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Filtering and stastistical learning for financial risk management

Tom Picard

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THÈSE

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Filtering and statistical learning for financial risk management

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Résumé

Cette thèse s'intéresse à la gestion du risque financier, étudié à travers le prisme de deux environnements que sont le crédit et l'investissement.

La première thématique abordée concerne la gestion du risque lié aux migrations des notations de crédit. Un rating ou une notation de crédit évalue la capacité d'une entité à respecter ses engagements envers ses créanciers. Ces notations sont constamment réévaluées et peuvent évoluer au cours du temps, notamment en fonction du cycle économique. Dans un premier temps, nous nous intéressons au rôle des notations de crédit dans la réglementation et aux risques que leurs migrations génèrent pour les finances des banques. À travers un état de l'art des méthodes existantes d'estimation Point-In-Time (PIT) des migrations de ratings et l'introduction d'une nouvelle méthode, cette partie cherche dans un second temps, à apporter un éclairage sur le lien qui existe entre la dynamique des migrations de rating et le risque systématique, étroitement lié à la santé de l'économie. En particulier, l'approche proposée considère que la dynamique des migrations de rating est régie par une chaîne de Markov latente non observée, représentant les états de l'économie. Cette approche permet de modéliser l'effet du cycle économique sans utiliser de variables macro-économiques et donc d'éviter toutes confusions ou ambiguïtés dans le choix de ces variables. S'appuyant sur la théorie de filtrage de processus ponctuels, nous expliquons comment l'état actuel du facteur caché peut être efficacement déduit à partir des observations historiques de notation. Nous adaptons l'algorithme classique de Baum-Welch à notre contexte afin d'estimer les paramètres du facteur latent. Une fois calibrée, notre méthode permet de détecter en temps réel, les changements économiques affectant la dynamique des migrations de notation. Deux versions sont proposées : une version discrète et une version continue. Nous démontrons et comparons l'efficacité des deux approches sur des données fictives et sur une base de données de notation de crédit d'entreprises.

La deuxième partie de cette thèse s'intéresse à la gestion du compromis entre rendement attendu et risque estimé dans le processus d'investissement. En particulier, nous étudions les moyens d'allouer la richesse au sein d'un portefeuille composé d'un panier d'actifs risqués. Nous décrivons les méthodes statiques et dynamiques et tentons de comprendre les motivations qui poussent les investisseurs à choisir une approche plutôt qu'une autre. L'approche mean-variance de Markowitz est le classique paradigme en allocation de portefeuille. Elle offre une stratégie de répartition de richesse pour laquelle le rendement attendu est maximal pour un niveau de risque donné. Lorsque l'objectif de l'investisseur est à un horizon de temps lointain, la stratégie classique mean-variance uni-période, ne peut pas se projeter au-delà de la prochaine période et ne peut pas rivaliser avec la stratégie optimale dynamique obtenue à partir de la version multi-période du problème. Les méthodes existantes, basées sur la programmation dynamique, ne sont pas totalement adaptées pour résoudre le problème multi-périodes mean-variance en présence de coûts de transactions. Elles reposent sur des approximations importantes ou s'avèrent avoir une trop grande complexité pour être applicable. En réponse à cette problématique, cette thèse propose une nouvelle approche qui consiste à représenter la valeur du portefeuille par sa décomposition

en chaos de Wiener. Cette méthode est numériquement raisonnable et peut prendre en compte les coûts de transactions. Nous comparons cette approche avec l'approche statique mean-variance et tentons de raviver le débat autour du consensus idéal à trouver entre simplicité et performance.

Abstract

This thesis aims to examine financial risk management within the contexts of credit and investment domains.

The first part of this thesis explores the risk associated with credit rating migrations. A credit rating evaluates the confidence in the ability of the borrower to comply with the credit's terms. We are interested in the role that credit ratings play in regulation and the risks they generate to banks' finances. In particular, the risk of credit rating migration can largely be explained by the evolution of the economic cycle. As a result of increasing pressure from regulators, analyzing the effect of the business cycle on rating transitions has become a subject of great interest for financial institutions. They seek to understand, estimate, and even forecast this risk in order to manage their credit portfolio efficiently. Through an overview of existing Point-in-time (PIT) rating transitions modeling methods and the introduction of a novel one, the first part aims to explore the link between rating migration dynamics and economic health. In the proposed approach, we consider that the dynamics of rating migrations in a pool of credit references, are governed by a common unobserved latent Markov chain, which describes the economic cycle. Under a point process filtering framework, we derive general discrete-time and continuous-time filtering formulas. We then apply the filtering frameworks to credit migrations and show how to infer the current state of the hidden factor from past rating transitions. We adapt an EM algorithm to estimate the parameters involved and assess both approaches on fictive data and on a corporate credit rating database. Our approach may be considered as a new Point-in-time (PIT) rating transitions modeling which does not use any macro-economic factors.

The second part of this thesis focuses on managing the trade-off between expected return and estimated risk in the decision investment process. The investment becomes increasingly attractive as the level of investment risk decreases. However, the general rule is that higher risk investments offer better returns. In this part, we study the way to allocate wealth within an investment portfolio composed of a basket of risky assets. We review static and dynamic methods and explore the motivations of investors for choosing one approach over another. Mean-variance framework offer to build a portfolio of assets such that the expected return is maximized for a given level of risk. When the time horizon increases, this single-period strategy which cannot see ahead of the next time period, cannot challenge the dynamic optimal portfolio obtained from the multi-period version of the problem. However, the existing dynamic methods, mostly based on dynamic programming, are not fully suitable for solving the multi-period mean-variance problem in the presence of transaction costs. They rely on significant approximations or are not computationally tractable. To bridge this gap, we propose a new approach that represents the portfolio value with its finite-dimensional Wiener chaos expansion. This method is computationally tractable and can handle transaction costs. This new approach is compared to a competitive benchmark based on the sequential single-period Markowitz allocation method. Through this comparison, we endeavor to revive the debate surrounding the ideal consensus to be reached between simplicity and performance.

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Chapter 1

Introduction

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This thesis was conducted under the supervision of Pr. Jérôme Lelong and Areski Cousin and within Nexialog Consulting, as part of the CIFRE program.

Financial risk refers to the possibility of adverse financial effects on individuals, groups, or companies due to internal or external changes. Managing financial risk involves evaluating potential losses and taking actions to reduce or eliminate these threats. It's an ever-present and complex challenge that individuals, organizations, and societies face in navigating the uncertainty of modern economic systems. The core concepts of risk management include avoidance, loss prevention, loss reduction, and diversification. Financial risks can be classified into four categories: credit risk, market risk, liquidity risk, and operational risk. Furthermore, two types of risks are distinguished: systematic risk and idiosyncratic risk. Generally, systematic risk, also known as non-diversifiable risk, cannot be eliminated as its source is primarily external, such as a pandemic or political climate. On the other hand, idiosyncratic risk, inherent to the individual, can be mitigated, for instance, within a portfolio through diversification. This thesis focuses on these two types of risks and aims to enhance our understanding of how their management can impact the decision-making process. Specifically, these risks are examined within the contexts of credit and investment domains.

The first part of this thesis explores credit risk management and the risks associated with credit rating migrations. Credit risk management aims to mitigate losses primarily associated with nonpayment of loans. We are interested in the role that credit ratings play in regulation and the risks they generate to banks' finances. In particular, the risk of credit rating migration can largely be explained by the evolution of the economic cycle.

As a result of increasing pressure from regulators, analyzing the effect of the business cycle on rating transitions has become a subject of great interest for financial institutions. They seek to understand, estimate, and even forecast this risk in order to manage their credit portfolio efficiently. Through an overview of existing methods and the introduction of a novel one, the first part aims to explore the link between rating migration dynamics and economic health. In the proposed approach, we consider that the dynamics of rating migrations in a pool of credit references, are governed by a common unobserved latent Markov chain, which describes the economic cycle. Note that this approach is described in Chapter 3 that is based on the article [Cousin et al. \(2023a\)](#).

The second part shifts focus to the field of investments and examines the intricacies of risk management within this domain. Asset or investment risk can be defined as the likelihood of incurring losses relative to the realized return on a particular investment. Most investors consider lower risk as more favorable when making an investment. The investment becomes increasingly attractive as the level of investment risk decreases. However, the general rule is that higher risk investments offer better returns. The second part of this thesis focuses on managing the trade-off between expected return and estimated risk in the decision investment process. In particular, we study the way to allocate wealth within an investment portfolio composed of a basket of risky assets. We review static and dynamic methods and explore the motivations of investors for choosing one approach over another. We highlight the challenges for dynamic approaches to integrate realistic constraints and transaction costs. To bridge this gap, we propose a new approach that represents the portfolio value with its finite-dimensional Wiener chaos expansion. This method is computationally tractable and can handle transaction costs. We compare this new approach with a competitive benchmark based on the sequential single-period Markowitz method. Note that this approach is described in Chapter 5 that is based on the article [Cousin et al. \(2023b\)](#), submitted to a peer-reviewed journal.

Through the exploration of existing and novel approaches, this study aims to contribute to the understanding of effective management of financial risk in a constantly evolving financial environment.

1 Introduction: Point-in-time estimation of credit rating migrations

Credit risk research has been on the rise over the last 20 years. In particular, the challenges that arose from the previous financial crisis prompted researchers to develop credit risk valuation models that take into account the evolution of the business cycle. Used by financial institutions, some risk indicators focus on the current situation and others attempt to look at likely developments over several years. The regulation refers to these contrasting approaches as Point-in-Time (PIT) and Through-The-Cycle (TTC). The Basel committee

provides a formal distinction between PIT and TTC approaches. A PIT measure covers a short horizon of a year or less, while a TTC measure covers a longer horizon that is sufficient for business-cycle effects to mostly dissipate. The evolution of banking supervisory regulations and accounting rules follow the trend underlying the first approach, of taking into account the economic cycle. Official guidelines of IFRS 9 as [Regulation \(2016\)](#) recommend the use of Point-In-Time estimation of credit risk, i.e., the use of macro-economic factors in the credit risk assessment process. Moreover, the EBA guidelines [Authority. \(2017\)](#) on LGD downturn, require to identify economic downturn periods to adjust the initial LGD estimations. In addition, the EBA stress testing methodology described in [Authority \(2018\)](#), strongly relies on past economical scenarios.

A credit rating system evaluates the confidence in the ability of the borrower to comply with the credit's terms. A default probability is associated with each rating, which under the capital rule, Basel regulations (see [Settlements \(2017\)](#)), impacts the amount of capital required for a credit. Ratings can be generated by internal rating systems (IRB) or issued when it is available, by external rating agencies (like Moody's, Standard and Poor's, Fitch Ratings, ...). After the assignment of an initial credit rating, reviews are performed either periodically or based on market events. In that way, a rating may evolve over time, according to the health of the rated entity and the economic cycle. Therefore, predicting the evolution of rating migrations is of primary importance for every financial institution. The rating migrations of a group of credit entities can be described by transition matrices, which determine the probabilities of moving from one rating category to another within a given time period. Given recent evolution in banking supervisory and accounting rules, financial institutions are facing a new challenge: explaining changes in transition probabilities due to the dynamic of the business cycle.

A popular and simple approach to estimate rating transitions consists in assuming that they follow homogeneous Markov chains. In the basic reduced intensity form model, a credit event corresponds to the first jump time of a Poisson process with a constant hazard rate. The reduced form approach has been widely studied in the credit risk literature, see, e.g., [Duffie et al. \(2007\)](#), [Jarrow et al. \(1997\)](#). We can point out a limit of this naive modeling: many studies show that rating migrations' dynamics first exhibit non-homogeneous property. Others studies even reject the Markovian behavior (migration data exhibit correlation among rating change dates, known as "rating drift", time-dependent default and transition probabilities, contagion effect, ...). [Altman and Kao \(1992\)](#), [Hamilton \(2004\)](#) and [Fledelius et al. \(2004\)](#) highlight that migration intensities vary over time. In their research [Lando and Skødeberg \(2002\)](#) and [Hamilton \(2004\)](#) come to the conclusion that the rating transition probabilities depend on whether the bond entered its current rating by an upgrade or a downgrade. [Lando and Skødeberg \(2002\)](#) also notice that the probability to leave a rating category tends to decrease with the time spent at that rating. [Bangia et al. \(2002\)](#) and [Carty and Fons \(1994\)](#) highlight *Rating Momentums*. In particular, a firm which has experienced a downgrade in the past has more chance to experience another than a firm which has never experienced one. Above all, [Bangia et al. \(2002\)](#), [Frydman](#)

and Schuermann (2008), Nickell et al. (2000) give strong evidence that credit risk exposure is considerably affected by the macroeconomic conditions and differ across different economic regimes. The first paper highlight quantified rating migrations' dependency on the industry, domicile of the obligor and on the stage of the business cycle. The second, by considering two economic states provided by The National Bureau of Economic Research (NBER), for conditioning migration matrices (one of expansion and one of contraction), show that the loss distribution of credit portfolios can differ greatly. Figlewski et al. (2012) observe and confirm those behaviours in their studies by incorporating and comparing a wide range of measures of the macroeconomic environment and of firm's rating histories into an intensity reduced Cox model.

Factor-based migration models provide a nice framework for capturing migration sensitivities to macro-economic changes. Factor migration models allow transition probabilities to depend on dynamic factors. Two main families of models are usually considered in the credit risk literature : the "ordered Probit" (or structural approach) introduced by Tobin (1958), popularized by Merton (1974) and studied for credit ratings , e.g., in Bangia et al. (2002), Feng et al. (2008), Gagliardini and Gouriéroux (2005), Nickell et al. (2000) and the "multi-state latent factor intensity model" (or intensity approach) studied, e.g., in Cousin and Khelouen (2015), Figlewski et al. (2012), Kavvathas (2001), Koopman et al. (2008). Structural models describe how a company's value process evolves over time. These models are based on sector-specific or macroeconomic indicators and assume that the company will default when its value process drops below a certain threshold. This threshold can be further divided into multiple levels, which represent changes in the company's rating category. In the factor intensity approach, the migration dynamics of each credit entity is described by a stochastic intensity matrix (or generator matrix) whose components depend on a pool of common factors. In both models, the factors may be considered observable or not.

The second approach has emerged in response to criticisms made against the first. As Gagliardini and Gouriéroux (2005) point out, the risk in selecting covariates lies in excluding others which could be more relevant. A natural alternative consists in directly filtering the hidden factor given rating transitions' past history. For a bond portfolio, the dynamics of rating migrations can mathematically be represented as a multivariate counting process, each component representing the cumulative number of transitions from one rating category to another. A realistic and standard setting assumes that the unobserved driving factor is given as a finite state Markov chain and that the rating transition process follows a Hidden Markov model (HMM). Hidden Markov Chain modeling (HMM) remains a popular approach in credit risk analysis (see e.g., Ching et al. (2009), Elliott et al. (2008), Elliott et al. (2014), Thomas et al. (2002)). The hidden process can have different interpretations according to the assumptions made and the way to filter. In the context of this thesis, the hidden factor is interpreted as a systematic and common factor, governing transitions of all firms.

Under a point process filtering framework, we first derive general discrete-time and continuous-time filtering formulas. We then apply the filtering frameworks to credit migrations, and show how to infer the current state of the hidden factor from past rating transitions. We adapt an EM algorithm to estimate the parameters involved. Contrary to [Damian et al. \(2018\)](#), [Korolkiewicz and Elliott \(2008\)](#), [Oh et al. \(2019\)](#), we assume that the dynamics of rating migrations in a pool of credit references, is governed by a common unobserved latent Markov chain, which aims to represent the economic cycle. Therefore the realization of the unobservable factor is assumed to be common to every firm whereas one hidden factor per bond is considered in [Damian et al. \(2018\)](#), [Korolkiewicz and Elliott \(2008\)](#), [Oh et al. \(2019\)](#). We believe that our approach which rather keeps the dependencies within the observations sample, is reliable and realistic. Indeed, rating entities should be affected by the same realization of the economic factor. This different consideration changes the way to calibrate and to filter: our filtering framework uses the whole history of aggregated number of jumps. Once calibrated, we may reveal and detect economic changes affecting the dynamics of rating migration, in real-time. By updating the filtered factor, we are able to forecast rating transitions according to these economic changes. Our approach may be considered as a new Point-in-time (PIT) rating transitions modeling which does not use any macro-economic factors.

Behind every model mentioned, choosing a continuous or discrete approach is crucial and is a matter of debate. This study aims at participating to this debate by presenting different results: we adapt filtering formulas, derived under special HMM assumptions in [Brémaud \(1981\)](#) and [Elliott et al. \(2008\)](#), to migration ratings context, both in a continuous-time and discrete-time setting. In particular, we show how to adapt the continuous-time filtering framework to handle discrete-time data and simultaneous jumps. We assess and compare both approaches on a fictive data set and on a Moody's ratings history [01/2000-05/2021] of a diversified portfolio of 5030 corporate entities.

In this part, we introduce the context of Point-in-time rating migration models both under a practical and a regulatory point of view. We present commonly used factor-based migration models from both structural and intensity model categories. The purpose of this review is to introduce our original contribution to the literature on credit rating dynamics, which is a new Point-In-Time approach based on counting process filtering. This model is both developed under a discrete and continuous-time setting. We assess and compare the efficiency of both versions in providing point-in-time estimations and forecasting of rating transition probabilities. We aim to show that this new model represents a viable alternative to existing approaches.

2 Introduction: portfolio allocation

Effective asset management goes beyond simply identifying a good investment universe; it also entails continually optimizing the portfolio structure over time. Building an effective

portfolio consists in enhancing its expected return while reducing the risk it generates. Diversification represents a natural means of reducing this risk. Portfolio allocation theory endeavors to offer effective solutions for diversification management. It proposes investment strategies that achieve a suitable balance between risk and reward. These strategies rely on modifying the allocation of each asset in the investment portfolio, based on the investor's investment time horizon, goals, and risk tolerance. Risk tolerance reflects the willingness and ability of an investor to accept the possibility of losing some or all of initial investment in exchange for potentially higher returns. Aggressive investors, with a high-risk tolerance, are more likely to risk losing money in order to get better results. In contrast, conservative investors, with a low-risk tolerance, prefer strategies that will preserve their original investment. The time horizon of investors refers to the expected number of months, years, or decades they will be investing to achieve a particular financial goal. Investors with a longer time horizon may feel more comfortable taking on riskier, or more volatile assets, as they can wait out economic cycles and the inevitable ups and downs of the markets. By contrast, investors with a shorter time horizon are likely to opt for lower-risk investments.

The degree of diversification in a portfolio relies on the allocation of wealth among various investment types within the portfolio. An adequately diversified portfolio should have diversification at two levels: between asset categories and within asset categories. Stocks, bonds, and cash are the most common asset categories. But investors may include in their portfolio other asset categories including real estate, commodities, and private equity. Investing in various asset categories can decrease a portfolio's exposure to category-specific risks. This approach constitutes the first level of diversification. Stocks have traditionally offered the highest returns but also carry the highest risk. Bonds are typically less volatile than stocks but provide more modest returns. Cash and cash equivalents — such as savings deposits, certificates of deposit, treasury bills, money market deposit accounts, and money market funds — are the safest investments, but offer the lowest return of the three major asset categories. The principal concern for investors investing in cash equivalents is inflation risk. The second level of diversification consists in spreading out investments within each asset category. Investors typically reconsider the entire investment basket when their risk aversion or financial goals change. However, if there are only changes in the relative performance of assets, they usually merely reallocate their portfolios. The principle of reallocation involves adjusting the allocation of wealth within the investment portfolio to stay aligned with the financial goal in response to changes in the market environment. Nonetheless, the rebalancing process can be challenging and restrictive, as it can trigger transaction costs or tax implications that must be carefully considered.

Markowitz mean-variance approach (MV), introduced in [Markowitz \(1952\)](#), is the classic paradigm of modern portfolio selection for efficiently allocating capital among risky assets. Given estimates of expected return, standard deviations and correlations of all assets in the investment universe, (MV) approach offers to build a portfolio composed with these assets such that the expected return is maximized for a given level of risk. Mean-variance optimization is recognized as superior to many ad hoc techniques in terms of

integration of portfolio objectives with client constraints and efficient use of information. However, Markowitz optimization was strongly criticized after the 2008 financial crisis, extremes declaring the death of Markowitz model. These extreme reactions can be explained by the fact that the strategy of diversification, traditionally associated with Markowitz optimization, has not been sufficient to protect portfolios against contagion effect. However, the poor performance of the mean-var efficient portfolio during crisis cannot be solely attributed to the methodology itself. In reality, a significant portion of the failure stems from inaccuracies in the input parameters, which were calibrated based on historical data and failed to account for the observed increase in risk levels and correlations during periods of stress. Despite being aware that errors in parameter estimation can lead to significant inaccuracies, investors persist in using Harry Markowitz's mean-variance model. The main reason is that this approach remains straightforward to use and interpret but above all, there is currently no alternative model capable of outperforming the mean-var optimal portfolio.

Over the past few years, several alternative approaches have been proposed to address the shortcomings of the mean-variance approach. The purpose of these methodologies is not to enhance the mean-variance model, but rather to provide alternative optimization techniques for constructing portfolios that exhibit distinct optimal traits, based on a varied range of input parameters or no parameters at all. Three such approaches are worth mentioning. The global minimum variance approach (GMVP), studied by [Clarke et al. \(2011\)](#), does not require any information about expected returns as it only aims to find the portfolio with the minimum of risk. In the same vein, the equal risk contribution portfolio (ERC), studied by [Roncalli \(2013\)](#), is designed such that each asset contributes equally to the overall risk of the portfolio. Lastly, the equal weight portfolio (EWP), as presented in [Maillard et al. \(2010\)](#), shares wealth in the same proportion among assets. Although this last method is quite naive, it does not need any input parameters. The selection of a particular method is closely linked to the information that is available and the level of confidence that the investor has in it. In addition, constructing an efficient portfolio that meets the investor's objectives can be challenging due to various factors that impact the allocation process. Portfolio allocation entails careful consideration of two crucial factors — realistic constraints and transaction costs. Realistic constraints refer to the practical limitations that investors face when constructing a portfolio. These constraints can include regulatory restrictions, liquidity requirements, tax considerations, and investment policies. For example, institutional investors such as pension funds may be subject to regulatory constraints that limit the types of assets they can invest in. Similarly, individual investors may have tax considerations that impact their portfolio construction decisions. Transaction costs, on the other hand, are the expenses incurred when buying or selling assets to adjust the portfolio's allocation. These costs can include brokerage fees, bid-ask spreads, and taxes. Transaction costs can significantly impact portfolio performance and reduce the efficiency of portfolio construction. Given the importance of realistic constraints and transaction costs in portfolio allocation, it is essential to incorporate these factors into the investment decision-making process.

Overall, effective portfolio allocation requires a thorough understanding of the investor's

objectives, constraints, and preferences. Moreover, the investor must also be able to identify and manage transaction costs to ensure that the portfolio's performance is not adversely affected. In this context, the development of innovative portfolio optimization techniques that account for realistic constraints and transaction costs has become an area of active research in the field of investment management.

All the approaches mentioned are also challenged by another family of models: dynamic asset allocations. These models aim to provide performing portfolios for a long-term investment horizon that can be adjusted over time based on changes in the investment environment. A dynamic portfolio is rebalanced periodically to ensure that it remains aligned with the investor's objectives and the prevailing market conditions. When considering a value or outcome that is significantly far into the future, myopic strategies that only focus on the immediate next time period are unable to compete with dynamic strategies that take into account the long-term value of interest. By not accounting for the potential consequences of future decisions, myopic strategies limit their ability to effectively address complex and dynamic problems. In contrast, dynamic strategies that consider the potential impacts of decisions on future outcomes are better positioned to adapt and thrive in changing environments over time. [Merton \(1969\)](#) is one of the first paper studying multi-period portfolio investment in a dynamic programming framework. In this seminal paper, the authors consider a problem with one risky asset and one risk-free asset. At each date, the investor can re-balance its wealth between the two assets, seeking to maximize an utility of the final time horizon wealth. They derive a simple closed-form expression for the optimal policy when there are no constraint or transaction cost. In a companion paper, [Samuelson \(1975\)](#) derives the discrete-time analog approach. The results presented in those studies and the innovative and promising aspect of the multi-period portfolio selection have stimulated the interest of the related scientific community. In the years that followed, the literature in multi-period portfolio selection has considerably grown, dominated by maximizing expected utility of terminal wealth of elementary forms as logarithm, exponential or CRRA functions. Dynamic programming techniques turn out to be the most suitable approaches to solve this kind of problems. Among the most noteworthy, [Brandt et al. \(2005\)](#) compare and highlight the conditions of equivalence between dynamic approaches and myopic strategies with CRRA utility functions.

However, important difficulties due the non separability of the problem in the sense of dynamic programming, have been reported in finding the optimal portfolio issued from the multi-period mean-variance approach. Nevertheless [Li and Ng \(2000\)](#) and [Zhou and Li \(2000\)](#) successfully provide the unconstrained multi-period mean-variance optimal portfolio explicit formulation in a discrete and continuous time setting, respectively. Suggested by Professor Markowitz himself, [Li et al. \(2002\)](#) derive the optimal portfolio policy for the continuous-time mean-variance model with no-shorting constraint. [Cui et al. \(2014\)](#) extend this work to provide a discrete framework. Even if these last studies become increasingly realistic, the ignorance of transaction costs, hinders its efficient extension for real-life application. Transaction costs impact a lot the optimal choice of policies and cannot be ignored.

The integration of transaction costs has been widely studied in the uni-period mean-var case (see [Best and Hlouskova \(2007\)](#), [Lobo et al. \(2007\)](#), [Pogue \(1970\)](#), [Xue et al. \(2006\)](#), [Yoshimoto \(1996\)](#)). In a multi-period context, the literature is also large on the subject. In a continuous setting, the problem is not recent, especially when the time horizon is considered infinite (see [Dai and Zhong \(2008\)](#), [Davis and Norman \(1990\)](#), [Dumas and Luciano \(1991\)](#), [Morton and Pliska \(1995\)](#), [Muthuraman and Kumar \(2006\)](#), [Topcu et al. \(2008\)](#)). Discrete time allocation strategies submitted to transaction costs have also been widely pursued (see [Bertsimas and Pachamanova \(2008\)](#), [Boyd et al. \(2013\)](#), [Calafiore \(2008\)](#), [Constantinides \(1979\)](#), [Holden and Holden \(2013\)](#), [Li et al. \(2022\)](#) and [Peng et al. \(2011\)](#)). But many difficulties have been reported by the literature to have an efficient and accurate method to compute optimal solutions. Furthermore, when solutions are proposed, they remain computationally heavy and hardly applicable. Some of them tractably solve the problem in several special cases. Others made various approximations. The most prevalent methods to tackle this kind of problem are based on stochastic control techniques. However, they also suffer from the same drawbacks (see [Al-Nator et al. \(2020\)](#), [Boyd et al. \(2013\)](#), [Cai et al. \(2013\)](#), [Cong and Oosterlee \(2016\)](#), [Genotte and Jung \(1994\)](#), [Pun and Ye \(2022\)](#), [Steinbach \(2001\)](#) and [Wang and Liu \(2013\)](#)).

It remains difficult to find a reliable and accurate approach that is also computationally tractable. Thus, there is no clear-cut selection of techniques. Instead, investors have to find a consensus between simplicity, tractability and performance. Due to the aforementioned challenges, multi-period models do not have the support of asset managers. In fact, the prevalent approach to date continues to be the use of single-period models.

The purpose of this part is to gain insight into the decision-making process behind selecting a particular approach for portfolio allocation. To achieve this, we will review the existing methods for portfolio allocation, focusing on the two main categories: myopic strategies, which focus on a single time period, and dynamic strategies, which adopt a long-term investment view. On one hand, single-time period models are simple to use and can easily handle realistic constraints and transaction costs. On the other hand, multi-period models should be theoretically more performing but may suffer from a higher computational complexity and are hardly tractable to handle constraints and costs. This study aims to contribute to this debate by proposing a new computational scheme that can solve multi-period portfolio allocation problems subject to transaction costs. We address this problem by proposing a new approach that relies on representing the set of admissible portfolios by their finite dimensional Wiener chaos expansion. The method is computationally tractable and can be extended to handle realistic constraints. This new approach is compared to a competitive benchmark based on the sequential single-period Markowitz allocation method. We aim to show that this new model represents a viable alternative to existing approaches.

3 Structure and content of the thesis

This thesis is organised in two main parts. The first part, Part I, explores credit risk management and the risks associated with credit rating migrations. The part is divided into two chapters.

- The first chapter, Chapter 2, reviews the existing regulation and methodologies surrounding credit rating migrations. This chapter is composed of two sections. Section 1, begins by introducing fundamental concepts in credit risk, such as the characteristics of credit ratings and recent regulatory definition of default. We then provide a brief overview of the present regulatory directives pertaining to the computation of default probability and the role of credit ratings in estimating banks' regulatory capital. A second section, Section 2, aims to introduce the two families of factor-based migration models, the structural and intensity approaches. We concisely present both categories and the related literature. We highlight the limits of considering observable factors but also the difficulties to integrate unobservable ones.
- The second chapter, Chapter 3, presents a novel approach for modeling Point-in-time rating migrations, based on filtering of point processes. This chapter is based on the article Cousin et al. (2023a). This chapter is structured into several sections. Firstly, in Section 2, a general filtering framework in a discrete-time setting is presented. Then, Section 3 describes the discrete-time filtering framework adapted to credit rating migrations. Correspondingly, Section 4 is dedicated to describe the filtering framework in a continuous-time setting while Section 5 presents its adaptation to the same context. Section 6 illustrates and validates the two filtering approaches on fictive data. Then, in Section 7, we compare the two filters on real data sets.

The second part of this thesis, Part II, explores portfolio allocation management and the risks associated. The part is divided in three chapters.

- The first chapter, Chapter 4 is an introductory chapter. It is composed of two sections. The first section, Section 1, provides an overview of existing and widely used approaches for portfolio allocation in a single-period time horizon. We discuss the fundamental Markowitz mean-variance approach and also review the various realistic constraints and cost models that are present in portfolio management. The second section, Section 2, focuses on dynamic multi-period strategies. We provide a brief overview of various methods, including those based on dynamic programming, and examine specific cases that can be solved efficiently. Additionally, we draw attention to potential challenges that these models may encounter. Among these difficulties, we highlight the absence of reliable method to find optimal mean-variance portfolios submitted to transactions costs.
- Chapter 5 is based on the article Cousin et al. (2023b). This chapter is organized as follows. Section 2 is dedicated to describe the mean-variance problem for multi-period

portfolio allocation submitted to transaction costs. Our methodology which aims at finding optimal portfolios in this context, is presented in Section 3. In section 4, we describe a numerical framework to investigate optimal solutions. We also study the link between risk aversion and those solutions and discuss the use of a finer time grid for the Wiener chaos expansion. Finally, in Section 6, we show the efficiency of our solution and investigate the impact of transaction cost by comparing performances of the presented models with benchmark models such as the sequential uni-period Markowitz approach, described in Section 5.

- In Chapter 6, we discuss extensions of our approach. In Section 1, we present a sub-optimal solution in the absence of transaction costs. Then, we show in Section 2.1 how to address the dynamic allocation problem with constraints, such as no shorting.

Part I

POINT-IN-TIME ESTIMATION OF CREDIT RATING MIGRATIONS

Chapter 2

Credit ratings: a review of regulation and models

Abstract

This chapter provides an overview of the regulation related to credit ratings and the commonly employed methods to estimate their dynamics.

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2	Rating migration models	16

1 Credit risk and ratings

In this section, we provide a general overview of credit risk, as well as a brief history of significant regulatory requirements pertaining to the estimation of banks' regulatory capital. While we cannot cover all regulatory aspects comprehensively, our main objective is to integrate our research into practical applications by emphasizing the role that credit ratings play in regulatory compliance. It should be noted that this introductory section is inspired by [Gouriéroux and Tiomo \(2007\)](#) which offers an overview of classic notions in credit risk. We also refer to the related regulation (see, e.g., [Authority \(2016\)](#), [Authority \(2018\)](#), [Regulation \(2016\)](#), [Settlements \(2017\)](#), [Settlements \(2023\)](#) and [Settlements \(2006\)](#)).

1.1 Credit Risk

Credit risk refers to the potential for financial loss resulting from a borrower's failure to fulfill their credit obligations, due to various factors such as economic downturns, changes in interest rates, or a decline in the borrower's financial health. It includes the risk of default, when a delay or failure of payment on the principal or the interests is observed, the risk on the rate of recovery in case of default and on the risk of a potential degradation of the quality of the credit portfolio. The credit risk may concerns different instruments. We can mention

- Instruments whose credit terms are fixed in the contract. For instance, the risk may relate to partial repayment, non-payment at maturity, or delayed payment. The risk may also imply the counterparty when it is not able to respect its obligations.
- Instruments whose the date or amount of repayment are not settled in the contract. The risk mainly concerns credit contracts with variable rate, which focus only on interests or instruments that can be re-purchased.
- Products whose payment is dependant of other conditions. We can mention the options, credits swaps or lease purchase contracts. The risk may arise from the counterpart and not from the instrument it self.
- The devaluation of the currency in which the debts are supposed to be repaid may generate a loss.
- When an instrument is listed on the stock exchange, a decrease in its value can have a negative impact by increasing the probability of default in the future. Although this risk can be covered by an increase in the value of another instrument or reduced by diversifying the portfolio, it cannot be eliminated. In addition, a high risk of contagion can increase the risk significantly. On financial markets, default risk is generally measured by spreads. The evolution of these spreads impacts the value of assets and highlights the presence of credit risk in the market.
- The risk associated with a credit line authorized to a client, may increase its exposition to default. In a bad financial situation, the client can increase its debts and therefore decrease its ability to repaid it.

1.2 Definition of default

The solvency of a firm is its capacity to repay all its debts. It mainly depends on the quality of its assets (in particular their liquidity), the amounts of its debts and its collaterals. The solvency can me measured with the ratio between the total debts and the total assets value. The default of a firm traduces its state of insolvency, noticed for a given time period. An economic, legal, even judicial process may be the next step of the procedure and may leads

to a total liquidation of the firm.

The Basel Committee laid the foundation for a common definition of default. This was an important step in promoting consistency and transparency in the assessment of credit risk across different financial institutions and markets (Article 178 of the Capital Requirement Regulation (CRR), June 2013). This definition has been updated by the EBA guidelines in [Authority \(2016\)](#) which aim to continue to harmonize the definition of default among the European financial intuitions. This report defines the common criteria to identify and account the default, and insists on the need for the banks to show a clear, transparent, coherent and effective governance. The principal trigger to downgrade a client to default is a delay of payment of more than 90 days. The due amounts considered encompass not only the capital, but also any interests or taxes owed. The thresholds of unpaid capital can be absolute or relative of the total amount of debts (1%). Others alarming signals can justify a reclassification to default. For instance an entity flagged UTP (Unlikeness To Pay) may be declared in default without necessarily exceeding the limit of 90 days. UTP may concern

- Distressed restructuring, that should be considered to occur when concessions have been extended towards a debtor facing or about to face difficulties in meeting its financial commitments.
- SCRA (Specific Credit Adjustment), which covers situations, not covered by the EBA-guidelines, and judged doubtful according to internal statements.
- When the situation is sufficiently critic to engage the collateral.
- When the credit is sold with a loss of more than 5%.
- When cases of bankruptcy, fraud and pulling effect have been proven.

1.3 Credit Ratings

A credit rating is an assessment of the creditworthiness of an individual, corporation, or government. It is an evaluation of the borrower's ability to fulfill its financial obligations, such as repaying a loan or bond. Formally, it mainly evaluates the probabilities that the borrower does not pay his debt, the totality of the principal and the interest, at the settled time. Credit ratings are determined internally by financial institutions or provided by credit rating agencies according to qualitative and quantitative information on the prospective debtor. Concretely, a rating is associated to a default probability in a fixed time horizon. A rating can be deduced from a score (quantitative valuation), or can be directly represented by a qualitative notation. The most three famous rating agencies, Moody's S&P and Fitch have different rating scales. The matching between the notations is presented in [Table 2.1](#). Financial institutions associate a default probability to each rating. This probability can be estimated historically, or directly provided by the agencies. Banks may

have their own rating system, based on human judgments or on models that use specific information on the borrower, such as its sector, its financial information and the presence of a collateral. Ratings are frequently reevaluated according to the the evolution of these specific information but also according to the economic cycle.

They represent powerful internal tools to decide whether to grant a credit, to define the terms of a contract, such as the amount, the rate of interest and the maturity. Requirements in international regulation [Settlements \(2017\)](#) and [Settlements \(2023\)](#), state that all borrowers and guarantors, corporate, sovereign or bank exposures included, must be assigned to a rating as part of the loan approval process. They are also mandatory when a bond is issued on the market. They aim to help the investors to measure the level of risk. Furthermore, they may be used to control and to constrain investments. For instance, pension funds cannot invest in speculative debt.

These ratings also play an important role in the computation of regulatory capital. They are used to group credit references with the same level of risk in order to analyze the risk on a credit portfolio with a high number of borrowers. Banks can use rating systems provided by external rating agencies or adopt an internal ratings-based approach (IRB). Reforms to the internal ratings-based approach to credit risk had to be introduced under Basel III, which aims to finalise post-crisis reforms standards. This point is further discussed later in this chapter, in [Section 1.5](#). The process by which a rating is assigned and the rating itself must be reviewed periodically by an independent party. Ratings must be reviewed at least once a year. Banks are also required to regularly stress test their rating systems by considering economic downturn scenarios, market risk based events or liquidity conditions that may increase the level of capital held by the bank. In general, a time horizon of one year or under is considered short term, and anything above that is considered long term. In the past, institutional investors preferred to consider long-term ratings. Nowadays, short-term ratings are also commonly used. It is important to note that credit rating agencies may use different rating scales for short-term and long-term ratings.

Table 2.1: Long-term ratings table for 3 rating agencies

Rating description	Moody's	S&P	Fitch
Prime	Aaa	AAA	AAA
High grade	Aa1	AA+	AA+
	Aa2	AA	AA
	Aa3	AA-	AA-
Upper medium grade	A1	A+	A+
	A2	A	A
	A3	A-	A-
Lower medium grade	Baa1	BBB+	BBB+
	Baa2	BBB	BBB
	Baa3	BBB-	BBB-
Non-investment grade speculative	Ba1	BB+	BB+
	Ba2	BB	BB
	Ba3	BB-	BB-
Highly speculative	B1	B+	B+
	B2	B	B
	B3	B-	B-
Substantial risks	Caa1	CCC+	C
Extremely speculative	Caa2	CCC	C
Default imminent with little prospect for recovery	Caa3	CCC-	C
	Ca	CC	C
	Ca	C	C
In default	C	D	DDD
	/	D	DD
	/	D	D

1.4 From Basel I to III: a new capital requirement

The regulation defines the rules and best practices for financial institutions in the management of their activities. The Basel Accords refer to the banking supervision accords issued by the Basel Committee on Banking Supervision. They aim to enhance financial stability by improving the quality of banking supervision worldwide, and to serve as a forum for regular cooperation between its member countries on banking supervisory matters. They also intend to improve the control and the harmonisation of banks' practices. Historically, three main agreements have been published by the Basel committee: Basel I in 1988, Basel II in 2004 and Basel III in 2010. This section presents major requirements and evolution of the regulation but does not aim to provide an exhaustive summary. We invite readers to refer to [Settlements \(2017\)](#), [Settlements \(2023\)](#), [Settlements \(2006\)](#) and [Verboomen and De Bel \(2011\)](#) for further details on the regulation.

The first Basel Accord mainly focus on the amount of regulatory capital that a bank has to save in regard of the risk of its credit contracts. The 1988 Accord call for a minimum ratio of capital to risk-weighted assets of 8% (Cook ratio). Many criticisms arose from these recommendations. In particular, the weighting in credit risk valuation did not sufficiently take into account the quality of the borrower. Furthermore, new exposition off banks' balance sheets as market risk, were not considered in the Cook ratio. The Committee refined the framework to address risks other than credit risk. In January 1996, the Committee issued an Amendment to the Capital Accord to incorporate market risks, which was intended to establish a capital requirement for risks arising from banks' exposures to foreign exchange, traded debt securities, equities, commodities, and options. Published in 2004, Basel II recommend a new capital framework to supersede the Basel I framework. It is based on "three pillars":

- Minimum capital requirements, which seek to develop and expand the standardised rules set out in Basel I. Notably, the threshold of 8% is retained incorporating the market and operational risk. Moreover, the risk is calibrated in regard of the quality of the debtors.
- Supervisory review of an institution's capital adequacy and internal assessment process. Banks have to establish procedures of back testing and stress testing to prove the validity and the reliability of their methods.
- Effective use of disclosure as a lever to strengthen market discipline and encourage sound banking practices.

In particular, the first pillar introduce a new solvency ratio, the Mcdonough ratio as

$$\frac{\text{Regulatory capital}}{\text{Credit risk} + \text{Market risk} + \text{Operational risk}} \geq 8\%. \quad (2.1)$$

The valuation of the credit risk is called RWA (Risk Weighted Asset). The estimation of RWA is based on the estimation of four risk parameters, PD, LGD, M, EAD. RWA can be estimated as

$$\text{RWA} = \text{RW} \times \text{EAD}, \quad (2.2)$$

where $\text{RW} = f(\text{PD}, \text{LGD}, \text{M})$ is a weighting function. The exposure at default (EAD) is an estimate of the outstanding amount (drawn amounts plus likely future drawdowns of yet undrawn lines) in the event of borrower default. The loss given default (LGD), complementary of the recovery rate, is the estimated rate of loss in case of default. It highly depends on the guarantees of the debtor. The probability of default of the debtor (PD) represents the probability that the debtor does not respect its contract in the course of one year (see definition of Default in Section 1.2). The maturity (M) of the contract is the last involved parameter.

In order to evaluate the RWA, and the capital requirement, financial institutions have the choice between two approaches. The first approach, called Standard Approach (SA), is

based on using external ratings and using risk weighting provided by the regulation (see Section 1.5.1). In the other alternative, banks are allowed to use their own estimated risk parameters for computing their regulatory capital. This approach is known as the internal ratings-based (IRB) approach (see 1.5.2).

Even before Lehman Brothers collapsed in September 2008, the need for a fundamental strengthening of the Basel II framework became apparent. The banking sector entered the financial crisis with too much leverage and inadequate liquidity buffers. Two main weaknesses of Basel II are highlighted: its negligence of liquidity risk and its procyclical effect on economy. The new standards or Basel III, issued by the Committee in mid-December 2010 and revised between 2013 and 2019, strengthens the three pillars established by Basel II, while also extending it in several areas. In particular, stricter requirements for the quality and quantity of regulatory capital are proposed. Basel III also enforces the liquidity requirements by the introduction of two ratios: a short-term ratio, the Liquidity Coverage Ratio (LCR), intended to provide enough cash to cover funding needs over a 30-day period of stress and a longer-term ratio, the Net Stable Funding Ratio (NSFR), designed to promote the resilience of banks by ensuring that they have sufficient stable funding to support their activities over a one-year time horizon. Moreover, the committee aims to limit the leverage effect by introducing a leverage ratio, as a minimum amount of loss-absorbing capital relative to all of a bank's assets and off-balance sheet exposures regardless of risk weighting. Basel III also encourages the banks to establish a countercyclical capital buffer during periods of economic prosperity.

1.5 Role of credit ratings in banking regulation

To determine the Basel capital requirement that covers the credit risk they support, financial institutions can choose between two regulatory approaches. The first approach is the Standard approach (SA) involves using an external rating system, provided by rating agencies. Alternatively, the Internal Ratings Based (IRB), allows banks to use their own rating system. For more details on these two regulatory approaches, we refer the reader to [Verboomen and De Bel \(2011\)](#).

1.5.1 Standard approach

The standard approach provides a credit risk weightings based on external ratings provided by peer rating agencies. Specifically, the regulator provides estimations of risk weightings according to the category of the debtor, their credit rating, and other relevant information. For a simple credit contract, the provided regulatory risk weighting (RW) , and the Mcdonough ratio (see 2.1) provides the regulatory capital required to cover the credit line. Debtors are classified into four categories: sovereign, banks, corporate and retail. We present the matching table between credit rating and RW, as provided by the regulator

in Table 2.2. Note that specific information, such as the currency, geographical area, or presence of a real estate mortgage to cover the credit, is omitted.

Table 2.2: Matching table between RW and Moody's ratings

Category	Aaa	A1	Baa1	Ba1	B1	Caa1	NR
	Aa2	A2	Baa2	Ba2	B2	or <	
	Aa2	A3	Baa3	Ba3	B3		
	Aa3						
Sovereign	0%	20%	50%	100%	100%	150%	100%
Bank	20%	50%	50%	100%	100%	150%	50%
Corporate	20%	50%	100%	100%	150%	150%	100%
Retail	x	x	x	x	x	x	75%

Since external agencies does not provide ratings for retail population and small firms, the regulation applies a uniform weighting of 75%. The provided risk weighting increases with default probabilities. The lower the entity's rating and the less likely it is to pay, the greater the capital charge for the bank. Note that risk weightings for the riskiest categories may even exceed 100%. These high considerations may intend to cover underlying risks such as the risk of contagion. This approach must be conservative for all entities therefore the risk and the capital to save tend to be overestimated. Although easy to apply, the standard approach is most of the time not the most advantageous approach for banks. Note that these weightings are available when the debtor is not in default. If the debtor is considered to be in default, then the capital requirement is not based on the credit rating but on the level of provisions.

1.5.2 IRB approach

The Basel II agreements allow banks to choose between two broad methodologies for calculating their capital requirements related to their credit risk. This section outlines the second alternative, based on internal rating system. Under certain conditions, banks may use their own estimated risk parameters for to calculate regulatory capital. This approach is known as the internal ratings-based (IRB) approach. However, this approach is only available to banks that meet certain minimum conditions, such as disclosure requirements and approval from their national supervisor. With this alternative approach, the risk weight is not only dependent on the quality of the debtor but also on the quality of the bank. Four risk parameters are involved in the computation of the loss attributed to a credit line. The exposition at the default date (EAD), the loss given default (LGD), the probability of default (PD) and the maturity (M) of the contract. We distinguish two approaches, the foundation and the advanced approach. In the first approach, financial institutions estimate their own PD parameter while the other risk parameters are provided by the bank's national supervisor. In the second approach, they calculate their own risk

parameters subject to meeting certain guidelines. It is important to note that the foundation approach is not available for retail exposures. In the IRB approach, the computation of the risk weighted assets (see function f in (2.2)), is realized through the estimation of the expected and unexpected loss.

1.5.3 Expected and unexpected loss

Before banks grant a credit, they always evaluate the expected amount of loss, the credit contract is expected to generate. This amount, called Expected loss (EL), is theoretically covered by the provisions. Nevertheless, these provision are not sufficient to cover all the risk. Indeed, the effective loss can be greater than the expected one. This difference is called unexpected loss (UL), and must also be estimated and covered by the banks. This distinction is crucial since the regulatory capital K , appearing in Mcdonough ratio (2.1), aims to only cover the unexpected loss. We briefly describe here how financial intuitions can estimate these two kinds of loss according to the current requirements, imposed by the financial European authorities. We highlight the role of credit ratings in the computation of the regulatory capital.

Granting credits may generate losses. It is impossible to predict the exact amount of loss, but financial institutions can estimate the mean level of this loss. This amount is inherent. It is integrated in the interest rate as risk premium and is recorded in provisions. Indeed, in the IRB approach, the expected loss does not generate any capital requirement. Nevertheless banks must show that their provisions effectively cover the loss. The way to compute the expected loss fall within the scope of official guidelines of IFRS 9 (Regulation (2016)). Formally, the Expected loss can be estimated as

$$EL = PD \times LGD. \quad (2.3)$$

IFRS 9 guidelines require an estimation of parameters which integrates a forward looking vision. In addition, default probabilities are also recommended to be estimated per rating grade. A rating migration matrix is rather estimated whose last column provides default probabilities. Sections 1.5.3.1 and 2.3.1.2 aim to further describe the recommended way by the regulator to compute this default probability.

1.5.3.1 The expected loss in IFRS9: a forward looking approach

The final version of IFRS 9, published in July 2014, marked the completion of replacing IAS 39. IFRS 9 aims to streamline and strengthen risk measurement and reporting of financial instruments in an efficient, forward-looking manner. In particular, the revised impairment model aims to provide more transparent and useful information regarding expected credit losses.

The process of estimation recommended by the regulation involves three stages.

- The classification; Financial instruments are classified into three main categories

based on their intended purpose, which determines how they will appear on the balance sheet. The classification of an instrument primarily depends on whether it is held to collect contractual cash flows, held for sale, or both.

- The impairment; this step aims to classify among three stages, instruments whose the amortized cost basis is used to determine profit and loss. If the credit is considered with low credit risk, the expected loss is projected for a time horizon of 12 months. If there is a significant increase in credit risk, the amount of provision is estimated as the discounted value of credit losses generated by the instrument, projected over its full lifetime. Criteria for monitoring the evolution of the credit quality is not clearly specified by the guidelines, but one can presume a significant increase in credit risk when the borrower is 30 days past-due, or if its rating experiences a significant downgrade. If the credit risk recovers, the provisions can once again be limited to the projected credit losses over the following 12 months. The last stage gathers the non-performing loans, whose default is detected. The expected loss is also projected over the full lifetime of the instrument, with a default probability equal to 1. The impairment process is summarized in Table 2.3.

Table 2.3: IFRS9 staging

Name	Form	EL	main indicator	interest revenue treatment
Bucket 1	performing	one-year	x	on gross carrying amount
Bucket 2	under-performing	lifetime	30 days past due	on gross carrying amount
Bucket 3	Non-performing	lifetime	90 days past due	on net carrying amount

- The expected credit loss measurement; every entity has to base its measurement of expected credit loss on reasonable effective data, that includes historical, current and forecast information. Moreover the measure of expected loss should be a probability-weighted that can be determined by a range of scenarios. Consequently the risk parameters PD and LGD, involved in the estimation of the expected loss must take into account the current and future states of the economic cycle. This framework is referred as forward-looking estimation of the risk parameters. Economical scenarios must be projected for 1-year for the first bucket, and for lifetime for the buckets 2 and 3. In order to project the default probability according to the forecasted dynamics of the economic cycle, the PD must be estimated in a point-in-time manner. In practice default probabilities are not estimated directly. Financial institutions are rather used to estimate point-in time rating transition matrices, whose last column provides a point-in-time default probability to each rating. e further detail the estimation process in Section 2.3.1.2. In addition, the model presented in Chapter 3 represents an alternative method to estimate point-in-time default probabilities. It may be considered as a response to a regulatory requirement.

1.5.3.2 The unexpected loss in IRB approach

Bank capital serves the important purpose of providing a buffer to protect against unexpected losses that exceed expected levels, commonly referred to as Unexpected Losses (UL). While such peak losses do not occur every year, when they do occur, they can potentially be significant. Therefore, capital is necessary to cover the risks of such unpredictable peak losses. There are various approaches to determine the amount of capital a bank should hold. The IRB approach focuses on the frequency of bank insolvencies arising from credit losses that supervisors are willing to accept. In order to determine the maximum amount of loss, the Basel committee fixes this frequency by recommending a confident interval at 99,9%. The maximum loss is then estimated by calculating the 99,9% Value at Risk of loss, which means that there is less than 0,01% of chance that a bank loses more than this estimated maximum value. Since the expected loss is presumed to already covered by provisions, the regulatory capital should only covers

$$UL = \text{Var}_{99,9\%}[\text{PD} \times \text{LGD}] - \text{EL} \quad (2.4)$$

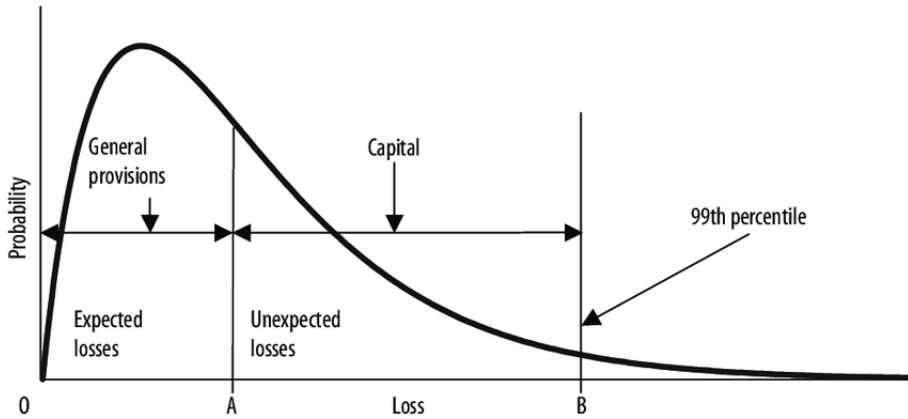


Figure 2.1: Expected and unexpected loss

According to the normality assumptions made, the regulatory form of the capital requirement is

$$UL = K = \left(\text{LGD}_d \Phi \left(\sqrt{1 - \rho} \Phi^{-1}(\text{PD}) + \frac{\sqrt{\rho}}{\sqrt{1 - \rho}} \Phi^{-1}(0, 999) \right) - \text{PD} \times \text{LGD} \right) M_a \quad (2.5)$$

In this formula LGD_d represents an estimation of the LGD downturn, which reflects the expected rate of loss in a period of economical recession. $\Phi \left(\sqrt{1 - \rho} \Phi^{-1}(\text{PD}) + \frac{\sqrt{\rho}}{\sqrt{1 - \rho}} \Phi^{-1}(0, 999) \right)$ is the computation of the conditional default probability at a level 99,9% issued from the so-called Asymptotic Single Risk Factor (ASRF). In this model, ρ represents the asset value correlation with the economic cycle. M_a is an adjustment factor due to the maturity of

the contract. This factor aims to anticipate a degradation of credit quality of the borrower over time. The ASRF model and the derivation of the conditional default probability is detailed later, in Section 2.3.1.2.

Since 2006, the capital requirement has also been adjusted with a scaling factor equal to 1,06 such that

$$K' = K \times 1,06. \quad (2.6)$$

The RW can be estimated as

$$RW = \frac{K'}{0,08}. \quad (2.7)$$

Finally, the Risk Weighted assets writes

$$RWA = \frac{\tilde{K}}{0,08}, \quad (2.8)$$

where $\tilde{K} = K' \times EAD$ is the real amount in cash of regulatory capital the bank has to save. This estimation is usually achieved per rating class. For retail population, the financial institution is required to justify its segmentation in homogeneous risk classes.

1.5.4 Regulatory stress-testing

A stress test is a simulation of extreme but plausible economic and financial conditions in order to observe the consequences on banks' models and to measure their ability to resist to such situations. Banking stress tests have been implemented by central banks and banking supervisors in the late 1990s but became an essential tool after 2008 in response to the financial crisis. Banks must conduct frequent stress tests and publish the results under the guidance of international financial authorities. These frameworks are applied by central banks (and supervisors) or by the banks themselves. These stress tests can be regulatory or internal, and they fall into various categories, but we will focus on macroeconomic stress tests, which aim to measure the impact of macroeconomic shocks on credit risk, asset value, and solvency ratios. If the results are unfavorable, banks must react either by increasing their capital, either by carrying out restructuring. Stress tests are specific to each entity and are based on four common steps, namely the choice of scenarios, the choice of models, the impact measurement and the analysis stage. The first step consists in defining several scenarios with different time horizons (generally from 1-year to 10-years). These scenarios may be historical or hypothetical. Historical scenarios are economic scenarios that have already occurred, such as old crises. The challenge in building historical scenarios is to clearly determine crisis periods. Hypothetical scenarios are imagined by economists and experts. Basel II requires that these scenarios must be plausible (their probabilities of occurrence must not be extremely weak) and their construction should obey to a rigorous economic logic.

Rating migration matrices represent a good indicator of the economic cycle. According to the regulation, they ensure the link between economic health and the amount of provision

and capital requirements making them relevant parameters for stress testing banks' capital capacity. They can be stressed with economic scenarios through PIT models presented in Section 2. The approach presented in Chapter 3 may provide hypothetical scenarios based on historical data. The plausibility of scenarios and the economic logical dynamics are well respected. Therefore, our approach can represent a new and interesting way to stress capital requirement models.

