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Julien Fixary

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UNIVERSITÉ PARIS 1 PANTHÉON-SORBONNE
Centre d'Économie de la Sorbonne

**Nouveaux résultats concernant les équilibres
de Nash et les réseaux stables par paires dans
le cas semi-algébrique: nombre, structure et
dynamiques**

Thèse présentée par

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pour obtenir le grade de

DOCTEUR EN MATHÉMATIQUES APPLIQUÉES

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Résumé

La théorie des réseaux est une branche des mathématiques appliquées à l'économie qui prend de l'ampleur depuis plusieurs années. La *stabilité par paires* en est un concept majeur qui permet de cibler certains réseaux¹ comme étant des états d'équilibre au sein d'un processus de formation (de réseaux). De manière informelle, un réseau est dit stable par paires si aucun des agents n'a d'intérêt à baisser un de ses poids, et si les agents d'aucune paire d'agents distincts (d'aucun *lien*) n'ont d'intérêt à augmenter leur poids commun. D'abord introduite en 1996 par Jackson et Wolinsky [32] dans le cadre des réseaux non-pondérés, la stabilité par paires a ensuite été adaptée en 2020 par Bich et Morhaim [6] au cas des réseaux pondérés. Plus précisément, une *société* est une structure de base de la théorie des réseaux; elle comprend implicitement un ensemble N d'agents, ainsi que pour chaque agent $i \in N$, une fonction de paiement v_i traduisant les préférences en termes de réseaux de l'agent i en question. Ainsi, de façon plus formelle, un réseau (pondéré) g est stable par paires par rapport à une société $(N, (v_i)_{i \in N})$ fixée si pour tout lien ij et pour tout poids w , les deux conditions suivantes sont satisfaites:

- (i) si $w \leq g_{ij}$, alors $v_i(w, g_{-ij}) \leq v_i(g_{ij}, g_{-ij})$ et $v_j(w, g_{-ij}) \leq v_j(g_{ij}, g_{-ij})$
(i.e. aucun agent n'a intérêt à baisser le poids g_{ij});
- (ii) si $w \geq g_{ij}$, alors $v_i(w, g_{-ij}) \leq v_i(g_{ij}, g_{-ij})$ ou $v_j(w, g_{-ij}) \leq v_j(g_{ij}, g_{-ij})$
(i.e. au moins un agent n'a pas intérêt à augmenter le poids g_{ij}).

Cette définition évoque un peu celle de l'équilibre de Nash en théorie des jeux, mais avec des déviations parfois unilatérales (quand les poids diminuent), et parfois bilatérales (quand les poids augmentent). Malgré cette analogie, il est remarquable de noter que la connaissance des propriétés structurelles des réseaux stables par paires est très réduite comparée à celle des propriétés structurelles des équilibres de Nash. En effet, dans le premier cas, on a principalement le résultat de Bich et Morhaim, qui ont démontré l'existence d'un tel réseau sous des hypothèses assez générales de continuité et de quasiconcavité des fonctions de paiement. Par contre, la structure du *graphe des équilibres de Nash* (l'ensemble des paires (u, x) , où u est un jeu sous forme stratégique, et x un équilibre de Nash de u) est un sujet qui a été exploré pendant plusieurs années. En particulier, quatre papiers sont au cœur de

¹Un réseau peut être vu comme un graphe dont les sommets sont les agents, et une arête entre deux agents est le poids qui mesure l'intensité de la relation entre ces deux agents. Quand les poids sont des nombres réels entre 0 et 1 (0 signifiant aucune connexion, et à l'autre extrême, 1 signifiant une connexion totale), on parle de *réseau pondéré*. Lorsque les poids sont uniquement égaux à 0 ou à 1, on parle de *réseau non-pondéré*.

cette thèse:

1. Wilson [43] a démontré en 1971 que *génériquement*, chaque jeu fini (sous forme stratégique) admet un nombre impair d'équilibres de Nash en stratégies mixtes.
2. Kohlberg et Mertens [33] ont démontré en 1986 que le graphe \mathcal{N}_F des équilibres de Nash mixtes associé aux jeux finis est homéomorphe à l'espace F des jeux finis, par un homéomorphisme qui est proprement homotopique à une projection (le résultat de Wilson étant vu comme un corollaire de ce dernier). Plus simplement, \mathcal{N}_F peut être déformé de façon continue en l'espace (plus simple) F , et cette déformation elle-même peut être déformée de façon continue en une application plus simple qu'est la projection $\pi_F : \mathcal{N}_F \rightarrow F, (u, x) \mapsto u$.
3. Demichelis et Germano [17] ont fourni en 2002 une généralisation du théorème de Kohlberg et Mertens en prouvant que le graphe \mathcal{N}_F des équilibres de Nash mixtes associé aux jeux finis *n'a pas de nœuds*, c'est-à-dire qu'il existe une isotopie ambiante (une déformation continue plus restrictive qu'un simple homéomorphisme) entre \mathcal{N}_F et une copie triviale de F de la forme $F \times \{\sigma_0\}$ (σ_0 étant un profil de stratégies mixtes arbitrairement fixé).
4. Predtetchinski [40] a fourni en 2009 un résultat plus vaste à la fois que le théorème de Kohlberg et Mertens, mais aussi que le théorème de Demichelis et Germano en s'intéressant au graphe $\mathcal{N}_{\mathcal{F}}$ des équilibres de Nash associé aux jeux \mathcal{C}^1 et concaves en stratégie propre (i.e. les profils $(u_i)_{i \in N}$ de fonctions de paiements continument différentiables et telles que, pour tout $i \in N$ et tout profil de stratégies x_{-i} autres que celui du joueur i , $u_i(\cdot, x_{-i})$ soit concave). Predtetchinski démontra que, tout comme le graphe des équilibres de Nash mixtes associé aux jeux finis, $\mathcal{N}_{\mathcal{F}}$ est également sans nœuds.

L'objectif principal de cette thèse est de contribuer à un programme d'étude en théorie des réseaux (pour la stabilité par paires) et en théorie des jeux (pour la notion d'équilibre de Nash) qui se rapporte à celui qui a d'ores et déjà été suivi en théorie des jeux depuis les années 1970-1980. Plus particulièrement, cette thèse consiste d'abord à poser les premières fondations de l'étude du *graphe des réseaux stables par paires* (l'ensemble des paires (v, g) , où v est une société, et g un réseau stable par paires par rapport à v) en s'intéressant à sa structure topologique et à son imparité générique, et ce dans un cadre plus général que le cas "mixte" (i.e. multilinéaire) en considérant des fonctions de paiements polynomiales satisfaisant des hypothèses "classiques"

de concavité. D'autre part, cette thèse a également pour but d'étendre ce qui a déjà été fait jusqu'à présent pour le graphe des équilibres de Nash (comme précédemment, du point de vue de sa structure topologique et de celui de son imparité générique), et ce dans le cas d'ensembles de stratégies semi-algébriques² et de fonctions de paiements polynomiales satisfaisant les hypothèses de concavité précédemment mentionnées.

Le premier chapitre est un travail en commun avec P. Bich dans lequel nous commençons par étudier la structure du graphe des réseaux stables par paires associé à tout sous-ensemble de sociétés \mathcal{C}^1 et concaves en poids propres (i.e. les profils $(v_i)_{i \in N}$ de fonctions de paiements continument différentiables et telles que, pour tout lien ij et tout profil de poids g_{-ij} autres que celui du lien ij , $v_i(\cdot, g_{-ij})$ et $v_j(\cdot, g_{-ij})$ soient concaves) de la forme $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$, et clos pour la somme avec un ensemble particulier \mathcal{A} de sociétés affines (nous appelons \mathcal{A} -réguliers ces ensembles de sociétés). Dans la lignée de Kohlberg-Mertens et de Predtetchinski, nous montrons que le graphe des réseaux stables par paires associé à tout ensemble \mathcal{A} -régulier de sociétés est homéomorphe à ce même-ensemble de sociétés, également par un homéomorphisme qui est proprement homotopique à une projection. Ensuite, nous considérons certains ensembles \mathcal{A} -réguliers de sociétés que nous appelons \mathcal{A} -semi-algébriquement réguliers; il s'agit de ceux tels que pour tout agent i , \mathcal{V}_i contient uniquement des fonctions polynomiales et dont l'ensemble de coefficients (réels) associé est semi-algébrique. Nous démontrons que *génériquement*, toute société appartenant à un ensemble \mathcal{A} -semi-algébriquement régulier de sociétés admet un nombre impair de réseaux stables par paires; ce résultat est dans l'esprit de celui de Wilson, mais avec des classes assez larges de sociétés.

Le second chapitre est également un travail en commun avec P. Bich dans lequel nous transposons le vocabulaire du premier chapitre (ensembles \mathcal{A} -réguliers, ensembles \mathcal{A} -semi-algébriquement réguliers) à la théorie des jeux, et dans lequel nous généralisons nos résultats. Tout d'abord, nous ne supposons plus que les ensembles considérés de jeux soient sous la forme $\prod_{i \in N} \mathcal{U}_i$; nous considérons des ensembles \mathcal{A} -réguliers arbitraires. Nous obtenons une légère amélioration du théorème de Predtetchinski en considérant ces ensembles \mathcal{A} -réguliers de jeux. De plus, nous considérons une régularité moins forte que la régularité \mathcal{A} -semi-algébrique: la *régularité fortement $\dim(\mathcal{L})$ -semi-algébrique*. A savoir, nous considérons un ensemble particulier \mathcal{L} de jeux

²Un ensemble est appelé *semi-algébrique* s'il peut s'écrire sous la forme $\bigcup_{p=1}^s \bigcap_{q=1}^{r_p} \{x \in \mathbb{R}^m : f_{p,q}(x) \star_{p,q} 0\}$, où $\star_{p,q}$ peut être remplacé soit par $<$ ou par $=$, et $f_{p,q} \in \mathbb{R}[x_1, \dots, x_m]$, pour tout $p = 1, \dots, s$ et tout $q = 1, \dots, r_p$.

linéaires, et nous nous intéressons aux sous-ensemble de jeux \mathcal{C}^1 et concaves en stratégie propre (plus nécessairement sous la forme d'un produit cartésien), clos pour la somme avec \mathcal{L} , et enfin, contenant uniquement des fonctions polynomiales à terme constant nul et dont l'ensemble de coefficients (réels) associé est semi-algébrique. Avec ces ensembles, nous améliorons le théorème de Wilson en démontrant la généricité de l'imparité du nombre d'équilibres de Nash, pour tout jeu dans n'importe quel ensemble $\dim(\mathcal{L})$ -fortement semi-algébriquement régulier de jeux.

Le dernier chapitre de cette thèse est un travail réalisé seul qui consiste en l'approfondissement de l'analyse faite dans le premier chapitre (en théorie des réseaux). Plus précisément, nous montrons que le graphe des réseaux stables par paires associé à n'importe quel ensemble \mathcal{A} -régulier \mathcal{V} de sociétés n'a pas de nœuds (i.e. il existe une isotopie ambiante entre le graphe des réseaux stables par paires associé à \mathcal{V} et une copie triviale de \mathcal{V} lui-même), suivant ainsi les résultats de Demichelis-Germano et de Predtetchinski. Dans un deuxième temps, nous introduisons la notion de *dynamique de réseaux* (analogue à celle de champ de Nash en théorie des jeux); une famille de champs de vecteurs sur un ensemble \mathcal{A} -régulier \mathcal{V} arbitraire de sociétés dont les points stationnaires coïncident avec les réseaux stables par paires associés aux sociétés de \mathcal{V} . Finalement, nous utilisons notre précédent résultat afin de montrer que n'importe quelles dynamiques de réseaux D et D' sur un ensemble \mathcal{A} -régulier \mathcal{V} arbitraire sont homotopes au sein même de l'ensemble de toutes les dynamiques de réseaux sur \mathcal{V} , suivant encore les résultats de Demichelis-Germano. De ce dernier théorème résulte enfin notre résultat d'égalité des indices des dynamiques de réseaux sur n'importe quel ensemble \mathcal{A} -régulier de sociétés.

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Part I

Introduction

$B \setminus A$	<i>Relative complement of a set A in a set B</i>
$\text{card}(A)$	<i>Cardinal of a set A</i>
$\mathcal{F}(A, B), B^A, \{A \rightarrow B\}$	<i>Set of all maps from a set A to a set B</i>
$\Psi : A \twoheadrightarrow B$	<i>A correspondence Ψ from a set A to a set B (i.e. a map from A to the power set of B).</i>
$\text{Cst}(A, B)$	<i>Set of all constant maps from a set A to a set B</i>
$\mathcal{C}^0(X, Y)$	<i>Set of all continuous maps from a topological space X to a topological space Y</i>
$L_n(E_1, \dots, E_n, F)$	<i>Set of all multilinear maps from a vector space $E_1 \times \dots \times E_n$ to a vector space F</i>
$A_n(E_1, \dots, E_n, F)$	<i>Set of all multiaffine maps from an affine space $E_1 \times \dots \times E_n$ to an affine space F</i>

Table 1: *Some mathematical notations*

Throughout all this thesis, if no details are provided:

1. Any subset of some topological space is endowed with the induced topology.
2. Any finite cartesian product of topological spaces is endowed with the product topology.
3. Any Euclidean space is endowed with the Euclidean topology.

Strategic network formation theory and pairwise stability

For several years, network formation theory is gaining importance in economic theory, and in particular, strategic network formation theory. In this context, the original structure of interest is the one of *unweighted society*, which is composed by:

- (i) a finite set N such that $\text{card}(N) \geq 2$, called *set of agents*, which induced the set

$$L = \{\{i, j\} : i, j \in N, j \neq i\},$$

called the *set of links (on N)*;³

- (ii) a family $(v_i)_{i \in N}$ of maps from the set

$$\mathbb{G}_u = \{L \rightarrow \{0, 1\}\}$$

of (*unweighted*) *networks (on N)* to the set \mathbb{R} of real numbers, where for every $i \in N$, v_i is called the *payoff function of agent i* .⁴

Informally, considering an unweighted society $(N, (v_i)_{i \in N})$, the set L of links can be seen as the set of possible relationships between two distinct agents, and the set \mathbb{G}_u of networks can be seen as the set of possible “intensity measures” of the different relationships (0 meaning a no strength relationship, 1 meaning a full strength relationship). For example, consider an unweighted society defined by a group of people, where networks measure friendship relationships between any two persons, or an unweighted society defined by a group of researchers, where networks measure co-author relationships between any two colleagues. On the other hand, payoff function of any agent $i \in N$ represents his or her linking preferences.

In 1996, Jackson and Wolinsky [32] introduced *pairwise stability* concept in the framework of unweighted societies, which is often applied in network formation theory in order to predict which unweighted networks are likely to arise in a strategic setting. In this context, a network is said to be pairwise stable if: (i) no single agent could gain by severing one of his or her link; (ii) no two agents could gain from linking. Formally, $g \in \mathbb{G}_u$ is *pairwise stable with respect to $(N, (v_i)_{i \in N})$* if for every link $ij \in L$:

- (i) either $g_{ij} = 1$, $v_i(0, g_{-ij}) \leq v_i(g)$ and $v_j(0, g_{-ij}) \leq v_j(g)$;

³Every link $\{i, j\} \in L$ is denoted ij .

⁴For every link $ij \in L$ and every network $g \in \mathbb{G}_u$, $g(ij)$ is denoted g_{ij} .

(ii) or $g_{ij} = 0$, $v_i(1, g_{-ij}) \leq v_i(g)$ or $v_j(1, g_{-ij}) \leq v_j(g)$.

In 2001, Jackson and Watts [31] established their existence theorem of a pairwise stable network.

Jackson-Watts' existence theorem. *Every unweighted society admits a pairwise stable network or a closed cycle of networks.*⁵

An important point of this theorem is that it does not state that a pairwise stable network always exists: there exists unweighted societies which does not admit any pairwise stable network (e.g. see [31], Example 5).

In 2020, Bich and Morhaim [6] extended pairwise stability concept to *weighted societies*, another kind of structure used in strategic network formation theory which is composed by:

- (i) a set N of agents (i.e. finite, with $\text{card}(N) \geq 2$);
- (ii) a family $(v_i)_{i \in N}$ of maps from the set

$$\mathbb{G} = \{L \rightarrow [0, 1]\}$$

of (*weighted*) *networks* (on N) to the set \mathbb{R} of real numbers, where for every $i \in N$, v_i is called the *payoff function of agent i* .

Throughout this introduction, we consider a fixed set N of agents. Then, the set of all weighted societies (with respect to N) can be identified to the set

$$\text{Soc} = \{N \rightarrow \{\mathbb{G} \rightarrow \mathbb{R}\}\}.$$

The difference with the previous structure (i.e. with the structure of unweighted society) is that weighted networks assign a weight between 0 and 1 to the different relationships (a weight closed to 0 means a nearly no strength relationship, a weight closed to 1 means a nearly full strength relationship), i.e. these weights can be different from 0 or 1. In this framework, a network is said to be pairwise stable if: (i) no single agent could gain by decreasing one of his or her weights; (ii) no two agents could gain from increasing their common weight. Formally, $g \in \mathbb{G}$ is *pairwise stable with respect to* $(N, (v_i)_{i \in N})$ if for every link $ij \in L$:

⁵An *improving path* in $C \subset \mathbb{G}_u$ from $g \in C$ to $g' \in C$ is a finite sequence $g^0 = g, g^1, \dots, g^{\ell-1}, g^\ell = g'$ of networks in C such that for every $k \in \{1, \dots, \ell - 1\}$: (i) either $g_{ij}^{k+1} < g_{ij}^k$ and $(v_i(g^{k+1}) > v_i(g^k)$ or $v_j(g^{k+1}) > v_j(g^k)$), for some unique $ij \in L$; (ii) or $g_{ij}^{k+1} > g_{ij}^k$ and $(v_i(g^{k+1}) > v_i(g^k)$ and $v_j(g^{k+1}) > v_j(g^k)$), for some unique $ij \in L$. A *closed cycle of networks* corresponds to a subset $C \subset \mathbb{G}_u$ of networks such that for every $g, g' \in C$, there exists an improving path in C from g to g' .

- (i) for every $w \in [0, g_{ij})$, $v_i(w, g_{-ij}) \leq v_i(g)$ and $v_j(w, g_{-ij}) \leq v_j(g)$;
- (ii) for every $w \in (g_{ij}, 1]$, $v_i(w, g_{-ij}) \leq v_i(g)$ or $v_j(w, g_{-ij}) \leq v_j(g)$.

Bich and Morhaim proved the existence of a pairwise stable weighted network for large classes of payoff functions.

Bich-Morhaim's existence theorem. *Every \mathcal{C}^0 own-weights quasiconcave weighted society admits a pairwise stable network.*⁶

Now, in order to introduce the general motivation of this thesis, let us make a detour to game theory, starting by one of the most famous result in this area: Kohlberg-Mertens' theorem [33].

Some useful definitions

Before to present Kohlberg-Mertens' theorem, let us introduce some useful definitions. A (*strategic-form*) *game* is composed by:

- (i) a nonempty finite set I , called the set of *players*;
- (ii) for every player $i \in I$, a nonempty set X_i , called the set of *strategies of player i* ;
- (iii) a family $(u_i)_{i \in I}$ of *payoff functions* from the set

$$X = \prod_{i \in I} X_i$$

of *strategy profiles* to the set \mathbb{R} of real numbers.

Recall that $x \in X$ is a *Nash equilibrium* of $(I, (X_i)_{i \in I}, (u_i)_{i \in I})$ if for every $i \in I$ and every $d_i \in X_i$, $u_i(d_i, x_{-i}) \leq u_i(x)$.

Throughout this introduction, we consider a fixed set I of players and a fixed family $(X_i)_{i \in I}$ of sets of strategies. Then, the set of all games (with respect to I and $(X_i)_{i \in I}$) can be identified to the set

$$\mathbf{Gam} = \{I \rightarrow \{X \rightarrow \mathbb{R}\}\}.$$

Furthermore, if for every $i \in I$, X_i is finite, then \mathbf{Gam} itself can be identified to \mathbb{R}^{np} , where $n = \text{card}(I)$ and $p = \prod_{i \in I} \text{card}(X_i)$.

Let $\mathcal{U} \subset \mathbf{Gam}$ be a set of games:

⁶A weighted society is: (i) \mathcal{C}^0 if payoff function of any agent is continuous; (ii) *own-weights quasiconcave* if for every agent $i \in N$, every $j \neq i$ and every $g_{-ij} : L \setminus ij \rightarrow [0, 1]$, the function $v_i(\cdot, g_{-ij}) : w \in [0, 1] \mapsto v_i(w, g_{-ij}) \in \mathbb{R}$ is quasiconcave.

- The *graph of Nash equilibria associated to \mathcal{U}* is defined as

$$\mathcal{N}_{\mathcal{U}} = \{(u, x) \in \mathcal{U} \times X : x \text{ is a Nash equilibrium of } u\}.$$

- Consider a fixed strategy profile $x^0 \in X$, and consider an arbitrary topology on the set \mathcal{U} . Suppose that for every $i \in I$, X_i is a convex compact subspace of \mathbb{R}^{m_i} (for some $m_i \in \mathbb{N}$). Furthermore, consider the space $\tilde{X} = \prod_{i \in I} \tilde{X}_i$, where for every $i \in I$, \tilde{X}_i is a nonempty convex compact subspace of \mathbb{R}^{m_i} which contains X_i in its relative interior (\tilde{X} can be seen as an “enlargement” of X). The *unknot associated to \mathcal{U} (and x^0)* is the topological embedding

$$\begin{cases} \kappa_{\mathcal{U}}^0 : \mathcal{U} & \rightarrow & \mathcal{U} \times \tilde{X} \\ u & \mapsto & (u, x^0) \end{cases}.$$

Now, suppose that there exists also a homeomorphism f from \mathcal{U} to $\mathcal{N}_{\mathcal{U}}$. The *knot of Nash equilibria associated to \mathcal{U} and f* is the topological embedding

$$\begin{cases} \kappa_f : \mathcal{U} & \rightarrow & \mathcal{U} \times \tilde{X} \\ u & \mapsto & f(u) \end{cases}.$$

Kohlberg-Mertens’ theorem

Suppose that $I = \{1, \dots, n\}$ and that for every $i \in I$, X_i corresponds to the unit $m_i - 1$ -simplex $\Delta_i^{m_i-1} \subset \mathbb{R}^{m_i}$ (for some $m_i \in \mathbb{N}$). Moreover, consider the set

$$\mathcal{L}^n(X, \mathbb{R}) = \{I \rightarrow L_n(X_1, \dots, X_n, \mathbb{R})\}$$

of *multilinear games*.⁷ Recall that

$$\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} = \{(u, x) \in \mathcal{L}^n(X, \mathbb{R}) \times X : x \text{ is a Nash equilibrium of } u\}.$$

Kohlberg-Mertens’ structure theorem. *The graph $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$ of Nash equilibria associated to $\mathcal{L}^n(X, \mathbb{R})$ is homeomorphic to $\mathcal{L}^n(X, \mathbb{R})$, through a homeomorphism $\eta_{\mathcal{L}^n(X, \mathbb{R})} : \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} \rightarrow \mathcal{L}^n(X, \mathbb{R})$ which is properly homotopic to the projection $\pi_{\mathcal{L}^n(X, \mathbb{R})} : \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} \rightarrow \mathcal{L}^n(X, \mathbb{R})$, $(u, x) \mapsto u$.*

Kohlberg-Mertens’ theorem [33] can be also stated in a more common form (however, the precedent formulation of the result will be more coherent with

⁷The vector space $L_n(X_1, \dots, X_n, \mathbb{R})$ of all multilinear maps from $X = \prod_{i \in I} X_i$ to \mathbb{R} (with its usual operations) is finite-dimensional. Hence, there exists a unique topology which turns it into a Hausdorff topological vector space.

the other results of this thesis). For every $i \in I$, suppose that X_i is a finite set and that $m_i = \text{card}(X_i)$ (for some $m_i \in \mathbb{N}$), and let $\Sigma_i \subset \mathbb{R}^{m_i}$ be the space of *mixed strategies of player i* and $\Sigma = \prod_{i \in I} \Sigma_i$ be the space of *mixed strategy profiles*. Then, the space

$$\mathcal{N} = \{(u, \sigma) \in \text{Gam} \times \Sigma : \sigma \text{ is a mixed Nash equilibrium of } u\}$$

is homeomorphic to the space Gam of all finite games (which itself can be identified to \mathbb{R}^{np}), through a homeomorphism which is properly homotopic to the projection $\pi : \mathcal{N} \rightarrow \text{Gam}, (u, \sigma) \mapsto u$.

Intuitively, Kohlberg-Mertens' theorem states that: (i) the graph $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$ of Nash equilibria associated to $\mathcal{L}^n(X, \mathbb{R})$ can be continuously deformed into some Euclidean space (which is topologically much simpler); (ii) the homeomorphism $\eta_{\mathcal{L}^n(X, \mathbb{R})} : \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} \rightarrow \mathcal{L}^n(X, \mathbb{R})$ itself can be continuously deformed into the simpler map $\pi_{\mathcal{L}^n(X, \mathbb{R})} : \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} \rightarrow \mathcal{L}^n(X, \mathbb{R})$.

In particular, applying topological degree to $\pi_{\mathcal{L}^n(X, \mathbb{R})}$, demonstrate the following result.

