathbb{R}), \\ & \dot{x}(t) = h(x(t), u(t)), \quad \text{a.e. } t \in [0,2], \\ & x(0) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1, 1], \quad \text{a.e. } t \in [0,2], \end{aligned} \tag{4.11}$$

where the spatially heterogeneous dynamics $h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by

$$h(x,u) := \begin{cases} \Big(1, ((1 - x_1)^+)^2\Big), & \text{if } x \in X_1, \\ \Big(u, ((1 - x_1)^+)^2\Big), & \text{if } x \in X_2, \end{cases}$$

for all $x = (x_1, x_2) \in X_1 \cup X_2$ and all $u \in \mathbb{R}$, and where $\rho > 96$.

A global solution (x^*, u^*) to Problem (4.11)

In view of the definition of h in the region X_1 and following Definition 4.2.1, any admissible pair (x, u) for Problem (4.11) has exactly one crossing time $\tau_1 = 1$, and satisfies $x_1(t) = t$ for all $t \in [0, 1]$ and $x_1(t) > 1$ for all $t \in (1, 2]$. Moreover an easy computation shows that

$$x_2(t) = \begin{cases} \frac{1}{3}((t-1)^3 + 1) & \text{if } t \in [0, 1], \\ \frac{1}{3} & \text{if } t \in [1, 2], \end{cases}$$

for all $t \in [0, 2]$. Since the value $x_2(2)$ is fixed to $\frac{1}{3}$ for any admissible pair, Problem (4.11) simply amounts to maximize the value of $x_1(2)$. In view of the definition of h in the region X_2, one can easily deduce that a global solution (x^*, u^*) to Problem (4.11) is given by

$$x_1^*(t) := t, \quad x_2^*(t) := \begin{cases} \frac{1}{3}((t-1)^3 + 1) & \text{if } t \in [0, 1], \\ \frac{1}{3} & \text{if } t \in [1, 2], \end{cases} \qquad u^*(t) := 1,$$

for all $t \in [0, 2]$, and the corresponding optimal cost is given by $\mathcal{C}^* := -\frac{\rho}{3}$. Furthermore one can observe that the pair (x^*, u^*) is a regular solution to the corresponding hybrid control system (Definition 4.3.2) with exactly one crossing time $\tau_1^* = 1$.

The corresponding triplet $(y^*, v^*, \mathbb{T}^*)$

Now consider the framework of Proposition 4.3.1. The corresponding triplet $(y^*, v^*, \mathbb{T}^*)$ is given by

$$(y_1^1)^*(s) = s, \quad (y_2^1)^*(s) = s + 1, \quad (y_1^2)^*(s) = \frac{1}{3}((s-1)^3 + 1), \quad (y_2^2)^*(s) = \frac{1}{3},$$

and $v_1^*(s) = v_2^*(s) = 1$ for all $s \in [0, 1]$, and $\mathbb{T}^* = \{0, 1, 2\}$. As expected the triplet $(y^*, v^*, \mathbb{T}^*)$ is admissible for the classical optimal control problem with parameter

$$\begin{aligned} \text{minimize} \quad & -(y_2^1(1) - 2)^3 - \rho y_2^2(1), \\ \text{subject to} \quad & y = (y_1^1, y_2^1, y_1^2, y_2^2) \in \mathrm{AC}([0, 1], \mathbb{R}^4),\ v = (v_1, v_2) \in \mathrm{L}^\infty([0, 1], \mathbb{R}^2),\ \mathbb{T} = (\tau_0, \tau_1, \tau_2) \in \mathbb{R}^3, \\ & \dot{y}_1^1(s) = \tau_1, \quad \text{a.e. } s \in [0, 1], \\ & \dot{y}_1^2(s) = \tau_1((1 - y_1^1(s))^+)^2, \quad \text{a.e. } s \in [0, 1], \\ & \dot{y}_2^1(s) = (2 - \tau_1)v_2(s), \quad \text{a.e. } s \in [0, 1], \\ & \dot{y}_2^2(s) = (2 - \tau_1)((1 - y_2^1(s))^+)^2, \quad \text{a.e. } s \in [0, 1], \\ & y_1^1(0) = 0, \quad y_1^2(0) = 0, \quad y_1^1(1) - 1 = 0, \quad y_2^1(0) - y_1^1(1) = 0, \quad y_2^2(0) - y_1^2(1) = 0, \\ & \tau_0 = 0, \quad \tau_1 \in [0, 2], \quad \tau_2 = 2, \\ & v_1(s),\ v_2(s) \in [-1, 1], \quad \text{a.e. } s \in [0, 1], \end{aligned} \tag{4.12}$$

with the cost $\mathcal{C}^* = -\frac{\rho}{3}$.

The triplet $(y^*, v^*, \mathbb{T}^*)$ is not a L^1–local solution to Problem (4.12)

For any $\varepsilon > 0$ small enough, we introduce the triplet $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$ defined by $(y_1^1)^\varepsilon := (y_1^1)^*$, $(y_1^2)^\varepsilon := (y_1^2)^*$, $v_1^\varepsilon = v_1^*$, $\mathbb{T}^\varepsilon = \mathbb{T}^*$, and by

$$(y_2^1)^\varepsilon(s) := \begin{cases} s+1, & \text{if } s \in [0,\varepsilon], \\ 2\varepsilon - s + 1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ s - 4\varepsilon + 1, & \text{if } s \in [3\varepsilon, 1], \end{cases} \qquad v_2^\varepsilon(s) := \begin{cases} 1, & \text{if } s \in [0,\varepsilon], \\ -1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ 1, & \text{if } s \in [3\varepsilon, 1], \end{cases}$$

and

$$(y_2^2)^\varepsilon(s) := \begin{cases} \frac{1}{3}, & \text{if } s \in [0, 2\varepsilon], \\ \frac{1}{3}((s-2\varepsilon)^3 + 1), & \text{if } s \in [2\varepsilon, 3\varepsilon], \\ \frac{1}{3}((s-4\varepsilon)^3 + 2\varepsilon^3 + 1), & \text{if } s \in [3\varepsilon, 4\varepsilon], \\ \frac{1}{3}(2\varepsilon^3 + 1), & \text{if } s \in [4\varepsilon, 1], \end{cases}$$

for all $s \in [0,1]$. One can easily conclude that the triplet (y^*, v^*, τ^*) is not a L^1–local solution to Problem (4.12) since:

– The triplet $(y^\varepsilon, v^\varepsilon, \tau^\varepsilon)$ is admissible for Problem (4.12) for any $\varepsilon > 0$.

– It holds that $\lim_{\varepsilon \to 0}(\|y^\varepsilon - y^*\|_{\mathrm{C}} + \|v^\varepsilon - v^*\|_{\mathrm{L}^1} + \|\mathbb{T}^\varepsilon - \mathbb{T}^*\|_{\mathbb{R}^3}) = 0$.

– For any $\varepsilon > 0$, the cost $\mathcal{C}^\varepsilon$ associated with the triplet $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$ is given by

$$\mathcal{C}^\varepsilon = -\frac{\rho}{3} - \left(\frac{2\rho}{3} - 64\right)\varepsilon^3 < -\frac{\rho}{3} = \mathcal{C}^*.$$

4.4 Proof of Proposition 4.3.1

Consider the framework of Proposition 4.3.1 and let us prove that the triplet $(y^*, v^*, \mathbb{T}^*)$ is a $\mathrm{L}^1_{[\varepsilon, 1-\varepsilon]}$–local solution to Problem (4.9) for any $0 < \varepsilon < \frac{1}{2}$. Therefore let $0 < \varepsilon < \frac{1}{2}$ and $R \geq \|v^*\|_{\mathrm{L}^\infty}$. Our aim is to prove that there exists $\eta > 0$ such that $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$ for any triplet $(y, v, \mathbb{T})$ that is admissible for Problem (4.9) and satisfying

$$\begin{cases} \|y - y^*\|_{\mathrm{C}} + \|v - v^*\|_{\mathrm{L}^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\ \|v\|_{\mathrm{L}^\infty} \leq R, \\ v(s) = v^*(s) \text{ a.e. } s \in [0,\varepsilon] \cup [1-\varepsilon, 1]. \end{cases} \tag{4.13}$$

To this aim we need to introduce several technical positive parameters:

($\mathfrak{P}_1$) Let $\underline{\theta} := \min_{k \in \{1,\dots,N\}} |\tau_k^* - \tau_{k-1}^*| > 0$ and $\overline{\theta} := \max_{k \in \{1,\dots,N\}} |\tau_k^* - \tau_{k-1}^*| > 0$.

($\mathfrak{P}_2$) From the transverse conditions (see Definition 4.3.2) and the (uniform) continuities of the functions ∇F_k^* and f_k^* on compact sets, there exist $0 < \nu \leq \min_{k \in \{1,\dots,N-1\}} \nu_k^*$ and $0 < \alpha \leq \min\{\frac{\theta}{3}, \min_{k \in \{1,\dots,N-1\}} \alpha_k^*\}$ such that

$$\begin{cases} \langle \nabla F_k^*(z), f_k^*(z, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in [\tau_k^* - \alpha, \tau_k^*), \\ \langle \nabla F_k^*(z), f_{k+1}^*(z, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in (\tau_k^*, \tau_k^* + \alpha], \end{cases}$$

for all $z \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and all $k \in \{1, \dots, N-1\}$.

($\mathfrak{P}_3$) From continuity of y^* over $[0,1]$, there exists $0 < \chi < \frac{1}{2}$ such that $\|y_k^*(s) - y_k^*(0)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for

all $s \in [0,\chi]$ and $\|y_k^*(s) - y_k^*(1)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for all $s \in [1-\chi, 1]$, for all $k \in \{1,\dots,N\}$.

($\mathfrak{P}_4$) Define $\gamma := \frac{\theta}{3}\min\{\varepsilon, \chi, \frac{\alpha}{\theta}\} > 0$ and $r := \frac{\gamma}{\overline{\theta}+\underline{\theta}} > 0$. Note that $0 < \gamma \leq \alpha \leq \frac{\theta}{3}$ and $0 < r < \frac{1}{2}$.

($\mathfrak{P}_5$) From continuity of y^*, from (4.8) and the openness of the regions E_k^*, there exists $\delta > 0$ such that

$$\begin{cases} \overline{\mathrm{B}}_{\mathbb{R}^n}(y_1^*(s),\delta) \subset E_1^*, & \forall s \in [0, 1-r], \\ \overline{\mathrm{B}}_{\mathbb{R}^n}(y_k^*(s),\delta) \subset E_k^*, & \forall s \in [r, 1-r], \quad \forall k \in \{2,\dots,N-1\}, \\ \overline{\mathrm{B}}_{\mathbb{R}^n}(y_N^*(s),\delta) \subset E_N^*, & \forall s \in [r, 1]. \end{cases}$$

We are now in a position to continue the proof. To this aim let $\eta := \min\{\frac{\theta}{3}, \frac{\nu}{2}, \delta\} > 0$ and $(y, v, \mathbb{T})$ be an admissible triplet for Problem (4.9) satisfying (4.13). Our aim is to prove that $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$.

Step 1 Since $0 = \tau_0^* < \tau_1^* < \cdots < \tau_{N-1}^* < \tau_N^* = T$ and $\mathbb{T} \in \Delta$ with $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, one can easily deduce that $0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = T$. Therefore we are in a position to define $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ by

$$x(t) := y_k\left(\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right) \quad \text{and} \quad u(t) := v_k\left(\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right), \tag{4.14}$$

for all $t \in [\tau_{k-1}, \tau_k]$ and all $k \in \{1,\dots,N\}$. Note that x is well defined since $y_{k+1}(0) = y_k(1)$ for all $k \in \{2,\dots,N\}$ (from admissibility of the triplet $(y, v, \mathbb{T})$). Observe that $(y_1(0), y_N(1)) = (x(0), x(T))$ and recall that $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$. Therefore, from the definition of ϕ^* (see Section 4.3.2) and since (x^*, u^*) is a global solution to Problem (4.2), to obtain that $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$, we only need to prove that the pair (x,u) is admissible for Problem (4.2).
From admissibility of the triplet $(y, v, \mathbb{T})$, it is clear that $g(x(0), x(T)) \in \mathrm{S}$ and $u(t) \in \mathrm{U}$ for almost every $t \in [0,T]$. Therefore it only remains to prove that (x,u) is a solution to the hybrid control system (4.3) (see Definition 4.3.1). From (4.14) and the admissibility of the triplet $(y, v, \mathbb{T})$, one can easily obtain that

$$\dot{x}(t) = f_k^*(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \tag{4.15}$$

for all $k \in \{1,\dots,N\}$. Therefore, to conclude the proof, we only need to prove that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k^*, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2,\dots,N-1\}, \\ x(t) \in E_N^*, & \forall t \in (\tau_{N-1}, \tau_N]. \end{cases}$$

This is exactly our goal in the next two steps.

Step 2 Since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, note that $\tau_k - \tau_{k-1} \leq \overline{\theta} + 2\eta \leq \overline{\theta} + \underline{\theta}$ for all $k \in \{1,\dots,N\}$. Hence, since moreover $r := \frac{\gamma}{\overline{\theta}+\underline{\theta}}$, observe that

$$\begin{cases} \frac{t-\tau_0}{\tau_1-\tau_0} \in [0, 1-r], & \forall t \in [\tau_0, \tau_1 - \gamma], \\ \frac{t-\tau_{k-1}}{\tau_k-\tau_{k-1}} \in [r, 1-r], & \forall t \in [\tau_{k-1}+\gamma, \tau_k - \gamma], \quad \forall k \in \{2,\dots,N-1\}, \\ \frac{t-\tau_{N-1}}{\tau_N-\tau_{N-1}} \in [r, 1], & \forall t \in [\tau_{N-1}+\gamma, \tau_N]. \end{cases}$$

As a consequence, from (4.14) and $(\mathfrak{P}_5)$, and since $\|y_k - y_k^*\|_{\mathrm{C}} \leq \|y - y^*\|_{\mathrm{C}} \leq \eta \leq \delta$, one can easily obtain that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1 - \gamma], \\ x(t) \in E_k^*, & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

Therefore, to conclude the proof, it only remains to prove that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k)$ and $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$, for all $k \in \{1, \dots, N-1\}$. This is the objective of the following last step.

Step 3 Let us start with two observations. First, since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, it holds that $|\tau_k - \tau_{k-1}| \geq \frac{\theta}{3}$ for all $k \in \{1, \dots, N\}$. Second, since $\gamma := \frac{\theta}{3}\min\{\varepsilon, \chi, \frac{\alpha}{\theta}\}$, one can get that

$$\tfrac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1-\varepsilon, 1], \quad \tfrac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1-\chi, 1], \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\tfrac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

for all $t \in [\tau_k - \gamma, \tau_k]$ and all $k \in \{1, \dots, N-1\}$. We deduce the following results:

(i) Since $v_k(s) = v_k^*(s)$ for almost every $s \in [1-\varepsilon, 1]$, one can easily obtain from (4.14) and (4.6) that $u(t) = u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}})$, with $\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*]$, for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

(ii) Since $\|y_k - y_k^*\|_{\mathrm{C}} \leq \|y - y^*\|_{\mathrm{C}} \leq \eta \leq \frac{\nu}{2}$, one can easily obtain from (4.14), from the equality $x^*(\tau_k^*) = y_k^*(1)$ and from $(\mathfrak{P}_3)$ that $x(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ for all $t \in [\tau_k - \gamma, \tau_k]$ and all $k \in \{1, \dots, N-1\}$.

(iii) We obtain from (4.15), from the previous two items and from $(\mathfrak{P}_2)$ that the derivative of $F_k^* \circ x$ satisfies

$$\left\langle \nabla F_k^*(x(t)), f_k^*\left(x(t), u^*\left(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right)\right)\right\rangle_{\mathbb{R}^n} > 0,$$

for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$. From admissibility of the triplet $(y, v, \mathbb{T})$ and (4.14), we know that $F_k^*(x(\tau_k)) = F_k^*(y_k(1)) = 0$ for all $k \in \{1, \dots, N-1\}$. As a consequence we obtain that $F_k^*(x(t)) < 0$ for all $t \in [\tau_k - \gamma, \tau_k)$ which implies from Definition 4.3.2, since $x(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and $\nu \leq \nu_k^*$, that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

Following the same strategy one can obtain that $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$ and all $k \in \{1, \dots, N-1\}$. The proof of Proposition 4.3.1 is complete.

4.5 Proof of Theorem 4.3.1

Let $(x^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ be a global solution to Problem (4.2), that is moreover a regular solution to (4.3), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, such that g is submersive at $(x^*(0), x^*(T))$. From Proposition 4.3.1, the corresponding triplet $(y^*, v^*, \mathbb{T}^*)$ constructed in Section 4.3.2 is a $L^1_\square$–local solution to Problem (4.9). Before applying Theorem 4.2.1, we need to verify that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$. From the definition of the function g^* (see Section 4.3.2), note that the matrix $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \in \mathbb{R}^{(nN+nN+(N+1))\times \ell^*}$ is given by

$$\left[\begin{array}{c|c|c|c} \nabla_1 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n\times n(N-1)}} & 0_{\mathbb{R}^{n\times(N-1)}} & 0_{\mathbb{R}^{n\times(N+1)}} \\ \hline 0_{\mathbb{R}^{n(N-1)\times \ell}} & \mathrm{Id}_{\mathbb{R}^{n(N-1)\times n(N-1)}} & 0_{\mathbb{R}^{n(N-1)\times(N-1)}} & 0_{\mathbb{R}^{n(N-1)\times(N+1)}} \\ \hline 0_{\mathbb{R}^{n(N-1)\times \ell}} & -\mathrm{Id}_{\mathbb{R}^{n(N-1)\times n(N-1)}} & \begin{array}{c}\nabla F_1^*(y_1^*(1)) \\ \ddots \\ \nabla F_{N-1}^*(y_{N-1}^*(1))\end{array} & 0_{\mathbb{R}^{n(N-1)\times(N+1)}} \\ \hline \nabla_2 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n\times n(N-1)}} & 0_{\mathbb{R}^{n\times(N-1)}} & 0_{\mathbb{R}^{n\times(N+1)}} \\ \hline 0_{\mathbb{R}^{(N+1)\times \ell}} & 0_{\mathbb{R}^{(N+1)\times n(N-1)}} & 0_{\mathbb{R}^{(N+1)\times(N-1)}} & \mathrm{Id}_{\mathbb{R}^{(N+1)\times(N+1)}} \end{array}\right]$$

From Definition 4.3.2, it holds that $\nabla F_k^*(y_k^*(1)) = \nabla F_k^*(x^*(\tau_k^*)) \neq 0_{\mathbb{R}^n}$ for all $k \in \{1, \ldots, N-1\}$. Since g is submersive at $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$, one can easily conclude that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$.