2 Rating migration models

This section aims to present models commonly used to estimate Point-In-Time rating transition matrices. It should be noted that this section is inspired of [Gouriéroux and Tiomo \(2007\)](#) which offers a review of the classic credit risk modeling approaches. We present several notations and assumptions before introducing the models.

2.1 Setting and environment

we describe here two environments, a discrete-time and a continuous-time setting.

2.1.1 Discrete-time setting

Let $T \in \mathbb{R}$ be a finite time horizon and let $\Gamma \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_\Gamma = T$ a discrete-time grid. We define the filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in \{0, \dots, \Gamma\}}, \mathbb{P})$. For every \mathbf{F} -adapted process O , we denote $O_n = O_{t_n}$.

We consider the list of rating categories $\bar{\Upsilon} = \{1, \dots, p\}$. This space represents different credit risk scores or ratings in descending order, p being the default state. Let $Q \in \mathbb{N}$ be the number of considered entities and let $Z_n^q \in \bar{\Upsilon}$, be the random variable, describing the state of entity q , $q \in \{1, \dots, Q\}$, at time t_n , $n \in \{0, \dots, \Gamma\}$ and let $Z^q = (Z_n^q)_{n \in \{0, \dots, \Gamma\}}$ be the migration process that describes its evolution. The migration counting process associated to Z^q , which counts the jump of the entity q from rating i to r , $(i, r) \in \bar{\Upsilon}^2$, on the discrete grid, is denoted by $N^{q,ir}$ and is such that, $\forall n \in \{1, \dots, \Gamma\}$,

$$\Delta N_n^{q,ir} = \mathbf{1}_{[Z_{n-1}^q=i, Z_n^q=r]}.$$

In addition, let us denote by the process $Y^{i,q}$, the indicator function

$$\forall i \in \bar{\Upsilon}, Y_n^{q,i} = \mathbf{1}_{[Z_{n-1}^q=i]}.$$

The counting process, which counts the total number of jumps of the entities, from rating i to r , $(i, r) \in \bar{\Upsilon}^2$, is denoted by N^{ir} and is such that, $\forall n \in \{1, \dots, \Gamma\}$,

$$\Delta N_n^{ir} = \sum_{q \leq Q} \Delta N_n^{q,ir}.$$

In addition, let us denote by the process Y^i , the number of observed and active entities that belong to rating i and such that

$$\forall i \in \bar{\Upsilon}, Y_n^i = \sum_{q=1}^Q Y_n^{q,i}.$$

2.1.2 Continuous-time setting

Let $T \in \mathbb{R}$ be a finite time horizon and let $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, be a filtered probability space satisfying the “usual conditions” of right-continuity and completeness needed to justify all operations to be made. All stochastic processes encountered are assumed to be adapted to the filtration \mathbf{F} and integrable on $[0, T]$. We consider the list of rating categories $\bar{\Upsilon} = \{1, \dots, p\}$. This space represents different credit risk scores or ratings in descending order, p being the default state. Let $Q \in \mathbb{N}$ be the number of considered entities and let $Z_t^q \in \bar{\Upsilon}$, be the random variable, describing the state of entity q , $q \in \{1, \dots, Q\}$, at time $t \in [0, T]$ and let $Z^q = (Z_t^q)_{t \in [0, T]}$ be the migration process that describes its evolution. We assume that, for any obligor $q = 1, \dots, Q$, the observed number of ratings visited during the period $[0, T]$, is denoted by Γ_q and $0 = t_0^q < t_1^q < \dots < t_{\Gamma_q}^q = T$ be the instants such that

$$\begin{cases} \forall 1 \leq n \leq \Gamma_q - 1, Z_{t_n^q}^q \neq Z_{t_{n-1}^q}^q \text{ and } \forall t \in [t_{n-1}^q, t_n^q), Z_t^q = Z_{t_{n-1}^q}^q \\ \forall t \in [t_{\Gamma_q-1}^q, t_{\Gamma_q}^q], Z_t^q = Z_{t_{\Gamma_q-1}^q}^q \end{cases}$$

For any $q = 1, \dots, Q$, for every process O^q associated to entity q , we denote for all $n = 1, \dots, \Gamma_q$, $O_n^q = O_{t_n^q}^q$. Formally, the time interval $[t_{n-1}^q, t_n^q)$ corresponds to the visiting of the rating z_{n-1}^q , where $t_0^q = 0$ and $t_{\Gamma_q}^q = T$.

The migration counting process associated to Z^q , which counts the jump of the entity q from rating i to r , $(i, r) \in \bar{\Upsilon}^2$, is denoted by $N^{q, ir}$ and is such that, $\forall t \in [0, T]$,

$$\Delta N_t^{q, ir} = \mathbf{1}_{[Z_{t-}^q = i, Z_t^q = r]}.$$

In addition, let us denote by the process $Y^{i, q}$, the indicator function

$$\forall i \in \bar{\Upsilon}, Y_t^{q, i} = \mathbf{1}_{[Z_{t-}^q = i]}.$$

The counting process, which counts the total number of jumps of the entities, from rating i to r , $(i, r) \in \Upsilon^2$, is denoted by N^{ir} and is such that, $\forall t \in [0, T]$,

$$\Delta N_t^{ir} = \sum_{q \leq Q} \Delta N_t^{q, ir}.$$

In addition, let us denote by the process Y^i , the number of observed and active entities that belong to rating i and such that ,

$$\forall i \in \bar{\Upsilon}, Y_t^i = \sum_{q=1}^Q Y_t^{q, i}.$$

2.2 Trough the cycle estimation

A Through-The-Cycle (TTC) measure covers a horizon long enough for business-cycle effects mostly to go away. It takes into account the full economic cycle. TTC estimation involves analyzing historical data over multiple economic cycles. We distinguish here two approaches, a discrete and continuous time approaches.

2.2.1 Homogeneous discrete time approach

We consider in this approach the discrete-time setting described in Section 2.1.1. In this approach, we assume that the rating processes $\{Z^l, q \leq Q\}$ are \mathbf{F} -adapted, independent and homogeneous Markov chains. According to this setting, we define the transition probabilities of $(Z^q)_q$ as

$$\text{For } (i, r) \in \bar{\Upsilon}^2, \forall n \in \{1, \dots, \Gamma\}, L^{ir} = \mathbb{P}(Z_n^q = r | Z_{n-1}^q = i).$$

Estimations of $\mathcal{P} = (L^{ir})_{i,r \in \bar{\Upsilon}}$, are obtained by maximizing the likelihood of the model as

$$\begin{aligned} \hat{\mathcal{P}} = \underset{\mathcal{P}}{\operatorname{argmax}} \quad & \prod_{n=1}^{\Gamma} \prod_{i,r \in \bar{\Upsilon}} (L^{ir})^{\Delta N_n^{ir}} \\ \text{subject to } \quad & \forall i \in \bar{\Upsilon}, \sum_{j \in \bar{\Upsilon}} L^{ij} = 1 \end{aligned} \quad (2.9)$$

The maximum of (2.9), is given by

$$\hat{L}^{ij} = \frac{\sum_{n=1}^{\Gamma} \Delta N_n^{ij}}{\sum_{n=1}^{\Gamma} Y_n^i}. \quad (2.10)$$

When Q is large, we can consider that this estimator provide a good estimation of the migration probabilities. This approach can be considered as Trough-The-Cycle (TTC) because the estimation covers the whole time-period.

Remark 1. *Applying this estimation framework on a small time interval provides a Point-In-Time estimation of rating transition probabilities. For instance the migration probabilities on the time interval $[t_{n-1}, t_n]$ can be estimated by*

$$\hat{L}_n^{ij} = \frac{\Delta N_n^{ij}}{Y_n^i}. \quad (2.11)$$

Although it is one the most popular model for financial institutions, it suffers of many criticisms. It can not capture rare events and ignores intermediates transitions.

2.2.2 Homogeneous continuous time approach

This approach is the alternative version of the homogeneous discrete-time approach, where we consider processes evolving in the continuous-time setting described in Section 2.1.2. In this approach, the rating processes $\{Z^q, q \leq Q\}$ are assumed to be continuous-time Markov chains which are independent and homogeneous. For $(i, r) \in \bar{\Upsilon}^2, i \neq r$, we denote as l^{ir} the \mathbf{G} -intensity of transition from rating i to r , such that

$$\text{For } i, r \in \bar{\Upsilon} \ i \neq r, \mathbb{P}(Z_{t+dt}^q = r | Z_t^q = i) = l^{ir} dt + o(dt) \text{ and For } i \in \bar{\Upsilon}, l^i = \sum_{j \neq i}^p l^{ij}.$$

We associate the intensity matrix as

$$l = \begin{pmatrix} -l^1 & \dots & l^{1p} \\ l^{21} & -l^2 & \vdots & \vdots \\ \vdots & & \ddots & \vdots \\ l^{p1} & l^{p2} & \dots & -l^p \end{pmatrix}.$$

For $0 \leq s < t \leq T$, the transition rating probabilities on the period $[s, t]$, is given by

$$L(s, t) = \exp((t - s)l).$$

We denote the intensities parameters as $\mathcal{I} = \{l^{ir}, i, r \in \bar{\Upsilon}, i \neq r\}$. Given the observed path of rating migration histories $(z_u^q)_{0 \leq u \leq T}$, the likelihood writes

$$\mathcal{L}(\mathcal{I}) = \prod_{q=1}^Q \prod_{n=1}^{\Gamma_q} l^{z_{n-1}^q, z_n^q} e^{(t_n^q - t_{n-1}^q) l^{z_{n-1}^q}}, \quad (2.12)$$

with the convention $l^{z_{\Gamma_q-1}^q, z_{\Gamma_q}^q} = 1$. The maximum likelihood estimator can be expressed as

$$\forall i \neq r \in \bar{\Upsilon}^2, \hat{l}^{ir} = \frac{\sum_{l=1}^Q \sum_{n=1}^{\Gamma^l} \Delta N_n^{q,ir}}{\sum_{q=1}^Q \sum_{n=1}^{\Gamma^l} Y_n^{q,i} (t_n^q - t_{n-1}^q)}. \quad (2.13)$$

Under this specification, if the time interval $[0, T]$ is long enough, we can consider that the dynamics of rating migrations does not depend on the business cycle. We can estimate TTC rating transition probabilities as

$$\forall i \neq r \in \bar{\Upsilon}^2, \hat{L}^{ir} = [\hat{L}(0, T)]_{ir} = [\exp(T\hat{l})]_{ir}. \quad (2.14)$$

Remark 2. *The intensities in (2.13) could be estimated on a time intervals $[t_{n-1}, t_n]$ in order to provide a past time series of Point-In-Time transition probabilities.*

2.3 Structural models

Structural models provide dynamic estimation of a credit risk parameter, generally according to the position of specific indicators in regard of pre-specified thresholds. They are initially called structural because they provided, in their first use, a relationship between the default migrations and the asset (capital) structure of the firm. Introduced by [Tobin \(1958\)](#), and popularised by [Black and Scholes \(1973\)](#) and [Merton \(1974\)](#), the first idea behind structural models consists in declaring the default of a firm if its asset value becomes inferior than its liabilities. [Merton \(1969\)](#) imagine the company's equity as a call option on its assets. The probability of default (PD) is estimated as the probability of the asset value falling below the liability threshold at the end of the time horizon. Instead of admitting only the possibility of default at maturity time, [Black and Cox \(1976\)](#) extend this model by postulating that default occurs at the first time that the firm's asset value drops below the liability barrier. This approach have been widely investigated in the literature (see, e.g., [Briys and De Varenne \(1997\)](#), [Fischer et al. \(1989\)](#), [Kim et al. \(1993\)](#), [Leland and Toft \(1996\)](#)). Naturally, structural approaches have then been extended to rating migrations. Nevertheless, [Figlewski et al. \(2012\)](#) highlight that many problems arise including the difficulty to value the firm's complex capital structure especially when the information are private. In response to that criticism, the specification of the latent process evolves differently in the last twenty years (see, e.g., [Bangia et al. \(2002\)](#), [Feng et al. \(2008\)](#), [Gagliardini and Gouriéroux \(2005\)](#), [Nickell et al. \(2000\)](#)). In particular, the ordered Probit model and the Asymptotic Single Risk Factor Model, assume that the latent factor is a function of a systematic and an idiosyncratic factor. The systematic factor may be observable and be represented by a set of macro-economic variables or may be unobservable. In this section, we detail these two approaches.

We describe structural models under the discrete-time setting described in [Section 2.1.1](#).

2.3.1 Observable factor

We decide here to focus on two quite similar and popular structural approaches respectively presented in [Vasicek \(2002\)](#) and [Gagliardini and Gouriéroux \(2005\)](#). In both approaches a latent process S^q , governing the rating transition of a firm q , is expressed as a function of a common factor Θ and an idiosyncratic factor ϵ^q . We assume here that Θ is observable. A natural way to model the systematic and observable factor Θ is to lie it with observable factors.

2.3.1.1 Ordered Probit model

A firm q may jump to a new rating category when a quantitative latent process S^q cross some pre-specified barriers. The rating at time t_n , $n \in \{0, \dots, \Gamma\}$, described by Z_n^q , is given by the position of the latent random variable S_n^q inside a partition of its space value,

$-\infty = C_{p+1} < C_p < \dots < C_1 = +\infty$. Formally,

$$Z_n^q = \sum_{i=1}^p i \mathbb{1}_{\{C_{i+1} \leq S_n^q < C_i\}}. \quad (2.15)$$

The latent process is expressed here as a deterministic affine transformation of Θ and an independent idiosyncratic factor ϵ^q such that

$$S_n^q = \sum_{i=1}^p (\alpha_i + \langle \beta_i, \Theta_{n-1} \rangle + \sigma_i \epsilon_n^q) \mathbb{1}_{\{Z_{n-1}^q = i\}}, \quad (2.16)$$

where α_i is the level of S^q at rating i , β_i represents the sensitivity of rating i of the latent factor to the common factor Θ . σ_i is the volatility of the idiosyncratic risk in rating i and $\{\epsilon_n^q, q = 1, \dots, Q\}$ are independent standard Gaussian variables. The conditional transition probabilities can be computed as, $\forall (i, r) \in \bar{\Upsilon}^2, n \in \{1, \dots, \Gamma\}$,

$$\begin{aligned} L^{ir}(\Theta_{n-1}) &= \mathbb{P} [Z_n^q = r | Z_{n-1}^q = i, \Theta_{n-1}] \\ &= \mathbb{P} [C_{r+1} \leq S_n^q < C_r | Z_{n-1}^q = i, \Theta_{n-1}] \\ &= \mathbb{P} \left[\frac{C_{r+1} - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \leq \epsilon_n^q < \frac{C_r - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \middle| \Theta_{n-1} \right]. \end{aligned}$$

Then we can deduce the PIT probabilities from rating $i \in \bar{\Upsilon}$, as $\forall n \in \{1, \dots, \Gamma\}$,

$$\begin{cases} L^{ir}(\Theta_{n-1}) = \Phi \left(\frac{C_r - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \right) - \Phi \left(\frac{C_{r+1} - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \right), \text{ for } 2 \leq r \leq p-1, \\ L^{i1}(\Theta_{n-1}) = 1 - \Phi \left(\frac{C_2 - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \right), \\ L^{ip}(\Theta_{n-1}) = \Phi \left(\frac{C_p - \alpha_i - \langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} \right). \end{cases} \quad (2.17)$$

The calibration of the parameters $\mathcal{W} = (C_i, \alpha_i, \beta_i, \sigma_i)_{i=1, \dots, p}$ is achieved by maximizing the conditional likelihood function of the model. We deduce that

$$\begin{aligned} \mathcal{W}^* &= \underset{\mathcal{W}}{\operatorname{argmax}} \left\{ \sum_{q=1}^Q \sum_{n=1}^{\Gamma} \sum_{i \neq j} \Delta N_n^{q,ij} \log L^{ij}(\Theta_{n-1}) \right\} \\ &\text{subject to } \sigma_i > 0 \\ &\quad -\infty < C_{p+1} < C_p < \dots < C_1 = +\infty \end{aligned} \quad (2.18)$$

2.3.1.2 Regulatory structural models

This section aims to present the Point-In-Time approach recommended by the regulation Basel II and IFRS9 and widely used by financial institutions. We present two versions. The first one, called Asymptotic Single Risk Factor Model (ASFR), only focuses on the

transition to default. In particular this approach is used for the estimation of unexpected loss in the framework of Basel II (see Section 1.5.3.2). The second one is a multivariate extension, which considers every rating transitions. This approach is used for the estimation of expected loss in the framework of IFRS9 (see Section 1.5.3.1).

Asymptotic Single Risk Factor Model

This model is intensively used by financial institutions. Firstly introduced by Vasicek (2002), it is recommended by Basel II to compute the conditional default probability in the estimation of the unexpected loss. A firm q may jump to default when a quantitative latent process S^q cross a threshold. We define the threshold b , such that an entity q , is considered to be in default if and only if the latent factor $S_n^q < b$. The latent factor can be expressed as

$$\forall n \in \{1, \dots, \Gamma\}, S_n^q = \sqrt{1 - \rho_q} \epsilon_n^q + \sqrt{\rho_q} \Theta_{n-1}. \quad (2.19)$$

where Θ_{n-1} and ϵ_n^q are independent and follow standard normal law, ρ_q is the correlation with the systematic risk. We deduce that S_n^q has a standard normal distribution.

Remark 3. *This framework is usually applied separately for each homogeneous risk classes. Most of the time, $\rho_q = \rho$ is constant for every entity belonging to the same class. We present this approach under this hypothesis.*

We compute the conditional default probability, as

$$\text{for } 1 \leq n < \Gamma, \text{ PD}(\Theta_{n-1}) = \Phi\left(\frac{b - \sqrt{\rho} \Theta_{n-1}}{\sqrt{1 - \rho}}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. The Through-the-cycle (TTC) default probability, denoted PD, is the unconditional default probability. It represents the mean of the default probability over the economic cycle.

$$\text{PD} = \mathbb{E}_{\Theta_{n-1}} [\mathbb{P}(S_{n-1}^q \leq b | \Theta_{n-1})] = \Phi(b).$$

The conditional default probability, or PIT default probability is obtained by deforming the TTC default probability such that

$$\text{PD}(\Theta_{n-1}) = \mathbb{E}[\mathbb{1}_{\{S_n^q \leq b\}} | \Theta_{n-1}] = \Phi\left(\frac{\Phi^{-1}(\text{PD}) - \sqrt{\rho} \Theta_{n-1}}{\sqrt{1 - \rho}}\right).$$

We can deduce the $(1 - \alpha)$ conditional default probability. This estimation is computed as the $1 - \alpha$ value at risk of the Point-in-time default probability.

$$\text{Var}_{1-\alpha}(\text{PD}) = \text{Var}_{1-\alpha}(\mathbb{E}[\mathbb{1}_{\{S_n^q < b\}} | \Theta_{n-1}]) = \Phi\left(\frac{\Phi(\text{PD})^{-1} - \sqrt{\rho} \Phi^{-1}(1 - \alpha)}{\sqrt{1 - \rho}}\right). \quad (2.20)$$

This risk value can be interpreted. There is less than α chance than the economic cycle put the default probability higher than this $1 - \alpha$ conditional default probability. This estimation intervenes in the computation of the unexpected loss (see Section 1.5.3.2).

Multivariate version of AFSR

This section presents the multivariate version of the AFSR model, extended by Yang and Du (2015) and Miu and Ozdemir (2008) for the context of rating migrations. This model is conventionally used to project default probabilities in a forward looking manner for the estimation of the expected loss (EL) according to the regulation IFRS9 (see Regulation (2016)). A firm q , with rating i , may jump to a new rating category when a quantitative latent process S^q crosses some pre-specified barriers. We define the threshold of migrations $-\infty = b_{i,p+1} < \dots < b_{i,1} = +\infty$. An entity q , with rating i , goes to a higher rating than r if and only if the latent factor $S_n^q < b_{i,r+1}$. The latent factor associated to firm q , rated i , can be expressed at time t_n as

$$S_n^q = \sqrt{1 - \rho_i} \epsilon_n^q + \sqrt{\rho_i} \Theta_{n-1}, \quad (2.21)$$

where Θ_{n-1} and ϵ_n^q are independent and follow standard normal law, ρ_i is the correlation of the class i with the systematic risk. We deduce that S_n^q follows a standard normal distribution. We compute the PIT probability of transition from rating $i \in \bar{\Upsilon}$ as

$$\begin{cases} L^{ir}(\Theta_{n-1}) &= \Phi\left(\frac{b_{i,r} - \sqrt{\rho_i} \Theta_{n-1}}{\sqrt{1 - \rho_i}}\right) - \Phi\left(\frac{b_{i,r+1} - \sqrt{\rho_i} \Theta_{n-1}}{\sqrt{1 - \rho_i}}\right), \quad 1 \leq r \leq p-1, \\ L^{ip}(\Theta_{n-1}) &= \Phi\left(\frac{b_{i,p} - \sqrt{\rho_i} \Theta_{n-1}}{\sqrt{1 - \rho_i}}\right) \end{cases} \quad (2.22)$$

The unconditional transition probabilities are given by

$$\begin{cases} L^{ir} &= \mathbb{E}_{\Theta_{n-1}} [\mathbb{P}(b_{i,r} \leq S_n^q \leq b_{i,r+1} | \Theta_{n-1})] = \Phi(b_{i,r}) - \Phi(b_{i,r+1}), \quad 1 \leq r \leq p-1, \\ L^{ip} &= \Phi(b_{i,p}) \end{cases} \quad (2.23)$$

The TTC transition probabilities $\{\hat{L}^{ir}, i, r \in \bar{\Upsilon}\}$ can be estimated with one of the two estimators (2.10) and (2.14). The threshold are then calibrated as

$$\forall (i, r) \in \bar{\Upsilon}, \hat{b}_{i,r} = \Phi^{-1} \left(\sum_{s \geq r} \hat{L}^{is} \right).$$

Then the approach suggests to calibrate the trajectory of the factor Θ and the correlations $(\rho_i)_{i \in \bar{\Upsilon}}$ by maximizing the likelihood of the model. Note that $\forall n \in \{1, \dots, \Gamma\}$, ϵ_n^q is independent with Θ_{n-1} . Therefore knowing the realisation of Θ_{n-1} , the ratings transitions

are independent. We call $\mathcal{W} = \{\Theta_n, \rho_i, 1 \leq n \leq \Gamma, i \in \bar{\Upsilon}\}$. We try to find

$$\mathcal{W}^* = \underset{\mathcal{W}}{\operatorname{argmax}} \left\{ \sum_{q=1}^Q \sum_{n=1}^{\Gamma} \sum_{i \neq r} \Delta N_n^{q,ir} \log L^{ir}(\Theta_{n-1}) \right\} \quad (2.24)$$

subject to $Var[\Theta_{n-1}] = 1, \text{ for } n = 1, \dots, \Gamma$

Then let us consider $Z = (Z_t)_t$ be a multivariate vector of observable macroeconomic variables. The estimate trajectory of the systematic factor Θ can be inferred with the macro-economic variables through a simple linear regression such that

$$\beta^* = \underset{\beta}{\operatorname{argmax}} \sum_{n=0}^{\Gamma-1} (\Theta_n - \langle Z_n, \beta \rangle)^2. \quad (2.25)$$

Finally, the estimation of β^* and equations (2.22) provide rating transition probabilities as functions of the economic cycle.

Remark 4. *In practice, the systemic factor and the coefficients β are sometimes directly estimated with the historic of 1-year default rates $(TD_n)_n$ such that*

$$\forall n \in \{0, \dots, \Gamma - 1\}, \Theta_n = \ln\left(\frac{TD_{n+1}}{1 - TD_{n+1}}\right) = \beta_0 + \sum_{i=0}^n a_i \Theta_{n-i} + \langle Z_n, \beta \rangle. \quad (2.26)$$

In that case, the maximisation of (2.24) is reduce to the estimation of the rating correlations $(\rho_i)_{i \in \bar{\Upsilon}}$.

This framework can be used to determine rating transitions in various states of the economy. Several economical scenarios may be used to project the value of Z and then the value of Θ (according to (2.25) or (2.26)). Rating transitions matrices are forecasted. The last column of these matrices are retained to be the forecasted default probabilities associated to each rating, for the projection of the expected loss in the context of IFRS9 (see (2.3)).

Remark 5. *This approach may be used to realize regulatory stress-testing.*

Remark 6. *Chapter 3 proposes an alternative of this model. We propose a new way to estimate and to project PIT rating migration matrices without using any external covariates.*

2.3.2 Unobservable factor

In this section, we present the ordered probit model where the underlying process Θ , is assumed to be unobservable. We refer to [Feng et al. \(2008\)](#) and [Gagliardini and Gouriéroux \(2005\)](#) for a detailed description of this approach. We consider the same environment as

in Section 2.3.1.1. The dynamic of Θ is assumed to follow auto-regressive dynamics such that

$$\forall n \in \{1, \dots, \Gamma\}, \Theta_n = A\Theta_{n-1} + \nu_n, \quad (2.27)$$

where the matrix A characterizes the covariates dynamics and ν_n are iid standard Gaussian variables. The estimation of parameters $\mathcal{W} = (A, C_{i+1}, \alpha_i, \beta_i, \sigma_i)_{i \in \bar{\Upsilon}}$ involves the computation of the unconditional likelihood such that

$$\begin{aligned} \mathcal{W}^* = \underset{\mathcal{W}}{\operatorname{argmax}} \quad & \mathbb{E}_{\Theta} \left[\left\{ \sum_{q=1}^Q \sum_{n=1}^{\Gamma} \sum_{i \neq r} \Delta N_n^{q,ir} \log L^{ir}(\Theta_{n-1}) \right\} \right] \\ \text{subject to} \quad & \sigma_i > 0 \\ & -\infty < C_{p+1} < C_p < \dots < C_1 = +\infty \end{aligned} \quad (2.28)$$

Estimating \mathcal{W}^* raises the issue of potentially high number of integrals to compute with respect to the variables $(\Theta_0, \dots, \Theta_{\Gamma-1})$. We briefly describe the method presented in [Gagliardini and Gouriéroux \(2005\)](#), which consists on transforming the probability of migrating towards a rating J or above during the time interval $[t_{n-1}, t_n)$, as a linear function of latent covariates. A linear Gaussian model is then obtained. We denote $q_{iJ,n}$ such a probability. Assuming an ordered Probit model, we have

$$\forall (i, J) \in \bar{\Upsilon}^2, n \in \{1, \dots, \Gamma\}, \Phi^{-1}(q_{iJ,n}) = \frac{C_J - \alpha_i}{\sigma_i} - \frac{\langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i}.$$

We can obtain an estimation of $q_{iJ,n}$, denoted $\hat{q}_{iJ,n}$ by using the estimators derived in (2.10) of (2.14) on the interval $[t_{n-1}, t_n]$. The asymptotic normality of the estimator $\hat{q}_{iJ,n}$ writes $\sqrt{Q}(\hat{q}_{iJ,n} - q_{iJ,n}) \xrightarrow{Q \rightarrow +\infty} \mathcal{N}(0, \Omega_{iJ,n}^2)$ as the number of entities Q (and observations of transitions) goes to infinity. Using the Delta method, we deduce that $\sqrt{Q}(\Phi^{-1}(\hat{q}_{iJ,n}) - \Phi^{-1}(q_{iJ,n})) \xrightarrow{Q \rightarrow +\infty} \mathcal{N}(0, \mathcal{O}_{iJ,n}^2)$, where $\mathcal{O}_{iJ,n}^2 \approx (\Phi^{-1})'(\hat{q}_{iJ,n})(\hat{q}_{iJ,n}(1 - \hat{q}_{iJ,n}))$. Indeed $\Omega_{iJ,n}^2$ can be approximated by $\hat{q}_{iJ,n}(1 - \hat{q}_{iJ,n})$ since the number of entities which experience a transition from rating i to rating J or above can be consider as a binomial variable. For a panel of observations sufficiently large, we obtain the following linear Gaussian representation, $\forall n \in \{1, \dots, \Gamma\}$,

$$\begin{cases} \Phi^{-1}(q_{iJ,n}) = \frac{C_J - \alpha_i}{\sigma_i} - \frac{\langle \beta_i, \Theta_{n-1} \rangle}{\sigma_i} + \frac{\mathcal{O}_{iJ,n}}{\sqrt{Q}} \mu_{iJ,n}, \text{ for } i, J \in \bar{\Upsilon}^2 \\ \Theta_n = A\Theta_{n-1} + \nu_n \end{cases} \quad (2.29)$$

where $\mu_{iJ,n}$ and ν_n are iid standard Gaussian variables. The first equation is the measurement equation and the second equation is the transition equation. We denote as $\Psi_n = (\Phi^{-1}(q_{iJ,n}))_{i,j \in \bar{\Upsilon}}$ the vector of information. Let $\hat{\Theta}_n = \mathbb{E}[\Theta_n | \Psi_n]$ be the filtered image of Θ . The Kalman filtering framework provides an estimation of the parameters $\hat{\mathcal{W}} = (\hat{A}, \hat{C}_{i+1}, \hat{\alpha}_i, \hat{\beta}_i, \hat{\sigma}_i)_{i \in \bar{\Upsilon}}$. Finally, the estimated Point-in-time probability of migration

from a rating i toward a rating J or above can be estimated by

$$\forall n \in \{1, \dots, \Gamma\}, \hat{q}_{i,J,n} = \Phi \left(\frac{\hat{C}_J - \hat{\alpha}_i}{\hat{\sigma}_i} - \frac{\langle \hat{\beta}_i, \hat{\Theta}_{n-1} \rangle}{\hat{\sigma}_i} \right). \quad (2.30)$$

2.4 Intensity models

Intensity models have been first introduced by [Jarrow and Turnbull \(1995\)](#) in order to price credit derivatives. In the basic reduced intensity form model, a credit event corresponds to the first jump time of cox process with a constant intensity or time-dependant intensity. The reduced form approach has been widely studied in the credit risk literature, see, e.g., [Duffie et al. \(2007\)](#), [Jarrow et al. \(1997\)](#). This approach has firstly been adapted to the context of rating migration by [Lando \(1998\)](#). In most of existing approaches, the migration dynamics of each credit entity is described by a stochastic intensity matrix (or generator matrix) whose components depends on a pool of macro-economic variables. The literature on the subject is now very extensive (see, e.g., [Damian et al. \(2018\)](#), [Figlewski et al. \(2012\)](#), [Kavvathas \(2001\)](#), [Koopman et al. \(2008\)](#), [Lando and Skødeberg \(2002\)](#), [Leijdekker and Spreij \(2011\)](#)). [Lando and Skødeberg \(2002\)](#) underline many advantages of using such models such as the ease to formulate, to test, and to integrate dependencies on external covariates. The intensity models are also well suited to manage censoring. Lastly, [Koopman et al. \(2008\)](#) extend the multi-factor intensity approach to integrate simultaneously obligor-specific information as well as common observable factors (macro-economic variables) or common unobserved latent factors.

In this section, we describe the multi-state factor intensity models. We refer to [Cousin and Kheliouen \(2015\)](#) and [Koopman et al. \(2008\)](#) for details and proofs on the presented results. We describe intensity models under the continuous-time setting described in Section [2.1.2](#).

Given the history of a process Θ , the rating migration process $\{Z^q, q \leq Q\}$ are assumed to be conditionally independent continuous-time Markov chains. We define the generator matrix

$$\forall t \in [0, T], l_{\Theta}(t) = \begin{pmatrix} -l^1(\Theta_t) & \dots & l^{1p}(\Theta_t) \\ l^{21}(\Theta_t) & -l^2(\Theta_t) & \vdots & \vdots \\ \vdots & & \ddots & \vdots \\ l^{p1}(\Theta_t) & l^{p2}(\Theta_t) & \dots & -l^p(\Theta_t) \end{pmatrix}$$

, where for $(i, r) \in \Upsilon^2$, $i \neq r$, the product $l^{ir} dt$ corresponds to the conditional probability of going from rating i to rating r in the small interval $[t, t + dt]$. Formally

$$\forall (i, r) \in \Upsilon^2, i \neq r, \mathbb{P}[Z_{t+dt}^q = r | Z_t^q = i, \Theta_t] \approx l^{ir}(\Theta_t) dt.$$

We distinguish the cases where Θ is observable and unobservable.

2.4.1 Observable factor

We assume here that the common factor Θ is a multivariate observable factor and that migration intensities are an exponential-affine transformation of Θ such as

$$\forall (i, r) \in \bar{\Upsilon}^2, i \neq r, l^{ir}(\Theta_t) = \exp(\alpha_{ir} + \langle \beta_{ir}, \Theta_t \rangle), \quad (2.31)$$

where α_{ir} is a constant parameter and β_{ir} accounts for the sensitivity of migration intensity to the common factor Θ . This setting has been investigated by among others [Cousin and Kheliouen \(2015\)](#), [Kavvathas \(2001\)](#), [Koopman et al. \(2008\)](#), [Lando and Skødeberg \(2002\)](#). We call $\mathcal{W} = \{\alpha_{ir}, \beta_{ir}, (i, r) \in \bar{\Upsilon}^2, i \neq r\}$. The parameters \mathcal{W} are estimated by maximizing the conditional likelihood of the model

$$\mathcal{L}_\Theta(\mathcal{W}) = \prod_{q=1}^Q \prod_{n=1}^{\Gamma^q} l^{Z_{n-1}^q, Z_n^q}(\Theta_{t_{n-1}^q}) e^{-\int_{t_{n-1}^q}^{t_n^q} l^{Z_{n-1}^q}(X_u) du}, \quad (2.32)$$

with the convention $l^{Z_{\Gamma^q-1}^q, Z_{\Gamma^q}^q} = 1$. Similar criticisms to those made against structural models with observable factors can be made against this approach. Choosing macro-economic variables presents a risk of excluding relevant others and losing information. Furthermore the importance of variables is not static but dynamic. Therefore, the set of relevant variables must be frequently updated.

2.4.2 Unobservable factor

We consider here that the migration intensities follow similar specifications than in [Section 2.4.1](#) but the driving process Θ is now assumed unobservable. The estimation of model parameters in [\(2.31\)](#) may be obtained by maximizing the likelihood of the model

$$\mathcal{L}(\mathcal{W}) = \mathbb{E}_\Theta \left[\prod_{q=1}^Q \prod_{n=1}^{\Gamma^q} l^{Z_{n-1}^q, Z_n^q}(\Theta_{t_{n-1}^q}) e^{-\int_{t_{n-1}^q}^{t_n^q} l^{Z_{n-1}^q}(X_u) du} \right]. \quad (2.33)$$

As for the structural approach, this expectation must be taken over the joint distribution of Θ which can be very intensive and not computationally tractable. Only numerical and approximated approaches have emerged to tackle this issue. For instance, [Koopman et al. \(2008\)](#) propose to use a sampling framework with Monte Carlo techniques to estimate such a likelihood. [Cousin and Kheliouen \(2015\)](#) try to apply the same framework applied in [Section 2.3.2](#) for structural models, to intensity models. We briefly describe here such an extension. This recursive method needs a discrete grid. We then consider the discrete-time setting described in [Section 2.1.1](#). We assume that Θ only changes at time $0 = t_0 < t_1 < \dots < t_\Gamma = T$. The dynamic of Θ is assumed to follow auto-regressive dynamics

$$\forall n \in \{1, \dots, \Gamma\}, \Theta_n = A\Theta_{n-1} + \nu_n, \quad (2.34)$$

where the matrix A characterizes the covariates dynamics and ν_n are iid standard Gaussian variables. The estimation of parameters $\mathcal{W} = (A, \alpha_{ir}, \beta_{ir})_{i,r \in \bar{\Upsilon}}$ can be obtained by transforming the log-intensities of jumps as a linear function of latent covariates and thus construct a linear Gaussian model. Let's remark that

$$\text{For } 1 \leq n \leq \Gamma, (i, r) \in \bar{\Upsilon}^2, i \neq r, \log(l_n^{ir}) = \alpha_{ir} + \langle \beta_{ir}, \Theta_{n-1} \rangle.$$

We can obtain estimation of $\log(l_n^{ir})$, denoted $\log(\hat{l}_n^{ir})$ by using the estimators derived in (2.14) on the interval $[t_{n-1}, t_n]$. A time series of log-intensities $\log(\hat{l}_n^{ir})$ can be built. The asymptotic normality of the maximum likelihood \hat{l}_n^{ir} writes $\sqrt{Q}(\hat{l}_n^{ir} - l_n^{ir}) \xrightarrow{Q \rightarrow +\infty} \mathcal{N}(0, \Omega_{ir,n}^2)$.

Using the Delta method, we deduce that $\sqrt{Q}(\log(\hat{l}_n^{ir}) - \log(l_n^{ir})) \xrightarrow{Q \rightarrow +\infty} \mathcal{N}(0, \mathcal{O}_{ir,n}^2)$, where

$\mathcal{O}_{ir,n}^2 \approx \frac{\Omega_{ir,n}^2}{(j_n^{ir})^2}$. Indeed $\Omega_{ir,n}^2$ can be approximated by $\hat{\Omega}_{ir,n}^2 = \frac{\Delta N_n^{ir}}{(Y_n^i)^2}$ according to Hougaard (2000). For a panel of observations sufficiently large, we obtain the following linear Gaussian representation

$$\begin{cases} \log(\hat{l}_n^{ir}) \approx \alpha_{ir} + \langle \beta_{ir}, \Theta_{n-1} \rangle + \mathcal{O}_{ir,n} \mu_{ir,n}, \quad \forall (i, r) \in \bar{\Upsilon}^2, i \neq r, \\ X_n = AX_{n-1} + \nu_n \end{cases} \quad (2.35)$$

where $\mu_{ir,n}$ and ν_n are iid standard Gaussian variables. The first equation is the measurement equation and the second equation is the transition equation. Using Kalman filtering, the hidden process Θ can be recursively filtered. We denote as $\Psi_n = \{\log(\hat{l}_n^{ir}), i, r \in \bar{\Upsilon}, i \neq r\}$ the vector of information. Let $\hat{\Theta}_n = \mathbb{E}[\Theta_n | \Psi_n]$ be the filtered image of Θ . The Kalman filtering framework provides an estimation of the parameters $\hat{\mathcal{W}} = (\hat{A}, \hat{\alpha}_{ir}, \hat{\beta}_{ir})_{i,r \in \bar{\Upsilon}}$. Finally we define the estimated Point-in-time migration intensities as

$$\text{For } i, r \in \bar{\Upsilon}, i \neq r, \hat{l}^{ir}(\hat{\Theta}_{n-1}) = \exp(\hat{\alpha}_{ir} + \langle \hat{\beta}_{ir}, \hat{\Theta}_{n-1} \rangle), \quad (2.36)$$

2.5 Comparison analysis

In conclusion, both structural models and intensity models have their advantages and disadvantages in predicting rating migrations. When it comes to the use of observable versus unobservable factors, there is a trade-off to consider. Models that incorporate observable factors, such as market prices or financial ratios, are more transparent and easier to interpret. However, they may not capture all of the relevant information about a borrower's creditworthiness.

According to Cousin and Kheliouen (2015), intensity models with observable factors fit very well the empirical transition probabilities especially during periods of stress. In contrast, structural models with observable factors tend to over-estimate transition probabilities during flat periods. They also observe that the intensity model with unobservable factors shows better adjustment and responsiveness to the business cycle than the structural model with unobservable factors. However, it still struggles to accurately replicate the

empirical dynamic transitions during periods of decline. The size of the sample may be too small to assume normality behaviour. In general, models that incorporate observable factors tend to outperform those based on unobservable factors, since the assumptions underlying the latter can be overly restrictive and limit their applicability. However, choosing macro-economic variables presents a risk of excluding relevant others and losing information. Additionally, the significance of variables is not static but dynamic, and the set of relevant variables must be updated frequently.

In this debate, we aim to propose an alternative approach in Chapter 3, where the governing factor is unobservable, and the hidden process is assumed to be a Markov Chain.

Chapter 3

Rating transitions forecasting: a filtering approach

Abstract

Analyzing the effect of business cycle on rating transitions has been a subject of great interest these last fifteen years, particularly due to the increasing pressure coming from regulators for stress testing. In this chapter, we consider that the dynamics of rating migrations, in a pool of credit references, is governed by a common unobserved latent Markov chain. We explain how the current state of the hidden factor, can be efficiently inferred from observations of rating histories. We then adapt the classical Baum-Welch algorithm to our setting and show how to estimate the latent factor parameters. Once calibrated, we may reveal and detect economic changes affecting the dynamics of rating migration, in real-time. The filtering formula is then used to predict future transition probabilities according to the economic cycle without using any external covariates. We propose two filtering frameworks: a discrete and a continuous version. We demonstrate and compare the efficiency of both approaches on fictive data and on a corporate credit rating database. The methods could also be applied to retail credit loans.

Note that this Chapter 3 that is based on the article [Cousin et al. \(2023a\)](#).

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1 Introduction

Accurate credit ratings stand out as central to Basel II and to most aspects of managing credit risk. In recent years, analysts have introduced the concepts of PIT and TTC ratings, concepts that purportedly distinguish between some of the existing ratings systems and models. PIT models can be classified in two categories, the structural and the intensity models. In both structural and intensity models, the factors may be considered observable or unobservable. The second approach has emerged in response to criticisms made against the first. As [Gagliardini and Gouriéroux \(2005\)](#) point out, the risk in selecting covariates lies in excluding others which could be more relevant. [Cousin and Kheliouen \(2015\)](#) provide an overview of usual modelling and estimation approaches and compare the estimation and the predictive performance of each approach on real data. When the underlying factors are unobservable, they adapt a method given in [Gagliardini and Gouriéroux \(2005\)](#) to represent the considered factor migration model as a linear Gaussian model, and apply a Kalman filter to predict the state of the underlying latent factor. This approximation lies on the hypothesis that the data set is large enough to apply asymptotic normality. This assumption may be too restrictive and may explain the poor quality of predictions obtained by [Cousin and Kheliouen \(2015\)](#).

A natural alternative consists in directly filtering the hidden factor given rating transitions' past history. For a bond portfolio, the dynamics of rating migrations can mathematically be represented as a multivariate counting process, each component representing the cumulative number of transitions from one rating category to another. Estimating the hidden factor dynamics by only using observations of the counting process has already been considered in the credit risk literature. For instance, [Fontana and Runggaldier \(2010\)](#) and [Frey and Schmidt \(2012\)](#) follow this approach for pricing derivatives under incomplete information.

A realistic and standard setting assumes that the unobserved driving factor is given as a finite state Markov chain and that the rating transition process follows a Hidden Markov model (HMM). [Brémaud \(1981\)](#) and [Elliott et al. \(2008\)](#) respectively present a detailed analysis of continuous-time and discrete-time filtering under special HMM assumptions. Hidden Markov Chain modeling (HMM) remains a popular approach in credit risk analysis (see e.g., [Ching et al. \(2009\)](#), [Elliott et al. \(2008\)](#), [Elliott et al. \(2014\)](#), [Thomas et al. \(2002\)](#)). The hidden process can have different interpretations according to the assumptions made and the way to filter. In the credit rating literature, [Korolkiewicz and Elliott](#)

(2008) assume that the observed rating of a firm is a noisy observation of its true credit rating, represented by a hidden Markov chain. They apply an Expectation Maximisation (EM) algorithm for hidden Markov models under a discrete-time setting. They calibrate the filtering formula applied to Markov chains, derived in Elliott et al. (2008), to infer from its rating evolution, the true credit quality of a firm. Damian et al. (2018) extend the parameter estimation via the EM algorithm to continuous-time hidden Markov models. Similarly, they infer the true credit quality from rating observations but also with credit spreads. In these studies, each firm has its own true rating process, therefore its rating dynamic is governed by its own hidden process. Then the dynamic of rating transition of entities are governed by independent and identically distributed hidden Markov chains. Therefore, rating observations are assumed to be independent and an aggregated calibration procedure can be made. In the context of this study, the hidden factor is interpreted as a systematic and common factor, governing transitions of all firms. Among the studies which share the same interpretation, Giampieri et al. (2005), also use the classical Baum-Welch algorithm (introduced in Baum et al. (1970)), for estimation of a two-state hidden factor driving occurrence of defaults. They obtain estimates for the model parameters and are able to reconstruct the most likely past sequence of the hidden factor. Their approach only holds for an unique transition and is not suitable for providing online estimations of the hidden factor state. In the same vein Deroose et al. (2008) and Oh et al. (2019) identify two states, one of expansion and the other of contraction. In particular, Oh et al. (2019) use an extension of the Baum-Welch algorithm adapted to “regime switching hidden Markov model” (RSMC) to forecast sovereign credit rating transitions. In a different scope, they also assume that every rating processes are governed by independent and identically distributed Markov chains.

In this section, under a point process filtering framework, we first derive general discrete-time and continuous-time filtering formulas. We then apply the filtering frameworks to credit migrations, and show how to infer the current state of the hidden factor from past rating transitions. An EM algorithm is adapted to estimate the parameters involved. Contrary to Damian et al. (2018), Korolkiewicz and Elliott (2008), Oh et al. (2019), we assume that the dynamics of rating migrations in a pool of credit references, is governed by a common unobserved latent Markov chain, which aims to represent the economic cycle. Therefore the realization of the unobservable factor is assumed to be common to every firm whereas one hidden factor per bond is considered in Damian et al. (2018), Korolkiewicz and Elliott (2008), Oh et al. (2019). We believe that our approach which rather keeps the dependencies within the observations sample, is reliable and realistic. Indeed, rating entities should be affected by the same realization of the economic factor. This different consideration changes the way to calibrate and to filter: our filtering framework uses the whole history of aggregated number of jumps. Once calibrated, we may reveal and detect economic changes affecting the dynamics of rating migration, in real-time. By updating the filtered factor, we are able to forecast rating transitions according to these economic changes. Our approach may be considered as a new Point-in-time (PIT) rating transitions modeling which does not use any macro-economic factors.

Behind every model mentioned, choosing a continuous or discrete approach is crucial and is a matter of debate. This study aims at participating to this debate by presenting different results: we adapt filtering formulas, derived under special HMM assumptions in [Brémaud \(1981\)](#) and [Elliott et al. \(2008\)](#), to migration ratings context, both in a continuous-time and discrete-time setting. In particular, we show how to adapt the continuous-time filtering framework to handle discrete-time data and simultaneous jumps. We assess and compare both approaches on a fictive data set and on a Moody's ratings history [01/2000-05/2021] of a diversified portfolio of 5030 corporate entities.

This chapter is structured into several sections. Firstly, in [Section 2](#), a general filtering framework in a discrete-time setting is presented. Then, [Section 3](#) describes the discrete-time filtering framework adapted to credit rating migrations. Correspondingly, [Section 4](#) is dedicated to describe the filtering framework in a continuous-time setting while [Section 5](#) presents its adaptation to the same context. [Section 6](#) illustrates and validates the two filtering approaches on fictive data. Then, in [Section 7](#), we compare the two filters on real data sets.

2 General discrete-time version of the filter

In this section, we present a discrete-time filtering framework both in a univariate and multivariate setting.

2.1 Univariate Form

Let $\Gamma \in \mathbb{N}$ be the discrete time horizon. We work with the filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in \{0, \dots, \Gamma\}}, \mathbb{P})$. Let N be a discrete-time \mathbf{F} -adapted counting process starting from 0. Let \mathbf{F}^N be the natural filtration of N , augmented with \mathbb{P} -null sets. We write

$$\forall n \in \{1, \dots, \Gamma\}, \Delta N_n = N_n - N_{n-1}.$$

We denote by \mathbb{J} , the support of the jumps of N ($\mathbb{J} = \mathbb{N}$ for a Poisson process). The support could vary over time but as this would not impact our results, it is assumed constant for the sake of clearness. To account for the times when N does not jump, we assume that $0 \in \mathbb{J}$ and we define $\bar{\mathbb{J}} = \mathbb{J} \setminus \{0\}$ as the support of the true jumps. Let Θ be a square integrable \mathbf{F} -adapted process such that conditionally on the σ -field \mathcal{F}_{n-1} , Θ_n and ΔN_n are independent. Let \mathbf{F}^Θ be the natural filtration of Θ , augmented with \mathbb{P} -null sets. Θ has a natural decomposition of the form

$$\Theta_n = A_n + M_n,$$

where A is a \mathbf{F} -predictable square integrable process and M is a square integrable \mathbf{F} -martingale.

We define the filtration $\mathbf{G} = (\mathcal{G}_n)_{n \in \{0, \dots, \Gamma\}}$ by $\mathcal{G}_n = \mathcal{F}_n^\theta \vee \mathcal{F}_{n-1}^N$. For $n \in \{1, \dots, \Gamma\}$, we introduce

$$\hat{O}_n = \mathbb{E}[O_n | \mathcal{F}_n^N], \quad \tilde{O}_{n-1} = \mathbb{E}[O_n | \mathcal{F}_{n-1}].$$

For $n \in \{1, \dots, \Gamma\}$ and $j \in \mathbb{J}$, we define

$$\epsilon_n^j = \mathbb{1}_{[\Delta N_n = j]}, \quad \lambda_{n-1}^j = \mathbb{E}[\epsilon_n^j | \mathcal{F}_{n-1}], \quad \hat{\lambda}_{n-1}^j = \mathbb{E}[\lambda_{n-1}^j | \mathcal{F}_{n-1}^N] = \mathbb{E}[\mathbb{E}[\epsilon_n^j | \mathcal{F}_{n-1}] | \mathcal{F}_{n-1}^N].$$

Note that, for all n , we have $\sum_{j \in \mathbb{J}} \epsilon_n^j = \sum_{j \in \mathbb{J}} \lambda_n^j = \sum_{j \in \mathbb{J}} \hat{\lambda}_n^j = 1$. Under this setting, we present a discrete time adaptation of the univariate filtering equation presented in [Brémaud \(1981\)](#).

Proposition 7. *The filtered process $\hat{\Theta}$ satisfies the following equation*

$$\hat{\Theta}_n = \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{\Theta} \lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} \mathbb{1}_{[\Delta N_n = j]}, \quad \forall n = 1, \dots, \Gamma. \quad (3.1)$$

The proof of Proposition 7 heavily relies on the following lemma.

Lemma 8. *Let K be a square integrable and \mathbf{G} -adapted process such that \hat{K} is a \mathbf{F}^N -martingale. Then, K is solution of the following recursive equation*

$$\hat{K}_n = K_0 + \sum_{k=1}^n \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{K} \lambda^j)_{k-1}}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j), \quad \forall n = 1, \dots, \Gamma. \quad (3.2)$$

Remark 9. *If K is also a \mathbf{F} -martingale then the previous equation also holds true with $\tilde{K} = K$.*

Proof (of Lemma 8). Let P be a square integrable \mathbf{F}^N -martingale. Then, there exists a measurable function g such that $P_n = g(n, N_0, \dots, N_n)$. It can equivalently be written $P_n = h(n, \Delta N_1, \dots, \Delta N_n)$.

We can write

$$P_n = \sum_{j \in \mathbb{J}} h(n, \Delta N_1, \dots, \Delta N_{n-1}, j) \epsilon_n^j.$$

Since P is a \mathbf{F}^N -martingale,

$$P_n - P_{n-1} = P_n - \mathbb{E}[P_n | \mathcal{F}_{n-1}^N] = \sum_{j \in \mathbb{J}} h(n, \Delta N_1, \dots, \Delta N_{n-1}, j) (\epsilon_n^j - \hat{\lambda}_{n-1}^j).$$

Then, P has the following martingale representation

$$P_n = P_0 + \sum_{k=1}^n \sum_{j \in \mathbb{J}} H_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j),$$

with H_{k-1}^j being \mathcal{F}_{k-1}^N -measurable. Conversely, it is easy to check that a process written like that is a \mathbf{F}^N -martingale.

Since $\epsilon_k^0 = 1 - \sum_{j \in \bar{\mathbb{J}}} \epsilon_k^j$ and $\hat{\lambda}_k^0 = 1 - \sum_{j \in \bar{\mathbb{J}}} \hat{\lambda}_k^j$, we can rewrite P with the following martingale representation

$$P_n = P_0 + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j), \quad (3.3)$$

with $W_{k-1}^j = H_{k-1}^j - H_{k-1}^0$.

Let K be a square integrable \mathbf{G} -adapted process such that \hat{K} is a \mathbf{F}^N -martingale. According to (3.3), we can write $\hat{K}_n = \zeta + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j)$, where the sequence $(W_k)_k$ is \mathcal{F}^N -adapted. For any square integrable \mathbf{F}^N -adapted process X , we have $\mathbb{E}[(K_n - \hat{K}_n)X_n] = 0$. Choosing X to be a \mathbf{F}^N -martingale with the decomposition $g + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j)$ with $(G_k)_k$ \mathbf{F}^N -adapted, we obtain

$$\mathbb{E} \left[\left(K_n - \zeta - \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right) \left(g + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right) \right] = 0.$$

Choosing $G_{k-1}^j = 0$ for all $j \in \bar{\mathbb{J}}$ leads to $\zeta = \mathbb{E}[\hat{K}_n] = \mathbb{E}[K_n] = K_0$.