Corollary - Oddness theorem. *Generically, every game in $\mathcal{L}^n(X, \mathbb{R})$ admits an odd number of Nash equilibria.*

Hereafter is a sketch of the proof: from homotopy invariance of (topological) degree, the degree of $\pi_{\mathcal{L}^n(X, \mathbb{R})}$ is equal to the degree of $\eta_{\mathcal{L}^n(X, \mathbb{R})}$, and since $\eta_{\mathcal{L}^n(X, \mathbb{R})}$ is a proper homeomorphism from $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$ to $\mathcal{L}^n(X, \mathbb{R})$ ($\mathcal{L}^n(X, \mathbb{R})$ being homeomorphic to some Euclidean space), its degree is equal to ± 1 . From some semi-algebraic properties of $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$, $\mathcal{L}^n(X, \mathbb{R})$ and $\pi_{\mathcal{L}^n(X, \mathbb{R})}$, there exists a *generic* subset G of $\mathcal{L}^n(X, \mathbb{R})$ such that for every $u \in G$, $\pi_{\mathcal{L}^n(X, \mathbb{R})}^{-1}(u) = \{u\} \times \{x \in X : x \text{ is a Nash equilibrium of } u\}$ is a finite set $\{(u, x^1), \dots, (u, x^{\ell_u})\}$ (for some integer ℓ_u).⁸ The same semi-algebraic properties ensure that $\pi_{\mathcal{L}^n(X, \mathbb{R})}$ is a local homeomorphism at each (u, x^k) , $k \in \{1, \dots, \ell_u\}$, hence that its local degree at (u, x^k) has to be equal to ± 1 . Last, the sum of the local degrees of $\pi_{\mathcal{L}^n(X, \mathbb{R})}$ has to be equal to the degree of $\pi_{\mathcal{L}^n(X, \mathbb{R})}$, which finally implies that u admits an odd number of mixed Nash equilibria.

Briefly, oddness theorem states that “almost every” multilinear games admit an odd number of Nash equilibria, or equivalently, that “almost every” finite games admit an odd number of mixed Nash equilibria.⁹ Actually, this

⁸The intuition behind this concept is that G fills “almost completely” $\mathcal{L}^n(X, \mathbb{R})$.

⁹The term “generically” will be explained in the core of this thesis. For the moment, try to keep in mind the intuitive meaning.

oddness result is well-known since Wilson’s paper [43]. Since then, several authors provided results in the spirit of Wilson’s oddness theorem throughout different proof techniques, but always for mixed extensions of finite games (e.g. see Blume-Zame [7], Govindan-Wilson [24], Harsanyi [27], Mas-Colell [37], Govindan-McLennan [22], Pimienta [39] or Herings-Peeters [29]).¹⁰

Demichelis-Germano’s theorems

Several years latter, Demichelis and Germano [17] proposed an interesting improvement of Kohlberg-Mertens’ theorem. Suppose that for every $i \in I$, X_i corresponds to the unit $m_i - 1$ -simplex $\Delta_i^{m_i-1} \subset \mathbb{R}^{m_i}$ (for some $m_i \in \mathbb{N}$). Furthermore, consider a fixed strategy profile $x^0 \in X = \prod_{i \in I} X_i$, and consider an “enlargement” \tilde{X} of X (i.e. $\tilde{X} = \prod_{i \in I} \tilde{X}_i$, where for every $i \in I$, \tilde{X}_i is a nonempty convex compact subspace of \mathbb{R}^{m_i} which contains X_i in its relative interior). Recall that $\mathcal{L}^n(X, \mathbb{R})$ is the space of multilinear games, that the unknot associated to $\mathcal{L}^n(X, \mathbb{R})$ is the topological embedding

$$\left\{ \begin{array}{lcl} \kappa_{\mathcal{L}^n(X, \mathbb{R})}^0 : \mathcal{L}^n(X, \mathbb{R}) & \rightarrow & \mathcal{L}^n(X, \mathbb{R}) \times \tilde{X} \\ u & \mapsto & (u, x^0) \end{array} \right. ,$$

and that the knot of Nash equilibria associated to $\mathcal{L}^n(X, \mathbb{R})$ and $\rho_{\mathcal{L}^n(X, \mathbb{R})}$ is the topological embedding

$$\left\{ \begin{array}{lcl} \kappa_{\rho_{\mathcal{L}^n(X, \mathbb{R})}} : \mathcal{L}^n(X, \mathbb{R}) & \rightarrow & \mathcal{L}^n(X, \mathbb{R}) \times \tilde{X} \\ u & \mapsto & \rho_{\mathcal{L}^n(X, \mathbb{R})}(u) \end{array} \right.$$

where $\rho_{\mathcal{L}^n(X, \mathbb{R})} = \eta_{\mathcal{L}^n(X, \mathbb{R})}^{-1}$ is the inverse of Kohlberg-Mertens’ structure theorem homeomorphism.

Demichelis-Germano’s unknottedness theorem. *The knot $\kappa_{\rho_{\mathcal{L}^n(X, \mathbb{R})}}$ of Nash equilibria associated to $\mathcal{L}^n(X, \mathbb{R})$ and $\rho_{\mathcal{L}^n(X, \mathbb{R})}$ is ambient isotopic to the unknot $\kappa_{\mathcal{L}^n(X, \mathbb{R})}^0$ associated to $\mathcal{L}^n(X, \mathbb{R})$ within the ambient space $\mathcal{L}^n(X, \mathbb{R}) \times \tilde{X}$.*¹¹

¹⁰In general equilibrium theory, we can also mention the famous works of Debreu [14], of Dierker [18] and of Balasko [3] which provided similar results on the structure of the graph of Walrasian equilibria.

¹¹Recall that two topological embeddings e^1, e^2 from a topological space X to another topological space Y are said to be *ambient isotopic within Y* if there exists a continuous map $\theta : [0, 1] \times Y \rightarrow Y$ such that $\theta(0, \cdot) = \text{id}_Y$, $\theta(1, \cdot) \circ e^1 = e^2$ and for every $t \in [0, 1]$, $\theta(t, \cdot)$ is a homeomorphism.

Demichelis-Germano's theorem can be interpreted in a simpler way (similarly to the previous remark following Kohlberg-Mertens' theorem, the precedent way of writing Demichelis-Germano's theorem will be more coherent with the other results of this thesis). For every $i \in I$, suppose that X_i is a finite set and that $m_i = \text{card}(X_i)$ (for some $m_i \in \mathbb{N}$), and let $\Sigma_i \subset \mathbb{R}^{m_i}$ be the space of mixed strategies of player i and $\Sigma = \prod_{i \in I} \Sigma_i$ be the space of mixed strategy profiles. Moreover, let $\sigma^0 \in \Sigma$ be a fixed strategy profile. Then, the space

$$\mathcal{N} = \{(u, \sigma) \in \mathbf{Gam} \times \Sigma : \sigma \text{ is a mixed Nash equilibrium of } u\}$$

can be continuously deformed into a trivial copy $\mathbf{Gam} \times \{\sigma^0\}$ of the space \mathbf{Gam} of all finite games associated to I and $(X_i)_{i \in I}$, within the ambient space $\mathbf{Gam} \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is an "enlargement" of the space Σ of mixed strategy profiles (in a similar sense as in the previous paragraph).

The key idea in this theorem is that not only the graph $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})} \subset \mathcal{L}^n(X, \mathbb{R}) \times \tilde{X}$ of Nash equilibria associated to $\mathcal{L}^n(X, \mathbb{R})$ is deformed, but the entire space $\mathcal{L}^n(X, \mathbb{R}) \times \tilde{X}$ itself. Hence, a part of Kohlberg-Mertens' theorem (more precisely, the " $\mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$ is homeomorphic to $\mathcal{L}^n(X, \mathbb{R})$ " part) can be seen as a corollary of Demichelis-Germano's unknottedness theorem.¹²

Also, Demichelis and Germano used their unknottedness result in order to obtain interesting insights regarding *Nash fields* (or *Nash dynamics*); this notion has been studied extensively by many authors (e.g. see Demichelis-Germano [15], Govindan-Wilson [23], Gul-Pearce-Stacchetti [26] or Ritzberger [41]).¹³

Demichelis-Germano's theorem on extended Nash fields. *Any two extended Nash fields are homotopic within the set of all extended Nash fields.*

More explicitly, the previous theorem asserts that two extended Nash fields are homotopic, through a homotopy H such that for every $t \in [0, 1]$, H_t is

¹²Remark that by definition, if two topological embeddings $e^1, e^2 : X \rightarrow Y$ are ambient isotopic within Y , then $e^1(X)$ and $e^2(X)$ are homeomorphic. Here, observe that $\kappa_{\rho_{\mathcal{L}^n(X, \mathbb{R})}}(\mathcal{L}^n(X, \mathbb{R})) = \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$ and $\kappa_{\mathcal{L}^n(X, \mathbb{R})}^0(\mathcal{L}^n(X, \mathbb{R})) = \mathcal{L}^n(X, \mathbb{R}) \times \{x^0\}$, where $\mathcal{L}^n(X, \mathbb{R}) \times \{x^0\}$ is trivially homeomorphic to $\mathcal{L}^n(X, \mathbb{R})$.

¹³Recall that *Nash fields* correspond to families $(D_u)_{u \in \mathcal{L}^n(X, \mathbb{R})}$ of vector fields on the space $X = \prod_{i \in I} X_i = \prod_{i \in I} \Delta_i^{m_i-1}$ such that for every $u \in \mathcal{L}^n(X, \mathbb{R})$: (i) the set of zeros of the vector field D_u corresponds to the set of Nash equilibria of u ; (ii) D_u weakly points to the interior of $\{u\} \times X$ along its boundary. Also, *extended Nash fields* correspond to families $(\tilde{D}_u)_{u \in \mathcal{L}^n(X, \mathbb{R})}$ of vector fields on the "enlarged" space \tilde{X} such that for every $u \in \mathcal{L}^n(X, \mathbb{R})$: (i) the set of zeros of the vector field \tilde{D}_u corresponds to the set of Nash equilibria of u ; (ii) \tilde{D}_u strongly points to the interior of $\{u\} \times X$ on $\{u\} \times (\tilde{X} \setminus X)$. For some reminders about Nash fields, see Laraki-Renault-Sorin [35], pp. 84-85, or Demichelis-Germano [17].

also an extended Nash field. In particular, this result implies that for every $u \in \mathcal{L}^n(X, \mathbb{R})$, the index of \tilde{D}_u at any of its zeros has to be equal to the index of \tilde{D}'_u at this same zero, for any two extended Nash fields \tilde{D} and \tilde{D}' (this corresponds to homotopy invariance of indices).

Predtetchinski's theorems

More recently, in 2006, Predtetchinski [40] proposed an important improvement regarding both Kohlberg-Mertens' structure theorem [33] and Demichelis-Germano's unknottedness theorem [17].

Suppose that for every $i \in I$, the set X_i of strategies of player i is any convex compact subspace of \mathbb{R}^{m_i} with nonempty interior, for some fixed integer $m_i \in \mathbb{N}$. Moreover, for every $i \in I$, consider the space

$$\begin{aligned} \mathcal{F}_i = & \{u_i \in \mathcal{F}(X, \mathbb{R}) : \forall x_{-i} \in X_{-i}, u_i(\cdot, x_{-i}) \text{ is concave}\} \\ & \cap \{u_i \in \mathcal{C}^0(X, \mathbb{R}) : \forall \ell \in \{1, \dots, m_i\}, \frac{\partial u_i}{\partial x_{i,\ell}} \text{ exists and is continuous}\} \end{aligned}$$

and consider the space

$$\mathcal{F} = \prod_{i \in I} \mathcal{F}_i$$

of *own-strategy \mathcal{C}^1 concave games*.¹⁴

Predtetchinski's first result is similar to Kohlberg-Mertens' structure theorem, but for a wider class of payoff functions.

Predtetchinski's structure theorem. *The graph $\mathcal{N}_{\mathcal{F}}$ of Nash equilibria associated to \mathcal{F} is homeomorphic to \mathcal{F} , through a homeomorphism $\eta_{\mathcal{F}}$ which is properly homotopic to the projection $\pi_{\mathcal{F}}$.*

Now, consider a fixed strategy profile $x^0 \in X$, and consider an "enlargement" \tilde{X} of X (i.e. $\tilde{X} = \prod_{i \in I} \tilde{X}_i$, where for every $i \in I$, \tilde{X}_i is a nonempty convex compact subspace of \mathbb{R}^{m_i} which contains X_i in its relative interior). Recall that the unknot associated to \mathcal{F} is the topological embedding

$$\left\{ \begin{array}{l} \kappa_{\mathcal{F}}^0 : \mathcal{F} \rightarrow \mathcal{F} \times \tilde{X} \\ u \mapsto (u, x^0) \end{array} \right. ,$$

¹⁴For every $i \in I$, the set \mathcal{F}_i is endowed with the topology generated by all subsets of \mathcal{F}_i of the form $\{u_i \in \mathcal{F}_i : \forall x = (x_i, x_{-i}) \in K, u_i(x) \in O \text{ and } \nabla_{x_i} u_i(\cdot, x_{-i}) \in O'\}$, where K is a compact subspace of X , O is an open subset of \mathbb{R} and O' is an open subset of \mathbb{R}^{m_i} , and where $\nabla_{x_i} u_i(\cdot, x_{-i})$ denotes the gradient of $u_i(\cdot, x_{-i})$ at x_i (for more details, see [40]).

and that the knot of Nash equilibria associated to \mathcal{F} and $\rho_{\mathcal{F}}$ is the topological embedding

$$\begin{cases} \kappa_{\rho_{\mathcal{F}}} : \mathcal{F} & \rightarrow & \mathcal{F} \times \tilde{X} \\ u & \mapsto & \rho_{\mathcal{F}}(u) \end{cases}$$

where $\rho_{\mathcal{F}} = \eta_{\mathcal{F}}^{-1}$ is the inverse of Predtetchinski's structure theorem homeomorphism.

Predtetchinski's second result is similar to Demichelis-Germano's unknottedness theorem, but again, for a bigger set of payoff functions.

Predtetchinski's unknottedness theorem. *The knot $\kappa_{\rho_{\mathcal{F}}}$ of Nash equilibria associated to \mathcal{F} and $\rho_{\mathcal{F}}$ is ambient isotopic to the unknot $\kappa_{\mathcal{F}}^0$ associated to \mathcal{F} within the ambient space $\mathcal{F} \times \tilde{X}$.*

Contributions of this thesis

Chapter one

In this first chapter, we come back to (strategic) network formation theory and to the notion of pairwise stability (in the case of weighted networks). In views of what we said before, an immediate idea is that it seems quite important to contribute to the analysis of the concept of pairwise stability, by taking to refer Nash equilibrium's story in game theory. We try to do so by providing two main results: our *structure theorem* and our *oddness theorem*. Recall that N is a fixed set of agents and that

$$\mathbf{Soc} = \{N \rightarrow \{\mathbb{G} \rightarrow \mathbb{R}\}\}$$

is the set of all societies (with respect to N). Also, for every $\mathcal{V} \subset \mathbf{Soc}$, the *graph of pairwise stable networks associated to \mathcal{V}* is defined as

$$\mathcal{P}_{\mathcal{V}} = \{(v, g) \in \mathcal{V} \times \mathbb{G} : g \text{ is pairwise stable with respect to } v\}.$$

Before presenting our theorems, we introduce the notions of *\mathcal{A} -regular set of societies* and of *\mathcal{A} -semi-algebraically regular set of societies* which are at the core of this thesis. First, for every $i \in N$, consider the space

$$\begin{aligned} \mathcal{F}_i &= \{v_i \in \mathcal{F}(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \text{ is concave}\} \\ &\cap \{v_i \in \mathcal{C}^0(X, \mathbb{R}) : \forall j \neq i, \frac{\partial v_i}{\partial g_{ij}} \text{ exists and is continuous}\} \end{aligned}$$

and consider the space

$$\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$$

of *own-weights* \mathcal{C}^1 *concave societies*.¹⁵ First, for every $i \in N$, let: (i) \mathcal{A}_i be the set of payoff functions of agent i which are affine in $(g_{ij})_{j \neq i}$ and which only depend on those weights;¹⁶ (ii) $\mathbb{R}_{\delta_i}[g]$ be the set of payoff functions of agent i which are polynomial functions of g (with coefficients in \mathbb{R}) whose degree is less or equal to some fixed integer δ_i (for any $\mathcal{V}_i \subset \mathbb{R}_{\delta_i}[g]$, the set of coefficients associated to polynomial payoff functions in \mathcal{V}_i is denoted $\mathbf{C}_{\mathcal{V}_i}$).¹⁷ Then, for every $i \in N$, consider $\mathcal{V}_i \subset \mathcal{F}(\mathbb{G}, \mathbb{R})$, $\delta_i \in \mathbb{N}$ and consider the following properties:

1. (**Concavity**). $\mathcal{V}_i \subset \mathcal{F}_i$.
2. (**\mathcal{A} -invariance**). For every $v_i \in \mathcal{V}_i$ and every $a_i \in \mathcal{A}_i$, $v_i + a_i \in \mathcal{V}_i$.
3. (**Semi-algebraicity**). $\mathcal{V}_i \subset \mathbb{R}_{\delta_i}[g]$ and $\mathbf{C}_{\mathcal{V}_i}$ is a semi-algebraic set.

The set $\prod_{i \in N} \mathcal{V}_i$ of societies is called:

- (i) \mathcal{A} -regular if it satisfies concavity and \mathcal{A} -invariance assumptions;
- (ii) \mathcal{A} -semi-algebraically regular if it satisfies concavity, \mathcal{A} -invariance and semi-algebraicity assumptions.

A \mathcal{A} -semi-algebraically regular set of societies is a particular case of \mathcal{A} -regular set of societies whose payoff functions are polynomial, with additional assumptions on their associated sets of coefficients allowing to define it using a finite number of polynomial equalities or inequalities.

Bich-Fixary's structure theorem. *The graph $\mathcal{P}_{\mathcal{V}}$ of pairwise stable networks associated to any \mathcal{A} -regular set $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$ of societies is homeomorphic to \mathcal{V} , through a homeomorphism $\eta_{\mathcal{V}}$ which is properly homotopic to the projection $\pi_{\mathcal{V}}$.*

¹⁵For every $i \in N$, the set $\mathcal{D}_i = \{v_i \in \mathcal{C}^0(X, \mathbb{R}) : \forall j \neq i, \frac{\partial v_i}{\partial g_{ij}} \text{ exists and is continuous}\}$ together with the map $\|\cdot\|_i : v_i \in \mathcal{D}_i \mapsto \max\{\max\{\|v_i\|_{\infty}, \|\frac{\partial v_i}{\partial g_{ij}}\|_{\infty}\} : j \neq i\} \in \mathbb{R}$ is a normed vector space. Hence, for every $i \in N$, the topology on $\mathcal{F}_i \subset \mathcal{D}_i$ is induced by the norm $\|\cdot\|_i$.

¹⁶E.g. Suppose that $N = \{1, 2, 3\}$. If for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$, $v_1(g) = 2g_{12} + 4g_{23} + 1$, $v_2(g) = 2g_{12} + 1$ and $v_3(g) = 5g_{13} + 3g_{23}$, then $v_1 \notin \mathcal{A}_1$, $v_2 \in \mathcal{A}_2$ and $v_3 \in \mathcal{A}_3$.

¹⁷E.g. Suppose that $N = \{1, 2, 3\}$, and that $\delta_1 = \delta_3 = 1$ and $\delta_2 = 2$. If for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$, $v_1(g) = -g_{12}g_{23} + 3g_{12}g_{13} - g_{13}^2$, $v_2(g) = -g_{12}^2 + g_{13} - 5g_{23}^2$ and $v_3(g) = 5g_{13} - 2g_{23} + 3$, then $v_1 \notin \mathbb{R}_{\delta_1}[g]$, $v_2 \in \mathbb{R}_{\delta_2}[g]$ and $v_3 \in \mathbb{R}_{\delta_3}[g]$.

The intuition behind our structure theorem is similar to the one of Kohlberg-Mertens' theorem [33] and to the one of Predtetchinski's structure theorem [40], but for pairwise stable networks instead of Nash equilibria. However, our result works with many sets of payoff functions (those included in \mathcal{F} and which are closed under addition of any society in \mathcal{A}).

Bich-Fixary's oddness theorem. *Generically, every society in any \mathcal{A} -semi-algebraically regular set $\prod_{i \in N} \mathcal{V}_i$ of societies admits an odd number of pairwise stable networks.*

Similarly as before with the oddness theorem derived from Kohlberg-Mertens' theorem, our oddness theorem could be summarize by saying that “almost every” society in any \mathcal{A} -semi-algebraically regular set admits an odd number of pairwise stable networks. The difference between these two results lies in the fact that our theorem works for many sets of polynomial payoff functions, not only for multilinear payoff functions.¹⁸ We will show that our oddness theorem is a consequence of the topological structure of the graph $\mathcal{P}_{\mathcal{V}}$ of pairwise stable networks associated to any \mathcal{A} -regular set $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$ (characterized by our structure theorem).

We would like to emphasize that our theorems are not applications of existing results in game theory:

- First, by nature, a pairwise stable network is not a Nash equilibrium: two agents who want to create a link together have to decide it simultaneously (i.e. deviations have to be bilateral in some cases), whereas deviations are always unilateral in Nash equilibrium concept. More precisely, assessing that there exists generically an odd number of Nash equilibria is equivalent to say that there are generically an odd number of fixed-points of the best-reply correspondence, but there is no natural and analogue formulation for pairwise stability concept.
- Second, a notion of “mixed pairwise stability”, comparable to the notion of mixed Nash equilibrium in game theory, seems less meaningful in network formation theory. This explains also - apart from its mathematical interest - why we consider general sets of polynomial payoff functions, going beyond the case of multilinear payoff functions. However, this also creates new technical difficulties: to prove our oddness

¹⁸In particular, as a byproduct, we encompass a recent work of Herings and Zhan [30] which states the same oddness result, but for multilinear payoff functions (as a matter of fact, Herings and Zhan's paper also treats the problem of computation of pairwise stable networks, an issue that we do not consider here).

theorem, we have to enter more deeply in the field of real semi-algebraic geometry (in particular, we will provide some new decomposition result for semi-algebraic sets).

- Third, we have to “extend” Kohlberg-Mertens’ theorem to the framework of network formation theory and pairwise stability. Our extension (i.e. our structure theorem) is not only a rewriting of existing proofs in game theory, since, in essence, we do not deal with standard non-cooperative strategic-form games.¹⁹

To conclude this chapter, we describe several standard models in network formation theoretical literature to which our oddness theorem can be applied. A first example is the public good model of Bramoullé and Kranton [10]: each agent $i \in N$ is characterized by some level of effort $e_i \in [0, +\infty)$ (e.g. it could be the amount of time a consumer spends researching a new product) and interacts in some network $g \in \mathbb{G}$ (the idea being that agents could benefit from other agents’ efforts, thanks to network externalities). Oddness theorem implies that, in this standard model, there exists an odd number of pairwise stable networks, generically with respect to some parameters of the model (i.e., in short, there exists an odd number of pairwise stable networks for “most” parameters of the model). We prove similar results in the information transmission model of Calvó-Armengol and İlkılıç [11] and the two-way flow model of Bala and Goyal [2].

Chapter two

Originally, the principal objective of this second chapter was to transpose the results we obtained in the framework of network formation theory to the one of game theory; i.e. to obtain both a topological structure result and an oddness result similar to the ones of Chapter 1. Hence, this chapter has some common points with the preceding one. However, it has also non negligible differences that will be explained in a moment.

First of all, recall that I is a fixed set of players, that for every player $i \in I$, X_i is the fixed set of strategies of player i and that

$$\text{Gam} = \{I \rightarrow \{X \rightarrow \mathbb{R}\}\}$$

is the set of all games (with respect to I and $(X_i)_{i \in I}$). In this chapter, suppose that for every $i \in I$, the set X_i of strategies of player i is a convex compact

¹⁹However, several important ingredients of our proof comes from the proof of Pre-tetchinski’s structure theorem.

semi-algebraic subset of \mathbb{R}^{m_i} with nonempty interior (for some fixed integer $m_i \in \mathbb{N}$). Also, recall that

$$\begin{aligned} \mathcal{F}_i &= \{u_i \in \mathcal{F}(X, \mathbb{R}) : \forall x_{-i} \in X_{-i}, u_i(\cdot, x_{-i}) \text{ is concave}\} \\ &\cap \{u_i \in \mathcal{C}^0(X, \mathbb{R}) : \forall \ell \in \{1, \dots, m_i\}, \frac{\partial u_i}{\partial x_{i,\ell}} \text{ exists and is continuous}\}, \end{aligned}$$

with $\mathcal{F} = \prod_{i \in I} \mathcal{F}_i$ being the space of own-strategy \mathcal{C}^1 concave games.

For every $i \in I$, let:

- (i) \mathcal{A}_i be the set of payoff functions of player i which are affine in x_i and which only depend on this strategy, and let $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$.
- (ii) \mathcal{L}_i be the set of payoff functions of player i which are linear in x_i and which only depend on this strategy, and let $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i$.
- (iii) $\mathbb{R}_{\delta_i}[x]$ be the set of payoff functions of player i which are polynomial functions of x (with coefficients in \mathbb{R}) whose degree is less or equal to some fixed integer δ_i , and let $\mathbb{R}_{\delta}[x] = \prod_{i \in I} \mathbb{R}_{\delta_i}[x]$ (for any $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$, the set of coefficients associated to profiles of polynomial payoff functions in \mathcal{U} is denoted $\mathbf{C}_{\mathcal{U}}$).

Then, consider $\mathcal{U} \subset \mathbf{Gam}$, $\delta = (\delta_i)_{i \in \mathbb{N}}$ and consider the following properties:

1. **(Concavity)**. $\mathcal{U} \subset \mathcal{F}$.
2. **(\mathcal{A} -invariance)**. $\mathcal{U} + \mathcal{A} = \mathcal{U}$.
3. **(Semi-algebraicity)**. $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$ and $\mathbf{C}_{\mathcal{U}}$ is a semi-algebraic set.
4. **(Strong semi-algebraicity)**. $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$, payoff functions in \mathcal{U} do not have constant part, and $\mathbf{C}_{\mathcal{U}}$ is a semi-algebraic set.
5. **(dim(\mathcal{A})-invariance)**. The (semi-algebraic) dimension of $\mathbf{C}_{\mathcal{U}+\mathcal{A}}$ is equal to the (semi-algebraic) dimension of $\mathbf{C}_{\mathcal{U}}$.
6. **(dim(\mathcal{L})-invariance)**. The (semi-algebraic) dimension of $\mathbf{C}_{\mathcal{U}+\mathcal{L}}$ is equal to the (semi-algebraic) dimension of $\mathbf{C}_{\mathcal{U}}$.