Application of Theorem 4.2.1. Let us introduce the Hamiltonian $\widetilde{H} : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN} \to \mathbb{R}$ associated with Problem (4.9) given by

$$\widetilde{H}(y, v, \mathbb{T}, q) := \langle q, f^*(y, v, \mathbb{T})\rangle_{\mathbb{R}^{nN}} = \sum_{k=1}^{N} \langle q_k, (\tau_k - \tau_{k-1}) f_k^*(y_k, v_k)\rangle_{\mathbb{R}^n},$$

for all $y = (y_1, \ldots, y_N) \in \mathbb{R}^{nN}$, $v = (v_1, \ldots, v_N) \in \mathbb{R}^{mN}$, $\mathbb{T} = \{\tau_0, \ldots, \tau_N\} \in \mathbb{R}^{N+1}$ and $q = (q_1, \ldots, q_N) \in \mathbb{R}^{nN}$. From Theorem 4.2.1, there exists a nontrivial pair $(q, q^0) \in \mathrm{AC}([0,1], \mathbb{R}^{nN}) \times \mathbb{R}_+$ satisfying:

(i) the Hamiltonian system

$$\dot{y}^*(s) = \nabla_q \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) \quad \text{and} \quad -\dot{q}(s) = \nabla_y \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)),$$

for almost every $s \in [0,1]$;

(ii) the endpoint transversality condition

$$\begin{pmatrix} q(0) \\ -q(1) \\ \displaystyle\int_0^1 \nabla_{\mathbb{T}} \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s))\, ds \end{pmatrix} = q^0 \nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*) + \nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)\tilde{\xi},$$

for some $\tilde{\xi} \in \mathrm{N}_{\mathrm{S}^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$;

(iii) the Hamiltonian maximization condition

$$v^*(s) \in \operatorname*{arg\,max}_{\widetilde{\omega} \in \mathrm{U}^N} \widetilde{H}(y^*(s), \widetilde{\omega}, \mathbb{T}^*, q(s)),$$

for almost every $s \in [0,1]$.

Introduction of the nontrivial pair (p, p^0). Since the pair (q, q^0) is not trivial, it is clear that the pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0,T], \mathbb{R}^n) \times \mathbb{R}_+$ defined by $p^0 := q^0$ and

$$p(t) := \begin{cases} q_1\left(\frac{t-\tau_0^*}{\tau_1^*-\tau_0^*}\right), & \forall t \in [\tau_0^*, \tau_1^*), \\ q_k\left(\frac{t-\tau_{k-1}^*}{\tau_k^*-\tau_{k-1}^*}\right), & \forall t \in (\tau_{k-1}^*, \tau_k^*), \quad \forall k \in \{2, \dots, N-1\}, \\ q_N\left(\frac{t-\tau_{N-1}^*}{\tau_N^*-\tau_{N-1}^*}\right), & \forall t \in (\tau_{N-1}^*, \tau_N^*], \end{cases}$$

is not trivial.

Hamiltonian system and Hamiltonian maximization condition of Theorem 4.3.1. From the above Items (i) and (iii) and from (4.7), one can easily obtain that (p, p^0) satisfies the Hamiltonian system and the Hamiltonian maximization condition of Theorem 4.3.1.

Endpoint transversality condition of Theorem 4.3.1. From the definitions of g^* and S^* (see Section 4.3.2) and since $\tilde{\xi} \in \mathrm{N}_{\mathrm{S}^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$, we can write $\tilde{\xi} := (\xi, \xi^2, \xi^3, \xi^4) \in \mathbb{R}^\ell \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^{N-1} \times \mathbb{R}^{N+1}$ with

$$\xi \in \mathrm{N}_{\mathrm{S}}[g(y_1^*(0), y_N^*(1))] \quad \text{and} \quad \xi^4 \in \mathrm{N}_\Delta[\mathbb{T}^*].$$

Since $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$, note that $\xi \in \mathrm{N}_{\mathrm{S}}[g(x^*(0), x^*(T))]$. Furthermore, from the first two components of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 4.5 and from the expression of $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$ (see Section 4.3.2 for the definition of ϕ^*), we obtain that

$$p(0) = q_1(0) = q^0 \nabla_1 \phi(y_1^*(0), y_N^*(1)) + \nabla_1 g(y_1^*(0), y_N^*(1))\xi = p^0 \nabla_1 \phi(x^*(0), x^*(T)) + \nabla_1 g(x^*(0), x^*(T))\xi,$$

and

$$\begin{aligned} -p(T) = -q_N(1) &= q^0 \nabla_2 \phi(y_1^*(0), y_N^*(1)) + \nabla_2 g(y_1^*(0), y_N^*(1))\xi \\ &= p^0 \nabla_2 \phi(x^*(0), x^*(T)) + \nabla_2 g(x^*(0), x^*(T))\xi. \end{aligned}$$

Therefore the endpoint transversality condition of Theorem 4.3.1 is proved.

Discontinuity condition of Theorem 4.3.1. From the first two components of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 4.5 and from the expression of $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$ (see Section 4.3.2 for the definition of ϕ^*), we obtain that

$$\forall k \in \{2, \dots, N\}, \quad q_k(0) = \xi_{k-1}^2 \qquad \text{and} \qquad \forall k \in \{1, \dots, N-1\}, \quad -q_k(1) = -\xi_k^2 + \xi_k^3 \nabla F_k^*(y_k^*(1)).$$

We deduce that

$$p^+(\tau_k^*) - p^-(\tau_k^*) = q_{k+1}(0) - q_k(1) = \xi_k^3 \nabla F_k^*(y_k^*(1)) = \xi_k^3 \nabla F_k^*(x^*(\tau_k^*)), \tag{4.16}$$

for all $k \in \{1, \dots, N-1\}$. Therefore the discontinuity condition of Theorem 4.3.1 is satisfied with $\sigma_k := \xi^3_k$ for all $k \in \{1, \dots, N-1\}$.

Hamiltonian constancy condition of Theorem 4.3.1. From the Hamiltonian system and the maximization condition and applying [63, Theorem 2.6.1] on each interval $[\tau^*_{k-1}, \tau^*_k]$, we obtain that, for all $k \in \{1, \dots, N\}$, there exists a constant $c_k \in \mathbb{R}$ such that

$$\langle p(t), f^*_k(x^*(t), u^*(t))\rangle_{\mathbb{R}^n} = c_k,$$

for almost every $t \in [\tau^*_{k-1}, \tau^*_k]$. Furthermore, from the definition of Δ (see Section 4.3.2) and since $0 = \tau^*_0 < \tau^*_1 < \ldots < \tau^*_{N-1} < \tau^*_N = T$, we deduce from $\xi^4 \in \mathrm{N}_\Delta[\mathbb{T}^*]$ that all components of ξ^4 are zero, except possibly the first and last components. Thus, from the third component of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Appendix 4.5 and from the expression of $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$ (see Section 4.3.2 for the definition of ϕ^*), we obtain that

$$\int_0^1 \langle q_{k+1}(s), f^*_{k+1}(y^*_{k+1}(s), v^*_{k+1}(s))\rangle_{\mathbb{R}^n}\, ds = \int_0^1 \langle q_k(s), f^*_k(y^*_k(s), v^*_k(s))\rangle_{\mathbb{R}^n}\, ds,$$

for all $k \in \{1, \dots, N-1\}$. From affine changes of time variable, we obtain that

$$\frac{1}{\tau^*_{k+1} - \tau^*_k} \int_{\tau^*_k}^{\tau^*_{k+1}} \langle p(t), f^*_{k+1}(x^*(t), u^*(t))\rangle_{\mathbb{R}^n}\, dt = \frac{1}{\tau^*_k - \tau^*_{k-1}} \int_{\tau^*_{k-1}}^{\tau^*_k} \langle p(t), f^*_k(x^*(t), u^*(t))\rangle_{\mathbb{R}^n}\, dt,$$

for all $k \in \{1, \dots, N-1\}$. From constancy of the above two integrands, we deduce that $c_{k+1} = c_k$ for all $k \in \{1, \dots, N-1\}$. Therefore the Hamiltonian constancy condition is satisfied and the proof of Theorem 4.3.1 is complete.

5

Loss control regions in optimal control problems

Contents

This chapter is based on the paper "Loss control regions in optimal control problems" by T. Bayen, A. Bouali, L. Bourdin, and O. Cots (see [17]). Here, we address general optimal control problems with loss control regions with general terminal state constraints. The objective of this chapter is twofold. First, we

reformulate the above problem into a spatially hybrid optimal control problem that involves a regionally switching parameter. Second, we propose a two-step numerical scheme to solve optimal control problems with loss control regions. The scheme is based on direct and indirect numerical methods. This numerical approach is applied to several illustrative examples: Zermelo-type problem and a version of the minimum time problem for the harmonic oscillator both involving loss control regions.

5.1 Introduction

General context of the present chapter. Our main objective in this chapter is to establish a Pontryagin maximum principle (in short, PMP) for general optimal control problem with loss control regions. Let us recall that in the literature, optimal control theory and, in particular, the PMP are usually concerned with a *permanent control*, in the sense that the control value is authorized to be modified at any instant of time. However, in practice, when acting on concrete processes, some constraints may prevent changing the control value in a full free way: we speak then of a *nonpermanent control* and the situations are diverse. For instance, in automatic, when the control is digital, the control value can be modified only in a discrete way in time, resulting into a piecewise constant control (also called *sampled-data control*). A version of the PMP has been obtained for sampled-data controls, in which the Hamiltonian maximization condition is replaced by the so-called *averaged Hamiltonian gradient condition* (see [1], [24], [34], [36], [41]). In aerospace, the control is not permanent in the presence of an eclipse constraint (see [69], [73]). Such a constraint applies to satellites that use solar power and that cannot be active when they are in a shadow region (i.e. an area that is not directly exposed to sunlight). In such a shadow region, the control input can only be set to zero due to the absence of power. Additionally, in viability problems or in epidemiology, we often encounter constraint sets or environment sets attached to the controlled dynamics (see, e.g., [27], [92] in the context of an optimal control analysis of covid-19). Such sets can be related to thresholds not to exceed and they can in general be described by a set of inequalities involving state variables. Depending on the application model, it is then desirable to maintain the system as much as possible outside this set. This is typically the case in time crisis problems [20] when the state belongs to the environment set. In this context, to reduce operating costs, it can be convenient to make use of constant controls. In the same way, constant controls may also be convenient whenever the system fulfills the constraints.

Objective and methodology. The objective of this chapter is to address optimal control problems with *loss control regions*. In that context the state space is partitioned into disjoint sets, referred to as *regions*, which are classified into two types: *control regions* and *loss control regions*. When the state belongs to a control region, the control is permanent (i.e. the control value is authorized to be modified at any time). On the contrary, when the state belongs to a loss control region, the control must remain constant, equal to the last assigned value before the state enters into the loss control region, and this value is kept until the state exits this region. As it was previously mentioned, the consideration of such problems is motivated by various applications. To address optimal control problems with loss control regions, we pursue the approach initiated in our previous works [13], [14], by considering the control function as a permanent control in control regions and as a constant parameter in loss control regions. With this point of view, our framework falls into the domain of *hybrid* optimal control theory which extends the classical theory to discontinuous dynamics. In that field, the so-called *hybrid maximum principle* (in short, HMP) extends the PMP to various hybrid settings (i.e. to discontinuities of various

natures, such as temporal or spatial, see below). In the HMP, the adjoint vector is usually piecewise absolutely continuous, admitting a discontinuity jump at each time the dynamics changes discontinuously. We refer to [46], [59], [68], [99], [108], [109], [111] and references therein.
To be specific, our methodology leads to a hybrid setting where the dynamics changes discontinuously (only) according to the state position in a given partition of the state space: we speak of a *spatially heterogeneous dynamics* and this setting corresponds to the spirit of previous works such as [10], [73] (in which transversal crossing assumptions are made to handle the boundary crossings of the optimal trajectory). In contrast, the change of dynamics may depend on time only, i.e. we fix in advance a certain number of instants (fixed or free) at which the dynamics changes. This discontinuity may be controlled or not (see, e.g. [46], [59]). In that case, we would rather speak of a temporal discontinuity. Note that this case is not under consideration throughout this paper which is devoted only to spatially heterogeneous dynamics. Finally we emphasize that our strategy, not only leads to a hybrid optimal control problem (with spatially heterogeneous dynamics), but moreover involving a *regionally switching parameter* (i.e. a parameter that can change its value when the trajectory moves from one region to another). To the best of our knowledge, this last concept has never been considered in the literature until our previous work [13].

Main results.

(a) **Theoretical contributions.** The main theoretical contribution of this chapter is the derivation of a PMP for optimal control problems with loss control regions (see Theorem 5.3.1). The necessary optimality condition is expressed as the usual Hamiltonian maximization condition whenever the state belongs to a control region, and as the averaged Hamiltonian gradient condition whenever the state belongs to a loss control region. Theorem 5.3.1 is actually a direct consequence of a more general result that we establish in Theorem 5.2.1: a HMP for hybrid optimal control problems (with spatially heterogeneous dynamics) involving a regionally switching parameter.
Theorem 5.2.1 is an extension of a HMP that can be found in our previous work [13] in the simpler context of a fixed initial condition and no final state constraint. Furthermore our methodology in [13] consists in a thorough sensitivity analysis of the hybrid control system, involving an inductive reasoning to handle the consecutive crossing times thanks to implicit function theorems. In the present work, Theorem 5.2.1, not only handles mixed terminal state constraints, but moreover is proved with a simpler method based on an *augmentation technique*. This approach consists in reducing the hybrid optimal control problem into a classical (non-hybrid) optimal control problem to which the classical PMP can be applied. It was initiated in [59] to deal with *hybrid* optimal control problems in which the word *hybrid* refers to the situation where the dynamics changes discontinuously but at fixed or free instants of time (above all, independently of the state position). As highlighted in our previous paper [16] with counterexamples, the augmentation technique must be carefully adapted to the present framework of spatially heterogeneous dynamics. In particular it requires the introduction of a new notion of local solution (called $\mathrm{L}^1_{\square}$*–local solution*) to classical (non-hybrid) optimal control problems and the derivation of a corresponding PMP (see [16, Theorem 2.1]).
We point out that, similarly to Theorem 5.2.1 that is an extension of the HMP that can be found in our previous paper [13], Theorem 5.3.1 is an extension of the PMP that can be found in our previous work [14] in the simpler context of a fixed initial condition and no final state constraint.
Finally, since our framework is related to the hybrid setting (with spatially heterogeneous dynamics), Theorem 5.2.1 and Theorem 5.3.1 are both results obtained under appropriate transversal crossing

assumptions and both ensure the existence of a piecewise absolutely continuous adjoint vector admitting a discontinuity jump at each crossing time.

(b) **Numerical contributions.** This chapter proposes a two-steps numerical approach to solve optimal control problems with loss control regions. First, we use a direct numerical method applied to a *regularized* problem: a regularization is required to overcome the discontinuities appearing when rewriting the optimal control problem with loss control regions as a hybrid optimal control problem with regionally switching parameter. This first step is useful to determine the *structure* of the optimal trajectory (i.e. the ordered sequence of regions that the optimal trajectory visits) and, second, to initialize an indirect numerical method applied to the original problem and based on the PMP stated in Theorem 5.3.1. The originality here is to incorporate the averaged Hamiltonian gradient condition, as well as the discontinuity jumps of the adjoint vector, to define an appropriate shooting function (in addition to the classical terms defining the shooting function for non-hybrid control problems, see [45], [53]).

For the sake of brevity, in the rest of this paper, the word *hybrid* will refer (only) to the situation of spatially heterogeneous dynamics.

Organization of the chapter. This chapter is organized as follows. In Section 5.2, a general hybrid optimal control problem with regionally switching parameter is introduced (see Problem (5.1)). Then Proposition 5.2.1 asserts that the augmentation of a global solution to Problem (5.1) leads to a $L^1_\square$–local solution to a classical (nonhybrid) augmented optimal control problem. Hence, applying the PMP for $L^1_\square$–local solutions (see [16, Theorem 2.1]) and reversing the augmentation procedure, a HMP for hybrid optimal control problems with regionally switching parameter is obtained (see Theorem 5.2.1). In Section 5.3, a general optimal control problem with loss control regions is introduced (see Problem (5.9)). By rewriting this problem as a hybrid optimal control problem with regionally switching parameter and applying the previous Theorem 5.2.1, a PMP for optimal control problems with loss control regions is obtained (see Theorem 5.3.1). In Section 5.4, a two-steps numerical scheme is proposed to solve optimal control problems with loss control regions. Afterwards it is applied to numerically solve some illustrative examples, precisely a Zermelo-type problem [6] and a version of the minimal time problem for the harmonic oscillator [107] both including loss control regions. Finally the technical proofs of Proposition 5.2.1 and Theorem 5.2.1 are provided in Sections 5.5 and 5.6 respectively.

5.2 Derivation of a HMP with regionally switching parameter

This section addresses hybrid optimal control problems with regionally switching parameter (which will allow us in the next Section 5.3 to address optimal control problems with loss control regions as a particular case). To this aim, Section 5.2.1, a general hybrid optimal control problem with regionally switching parameter is introduced, together with terminology and assumptions. In Section 5.2.2, we discuss the notion of *regular* solution to the corresponding hybrid control system. In Section 5.2.3, thanks to an augmentation technique, we establish in Proposition 5.2.1 the correspondence between a solution to the hybrid optimal control problem with regionally switching parameter, that is regular, and a $L^1_\square$–*local solution* (notion that was previously introduced in [16]) to a classical optimal control problem with (constant) parameter. In Section 5.2.4, applying a version of the PMP that is adapted to $L^1_\square$–local solutions (extracted from [16, Theorem 2.1]), we derive the main result of this section (Theorem 5.2.1)

which is a HMP with regionally switching parameter.

5.2.1 A hybrid optimal control problem with regionally switching parameter: terminology and assumptions

Let n, m, d, $\ell \in \mathbb{N}^*$ be four fixed positive integers and $T > 0$ be a fixed positive real number. In this section we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where $\mathcal{J}$ is a (possibly infinite) family of indexes and the nonempty open subsets $X_j \subset \mathbb{R}^n$, called *regions*, are disjoint. Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the *hybrid* optimal control problem with *regionally switching parameter* given by

$$\begin{array}{ll} \text{minimize} & \phi(x(0), x(T)), \\ \text{subject to} & (x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m), \\ & \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0,T], \\ & g(x(0), x(T)) \in \mathrm{S}, \\ & (\lambda(t), u(t)) \in \Lambda \times \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\ & \lambda \text{ is a regionally switching parameter associated with } x, \end{array} \tag{5.1}$$

where both the Mayer cost function $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$ are of class C^1, where both subsets $\mathrm{S} \subset \mathbb{R}^\ell$ and $\Lambda \subset \mathbb{R}^d$ are nonempty closed convex subsets, where $\mathrm{U} \subset \mathbb{R}^m$ is a nonempty subset and where the *spatially heterogeneous* dynamics $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ is defined *regionally* by

$$\forall (x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m, \quad h(x, \lambda, u) := h_j(x, \lambda, u) \quad \text{if } x \in X_j,$$

where the maps $h_j : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ are of class C^1. Note that $h(x, \lambda, u)$ is not defined when $x \notin \cup_{j \in \mathcal{J}} X_j$ but this fact will have no impact on the rest of this paper thanks to transverse crossing assumptions (see Definition 5.2.2).