With no loss of generality, we can consider that $K_0 = 0$. By choosing $g = 0$ we obtain:

$$\mathbb{E} \left[K_n \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] - \mathbb{E} \left[\sum_{j_1, j_2 \in \bar{\mathbb{J}}} \sum_{k_1, k_2 \geq 1} W_{k_1-1}^{j_1} (\epsilon_{k_1}^{j_1} - \hat{\lambda}_{k_1-1}^{j_1}) G_{k_2-1}^{j_2} (\epsilon_{k_2}^{j_2} - \hat{\lambda}_{k_2-1}^{j_2}) \right] = 0. \quad (3.4)$$

For $k_1 < k_2$,

$$\mathbb{E} \left[\mathbb{E} \left[W_{k_1-1}^{j_1} (\epsilon_{k_1}^{j_1} - \hat{\lambda}_{k_1-1}^{j_1}) G_{k_2-1}^{j_2} (\epsilon_{k_2}^{j_2} - \hat{\lambda}_{k_2-1}^{j_2}) \mid \mathcal{F}_{k_2-1}^N \right] \right] = 0$$

Noticing that $\mathbb{E} [K_n \mid \mathcal{F}_k^N] = \mathbb{E} [\mathbb{E} [K_n \mid \mathcal{F}_n^N] \mid \mathcal{F}_k^N] = \hat{K}_k$, we compute the term

$$\begin{aligned} S_{n,k} &= \mathbb{E} \left[K_n G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] = \mathbb{E} \left[\mathbb{E} [K_n \mid \mathcal{F}_k^N] G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] \\ &= \mathbb{E} \left[\hat{K}_k G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] = \mathbb{E} \left[K_k G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right]. \end{aligned}$$

Now, let us compute

$$\begin{aligned}\mathbb{E}[K_k(\epsilon_k^j - \hat{\lambda}_{k-1}^j)|\mathcal{F}_{k-1}^N] &= \mathbb{E}[K_k\epsilon_k^j|\mathcal{F}_{k-1}^N] - \mathbb{E}[K_k\hat{\lambda}_{k-1}^j|\mathcal{F}_{k-1}^N] \\ &= \mathbb{E}[\mathbb{E}[K_k\epsilon_k^j|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}^N] - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j.\end{aligned}$$

Remember that K is \mathbf{G} -measurable and that $\mathcal{G}_k = \mathcal{F}_k^\theta \vee \mathcal{F}_{k-1}^N$. The conditional independence of Θ_k and ΔN_k knowing \mathcal{F}_{k-1} yields the conditional independence of K_k and ϵ_k . Hence, we obtain

$$\begin{aligned}\mathbb{E}[\mathbb{E}[K_k\epsilon_k^j|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}^N] - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j &= \mathbb{E}[\mathbb{E}[K_k|\mathcal{F}_{k-1}]\mathbb{E}[\epsilon_k^j|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}^N] - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j \\ &= \widehat{(\tilde{K}\lambda^j)}_{k-1} - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j.\end{aligned}$$

From 3.4, we obtain

$$\mathbb{E}\left[\sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} G_{k-1}^j \left(\widehat{(\tilde{K}\lambda^j)}_{k-1} - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j - \mathbb{E}\left[\sum_{i \in \bar{\mathbb{J}}} W_{k-1}^i(\epsilon_k^i - \hat{\lambda}_{k-1}^i)(\epsilon_k^j - \hat{\lambda}_{k-1}^j)|\mathcal{F}_{k-1}^N\right]\right)\right] = 0$$

As this holds true for any \mathbf{F}^N -adapted process $(G_k)_k$, we can choose

$$G_{k-1}^j = \widehat{(\tilde{K}\lambda^j)}_{k-1} - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j - \mathbb{E}\left[\sum_{i \in \bar{\mathbb{J}}} W_{k-1}^i(\epsilon_k^i - \hat{\lambda}_{k-1}^i)(\epsilon_k^j - \hat{\lambda}_{k-1}^j)|\mathcal{F}_{k-1}^N\right].$$

Then, we deduce the following system of linear equations, $\forall j \in \bar{\mathbb{J}}, k \in \{1, \dots, n\}$,

$$\sum_{i \in \bar{\mathbb{J}}} W_{k-1}^i \mathbb{E}\left[(\epsilon_k^i - \hat{\lambda}_{k-1}^i)(\epsilon_k^j - \hat{\lambda}_{k-1}^j)|\mathcal{F}_{k-1}^N\right] = \widehat{(\tilde{K}\lambda^j)}_{k-1} - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j \quad a.s. \quad (3.5)$$

Let $c_j^{(k-1)} = \widehat{(\tilde{K}\lambda^j)}_{k-1} - \hat{K}_{k-1}\hat{\lambda}_{k-1}^j$. Note that

$$\mathbb{E}\left[(\epsilon_k^i - \hat{\lambda}_{k-1}^i)(\epsilon_k^j - \hat{\lambda}_{k-1}^j)|\mathcal{F}_{k-1}^N\right] = \begin{cases} -\hat{\lambda}_{k-1}^i \hat{\lambda}_{k-1}^j & \text{for } i \neq j \\ \hat{\lambda}_{k-1}^j(1 - \hat{\lambda}_{k-1}^j) & \text{for } i = j \end{cases}$$

The equations in (3.5) can be written for all $j \in \bar{\mathbb{J}}$

$$\hat{\lambda}_{k-1}^j(1 - \hat{\lambda}_{k-1}^j)W_{k-1}^j - \sum_{\substack{i \in \bar{\mathbb{J}} \\ i \neq j}} \hat{\lambda}_{k-1}^i \hat{\lambda}_{k-1}^j W_{k-1}^i = c_j^{(k-1)}.$$

Then, we obtain for all $j \in \bar{\mathbb{J}}$

$$\hat{\lambda}_{k-1}^j W_{k-1}^j - \hat{\lambda}_{k-1}^j \sum_{i \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^i W_{k-1}^i = c_j^{(k-1)}. \quad (3.6)$$

By summing for $j \in \bar{\mathbb{J}}$, we deduce from (3.6),

$$\sum_{j \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^j W_{k-1}^j - \sum_{j \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^j \sum_{i \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^i W_{k-1}^i = \sum_{j \in \bar{\mathbb{J}}} c_j^{(k-1)}.$$

Then

$$\hat{\lambda}_{k-1}^0 \sum_{j \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^j W_{k-1}^j = \sum_{j \in \bar{\mathbb{J}}} c_j^{(k-1)}.$$

Inserting the expression of $\sum_{i \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^i W_{k-1}^i$ in (3.6) gives

$$W_{k-1}^j = \frac{c_j^{(k-1)}}{\hat{\lambda}_{k-1}^j} + \frac{\sum_{i \in \bar{\mathbb{J}}} c_i^{(k-1)}}{\hat{\lambda}_{k-1}^0}.$$

Then, we obtain

$$W_{k-1}^j = \frac{1}{\hat{\lambda}_{k-1}^j} (\widehat{(\tilde{K} \lambda^j)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j) + \frac{1}{\hat{\lambda}_{k-1}^0} \sum_{i \in \bar{\mathbb{J}}} (\widehat{(\tilde{K} \lambda^i)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^i).$$

Finally, by replacing $\sum_{j \in \bar{\mathbb{J}}} \epsilon_k^j$, $\sum_{j \in \bar{\mathbb{J}}} \hat{\lambda}_{k-1}^j$ by $1 - \epsilon_k^0$ and $1 - \hat{\lambda}_{k-1}^0$, we derive the general filtering formula

$$\begin{aligned} \hat{K}_n &= K_0 + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} \left(\frac{\widehat{(\tilde{K} \lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} - \frac{\widehat{(\tilde{K} \lambda^0)}_{k-1}}{\hat{\lambda}_{k-1}^0} \right) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \\ &= K_0 + \sum_{k=1}^n \sum_{j \in \bar{\mathbb{J}}} \frac{\widehat{(\tilde{K} \lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j), \end{aligned}$$

This finishes the proof. ■

Proof (of Proposition 7). We define

$$a_n = A_n - A_{n-1} = \mathbb{E}[\Theta_n | \mathcal{F}_{n-1}] - \Theta_{n-1}, \quad (3.7)$$

$$B_n = \sum_{k=1}^n \mathbb{E}[a_k | \mathcal{F}_{k-1}^N]$$

$$L_n = \sum_{k=1}^n a_k - \mathbb{E}[a_k | \mathcal{F}_{k-1}^N].$$

Note that

$$\Theta_n - B_n = M_n + L_n.$$

We know that \hat{M} is a \mathbf{F}^N -martingale and \hat{L} is clearly a \mathbf{F}^N -martingale too. Then, we can apply Lemma 8 to $\Theta - B$. Note that B is \mathbf{F}^N predictable, then $\tilde{B}_{k-1} = B_k$ and so $\widehat{\tilde{B}\lambda^j}_{k-1} = B_k \hat{\lambda}_{k-1}^j$. Then,

$$\begin{aligned} \hat{\Theta}_n - \hat{B}_n &= \hat{\Theta}_0 - \hat{B}_0 + \sum_{k=1}^n \sum_{j \in \mathbb{J}} \left(\frac{\widehat{((\tilde{\Theta} - \tilde{B})\lambda^j)_{k-1}}}{\hat{\lambda}_{k-1}^j} \right) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \\ &= \hat{\Theta}_0 - \hat{B}_0 + \sum_{k=1}^n \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{\Theta}\lambda^j)_{k-1}}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j). \end{aligned}$$

We compute $\hat{\Theta}_n - \hat{\Theta}_{n-1} = \hat{B}_n - \hat{B}_{n-1} + f(\lambda_{n-1}, \epsilon_n, \tilde{\Theta}_{n-1}) = \mathbb{E}[a_n | \mathcal{F}_{n-1}^N] + f(\lambda_{n-1}, \epsilon_n, \tilde{\Theta}_{n-1})$. From (3.7), $\mathbb{E}[a_n | \mathcal{F}_{n-1}^N] = \mathbb{E}[\Theta_n | \mathcal{F}_{n-1}^N] - \hat{\Theta}_{n-1}$ and using that

$$f(\lambda_{n-1}, \epsilon_n, \tilde{\Theta}_{n-1}) = \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{\Theta}\lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} (\epsilon_n^j - \hat{\lambda}_{n-1}^j) = \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{\Theta}\lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} \mathbf{1}_{[\Delta N_n = j]} - \mathbb{E}[\Theta_n | \mathcal{F}_{n-1}^N],$$

we deduce the final form of the filtering formula

$$\hat{\Theta}_n = \sum_{j \in \mathbb{J}} \frac{\widehat{(\tilde{\Theta}\lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} \mathbf{1}_{[\Delta N_n = j]}.$$

Note that, from (3.7), the previous formula can also be stated as

$$\hat{\Theta}_n = \sum_{j \in \mathbb{J}} \left(\frac{\widehat{(a\lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} + \frac{\widehat{(\Theta\lambda^j)_{n-1}}}{\hat{\lambda}_{n-1}^j} \right) \mathbf{1}_{[\Delta N_n = j]}.$$

where $\widehat{(a\lambda^j)_{n-1}} = \mathbb{E}[a_n \lambda_{n-1}^j | \mathcal{F}_{n-1}^N]$. ■

This formula can be extended to a multivariate setting.

2.2 Multivariate form

Let $\Gamma \in \mathbb{N}$ be the discrete time horizon. We work with the filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in \{0, \dots, \Gamma\}}, \mathbb{P})$. Let $N = (N^1, \dots, N^\rho)$ be a discrete-time multivariate counting process where for $i = 1, \dots, \rho$, $N^i = (N_n^i)_{n \in \{0, \dots, \Gamma\}}$, is a be a simple \mathbf{F} -adapted counting processes starting from 0. Let \mathbf{F}^N be the natural filtration of N , augmented with \mathbb{P} -null

sets. We write

$$\forall i \in \{1, \dots, \rho\}, \forall n \in \{1, \dots, \Gamma\}, \Delta N_n^i = N_n^i - N_{n-1}^i.$$

We denote by \mathbb{J}_i the support of the jumps of N^i , for $i = 1, \dots, \rho$. The support may vary over time but as this would not impact our results, it is assumed constant for the sake of clearness. To account for the times when N^i does not jump, we assume that $0 \in \mathbb{J}_i$ and we define $\bar{\mathbb{J}}_i = \mathbb{J}_i \setminus \{0\}$ as the support of the true jumps. Let us define the product spaces $\mathbb{J}^\otimes = \prod_{i=1}^\rho \mathbb{J}_i$ and $\bar{\mathbb{J}}^\otimes = \prod_{i=1}^\rho \bar{\mathbb{J}}_i$.

Let Θ be a square integrable \mathbf{F} -adapted process such that conditionally on the σ -field \mathcal{F}_{n-1} , Θ_n and $(\Delta N_n^i)_{i=1, \dots, \rho}$ are independent. Let \mathbf{F}^Θ be the natural filtration of Θ , augmented with \mathbb{P} -null sets. We define the filtration $\mathbf{G} = (\mathcal{G}_n)_{n \in \{0, \dots, \Gamma\}}$ by $\mathcal{G}_n = \mathcal{F}_n^\Theta \vee \mathcal{F}_{n-1}^N$.

We extend the previous setting to multivariate case,

$$\forall \delta \in \mathbb{J}^\otimes, \epsilon_n^\delta = \mathbb{1}_{[\Delta N_n = \delta]} = \mathbb{1}_{[\bigcap_{i=1}^\rho \Delta N_n^i = \delta_i]}, \lambda_{n-1}^\delta = \mathbb{E}[\epsilon_n^\delta | \mathcal{F}_{n-1}].$$

Proposition 10. *The filtered process $\hat{\Theta}$ satisfies for all $n = 1, \dots, \Gamma$,*

$$\hat{\Theta}_n = \sum_{\delta \in \mathbb{J}^\otimes} \frac{\widehat{(\tilde{\Theta} \lambda^\delta)_{n-1}}}{\hat{\lambda}_{n-1}^\delta} \mathbb{1}_{[\Delta N_n = \delta]}. \quad (3.8)$$

Proof. We leave the proof to the reader as it goes along the same lines as the proof of Prop. 7. ■

Remark 11. *With adequate assumptions, this filtering formula could cover non Markovian case. These considerations are left for future research.*

3 Discrete-time filtering for rating migrations

We aim to adapt the discrete-time filtering framework of Section 2, to the context of rating migrations. We consider that a common Markov chain governs the dynamics of all transitions. This hidden process may carry the systematic risk shared by rating transitions and might be interpreted as the economic factor. We first present the formula in the context of a single pair of rating categories (a single transition from one given rating, to another). Then we extend the approach to multiple rating transitions.

Let $\Gamma \in \mathbb{N}$ be a discrete time horizon. We work with the filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in \{0, \dots, \Gamma\}}, \mathbb{P})$. Let Θ be a Markov chain with finite number of states in $\mathbb{T} = \{1, \dots, m\}$. Let \mathbf{F}^Θ be the natural filtration of Θ , augmented with \mathbb{P} -null sets. Let's define, for $h \in \mathbb{T}$, $n \in \{0, \dots, \Gamma\}$, $I_n^h = \mathbb{1}_{[\Theta_n = h]}$, the indicator function of Θ on state h , at time n .

We consider the list of rating categories $\bar{\mathbb{Y}} = \{1, \dots, p\}$. This space represents different

credit risk scores or ratings in descending order, p being the default state. For example, Standard and Poor's long-term investment ratings can be translated to AAA = 1, AA = 2, A = 3, BBB = 4, ..., D (Default) = 10. In practice the number of credit entities monitored over time may vary, either because some names are censored or simply because of missing data. This consideration is deeply discussed in Section 7.1. We attribute the rating 0 to an entity in this case. Then, it is clear that a transition involving the rating of censure 0, is assumed to be independent with the states of the hidden factor. Then we call $\Upsilon = \{0, \dots, p\}$, the completed list of ratings. Note that, with this setting, the number of entities observed on Υ is constant over time and equal to Q . Let $Z_n^q \in \Upsilon$, be the random variable, describing the state of bond q , $q \in \{1, \dots, Q\}$, at time $n \in \{0, \dots, \Gamma\}$ and let $Z^q = (Z_n^q)_{n \in \{0, \dots, \Gamma\}}$ be the migration process that describes its evolution. The counting process, which counts the total number of jumps of the entities, from rating i to r , $(i, r) \in \Upsilon^2$, is denoted by N^{ir} and is such that, $\forall n \in \{1, \dots, \Gamma\}$,

$$\Delta N_n^{ir} = \sum_{q \leq Q} \mathbb{1}_{[Z_{n-1}^q = i, Z_n^q = r]}.$$

Let \mathbf{F}^N be the natural filtration of N , augmented with \mathbb{P} -null sets. For $n \in \{1, \dots, \Gamma\}$, we introduce for every process O , the notation

$$\hat{O}_n = \mathbb{E}[O_n | \mathcal{F}_n^N].$$

In addition, let us denote by the process Y^i representing the number of observed and active entities that belong to rating i , which may jump to another one. It may evolve over time, according to censorship, arrivals of new entities on the market with initial rating i , rating transitions or bankruptcies. This process is assumed to be \mathbf{F}^N -predictable. In this framework, Θ aims to represent the systematic risk factor. It is unique and governs dynamics of all rating transitions. Furthermore, for the sake of tractability, it is assumed that Θ impacts entities with the same rating in the same way. Consequently, we consider that entities with the same rating, are perfectly indistinguishable. Under this exchangeable setting, to infer information on the underlying hidden factor Θ , it is sufficient to observe the aggregated counting processes $(\Delta N^{ir})_{i,r \in \Upsilon}$ and the processes $(Y_i)_{i \in \Upsilon}$. The number of jumps from i to r can not exceed the number of active entities. Then, the support of ΔN_n^{ir} is $\mathbb{J}_n^i = \{0, \dots, Y_n^i\}$. Let us define the support of ΔN_n , the product spaces $\mathbb{J}_n^\otimes = \prod_{i=1}^p \mathbb{J}_n^i$. We define the transition probabilities of Θ as

$$\forall (s, h) \in \mathbb{T}^2, \forall n \in \{1, \dots, \Gamma\}, K^{sh} = \mathbb{P}(\Theta_n = h | \Theta_{n-1} = s), \quad \Pi^h = \mathbb{P}(\Theta_0 = h).$$

3.1 Unique rating transition

In a first framework, we present a specific case of the general setting presented above. we consider a unique transition from a rating called $r_0 \in \tilde{\Upsilon}$ to another, called $r_1 \in \tilde{\Upsilon}$. N is reduced to $N = N^{r_0, r_1}$, the associated univariate counting process which counts the total

number of jumps of the entities, from rating r_0 to r_1 , such that, $\forall n \in \{1, \dots, \Gamma\}$,

$$\Delta N_n = \sum_{q \leq Q} \mathbb{1}_{[Z_{n-1}^q = r_0, Z_n^q = r_1]}.$$

Let \mathbf{F}^N be the natural filtration of N , augmented with \mathbb{P} -null sets. For $n \in \{1, \dots, \Gamma\}$, Y_n represents the number of active and observed entities with rating r_0 at time $n-1$, that may jump to rating r_1 at time n . We present here the recursive equation satisfied by $\hat{I}_n^h = \mathbb{E}[\mathbb{1}_{[\Theta_n = h]} | \mathcal{F}_n^N]$. We define, for any $n \in \{1, \dots, \Gamma\}$ and $s \in \mathbb{T}$, the conditional transition probabilities as

$$L^s = \mathbb{P}(Z_n^s = r_1 | Z_{n-1}^s = r_0, \Theta_{n-1} = s).$$

According to the previous notations, the number of jumps from r_0 to r_1 cannot exceed the number of active entities. Then, the support of ΔN_n is $\mathbb{J}_n = \{0, \dots, Y_n\}$. Knowing that $\{\Theta_{n-1} = h, Y_n = y_n\}$, we assume that the conditional distribution of the random variable ΔN_n is binomial with parameters (y_n, L^h) . Similar settings can be found in [Caja et al. \(2015\)](#) and [Giampieri et al. \(2005\)](#).

Proposition 12. *With these assumptions, the filtered process \hat{I}_n^h solves the following recursive equation. For $n = 1, \dots, \Gamma$,*

$$\hat{I}_n^h = \sum_{j \in \mathbb{J}_n} \frac{\sum_{s=1}^m K^{sh} (L^s)^j (1-L^s)^{Y_n-j} \hat{I}_{n-1}^s}{\sum_{s=1}^m (L^s)^j (1-L^s)^{Y_n-j} \hat{I}_{n-1}^s} \mathbb{1}_{[\Delta N_n = j]}. \quad (3.9)$$

Proof. This formula can be derived from the general discrete-time filtering formula, in Section 2. This formula can also be derived from the filtering formula ([Elliott et al., 2008](#), Chapter 2-Theorem 4.3). However, the filtering formula must be applied to the context described above. We apply the formula to the counting process ΔN , considered as an observable Markov chain with finite number of states, \mathbb{J}_n at time $n \in \{1, \dots, \Gamma\}$, and governed by the common hidden Markov chain Θ_n . We describe here the first approach. According to the setting described above, we have

$$\mathbb{E}(\mathbb{1}_{[\Delta N_n = j]} | \mathcal{F}_{n-1}) = \mathbb{E}(\mathbb{1}_{[\Delta N_n = j]} | N_{n-1}, \Theta_{n-1}, Y_n),$$

and

$$\mathbb{P}(\Delta N_n = j | \Theta_{n-1} = h, Y_n = y_n) = \binom{y_n}{j} (L^h)^j (1-L^h)^{y_n-j}.$$

Then we compute the following expressions

$$\hat{\lambda}_{n-1}^j = \mathbb{E}[\mathbb{1}_{[\Delta N_n = j]} | \mathcal{F}_{n-1}^N] = \sum_{h=1}^m \binom{Y_n}{j} (L^h)^j (1-L^h)^{Y_n-j} \hat{I}_{n-1}^h,$$

$$\widehat{(I^h \lambda^j)}_{n-1} = \binom{Y_n}{j} (L^h)^j (1-L^h)^{Y_n-j} \hat{I}_{n-1}^h$$

and

$$\begin{aligned}
\widehat{(\tilde{I}^h \lambda^j)}_{n-1} &= \mathbb{E} \left[\mathbb{E}[I_n^h | \mathcal{F}_{n-1}] \mathbb{E}[\mathbf{1}_{[\Delta N_n = j]} | \mathcal{F}_{n-1}] | \mathcal{F}_{n-1}^N \right] \\
&= \mathbb{E} \left[\left(\sum_{\mu=1}^m K^{\mu h} I_{n-1}^\mu \right) \left(\sum_{s=1}^m \binom{Y_n}{j} (L^s)^j (1 - L^s)^{Y_n - j} I_{n-1}^s \right) | \mathcal{F}_{n-1}^N \right] \\
&= \sum_{s=1}^m K^{sh} \binom{Y_n}{j} (L^s)^j (1 - L^s)^{Y_n - j} \hat{I}_{n-1}^s.
\end{aligned}$$

Finally using (3.1) we obtain the desired equation. \blacksquare

Remark 13. *For the sake of interpretability, our setting assumes that the entities should be affected by the same realisation of the economic factor. Therefore, our filtering framework uses the whole history of aggregated number of jumps, keeping the dependencies within the observations sample.*

In this framework, the hidden factor governs a unique transition. It might be more realistic to assume that it affects all transitions. Then, we naturally extend the previous equation to a multivariate setting.

3.2 Multiple Rating transitions

In this application, we extend the previous result by considering multiple rating transitions. We present now the recursive equation satisfied by $\hat{I}_n^h = \mathbb{E}[\mathbf{1}_{[\Theta_n = h]} | \mathcal{F}_n^N]$, where $N = (N^{ir})_{ir \in \Upsilon}$. According to this setting, we define the conditional transition probabilities of $(Z^q)_q$ as

$$\text{For } (i, r) \in \Upsilon^2, s \in \mathbb{T} \text{ and } \forall n \in \{1, \dots, \Gamma\}, L^{s,ir} = \mathbb{P}(Z_n^q = r | Z_{n-1}^q = i, \Theta_{n-1} = s).$$

Note that if $i = 0$ or $r = 0$ then $\forall s \in \mathbb{T}, L^{s,ir} = \mathbb{P}(Z_n^q = r | Z_{n-1}^q = i)$. Indeed, the transitions from or to the rating 0 are assumed to be independent of the hidden factor because censorship is non-informative.

Since only one realisation of trajectory of Θ governs observed rating processes, the random variables $\{Z_n^q, q \leq Q\}$, for $n \in \{1, \dots, \Gamma\}$, are still not independent. Nevertheless, knowing the state of Θ_n , they are independent. Then, the conditional distribution of the multivariate random variable ΔN_n^{ir} , knowing that $\{\Theta_{n-1} = s, Y_n^i = y_n^i\}$, is multinomial with parameters $(y_n^i, (L^{s,ir})_r)$.

Proposition 14. *With such assumptions, the filtered process \hat{I}_n^h is solution of the following recursive equation*

$$\hat{I}_n^h = \sum_{\delta \in \mathbb{J}_n^\otimes} \frac{\sum_s K^{sh} \prod_{i,r=1}^p (L^{s,ir})^{\delta_{ir}} \hat{I}_{n-1}^s}{\sum_s \prod_{i,r=1}^p (L^{s,ir})^{\delta_{ir}} \hat{I}_{n-1}^s} \mathbf{1}_{[\Delta N_n = \delta]}. \quad (3.10)$$

Proof. We leave the proof to the reader as it goes along the same lines as the proof of Proposition 12. ■

Once the hidden factor filtered state is obtained, it is then possible to predict the future migration probabilities.

3.3 Transition probability prediction

We define for $(i, r) \in \tilde{\Upsilon}^2$, the process ν^{ir} , which forecasts the transition probability from rating i to rating r , for the next time step.

$$\forall (i, r) \in \tilde{\Upsilon}^2, \forall n \in \{1, \dots, \Gamma\} : \nu_{n-1}^{ir} = \mathbb{E} [\mathbb{1}_{[Z_n^q=r]} | Z_{n-1}^q = i, \mathcal{F}_{n-1}] = \sum_{h \in \mathbb{T}} L^{h,ir} I_{n-1}^h.$$

With the filtered current hidden factor, we can forecast the future transition probabilities

$$\forall (i, r) \in \tilde{\Upsilon}^2, \forall n \in \{1, \dots, \Gamma\} : \hat{\nu}_{n-1}^{ir} = \mathbb{E} [\mathbb{1}_{[Z_n^q=r]} | Z_{n-1}^q = i, \mathcal{F}_{n-1}^N] = \sum_{h \in \mathbb{T}} L^{h,ir} \hat{I}_{n-1}^h. \quad (3.11)$$

3.4 Calibration

In this section, we explain how to estimate model parameters involved in the filtering equation (3.10). We apply the so-called Baum-Welch algorithm to our discrete-time framework.

3.4.0.1 A Baum-Welch algorithm adapted for the discrete framework

The proposed method is a maximisation expectation (EM) algorithm for hidden Markov chains (HMM), adapted to the model. We can find studies on the classical model in Bishop and Nasrabadi (2006), Özkan et al. (2014), Rabiner (1989) and Tenyakov (2014). However the classical algorithm is not totally suitable for calibration of the discrete filtering equation (3.10). We highlight one inconsistency between the classical algorithm and our model. Rating process trajectories of each entity must be independent whereas in our framework, they are dependent through the common factor Θ .

The first step of the algorithm assigns initial values to the parameters we want to estimate. Then the algorithm replaces the missing data (states of Θ) with Bayesian estimators using the observations and the current parameters estimated values.

The second one consists in improving a conditional likelihood. Better parameters are estimated. Then these new estimates are used to repeat the first step. We iterate this process to converge to a local maximum.

Let $Z = (Z^q)_{q \leq Q}$ be the multivariate rating process and we call for $(n_1, n_2) \in \{0, \dots, \Gamma\}^2$, $(Z)_{n \in \{n_1, \dots, n_2\}} = Z_{n_1|n_2}$, the rating trajectories between time n_1 and n_2 . As the new rating

does not only depend on the economic cycle (state of Θ) but also on the previous rating, we apply the Baum-Welch algorithm by considering that

$$\forall n \in \{1, \dots, \Gamma\}, \mathbb{P}(Z_n | Z_0, \dots, Z_{n-1}, \Theta_0, \dots, \Theta_{n-1}) = \mathbb{P}(Z_n | Z_{n-1}, \Theta_{n-1}).$$

Furthermore, as rating history of all entities are dependent on a same realization of Θ , we must adapt our algorithm differently from [Oh et al. \(2019\)](#) who considered that each rating process is governed by its own and independent trajectory of Θ .

3.4.1 Initialization

The calibration algorithm presented is based on iterative improvement of a likelihood. This expectation maximization algorithm (EM) (see [Dempster et al. \(1977\)](#)), as most of iterative maximisation algorithms, might be trapped in a local maximum. Obtained parameters may not be relevant when the global maximum is not found. This success is deeply dependant on the initialization. Several empirical and analytical methods have been proposed to deal with this matter. In [Liu et al. \(2014\)](#), transition probabilities are initiated using empirical frequencies. They succeed to considerably reduce the number of iterations to find their local maximum. By noticing that the transition matrices have strong diagonals, [Oh et al. \(2019\)](#) initialized their model by adding small perturbations to identity matrix or to uniform distributions. In our study we choose a third option which seems to be more reliable: we test a high number of initial values (picked at random) in order to find the global maximum. In order to guarantee almost surely convergence to the global maximum, initial values are chosen according to a uniform distribution on the parameters space.

3.4.2 Bayesian estimators

We respectively define the forward and the backward probabilities as,

$$\forall s \in \mathbb{T}, \forall n \in \{1, \dots, \Gamma\} : \alpha_n(s) = \mathbb{P}(Z_{0|n} = z_{0|n}, \Theta_{n-1} = s)$$

$$\forall s \in \mathbb{T}, \forall n \in \{1, \dots, \Gamma - 1\} : \beta_n(s) = \mathbb{P}(Z_{n+1|\Gamma} = z_{n+1|\Gamma} | Z_n = z_n, \Theta_{n-1} = s).$$

Let u and v be two processes describing Θ such that

$$u_n(h) = \mathbb{1}_{[\Theta_n=h]}, \quad v_n(s, h) = \mathbb{1}_{[\Theta_n=h, \Theta_{n-1}=s]}.$$

Then we can derive the following recursive formulas

$$\left\{ \begin{array}{l} \alpha_n(s) = \sum_{l=1}^m \alpha_{n-1}(l) K^{ls} \prod_{i,r \in \Upsilon} (L^{s,ir})^{\Delta N_n^{ir}}, \\ \beta_n(s) = \sum_{l=1}^m \beta_{n+1}(l) K^{sl} \prod_{i,r \in \Upsilon} (L^{l,ir})^{\Delta N_{n+1}^{ir}}, \\ \check{u}_n(h) = \mathbb{P}(\Theta_n = h | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{n+1}(h) \alpha_{n+1}(h)}{\sum_{j=1}^m \alpha_{\Gamma}(j)}, \\ \check{v}_n(s, h) = \mathbb{P}(\Theta_n = h, \Theta_{n-1} = s | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{n+1}(h) K^{sh} \alpha_n(s) \prod_{i,r \in \Upsilon} (L^{h,ir})^{\Delta N_n^{ir}}}{\sum_{j=1}^m \alpha_{\Gamma}(j)}. \end{array} \right.$$

Proof (Recursive formulas). We derive $\forall n \in \{2, \dots, \Gamma - 1\}$ and $\forall s \in \{1, \dots, m\}$,

$$\begin{aligned}
\alpha_n(s) &= \mathbb{P}(Z_{0|n} = z_{0|n}, \Theta_{n-1} = s) = \sum_{l=1}^m \mathbb{P}(Z_{0|n} = z_{0|n}, \Theta_{n-1} = s, \Theta_{n-2} = l) \\
&= \sum_{l=1}^m \prod_{d=1}^Q \mathbb{P}(Z_n^d = z_n^d | Z_{0|n-1} = z_{0|n-1}, \Theta_{n-1} = s, \Theta_{n-2} = l) \mathbb{P}(Z_{0|n-1} = z_{0|n-1}, \Theta_{n-1} = s, \Theta_{n-2} = l) \\
&= \sum_{l=1}^m \prod_{d=1}^Q \mathbb{P}(Z_n^d = z_n^d | Z_{n-1} = z_{n-1}, \Theta_{n-1} = s) \mathbb{P}(\Theta_{n-1} = s | \Theta_{n-2} = l, Z_{0|n-1} = z_{0|n-1}) \alpha_{n-1}(l) \\
&= \sum_{l=1}^m \prod_{d=1}^Q \mathbb{P}(Z_n^d = z_n^d | Z_{n-1}^d = z_{n-1}^d, \Theta_{n-1} = s) \mathbb{P}(\Theta_{n-1} = s | \Theta_{n-2} = l) \alpha_{n-1}(l) \\
&= \sum_{l=1}^m \alpha_{n-1}(l) K^{ls} \prod_{d=1}^Q L^{s, z_{n-1}^d z_n^d} = \sum_{l=1}^m \alpha_{n-1}(l) K^{ls} \prod_{i,r \in \Upsilon} (L^{s, ir})^{\Delta N_n^{ir}}.
\end{aligned}$$

For $n=1$, we do not know the state of the individuals before the simulation. To tackle this issue, we use the initial proportion of the ratings. We have

$$\begin{aligned}
\alpha_1(s) &= \mathbb{P}(Z_1 = z_1, \Theta_0 = s) = \prod_{d=1}^Q \mathbb{P}(Z_1^d = z_1^d | \Theta_0 = s) \Pi(s) \\
&= \prod_{d=1}^Q \sum_{j \in \Upsilon} \mathbb{P}(Z_1^d = z_1^d | Z_0^d = j, \Theta_0 = s) \mathbb{P}(Z_0^d = j | \Theta_0 = s) \Pi(s) \\
&= \prod_{d=1}^Q \sum_{j \in \Upsilon} \mathbb{P}(Z_1^d = z_1^d | Z_0^d = j, \Theta_0 = s) \mathbb{P}(Z_0^d = j) \Pi(s) = \prod_{d=1}^Q \sum_{j \in \Upsilon} L^{s, j z_1^d} \mathbb{P}(Z_0^d = j) \Pi(s).
\end{aligned}$$

Similarly, we recursively derive the backward probability for all $n \in \{1, \dots, \Gamma - 2\}$ and

$s \in \{1, \dots, m\}$,

$$\begin{aligned}
\beta_n(s) &= \mathbb{P}(Z_{n+1|\Gamma} = z_{n+1|\Gamma} | \Theta_{n-1} = s, Z_n = z_n) \\
&= \sum_{l=1}^m \mathbb{P}(Z_{n+1|\Gamma} = z_{n+1|\Gamma}, \Theta_n = l | \Theta_{n-1} = s, Z_n = z_n) \\
&= \sum_{l=1}^m \mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | \Theta_n = l, Z_{n+1} = z_{n+1}, Z_n = z_n, \Theta_{n-1} = s) \\
&\quad \times \mathbb{P}(Z_{n+1} = z_{n+1}, \Theta_n = l | \Theta_{n-1} = s, Z_n = z_n) \\
&= \sum_{l=1}^m \mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | \Theta_n = l, Z_{n+1} = z_{n+1}) \\
&\quad \times \prod_{d=1}^Q (\mathbb{P}(Z_{n+1}^d = z_{n+1}^d | \Theta_n = l, Z_n^d = z_n^d)) \mathbb{P}(\Theta_n = l | \Theta_{n-1} = s) \\
&= \sum_{l=1}^m \beta_{n+1}(l) K^{sl} \prod_{d=1}^Q L^{l, z_n^d z_{n+1}^d} = \sum_{l=1}^m \beta_{n+1}(l) K^{sl} \prod_{i,r \in \Upsilon} (L^{l,ir})^{\Delta N_{n+1}^{ir}}.
\end{aligned}$$

For $n = \Gamma - 1$, we take $\forall s \in \{1, \dots, m\}$, $\beta_{\Gamma-1}(s) = 1$. We derive expression of these Bayesian estimators with the forward and the backward probabilities.

For all $h \in \{1, \dots, m\}$ and all $n \in \{1, \dots, \Gamma - 2\}$,

$$\begin{aligned}
\check{u}_n(h) &= \mathbb{P}(\Theta_n = h | Z_{0|\Gamma} = z_{0|\Gamma}) \\
&= \frac{\mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | \Theta_n = h, Z_{0|n+1} = z_{0|n+1}) \alpha_{n+1}(h)}{\mathbb{P}(Z_{0|\Gamma} = z_{0|\Gamma})} \\
&= \frac{\mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | \Theta_n = h, Z_{n+1} = z_{n+1}) \alpha_{n+1}(h)}{L_\Gamma} \\
&= \frac{\beta_{n+1}(h) \alpha_{n+1}(h)}{L_\Gamma}.
\end{aligned}$$

$$\begin{aligned}
\check{v}_n(s, h) &= \mathbb{P}(\Theta_n = h, \Theta_{n-1} = s | Z_{0|\Gamma} = z_{0|\Gamma}) \\
&= \frac{\mathbb{P}(\Theta_n = h, Z_{n+1|\Gamma} = z_{n+1|\Gamma} | Z_{0|n} = z_{0|n}, \Theta_{n-1} = s) \alpha_n(s)}{L_\Gamma} \\
&= \frac{\mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | Z_{0|n+1} = z_{0|n+1}, \Theta_{n-1} = s, \Theta_n = h)}{L_\Gamma} \\
&\times \mathbb{P}(\Theta_n = h, Z_{n+1} = z_{n+1} | Z_{0|n} = z_{0|n}, \Theta_{n-1} = s) \alpha_n(s) \\
&= \frac{\mathbb{P}(Z_{n+2|\Gamma} = z_{n+2|\Gamma} | Z_{n+1} = z_{n+1}, \Theta_n = h)}{L_\Gamma} \\
&\times \prod_{d=1}^Q (\mathbb{P}(Z_{n+1}^d = z_{n+1}^d | \Theta_n = h, Z_n^d = z_n^d)) \mathbb{P}(\Theta_n = h | \Theta_{n-1} = s) \alpha_n(s) \\
&= \frac{\beta_{n+1}(h) K^{sh} \alpha_n(s) \prod_{d=1}^Q L^{h, z_n^d z_{n+1}^d}}{L_\Gamma} = \frac{\beta_{n+1}(h) K^{sh} \alpha_n(s) \prod_{i,r \in \Upsilon} (L^{h,ir})^{\Delta N_n^{ir}}}{L_\Gamma},
\end{aligned}$$

with L_Γ being the likelihood on the whole sample, $L_\Gamma = \mathbb{P}(Z_{0|\Gamma} = z_{0|\Gamma}) = \sum_j \alpha_\Gamma(j)$. \blacksquare

3.4.3 Parameters estimation

The maximum likelihood estimators of the EM algorithm described above can be computed as

$$\Pi^h = \check{u}_0(h); \quad L^{s,ir} = \frac{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s) \Delta N_n^{ir}}{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s) Y_n^i}; \quad K^{sh} = \frac{\sum_{n=1}^{\Gamma} \check{v}_n(s, h)}{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s)}. \quad (3.12)$$

Proof (Derivation of the estimators). Using the concavity of the log function, let's first notice that for two strictly positive sequences w and w' ,

$$\log \left(\frac{\sum_i w'_i}{\sum_k w_k} \right) = \log \left(\sum_i \frac{w_i w'_i}{\sum_k w_k w_i} \right) \geq \frac{1}{\sum_k w_k} \left(\sum_i (w_i \log(w'_i) - w_i \log(w_i)) \right).$$

The maximization step consists in finding better parameters than those of the previous

iteration. We call $M^{(\gamma)} = (\Pi^{(\gamma)}, L^{(\gamma)}, K^{(\gamma)})$, the parameters of the current iteration (γ). We are seeking new parameters $M^{(\gamma+1)} = (\Pi^{(\gamma+1)}, L^{(\gamma+1)}, K^{(\gamma+1)})$. Let's consider the finite spaces $\Xi_k = \{1, \dots, m\}^k, k \in \{1, \dots, \Gamma\}$. We call a possible trajectory of Θ, θ , belonging to the finite space Ξ_Γ . Using $w_\theta = \mathbb{P}(Z = z, \Theta = \theta | M^{(\gamma)})$ and $w'_\theta = \mathbb{P}(Z = z, \Theta = \theta | M^{(\gamma+1)})$ in the previous inequality and defining $Q(M^{(\gamma)}, M^{(\gamma+1)}) = \sum_{\theta \in \Xi_\Gamma} w_\theta \log(w'_\theta)$ and $Q(M^{(\gamma)}, M^{(\gamma)}) = \sum_{\theta \in \Xi_\Gamma} w_\theta \log(w_\theta)$, we obtain

$$\begin{aligned} \log \left(\frac{\sum_{\theta \in \Xi_\Gamma} w'_\theta}{\sum_{\theta \in \Xi_\Gamma} w_\theta} \right) &= \log \left(\frac{\mathbb{P}(Z = z | M^{(\gamma+1)})}{\mathbb{P}(Z = z | M^{(\gamma)})} \right) \\ &\geq \frac{1}{\mathbb{P}(Z = z | M^{(\gamma)})} (Q(M^{(\gamma)}, M^{(\gamma+1)}) - Q(M^{(\gamma)}, M^{(\gamma)})). \end{aligned}$$

This last inequality shows that we obtain $\mathbb{P}(Z = z | M^{(\gamma+1)}) \geq \mathbb{P}(Z = z | M^{(\gamma)})$ by maximizing

$$Q(M^{(\gamma)}, M^{(\gamma+1)}) = \sum_{\theta \in \Xi_\Gamma} \mathbb{P}(\Theta = \theta, Z = z | M^{(\gamma)}) \log \mathbb{P}(\Theta = \theta, Z = z | M^{(\gamma+1)}).$$

We cut $\log(\mathbb{P}(Z = z, \Theta = \theta | M^{(\gamma+1)})) = \log(\mathbb{P}(Z = z | \Theta = \theta, M^{(\gamma+1)})) + \log(\mathbb{P}(\Theta = \theta | M^{(\gamma+1)}))$. Since the processes $(Z^d)_d$ are independent knowing the unobserved factor, we have

$$\begin{aligned} \log(\mathbb{P}(Z = z, \Theta = \theta | M^{(\gamma+1)})) &= \log(\mathbb{P}(\theta_0)) + \sum_{n=1}^{\Gamma} \log(\mathbb{P}(\theta_n | \theta_{n-1})) + \sum_{d=1}^Q \log(\mathbb{P}(z_{0|\Gamma}^d | \theta)) \\ &= \log(\mathbb{P}(\theta_0)) + \sum_{n=1}^{\Gamma} \log(\mathbb{P}(\theta_n | \theta_{n-1})) + \sum_{d=1}^Q \sum_{n=1}^{\Gamma} \log \mathbb{P}(z_n^d | z_{n-1}^d, \theta_{n-1}). \end{aligned}$$

So,

$$\begin{aligned}
Q(M^{(\gamma)}, M^{(\gamma+1)}) &= \sum_{\theta \in \Xi_\Gamma} \log(\mathbb{P}(\theta_0)) \mathbb{P}(\theta, z | M^{(\gamma)}) + \sum_{\theta \in \Xi_\Gamma} \sum_{n=1}^{\Gamma} \log(\mathbb{P}(\theta_n | \theta_{n-1})) \mathbb{P}(\theta, z | M^{(\gamma)}) \\
&+ \sum_{\theta \in \Xi_\Gamma} \sum_{d=1}^Q \sum_{n=1}^{\Gamma} \log \mathbb{P}(z_n^d | z_{n-1}^d, \theta_{n-1}) \mathbb{P}(\theta, z | M^{(\gamma)}) \\
&= \sum_{\substack{\theta \setminus \theta_0 \\ \in \Xi_{\Gamma-1}}} \sum_{h=1}^m \log(\mathbb{P}(\Theta_0 = h)) \mathbb{P}(\Theta_0 = h, \theta \setminus \theta_0, z | M^{(\gamma)}) \\
&+ \sum_{n=1}^{\Gamma} \sum_{\theta \setminus (\theta_n, \theta_{n-1}) \in \Xi_{\Gamma-2}} \sum_{h,s=1}^m \log(\mathbb{P}(\Theta_n = h | \Theta_{n-1} = s)) \mathbb{P}(\theta \setminus (\theta_n, \theta_{n-1}), \Theta_n = h, \Theta_{n-1} = s, z | M^{(\gamma)}) \\
&+ \sum_{n=1}^{\Gamma} \sum_{\theta \setminus \theta_{n-1} \in \Xi_{\Gamma-1}} \sum_{s=1}^m \sum_{d=1}^Q \log(\mathbb{P}(z_n^d | z_{n-1}^d, \Theta_{n-1} = s)) \mathbb{P}(\theta \setminus \theta_{n-1}, \Theta_{n-1} = s, z | M^{(\gamma)}) \\
&= \sum_{h=1}^m \log(\mathbb{P}(\Theta_0 = h, z | M^{(\gamma)})) \Pi^h + \sum_{n=1}^{\Gamma} \sum_{h,s=1}^m \log(K^{sh}) \mathbb{P}(\Theta_n = h, \Theta_{n-1} = s, z | M^{(\gamma)}) \\
&+ \sum_{d=1}^Q \sum_{n=1}^{\Gamma} \sum_{s=1}^m \sum_{i,r \in \Upsilon} \log(L^{h,ir}) \mathbb{P}(\Theta_{n-1} = s, z | M^{(\gamma)}) \mathbf{1}_{[Z_n^d=r, Z_{n-1}^d=i]}.
\end{aligned}$$

Then, we can maximize by considering the three terms independently. We obtain

$$\Pi^h = \frac{\mathbb{P}(\Theta_0 = h, Z | M^{(\gamma)})}{\sum_{j=1}^m \mathbb{P}(\Theta_0 = j, Z | M^{(\gamma)})} = \mathbb{P}(\Theta_1 = h | Z, M^{(\gamma)}) = \check{u}_0(h),$$

$$L^{s,ir} = \frac{\sum_{d=1}^Q \sum_{n=1}^{\Gamma} \mathbb{P}(\Theta_{n-1} = s, Z | M^{(\gamma)}) \mathbf{1}_{[Z_n^d=r, Z_{n-1}^d=i]}}{\sum_{d=1}^Q \sum_{n=1}^{\Gamma} \sum_{j \in \Upsilon} \mathbb{P}(\Theta_{n-1} = s, Z | M^{(\gamma)}) \mathbf{1}_{[Z_n^d=j, Z_{n-1}^d=i]}} = \frac{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s) \Delta N_n^{ir}}{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s) Y_n^i},$$

$$K^{sh} = \frac{\sum_{n=1}^{\Gamma} \mathbb{P}(\Theta_{n-1} = s, \Theta_n = h, Z | M^{(\gamma)})}{\sum_{n=1}^{\Gamma} \sum_{j=1}^m \mathbb{P}(\Theta_{n-1} = s, \Theta_n = j, Z | M^{(\gamma)})} = \frac{\sum_{n=1}^{\Gamma} \check{v}_n(s, h)}{\sum_{n=1}^{\Gamma} \check{u}_{n-1}(s)}. \quad \blacksquare$$

4 Continuous-time version of the filter

In this section, we present the filtering framework in a continuous-time setting.

4.1 Framework and statements

Let $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, be a filtered probability space satisfying the “usual conditions” of right-continuity and completeness needed to justify all operations to be made. All stochastic processes encountered are assumed to be adapted to the filtration \mathbf{F} and integrable on $[0, T]$. In particular, we have $\mathbf{A} = \mathcal{F}_T$. The time horizon T is supposed to be finite.

The problem is to estimate the states of an unobserved process Θ using only the information \mathbf{F}^N , resulting from the observation of a counting process N . We also define \mathbf{F}^Θ , the natural filtration of Θ . By definition of the conditional expectation,

$$\hat{\Theta}_t = \mathbb{E}[\Theta_t | \mathcal{F}_t^N].$$

is the L^2 approximation of Θ knowing N . With the same notation, all the processes O filtered by \mathcal{F}_t^N are written

$$\hat{O}_t = \mathbb{E}[O_t | \mathcal{F}_t^N].$$

Unfortunately, the explicit computation of $\hat{\Theta}$ (by the Bayes formula) is only possible in very simple cases, for example, when N is a mixture of Poisson processes. Moreover, numerical approaches are extremely heavy to implement as soon as non-trivial models are considered. The filtering approach is different. It consists in writing $\hat{\Theta}$ as the solution of a stochastic differential equation (SDE). The main result on univariate point process filtering can be stated in the following way (see [Brémaud \(1981\)](#), [Karr \(2017\)](#), [Leijdekker and Spreij \(2011\)](#), [Van Schuppen \(1977\)](#)). In order to study the dynamics of ratings in a credit portfolio or the dynamics of rating migrations, this filtering equation must be extended to a multivariate case.

4.2 Filtering multivariate point process

In this section, we show that the previous filtering equation admits a natural multivariate extension. Let $N = (N^1, \dots, N^\rho)$ be a multivariate counting process where $N^j = (N_t^j)_{t \in [0, T]}$, $j = 1, \dots, \rho$, is a set of simple counting processes, such that, $N_t^j = \sum_{0 < s \leq t} \Delta N_s^j < \infty$ and $\Delta N_s^j \in \{0, 1\}$, for any $j = 1, \dots, \rho$. It is assumed that these processes admit a predictable \mathbf{F} -intensity $\nu^j = (\nu_t^j)_{t \in [0, T]}$, and that they do not have any common jumps, i.e., $\Delta N_t^j \Delta N_t^k = \delta_{jk} \Delta N_t^j$ ie $[N^j, N^k]_t = 0$ (the continuous martingale part of a counting process being null). We introduce $\mathbf{F}^N = (\mathcal{F}_t^N)_{t \in [0, T]}$ the natural filtration of the multivariate counting process $N = (N^1, \dots, N^\rho)$.

Proposition 15. Let Θ be a square integrable process of the form

$$\Theta_t = \int_0^t a_s ds + M_t, \quad (3.13)$$

where a is a \mathbf{F} -adapted process and M is a \mathbf{F} -square integrable martingale with no jumps in common with N . Therefore, the process $\hat{\Theta}$ is solution of the SDE

$$d\hat{\Theta}_t = \hat{a}_t dt + \sum_j \eta_t^j (dN_t^j - \hat{\nu}_{t-}^j dt), \quad (3.14)$$

with

$$\eta_t^j = \frac{\widehat{(\Theta \nu^j)}_{t-}}{\hat{\nu}_{t-}^j} - \hat{\Theta}_{t-}, \quad (3.15)$$

and initial condition

$$\hat{\Theta}_0 = \mathbb{E}[\Theta_0]. \quad (3.16)$$

Proof. This equation can be derived from the univariate filtering framework presented in (Brémaud, 1981, Sec.IV.1). For a sake of completeness and consistence with the discrete-time derivation, we propose here a second approach to derive this filtering equation.

Let g and h be two \mathbf{F} predictable processes such that $\mathbb{E} \left[\int_0^T (g_s^2 + h_s^2) \nu_s^j ds \right] \leq \infty$. We introduce the processes X and Y defined by $X_t = \int_0^t g_r (dN_r^j - \nu_r^j dr)$ and $Y_t = \int_0^t h_s (dN_s^k - \nu_s^k ds)$ for all $t \leq T$. X and Y are two \mathbf{F} -martingales. The Itô formula applied to XY yields

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + \Delta X_t \Delta Y_t.$$

Since N^j and N^k have no common jumps, $\Delta X_t \Delta Y_t = g_t h_t \Delta N_t^j \delta_{jk}$. Then, we obtain

$$\mathbb{E}[X_T Y_T - X_t Y_t | \mathcal{F}_t] = \delta_{jk} \mathbb{E} \left[\int_t^T g_s h_s dN_s^j | \mathcal{F}_t \right]. \quad (3.17)$$

Note that $X_T Y_T - X_t Y_t = (X_T - X_t)(Y_T - Y_t) - 2X_t Y_t + X_t Y_T + X_T Y_t$. Then,

$$\mathbb{E}[X_t Y_T - X_t Y_t | \mathcal{F}_t] = \mathbb{E}[(X_T - X_t)(Y_T - Y_t) | \mathcal{F}_t]. \quad (3.18)$$

Note that the process $Z = \left(\int_0^t g_s h_s (dN_s^j - \nu_s^j ds) \right)_{0 \leq t \leq T}$ is a \mathbf{F} -martingale.

So, $\mathbb{E}[Z_T - Z_t | \mathcal{F}_t] = 0$. Combining this remark with Equations (3.17) and (3.18), we finally obtain

$$\mathbb{E} \left[\int_t^T g_r (dN_r^j - \nu_r^j dr) \int_t^T h_s (dN_s^k - \nu_s^k ds) \middle| \mathcal{F}_t \right] = \delta_{jk} \mathbb{E} \left[\int_t^T g_s h_s \nu_s^j ds \middle| \mathcal{F}_t \right]. \quad (3.19)$$

The innovation theorem says that the \mathbf{F}^N -intensities of the counting processes N^j exist

and are

$$\hat{\nu}_{t-}^j = \mathbb{E}[\nu_{t-}^j | \mathcal{F}_{t-}^N] = \mathbb{E}[\nu_t^j | \mathcal{F}_{t-}^N]. \quad (3.20)$$

For any \mathbf{F}^N predictable process h satisfying $\mathbb{E}\left[\int_0^T |h_s \nu_s| ds\right] < \infty$, we have

$$\mathbb{E}\left[\int_0^\infty h_t dN_t\right] = \mathbb{E}\left[\int_0^\infty h_t \nu_t dt\right] = \mathbb{E}\left[\int_0^\infty h_t \mathbb{E}[\nu_t | \mathcal{F}_{t-}^N] dt\right] = \mathbb{E}\left[\int_0^\infty h_t \hat{\nu}_{t-} dt\right].$$

Now, rewrite (3.13) as

$$\Theta_t = \int_0^t \hat{a}_s ds + L_t + M_t, \quad (3.21)$$

with

$$L_t = \int_0^t (a_s - \hat{a}_s) ds. \quad (3.22)$$

Taking conditional expectation, w.r.t. \mathcal{F}_t^N , in (3.21) yields

$$\hat{\Theta}_t = \int_0^t \hat{a}_s ds + \hat{L}_t + \hat{M}_t. \quad (3.23)$$

While L need not be a \mathbf{F} -martingale, it is clear that \hat{L} is an \mathbf{F}^N -martingale. For $r < t$,

$$\mathbb{E}[\hat{L}_t - \hat{L}_r | \mathcal{F}_r^N] = \mathbb{E}\left[\int_r^t (a_s - \mathbb{E}[a_s | \mathcal{F}_s^N]) ds \middle| \mathcal{F}_r^N\right] = 0.$$

From the tower property, we deduce that \hat{M} is also a \mathbf{F}^N -martingale. Introduce

$$K_t = L_t + M_t = \Theta_t - \int_0^t \hat{a}_s ds. \quad (3.24)$$

Since Θ and N have no common jumps, we can deduce that K and N have any either. It has been shown that $\hat{K} = \hat{L} + \hat{M}$ is a \mathbf{F}^N -martingale. Therefore it has a predictable representation,

$$\hat{K}_t = \gamma + \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds), \quad (3.25)$$

where $\gamma = \hat{K}_0$ is \mathcal{F}_0^N -measurable and the η^j are \mathbf{F}^N -predictable processes (see Brémaud (1981)). Note that $\hat{K}_0 = \mathbb{E}[\Theta_0]$. Now, any integrable \mathbf{F}^N -measurable random variable has a representation $g + \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_s^j ds)$, with g constant and the h^j are \mathbf{F}^N -predictable. Therefore, since \hat{K}_t is the L^2 projection of K_t onto the space of square integrable \mathcal{F}_t^N -measurable random variables, the coefficients in the representation (3.25) are uniquely

determined by the normal equations

$$\mathbb{E} \left[\left(K_t - \gamma - \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) \right) \left(g + \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) \right) \right] = 0,$$

for all constants g and all \mathbf{F}^N -predictable processes h^j . Setting $g = 0$ and using (3.19) give

$$\mathbb{E} \left[K_t \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) - \sum_j \int_0^t h_s^j \eta_s^j \hat{\nu}_{s-}^j ds \right] = 0. \quad (3.26)$$

For $j \in \{1, \dots, \rho\}$, we compute $\mathbb{E} \left[K_t \int_0^t h_s^j dN_s^j \right]$. Using that \hat{K} is a \mathbf{F}^N -martingale and that K and N^j have no common jumps, we have

$$\begin{aligned} \mathbb{E} \left[K_t \int_0^t h_s^j dN_s^j \right] &= \mathbb{E} \left[K_t \sum_{s \leq t} h_s^j \Delta N_s^j \right] = \sum_{s \leq t} \mathbb{E} \left[\mathbb{E} [K_t | \mathcal{F}_s^N] h_s^j \Delta N_s^j \right] \\ &= \sum_{s \leq t} \mathbb{E} \left[\hat{K}_s h_s^j \Delta N_s^j \right] = \mathbb{E} \left[\int_0^t K_{s-} h_s^j dN_s^j \right] \\ &= \mathbb{E} \left[\int_0^t K_{s-} h_s^j \nu_s^j ds \right] = \int_0^t \mathbb{E} \left[h_s^j \mathbb{E} [K_{s-} \nu_s^j | \mathcal{F}_{s-}^N] \right] ds \\ &= \mathbb{E} \left[\int_0^t h_s^j \widehat{\Theta \nu}_{s-}^j ds \right] - \mathbb{E} \left[\int_0^t h_s^j \hat{\nu}_{s-}^j \int_0^s \hat{a}_u du ds \right]. \end{aligned}$$

Using similar arguments, we compute the second term

$$\begin{aligned} \mathbb{E} \left[K_t \int_0^t h_s^j \hat{\nu}_{s-}^j ds \right] &= \int_0^t \mathbb{E} \left[h_s^j \mathbb{E} [K_t | \mathcal{F}_{s-}^N] \hat{\nu}_{s-}^j \right] ds = \int_0^t \mathbb{E} \left[h_s^j \hat{K}_{s-} \hat{\nu}_{s-}^j \right] ds \\ &= \mathbb{E} \left[\int_0^t h_s^j \hat{\Theta}_{s-} \hat{\nu}_{s-}^j ds \right] - \mathbb{E} \left[\int_0^t h_s^j \hat{\nu}_{s-}^j \int_0^s \hat{a}_u du ds \right]. \end{aligned}$$

Inserting these expressions into (3.26), gives

$$\sum_j \mathbb{E} \left[\int_0^t h_s^j \left(\widehat{\Theta \nu}_{s-}^j - \hat{\Theta}_{s-} \hat{\nu}_{s-}^j - \eta_s^j \hat{\nu}_{s-}^j \right) ds \right] = 0.$$

Choosing h_s^j equal to the expression in the parentheses, gives $\sum_j \mathbb{E} \left[\int_0^t (h_s^j)^2 ds \right] = 0$ hence all h^j vanish and $\forall j = 1, \dots, \rho$:

$$\eta_s^j = \frac{\widehat{\Theta_{s-} \nu_{s-}^j}}{\hat{\nu}_{s-}^j} - \hat{\Theta}_{s-}. \quad (3.27)$$

From (3.25), (3.23), and the equality $\hat{K} = \hat{L} + \hat{M}$, it follows that

$$\hat{\Theta}_t = \mathbb{E}[\Theta_0] + \int_0^t \hat{a}_s ds + \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds).$$

with the η^j are given by (3.27). This finishes the proof of the proposition. ■

5 Continuous-time filtering for rating migrations

In this section, we present the application of the filtering framework in continuous-time, presented in Section 4, to the context of credit migrations. In particular, we describe how to adapt the continuous-time setting with the discrete-time format of the rating data.