The set \mathcal{U} of games is called:

- (i) \mathcal{A} -regular if it satisfies concavity and \mathcal{A} -invariance assumptions;
- (ii) \mathcal{A} -semi-algebraically regular if it satisfies concavity, semi-algebraicity and \mathcal{A} -invariance assumptions.
- (iii) dim(\mathcal{A})-semi-algebraically regular if it satisfies concavity, semi-algebraicity and dim(\mathcal{A})-invariance assumptions.

(iv) $\dim(\mathcal{L})$ -strongly semi-algebraically regular if it satisfies concavity, strong semi-algebraicity and $\dim(\mathcal{L})$ -invariance assumptions.

Observe that \mathcal{A} -semi-algebraically regular sets of games are particular cases of \mathcal{A} -regular set of games whose payoff functions are polynomial, with additional assumptions on their associated sets of coefficients. However, a $\dim(\mathcal{A})$ -semi-algebraically regular set of games might not verify \mathcal{A} -invariance assumption, which would imply that it might not be \mathcal{A} -semi-algebraically regular. Thus, $\dim(\mathcal{A})$ -semi-algebraic regularity is weaker than \mathcal{A} -semi-algebraic regularity.

Finally, even if $\dim(\mathcal{L})$ -strongly semi-algebraic regularity is not, strictly speaking, weaker than $\dim(\mathcal{A})$ -semi-algebraic regularity (from a logical point of view), it has the advantage to not take into account the constant part of any game, which is irrelevant regarding its set of Nash equilibria (i.e. adding constant terms to a game does not modify its set of Nash equilibria).

Bich-Fixary's structure theorem. *The graph $\mathcal{N}_{\mathcal{U}}$ of Nash equilibria associated to any \mathcal{A} -regular set \mathcal{U} of games is homeomorphic to \mathcal{U} , through a homeomorphism $\eta_{\mathcal{U}}$ which is properly homotopic to the projection $\pi_{\mathcal{U}}$.*

Bich-Fixary's oddness theorem. *Generically, every game in any $\dim(\mathcal{L})$ -strongly semi-algebraically regular set \mathcal{U} of games admits an odd number of Nash equilibria.*

As mentioned before, our results are closely related to the ones of Chapter 1. Indeed, analogously to our network theoretical oddness theorem in Chapter 1, our game theoretical oddness theorem states that “most” games should have an odd number of Nash equilibria, when payoff functions are polynomial and satisfy the “standard” concavity assumption. However, our game theoretical theorem is less assumptions demanding and the proof of this result is also slightly simplified at some points: (i) we do not require anymore \mathcal{U} to be a cartesian product of spaces $(\mathcal{U}_i)_{i \in I}$; (ii) we now drop the constant parts (which are not relevant when one deals with Nash equilibria); (iii) $\dim(\mathcal{A})$ -invariance assumption is weaker than \mathcal{A} -invariance assumption; (iv) we use topological degree of proper continuous maps between topological oriented m -manifolds instead of topological degree of continuous maps from the unit m -sphere to itself - a more technical approach, but which avoids complex details (which are in fact not necessary).

Very important point: every improvements that have been done in this chapter (compared to Chapter 1) could be transpose in network formation

theory without too much difficulty.²⁰

To conclude this chapter, we provide examples of *games on networks* (introduced in network formation theory) to which our oddness theorem can be applied. The main example being Jackson-Zenou’s benchmark quadratic model [44], from which several other models are derived from: Patacchini-Zenou’s model [38] about juvenile delinquency and conformism, Calvó-Armengol-Patacchini-Zenou’s model [12] about social networks in education, Konig-Liu-Zenou’s model [34] about R&D networks, Helsley-Zenou’s model [28] about social networks and interactions in cities, etc.

Chapter three

This thesis ends with this last chapter which constitutes a final contribution in network formation theory. This last apport consists in two mains results which are in the spirit of Demichelis and Germano’s theorems: our *unknottedness theorem* and our *dynamics equivalence theorem*.

Recall that N is a fixed set of agents, that $\mathbf{Soc} = \{N \rightarrow \{\mathbb{G} \rightarrow \mathbb{R}\}\}$ is the set of all societies (with respect to N), and that for every set $\mathcal{V} \subset \mathbf{Soc}$ of societies, $\mathcal{P}_{\mathcal{V}} = \{(v, g) \in \mathcal{V} \times \mathbb{G} : g \text{ is pairwise stable with respect to } v\}$ is the graph of pairwise stable networks associated to \mathcal{V} , and that $\rho_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{P}_{\mathcal{V}}$ is the inverse of the homeomorphism $\eta_{\mathcal{V}} : \mathcal{P}_{\mathcal{V}} \rightarrow \mathcal{V}$ of Bich-Fixary’s structure theorem (cf. Chapter 1).

Now, consider a fixed network $g^0 \in \mathbb{G}$ and a fixed $\varepsilon \in (0, +\infty)$. For any subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies, the space $\mathcal{V} \times \mathbb{G}^{\varepsilon}$, where $\mathbb{G}^{\varepsilon} = [-\varepsilon, 1 + \varepsilon]^L$, is called the (ε) -ambient space (associated to \mathcal{V}); it can be seen as an “enlargement” of $\mathcal{V} \times \mathbb{G}$. Also, the *unknot associated to \mathcal{V} (and g^0)* is defined as the topological embedding

$$\left\{ \begin{array}{l} \kappa_{\mathcal{V}}^0 : \mathcal{V} \rightarrow \mathcal{V} \times \mathbb{G}^{\varepsilon} \\ v \mapsto (v, g^0) \end{array} \right. .$$

Moreover, if f is a homeomorphism from \mathcal{V} to $\mathcal{P}_{\mathcal{V}}$, then the *knot of pairwise stable networks associated to \mathcal{V} and f* is defined as the topological embedding

$$\left\{ \begin{array}{l} \kappa_f : \mathcal{V} \rightarrow \mathcal{V} \times \mathbb{G}^{\varepsilon} \\ v \mapsto f(v) \end{array} \right. .$$

²⁰Chronologically, this chapter has been written after Chapter 1. In my opinion, it was very important to write Chapter 1 without the “upgrade” of Chapter 2; I think that this allows a better appreciation of the general progression of this thesis (“*Ce qui compte, c’est pas l’arrivée, c’est la quête*” - Orelsan).

The first goal of this chapter is to strengthen Bich-Fixary's structure theorem by showing that the graph $\mathcal{P}_{\mathcal{V}}$ of pairwise stable networks associated to any \mathcal{A} -regular set \mathcal{V} of societies is not only homeomorphic to \mathcal{V} , but that it can be continuously deformed to a trivial copy $\mathcal{V} \times \{g^0\}$ of the space \mathcal{V} within the ambient space $\mathcal{V} \times \mathbb{G}^\varepsilon$. Namely, we show that $\kappa_{\rho_{\mathcal{V}}}$ is ambient isotopic to $\kappa_{\mathcal{V}}^0$ within $\mathcal{V} \times \mathbb{G}^\varepsilon$.

Fixary's unknottedness theorem. *The knot $\kappa_{\rho_{\mathcal{V}}}$ of pairwise stable networks associated to any \mathcal{A} -regular set \mathcal{V} of societies and $\rho_{\mathcal{V}}$ is ambient isotopic to the unknot $\kappa_{\mathcal{V}}^0$ associated to \mathcal{V} within the ambient space $\mathcal{V} \times \mathbb{G}^\varepsilon$.*

This result is parallel to those of game theory, and in particular to the work of Predtetchinski [40] and to the work of Demichelis and Germano [17].

Now, in order to achieve our last goal, we introduce what we call *network dynamics* and *extended network dynamics*. Briefly, for any subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies, a network dynamic (resp. extended network dynamic) on \mathcal{V} is a family of vector fields $(D_v)_{v \in \mathcal{V}}$ on the set \mathbb{G} of networks (resp. $(D_v^\varepsilon)_{v \in \mathcal{V}}$ on the "enlarged" set $\mathbb{G}^\varepsilon = [-\varepsilon, 1 + \varepsilon]^L$ of networks) whose zeros coincide with the set of pairwise stable networks of v ($v \in \mathcal{V}$). The idea is to use the unknottedness theorem to demonstrate the second main result of this chapter: our dynamics equivalence theorem. This theorem states that for every \mathcal{A} -regular set \mathcal{V} of societies, any two *strongly inward-pointing* extended network dynamics on \mathcal{V} are homotopic within the set of all extended network dynamics on \mathcal{V} . In other words, any extended network dynamic that points toward \mathbb{G} outside of it (i.e. on $\mathbb{G}^\varepsilon \setminus \mathbb{G}$) can be continuously deformed into any other extended network dynamic with the same property, without adding additional zeros.

Fixary's dynamics equivalence theorem. *Any two strongly inward-pointing extended network dynamics on any \mathcal{A} -regular set \mathcal{V} of societies are homotopic within the set of all extended network dynamics on \mathcal{V} .*

To conclude this chapter, we provide some examples of network dynamics/extended network dynamics, and some consequences of the dynamics equivalence theorem.

A first result concerns the indices of extended network dynamics at any isolated zero. This result is called *indices equality theorem*.

Fixary's indices equality theorem. *Consider any \mathcal{A} -regular set \mathcal{V} of societies. For every strongly inward-pointing extended network dynamics \bar{D}*

and \tilde{D}' on \mathcal{V} and every $v \in \mathcal{V}$, the index of \tilde{D}_v at g is equal to the index of \tilde{D}'_v at g , for every isolated zero g of \tilde{D}_v (resp. \tilde{D}'_v).

Another consequence of the dynamics equivalence theorem comes from the following proposition: for every subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies, any *inward-pointing* network dynamic D on \mathcal{V} admits an *extension to the ambient space* $\mathcal{V} \times \mathbb{G}^\varepsilon$ (i.e. a strongly inward-pointing extended network dynamic D^ε on \mathcal{V} such that its restriction to $\mathcal{V} \times \mathbb{G}$ corresponds to D).²¹ Hence, what follows from dynamics equivalence theorem/indices equality theorem and from the last proposition is that for every \mathcal{A} -regular set \mathcal{V} of societies, every inward-pointing network dynamic D on \mathcal{V} and every extension D^ε of D to the ambient space $\mathcal{V} \times \mathbb{G}^\varepsilon$, the index of D^ε_v ($v \in \mathcal{V}$) at any isolated zero g (of D_v) does not depend on the choice of this extension.²² Thus, even if we technically cannot talk about the index of D_v ($v \in \mathcal{V}$) at an isolated zero g on the boundary $\partial\mathbb{G}$ of \mathbb{G} , we can still describe how D_v behaves around g using any extension D^ε .

Last, another version of the indices equality theorem is proposed, when one considers \mathcal{A} -semi-algebraically regular sets of societies instead of \mathcal{A} -regular sets of societies, and which is based on Bich-Fixary's oddness theorem.

Fixary's indices equality theorem with semi-algebraic regularity.

Consider any \mathcal{A} -semi-algebraically regular set \mathcal{V} of societies. Generically, for every strongly inward-pointing extended network dynamic \tilde{D} and \tilde{D}' on \mathcal{V} and every society v in \mathcal{V} , the index of \tilde{D}_v at g is equal to the index of \tilde{D}'_v at g , for every zero g of \tilde{D}_v (resp. \tilde{D}'_v).

The important difference between the two versions of the indices equality theorem is the following: if one considers a \mathcal{A} -semi-algebraically regular set \mathcal{V} of societies instead of a \mathcal{A} -regular set of societies which is not \mathcal{A} -semi-algebraically regular, then for every strongly inward-pointing extended network dynamic \tilde{D} and \tilde{D}' on \mathcal{V} and "almost every" society $v \in \mathcal{V}$, we are now able to talk about the equality of the indices of \tilde{D}_v and \tilde{D}'_v at any zero, since in that case, each zero is by definition isolated.

²¹An inward-pointing network dynamic on \mathcal{V} is a dynamic which points weakly to the interior of $\{v\} \times \mathbb{G}$ along the boundary $\{v\} \times \partial\mathbb{G}$, for every $v \in \mathcal{V}$.

²²Recall that for a given vector field V on an arbitrary smooth manifold, and a given isolated zero z of V , the index of V at z is an indicator which helps to quantify the behavior of V around z (i.e. V may circulate around z , it may have a source, a sink, a saddle, etc.).

Part II

Thesis chapters

Chapter 1

Topological Structure and Generic Oddness of the Graph of Pairwise Stable Networks

\mathbb{G}	Set of weighted networks on N
$\mathcal{P}_{\mathcal{V}}$	Graph of pairwise stable networks associated to the set \mathcal{V} of societies
$\pi_{\mathcal{V}}$	Projection from $\mathcal{P}_{\mathcal{V}}$ to \mathcal{V} , where \mathcal{V} is a set of societies
\mathcal{F}_i	Set of continuous payoff functions of agent i which are concave in g_{ij} and with continuous first-order derivative with respect to the ij -th variable, for every $j \neq i$ ($\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$)
\mathcal{A}_i	Set of payoff functions of agent i which are affine in $(g_{ij})_{j \neq i}$ and which only depend on those weights ($\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$)
$\eta_{\mathcal{R}}$	Homeomorphism from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} of structure theorem, where \mathcal{R} is a \mathcal{A} -regular set of societies
$\mathbb{R}_{\delta_i}[g]$	Set of payoff functions of agent i which are polynomial of g and whose degree is less or equal to δ_i ($\mathbb{R}_{\delta}[g] = \prod_{i \in N} \mathbb{R}_{\delta_i}[g]$)
φ_i	Vector space isomorphism which assigns to each payoff function of agent i in $\mathbb{R}_{\delta_i}[g]$ its coefficients in \mathbb{R}^{m_i} , where $m_i \in \mathbb{N}$ depends on δ_i ($m = \sum_{i \in N} m_i$, $\varphi = \times_{i \in N} \varphi_i : \mathbb{R}_{\delta}[g] \rightarrow \mathbb{R}^m$)
$\mathbf{C}_{\mathcal{V}_i}$	Set of coefficients of polynomial payoff functions of agent i in \mathcal{V}_i (i.e. $\mathbf{C}_{\mathcal{V}_i} = \varphi_i(\mathcal{V}_i)$), where $\mathcal{V}_i \subset \mathbb{R}_{\delta_i}[g]$
$\mathbf{C}_{\mathcal{V}}$	Set of coefficients of polynomial societies in \mathcal{V} (i.e. $\mathbf{C}_{\mathcal{V}} = \varphi(\mathcal{V})$), where $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i \subset \mathbb{R}_{\delta}[g]$
v^x	Polynomial society in \mathcal{V} whose coefficients correspond to x (i.e. $v^x = \varphi^{-1}(x)$), where $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i \subset \mathbb{R}_{\delta}[g]$ and where $x \in \mathbf{C}_{\mathcal{V}}$
$\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}$	Linear subspace of $\mathbb{R}_{\delta_i}[g]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[g]$, except the ones in \mathcal{A}_i
$\Pi_{-\mathcal{A}_i}$	Linear projection from $\mathbb{R}^{m_i} = \mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}} \oplus \mathbf{C}_{\mathcal{A}_i}$ to $\mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}}$
$(T_k^i)_{k=1}^{r_i}$	Semi-algebraic decomposition of $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})$, where $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$ is a \mathcal{A} -semi-algebraically regular set of societies and where $r_i \in \mathbb{N}$ depends on \mathcal{S}_i
$\mathcal{V}_{\lambda_i}^i$	Minkowski sum of $\varphi_i^{-1}(T_{\lambda_i}^i)$ and \mathcal{A}_i ($\mathcal{V}_{\lambda} = \prod_{i \in N} \mathcal{V}_{\lambda_i}^i$), where $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$ is a \mathcal{A} -semi-algebraically regular set of societies and where $\lambda \in \Lambda = \prod_{i \in N} \{1, \dots, r_i\}$
Λ^{op}	Subset of indicies $\lambda \in \Lambda$ such that for every $i \in N$, $T_{\lambda_i}^i$ is open in $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})$, where $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$ is a \mathcal{A} -semi-algebraically regular set of societies

Table 1.1: Table of notations of Chapter 1

This chapter is based on the research paper *Network formation and pairwise stability: A new oddness theorem*, published in *Journal of Mathematical Economics* (December 2022) by Philippe Bich and Julien Fixary [5].

1.1 Introduction

In this first chapter, we provide our *structure theorem* and our *oddness theorem* in network formation theory. The chapter is organized as follows: (i) in Subsection 1.2.1 (of Section 1.2), we first recall some basic definitions and notations about strategic network formation theory - in particular, *pairwise stability* concept - and we define the *graph of pairwise stable networks associated to any set of societies*; (ii) in Subsection 1.2.2, we introduce the notion of \mathcal{A} -regular set of societies and we present our structure theorem (Theorem 1.2.1); (iii) in Subsection 1.3.1 (of Section 1.3), we introduce the notion of \mathcal{A} -semi-algebraically regular set of societies and we present our oddness theorem (Theorem 1.3.1); (iv) in Subsection 1.3.2, we provide several examples of applications of oddness theorem; (v) in Section 1.4 (Appendix), we provide first the necessary reminders about specific notions of general topology and of differential calculus (Subsection 1.4.1) and about real semi-algebraic geometry (Subsection 1.4.2), and we provide next the proofs of structure theorem (Subsection 1.4.3) and of oddness theorem (Subsection 1.4.4).

1.2 Topological structure of the graph of pairwise stable networks

1.2.1 The graph of pairwise stable networks

First of all, let us recall some elementary definitions and notations from network formation theory.

Definition 1.2.1. A *set of agents* is a finite set N such that $\text{card}(N) \geq 2$. For every set N of agents, the set

$$L = \{\{i, j\} : (i, j) \in N \times N, i \neq j\}$$

is called the *set of links (on N)* and the set

$$\mathbb{G} = \{L \rightarrow [0, 1]\}$$

is called the *set of (weighted) networks (on N)*.

Remark 1.2.1. The vector space \mathbb{R}^L (with its usual operations) is endowed with the Euclidean norm, i.e.

$$\|\cdot\| : g \in \mathbb{R}^L \mapsto \sqrt{\sum_{ij \in L} g_{ij}^2} \in \mathbb{R},$$

and \mathbb{G} is endowed with the induced topology.

Definition 1.2.2. A (*weighted*) *society* is a couple $(N, (v_i)_{i \in N})$ composed by a set N of agents and a family $(v_i)_{i \in N}$ of maps from \mathbb{G} to \mathbb{R} , where for every $i \in N$, v_i is called the *payoff function of agent i* . For every set N of agents, the set of all societies whose the set of agents is equal to N can be identified to the set

$$\text{Soc} = \{N \rightarrow \{\mathbb{G} \rightarrow \mathbb{R}\}\}.$$

Throughout this chapter, we consider a fixed set N of agents.

Notations. Every link $\{i, j\} \in L$ is denoted ij . For every network $g \in \mathbb{G}$ and every link $ij \in L$, $g(ij)$ is denoted g_{ij} and is called the *weight associated to ij (in g)*. For every link $ij \in L$, $L_{-ij} = L \setminus ij$ and $\mathbb{G}_{-ij} = \{L_{-ij} \rightarrow [0, 1]\}$. For every link $ij \in L$, every $g_{-ij} = (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$ and every $w \in [0, 1]$, $g' = (w, g_{-ij}) \in \mathbb{G}$ is the network defined by $g'_{kl} = g_{kl}$, for every $kl \neq ij$, and $g'_{ij} = w$. For every network $g \in \mathbb{G}$ and every link $ij \in L$, $g_{-ij} = (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$.

Pairwise stability is one of the most important concept of network formation theory. It has been introduced by Jackson and Wolinsky [32] for unweighted societies,¹ and has been extended by Bich and Morhaim [6] to weighted societies.

Definition 1.2.3. Let $v \in \text{Soc}$ be a society. A network $g \in \mathbb{G}$ is *pairwise stable (with respect to v)* if for every $ij \in L$, the two following conditions hold:

1. For every $w \in [0, g_{ij})$, $v_i(w, g_{-ij}) \leq v_i(g)$ and $v_j(w, g_{-ij}) \leq v_j(g)$.
2. For every $w \in (g_{ij}, 1]$, $v_i(w, g_{-ij}) \leq v_i(g)$ or $v_j(w, g_{-ij}) \leq v_j(g)$.

¹For every set N' of agents, the *set of unweighted networks (on N')* is defined by $\mathbb{G}_u = \{L' \rightarrow \{0, 1\}\}$ (with $L' = \{\{i, j\} : (i, j) \in N' \times N', i \neq j\}$), and an *unweighted society* is a couple $(N', (v'_i)_{i \in N'})$ composed by a set N' of agents and a family $(v'_i)_{i \in N'}$ of maps from \mathbb{G}_u to \mathbb{R} , where for every $i \in N'$, v'_i is called the *payoff function of agent i* .

Definition 1.2.4. Let $\mathcal{V} \subset \text{Soc}$ be a set of societies. The *pairwise stable networks correspondence associated to \mathcal{V}* is the correspondence

$$\left\{ \begin{array}{l} \Psi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbb{G} \\ v \mapsto \{g \in \mathbb{G} : g \text{ is pairwise stable with respect to } v\} \end{array} \right. .$$

The graph of the pairwise stable networks correspondence associated to \mathcal{V} is called the *graph of pairwise stable networks associated to \mathcal{V}* and is denoted $\mathcal{P}_{\mathcal{V}}$, i.e.

$$\mathcal{P}_{\mathcal{V}} = \text{Gr}(\Psi_{\mathcal{V}}) = \{(v, g) \in \mathcal{V} \times \mathbb{G} : g \text{ is pairwise stable with respect to } v\}.$$

The projection from $\mathcal{P}_{\mathcal{V}}$ to \mathcal{V} is denoted $\pi_{\mathcal{V}}$, i.e. $\pi_{\mathcal{V}}(v, g) = v$, for every $(v, g) \in \mathcal{P}_{\mathcal{V}}$.

1.2.2 \mathcal{A} -regular sets of societies and structure theorem

Payoff functions which are considered in this chapter have to satisfy some differentiability and some concavity properties.

Definition 1.2.5. For every $i \in N$,

$$\mathcal{C}_i = \{v_i \in \mathcal{F}(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \text{ is concave}\},$$

$$\mathcal{D}_i = \{v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \partial_{ij} v_i \text{ exists and is continuous}\},$$

where for every $v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R})$ and every $j \neq i$,

$$\partial_{ij} v_i : g = (g_{ij}, g_{-ij}) \in \mathbb{G} \mapsto \frac{\partial v_i}{\partial g_{ij}}(g) \in \mathbb{R},$$

and

$$\mathcal{F}_i = \mathcal{C}_i \cap \mathcal{D}_i.$$

The set

$$\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$$

is called the *set of own-weights \mathcal{C}^1 concave societies*.

Remark 1.2.2. Since \mathbb{G} is a closed convex subspace of \mathbb{R}^L such that $\text{int}(\mathbb{G}) \neq \emptyset$, for every $i \in N$, every $v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R})$ and every $j \neq i$, $\partial_{ij} v_i$ is well-defined (see Proposition 1.4.5 and Definition 1.4.5 in Appendix 1.4.1).

Remark 1.2.3. For every $i \in N$, every $v_i \in \mathcal{D}_i$, every $j \neq i$ and every $g_{-ij} \in \mathbb{G}_{-ij}$, $v_i(\cdot, g_{-ij})$ is a \mathcal{C}^1 function (see Definition 1.4.4 in Appendix 1.4.1).

Example 1.2.1. Suppose that $N = \{1, 2, 3\}$:

- Suppose that

$$v_1(g) = 2\ln(g_{12})g_{13}^3 + g_{23}^2,$$

for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_1 \in \mathcal{F}_1$.

- Suppose that

$$v_2(g) = 2\ln(g_{12})g_{13}^3 + g_{23}^2,$$

for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_2 \notin \mathcal{F}_2$, since $v_2 \notin \mathcal{C}_2$ (for every $g_{-23} \in \mathbb{G}_{-23}$, the map $v_2(\cdot, g_{-23})$ is not concave). However, if $v_2(g) = 2\ln(g_{12})g_{13}^3 - g_{23}^2$, then $v_2 \in \mathcal{F}_2$ (since $g_{23} > 0$).

- Suppose that

$$v_3(g) = -\sqrt{|g_{12} - \frac{1}{2}|^2 + |g_{13} - \frac{1}{2}|^2},$$

for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_3 \notin \mathcal{F}_3$, since $v_3 \notin \mathcal{D}_3$ (for every $g \in \mathbb{G}$ such that $g_{12} = g_{13} = \frac{1}{2}$, $\partial_{13}v_3(g)$ is not defined). However, if $v_3(g) = -\sqrt{|g_{12}|^2 + |g_{13}|^2}$, then $v_3 \in \mathcal{F}_3$ (since $(g_{12}, g_{13}, 0)$ belongs to the boundary of \mathbb{G}).

- Suppose that for some $i \in N$,

$$v_i : g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G} \mapsto \begin{cases} g_{12}g_{13}g_{23} & \text{if } g \neq (0, 0, 0) \\ 1 & \text{otherwise} \end{cases} \in \mathbb{R}.$$

Then, $v_i \notin \mathcal{F}_i$, since $v_i \notin \mathcal{D}_i$ (v_i is not a continuous map).