In Problem (5.1), as usual in the literature, $x \in \mathrm{AC}([0,T], \mathbb{R}^n)$ is called the *state* (or the *trajectory*) and $u \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is called the *control*. Additionally we consider a regionally switching parameter $\lambda \in \mathrm{PC}([0,T], \mathbb{R}^d)$, meaning that the parameter λ stays constant as long as the trajectory x stays within a region, but it is allowed to change its value (i.e. to switch) when the trajectory x crosses a boundary and moves from one region to another (see Definition 5.2.1). A triplet $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is said to be *admissible* for Problem (5.1) if it satisfies all the constraints of Problem (5.1). Such an admissible triplet is said to be a *global solution* to Problem (5.1) if it minimizes the Mayer cost $\phi(x(0), x(T))$ among all admissible triplets.

Remark 5.2.1. In the whole chapter (not only for Problem (5.1)), we will consider problems with (only) Mayer cost, fixed final time and autonomous dynamics (i.e. independent of t). It is well known in the literature (see, *e.g.*, [28], [44], [48]) that standard techniques, such as augmentation techniques or

changes of variables, allow to deal with more general problems that include Bolza cost, free final time and time-dependent dynamics. Also we assume for simplicity that the maps ϕ, f and g are of class C^1 and some topological conditions on the sets S and Λ. However the results presented in this work can be extended to weaker assumptions (as in [49], [116]). Overall our aim in this chapter is not to consider the most general framework possible. We keep our setting as simple as possible to stay focused on the main novelties of this work.

5.2.2 Regular solution to the hybrid control system with regionally switching parameter

Consider the hybrid control system with regionally switching parameter associated with Problem (5.1) given by

$$\begin{cases} \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{for a.e. } t \in [0,T], \\ \lambda \text{ is a regionally switching parameter associated with } x. \end{cases} \tag{5.2}$$

Due to the discontinuities of the spatially heterogeneous dynamics h and to the presence of a regionally switching parameter, we need to precise the definition of a solution to (5.2).

Definition 5.2.1 (Solution to (5.2))**.** *A triplet $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is said to be a solution to (5.2) if there exists a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0,T]$ such that:*

(i) It holds that

$$\forall k \in \{1,\dots,N\}, \quad \exists j(k) \in \mathcal{J}, \quad \forall t \in (\tau_{k-1}, \tau_k), \quad x(t) \in X_{j(k)},$$

where $j(k) \neq j(k-1)$ for all $k \in \{2,\dots,N\}$. The sequence $\{j(1),\dots,j(N)\}$ is called the switching sequence.

(ii) It holds that $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.

(iii) λ is a regionally switching parameter associated with x, that is, $\lambda \in \mathrm{PC}_{\mathbb{T}}([0,T], \mathbb{R}^d)$.

(iv) It holds that $\dot{x}(t) = h_{j(k)}(x(t), \lambda_k, u(t))$ for almost every $t \in (\tau_{k-1}, \tau_k)$ and all $k \in \{1,\dots,N\}$.

In that case, to ease notation, we set $f_k := h_{j(k)}$ and $E_k := X_{j(k)}$ for all $k \in \{1,\dots,N\}$. With this system of notations, we have

$$\begin{cases} x(t) \in E_1, \quad \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k, \quad \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2,\dots,N-1\}, \\ x(t) \in E_N, \quad \forall t \in (\tau_{N-1}, \tau_N], \\ \dot{x}(t) = f_k(x(t), \lambda_k, u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{1,\dots,N\}. \end{cases}$$

The times τ_k for $k \in \{1,\dots,N-1\}$, called crossing times, correspond to the instants at which the trajectory x goes from the region E_k to the region E_{k+1}, and thus $x(\tau_k) \in \partial E_k \cap \partial E_{k+1}$.

The main result of this section (Theorem 5.2.1 stated in Section 5.2.4) is based on some regularity assumptions made on the optimal triplet of Problem (5.1) at each of its crossing times. These hypotheses are precised in the next definition.

Definition 5.2.2 (Regular solution to (5.2))**.** *Following the notations introduced in Definition 5.2.1, a solution $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ to (5.2), associated with a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$, is said to be regular if the following conditions are both satisfied:*

(i) At each crossing time τ_k, there exists a C^1 function $F_k : \mathbb{R}^n \to \mathbb{R}$ such that

$$\exists \nu_k > 0, \quad \forall z \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in E_k & \Leftrightarrow \quad F_k(z) < 0, \\ z \in \partial E_k \cap \partial E_{k+1} & \Leftrightarrow \quad F_k(z) = 0, \\ z \in E_{k+1} & \Leftrightarrow \quad F_k(z) > 0. \end{cases}$$

In particular it holds that $F_k(x(\tau_k)) = 0$.

(ii) At each crossing time τ_k, there exists $\alpha_k > 0$ and $\beta_k > 0$ such that the transverse conditions

$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), \lambda_k, u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), \lambda_{k+1}, u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k, \tau_k + \alpha_k], \end{cases} \tag{5.3}$$

are both satisfied. We refer to Figure 5.1 for for a geometrical illustration.

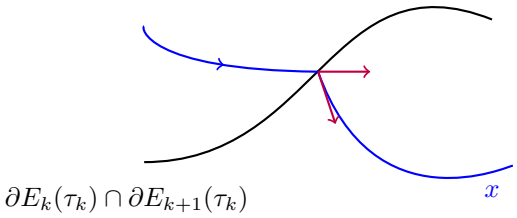


Figure 5.1: Geometrical illustration of a transversal boundary crossing.

Remark 5.2.2. In this remark we comment some aspects related to Definitions 5.2.1 and 5.2.2.

(i) In Definition 5.2.1, we exclude a few possibilities, such as having an infinite number of crossing times (no Zeno phenomenon [47]), bouncing against a boundary or moving along a boundary (as considered in [10]). Also we (only) consider trajectories that start and finish inside the regions (and not on a boundary). Nevertheless we discuss in Remark 5.2.8 possible issues to address the case where trajectories start or finish on a boundary of a stratum.

(ii) The transverse condition introduced in Definition 5.2.2 has a geometrical interpretation. It means that the trajectory does not cross the boundary $\partial E_k \cap E_{k+1}$ tangentially (see Figure 5.1). Note that, in the present work, we consider a transverse condition that is (slightly) weaker than the one examined in previous works such as [13], [73] and given by

$$\begin{cases} u \text{ admits left and right limits at } \tau_k \text{ denoted by } u^-(\tau_k) \text{ and } u^+(\tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), \lambda_k, u^-(\tau_k)) \rangle_{\mathbb{R}^n} > 0, \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), \lambda_{k+1}, u^+(\tau_k)) \rangle_{\mathbb{R}^n} > 0. \end{cases} \tag{5.4}$$

5.2.3 Reduction to a classical optimal control problem with (constant) parameter

To establish a correspondence from the hybrid optimal control problem with regionally switching parameter (5.1) to a classical optimal control problem with (constant) parameter, we proceed to simple affine changes of time variable. Precisely let $(x^*, \lambda^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ be a solution to (5.2), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, and let us denote by E_k^*, f_k^* and λ_k^* the

corresponding regions, functions and constant values of λ^* whenever x^* belongs to E_k^* (see Definition 5.2.1). We introduce $(y^*, v^*) \in \mathrm{AC}([0,1], \mathbb{R}^{(n+d)N}) \times \mathrm{L}^\infty([0,1], \mathbb{R}^{mN})$ defined by

$$y_k^*(s) := x^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s),\ y_{N+k}^*(s) := \lambda^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s) \text{ and } v_k^*(s) := u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s), \tag{5.5}$$

for all $s \in [0,1]$ and all $k \in \{1, \dots, N\}$. Note that the above affine changes of time variable allow mapping each time interval $[\tau_{k-1}^*, \tau_k^*]$ to the common time interval $[0,1]$ and, therefore, augment the state dimension to $(n+d)N$ and the control dimension to mN, respectively. To reverse the above changes of the time variable, one simply has

$$x^*(t) = y_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right), \quad \lambda^*(t) = y_{N+k}^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right) \quad \text{and} \quad u^*(t) = v_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right), \tag{5.6}$$

for all $t \in [\tau_{k-1}^*, \tau_k^*]$ and all $k \in \{1, \dots, N\}$. In particular note that $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$. From a more general point of view, it holds that $x^*(\tau_k^*) = y_{k+1}^*(0)$ for all $k \in \{0, \dots, N-1\}$, and $x^*(\tau_k^*) = y_k^*(1)$ and $\lambda_k^* = y_{N+k}^*(s)$ for all $s \in [0,1]$ and all $k \in \{1, \dots, N\}$.
With the above context note that the triplet $(y^*, v^*, \mathbb{T}^*)$ satisfies

$$\dot{y}^*(s) = f^*(y^*(s), v^*(s), \mathbb{T}^*), \quad \text{a.e. } s \in [0,1],$$

where $f^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \to \mathbb{R}^{(n+d)N}$ is the C^1 function defined by

$$f^*(y, v, \mathbb{T}) := \Big((\tau_1 - \tau_0) f_1^*(y_1, y_{N+1}, v_1), \dots, (\tau_N - \tau_{N-1}) f_N^*(y_N, y_{2N}, v_N), 0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}\Big),$$

for all $y = (y_1, \dots, y_N, y_{N+1}, \dots, y_{2N}) \in \mathbb{R}^{(n+d)N}$, $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$ and $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$. Furthermore, it holds that

$$\begin{cases} y_1^*(s) \in E_1^*, & \forall s \in [0,1), \\ y_k^*(s) \in E_k^*, & \forall s \in (0,1), \quad \forall k \in \{2, \dots, N-1\}, \\ y_N^*(s) \in E_N^*, & \forall s \in (0,1], \end{cases} \tag{5.7}$$

and $y_{k+1}^*(0) = y_k^*(1) \in \partial E_k^* \cap \partial E_{k+1}^*$ for all $k \in \{1, \dots, N-1\}$. Also note that $\mathbb{T}^* \in \Delta$ where $\Delta \subset \mathbb{R}^{N+1}$ is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0,\dots,N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T\}.$$

Now assume that the triplet (x^*, λ^*, u^*) is moreover a regular solution to (5.2) and denote by F_k^* and $\nu_k^* > 0$ the corresponding functions and positive radii (see Definition 5.2.2). In that context note that $F_k^*(x(\tau_k^*)) = F_k^*(y_k^*(1)) = 0$ for all $k \in \{1, \dots, N-1\}$.
Finally it is clear that, if the triplet (x^*, λ^*, u^*) is furthermore admissible for Problem (5.1), then the

triplet $(y^*, v^*, \mathbb{T}^*)$ is admissible for the classical optimal control problem with (constant) parameter given by

$$\begin{array}{ll} \text{minimize} & \phi^*(y(0), y(1)), \\ \text{subject to} & (y, v, \mathbb{T}) \in \mathrm{AC}([0,1], \mathbb{R}^{(n+d)N}) \times \mathrm{L}^\infty([0,1], \mathbb{R}^{mN}) \times \mathbb{R}^{N+1}, \\ & \dot{y}(s) = f^*(y(s), v(s), \mathbb{T}), \quad \text{a.e. } s \in [0,1], \\ & g^*(y(0), y(1), \mathbb{T}) \in \mathrm{S}^*, \\ & v(s) \in \mathrm{U}^N, \quad \text{a.e. } s \in [0,1], \end{array} \tag{5.8}$$

where $\phi^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{(n+d)N} \to \mathbb{R}$ and $g^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{(n+d)N} \times \mathbb{R}^{N+1} \to \mathbb{R}^{\ell^*}$ are the C^1 functions defined by $\phi^*(y^0, y^1) := \phi(y_1^0, y_N^1)$ and

$$g^*(y^0, y^1, \mathbb{T}) := (g(y_1^0, y_N^1), y_2^0 - y_1^1, \ldots, y_N^0 - y_{N-1}^1, F_1^*(y_1^1), \ldots, F_{N-1}^*(y_{N-1}^1), y_{N+1}^1, \ldots, y_{2N}^1, \mathbb{T}),$$

for all $y^0 = (y_1^0, \ldots, y_N^0, y_{N+1}^0, \ldots, y_{2N}^0)$, $y^1 = (y_1^1, \ldots, y_N^1, y_{N+1}^1, \ldots, y_{2N}^1) \in \mathbb{R}^{(n+d)N}$ and $\mathbb{T} = \{\tau_0, \ldots, \tau_N\} \in \mathbb{R}^{N+1}$, where $\ell^* := \ell + n(N-1) + (N-1) + dN + (N+1)$, and where $\mathrm{S}^* \subset \mathbb{R}^{\ell^*}$ stands for the nonempty closed convex set defined by

$$\mathrm{S}^* := \mathrm{S} \times \{0_{\mathbb{R}^n}\}^{N-1} \times \{0\}^{N-1} \times \Lambda^N \times \Delta.$$

Before stating the main result of this section, we first need to recall some notions of local solution, extracted from our previous paper [16], which will play central roles in the present work.

Definition 5.2.3 (L^1_A–local solution)**.** *The triplet $(y^*, v^*, \mathbb{T}^*)$ is said to be a L^1_A–local solution to Problem (5.8), for a measurable subset $A \subset [0,1]$, if, for all $R \geq \|v^*\|_{\mathrm{L}^\infty}$, there exists $\eta > 0$ such that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ for all admissible triplets $(y, v, \mathbb{T})$ satisfying*

$$\begin{cases} \|y - y^*\|_{\mathrm{C}} + \|v - v^*\|_{\mathrm{L}^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\ \|v\|_{\mathrm{L}^\infty} \leq R, \\ v(t) = v^*(t) \ \text{a.e. } t \in [0,T] \backslash A. \end{cases}$$

With $A = [0,T]$, we recover the standard notion of L^1–local solution that can be found in [29], [93].

Definition 5.2.4 ($\mathrm{L}^1_\square$–local solution)**.** *The triplet $(y^*, v^*, \mathbb{T}^*)$ is said to be a $\mathrm{L}^1_\square$–local solution to Problem (5.8), if there exists an increasing family $(A_\varepsilon)_{\varepsilon>0}$ of measurable subsets of $[0,1]$, satisfying $\lim_{\varepsilon \to 0} \mu(A_\varepsilon) = 1$, such that $(y^*, v^*, \mathbb{T}^*)$ is a $\mathrm{L}^1_{A_\varepsilon}$–local solution to Problem (5.8) for all $\varepsilon > 0$.*

Proposition 5.2.1. *If $(x^*, \lambda^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is a global solution to Problem (5.1), that is moreover a regular solution to (5.2), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\ldots,N}$, then the triplet $(y^*, v^*, \mathbb{T}^*)$ constructed above is a $\mathrm{L}^1_\square$–local solution to Problem (5.8).*

Proof. The detailed proof of Proposition 5.2.1 can be found in Section 5.5. Precisely we prove that the triplet $(y^*, v^*, \mathbb{T}^*)$ is a $\mathrm{L}^1_{[\varepsilon, 1-\varepsilon]}$–local solution to Problem (4.9) for any $0 < \varepsilon < \frac{1}{2}$. □

Remark 5.2.3. Consider the setting of Proposition 5.2.1.

(i) First of all, it is worth mentioning that the triplet $(y^*, v^*, \mathbb{T}^*)$ is not a L^1–local solution to Problem (5.8) in general. A counterexample is provided in our previous paper [16]. As a consequence one

cannot use the classical PMP on the triplet $(y^*, v^*, \mathbb{T}^*)$. Nevertheless, thanks to Proposition 5.2.1, we can apply a version of the PMP that is adapted to $\mathrm{L}^1_\square$–local solutions (see [16, Theorem 2.1]).

(ii) Now let us briefly comment on why the standard notion of L^1–local solution fails, and why the notion of $\mathrm{L}^1_\square$–local solution is well-suited, when using an augmentation technique in our hybrid setting. Given an admissible triplet $(y, v, \mathbb{T})$ for Problem (5.8), when we invert the augmentation procedure, we obtain a pair (x, u) that satisfies all the constraints of Problem (5.1), except one. Specifically, even if x follows the same dynamics sequence $(f_k^*)_{k=1,\dots,N}$ as the nominal state x^*, it may not visit the same regions $(E_k^*)_{k=1,\dots,N}$. As a result (x, u) may be not admissible for Problem (5.1). To overcome this issue, we rely on the notion of $\mathrm{L}^1_\square$–local solution. This approach allows L^1-perturbations of the control u^*, but only outside the neighborhoods of the crossing times τ_k^*. This strategy ensures that the perturbed pair (x, u) satisfies a transverse condition at each perturbed crossing time τ_k, making it admissible for Problem (5.1) since x visits the same regions as the nominal trajectory x^*. We refer to [16] for details.

5.2.4 HMP with regionally switching parameter and comments

The Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (5.1) is defined by

$$\mathcal{H}(x, \lambda, u, p) := \langle p, h(x, \lambda, u)\rangle_{\mathbb{R}^n},$$

for all $(x, \lambda, u, p) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to state the main result of this section.

Theorem 5.2.1 (HMP with regionally switching parameter)**.** *Suppose that $(x^*, \lambda^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ is a global solution to Problem (5.1), that it is moreover a regular solution to (5.2), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, and that it is such that g is submersive at $(x^*(0), x^*(T))$. Then there exists a nontrivial pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0,T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:*

(i) the Hamiltonian system

$$\dot{x^*}(t) = \nabla_p \mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)) \quad \text{and} \quad -\dot{p}(t) = \nabla_x \mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)),$$

for almost every $t \in [0,T]$;

(ii) the transversality condition

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T))\xi,$$

for some $\xi \in \mathrm{N_S}[g(x^*(0), x^*(T))]$;

(iii) the discontinuity condition

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*)),$$

for some $\sigma_k \in \mathbb{R}$, for all $k \in \{1, \dots, N-1\}$;

(iv) the Hamiltonian maximization condition

$$u^*(t) \in \arg\max_{\omega \in \mathrm{U}} \mathcal{H}(x^*(t), \lambda^*(t), \omega, p(t)),$$

for almost every $t \in [0,T]$;

(v) the averaged Hamiltonian gradient condition

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_\lambda \mathcal{H}(x^*(t), \lambda_k^*, u^*(t), p(t))\, dt \in \mathrm{N}_\Lambda[\lambda_k^*],$$

for all $k \in \{1, \dots, N\}$*;*

(vi) the Hamiltonian constancy condition

$$\mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)) = c,$$

for almost every $t \in [0, T]$*, for some* $c \in \mathbb{R}$*.*

Proof. The detailed proof of Theorem 5.2.1 can be found in Section 5.6. It is based on Proposition 5.2.1 and on the application of a PMP for $\mathrm{L}^1_\square$–local solutions [16, Theorem 2.1]. □

Remark 5.2.4. As usual in the literature on optimal control theory, the nontrivial pair (p, p^0) provided in Theorem 5.2.1 is defined up to a positive multiplicative constant. It is said to be *normal* whenever $p^0 > 0$, and *abnormal* whenever $p^0 = 0$. In the normal case $p^0 > 0$, it is usual to renormalize it so that $p^0 = 1$.

Remark 5.2.5. As explained in [24], [35], the submersiveness hypothesis made in Theorem 5.2.1 can be removed. In that case, all items of Theorem 5.2.1 remain valid except Item (ii).