5.1 Statements and filtering framework

In order to remain realistic and to fix the terminology, a bond market containing a finite number of individual bonds is considered. All bonds are affected by variable and random market conditions represented by the same latent process Θ . The hidden factor driving process Θ is assumed to be a Markov chain with finite number of states in \mathbb{T} and with constant transition intensities k^{rh} , $r \neq h$ and such that $k^{rr} = -\sum_{l;l \neq r} k^{rl}$, so that, for small enough dt ,

$$\forall r \neq h \in \mathbb{T} : \mathbb{P}(\Theta_{t+dt} = h | \Theta_t = r) \approx k^{rh} dt. \quad (3.28)$$

The initial distribution of Θ is defined as

$$\forall h \in \mathbb{T} : \Pi_h = \mathbb{P}(\Theta_0 = h).$$

Let us introduce the state indicator processes I^h , $h \in \mathbb{T} = \{1, \dots, m\}$, defined by

$$I_t^h = \mathbf{1}_{[\Theta_t = h]}, \quad h \in \mathbb{T}.$$

A bond q of the sample is observed between the dates s^q and u^q , $0 \leq s^q \leq u^q \leq T$. We consider that the bond q may evolve in the same credit rating categories space, $\bar{\mathbb{Y}} = \{1, \dots, p\}$, than for the discrete-time framework.

Let $\mathbb{Y} = \{(i, j) \in \bar{\mathbb{Y}}^2, i \neq j\}$, be the space of possible migrations. Let $Z_t^q \in \bar{\mathbb{Y}}$ be the rating state of bond q at time t and $Z^q = (Z_t^q)_{t \in [s^q, u^q]}$ be the rating process describing its evolution. The migration counting process associated with Z^q , which counts the number of jumps of the entity q from rating i to j , is denoted by $N^{q,ij}$ and is such that, $\forall t \in [s^q, u^q]$,

$$\Delta N_t^{q,ij} = \mathbf{1}_{[Z_{t-}^q = i, Z_t^q = j]}.$$

We introduce $\mathbf{F}^N = (\mathcal{F}_t^N)_{t \in [0, T]}$ the natural filtration of the multivariate counting process $N = (N^{ij})_{i, j \in \tilde{\Upsilon}}$. We assume that two entities cannot jump at the same time, i.e., they do not have any common jumps, i.e., $\forall i, j, r, k \in \tilde{\Upsilon}, \Delta N_t^{ij} \Delta N_t^{rk} = \delta_{ir} \delta_{jk} \Delta N_t^{ij}$. We also assume that there is no common jump with Θ .

In this study, we assume that $(Z_t^q, t \in [0, T])_q$ are described within a factor migration model. More specifically, knowing \mathcal{F}_T^Θ , the rating processes $(Z^q)_q$ are assumed to be conditionally independent Markov chains with the same generator matrix. In reality the change of rating of a bond may also induce the change of state of other bonds but this contagion effect is not considered in this study. Moreover, the censorship mechanism governing (s^q, u^q) is assumed to be non-informative and can therefore be considered deterministic and belonging to \mathcal{F}_0^N . Under this exchangeable setting, to infer information on the underlying hidden factor Θ , it is sufficient to observe the aggregated counting processes N^{ij} , $(i, j) \in \mathbb{Y}$, defined by

$$N_t^{ij} = \sum_{q; s^q \leq t < u^q} N_t^{q, ij}.$$

and the exposure processes Y^i , $i \in \tilde{\Upsilon}$ defined by

$$Y_t^i = \sum_{q; s^q \leq t < u^q} \mathbf{1}_{[Z_{t-}^q = i]}.$$

Y_t^i represents the number of observed entities with rating i at time t , which may jump. Note that the exposure process Y^i is left continuous. It increases by 1 when N^{ji} jumps for any $j \neq i$, or when a new bond enters the pool with rating i . It decreases by 1 when a bond jumps outside rating i , i.e., whenever N^{ij} jumps for $j \neq i$ or when a bond expires with rating i .

This framework aims to determine the recursive equation satisfied by $\hat{I}_n^h = \mathbb{E}[\mathbf{1}_{\{\Theta_n = h\}} | \mathcal{F}_n^N]$. We denote by $(\nu_t^{ij})_{(i, j) \in \mathbb{Y}}$ the \mathbf{F} intensity of N . We assume that the intensities $(\nu_t^{ij})_{(i, j) \in \mathbb{Y}}$ are governed by the finite state hidden Markov chain Θ . With these assumptions, the processes $(Z^q)_q$ are governed by their common intensity matrices $(l^h)_{h \in \mathbb{T}}$, such as for small enough dt

$$\mathbb{P}[Z_{t+dt}^q = j \mid Z_t^q = i, \Theta_t = h] \approx \ell^{h, ij} dt. \quad (3.29)$$

Then the counting processes N^{ij} are governed by the \mathbf{F} intensities

$$\nu_t^{ij} = Y_t^i \sum_{r \in \mathbb{T}} \ell^{r, ij} I_{t-}^r.$$

As Y^i is \mathbf{F}^N -predictable, the \mathbf{F}^N -intensity of N^{ij} , may be written as

$$\hat{\nu}_{t-}^{ij} = Y_t^i \sum_{r \in \mathbb{T}} \ell^{r, ij} \hat{I}_{t-}^r. \quad (3.30)$$

We present the multivariate filtering formula satisfied by the process $(I_t^h)_{t \in [0, T]}$ in the following proposition.

Proposition 16. *With the previous assumptions, the unobserved indicator process filtered with rating jumps satisfies the following recursive equation*

$$d\hat{I}_t^h = \sum_{r=1}^m k^{rh} \hat{I}_{t-}^r dt + \sum_{i \neq j} \left(\frac{l^{h,ij} \hat{I}_{t-}^h}{\sum_r l^{r,ij} \hat{I}_{t-}^r} - \hat{I}_{t-}^h \right) \left(dN_t^{ij} - Y_t^i \sum_{r=1}^m l^{r,ij} \hat{I}_{t-}^r dt \right) \quad (3.31)$$

Proof. This result stems from the general continuous-time filtering theory developed in (Brémaud, 1981, Sec.IV.1). Compared to the setting of Brémaud (1981), two adaptations are necessary. First, we deal here with an aggregated multivariate process over the entire portfolio and secondly, we take censorship into account though the processes of risk exposure Y^i . For the sake of completeness, we provide a self-consistent proof yielding a general explicit filtering formula with no simultaneous jumps.

In order to apply Prop. 15, one needs to find the representation (3.13) for I_t^h . Let Ψ^{hr} , $h \neq r$, $h, r \in \mathbb{T}$, be the counting processes defined by $\Psi_t^{hr} = \#\{s \in (0, t]; \Theta_{s-} = h, \Theta_s = r\}$. The starting point is the expression

$$I_t^h = I_0^h + \sum_{r:r \neq h} (\Psi_t^{rh} - \Psi_t^{hr}),$$

which comes from the obvious dynamics

$$dI_t^h = \sum_{r:r \neq h} (d\Psi_t^{rh} - d\Psi_t^{hr}).$$

The counting processes Ψ^{hr} have intensities of the form $I_{t-}^h \kappa^{hr}$. Reshaping the last expression as

$$\begin{aligned} I_t^h &= I_0^h + \int_0^t \sum_{r:r \neq h} (I_{s-}^r \kappa^{rh} - I_{s-}^h \kappa^{hr}) ds \\ &\quad + \int_0^t \sum_{r:r \neq h} [(d\Psi_s^{rh} - I_{s-}^r \kappa^{rh} ds) - (d\Psi_s^{hr} - I_{s-}^h \kappa^{hr} ds)], \end{aligned}$$

shows that I^h is of the form $I_t^h = \int_0^t a_s^h ds + M_t^h$, with

$$a_t^h = \sum_{r:r \neq h} (I_{t-}^r \kappa^{rh} - I_{t-}^h \kappa^{hr}) = \sum_r \kappa^{rh} I_{t-}^r, \quad (3.32)$$

and M^h is a martingale commencing at $M_0^h = I_0^h$. This martingale has no jumps in common with N . Now, let I^h take the role of Θ in Prop. 15. Then the role of a_t is taken

by a_t^h in (3.32), the role of $(\Theta \nu^i)_{t-}$ is taken by

$$(-I^h \nu^{ij})_{t-} = I_{t-}^h Y_t^i \sum_r \ell^{r,ij} I_{t-}^r = Y_t^i \ell^{h,ij} I_{t-}^h,$$

and the \mathbf{F}^N -intensities of N are given in (3.30). Inserting these expressions into (3.14), gives the desired equation. \blacksquare

Remark 17. *Despite being non-linear, the filtering equations (3.31) can be decomposed into a prediction term and an innovation term, as for the classical Kalman filter.*

$$\mathbb{E}[I_t^h | F_{t-}^N] = \hat{I}_{t-}^h + \sum_{r=1}^m k^{rh} \hat{I}_{t-}^r dt \quad (3.33a)$$

$$\mathbb{E}[I_t^h | F_t^N] = \underbrace{\mathbb{E}[I_t^h | F_{t-}^N]}_{\text{prediction}} + \hat{I}_{t-}^h \sum_{i \neq j} \left(\frac{l^{h,ij}}{\sum_r l^{r,ij} \hat{I}_{t-}^r} - 1 \right) \underbrace{\left(dN_t^{ij} - Y_t^i \sum_{r=1}^m l^{r,ij} \hat{I}_{t-}^r dt \right)}_{\text{new Information}} \quad (3.33b)$$

We can identify (3.33a) as the prediction equation and (3.33b) as the correction equation.

Usually, information on rating migrations are only available to public on a daily basis. For large credit portfolios, it is then frequent to observe multiple transitions (of several entities) occurring at the same day. In addition, clustering of rating migrations may also happen following the disclosure of a major economic events. Then, the presented continuous-time filtering approach is not fully compliant with migration data since it precludes simultaneous jumps between counting processes. This has lead us to preprocess the data and adapt the calibration algorithm.

5.2 Adaptation of the continuous-time setting to discrete migration data

This part aims to explain how to adapt continuous filtering to discrete rating migration framework. We propose an adaptation of the calibration algorithm in order to be compliant with the continuous filtering formula (3.31). Previous adaptations done for the discrete framework are still required but are not sufficient: Baum-Welch algorithm is an estimation in discrete time which is not compliant with the continuous version of the filter. Furthermore the continuous-time filtering framework assumes the absence of simultaneous jumps. For the first deviation, we propose to calibrate discrete-time parameters. Then it is necessary to switch to continuous time dimension for filtering. The transition between probabilities to intensities turns out to be easy when the time interval chosen is small enough according to (3.28) and (3.29).

The second deviation is also essential. There is no common jumps among the rating processes. However, ratings are not natural processes. Human decisions and algorithm appreciations are reported at the same moment in a day. Therefore we have to deal with simultaneous daily observations. Our solution consists in considering a different time grid. Each day is cut into small intervals and jumps are randomly spread on these time intervals. We insure to cut enough finely to have a maximum of one jump per interval. This manipulation has two drawbacks. First, the conditional independence between rating entities might be lost if the non-simultaneity of jumps is enforced. Then, distributing simultaneous jumps on a finer time grid may ultimately modify the original information. The second effect is studied in a testing benchmark at Section 6.2.

Once the data have been modified, we adapt our calibration algorithm to respect continuous structure. To this end, we use a prior law of jumps which respects the constraint of no simultaneity (that the number of observed jumps is only 0 or 1). The main idea consists on assuming that one entity is randomly chosen to be allowed to jump. Then, the entity may jump according the common migration matrices $\{(L^{h,ij})_h, (i, j) \in \bar{\Upsilon}\}$. In practice the number of entities monitored over time may vary: either because some names appear or disappear or simply because of missing data. This happens when the data is missing, censored or when it is not appeared yet. We attribute the rating 0 to an entity in this case. Then it is clear that a transition involving the rating of censure 0, is assumed to be independent with the states of the hidden factor. Let consider the list of ratings $\bar{\Upsilon} = \{1, \dots, p\}$ and $\Upsilon = \{0, \dots, p\}$, the completed list of ratings. Note that the total number of entities observed on Υ is constant equal to Q . Let Q_t , be the number of entities which have their rating in $\bar{\Upsilon}$ (have a real rating) at time t .

We propose a calibration algorithm which assumes that no more than one entity may jump at a given time step. In order to make the model identifiable while considering the impact of the size of the sample (which may evolve), we define an independent process I , with values in $\{0, \dots, Q\}$, which uniformly picks the entity that may jump. If I picks an entity which is rated 0, (because not already rated or censored), we do not observe jumps. Otherwise the entity jumps according to the transition matrices $(L^h)_h$.

We have $\forall t \in \{0, \dots, \Gamma\}$, $(i, j) \in \bar{\Upsilon}^2$, $h \in \mathbb{T}$, $q \in \{0, \dots, Q\}$, $\mathbb{P}(Z_t^q = j | Z_{t-1}^q = i, I_{t-1} = q, \Theta_{t-1} = h) = L^{h,ij}$. For $z_t, z_{t-1} \in \Upsilon^Q$, we define $W_{t-1}^h = \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, \Theta_{t-1} = h)$,

where $Z_t = (Z_t^q)_{q \leq Q_t}$. We compute

$$\begin{aligned} W_{t-1}^h &= \sum_{d=1}^Q \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \mathbb{P}(I_{t-1} = d) \\ &= \sum_{d=1}^{Q_{t-1}} \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \mathbb{P}(I_{t-1} = d) \\ &\quad + \sum_{d=Q_{t-1}+1}^Q \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \mathbb{P}(I_{t-1} = d). \end{aligned}$$

Let focus on the first sum, describing the situation when the chosen entity has a rating at current time.

$$\begin{aligned} &\mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \\ &= \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \mathbb{1}_{[z_t^d = z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]} \\ &\quad + \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) \mathbb{1}_{[z_t^d \neq z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]} \\ &= L^{h, z_{t-1}^d, z_t^d} \mathbb{1}_{[z_t = z_{t-1}]} + L^{h, z_{t-1}^d, z_t^d} \mathbb{1}_{[z_t^d \neq z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]}. \end{aligned}$$

For the second sum, we have: $\mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1} = h) = \mathbb{1}_{[z_t = z_{t-1}]}$.

$$\text{So finally, } W_{t-1}^h = (1 - \frac{Q_{t-1}}{Q}) \mathbb{1}_{[z_t = z_{t-1}]} + \sum_{d=1}^{Q_{t-1}} \frac{1}{Q} \mathbb{1}_{[|z_t - z_{t-1}|_0 \leq 1]} \mathbb{1}_{[\forall l \neq d: z_t^l = z_{t-1}^l]} L^{h, z_{t-1}^d, z_t^d},$$

where $|x|_0 = \#\{x_i \neq 0\}$.

Then, it is easy to check that the previous algorithm can be adapted to the new framework

$$\alpha_t(h) = \sum_{s=1}^m \alpha_{t-1}(s) K^{sh} W_{t-1}^h, \quad \beta_t(h) = \sum_{l=1}^m \beta_{t+1}(l) K^{hl} W_t^h,$$

$$\check{\alpha}_t(h) = \mathbb{P}(\Theta_t = h | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{t+1}(h) \alpha_{t+1}(h)}{L_\Gamma},$$

$$\check{\alpha}_t(s, h) = \mathbb{P}(\Theta_t = h, \Theta_{t-1} = s | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{t+1}(h) K^{sh} \alpha_t(s) W_t^h}{L_\Gamma}.$$

The forms of the transitions matrices $(L^h)_h$, are a lot impacted by this adaptation. The maximisation does not run as simply as it does for the discrete setting. Explicit forms are heavy to derive. Then, these parameters are directly estimated with optimization algorithms.

6 Filtering on simulated data

The purpose of this section is to test and validate the continuous-time and discrete-time versions of the filter using simulated data: we build two rating migration databases from the two underlying credit migration models: the discrete-time model (as described in Section 3) and the continuous-time factor migration model (as described in Section 5). Since the structure of the two models are different, different data sample are used for testing the two approaches. The inputs of the filtering models are the evolution of the number of transitions, described by $\{\Delta N_n^{ir}, n \in \{1, \dots, \Gamma\}, i, r \in \Upsilon\}$. Our testing framework aims to compare the filtered trajectory of the hidden factor, $(\hat{\Theta}_n)_{n \in \{1, \dots, \Gamma\}}$ and the real simulated one $(\Theta_n)_{n \in \{1, \dots, \Gamma\}}$. Then it compares estimated point-in-time transition probabilities to real observed transition rates. The real observed rate of a transition is the ratio between the number of observed jumps during a time interval and the number of entities at the beginning of this time interval, susceptible to jump. Estimations of transition probabilities are respectively given by $\{\hat{\nu}_n^{ir}, n \in \{1, \dots, \Gamma\}, i, r \in \Upsilon\}$, in (3.11) for the discrete-time setting and $\{\hat{\nu}_n^{ir} dt, t \in [0, T], (i, r) \in \Upsilon\}$, in (3.30) for the continuous-time setting. The value of dt is chosen equal to $\frac{1}{1000}$, 1000 being the number of small intervals in which we have cut each day (see Section 5.2). To calibrate the models, the EM algorithm of Section 3.4 is run 1000 times by sampling random initial values (see Section 3.4.1). We keep the solution, which provides the highest likelihood estimation.

6.1 Discrete time filtering approach

To build the discrete-time database, we assume that the hidden factor is described by a finite state space Markov chain with 7 states. We consider a given set of model parameters. Each state of the hidden factor is associated with a specific rating migration matrix. We try to choose matrices compliant with filtering: we must have sufficient variability among conditional transition probabilities $(L^{h,ij})_h, (i, j) \in \{1, \dots, p\}$. We work with 3 ratings categories $\{A, B, C\}$ and initialize our sample with 1000 entities per rating. Then the hidden Markov chain is simulated on 300 time steps. According to the hidden factor's sample path, we simulate transitions using conditional transition matrices. We use the first 200 time-steps to calibrate the model and the remainder to test is. We perform the calibration as described in Section 3.4. Parameters chosen for sampling the model and estimated parameters are presented in the following section.

6.1.0.1 Calibration

We present the parameters chosen for the simulation in the testing framework described in Section 6, with the estimated parameters issued from the EM algorithm.

Table 3.1: Initial and estimated rating transition matrix for $\Theta = 0$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.98	0.01	0.01
<i>B</i>	0.29	0.7	0.01
<i>C</i>	0.1	0.3	0.6

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.9799	0.0099	0.0102
<i>B</i>	0.2923	0.6977	0.0100
<i>C</i>	0.1023	0.2962	0.6016

Table 3.2: Initial and estimated rating transition matrix for $\Theta = 1$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.98	0.01	0.01
<i>B</i>	0.39	0.6	0.01
<i>C</i>	0.2	0.3	0.5

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.9803	0.0100	0.0097
<i>B</i>	0.3887	0.6018	0.0095
<i>C</i>	0.2002	0.3003	0.4995

Table 3.3: Initial and estimated rating transition matrix for $\Theta = 2$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.5	0.3	0.2
<i>B</i>	0.01	0.6	0.39
<i>C</i>	0.01	0.01	0.98

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.5072	0.3002	0.1926
<i>B</i>	0.0095	0.6004	0.3901
<i>C</i>	0.0099	0.0103	0.9798

Table 3.4: Initial and estimated rating transition matrix for $\Theta = 3$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.98	0.01	0.01
<i>B</i>	0.39	0.6	0.01
<i>C</i>	0.2	0.3	0.5

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.9803	0.0100	0.0097
<i>B</i>	0.3887	0.6018	0.0095
<i>C</i>	0.2002	0.3003	0.4995

Table 3.5: Initial and estimated rating transition matrix for $\Theta = 4$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.6	0.3	0.1
<i>B</i>	0.01	0.7	0.29
<i>C</i>	0.01	0.01	0.98

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.5993	0.2992	0.1015
<i>B</i>	0.0104	0.6983	0.2913
<i>C</i>	0.0099	0.0095	0.9806

Table 3.6: Initial and estimated rating transition matrix for $\Theta = 5$

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.8	0.15	0.05
<i>B</i>	0.01	0.9	0.09
<i>C</i>	0.01	0.01	0.98

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.8001	0.1493	0.0506
<i>B</i>	0.0099	0.8996	0.0904
<i>C</i>	0.0101	0.0098	0.9801

Table 3.7: Initial and estimated rating transition matrix for $\Theta = 6$

	<i>A</i>	<i>B</i>	<i>C</i>		<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.98	0.01	0.01	<i>A</i>	0.9799	0.0102	0.0099
<i>B</i>	0.09	0.9	0.01	<i>B</i>	0.0908	0.8991	0.0101
<i>C</i>	0.05	0.15	0.8	<i>C</i>	0.0500	0.1517	0.7984

Table 3.8: Θ 's transition matrix

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$	$\Theta = 5$	$\Theta = 6$
$\Theta = 0$	0.6	0.3	0.1	0	0	0	0
$\Theta = 1$	0.25	0.4	0.25	0.1	0	0	0
$\Theta = 2$	0.05	0.15	0.6	0.15	0.05	0	0
$\Theta = 3$	0	0.03	0.12	0.7	0.12	0.03	0
$\Theta = 4$	0	0	0.05	0.15	0.6	0.15	0.05
$\Theta = 5$	0	0	0	0.1	0.25	0.4	0.25
$\Theta = 6$	0	0	0	0	0.1	0.3	0.6

Table 3.9: Estimated Θ 's transition matrix

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$	$\Theta = 5$	$\Theta = 6$
$\Theta = 0$	0.6029	0.3426	0.0544	0	0	0	0
$\Theta = 1$	0.2827	0.4015	0.2526	0.0632	0	0	0
$\Theta = 2$	0.0825	0.1546	0.5773	0.1443	0.0412	0	0
$\Theta = 3$	0	0.0583	0.1083	0.7000	0.0750	0.0583	0
$\Theta = 4$	0	0	0.0716	0.1592	0.5188	0.2124	0.0381
$\Theta = 5$	0	0	0	0.1765	0.2475	0.3053	0.2708
$\Theta = 6$	0	0	0	0	0.1237	0.2851	0.5912

The average difference is equal to 0,0014 (9,8% of average relative error on superior and inferior diagonals) for the rating transitions $\{(L^{h,ij})_h, (i, j) \in \{1, \dots, p\}\}$ and equal to 0,0159 (5,5% of average of relative error) for the hidden factor's transition probabilities $\{K^{rh}, r, h \in \mathbb{T}\}$. Despite the high dimensionality of the problem, these indicators demonstrate that the estimation algorithm is able to recover the parameters.

6.1.1 Filtering

Figure 3.1 shows the real trajectory and the filtered trajectory of the hidden factor computed on the testing sample (of 100 time-steps).

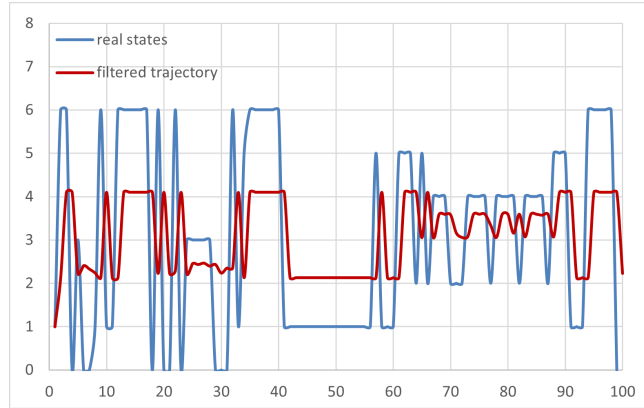


Figure 3.1: Simulated and filtered trajectories of the hidden factor on the last 100 testing dates

We can notice that the filter is able to detect the changes of states. It faithfully follows the real trajectory and respects the different phases and trends. However it does not exactly mimic the true value, since the filtering formula is a weighted average of the state values. We can also observe a small delay in the estimation. The explanation is theoretical: it is caused by the effect of delay in the filtering model: the impact of the hidden factor at time $t - 1$ is observable on ratings at time t . Therefore, when the hidden factor at time t is filtered, the freshest observations available at this time, is the rating jumps at time t which have been governed by the hidden factor at time $t - 1$. Consequently we infer the current hidden factor state with information generated by its previous value.

We can easily understand that the calibration plays a crucial role to make the filtering efficient. In order to forecast in time, states need to be strongly linked at least to another state. Let's imagine a rare and very unstable state. Since it is hardly visited from other states, it will never influence the direction of the filter and will be difficult to predict. Once the filter realizes that the hidden factor jumps to this state, it is too late, the hidden factor has already returned to another state. Finally, the filter is unable to capture rare events to unstable states. This remains acceptable since our main purpose is to detect transitions to stable regimes. Visiting a state for a brief period of time does not represent useful information for long term forecasting.

The following Figures 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, represent real ratios of observed transitions with the predicted transitions dynamics, obtained from (3.10) and (3.11), between the three considered rating categories $\{A, B, C\}$.

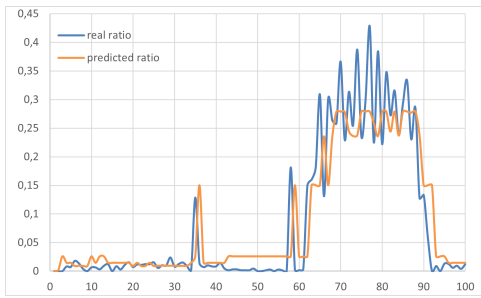


Figure 3.2: Real and predicted ratios from A to B

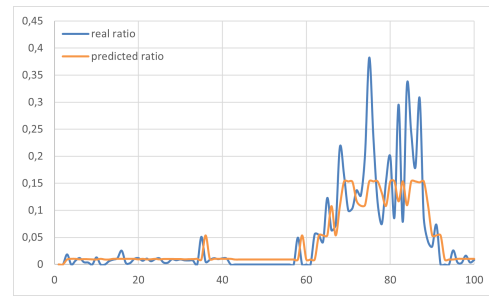


Figure 3.3: Real and predicted ratios from A to C

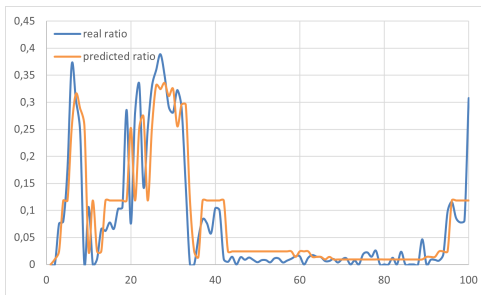


Figure 3.4: Real and predicted ratios from B to A

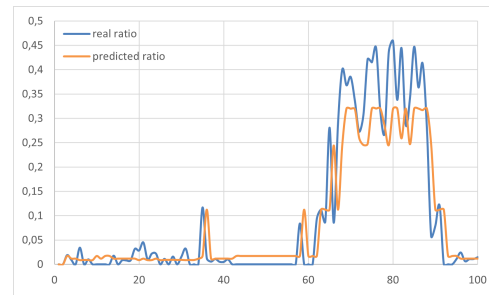


Figure 3.5: Real and predicted ratios from B to C

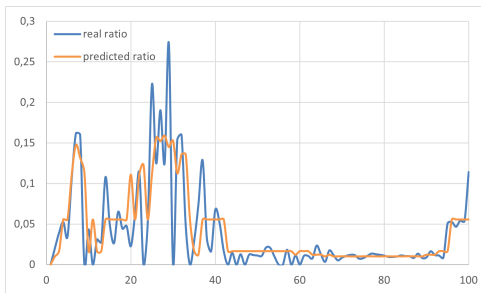


Figure 3.6: Real and predicted ratios from C to A

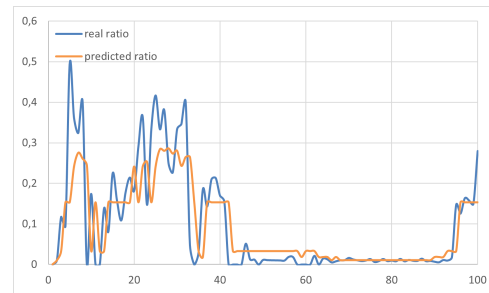


Figure 3.7: Real and predicted ratios from C to B

The results are very encouraging. The filter provides good predictions of future jumps. The predictions vary as a function of the regime cycle. Even when the real ratios sharply increase or decrease, the prediction are immediately corrected.

Although, one of our previous intuitions is confirmed, the filtering approach can not capture extreme variations since our approach forecasts an average of the rating transition probabilities.

6.2 Continuous time filtering approach

In order to validate the continuous-time filtering approach, we generate a data set using the migration model described in Section 5. The simulated rating processes exhibit no simultaneous jumps. We consider 3 rating categories and 1000 entities per class. We build a continuous Markov chain with 5 states. Note that we reduce the number of states compared to the discrete-time framework as the continuous framework is much more computationally demanding. Since the data is fictive and specific to the continuous-time model, this choice has no impact on our validation experiment. We directly applied the continuous-time filtering approach on the simulated data set, which does not contain simultaneous jumps. Then, in order to challenge the relevance of the use of the continuous model on discrete data, we transform the data set. Jumps are aggregated and randomly spread before filtering as described in Section 5.2. We apply the continuous-time filtering approach and compare the two predicted ratios dynamics. This comparison highlights the effect of the random re-distribution of jumps.

Figures 3.8 and 3.9 show the dynamics of the proportion of transitions predicted against the real observed ratios from rating A to rating B, respectively without and with redistribution.

The predicted ratios dynamics in Figure 3.8, validate the use of the continuous-time filtering approach: the predicted ratios follow the real trajectory of ratios.

By comparing with Figure 3.9, we deduce that spreading information (to avoid the simultaneity of jumps) does not alter the predictions. Thanks to this comparison exercise, we can apply continuous framework to real data without concern that the results are altered by this action.

Even if the data samples used are different, we can notice that the changes in both predicted ratios dynamics are less brutal than in the discrete-time filtering framework applied in Section 6.1. The continuous filter is updated with progressive information (due to the absence of simultaneous jumps) and is more flexible than the discrete filter to anticipate regime changes. Assimilating jumps one by one, seems to improve the quality of predictions. Nevertheless, the effect of delay is still observable.

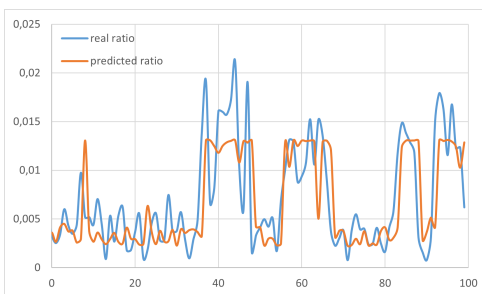


Figure 3.8: Real and predicted ratios from A to B without redistribution of common jumps

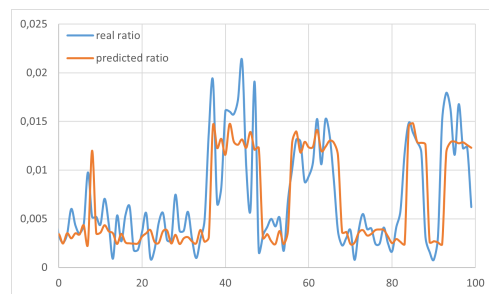


Figure 3.9: Real and predicted ratios from A to B with redistribution of common jumps

7 Application on real data

This section compares the results of our different models on a real rating database. We consider two discrete-time versions of the filter (one univariate and one multivariate) and a continuous-time multivariate alternative approach.

7.1 Data Description

Credit ratings are forward-looking opinions about the creditworthiness of an obligor with respect to a specific financial obligation. We build a transitions rating database from Moody's credit rating disclosure. We only use aggregated data (number of transitions). The considered sample contains 7791 days from January 2000 to May 2021. We study the evolution of Long Term ratings of 5030 corporate entities during this period without sector consideration. For specific experiments (analyses, validation, comparison), we consider the whole sample to calibrate the models. For others, such as testing the predictive power of model, we proceed to a cross validation. We choose a 5 states hidden factor for each experiment.

Moody's rating system relates 21 ratings categories. Keeping this granularity means estimating more than 420 transitions. Therefore, many studies ([Elliott et al. \(2014\)](#), [Koopman et al. \(2008\)](#)) reduce the number of rating categories. In the same way, we decide to aggregate the 22 ratings to 6 : A, Baa, Ba, B, C and W. An obligation is rated W when it has no rating. We will also rate W the entity whose rating is not observed. This happens when the data is missing, censored or when it is not appeared yet. There exists many ways to manage not rated status (W). It can be considered as bad information, good information, no information for the credit or not considering them at all. According to [Carty \(1997\)](#), only few (roughly 13 percent) of the migration to the not rated category are related to changes in credit quality. This argument motivated [Nickell et al. \(2000\)](#) to use the last method, consisting in removing from the sample all the entities that experiences a not rated status. But this approach is dubious in regard of the loss of information. In this study, we will consider no rated status as censorship. This is achieved by progressively eliminating companies whose rating is not known or withdrawn and adding them when a new rating is provided.

A reference time-step is chosen for each experiment. The daily data are aggregated in order to observe and to predict rating transitions on a larger time window.

7.2 Discrete time filtering in sample

In order to observe and interpret the effect of the discrete-time framework on a real credit rating database, we present in this section, the main results of a univariate and a multivariate filters, calibrated on the whole period.

7.2.1 Univariate discrete time filtering

In this part, we assume that each transition is governed by its own hidden factor. Under this assumption, each transition evolves according to the evolution of its own latent factor, independently from the others. This modeling is meaningful to integrate rating specificities in the predictions.

On the data set described above, we focus on a single transition: from rating B to C. We choose this transition because it could be identified as “transition to default” and witness of crisis. This will entail the use and calibration of the univariate form of the discrete-time filter (3.9).

A first step consists in calibrating the models with the past history of the involved transition. The reference time step, at stake in every transition, is 30 days. We highlight the efficiency of our approach without cross validation: all past transition history available (from January 2000 to May 2021) is used to calibrate the model.

We obtain in Table 3.10, the calibrated 30 days transition matrix of the hidden factor Θ . Table 3.11 presents the conditional transition probabilities from rating B to C in each state.

Table 3.10: Θ 's transition matrix

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$\Theta = 0$	0.90598	0.074109	0.018316	0	0.001595
$\Theta = 1$	0.230415	0.715919	0.040626	0	0.013040
$\Theta = 2$	0.000304	0.381375	0.540412	0.077909	0
$\Theta = 3$	0	0	0.740452	0.259548	0
$\Theta = 4$	0.491597	0	0	0.508403	0

Table 3.11: 30 days transition probabilities from B to C

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$B \rightarrow C$	0.001814	0.0050001	0.0158818	0.0451715	0.085771

Table 3.10 highlights two stable states, 0 and 1 and an unstable and rare state, state 4. By analysing Table 3.11, we notice a hierarchy of risk between the states of Θ . State 4 is clearly identified as the riskiest state with a downgrade probability fifty time greater than in state 0, the most favourable state. State 3 is also a state of crisis which is more stable. State 2 can be interpreted as an intermediate state between favourable and unfavourable

situation. Consequently we can expect that the economy often remains in a calm and favourable situation and experiences sometimes brief transitions to stressed states when downgrade probability B to C increases a lot.

Figure 3.10 presents the filtered indicator function trajectories of the own hidden factor of the transition B to C, $\{\hat{I}_n^h, n \in \{0, \dots, \Gamma\}, h \in \mathbb{T}\}$, without cross validation. Figure 3.11 shows the dynamics of 30 days forecasted ratios from rating B to C, $\{\hat{\nu}_{n-1}^{BC}, n \in \{1, \dots, \Gamma\}\}$, given in (3.11).

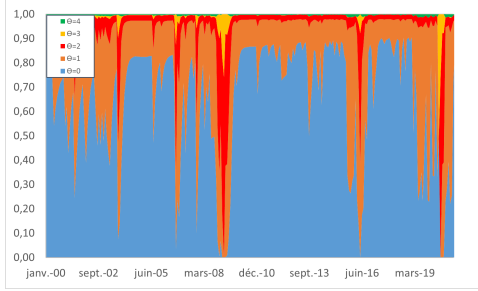


Figure 3.10: Filtered trajectories of the hidden factor indicator functions

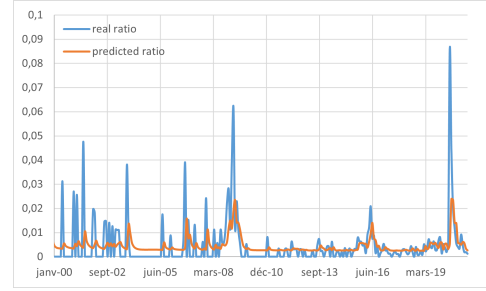


Figure 3.11: Real and predicted ratios for transition B to C

Figure 3.10 shows that the dominant state changes across time and highlights regime switching. Our intuitions are confirmed, the filter is often “closed” to favourable states 0 and 1. The dominant state is sometimes, for a brief moment, state 2, an intermediate state, where the downgrade probability from B to C increases. After periods when state 2 is dominant, the filter sometimes indicates that a state of true crisis, state 3, becomes dominant. Transitions from periods where state 0 or 1 are dominant to periods where state 4 is dominant may be sudden but remain rare. Fortunately this state of extreme “crisis” is only dominant for very brief periods. By analysing Figure 3.11, it can be noted that the predicted ratios from B to C reflect the general trend of real ratios with the same “lag” effect observed than on fictive data. The filter is able to detect regimes and transition phases but cannot capture brutal and short transitions. Finally the filter infers that the economic cycle experiences long periods of favourable situations and brief transitions to stress states.

Note that the hidden factor is specific to the involved transition. It may cover systematic risk but also the risk which might be specific to the ratings at stake.

We now consider the multivariate case where the hidden factor is shared by several transitions.

7.2.2 Multivariate discrete time filtering

Using multiple transitions to infer the hidden factor assumes that the later is shared by those transitions. This approach should bring more information to forecast the dynamics of these transitions but presents several difficulties. The calibration algorithm finds

centroids in the parameters space which might be far from each other due to the high dimension of the parameters space. Consequently the predicted number of transitions may be very different from the realized one. Furthermore rating transition events may not be sufficiently correlated. Indeed certain transitions are weakly correlated and might bring noise. We must only consider the most correlated transitions to extract the global factor dynamics. Therefore we decide to only focus on adjacent downgrade transitions (the upper diagonal). Indeed empirical results from [Cousin and Khelouen \(2015\)](#) show that the upgrades are more subject to idiosyncratic shocks than downgrades. To remove the impact of the remaining transitions on the model, we assign them the same probability for each state of the hidden factor: we use the time-homogeneous intensity estimators to compute these probabilities (see, e.g., [Cousin and Khelouen \(2015\)](#), [Duffie et al. \(2007\)](#), [Jarrow et al. \(1997\)](#), [Koopman et al. \(2008\)](#), [Lando and Skødeberg \(2002\)](#)). Consequently we reduce the number of transitions to calibrate to four.

We achieve two experiments. First we consider a time step reference of 30 days. We calibrate on whole period of the data set to observe the behaviour of the multivariate model. Then, along a second experiment, we will proceed to a cross validation to faithfully assess the predictive power of the model. For this experiment which is computationally more expensive, we will choose a larger time window, with a time step of 50 days.

For the first experiment, as in [Section 7.2.1](#), we again consider 5 states for the hidden factor, a time step of 30 days and we do not proceed to cross validation.

[Table 3.12](#) gives the calibrated transition matrix of the hidden factor. [Table 3.13](#) presents the conditional downgrade probabilities for a time step of 30 days.

Table 3.12: Θ 's transition matrix

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$\Theta = 0$	0.9499	0.0418	0.0010	0	0.0073
$\Theta = 1$	0.1075	0.7661	0.1264	0	0
$\Theta = 2$	0.0004	0.2685	0.6340	0.0503	0.0469
$\Theta = 3$	0	0	0.5133	0.4867	0
$\Theta = 4$	0	0	1	0	0

Table 3.13: Adjacent 30 days downgrade probabilities

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$A \rightarrow Baa$	0.00297589	0.00224944	0.00838262	0.00801885	0.0194804
$Baa \rightarrow Ba$	0.00125687	0.00146192	0.00492593	0.00985172	0.031583
$Ba \rightarrow B$	0.00326413	0.00633207	0.0150595	0.0282736	0.0228716
$B \rightarrow C$	0.00189228	0.00492691	0.0128149	0.0641203	0.0114155

By analysing the tables, it is noteworthy that states 0 and 1 are stable states which induce a “favourable” situation, where downgrade probabilities are quite low. States 3 and 4 can

be interpreted as a stressed economy, where downgrade probabilities are higher. Note that state 4 is totally unstable and transitory. The transition between favourable periods (state 0 and 1) and stable stressed periods (state 3) is exclusively achieved through state 2.

Figure 3.12 shows the filtered trajectories of state probabilities according to (3.10). Figure 3.13 presents the dynamics of the predicted ratio from rating B to C, within a multivariate framework, without cross validation. We focus on transition B to C to compare with Section 7.2.1.

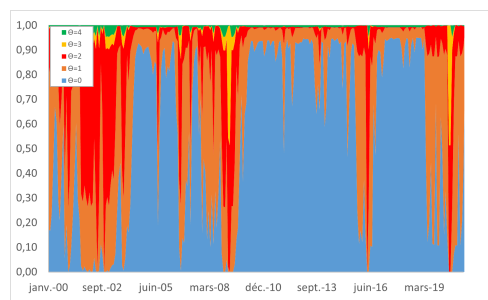


Figure 3.12: Filtered trajectories of hidden factor indicator functions

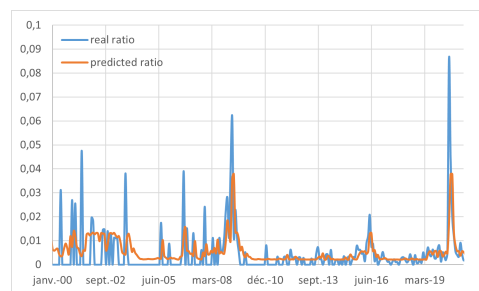


Figure 3.13: Real and predicted ratios for 30 days transition from B to C

Figure 3.12 brings us new information on the evolution of the predicted hidden state. Periods of crisis when state 3 and 4 dominant, are pretty rare and brief. By analyzing Figure 3.13, we can first notice that the multivariate framework is also a good predictor. The forecasted transition ratios follow the trend of observed ratios and fit with different regimes. Comparing with the univariate case (see Figure 3.10), the multivariate model seems to be more sensitive to events: the multivariate model better captures the crisis of sep-2000 compared to the univariate model. The forecasted rating transition B to C is not only based on its own past evolution but also stem from the history of others.

7.3 Comparison of the filters out of sample: annual recalibration

We use a cross-validation approach to assess the predictive power of the multivariate models both in the continuous-time and discrete-time frameworks. To this end, we use data from 2000 to 2008 to perform a first calibration and to initialize our parameters. Then, from January 2008 to may 2021, we predict the dynamics of the 50 days transition rates. The model is re-calibrated every year, integrating the new observations of the last year. Note that we changed the reference time step to 50 days for a sake of computational speed. Note also that since we re-calibrate the model yearly, parameters and states structure vary over time.

7.3.1 Multivariate continuous time filtering

In this section, we apply the continuous filtering framework, presented in Section 5 and its adaptations, described in Section 5.2, to real data. We choose a reference time step equal to 50 days. The real and predicted 50 days rating transition ratios are presented in Figures 3.14, 3.15, 3.16 and 3.17.

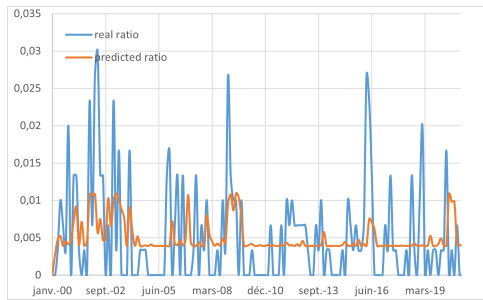


Figure 3.14: Real and predicted ratios 50 days transition from A to Baa

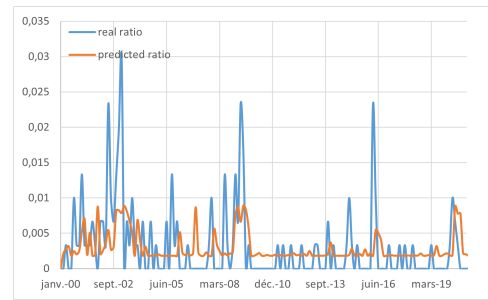


Figure 3.15: Real and predicted ratios 50 days transition from Baa to Ba

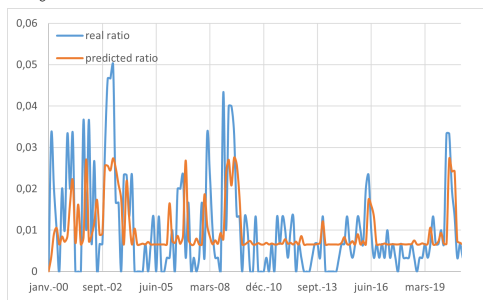


Figure 3.16: Real and predicted ratios 50 days transition from Ba to B

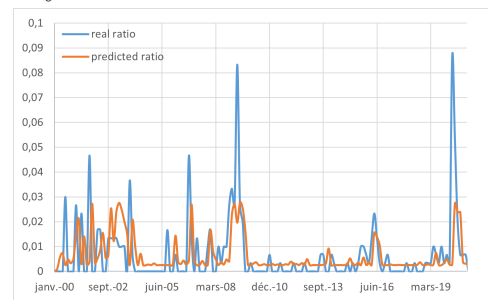


Figure 3.17: Real and predicted ratios 50 days transition from B to C

7.3.2 Multivariate discrete time filtering

Here, we apply the discrete-time filtering framework, presented in Section 3 to real data. The model is applied on the same sample used for the continuous-time filtering approach, with annual recalibration as in Section 7.3.1. We keep a reference time step equal to 50 days. Figures 3.18, 3.19, 3.20, 3.21 compare the dynamics of predicted transition ratios to observed one.

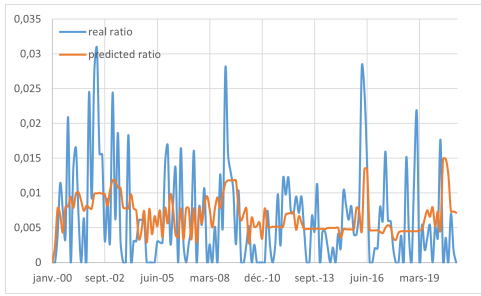


Figure 3.18: Real and predicted 50 days-transition ratios from A to Baa

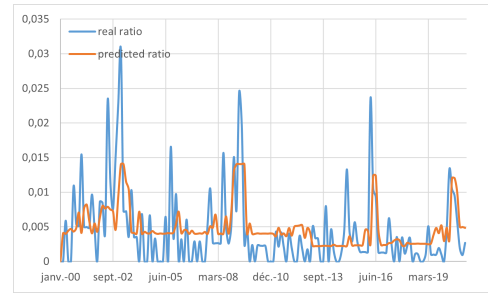


Figure 3.19: Real and predicted 50 days-transition ratios from Baa to Ba

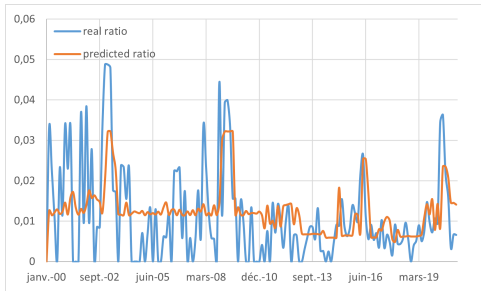


Figure 3.20: Real and predicted 50 days-transition ratios from Ba to B

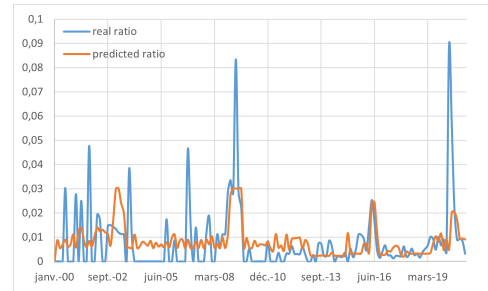


Figure 3.21: Real and predicted 50 days-transition ratios from B to C

7.3.3 Comparisons and analyses

The results look almost similar in both approaches. The dynamics of predicted ratios follow the trend of realized ratios. The forecasts also evolve when noteworthy crisis occurs. We notice that transitions are more correlated during specific periods like crisis. Four crisis periods can be identified: a first small one around 2002, a moderated one in 2016 and two significant in 2008 and 2020. These latter are clearly identified as the subprime crisis and the health crisis caused by the Covid 19. The two others, moderated, would be respectively the consequences of the dot-com bubble in 2000 and the China stock market crash in June, 2015. During these periods the downgrades probabilities increase.

Both models are able to detect the evolution of the economic cycle from observations of rating migrations. The forecasts are adapted to the inferred economic state. During crisis periods, the models are able to predict adapted and higher downgrade probabilities.

We can underline three advantages of the continuous-time version compared to the discrete-time one.

- The effect of delay (or lag effect) is less significant than in the discrete-time framework. By spreading simultaneous jumps in small time intervals, we make last infor-

mation used for filtering fresher than it actually is. This fictive operation, however, improves the predictions.

- We note that the discrete-time model struggles to capture brief and brutal variations. As we observed in Section 6.2, continuous-time filtering approach has the advantage of assimilating jumps one by one and of being more flexible and suitable to anticipate sudden transitions. Since information is spread and distributed in fictive intervals, the filter progressively assimilates information and is therefore quicker to adapt its predictions.

Nevertheless we can see that this framework is not fully adapted to rating migrations. The discrete version is easier and faster to compute: manipulations described in Section 5.2 increases considerably the number of time intervals to consider, the complexity and remains laborious. Above all, the discrete model is more consistent with the data and finally, provides predictions of a better quality. The continuous approach deals with continuous-time Markov chains. Therefore it could be improved by using an EM algorithm which estimates intensities directly. The effectiveness of this method would rather be highlighted by filtering a continuous phenomenon, where observations cannot occur simultaneously and exact occurrence dates are known. This intuition is confirmed by the following experiment. We compute the R^2 coefficient in the sample, to compare the forecasting power of the considered predictive models. We keep a reference time step of 50 days. We respectively compare the R^2 of the constant generator intensity model, the univariate models, and the multivariate discrete models and the continuous model in Table 3.14.

Table 3.14: R^2 in the sample

	$A \rightarrow Baa$	$Baa \rightarrow Ba$	$Ba \rightarrow B$	$B \rightarrow C$
Constant	0.463012	0.250668	0.483855	0.250514
Univ.Discrete	0.431184	0.346886	0.608684	0.22263515
Mult.Discrete	0.494395	0.479324	0.644975	0.367094
Mult.Continuous	0.2022	0.331062	0.49396	0.279736

We can directly notice that the multivariate discrete-time filter is the most accurate whatever the transition. The R^2 of the continuous filter is closed to discrete filter for transitions Baa to Ba and B to C but is lower for the transitions A to Ba, even lower than the R^2 from the constant generator model. This phenomenon can be explained by the poorer calibration achieved for the continuous-time filtering (for a sake of rapidity) and the inconsistency with the format of the data. The adapted continuous-time version can be applied to rating transitions framework and provides satisfactory predictions but can not reach the performance of the discrete-time version. Note that, in the univariate case, each transition has its specific model.

8 Conclusion of the Chapter

In this chapter, we present two filtering frameworks, respectively in a discrete-time and in a continuous-time setting. We adapt the filtering frameworks applied to Markov chains, to the context of rating migrations. For both approaches, we assume that rating transitions in a pool of obligors are driven by the same systematic hidden factor. The two alternatives are studied and compared. We discussed calibration issues and compared the predicted future rating transition probabilities on fictive and real data.

As illustrated in Sections 6 and 7, our methodology provides predictors adapted to the evolution of the economical cycle. We believe that our approaches can be used for PIT-estimations of transitions and detection of regimes. During crisis periods, our models are able to predict higher downgrade probabilities. Compared to other PIT-estimation models, our approaches base their predictions on the business cycle without concern of macro economic factors. From a practical point of view, our approaches also have the advantage to be interpretable. Observing the risk profile of each state and their filtered trajectories allows us to better understand the dynamics of the economic cycle as well as its systematic effect on rating migrations.

However, both approaches cannot capture idiosyncratic information: indeed [Schwaab et al. \(2017\)](#) found that only 18% to 26% of global default risk variation is systematic while the reminder is idiosyncratic. The share of systematic default risk is higher (39% to 51%) if industry-specific variation is counted as systematic.

In addition, applying the continuous framework to discrete time data is tedious and presents a risk of altering information and the quality of predictions. Since its complexity is much more important, the execution results of the continuous-time algorithm is very slow. Therefore, it suffers from poorer calibration than the discrete-time version. Thanks to the adaptations presented in Section 7.3, the continuous-time version is able to provide satisfactory predictions but can not reach the performances of the discrete-time model. For these reasons, the discrete-time approach turns out to be more adapted and efficient for the context of rating migrations.

Several improvements could be made to our framework. Both models could consider additional idiosyncratic observable factors as in [Koopman et al. \(2008\)](#). Moreover, the continuous-time framework could be improved by using an EM algorithm which estimates continuous time parameters directly (as, e.g., in [Damian et al. \(2018\)](#), [Liu et al. \(2017\)](#) and [Nodelman et al. \(2012\)](#)). This algorithm will still have to deal with simultaneous jumps and be computationally tractable. Furthermore, many studies show that rating migrations' dynamics first exhibit a non-Markovian behavior (migration data exhibit correlation among rating change dates, known as “rating drift”, contagion effect, ...) that cannot be captured by our models. The integration of these effects may represent a subject of reflection. Finally, the general derived filtering formulas could be applied to the context of rating migrations, possibly driven by non-Markovian hidden process. These considerations are left for future researches.

Conclusion of the part

In this part, we have introduced fundamental concepts in credit risk and provided a summary of the regulatory developments over the past 20 years regarding the role of credit ratings in estimating banks' regulatory capital. Specifically, we have emphasized the PIT property requirements for modeling rating migrations. The two main families of Point-in-time rating models have been described, distinguishing between whether the factors are observable or not and highlighting the limitations of each approach. This review of the existing literature and regulation has introduced our contribution to this field. We have developed an intensity-based factor migration model where transitions are driven by a common unobservable Markov chain. This approach has been developed for both discrete and continuous-time settings. We have demonstrated how to adapt the Baum-Welsh algorithm to calibrate model parameters based on past rating transition history and we have shown that the common factor can be estimated recursively using a filtering formula. Both approaches have been demonstrated to fit empirical transition probability dynamics very well, on both fictive and real datasets. This comparison reopens the debate about whether to use a continuous or discrete-time setting in modeling. While continuous-time modeling may better respect the underlying information, it is important for the user to prioritize compatibility with the format of the input data. In both settings, this model is a promising alternative to the popular approaches described earlier in this chapter, as it provides estimation of point-in-time migration rating probabilities without using any covariates. It is also tractable and does not require strong assumptions. Moreover, the model is interpretable and can be a useful tool for experts, as well as for detecting downturn periods and building stress test scenarios. However, the model is unable to account for idiosyncratic risk, which represents a significant loss of information. Overall, the choice of modeling approach depends on the specific needs and objectives of the investor or institution. It may be appropriate to use a combination of both structural and intensity models, as well as observable and unobservable factors, to achieve the best possible predictions of rating migration.

Part II

PORTFOLIO ALLOCATION

Chapter 4

Portfolio allocation: a review of methods

Abstract

In this chapter, we provide an overview of the portfolio allocation environment and the approaches commonly employed. We distinguish static and dynamic strategies and delve into the reason why investors may choose one method over another.

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1 Static portfolio allocation

We define the probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in [0,1]}, \mathbb{P})$. We study the problem of portfolio allocation for a single time period. An investor with initial wealth V_0 , aims to find a strategy of allocation at date $t = 0$ between d risky assets $S = (S^1, \dots, S^d)$ and a risk free asset S^0 which maximizes the expected utility of wealth at a terminal date $t = 1$. Let $R = (R^1, \dots, R^d)$, denotes the vector of rate of returns of risky assets. Let $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}[(R - \mu)(R - \mu)^T] = [\rho_{i,j} \hat{\sigma}_i \hat{\sigma}_j]_{1 \leq i, j \leq d}$ be the the vector of expected returns and the covariance matrix of risky asset returns. Let $R^0 = \mu_0$, denotes the rate of return of the risk free asset and $\bar{R} = \{R^0, \dots, R^d\}$, $\bar{\mu} = \mathbb{E}[\bar{R}]$, denote the augmented vectors R and μ with the risk free asset. Let V be the portfolio value at the final date $t = 1$. We define $(\alpha, \alpha^0) \in \mathbb{R}^d \times \mathbb{R}$, the respective risky and risk free asset quantities hold by the investor, and equivalently $(x, x^0) \in \mathbb{R}^d \times \mathbb{R}$, the proportion of the portfolio value invested respectively in the risky and risk free assets.