Definition 1.2.6. For every $i \in N$, the vector space \mathcal{D}_i (with its usual operations) is endowed with the following norm:

$$\|\cdot\|_i : v_i \in \mathcal{D}_i \mapsto \max\{\max\{\|v_i\|_\infty, \|\partial_{ij}v_i\|_\infty\} : j \neq i\} \in \mathbb{R},$$

where for every $v_i \in \mathcal{D}_i$ and every $j \neq i$, $\|v_i\|_\infty = \sup_{g \in \mathbb{G}} |v_i(g)|$ and $\|\partial_{ij}v_i\|_\infty = \sup_{g \in \mathbb{G}} |\partial_{ij}v_i(g)|$. Furthermore, any subset of \mathcal{D}_i is endowed with the induced topology.

Remark 1.2.4. For every $i \in N$, every $v_i \in \mathcal{D}_i$ and every $j \neq i$, $\|v_i\|_\infty = \max_{g \in \mathbb{G}} |v_i(g)|$ and $\|\partial_{ij}v_i\|_\infty = \max_{g \in \mathbb{G}} |\partial_{ij}v_i(g)|$, since both v_i and $\partial_{ij}v_i$ are continuous maps and since \mathbb{G} is a compact subspace of \mathbb{R}^L .

For every $i \in N$, we consider the set of payoff functions of agent i which are affine in $(g_{ij})_{j \neq i}$ and which only depend on those weights.

Definition 1.2.7. For every $i \in N$,

$$\mathcal{A}_i = \{g \in \mathbb{G} \mapsto \sum_{j \neq i} \alpha_{ij} g_{ij} + c \in \mathbb{R} : \forall j \neq i, \alpha_{ij} \in \mathbb{R}, c \in \mathbb{R}\}.$$

Moreover, $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$.

Example 1.2.2. Suppose that $N = \{1, 2, 3\}$:

- Suppose that $v_1(g) = 2g_{12} + 4g_{13} + 1$, for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_1 \in \mathcal{A}_1$, since $v_1(g) = \sum_{j \neq 1} \alpha_{1j} g_{1j} + c$, with $\alpha_{12} = 2$, $\alpha_{13} = 4$ and $c = 1$.
- Suppose that $v_2(g) = 2g_{12} + 4g_{13} + 1$, for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_2 \notin \mathcal{A}_2$, since it depends on g_{13} . However, if $v_2(g) = 2g_{12} + 1$, then $v_2(g) = \sum_{j \neq 2} \alpha_{2j} g_{2j} + c$, with $\alpha_{12} = 2$, $\alpha_{23} = 0$ and $c = 1$. Hence $v_2 \in \mathcal{A}_2$.
- Suppose that $v_3(g) = 5g_{13} + 3g_{23}$, for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then, $v_3 \in \mathcal{A}_3$, since $v_3(g) = \sum_{j \neq 3} \alpha_{3j} g_{3j} + c$, with $\alpha_{13} = 5$, $\alpha_{23} = 3$ and $c = 0$.

We now introduce the notion of \mathcal{A} -regular set of societies which is at the core of our structure theorem.

Definition 1.2.8. For every $i \in N$, let $\mathcal{R}_i \subset \mathcal{F}(\mathbb{G}, \mathbb{R})$. The set $\prod_{i \in N} \mathcal{R}_i$ of societies is \mathcal{A} -regular if for every $i \in N$, the two following conditions hold:

1. (**Concavity**). $\mathcal{R}_i \subset \mathcal{F}_i$.
2. (**\mathcal{A} -invariance**). For every $v_i \in \mathcal{R}_i$ and every $a_i \in \mathcal{A}_i$, $v_i + a_i \in \mathcal{R}_i$.

Our following result characterizes the topological structure of the graph of pairwise stable networks associated to any \mathcal{A} -regular set of societies (e.g. to \mathcal{F} itself).

Theorem 1.2.1. (*Structure theorem*)

For every \mathcal{A} -regular set \mathcal{R} of societies, the projection $\pi_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$.

This theorem provides three important informations:

1. Thanks to $\eta_{\mathcal{F}}$, the graph $\mathcal{P}_{\mathcal{F}}$ of pairwise stable networks associated to \mathcal{F} is homeomorphic to \mathcal{F} , which corresponds to the intuition that $\mathcal{P}_{\mathcal{F}}$ can be continuously deformed into the simpler space \mathcal{F} .

2. The projection $\pi_{\mathcal{F}}$ is properly homotopic to the homeomorphism $\eta_{\mathcal{F}}$, which corresponds to the intuition that $\eta_{\mathcal{F}}$ itself can be continuously deformed into the simpler map $\pi_{\mathcal{F}}$.
3. The two previous properties remain true if \mathcal{F} is replaced by any \mathcal{A} -regular set \mathcal{R} of societies.

Structure theorem is a key ingredient in the proof of our oddness theorem (Theorem 1.3.1 in Section 1.3.1), which is itself very important for applications (see Section 1.3.2). Indeed, Theorem 1.3.1 relies partly on some properties of topological degree (see Appendix 1.4.1 for some reminders) which can be derived from Theorem 1.2.1; for every \mathcal{A} -regular set \mathcal{R} of societies, the fact that the homotopy between $\pi_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ is proper means intuitively that it maps points “close to infinity” to points “close to infinity”, which plays an important role in the proof of Theorem 1.3.1. Figure 1.1 provides a simple representation of the graph $\mathcal{P}_{\mathcal{R}}$ of pairwise stable networks associated to some \mathcal{A} -regular set \mathcal{R} of societies, of the projection $\pi_{\mathcal{R}}$ and of the homeomorphism $\eta_{\mathcal{R}}$.

In the field of game theory, Kohlberg and Mertens [33] provided a similar result in the case of mixed Nash equilibria of finite strategic-form games. More recently, Predtetchinski [40] provided a generalization of Kohlberg-Mertens’s structure theorem in the case of Nash equilibria of own-strategy \mathcal{C}^1 concave games. We can also mention the interesting works of Demichelis and Germano [17, 16], which provided sharper results on the topological structure of the graph of Nash equilibria associated to the set of mixed extensions of finite strategic-form games and on the topological structure of the graph of Walrasian equilibria.

Sketch of proof

The full proof is provided in Appendix 1.4.3.

First, we construct the homeomorphism $\eta_{\mathcal{F}}$ from $\mathcal{P}_{\mathcal{F}}$ to \mathcal{F} (Step I). To do so, we have to associate to a couple (v, g) , where v is an own-weights \mathcal{C}^1 concave society (i.e. a profile of payoff functions in \mathcal{F}) and g is a network which is pairwise stable with respect to v , another own-weights \mathcal{C}^1 concave society $\eta_{\mathcal{F}}(v, g)$. Moreover, $\eta_{\mathcal{F}}(v, g)$ has to contain all the information conveyed by v and g , since we want to be able to define an inverse map $\eta_{\mathcal{F}}^{-1} : \mathcal{F} \rightarrow \mathcal{P}_{\mathcal{F}}$ (Steps II-IV). The idea is to define $\eta_{\mathcal{F}}(v, g)$ by adding to $v = (v_i)_{i \in N}$ a profile of affine payoff functions whose coefficients contain some weights of $g = (g_{ij})_{ij \in L}$, as well as first-order derivatives of each v_i at g and at (g_{ij}, g_{-ij}^0) (where g^0 is an arbitrary fixed network). These coefficients are chosen so

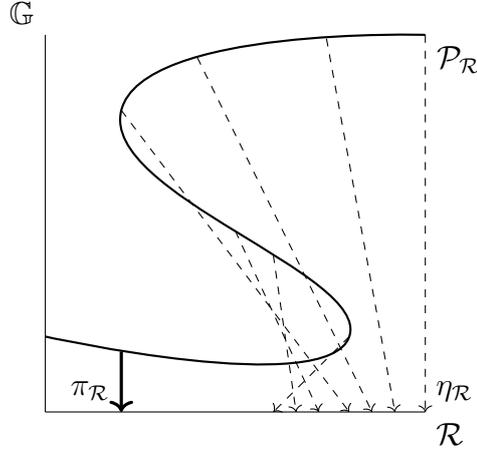


Figure 1.1: $\mathcal{P}_{\mathcal{R}}$, $\pi_{\mathcal{R}}$ (in thick line) and $\eta_{\mathcal{R}}$ (in dashed line)

that for every $v \in \mathcal{F}$, $\eta_{\mathcal{F}}^{-1}(v)$ has a simple form (\underline{u}^v, g^v) :

- For every $ij \in L$, g_{ij}^v is defined from v by taking the minimum of the two (unique) solutions of

$$\max_{w \in [0,1]} v_k(w, g_{-ij}^0) - \frac{w^2}{2},$$

for $k \in ij$ (uniqueness is guaranteed by concavity of $v_k(\cdot, g_{-ij}^0)$). The intuition is that for every $k \in ij$, from the first-order necessary and sufficient conditions (see Proposition 1.4.6 in Appendix 1.4.1), the solution of the above maximization problem should depend on some first-order derivatives of v_k : we precisely fix the coefficients in the affine part added to v in $\eta_{\mathcal{F}}(v, g)$ (see the discussion above) in accordance to this solution in order to guarantee that $g_{ij}^{\eta_{\mathcal{F}}(v, g)} = g_{ij}$.

- \underline{u}^v is defined from v in a very similar way as $\eta_{\mathcal{F}}(v, g)$, simply by reversing some signs (in particular, \underline{u}^v is also equal to v up to some profile of affine payoff functions).

In Step V and Step VI, we prove that both $\eta_{\mathcal{F}}$ and $\eta_{\mathcal{F}}^{-1}$ are continuous maps. In Step VII, we prove that the straight-line homotopy $(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{F}} \mapsto (1-t)\pi_{\mathcal{F}}(v, g) + t\eta_{\mathcal{F}}(v, g) \in \mathcal{F}$ is a proper homotopy between the projection $\pi_{\mathcal{F}} : \mathcal{P}_{\mathcal{F}} \rightarrow \mathcal{F}$ and $\eta_{\mathcal{F}}$. Finally, in Step VIII, we show that for every \mathcal{A} -regular set \mathcal{R} of societies, the restriction of $\eta_{\mathcal{F}}$ from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} , which is denoted $\eta_{\mathcal{R}}$, is a homeomorphism (its inverse corresponds to the restriction of $\eta_{\mathcal{F}}^{-1}$ from \mathcal{R} to $\mathcal{P}_{\mathcal{R}}$). Furthermore, thanks to \mathcal{A} -invariance assumption,

we show that the straight-line homotopy $(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{R}} \mapsto (1 - t)\pi_{\mathcal{R}}(v, g) + t\eta_{\mathcal{R}}(v, g) \in \mathcal{R}$ is a proper map.

1.3 Generic oddness of the graph of pairwise stable networks

1.3.1 \mathcal{A} -semi-algebraically regular sets of societies and oddness theorem

In this section, we are interested by sets of societies whose payoff functions are polynomial functions of g (with coefficients in \mathbb{R}), and by their corresponding sets of coefficients.

Definition 1.3.1. Let

$$\mathbb{R}[g] = \{g \in \mathbb{G} \mapsto \sum_{k \in \mathbb{N}^L} (\alpha_k \prod_{ij \in L} g_{ij}^{k_{ij}}) \in \mathbb{R} : \forall k \in \mathbb{N}^L, \alpha_k \in \mathbb{R}\}.$$

For every $i \in N$ and every $\delta_i \in \mathbb{N}$,

$$\mathbb{R}_{\delta_i}[g] = \{v_i \in \mathbb{R}[g] : \deg(v_i) \leq \delta_i\},$$

where for every $v_i \in \mathbb{R}[g]$, $\deg(v_i) = \max\{\deg(k) : \alpha_k \neq 0\}$, with $\deg(k) = \sum_{ij \in L} k_{ij}$, for every $k \in \mathbb{N}^L$. For every $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, the set

$$\mathbb{R}_{\delta}[g] = \prod_{i \in N} \mathbb{R}_{\delta_i}[g]$$

is called the *set of (δ) -polynomial societies*.

Example 1.3.1. Suppose that $N = \{1, 2, 3\}$ and that for some $i \in N$, $\delta_i = 12$ and

$$v_i(g) = -g_{12}^2 g_{23} + 3g_{12} g_{13}^5 g_{23}^4 - g_{13}^2 g_{23},$$

for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Then,

$$v_i(g) = \alpha_{k^1} \prod_{jl \in L} g_{jl}^{k_{jl}^1} + \alpha_{k^2} \prod_{jl \in L} g_{jl}^{k_{jl}^2} + \alpha_{k^3} \prod_{jl \in L} g_{jl}^{k_{jl}^3} \in \mathbb{R}_{\delta_i}[g],$$

with $k^1 = (2, 0, 1)$, $k^2 = (1, 5, 4)$ and $k^3 = (0, 2, 1)$, and with $\alpha_{k^1} = -1$, $\alpha_{k^2} = 3$ and $\alpha_{k^3} = -1$. Indeed, $\deg(v_i) = 10$, since $\deg(k^1) = 3$, $\deg(k^2) = 10$ and $\deg(k^3) = 3$.

Remark 1.3.1. For every $i \in N$ and every $\delta_i \in \mathbb{N}$, $\mathbb{R}_{\delta_i}[g]$ (with its usual operations) is a finite-dimensional vector space. Thus, $\mathbb{R}_{\delta_i}[g]$ is endowed with the unique topology which makes it a Hausdorff topological vector space. Since \mathcal{D}_i is Hausdorff (see Definition 1.2.6), this topology corresponds also to the one induced by \mathcal{D}_i on $\mathbb{R}_{\delta_i}[g]$. Furthermore, with this topology, note that every linear map from $\mathbb{R}_{\delta_i}[g]$ to any other topological vector space is also continuous.

Definition 1.3.2. Let $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, and consider an order on the set L and an order on the set \mathbb{N}^L . For every $i \in N$, there exists a unique $m_i \in \mathbb{N}$ such that the map

$$\varphi_i : v_i \in \mathbb{R}_{\delta_i}[g] \mapsto (\alpha_k)_{k \in \mathbb{N}^L} \in \mathbb{R}^{m_i}$$

is a well-defined vector space isomorphism. Furthermore, the map

$$\varphi : v \in \mathbb{R}_{\delta}[g] \mapsto \times_{i \in N} \varphi_i(v_i) \in \mathbb{R}^m,$$

where $m = \sum_{i \in N} m_i$, is also a well-defined vector space isomorphism.

Example 1.3.2. Suppose that $N = \{1, 2, 3\}$, that for every $i \in N$, $\delta_i = 2$, and that

$$v_1(g) = -g_{12}g_{23} + 3g_{12}g_{13} - g_{13}^2, v_2(g) = -g_{12}^2 + g_{13} - 5g_{23}^2, v_3(g) = 5g_{13} - 2g_{23}^2 + 3,$$

for every $g = (g_{12}, g_{13}, g_{23}) \in \mathbb{G}$. Moreover consider the order $g_{12} \leq g_{13} \leq g_{23}$ on L and the order

$$\begin{aligned} (0, 0, 0) &\preceq (1, 0, 0) \preceq (0, 1, 0) \preceq (0, 0, 1) \\ &\preceq (1, 1, 0) \preceq (1, 0, 1) \preceq (0, 1, 1) \\ &\preceq (2, 0, 0) \preceq (0, 2, 0) \preceq (0, 0, 2) \end{aligned}$$

on $\{0, 1, 2\}^L$ (for simplicity). Then,

$$\begin{aligned} \varphi_1(v_1) &= (0, 0, 0, 0, 3, -1, 0, 0, -1, 0), \\ \varphi_2(v_2) &= (0, 0, 1, 0, 0, 0, 0, -1, 0, -5), \\ \varphi_3(v_3) &= (3, 0, 5, 0, 0, 0, 0, 0, 0, -2), \end{aligned}$$

with $m_i = 10$, for every $i \in N$, and $\varphi(v_1, v_2, v_3) = (\varphi_1(v_1), \varphi_2(v_2), \varphi_3(v_3))$.

Throughout the rest of this chapter, we consider a fixed $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, a fixed order on the set L and a fixed order on the set \mathbb{N}^L (in particular, we consider also the vector space isomorphism φ_i of Definition 1.3.2, for every $i \in N$).

Notations. For every $i \in N$, let $\mathcal{V}_i \subset \mathbb{R}_{\delta_i}[g]$, and consider the subset $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i \subset \mathbb{R}_{\delta}[g]$ of polynomial societies. For every $i \in N$, the set $\varphi_i(\mathcal{V}_i)$ of coefficients of polynomial payoff functions in \mathcal{V}_i is denoted $\mathbf{C}_{\mathcal{V}_i}$, and the set $\varphi(\mathcal{V})$ of coefficients of polynomial societies in \mathcal{V} is denoted $\mathbf{C}_{\mathcal{V}}$. By abuse of notation, for every $i \in N$, both the restriction of φ_i from \mathcal{V}_i to \mathbb{R}^{m_i} and the restriction of φ_i from \mathcal{V}_i to $\mathbf{C}_{\mathcal{V}_i}$ are denoted φ_i (however, note that $\varphi_i : \mathcal{V}_i \rightarrow \mathbf{C}_{\mathcal{V}_i}$ is a homeomorphism). Similarly, by abuse of notation, both the restriction of φ from \mathcal{V} to \mathbb{R}^m and the restriction of φ from \mathcal{V} to $\mathbf{C}_{\mathcal{V}}$ are denoted φ (however, note that $\varphi : \mathcal{V} \rightarrow \mathbf{C}_{\mathcal{V}}$ is a homeomorphism). For every $x \in \mathbf{C}_{\mathcal{V}}$, the polynomial society in \mathcal{V} whose coefficients correspond to x is denoted v^x , i.e. $v^x = \varphi^{-1}(x)$.

In the following, we introduce the notion of \mathcal{A} -semi-algebraically regular set of societies; a particular case of \mathcal{A} -regular set of societies whose payoff functions are polynomial, with an additional assumption on its associated set of coefficients.

Definition 1.3.3. For every $i \in N$, let $\mathcal{S}_i \subset \mathcal{F}(\mathbb{G}, \mathbb{R})$. The set $\prod_{i \in N} \mathcal{S}_i$ of societies is \mathcal{A} -semi-algebraically regular if for every $i \in N$, the three following conditions hold:

1. **(Concavity).** $\mathcal{S}_i \subset \mathcal{F}_i$.
2. **(\mathcal{A} -invariance).** For every $v_i \in \mathcal{S}_i$ and every $a_i \in \mathcal{A}_i$, $v_i + a_i \in \mathcal{S}_i$.
3. **(Semi-algebraicity).** $\mathcal{S}_i \subset \mathbb{R}_{\delta_i}[g]$ and $\mathbf{C}_{\mathcal{S}_i}$ is a semi-algebraic set.

For every $i \in N$, consider $\mathcal{V}_i \subset \mathbb{R}_{\delta_i}[g]$. To say that $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$ satisfies semi-algebraicity assumption means that for every $i \in N$, the set $\mathbf{C}_{\mathcal{V}_i}$ of coefficients of polynomial payoff functions in \mathcal{V}_i can be defined using a finite number of polynomial equalities or inequalities (see Appendix 1.4.2 for some reminders about real semi-algebraic geometry).

Recall that a semi-algebraic subset G of a semi-algebraic set S is said to be a *generic subset of S* if $\dim(S \setminus G) < \dim(S)$, and if G is open in S (again, see Appendix 1.4.2 for some reminders). The intuition behind this definition is that a generic subset of S fills “almost completely” S . Now, we present the second main theorem of this chapter: our oddness theorem.

Theorem 1.3.1. (*Oddness theorem*)

For every \mathcal{A} -semi-algebraically regular set \mathcal{S} of societies, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}}$ such that for every $x \in G$, the society v^x has an odd number of pairwise stable networks.

Briefly, our result states that “most” societies in network formation theory should have an odd number of pairwise stable networks, when payoff functions are polynomial and satisfy the “standard” concavity assumption.

Sketch of proof

The full proof is provided in Appendix 1.4.4.

Recall that from Theorem 1.2.1, for every \mathcal{A} -regular set \mathcal{R} of societies, the projection $\pi_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$. Since \mathcal{S} is \mathcal{A} -semi-algebraically regular, it is also \mathcal{A} -regular (by definition), which implies that $\pi_{\mathcal{S}} : \mathcal{P}_{\mathcal{S}} \rightarrow \mathcal{S}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{S}} : \mathcal{P}_{\mathcal{S}} \rightarrow \mathcal{S}$.

Following a tradition of existence proofs in game theory or in general equilibrium theory, one could try to apply topological degree to the projection map $\pi_{\mathcal{S}}$ in order to obtain that generically, $\pi_{\mathcal{S}}^{-1}(v)$ (which “counts” the number of pairwise stable networks of the society $v \in \mathcal{S}$) has an odd number of elements. The idea would be, first, to prove that the degree of $\eta_{\mathcal{S}}$ is equal to 1, which would imply that the degree of $\pi_{\mathcal{S}}$ is also equal to 1, by homotopy invariance of topological degree (see Proposition 1.4.2 in Appendix 1.4.1). Also, from some covering space properties (see Theorem 1.4.5 in Appendix 1.4.2), we could obtain that $\pi_{\mathcal{S}}^{-1}(v) = \{(v, g_1), \dots, (v, g_n)\}$ is generically finite, where for every $k \in \{1, \dots, n\}$, (v, g_k) has some open neighborhood in $\mathcal{P}_{\mathcal{S}}$ which is mapped by $\pi_{\mathcal{S}}$ homeomorphically onto its image. This would imply that the local degree over v of the restriction of $\pi_{\mathcal{S}}$ to such a neighborhood of (v, g_k) is equal to ± 1 (see Proposition 1.4.4 in Appendix 1.4.1). Thus, by additivity of topological degree, the sum of these local degrees over v being equal to the degree of $\pi_{\mathcal{S}}$ (which is equal to 1), this would imply that n is odd (see Theorem 1.4.1 in Appendix 1.4.1), i.e. that v admits an odd number of pairwise stable networks.

One of the main problem with this approach is that to be able to apply topological degree on $\pi_{\mathcal{S}}$, $\mathcal{P}_{\mathcal{S}}$ and \mathcal{S} have to be topological manifolds, which is not verified in general. From Step I to Step VI, we first skip this difficulty by assuming that the space $\mathbf{C}_{\mathcal{S}} = \varphi(\mathcal{S})$ of coefficients of polynomial societies in \mathcal{S} is equal to \mathbb{R}^m (the whole reasoning is similar if we only assume that $\mathbf{C}_{\mathcal{S}}$ is homeomorphic to \mathbb{R}^p , for some $p \leq m$).

Under this assumption, φ is a global chart which allows to identify \mathcal{S} with \mathbb{R}^m (i.e. \mathcal{S} is now a topological manifold). Hence, the two maps $\pi_{\mathcal{S}}$ and $\eta_{\mathcal{S}}$, from $\mathcal{P}_{\mathcal{S}}$ to \mathcal{S} , can be read in this chart as continuous maps $\pi_{\mathbf{C}_{\mathcal{S}}}$ and $\eta_{\mathbf{C}_{\mathcal{S}}}$

from $\mathcal{M}_{\mathcal{S}}$ to \mathbb{R}^m , where

$$\mathcal{M}_{\mathcal{S}} = \{(\varphi(v), g) \in \mathbf{C}_{\mathcal{S}} \times \mathbb{G} : (v, g) \in \mathcal{P}_{\mathcal{S}}\} \subset \mathbb{R}^m \times \mathbb{G},$$

these two spaces being (semi-algebraically) homeomorphic (Step I).

Also, since $\pi_{\mathcal{S}}$ and $\eta_{\mathcal{S}}$ are properly homotopic, we show that this homotopy (denoted $H_{\mathcal{S}}$) can be read in the global chart φ as a proper homotopy between $\pi_{\mathbf{C}_{\mathcal{S}}}$ and $\eta_{\mathbf{C}_{\mathcal{S}}}$ (denoted $H_{\mathbf{C}_{\mathcal{S}}}$). However, a second issue which could prevent from using topological degree is the possibility to have pairwise stable networks “at infinity”. Formally, we avoid that problem by considering the compactifications $\mathcal{M}_{\mathcal{S}}^{\infty} = \mathcal{M}_{\mathcal{S}} \cup \{\infty\}$ of $\mathcal{M}_{\mathcal{S}}$ and $(\mathbb{R}^m)^{\infty} = \mathbb{R}^m \cup \{\infty\}$ of \mathbb{R}^m (see Appendix 1.4.1 for some reminders about topology). The fact that $H_{\mathbf{C}_{\mathcal{S}}}$ is a proper homotopy between $\pi_{\mathbf{C}_{\mathcal{S}}}$ and $\eta_{\mathbf{C}_{\mathcal{S}}}$ allows us to: (i) (uniquely) extend the map $\pi_{\mathbf{C}_{\mathcal{S}}}$ (resp. $\eta_{\mathbf{C}_{\mathcal{S}}}$) to a continuous map $\pi_{\mathbf{C}_{\mathcal{S}}}^{\infty}$ (resp. $\eta_{\mathbf{C}_{\mathcal{S}}}^{\infty}$) from the compactification $\mathcal{M}_{\mathcal{S}}^{\infty}$ to the compactification $(\mathbb{R}^m)^{\infty}$ (see Proposition 1.4.1 in Appendix 1.4.1), where $\pi_{\mathbf{C}_{\mathcal{S}}}^{\infty}(\infty) = \eta_{\mathbf{C}_{\mathcal{S}}}^{\infty}(\infty) = \infty$; (ii) construct a homotopy between $\pi_{\mathbf{C}_{\mathcal{S}}}^{\infty}$ and $\eta_{\mathbf{C}_{\mathcal{S}}}^{\infty}$ (denoted $H_{\mathbf{C}_{\mathcal{S}}}^{\infty}$). Furthermore, the compactifications $\mathcal{M}_{\mathcal{S}}^{\infty}$ and $(\mathbb{R}^m)^{\infty}$ being homeomorphic to the unit m -sphere \mathbb{S}^m , we can “transport” the map $\pi_{\mathbf{C}_{\mathcal{S}}}^{\infty}$ (resp. $\eta_{\mathbf{C}_{\mathcal{S}}}^{\infty}$) to a continuous map $\pi_{\mathbb{S}^m}$ (resp. $\eta_{\mathbb{S}^m}$) from \mathbb{S}^m to itself, and the homotopy $H_{\mathbf{C}_{\mathcal{S}}}^{\infty}$ to a homotopy $H_{\mathbb{S}^m}$ between $\pi_{\mathbb{S}^m}$ and $\eta_{\mathbb{S}^m}$ (Step II and Step III). These constructions are summarized by the following diagram:

$$\begin{array}{ccccccc} \mathcal{P}_{\mathcal{S}} & \longrightarrow & \mathcal{M}_{\mathcal{S}} & \longrightarrow & \mathcal{M}_{\mathcal{S}}^{\infty} & \longrightarrow & \mathbb{S}^m \\ \pi_{\mathcal{S}} \downarrow & & \downarrow \pi_{\mathcal{S}} & & \downarrow \pi_{\mathbf{C}_{\mathcal{S}}} & & \downarrow \eta_{\mathbb{S}^m} \\ \mathcal{S} & \longrightarrow & \mathbf{C}_{\mathcal{S}} & \longrightarrow & (\mathbb{R}^m)^{\infty} & \longrightarrow & \mathbb{S}^m \end{array}$$

In particular, all left vertical arrows (which “represent” $\pi_{\mathcal{S}}$) should be thought as very similar to each other, and similarly for all right vertical arrows (which “represent” $\eta_{\mathcal{S}}$). However, one great advantage of this construction is that it allows us to apply topological degree to the last couple $(\pi_{\mathbb{S}^m}, \eta_{\mathbb{S}^m})$ (which will also provides informations about $\pi_{\mathcal{S}}$).