Remark 5.2.6. In Theorem 5.2.1, the costate p admits a discontinuity jump at each crossing time τ_k^*, satisfying Item (iii). Under the (slightly) stronger transverse condition (5.4) and using the Hamiltonian constancy condition, one can easily prove that

$$\begin{aligned}\sigma_k &= -\frac{\Big\langle p^-(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}}{\Big\langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*))\Big\rangle_{\mathbb{R}^n}} \\ &= -\frac{\Big\langle p^+(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}}{\Big\langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*))\Big\rangle_{\mathbb{R}^n}},\end{aligned}$$

for all $k \in \{1, \dots, N-1\}$. Therefore, in that context, the discontinuity conditions of Theorem 5.2.1 can be expressed as forward (or backward) discontinuity jumps. Let us recall that discontinuity jumps of the costate are common in general hybrid optimal control theory (see, *e.g.*, [25], [99]). In particular, the previous papers [10], [73] address these discontinuity jumps in the context of spatially heterogeneous dynamics, but in a setting without regionally switching parameter and, above all, their proofs are unsatisfactory (see [16] for details).

Remark 5.2.7. Consider the setting of Proposition 5.2.1. From Remark 5.2.3, we know that $(y^*, v^*, \mathbb{T}^*)$ is not a L^1–local solution to Problem (5.8) in general. Nevertheless it is possible to avoid the use of the notion of $\mathrm{L}^1_\square$–local solution. However, to our best knowledge, this would not be possible without obtaining a weaker result and/or without restricting the framework. We refer to [16, Remark 3.4 Item (i)] for details. Roughly speaking the choice of the transverse conditions (more or less strong) influences the *local quality* (L^1, L^∞ or $\mathrm{L}^1_\square$) of the solution $(y^*, v^*, \mathbb{T}^*)$ to Problem (5.8) and thus the version of the PMP that can be applied to it, and finally the version of the HMP obtained on the original triplet (x^*, λ^*, u^*).

Remark 5.2.8. In this remark we would like to emphasize a few relaxations and extensions that can

be considered to the present work. They have already been discussed in [16, Remark 3.4] (in a setting without regionally switching parameter).

(i) To handle trajectories possibly starting or finishing on boundaries, adjustments need to be made in Definition 5.2.2. For instance, in the context of Theorem 5.2.1 with $x^*(0) \in E_1^*$ and $x^*(T) \in \partial E_N^*$, additional conditions involving a local C^1 description of ∂E_N^* near $x^*(T)$ and an adapted transverse condition should be included in Definition 5.2.2. Moreover Problem (5.8) requires careful adjustment to maintain the validity of Proposition 5.2.1. Finally it is necessary to adapt the submersiveness hypothesis to apply [16, Theorem 2.1] before inverting the augmentation procedure to obtain the corresponding HMP.

(ii) One may consider a setting where the subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^{m_j} \to \mathbb{R}^n$ have possibly different control dimensions $m_j \in \mathbb{N}^*$ and with possibly different control constraint sets $\mathrm{U}_j \subset \mathbb{R}^{m_j}$. This generalized context can be useful to set specific control values in particular regions (for example, by taking $\mathrm{U}_j = \{0_{\mathbb{R}^{m_j}}\}$ for some $j \in \mathcal{J}$). We are confident that our methodology can be easily adapted to this framework without significant difficulties.

5.3 Derivation of a PMP with loss control regions

This section is organized as follows. In Section 5.3.1, we introduce a general optimal control problem with loss control regions, along with terminology and assumptions. In Section 5.3.2, we discuss the notion of *regular* solution to the corresponding control system. In Section 5.3.3, we state and prove the main theoretical result of this paper (Theorem 5.3.1) which is a PMP with loss control regions. Its proof is based on Theorem 5.2.1 that was established in the previous section.

5.3.1 An optimal control problem with loss control regions: terminology and assumptions

Let n, m, $\ell \in \mathbb{N}^*$ be three fixed positive integers and $T > 0$ be a fixed positive real number. In this section we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where $\mathcal{J}$ is a (possibly infinite) family of indexes and the nonempty open subsets $X_j \subset \mathbb{R}^n$, called *regions*, are disjoint. We introduce an indexation $q_j \in \{0, 1\}$ allowing us to separate *control regions* and *loss control regions* (see Introduction for details) as follows

$$q_j := \begin{cases} 1 \text{ if } X_j \text{ is a control region,} \\ 0 \text{ if } X_j \text{ is a loss control region.} \end{cases}$$

Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the optimal control problem with *loss control regions* given by

$$\begin{array}{ll} \text{minimize} & \phi(x(0),x(T)),\\ \text{subject to} & (x,u)\in \mathrm{AC}([0,T],\mathbb{R}^n)\times \mathrm{L}^\infty([0,T],\mathbb{R}^m),\\ & \dot{x}(t)=f(x(t),u(t)),\quad \text{a.e. } t\in[0,T],\\ & g(x(0),x(T))\in \mathrm{S},\\ & u(t)\in \mathrm{U},\quad \text{a.e. } t\in[0,T],\\ & u \text{ is constant when } x \text{ is in a loss control region,} \end{array} \tag{5.9}$$

where the Mayer cost function $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$ are of class C^1, and where both subsets $\mathrm{S} \subset \mathbb{R}^\ell$ and $\mathrm{U} \subset \mathbb{R}^m$ are nonempty closed convex subsets.
A pair $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is said to be *admissible* for Problem (5.9) if it satisfies all the constraints of Problem (5.9). Such an admissible pair is said to be a *global solution* to Problem (5.9) if it minimizes the Mayer cost $\phi(x(0),x(T))$ among all admissible pairs.

5.3.2 Regular solution to the control system with loss control regions

Consider the control system with loss control regions associated with Problem (5.9) given by

$$\begin{cases} \dot{x}(t)=f(x(t),u(t)), \quad \text{for a.e. } t\in[0,T],\\ u \text{ is constant when } x \text{ is in a loss control region.} \end{cases} \tag{5.10}$$

Let us precise the definition of a solution to (5.10).

Definition 5.3.1 (Solution to (5.10))**.** *A pair $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is said to be a solution to* (5.10) *if there exists a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0,T]$ such that:*

(i) It holds that

$$\forall k\in\{1,\dots,N\},\quad \exists j(k)\in\mathcal{J},\quad \forall t\in(\tau_{k-1},\tau_k),\quad x(t)\in X_{j(k)},$$

where $j(k)\neq j(k-1)$ for all $k\in\{2,\dots,N\}$. The sequence $\{j(1),\dots,j(N)\}$ is called the switching sequence.

(ii) It holds that $x(0)\in X_{j(1)}$ and $x(T)\in X_{j(N)}$.

(iii) For all $k\in\{1,\dots,N\}$ such that $q_{j(k)}=0$, the control u is constant over (τ_{k-1},τ_k) (the constant value being denoted by u_k in the sequel).

(iv) It holds that $\dot{x}(t)=f(x(t),u(t))$ for almost every $t\in[0,T]$.

The times τ_k for $k\in\{1,\dots,N-1\}$, called crossing times, correspond to the instants at which the trajectory x goes from the region $X_{j(k)}$ to the region $X_{j(k+1)}$, and thus $x(\tau_k)\in\partial X_{j(k)}\cap\partial X_{j(k+1)}$.

Similarly to Section 5.2.2, the main result of this section (Theorem 5.3.1 stated in Section 5.3.3) is based on some regularity assumptions made on the optimal pair of Problem (5.9) at each of its crossing times. These hypotheses are made more precise in the next definition.

Definition 5.3.2 (Regular solution to (5.10))**.** *Following the notations introduced in Definition 5.3.1, a solution $(x,u)\in\mathrm{AC}([0,T],\mathbb{R}^n)\times\mathrm{L}^\infty([0,T],\mathbb{R}^m)$ to* (5.10)*, associated with a partition $\mathbb{T}=\{\tau_k\}_{k=0,\dots,N}$ and a switching sequence $\{j(1),\dots,j(N)\}$, is said to be regular if the following conditions are both satisfied:*

(i) At each crossing time τ_k, there exists a C^1 function $F_k : \mathbb{R}^n \to \mathbb{R}$ such that

$$\exists \nu_k > 0, \quad \forall z \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in X_{j(k)} & \Leftrightarrow \quad F_k(z) < 0, \\ z \in \partial X_{j(k)} \cap \partial X_{j(k+1)} & \Leftrightarrow \quad F_k(z) = 0, \\ z \in X_{j(k+1)} & \Leftrightarrow \quad F_k(z) > 0. \end{cases}$$

In particular it holds that $F_k(x(\tau_k)) = 0$.

(ii) At each crossing time τ_k, there exists $\alpha_k > 0$ and $\beta_k > 0$ such that the transverse condition

$$\langle \nabla F_k(x(\tau_k)), f(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, \quad \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k + \alpha_k], \tag{5.11}$$

is satisfied.

Remark 5.3.1. Similar comments than the ones developed in Remark 5.2.2 remain true for Definitions 5.3.1 and 5.3.2.

5.3.3 PMP with loss control regions and comments

The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (5.9) is defined by

$$H(x, u, p) := \langle p, f(x, u) \rangle_{\mathbb{R}^n},$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to state the main result of this section.

Theorem 5.3.1 (PMP with loss control regions)**.** *Suppose that $(x^*, u^*) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m)$ is a global solution to Problem (5.9), that it is moreover a regular solution to (5.10), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$ and a switching sequence $\{j(1), \dots, j(N)\}$, and that it is such that g is submersive at $(x^*(0), x^*(T))$. Then, there exists a nontrivial pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:*

(i) the Hamiltonian system

$$\dot{x^*}(t) = \nabla_p H(x^*(t), u^*(t), p(t)) \quad \text{and} \quad -\dot{p}(t) = \nabla_x H(x^*(t), u^*(t), p(t)),$$

for almost every $t \in [0, T]$;

(ii) the transversality condition

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

for some $\xi \in \mathrm{N_S}[g(x^*(0), x^*(T))]$;

(iii) the discontinuity condition

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*)),$$

for some $\sigma_k \in \mathbb{R}$ and for all $k \in \{1, \dots, N-1\}$;

(iv) for all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 1$, the Hamiltonian maximization condition

$$u^*(t) \in \underset{\omega \in \mathrm{U}}{\arg\max}\, H(x^*(t), \omega, p(t)),$$

for almost every $t \in (\tau^*_{k-1}, \tau^*_k)$*;*

(v) for all $k \in \{1, \dots, N\}$ *such that* $q_{j(k)} = 0$*, the averaged Hamiltonian gradient condition*

$$\int_{\tau^*_{k-1}}^{\tau^*_k} \nabla_u H(x^*(t), u^*_k, p(t))\, dt \in \mathrm{N}_{\mathrm{U}}[u^*_k],$$

for all $k \in \{1, \dots, N\}$*;*

(vi) the Hamiltonian constancy condition

$$H(x^*(t), u^*(t), p(t)) = c,$$

for almost every $t \in [0, T]$*, for some* $c \in \mathbb{R}$*.*

Proof. Consider the framework of Theorem 5.3.1. The proof of Theorem 5.3.1 is based on the application of Theorem 5.2.1 to the hybrid optimal control problem with regionally switching parameter given by

$$\begin{array}{ll} \text{minimize} & \phi(x(0), x(T)), \\ \text{subject to} & (x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^m) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m), \\ & \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0,T], \\ & g(x(0), x(T)) \in \mathrm{S}, \\ & (\lambda(t), u(t)) \in \mathrm{U}^2, \quad \text{a.e. } t \in [0,T], \\ & \lambda \text{ is a regionally switching parameter associated with } x, \end{array} \tag{5.12}$$

where the spatially heterogeneous dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is regionally defined by

$$h(x, \lambda, u) := \begin{cases} f(x, u) & \text{if } x \in X_j \text{ with } q_j = 1, \\ f(x, \lambda) & \text{if } x \in X_j \text{ with } q_j = 0, \end{cases}$$

for all $(x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$. The proof will be done in two steps.

Step 1. Consider some $\omega_0 \in \mathrm{U}$ and introduce $\lambda^* \in \mathrm{PC}_{\mathbb{T}^*}([0,T], \mathbb{R}^m)$ defined by

$$\lambda^*(t) := \begin{cases} \omega_0 & \text{for all } t \in (\tau^*_{k-1}, \tau^*_k) \text{ such that } q_{j(k)} = 1, \\ u^*_k & \text{for all } t \in (\tau^*_{k-1}, \tau^*_k) \text{ such that } q_{j(k)} = 0. \end{cases}$$

One can easily see that (x^*, λ^*, u^*) is admissible for Problem (5.12) associated with the partition $\mathbb{T}^*$. Now let us prove that the triplet (x^*, λ^*, u^*) is a global solution to Problem (5.12). To this aim let $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^m) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ be an admissible triplet for Problem (5.12), associated with a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N'}$ and a switching sequence $\{j'(1), \dots, j'(N')\}$. Let us introduce $v \in \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ defined by

$$v(t) := \begin{cases} u(t) & \text{for all } t \in (\tau_{k-1}, \tau_k) \text{ such that } q_{j'(k)} = 1, \\ \lambda_k & \text{for all } t \in (\tau_{k-1}, \tau_k) \text{ such that } q_{j'(k)} = 0. \end{cases}$$

One can easily see that (x, v) is admissible for Problem (5.9) associated with the partition $\mathbb{T}$. Therefore, using the optimality of (x^*, u^*), we obtain that $\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T))$ which completes our

intermediate goal. Furthermore one can easily see that, since (x^*, u^*) is a regular solution to (5.10), then the triplet (x^*, λ^*, u^*) is a regular solution to the hybrid control system associated with Problem (5.12). Finally recall that g is submersive at $(x^*(0), x^*(T))$ from hypotheses of Theorem 5.3.1.

Step 2. From Step 1, we are in a position to apply Theorem 5.2.1 on the triplet (x^*, λ^*, u^*). Therefore there exists a nontrivial pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0,T], \mathbb{R}^n) \times \mathbb{R}_+$ which satisfies items from (i) to (vi) of Theorem 5.2.1, where the Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\mathcal{H}(x, \lambda, u, p) = \begin{cases} H(x, u, p) & \text{if } x \in X_j \text{ with } q_j = 1, \\ H(x, \lambda, p) & \text{if } x \in X_j \text{ with } q_j = 0, \end{cases}$$

for all $(x, \lambda, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. One can easily deduce that items from (i) to (vi) of Theorem 5.3.1 are satisfied. The proof is complete. □

Remark 5.3.2. Similar comments than the ones developed in Remarks 5.2.4, 5.2.5, 5.2.6 and 5.2.8 also apply to Theorem 5.3.1.

5.4 A numerical approach for optimal control problems with loss control regions and application to illustrative examples

In this section we consider the framework outlined in Section 5.3 and our objective is to introduce a numerical approach that can compute an optimal control for Problem (5.9) on illustrative examples[1]. In Section 5.4.1, we present this approach which is composed of two steps: a direct method followed by an indirect method. Additionally we will highlight their pros and cons. In Section 5.4.2, we numerically solve a Zermelo-type problem with loss control regions (in two cases with different state space partitions). In Section 5.4.3, we numerically solve a minimal time problem for the harmonic oscillator with a loss control region.

5.4.1 A numerical approach for optimal control problems with loss control regions

In optimal control theory, there are several ways for solving numerically an optimal control problem. Direct and indirect methods represent an important class of methods that we will use hereafter. Direct methods involve discretizing the state and control variables, simplifying the problem into a nonlinear optimization problem. On the other hand, indirect methods tackle the problem by solving a boundary value problem through the use of a shooting method, which is based on the maximum principle. It is important to note that neither of these methods is better than the other.

Moreover, each of these methods has its pros and cons. For instance, although the direct method is simple to implement, more robust, and less sensitive to the choice of the initial condition, it should be noted that it yields less precise results and can converge to local minima that significantly deviate from the optimal solution. Additionally, this method requires a large amount of memory. On the other hand, the indirect method is known for its extreme precision. However, it is based only on necessary optimality conditions (maximum principle) and often requires knowledge of the structure of the optimal

[1]The scripts for reproducing the numerical experiments in this paper are published in the repository: https://github.com/control-toolbox/control-loss

solution. Moreover, it is quite sensitive to the choice of the initial condition, which must be chosen carefully to ensure convergence. Therefore, both methods have their own strengths and limitations, and it is not accurate to say that one is better than the other.
Often in the literature, we proceed in two steps. The first step is to implement a direct method to determine the optimal solution's structure and extract the associated adjoint vectors. The second step involves constructing an indirect (shooting) method, where the initial condition is based on the results obtained from the direct method.

Description of the direct method. For some $\omega_0 \in \mathrm{U}$ and some $\varepsilon > 0$ small enough, we introduce the regularized problem given by

$$\begin{aligned}
\text{minimize} \quad & \phi(x(0), x(T)) + \varepsilon \int_0^T v^2(t)\, dt + \int_0^T (1 - \Psi_\varepsilon(x(t)))\|u(t) - \omega_0\|_{\mathbb{R}^m}^2\, dt, \\
\text{subject to} \quad & (x, \lambda, u, v) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{AC}([0,T], \mathbb{R}^m) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times \mathrm{L}^\infty([0,T], \mathbb{R}), \\
& \dot{x}(t) = \Psi_\varepsilon(x(t)) f(x(t), u(t)) + (1 - \Psi_\varepsilon(x(t))) f(x(t), \lambda(t)), \quad \text{a.e. } t \in [0,T], \\
& \dot{\lambda}(t) = \Psi_\varepsilon(x(t)) v(t), \quad \text{a.e. } t \in [0,T], \\
& g(x(0), x(T)) \in \mathrm{S}, \\
& \lambda(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\
& (u(t), v(t)) \in \mathrm{U} \times \mathbb{R}, \quad \text{a.e. } t \in [0,T],
\end{aligned} \tag{5.13}$$

where $\Psi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is the regularization of the characteristic function of $\cup_{q_j=1} \overline{X_j}$ given by

$$\Psi_\varepsilon(x) := \sum_{q_j=1} e^{-\frac{1}{2\varepsilon}\mathrm{d}_j^2(x)},$$

for all $x \in \mathbb{R}^n$, where $\mathrm{d}_j : \mathbb{R}^n \to \mathbb{R}$ stands for the distance function to the set $\overline{X_j}$ defined by $\mathrm{d}_j(x) := \inf_{y \in \overline{X_j}} \|x - y\|_{\mathbb{R}^n}$ for all $x \in \mathbb{R}^n$ and every $j \in \mathcal{J}$. Note that the above regularized problem arises from considering the hybrid optimal problem with regionally switching parameter associated with Problem (5.9), as outlined in the proof of Theorem 5.3.1. Before presenting the direct method we would like to explain the numerical role of certain quantities that are considered in Problem (5.13):

(i) Comment on the dynamics of x and λ. Since Ψ_ε (approximately) equals to one in control regions, it follows that the dynamics of x is described by $\dot{x}(t) = f(x(t), u(t))$ in these regions. On the contrary, since Ψ_ε (approximately) vanishes in loss control regions, it follows that λ remains constant and that the dynamics of x is described by $\dot{x}(t) = f(x(t), \lambda(t))$ in these regions.

(ii) Comment on the additional control v and the penalization cost $\varepsilon \int_0^T v^2(t)\, dt$. Since Ψ_ε (approximately) equals to one in control regions, the additional control v can operate in these regions to allow λ to change its constant value between two consecutive loss control regions. On the other hand, the penalization cost is introduced to ensure the convergence of the direct method and to guarantee the uniqueness of the optimal control v. In addition, the multiplicative parameter $\varepsilon > 0$ (small enough) ensures that the penalization cost does not influence (too much) the original cost of Problem (5.9). Finally, note that, since Ψ_ε (approximately) vanishes in loss control regions, the control system is independent of v in these regions and thus, due to the penalization cost, the optimal control v

vanishes in these regions.