1.1 An Overview of Methods

In the endeavor to identify an investment strategy, an investor must navigate through an environment that is inherently characterized by uncertainty. In addition, the chosen strategy should fit with its profile and its financial goal.

There exist various approaches to pursue strategies that align with the investor's preferences. Among the commonly employed methods is the maximization of the investor's utility. Utility functions measure the level of satisfaction of an investor for different levels of its wealth. They are twice differentiable and monotone increasing. The second derivative determines the profile of the agent. If the utility function is convex, the agent exhibits risk-seeking behavior. Conversely, a concave utility function indicates risk aversion in the agent. Absolute and relative risk aversions are important concepts in the field of finance and economics. They are measures of an individual's willingness to take risks when making investment decisions. Given a utility function U , the absolute risk aversion (Ab) and the relative risk aversion (Re) are defined as follows

$$Ab(V) = \frac{U''(V)}{U'(V)} \text{ and } Re(V) = V \frac{U''(V)}{U'(V)}.$$

If the investor experiences an increase in wealth, he or she will choose to increase (or keep unchanged, or decrease) the money in the risky assets held in the portfolio if absolute risk aversion is decreasing (or constant, or increasing). Similarly, if the investor experiences an increase in wealth, he or she will choose to increase (or keep unchanged, or decrease) the proportion of the portfolio held in the risky assets if relative risk aversion is decreasing (or constant, or increasing).

The investor selects a financial portfolio that maximizes its utility function, or alternatively, minimizes its risk measure. Formally, the investor with utility U aims to solve

$$\sup_{\alpha, \alpha^0 \in \mathbb{R}^d \times \mathbb{R}} \mathbb{E}[U(V)]. \quad (4.1)$$

This section provides an overview of the existing portfolio allocation methods.

1.1.1 HARA utility

We say that a utility function U exhibits hyperbolic absolute risk aversion (HARA) if it can be expressed in the form

$$U[V] = \frac{1-\gamma}{\gamma} \left(\frac{aV}{1-\gamma} + b \right), \quad a > 0, \quad \frac{aV}{1-\gamma} + b > 0. \quad (4.2)$$

The HARA utility function exhibits decreasing absolute risk aversion. Noteworthy HARA utility functions include the quadratic utility functions, CRRA utility functions, the exponential and log utility functions.

1.1.1.1 CRRA Utility

The Constant Relative risk Aversion (CRRA), also known as Power utility function, has the same level of relative risk aversion for every level of wealth. It takes the following form

$$U[V] = \begin{cases} \frac{V^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1 \\ \ln V, & \gamma = 1 \end{cases} \quad (4.3)$$

The CRRA utility function has several important properties that make it a useful tool for modeling economic behavior. The CRRA utility function is a concave function and is a normalized function, which means that it is scaled in such a way that the utility of a person with zero consumption is zero.

Remark 18. *CRRA utility functions illustrate the difference between the dynamic and myopic (single-period) portfolio choice. If asset returns are independent, solving the multi-period allocation problem is equivalent to sequentially solve the uni-period one (see [Brandt et al. \(2005\)](#)).*

1.1.1.2 Exponential utility

An investor with an exponential utility U of parameter γ , aims to maximize

$$U[V] = -\gamma e^{-\gamma V}. \quad (4.4)$$

The exponential utility function exhibits constant absolute risk aversion, indicating that individuals are equally risk-averse across all levels of wealth.

1.1.2 Markowitz mean-var portfolio

In the famous Markowitz mean-var approach, investors are assumed to be risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. An investor will take on increased risk only if compensated by higher expected returns. Conversely, an investor who wants higher expected returns must accept more risk. The risk-reward trade-off is not the same for all investors, depending on their own individual risk aversion characteristics. In contrast with previous models, the objective function for obtaining the Markowitz mean-variance portfolio does not write an expected utility of wealth, $\mathbb{E}[U(V)]$. Indeed, the objective function aims to capture the balance between risk and reward. As a result, the satisfaction of the investor is not solely based on the final wealth, but also on the distribution of wealth. There are various ways to express an individual's risk aversion. One common method is to use a parameter, denoted by γ ,

such that an agent with risk aversion γ aims to maximize

$$\begin{aligned} & \sup_x \bar{\mu}^T x - \gamma x^T \Sigma x \\ & \text{s.t.} \quad \sum_{i=0}^d x^i = 1 \end{aligned} \quad (P_1(\gamma))$$

The risk aversion can also be translated into a return constraint ($P_2(\rho)$) or a variance constraint ($P_3(\eta)$).

$$\begin{aligned} & \inf_x x^T \Sigma x \\ & \text{s.t.} \quad \sum_{i=0}^d x^i = 1 \\ & \quad \bar{\mu}^T x = \rho \end{aligned} \quad (P_2(\rho))$$

$$\begin{aligned} & \sup_x \bar{\mu}^T x \\ & \text{s.t.} \quad \sum_{i=0}^d x^i = 1 \\ & \quad x^T \Sigma x = \eta \end{aligned} \quad (P_3(\eta))$$

We refer to [Roncalli \(2013\)](#) and [Steinbach \(2001\)](#) for details on this approach.

Let's first solve the mean-var allocation problem when the agent cannot invest in the risk free asset. The problem becomes

$$\begin{aligned} & \sup_x \mu^T x - \gamma x^T \Sigma x \\ & \text{s.t.} \quad \sum_{i=1}^d x^i = 1 \end{aligned} \quad (P_0(\gamma))$$

We deduce the Lagrange function $\mathcal{L}(x, \lambda) = x^T \mu - \gamma x^T \Sigma x - \lambda(\mathbf{1}^T x - 1)$. The first order conditions are

$$\begin{cases} d_x \mathcal{L}(x, \lambda) = \mu - 2\gamma \Sigma x - \lambda \mathbf{1} = 0 \\ d_\lambda \mathcal{L}(x, \lambda) = -\mathbf{1}^T x + 1 = 0 \end{cases}$$

The solution is given by

$$\begin{cases} x^* = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} + \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) \Sigma^{-1} \mu - (\mathbf{1}^T \Sigma^{-1} \mu) \Sigma^{-1} \mathbf{1}}{2\gamma \mathbf{1}^T \Sigma^{-1} \mathbf{1}} \\ \lambda^* = \frac{\mathbf{1}^T \Sigma^{-1} \mu - 2\gamma}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \end{cases}$$

By calling $A = (\mathbf{1}^T \Sigma^{-1} \mathbf{1})(\mu^T \Sigma^{-1} \mu) - (\mathbf{1}^T \Sigma^{-1} \mu)^2$ and $B = (\mu^T \Sigma^{-1} \mu)(\mathbf{1}^T \Sigma^{-1} \mathbf{1})^2 - (\mathbf{1}^T \Sigma^{-1} \mu)^2 (\mathbf{1}^T \Sigma^{-1} \mathbf{1})$, we obtain the form of the efficient frontier

$$\text{Var}[V^*] = \frac{B}{A^2} \left(\mathbb{E}[V^*] - \frac{\mu^T \Sigma^{-1} \mu}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \right)^2 + \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

The feature of this efficient frontier is represented by the blue curve in Figure 4.1.

Now let's return to the general case, where the agent can invest in the risk-free asset. We propose to solve $(P_1(\gamma))$. We note that the problem can be written as

$$\begin{aligned} \sup_x \quad & x^T \mu + x^0 \mu^0 - \gamma x^T \Sigma x \\ \text{s.t.} \quad & \mathbf{1}^T x + x^0 = 1 \end{aligned}$$

We deduce the Lagrange function $\mathcal{L}(x, x^0, \lambda) = x^T \mu + x^0 \mu^0 - \gamma x^T \Sigma x - \lambda(\mathbf{1}^T x + x^0 - 1)$. The first order conditions are

$$\begin{cases} d_x \mathcal{L}(x, x^0, \lambda) = \mu - 2\gamma \Sigma x - \lambda \mathbf{1} = 0 \\ d_{x^0} \mathcal{L}(x, x^0, \lambda) = \mu^0 - \lambda = 0 \\ d_\lambda \mathcal{L}(x, x^0, \lambda) = -\mathbf{1}^T x - x^0 + 1 = 0 \end{cases}$$

The solution is then

$$\begin{cases} x^* = \frac{1}{2\gamma} \Sigma^{-1} (\mu - \mu^0 \mathbf{1}) \\ x^{*,0} = 1 - \frac{1}{2\gamma} \mathbf{1}^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1}) \\ \lambda^* = \mu^0 \end{cases}$$

Let's consider the portfolio $x_T = \frac{\Sigma^{-1}(\mu - \mu^0 \mathbf{1})}{\mathbf{1}^T \Sigma^{-1}(\mu - \mu^0 \mathbf{1})}$. The solution can be rewritten as

$$\begin{cases} x^* = \alpha x_T \\ x^{*,0} = 1 - \alpha \\ \alpha = \frac{1}{2\gamma} \mathbf{1}^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1}) \end{cases}$$

The optimal portfolio is the composition of a risky portfolio x_T and the risk free asset. Note that the risky portfolio is independent of the risk aversion. We deduce that the risk aversion only fixes the proportions between the risk free asset and a risky portfolio. The expectation and standard deviation of the optimal portfolio can be expressed as

$$\mathbb{E}[V^*] = \mu^T x^* V_0 + x^{*,0} \mu^0 V_0 = \alpha (\mu^T x_T V_0) + (1 - \alpha) \mu^0 V_0,$$

$$\text{Var}[V^*]^{\frac{1}{2}} = \sqrt{(x^*)^T \Sigma x^*} V_0 = \alpha V_0 \sqrt{x_T^T \Sigma x_T}.$$

Then the Sharpe ratio of the optimal portfolio is

$$\text{Sharpe}[V^*] = \frac{\mathbb{E}[V^*] - \mu^0 V_0}{\text{Var}[V^*]^{\frac{1}{2}}} = \frac{\mu^T x_T - \mu^0}{\sqrt{x_T^T \Sigma x_T}} = \sqrt{(\mu - \mu^0 \mathbf{1})^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1})}.$$

We notice that the Sharpe ratio of the optimal portfolio is equal to the Sharpe ratio of the portfolio x_T . In addition, it is independent of the risk aversion. The risky portfolio x_T is the optimal portfolio for a risk aversion $\gamma = \frac{1}{2} \mathbf{1}^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1})$, where there is no cash invested in the risk free asset. This portfolio is called the Tangency portfolio. It belongs to the efficient frontier of the mean-var problem in absence of risky free asset. It has the highest Sharpe ratio on that efficient frontier. Figure 4.1 illustrates the relationship between optimal portfolios and risk aversion, by presenting the features of the efficient frontiers. Three portfolios stand out as particularly noteworthy: the Maximum Return Portfolio, the Minimum Variance Portfolio, and the Tangency Portfolio. The two first portfolios are efficient only when a risk-free asset is not available, while the Tangency Portfolio is efficient in both cases. It is also noteworthy that the efficient frontier with a risk-free asset is represented by a straight line with a slope equal to the Sharpe ratio.

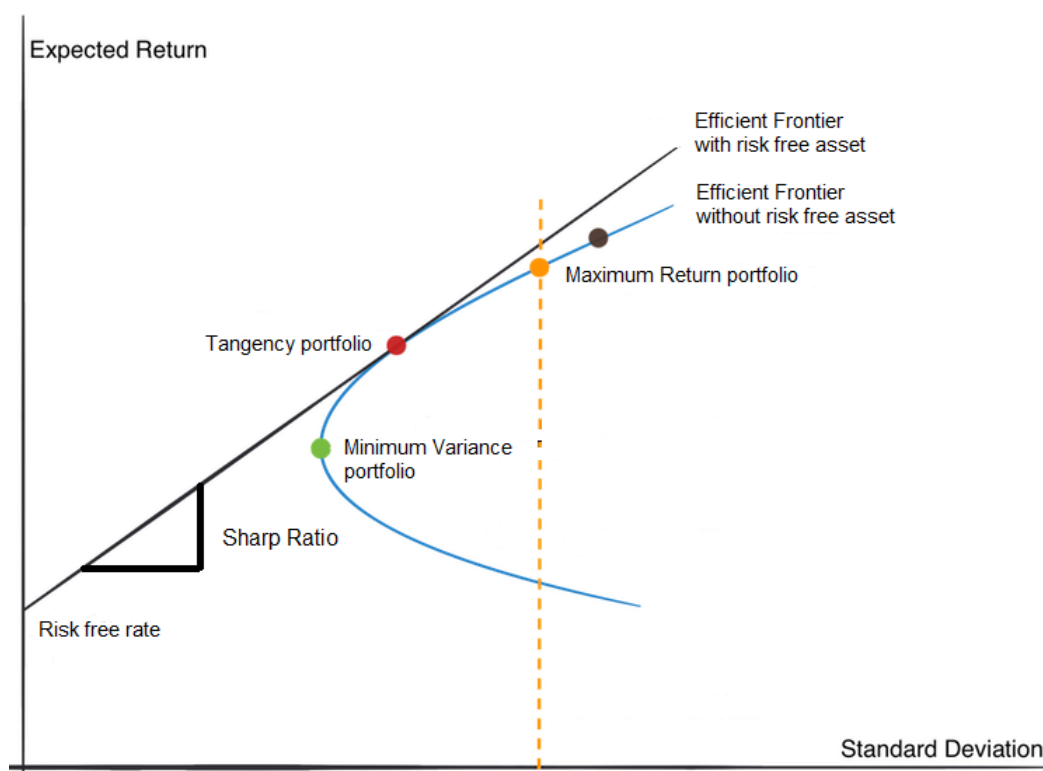


Figure 4.1: Efficient frontier and Tangency portfolio

Remark 19. *When the market is at equilibrium, the Tangency portfolio is equal to the Market portfolio. The Market portfolio is a diversified portfolio that contains all assets in*

the market, weighted according to their market value. The Capital Asset Pricing Model (CAPM) uses the Market portfolio as a key component to estimate expected returns on individual assets and the overall market.

By similar techniques, we can solve $(P_2(\rho))$ and $(P_3(\eta))$. The optimal solution being unique, we can equalize the solutions, to express an equivalence between the risk aversion and the minimum expected return ρ in $(P_2(\rho))$ and the maximum level of risk η in $(P_3(\eta))$.

Proposition 20. *Problems $(P_1(\gamma))$ and $(P_2(\rho))$ are equivalent if and only if*

$$\rho = \mu^0 + \frac{(\mu - \mu^0 \mathbf{1})^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1})}{2\gamma}.$$

Proposition 21. *Problems $(P_1(\gamma))$ and $(P_3(\eta))$ are equivalent if and only if*

$$\eta = \frac{(\mu - \mu^0 \mathbf{1})^T \Sigma^{-1} (\mu - \mu^0 \mathbf{1})}{4\gamma^2}.$$

1.1.3 Popular objective functions and portfolios

In this section, we present other strategies of allocation commonly used by professionals. Table 4.1 summarizes the objective functions related to these strategies. Besides the mean-variance optimized portfolio (MVO) and (HARA) utility functions, we find basic allocation methods as, the equal weight portfolio (EWP), the inverse volatility portfolio (IVOP) and the inverse variance portfolio (IVAP). We also find more sophisticated approaches as the global minimum variance portfolio (GMVP), the maximum decorrelation portfolio (MDP), the maximum diversified portfolio (MDPC), the equal risk contribution portfolio (ERC), the risk budgeting portfolio (RB). We refer to [Roncalli \(2013\)](#) for the proper definition of \mathcal{R} , a convex and coherent risk measure and $\mathcal{R}B$, the vector of risk budgets. We also include the (KL) portfolio, based on the Kullback-Leibler (KL) information criteria.

Table 4.1: Summary of popular objective functions

Name	Form	Reference
MVO	$\bar{\mu}^T x - \gamma x^T \Sigma x$	Markowitz (1952)
EWP	$\frac{1}{d} \sum_{i=1}^d \ln x^i$	Maillard et al. (2010)
IVOP	$\sum_{i=1}^d \hat{\sigma}_i \ln x^i$	Shimizu and Shiohama (2020)
IVAP	$\sum_{i=1}^d \hat{\sigma}_i^2 \ln x^i$	Arévalo et al. (2019)
GMVP	$\frac{1}{2} x^T \Sigma x$	Clarke et al. (2011)
MDP	$\frac{1}{2} x^T \rho x$	Errunza et al. (2010)
MDPC	$\ln(\sqrt{x^T \Sigma x}) - \ln(x^T \hat{\sigma})$	Choueifaty and Coignard (2008)
ERC	$\frac{1}{2} x^T \Sigma x - \lambda \sum_{i=1}^d \ln x^i$	Maillard et al. (2010)
RB	$\mathcal{R}(x) - \lambda \sum_{i=1}^d \mathcal{R}B_i \ln x^i$	Roncalli (2013)
KL	$\sum_{i=1}^d x_i \ln\left(\frac{x^i}{x_0^i}\right)$	Bera and Park (2008)

The Equal Weight Portfolio (EWP) strategy involves distributing wealth equally among assets in the portfolio, assigning the same weight to each asset and ensuring that every asset has the same level of importance. Formally, the weights are given by $\forall i \in \{1, \dots, d\}$, $x^i = \frac{1}{d}$. This approach expresses the view that market returns are totally random, with no available information. The Equal-Weighted Portfolio (EWP) is only considered optimal when there is no relationship between expected return and risk. In an extreme scenario where assets have the same expected return, volatility, and homogeneous correlations, this approach can be effective. This naive methods offers the advantage of totally ignoring of estimation errors and market information.

Inverse volatility (IVOP) and respectively inverse variance (IVAP) provides an optimal portfolio weighted in proportion to the inverse of the assets' volatilities, respectively to the inverse of the assets' variances. The weights of the inverse volatility and inverse variance portfolio are given by

$$\forall i \in \{1, \dots, d\}, x^i(IVO) = \frac{\frac{1}{\hat{\sigma}_i}}{\sum_{j=1}^d \frac{1}{\hat{\sigma}_j}}, x^i(IVA) = \frac{\frac{1}{\hat{\sigma}_i^2}}{\sum_{j=1}^d \frac{1}{\hat{\sigma}_j^2}}.$$

This approach makes the volatility, respectively the variance, of each weighted assets equal to $\frac{1}{d}$. These methods ignore the assets correlations.

The global minimum variance portfolio (GMVP) seeks the allocation which provides the risk-less portfolio. In a scenario where assets have identical expected return , the (GMVP) would be completely efficient. Nevertheless, a global low variance implies low chances of profit. The strategy of this portfolio is defensive and aims to be resilient in the wake of financial crisis rather than making profit.

The most diversified portfolio (MDP) aims to maximise the diversification ratio (DR) defined by

$$DR = \frac{x^T \hat{\sigma}}{\sqrt{x^T \Sigma x}}.$$

This approach tends to invest in assets that are individually risky but less sensible to contagion effect in the portfolio. This approach may be relevant if the market is assumed to be risk-efficient, meaning that investments will generate returns in proportion to their total risk as measured by volatility. In this scenario, (MDP) approach maximizes the Sharpe ratio.

The maximum decorrelation portfolio (MDCP) is closely related to (GMVP) and (MDP), but applies to the case where the investor believes that all assets have similar returns and volatility, but heterogeneous correlations. This approach involves applying a Minimum Variance optimization to the correlation matrix rather than the covariance matrix. The diversification ratio (DR) is maximized when all assets have the same volatility. The

Sharpe ratio (SR) is maximized when all the assets have the same volatility and the same expected return.

The equal risk portfolio (ERC) is satisfied when each asset contributes equally to the portfolio's overall volatility. We define the marginal risk contribution of asset $i \in \{1, \dots, d\}$ as $\frac{\partial \hat{\sigma}(x)}{\partial x^i}$. Then the risk contribution in the portfolio of asset i is $\mathcal{RC}_i = x^i \frac{\partial \hat{\sigma}(x)}{\partial x^i} = x^i \frac{[\Sigma x]_i}{\sqrt{x^T \Sigma x}}$ (see [Roncalli \(2013\)](#)). This approach aims to equalize the values $(\mathcal{RC}_i)_{i \in \{1, \dots, d\}}$, ie $\forall i, j \in \{1, \dots, d\}, x^i [\Sigma x]_i = x^j [\Sigma x]_j$. As a result, the solution can be found by numerically minimizing $\sum_{i,j=1}^d (x_i [\Sigma x]_i - x_j [\Sigma x]_j)^2$ (see [Maillard et al. \(2010\)](#), [Mausser and Romanko \(2014\)](#)). An equivalent alternative approach, proposed by [Maillard et al. \(2010\)](#), consists in minimizing the objective function presented in [Table 4.1](#). The (ERC) portfolio does not require the investor to express a view on future expected returns. Note that the equal risk portfolio is a compelling balance between the equal portfolio and the minimum variance portfolios. It is also noteworthy that this approach can be extended to other risk measures.

The risk budgeting allocation (RB) is an extension of the (ERC). It consists in finding the portfolio such that risk contributions match with fixed risk budgets, $(\mathcal{RB}_i)_{i \in \{1, \dots, d\}}$, ie $\forall i \in \{1, \dots, d\}, \mathcal{RC}_i = \mathcal{RB}_i$. The main objective of this allocation method is not to maximize performance, but rather to ensure that the portfolio is diversified enough to cover the risk according to pre-determined budget constraints.

We present in the following figure, an example of decision tree, extracted from [asset management \(2015\)](#). This tree aims to illustrate the procedure that can be followed by investors to choose their approach, according to the information and views they have on the market. This tree illustrates the decision-making process that an agent may undergo when selecting an approach. The governing principle is investors' perspectives on the market. If investors have insights on both returns and volatilities, the mean-variance approach is the natural preference. When investors' views are limited to volatilities alone, the chosen approach will be determined by the kind of risk the investors believe will be rewarded by the market. In absence of views on volatilities but with insights on correlation, the investors seek decorrelating their portfolio. When investors have no view on the market, choosing the less sensitive portfolio, the equally-weighted portfolio is the recommended choice. Opting for a sophisticated approach that is highly sensitive to input parameters can lead to poor outcomes if there are significant errors in the estimation of these parameters. Investors should assess their confidence in the input parameters and choose their approach accordingly.

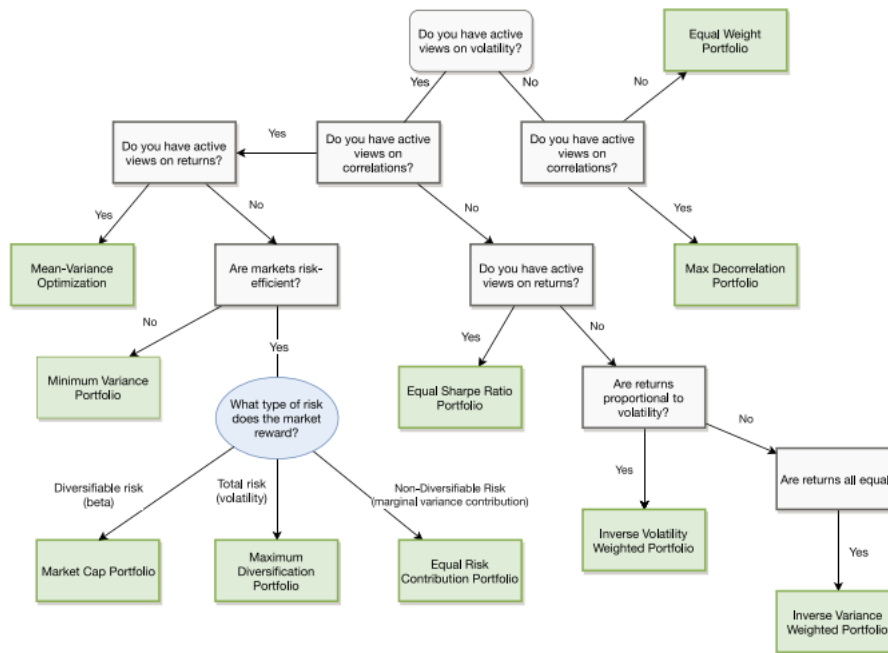


Figure 4.2: Decision tree for portfolio allocation method

1.1.4 Black-Litterman model

In this section we briefly present the concept the formalisation of Black-Litterman allocation approach. More details can be found in [Satchell and Scowcroft \(2000\)](#).

It was developed by Fischer Black and Robert Litterman in the early 1990s (see [Black and Litterman \(1992\)](#)).

Black-Litterman allocation is a powerful asset allocation technique that can help investors to incorporate their views and beliefs into the investment process. It is a valuable tool for investors seeking to build portfolios that are aligned with their investment objectives and risk tolerance. Indeed, while modern portfolio theory is an important theoretical advance, its application has universally encountered a problem: although the covariances of a few assets can be adequately estimated, it is difficult to come up with reasonable estimates of expected returns. Black-Litterman's model overcomes this problem by not requiring the user to input estimates of expected return. The investor begins by specifying their views on the expected returns of certain assets. These views are often based on fundamental analysis, macroeconomic factors, or technical analysis.

The contribution of Black-Litterman model was to place this problem into a tractable form with a prior distribution that was both sensible and communicable to investors. In the Black-Litterman model, the CAPM equilibrium distribution is the prior, the investor's views are the additional information.

Remark 22. In the CAPM, the expected excess return of asset i is given by

$$\forall i \in \{1, \dots, d\}, \Pi_i = \mu_i - r = \beta_i (\mathbb{E}[R_m] - r),$$

where \mathbb{R}_m is the return on the global market, or the return of the market portfolio, $\beta_i = \frac{\text{COV}[R_i, R_m]}{\text{VAR}[R_m]}$, the sensitivity of the expected excess asset returns to the expected excess market returns. The Market portfolio is a portfolio consisting of a weighted sum of every asset in the market, with weights in the proportions that they exist in the market. We denote as x_m the weights of assets on the global market, determined by their market values. Algebraically, assuming the validity of the CAPM,

$$\Pi = \delta \Sigma x_m,$$

where $\delta = \frac{(\mathbb{E}[R_m] - r)}{\text{VAR}[R_m]}$ is a positive constant. δ can be interpreted as the risk aversion parameter representing the world average risk tolerance.

In an equilibrium state, the market portfolio is held collectively by all investors. The equilibrium expected excess return Π , is such that if all investors hold the same view and invest in a CAPM-type world, the demand for these assets exactly equals to the outstanding supply and Π represents the equilibrium returns conditional upon the individuals' common beliefs. We then assume that the equilibrium excess return conditional upon individual views equals the individual views on average. $(\Pi|\mu) \sim N(\mu, t\Sigma)$, where t is a scalar indicating the uncertainty of the CAPM prior.

The investor has a set of K beliefs represented as linear relationships. Let $P = (P_1, \dots, P_K)$ representing these relationships. We denote as $Q = (Q_1, \dots, Q_K)$ the views on returns of P . We assume that $P\mu \sim N(Q, \mathbb{J})$. For $1 \leq i \neq j \leq K$, \mathbb{J}_{ij} is the covariance between the view i and the view j . \mathbb{J}_{ii} traduces the degree of confidence of the investor in the view i . CAPM equilibrium and views can be combined in

$$f(\mu|\Pi) \propto f(\Pi|\mu)f(\mu),$$

where $f(\cdot)$ means probability density function. $f(\mu|\Pi)$ is the combined return or posterior forecast given the equilibrium information, $f(\mu)$ is the prior probability function of prior views of the investor, $f(\Pi|\mu)$ is the conditional excess return equilibrium given the views of investors. We can deduce that $(\mu|\Pi) \sim N(\hat{\mu}, \hat{\Sigma})$, where

$$\begin{cases} \hat{\mu} = [(t\Sigma)^{-1} + P^T \mathbb{J}^{-1} P]^{-1} [(t\Sigma)^{-1} \Pi + P^T \mathbb{J}^{-1} Q] \\ \hat{\Sigma} = [(t\Sigma)^{-1} + P^T \mathbb{J}^{-1} P]^{-1} \end{cases}$$

We can then solve the mean-var allocation problem ($P_3(\eta)$) with parameters $(\hat{\mu}, \hat{\Sigma})$.

1.2 Allocation Constraints

This section provides a brief overview of the prominent constraints encountered in portfolio allocation management. We refer the reader to [Perrin and Roncalli \(2020\)](#).

Table 4.2: Constraints in portfolio allocation

Name	Form
No cash and leverage	$\sum_{i=1}^d x^i = 1$
No short selling	$x^i \geq 0$
Weight bounds	$x^i_- \leq x^i \leq x^i_+$
Asset class limits	$\tau_-^j \leq \sum_{i \in C_j} x^i \leq \tau_+^j$
Turnover	$\sum_{i=1}^d x^i - x_0^i \leq \tau$
Transaction cost	$\sum_{i=1}^d c^+(x^i - x_0^i)_+ + c^-(x^i - x_0^i)_- \leq \tau$
Leverage limit	$\sum_{i=1}^d x^i \leq \tau$
Long short exposure	$\tau_- \leq \sum_{i=1}^d x^i \leq \tau_+$
Number of active bets	$(x^T x)^{-1} \geq \tau$
Benchmark	$\sqrt{(x - y)^T \Sigma (x - y)} \leq \tau$
Tracking error floor	$\sqrt{(x - y)^T \Sigma (x - y)} \geq \tau$
Active share floor	$\frac{1}{2} \sum_{i=1}^d x^i - y^i \geq \tau$

Portfolio management involves numerous constraints that are often applied to ensure that portfolios remain within acceptable risk and investment parameters. One well-known constraint in portfolio management is the no-short selling which forbids short positions. Weight bounds and asset class limits are also extensively used by professionals. Turnover management is another common constraint, which limits an investor's ability to reverse his positions. Managing long/short portfolios often involves imposing leverage or long/short exposure limits. Diversification is also an important aspect of portfolio management, with the number of active bets or the inverse of the Herfindahl index commonly used to monitor it. For benchmarked strategies, a tracking error limit with respect to the benchmark y can be imposed. A minimum tracking error or active share can also be required for active management.

1.3 Transaction and position costs

This section provides a brief overview of the prominent costs encountered in portfolio allocation management. For further details, the reader may consult [Perrin and Roncalli \(2020\)](#).

1.3.1 Broker commission

When trades are executed through a broker, the agent can be charged a commission for carrying out the trade on its behalf. We found fixed cost and proportional costs, proportional to the total trade volume. The agent pays the costs every time he rebalances its portfolio.

$$C = \sum_{i=1}^d \nu_{buy}^i (x^i - x_0^i)^+ V_0 + \nu_{sell}^i (x^i - x_0^i)^- V_0 + \nu^f \mathbb{1}_{\{x \neq x_0\}}. \quad (4.5)$$

We can also find the simplified form when $\nu_{buy}^i = \nu_{sell}^i$,

$$C = \sum_{i=1}^d \nu^i |x^i - x_0^i| V_0 + \nu^f \mathbb{1}_{\{x \neq x_0\}}. \quad (4.6)$$

Integrating transaction costs into models can often be challenging due to the form of the terms involved.

1.3.2 Bid-ask spread

The price at which one buys and sells assets are different. If the mid-price is used as reference for determining the allocation strategy, it can result into an error that may be offset by incorporating extra transaction costs as

$$C = \sum_{i=1}^d \kappa^i |x^i - x_0^i| V_0.$$

where κ^i is one half the bid-ask spread for asset i .

1.3.3 Price impact

When a large order is filled, the price moves against the trader as orders in the book are filled. This is known as price impact. A widespread way for modeling this impact uses a general convex piecewise linear transaction costs, which models the depth of the order book at each price level. For instance, the following transaction costs can be used

$$C = \sum_{i=1}^d (\alpha_i S^i)^2 S^i.$$

2 Multi-period portfolio allocation

The aim of this section is to introduce the dynamic allocation problem. A general problem formulation is presented, followed by a detailed discussion of specific cases and commonly used techniques.

2.1 General formulation

Rather than simply buying and holding a fixed set of investments, dynamic portfolio allocation relies on investment strategies which adjust the portfolio composition over time in response to changes in economic conditions, market trends, or other relevant factors. The frequency of the adjustments depends on the specific strategy and the investment horizon. Some strategies may involve more frequent adjustments, such as monthly or quarterly, while others may be adjusted annually or even less frequently. The goal is to capture market opportunities and adjust the portfolio's risk exposure to achieve the desired risk and return objectives while respecting investors' preferences and realistic constraints.

We consider a time interval $[0, T]$ and for $N \in \mathbb{N}$, $N > 0$, we introduce the discrete-time grid $0 = t_0 < t_1 < \dots < t_N = T$. We define the filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_n)_{n \in [0, N]}, \mathbb{P})$. Let $(\Theta_n)_n$ be a discrete time process adapted to the filtration \mathbf{F} . For $k, n \in \mathbb{N}$, $k < n$, we write $\Theta_{k:n} = (\Theta_k, \Theta_{k+1}, \dots, \Theta_{n-1}, \Theta_n)$. Let Δ be the operator which associates to Θ_n , $\Delta\Theta_n = \Theta_n - \Theta_{n-1}$. Let Diag be the operator which

associates to the vector $(\Theta_n^1, \dots, \Theta_n^i)$, the matrix
$$\begin{bmatrix} \Theta_n^1 & & \\ & \ddots & \\ & & \Theta_n^i \end{bmatrix}.$$

We consider a portfolio V of initial wealth V_0 , composed of d risky assets $S = (S^1, \dots, S^d)$ and a risk free asset S^0 . An investor, with utility U , can re-allocate its portfolio V at discrete times (t_1, \dots, t_N) . We define $(\alpha_n, \alpha_n^0)_{n \in \{1, N\}}$, the respective risky and risk free asset quantities. $(\alpha_n, \alpha_n^0)_{n \in \{1, N\}}$ are assumed to be \mathbf{F} -predictable. We associate $(x_n, x_n^0)_{n \in \{1, N\}}$, the proportion of wealth invested respectively to the risky and risk free assets in the portfolio. At time 0, we assume that the owner has all its wealth in cash, ie $\alpha_0^0 S_0^0 = V_0$ and $\forall i \geq 1$, $\alpha_0^i = 0$. We denote $R = (R^1, \dots, R^d)$ as the vector of risky asset returns and R^0 the risk free asset return such that $\forall i \in \{0, \dots, d\}$, $n \in \{1, \dots, N\}$, $R_n^i = \frac{S_n^i}{S_{n-1}^i}$. In this section, we assume that $\{R_n, n \leq N\}$ are independent random (vector) variables, with known distributions

$$\mu_n = \mathbb{E}[R_n] \text{ and } \Sigma_n = \mathbb{E}[(R_n - \mu_n)(R_n - \mu_n)^T] = [\rho_n^{ij} \hat{\sigma}_n^i \hat{\sigma}_n^j]_{1 \leq i, j \leq d}.$$

We denote as $\bar{R} = \{R^0, \dots, R^d\}$ and $\bar{S} = \{S^0, \dots, S^d\}$, the completed vectors R and S with the risk free asset with associated parameters $(\bar{\mu}_n)_n$ and $(\bar{\Sigma}_n)_n$. Let's consider $\{\mathcal{L}_n, 1 \leq n \leq N\}$, a set of investment constraints. These constraints may take the form

of those presented in Section 1.2. We denote \mathcal{C}_n , the set of \mathcal{F}_{n-1} -adapted random variables which verifies the constraints \mathcal{L}_n .

An investor is seeking the best investment strategy $(\alpha_n, \alpha_n^0)_{n \in \{1, N\}}$, such that the expected value of its utility wealth, $\mathbb{E}[U(V_N)]$, is maximal. Let \mathcal{T} , be a deterministic function describing the evolution of the portfolio value such that

$$V_{n+1} = \mathcal{T}(V_n, V_{0:n-1}, \alpha_{n+1}, \alpha_{0:n}, S_{n+1}, S_{0:n}). \quad (4.7)$$

The dynamic allocation problem can be formulated as

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_N} \mathbb{E}[U(V_N)] \\ \text{s.t.} \quad & V_0 = V_0, (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, V_{n+1} = \mathcal{T}(V_n, V_{0:n-1}, \alpha_{n+1}, \alpha_{0:n}, S_{n+1}, S_{0:n}), \\ & \alpha_{n+1} \in \mathcal{C}_n \end{aligned} \quad (E_U)$$

2.2 General dynamic programming equation

Dynamic programming is a powerful mathematical technique used in economics and finance to optimize decision-making in situations where decisions made today will have an impact on future outcomes. Dynamic programming comes into play by breaking down the multi-period problem into a series of smaller, simpler sub-problems. The key idea is to solve the problem backwards, starting with the final period and working backwards to the first period. This approach, known as backward induction, allows in our case, the investor to determine the optimal portfolio allocation for each period, given their expectations about future returns and risks, and then use that information to make decisions in the preceding periods. One important aspect of dynamic programming is the concept of the state variable. In multi-period portfolio allocation, the state variable represents the information that investors have at each date, such as the current portfolio composition and the realized returns of the assets. The state variable is used to define the sub-problems that must be solved at each period, and it enables investors to take into account the impact of their decisions on future outcomes. Another key element of dynamic programming is value functions. Value functions represent the expected maximum objective value that can be achieved starting from a given state variable. By solving the backward sub-problems, investors can derive a value function for each period and use it to determine the optimal portfolio allocation for that period.

We can rewrite the problem (E_U) with backward equations and value functions. We recall that (R_{n+1}) is independent of \mathcal{F}_n . The state variable at time $n \in \{0, \dots, N\}$ is composed of the history of wealth, positions and assets returns: $\{V_k, \alpha_k, S_k, k \leq n\}$. The problem

can be expressed with value functions $(b_n)_{1 \leq n \leq N}$ such that $\forall n \in \{1, \dots, N\}$,

$$b_n(V_n, V_{0:n-1}, \alpha', \alpha'_{0:n-1}, S_n, S_{0:n-1}) = \begin{cases} U(V_N), & \text{if } n = N, \\ \sup_{\alpha_{n+1} = \alpha \in \mathcal{C}_n} \mathbb{E} [b_{n+1}(\mathcal{T}(V_n, V_{0:n-1}, \alpha, \alpha'_{0:n}, S_n, S_{0:n}), \alpha, \alpha'_{0:n}, \text{Diag}(\bar{R}_{n+1})S_n, S_{0:n})], & \end{cases}$$

The above formulation does not necessarily imply that the problem can be solved efficiently in a backward fashion. The transition function may be complex, and the large number of state variables can create issues of complexity. Estimating the value functions for all state variables can become computationally infeasible, resulting in an unmanageable number of maximizations to be performed.

2.3 Finding optimal solutions

Finding optimal solutions to (E_U) either by explicit derivation or numerical computation is only feasible in a few limited cases. The goal of this section is to present some of these cases.

2.3.1 Quadratic case

The quadratic case is a special case where (E_U) can be solved explicitly. We assume that the utility function U is concave quadratic in V_N and the position are submitted to linear constraints. We assume that the portfolio is self financing such that

$$\forall 0 \leq n \leq N - 1, V_n = \alpha_{n+1} \cdot S_n + \alpha_{n+1}^0 S_n^0.$$

For $n \leq N - 1$, we denote as $u_{n+1} = ((\alpha_{n+1}^i - \alpha_n^i)S_n^i)_{i \in \{0, \dots, d\}}$ and $v_n = (\alpha_n^i S_n^i)_{i \in \{0, \dots, d\}}$, the respectively amount of money invested and the amount of money hold by the investor per asset at time n . With these notations, we have $\forall n \leq N - 1, v_{n+1} = \text{Diag}(\bar{R}_{n+1})(u_{n+1} + v_n)$ and $\forall n \leq N, \mathbf{1} \cdot v_n = V_n$. For $n \leq N - 1$, let's call the space $\hat{\mathcal{C}}_n$, such that $u_{n+1} \in \hat{\mathcal{C}}_n \iff \alpha_{n+1} \in \mathcal{C}_n$. The dynamic equations of the problem writes

$$b_n(v_n) = \begin{cases} U(\mathbf{1} \cdot v_N), & \text{if } n = N \\ \sup_{u_{n+1} = u \in \hat{\mathcal{C}}_n} \mathbb{E}_P [b_{n+1}(\text{Diag}(\bar{R}_{n+1})(u + v_n))] \end{cases}$$

By induction, we show that b_n is concave quadratic. The function b_N is concave quadratic in v_N by assumption. We assume that b_{n+1} is concave quadratic. The function $u \rightarrow \text{Diag}(\bar{R}_{n+1})(u + v_n)$ is monotone increasing so $b_{n+1}(\text{Diag}(\bar{R}_{n+1})(u + v_n))$ is concave in u . Expectation and partial maximisation on u under linear constraints, conserve concavity and quadratic form so b_n is concave quadratic. Solving the dynamic problem is equivalent to maximize for $n \in \{0, \dots, N - 1\}$, over $u \in \hat{\mathcal{C}}_n, \mathbb{E}_P [b_{n+1}(\text{Diag}(\bar{R}_{n+1})(u + v_n))]$, which

is a maximisation of a concave quadratic function under linear constraints. According to [Boyd et al. \(2013\)](#), $\exists J_n \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, $j_n \in \mathbb{R}^{d+1}$ such that $u_{n+1}^*(v_n) = J_n v_n + j_n$. We can then deduce the optimal solution

$$\forall n \leq N - 1, \alpha_{n+1}^* = \alpha_n^* + \text{Diag}(\bar{S}_n)^{-1} u_{n+1}^*.$$

Although we have an explicit solution, adopting a quadratic form for the utility is overly limiting.

2.3.2 CRRA utility without cost without constraint

We consider here the case where the utility function is CRRA and the portfolio allocation is unconstrained and not subject to transaction costs. We refer to [Brandt et al. \(2005\)](#) for further details on this approach. The portfolio is self-financing, then $\forall n \leq N - 1$, $V_n = \alpha_{n+1}^T S_n + \alpha_{n+1}^0 S_n^0$. The transition function writes

$$V_{n+1} = \mathcal{T}(V_n, V_{0:n-1}, \alpha_{n+1}, \alpha_{0:n}, S_{n+1}, S_{0:n}) = V_n + \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0. \quad (4.8)$$

We can rewrite this equation in terms of rate of return.

$$\begin{aligned} V_n \frac{V_{n+1}}{V_n} &= V_n \left(1 + \frac{\alpha_{n+1} \cdot S_{n+1}}{V_n} - \frac{\alpha_{n+1} \cdot S_n}{V_n} + \frac{\alpha_{n+1}^0 S_{n+1}^0}{V_n} - \frac{\alpha_{n+1}^0 S_n^0}{V_n} \right) \\ &= V_n (x_{n+1} \cdot R_{n+1} + (1 - x_{n+1}) R_{n+1}^0). \end{aligned}$$

By an immediate recurrence, we obtain

$$V_N = V_0 \prod_{i=0}^{N-1} (x_{i+1} \cdot R_{i+1} + (1 - x_{i+1}) R_{i+1}^0). \quad (4.9)$$

With this form and using the independence of the rates of return we can compute the value function under the following form. Please note that the inversion of the sup and expectation operations performed in the subsequent derivation is highly technical to prove and has little relevance to our main point. Hence, for the sake of clearness, we present here

the formal derivation of the dynamic principle. $\forall n \leq N - 1$,

$$\begin{aligned}
b_n(V_n) &= \sup_{x_N, \dots, x_{n+1}} \mathbb{E} \left[\frac{V_N^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_n \right] = \sup_{x_{n+1}} \mathbb{E} \left[\sup_{x_N, \dots, x_{n+2}} \mathbb{E} \left[\frac{V_N^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_{n+1} \right] \middle| \mathcal{F}_n \right] \\
&= \sup_{x_{n+1}} \mathbb{E} \left[\sup_{x_N, \dots, x_{n+2}} \mathbb{E} \left[\frac{\left(V_n \prod_{i=n}^{N-1} (x_{i+1} \cdot R_{i+1} + (1 - x_{i+1}) R_{i+1}^0) \right)^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_{n+1} \right] \middle| \mathcal{F}_n \right] \\
&= \sup_{x_{n+1}} \mathbb{E}_n \left[\frac{\left(V_n (x_{n+1} \cdot R_{n+1} + (1 - x_{n+1}) R_{n+1}^0) \right)^{1-\gamma}}{1-\gamma} \right] \\
&\times \mathbb{E}_n \left[\sup_{x_N, \dots, x_{n+2}} \left(\prod_{i=n+1}^{N-1} (x_{i+1}^T R_{i+1} + (1 - x_{i+1}) R_{i+1}^0) \right)^{1-\gamma} \right]
\end{aligned}$$

The second expectation of the product being independent with x_{n+1} , we deduce that finding the optimal multi-period strategy is equivalent to sequentially solve an uni-period allocation problem. According to the dynamic of the rate of return R , it may be possible to compute the optimal strategy explicitly or to use convex optimization algorithms to numerically determine it.

2.3.3 Approximate Dynamic Programming

Using exact dynamic programming algorithm to solve (E_U) is often computationally infeasible when the transition function is complex or in presence of constraints and transaction costs, as it requires estimating value functions and controls for each state variable. Approximated dynamic programming (ADP) is an approach category that aims to streamline the process. Formally, ADP techniques suggest to solve the problem by replacing the value functions $(b_n)_n$ or controls $(\alpha_n)_n$ with surrogate or approximate functions. The policy should be easily evaluated and attain near optimal performance. For instance, the approach of ADP involves approximating the value or policy functions through the use of a parametric function, such as a neural network or decision tree. A variety of methods can be used to choose approximate value functions (see e.g., Bertsekas and Tsitsiklis (1996), Boyd et al. (2013), De Farias and Van Roy (2003), Lincoln and Rantzer (2006), Powell (2007)). A straightforward approach involves using the value function of the problem, which relaxes constraints or ignores transaction costs. It is also common to use value functions issued from sub-optimal strategies to run approximate dynamic programming. As described in Section 2.3.1, a problem with quadratic value functions can be explicitly solved. That explicit resolution motivates Boyd et al. (2013) to seek for a set of concave quadratic value functions which closely underestimate the real value functions.

We decide here to focus on two examples of Approximate Dynamic Programming (ADP) algorithms for solving multi-period allocation problems subject to transaction costs. The

general form of these stochastic control algorithms is presented and described in [Bachouch et al. \(2021\)](#) and [Huré et al. \(2021\)](#). In the first approach, control variables are represented with neural networks. In the second approach, both control variables and value functions are represented using neural networks.

We assume that the portfolio is self financing and submitted to transaction costs, then

$$\begin{aligned} V_{n+1} &= \mathcal{T}(V_n, V_{0:n-1}, \alpha_{n+1}, \alpha_{0:n}, S_{n+1}, S_{0:n}) \\ &= V_n + \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i. \end{aligned}$$

In this framework, it is easier to work with the discounted processes.

$$\tilde{V}_{n+1} = \tilde{\mathcal{T}}(\tilde{V}_n, \tilde{V}_{0:n-1}, \alpha_{n+1}, \alpha_{0:n}, \tilde{S}_{n+1}, \tilde{S}_{0:n}) = \tilde{V}_n + \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| \tilde{S}_n^i.$$

We identify $\{\tilde{V}_n, \tilde{S}_n, \alpha_n\}$, the state variable at time n . The control is α_{n+1} . The following proposed algorithm needs to explore the state variables at each iteration. Let M be the number of trajectories for the exploration. We denote the set of trajectories for the exploration as $\{\tilde{V}_n^{(m)}, \tilde{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, m \leq M, n \leq N\}$.

2.3.3.1 A stochastic control algorithm to solve portfolio allocation submitted to transactions costs

In this framework, we choose to represent the controls with neural networks such that

$$\alpha_{n+1}^* (\tilde{V}_n, \tilde{S}_n, \alpha_n) : (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \theta_n^*). \quad (4.10)$$

Where θ_n^* is the neural network vector parameters. We solve the dynamic problem in a backward fashion.

Algorithm 1

Input: $\{\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, m \leq M, n \leq N\}$

Output: $\{\alpha_n^*, n \leq N\}$

1: $n \leftarrow N - 1$

2: **while** $n \geq 0$ **do**

3: **find** θ_n^* solution of $\sup_{\theta_n} \frac{1}{M} \sum_{m=1}^M U(\hat{V}_N^{(m)} S_N^0)$

4: **with** $k=n$

5: $\hat{V}_{k+1}^{(m)} = \bar{V}_k^{(m)} + \Delta \bar{S}_{k+1}^{(m)} \cdot \Phi(\bar{V}_k^{(m)}, \bar{S}_k^{(m)}, \bar{\alpha}_k^{(m)}, \theta_k) - \sum_{i=1}^d \nu |\Phi_i(\bar{V}_k^{(m)}, \bar{S}_k^{(m)}, \bar{\alpha}_k^{(m)}, \theta_k) - \bar{\alpha}_k^{i,(m)}| \bar{S}_k^{i,(m)}$

6: **with** $k \geq n + 1$

7: $\hat{V}_{k+1}^{(m)} = \hat{V}_k^{(m)} + \Delta \bar{S}_{k+1}^{(m)} \cdot (\alpha_{k+1}^*(\hat{V}_k^{(m)}, \bar{S}_k^{(m)}, \alpha_k^{*(m)})) - \sum_{i=1}^d \nu |\alpha_{k+1}^{i,*(m)} - \alpha_k^{i,*(m)}| \bar{S}_k^{i,(m)}$

8: **set** $\alpha_{n+1}^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) = (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \theta_n^*)$.

9: $n \leftarrow n - 1$

10: **end while**

2.3.3.2 Another stochastic control algorithm to solve portfolio allocation submitted to transactions costs

In this version, we do not project all the trajectories until the final date. We rather approximate value functions with neural networks.

$$\alpha_{n+1}^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) : (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \theta_n^*). \quad (4.11)$$

$$b_n^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) : (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Psi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \epsilon_n^*). \quad (4.12)$$

Algorithm 2

Input: $\{\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, m \leq M, n \leq N\}$ of $\{\tilde{V}_n, \tilde{S}_n, \alpha_n\}$
Output: $\{\alpha_n^*, n \leq N\}$

- 1: **set** $b_N^*(\tilde{V}_N) = U(\tilde{V}_N S_N^0)$
- 2: $n \leftarrow N - 1$
- 3: **while** $n \geq 0$ **do**
- 4: **find** θ_n^* solution to
- 5: $\sup_{\theta_n} \frac{1}{M} \sum_{m=1}^M b_{n+1}^*(\hat{V}_{n+1}^{(m)}, \bar{S}_{n+1}^{(m)}, \alpha_{n+1}^{(m)}) = \sum_{m=1}^M \Psi(\hat{V}_{n+1}^{(m)}, \bar{S}_{n+1}^{(m)}, \Phi(\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \theta_n), \epsilon_{n+1}^*)$
- 6: **with**
- 7: $\hat{V}_{n+1}^{(m)} = \bar{V}_n^{(m)} + \Delta \bar{S}_{n+1}^{(m)} \cdot \Phi(\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \theta_n) - \sum_{i=1}^d \nu_i |\Phi_i(\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \theta_n) - \bar{\alpha}_n^{i,(m)}| \bar{S}_n^{i,(m)}$
- 8: **set** $\alpha_{n+1}^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) = (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \theta_n^*)$.
- 9: **find** ϵ_n^* solution to $\sup_{\epsilon_n} \frac{1}{M} \sum_{m=1}^M (b_{n+1}^*(\hat{V}_{n+1}^{(m)}, \bar{S}_{n+1}^{(m)}, \alpha_{n+1}^{*(m)}) - \Psi(\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \epsilon_n^*))^2$
- 10: **set** $b_n^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) = (\tilde{V}_n, \tilde{S}_n, \alpha_n) \rightarrow \Psi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \epsilon_n^*)$.
- 11: **end while**

Both algorithms are designed to approximate an optimal strategy, but they can still be computationally demanding. Additionally, these algorithms require an exploration of the state space, which can be a challenging task. Indeed, the exploration phase should smartly sample the state space or should have an idea of the form of the solution to avoid increasing complexity or introducing a bias. In the next section, we described, another alternative category of approach which seeks sub-optimal solutions.

2.4 Finding sub-optimal solutions

In multi-period portfolio allocation, sub-optimal strategies refer to approaches that do not fully optimize portfolio allocation over the entire investment horizon. Instead, they rely on simpler, more straightforward methods that may not account for all the relevant information or market factors. While sub-optimal strategies may be less sophisticated than optimal ones, they can still be effective in certain situations. Indeed sub-optimal strategies may be favored over optimal strategies due to the complexity of the latter. For instance, methods like stochastic control (as discussed in Section 2.3.3) can be time-consuming and expensive to implement. Sub-optimal strategies may also be useful for investors who do not have access to advanced analytical tools or who are not comfortable with more complex investment strategies. We briefly present here, two popular sub-optimal strategies.

2.4.1 Use of scenario trees

Scenario trees are a useful tool for finding sub-optimal solutions in multi-period portfolio allocation. They involve creating a tree-like structure that represents the different possible scenarios and outcomes that could occur over the investment horizon. To construct a scenario tree, key factors that are likely to influence the investment outcomes, such as economic indicators, market trends, or geopolitical events, are identified. Then, a set of scenarios that capture the different ways these factors could play out over time, are defined. For example, a scenario tree might include scenarios for a recession, a market boom, or a geopolitical crisis. Next, probabilities are assigned to each scenario based on their assessment of the likelihood of each outcome. These probabilities are used to weight the potential returns and risks associated to each scenario. Finally, the scenario tree are used to simulate different portfolio allocation strategies over the investment horizon. By considering the potential outcomes for each scenario, investors can identify sub-optimal strategies that are less complex and easier to implement than optimal solutions. Studies using this approach can be found in [Gennotte and Jung \(1994\)](#), [Steinbach \(2001\)](#), and [Wang and Liu \(2013\)](#). Building scenarios trees may also represent an useful tool for the exploration phase of stochastic control approaches described in [Section 2.3.3](#).

2.4.2 Model predictive control

Model predictive control (MPC) is a method which transforms the stochastic problem to be deterministic. Model predictive control can easily handle constraints on portfolio allocation, such as transaction costs or holding limits, and can provide a flexible and computationally efficient solution to the multi-period allocation problem. To determine the policy at time n , all future random values $\{S_n, n = k, \dots, N\}$ are replaced by their expected values $\{\mathbb{E}[V_n | \mathcal{F}_k], n = k, \dots, N\}$. The stochastic control problem turns into a standard optimization problem. For further details on the approach, please refer to [Boyd et al. \(2013\)](#).

Algorithm 3

Input:

Output: $\{\alpha_n^*, n \leq N\}$

1: $n \leftarrow 0$

2: **while** $n \leq N - 1$ **do**

3: **find** $(\alpha_{n+1}, \dots, \alpha_N)$ solution of

$$\sup_{\alpha_{n+1}, \dots, \alpha_N} U(V_N)$$

$$\text{s.t.} \quad V_n = V_n^*, \quad \alpha_n = \alpha_n^*$$

$$\forall n \leq k \leq N - 1,$$

$$V_{k+1} = \mathcal{T}(V_k, (V_{0:n}^*, V_{n+1:k}), \alpha_{n+1}, (\alpha_{0:n}^*, \alpha_{n+1,k}), \mathbb{E}[S_{k+1} | \mathcal{F}_n], \mathbb{E}[S_{0:k} | \mathcal{F}_n]),$$

$$\alpha_{k+1} \in \mathcal{C}_k$$

4: **set** $\alpha_{n+1}^* = \alpha_{n+1}$

5: **set** $V_{n+1}^* = \mathcal{T}(V_n^*, V_{0:n-1}^*, \alpha_{n+1}^*, \alpha_{0:n}^*, S_{n+1}, S_{0:n})$

6: $n \leftarrow n + 1$

7: **end while**

Formally, the first trade in the planned sequence of trades is executed. At the next step, the process starting from the new portfolio V^* , is repeated. This category of approach is highly valued, despite its tendency to only yield sub-optimal solutions. The literature contains numerous sophisticated extensions of this approach. For example, [Li et al. \(2022\)](#) use a hidden Markov model to enhance predictions of undetermined values.