In Step IV, we show that the degree of $\pi_{\mathbb{S}^m} : \mathbb{S}^m \rightarrow \mathbb{S}^m$ is equal to 1, because $\pi_{\mathbb{S}^m}$ is homotopic to the homeomorphism $\eta_{\mathbb{S}^m}$, which corresponds in fact to the identity map on \mathbb{S}^m (see Proposition 1.4.2 in Appendix 1.4.1). This implies that $\pi_{\mathbb{S}^m}$ is a surjective map (see Proposition 1.4.3 in Appendix 1.4.1), hence that both $\pi_{\mathbf{C}_{\mathcal{S}}}$ and $\pi_{\mathcal{S}}$ are also surjective (from their definitions). In Step V, we use the fact that $\pi_{\mathbf{C}_{\mathcal{S}}}$ is a continuous semi-algebraic surjective map from $\mathcal{M}_{\mathcal{S}}$ (whose dimension is equal to m , from Step I) to \mathbb{R}^m in order to apply some semi-algebraic trivialization result (see Theorem 1.4.5 in

Appendix 1.4.2) and to obtain that generically, every society in \mathcal{S} admits a strictly positive finite number of pairwise stable networks. In Step VI, we use additivity of topological degree (as discussed above) to obtain the oddness theorem, when $\mathbf{C}_{\mathcal{S}} = \mathbb{R}^m$.

Finally, in the last step of the proof (Step VII), which is perhaps the most important in terms of providing new proof techniques, we want to get rid of the assumption that $\mathbf{C}_{\mathcal{S}}$ is equal to \mathbb{R}^m . The idea is to decompose \mathcal{S} into a finite union of sets $(\mathcal{V}_{\lambda})_{\lambda \in \Lambda}$ to which we can apply the previous steps (Steps I-VI). This requires that for every $\lambda \in \Lambda$, \mathcal{V}_{λ} : (i) is \mathcal{A} -semi-algebraically regular; (ii) is homeomorphic to $\mathbb{R}^{p_{\lambda}}$, for some $p_{\lambda} \leq m$. To construct $(\mathcal{V}_{\lambda})_{\lambda \in \Lambda}$, we could try, using semi-algebraicity of $\mathbf{C}_{\mathcal{S}}$, to decompose $\mathbf{C}_{\mathcal{S}}$ in a finite union of sets $(S_k)_{k=1}^n$ (for some $n \in \mathbb{N}$), each one being (semi-algebraically) homeomorphic to some Euclidean space (see Proposition 1.4.9 in Appendix 1.4.2). The problem with this method is that there is no guarantee that the set $\varphi^{-1}(S_k)$ of societies associated to S_k ($k \in \{1, \dots, n\}$) satisfies \mathcal{A} -invariance assumption. Instead, the idea is to proceed as follows:

1. In Substep VII.1, for every $i \in N$, we remove the “ \mathcal{A}_i ” part of the set $\mathbf{C}_{\mathcal{S}_i}$ of coefficients of polynomial payoff functions in \mathcal{S}_i (i.e. we remove the coefficients of the constant monomial and of the monomials $(g_{ij})_{j \neq i}$ from $\mathbf{C}_{\mathcal{S}_i}$) using some linear projection $\Pi_{-\mathcal{A}_i}$ (n.b.: this step is important for the next ones; the rest of the proof would not hold otherwise).
2. In Substep VII.2, for every $i \in N$, we semi-algebraically decompose the outcome $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})$ of Substep VII.1 using Proposition 1.4.9 (in Appendix 1.4.2).
3. In Substep VII.3, for every $i \in N$, we re-introduce the coefficients that were removed in Substep VII.1 by summing elements of the previous decomposition of $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})$ with the set $\mathbf{C}_{\mathcal{A}_i}$ of coefficients corresponding to \mathcal{A}_i ; by doing so, we construct the family $(\mathcal{V}_{\lambda})_{\lambda \in \Lambda}$ mentioned above (the definition of the finite set Λ is detailed in the proof). Indeed, we show in Substep VII.4 that for every $\lambda \in \Lambda$, \mathcal{V}_{λ} is \mathcal{A} -semi-algebraically regular and is homeomorphic to $\mathbb{R}^{p_{\lambda}}$, for some $p_{\lambda} \leq m$.

The end of the proof consists in applying Steps I-VI to each set \mathcal{V}_{λ} ($\lambda \in \Lambda$): this provides a generic semi-algebraic subset G_{λ} of the set $\mathbf{C}_{\mathcal{V}_{\lambda}}$ of coefficients corresponding to \mathcal{V}_{λ} such that for every $x \in G_{\lambda}$, the society $v^x = \varphi^{-1}(x)$ has an odd number of pairwise stable networks. To finish, a generic semi-algebraic subset of $\mathbf{C}_{\mathcal{S}}$ can be obtained by considering the union G of the sets $(G_{\lambda})_{\lambda \in \Lambda}$, retaining only the indices λ for which \mathcal{V}_{λ} is “thick enough”, in

order to be sure that G is open in \mathbf{C}_S (Substeps VII.5-VII.7).

1.3.2 Some applications of oddness theorem

Polynomial own-weights concave societies

The following proposition states that polynomial societies which are own-weights concave admit generically an odd number of pairwise stable networks.

Proposition 1.3.1. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \mathcal{F}_i \cap \mathbb{R}_{\delta_i}[g],$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of \mathbf{C}_S such that for every $x \in G$, the society v^x has an odd number of pairwise stable networks.

Proof. Let $i \in N$. By definition, \mathcal{S}_i satisfies both concavity assumption (i.e. $\mathcal{S}_i \subset \mathcal{F}_i$) and \mathcal{A} -invariance assumption (i.e. $\mathcal{S}_i + \mathcal{A}_i = \mathcal{S}_i$). Now, observe that \mathcal{S}_i also satisfies semi-algebraicity assumption. Indeed, recall that any payoff function $v_i \in \mathcal{S}_i$ can be written as

$$v_i(g) = \sum_{k \in \mathbb{N}^L} (\alpha_k \prod_{jl \in L} g_{jl}^{k_{jl}}),$$

for every $g \in \mathbb{G}$ (by definition of $\mathbb{R}_{\delta_i}[g]$). Consider the polynomial function

$$P_i : ((\alpha_k)_{k \in \mathbb{N}^L}, g) \in \mathbb{R}^{m_i} \times \mathbb{G} \mapsto \sum_{k \in \mathbb{N}^L} (\alpha_k \prod_{jl \in L} g_{jl}^{k_{jl}}) \in \mathbb{R}.$$

Thus, remark that

$$\mathbf{C}_{\mathcal{S}_i} = \{\alpha = (\alpha_k)_{k \in \mathbb{N}^L} \in \mathbb{R}^{m_i} : \forall j \neq i, \forall g = (g_{ij}, g_{-ij}) \in \mathbb{G}, \frac{\partial^2 P_i(\alpha, g)}{\partial g_{ij}^2} \leq 0\},$$

which is a semi-algebraic set, since \mathbb{G} is semi-algebraic and from Proposition 1.4.7 (in Appendix 1.4.2). Therefore, \mathcal{S} is a \mathcal{A} -semi-algebraically regular set of societies, and from oddness theorem (Theorem 1.3.1), there exists a generic semi-algebraic subset G of \mathbf{C}_S such that for every $x \in G$, the society v^x has an odd number of pairwise stable networks. \square

Network formation with linear costs

For every $i \in N$, consider a fixed payoff function $\bar{v}_i \in \mathbb{R}_{\delta_i}[g] \cap \mathcal{F}_i$, i.e. polynomial with a degree less or equal to δ_i and concave in g_{ij} , for every $j \neq i$. Moreover, let $L_d = \{(i, j) \in N^2 : i \neq j\}$ be the *set of directed links (on N)* (every directed link $(i, j) \in L_d$ is denoted i, j).

Now, for every $\alpha = (\alpha_{i,j})_{i,j \in L_d} \in \mathbb{R}^{L_d}$ and every $c = (c_i)_{i \in N} \in \mathbb{R}^N$, consider the society

$$v^\alpha = \left(g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R} \right)_{i \in N}$$

parameterized by α . For every $i, j \in L_d$, $\alpha_{i,j} \in \mathbb{R}$ can be interpreted as the marginal cost for agent i of maintaining the weight g_{ij} of link ij (at least when $\alpha_{i,j} \geq 0$).

Remark 1.3.2. Notice that the society v^α is also parametrized by the constant coefficients c : it can be proved that the result that will follow (Proposition 1.3.2) still holds without taking into account these coefficients, that is why the parameter c is dropped in the previous notation.²

The following proposition states that polynomial societies of the form v^α ($\alpha \in \mathbb{R}^{L_d}$) admit generically an odd number of pairwise stable networks.

Proposition 1.3.2. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \left\{ g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R} : \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R} \right\},$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of $\mathbb{R}^{L_d} \times \mathbb{R}^N$ such that for every $(\alpha, c) \in G$, the society v^α has an odd number of pairwise stable networks.

Proof. Let $i \in N$. By definition, \mathcal{S}_i satisfies both concavity assumption (i.e. $\mathcal{S}_i \subset \mathcal{F}_i$) and \mathcal{A} -invariance assumption (i.e. $\mathcal{S}_i + \mathcal{A}_i = \mathcal{S}_i$). Now, observe that \mathcal{S}_i also satisfies semi-algebraicity assumption. Indeed, the set $\mathbf{C}_{\mathcal{S}_i}$ is a finite product of copies of \mathbb{R} and of singletons,³ thus is a semi-algebraic set. Thus,

²Even if it may seem obvious, the proof of the general case requires in fact a little more work and is not treated in this chapter. However, an analogous proof will be presented in Chapter 2, in the field of game theory.

³More precisely, $\ell_i + 1$ copies of \mathbb{R} (corresponding to the “ \mathcal{A}_i part” of the whole map $g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R}$) and $\max\{m_i - (\ell_i + 1), 0\}$ singletons (corresponding to the “non- \mathcal{A}_i part” of the whole map $g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R}$, which depends on \bar{v}), where $\ell_i = \text{card}(\{ij \in L : j \neq i\})$.

\mathcal{S} is a \mathcal{A} -semi-algebraically regular set of societies, and from Theorem 1.3.1, there exists a generic semi-algebraic subset $\mathbf{C}_{\mathcal{S}}^0$ of $\mathbf{C}_{\mathcal{S}}$ such that for every $x \in \mathbf{C}_{\mathcal{S}}^0$, the society v^x has an odd number of pairwise stable networks. Now, since the trivial map f which associates to any $(\alpha, c) \in \mathbb{R}^{L_d} \times \mathbb{R}^N$ the “same” element in $\mathbf{C}_{\mathcal{S}}$ (up to singletons) is a semi-algebraic homeomorphism, for every $(\alpha, c) \in G = f^{-1}(\mathbf{C}_{\mathcal{S}}^0)$, the society v^α has an odd number of pairwise stable networks, where G is a generic subset of $\mathbb{R}^{L_d} \times \mathbb{R}^N$. \square

Network formation with quadratic costs

For every $i \in N$, consider a fixed payoff function $\bar{v}_i \in \mathbb{R}_{\delta_i}[g] \cap \mathcal{F}_i$, i.e. polynomial with a degree less or equal to δ_i and concave in g_{ij} , for every $j \neq i$. Moreover, recall that $L_d = \{(i, j) \in N^2 : i \neq j\}$ is the set of directed links and that every directed link $(i, j) \in L_d$ is denoted i, j .

Now, for every $\alpha = (\alpha_{i,j})_{i,j \in L_d} \in \mathbb{R}^{L_d}$, every $\beta = (\beta_{i,j})_{(i,j) \in L_d} \in [0, +\infty)^{L_d}$ and every $c = (c_i)_{i \in N} \in \mathbb{R}^N$, consider the society

$$v^{\alpha, \beta} = \left(g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \beta_{i,j} g_{ij}^2 - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R} \right)_{i \in N}$$

parameterized by α and β .

The following proposition states that polynomial societies of the form $v^{\alpha, \beta}$ ($\alpha \in \mathbb{R}^{L_d}$, $\beta \in [0, +\infty)^{L_d}$) admit generically an odd number of pairwise stable networks.

Proposition 1.3.3. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \left\{ g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j \neq i} \beta_{i,j} g_{ij}^2 - \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i \in \mathbb{R} : \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, \beta_{i,j} \in [0, +\infty), c_i \in \mathbb{R} \right\},$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of $\mathbb{R}^{L_d} \times [0, +\infty)^{L_d} \times \mathbb{R}^N$ such that for every $(\alpha, \beta, c) \in G$, the society $v^{\alpha, \beta}$ has an odd number of pairwise stable networks.

Proof. The proof is similar to the one of Proposition 1.3.2 (network formation with linear costs). \square

Bramoullé-Kranton’s public good model [10]

In their paper, Bramoullé and Kranton suppose that agents interact in an exogenous unweighted network $g \in \mathbb{G}$ (i.e. for every $ij \in L$, $g_{ij} \in \{0, 1\}$)

and are characterized by some endogenous levels of efforts $e \in [0, +\infty)^N$, and that the payoff function of agent $i \in N$ is of the form

$$v_i(e) = b(e_i + \sum_{\substack{j \neq i \\ g_{ij}=1}} e_j) - \bar{c}e_i,$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a twice-differentiable strictly concave benefit function and $\bar{c} \in (0, +\infty)$ is the marginal cost of effort for any agent.

Here, we consider a slight modification of Bramoullé-Kranton's model: we suppose that agents interact in an endogenous weighted network $g \in \mathbb{G}$ (i.e. for every $ij \in L$, $g_{ij} \in [0, 1]$) and are characterized by some exogenous levels of efforts $e \in [0, +\infty)^N$, and that the payoff function of agent $i \in N$ is of the form

$$v_i^\alpha(g) = \tilde{b}(e_i + \sum_{j \neq i} e_j g_{ij}) - \bar{c}e_i - \sum_{j \neq i} \alpha_{i,j} g_{ij} - c_i,$$

where $\tilde{b} : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial concave benefit function, $\alpha_{i,j} \in \mathbb{R}$ ($j \neq i$) is the marginal cost for agent i of maintaining the weight g_{ij} of link ij (with $\alpha = (\alpha_{k,l})_{k,l \in L_d}$) and $c_i \in \mathbb{R}$ is a constant cost for agent i (with $c = (c_j)_{j \in N}$).

The following corollary of Proposition 1.3.2 (network formation with linear costs) states that societies with payoff functions of the previous form admit generically an odd number of pairwise stable networks.

Corollary 1.3.1. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \{g \in \mathbb{G} \mapsto v_i^\alpha(g) \in \mathbb{R} : \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R}\},$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of $\mathbb{R}^{L_d} \times \mathbb{R}^N$ such that for every $(\alpha, c) \in G$, the society $v^\alpha = (v_i^\alpha)_{i \in N}$ has an odd number of pairwise stable networks.

Calvó-Armengol-İlkkılıç's information transmission model [11]

In their paper, Calvó-Armengol and İlkkılıç suppose that agents interact in an unweighted network $g \in \mathbb{G}$ (i.e. $g_{ij} \in \{0, 1\}$, for every $ij \in L$) and that the payoff function of agent $i \in N$ is of the form

$$v_i(g) = 1 - \prod_{\substack{j \neq i \\ g_{ij}=1}} p_{ij} - \bar{c}n_i(g),$$

where $p_{ij} \in (0, 1)$ ($j \neq i$) is the probability that some information can be transmitted from agent i to agent j , $\bar{c} \in (0, +\infty)$ is the marginal cost for any agent of maintaining any link and $n_i(g) = \text{card}(\{j \in N \setminus \{i\} : g_{ij} = 1\})$.

Here, we consider a slight modification of Calvó-Armengol-İlkılıç's model: we suppose that agents interact in a weighted network $g \in \mathbb{G}$ (i.e. $g_{ij} \in [0, 1]$, for every $ij \in L$) and that the payoff function of agent $i \in N$ is of the form

$$v_i^\alpha(g) = 1 - \prod_{j \neq i} p_{ij} g_{ij} - \sum_{j \neq i} \alpha_{i,j} g_{ij} - c_i$$

where $\alpha_{i,j} \in \mathbb{R}$ ($j \neq i$) is the marginal cost for agent i of maintaining the weight g_{ij} of link ij (with $\alpha = (\alpha_{k,l})_{k,l \in L_d}$) and $c_i \in \mathbb{R}$ is a constant cost for agent i (with $c = (c_j)_{j \in N}$).

The following corollary of Proposition 1.3.2 (network formation with linear costs) states that societies with payoff functions of the previous form admit generically an odd number of pairwise stable networks.

Corollary 1.3.2. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \{g \in \mathbb{G} \mapsto v_i^\alpha(g) \in \mathbb{R} : \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R}\},$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of $\mathbb{R}^{L_d} \times \mathbb{R}^N$ such that for every $(\alpha, c) \in G$, the society $v^\alpha = (v_i^\alpha)_{i \in N}$ has an odd number of pairwise stable networks.

Bala-Goyal's two-way flow model [2]

In their paper, Bala and Goyal suppose that agents interact in an unweighted network $g \in \mathbb{G}$ (i.e. $g_{ij} \in \{0, 1\}$, for every $ij \in L$) and that the payoff function of agent $i \in N$ is of the form

$$v_i(g) = \mu_i(g) - \bar{c} n_i(g),$$

where $\mathbf{P}_{i \rightarrow j}^g$ ($j \neq i$) is the set of all (finite) paths from agent i to agent j with respect to g (i.e. the set of all finite sequences $p_0 = i, p_1, \dots, p_{n-1}, p_n = j$ of distinct agents such that the weights $g_{ip_1}, \dots, g_{p_{n-1}p_n}$ are not null), $\mu_i(g) = \text{card}(\{j \in N : \mathbf{P}_{i \rightarrow j}^g \neq \emptyset\})$, $\bar{c} \in (0, +\infty)$ is the marginal cost for any agent of maintaining any link and $n_i(g) = \text{card}(\{j \in N \setminus \{i\} : g_{ij} = 1\})$.

Here, we consider a slight modification of Bala-Goyal's model: we suppose that agents interact in a weighted network $g \in \mathbb{G}$ (i.e. $g_{ij} \in [0, 1]$, for every $ij \in L$) and that the payoff function of agent $i \in N$ is of the form

$$v_i^\alpha(g) = \sum_{j \neq i} \tilde{\mu}_{i,j}(g) - \sum_{j \neq i} \alpha_{i,j} g_{ij} - c_i$$

where $\mathbf{P}_{i \rightarrow j}$ is the set of all (finite) paths from agent i to agent j (i.e. the set of all finite sequences $p_0 = i, p_1, \dots, p_{n-1}, p_n = j$ of distinct agents), $\tilde{\mu}_{i,j}(g)$

$(j \neq i)$ is the sum on all paths in $\mathbf{P}_{i \rightarrow j}$ of the product of the weights along these paths (the quantity $\sum_{j \neq i} \tilde{\mu}_{i,j}(g)$ can be interpreted as the benefit that agent i receives from his links), $\alpha_{i,j} \in \mathbb{R}$ ($j \neq i$) is the marginal cost for agent i of maintaining the weight g_{ij} of link ij (with $\alpha = (\alpha_{k,l})_{k,l \in L_d}$) and $c_i \in \mathbb{R}$ is a constant cost for agent i (with $c = (c_j)_{j \in N}$).

The following corollary of Proposition 1.3.2 (network formation with linear costs) states that societies with payoff functions of the previous form admit generically an odd number of pairwise stable networks.

Corollary 1.3.3. *The set $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$, where for every $i \in N$,*

$$\mathcal{S}_i = \{g \in \mathbb{G} \mapsto v_i^\alpha(g) \in \mathbb{R} : \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R}\},$$

is a \mathcal{A} -semi-algebraically regular set of societies. Furthermore, there exists a generic semi-algebraic subset G of $\mathbb{R}^{L_d} \times \mathbb{R}^N$ such that for every $(\alpha, c) \in G$, the society $v^\alpha = (v_i^\alpha)_{i \in N}$ has an odd number of pairwise stable networks.

1.4 Appendix

1.4.1 Reminders about topology and differential calculus

Elementary concepts of topology

Definition 1.4.1. Let X, Y be two topological spaces:

- A map $f : X \rightarrow Y$ is *proper* if for every compact subspace K of Y , $f^{-1}(K)$ is a compact subspace of X .
- A map $f : X \rightarrow Y$ is a *homeomorphism* if f is a bijection, and if f and f^{-1} are continuous maps. If such a map exists, then X and Y are said to be *homeomorphic*.
- A map $f : X \rightarrow Y$ is a *topological embedding* if the map $x \in X \mapsto f(x) \in f(X)$ is a homeomorphism.
- Let $f, g : X \rightarrow Y$ be two continuous maps. A *homotopy between f and g* is a continuous map $H : [0, 1] \times X \rightarrow Y$ such that $H(0, \cdot) = f$ and $H(1, \cdot) = g$. If such a map exists, then f and g are said to be *homotopic*. Furthermore, if such a map is proper, then f and g are said to be *properly homotopic*.

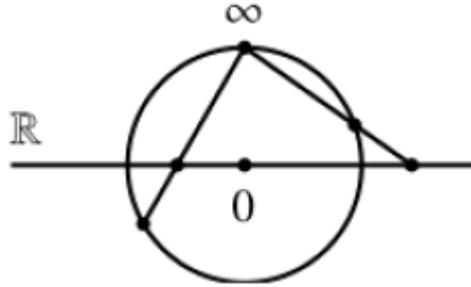


Figure 1.2: \mathbb{S}^1 is (homeomorphic to) the compactification of \mathbb{R} (Aliprantis-Border [1], p. 57)

Alexandroff one-point compactification of a topological space

Definition 1.4.2. Let (X, τ) be a noncompact locally compact Hausdorff topological space and $X^\infty = X \cup \{\infty\}$, where $\infty \notin X$. The set

$$\tau^\infty = \tau \cup \{X^\infty \setminus K : K \subset X \text{ is compact}\}$$

is a topology on X^∞ , i.e. a set $O \subset X^\infty$ is open in X^∞ if: (i) either $\infty \notin O$ and O is open in X ; (ii) or $\infty \in O$ and the complement of O in X^∞ is compact for the topology induced by τ . The space (X^∞, τ^∞) is compact and is called the (Alexandroff one-point) compactification of X .

As a well-known example, Figure 1.2 illustrates the compactification of the set \mathbb{R} of real numbers.

Proposition 1.4.1. Let X, Y be two noncompact locally compact Hausdorff topological spaces and $f : X \rightarrow Y$ be a continuous map. Then, f can be extended to a continuous map from X^∞ to Y^∞ if and only if f is a proper map.

Proof. Consider the map

$$\left\{ \begin{array}{l} f^\infty : X^\infty \rightarrow Y^\infty \\ x \mapsto \begin{cases} f(x) & \text{if } x \in X \\ \infty & \text{otherwise} \end{cases} \end{array} \right. ,$$

which is the only possible extension of f .

Suppose that f^∞ is a continuous map, and suppose that f is not a proper map. By assumption, there exists a compact subspace K of Y such that $f^{-1}(K)$ is not a compact subspace of X . Note that $Y^\infty \setminus K$ is open in Y^∞ .

However, $(f^\infty)^{-1}(Y^\infty \setminus K) = X^\infty \setminus f^{-1}(K)$ is not open in X^∞ , since $f^{-1}(K) \subset X$ is not compact. This contradicts the continuity of f^∞ .

Suppose that f is a proper map. Let O be an open subset of Y^∞ . First, suppose that $\infty \notin O$. By definition, O is open in Y . Then, $(f^\infty)^{-1}(O) = f^{-1}(O)$ is open in X^∞ , since $\infty \notin (f^\infty)^{-1}(O)$ and since f is a continuous map. Last, suppose that $\infty \in O$. By definition, there exists a compact subspace K of Y such that $O = Y^\infty \setminus K$. Then, $(f^\infty)^{-1}(O) = (f^\infty)^{-1}(Y^\infty \setminus K) = X^\infty \setminus f^{-1}(K)$ is open in X^∞ , since $K \subset Y$ is compact and since f is a proper map. Therefore, f^∞ is a continuous map. \square

Topological degree of a continuous map from \mathbb{S}^m to \mathbb{S}^m

For every continuous map $f : \mathbb{S}^m \rightarrow \mathbb{S}^m$, one can associate to f an integer $\deg(f) \in \mathbb{Z}$ called the *degree of f* (Dold [19], Definition 4.1, p. 62).