(iii) Comment on the running cost $\int_0^T (1 - \Psi_\varepsilon(x(t)))\|u(t) - \omega_0\|^2_{\mathbb{R}^m}\, dt$. Since Ψ_ε (approximately) vanishes in loss control regions, the control system is independent of u in these regions and thus, due to this running cost, the optimal control u is unique (given by $u(t) = \omega_0$) in these regions. Furthermore, since Ψ_ε (approximately) equals to one in control regions, we deduce that this running cost is (approximately) equal to zero at the optimal solution.

We are now in a position to present the direct method for solving Problem (5.13). After rewriting the problem in a Mayer form, we discretize the time interval and approximate the state and the control at discrete time points, resulting in a finite-dimensional nonlinear optimization problem. This problem is solved using `JuMP`, a modeling language for mathematical optimization embedded in `Julia`, with the Mayer cost function as objective function. To handle the dynamics, we employ the *Crank-Nicholson* scheme, and the initial, terminal, and control constraints are formulated as optimization problem constraints. We use the `IPOPT` solver (whose precision is set to 10^{-8}) to solve the resulting optimization problem and extract the adjoint vectors using the predefined `dual` function.

Description of the indirect method. From the direct method described above, we extract the structure of the optimal solution (x^*, u^*) to Problem (5.9). This latter contains an initialization of the adjoint vector p, crossing times, switching times, constant values of the control (when the state belongs to loss control regions) and discontinuity jumps of p at each crossing time. These elements allow us to construct a shooting method based on Theorem 5.3.1 which consists in two parts described below. In the rest of this section, for simplicity, we will assume that, when x^* belongs to a loss control region, then the constant value u_k^* of the control belongs to the interior of U (see Remark 5.4.1 for details).

(i) **Part 1.** Recall that the direct method has captured the structure of the optimal pair (x^*, u^*). Therefore, in the indirect method we will address each arc (bang-bang, constant (interior value to U), feedback etc.) of the optimal solution separately. Indeed, we begin by defining the flow of the Hamiltonian associated with each arc. To accomplish this, we use the function `Flow`[2]. This latter allows to solve the Hamiltonian system over a given time interval from given initial values of the state and the adjoint vector. This function requires necessary libraries such as `ForwardDiff` for calculating gradients and Jacobians and `OrdinaryDiffEq` for solving ordinary differential equations. In the setting of loss control regions, we distinguish between two types of Hamiltonian flows:

- **Hamiltonian flows in control regions.** Consider the setting of Theorem 5.3.1, we recall the Hamiltonian associated with Problem (5.9) as

$$H(x, u, p) = \langle p, f(x, u)\rangle_{\mathbb{R}^n},$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. Using Theorem 5.3.1, specifically the maximization condition, we obtain the expression of the control u^* (which can generate a sequence of arcs). Thus, it remains to define a *pseudo-Hamiltonian*[3] associated with each arc. Finally, we define the flow associated with each arc, which allows the resolution of the boundary value problem on a time interval satisfied by the pair (x^*, p) with an initial condition.

- **Hamiltonian flows in loss control regions.** In these regions, we recall that u^* satisfies an averaged Hamiltonian gradient condition (instead of a maximization condition). Here, the

[2] the `Flow` function can be found in the `CTFlows.jl` package: https://github.com/control-toolbox/CTFlows.jl
[3] the pseudo-Hamiltonian stands for the Hamiltonian flow associated with each arc.

difficulty lies in the fact that this condition is given in an integral and implicit form. Therefore, to overcome this difficulty, we first introduce new states λ and y. The state λ comes from the augmentation technique to handle the constant value of u^*, for example u_k^*, so it satisfies the dynamics $\dot{\lambda}(t) = 0$ and $\lambda(\tau_{k-1}^*) = u_k^*$. Second, the state y satisfies $\dot{y}(t) = 0$ and $y(\tau_{k-1}^*) = 0$ (and thus $y = 0$). Now, we can define the new Hamiltonian as follows:

$$\begin{aligned}\tilde{H}(x, \lambda, y, p) = &\quad H(x, \lambda, p) - y\nabla_\lambda H(x, \lambda, p) \\ = &\quad \langle p, f(x, \lambda)\rangle_{\mathbb{R}^n} - y\nabla_\lambda H(x, \lambda, p),\end{aligned}$$

for all $(x, \lambda, y, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n$. It is important to note that $y = 0$ is necessary to recover the same Hamiltonian H. But, the actual utility of introducing the state y is that it allows us to rewrite the integral expressed in the averaged Hamiltonian gradient condition as a terminal value of an adjoint vector. This makes it easier to take into account in the shooting function. Indeed, we define p_y as the solution to the following system:

$$\begin{cases} \dot{p}_y(t) = -\nabla_y \tilde{H}(x^*(t), u_k^*, y(t), p(t)), \quad \text{for a.e. } t \in [\tau_{k-1}^*, \tau_k^*], \\ p_y(\tau_{k-1}^*) = 0_{\mathbb{R}^n}. \end{cases}$$

Since u_k^* is assumed to be an interior value to U then we get that

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_u H(x^*(t), u_k^*, p(t))\, dt = p_y(\tau_k^*) = 0,$$

so that there is no need to compute an integral in order to take into account the averaged gradient condition.

(ii) **Part 2.** It consists in determining the appropriate shooting function which includes novel elements such as crossing times, the discontinuity condition of the adjoint vector at each crossing time and (mainly) the averaged Hamiltonian gradient condition. In addition, it also includes standard elements such as the switching times, terminal state constraints and transversality conditions. Once the shooting function is constructed, we use the predefined function `NLsolve` (whose precision is set to 10^{-8}) with an initialization based on the direct method in order to find a zero of the corresponding shooting function.

Remark 5.4.1. In this section we assumed that, when x^* belongs to a loss control region, then the constant value u_k^* of the control belongs to the interior of U. This was made for simplicity since the averaged Hamiltonian gradient condition simplifies into an equality. Note that the case where u_k^* belongs to the boundary of U can be treated easily when $m = 1$ (typically if $\mathrm{U} = [-1, 1]$ for example), since the boundary is finite and the optimal value can be deduced from the direct method (see Remark 5.4.2 in Section 5.4.3). When u_k^* belongs to the boundary of U and $m \geq 2$, the averaged Hamiltonian gradient condition is more involved to handle in the shooting function since it only provides an inclusion. A possible way could be to combine this inclusion together with a description of the set U to get additional equations. The consideration of this case is out of the scope of the paper and could be the matter of future works.
In the remaining sections, we will solve numerically a Zermelo-type problem with two different state partitions that falls within the framework described in Section 5.3. Next, we will study numerically the harmonic oscillator problem including a loss control region. This is a minimum time problem that can be transformed into the framework of Section 5.3 using a change of variables and introducing a new state

variable $\dot{x}_{n+1} = T$ for almost every $s \in [0,1]$ with $x_{n+1}(0) = 0$, in such a way that the time interval is fixed. However, we obtain an optimal control problem with loss control regions, but with an additional constant parameter. So, to fall in the framework of Section 5.3, we rely on the augmentation technique.

5.4.2 Application to a Zermelo-type problem with loss control regions

Our first example is based on Zermelo's problem, which is well-known in viability theory [6] and also in geometric optimal control theory [32]. One interesting issue related to this problem is that, depending on the parameter values, it highlights the link between non-controllability and abnormal curves. However, we do not investigate this point. Instead, we focus on optimizing one coordinate whenever a loss control region is taken into account (which can model various behaviors of the flow in this navigation problem). Such a problem will be formulated as a Mayer optimal control problem. Specifically, we consider the Zermelo-type optimal control problem with loss control regions given by

$$\begin{aligned}
&\text{minimize} && -x_1(8),\\
&\text{subject to} && (x,u) \in \mathrm{AC}([0,8],\mathbb{R}^2) \times \mathrm{L}^\infty([0,8],\mathbb{R}),\\
&&& \dot{x}_1(t) = x_2(t) + \cos(u(t)), \quad \text{a.e. } t \in [0,8],\\
&&& \dot{x}_2(t) = \sin(u(t)), \quad \text{a.e. } t \in [0,8],\\
&&& x(0) = 0_{\mathbb{R}^2}, \quad x_2(8) = 4,\\
&&& u(t) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad \text{a.e. } t \in [0,8],\\
&&& u \text{ is constant when } x \text{ is in a loss control region.}
\end{aligned} \tag{5.14}$$

In the sequel, based on the numerical approach developed in Section 5.4, we solve Problem (5.14) on two different situations of space partition of $\mathbb{R}^2$ (see Example 1 and Example 2). Note that when applying Theorem 5.3.1 to Problem (5.14) we remain in a normal situation, i.e., $p^0 = 1$.

Example 1. Consider the space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3}$ with

$$X_1 := \left\{ x \in \mathbb{R}^2 \mid x_2 < \frac{1}{2} \right\}, \quad X_2 := \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2} < x_2 < \frac{7}{2} \right\} \quad \text{and} \quad X_3 := \left\{ x \in \mathbb{R}^2 \mid x_2 > \frac{7}{2} \right\},$$

with $q_1 = q_3 = 1$ and $q_2 = 0$ (see Figure 5.2).

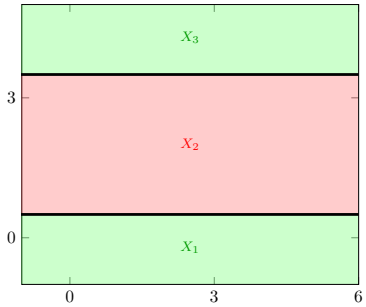


Figure 5.2: Partition of $\mathbb{R}^2$ into one loss control region (in red) and two control regions (in green).

Application of the direct method. We obtain an optimal triplet (x, λ, u) that visits the loss control region X_2 once with $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, the adjoint vector p_2 has discontinuity jumps at each crossing time as illustrated in Figure 5.3.

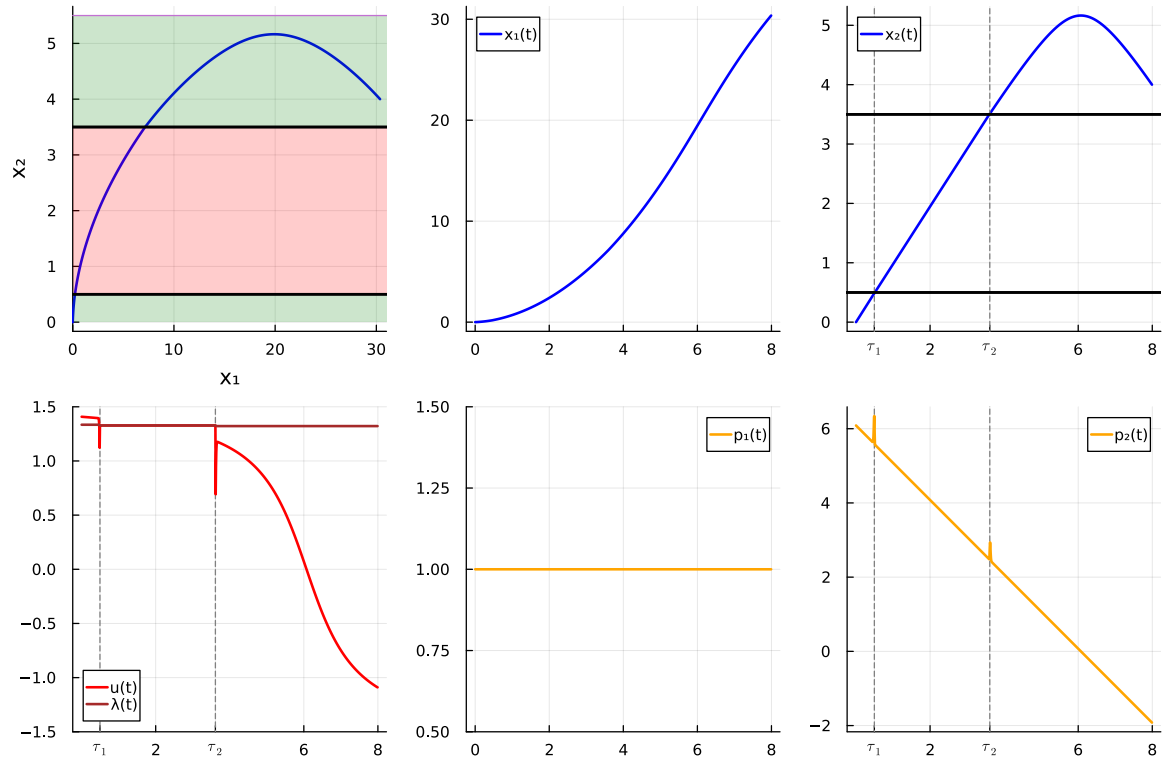


Figure 5.3: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (5.14) (Example 1).

Application of the indirect method. From the above results, we deduce that the optimal solution to Problem 5.14 (Example 1), denoted by (x^*, u^*), has three arcs: feedback, then a constant value in $(-\frac{\pi}{2}, \frac{\pi}{2})$, then feedback. From the adjoint equation and the transversality condition we get that $p_1(t) = 1$ for all $t \in [0, 8]$. Therefore, we can express the control u^* in a feedback form using the Hamiltonian maximization condition.

$$u^*(t) \in \underset{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]}{\arg\max}\, p_1(t)\cos(\omega) + p_2(t)\sin(\omega) = \underset{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]}{\arg\max} \left\langle \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}, \begin{pmatrix} 1 \\ p_2(t) \end{pmatrix} \right\rangle_{\mathbb{R}^2},$$

which is satisfied for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, 8]$. From the classical Cauchy-Schwartz inequality, we get that

$$\begin{pmatrix} \cos(u^*(t)) \\ \sin(u^*(t)) \end{pmatrix} = \frac{1}{\|(1, p_2(t))\|_{\mathbb{R}^2}} \begin{pmatrix} 1 \\ p_2(t) \end{pmatrix},$$

for almost every $[0, \tau_1^*] \cup [\tau_2^*, 8]$, we deduce that $\cos(u^*(t)) \neq 0$ and thus $u^*(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ over $[0, \tau_1^*] \cup [\tau_2^*, 8]$. We conclude that $\tan(u^*(t)) = p_2(t)$ and thus $u^*(t) = \arctan(p_2(t))$ for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, 8]$.

Now, we give the shooting function for Problem (5.14) (Example 1).

$$
\begin{aligned}
\mathcal{S}_1 : \mathbb{R}^7 &\rightarrow \mathbb{R}^7 \\
(p_1^0, p_2^0, \tau_1, \tau_2, a_1, a_2, u_1) &\mapsto \begin{pmatrix}
x_2(8) - 4 \\
p_1(8) - 1 \\
x_2(\tau_1) - \frac{1}{2} \\
x_2(\tau_2) - \frac{7}{2} \\
p_y(\tau_2) \\
a_1 - \dfrac{p_1(\tau_1)\big(\cos(u_1) - \cos(u^-(\tau_1))\big) + p_2^-(\tau_1)\big(\sin(u_1) - \sin(u^-(\tau_1))\big)}{\sin(u_1)} \\
a_2 - \dfrac{p_1(\tau_2)\big(\cos(u^+(\tau_2)) - \cos(u_1)\big) + p_2^-(\tau_2)\big(\sin(u^+(\tau_2)) - \sin(u_1)\big)}{\sin(u^+(\tau_2))}
\end{pmatrix}
\end{aligned}
$$

The numerical results presented in Figure 5.4 depict an optimal pair (x^*, u^*) of Problem (5.14) (Example

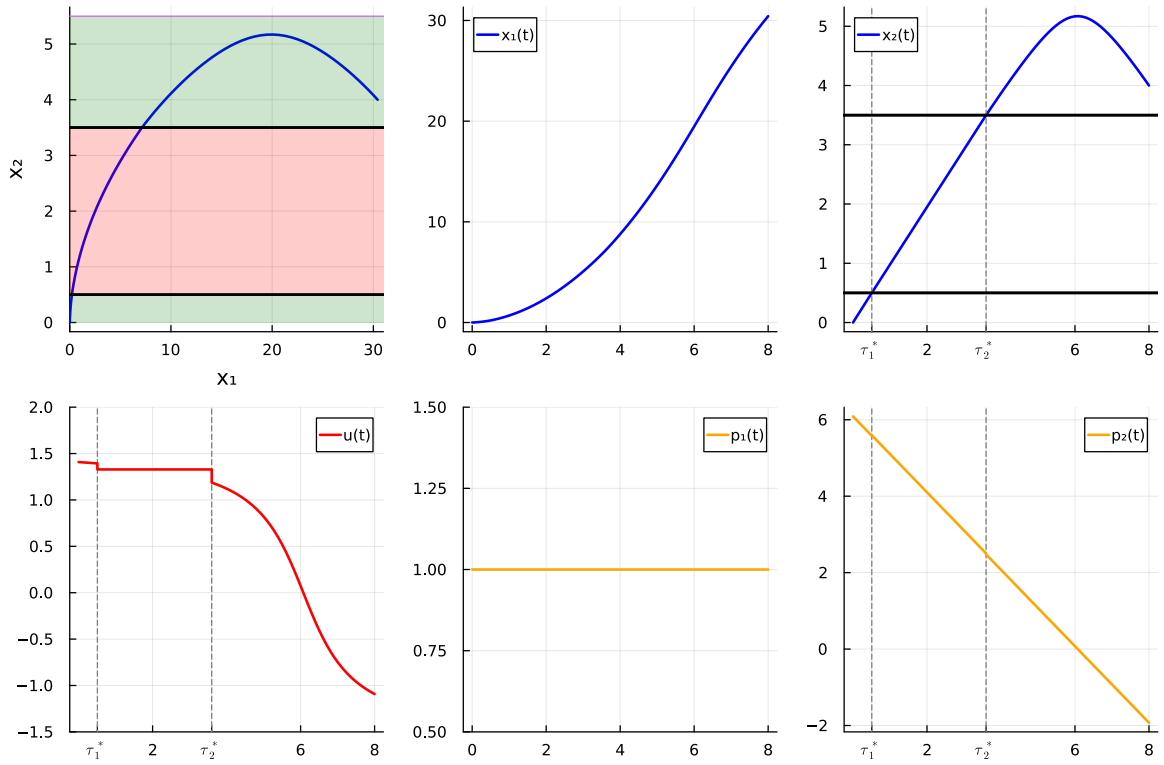


Figure 5.4: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (5.14) (Example 1).

1) together with an adjoint vector p. We observe (small) jumps in the adjoint vector p_2 at both crossing times. In summary, the indirect method confirms the solution obtained using the direct method. Finally, note in the direct method we notice that constancy is satisfied (see Figure 5.5).

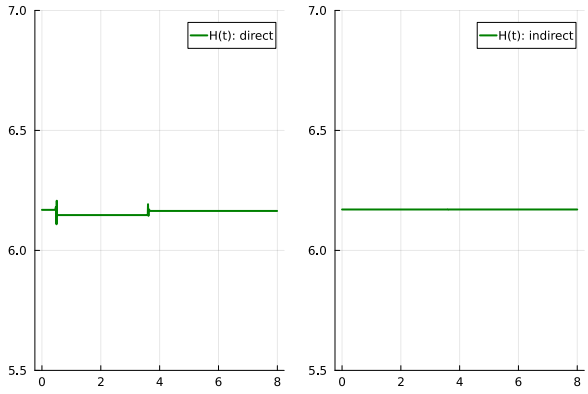


Figure 5.5: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (5.14) (Example 1).