2.5 Multi-period mean-var portfolio allocation

This section is dedicated to describe the Markowitz mean-var multi-period allocation problem.

2.5.1 General formulation

The objective function of this problem aims to represent the trade-off between risk and reward. As a result, it is not formulated with a utility function as outlined for the single-period case in Section 1.1.2. We assume that transactions are submitted to transaction costs. At time $n \in \{1, \dots, N-1\}$, the portfolio is re-balanced from (α_n, α_n^0) to $(\alpha_{n+1}, \alpha_{n+1}^0)$, and pays the cost $\sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i$. The self-financing condition is

$$\tilde{V}_n = \alpha_n \tilde{S}_n + \alpha_n^0 = \alpha_{n+1} \cdot \tilde{S}_n + \alpha_{n+1}^0 + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}.$$

So we have

$$\Delta \tilde{V}_{n+1} = \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}. \quad (4.13)$$

In addition we assume that the controls are subject to a set of constraints \mathcal{C}_n . The system to solve is

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_N} \mathbb{E}[\tilde{V}_N S_N^0] - \gamma \text{Var}[\tilde{V}_N S_N^0] \\ \text{s.t.} \quad & V_0 = V_0, (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, \tilde{V}_{n+1} = \tilde{V}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}, \\ & \alpha_{n+1} \in \mathcal{C}_n \end{aligned} \quad (E(\gamma))$$

Note that the problem $(E(\gamma))$ is equivalent to the following problems

$$\begin{aligned} & \inf_{\alpha_1, \dots, \alpha_N} \text{Var}[V_N S_N^0] \\ \text{s.t.} \quad & V_0 = V_0, (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, \tilde{V}_{n+1} = \tilde{V}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}, \\ & \alpha_{n+1} \in \mathcal{C}_n \\ & \mathbb{E}[V_N] \geq \rho \end{aligned} \quad (E(\rho))$$

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_N} \mathbb{E}[V_N S_N^0] \\ \text{s.t.} \quad & V_0 = V_0, (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, \tilde{V}_{n+1} = \tilde{V}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}, \\ & \alpha_{n+1} \in \mathcal{C}_n \\ & \text{Var}[V_N] \leq \eta \end{aligned} \quad (E(\eta))$$

Problems $(E(\gamma))$, $(E(\rho))$ and $(E(\eta))$ cannot be directly solved with dynamic programming. The problem cannot be expressed with value functions in Section 2.2. Indeed, by denoting I_t an information set available at time t and $I_{t-1} \subset I_t$, we notice that while the expectation operator satisfies the smoothing property, $\mathbb{E}[\mathbb{E}[\cdot | I_j] | I_k] = \mathbb{E}[\cdot | I_j]$, $\forall j < k$, the variance operator does not, $\text{Var}[\text{Var}[\cdot | I_j] | I_k] \neq \text{Var}[\cdot | I_j]$. The next section describes an embedding technique that enables the resolution of this problem without cost or constraint.

2.5.2 Mean-variance multi-period portfolio selection without transaction cost or constraint

Here, we assume that there is no transaction cost or constraint on the portfolio composition. We detail the derivation presented in [Li and Ng \(2000\)](#), which provides an explicit formulation of the optimal policy. Let us introduce the following useful notations, $\forall n \in \{1, \dots, N\}$,

$$P_n = (R_n^1 - R_n^0, \dots, R_n^d - R_n^0), \quad A_n = \mathbb{E}^{-1}[P_n^T P_n], \quad C_n = \mathbb{E}[P_n],$$

$$B_n = C_n^T A_n C_n, \quad \mathcal{S}_n = \text{Diag}(S_n).$$

The unconstrained and cost-free version of $(E(\gamma))$ writes

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_N} \quad \mathbb{E}[V_N] - \gamma \text{Var}[V_N] \\ \text{s.t.} \quad & V_0 = V_0, \quad (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, \quad \Delta V_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0. \end{aligned} \quad (E_0(\gamma))$$

Even without constraint and transaction cost, Problem $(E_0(\gamma))$ is still not separable in the sense of dynamic programming and can not be solved directly but can be embedded into a tractable auxiliary problem, which is separable. An analytical optimal portfolio policy is derived for the multiperiod mean-variance formulation along with the analytical expression of the mean-variance efficient frontier. Let us consider the auxiliary problem for $\lambda \in \mathbb{R}$,

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_N} \quad \mathbb{E}[\lambda V_N - \gamma V_N^2] \\ \text{s.t.} \quad & V_0 = V_0, \quad (\alpha_n^0, \alpha_n)_n \mathbf{F} - \text{Pred} \\ & \forall 1 \leq n \leq N-1, \quad V_{n+1} = V_n + \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0. \end{aligned} \quad (A(\gamma, \lambda))$$

We define $\Pi_E(\gamma) = \{\alpha \mid \alpha \text{ solution to } (E_0(\gamma))\}$, $\Pi_A(\gamma, \lambda) = \{\alpha \mid \alpha \text{ solution to } (A(\gamma, \lambda))\}$.

Proposition 23. $\Pi_E(\gamma) \subseteq \bigcup_{\lambda} \Pi_A(\gamma, \lambda)$.

Proof. We define $\hat{U}(\mathbb{E}[V_N^2], \mathbb{E}[V_N]) = \mathbb{E}[V_N] - \gamma \text{Var}[V_N] = -\gamma \mathbb{E}[V_N^2] + (\gamma \mathbb{E}^2[V_N] + \mathbb{E}[V_N])$. U is obviously convex and

$$d(\alpha, \gamma) = \frac{\partial \hat{U}(\mathbb{E}[V_N^2], \mathbb{E}[V_N])}{\partial \mathbb{E}[V_N]} = 1 + 2\gamma \mathbb{E}_{\alpha}[V_N], \quad (4.14)$$

where $\mathbb{E}_{\alpha}[V_N]$ is the expected final value of the portfolio built with the strategy α . We prove that for $\alpha^* \in \Pi_E(\gamma)$, then $\alpha^* \in \Pi_A(\gamma, d(\alpha^*, \gamma))$.

We assume that $\alpha^* \notin \Pi_A(\gamma, d(\alpha^*, \gamma))$. Then $\exists \alpha'$ such that

$$-\gamma \mathbb{E}_{\alpha'}[V_N^2] + d(\alpha^*, \gamma) \mathbb{E}_{\alpha'}[V_N] > -\gamma \mathbb{E}_{\alpha^*}[V_N^2] + d(\alpha^*, \gamma) \mathbb{E}_{\alpha^*}[V_N].$$

\hat{U} is convex so we also have

$$\hat{U}(\mathbb{E}_{\alpha'}[V_N^2], \mathbb{E}_{\alpha'}[V_N]) \geq \hat{U}(\mathbb{E}_{\alpha^*}[V_N^2], \mathbb{E}_{\alpha^*}[V_N]) + [-\gamma, d(\alpha^*, \gamma)] \left(\begin{bmatrix} \mathbb{E}_{\alpha'}[V_N^2] \\ \mathbb{E}_{\alpha'}[V_N] \end{bmatrix} - \begin{bmatrix} \mathbb{E}_{\alpha^*}[V_N^2] \\ \mathbb{E}_{\alpha^*}[V_N] \end{bmatrix} \right).$$

Then $\hat{U}(\mathbb{E}_{\alpha'}[V_N^2], \mathbb{E}_{\alpha'}[V_N]) > \hat{U}(\mathbb{E}_{\alpha^*}[V_N^2], \mathbb{E}_{\alpha^*}[V_N])$, that contradicts the assumption. \blacksquare

According to Proposition 23, all solutions of $(E_0(\gamma))$ are in $\bigcup_{\lambda} \Pi_A(\gamma, \lambda)$, whose elements can be parameterized by (γ, λ) . The portfolio values can also be parameterized with (γ, λ) . Then $(E_0(\gamma))$ can be reduced to the following equivalent form

$$\begin{aligned} & \sup_{\lambda \in \mathbb{R}} \hat{U}(\mathbb{E}[V_N^2(\gamma, \lambda)], \mathbb{E}[V_N(\gamma, \lambda)]) \\ & \text{s.t. } \forall 1 \leq n \leq N-1, \Delta V_{n+1}(\gamma, \lambda) = \alpha_{n+1} \cdot (\gamma, \lambda) \Delta S_{n+1} + \alpha_{n+1}^0(\gamma, \lambda) \Delta S_{n+1}^0 \\ & \quad \alpha(\gamma, \lambda) \in \Pi_A(\gamma, \lambda) \end{aligned}$$

$(A(\lambda, \gamma))$ is a linear quadratic problem. According to Section 2.3.1 and Li et al. (1998), the optimal solution can be derived analytically using dynamic programming. The optimal policy is given by

$$\alpha_{n+1}(\alpha_n, S_{0|n}, \gamma, \lambda) = \mathcal{S}_n^{-1} [-K_{n+1}(\alpha_n^T S_n + \alpha_n^0 S_n^0) + f_{n+1}(\gamma, \lambda)]. \quad (4.15)$$

where

$$K_n = A_n C_n R_n^0, \quad f_n(\gamma, \lambda) = \frac{\lambda}{2\gamma} \left(\prod_{i=n+1}^N \frac{R_i^0(1-B_i)}{(R_i^0)^2(1-B_i)} A_n C_n \right), \quad f_N(\gamma, \lambda) = \frac{\lambda}{2\gamma} A_N C_N.$$

Then the optimal portfolio value can be derived as

$$\begin{aligned} V_{n+1}(\alpha, \gamma) &= \alpha_{n+1}^T S_{n+1} + \alpha_{n+1}^0 S_{n+1}^0 \\ &= \alpha_{n+1}^T \mathcal{S}_n R_{n+1} + (V_n - \alpha_{n+1}^T S_n) R_{n+1}^0 \\ &= (\mathcal{S}_n \alpha_{n+1}) P_{n+1}^T + V_n R_{n+1}^0 \\ &= (-K_{n+1}(\alpha_n^T S_n + \alpha_n^0 S_n^0) + f_{n+1}(\gamma, \lambda)) P_{n+1}^T + V_n R_{n+1}^0. \end{aligned}$$

Since R_{n+1} is independent from α_n and S_n , we can compute

$$\mathbb{E}[V_{n+1}(\gamma, \lambda)] = \mathbb{E}[V_n(\gamma, \lambda)](-K_{n+1} C_{n+1}^T + R_{n+1}^0) + f_{n+1}(\gamma, \lambda) C_{n+1}^T.$$

We finally obtain

$$\mathbb{E}[V_{n+1}(\gamma, \lambda)] = R_{n+1}^0(1-B_{n+1})\mathbb{E}[V_n(\gamma, \lambda)] + \frac{\lambda}{2\gamma} \left(\prod_{i=n+2}^N \frac{1}{R_i^0} \right) B_{n+1}.$$

With the same argument we can compute

$$\mathbb{E}[V_{n+1}^2(\gamma, \lambda)] = (R_{n+1}^0)^2(1 - B_{n+1})\mathbb{E}[V_n^2(\gamma, \lambda)] + \frac{\lambda^2}{4\gamma^2} \left(\prod_{i=n+2}^N \left(\frac{1}{R_i^0} \right)^2 \right) B_{n+1}.$$

By induction, we can prove that

$$\begin{aligned} \mathbb{E}[V_{n+1}(\gamma, \lambda)] &= \prod_{k=1}^{n+1} R_k^0(1 - B_k)V_0 + \frac{\lambda}{2\gamma} \prod_{i=n+2}^N \frac{1}{R_i^0} \left(1 - \prod_{k=1}^{n+1} (1 - B_k) \right) \\ &= aV_0 + \frac{\lambda}{2\gamma}b. \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[V_{n+1}^2(\gamma, \lambda)] &= \prod_{k=1}^{n+1} (R_k^0)^2(1 - B_k)V_0^2 + \frac{\lambda^2}{4\gamma^2} \prod_{i=n+2}^N \frac{1}{(R_i^0)^2} \left(1 - \prod_{k=1}^{n+1} (1 - B_k) \right) \\ &= cV_0^2 + \frac{\lambda^2 b}{\gamma^2 2}. \end{aligned}$$

Then we can compute

$$\tilde{U}(\mathbb{E}[V_N^2(\gamma, \lambda)], \mathbb{E}[V_N(\gamma, \lambda)]) = aV_0 + \frac{\lambda}{2\gamma}b - \gamma cV_0^2 + \frac{\lambda^2}{4\gamma^2}b - a^2V_0^2 - \frac{\lambda^2}{4\gamma^2}b^2 - \frac{2ab\lambda}{2\gamma}. \quad (4.16)$$

\hat{U} being a concave function of λ , the optimal λ^* must satisfy the optimally condition $\frac{\partial \hat{U}}{\partial \lambda}(\lambda^*, \gamma) = 0$, then

$$\lambda^* = \frac{2\gamma a}{1 - b} + \frac{1}{1 - b} = 2\gamma \prod_{k=1}^N R_k^0 V_0 + \frac{1}{\prod_{k=1}^N (1 - B_k)}.$$

Finally substituting the optimal λ^* into (4.15) provides the optimal solution

$$\begin{aligned} \alpha_{n+1}^*(\alpha_n^*, S_{0|n}, \gamma) &= \mathcal{S}_n^{-1} [-K_{n+1}(\alpha_n^{*T} S_n + \alpha_n^{0*} S_n^0)] \\ &+ \mathcal{S}_n^{-1} \left[\left(\prod_{k=1}^N R_k^0 V_0 + \frac{1}{2\gamma \prod_{k=1}^N (1 - B_k)} \right) \left(\prod_{i=n+2}^N \frac{1}{(R_i^0)} A_{n+1} C_{n+1} \right) \right]. \end{aligned}$$

It is noteworthy that this approach can not provide explicit solution if transactions are submitted to costs due to its non linear form. Numerical approaches are preferred to find optimal portfolios submitted to transaction costs.

2.5.3 A stochastic control algorithm to solve mean-variance multi-period portfolio allocation with transaction costs and constraints

In this section, we show how to adapt the approach described in 2.3.3.1, to solve $(E(\gamma))$.

Problem $(E(\gamma))$ is not separable in the sense of dynamic programming. The algorithm presented in Section 2.3.3.1 cannot be directly applied. We draw inspiration from the embedding techniques in Li and Ng (2000) to propose a framework capable of numerically finding the optimal strategy. More precisely, we adapt the framework presented in section 2.3.3.1, to solve an auxiliary problem. Formally, we apply the algorithm to the utility function $U(V_N) = \lambda V_N - \gamma V_N^2$. The state variables vector is completed with a new parameter λ .

$$\alpha_{n+1}^*(V_n, \tilde{S}_n, \alpha_n, \lambda) : (\tilde{V}_n, \tilde{S}_n, \alpha_n, \lambda) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \lambda, \theta_n^*). \quad (4.17)$$

Where θ_n^* is the neural network vector parameters. We solve the dynamic problem in a backward fashion.

Algorithm 4

Input: $\{\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \bar{\lambda}^{(m)}, m \leq M, n \leq N\}$.

Output: $\{\alpha_n^*, n \leq N\}$

- 1: $n \leftarrow N - 1$
- 2: **while** $n \geq 0$ **do**
- 3: **find** θ_n^* solution of θ_n^* solution to

$$\begin{aligned} & \sup_{\theta_n} \quad \frac{1}{M} \sum_{m=1}^M U(\hat{V}_N^{(m)} S_N^0) \\ & \text{subject to} \quad \Phi(\bar{V}_n^{(m)}, \bar{S}_n^{(m)}, \bar{\alpha}_n^{(m)}, \bar{\lambda}^{(m)}, \theta_n) \in \mathcal{C}_n \end{aligned}$$

- 4: **with** $k=n$

- 5: $\hat{V}_{k+1}^{(m)} = \bar{V}_k^{(m)} + \Delta \bar{S}_{k+1}^{(m)} \cdot \Phi(\bar{V}_k^{(m)}, \bar{S}_k^{(m)}, \bar{\alpha}_k^{(m)}, \bar{\lambda}^{(m)}, \theta_k)$

- 6: $-\sum_{i=1}^d \nu |\Phi_i(\bar{V}_k^{(m)}, \bar{S}_k^{(m)}, \bar{\alpha}_k^{(m)}, \bar{\lambda}^{(m)}, \theta_k) - \bar{\alpha}_k^{i,(m)}| \bar{S}_k^i$

- 7: **with** $k \geq n + 1$

- 8: $\hat{V}_{k+1}^{(m)} = \bar{V}_k^{(m)} + \Delta \bar{S}_{k+1}^{(m)} \cdot (\alpha_{k+1}^*(\hat{V}_k^{(m)}, \tilde{S}_k^{(m)}, \alpha_k^{*,(m)}, \bar{\lambda}^{(m)})) - \sum_{i=1}^d \nu |\alpha_{k+1}^{i,*,(m)} - \alpha_k^{i,*,(m)}| \bar{S}_k^i$

- 9: **find** λ^* solution of $\sup_{\lambda \in \mathbb{R}} \frac{1}{M} \sum_{m=1}^M U(\hat{V}_N^{(m)} S_N^0)$.

- 10: **set** $\alpha_{n+1}^*(\tilde{V}_n, \tilde{S}_n, \alpha_n) = (\tilde{V}_n, \tilde{S}_n, \alpha_n, \lambda) \rightarrow \Phi(\tilde{V}_n, \tilde{S}_n, \alpha_n, \lambda^*, \theta_n^*)$.

- 11: $n \leftarrow n - 1$

- 12: **end while**

This method is theoretically able to find the optimal portfolio but remains computationally heavy. Indeed we need a large set of exploring variables and must proceed to a high number of maximizations in order to set the neural networks. The next section presents a more

straightforward alternative approach to solving dynamic mean-variance portfolio allocation with transaction costs.

Chapter 5

Wiener chaos expansion for multi-period portfolio allocation

Abstract

This chapter studies the multi-period mean-variance portfolio allocation problem. Many methods have been proposed these last years to challenge the famous uni-period Markowitz strategy. But these methods cannot integrate transaction costs or become computationally heavy and hardly applicable. In this study, we try to tackle this allocation problem by proposing a new approach which relies on representing the set of admissible portfolios by a finite dimensional Wiener chaos expansion. This numerical method is able to find an optimal strategy for the allocation problem subject to transaction costs. To complete the study, the link between optimal portfolios submitted to transaction costs and the underlying risk aversion is investigated. Then a competitive and compliant benchmark based on the sequential uni-period Markowitz strategy is built to highlight the efficiency of our approach.

Note that this Chapter 5 that is based on the article [Cousin et al. \(2023b\)](#), submitted to a peer-reviewed journal.

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1 Introduction

Dynamic portfolio selection is one of the most studied topic in financial economics. The problem consists in allocating the wealth of an investor, among a basket of assets, over time. Finding the optimal portfolio is a difficult challenge since it depends on the objective of the investor. The Markowitz's mean-variance formulation represents a first answer to this problem by providing fundamental basics for static portfolio allocation in a uni-period case (see [Markowitz \(1952\)](#)). Mean-variance framework offer to build a portfolio of assets such that the expected return is maximized for a given level of risk. This portfolio theory is based on the assumption that the parameters of the underlying stochastic model are known and contain no estimation error. This method is easy to apply and has the favor of asset managers. Nevertheless, when the time horizon increases, this myopic strategy which cannot see ahead of the next time period, cannot challenge the dynamic optimal portfolio obtained from the multi-period version of the problem. [Merton \(1969\)](#) is one of the first paper studying multi-period portfolio investment in a dynamic programming framework. In this seminal paper, the authors consider a problem with one risky asset and one risk-free asset. At each date, the investor can re-balance its wealth between the two assets, seeking to maximize an utility of the final time horizon wealth. They derive a simple closed-form expression for the optimal policy when there are no constraint or transaction cost. In a companion paper, [Samuelson \(1975\)](#) derive the discrete-time analog approach. The results presented in those studies and the innovative and promising aspect of the multi-period portfolio selection have stimulated the interest of the related scientific community. In the following years, the literature in multi-period portfolio selection has considerably grown, dominated by maximizing expected utility of terminal wealth of elementary forms as logarithm, exponential or CRRA functions. Dynamic programming techniques turn out to be the most suitable approaches to solve this kind of problems. Among the most noteworthy articles, [Brandt et al. \(2005\)](#) compares and highlights the conditions of equivalence between dynamic approaches and myopic strategies with CRRA utility functions.

However, important difficulties due the non separability of the problem in the sense of dynamic programming, have been reported in finding the optimal portfolio issued from the multi-period mean-variance approach. Nevertheless [Li and Ng \(2000\)](#) and [Zhou and Li \(2000\)](#) provide explicit formulation for the unconstrained multi-period mean-variance optimal portfolio both in a discrete and continuous time setting. Suggested by Professor Markowitz him self, [Li et al. \(2002\)](#) derive the optimal portfolio policy for the continuous-time mean-variance model with no-shorting constraint. [Cui et al. \(2014\)](#) extend this work to provide a discrete framework. Even if these last studies become increasingly realistic, the ignorance of transaction costs, hinders its efficient extension for real-life application. Transaction costs impact a lot the optimal choice of policies and cannot be ignored.

The integration of transaction cost has been widely studied in the uni-period mean-var

case (see [Best and Hlouskova \(2007\)](#), [Lobo et al. \(2007\)](#), [Pogue \(1970\)](#), [Xue et al. \(2006\)](#), [Yoshimoto \(1996\)](#)). In a multi-period context, the literature is also large on the subject. In a continuous setting, the problem is not recent, especially when the time horizon is considered infinite (see [Dai and Zhong \(2008\)](#), [Davis and Norman \(1990\)](#), [Dumas and Luciano \(1991\)](#), [Morton and Pliska \(1995\)](#), [Muthuraman and Kumar \(2006\)](#)). In a mean-var and finite time horizon setting, [Dai et al. \(2010\)](#) study properties on optimal strategies and boundaries which define the buy, sell and no trade-regions. Discrete time allocation strategies submitted to transaction costs have also been widely pursued. [Constantinides \(1979\)](#) and [Holden and Holden \(2013\)](#) investigate optimal investment policies with proportional costs, accompanied by fixed costs for the second. They also describe them in terms of a no-trade region in which it is optimal leave the portfolio allocation unchanged. But many difficulties have been reported by the literature to have an efficient and accurate method to compute optimal solutions. Furthermore, when solutions are proposed, they remain computationally heavy and hardly applicable. Some of them tractably solve the problem in several special cases. [Boyd et al. \(2013\)](#) and [Peng et al. \(2011\)](#) assume that costs are convex quadratic. Other made various approximations. For instance, [Draviam and Chellathurai \(2002\)](#) assume that the number of shares in risky assets are deterministic before optimizing. [Calafiore \(2008\)](#) and [Topcu et al. \(2008\)](#) assume an affine structure for the strategies. From another perspective, both [Bertsimas and Pachamanova \(2008\)](#) and [Li et al. \(2022\)](#) rely a rolling horizon philosophy, with the latter utilizing a Model Predictive Control (MPC) approach to identify sub-optimal strategies.

The most prevalent methods to tackle this kind of problem are based on stochastic control algorithm. However, they also suffer from these criticisms. In order to limit the dimensionality, true curse in dynamic programming, [Cai et al. \(2013\)](#) use Chebyshev polynomials to interpolate value functions on a sparse grid of the space. Many of other related studies, as [Genotte and Jung \(1994\)](#), [Steinbach \(2001\)](#) and [Wang and Liu \(2013\)](#), rely on trees for modeling scenarios of rates of return. [Al-Nator et al. \(2020\)](#) go further by assuming that every future rates of return are known by the investor. Under the same assumption, [Brown and Smith \(2011\)](#) derive an upper bound to measure and highlight the good performances of heuristic strategies. [Boyd et al. \(2013\)](#) perform an ADP (Approximate dynamic programming) by using sub-optimal solutions to approximate value functions. These sub-optimal solutions are issued from a quadratic version of the problem or from MPC method (model predictive control). [Cong and Oosterlee \(2016\)](#) use an other sub-optimal solution, called multi-stage strategy to tune the exploring phase of its backward recursion algorithm. This method provides a solution at least, as good as the sub-optimal one. Recently, [Pun and Ye \(2022\)](#) also adopt dynamic programming but is forced to deal different cases separately, which makes their method hardly computationally applicable.

As a result, it is difficult to build intuition about the nature of optimal policies. In this section, we attempt to fill this gap in the literature by providing a new computational scheme to solve multi-periods portfolio allocation problem submitted to transaction cost. We address this problem by proposing a new approach that relies on representing the set admissible portfolios by a finite dimensional Wiener chaos expansion. This nu-

numerical method estimates optimal portfolios submitted to proportional transaction costs. The policies are computed thanks to a stochastic gradient descent algorithm, and require no exploration framework, which was an major step of the methods based on stochastic control algorithms (see [Bachouch et al. \(2021\)](#)). Then a competitive benchmark, based on the sequential uni-period Markowitz strategy is built to highlight the efficiency of our approach. This benchmark relies on the independence between the sharp ratio and the risk aversion.

The main contribution of this study is threefold. We introduce a new and efficient numerical method to get optimal portfolios submitted to transaction costs. Then, we study the links between risk aversion and multi-period optimal portfolios submitted to transaction cost. Finally, we provide a reliable benchmark with sequential mean-variance uni-period models in the context of transaction costs mentioned above. To the best of our knowledge, our study is the first to provide this kind of benchmark.

The remaining of this chapter is organized as follows. Section 2 is dedicated to describe the mean-variance problem for multi-period portfolio allocation submitted to transaction costs. Our methodology which aims at finding optimal portfolios in this context, is presented in Section 3. In section 4, we describe a numerical framework to investigate optimal solutions. We also study the link between risk aversion and those solutions and discuss the use of a finer time grid for the Wiener chaos expansion. Finally, in Section 6, we show the efficiency of our solution and investigate the impact of transaction cost by comparing performances of the presented models with benchmark models such as the sequential uni-period Markowitz approach, described in Section 5.

2 Environment

This section defines the environment in which we address the mean-variance allocation problem. In particular, we present the general mean-variance formulation for multi-period portfolio allocation with transaction costs. Then, we specify the dynamics of risky assets and reformulate the initial problem.

2.1 Mean-variance formulation for multi-period portfolio allocation

We define the filtered probability space $(\Omega, \mathbf{A}, \Gamma = (\mathcal{G}_t)_{t \in [0, T]}, \mathbb{P})$, with $\Gamma = \sigma(W)$, where $W = (W^1, \dots, W^d)$ is a Brownian motion defined on $[0, T]$ with values in \mathbb{R} . For $N \in \mathbb{N}$, $N > 0$, we introduce the discrete-time grid $0 = t_0 < t_1 < \dots < t_N = T$. Let \mathbf{F} , be the discrete time filtration generated by the Brownian increments on this grid, $\mathcal{F}_n = \sigma(W_{t_k}, k \leq n)$ for $0 \leq n \leq N$. We denote as $\mathcal{L} = \bigcap_{p \geq 1} L^p(\Omega, \mathcal{F}_N, \mathbb{P})$.

We consider a portfolio V , of initial wealth v_0 , composed of the d risky assets $S =$

(S^1, \dots, S^d) and the risk-free asset S^0 . An investor, with a risk aversion γ can re-allocate its portfolio V at discrete times $t_1 < \dots < t_N$. We define $(\alpha_n, \alpha_n^0)_{n \in \{1, N\}}$, the quantities of risky and risk free assets hold in the portfolio, such that $\forall n \in \{0, \dots, N\}$, $V_n = \alpha_n \cdot S_n + \alpha_n^0 S_n^0$. The processes α and α^0 are assumed to be \mathbf{F} -predictable. The agent aims to find the strategy $(\alpha_n, \alpha_n^0)_{n \in \{1, N\}}$, such that the generated portfolio V , maximizes $\mathbb{E}_{\mathbb{P}}[V_N] - \gamma \mathbb{E}_{\mathbb{P}}[(V_N - \mathbb{E}_{\mathbb{P}}[V_N])^2]$.

At time 0, we assume that the owner has all its wealth in cash, ie $\alpha_0^0 S_0^0 = v_0$ and $\forall i \in \{1, \dots, d\}$, $\alpha_0^i = 0$. At time $n \in \{1, \dots, N-1\}$, the investor re-balances his portfolio from (α_n, α_n^0) to $(\alpha_{n+1}, \alpha_{n+1}^0)$. He pays the transaction costs, proportional to the trade cash volume per asset and equal to $\sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i$. The self-financing condition is

$$V_n = \alpha_{n+1} \cdot S_n + \alpha_{n+1}^0 S_n^0 + \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i.$$

The dynamic mean-variance portfolio allocation with transaction costs can be written as

$$\begin{aligned} & \sup_{(\alpha_n, \alpha_n^0)_n} \mathbb{E}_{\mathbb{P}} [V_N - \gamma (V_N - \mathbb{E}_{\mathbb{P}}[V_N])^2] \\ \text{s.t.} \quad & V_0 = v_0, (\alpha_n, \alpha_n^0)_n \mathbf{F} - \text{Pred} \\ & V_n = \alpha_n S_n + \alpha_n^0 S_n^0 = \alpha_{n+1} \cdot S_n + \alpha_{n+1}^0 S_n^0 + \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i. \end{aligned} \tag{E^\gamma}$$

2.2 Framework and assumptions

For any discrete time process Θ , we write for $k, n \in \mathbb{N}$, $k < n$ $\Theta_{k:n} = (\Theta_k, \Theta_{k+1}, \dots, \Theta_{n-1}, \Theta_n)$. Let Δ be the operator which associates its increments process $\Delta \Theta_n = \Theta_n - \Theta_{n-1}$. We also define the normalized Brownian increments by $\Delta \widehat{W}_k = \left(\frac{W_{t_k}^j - W_{t_{k-1}}^j}{\sqrt{t_k - t_{k-1}}} \right)_{j \in \{1, \dots, d\}}$ for $1 \leq k \leq N$.

We assume that the financial market they represent is complete. The risk free rate $(r_n)_{n \leq N}$ is assumed to be deterministic and $\forall n \in \{0, \dots, N\}$, $\log \left(\frac{S_{n+1}^0}{S_n^0} \right) = r_n (t_{n+1} - t_n)$. We assume that the risky assets are defined by

$$\forall n \in \{1, \dots, N\}, i \in \{1, \dots, d\}, \log \left(\frac{S_{n+1}^i}{S_n^i} \right) = \left(\mu_n^i - \frac{(\sigma_n^i)^T \sigma_n^i}{2} \right) (t_{n+1} - t_n) + \sigma_n^i \cdot \Delta W_{n+1}, \tag{5.1}$$

where μ and σ are \mathbf{F} -adapted processes with values in \mathbb{R}^d and $\mathbb{R}^{d \times d}$ respectively. The process σ is called the volatility process and we assume that $\forall n \in \{1, \dots, N\}$, σ_n is a.s. invertible, $\mathbb{E}_{\mathbb{P}}[|\sigma_n|^2] < \infty$ and that $S_n \in \mathcal{L}$. This model implies that knowing \mathcal{F}_n , the

returns $\left\{ \frac{S_{n+1}^i}{S_n^i}, i \in \{1, \dots, d\} \right\}$ are independent of \mathcal{F}_n . Note that local volatility models fit in this framework by considering the Euler scheme of $\log(S)$. Let $\mathbf{F}^S = (\mathcal{F}_n^S)_{0 \leq n \leq N}$ be the natural filtration of the risky assets, $\mathcal{F}_n^S = \sigma(S_{t_k}, k \leq n)$. Since $(\mu^i)_{i \in \{1, \dots, d\}}$ and $(\sigma^i)_{i \in \{1, \dots, d\}}$ are \mathbf{F} -adapted processes and σ_n is a.s. invertible, we can easily prove by induction that $\mathbf{F} = \mathbf{F}^S$.

For $0 \leq n \leq N$ we define $\Phi_n = \sigma_n^{-1}(\mu_n - r_n)$. Let $\tau(t) = \sup\{t_n < t : n \leq N\}$. We assume that $\mathbb{E}_{\mathbb{P}} \left[\exp \left(\int_0^T \frac{1}{2} |\Phi_{\tau(u)}|^2 du \right) \right] < \infty$ such that the process $\left(\exp \left(- \int_0^t \Phi_{\tau(u)} \cdot dW_u - \frac{1}{2} \int_0^t |\Phi_{\tau(u)}|^2 du \right) \right)_{0 \leq t \leq T}$ is a martingale. Therefore, we know from Girsanov's theorem that there exists a probability measure \mathbb{Q} equivalent to \mathbb{P} , defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(- \int_0^T \Phi_{\tau(u)} \cdot dW_u - \frac{1}{2} \int_0^T |\Phi_{\tau(u)}|^2 du \right).$$

We denote as $\mathcal{Z} = \left(\exp \left(- \int_0^{t_n} \Phi_{\tau(u)} \cdot dW_u - \frac{1}{2} \int_0^{t_n} |\Phi_{\tau(u)}|^2 du \right) \right)_{0 \leq n \leq N}$. Let $(W_t^{\mathbb{Q}})_{t \in [0, T]}$ be a Γ -adapted process defined by $W_0^{\mathbb{Q}} = 0$ and

$$dW_t^{\mathbb{Q}} = dW_t + \Phi_{\tau(t)} dt,$$

then $W^{\mathbb{Q}}$ is a Brownian motion under \mathbb{Q} . Note that

$$\forall i \in \{1, \dots, d\}, \Delta W_{n+1}^{\mathbb{Q}, i} \stackrel{\mathbb{P}}{=} \Delta W_{n+1}^i + \Phi_n^i (t_{n+1} - t_n). \quad (5.2)$$

Let $\mathbf{F}^{\mathbb{Q}} = (\mathcal{F}_n^{\mathbb{Q}})_{0 \leq n \leq N}$, be the discrete time filtration generated by $W^{\mathbb{Q}}$, $\mathcal{F}_n^{\mathbb{Q}} = \sigma(W_{t_k}^{\mathbb{Q}}, k \leq n)$. We notice that $\forall n \in \{1, \dots, N\}, i \in \{1, \dots, d\}$,

$$\log \left(\frac{S_{n+1}^i}{S_n^i} \right) = \left(r_n - \frac{(\sigma_n^i)^T \sigma_n^i}{2} \right) (t_{n+1} - t_n) + \sigma_n^i \cdot \Delta W_{n+1}^{\mathbb{Q}},$$

We can see by induction that we also have $\mathbf{F}^{\mathbb{Q}} = \mathbf{F}^S$.

We use the tilde notation to denote discounting and we notice that $\tilde{S}^i = S^i/S^0$ is a \mathbf{F} -martingale under the probability \mathbb{Q} . Lastly, we assume that the process $\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]$ is a.s. invertible and $\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1}, \mathcal{Z}_n \in \mathcal{L}$.

2.3 Reformulation of the problem

With the notations and assumptions of Section 2.2, the self-financing condition can be reformulated as

$$\Delta \tilde{V}_{n+1} = \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| \tilde{S}_n^i. \quad (5.3)$$

We define the cumulative cost process \mathcal{C} by

$$\forall n \in \{1, \dots, N\}, \mathcal{C}_n = \sum_{k=0}^{n-1} \sum_{i=1}^d \nu |\alpha_{k+1}^i - \alpha_k^i| \tilde{S}_k^i, \mathcal{C}_0 = 0. \quad (5.4)$$

We decide to work with the \mathbf{F} -martingale under \mathbb{Q} , $\tilde{X}_n = \tilde{V}_n + \mathcal{C}_n$. Let \mathcal{M} be the space of squared integrable \mathbf{F} -martingales under \mathbb{Q} and \mathcal{M}_S the sub-space of \mathcal{M} defined by

$$\mathcal{M}_S = \left\{ \begin{array}{l} M \in \mathcal{M} : \exists (\alpha_k)_{1 \leq k \leq N} \mathbf{F} \text{ - predictable s.t. } \forall n \in \{0, \dots, N\}, \\ M_n = h + \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \end{array} \right\}.$$

\mathcal{M}_S is the set of martingales which are a martingale transformations of \tilde{S} . The space \mathcal{M}_S also represents the space of admissible portfolios in discrete-time. The mean-variance multi-period portfolio allocation problem with transaction costs (\mathcal{E}^γ), can be reformulated as

$$\begin{aligned} & \sup_{\tilde{X} \in \mathcal{M}_S} \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N) S_N^0 - \mathbb{E}_{\mathbb{P}} [(\tilde{X}_N - \mathcal{C}_N) S_N^0] \right)^2 \right] \\ \text{s.t.} \quad & \tilde{X}_0 = v_0, \\ & \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| \tilde{S}_n^i \end{aligned} \quad (\mathcal{E}_{\mathcal{M}_S}^\gamma)$$

In the next section, we present a novel approach to solve the dynamic mean-var allocation problem in presence of transaction costs.

3 Main results

The main contribution of this study is to propose and study a numerically tractable approximation of ($\mathcal{E}_{\mathcal{M}_S}^\gamma$). First, we embed the original problem into a more standard stochastic optimization framework ($\mathcal{E}_{\mathcal{M}}^\gamma$). Then, we use Wiener chaos polynomials to obtain a finite dimensional formulation (\mathcal{E}_K^γ).

3.1 Embedding representation

The main difficulty remains to parameterize \mathcal{M}_S . We propose to find optimal portfolios on \mathcal{M}_S by exploring \mathcal{M} . Furthermore, the objective function $(\tilde{X}_N - \mathcal{C}_N)S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N)S_N^0 - \mathbb{E}_{\mathbb{P}}[(\tilde{X}_N - \mathcal{C}_N)S_N^0] \right)^2$ depends on the law of the portfolio value. This non-separable form is not easy to manipulate. In order to solve $(E_{\mathcal{M}_S}^\gamma)$, we would like to embed it into a tractable equivalent one.

Proposition 24. *The problem $(E_{\mathcal{M}_S}^\gamma)$ is equivalent to*

$$\begin{aligned} & \sup_{Z \in \mathcal{M}, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F^\gamma(Z, \theta)] \\ \text{s.t.} \quad & Z_0 = v_0, \\ & \mathcal{C}_{n+1}(Z) = \mathcal{C}_n(Z) + \sum_{i=1}^d \nu |\alpha_{n+1}^i(Z) - \alpha_n^i(Z)| \tilde{S}_n^i \end{aligned} \tag{\mathcal{E}_{\mathcal{M}}^\gamma}$$

With

$$F^\gamma(Z, \theta) = \mathcal{R}(Z)S_N^0 - \gamma \left((\mathcal{R}(Z) - \theta)S_N^0 \right)^2, \tag{5.5}$$

$$\mathcal{R}(Z) = \text{Pr}(Z)_N - \mathcal{C}_N(Z), \tag{5.6}$$

$$\text{Pr}(Z) = \left(Z_0 + \sum_{k=0}^{n-1} \alpha_{k+1}(Z) \cdot \Delta \tilde{S}_{k+1} \right)_{0 \leq n \leq N}, \tag{5.7}$$

$$\alpha_{n+1}(Z) = \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right] \right)^{-1} \mathbb{E}_{\mathbb{Q}} \left[\Delta Z_{n+1} \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right]. \tag{5.8}$$

The proof of Proposition 24 relies on the following Lemma.

Lemma 25. *The problem $(E_{\mathcal{M}_S}^\gamma)$ is equivalent to*

$$\begin{aligned} & \sup_{\tilde{X} \in \mathcal{M}_S, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N)S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \theta)S_N^0 \right)^2 \right] \\ \text{s.t.} \quad & \tilde{X}_0 = v_0, \\ & \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| \tilde{S}_n^i \end{aligned} \tag{\mathcal{E}_{\mathcal{M}_S}^\gamma}$$

Proof (of Lemma 25). Let $(T_n)_n$ be a \mathbf{F} -adapted process. We define the function

$$\begin{aligned} \mathcal{U} : \mathbb{R} &\rightarrow \mathbb{R} \\ \theta &\longmapsto \mathbb{E}_{\mathbb{P}}[T_N] - \gamma \mathbb{E}_{\mathbb{P}} \left[\left((T_N - \theta)S_N^0 \right)^2 \right] \end{aligned}$$

The function \mathcal{U} is a second order polynomial such that $\lim_{|\theta| \rightarrow \infty} \mathcal{U}(\theta) = -\infty$. Then, \mathcal{U} attains its maximum at a unique point θ^* defined by $\nabla \mathcal{U}(\theta^*) = 0$, $\theta^* = \mathbb{E}_{\mathbb{P}}[T_N]$. ■

Proof (of Proposition 24). Let $M \in \mathcal{M}$. The projection of M on \mathcal{M}_S writes as

$$\left(h + \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right)_{0 \leq n \leq N},$$

with α_k^j \mathbf{F} -predictable, $h \in \mathbb{R}$. Let $B \in \mathcal{M}_S$ such that for $n \in \{0, \dots, N\}$, $B_n = g + \sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1}$, $g \in \mathbb{R}$, c_{k+1}^j \mathbf{F} -predictable. Then we have for $n \in \{1, \dots, N\}$,

$$\mathbb{E}_{\mathbb{Q}} \left[\left(M_n - h - \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \left(g + \sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \right] = 0. \quad (5.9)$$

By taking $c_{k+1}^j = 0$, for all $j \in \{1, \dots, d\}$, we get $h = M_0$. Setting $g = 0$, we have

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\sum_{l=0}^{n-1} \Delta M_{l+1} - \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \left(\sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \right] = 0. \quad (5.10)$$

By developing the expression and using that M and \tilde{S}^j are \mathbf{F} -martingale, we obtain for $l \neq k$,

$$\mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \Delta M_{l+1} \Delta \tilde{S}_{k+1}^j \right] = \mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \mathbb{E}_{\mathbb{Q}} \left[\Delta M_{l+1} \Delta \tilde{S}_{k+1}^j \mid \mathcal{F}_{l \vee k} \right] \right] = 0. \quad (5.11)$$

Then, for $1 \leq k \leq n-1$,

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{l=0}^{n-1} \Delta M_{l+1} c_{k+1}^j \Delta \tilde{S}_{k+1}^j \right] = \mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \Delta M_{k+1} \Delta \tilde{S}_{k+1}^j \right]. \quad (5.12)$$

Finally, inserting (5.12) in (5.10) yields

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \Delta M_{k+1} - \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} (\alpha_{k_1+1} \cdot \Delta \tilde{S}_{k_1+1}) (c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) \right] = 0 \quad (5.13)$$

In the same way, for $k_1 \neq k_2$,

$$\mathbb{E}_{\mathbb{Q}} \left[(\alpha_{k_1+1} \cdot \tilde{S}_{k_1+1}) (c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) \right] = \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E} \left[(\alpha_{k_1+1} \cdot \Delta \tilde{S}_{k_1+1}) (c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) \mid \mathcal{F}_{k_1 \vee k_2} \right] \right] = 0. \quad (5.14)$$

Then

$$\sum_{k=0}^{n-1} \mathbb{E}_{\mathbb{Q}} \left[c_{k+1} \cdot \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta M_{k+1} \mid \mathcal{F}_k \right] - c_{k+1} \cdot \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta \tilde{S}_{k+1}^T \mid \mathcal{F}_k \right] \alpha_{k+1} \right) \right] = 0. \quad (5.15)$$

So

$$\sum_{k=0}^{n-1} \mathbb{E}_{\mathbb{Q}} \left[c_{k+1} \cdot \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta M_{k+1} | \mathcal{F}_k \right] - \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta \tilde{S}_{k+1}^T | \mathcal{F}_k \right] \alpha_{k+1} \right) \right] = 0. \quad (5.16)$$

As this equation holds for all $(c_k)_{1 \leq k \leq N}$, we deduce that $(\alpha_n)_n$ solves

$$\mathbb{E}_{\mathbb{Q}} \left[\Delta M_{n+1} \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right] = \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right] \alpha_{n+1}.$$

Then, we conclude that $(\alpha_n)_n$ is defined by (5.8) and that $(\mathcal{E}_{\mathcal{M}_S}^\gamma)$ can be reformulated as $(\mathcal{E}_{\mathcal{M}}^\gamma)$. \blacksquare

3.2 Wiener chaos parametrization

We will parametrize the elements of \mathcal{M} through the Wiener chaos expansion of their terminal value. The basic theory of Wiener chaos expansion and the properties used here are presented in Appendix **Tool box: Wiener chaos expansion properties**. According to (Akahori et al., 2017, Theorem 2.1), every element Y of $L^2(\Omega, \mathcal{F}_N, \mathbb{Q})$ can be represented by its Wiener chaos expansion as

$$Y = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}^{\mathbb{Q}}), \quad (5.17)$$

where

$$H_\lambda^\otimes(\Delta \widehat{W}^{\mathbb{Q}}) = \prod_{j=1}^d \prod_{i=1}^N H_{\lambda_i^j} \left(\frac{W_{t_i}^{\mathbb{Q},j} - W_{t_{i-1}}^{\mathbb{Q},j}}{\sqrt{t_i - t_{i-1}}} \right).$$

We define the truncated expansion of Y of order K under \mathbb{Q} by $\mathcal{C}_K(Y)$ such that

$$\mathcal{C}_K(Y) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}^{\mathbb{Q}}).$$

It is well-known that $\lim_{K \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}}[|Y - \mathcal{C}_K(Y)|^2] = 0$. Proposition 51 states that for $n \leq N$, $\mathbb{E}_{\mathbb{Q}}[\mathcal{C}_K(Y) | \mathcal{F}_n] = \mathcal{C}_K(\mathbb{E}_{\mathbb{Q}}[Y | \mathcal{F}_n])$, which can be obtained by removing the non \mathcal{F}_n -measurable terms from the chaos expansion of Y . By identifying the martingale $Z \in \mathcal{M}$ with the chaos expansion of its terminal value Z_N , we slightly abuse the notation \mathcal{C}_K to define the process $\mathcal{C}_K(Z) = (\mathcal{C}_K(Z_n))_n$.

Let's consider the analog problem for $K > 0$

$$\begin{aligned}
& \sup_{Z \in \mathcal{M}, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z, \theta)] \\
& \text{s.t.} \quad Z_0 = v_0, \\
& \mathcal{C}_{n+1}(\mathcal{C}_K(Z)) = \mathcal{C}_n(\mathcal{C}_K(Z)) + \sum_{i=1}^d \nu |\alpha_{n+1}^i(\mathcal{C}_K(Z)) - \alpha_n^i(\mathcal{C}_K(Z))| \tilde{S}_n^i
\end{aligned} \tag{\mathcal{E}_K^\gamma}$$

$$\text{with } F_K^\gamma(Z, \theta) = F^\gamma(\mathcal{C}_K(Z), \theta) = \mathcal{R}(\mathcal{C}_K(Z)) S_N^0 - \gamma ((\mathcal{R}(\mathcal{C}_K(Z)) - \theta) S_N^0)^2.$$

The following proposition states that the optimum of (\mathcal{E}_K^γ) converges to the optimum of $(\mathcal{E}_{\mathcal{M}}^\gamma)$. As a result, we can approach the optimum of $(\mathcal{E}_{\mathcal{M}}^\gamma)$ by solving (\mathcal{E}_K^γ) .

Proposition 26. *Let (Z^*, θ^*) be a solution to $(\mathcal{E}_{\mathcal{M}}^\gamma)$. If $\forall \mathcal{X} \in \mathcal{L}$, $\sup_{K>1} [\mathcal{X} \mathcal{C}_K(Z^*)^2] < \infty$, then, there exists $\eta > 0$ such that for all $K > 0$ and (Z^K, θ^K) solution to (\mathcal{E}_K^γ) , we have*

$$|\mathbb{E}_{\mathbb{P}} [F^\gamma(Z^*, \theta^*)] - \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^K, \theta^K)]| \leq \eta \mathbb{E}_{\mathbb{P}} [|Z_N^* - \mathcal{C}_K(Z_N^*)|^2]^{\frac{1}{2}}.$$

The proof of the proposition is based on the following lemmas.

Lemma 27. *Let (Z^*, θ^*) be a solution to $(\mathcal{E}_{\mathcal{M}}^\gamma)$. If $\forall \mathcal{X} \in \mathcal{L}$, $\sup_{K>1} [\mathcal{X} \mathcal{C}_K(Z^*)^2] < \infty$, then there exists $\eta > 0$ such that*

$$|\mathbb{E}_{\mathbb{P}} [F^\gamma(Z^*, \theta^*)] - \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^*, \theta^*)]| \leq \eta \mathbb{E}_{\mathbb{P}} [|Z_N^* - \mathcal{C}_K(Z_N^*)|^2]^{\frac{1}{2}}.$$

Lemma 28. *Let $1 \leq p < 2$, then $\exists \zeta > 0$, such that $\forall Z, Z' \in \mathcal{M}$, $\forall n \in \{0, \dots, N-1\}$,*

$$\mathbb{E}_{\mathbb{P}} [|\alpha_{n+1}(Z) - \alpha_{n+1}(Z')|^p] \leq \zeta \mathbb{E}_{\mathbb{Q}} [|\Delta Z_{n+1} - \Delta Z'_{n+1}|^2]^{\frac{p}{2}}, \tag{5.18}$$

$$\mathbb{E}_{\mathbb{P}} [|\text{Pr}(Z^*)_N - \text{Pr}(\mathcal{C}(Z^*))_N|] \leq \zeta \mathbb{E}_{\mathbb{Q}} \left[|Z_N - Z'_N|^2 \right]^{\frac{1}{2}}. \tag{5.19}$$

Proof (of Lemma 28). Using consecutively Jensen's inequality and Holder's inequality with the coefficients $\frac{2}{2-p}$ and $\frac{2}{p}$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}} [|\alpha_{n+1}(Z) - \alpha_{n+1}(Z')|^p] &= \mathbb{E}_{\mathbb{P}} \left[\left| \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1} \mathbb{E}_{\mathbb{Q}} \left[(\Delta Z_{n+1} - \Delta Z'_{n+1}) \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right] \right|^p \right] \\
&\leq \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\mathbb{Q}} \left[|\mathcal{Z}_n| \left| (\Delta Z_{n+1} - \Delta Z'_{n+1}) \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1} \Delta \tilde{S}_{n+1} \right|^p | \mathcal{F}_n \right] \right] \\
&\leq \mathbb{E}_{\mathbb{Q}} \left[\left[\mathcal{Z}_n^{\frac{1}{p}} \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1} \Delta \tilde{S}_{n+1} \right]^{\frac{2-p}{2}} \mathbb{E}_{\mathbb{Q}} \left[|\Delta Z_{n+1} - \Delta Z'_{n+1}|^2 \right]^{\frac{p}{2}} \right].
\end{aligned}$$

By recalling that \tilde{S}_{n+1} , $\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1}$, $\mathcal{Z}_n \in \mathcal{L}$, we find $\zeta > 0$ such that

$\mathbb{E}_{\mathbb{Q}} \left[\left[\mathcal{Z}_n^{\frac{1}{p}} \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right]^{-1} \Delta \tilde{S}_{n+1} \right] \right]^{\frac{2-p}{2}} < \zeta$, and obtain (5.18). For the second inequality (5.19), we use Holder's inequality with the coefficients $p = \frac{3}{2}$, $q = 3$, and (5.18). The constant $\zeta' > 0$ may change over the inequalities.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [|Pr(Z^*)_N - Pr(\mathcal{C}(Z^*))_N|] &\leq \sum_{k=0}^{N-1} \mathbb{E}_{\mathbb{P}} \left[\left| \alpha_{k+1}(Z^*) \cdot \Delta \tilde{S}_{k+1} - \alpha_{k+1}(\mathcal{C}_K(Z^*)) \cdot \Delta \tilde{S}_{k+1} \right| \right] \\ &\leq \sum_{k=0}^{N-1} \mathbb{E}_{\mathbb{P}} [|\alpha_{k+1}(Z^*) - \alpha_{k+1}(\mathcal{C}_K(Z^*))|^p]^{\frac{1}{p}} \mathbb{E}_{\mathbb{P}} \left[\left| \Delta \tilde{S}_{k+1} \right|^q \right]^{\frac{1}{q}} \\ &\leq \sum_{k=0}^{N-1} \zeta' \mathbb{E}_{\mathbb{Q}} [|\Delta Z_{k+1} - \Delta \mathcal{C}_K(Z^*)_{k+1}|^2]^{\frac{1}{2}}. \\ &\leq \zeta' \left(\sum_{k=0}^{N-1} \mathbb{E}_{\mathbb{Q}} [|\Delta Z_{k+1} - \Delta \mathcal{C}_K(Z^*)_{k+1}|^2] \right)^{\frac{1}{2}} = \zeta' \mathbb{E}_{\mathbb{Q}} [|Z_N - \mathcal{C}_K(Z^*)_N|^2]^{\frac{1}{2}}. \end{aligned}$$

The last equality stems from the martingale property of $Z^* - \mathcal{C}_K(Z^*)$. ■

Proof (of Lemma 27). We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [|F^\gamma(Z^*, \theta^*) - F_K^\gamma(Z^*, \theta^*)|] &\leq \mathbb{E}_{\mathbb{P}} [|\mathcal{R}(Z^*) - \mathcal{R}(\mathcal{C}_K(Z^*))|] S_N^0 \\ &\quad + \gamma \mathbb{E}_{\mathbb{P}} [|(\mathcal{R}(\mathcal{C}_K(Z^*)) - \theta^*)^2 - (\mathcal{R}(Z^*) - \theta^*)^2|] (S_N^0)^2. \end{aligned}$$

For the first term, we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} [|Pr(Z^*)_N - \mathcal{C}_N(Z^*) - (Pr(\mathcal{C}_K(Z^*))_N - \mathcal{C}_N(\mathcal{C}_K(Z^*)))|] \\ &\leq \mathbb{E}_{\mathbb{P}} [|Pr(Z^*)_N - Pr(\mathcal{C}_K(Z^*))_N|] \\ &\quad + \sum_{i=1}^d \sum_{k=0}^{N-1} \nu \mathbb{E}_{\mathbb{P}} \left[|\alpha_{k+1}^i(\mathcal{C}_K(Z^*)) - \alpha_k^i(\mathcal{C}_K(Z^*)) - (\alpha_{k+1}^i(Z^*) - \alpha_k^i(Z^*))| |\tilde{S}_k^i| \right]. \end{aligned}$$

Using Lemma 28 and the same arguments as to prove (5.19), we find $\eta_1 > 0$, such that

$$\mathbb{E}_{\mathbb{P}} [|Pr(Z^*)_N - \mathcal{C}_N(Z^*) - (Pr(\mathcal{C}_K(Z^*))_N - \mathcal{C}_N(\mathcal{C}_K(Z^*)))|] \leq \eta_1 \mathbb{E}_{\mathbb{Q}} [|Z_N - \mathcal{C}_K(Z^*)_N|^2]^{\frac{1}{2}}.$$

For the second term, we rewrite

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} [|(\mathcal{R}(\mathcal{C}_K(Z^*)) - \theta^*)^2 - (\mathcal{R}(Z^*) - \theta^*)^2|] \\ &= \mathbb{E}_{\mathbb{P}} [|\mathcal{R}(\mathcal{C}_K(Z^*)) + \mathcal{R}(Z^*) - 2\theta^*| |\mathcal{R}(\mathcal{C}_K(Z^*)) - \mathcal{R}(Z^*)|] \\ &\leq \mathbb{E}_{\mathbb{P}} [|\mathcal{R}(\mathcal{C}_K(Z^*)) + \mathcal{R}(Z^*)| |\mathcal{R}(\mathcal{C}_K(Z^*)) - \mathcal{R}(Z^*)|] + 2|\theta^*| \mathbb{E}_{\mathbb{P}} [|\mathcal{R}(\mathcal{C}_K(Z^*)) - \mathcal{R}(Z^*)|] \end{aligned}$$

Let $i, j \in \{1, \dots, d\}$, $k, l \in \{0, \dots, N-1\}$ and $\mathcal{Y} \in \{\Delta \tilde{S}_{k+1}^i \Delta \tilde{S}_{l+1}^j, \Delta \tilde{S}_{k+1}^i \tilde{S}_{l+1}^j, \tilde{S}_{k+1}^i \tilde{S}_{l+1}^j, \tilde{S}_{k+1}^i \Delta \tilde{S}_{l+1}^j\}$. By using the same decomposition and arguments as to prove Lemma 28 and Cauchy Schwartz' inequality, we find $\mathcal{X} \in \mathcal{L}$ such that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} [|\alpha_{k+1}^i(Z^*) - \alpha_{k+1}^i(\mathcal{C}_K(Z^*))| |\alpha_{l+1}^j(\mathcal{C}_K(Z^*))| |\mathcal{Y}|] \\ &= \mathbb{E}_{\mathbb{P}} [|\Delta Z_{k+1}^* - \Delta \mathcal{C}_K(Z^*)_{k+1}| |\Delta \mathcal{C}_K(Z^*)_{l+1}| |\mathcal{X}|] \\ &\leq \mathbb{E}_{\mathbb{P}} [|\Delta Z_{k+1}^* - \Delta \mathcal{C}_K(Z^*)_{k+1}|^2]^{\frac{1}{2}} \mathbb{E}_{\mathbb{P}} [|\Delta \mathcal{C}_K(Z^*)_{l+1}|^2 |\mathcal{X}|^2]^{\frac{1}{2}}. \end{aligned}$$

The same result can be obtain for $\mathbb{E}_{\mathbb{P}} [|\alpha_{k+1}^i(Z^*) - \alpha_{k+1}^i(\mathcal{C}_K(Z^*))| |\alpha_{l+1}^j(Z^*)| |\mathcal{Y}|]$. By using the main assumption of the proposition, and considering the form of $R(\mathcal{C}_K(Z^*))$ and $R(Z^*)$, we find $\zeta > 0$ such that

$$\mathbb{E}_{\mathbb{P}} [|\mathcal{R}(\mathcal{C}_K(Z^*)) + \mathcal{R}(Z^*)| |\mathcal{R}(\mathcal{C}_K(Z^*)) - \mathcal{R}(Z^*)|] \leq \zeta \mathbb{E}_{\mathbb{P}} [|\mathcal{Z}_N^* - \mathcal{C}_K(\mathcal{Z}_N^*)|^2]^{\frac{1}{2}}.$$

Finally, by using Lemma 28 to bound the term $2|\theta^*| \mathbb{E}_{\mathbb{P}} [|\mathcal{R}(\mathcal{C}_K(Z^*)) - \mathcal{R}(Z^*)|]$, we find $\eta_2 > 0$ such that

$$\mathbb{E}_{\mathbb{P}} [|\text{Pr}(\mathcal{C}_K(Z^*))_N - \mathcal{C}_N(\mathcal{C}_K(Z^*)) - \text{Pr}(Z^*)_N + \mathcal{C}_N(Z^*)|^2] \leq \eta_2 \mathbb{E}_{\mathbb{P}} [|\mathcal{Z}_N^* - \mathcal{C}_K(\mathcal{Z}_N^*)|^2]^{\frac{1}{2}}.$$

Finally setting $\eta = \eta_1 S_N^0 + \gamma \eta_2 (S_N^0)^2$ gives the result. \blacksquare

Proof (of Proposition 26). Let (Z^K, θ^K) be a solution to (\mathcal{E}_K^γ) . $(\mathcal{C}_K(Z^K), \theta^K)$ is an admissible solution for (\mathcal{E}_M^γ) , then

$$\mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^*, \theta^*)] \leq \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^K, \theta^K)] \leq \mathbb{E}_{\mathbb{P}} [F^\gamma(Z^*, \theta^*)]. \quad (5.20)$$

According to Lemma 27, we find $\eta > 0$ independent of K such that

$$\begin{aligned} |\mathbb{E}_{\mathbb{P}} [F^\gamma(Z^*, \theta^*)] - \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^K, \theta^K)]| &\leq |\mathbb{E}_{\mathbb{P}} [F^\gamma(Z^*, \theta^*)] - \mathbb{E}_{\mathbb{P}} [F_K^\gamma(Z^*, \theta^*)]| \\ &\leq \mathbb{E}_{\mathbb{P}} [|\mathcal{F}^\gamma(Z^*, \theta^*) - \mathcal{F}_K^\gamma(Z^*, \theta^*)|] \\ &\leq \eta \mathbb{E}_{\mathbb{P}} [|\mathcal{Z}_N^* - \mathcal{C}_K(\mathcal{Z}_N^*)|^2]^{\frac{1}{2}}. \end{aligned} \quad (5.21) \quad \blacksquare$$

We denote m as the number of coefficients β_λ appearing in the chaos expansion of order K , $m = \#\{\lambda \in (\mathbb{N}^N)^d | |\lambda|_1 \leq K\}$. By identifying the martingale $Z \in \mathcal{M}$ with the coefficients of the chaos expansion of its terminal value Z_N , we slightly abuse the notation \mathcal{C}_K to write

$$\forall \beta \in \mathbb{R}^m, \mathcal{C}_K(\beta) = \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^\otimes (\Delta \widehat{W}^{\mathbb{Q}}).$$

We can also slightly abuse the definition of the controls in (5.8). Recalling that $(\beta_\lambda)_\lambda$ define

the expansion of Z_N , we define

$$\alpha_{n+1}(\beta) = \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta (\tilde{S}_{n+1})^T | \mathcal{F}_n \right]^{-1} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right]. \quad (5.22)$$

We notice that α_{n+1} can be expressed as a linear combination of the chaos coefficients of Z_N . Similarly, we also extend the cost function to

$$\mathcal{C}_n(\beta) = \sum_{k=0}^{n-1} \sum_{i=1}^d \nu |\alpha_{k+1}^i(\beta) - \alpha_k^i(\beta)| \tilde{S}_k^i. \quad (5.23)$$

Let's extend in this context, the portfolio value function as

$$\forall \beta \in \mathbb{R}^m, \mathcal{R}(\beta) = Pr(\mathcal{C}_K(\beta))_N - \mathcal{C}_N(\beta). \quad (5.24)$$

Note that $\beta \rightarrow \mathcal{R}(\beta) - v_0$ is a positive homogeneous function, ie $\forall u > 0, \mathcal{R}(u\beta) - v_0 = u(\mathcal{R}(\beta) - v_0)$. We also express the objective function as

$$\forall \beta \in \mathbb{R}^m, F^\gamma(\beta, \theta) = \mathcal{R}(\beta) S_N^0 - \gamma ((\mathcal{R}(\beta) - \theta) S_N^0)^2. \quad (5.25)$$

Note that F^γ is a random function. With these new functions, we can approximate the original problem ($E_{\mathcal{M}_S}^\gamma$) by a finite dimensional optimisation problem

$$\begin{aligned} & \sup_{\beta \in \mathbb{R}^m, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \theta)] \\ \text{s.t.} & \quad \mathbb{E}_{\mathbb{Q}} [\mathcal{C}_K(\beta)] = v_0 \end{aligned} \quad (J^\gamma)$$

The constraint $\mathbb{E}_{\mathbb{Q}} [\mathcal{C}_K(\beta)] = v_0$ is easily satisfied by setting the first coefficient equal to v_0 , $\beta_0 = v_0$.