Proposition 1.4.2. (Dold [19], Proposition 4.2, pp. 62-63)

- $\deg(\text{id}_{\mathbb{S}^m}) = 1$.
- Let $f, g : \mathbb{S}^m \rightarrow \mathbb{S}^m$ be two continuous maps. If f and g are homotopic, then $\deg(f) = \deg(g)$.
- Let $f : \mathbb{S}^m \rightarrow \mathbb{S}^m$ be a homeomorphism. Then, $\deg(f) = \pm 1$.

For every open subset V of \mathbb{S}^m , every continuous map $f : V \rightarrow \mathbb{S}^m$ and every $y \in \mathbb{S}^m$ such that $f^{-1}(y) \subset \mathbb{S}^m$ is compact, one can associate to f an integer $\deg_y(f) \in \mathbb{Z}$ called the *local degree of f over y* (Dold [19], Definition 5.1, pp. 66-67).

Proposition 1.4.3. Let $f : \mathbb{S}^m \rightarrow \mathbb{S}^m$ be a continuous map. If $\deg(f) \neq 0$, then f is surjective.

Proof. The proof follows directly from Dold [19], Examples 5.4, p. 67 and from Dold [19], Corollary 5.6, p. 67. \square

Proposition 1.4.4. (Dold [19], Examples 5.4, p. 67)

Let V be an open subset of \mathbb{S}^m , $f : V \rightarrow \mathbb{S}^m$ be a continuous map and $y \in f(V)$ such that $f^{-1}(y) \subset \mathbb{S}^m$ is compact. If f is a topological embedding, then $\deg_y(f) = \pm 1$.

Theorem 1.4.1. Let $f : \mathbb{S}^m \rightarrow \mathbb{S}^m$ be a continuous map and $y \in \mathbb{S}^m$ such that $f^{-1}(y) = \{x_1, \dots, x_n\}$, where $n > 0$. Moreover, let $V = \bigcup_{i=1}^n V_i$, where

$(V_i)_{i=1}^n$ is a family of pairwise disjoint open subsets of \mathbb{S}^m such that for every $i \in \{1, \dots, n\}$, $x_i \in V_i$. Then,

$$\deg(f) = \sum_{i=1}^n \deg_y(f|_{V_i}).$$

Furthermore, if $\deg(f) = \pm 1$, and if for every $i \in \{1, \dots, n\}$, $f|_{V_i}$ is a topological embedding, then n is odd.

Proof. From Dold [19], Proposition 5.8, p. 68, $\deg_y(f|_V) = \sum_{i=1}^n \deg_y(f|_{V_i})$. From Dold [19], Corollary 5.6, p. 67 and still from Dold [19], Proposition 5.8, p. 68, $\deg(f) = \deg_y(f) = \deg_y(f|_V) + \deg_y(f|_{V'})$, where V' is any open subset of \mathbb{S}^m which contains $\mathbb{S}^m \setminus V$ and such that $x_i \notin V'$, for every $i \in \{1, \dots, n\}$ (such a set always exists if for every $i \in \{1, \dots, n\}$, V_i is small enough). Then, from Dold [19], Examples 5.4, p. 67, $\deg_y(f|_{V'}) = 0$, which implies that $\deg(f) = \sum_{i=1}^n \deg_y(f|_{V_i})$.

Now, suppose that $\deg(f) = \pm 1$, and that for every $i \in \{1, \dots, n\}$, $f|_{V_i}$ is a topological embedding. From Proposition 1.4.4, for every $i \in \{1, \dots, n\}$, $\deg_y(f|_{V_i}) = \pm 1$. If $\deg(f) = 1$, then

$$\text{card}(\{x_i : \deg_y(f|_{V_i}) = -1\}) = \text{card}(\{x_i : \deg_y(f|_{V_i}) = 1\}) + 1.$$

Therefore,

$$\begin{aligned} n &= \text{card}(\{x_i : \deg_y(f|_{V_i}) = -1\}) + \text{card}(\{x_i : \deg_y(f|_{V_i}) = 1\}) \\ &= 2\text{card}(\{x_i : \deg_y(f|_{V_i}) = 1\}) + 1 \end{aligned}$$

is odd (the proof is similar if $\deg(f) = -1$). □

Covering spaces

Definition 1.4.3. Let E be a topological space, B be a connected topological space, $p : E \rightarrow B$ be a continuous map and F be a discrete topological space. The tuple (E, B, p, F) is a *covering space* (of total space E , of base space B , of projection p and of fiber F) if for every $b \in B$, there exists an open subset V of B which contains b and a homeomorphism ψ making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow{\psi} & V \times F \\ \downarrow p & \swarrow \text{pr}_V & \\ V & & \end{array}$$

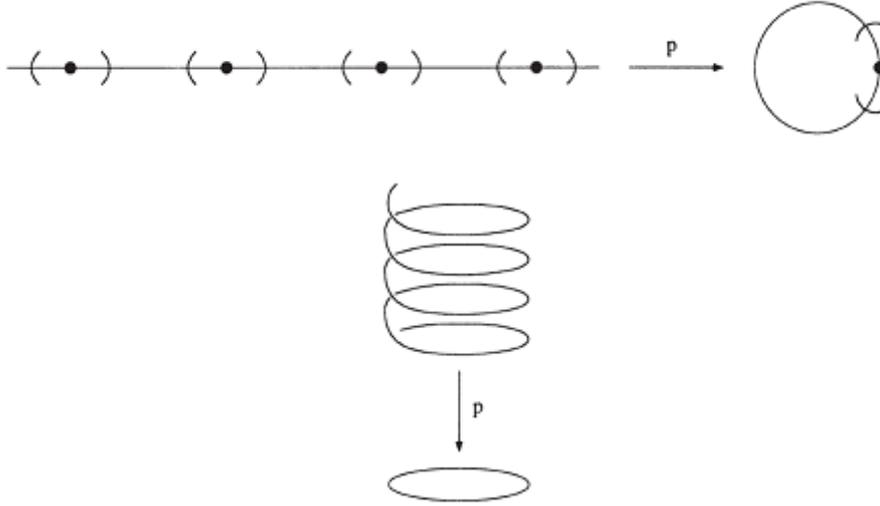


Figure 1.3: Covering of \mathbb{S}^1 by $p : \mathbb{R} \rightarrow \mathbb{S}^1$ (Fulton [21], p. 154)

where $\text{pr}_V : (x, y) \in V \times F \mapsto x \in V$.

Theorem 1.4.2. *Let E be a topological space, B be a connected topological space, $p : E \rightarrow B$ be a continuous map and F be a discrete topological space. Then, (E, B, p, F) is a covering space if and only if for every $b \in B$, there exists an open subset V of B which contains b and such that $p^{-1}(V) = \bigcup_{i \in F} V_i$, where $(V_i)_{i \in F}$ is a family of pairwise disjoint open subsets of E such that for every $i \in F$, the map $x \in V_i \mapsto p(x) \in V$ is a homeomorphism.*

Figure 1.3 illustrates the covering of the unit circle of \mathbb{S}^1 by the projection $p : x \in \mathbb{R} \mapsto (\cos(2\pi x), \sin(2\pi x)) \in \mathbb{S}^1$.

Proof. Suppose that (E, B, p, F) is a covering space. Let $b \in B$. By assumption, there exists an open subset V of B which contains b and a homeomorphism $\psi : p^{-1}(V) \rightarrow V \times F$ such that for every $x \in p^{-1}(V)$, $p(x) = (\text{pr}_V \circ \psi)(x)$. Then, $(\psi^{-1}(V \times \{i\}))_{i \in F}$ is a family of pairwise disjoint open subsets of E such that for every $i \in F$, the map $x \in \psi^{-1}(V \times \{i\}) \mapsto p(x) \in V$ is a homeomorphism. Moreover,

$$\bigcup_{i \in F} \psi^{-1}(V \times \{i\}) = \psi^{-1}\left(\bigcup_{i \in F} V \times \{i\}\right) = \psi^{-1}(\text{pr}_V^{-1}(V)) = (\text{pr}_V \circ \psi)^{-1}(V) = p^{-1}(V).$$

Suppose that for every $b \in B$, there exists an open subset V of B which contains b and such that $p^{-1}(V) = \bigcup_{i \in F} V_i$, where $(V_i)_{i \in F}$ is a family of pairwise disjoint open subsets of E such that for every $i \in F$, the map

$x \in V_i \mapsto p(x) \in V$ is a homeomorphism. Let $b \in B$. The map $\psi : x \in \bigcup_{i \in F} V_i \mapsto (p(x), i_x) \in V \times F$, where i_x is the only element of F such that $x \in V_{i_x}$, is a homeomorphism, since for every $i \in F$, the map $x \in V_i \mapsto p(x) \in V$ is a homeomorphism. Moreover, for every $x \in \bigcup_{i \in F} V_i$, $(\text{pr}_V \circ \psi)(x) = \text{pr}_V(p(x), i_x) = p(x)$. \square

Differential calculus and mathematical optimization

Definition 1.4.4. A function f from an arbitrary subset A of \mathbb{R}^m to \mathbb{R} is said to be \mathcal{C}^1 if for every $a \in A$, there exists an open subset O of \mathbb{R}^m which contains a and a \mathcal{C}^1 function $g : O \rightarrow \mathbb{R}$ such that $g|_{O \cap A} = f|_{O \cap A}$.

Proposition 1.4.5. Let C be a closed convex subset of \mathbb{R}^m such that $\text{int}(C) \neq \emptyset$, $f : C \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function, $c \in C$ and $g_1 : O_1 \rightarrow \mathbb{R}$, $g_2 : O_2 \rightarrow \mathbb{R}$ be two \mathcal{C}^1 extensions of f around c . Then, for every $i = 1, \dots, m$,

$$\frac{\partial g_1}{\partial x_i}(c) = \frac{\partial g_2}{\partial x_i}(c).$$

Proof. First, note that $\text{cl}(\text{int}(C)) = C$, from C being a closed convex subset of \mathbb{R}^m . By definition, there exists a sequence $(c^\ell)_{\ell \in \mathbb{N}}$ in $\text{int}(C)$ ($\neq \emptyset$) such that $(c^\ell)_{\ell \in \mathbb{N}}$ converges to c . Thus, there exists $\ell^* \in \mathbb{N}$ such that for every $\ell \geq \ell^*$, $c^\ell \in O_1 \cap O_2$. Then, for every $\ell \geq \ell^*$,

$$\frac{\partial g_1}{\partial x_i}(c^\ell) = \frac{\partial f}{\partial x_i}(c^\ell) = \frac{\partial g_2}{\partial x_i}(c^\ell).$$

Since both g_1 and g_2 are \mathcal{C}^1 functions, their first-order derivative with respect to the i -th variable are continuous, which finally implies that

$$\frac{\partial g_1}{\partial x_i}(c) = \lim_{\substack{c^\ell \rightarrow c \\ \ell \geq \ell^*}} \frac{\partial g_1}{\partial x_i}(c^\ell) = \lim_{\substack{c^\ell \rightarrow c \\ \ell \geq \ell^*}} \frac{\partial g_2}{\partial x_i}(c^\ell) = \frac{\partial g_2}{\partial x_i}(c).$$

\square

Definition 1.4.5. Let C be a closed convex subset of \mathbb{R}^m such that $\text{int}(C) \neq \emptyset$, $f : C \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function, $c \in C$ and $g : O \rightarrow \mathbb{R}$ be a \mathcal{C}^1 extension of f around c . For every $i = 1, \dots, m$,

$$\frac{\partial f}{\partial x_i}(c) = \frac{\partial g}{\partial x_i}(c).$$

Proposition 1.4.6. (First-order necessary and sufficient conditions)

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a \mathcal{C}^1 concave function. Then, $x \in [0, 1]$ is a maximizer of f if and only if one of the three following conditions holds:

1. $f'(x) = 0$ and $x \in (0, 1)$.
2. $f'(x) \geq 0$ and $x = 1$.
3. $f'(x) \leq 0$ and $x = 0$.

Theorem 1.4.3. (*Berge's theorem*)

Let X, Y be two topological spaces, $\Psi : X \rightarrow Y$ be a continuous correspondence such that for every $x \in X$, $\Psi(x)$ is a nonempty compact subspace of Y , and $f : \text{Gr}(\Psi) \rightarrow \mathbb{R}$ be a continuous map. Then, the map

$$x \in X \mapsto \max_{y \in \Psi(x)} f(x, y) \in \mathbb{R}$$

is continuous. Moreover, if Y is Hausdorff, then the correspondence

$$x \in X \mapsto \text{argmax}_{y \in \Psi(x)} f(x, y)$$

is upper hemicontinuous.

1.4.2 Reminders about real semi-algebraic geometry

Elementary concepts

Definition 1.4.6. A *semi-algebraic subset* of \mathbb{R}^m is a set of the form

$$\bigcup_{i=1}^s \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^m : f_{i,j}(x) \star_{i,j} 0\},$$

where $\star_{i,j}$ denotes either $<$ or $=$ and $f_{i,j} \in \mathbb{R}[X_1, \dots, X_m]$, for every $i = 1, \dots, s$ and every $j = 1, \dots, r_i$. A set S is said to be *semi-algebraic* if it is a semi-algebraic subset of \mathbb{R}^m , for some $m \in \mathbb{N}$.

Example 1.4.1.

- The unit m -disk $\mathbb{D}^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : \sum_{k=1}^{m+1} x_k^2 \leq 1\}$, the unit m -sphere $\mathbb{S}^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : \sum_{k=1}^{m+1} x_k^2 = 1\}$ and the unit m -simplex $\Delta^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : \sum_{k=1}^{m+1} x_k = 1 \text{ and } \forall k \in [1, m+1], x_k \geq 0\}$ are semi-algebraic.
- The graph $\{(x, y) \in \mathbb{R}^2 : y = \cos(x)\}$ of the cosine function is not semi-algebraic.
- The “infinite fan” $\{(x, y) \in \mathbb{R}^2 : \exists n \in \mathbb{N}, y = nx\}$ is not semi-algebraic (cf. Figure 1.4).

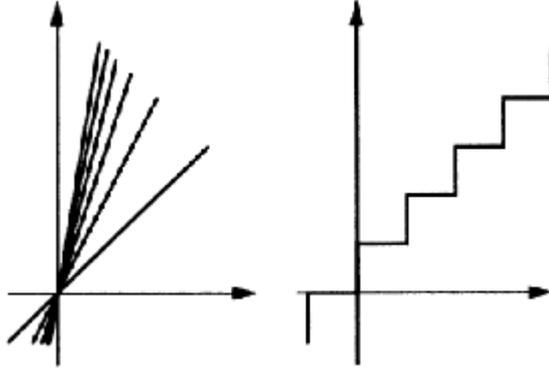


Figure 1.4: *Infinite fan (left) and infinite staircase (right)* (Bochnak et al. [8], pp. 24-25)

- The “infinite staircase” $\{(x, y) \in \mathbb{R}^2 : y = \lfloor x \rfloor \text{ or } [x \in \mathbb{Z} \text{ and } x \leq y \leq x + 1]\}$ (where $\lfloor \cdot \rfloor$ denotes the floor function) is not semi-algebraic (cf. Figure 1.4).

Starting from semi-algebraic sets, one can create new ones by taking finite unions, finite intersections and complements (by definition). However, it is also the case by taking projections:

Theorem 1.4.4. (*Tarski-Seidenberg’s theorem*)

Let S be a semi-algebraic subset of \mathbb{R}^{m+1} and $\text{pr}_{\mathbb{R}^m} : (x_1, \dots, x_m, y) \in \mathbb{R}^{m+1} \mapsto (x_1, \dots, x_m) \in \mathbb{R}^m$. Then, $\text{pr}_{\mathbb{R}^m}(S)$ is a semi-algebraic subset of \mathbb{R}^m .

Remark 1.4.1. In this chapter, even if Tarski-Seidenberg’s theorem will not be used *per se*, it remains one of the most important result of real semi-algebraic geometry, since many results (including the ones which follow in this section) are derived from this theorem.

Definition 1.4.7. A *first-order formula of the language of ordered fields with parameters in \mathbb{R}* is a formula written with a finite number of conjunctions, disjunctions, negations, and universal or existential quantifiers on variables, starting from atomic formulas which are formulas of the kind $f(x_1, \dots, x_m) = 0$ or $g(x_1, \dots, x_m) > 0$, where f and g are polynomials with coefficients in \mathbb{R} . The free variables of a formula are those variables of the polynomials appearing in the formula, which are not quantified.

Example 1.4.2. Here are some examples of first-order formula of the language of ordered fields:

- $x > 0$.
- $(x^2 + 4x = y) \wedge (y^3 = 3)$.
- $\forall y, \exists x, (x + y = 1) \wedge (x = y)$.
- $(\exists y, (y > 0) \wedge (xy = 3)) \Rightarrow (x > 0)$.

Proposition 1.4.7. (Bochnak et al. [8], Proposition 2.2.4, p. 28)

Let $\Phi(x_1, \dots, x_m)$ be a first-order formula of the language of ordered fields with parameters in \mathbb{R} , with free variables x_1, \dots, x_m . Then,

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m : \Phi(x_1, \dots, x_m)\}$$

is a semi-algebraic set.

Remark 1.4.2. In the previous notation, “ $\Phi(x_1, \dots, x_m)$ ” is implicitly considered as a first-order formula of the language of ZF(C) set theory. In particular, quantifiers are only allowed on variables that range over \mathbb{R} . For example, if $\Phi(x, y)$ corresponds to

$$\exists z, (x + y + z = 3) \wedge (xy + z^2 = 5),$$

then $\{(x, y) \in \mathbb{R}^2 : \Phi(x, y)\}$ corresponds to

$$\{(x, y) \in \mathbb{R}^2 : \exists z \in \mathbb{R}, x + y + z = 3 \text{ and } xy + z^2 = 5\}.$$

Example 1.4.3. If f is a polynomial function of four variables with coefficients in \mathbb{R} , then $S = \{(x, y) \in \mathbb{R}^2 : \forall (z, t) \in \mathbb{S}^1, f(x, y, z, t) \geq 0\}$ is a semi-algebraic set. Indeed, S can be written as $\{(x, y) \in \mathbb{R}^2 : \Phi(x, y)\}$, where $\Phi(x, y)$ corresponds to

$$\forall z, \forall t, (z^2 + t^2 = 1) \Rightarrow (f(x, y, z, t) \geq 0).$$

More generally, if S is a semi-algebraic set, and if $\Phi(x_1, \dots, x_m)$ is a first-order formula of the language of ordered fields with parameters in \mathbb{R} , then both the sets

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m : \forall x \in S, \Phi(x_1, \dots, x_m)\}$$

and

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m : \exists x \in S, \Phi(x_1, \dots, x_m)\}$$

are semi-algebraic.

Definition 1.4.8. Let S be a semi-algebraic subset of \mathbb{R}^m and T be a semi-algebraic subset of \mathbb{R}^p . A map $f : S \rightarrow T$ is *semi-algebraic* if its graph $\text{Gr}(f) = \{(x, f(x)) : x \in S\}$ is a semi-algebraic subset of $\mathbb{R}^m \times \mathbb{R}^p$. A map $f : S \rightarrow T$ is a *semi-algebraic homeomorphism* if f is both a semi-algebraic map and a homeomorphism. If such a map exists, then S and T are said to be *semi-algebraically homeomorphic*.

Example 1.4.4.

- If $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^p$ are semi-algebraic, and if $f : S \rightarrow T$ is a polynomial function, then f is semi-algebraic.
- The map $f : x \in \mathbb{R} \setminus \{0\} \mapsto \frac{1}{x} \in \mathbb{R}$ is semi-algebraic since

$$\text{Gr}(f) = \{(x, y) \in \mathbb{R} \setminus \{0\} \times \mathbb{R} : y = \frac{1}{x}\} = \{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } xy = 1\}.$$

- If $S \subset \mathbb{R}^m$ is semi-algebraic, and if $f : S \rightarrow [0, +\infty)$ is semi-algebraic, then $\sqrt{f} : S \rightarrow [0, +\infty)$ is semi-algebraic.
- The maps \ln , \exp , \cos and \sin are not semi-algebraic.

Proposition 1.4.8. (Bochnak et al. [8], Proposition 2.2.7, p. 29)

Let S, T be two semi-algebraic sets and $f : S \rightarrow T$ be a semi-algebraic map. For every semi-algebraic set $A \subset S$, $f(A)$ is a semi-algebraic set, and for every semi-algebraic set $B \subset T$, $f^{-1}(B)$ is a semi-algebraic set.

Corollary 1.4.1. Let S, T be two semi-algebraic sets and $f : S \rightarrow T$ be a semi-algebraic bijective map. Then, f^{-1} is a semi-algebraic map.

Proof. The proof follows directly from Proposition 1.4.7 and from Proposition 1.4.8. \square

Corollary 1.4.2. Let S, T be two semi-algebraic subsets of \mathbb{R}^m . Then, $S + T$ is a semi-algebraic set.

Proof. Note that $S + T = f(S \times T)$, where $f : (x, y) \in S \times T \mapsto x + y \in \mathbb{R}^m$ is a semi-algebraic map. The proof follows from Proposition 1.4.8. \square

Dimension of a semi-algebraic set

For every semi-algebraic set S , one can associate to S an integer $\dim(S) \in \mathbb{N}$ called the *dimension of S* (Bochnak et al. [8], Definition 2.8.1, p. 50).

Proposition 1.4.9. (Bochnak et al. [8], Theorem 2.3.6, p. 33)

Every semi-algebraic subset S of \mathbb{R}^m is the union of a finite number of pairwise disjoint semi-algebraic sets $(S_i)_{i=1}^n$. Furthermore, for every $i \in \{1, \dots, n\}$, S_i is semi-algebraically homeomorphic to an open hypercube $(0, 1)^{d_i}$, for some $d_i \in \mathbb{N}$.

Proposition 1.4.10. (Bochnak et al. [8], Corollary 2.8.9, p. 53)

Let S be a semi-algebraic subset of \mathbb{R}^m . If S is a union of a finite number of semi-algebraic sets $(S_i)_{i=1}^n$ such that for every $i \in \{1, \dots, n\}$, S_i is semi-algebraically homeomorphic to an open hypercube $(0, 1)^{d_i}$ ($d_i \in \mathbb{N}$), then

$$\dim(S) = \max\{d_1, \dots, d_n\}.$$

Example 1.4.5. $\dim(\mathbb{S}^1) = 1$ (by removing $(0, 1)$ and $(0, -1)$). More generally, $\dim(\mathbb{D}^m) = \dim(\mathbb{S}^m) = \dim(\Delta^m) = m$.

Corollary 1.4.3. Let $S = \bigcup_{i=1}^n S_i$, where $(S_i)_{i=1}^n$ is a family of semi-algebraic subsets of \mathbb{R}^m such that for every $i \in \{1, \dots, n\}$, S_i is semi-algebraically homeomorphic to an open hypercube $(0, 1)^{d_i}$ ($d_i \in \mathbb{N}$). For every $i \in \{1, \dots, n\}$, if $\dim(S_i) = \dim(S)$, then S_i is open in S .

Proof. The result follows from Bochnak et al. [8], proof of Theorem 2.3.6, pp. 33-34. \square

Proposition 1.4.11. (Bochnak et al. [8], Proposition 2.8.5, p. 51)

- Let $S = \bigcup_{i=1}^n (S_i)_{i=1}^n$, where for every $i \in \{1, \dots, n\}$, S_i is a semi-algebraic set. Then, $\dim(S) = \max\{\dim(S_1), \dots, \dim(S_n)\}$.
- Let S, T be two semi-algebraic sets. Then, $\dim(S \times T) = \dim(S) + \dim(T)$.

Proposition 1.4.12. (Bochnak et al. [8], Theorem 2.8.8, p. 52)

Let S, T be two semi-algebraic sets and $f : S \rightarrow T$ be a semi-algebraic map. Then, $\dim(S) \geq \dim(f(S))$. Furthermore, if f is a bijective map, then $\dim(S) = \dim(T)$.

Corollary 1.4.4. Let S, T be two semi-algebraic sets. If $S \subset T$, then $\dim(S) \leq \dim(T)$.

Proof. The proof follows directly from Proposition 1.4.12. \square

Corollary 1.4.5. *Let S, T be two semi-algebraic subsets of \mathbb{R}^m . If $\text{Span}(S)$ and $\text{Span}(T)$ are in direct sum, then $\dim(S + T) = \dim(S) + \dim(T)$.*

Proof. Since $\text{Span}(S)$ and $\text{Span}(T)$ are in direct sum, the semi-algebraic map $f : (x, y) \in \text{Span}(S) \times \text{Span}(T) \mapsto x + y \in \text{Span}(S) + \text{Span}(T)$ is a vector space isomorphism. Thus, the restriction of f from $S \times T$ to $f(S \times T) = S + T$ is a semi-algebraic bijective map, and the result follows from Proposition 1.4.11 and from Proposition 1.4.12. \square

Generic semi-algebraic sets and semi-algebraic fiber bundles

Definition 1.4.9. Let S be a semi-algebraic set. A semi-algebraic set $G \subset S$ is said to be *generic in S* (or to be a *generic subset of S*) if $\dim(S \setminus G) < \dim(S)$, and if G is open in S .

Example 1.4.6. The complement of a line in the plane \mathbb{R}^2 is a generic subset of \mathbb{R}^2 .