Example 2. We study now a variant of the Problem (5.14) including several loss control regions. Consider the space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3} \cup \overline{X_4} \cup \overline{X_5}$ with

$$X_1 := \left\{x \in \mathbb{R}^2 \mid x_1 < 2\right\}, \quad X_2 := \left\{x \in \mathbb{R}^2 \mid 2 < x_1 < 16\right\}, \quad X_3 := \left\{x \in \mathbb{R}^2 \mid 16 < x_1 < 20\right\},$$

$$X_4 := \{x \in \mathbb{R}^2 \mid 20 < x_1 < 25\} \quad \text{and} \quad X_5 := \{x \in \mathbb{R}^2 \mid 25 < x_1\},$$

with $q_1 = q_3 = q_5 = 1$ and $q_2 = q_4 = 0$ (see Figure 5.6).

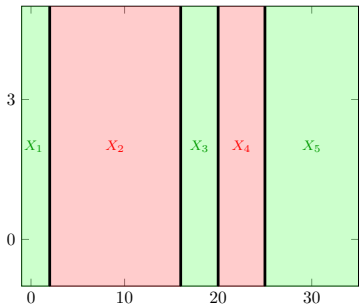


Figure 5.6: Partition of $\mathbb{R}^2$ into two loss control regions (in red) and three control regions (in green).

Application of the direct method. We obtain an optimal triplet (x, λ, u) that visits the loss control regions X_2 and X_4 with $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, the adjoint vector p_1 has discontinuity jumps at each crossing time as illustrated in Figure 5.7.

Application of the indirect method. We use similar arguments as in Example 1, we get that in control regions one has $u^*(t) = \arctan(p_2(t))$ for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, \tau_3^*] \cup [\tau_4^*, 8]$. Now, we

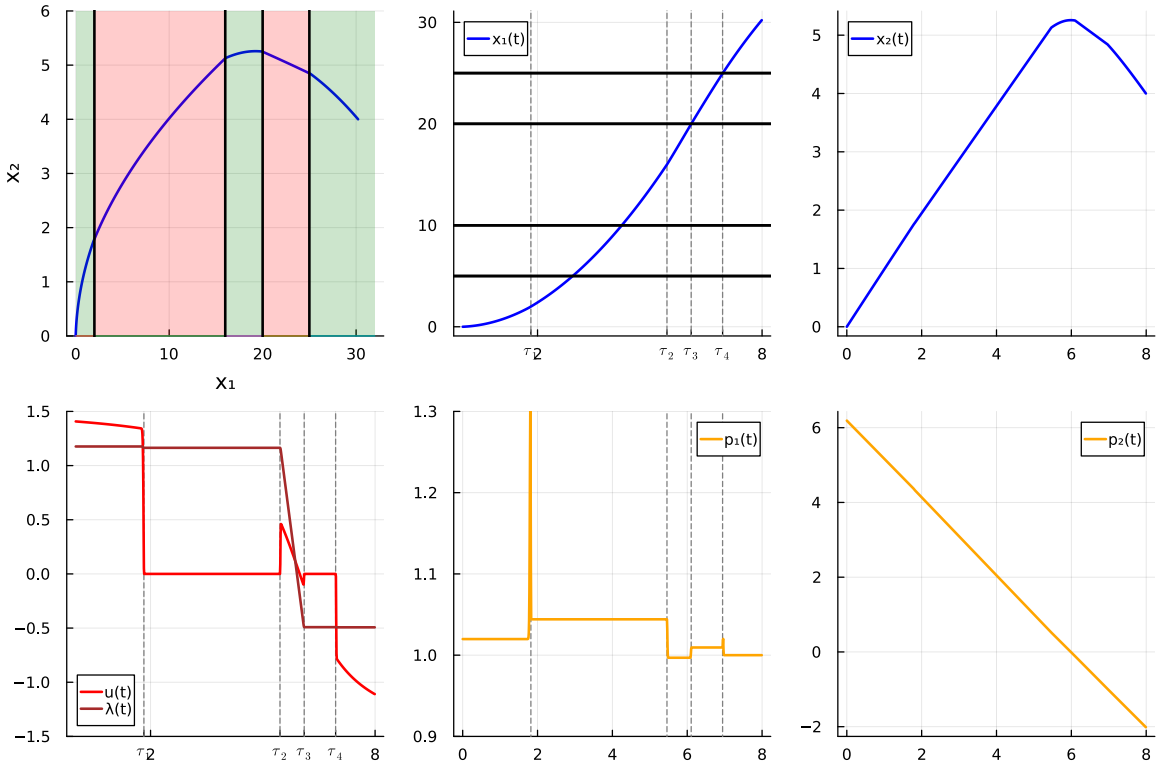


Figure 5.7: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (5.14) (Example 2).

give the shooting function for Problem (5.14) (Example 2).

$$\mathcal{S}_2 : \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}$$

$$(p_1^0, p_2^0, \tau_1, \tau_2, \tau_3, \tau_4, a_1, a_2, a_3, a_4, u_1, u_2) \mapsto \begin{pmatrix} x_2(8) - 4 \\ p_1(8) - 1 \\ x_1(\tau_1) - 2 \\ x_1(\tau_2) - 16 \\ x_1(\tau_3) - 20 \\ x_1(\tau_4) - 25 \\ p_y(\tau_2) \\ p_y(\tau_4) \\ a_1 - \frac{p_1^-(\tau_1)\big(\cos(u_1) - \cos(u^-(\tau_1))\big) + p_2(\tau_1)\big(\sin(u_1) - \sin(u^-(\tau_1))\big)}{\sin(u_1)} \\ a_2 - \frac{p_1^-(\tau_2)\big(\cos(u^+(\tau_2)) - \cos(u_1)\big) + p_2(\tau_2)\big(\sin(u^+(\tau_2)) - \sin(u_1)\big)}{\sin(u^+(\tau_2))} \\ a_3 - \frac{p_1^-(\tau_3)\big(\cos(u_2) - \cos(u^-(\tau_3))\big) + p_2(\tau_3)\big(\sin(u_2) - \sin(u^-(\tau_3))\big)}{\sin(u_2)} \\ a_4 - \frac{p_1^-(\tau_4)\big(\cos(u^+(\tau_4)) - \cos(u_2)\big) + p_2(\tau_4)\big(\sin(u^+(\tau_4)) - \sin(u_2)\big)}{\sin(u^+(\tau_4))} \end{pmatrix}$$

The numerical results presented in Figure 5.8 depict an optimal pair (x^*, u^*) of Problem (5.14) (Example 2) together with an adjoint vector p. We observe jumps in the adjoint vector p_1 at each crossing time

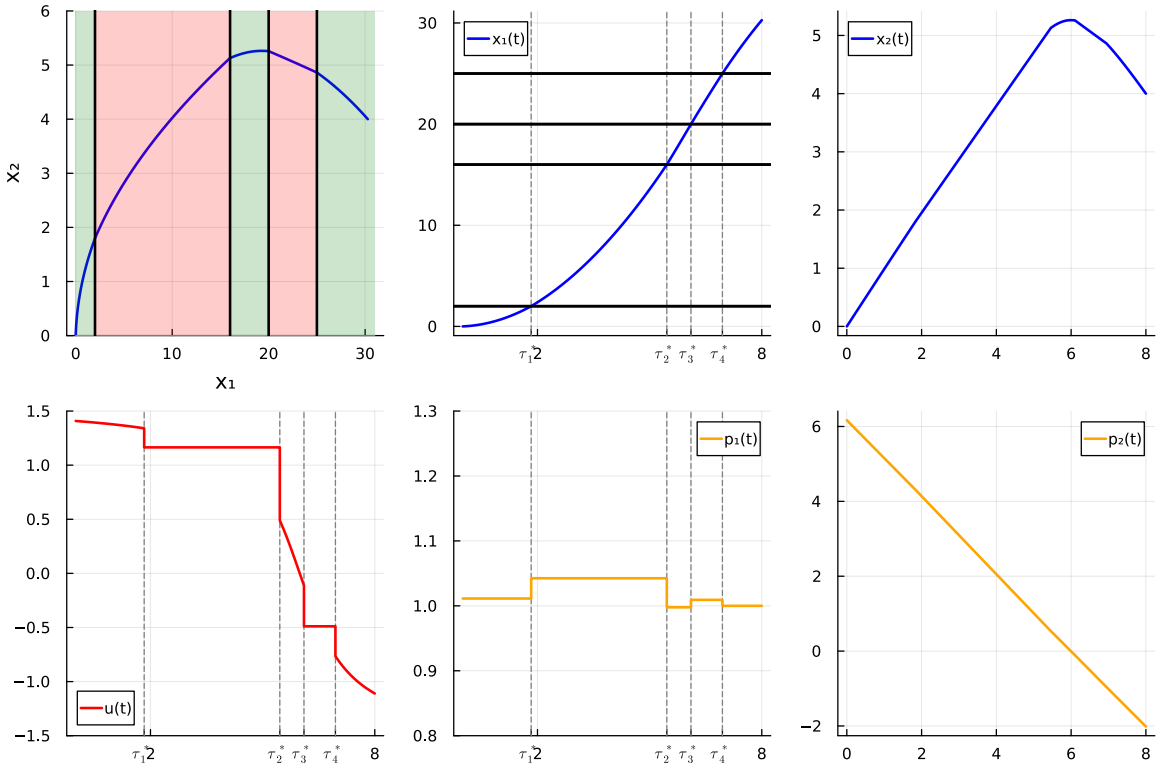


Figure 5.8: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (5.14) (Example 2).

(see Figures 5.8). Again, our simulations via the indirect method yields a solution (x^*, u^*) that is similar to the one obtained by the direct method. Finally, note in the direct method we notice that

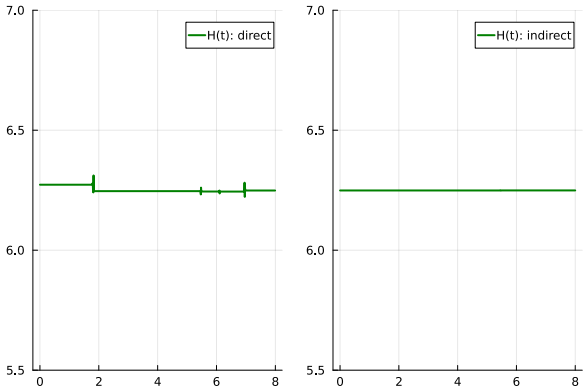


Figure 5.9: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (5.14) (Example 2).

constancy is satisfied (see Figure 5.9).

5.4.3 Minimum time problem for the harmonic oscillator with a loss control region

This example is in line with [15] in which we have computed theoretically optimal paths for the classical minimum time control problem governed by the double integrator with a loss control region. Here, we have considered a variant of the minimum time control problem governed by the harmonic oscillator (the classical version is treated in [107]) including a loss control region. One main issue (in contrast with the double integrator, as in [15]) is that trajectories spiral around the target (origin) in a finite number in order to reach this target. Hence, depending on the choice of a loss control region, we expect trajectories to visit this region several times. Thus, the constant value of the control at each visit could be modified. Specifically, we consider the minimum time problem for the harmonic oscillator with a loss control region given by

$$\begin{aligned}
&\text{minimize} && T,\\
&\text{subject to} && (x,u,T)\in \mathrm{AC}([0,T],\mathbb{R}^2)\times \mathrm{L}^\infty([0,T],\mathbb{R})\times(0,+\infty),\\
&&& \dot{x}_1(t)=x_2(t), \quad \text{a.e. } t\in[0,T],\\
&&& \dot{x}_2(t)=u(t)-x_1(t), \quad \text{a.e. } t\in[0,T],\\
&&& x(0)=(\tfrac{5}{2},4), \quad x(T)=0_{\mathbb{R}^2},\\
&&& u(t)\in[-1,1], \quad \text{a.e. } t\in[0,T],\\
&&& u \text{ is constant when } x \text{ is in a loss control region},
\end{aligned} \tag{5.15}$$

with the space partition $\mathbb{R}^2 = \overline{X_1}\cup\overline{X_2}$ with

$$X_1 := \{x\in\mathbb{R}^2 \mid x_2>0\}, \quad \text{and} \quad X_2 := \{x\in\mathbb{R}^2 \mid x_2<0\},$$

with $q_1=1$ and $q_2=0$ (see Figure 5.10). Note that when applying Theorem 5.3.1 to Problem (5.15) we remain in a normal situation i.e. $p^0=1$.

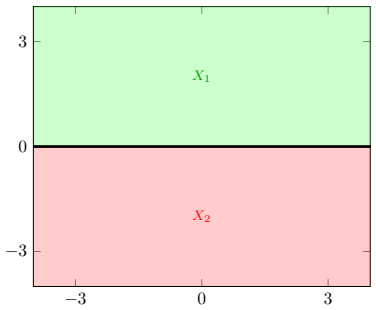


Figure 5.10: Partition of $\mathbb{R}^2$ into a "loss control regions" (in red) and a "control region" (in green).

Application of the direct method. We obtain an optimal triplet (x,λ,u) that visits the loss control regions X_2 twice. In the first visit, λ is equal to 1, and in the second visit, λ belongs to $(-1,1)$. Moreover, we observe that the control is bang-bang during the second visit of the control region as illustrated in in Figure 5.11.

Application of the indirect method. Based on the above results, we deduce that the optimal solution to Problem (5.15) consists of four arcs. These arcs include sequences of bang arcs and one arc with a

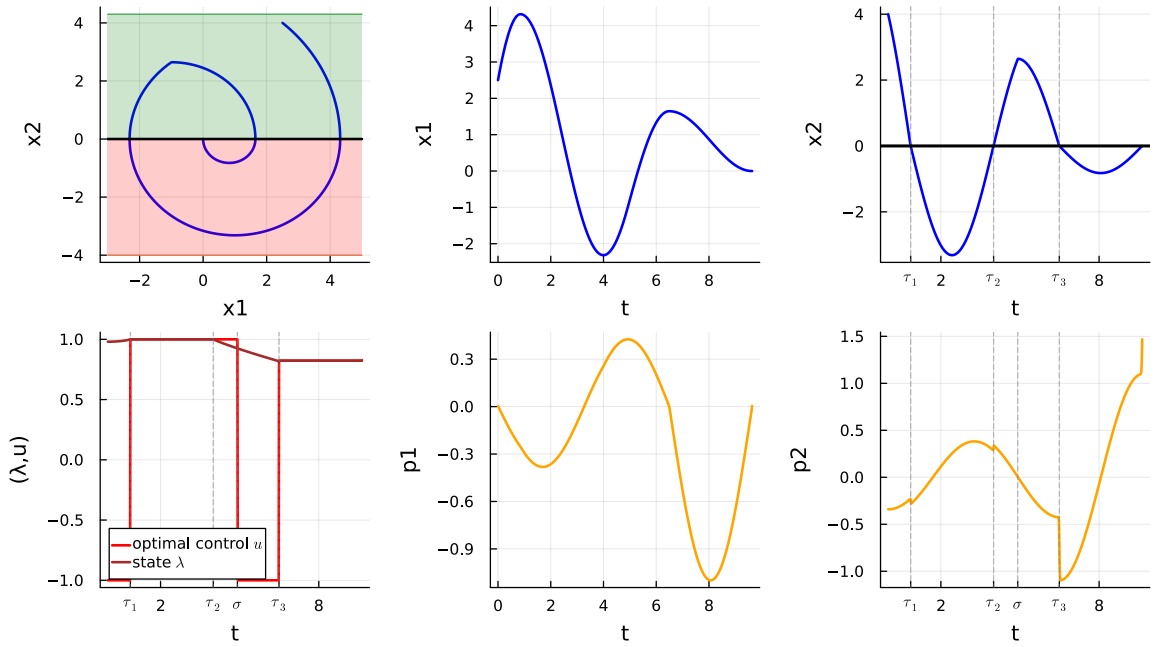


Figure 5.11: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (5.15).

constant value in the range of $(-1, 1)$. Now, we give the shooting function for Problem (5.15).

$$\mathcal{S}_3 : \mathbb{R}^9 \rightarrow \mathbb{R}^{10}$$

$$(p_1^0, p_2^0, \tau_1, \tau_2, \tau_3, \sigma, a_1, a_2, u_2) \mapsto \begin{pmatrix} x_1(T) \\ x_2(T) \\ x_2(\tau_1) \\ x_2(\tau_2) \\ x_2(\tau_3) \\ p_y(T) \\ a_1 - \dfrac{2p_1^-(\tau_1)}{1 - x_1(\tau_1)} \\ a_2 - \dfrac{p_1^-(\tau_3)(u_2 + 1)}{u_2 - x_1(\tau_3)} \\ p_2(\sigma) \\ p_1(0)x_2(0) - p_2(0)(1 + x_1(0)) - 1 \end{pmatrix}$$

The numerical results presented in Figure 5.12 depict an optimal pair (x^*, u^*) of Problem (5.15) together with an adjoint vector p. Contrary to the direct method, we observe jumps in the adjoint vector p_2 at τ_1^* and τ_3^*. There are no discontinuity jumps at τ_2^* because $(u^*)^+(\tau_2^*) = u_1^*$. Finally, note in the direct method we notice that constancy is satisfied (see Figure 5.13).

Remark 5.4.2. As discussed in Remark 5.4.1, this example shows that an extremal value of the control (namely, $u_1^* = 1$) is possible whenever the trajectory visits the loss control region. This value was expected from the numerical solution obtained via the direct method. Observe that it requires the verification of the averaged Hamiltonian gradient condition (here as an inequality). However, whenever the trajectory

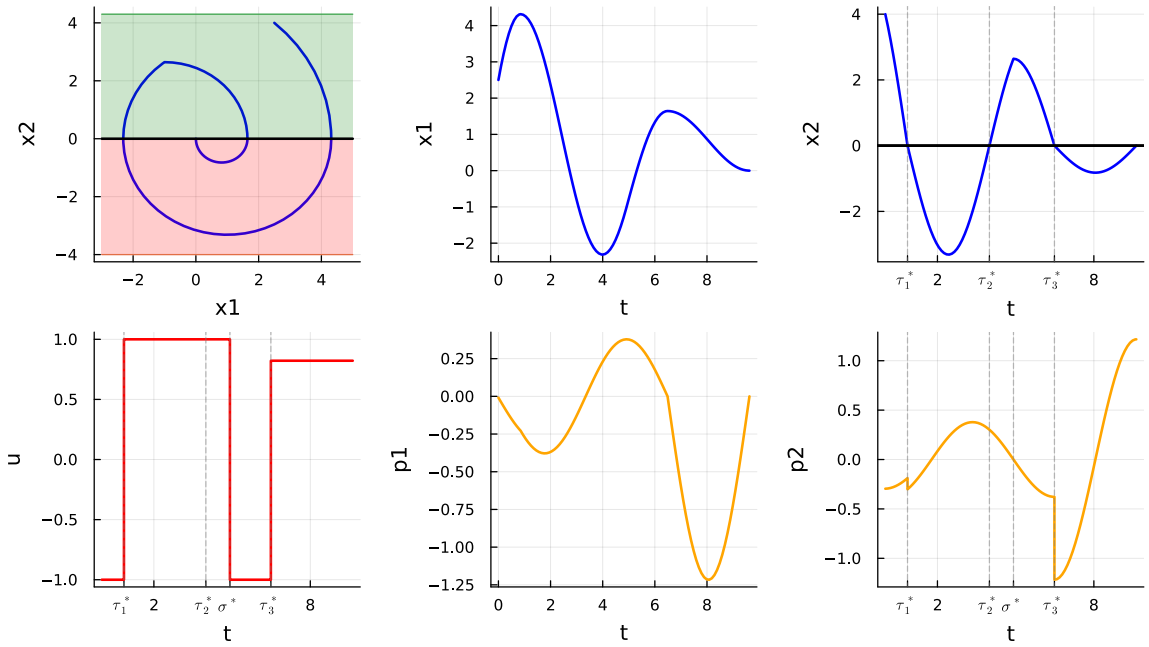


Figure 5.12: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (5.15).