Proposition 29. *The problem (J^γ) admits a solution.*

Proof. According to Lemma 25, for $\beta \in \mathbb{R}^m$, $\sup_{\theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \theta)]$ always exists and is attained for $\theta^* = \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]$. Let $\beta \in \mathbb{R}^m \neq 0_{\mathbb{R}^m}$. We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]) - v_0] &= |\beta| \mathbb{E}_{\mathbb{P}} \left[\mathcal{R} \left(\frac{\beta}{|\beta|} \right) - v_0 \right] - \gamma |\beta|^2 \text{Var} \left[\mathcal{R} \left(\frac{\beta}{|\beta|} \right) - v_0 \right] \\ &= |\beta| \mathbb{E}_{\mathbb{P}} \left[F^\gamma \left(\frac{\beta}{|\beta|}, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\frac{\beta}{|\beta|})] \right) - v_0 \right] + \gamma (|\beta| - |\beta|^2) \text{Var} \left[\mathcal{R} \left(\frac{\beta}{|\beta|} \right) \right]. \end{aligned}$$

By considering the optimums of continuous functions on compact sets, we define

$$v = \inf_{|\beta|=1} \text{Var} [\mathcal{R}(\beta)] \quad , \quad u = \sup_{|\beta|=1} \mathbb{E}_{\mathbb{P}} \left[F^\gamma \left(\frac{\beta}{|\beta|}, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\frac{\beta}{|\beta|})] \right) - v_0 \right].$$

Since the market is assumed to be complete, it is not possible to build a risk free portfolio

with risky assets, $v > 0$. Finally, $\forall \beta \in \mathbb{R}^m$, $|\beta| > 1$,

$$\mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]) - v_0] \leq |\beta| u + \gamma (|\beta| - |\beta|^2) v.$$

With this inequality, we deduce that $\lim_{|\beta| \rightarrow +\infty} \mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)])] = -\infty$. The objective function $\beta \rightarrow \mathbb{E}_{\mathbb{P}} [F^\gamma(\beta, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)])]$ is continuous and $\mathbb{E}_{\mathbb{P}} [F^\gamma(0, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(0)])] = v_0 > 0$, then it attains its maximum. We can conclude that (J^γ) admits a solution. \blacksquare

4 Numerical resolution

In this section, we describe the numerical framework to solve (J^γ) . We define the space where optimal strategies are investigated and we study the link between risk aversion and those solutions. In particular, we insure that risk aversion parametrization has no impact on performances. Lastly, we discuss the use of a finer time grid for the Wiener chaos expansion. We assume for the numerical resolution that σ is deterministic (Black-Scholes case).

Let's introduce

$$\mathcal{Z} = \{\beta \in \mathbb{R}^m, \forall n \in \{1, \dots, N\}, i \in \{1, \dots, d\} \alpha_n^i(\beta) \neq 0 \text{ a.s.}\}$$

$\mathbb{R}^m \setminus \mathcal{Z}$ is a closed set of \mathbb{R}^m of measure null. Elements of this set represents strategies where there is almost surely, no investment, during a time period between t_1 and t_{N-1} , in a risky asset. These strategies are not realistic, not optimal and therefore are ignored. Optimal strategies are investigated on \mathcal{Z} . We aim to apply a stochastic descent gradient algorithm to find such solutions.

4.1 Differentiability

We study the differentiability of $(\beta, \theta) \rightarrow \mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)] = \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}(\beta) S_N^0 - \gamma ((\mathcal{R}(\beta) - \theta) S_N^0)^2 \right]$.

Proposition 30. \mathcal{R} is almost surely differentiable on \mathcal{Z} .

Proof. Let us prove that the $\mathbb{Q}(\alpha_{n+1}^i = \alpha_n^i) = 0, \forall 1 \leq i \leq d, 0 \leq n \leq N-1$. We recall that

$$\alpha_{n+1}(\beta) = \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta (\tilde{S}_{n+1})^T | \mathcal{F}_n \right]^{-1} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right].$$

Let Diag be the operator which associates to a vector (d^1, \dots, d^k) the matrix $\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_k \end{bmatrix}$.

We define

$$Y_n = \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1 \right) \left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1 \right)^T \middle| \mathcal{F}_n \right] = \left[e^{\sigma_n^k \cdot \sigma_n^j (t_{n+1} - t_n)} - 1 \right]_{k,j}.$$

Note that Y_n is deterministic and invertible. We slightly abuse of the notation \mathcal{C}_K to denote the vector $\mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{S_{n+1}}{S_n} - 1 \right) \middle| \mathcal{F}_n \right] = \left(E_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \middle| \mathcal{F}_n \right] \right)_{j \in \{1, \dots, d\}}$. We have

$$\forall \beta \in \mathcal{Z}, \alpha_{n+1}(\beta) = \text{Diag}(\tilde{S}_n)^{-1} Y_n^{-1} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1 \right) \middle| \mathcal{F}_n \right].$$

The term $\mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \middle| \mathcal{F}_n \right]$ can be expressed as a linear combination of the chaos coefficients of Z_N such that

$$\mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \middle| \mathcal{F}_n \right] = \sum_{\substack{\lambda \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 \leq K, \\ \lambda_{n+1} \neq 0}} \beta_{\lambda} H_{\lambda_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}^{\mathbb{Q}}) \mathbb{E}_{\mathbb{Q}} \left[H_{\lambda_{n+1}}(\Delta \widehat{W}_{n+1}^{\mathbb{Q}}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \right] \quad (5.26)$$

Note that the terms $\mathbb{E}_{\mathbb{Q}} \left[H_{\lambda_{n+1}}(\Delta \widehat{W}_{n+1}^{\mathbb{Q}}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \right]$ can be estimated with Monte-Carlo simulations. For $i \in \{1, \dots, d\}$, we obtain

$$\mathbb{Q}(\alpha_{n+1}^i(\beta) = 0) \iff \mathbb{Q} \left(\left[Y_n^{-1} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1 \right) \middle| \mathcal{F}_n \right] \right]_i = 0 \right).$$

Note that $\left[Y_n^{-1} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{C}_K(\Delta Z_{n+1}) \left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1 \right) \middle| \mathcal{F}_n \right] \right]_i$ can be written as

$$\sum_{j=1}^d (Y_n^{-1})_{ij} \sum_{k' \leq K} \sum_{\substack{\lambda_{1:n} \in (\mathbb{N}^n)^d \\ |\lambda_{1:n}|_1 = k'}} \left(\sum_{\substack{\lambda_{n+1} \in \mathbb{N}^d \\ 1 \leq |\lambda_{n+1}|_1 \leq K - k'}} \beta_{\lambda} \mathbb{E}_{\mathbb{Q}} \left[H_{\lambda_{n+1}}(\Delta \widehat{W}_{n+1}^{\mathbb{Q}}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \right] \right) H_{\lambda_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}^{\mathbb{Q}}). \quad (5.27)$$

With the definition of \mathcal{Z} , we deduce that $\forall \beta \in \mathcal{Z}, \mathbb{Q}(\alpha_{n+1}^i(\beta) = 0) = 0$.

Let us introduce the following two spaces for $n \leq 1$,

$$\mathcal{P}_n = \left\{ \sum_{k=1}^K b_k H_k(\Delta W_n), (b_k)_k \mathcal{F}_{n-1} \text{ - measurable, } K \in \mathbb{N}^*, \exists k \in \{1, \dots, K\}, \mathbb{Q}(b_k = 0) = 0 \right\},$$

$$\mathcal{E}_n = \left\{ \begin{array}{l} \sum_{l=0}^L a_l e^{c_l \Delta W_n}, (a_l)_l, (c_l)_l \text{ } \mathcal{F}_{n-1} \text{-measurable,} \\ L \in \mathbb{N}^*, \exists l \in \{1, \dots, L\}, \mathbb{Q}(a_l = 0) = \mathbb{Q}(c_l = 0) = 0 \end{array} \right\}.$$

We can deduce from (5.27) that $\exists (p_n, e_n) \in \mathcal{P}_n \times \mathcal{E}_n$ such that $\alpha_{n+1}^i(\beta) = p_n e_n$. Finally, we conclude that $\forall \beta \in \mathcal{Z}$, $\mathbb{Q}(\alpha_{n+1}^i(\beta) = \alpha_n^i(\beta)) = \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\mathbb{Q}} \left[\mathbb{1}_{[\alpha_{n+1}^i(\beta) = \alpha_n^i(\beta)]} | \mathcal{F}_{n-1} \right] \right] = 0$. ■

Remark 31. *The differentiability can be verified in the general case, where the volatility matrix σ is stochastic and verifies the conditions of Section 2.2. In that case if $\forall i \in \{1, \dots, d\}$, $\forall X \in \mathcal{P}_{n-1}$, $(Y_n^{-1})_i \cdot X \neq 0$ a.s. then \mathcal{R} is a.s. differentiable on \mathcal{Z} .*

Proposition 32. *The function*

$$(\beta, \theta) \longmapsto \mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)] = \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}(\beta) S_N^0 - \gamma \left((\mathcal{R}(\beta) - \theta) S_N^0 \right)^2 \right]$$

is differentiable on $\mathcal{Z} \times \mathbb{R}$.

Lemma 33. *There exist a \mathcal{F}_N -measurable random variable D and \mathbf{F} -adapted processes B^i, K^i for $i \in \{1, \dots, d\}$ and taking values in \mathbb{R}^m , such that $\forall i \in \{1, \dots, d\}$, $n \in \{1, \dots, N\}$, $\mathbb{E}_{\mathbb{P}}[|B_n^i|^p] + \mathbb{E}_{\mathbb{P}}[|K_n^i|^p] + \mathbb{E}_{\mathbb{P}}[|D|^p] < \infty$ for any $p \geq 1$ and*

$$\alpha_n^i(\beta) = B_n^i \cdot \beta; \quad \mathcal{R}(\beta) = v_0 + D \cdot \beta - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i \cdot \beta|. \quad (5.28)$$

Proof (of Lemma 33). According to the linearity in β of the control functions, defined in (5.22), there exists a \mathcal{F}_N -measurable random variable D with values in \mathbb{R}^m such that $\Pr(\mathcal{C}_K(\beta))_N = v_0 + D \cdot \beta$ and $\exists \{B^i, i \in \{1, \dots, d\}\}$, \mathbf{F} -adapted processes with values in \mathbb{R}^m such that $\forall i \in \{1, \dots, d\}$, $n \in \{0, \dots, N-1\}$, $\alpha_n^i(\beta) = B_n^i \cdot \beta$. By calling $K_n^i = B_{n+1}^i - B_n^i$, we obtain $\mathcal{C}_N(\beta^*) = \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i \cdot \beta|$.

$\forall i \in \{1, \dots, d\}$, $n \in \{0, \dots, N-1\}$, $j \in \{1, \dots, m\}$, $(B_n^i)_j \in \text{span} \left\{ H_\lambda^\otimes(\Delta \widehat{W}^{\mathbb{Q}}), \lambda \in \mathbb{R}^m \right\}$, $(K_n^i)_j \in \text{span} \left\{ H_\lambda^\otimes(\Delta \widehat{W}^{\mathbb{Q}}), \lambda \in \mathbb{R}^m \right\}$ so $\forall p \geq 1$, $\mathbb{E}_{\mathbb{P}}[|B_n^i|^p]$ and $\mathbb{E}_{\mathbb{P}}[|K_n^i|^p] < \infty$. And we have $\forall j \in \{1, \dots, m\}$, $D_j = \sum_{n=0}^{N-1} \sum_{i=1}^d (B_{n+1}^i)_j \Delta \tilde{S}_{n+1}^i$. $\tilde{S}_n \in L^p(\Omega, \mathcal{F}_n, \mathbb{P})$, then $\mathbb{E}_{\mathbb{P}}[|D_j|^p] < \infty$. ■

Proof (of Proposition 32). Let $r > 0$, $r_\theta > 0$ and $\beta \in \mathbb{R}^m$, $\theta \in \mathbb{R}$ such that $|\beta| \leq r$, $|\theta| \leq r_\theta$. According to Lemma 33, we have

$$|\mathcal{R}(\beta)| \leq V_0 + |D| |\beta| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i| |\beta|.$$

Since we have $\forall i \in \{1, \dots, d\}$, $n \in \{0, \dots, N-1\}$, $\forall p \geq 1$, $\mathbb{E}_{\mathbb{P}}[|K_n^i|^p] + \mathbb{E}_{\mathbb{P}}[|D|^p] < \infty$,

we have $\mathbb{E}_{\mathbb{P}} \left[\sup_{|\beta| \leq r} |\mathcal{R}(\beta)| \right] < \infty$ and $\mathbb{E}_{\mathbb{P}} \left[\sup_{|\beta| \leq r} |\mathcal{R}(\beta)|^2 \right] < \infty$. We bound

$$\begin{aligned} |F^\gamma(\beta, \theta)| &\leq |\mathcal{R}(\beta)| + 2\gamma|\mathcal{R}(\beta)|^2 + 2\gamma\theta^2 \\ &\leq \sup_{|\beta| \leq r} |\mathcal{R}(\beta)| + 2\gamma \sup_{|\beta| \leq r} |\mathcal{R}(\beta)|^2 + 2\gamma r_\theta^2. \end{aligned}$$

So $\mathbb{E}_{\mathbb{P}} [|F^\gamma(\beta, \theta)|] < \infty$. According to Proposition 30, \mathcal{R} is a.s. differentiable on \mathcal{Z} , then F is also a.s. differentiable on $\mathcal{Z} \times \mathbb{R}$. We also have

$$|\nabla \mathcal{R}(\beta)| = |D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \text{sign}(K_n^i \cdot \beta) K_n^i| \leq |D| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i|.$$

Following the same arguments, we prove that $\forall j \in \{1, \dots, m\}$

$$\begin{aligned} |\nabla_{\beta_j} F^\gamma(\beta, \theta)| &\leq |\nabla \mathcal{R}(\beta)|_j + 2\gamma |\nabla \mathcal{R}(\beta)|_j |\mathcal{R}(\beta) - \theta| \\ &\leq |D_j| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |(K_n^i)_j| + 2\gamma \left(|D_j| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |(K_n^i)_j| \right) \left(\sup_{|\beta| \leq r} |\mathcal{R}(\beta)| + r_\theta \right). \end{aligned}$$

We conclude using Cauchy Schwartz' inequality that $\mathbb{E}_{\mathbb{P}}[\sup_{|\beta| \leq r, |\theta| \leq r_\theta} |\nabla_{\beta_j} F^\gamma(\beta, \theta)|] < \infty$. Similarly, we bound

$$|\nabla_\theta F^\gamma(\beta, \theta)| \leq 2\gamma |\mathcal{R}(\beta) - \theta| \leq 2\gamma \sup_{|\beta| \leq r} |\mathcal{R}(\beta)| + r_\theta.$$

and apply Lebesgue's theorem to conclude that $(\beta, \theta) \rightarrow \mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)]$ is differentiable on $\mathcal{Z} \times \mathbb{R}$ and $\nabla(\mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)]) = \mathbb{E}_{\mathbb{P}}[\nabla F^\gamma(\beta, \theta)]$. \blacksquare

4.2 Finding locally optimal solutions

We have seen that $\mathbb{E}_{\mathbb{P}}[\nabla_\theta F^\gamma(\beta, \theta^*)] = 0 \iff \theta^* = \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]$. We can then deduce according to Proposition 32 that if $\beta^* \in \mathcal{Z}$ is the chaos expansion decomposition of an optimal portfolio then β^* is solution to

$$\mathbb{E}_{\mathbb{P}}[\nabla_{\beta, \theta} F^\gamma(\beta^*, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)])] = 0. \quad (T^\gamma)$$

Proposition 34. *The solutions to (T^γ) are locally optimal.*

Proof. \mathcal{R} is concave w.r.t. β . $F_y : (Y, \theta) \in \mathbb{R}^2 \rightarrow \mathbb{E}_{\mathbb{P}} \left[Y S_N^0 - \gamma ((Y - \theta) S_N^0)^2 \right]$ is concave. For $\theta \in \mathbb{R}$, if $\mathbb{E}_{\mathbb{P}}[F_y(\cdot, \theta)]$ is monotone increasing then $\mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)]$ is concave in β . $\mathbb{E}_{\mathbb{P}}[F_y(\cdot, \theta)]$ is increasing in Y if and only if $\mathbb{E}_{\mathbb{P}}[Y] \leq \frac{1}{2\gamma} + \theta$.

Let β^* be a solution to (T^γ) . \mathcal{R} is continuous then we can find $\varepsilon > 0$ such that $\forall \beta \in \mathbb{R}^m$

s.t. $|\beta - \beta^*| \leq \varepsilon$, we have $|\mathcal{R}(\beta) - \mathcal{R}(\beta^*)| \leq \frac{1}{4\gamma}$. Then $\forall(\beta, \theta)$ such that $|\beta - \beta^*| \leq \varepsilon$, $|\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]| \leq \frac{1}{4\gamma}$, we have

$$|\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]| \leq |\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]| + \mathbb{E}_{\mathbb{P}}[|\mathcal{R}(\beta^*) - \mathcal{R}(\beta)|] \leq \frac{1}{2\gamma}.$$

Then $\mathbb{E}_{\mathbb{P}}[F^\gamma(\beta, \theta)]$ is concave on $\{(\beta, \theta) : |\beta - \beta^*| \leq \varepsilon, |\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]| \leq \frac{1}{4\gamma}\}$. \blacksquare

Using the previous results, we can apply a gradient descent algorithm to find a local optimum of (J^γ) .

4.3 Influence of risk aversion

In this section, we discuss the influence of the risk aversion on optimal multi-period portfolios submitted to transaction costs. One main result claims that the Sharp ratio of a zeros gradient portfolio submitted to transaction costs is independent from its risk aversion. A second main result affirms that all optimal portfolios have the same Sharp ratio. This study insures that the choice of risk aversion parameter does not influence performances. In this section, we consider the following formulation of the allocation problem

$$\begin{aligned} & \sup_{\beta \in \mathcal{Z}} \mathbb{E}_{\mathbb{P}} [G^\gamma(\beta)] \\ \text{s.t.} \quad & \mathbb{E}_{\mathbb{Q}} [\mathcal{C}_K(\beta)] = v_0 \end{aligned} \tag{J^\gamma}$$

where $G^\gamma(\beta) = F^\gamma(\beta, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)])$. We have seen that (β^*, θ^*) solves (J^γ) if and only if $\theta^* = \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]$ and β^* solves (J^γ) .

Definition 35. A γ -optimal strategy is a strategy $(\alpha_n(\beta^*))_{0 \leq n \leq N}$ such that β^* solves (J^γ) . The portfolio associated to a γ -optimal strategy is called a γ -optimal portfolio.

Definition 36. A γ -zero gradient strategy or γ -locally optimal strategy is a strategy $(\alpha_n(\beta^*))_{0 \leq n \leq N}$ such that β^* solves (T^γ) . The portfolio associated to a γ -zero gradient strategy is called a γ -zero gradient portfolio.

Proposition 37. The risk aversion, the Sharpe ratio and the volatility of a γ -zero gradient portfolio V^* submitted to transaction costs are related by

$$\gamma = \frac{\text{Sharpe}(V_N^*)}{2\text{Var}[V_N^*]^{\frac{1}{2}}}. \tag{5.29}$$

Proof. We have seen in (T^γ) that if V^* is a γ -zero gradient portfolio, then $\exists \beta^* \in \mathcal{Z}$ such that $V_N^* = \mathcal{R}(\beta^*)S_N^0$ and $\mathbb{E}_{\mathbb{P}}[\Psi(\beta^*, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)])] = 0$. This equality leads to

$$\mathbb{E}_{\mathbb{P}}[\nabla \mathcal{R}(\beta^*) S_N^0 (1 - 2\gamma S_N^0 (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]))] = 0.$$

We recall that $\mathcal{R}(\beta^*) = \Pr(\mathcal{C}_K(\beta^*)) - \mathcal{C}_N(\beta^*)$. As \mathcal{R} is a.s differentiable and using the decomposition of Lemma 33,

$$\forall \beta \in \mathcal{Z}, \nabla \mathcal{R}(\beta) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \text{sign}(K_n^i \cdot \beta) K_n^i. \quad (5.30)$$

We deduce that

$$\forall \beta \in \mathcal{Z}, \beta \cdot \nabla \mathcal{R}(\beta) = \mathcal{R}(\beta) - v_0.$$

Then with

$$\beta^* \cdot \mathbb{E}_{\mathbb{P}} [\nabla \mathcal{R}(\beta^*) S_N^0 [1 - 2\gamma S_N^0 (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*)])]] = 0,$$

we obtain

$$\mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*) S_N^0 [1 - 2\gamma S_N^0 (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*)])]] = v_0 S_N^0.$$

We deduce that

$$\mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*) S_N^0] - 2\gamma \mathbb{E}_{\mathbb{P}} [(\mathcal{R}(\beta^*) S_N^0)^2] + 2\gamma \mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*) S_N^0] = v_0 S_N^0.$$

Finally,

$$\gamma = \frac{\text{Sharpe}(V_N^*)}{2\text{Var}[V_N^*]^{\frac{1}{2}}}. \quad (5.31) \quad \blacksquare$$

Corollary 38. *All γ -optimal portfolios have the same Sharpe ratio.*

Proof. Let V_N^* be such a γ -optimal portfolio. As an optimal portfolio, V_N^* is also a γ -zero gradient portfolio. Then according to Proposition 37,

$$\mathbb{E}_{\mathbb{P}} [V_N^*] - v_0 S_N^0 = 2\gamma \text{Var} [V_N^*]$$

Inserting this term in the mean-variance objective function leads to

$$\mathbb{E}_{\mathbb{P}} [V_N^* - \gamma (V_N^* - \mathbb{E}_{\mathbb{P}}[V_N^*])^2] = \frac{\mathbb{E}_{\mathbb{P}} [V_N^*]}{2} - \frac{v_0 S_N^0}{2}. \quad (5.32)$$

We deduce that if V'_N is another γ -optimal portfolio then (5.32) gives $\mathbb{E}_{\mathbb{P}}[V_N^*] = \mathbb{E}_{\mathbb{P}}[V'_N]$. But the two portfolios also attain the same mean-variance value then

$$\mathbb{E}_{\mathbb{P}} [V_N^*] - \gamma \text{Var} [V_N^*] = \mathbb{E}_{\mathbb{P}} [V'_N] - \gamma \text{Var} [V'_N].$$

With this last equality, we can conclude they have the same variance and as a result, the same Sharpe ratio. \blacksquare

Remark 39. *Maximizing the mean-variance objective function of zero gradient portfolios, is equivalent to maximizing their Sharpe ratio.*

Now, let's prove this important proposition on the impact of the risk aversion.

Proposition 40. *Let $(\gamma, \gamma') \in \mathbb{R}^*$, then $(\alpha_n^*)_n$ is a γ -zero gradient strategy if and only if $(\frac{\gamma}{\gamma'}\alpha_n^*)_n$ is a γ' -zero gradient strategy.*

Proof. Let $(\alpha_n^*)_n$ be a γ -zero gradient strategy with the associated chaos coefficients β^* . Then β^* is solution to (T^γ) . Then with the previous notations

$$\mathbb{E}_{\mathbb{P}} [\nabla \mathcal{R}(\beta^*) S_N^0 [1 - 2\gamma S_N^0 (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*)])]] = 0. \quad (5.33)$$

By using the decomposition of Lemma 33, \mathcal{R} is a.s differentiable on \mathcal{Z} and

$$\nabla \mathcal{R}(\beta) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \text{sign}(K_n^i \cdot \beta) K_n^i. \quad (5.34)$$

Let's now consider the portfolio

$$\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) = v_0 + \frac{\gamma}{\gamma'} D \cdot \beta^* - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i \cdot \frac{\gamma}{\gamma'}\beta^*|.$$

We obtain the following equality

$$\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) - \frac{\gamma}{\gamma'} \mathcal{R}(\beta^*) = v_0 \left(1 - \frac{\gamma}{\gamma'}\right).$$

We deduce that

$$\frac{\gamma}{\gamma'} (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}} [\mathcal{R}(\beta^*)]) = \mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) - \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) \right]. \quad (5.35)$$

By using the expression (5.34), we also notice that

$$\nabla \mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \text{sign}\left(K_n^i \cdot \frac{\gamma}{\gamma'}\beta^*\right) K_n^i = \nabla \mathcal{R}(\beta^*). \quad (5.36)$$

Finally, by inserting (5.35) and (5.36) in equality (5.33), we obtain

$$\mathbb{E}_{\mathbb{P}} \left[\nabla \mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) S_N^0 \left[1 - 2\gamma' S_N^0 \left(\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) - \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^*\right) \right] \right) \right] \right] = 0, \quad (5.37)$$

which finishes the proof. ■

Corollary 41. *Let $(\gamma, \gamma') \in \mathbb{R}^*$, then $(\alpha_n^*)_n$ is a γ -optimal strategy if and only if $(\frac{\gamma}{\gamma'}\alpha_n^*)_n$ is a γ' -optimal strategy.*

Proof. Let $(\alpha_n^*)_n$ be a γ -optimal strategy with the associated chaos coefficients β^* . Then β^* is a solution to (\mathcal{J}^γ) . Firstly, according to Proposition 40, $\frac{\gamma}{\gamma'}\beta^*$ is a solution to $(T^{\gamma'})$ with

$$\mathbb{E}_{\mathbb{P}}[G^{\gamma'}(\frac{\gamma}{\gamma'}\beta^*)] = \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}(\frac{\gamma}{\gamma'}\beta^*)S_N^0 - \gamma' \left(\left(\mathcal{R}(\frac{\gamma}{\gamma'}\beta^*) - \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}(\frac{\gamma}{\gamma'}\beta^*) \right] \right) S_N^0 \right)^2 \right].$$

But as $\beta \rightarrow \mathcal{R}(\beta) - v_0$ is a positive homogeneous function, ie $\forall u > 0$, $\mathcal{R}(u\beta) - v_0 = u(\mathcal{R}(\beta) - v_0)$, then

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[G^{\gamma'}(\frac{\gamma}{\gamma'}\beta^*)] - v_0 S_N^0 &= \mathbb{E}_{\mathbb{P}} \left[\frac{\gamma}{\gamma'} (\mathcal{R}(\beta^*) - v_0) S_N^0 - \gamma' \left(\frac{\gamma}{\gamma'} (\mathcal{R}(\beta^*) - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^*)]) S_N^0 \right)^2 \right] \\ &= \frac{\gamma}{\gamma'} (\mathbb{E}_{\mathbb{P}}[G^\gamma(\beta^*)] - v_0 S_N^0). \end{aligned}$$

If $\exists \beta^1 \in \mathbb{R}^m$ such that $\mathbb{E}_{\mathbb{P}}[G^{\gamma'}(\frac{\gamma}{\gamma'}\beta^*)] < \mathbb{E}_{\mathbb{P}}[G^{\gamma'}(\beta^1)]$ then using the previous equality $\mathbb{E}_{\mathbb{P}}[G^\gamma(\beta^*)] < \mathbb{E}_{\mathbb{P}}[G^\gamma(\beta^1)]$, which is impossible with the optimality of β^* . We deduce that $\frac{\gamma}{\gamma'}\beta^*$ is a solution to $(\mathcal{J}^{\gamma'})$. The converse statement is obvious.

We have seen in (5.22), that the controls $(\alpha_n^*)_n$ can be expressed as linear functions of β^* . We can conclude that $(\alpha_n^*)_n$ is an optimal solution for (E^γ) if and only if $(\frac{\gamma}{\gamma'}\alpha_n^*)_n$ is an optimal solution for $(E^{\gamma'})$. ■

Proposition 42. *Let $(\alpha_n^*)_n$ be a γ -zero gradient strategy and $(V_n^*)_n$ the associated γ -zero gradient portfolio. Then for all $u > 0$, $\text{Sharpe}(V_N^*) = \text{Sharpe}(V_N^u)$, where $(V_n^u)_n$ is the portfolio associated to the strategy $(u\alpha_n^*)_n$.*

Proof. By using that $\beta \rightarrow \mathcal{R}(\beta) - v_0$ is a positive homogeneous function, we have

$$\text{Sharpe}(V_N^*) = \frac{\mathbb{E}_{\mathbb{P}} \left[\tilde{V}_N^* S_N^0 \right] - v_0 S_N^0}{\text{Var} \left[\tilde{V}_N^* S_N^0 \right]^{\frac{1}{2}}} = \frac{u \mathbb{E}_{\mathbb{P}} \left[\mathcal{R}(\beta^*) S_N^0 \right]}{u \text{Var} \left[\mathcal{R}(\beta^*) S_N^0 \right]^{\frac{1}{2}}} = \text{Sharpe}(V_N^u),$$

that finishes the proof. ■

Proposition 43. *The Sharpe ratio of a zeros gradient portfolio is independent of its risk aversion.*

Proof. A direct consequence of Propositions 40 and 42. ■

Proposition 44. *Let $\mathcal{V} > 0$ and $(V_n^*)_n$ be a γ -zero gradient portfolio. Then there exists γ' and a γ' -zeros gradient portfolio V' with $\text{Sharpe}(V_N') = \text{Sharpe}(V_N^*)$ and with volatility \mathcal{V} .*

Proof. According to Propositions 40 and 42, for all $u > 0$, $(u\alpha_n^*)_n$ is a $\frac{\gamma}{u}$ -zeros gradient strategy and the associated $\frac{\gamma}{u}$ -zeros gradient portfolio V_N^u has the same Sharpe ratio as V_N^* . Consequently, according to Proposition 37,

$$\frac{\gamma}{u} = \frac{\text{Sharpe}(V_N^*)}{2\text{Var}(V_N^u)^{\frac{1}{2}}}.$$

Then by choosing $u = \frac{2\gamma\mathcal{V}}{\text{Sharpe}(V_N^*)}$, the portfolio $(u\alpha_n^*)_n$ verifies the conditions. \blacksquare

Proposition 45. *Let $\gamma, \gamma' > 0$ s.t. $\gamma \neq \gamma'$. Let V^* (resp. V') be a γ -optimal portfolio (resp. γ' -optimal portfolio). Then, $\text{Sharpe}(V_N^*) = \text{Sharpe}(V_N')$.*

Proof. Let $(\alpha'_n)_n$ be the γ' -optimal strategy associated to V' . Let $V^\#$ the portfolio associated to the strategy $(\frac{\gamma'}{\gamma}\alpha'_n)_n$. According to Proposition 41, $V^\#$ is a γ -optimal portfolio. Furthermore, using Proposition 42, we have $\text{Sharpe}(V_N') = \text{Sharpe}(V_N^\#)$. According to Proposition 38, γ -optimal portfolios have the same Sharpe ratio. So $\text{Sharpe}(V_N^*) = \text{Sharpe}(V_N^\#)$, and finally $\text{Sharpe}(V_N^*) = \text{Sharpe}(V_N')$. \blacksquare

4.4 Use of a finer grid for chaos expansion

In this section we assume that N is a multiple of $p \in \mathbb{N}$. Let us consider two time grids, $\mathcal{T} = \{t_0, \dots, t_N\}$ and $\tilde{\mathcal{T}} = \{t_0, t_p, \dots, t_{kp}, \dots, t_N\}$. Transactions are only possible at time in $\tilde{\mathcal{T}}$, every p date time in \mathcal{T} but processes are observable in between. We define two associated filtrations

$$G_n = \sigma\{W_k - W_{k-1}, 1 \leq k \leq n\}, \mathcal{F}_n = \sigma\{W_{kp} - W_{(k-1)p}, 1 \leq k \leq \lfloor \frac{n}{p} \rfloor\}.$$

We rewrite the problem (E^γ) according to the finest grid.

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_{kp}, \dots, \alpha_N} \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \mathbb{E}_{\mathbb{P}}[\tilde{X}_N - \mathcal{C}_N]) S_N^0 \right)^2 \right] \\ \text{subject to} \quad & \tilde{X}_0 = V_0, (\alpha_n)_n \mathbf{G} - \text{Pred} \\ & \tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \\ & \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu |\alpha_{\lfloor \frac{n+1}{p} \rfloor p}^i - \alpha_{\lfloor \frac{n}{p} \rfloor p}^i| \tilde{S}_n^i \end{aligned} \tag{U^\gamma}$$

We call $(\alpha_1^*, \dots, \alpha_{kp}^*, \dots, \alpha_N^*)$ a solution of (U^γ). Since the transactions are only available at time in $\tilde{\mathcal{T}}$, we want to know whether representing processes with chaos on grid $\tilde{\mathcal{T}}$ is a loss of information or not.

We assume here that the assets follows the dynamics presented in Section 2.2 with constant

drift parameters and volatility matrix, ie. $\forall n \in \{0, \dots, N\}$, $\sigma_n = \sigma$, $r_n = r$, $\mu_n = \mu$. Under this assumption and according to the following proposition, the response is no.

Proposition 46. α_n^* is \mathcal{F}_{n-1} measurable.

Proof. Let us define for all $p \leq n \leq N$, the diagonal matrices $\tilde{R}d_n = \text{Diag} \left(\frac{\tilde{S}_n^i}{\tilde{S}_{n-p}^i} \right)_{1 \leq i \leq d}$.

First we can prove by recurrence that for all $k \leq \lfloor \frac{N}{p} \rfloor$, \tilde{S}_{kp} is \mathcal{F}_{kp} measurable.

\mathcal{H}_k : \tilde{S}_{kp} is \mathcal{F}_{kp} measurable.

\mathcal{H}_0 is true since \tilde{S}_0 is deterministic and therefore $G_0 = \mathcal{F}_0$ -measurable.

Let us assume $\mathcal{H}_1, \dots, \mathcal{H}_k$ true for $k > 1$. Then $\forall 1 \leq i \leq d$, $\tilde{S}_{(k+1)p}^i = \tilde{S}_{kp}^i e^{(\sigma^i) \cdot \Delta W_{(k+1)p}^Q}$.

\tilde{S}_{kp}^i is \mathcal{F}_{kp} measurable, so $\tilde{S}_{(k+1)p}^i$ is $\mathcal{F}_{(k+1)p}$ measurable and \mathcal{H}_{k+1} is true.

Now, Let us prove by recurrence the proposition $\forall n \geq 1$, \mathcal{H}_n : α_n^* is \mathcal{F}_{n-1} measurable.

\mathcal{H}_1 is true because α_1 is deterministic and therefore $G_0 = \mathcal{F}_0$ measurable. Lets now assume $\mathcal{H}_2, \dots, \mathcal{H}_n$ true for $n > 1$.

If $n \not\equiv 0 [p]$ then $\alpha_n^* = \alpha_{n+1}^*$. We deduce that α_{n+1}^* is \mathcal{F}_{n-1} measurable but $\mathcal{F}_{n-1} = \mathcal{F}_n$ so α_{n+1}^* is \mathcal{F}_n measurable.

If $n \equiv 0 [p]$, $\exists k \leq \lfloor \frac{N}{p} \rfloor - 1$ such that $n = kp$. Its has been proven in Proposition 25 that the problem (\mathcal{U}^γ) is equivalent to the following

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_{kp}, \dots, \alpha_N, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \theta) S_N^0 \right)^2 \right] \\ & \text{subject to} \quad \tilde{X}_0 = V_0, (\alpha_n)_n \mathbf{G} - \text{Pred} \\ & \quad \tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \\ & \quad \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu |\alpha_{\lfloor \frac{n+1}{p} \rfloor p}^i - \alpha_{\lfloor \frac{n}{p} \rfloor p}^i| \tilde{S}_n^i \end{aligned} \quad (\mathcal{U}^\gamma)$$

For $\theta \in \mathbb{R}$, we consider the analog problem

$$\begin{aligned} & \sup_{\alpha_1, \dots, \alpha_{kp}, \dots, \alpha_N} \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \theta) S_N^0 \right)^2 \right] \\ & \text{subject to} \quad \tilde{X}_0 = V_0, (\alpha_n)_n \mathbf{G} - \text{Pred} \\ & \quad \tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \\ & \quad \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu |\alpha_{\lfloor \frac{n+1}{p} \rfloor p}^i - \alpha_{\lfloor \frac{n}{p} \rfloor p}^i| \tilde{S}_n^i \end{aligned} \quad (\mathcal{U}_\theta^\gamma)$$

This stochastic control problem can be rewritten with value functions b , such that $\alpha_{(k+1)p}^*$ is solution of

$$b_{kp}(\tilde{X}_{kp}, \tilde{S}_{kp}, \mathcal{C}_{kp}, \alpha_{kp}, \theta) = \sup_{\alpha_{(k+1)p} \in \mathbb{R}} q_{kp}(\tilde{X}_{kp}, \tilde{S}_{kp}, \mathcal{C}_{kp}, \alpha_{(k+1)p}, \alpha_{kp}, \theta), \text{ with}$$

$$\left\{ \begin{array}{l} q_N(x, s, c, \alpha, \alpha', \theta) = (x - c)S_N^0 - \gamma((x - c - \theta)S_N^0)^2, \\ q_{N-p}(x, s, c, \alpha, \alpha', \theta) = \mathbb{E}_{\mathbb{P}} \left[b_N(x + \alpha \cdot (\tilde{R}d_N - Id)s, \tilde{R}d_N s, c + \nu|\alpha - \alpha'| \cdot s, \alpha, \theta) \right], \\ q_l(x, s, c, \alpha, \alpha', \theta) = \mathbb{E}_{\mathbb{P}} \left[b_{(l+1)p}(x + \alpha \cdot (\tilde{R}d_{(l+1)p} - Id)s, \tilde{R}d_{(l+1)p} s, c + \nu|\alpha - \alpha'| \cdot s, \alpha, \theta) \right], \\ \text{For } l < \lfloor \frac{N}{p} \rfloor. \end{array} \right.$$

With these three possible forms, and using that $\tilde{R}d_{(k+1)p}$ is independent from G_{kp} , we can say that $\alpha_{(k+1)p}^*(\theta)$ is a deterministic function of $(\tilde{S}_{kp}, \tilde{X}_{kp}, \alpha_{kp}, \theta)$. We also notice that $\tilde{X}_{kp} = V_0 + \sum_{i=0}^{k-1} \alpha_{(i+1)p}(\tilde{S}_{(i+1)p} - \tilde{S}_{ip})$. Then we can deduce that $\alpha_{(k+1)p}^*(\theta)$ is a deterministic function of $(\tilde{S}_{kp}, \tilde{S}_{(k-1)p}, \dots, S_0, \alpha_{kp}, \alpha_{(k-1)p}, \dots, \alpha_0)$. By applying Lemma 25, $\exists \theta^* \in \mathbb{R}$ such that $(\alpha_1^*(\theta^*), \dots, \alpha_N^*(\theta^*))$ is solution of the main problem (U^γ) . Equivalently, one can say that $\exists \theta^*$ such that $(\alpha_1^*(\theta^*), \dots, \alpha_{kp}^*(\theta^*), \dots, \alpha_N^*(\theta^*))$, a solution of $(\mathcal{U}_{\theta^*}^\gamma)$, is equal to $(\alpha_1^*, \dots, \alpha_{kp}^*, \dots, \alpha_N^*)$, solution of (U^γ) . θ^* is deterministic then $\alpha_{(k+1)p}^*(\theta^*)$ is a deterministic function of

$(\tilde{S}_{kp}, \tilde{S}_{(k-1)p}, \dots, S_0, \alpha_{(k-1)p}(\theta^*), \alpha_{(k-2)p}(\theta^*), \dots, \alpha_0)$. By applying the assumption of recurrence, we can conclude that $\alpha_{(k+1)p}^*(\theta^*) = \alpha_{(k+1)p}^* = \alpha_{kp+1}^*$ is \mathcal{F}_{kp} measurable. But $\alpha_{kp+1}^* = \alpha_{n+1}^*$ and $\mathcal{F}_{kp} = \mathcal{F}_n$ so H_{n+1} is true. The proposition is then true. \blacksquare

5 Benchmark models

Here, we present different approaches used to benchmark and challenge the multi-period one, described in Section 3. A first element of comparison is the multi-period approach itself but which ignores transaction costs. A second comparison is done with the sequential uni-period Markowitz framework. We present two versions of this approach, when cost are ignored and considered. We also study the link of risk aversion on optimal solutions for those benchmark models.

5.1 Benchmark: Multi-period allocation ignoring transaction costs

A natural and simplified approach consists in applying the framework described in Section 3, while ignoring costs. Costs are removed from the portfolio value but do not impact the strategy. Theoretically, we look for an optimal strategy whose costs are refunded. The control $(\alpha_n^*)_n$, are solution to (E^γ) by temporally setting $\nu = 0$ during the resolution. Formally, we solve

$$\begin{array}{ll} \sup_{(\alpha_n)_{n \in \{1, N\}}} & \mathbb{E}_{\mathbb{P}} \left[U(\tilde{X}_N S_N^0) \right] \\ \text{s.t.} & \tilde{X}_0 = V_0, (\alpha_n)_n \mathbf{F} - \text{Pred} \\ & \tilde{X}_{n+1} = \tilde{X}_n + \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} \end{array} \quad (E_0^\gamma)$$

Then, the real quantity in risk free asset at stake is deduced with the auto-financing relation and by removing costs generated by the real value of ν . Formally, after maximizing (E_0^γ), the generated cost \mathcal{C}_N is computed and removed to obtain the real portfolio value $\tilde{V}_N = \tilde{X}_N - \mathcal{C}_N$. This strategy is implemented in order to observe the effect of costs on optimal strategies.

5.2 Benchmark: Sequential Uni-period Markowitz portfolio allocation

In this section, we present an alternative approach, intended to serve as a benchmark. We would like to compare the performance of our method to a method traditionally applied by asset managers. We propose to implement the sequential uni-period mean–variance Markowitz framework. This approach also corresponds to the one-time-step Model Predictive Control (MPC) described in Li et al. (2022). In order to maximise the rate of return of a portfolio while controlling its volatility at time T , an agent consecutively maximizes at time $0 = t_0 < \dots < t_N = T$, its uni-period mean-var objective function. We assume that the agent has an uni-period risk aversion parameter γ_u . The agent has to consecutively solve for $n \in \{0, \dots, N - 1\}$,

$$\begin{aligned} & \sup_{\alpha_{n+1} \in \mathbb{R}^d, \alpha_{n+1}^0 \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [V_{n+1} | \mathcal{F}_n] - \gamma_u \text{Var}_{\mathbb{P}} [V_{n+1} | \mathcal{F}_n] \\ \text{s.t.} \quad & \Delta V_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i \end{aligned} \quad (\mathcal{Y}_n^{\gamma_u})$$

We present two versions of this model. A naive version, where the asset manager pays the cost but does not take the amount into account in its strategy (by ignoring them), is presented in Section 5.2.1. In a second version, described in Section 5.2.2, the investor consecutively solves ($\mathcal{Y}_n^{\gamma_u}$) and bases his strategy according to the cost he will have to pay.

5.2.1 Ignoring costs

We aim to propose here, a simplified version of the initial sequential uni-period problem ($\mathcal{Y}_n^{\gamma_u}$). The underlying idea is the same as the multi-period version described in Section 5.1. At each time step, the agent maximizes its mean-var utility function as though costs are refunded. Costs do not impact the strategy and are just removed from the portfolio value at the end. The agent has to consecutively solve for $n \in \{0, \dots, N - 1\}$,

$$\begin{aligned} & \sup_{\alpha_{n+1} \in \mathbb{R}^d, \alpha_{n+1}^0 \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [X_{n+1} | \mathcal{F}_n] - \gamma_u \text{Var}_{\mathbb{P}} [X_{n+1} | \mathcal{F}_n] \\ \text{s.t.} \quad & \Delta X_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 \end{aligned} \quad (\mathcal{Y}_n^{\gamma_u})$$

Let $(\alpha_{n+1}^*, \alpha_{n+1}^{0*})$ be a solution and (X_n^*) , the associated portfolio. The real portfolio is computed after removing the generated costs such that $V_{n+1}^* = X_{n+1}^* - \mathcal{C}_{n+1} S_{n+1}^0$. According to the dynamics of the assets, we have for $n \in \{0, \dots, N-1\}$,

$$\begin{cases} \mathbb{E}_{\mathbb{P}} [S_{n+1}^i | \mathcal{F}_n] = S_n^i e^{\mu_n^i (t_{n+1} - t_n)}, \quad \forall 1 \leq i \leq d, \\ \text{Cov} [S_{n+1}^i, S_{n+1}^j | \mathcal{F}_n] = S_n^i S_n^j e^{(\mu_n^i + \mu_n^j)(t_{n+1} - t_n)} (e^{(\sigma_n^i)^T \sigma_n^j (t_{n+1} - t_n)} - 1), \quad \forall 1 \leq i, j \leq d. \end{cases}$$

By calling $A_n = \text{Diag}((e^{\mu_n^i (t_{n+1} - t_n)})_{1 \leq i \leq d})$ and $B_n = [e^{(\mu_n^i + \mu_n^j)(t_{n+1} - t_n)} (e^{(\sigma_n^i)^T \sigma_n^j (t_{n+1} - t_n)} - 1)]_{ij}$, We have

$$\begin{aligned} \alpha_{n+1}^* &= \arg \sup_{\alpha_{n+1} \in \mathbb{R}^d} \alpha_{n+1}^T A_n S_n + (X_n - \alpha_{n+1}^T S_n) \frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha_{n+1}^T S_n^T B_n S_n \alpha_{n+1} \\ \Leftrightarrow A_n S_n - S_n \frac{S_{n+1}^0}{S_n^0} - 2\gamma_u (S_n^T B_n S_n) \alpha_{n+1}^* &= 0 \quad \Leftrightarrow \alpha_{n+1}^* = \frac{A_n S_n - S_n e^{r_n (t_{n+1} - t_n)}}{2\gamma_u S_n^T B_n S_n}. \end{aligned} \quad (5.38)$$

Remark 47. *It is equivalent to remove costs at each time step or to remove them at the end because the quantity in the risk free asset does not impact the strategy.*

Comparing the multi-period strategy with this framework is a difficult task. The risk aversion parameters γ and γ_u in the two models, do not refer to the same risk aversion for the agent. We need to find a correct matching, or a measure of performance, independent from risk aversion. Obviously, this measure is the Sharpe ratio.

Proposition 48. *The Sharpe ratio of the sequential uni-period Markowitz strategy which ignores costs, does not depend on the risk aversion parameter γ_u .*

Proof. We prove by induction $\forall 0 \leq n \leq N$, $\mathcal{H}_n : \exists$ a \mathcal{F}_n -measurable random variable \mathcal{X}_n not function of γ_u such that $V_n^* = V_0 S_n^0 + \frac{\mathcal{X}_n}{2\gamma_u}$.

\mathcal{H}_0 is true with $\mathcal{X}_0 = 0$. We assume \mathcal{H}_n true for $n > 0$. We have

$$\begin{aligned} \tilde{V}_{n+1}^* &= \tilde{V}_n^* + \alpha_{n+1}^* \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^{i*} - \alpha_n^{i*}| S_n^i}{S_n^0} \\ &= \frac{1}{S_n^0} \left(V_0 S_n^0 + \frac{\mathcal{X}_n}{2\gamma_u} \right) + (\alpha_{n+1}^*)^T \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^{i*} - \alpha_n^{i*}| S_n^i}{S_n^0}. \end{aligned}$$

By using that $\tilde{R}d_n = \text{Diag}(\frac{\tilde{S}_n^i}{\tilde{S}_{n-1}^i})$, and the form of the solution in (5.38), we rewrite

$$\begin{aligned} (\alpha_{n+1}^*)^T \Delta \tilde{S}_{n+1} &= (\alpha_{n+1}^*)^T (\tilde{R}d_{n+1} - Id) \tilde{S}_n = \tilde{S}_n^T (\tilde{R}d_{n+1} - Id) \alpha_{n+1}^* \\ &= \frac{1}{S_n^0} \frac{S_n^T (\tilde{R}d_{n+1} - Id) (A_n S_n - S_n e^{r_n (t_{n+1} - t_n)})}{2\gamma_u S_n^T B_n S_n}. \end{aligned}$$

We also have

$$\sum_{i=1}^d \nu \left(\frac{|\alpha_{n+1}^{i*} - \alpha_n^{i*}| S_n^i}{S_n^0} \right) = \sum_{i=1}^d \nu \left| \frac{A_n^i S_n^i - S_n^i e^{r_n(t_{n+1}-t_n)}}{2\gamma_u S_n^T B_n S_n} - \frac{A_{n-1}^i S_{n-1}^i - S_{n-1}^i e^{r_{n-1}(t_n-t_{n-1})}}{2\gamma_u S_{n-1}^T B_{n-1} S_{n-1}} \right| \frac{S_n^i}{S_n^0}.$$

We prove \mathcal{H}_{n+1} by denoting

$$\begin{aligned} \mathcal{X}_{n+1} &= \frac{S_{n+1}^0}{S_n^0} \left(\mathcal{X}_n + \frac{S_n^T (Rd_{n+1} - Id)(A_n S_n - S_n e^{r_n(t_{n+1}-t_n)})}{S_n^T B_n S_n} \right) \\ &\quad - \frac{S_{n+1}^0}{S_n^0} \sum_{i=1}^d \nu \left| \frac{A_n^i S_n^i - S_n^i e^{r_n(t_{n+1}-t_n)}}{S_n^T B_n S_n} - \frac{A_{n-1}^i S_{n-1}^i - S_{n-1}^i e^{r_{n-1}(t_n-t_{n-1})}}{S_{n-1}^T B_{n-1} S_{n-1}} \right| S_n^i. \end{aligned}$$

Using this form we deduce that $\mathbb{E}_{\mathbb{P}}[V_N^*] - V_0 S_N^0 = \frac{\mathbb{E}_{\mathbb{P}}[\mathcal{X}_N]}{2\gamma_u}$, and $\text{Var}[V_N^*] = \frac{\text{var}[\mathcal{X}_N]}{4\gamma_u^2}$. Finally

$$\text{Sharpe}[V_N^*] = \frac{\frac{\mathbb{E}_{\mathbb{P}}[V_N^*] - V_0}{V_0} - \frac{V_0 S_N^0 - V_0}{V_0}}{\text{Var}\left[\frac{V_N^* - V_0}{V_0}\right]^{\frac{1}{2}}} = \frac{\mathbb{E}_{\mathbb{P}}[\mathcal{X}_N]}{\text{Var}[\mathcal{X}_N]^{\frac{1}{2}}},$$

which does not depend on γ_u . ■

5.2.2 Taking costs into account

In this version, costs are taken into account in the objective function but the strategy remains myopic. We recall that the agent has to consecutively solve $(Y_n^{\gamma_u})$. In the presence of costs, we do not provide an explicit solution of $(Y_n^{\gamma_u})$.