Definition 1.4.10. Let E and F be two semi-algebraic sets, B be a semi-algebraically connected semi-algebraic set and $p : E \rightarrow B$ be a continuous semi-algebraic map. The tuple (E, B, p, F) is a *semi-algebraic fiber bundle* (of *total space E* , of *base space B* , of *projection p* and of *fiber F*) if there exists a semi-algebraic homeomorphism ψ making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(B) & \xrightarrow{\psi} & B \times F \\ \downarrow p & \swarrow \text{pr}_B & \\ B & & \end{array}$$

where $\text{pr}_B : (x, y) \in B \times F \mapsto x \in B$.

Proposition 1.4.13. *Let S, T be two semi-algebraic sets and $f : S \rightarrow T$ be a continuous semi-algebraic map. There exists a generic subset G of T such that for every connected component C of G , there exists a semi-algebraic set F_C such that $(f^{-1}(C), C, f, F_C)$ is a semi-algebraic fiber bundle.*

Proof. The result follows from Bochnak et al. [8], Theorem 2.4.5, p. 35 and from Bochnak et al. [8], Corollary 9.3.3, p. 224. \square

Theorem 1.4.5. *Let S, T be two semi-algebraic sets such that $\dim(S) = \dim(T)$ and $f : S \rightarrow T$ be a continuous semi-algebraic surjective map. There exists a generic subset G of T such that for every connected component C of G , there exists a nonempty finite set F_C such that $(f^{-1}(C), C, f, F_C)$ is a covering space.*

Proof. From Proposition 1.4.13, there exists a generic subset T_0 of T such that for every connected component C of T_0 , there exists a semi-algebraic set F_C such that $(f^{-1}(C), C, f, F_C)$ is a semi-algebraic fiber bundle (denote by ψ_C the semi-algebraic homeomorphism such that $\text{pr}_C \circ \psi_C = f$, where f is restrained from $f^{-1}(C)$ to C , by abuse of notation).

First, consider the sets

$$\mathcal{C}^=(T_0) = \{C \subset T_0 : C \text{ is a connected component of } T_0 \text{ and } \dim(C) = \dim(T)\},$$

$$\mathcal{C}^<(T_0) = \{C \subset T_0 : C \text{ is a connected component of } T_0 \text{ and } \dim(C) < \dim(T)\}$$

and

$$G = \bigcup_{C \in \mathcal{C}^=(T_0)} C.$$

Remark that $\mathcal{C}^=(T_0) \neq \emptyset$. Indeed, since $T = T_0 \cup (T \setminus T_0)$,

$$\dim(T) = \max\{\dim(T_0), \dim(T \setminus T_0)\} = \dim(T_0)$$

(from Proposition 1.4.11 and from T_0 being generic in T), which implies that at least one of the connected component of T_0 has the same dimension as T (again from Proposition 1.4.11, but applied to T_0). We show that G is a generic subset of T :

1. Since G and $\bigcup_{C \in \mathcal{C}^<(T_0)} C$ are disjoint, notice that

$$T \setminus G = (T \setminus (G \cup \bigcup_{C \in \mathcal{C}^<(T_0)} C)) \cup \bigcup_{C \in \mathcal{C}^<(T_0)} C.$$

Hence,

$$\dim(T \setminus G) = \max\{\dim((T \setminus (G \cup \bigcup_{C \in \mathcal{C}^<(T_0)} C))), \dim(\bigcup_{C \in \mathcal{C}^<(T_0)} C)\},$$

from Proposition 1.4.11. Observe that

$$\dim((T \setminus (G \cup \bigcup_{C \in \mathcal{C}^<(T_0)} C))) = \dim(T \setminus T_0) < \dim(T),$$

from T_0 being generic in T , and that

$$\dim(\bigcup_{C \in \mathcal{C}^<(T_0)} C) < \dim(T),$$

by definition and from Proposition 1.4.11. Therefore, $\dim(T \setminus G) < \dim(T)$.

2. Note that each connected component of T_0 is open in T , since T_0 is itself open in T (from T_0 being generic in T) and since T is locally connected (as a subspace of some Euclidean space). Thus, $G = \bigcup_{C \in \mathcal{C}^=(T_0)} C$ is also open in T as an arbitrary union of open subsets of T .

Now, for every $C \in \mathcal{C}^=(T_0)$, we show that there exists a nonempty finite set F_C such that $(f^{-1}(C), C, f, F_C)$ is a covering space. Let $C \in \mathcal{C}^=(T_0)$ be a connected component of G and $t \in C$. Since $f^{-1}(C)$ and $C \times F_C$ are semi-algebraically homeomorphic,

$$\dim(C \times F_C) = \dim(C) + \dim(F_C) = \dim(f^{-1}(C)),$$

from Proposition 1.4.11 and from Proposition 1.4.12. Also, note that $f^{-1}(t)$ is semi-algebraically homeomorphic to F_C ($\psi_C|_{f^{-1}(t)}$ is a semi-algebraic homeomorphism from $f^{-1}(t)$ to $\{t\} \times F_C$, which is itself semi-algebraically homeomorphic to F_C , and the result follows by composition), which implies that

$$\dim(f^{-1}(t)) = \dim(F_C),$$

again from Proposition 1.4.12. Moreover, $\dim(C) = \dim(T)$ (by definition of G). Thus, since $f^{-1}(C) \subset S$,

$$\dim(f^{-1}(t)) = \dim(f^{-1}(C)) - \dim(C) \leq \dim(S) - \dim(T),$$

from Corollary 1.4.4. By assumption, since $\dim(S) = \dim(T)$, notice that $\dim(f^{-1}(t)) = \dim(F_C) = 0$, hence that F_C is a finite subspace of some Euclidean space (i.e. a discrete space). Nonemptiness of F_C comes from surjectivity assumption on f and from the fact that $f^{-1}(t)$ and F_C are semi-algebraically homeomorphic (which implies that $\text{card}(f^{-1}(t)) = \text{card}(F_C)$). Finally, consider the set $C \ni t$ (which is open in itself) and the (semi-algebraic) homeomorphism ψ_C , and observe that $(f^{-1}(C), C, f, F_C)$ is a covering space with nonempty finite fiber F_C , since by assumption, $(\text{pr}_C \circ \psi_C)(x) = f(x)$, for every $x \in f^{-1}(C)$. \square

1.4.3 Proof of structure theorem

From now on, consider a fixed network $g^0 \in \mathbb{G}$.

Step I. Constructions of $\eta_{\mathcal{F}}$ and $\rho_{\mathcal{F}}$.

Consider the map

$$\begin{cases} \eta_{\mathcal{F}} : \mathcal{P}_{\mathcal{F}} & \rightarrow \mathcal{F} \\ & (v, g) \mapsto \bar{u}^{v,g} \end{cases},$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\bar{u}_i^{v,g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}. \quad (1.1)$$

Moreover, consider the map

$$\begin{cases} \rho_{\mathcal{F}} : \mathcal{F} & \rightarrow \mathcal{F} \times \mathbb{G} \\ v & \mapsto (\underline{u}^v, g^v) \end{cases},$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\underline{u}_i^v(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} g_{ij}^v \gamma_{ij}, \quad (1.2)$$

and where for every link $ij \in L$, $g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\}$, with $w_{i,j}^v \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_i[v] : [0, 1] & \rightarrow \mathbb{R} \\ w & \mapsto v_i(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases}$$

(since $v_i(\cdot, g_{-ij}^0)$ is concave) and $w_{j,i}^v \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_j[v] : [0, 1] & \rightarrow \mathbb{R} \\ w & \mapsto v_j(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases}$$

(since $v_j(\cdot, g_{-ij}^0)$ is concave).

Step II. $\rho_{\mathcal{F}}(\mathcal{F}) \subset \mathcal{P}_{\mathcal{F}}$.

Let $v \in \mathcal{F}$. We have to prove that the network g^v is pairwise stable with respect to \underline{u}^v . Let $ij \in L$, and without loss of generality, suppose that

$$g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\} = w_{i,j}^v$$

(i.e. $w_{i,j}^v \leq w_{j,i}^v$).

First, we show that g_{ij}^v maximizes the map $\underline{u}_i^v(\cdot, g_{-ij}^v)$ (which ensures that first condition of pairwise stability is fulfilled for agent i and that second condition of pairwise stability is also fulfilled). For every $w \in [0, 1]$,

$$\partial_{ij} \underline{u}_i^v(w, g_{-ij}^v) = \partial_{ij} v_i(w, g_{-ij}^v) - (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0)) - g_{ij}^v,$$

and in particular, note that

$$\partial_{ij} \underline{u}_i^v(g_{ij}^v, g_{-ij}^v) = \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0) - g_{ij}^v. \quad (1.3)$$

However, remark that for every $w \in [0, 1]$,

$$q_i[v]'(w) = \partial_{ij}v_i(w, g_{-ij}^0) - w,$$

and in particular, that

$$q_i[v]'(g_{ij}^v) = \partial_{ij}v_i(g_{ij}^v, g_{-ij}^0) - g_{ij}^v. \quad (1.4)$$

By definition, $w_{i,j}^v = g_{ij}^v$ is the unique maximizer of the function $q_i[v]$. Hence, from Equation (1.3) and Equation (1.4), and from Proposition 1.4.6 (applied first to $q_i[v]$, and then to $\underline{u}_i^v(\cdot, g_{-ij}^v)$), one obtains that g_{ij}^v also maximizes the map $\underline{u}_i^v(\cdot, g_{-ij}^v)$.

Last, it remains to show that first condition of pairwise stability is fulfilled for agent j . It is clearly the case if $g_{ij}^v = 0$, so suppose that $g_{ij}^v > 0$. We show that the map $\underline{u}_j^v(\cdot, g_{-ij}^v)$ is nondecreasing on $[0, g_{ij}^v]$, or equivalently (since $\underline{u}_j^v(\cdot, g_{-ij}^v)$ is concave), that $\partial_{ij}\underline{u}_j^v(g_{ij}^v, g_{-ij}^v) \geq 0$. After a computation similar to the one in Equation (1.3), note that $\partial_{ij}\underline{u}_j^v(g_{ij}^v, g_{-ij}^v) \geq 0$ if and only if $\partial_{ij}v_j(g_{ij}^v, g_{-ij}^0) - g_{ij}^v \geq 0$. However, after a computation similar to the one in Equation (1.4), remark that $q_j[v]'(g_{ij}^v) = \partial_{ij}\underline{u}_j^v(g_{ij}^v, g_{-ij}^v)$. By definition, $w_{j,i}^v \geq g_{ij}^v$ is the unique maximizer of the function $q_j[v]$. Hence, from Proposition 1.4.6 (applied to $q_j[v]$) and from concavity of $q_j[v]$, $\partial_{ij}\underline{u}_j^v(g_{ij}^v, g_{-ij}^v) = q_j[v]'(g_{ij}^v) \geq 0$.

Therefore, g^v is pairwise stable with respect to \underline{u}^v , which implies that $\rho_{\mathcal{F}}(\mathcal{F}) \subset \mathcal{P}_{\mathcal{F}}$.

From now on, by abuse of notation, the map $v \in \mathcal{F} \mapsto (\underline{u}^v, g^v) \in \mathcal{P}_{\mathcal{F}}$ is also denoted $\rho_{\mathcal{F}}$.

Step III. $\eta_{\mathcal{F}} \circ \rho_{\mathcal{F}} = \text{id}_{\mathcal{F}}$.

Let $v \in \mathcal{F}$, and consider

$$(\eta_{\mathcal{F}} \circ \rho_{\mathcal{F}})(v) = \eta_{\mathcal{F}}(\underline{u}^v, g^v) = \overline{u}^{\underline{u}^v, g^v}.$$

By definition of $\eta_{\mathcal{F}}$ (see Equation (1.1)) and of $\rho_{\mathcal{F}}$ (see Equation (1.2)), for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\overline{u}_i^{\underline{u}^v, g^v}(\gamma) = \underline{u}_i^v(\gamma) + \sum_{j \neq i} (\partial_{ij}\underline{u}_i^v(g_{ij}^v, g_{-ij}^v) - \partial_{ij}\underline{u}_i^v(g_{ij}^v, g_{-ij}^0)(\gamma_{ij} - g_{ij}^v)) + \sum_{j \neq i} g_{ij}^v \gamma_{ij}. \quad (1.5)$$

Then,

$$\begin{aligned}
& \bar{u}_i^{\underline{u}^v, g^v}(\gamma) - v_i(\gamma) \\
&= \sum_{j \neq i} (\partial_{ij} \underline{u}_i^v(g_{ij}^v, g_{-ij}^v) - \partial_{ij} \underline{u}_i^v(g_{ij}^v, g_{-ij}^0)(\gamma_{ij} - g_{ij}^v)) \\
&\quad - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0)(\gamma_{ij} - g_{ij}^v)) \\
&= \sum_{j \neq i} (\partial_{ij} (\underline{u}_i^v - v_i)(g_{ij}^v, g_{-ij}^v) - \partial_{ij} (\underline{u}_i^v - v_i)(g_{ij}^v, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v),
\end{aligned}$$

by summing Equation (1.2) and Equation (1.5). However, remark that the map $g \in \mathbb{G} \mapsto (\underline{u}_i^v - v_i)(g)$ is affine, which implies that for every $j \neq i$ and every $g \in \mathbb{G}$, $\partial_{ij} (\underline{u}_i^v - v_i)(g_{ij}, g_{-ij})$ does not depend on g_{-ij} . Thus, each term in the previous sum is null, which implies that $\bar{u}^{\underline{u}^v, g^v} = v$.

Step IV. $\rho_{\mathcal{F}} \circ \eta_{\mathcal{F}} = \text{id}_{\mathcal{P}_{\mathcal{F}}}$.

Let $(v, g) \in \mathcal{P}_{\mathcal{F}}$, and consider

$$(\rho_{\mathcal{F}} \circ \eta_{\mathcal{F}})(v, g) = \rho_{\mathcal{F}}(\bar{u}^{v, g}) = (\underline{u}^{\bar{u}^{v, g}}, g^{\bar{u}^{v, g}}),$$

where for every link $ij \in L$, $g_{ij}^{\bar{u}^{v, g}} = \min\{w_{i,j}^{\bar{u}^{v, g}}, w_{j,i}^{\bar{u}^{v, g}}\}$, with $w_{i,j}^{\bar{u}^{v, g}} \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_i[\bar{u}^{v, g}] : [0, 1] & \rightarrow & \mathbb{R} \\ w & \mapsto & \bar{u}_i^{v, g}(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases}$$

(since $\bar{u}_i^{v, g}(\cdot, g_{-ij}^0)$ is concave) and $w_{j,i}^{\bar{u}^{v, g}} \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_j[\bar{u}^{v, g}] : [0, 1] & \rightarrow & \mathbb{R} \\ w & \mapsto & \bar{u}_j^{v, g}(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases}$$

(since $\bar{u}_j^{v, g}(\cdot, g_{-ij}^0)$ is concave). We have to prove that $g^{\bar{u}^{v, g}} = g$ and that $\underline{u}^{\bar{u}^{v, g}} = v$.

Substep IV.1. $g^{\bar{u}^{v, g}} = g$.

Recall the following lemma, which provides a necessary condition for a network to be pairwise stable with respect to a continuous own-weights concave society.

Lemma 1.4.1. (*Bich-Morhaim [6]*)

Let v be a continuous own-weights concave society and g be a pairwise stable network with respect to v . Moreover, for every $ij \in L$, let

$$\Omega_i(g_{-ij}) = [\check{\omega}_i(g_{-ij}), \hat{\omega}_i(g_{-ij})] = \operatorname{argmax}_{w \in [0,1]} v_i(w, g_{-ij})$$

and

$$\Omega_j(g_{-ij}) = [\check{\omega}_j(g_{-ij}), \hat{\omega}_j(g_{-ij})] = \operatorname{argmax}_{w \in [0,1]} v_j(w, g_{-ij}).$$

Then, for every $ij \in L$,

$$g_{ij} \in [\min\{\check{\omega}_i(g_{-ij}), \check{\omega}_j(g_{-ij})\}, \min\{\hat{\omega}_i(g_{-ij}), \hat{\omega}_j(g_{-ij})\}].$$

Let $ij \in L$, and without loss of generality, suppose that $\check{\omega}_i(g_{-ij}) \leq \check{\omega}_j(g_{-ij})$ (recall that $(v, g) \in \mathcal{P}_{\mathcal{F}}$). In that case, from Lemma 1.4.1, observe that $g_{ij} \in \Omega_i(g_{-ij})$, i.e. g_{ij} is a maximizer of the map $v_i(\cdot, g_{-ij})$. Hence, $v_i(\cdot, g_{-ij})$ satisfies one of the three conditions of Proposition 1.4.6. Now, note that

$$q_i[\bar{u}^{v,g}]'(g_{ij}) = \partial_{ij} u_i^{v,g}(g_{ij}, g_{-ij}^0) - g_{ij} = \partial_{ij} v_i(g_{ij}, g_{-ij}), \quad (1.6)$$

computing $\partial_{ij} u_i^{v,g}(g_{ij}, g_{-ij}^0)$ with Equation (1.1). Then, $q_i[\bar{u}^{v,g}]$ also satisfies one of the three conditions of Proposition 1.4.6, which implies that g_{ij} is the unique maximizer of this function (from its strict concavity), i.e. $g_{ij} = w_{i,j}^{\bar{u}^{v,g}}$.

To finish, it only remains to prove that $g_{ij}^{\bar{u}^{v,g}} = w_{i,j}^{\bar{u}^{v,g}}$, or equivalently, that $w_{i,j}^{\bar{u}^{v,g}} \leq w_{j,i}^{\bar{u}^{v,g}}$. This inequality is clearly verified when $w_{i,j}^{\bar{u}^{v,g}} = 0$ or when $w_{j,i}^{\bar{u}^{v,g}} = 1$, so suppose that $w_{i,j}^{\bar{u}^{v,g}} > 0$ and that $w_{j,i}^{\bar{u}^{v,g}} < 1$. Since $w_{j,i}^{\bar{u}^{v,g}} < 1$, note that

$$q_j[\bar{u}^{v,g}]'(w_{j,i}^{\bar{u}^{v,g}}) \leq 0,$$

from Proposition 1.4.6. Also, a computation similar to the one in Equation (1.6) provides that $q_j[\bar{u}^{v,g}]'(g_{ij}) = \partial_{ij} v_j(g_{ij}, g_{-ij})$. Hence, since $v_j(\cdot, g_{-ij})$ is concave and since $0 < w_{i,j}^{\bar{u}^{v,g}} = g_{ij} \leq \hat{\omega}_j(g_{-ij})$ (because g_{ij} is less or equal to $\min\{\hat{\omega}_i(g_{-ij}), \hat{\omega}_j(g_{-ij})\}$, from Lemma 1.4.1), remark that $\partial_{ij} v_j(w_{i,j}^{\bar{u}^{v,g}}, g_{-ij}) \geq 0$, i.e.

$$q_j[\bar{u}^{v,g}]'(w_{i,j}^{\bar{u}^{v,g}}) \geq 0.$$

Finally, $q_j[\bar{u}^{v,g}]'(w_{j,i}^{\bar{u}^{v,g}}) \leq q_j[\bar{u}^{v,g}]'(w_{i,j}^{\bar{u}^{v,g}})$, which implies that $w_{i,j}^{\bar{u}^{v,g}} \leq w_{j,i}^{\bar{u}^{v,g}}$, because the map $q_j[\bar{u}^{v,g}]'$ is strictly decreasing from strict concavity of $q_j[\bar{u}^{v,g}]$. Therefore, $g^{\bar{u}^{v,g}} = g$.

Substep IV.2. $\underline{u}^{\bar{u}^{v,g}} = v$.

The proof is similar to the one of Step III. For every $i \in N$ and every $\gamma \in \mathbb{G}$, by definition of $\rho_{\mathcal{F}}$ and from Substep IV.1 (i.e $g^{\bar{u}^{v,g}} = g$),

$$\underline{u}_i^{\bar{u}^{v,g}}(\gamma) = \bar{u}_i^{v,g}(\gamma) - \sum_{j \neq i} (\partial_{ij} \bar{u}_i^{v,g}(g_{ij}, g_{-ij}) - \partial_{ij} \bar{u}_i^{v,g}(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) - \sum_{j \neq i} g_{ij} \gamma_{ij},$$

and by definition of $\eta_{\mathcal{F}}$,

$$\bar{u}_i^{v,g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}.$$

Then,

$$\underline{u}_i^{\bar{u}^{v,g}}(\gamma) - v_i(\gamma) = \sum_{j \neq i} (\partial_{ij} (v_i - \bar{u}_i^{v,g})(g_{ij}, g_{-ij}) - \partial_{ij} (v_i - \bar{u}_i^{v,g})(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}),$$

by summing the two above equations. However, remark that the map $g \in \mathbb{G} \mapsto (v_i - \bar{u}_i^{v,g})(g)$ is affine, which implies that for every $j \neq i$ and every $g \in \mathbb{G}$, $\partial_{ij} (v_i - \bar{u}_i^{v,g})(g_{ij}, g_{-ij})$ does not depend on g_{-ij} . Thus, each term in the previous sum is null, which implies that $\underline{u}^{\bar{u}^{v,g}} = v$.

Step V. $\eta_{\mathcal{F}}$ is a continuous map.

Recall that for every $(v, g) \in \mathcal{P}_{\mathcal{F}}$, $\eta_{\mathcal{F}}(v, g) = \bar{u}^{v,g}$, where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\bar{u}_i^{v,g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}$$

(see Equation (1.1)). Since \mathcal{F} is endowed with the product topology, the continuity of $\eta_{\mathcal{F}}$ is obtained from the three following points:

1. For every $i \in N$ and every continuous maps $f_i^1, f_i^2 : \mathcal{P}_{\mathcal{F}} \rightarrow \mathcal{F}_i$, the map

$$f_i : (v, g) \in \mathcal{P}_{\mathcal{F}} \mapsto f_i^1(v, g) + f_i^2(v, g) \in \mathcal{F}_i$$

is continuous. Indeed, consider $(v^*, g^*) \in \mathcal{P}_{\mathcal{F}}$ and consider a sequence $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ in $\mathcal{P}_{\mathcal{F}}$ which converges to (v^*, g^*) . By definition, for every $(v, g) \in \mathcal{P}_{\mathcal{F}}$, $\|f_i(v, g) - f_i(v^*, g^*)\|_i$ is equal to

$$\max\{\max\{\|f_i(v, g) - f_i(v^*, g^*)\|_\infty, \|\partial_{ik} f_i(v, g) - \partial_{ik} f_i(v^*, g^*)\|_\infty\} : k \neq i\},$$

where

$$\begin{aligned} f_i(v, g) - f_i(v^*, g^*) &= (f_i^1(v, g) + f_i^2(v, g)) - (f_i^1(v^*, g^*) + f_i^2(v^*, g^*)) \\ &= (f_i^1(v, g) - f_i^1(v^*, g^*)) + (f_i^2(v, g) - f_i^2(v^*, g^*)). \end{aligned}$$

Thus,

$$\|f_i(v^\ell, g^\ell) - f_i(v^*, g^*)\|_i \leq \|f_i^1(v^\ell, g^\ell) - f_i^1(v^*, g^*)\|_i + \|f_i^2(v^\ell, g^\ell) - f_i^2(v^*, g^*)\|_i,$$

which implies that

$$\lim_{(v^\ell, g^\ell) \rightarrow (v^*, g^*)} \|f_i(v^\ell, g^\ell) - f_i(v^*, g^*)\|_i = 0,$$

since $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, g^*) and since both f_i^1 and f_i^2 are continuous. Hence, f_i is a continuous map.

2. For every $i \in N$, the map $(v, g) \in \mathcal{P}_{\mathcal{F}} \mapsto v_i \in \mathcal{F}_i$ is continuous, as the restriction of a canonical projection.
3. For every $i \in N$, the map

$$(v, g) \in \mathcal{P}_{\mathcal{F}} \mapsto (\gamma \mapsto \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}) \in \mathcal{F}_i$$

is continuous. Indeed, we show that the map

$$f_{ij} : (v, g) \in \mathcal{P}_{\mathcal{F}} \mapsto (\gamma \mapsto \partial_{ij} v_i(g_{ij}, g_{-ij}) \gamma_{ij}) \in \mathcal{F}_i$$

is continuous, for every $j \neq i$ (the reasoning is similar for the other maps of the previous sum). Consider $(v^*, g^*) \in \mathcal{P}_{\mathcal{F}}$ and consider a sequence $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ in $\mathcal{P}_{\mathcal{F}}$ which converges to (v^*, g^*) . By definition, for every $(v, g) \in \mathcal{P}_{\mathcal{F}}$ and every $\gamma \in \mathbb{G}$,

$$\begin{aligned} f_{ij}(v, g)(\gamma) - f_{ij}(v^*, g^*)(\gamma) &= \partial_{ij} v_i(g_{ij}, g_{-ij}) \gamma_{ij} - \partial_{ij} v_i^*(g_{ij}^*, g_{-ij}^*) \gamma_{ij} \\ &= \partial_{ij} (v_i - v_i^*)(g_{ij}, g_{-ij}) \gamma_{ij} \\ &\quad + (\partial_{ij} v_i^*(g_{ij}, g_{-ij}) - \partial_{ij} v_i^*(g_{ij}^*, g_{-ij}^*)) \gamma_{ij}. \end{aligned}$$

This implies that

$$\begin{aligned} &\|f_{ij}(v^\ell, g^\ell) - f_{ij}(v^*, g^*)\|_\infty \\ &\leq |\partial_{ij} (v_i^\ell - v_i^*)(g_{ij}^\ell, g_{-ij}^\ell)| \\ &\quad + |\partial_{ij} v_i^*(g_{ij}^\ell, g_{-ij}^\ell) - \partial_{ij} v_i^*(g_{ij}^*, g_{-ij}^*)| \\ &\leq \|v_i^\ell - v_i^*\|_i + |\partial_{ij} v_i^*(g_{ij}^\ell, g_{-ij}^\ell) - \partial_{ij} v_i^*(g_{ij}^*, g_{-ij}^*)|. \end{aligned}$$

Then,

$$\lim_{(v^\ell, g^\ell) \rightarrow (v^*, g^*)} \|f_{ij}(v^\ell, g^\ell) - f_{ij}(v^*, g^*)\|_\infty = 0, \quad (1.7)$$

since $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, g^*) and since $\partial_{ij}v_i^*$ is continuous. Moreover, for every $(v, g) \in \mathcal{P}_{\mathcal{F}}$, every $k \neq i$ and every $\gamma \in \mathbb{G}$,

$$\begin{aligned} & \partial_{ik}f_{ij}(v, g)(\gamma) - \partial_{ik}f_{ij}(v^*, g^*)(\gamma) \\ &= \begin{cases} \partial_{ij}v_i(g_{ij}, g_{-ij}) - \partial_{ij}v_i^*(g_{ij}^*, g_{-ij}^*) & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Similarly as above, this implies that for every $k \neq i$,

$$\|\partial_{ik}f_{ij}(v^\ell, g^\ell) - \partial_{ik}f_{ij}(v^*, g^*)\|_\infty \leq \|v_i^\ell - v_i^*\|_i + |\partial_{ij}v_i^*(g_{ij}^\ell, g_{-ij}^\ell) - \partial_{ij}v_i^*(g_{ij}^*, g_{-ij}^*)|,$$

hence that

$$\lim_{(v^\ell, g^\ell) \rightarrow (v^*, g^*)} \|\partial_{ik}f_{ij}(v^\ell, g^\ell) - \partial_{ik}f_{ij}(v^*, g^*)\|_\infty = 0, \quad (1.8)$$

since $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, g^*) and since $\partial_{ij}v_i^*$ is continuous. Therefore, from Equation (1.7) and Equation (1.8),

$$\lim_{(v^\ell, g^\ell) \rightarrow (v^*, g^*)} \|f_{ij}(v^\ell, g^\ell) - f_{ij}(v^*, g^*)\|_i = 0,$$

which implies that f_{ij} is a continuous map, for every $j \neq i$.