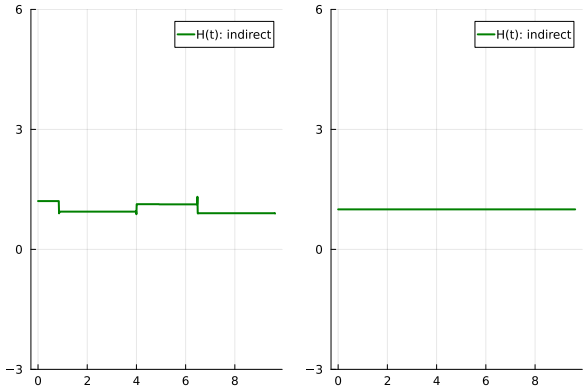


Figure 5.13: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (5.15).

visits the loss control region for the second time, we obtain that $u_2^* \approx 0.89 \in (-1, 1)$. Thus, finding the accurate value of u_2^* requires the use of an additional equation in the shooting function (see Section 5.4.1).

5.5 Proof of Proposition 5.2.1

Consider the framework of Proposition 5.2.1 and let us prove that the triplet $(y^*, v^*, \mathbb{T}^*)$ is a $\mathrm{L}^1_{[\varepsilon,1-\varepsilon]}$–local solution to Problem (5.8) for any $0 < \varepsilon < \frac{1}{2}$. Therefore let us fix some $0 < \varepsilon < \frac{1}{2}$ and some $R \geq \|v^*\|_{\mathrm{L}^\infty}$. Our aim is to prove that there exists $\eta > 0$ such that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ for any triplet $(y, v, \mathbb{T})$ that is admissible for Problem (5.8) and satisfying

$$\begin{cases} \|y-y^*\|_{\mathrm{C}} + \|v-v^*\|_{\mathrm{L}^1} + \|\mathbb{T}-\mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\ \|v\|_{\mathrm{L}^\infty} \leq R, \\ v(s) = v^*(s) \text{ a.e. } s \in [0,\varepsilon] \cup [1-\varepsilon, 1]. \end{cases} \tag{5.16}$$

To this aim we need to introduce several technical positive parameters:

($\mathfrak{P}_1$) First we introduce $\underline{\theta} := \min_{k\in\{1,\dots,N\}} |\tau_k^* - \tau_{k-1}^*| > 0$ and $\overline{\theta} := \max_{k\in\{1,\dots,N\}} |\tau_k^* - \tau_{k-1}^*| > 0$.

($\mathfrak{P}_2$) From the transverse conditions (see Definition 5.2.2) and the continuities of the functions ∇F_k^* and f_k^* (and thus the uniform continuities of the functions $\langle \nabla F_k^*, f_k^* \rangle_{\mathbb{R}^n}$ on compact sets), there exist $0 < \nu \leq \min_{k\in\{1,\dots,N-1\}} \nu_k^*$ and $0 < \alpha \leq \min\{\frac{\theta}{3}, \min_{k\in\{1,\dots,N-1\}} \alpha_k^*\}$ such that

$$\begin{cases} \langle \nabla F_k^*(z), f_k^*(z, \lambda^-, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in [\tau_k^* - \alpha, \tau_k^*), \\ \langle \nabla F_k^*(z), f_{k+1}^*(z, \lambda^+, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in (\tau_k^*, \tau_k^* + \alpha], \end{cases}$$

for all $z \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$, all $\lambda^- \in \overline{\mathrm{B}}_{\mathbb{R}^d}(\lambda_k^*, \nu)$, all $\lambda^+ \in \overline{\mathrm{B}}_{\mathbb{R}^d}(\lambda_{k+1}^*, \nu)$ and all $k \in \{1, \dots, N-1\}$.

($\mathfrak{P}_3$) From continuity of y^* over $[0,1]$, there exists $0 < \chi < \frac{1}{2}$ such that $\|y_k^*(s) - y_k^*(0)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for all $s \in [0, \chi]$ and $\|y_k^*(s) - y_k^*(1)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for all $s \in [1-\chi, 1]$, for all $k \in \{1, \dots, N\}$.

($\mathfrak{P}_4$) Define $\gamma := \frac{\theta}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\overline{\theta}}\} > 0$ and $r := \frac{\gamma}{\overline{\theta}+\underline{\theta}} > 0$. Note that $0 < \gamma \leq \alpha \leq \frac{\theta}{3}$ and $0 < r < \frac{1}{2}$.

($\mathfrak{P}_5$) From continuity of y^*, from (5.7) and the openness of the regions E_k^*, there exists $\delta > 0$ such that

$$\begin{cases} \overline{\mathrm{B}}_{\mathbb{R}^n}(y_1^*(s), \delta) \subset E_1^*, & \forall s \in [0, 1-r], \\ \overline{\mathrm{B}}_{\mathbb{R}^n}(y_k^*(s), \delta) \subset E_k^*, & \forall s \in [r, 1-r], \quad \forall k \in \{2, \dots, N-1\}, \\ \overline{\mathrm{B}}_{\mathbb{R}^n}(y_N^*(s), \delta) \subset E_N^*, & \forall s \in [r, 1]. \end{cases}$$

We are now in a position to continue the proof. To this aim let us fix $\eta := \min\{\frac{\theta}{3}, \frac{\nu}{2}, \delta\} > 0$ and let $(y, v, \mathbb{T})$ be an admissible triplet for Problem (5.8) satisfying (5.16). Our aim is to prove that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$.

Step 1. Since $0 = \tau_0^* < \tau_1^* < \cdots < \tau_{N-1}^* < \tau_N^* = T$ and $\mathbb{T} \in \Delta$ with $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, one can easily deduce that $0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = T$. Therefore we are in a position to define $(x, \lambda, u) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ by

$$x(t) := y_k\left(\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right), \quad \lambda(t) = \lambda_k := y_{N+k}\left(\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right) \quad \text{and} \quad u(t) := v_k\left(\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right), \tag{5.17}$$

for all $t \in [\tau_{k-1}, \tau_k]$ and all $k \in \{1, \dots, N\}$. Note that x is well defined since $y_{k+1}(0) = y_k(1)$ for all $k \in \{1, \dots, N-1\}$ and λ is a piecewise constant function with respect to $\mathbb{T}$ (from admissibility of the triplet $(y, v, \mathbb{T})$). Observe that $(y_1(0), y_N(1)) = (x(0), x(T))$ and recall that $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$. Therefore, from the definition of ϕ^* (see Section 5.2.3) and since (x^*, λ^*, u^*) is a global solution to Problem (5.1), to obtain that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$, we only need to prove that the pair (x, λ, u) is admissible for Problem (5.1).

From admissibility of the triplet $(y, v, \mathbb{T})$, it is clear that $g(x(0), x(T)) \in \mathrm{S}$ and $(\lambda(t), u(t)) \in \Lambda \times \mathrm{U}$ for almost every $t \in [0, T]$. Therefore it only remains to prove that (x, λ, u) is a solution to the hybrid control system with regionally switching parameter (5.2) (see Definition 5.2.1). From (5.17) and the

admissibility of the triplet $(y, v, \mathbb{T})$, one can easily obtain that

$$\dot{x}(t) = f_k^*(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \tag{5.18}$$

for all $k \in \{1, \dots, N\}$. Therefore, to conclude the proof, we only need to prove that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k^*, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in (\tau_{N-1}, \tau_N]. \end{cases}$$

This is exactly our goal in the next two steps.

Step 2. Since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, note that $\tau_k - \tau_{k-1} \leq \overline{\theta} + 2\eta \leq \overline{\theta} + \underline{\theta}$ for all $k \in \{1, \dots, N\}$. Hence, since moreover $r := \frac{\gamma}{\overline{\theta}+\underline{\theta}}$, observe that

$$\begin{cases} \dfrac{t - \tau_0}{\tau_1 - \tau_0} \in [0, 1-r], & \forall t \in [\tau_0, \tau_1 - \gamma], \\ \dfrac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [r, 1-r], & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ \dfrac{t - \tau_{N-1}}{\tau_N - \tau_{N-1}} \in [r, 1], & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

As a consequence, from (5.17) and $(\mathfrak{P}_5)$, and since $\|y_k - y_k^*\|_{\mathrm{C}} \leq \|y - y^*\|_{\mathrm{C}} \leq \eta \leq \delta$, one can easily obtain that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1 - \gamma], \\ x(t) \in E_k^*, & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

Therefore, to conclude the proof, it only remains to prove that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k)$ and $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$, for all $k \in \{1, \dots, N-1\}$. This is the objective of the following step.

Step 3. Let us start with two observations. First, since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, it holds that $|\tau_k - \tau_{k-1}| \geq \frac{\theta}{3}$ for all $k \in \{1, \dots, N\}$. Second, since $\gamma := \frac{\theta}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\theta}\}$, one can get that

$$\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1-\varepsilon, 1], \quad \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1-\chi, 1] \quad \text{and} \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

for all $t \in [\tau_k - \gamma, \tau_k]$ and all $k \in \{1, \dots, N-1\}$. We deduce the following results:

(i) Since $v_k(s) = v_k^*(s)$ for almost every $s \in [1-\varepsilon, 1]$, one can easily obtain from (5.17) and (5.5) that

$$u(t) = u^*\left(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}\right) \quad \text{with} \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

(ii) Since $\|y_k - y_k^*\|_{\mathrm{C}} \leq \|y - y^*\|_{\mathrm{C}} \leq \eta \leq \frac{\nu}{2} \leq \nu$ for all $k \in \{1, \dots, 2N\}$, one can easily obtain from (5.17), from the equality $x^*(\tau_k^*) = y_k^*(1)$ and from $(\mathfrak{P}_3)$ that $x(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and

$\lambda(t) = \lambda_k \in \overline{\mathrm{B}}_{\mathbb{R}^d}(\lambda_k^*, \nu)$ for all $t \in [\tau_k - \gamma, \tau_k]$ and all $k \in \{1, \dots, N-1\}$.

(iii) We obtain from (5.18), from the previous two items and $(\mathfrak{P}_2)$ that the derivative of $F_k^* \circ x$ satisfies

$$\left\langle \nabla F_k^*(x(t)), f_k^*\left(x(t), \lambda_k, u^*\left(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)\frac{t-\tau_{k-1}}{\tau_k - \tau_{k-1}}\right)\right)\right\rangle_{\mathbb{R}^n} > 0,$$

for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$. From admissibility of the triplet $(y, v, \mathbb{T})$ and (5.17), we know that $F_k^*(x(\tau_k)) = F_k^*(y_k(1)) = 0$ for all $k \in \{1, \dots, N-1\}$. As a consequence we obtain that $F_k^*(x(t)) < 0$ for all $t \in [\tau_k - \gamma, \tau_k)$ which implies from Definition 5.2.2, since $x(t) \in \overline{\mathrm{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and $\nu \leq \nu_k^*$, that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

Following the same strategy one can obtain that $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$ and all $k \in \{1, \dots, N-1\}$. The proof of Proposition 5.2.1 is complete.

5.6 Proof of Theorem 5.2.1

Let $(x^*, \lambda^*, u^*) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{PC}([0,T], \mathbb{R}^d) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m)$ be a global solution to Problem (5.1), that is moreover a regular solution to (5.2), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, such that g is submersive at $(x^*(0), x^*(T))$. From Proposition 5.2.1, the corresponding triplet $(y^*, v^*, \mathbb{T}^*)$ constructed in Section 5.2.3 is a $\mathrm{L}^1_\square$–local solution to Problem (5.8). Before applying [16, Theorem 2.1], we need to verify that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$. From the definition of the function g^* (see Section 5.2.3), note that the matrix $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \in \mathbb{R}^{((n+d)N+(n+d)N+(N+1))\times \ell^*}$ is given by

$$\left[\begin{array}{c|c|c|c|c}
\nabla_1 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n\times n(N-1)}} & 0_{\mathbb{R}^{n\times(N-1)}} & 0_{\mathbb{R}^{n\times dN}} & 0_{\mathbb{R}^{n\times(N+1)}} \\
\hline
0_{\mathbb{R}^{n(N-1)\times \ell}} & \mathrm{Id}_{\mathbb{R}^{n(N-1)\times n(N-1)}} & 0_{\mathbb{R}^{n(N-1)\times(N-1)}} & 0_{\mathbb{R}^{n(N-1)\times dN}} & 0_{\mathbb{R}^{n(N-1)\times(N+1)}} \\
\hline
0_{\mathbb{R}^{dN\times \ell}} & 0_{\mathbb{R}^{dN\times n(N-1)}} & 0_{\mathbb{R}^{dN\times(N-1)}} & 0_{\mathbb{R}^{dN\times dN}} & 0_{\mathbb{R}^{dN\times(N+1)}} \\
\hline
0_{\mathbb{R}^{n(N-1)\times \ell}} & -\mathrm{Id}_{\mathbb{R}^{n(N-1)\times n(N-1)}} & \begin{array}{c}\nabla F_1^*(y_1^*(1)) \\ \ddots \\ \nabla F_{N-1}^*(y_{N-1}^*(1))\end{array} & 0_{\mathbb{R}^{n(N-1)\times dN}} & 0_{\mathbb{R}^{n(N-1)\times(N+1)}} \\
\hline
\nabla_2 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n\times n(N-1)}} & 0_{\mathbb{R}^{n\times(N-1)}} & 0_{\mathbb{R}^{n\times dN}} & 0_{\mathbb{R}^{n\times(N+1)}} \\
\hline
0_{\mathbb{R}^{dN\times \ell}} & 0_{\mathbb{R}^{dN\times n(N-1)}} & 0_{\mathbb{R}^{dN\times(N-1)}} & \mathrm{Id}_{\mathbb{R}^{dN\times dN}} & 0_{\mathbb{R}^{dN\times(N+1)}} \\
\hline
0_{\mathbb{R}^{(N+1)\times \ell}} & 0_{\mathbb{R}^{(N+1)\times n(N-1)}} & 0_{\mathbb{R}^{(N+1)\times(N-1)}} & 0_{\mathbb{R}^{(N+1)\times dN}} & \mathrm{Id}_{\mathbb{R}^{(N+1)\times(N+1)}}
\end{array}\right]$$

From Definition 5.2.2, it holds that $\nabla F_k^*(y_k^*(1)) = \nabla F_k^*(x^*(\tau_k^*)) \neq 0_{\mathbb{R}^n}$ for all $k \in \{1, \dots, N-1\}$. Since g is submersive at $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$, one can easily conclude that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$.

Application of [16, Theorem 2.1]. Let us introduce the Hamiltonian $\widetilde{H} : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{(n+d)N} \to \mathbb{R}$ associated with Problem (5.8) given by

$$\widetilde{H}(y, v, \mathbb{T}, q) := \langle q, f^*(y, v, \mathbb{T})\rangle_{\mathbb{R}^{(n+d)N}} = \sum_{k=1}^{N} \langle q_k, (\tau_k - \tau_{k-1}) f_k^*(y_k, y_{N+k}, v_k)\rangle_{\mathbb{R}^n},$$

for all $y = (y_1, \dots, y_N, y_{N+1}, \dots, y_{2N}) \in \mathbb{R}^{(n+d)N}$, $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$, $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$ and $q = (q_1, \dots, q_N, q_{N+1}, \dots, q_{2N}) \in \mathbb{R}^{(n+d)N}$. From [16, Theorem 2.1], there exists a nontrivial pair $(q, q^0) \in \mathrm{AC}([0,1], \mathbb{R}^{(n+d)N}) \times \mathbb{R}_+$ satisfying:

(i) the Hamiltonian system

$$\dot{y}^*(s) = \nabla_q \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) \quad \text{and} \quad -\dot{q}(s) = \nabla_y \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)),$$

for almost every $s \in [0,1]$;

(ii) the transversality condition

$$\begin{pmatrix} q(0) \\ -q(1) \\ \displaystyle\int_0^1 \nabla_{\mathbb{T}} \widetilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s))\, ds \end{pmatrix} = q^0 \nabla \phi^*(y^*(0), y^*(1)) + \nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)\tilde{\xi},$$

for some $\tilde{\xi} \in \mathrm{N}_{\mathrm{S}^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$;

(iii) the Hamiltonian maximization condition

$$v^*(s) \in \operatorname*{arg\,max}_{\widetilde{\omega} \in \mathrm{U}^N} \widetilde{H}(y^*(s), \widetilde{\omega}, \mathbb{T}^*, q(s)),$$

for almost every $s \in [0,1]$.

Introduction of the nontrivial pair (p, p^0). Since the pair (q, q^0) is not trivial, it is clear that the pair $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^*}([0,T], \mathbb{R}^n) \times \mathbb{R}_+$ defined by $p^0 := q^0$ and

$$p(t) := \begin{cases} q_1\left(\frac{t-\tau_0^*}{\tau_1^*-\tau_0^*}\right), & \forall t \in [\tau_0^*, \tau_1^*), \\ q_k\left(\frac{t-\tau_{k-1}^*}{\tau_k^*-\tau_{k-1}^*}\right), & \forall t \in (\tau_{k-1}^*, \tau_k^*), \quad \forall k \in \{2, \dots, N-1\}, \\ q_N\left(\frac{t-\tau_{N-1}^*}{\tau_N^*-\tau_{N-1}^*}\right), & \forall t \in (\tau_{N-1}^*, \tau_N^*], \end{cases}$$

is not trivial.

Hamiltonian system and Hamiltonian maximization condition of Theorem 5.2.1. From the above Items (i) and (iii) and from (5.6), one can easily obtain that (p, p^0) satisfies the Hamiltonian system and the Hamiltonian maximization condition of Theorem 5.2.1.