In order to use this model as a benchmark, let us prove an analog result to Proposition 48.

Proposition 49. *The Sharpe ratio of the sequential uni-period Markowitz strategy considering costs, does not depend on the risk aversion parameter γ_u .*

Proof. With the notation of the previous section, $(Y_n^{\gamma_u})$ can be rewritten as

$$\begin{aligned} &\sup_{\alpha_{n+1} \in \mathbb{R}^d} \alpha_{n+1}^T A_n S_n + \left(V_n - \alpha_{n+1}^T S_n - \sum_{i=1}^d -\nu(|\alpha_{n+1}^i - \alpha_n^i| S_n^i) \right) \frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha_{n+1}^T S_n^T B_n S_n \alpha_{n+1} \\ &\text{subject to } V_{n+1} = \alpha_{n+1} S_{n+1} + \alpha_{n+1}^0 S_{n+1}^0 \end{aligned}$$

Let us consider the objective functions

$$\forall n \leq N-1, f_n(\alpha) = \alpha^T A_n S_n + \left(V_n - \alpha^T S_n - \sum_{i=1}^d -\nu(|\alpha^i - \alpha_n^i| S_n^i) \right) \frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha^T S_n^T B_n S_n \alpha.$$

f_n is strictly concave and $\lim_{|\alpha| \rightarrow +\infty} f_n(\alpha) = -\infty$, then $(Y_n^{\gamma_u})$ has an unique solution. We call α_{n+1}^* the solution of $(Y_n^{\gamma_u})$. f_n is differentiable on $\mathbb{R}^d \setminus \mathcal{O}_n$, where $\mathcal{O}_n = \{\alpha \in \mathbb{R}^d, \exists i \in \{1, \dots, d\} \text{ such that } \alpha^i = \alpha_n^i\}$. f_n admits a sub-differential ∂f_n , at any point $\alpha \in \mathbb{R}^d$

$$\partial f_n(\alpha) = \left\{ \begin{array}{l} A_n S_n - 2\gamma_u (S_n^T B_n S_n) \alpha - \text{Diag}(1 + \nu \epsilon) S_n \frac{S_{n+1}^0}{S_n^0}, \text{ with} \\ \epsilon^i = \text{sign}(\alpha^i - \alpha_n^i) \text{ if } \alpha^i \neq \alpha_n^i, \epsilon^i \in [-1, 1] \text{ otherwise} \end{array} \right\}.$$

We have $0 \in \partial f_n(\alpha_{n+1}^*)$. Let us show by induction $\forall 0 \leq n \leq N$,

$\mathcal{H}_n : \exists$ a \mathcal{F}_n -measurable random variable \mathcal{X}_n

not function of γ_u such that $\alpha_n^* = \frac{\mathcal{X}_n}{2\gamma_u}$.

\mathcal{H}_0 is true with $\mathcal{X}_0 = 0$. We assume \mathcal{H}_n true for $n > 1$. If $\exists i \in \{1, \dots, d\}$ such that $\alpha_{n+1}^{i*} = \alpha_n^{i*}$ then it is sufficient to choose $\mathcal{X}_{n+1}^i = \mathcal{X}_n^i$. If $\alpha_{n+1}^{i*} \neq \alpha_n^{i*}$, then

$$\alpha_{n+1}^{i*} = \frac{A_n^{ii} S_n^i - (1 - \nu \text{sign}(\alpha_{n+1}^{i*} - \alpha_n^{i*})) S_n^i e^r}{2\gamma_u S_n^T B_n S_n}.$$

Let us define \mathcal{Z} a \mathcal{F}_n -measurable random variable such that $\alpha_{n+1}^i = \frac{\mathcal{Z}}{2\gamma_u}$. Then with the form of α_n^{i*} , we have

$$\mathcal{Z} = \frac{A_n^{ii} S_n^i - (1 - \nu \text{sign}(\mathcal{Z} - \mathcal{X}_n^i)) S_n^i e^r}{S_n^T B_n S_n}.$$

We can deduce that \mathcal{Z} is not a function of γ_u . Therefore \mathcal{Z} is the chosen candidate to be \mathcal{X}_{n+1} . \mathcal{H}_{n+1} is then true, and \mathcal{H}_n true for all $n \geq 0$. Then it is easy to check that

$$\begin{aligned} \text{Sharpe}(V_N^*) &= \frac{\mathbb{E}_{\mathbb{P}} \left[S_N^0 \left(V_0 + \sum_{n=0}^N (\alpha_{n+1}^*)^T \tilde{S}_n - \sum_{i=1}^d \nu |\alpha_{n+1}^{i*} - \alpha_n^{i*}| \tilde{S}_n^i \right) \right] - V_0 S_N^0}{\text{Var} \left[S_N^0 \left(V_0 + \sum_{n=0}^N (\alpha_{n+1}^*)^T \tilde{S}_n - \sum_{i=1}^d \nu |\alpha_{n+1}^{i*} - \alpha_n^{i*}| \tilde{S}_n^i \right) \right]^{\frac{1}{2}}} \\ &= \frac{\mathbb{E}_{\mathbb{P}} \left[\left(\sum_{n=0}^N (\mathcal{X}_{n+1})^T \tilde{S}_n - \sum_{i=1}^d \nu |\mathcal{X}_{n+1}^i - \mathcal{X}_n^i| \tilde{S}_n^i \right) \right]}{\text{Var} \left[\left(\sum_{n=0}^N \mathcal{X}_{n+1}^T \tilde{S}_n - \sum_{i=1}^d \nu |\mathcal{X}_{n+1}^i - \mathcal{X}_n^i| \tilde{S}_n^i \right) \right]^{\frac{1}{2}}}, \end{aligned}$$

which is not a function of γ_u . ■

6 Numerical illustration

In this section, we implement the approach presented in Section 3 and the sequential uni-period Markowitz model which is used as benchmark, described in Section 5.2. We consider the cases where costs are ignored and considered. Further details on the implemented benchmark model and its link with risk aversion are presented in Section 5.2.

We assume that the risky assets follow the dynamics (5.1) with constant parameters. For-

mally, the drift terms, the volatility matrices and the risk free rate are chosen deterministic and constant, ie. $\forall n \in \{0, \dots, N\} \mu_n = \mu, \sigma_n = \sigma, r_n = r$. Along a first part, we compare the performances by choosing the same risk aversion parameter $\gamma = \gamma_u$. As the Sharpe ratios of the estimated portfolios is independent of the aversion parameter (see Propositions 43, 45, 48, 49), it can be used as an indicator of performance. Then, along a second part, we apply a framework for matching the risk aversion between uni-period and multi-period models. We are able to illustrate and compare the behaviour of our solutions on two realisations.

6.1 Model parameters

We consider $d = 3$ assets, evolving during $N = 368$ days. Transactions are only available every 92 days. The model is described in Section 2.2. Instead of specifying a volatility matrix, we fix the marginal volatilities $(\hat{\sigma}^i)_{i \in \{1,2,3\}}$ and a correlation matrix ρ as in Tables 5.1 and 5.2.

Table 5.1: Model parameters

	S_1	S_2	S_3
μ	0.06	0.02	0.14
$\hat{\sigma}$	0.1	0.06	0.2

Table 5.2: Correlation matrix ρ of risky assets

	S_1	S_2	S_3
S_1	1	-0.2	0.3
S_2	-0.2	1	-0.2
S_3	0.3	-0.2	1

We fix a constant risk free rate $r = 0.001$. The initial portfolio wealth is $v_0 = 100$. The implementation parameters are summarized in the following table

Table 5.3: Implementation parameters

nb.traj MonteCarlo	nb.traj. calibration	nb.traj. test	N	p	Chaos degree
10^7	10^5	10^5	368	92	2

A sample of trajectories is necessary to estimate the expectations $\mathbb{E}_{\mathbb{Q}} \left[H_{\lambda_{n+1}}(\Delta \widehat{W}_{n+1}^{\mathbb{Q}}) \left(\frac{\tilde{S}_{n+1}^j}{\tilde{S}_n^j} - 1 \right) \right]$.

These terms appear in the computation of the controls (see for instance (5.26)) and are estimated through Monte-Carlo simulations. Then, we split a new sample into two parts.

The first part is used to run the descent gradient algorithm to calibrate and to find an optimal portfolio, while the second part is used to compute performance indicators. In presence of costs, we assume that the first position is free of charge.

6.2 Same risk aversion parameter

In this first experiment, we choose the same risk aversion parameter $\gamma = \gamma_u$ for the different approaches. Since the objective functions are not the same between multi-period and uni-period models, the agents have not the same risk aversion. Therefore volatility and rates of return are not comparable. Nevertheless the Sharpe ratio is a relevant indicator for comparing performances of optimal portfolios based on different risk aversions. Indeed, the Sharpe ratios of estimated portfolios are independent from the risk aversion in every models according to Propositions 43, 45, 48 and 49.

The implementation parameters are summarized in the following table.

Table 5.4: Implementation parameters for Multiperiod model

risk aversion	batch size	iteration	learning rate	cost(%)
0.05	100	1000	8.5	1

We compare five models; two sequential uni-period versions and two multi-period versions, where cost are on one hand ignored and on the other, considered. Moreover, we add in the benchmark the famous equal weight portfolio to measure the performance of the other approaches. We refer to our approach as *Multi-period with costs*. We refer to Section 5 for further details on the four other benchmark models. Formally, our aim is to evaluate the performance of our method compared to the more basic approaches commonly used. The main results can be found in Table 5.5.

Table 5.5: Estimated metrics for evaluating models performances

	rate of return(%)	vol(%)	Min-Var	Sharpe ratio
Multi-period ignoring cost	13.24	12.17	105.83087	1.07939
Multi-period with costs	12.31	11.00	106.26365	1.11033
Sequential uni-period ignoring cost	10.44	10.00	105.43191	1.03316
Sequential uni-period with costs	11.00	10.72	105.24599	1.01606
equal weight	5.69	5.70	104.06893	0.98125

We analyse here, the difference in Sharpe ratios. By focusing on uni-period models, it is interesting to notice that ignoring costs seems to be better than considering them. The myopic effect, joined with the consideration of costs, may make the optimal strategy rigid and inflexible. The agent’s myopic behavior explains why they do not see the benefit of paying costs for short-term positions. Therefore considering cost almost freezes the strategy and can explain that performances are not as good as if we have ignored them. Obviously

these remarks, are dependent of the chosen parameter ν , which determines the weight of costs in transactions.

In contrast, considering costs in the multi-period version is a significant improvement. According to the choices of parameters, a difference of 0.03 in Sharpe ratio is not negligible. A multi-period model targets a final value and must adapt its positions according to the variations of the environment. These changes in positions are typically more significant than in myopic strategies, and thus, the impact of costs becomes more critical. Ignoring costs can have a considerable impact on strategies, leading to performance deterioration. Undoubtedly, the myopic effect has a negative impact on performances. The difference in Sharpe ratios between uni-period and multi-period models is not negligible. Therefore, we advise to use multi-period models, despite their greater complexity. In that case, costs must not be ignored.

6.3 Same risk aversion level

In a second experiment, we want to compare the optimal portfolios submitted to transaction costs in uni-period and multi-period settings with the same level of risk. According to Proposition 44, we can link the risk aversions γ and γ_u in both models to ensure the same level of risk. This experiment aims to illustrate the comparison in terms of rates of returns. We can also directly compare portfolio trajectories.

6.3.1 Performances

Our aim is to obtain a locally optimal multi-period portfolio that carries the same level of risk as the optimal sequential uni-period portfolio. We adopt the approach described in Proposition 44. We use the $\gamma = 0.05$ locally optimal multi-period portfolio estimated in the previous section to build another locally optimal multi-period portfolio with the same volatility as the optimal uni-period portfolio. With a uni-period risk aversion $\gamma_u = 0.05$, we have obtained a volatility of 10.72% for the optimal uni-period portfolio. The γ_u locally optimal multi-period portfolio estimated in the previous experiment has a Sharpe ratio equal to 1.11033. According to Proposition 44, we can build another locally optimal multi-period portfolio with the same Sharpe ratio and a volatility of 10.72%. To reach this volatility, a multi-period risk aversion parameter $\gamma = 0.0518$ is estimated by using (5.29). The main results can be found in Table 5.6

Table 5.6: Results

	risk aversion	rate of return(%)	Min-Var	Sharpe ratio
Multi-period with costs	0.0518	11.89	105.9372	1.11033
Sequential uni-period with costs	0.0500	11.00	105.24599	1.01606

The multi-period model is obviously the most performing. The difference between the rates of return is almost 1%. This difference is important according to a level of risk of 12.72%.

After comparing the performances of the optimal portfolios with the same risk aversion, we illustrate their behaviour by showing trajectories from two different situations, A and B. To ensure consistency, we use the same framework as before to match the risk aversions.

6.3.2 Behaviour on realisation A

To illustrate the comparison we present in Figures 5.1, one particular sample path of assets.

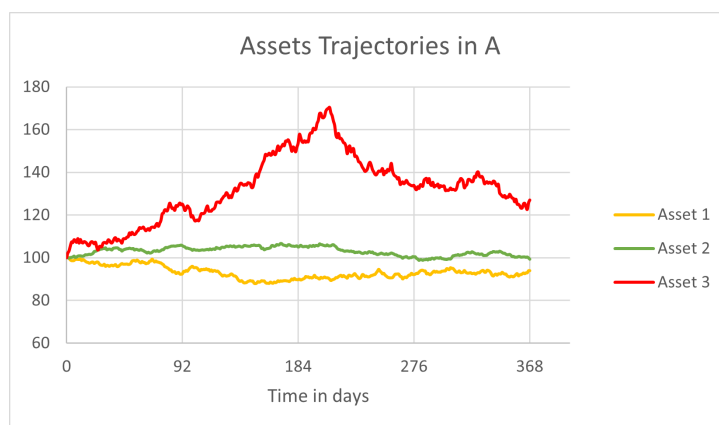


Figure 5.1: Asset trajectories in A

It can be observed that the curves for assets 1 and 2 are relatively flat and symmetric. Asset 2 remains above its initial value. On the other hand, asset 1 remains below its initial value. Asset 3 experiences significant growth, reaching a peak of 170 before collapsing towards the end of the period to finish just above 120.

We respectively present in Figures 5.2, 5.3, 5.4, 5.5, the portfolios values, the cumulative costs and the controls of the uni-period model considering cost against the multi-period model considering cost, whose performances have been presented in Table 5.6.

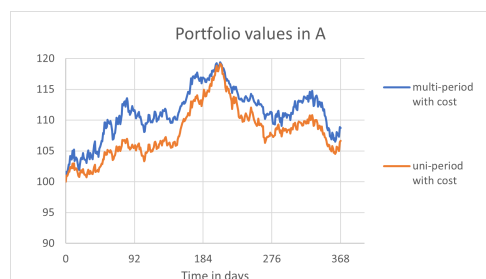


Figure 5.2: Portfolio values in A

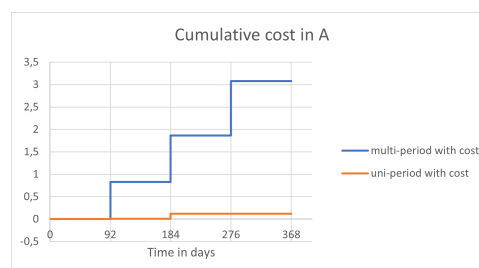


Figure 5.3: Cumulative cost in A

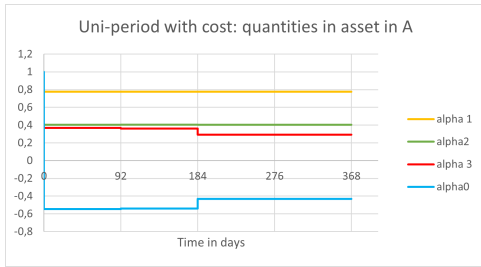


Figure 5.4: Controls of uni-period model with costs in A

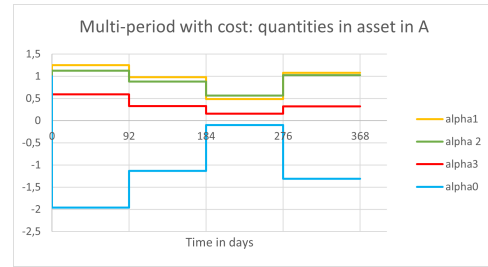


Figure 5.5: Controls of multi-period model with costs in A

By analysing Figure 5.2, both portfolio trajectories follow the trend of asset 3. This behavior is not surprising given the flat evolution of assets 1 and 2. The multi-period portfolio outperforms at every moment, for this particular realisation.

Figure 5.3 highlights that the uni-period strategy pays few cost compared to the multi-period one. This can be attributed to the myopic vision of the uni-period strategy, which results in a very flat strategy that is not very sensitive to asset and wealth variations. It can not embrace the benefit of sacrificing money in paying costs to anticipate future evolution. The high costs required to make a reversal of strategy, further reinforce this inflexibility, which partly explains the low Sharpe ratio estimated in Section 6.2. The long position of this portfolio explains the high dependence with asset 3 and the decline of its value after the middle of the period.

The multi-period portfolio follows a completely different policy. While initially adopting a more aggressive long strategy than the uni-period portfolio, the level of risk taken is significantly higher. The portfolio performs well during the growth of asset 3 until day 276, at which point the strategy starts to reverse as assets are gradually sold. Subsequently, in response to the decline of asset 3, the strategy undergoes another shift, with new asset quantities being purchased. This strategy can be explained by the decline of asset 3 and the portfolio value. The optimal strategy would see an opportunity to take some risk again. Consequently, new investments are made.

6.3.3 Behaviour on realisation B

To illustrate the comparison, we present in Figure 5.6, a new realisation of the assets. This scenario looks like the previous one, as Asset 1 and 2 exhibit minimal changes while Asset 3 experiences a large increase without any significant decrease.

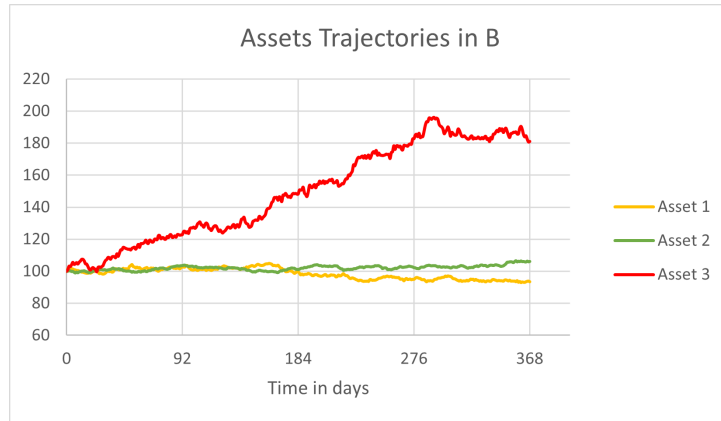


Figure 5.6: Asset trajectories in B

We present respectively in Figures 5.7, 5.8, 5.9, 5.10, the portfolios values, the cumulative costs and the controls of the uni-period model against the multi-period model with costs, whose performance results have been presented in Table 5.6.

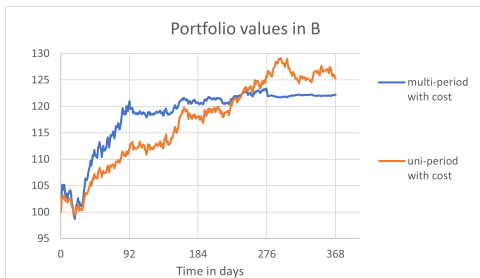


Figure 5.7: Portfolio values in B

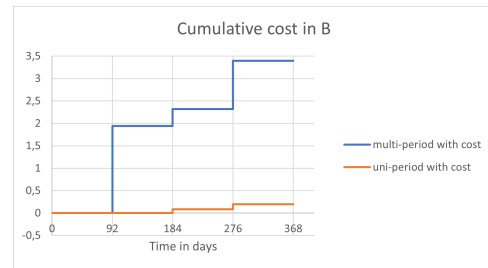


Figure 5.8: Cumulative cost in B

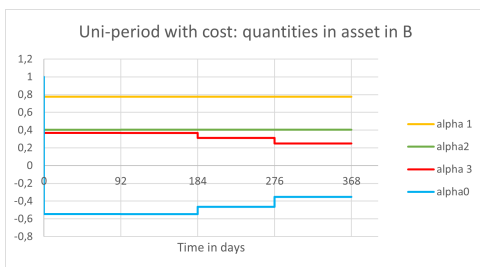


Figure 5.9: Controls of uni-period model with costs in B

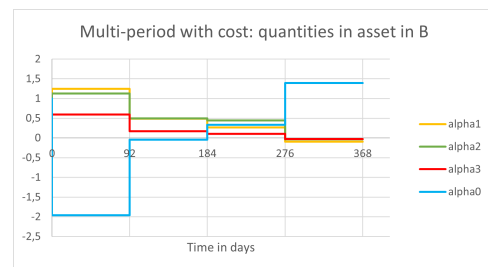


Figure 5.10: Controls of multi-period model with costs in B

Although both portfolios experience growth over the period, the uni-period portfolio with costs outperforms the multi-period portfolio at the end. It is important to note that

this outcome is specific to this particular trajectory but cannot be generalised. The uni-period portfolio appears to closely track the trend of asset 3, which continues to perform toward the end of the period. As for the realisation A, the positions of the uni-period strategy maintains predominantly long and flat positions. The multi-period strategy is considerably different and more versatile. Following 276 days of long positions, all assets are sold. This reverse of positions explains the plateau reached. After performing , the portfolio eliminates all risk to insure a positive performance. Even if Asset 3 were to continue its upward trend, this approach remains prudent. The portfolio achieves a higher value than anticipated, justifying its wish to protect its gains.

7 Conclusion of the chapter

In this chapter, we presented an efficient numerical method for addressing multi-stage portfolio optimization problems that involve multiple assets and transaction costs. We applied a stochastic descent gradient algorithm to find the Wiener chaos expansion of a locally optimal portfolio. The method can be extended to handle realistic constraints as no-shorting constraint and is computationally tractable. Our research explored the link between risk aversion and optimal portfolios subject to transaction costs, with two findings standing out. Firstly, we proved that the Sharpe ratio of a locally optimal portfolio with transaction costs does not depend on its risk aversion. Secondly, we established that optimal portfolios have the same Sharpe ratio. We used this result to compare our approach to a competitive benchmark, based on the sequential uni-period mean-variance strategy. We highlighted the efficiency of our approach and we showcased our benchmark by analyzing the performance of our models on two selected trajectories. Costs must not be ignored in multi-period setting since reverses of strategy are frequent. On the other hand, considering costs for uni-period models is still a topic of debate. The myopic effect induced by costs may freeze the strategy and negatively impact performance. As expected, the multi-period model is more intricate, but outperforms the uni-period models. We hope that this study contributes to challenging the conventional skepticism surrounding the use of multi-period models. Future research could extend the model to incorporate fix costs, or handle more intricate stochastic dynamics for asset modeling. Additionally, integrating the impact of the economic cycle with a Hidden Markov setting would significantly enhance the model.

Chapter 6

Extensions and variants of the approach with chaos expansion

Abstract

This chapter presents potential extensions of the approach presented in Chapter 5. We notably show how the approach can handle no-shorting constraint. We also propose a variant in absence of transaction costs and constraint where a sub-optimal portfolio is sought. If certain assumptions are met, this method can result in a deterministic optimization, reducing complexity.

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1 Mean-variance multi-period allocation problem without transaction costs

This section examines the case in which transactions are not subject to transaction costs. We compare the approach outlined in Chapter 5 to a variant where a sub-optimal portfolio is sought. We consider the same setting as in Chapter 5. According to this setting, $\forall n \in \{1, \dots, N\}$ knowing \mathcal{F}_{n-1} , R_n is independent from \mathcal{F}_{n-1} . Note the difference with the explicit approach described in Section 2.5.2, where $\{R_n, n \in \{1, \dots, N\}\}$ are assumed to be independent.

1.1 A sub-optimal strategy

This section presents an alternative to the approach described in Chapter 5. We consider here, that the portfolio is not submitted to transaction costs. By pursuing a sub-optimal solution, this approach achieves a significant simplification of the framework. Indeed, the optimisation can be deterministic in specific cases. We recall several notations. Let \mathcal{M} be the space of squared integrable \mathbf{F} -martingales under \mathbb{Q} and \mathcal{M}_S the sub-space of \mathcal{M} defined by

$$\mathcal{M}_S = \left\{ M \in \mathcal{M} : \exists (\alpha_k)_{1 \leq k \leq N} \mathbf{F} - \text{predictable s.t. } \forall n \in \{0, \dots, N\}, \right. \\ \left. M_n = h + \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right\}.$$

\mathcal{M}_S is the set of martingales which are a martingale transformation of \tilde{S} . \mathcal{M}_S can also be interpreted as the space of admissible portfolios in discrete-time. Equivalently, the problem to solve can be rewritten as

$$\begin{aligned} & \sup_{\tilde{X} \in \mathcal{M}_S} \mathbb{E}_{\mathbb{P}} \left[\tilde{X}_N S_N^0 - \gamma \left(\tilde{X}_N S_N^0 - \mathbb{E}_{\mathbb{P}}[\tilde{X}_N S_N^0] \right)^2 \right] \\ & \text{subject to } \tilde{X}_0 = V_0 \end{aligned} \quad (\mathcal{A}_{\mathcal{M}_S}^{\gamma})$$

While the approach described in Chapter 5 consists in finding optimal portfolios on \mathcal{M}_S by exploring \mathcal{M} , we will here, proceed differently. We preliminary seek the optimal solution on \mathcal{M} and then try to hedge this solution with an element of \mathcal{M}_S . Formally we find $(Z^*, \theta^*) \in \mathcal{M} \times \mathbb{R}$ solution to

$$\begin{aligned} & \sup_{Z \in \mathcal{M}, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} \left[Z_N S_N^0 - \gamma (Z_N S_N^0 - \theta)^2 \right] \\ & \text{subject to } Z_0 = V_0 \end{aligned} \quad (\mathcal{A}_{\mathcal{M}}^{\gamma})$$

By using the martingale representation theorem, Z^* could have been perfectly hedged by

$\left(h + \int_0^n \alpha_u \cdot d\tilde{S}_u, \alpha \Gamma - \text{predictable} \right)$. But this strategy is not realistic since we can only re-balance at discrete dates. Z^* solution to $(\mathcal{A}_{\mathcal{M}}^{\gamma})$, is not necessarily an admissible solution to $(\mathcal{A}_{\mathcal{M}_S}^{\gamma})$. The natural approach is to hedge this solution with an admissible portfolio $\Theta^* \in \mathcal{M}_S$. We have seen in Proposition 24 of Chapter 5 that the orthogonal projection of Z^* on \mathcal{M}_S under \mathbb{Q} is $\Theta^* = \text{Pr}(Z^*)$. By approximating elements of \mathcal{M} with their chaos expansion of order K , we aim to find β^*, θ^* solution to the analog problem

$$\begin{aligned} & \sup_{\beta \in \mathbb{R}^m, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} \left[\mathcal{C}_K(\beta) S_N^0 - \gamma (\mathcal{C}_K(\beta) S_N^0 - \theta)^2 \right] \\ & \text{subject to } \mathbb{E}_{\mathbb{Q}}[\mathcal{C}_K(\beta)] = V_0 \end{aligned} \quad (\mathcal{A}_K^{\gamma})$$

Finally we set $Z^* = \mathcal{C}_K(\beta^*)$, and build a sub-optimal portfolio $\Theta^* = \Pr(Z^*)$.

In some case where Φ , defined by $\forall i \in \{1, \dots, d\}$, $\Delta W_{n+1}^{Q,i} \stackrel{P}{=} \Delta W_{n+1}^i + \Phi_n^i(t_{n+1} - t_n)$, is deterministic (as when assets are governing by a Black-Scholes model), we can explicitly compute $\mathbb{E}_{\mathbb{P}} \left[\mathcal{C}_K(\beta) S_N^0 - \gamma (\mathcal{C}_K(\beta) S_N^0 - \mathbb{E}_{\mathbb{P}}[\mathcal{C}_K(\beta) S_N^0])^2 \right]$, according to Proposition 55. This derivation offers a deterministic form of the problem.

In the next section, we implement this solution and compare the sub-optimal portfolio to the optimal portfolio issued from the original approach described in Chapter 5.

1.2 Numerical application without transaction cost

In this section, we compare the performances of four models when there is no transaction cost. We consider $d=3$ assets, with the same dynamics assumed in Section 6. Table 6.1 provides the implementation parameters used for running the multi-period model without transaction cost presented in Chapter 5.

Table 6.1: Implementation parameters for multi-period model

risk aversion	batch size	iteration	learning rate	cost(%)
0.05	100	1000	25	0

In absence of transaction cost, we compare four models, the sub-optimal multi-period presented in Section 1.1, the optimal multi-period presented in Section 3, the sequential uni-period, presented in Section 5.2 and the famous equal weight. The main results can be found in Table 6.2

Table 6.2: Results

	rate of return(%)	vol(%)	Min-Var	Sharpe ratio
Sub-optimal multi-period	17,71	14,76	106,81390	1,19288
optimal multi-period	16,35	12,70	108,27987	1,27908
equal weight	5,69	5,70	104,06893	0,98125
Sequential uni-period	10,66	10,03	105,62782	1,05262

As expected, the optimal multi-period portfolio significantly outperforms the other portfolios in terms of performance. It is also noteworthy that while the sub-optimal portfolio exhibits a substantial loss of performance, it remains relatively acceptable when compared to the other two portfolios. Equal weight strategy is a very naive method and does not present a lot of interest when inputs parameters are known. We realize that the myopic effect in the uni-period model, strongly deteriorates the performance of the strategy.

2 Multi-period portfolio selection with transaction costs and no-shorting constraint

In this section, we show how our method can be easily adapted if short positions are not allowed.

2.1 Model

The constraint of no shorting requires the controls to be positive, ie $\forall 1 \leq n \leq N, 1 \leq i \leq d, \alpha_n^i \geq 0$. We still consider proportional transaction costs. At time $n \leq N - 1$, the portfolio is re-balanced from (α_n, α_n^0) to $(\alpha_{n+1}, \alpha_{n+1}^0)$ and the costs $\sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i$ is paid. The allocation problem writes

$$\begin{aligned} & \sup_{(\alpha_n)_n} \mathbb{E}_{\mathbb{P}} \left[\tilde{V}_N S_N^0 - \gamma \left(\tilde{V}_N S_N^0 - \mathbb{E}_{\mathbb{P}}[\tilde{V}_N S_N^0] \right)^2 \right] \\ & \text{subject to } \tilde{V}_0 = V_0, (\alpha_n)_n \mathbf{F} - \text{Pred} \\ & \Delta \tilde{V}_{n+1} = \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0} \\ & \alpha_{n+1} \geq 0 \end{aligned} \tag{I_0'}$$

Following the same arguments as in Chapter 5, we solve the problem

$$\begin{aligned} & \sup_{\beta \in \mathcal{Z}, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F_{\gamma}(\beta, \theta)] \\ & \text{subject to } \mathbb{E}_{\mathbb{Q}} [\mathcal{C}_K(\beta)] = V_0, \\ & \alpha_n(\beta) \geq 0 \text{ a.s.} \end{aligned} \tag{I_K'}$$

In the stochastic gradient descent, the constraint of no-shorting can be numerically integrated by penalizing negative controls. Let us define the penalty function

$$\Pi_{\kappa}(\beta) = \kappa \sum_{n=1}^N \sum_{i=1}^d |\alpha_n(\beta)|_-, \text{ for } \kappa > 0.$$

We integrate this penalty function to avoid short investments such that (I_K') writes

$$\begin{aligned} & \sup_{\beta \in \mathcal{Z}, \theta \in \mathbb{R}} \mathbb{E}_{\mathbb{P}} [F_{\gamma}(\beta, \theta) - \Pi_{\kappa}(\beta)] \\ & \text{subject to } \mathbb{E}_{\mathbb{Q}} [\mathcal{C}_K(\beta)] = V_0, \\ & \alpha_n(\beta) \geq 0 \text{ a.s.} \end{aligned} \tag{I_K''}$$

We can apply a stochastic descent algorithm to solve (\mathcal{I}'_K) .

2.2 Performance analysis

In this section, we compare the performances of the optimal portfolio with transaction costs with the optimal portfolio with transaction costs and no shorting constraint. We aim to observe how no-shorting constraint deteriorates the performance of the portfolios. The implementation parameters are summarized in Table 6.3

Table 6.3: Implementation parameters for Multiperiod model

risk aversion	batch size	iteration	learning rate	cost(%)	Penalty κ
0.05	100	1000	8,5	1	100

The main results can be found in Table 6.4.

Table 6.4: Results

	rate of return(%)	vol(%)	Min-Var	Sharpe ratio
Multi-period with costs	12,31	11,00	106,26365	1,11033
Multi-period with costs with no-shorting	6,77	6,04	104,94730	1,10475

When examining the Sharpe ratios, it is not surprising to observe that the no-short constraint results in a deterioration of performance. Due to the reduced set of potential strategies, the reachable optimal strategy can only offer inferior performance compared to an optimal strategy that allows shorting of assets. It is still interesting to notice that the behaviour of the optimal strategy with no-shorting is totally different. The no-sorting constraint almost divides the expected rate of return by two but also substantially reduced the risk. This behavior can be rationalized as follows: since the strategy without shorting has fewer degrees of freedom and cannot be reversed by inverting positions, the model recognizes that the strategy will face more difficulty adapting to changes. Consequently, the optimal strategy will tend to be more stable and assume less risk than if shorting positions were allowed.

To illustrate the comparison we present in Figures 6.1, a following realisation of assets.

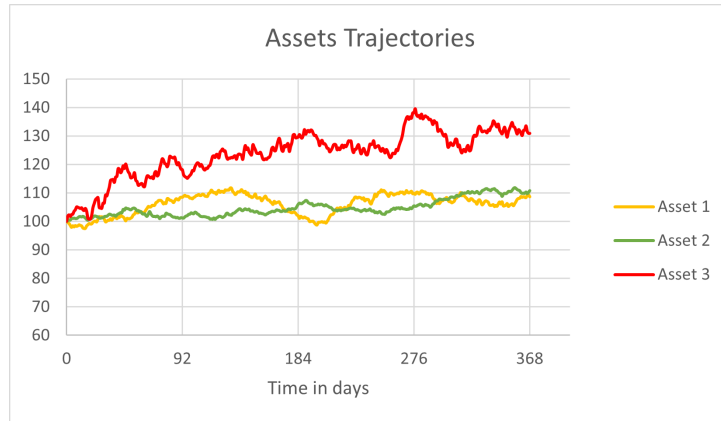


Figure 6.1: Asset trajectories

Overall, the assets increase in this situation. The variations of Asset 1 and 2 are quite symmetric and Asset 3 performs far more the two others. We respectively present in Figures 6.2, 6.3, 6.4, 6.5, the portfolios values, the cumulative costs and the controls of the multi-period model with costs against the multi-period model with costs and no shorting constraint.

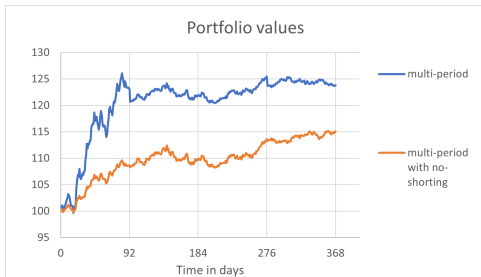


Figure 6.2: Portfolio values

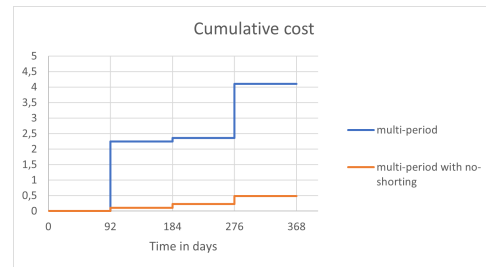


Figure 6.3: Cumulative cost

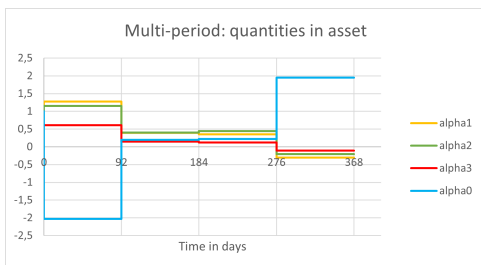


Figure 6.4: Controls of multi-period model with transaction costs

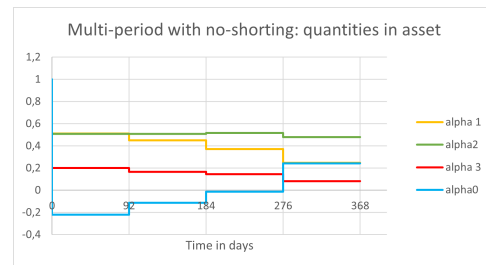


Figure 6.5: Controls of multi-period model with transaction costs and no-shorting

In this situation, the multi-period portfolio which can take short positions obviously performs better than the one which cannot. The strategy consists in buying asset at the beginning of the period. Then the positions are progressively reversed, assets are sold and are shorted at the end of the period. The strategy with no-shorting constraint is different. The quantity of assets purchased under this strategy is comparatively lower than that of the other strategy. Assets 1 and 3 are slowly sold whereas positions in asset 2 is kept during the whole period. At the end of the period, positions on assets are still long. It is interesting to notice that the optimal strategy with no-shorting does not seek to copy the strategy with short positions allowed. One might have expected that assets are totally sold when short positions are taken in the other portfolio, but it is not the case.

Conclusion of the part

This part provided an overview of existing approaches for portfolio allocation. The selection of a particular method mainly relies on market perspective, available data and resources, investor profile, and investment goals. When investors have reliable estimation of asset dynamic parameters, they have to choose between two main families, the single-period and the multi-period approaches. Single-period models, are user-friendly, computationally tractable but they may not be optimal for long investment horizon. On the other hand, multi-period models require more information, assumptions, and computations. They suffer of the curse of dimensionality and are often computationally intractable, especially when accounting for constraints and transaction costs. As a result, we have established the absence of reliable method to find optimal mean-variance portfolios submitted to transactions costs. We filled this gap by proposing a new computational scheme that relies on representing the portfolio value by its finite dimensional Wiener chaos expansion. This method is computationally feasible and can be extended to handle realistic constrain such as no-shorting selling. Note that this method can also be easily applied for a concave utility function. In summary, we hope that this study challenges the prevailing skepticism regarding the use of multi-period models and offers a promising strategy for mean-variance allocation in presence of transaction costs.

Tool box: Wiener chaos expansion properties

In this Appendix, we present several fundamental properties on Wiener chaos expansion and Hermite polynomials, use-full in our study. We refer to [Akahori et al. \(2017\)](#), [Lelong \(2018\)](#) [Nualart \(2006\)](#) for theoretical details.

Let H_i be the i -th Hermite polynomial defined by

$$H_0(x) = 1 : H_i(x) = (-1)^i e^{\frac{x^2}{2}} \frac{d^i}{dx^i} (e^{-\frac{x^2}{2}}), \text{ for } i \geq 1. \quad (6.1)$$

For $\lambda \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, we define

$$H_\lambda^\otimes(x) = \prod_{i=1}^n H_{\lambda_i}(x_i).$$

We have the following properties

- For $i \geq 1$, $H_i' = iH_{i-1}$ with $H_{-1} = 0$.
- For $x, y \in \mathbb{R}$,

$$H_i(x + y) = \sum_{r=0}^i \binom{i}{r} x^r H_{i-r}(y). \quad (6.2)$$

- Let X, Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1$. Then $\forall i, j \geq 0$, we have

$$\mathbb{E}[H_i(X)H_j(Y)] = 1_{(i=j)} i! (\mathbb{E}[XY])^i. \quad (6.3)$$

- Let $\lambda \in \mathbb{R}^n \neq 0_{\mathbb{R}^n}$ and X be a n -multivariate i.i.d standard normal vector. Then we have

$$\mathbb{E}[H_\lambda^\otimes(X)] = 0. \quad (6.4)$$

- Let $\lambda, \lambda' \in \mathbb{R}^n$ and X a n-multivariate i.i.d standard normal vector, then

$$\mathbb{E} \left[H_{\lambda}^{\otimes} (X) H_{\lambda'}^{\otimes} (X) \right] = 1_{(\lambda=\lambda')} \prod_{i=1}^n \lambda_i! . \quad (6.5)$$

The following theorem represents the base of our approach. It provides a discrete-time general representation with a convergent serie expansion. This expansion is introduced and proved in [Akahori et al. \(2017\)](#)[Theorem 2.1].

Theorem 50. *Let Y be a square integrable random variable, \mathcal{F}_N -measurable, then Y admits we have the following L^2 convergent series Wiener chaos expansion*

$$\begin{aligned} Y &= \mathbb{E}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_{\lambda} \prod_{j=1}^d \prod_{i=1}^N H_{\lambda_i^j} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} \right) \\ &= \mathbb{E}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_{\lambda} H_{\lambda}^{\otimes} (\Delta \widehat{W}). \end{aligned}$$

By truncating the decomposition of Y , we call $C_K(Y)$ the function which associates to a \mathcal{F}_N -measurable and square integrable random variable, its Wiener chaos expansion of order K and we have

$$\lim_{K \rightarrow +\infty} \mathbb{E} [|Y - C_K(Y)|^2] = 0, \text{ with } C_K(Y) = \mathbb{E}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_{\lambda} H_{\lambda}^{\otimes} (\Delta \widehat{W}).$$

We also call $C_K^{\mathbb{Q}}(Y)$ the random variable of the form

$$C_K^{\mathbb{Q}}(Y) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_{\lambda} H_{\lambda}^{\otimes} (\Delta \widehat{W}^{\mathbb{Q}}),$$

which has the same law than $C_K(Y)$ under \mathbb{Q} . By abuse of language, we call this variable, the Wiener chaos expansion of Y under \mathbb{Q} . Let us define $m \in \mathbb{N}$ be the number of coefficients λ appearing in the chaos decomposition. We have $m = \#\{\lambda \in (\mathbb{N}^N)^d | |\lambda|_1 \leq K\} = \binom{Nd+K}{Nd}$. By extension we also call the operator C_K on \mathbb{R}^m that associates to coefficients β , the expansion of order K such that

$$C_K(\beta) = \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_{\lambda} H_{\lambda}^{\otimes} (\Delta \widehat{W}).$$

Proposition 51. *Let $Y \in L^2(\Omega, \mathcal{F}_N, \mathbb{P})$, then*

$$E [C_K(Y) | \mathcal{F}_n] = C_K(\mathbb{E} [Y | \mathcal{F}_n]) = C_{K,n}(Y), \quad (6.6)$$

where

$$C_{K,n}(Y) = \mathbb{E}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K, \\ \forall j \leq d, \forall l \geq n+1: \lambda_l^j = 0}} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}) = \mathbb{E}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^n)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}).$$

The Wiener chaos expansion of the conditional expectation of the variable Y is easily obtained by truncating the terms which are not \mathcal{F}_n -measurable.

Proposition 52. *Let A, B two random variables of $L^4(\Omega, \mathcal{F}_N, \mathbb{P})$, with respective Wiener chaos expansions such that*

$$A = \sum_{\lambda_1 \in (\mathbb{N}^N)^d} \eta_{\lambda_1} H_{\lambda_1}^\otimes(\Delta \widehat{W}), \quad B = \sum_{\lambda_2 \in (\mathbb{N}^N)^d} \beta_{\lambda_2} H_{\lambda_2}^\otimes(\Delta \widehat{W}).$$

Let us write $(C_K(A)C_K(B))_K$ for the truncated expansion of order K of $C_K(A)C_K(B)$. Then $C_K(AB)$ is exactly $(C_K(A)C_K(B))_K$.

Proof. Let us write the chaos expansion of AB such that $AB = \sum_{\lambda \in (\mathbb{N}^N)^d} \mu_\lambda H_\lambda^\otimes(\Delta \widehat{W})$. AB also writes $\sum_{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d} \eta_{\lambda_1} \beta_{\lambda_2} H_{\lambda_1}^\otimes(\Delta \widehat{W}) H_{\lambda_2}^\otimes(\Delta \widehat{W})$. $\{H_\lambda^\otimes(\Delta \widehat{W}), \lambda \in (\mathbb{N}^N)^d\}$ is an orthonormal basis so the decomposition is unique. We can then identify

$$\forall \lambda \in (\mathbb{N}^N)^d, \quad \mu_\lambda = \sum_{\substack{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d \\ |\lambda_1|_1 + |\lambda_2|_1 = |\lambda|_1}} \eta_{\lambda_1} \beta_{\lambda_2}.$$

Therefore, we deduce that

$$C_K(AB) = \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \mu_\lambda H_\lambda^\otimes(\Delta \widehat{W}) = \sum_{\substack{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d \\ |\lambda_1|_1 + |\lambda_2|_1 \leq K}} \eta_{\lambda_1} \beta_{\lambda_2} H_{\lambda_1}^\otimes(\Delta \widehat{W}) H_{\lambda_2}^\otimes(\Delta \widehat{W}) = (C_K(A)C_K(B))_K.$$

Proposition 53. *Let A, B two random variables of $L^4(\Omega, \mathcal{F}_N, \mathbb{P})$, with respectively Wiener chaos expansions $C_K(A)$ and $C_K(B)$ such that*

$$C_K(A) = \mathbb{E}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda H_\lambda^\otimes(\Delta \widehat{W}), \quad C_K(B) = \mathbb{E}[B] + \sum_{\substack{\lambda' \in (\mathbb{N}^N)^d \\ |\lambda'|_1 \leq K}} \beta_{\lambda'} H_{\lambda'}^\otimes(\Delta \widehat{W}).$$

Then $C_K(E[AB|\mathcal{F}_n])$ writes

$$C_K(E[AB|\mathcal{F}_n]) = \mathbb{E}[A]C_{K,n}(B) + \mathbb{E}[B]C_{K,n}(A) \\ + \sum_{\substack{\lambda, \lambda' \in (\mathbb{N}^N)^d \\ |\lambda|_1 + |\lambda'|_1 \leq K}} \eta_\lambda \beta_{\lambda'} H_{\lambda_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}) H_{\lambda'_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}) 1_{(\lambda_{n+1:N} = \lambda'_{n+1:N})} \prod_{i=n+1}^N \prod_{j=1}^d \lambda_{ij}!$$

Proof. This result simply stems from Proposition 52, properties of Hermite polynomial and the linearity of the expectation. \blacksquare

Proposition 54. *Let A and B , two \mathbf{F} -martingales. Let's consider the following Wiener chaos expansion of their final value.*

$$C_K(A_N) = \mathbb{E}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}),$$

$$C_K(B_N) = \mathbb{E}[B] + \sum_{\substack{\lambda' \in (\mathbb{N}^N)^d \\ |\lambda'|_1 \leq K}} \beta_{\lambda'} H_{\lambda'}^{\otimes}(\Delta \widehat{W}).$$

Then we have the following decomposition of order K ,

$$C_K(\mathbb{E}[\Delta A_{n+1} \Delta B_{n+1} | \mathcal{F}_n]) = \sum_{\substack{\lambda, \lambda' \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 + |\lambda'|_1 \leq K}} \eta_\lambda \beta_{\lambda'} H_{\lambda_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}) H_{\lambda'_{1:n}}^{\otimes}(\Delta \widehat{W}_{1:n}) 1_{(\lambda_{n+1} = \lambda'_{n+1} \neq 0)} \prod_{j=1}^d \lambda_{n+1j}!.$$

Proof. As A and B are \mathbf{F} -martingales and using Proposition 51, we have

$$C_K(\Delta A_{n+1}) = \mathbb{E}[C_K(A_N) | \mathcal{F}_{n+1}] - \mathbb{E}[C_K(A_N) | \mathcal{F}_n] = C_{K,n+1}(A_N) - C_{K,n}(A_N).$$

This equality also holds for ΔB_n . We deduce the following decomposition

$$C_K(\Delta A_{n+1}) = \sum_{\substack{\lambda \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 \leq K, \\ \lambda_{n+1} \neq 0}} \eta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}), \text{ and } C_K(\Delta B_{n+1}) = \sum_{\substack{\lambda \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 \leq K, \\ \lambda_{n+1} \neq 0}} \beta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}).$$

The announced formula yields from Proposition 52 and the conditional expectation property of Wiener chaos expansions. \blacksquare

Proposition 55. *Let A , a random variables \mathcal{F}_N measurable associated to the following Wiener chaos expansion of order K .*

$$C_K(A) = \mathbb{E}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W})$$

Then $\forall \phi \in (\mathbb{R}^N)^d$,

$$\mathbb{E}[C_K(A)(\Delta W + \phi)] = \mathbb{E}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda \prod_{i=1}^N \prod_{j=1}^d (\phi_i^j)^{\lambda_{ij}}, \quad (6.7)$$

$$\begin{aligned} \mathbb{E}[C_K(A)^2(\Delta W + \phi)]_K &= \mathbb{E}[A]^2 + 2\mathbb{E}[A] \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda \prod_{i=1}^N \prod_{j=1}^d (\phi_i^j)^{\lambda_{ij}} \\ &+ \sum_{\substack{\lambda, \lambda' \in (\mathbb{N}^N)^d \\ |\lambda|_1 + |\lambda'|_1 \leq K}} \eta_\lambda \eta_{\lambda'} \prod_{(i,j) \in (\mathbb{N}^N)^d} R(\lambda_{ij}, \lambda'_{ij}), \end{aligned} \quad (6.8)$$

where $R(\lambda_{ij}, \lambda'_{ij}) = \sum_{s=0}^{\min(\lambda_{ij}, \lambda'_{ij})} s! \binom{\lambda_{ij}}{s} \binom{\lambda'_{ij}}{s} (\phi_i^j)^{\lambda_{ij}-s} (\phi_i^j)^{\lambda'_{ij}-s}$.

Proof. To prove (6.7), we use the properties (6.2) and (6.4) to compute for $(i, j) \in \{1, \dots, N\} \times \{1, \dots, d\}$

$$\begin{aligned} \mathbb{E} \left[H_{\lambda_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) \right] &= \sum_{r=0}^{\lambda_{ij}} \binom{\lambda_{ij}}{r} (\phi_i^j)^r \mathbb{E} \left[H_{\lambda_{ij}-r} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} \right) \right] \\ &= (\phi_i^j)^{\lambda_{ij}}. \end{aligned}$$

To derive (6.8), we also use (6.7) to compute

$$\begin{aligned} &E \left[\left(\mathbb{E}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda \prod_{j=1}^d \prod_{i \geq 1}^N H_{\lambda_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) \right)^2 \right] \\ &= \mathbb{E}[A]^2 + 2\mathbb{E}[A] \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda \prod_{i=1}^N \prod_{j=1}^d (\phi_i^j)^{\lambda_{ij}} \\ &+ \sum_{\substack{\lambda, \lambda' \in (\mathbb{N}^N)^d \\ |\lambda|_1 + |\lambda'|_1 \leq K}} \eta_\lambda \eta_{\lambda'} \prod_{(i,j) \in U_{N,d}} \mathbb{E} \left[H_{\lambda_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) H_{\lambda'_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) \right] \\ &\times \prod_{(i,j) \notin U_{N,d}} (\phi_i^j)^{\lambda_{ij}} (\phi_i^j)^{\lambda'_{ij}}, \end{aligned}$$

where $U_{N,d} = \{(i, j) \in (\mathbb{N}^N)^d / \lambda_{ij} \neq 0 \cap \lambda'_{ij} \neq 0\}$. Using (6.2), we can compute the term

$$\begin{aligned}
& \mathbb{E} \left[H_{\lambda_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) H_{\lambda'_{ij}} \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} + \phi_i^j \right) \right] \\
&= \sum_{s=0}^{\lambda_{ij}} \sum_{r=0}^{\lambda'_{ij}} \binom{\lambda_{ij}}{s} \binom{\lambda'_{ij}}{r} (\phi_i^j)^{\lambda_{ij}-s} (\phi_i^j)^{\lambda'_{ij}-r} \mathbb{E} \left[H_s \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} \right) H_r \left(\frac{W_{t_i}^j - W_{t_{i-1}}^j}{\sqrt{t_i - t_{i-1}}} \right) \right] \\
&= \sum_{s=0}^{\min(\lambda_{ij}, \lambda'_{ij})} s! \binom{\lambda_{ij}}{s} \binom{\lambda'_{ij}}{s} (\phi_i^j)^{\lambda_{ij}-s} (\phi_i^j)^{\lambda'_{ij}-s}.
\end{aligned}$$

This expression remaining true for $(i, j) \notin U_{N,d}$, we obtain (6.8). ■

General conclusion

Through the exploration of existing and novel approaches, this thesis tried to contribute to the understanding of effective management of financial risk in two environments. In a first chapter, we studied the effect of systemic risk on credit rating migrations. We reviewed the two main families of Point-In-Time rating migration models, and highlighted the limitations of each approach. Then, we proposed a promising alternative to the standard approaches, as it enables estimation of Point-In-Time migration rating probabilities without using any covariates. The model is a promising alternative to the popular approaches for predicting rating transitions, detecting periods of crisis or generating historical stress testing scenarios. However, the model is unable to account for idiosyncratic risk, which represents a significant loss of information.

In the second part, our focus was on managing investment risk through portfolio allocation. The method chosen by investors to reallocate their portfolio determines the way they handle risks. We examined two families of methods: the single-period and multi-period approaches, and highlighted the limitations of each. While single-period models remain popular among investors, we aim to revive the debate by proposing a promising and computationally feasible solution for optimizing multi-period mean-variance portfolios in the presence of transaction costs. We compared the performance of our approach to sequential single-period Markowitz portfolios, within the broader debate of how much we should prioritize simple models over performing models. Our conclusion is that our multi-period approach is more intricate, but outperforms uni-period models, especially when cost are considered.

Although we have previously addressed the technical and foreseeable expansions of both proposed approaches, we aim to expand the scope of such studies in this discussion. The current definition of financial risk fails to completely explain current decision-making in finance, particularly in light of climate threat and increased awareness among the public. Therefore, it is imperative to explore how behavior, in response to climate threats, can complete the current definition of financial risk.

Climate risk refers to the potential negative impacts that climate change can have on businesses, economies and societies. These risks can include physical risks such as increased frequency and severity of natural disasters, as well as transition risks such as changes in policy, regulation, and market sentiment towards low-carbon technologies. Financial authorities are becoming increasingly aware of this new form of risk and are recommending

that financial institutions incorporate climate scenarios into their forward-looking projections of rating migrations. However, we acknowledge that the models presented, which are calibrated with historical data, are not designed to incorporate this new type of risk. Therefore, a significant revision of credit rating models is necessary.

The consideration of climate change and evolving societal attitudes in investment landscape is reflected through ESG investments. ESG refers to a set of environmental, social, and governance factors that companies are increasingly being evaluated on by investors and stakeholders. These factors can include company's carbon footprint, social impact, board composition, and management practices. Investors are no longer solely concerned with the balance between risk and return, but also with the social and environmental impact of their investments. ESG investing allows investors to align their financial goals with their personal values and principles, while also promoting positive change in society and the environment. Additionally, studies have shown that companies with strong ESG practices tend to outperform their competitors in the long run, making ESG investing an attractive option for investors seeking both financial returns and a positive impact. As a result, modern portfolio theory needs to be revised, as it is no longer based solely on the trade-off between risk and return. A viable strategy should now incorporate investors' aversions as well as their preferences for environmental and social considerations, while alignment with their financial objectives. The inclusion of environmental and social preferences can be directly integrated into our approach through the objective function or constraints. However, further research is needed in this area.

By considering both climate risks and ESG factors, investors and companies can better understand the potential impacts of climate change on their operations and make informed decisions to mitigate these risks and build a more sustainable future.

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