Transversality condition of Theorem 5.2.1. From the definitions of g^* and S^* (see Section 5.2.3) and since $\tilde{\xi} \in \mathrm{N}_{\mathrm{S}^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$, we can write $\tilde{\xi} := (\xi, \xi^2, \xi^3, \xi^4, \xi^5) \in \mathbb{R}^\ell \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^{N-1} \times \mathbb{R}^{dN} \times \mathbb{R}^{N+1}$ with

$$\xi \in \mathrm{N}_{\mathrm{S}}[g(y_1^*(0), y_N^*(1))] \quad \text{and} \quad \xi^4 \in \mathrm{N}_{\Lambda^N}[(y_{N+1}^*(1), \dots, y_{2N}^*(1))] \quad \text{and} \quad \xi^5 \in \mathrm{N}_{\Delta}[\mathbb{T}^*].$$

Since $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$ and $y_{N+k}^*(1) = \lambda_k^*$, note that $\xi \in \mathrm{N}_{\mathrm{S}}[g(x^*(0), x^*(T))]$ and $\xi_k^4 \in \mathrm{N}_{\Lambda}[\lambda_k^*]$ for all $k \in \{1, \dots, N\}$. Furthermore, from the first two components of the above Item (ii), from

the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 5.6 and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 5.2.3 for the definition of ϕ^*), we obtain that

$$\begin{aligned} p(0) = q_1(0) = q^0 \nabla_1 \phi(y_1^*(0), y_N^*(1)) + \nabla_1 g(y_1^*(0), y_N^*(1))\xi \\ = p^0 \nabla_1 \phi(x^*(0), x^*(T)) + \nabla_1 g(x^*(0), x^*(T))\xi, \end{aligned}$$

and

$$\begin{aligned} -p(T) = -q_N(1) = q^0 \nabla_2 \phi(y_1^*(0), y_N^*(1)) + \nabla_2 g(y_1^*(0), y_N^*(1))\xi \\ = p^0 \nabla_2 \phi(x^*(0), x^*(T)) + \nabla_2 g(x^*(0), x^*(T))\xi. \end{aligned}$$

Therefore the transversality condition of Theorem 5.2.1 is proved.

Discontinuity condition of Theorem 5.2.1. From the first two components of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 5.6 and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 5.2.3 for the definition of ϕ^*), we obtain that

$$\forall k \in \{2, \dots, N\}, \quad q_k(0) = \xi_{k-1}^2 \qquad \text{and} \qquad \forall k \in \{1, \dots, N-1\}, \quad -q_k(1) = -\xi_k^2 + \xi_k^3 \nabla F_k^*(y_k^*(1)).$$

We deduce that

$$p^+(\tau_k^*) - p^-(\tau_k^*) = q_{k+1}(0) - q_k(1) = \xi_k^3 \nabla F_k^*(y_k^*(1)) = \xi_k^3 \nabla F_k^*(x^*(\tau_k^*)), \tag{5.19}$$

for all $k \in \{1, \dots, N-1\}$. Therefore the discontinuity condition of Theorem 5.2.1 is satisfied with $\sigma_k := \xi_k^3$ for all $k \in \{1, \dots, N-1\}$.

Averaged Hamiltonian gradient condition of Theorem 5.2.1. From the first two components of Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 5.6 and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 5.2.3 for the definition of ϕ^*), we obtain that

$$\forall k \in \{1, \dots, N\}, \quad q_{N+k}(0) = 0_{\mathbb{R}^d} \qquad \text{and} \qquad \forall k \in \{1, \dots, N\}, \quad -q_{N+k}(1) = \xi_k^4.$$

From the Hamiltonian system, we deduce that

$$\int_0^1 \nabla_{y_{N+k}} \tilde{H}(y^*(s), v^*(s), \mathbb{T}, q(s))\, ds = \xi_k^4 \in \mathrm{N}_\Lambda[y_{N+k}^*(1)].$$

for all $k \in \{1, \dots, N\}$. From affine changes of time variable, we obtain that

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_\lambda \mathcal{H}(x^*(t), \lambda_k^*, u^*(t), p(t))\, dt \in \mathrm{N}_\Lambda[\lambda_k^*],$$

for all $k \in \{1, \dots, N\}$.

Hamiltonian constancy condition of Theorem 5.2.1. From the Hamiltonian system and the maximization condition, and applying [63, Theorem 2.6.1] on each interval $[\tau_{k-1}^*, \tau_k^*]$, we obtain that,

for all $k \in \{1, \dots, N\}$, there exists a constant $c_k \in \mathbb{R}$ such that

$$\langle p(t), f_k^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} = c_k,$$

for almost every $t \in [\tau_{k-1}^*, \tau_k^*]$. Furthermore, from the definition of Δ (see Section 5.2.3) and since $0 = \tau_0^* < \tau_1^* < \ldots < \tau_{N-1}^* < \tau_N^* = T$, we deduce from $\xi^5 \in \mathrm{N}_\Delta[\mathbb{T}^*]$ that all components of ξ^5 are zero, except possibly the first and last components. Thus, from the third component of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Section 5.6 and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 5.2.3 for the definition of ϕ^*), we obtain that

$$\int_0^1 \langle q_{k+1}(s), f_{k+1}^*(y_{k+1}^*(s), y_{N+k+1}^*(s), v_{k+1}^*(s)) \rangle_{\mathbb{R}^n} \, ds = \int_0^1 \langle q_k(s), f_k^*(y_k^*(s), y_{N+k}^*(s), v_k^*(s)) \rangle_{\mathbb{R}^n} \, ds,$$

for all $k \in \{1, \dots, N-1\}$. From affine changes of time variable, we obtain that

$$\frac{1}{\tau_{k+1}^* - \tau_k^*} \int_{\tau_k^*}^{\tau_{k+1}^*} \langle p(t), f_{k+1}^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} \, dt = \frac{1}{\tau_k^* - \tau_{k-1}^*} \int_{\tau_{k-1}^*}^{\tau_k^*} \langle p(t), f_k^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} \, dt,$$

for all $k \in \{1, \dots, N-1\}$. From constancy of the above two integrands, we deduce that $c_{k+1} = c_k$ for all $k \in \{1, \dots, N-1\}$. Therefore the Hamiltonian constancy condition is satisfied and the proof of Theorem 5.2.1 is complete.

6

General conclusion

In this general conclusion, we first provide a review of the findings from our research carried out in this manuscript. Afterwards, we present several additional personal research projects to be pursued within both frameworks: spatially hybrid optimal control problems and optimal control problems with loss control regions.

Part I: findings from our research

The findings presented in this manuscript provide the first-order necessary optimality conditions for spatially hybrid optimal control problems in the form of a Pontryagin maximum principle (in short, PMP). Additionally, it provides the first-order necessary optimality conditions for optimal control problems with loss control regions, also in a PMP form. This work was initially motivated by the paper [73], which provides the statement of a spatially hybrid optimal control problem. In this manuscript, our aim was to address this family of problems that has not been extensively studied in the literature. By delving deeply into this setting, we were able to identify crucial issues. In the following items, we summarize our findings as follows:

(i) In the initial phase of our research, we focused on a classification of various situations and problem types that have been examined in the literature. This has lead us to separate this broad framework of hybrid systems with heterogeneous dynamics into two frameworks, namely, the temporally hybrid setting and the spatially hybrid setting. As a result, we concluded that establishing a *unified* framework (capable of accommodating all types of hybrid problems) is probably not feasible (for instance, both setting rely on different notions local solutions after applying the augmentation procedure).

(ii) Secondly, within a spatially hybrid setting, we have identified two crucial challenges that were not mentioned previously in the literature (to our best knowledge). The first challenge is related to the nonadmissibility of needle-like perturbations, while the second challenge occurs while adapting the

augmentation technique in a spatially hybrid context.

(iii) Thirdly, we introduced new tools to address these two challenges. The first tool, known as *auxiliary controls*, allows us to overcome the nonadmissibility of needle-like perturbations. The second tool is the novel notion of $\mathrm{L}^1_{\square}$–local solution, which lead to a correct adaptation of the augmentation technique and the resolution of the second challenge.

(iv) Fourthly, since the introduced tools allows us to overcome the main issues, we proceed to derive a spatially hybrid maximum principle (HMP, in short). Moreover, we considered a new type of parameter in the spatially hybrid context, which we referred to as a *regionally switching parameter*. This led us to obtain the *averaged Hamiltonian gradient* condition (which already appeared in [28] when considering a constant parameter or in the framework of sampled-data controls in [34], [35], [41]) in a spatially hybrid setting. As a result, we established first-order necessary optimality condition for optimal control problems with loss control regions, which was the motivation behind this work. We refer to this as a PMP with loss control regions.

(v) Finally, we introduced a numerical approach based on a two-step method. The first step involved a direct method aimed at determining the structure of the optimal solution, specifically the sequence of visited regions. In the second step, we constructed an indirect method, which relied on the PMP with loss control regions. The direct method is based on a slight modification of the initial non-smooth optimal control problem. Precisely, we proceed by performing a regularization of the initial problem. Its goal was to provide a good initialization for the indirect method. This latter was based on a shooting function that incorporated two new elements: the averaged Hamiltonian gradient condition (which enabled the determination of constant control values in loss control regions) and the discontinuity jumps of the adjoint vector. These jumps of the adjoint vector represented the characteristic features of the spatially hybrid HMP and (consequently) the PMP with loss control regions.

Part II: research perspectives

Hereafter, we provide a series of research perspectives and possible extensions to the frameworks considered in this manuscript.

(i) The consideration of *linear quadratic* optimal control problems has been widely studied in the literature, in various contexts. The importance of considering such problems lies in their practical applications. Take, for instance, *trajectory tracking* (see, for example, [90]). Therefore, one possible extension of our work is to tackle this type of problem in a spatially hybrid setting. Let us mention that in the setting of *switched systems*, which is a particular case of a temporally hybrid setting, such problems have been addressed [104], [119]. However, we believe that tackling linear quadratic problems in a spatially hybrid setting has never been done, and it would be of quite interest to generalize the *Riccati theory* in this setting. The interest behind such an extension is to be able to consider linear quadratic problems with loss control regions. As explained in the general introduction, such consideration comes naturally since maintaining permanent controls is not always possible.

(ii) In optimal control theory, we know that under certain assumptions, a relationship can be established between necessary optimality conditions, such as PMP, and sufficient conditions, such as dynamic programming. Roughly speaking, let us denote by $V(t,x)$ the *value function*, which represents the minimum cost of an optimal control problem. This value function is viewed as a function of the initial time and state (t,x). The relationship between PMP and dynamic programming is that the

gradient is equal to the negative value of the adjoint vector p provided by PMP [51]. In other words:

$$p(t) = -\nabla_x V(t, x^*(t)),$$

where x^* stands for the minimizing trajectory. In a temporally hybrid setting, a similar relation has been established in [98]. So, we could ask the following: in a spatially hybrid setting, does such a relation hold between the gradient of the value function and the adjoint vector?

(iii) Following above notations, it is well-known that (under certain assumptions) if the value function $V(t,x)$ is C^1 then it is a solution of the *Hamilton-Jacobi-Bellman* equation. Roughly speaking, V satisfies

$$\partial_t V(t,x) + \min_{\omega \in \mathrm{U}} H(x, \omega, \nabla_x V(t,x)) = 0, \ (t,x) \in (0, +\infty) \times \Omega,$$

where $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ stands for the Hamiltonian function, U is a nonempty set of $\mathbb{R}^m$ and Ω is a nonempty open set of $\mathbb{R}^n$. The question that remains is: is there an analogous statement when dealing with a spatially hybrid framework?

(iv) Throughout this manuscript, when considering a partition of the state space, we have (only) defined the *hybrid dynamics* regionally on open sets but not at the boundaries. This fact had no impact on the work presented in this manuscript, thanks to the transverse crossing assumptions made on the optimal trajectory. However, the question of considering optimal trajectories that stay on the boundary leads us to invoke the notion of *sliding modes*. In the literature, this question has been explored only to a limited extent and can be found in a few works such as [10], [83]. In view of this fact, it could be interesting revisiting such a framework (that allows trajectories to stay on boundaries) using the new tools that we have introduced such as: auxiliary controls and the notion of $\mathrm{L}^1_{\square}$–local solution.

(v) The new framework that we introduce in this manuscript, concerning loss control regions in optimal control problems, appears to be very interesting. Indeed, as we mentioned in the general introduction, the consideration of loss control regions arises naturally due to the limitations of finite resources, which cannot be sustained indefinitely. In this manuscript, we tackled (only) academic examples with loss control regions. However, it remains a good first step to understand the challenges that can arise from imposing such constraints (we refer to the example of the double integrator in [15]). In future works, integrating such constraints into the modeling can be explored in various domains, such as epidemiology (for instance, S.I.R. models) and viability theory (for example, time crisis problems [20]), among others.

(vi) Another perspective related to this thesis concerns optimal control problems arising in the field of bioprocesses. Several works considered for instance the problem of maximizing bio-gas production in a continuous-stirred bioreactor [12], [74]. It is also well-known that bioreactors can also be driven in fed-batch mode (i.e., with a variable volume until reaching the maximal capacity of the tank reactor), see, e.g., [21], [76]. This is of particular interest for wastewater treatment. Now, we can observe that the combination of these two modes (continuous and fed-batch) leads to a (temporally) hybrid control system involving a switching manifold. Therefore, it could be interesting to study optimal control problems arising in the field of bioprocesses governed by such a hybrid control system.

(vii) From a numerical standpoint, our aim is to further develop the two-step method for optimal control problems with loss control regions. More specifically, we plan to:

- Improve the regularization method for the direct method to obtain more accurate adjoint

vectors when dealing with more complex problems than the ones considered in Chapter 5.

- Numerically implement the averaged Hamiltonian condition (in loss control regions) for the indirect method, taking into account situations where a constant value of the control belongs to the boundary of the constraint set. This makes this condition complex to integrate into the shooting function.

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Contrôle optimal hybride: conditions d'optimalité et applications

Résumé : Ce manuscrit aborde le domaine mathématique de la théorie du contrôle optimal en se concentrant spécifiquement sur les problèmes de contrôle optimal hybrides spatiaux. Ici, le terme *spatial* indique que nous considérons un système de contrôle hybride défini sur une partition de l'espace d'état qui est divisée en régions disjointes. De plus, nous supposons que le système de contrôle dépend d'un *paramètre régional* qui reste constant à l'intérieur de chaque région, mais peut changer sa valeur lorsque la position de l'état traverse les frontières. Ce nouveau cadre nous permet de traiter des systèmes de contrôle qui incluent des *régions de perte contrôle*, ce qui constitue notre motivation initiale. Dans ce type de système, étant donné une partition de l'espace d'état, le comportement du contrôle varie en fonction de la position de l'état. Il peut être modifié à tout moment (appelé contrôles *permanents*) lorsque l'état appartient à des régions appelées *régions de contrôle*, ou il peut rester constant lorsque l'état appartient à des régions appelées *régions de perte contrôle*.

Dans les deux cadres, nos objectifs sont les suivants:

(i) dériver un principe maximum hybride spatial (abrégé en HMP) avec un paramètre régional.
(ii) dériver un principe maximum de Pontryagin avec des régions de de perte de contrôle.
(iii) fournir une approche numérique permettant de résoudre des problèmes de contrôle optimal avec des régions de perte de contrôle.

Ce manuscrit est composé de 6 chapitres:

(1) Le chapitre 1 est consacré aux notations et au cadre fonctionel nécessaires pour décrire le problème de contrôle optimal hybride avec paramètre régional et les problèmes de contrôle optimal avec des régions de perte de contrôle rencontrés dans le manuscrit.
(2) Le chapitre 2 est consacré à la dérivation d'un HMP spatial avec un paramètre régional, pour les problèmes de contrôle optimal hybride avec des conditions initiales fixes. Nous fournissons également les conditions nécessaires du premier ordre pour les problèmes de contrôle optimal avec des régions de perte de contrôle.
(3) Le chapitre 3 est consacré à l'étude d'une variante du problème de temps minimal pour le double intégrateur avec une région de perte de contrôle. Cette dernière est basée sur un PMP adapté aux problèmes de temps minimal avec une région de perte de contrôle.
(4) Le chapitre 4 se concentre sur la dérivation d'un HMP spatial basé sur une utilisation soigneuse de la technique d'augmentation. Cette dernière est basée sur l'introduction d'un nouveau concept de minimum local, puis sur la dérivation d'un PMP correspondant.
(5) Le chapitre 5 est consacré à la dérivation d'un PMP avec des régions de perte de contrôle, ainsi qu'à une approche numérique en deux étapes pour résoudre ce type de problèmes. Pour ce faire, nous commençons par fournir un HMP spatial avec un paramètre régional (de manière similaire au chapitre 4).
(6) Dans le chapitre 6, nous présentons une conclusion générale qui expose brièvement nos résultats de recherche et offre également des perspectives pour de futurs travaux.

Mots clés : contrôle optimal, conditions nécessaires d'optimalité, principe maximum de Pontryagin, conditions de transversalité, analyse de sensibilité, perturbation en aiguille, paramètre régional, régions de perte de contrôle, systèmes hybrides, dynamiques hétérogènes, principe maximum hybride, technique d'augmentation, minimum local, méthode directe, méthode indirecte, méthode de tir.

Hybrid optimal control: optimality conditions and applications

Abstract: This manuscript deals with the mathematical field of optimal control theory, specifically focusing on spatially hybrid optimal control problems. Here, the term *spatially* indicates that we consider a hybrid control system defined over a partition of the state space that is divided into disjoint regions. Furthermore, we assume that the control system depends on a *regionally switching parameter*, which remains constant within each region but can change its value when the state position crosses boundaries. This new framework allows us to address control systems that includes *loss control regions*, which presents our initial motivation. In such systems, given a partition of the state space, the control behavior varies depending on the position of the state. It can be modified at any time (referred to as *permanent* controls) when the state belongs to regions referred to as *control regions*, or it can remain constant when the state belongs to regions referred to as *loss control regions*.

In both frameworks, our goals are:

(i) to derive a spatially hybrid maximum principle (in short, HMP) with regionally switching parameter.
(ii) to derive a Pontryagin maximum principle with loss control regions.
(iii) provide a numerical approach allowing to solve optimal control problems with loss control regions.

To achieve these purposes, we introduce new tools and concepts that address certain challenges that can arise in a spatially hybrid setting. Specifically, based on careful investigation, we identify two main challenges: the nonadmissibility of needle-like perturbations and the inability to directly apply the well-known augmentation technique in a spatially hybrid setting.

This manuscript is made up of 6 chapters:

(1) Chapter 1 is devoted to notations and the basic framework needed to describe the hybrid optimal control problem with regionally switching parameters and optimal control problems with loss control regions encountered in the manuscript.
(2) Chapter 2 is devoted to deriving a spatially HMP with regionally switching parameters for hybrid optimal control problems with fixed initial conditions. We also provide first-order necessary conditions for optimal control problems with loss control regions.
(3) Chapter 3 is devoted to the study of a variant of the minimum time problem for the double integrator with a loss control region. The latter is based on a PMP adapted to a minimum time problem with a loss control region.
(4) Chapter 4 focuses on the derivation of a spatially HMP based on a careful use of the augmentation technique. The latter is based on the introduction of a new concept of local minimum and the subsequent derivation of a corresponding PMP.
(5) Chapter 5 is devoted to the derivation of a PMP with loss control regions, as well as a two-step numerical approach to solve this type of problem. To do so, we first provide a HMP with regionally switching parameters in a similar manner to Chapter 4.
(6) In Chapter 6, we provide a general conclusion that briefly presents our research findings and provides some perspectives for future research work.

Keywords: optimal control, necessary optimality conditions, Pontryagin maximum principle, transversality conditions, sensitivity analysis, needle-like perturbation, regionally switching parameter, loss control regions, hybrid systems, heterogeneous dynamics, hybrid maximum principle, augmentation technique, local minimum, direct method, indirect method, shooting method.

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