Step VI. $\rho_{\mathcal{F}}$ is a continuous map.

Recall that for every $v \in \mathcal{F}$, $\rho_{\mathcal{F}}(v) = (\underline{v}^v, g^v)$, where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\underline{v}_i^v(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij}v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij}v_i(g_{ij}^0, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} g_{ij}^v \gamma_{ij}$$

(see Equation (1.2)), where for every link $ij \in L$, $g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\}$, with $w_{i,j}^v \in [0, 1]$ being the unique maximizer of the strictly concave function $q_i[v] : w \in [0, 1] \mapsto v_i(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$ and $w_{j,i}^v \in [0, 1]$ being the unique maximizer of the strictly concave function $q_j[v] : w \in [0, 1] \mapsto v_j(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$. The continuity of ρ can be obtained similarly as the continuity of η . The only additional point is to prove that the map

$$v \in \mathcal{F} \mapsto g^v \in \mathbb{G}$$

is continuous. This follows from the continuity of the maps $v \in \mathcal{F} \mapsto w_{i,j}^v \in [0, 1]$ and $v \in \mathcal{F} \mapsto w_{j,i}^v \in [0, 1]$, for every $ij \in L$, which is a consequence of Theorem 1.4.3 (Berge's theorem). Indeed, let $ij \in L$, and consider the constant correspondence

$$\Psi : v \in \mathcal{F} \mapsto [0, 1] \subset [0, 1]$$

and the map

$$q_i : (v, w) \in \text{Gr}(\Psi) \mapsto q_i[v](w) = v_i(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}.$$

We show that the map q_i is continuous. Consider $(v^*, w^*) \in \mathcal{F} \times [0, 1]$ and consider a sequence $(v^\ell, w^\ell)_{\ell \in \mathbb{N}}$ in $\mathcal{F} \times [0, 1]$ which converges to (v^*, w^*) . By definition, for every $(v, w) \in \mathcal{F} \times [0, 1]$,

$$\begin{aligned} q_i[v](w) - q_i[v^*](w^*) &= (v_i(w, g_{-ij}^0) - \frac{w^2}{2}) - (v_i^*(w^*, g_{-ij}^0) - \frac{(w^*)^2}{2}) \\ &= (v_i(w, g_{-ij}^0) - v_i^*(w^*, g_{-ij}^0)) + (\frac{w^2}{2} - \frac{(w^*)^2}{2}) \\ &= (v_i - v_i^*)(w, g_{-ij}^0) + (v_i^*(w, g_{-ij}^0) - v_i^*(w^*, g_{-ij}^0)) \\ &\quad + (\frac{w^2}{2} - \frac{(w^*)^2}{2}). \end{aligned}$$

This implies that

$$\begin{aligned} |q_i[v^\ell](w^\ell) - q_i[v^*](w^*)| &\leq |(v_i^\ell - v_i^*)(w^\ell, g_{-ij}^0)| + |v_i^*(w^\ell, g_{-ij}^0) - v_i^*(w^*, g_{-ij}^0)| \\ &\quad + |\frac{(w^\ell)^2}{2} - \frac{(w^*)^2}{2}| \\ &\leq \|v_i^\ell - v_i^*\|_i + |v_i^*(w^\ell, g_{-ij}^0) - v_i^*(w^*, g_{-ij}^0)| \\ &\quad + |\frac{(w^\ell)^2}{2} - \frac{(w^*)^2}{2}|. \end{aligned}$$

Then,

$$\lim_{(v^\ell, w^\ell) \rightarrow (v^*, w^*)} |q_i[v^\ell](w^\ell) - q_i[v^*](w^*)| = 0,$$

since $(v^\ell, w^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, w^*) and since v_i^* is continuous. Therefore, from Berge's theorem, the map

$$v \in \mathcal{F} \mapsto \operatorname{argmax}_{w \in [0, 1]} q_i[v](w) = w_{i,j}^v \in [0, 1]$$

is continuous (the reasoning is similar for the map $v \in \mathcal{F} \mapsto w_{j,i}^v \in [0, 1]$).

Step VII. $\pi_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$ are properly homotopic.

We show that the map

$$\begin{cases} H_{\mathcal{F}} : [0, 1] \times \mathcal{P}_{\mathcal{F}} & \rightarrow & \mathcal{F} \\ (t, (v, g)) & \mapsto & (1-t)\pi_{\mathcal{F}}(v, g) + t\eta_{\mathcal{F}}(v, g) \end{cases}$$

is a proper homotopy between $\pi_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$. Note that $H_{\mathcal{F}}(0, (\cdot, \cdot)) = \pi_{\mathcal{F}}$ and $H_{\mathcal{F}}(1, (\cdot, \cdot)) = \eta_{\mathcal{F}}$, and that $H_{\mathcal{F}}$ is a continuous map (the proof is similar to the one of Step V).

Thus, it remains to show that $H_{\mathcal{F}}$ is a proper map. To do so, define the map

$$\begin{cases} G_{\mathcal{F}}^+ : [0, 1] \times \mathcal{F} \times \mathbb{G} & \rightarrow & \mathcal{F} \\ (t, (v, g)) & \mapsto & (G_{\mathcal{F}}^+(t, (v, g)))_{i \in N} \end{cases},$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\begin{aligned} & G_{\mathcal{F}}^+(t, (v, g))_i(\gamma) \\ &= v_i(\gamma) + t \left(\sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij} \right). \end{aligned}$$

Note that the restriction of $G_{\mathcal{F}}^+$ from $[0, 1] \times \mathcal{P}_{\mathcal{F}}$ to \mathcal{F} is equal to $H_{\mathcal{F}}$. Moreover, consider the map

$$\begin{cases} \Phi_{\mathcal{F}} : [0, 1] \times \mathcal{F} \times \mathbb{G} & \rightarrow & [0, 1] \times \mathcal{F} \times \mathbb{G} \\ (t, (v, g)) & \mapsto & (t, (G_{\mathcal{F}}^+(t, (v, g))), g) \end{cases},$$

and remark that this map is invertible: its inverse is the map

$$\begin{cases} \Phi_{\mathcal{F}}^{-1} : [0, 1] \times \mathcal{F} \times \mathbb{G} & \rightarrow & [0, 1] \times \mathcal{F} \times \mathbb{G} \\ (t, (v, g)) & \mapsto & (t, (G_{\mathcal{F}}^-(t, (v, g))), g) \end{cases},$$

where

$$\begin{cases} G_{\mathcal{F}}^- : [0, 1] \times \mathcal{F} \times \mathbb{G} & \rightarrow & \mathcal{F} \\ (t, (v, g)) & \mapsto & (G_{\mathcal{F}}^-(t, (v, g)))_{i \in N} \end{cases},$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\begin{aligned} & G_{\mathcal{F}}^-(t, (v, g))_i(\gamma) \\ &= v_i(\gamma) - t \left(\sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij} \right). \end{aligned}$$

Both $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{F}}^{-1}$ are continuous maps (proofs are similar to the one of Step V), which implies that $\Phi_{\mathcal{F}}$ is a homeomorphism. Now, for every compact subspace K of \mathcal{F} , observe that

$$\begin{aligned} H_{\mathcal{F}}^{-1}(K) &= \{(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{F}} : H_{\mathcal{F}}(t, (v, g)) \in K\} \\ &= \{(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{F}} : \Phi_{\mathcal{F}}(t, (v, g)) \in [0, 1] \times K \times \mathbb{G}\} \\ &= \Phi_{\mathcal{F}}^{-1}([0, 1] \times K \times \mathbb{G}) \cap ([0, 1] \times \mathcal{P}_{\mathcal{F}}). \end{aligned}$$

Moreover, consider the following lemma.

Lemma 1.4.2. $\mathcal{P}_{\mathcal{F}}$ is a closed subset of $\mathcal{F} \times \mathbb{G}$.

Proof. Consider $(v^*, g^*) \in \mathcal{F} \times \mathbb{G}$ and consider a sequence $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ in $\mathcal{P}_{\mathcal{F}}$ which converges to (v^*, g^*) , and suppose that $(v^*, g^*) \notin \mathcal{P}_{\mathcal{F}}$.

Suppose first that there exists $ij \in L$ and $w < g_{ij}^*$ such that $v_i^*(w, g_{-ij}^*) > v_i^*(g^*)$ (without loss of generality). By definition of the topology on \mathcal{F}_i , since $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, g^*) and since v_i^* is continuous, remark that the sequence $(v_i^\ell(g^\ell))_{\ell \in \mathbb{N}}$ converges to $v_i^*(g^*)$:

$$\begin{aligned} |v_i^\ell(g^\ell) - v_i^*(g^*)| &\leq |v_i^\ell(g^\ell) - v_i^*(g^\ell)| + |v_i^*(g^\ell) - v_i^*(g^*)| \\ &\leq \|v_i^\ell - v_i^*\|_\infty + |v_i^*(g^\ell) - v_i^*(g^*)| \\ &\leq \|v_i^\ell - v_i^*\|_i + |v_i^*(g^\ell) - v_i^*(g^*)|, \end{aligned}$$

and similarly, that the sequence $(v_i^\ell(w, g_{-ij}^\ell))_{\ell \in \mathbb{N}}$ converges to $v_i^*(w, g_{-ij}^*)$. Then, there exists $\ell \in \mathbb{N}$ such that $v_i^\ell(w, g_{-ij}^\ell) > v_i^\ell(g^\ell)$, which contradicts the fact that g^ℓ is pairwise stable with respect to v^ℓ .

Now, suppose that there exists $ij \in L$ and $w > g_{ij}^*$ such that $v_i^*(w, g_{-ij}^*) > v_i^*(g^*)$ and $v_j^*(w, g_{-ij}^*) > v_j^*(g^*)$. In a similar way as above, since $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ converges to (v^*, g^*) and since v_i^* and v_j^* are continuous, this means that there exists $\ell \in \mathbb{N}$ such that $v_i^\ell(w, g_{-ij}^\ell) > v_i^\ell(g^\ell)$ and $v_j^\ell(w, g_{-ij}^\ell) > v_j^\ell(g^\ell)$, which contradicts the fact that g^ℓ is pairwise stable with respect to v^ℓ . \square

From Lemma 1.4.2, note that $[0, 1] \times \mathcal{P}_{\mathcal{F}}$ is also closed in $[0, 1] \times \mathcal{F} \times \mathbb{G}$, thus that $H_{\mathcal{F}}^{-1}(K)$ is closed in $\Phi_{\mathcal{F}}^{-1}([0, 1] \times K \times \mathbb{G})$, which is a compact subspace of $[0, 1] \times \mathcal{F} \times \mathbb{G}$ (since K is compact, and since $\Phi_{\mathcal{F}}$ is a homeomorphism). Therefore, this finally implies that $H_{\mathcal{F}}^{-1}(K)$ is a compact subspace of $[0, 1] \times \mathcal{P}_{\mathcal{F}}$, thus that $H_{\mathcal{F}}$ is a proper map.

Step VIII. For every \mathcal{A} -regular set \mathcal{R} of societies, the projection $\pi_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$.

Let $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$ be a \mathcal{A} -regular set of societies (i.e. \mathcal{R} satisfies concavity assumption and \mathcal{A} -invariance assumption in Definition 1.2.8).

Consider the maps

$$\begin{cases} \eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} & \rightarrow & \mathcal{F} \\ (v, g) & \mapsto & \eta_{\mathcal{F}}(v, g) = \bar{u}^{v, g} \end{cases}$$

and

$$\begin{cases} \rho_{\mathcal{R}} : \mathcal{R} & \rightarrow & \mathcal{P}_{\mathcal{F}} \\ v & \mapsto & \rho_{\mathcal{F}}(v) = (\underline{u}^v, g^v) \end{cases}$$

(which are well-defined, from concavity assumption). From the previous steps, it is sufficient to prove that $\eta_{\mathcal{R}}(\mathcal{P}_{\mathcal{R}}) \subset \mathcal{R}$ and that $\rho_{\mathcal{R}}(\mathcal{R}) \subset \mathcal{P}_{\mathcal{R}}$ in order to obtain that $\mathcal{P}_{\mathcal{R}}$ and \mathcal{R} are homeomorphic (in that case, the restriction of $\eta_{\mathcal{R}}$ from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} is such a homeomorphism and its inverse is the restriction of $\rho_{\mathcal{R}}$ from \mathcal{R} to $\mathcal{P}_{\mathcal{R}}$).

Substep VIII.1. $\eta_{\mathcal{R}}(\mathcal{P}_{\mathcal{R}}) \subset \mathcal{R}$.

Recall that for every $(v, g) \in \mathcal{P}_{\mathcal{R}}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\bar{u}_i^{v, g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}^0, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}$$

(see Equation (1.1)). Remark that $\bar{u}_i^{v, g}$ is equal to v_i up to an element of \mathcal{A}_i . Thus, \mathcal{A} -invariance assumption directly implies that $\eta_{\mathcal{R}}(\mathcal{P}_{\mathcal{R}}) \subset \mathcal{R}$.

From now on, by abuse of notation, the map $(v, g) \in \mathcal{P}_{\mathcal{R}} \mapsto \eta_{\mathcal{F}}(v, g) \in \mathcal{R}$ is also denoted $\eta_{\mathcal{R}}$.

Substep VIII.2. $\rho_{\mathcal{R}}(\mathcal{R}) \subset \mathcal{P}_{\mathcal{R}}$.

Recall that for every $v \in \mathcal{R}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\underline{u}_i^v(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^0, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} g_{ij}^v \gamma_{ij}$$

(see Equation (1.2)), where for every link $ij \in L$, $g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\}$, with $w_{i,j}^v \in [0, 1]$ (resp. $w_{j,i}^v \in [0, 1]$) being the unique maximizer of the strictly concave function $q_i[v] : [0, 1] \mapsto v_i(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$ (resp. $q_j[v] : [0, 1] \mapsto v_j(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$). Similarly as in the previous step, observe that \underline{u}_i^v is

equal to v_i up to an element of \mathcal{A}_i . Thus, \mathcal{A} -invariance assumption directly implies that $\rho_{\mathcal{R}}(\mathcal{R}) \subset \mathcal{P}_{\mathcal{R}}$.

From now on, by abuse of notation, the map $v \in \mathcal{R} \mapsto \rho_{\mathcal{F}}(v) \in \mathcal{P}_{\mathcal{R}}$ is also denoted $\rho_{\mathcal{R}}$.

Substep VIII.3. $\pi_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ are properly homotopic.

Consider the map

$$\begin{cases} H_{\mathcal{R}} : [0, 1] \times \mathcal{P}_{\mathcal{R}} & \rightarrow & \mathcal{F} \\ (t, (v, g)) & \mapsto & (1-t)\pi_{\mathcal{R}}(v, g) + t\eta_{\mathcal{R}}(v, g) \end{cases}$$

(which is well-defined, from concavity assumption). Remark that, for every $(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{R}}$, there exists $a \in \mathcal{A}$ such that

$$H_{\mathcal{R}}(t(v, g)) = (1-t)\pi_{\mathcal{R}}(v, g) + t\eta_{\mathcal{R}}(v, g) = (1-t)v + t(v+a) = v + ta \in \mathcal{R} + \mathcal{A},$$

since $\eta_{\mathcal{R}}(v, g)_i$ is equal to v_i up to an element of \mathcal{A}_i . Hence, \mathcal{A} -invariance assumption implies that $H_{\mathcal{R}}([0, 1] \times \mathcal{P}_{\mathcal{R}}) \subset \mathcal{R}$.

From now on, by abuse of notation, the map $(t, (v, g)) \in [0, 1] \times \mathcal{P}_{\mathcal{R}} \mapsto (1-t)\pi_{\mathcal{R}}(v, g) + t\eta_{\mathcal{R}}(v, g) \in \mathcal{R}$ is also denoted $H_{\mathcal{R}}$. Moreover, the proof that $H_{\mathcal{R}}$ is a proper homotopy between $\pi_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ is similar to the one of Step VII.

This step ends the proof of the structure theorem. \square

1.4.4 Proof of oddness theorem

From now on, consider a fixed network $g^0 \in \mathbb{G}$. Recall that for every \mathcal{A} -regular set \mathcal{R} of societies, $\eta_{\mathcal{R}}$ is the homeomorphism from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} of structure theorem (Theorem 1.2.1): for every $(v, g) \in \mathcal{P}_{\mathcal{R}}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\eta_{\mathcal{R}}(v, g)_i(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}$$

(see Equation (1.1) in Appendix 1.4.3). Also, for every subset $\mathcal{V} \subset \mathbb{R}_{\delta}[g]$ of polynomial societies, consider the set

$$\mathcal{M}_{\mathcal{V}} = \{(\varphi(v), g) \in \mathbf{C}_{\mathcal{V}} \times \mathbb{G} : (v, g) \in \mathcal{P}_{\mathcal{V}}\} \subset \mathbb{R}^m \times \mathbb{G}.$$

Let $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$ be a \mathcal{A} -semi-algebraically regular set of societies (i.e. \mathcal{S} satisfies concavity assumption, \mathcal{A} -invariance assumption and semi-algebraicity

assumption in Definition 1.3.3). In particular, note that \mathcal{S} is also \mathcal{A} -regular (by definition). From Step I to Step VI, suppose that the set $\mathbf{C}_{\mathcal{S}} = \varphi(\mathcal{S})$ is equal to \mathbb{R}^m (n.b.: the proof is similar if we only assume that $\mathbf{C}_{\mathcal{S}}$ is homeomorphic to \mathbb{R}^p , for some integer $p \leq m$). The general case is treated in Step VII.

Step I. The set $\mathcal{M}_{\mathcal{S}}$ is semi-algebraic and its dimension is equal to m .

First, we show that $\mathcal{M}_{\mathcal{S}}$ is a semi-algebraic set. From concavity assumption, remark that for every $(x, g) \in \mathbb{R}^m \times \mathbb{G}$, $(x, g) \in \mathcal{M}_{\mathcal{S}}$ if and only if for every $ij \in L$:

- (i) either $g_{ij} \in (0, 1)$, $\partial_{ij}v_i^x(g_{ij}, g_{-ij}) = 0$ and $\partial_{ij}v_j^x(g_{ij}, g_{-ij}) \geq 0$;
- (ii) or $g_{ij} \in (0, 1)$, $\partial_{ij}v_j^x(g_{ij}, g_{-ij}) = 0$ and $\partial_{ij}v_i^x(g_{ij}, g_{-ij}) \geq 0$;
- (iii) or $g_{ij} = 0$ and $[\partial_{ij}v_i^x(g_{ij}, g_{-ij}) \leq 0$ or $\partial_{ij}v_j^x(g_{ij}, g_{-ij}) \leq 0]$;
- (iv) or $g_{ij} = 1$ and $[\partial_{ij}v_i^x(g_{ij}, g_{-ij}) \geq 0$ and $\partial_{ij}v_j^x(g_{ij}, g_{-ij}) \geq 0]$.

These conditions involve a finite number of equalities and of inequalities with semi-algebraic maps, thus $\mathcal{M}_{\mathcal{S}}$ is a semi-algebraic set.

Last, we show that the map

$$\left\{ \begin{array}{l} \eta_{\mathbf{C}_{\mathcal{S}}} : \mathcal{M}_{\mathcal{S}} \rightarrow \mathbb{R}^m \\ (x, g) \mapsto (\varphi \circ \eta_{\mathcal{S}})(v^x, g) \end{array} \right.$$

is a semi-algebraic homeomorphism, which implies that $\dim(\mathcal{M}_{\mathcal{S}}) = m$, from Proposition 1.4.12 (in Appendix 1.4.2). From its definition and from the definition of $\eta_{\mathcal{S}}$, the map $\eta_{\mathbf{C}_{\mathcal{S}}}$ is semi-algebraic if and only if for every $i \in N$, the map

$$(x, g) \in \mathcal{M}_{\mathcal{S}} \mapsto \varphi_i(\gamma \mapsto v_i^x(\gamma) + \sum_{j \neq i} (\partial_{ij}v_i^x(g_{ij}, g_{-ij}) - \partial_{ij}v_i^x(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}) \in \mathbb{R}^{m_i}$$

is semi-algebraic. Observe that this is the case, since each coefficient of the polynomial function

$$\gamma \in \mathbb{G} \mapsto v_i^x(\gamma) + \sum_{j \neq i} (\partial_{ij}v_i^x(g_{ij}, g_{-ij}) - \partial_{ij}v_i^x(g_{ij}, g_{-ij}^0))(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij} \in \mathbb{R}$$

is itself a polynomial function of (x, g) , for every $(x, g) \in \mathcal{M}_{\mathcal{S}}$. In order to understand why the map $\eta_{\mathbf{C}_{\mathcal{S}}}$ is a homeomorphism, consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{P}_{\mathcal{F}} & \longleftarrow & \mathcal{P}_{\mathcal{S}} & \xrightarrow{\varphi \times \text{id}_{\mathbb{G}}} & \mathcal{M}_{\mathcal{S}} \\
\pi_{\mathcal{F}} \downarrow & \eta_{\mathcal{F}} \downarrow & \pi_{\mathcal{S}} \downarrow & \eta_{\mathcal{S}} \downarrow & \pi_{\mathcal{C}_{\mathcal{S}}} \downarrow \eta_{\mathcal{C}_{\mathcal{S}}} \downarrow \\
\mathcal{F} & \longleftarrow & \mathcal{S} & \xrightarrow{\varphi} & \mathbb{R}^m
\end{array}$$

where $\pi_{\mathcal{C}_{\mathcal{S}}}$ is such that the right square commutes considering the map $\pi_{\mathcal{S}}$, i.e.

$$\pi_{\mathcal{C}_{\mathcal{S}}} = \varphi \circ \pi_{\mathcal{S}} \circ (\varphi \times \text{id}_{\mathbb{G}})^{-1}$$

(which is the restriction of the canonical projection $\mathbb{R}^m \times \mathbb{R}^L \rightarrow \mathbb{R}^m$ from $\mathcal{M}_{\mathcal{S}}$ to \mathbb{R}^m). Now, notice that by definition, $\eta_{\mathcal{C}_{\mathcal{S}}}$ makes commute the same square considering the map $\eta_{\mathcal{S}}$, i.e.

$$\eta_{\mathcal{C}_{\mathcal{S}}} = \varphi \circ \eta_{\mathcal{S}} \circ (\varphi \times \text{id}_{\mathbb{G}})^{-1}.$$

Therefore, $\eta_{\mathcal{C}_{\mathcal{S}}}$ is a homeomorphism as the composition of the three homeomorphisms φ , $\eta_{\mathcal{S}}$ and $(\varphi \times \text{id}_{\mathbb{G}})^{-1}$.

Step II. Extensions of $\pi_{\mathcal{C}_{\mathcal{S}}}$ and $\eta_{\mathcal{C}_{\mathcal{S}}}$ to the compactifications of $\mathcal{M}_{\mathcal{S}}$ and \mathbb{R}^m .

From Step I, since $\mathcal{M}_{\mathcal{S}}$ is homeomorphic to \mathbb{R}^m , it is a noncompact locally compact Hausdorff space, which implies that it admits a compactification $\mathcal{M}_{\mathcal{S}}^{\infty}$. Recall that from structure theorem (Theorem 1.2.1), both $\pi_{\mathcal{S}}$ and $\eta_{\mathcal{S}}$ are proper maps. Furthermore, by definition, both $\pi_{\mathcal{C}_{\mathcal{S}}}$ and $\eta_{\mathcal{C}_{\mathcal{S}}}$ are also proper maps, since φ and $(\varphi \times \text{id}_{\mathbb{G}})^{-1}$ are homeomorphisms. Hence, from Proposition 1.4.1 (in Appendix 1.4.1), $\pi_{\mathcal{C}_{\mathcal{S}}}$ and $\eta_{\mathcal{C}_{\mathcal{S}}}$ can be (uniquely) extended to continuous maps

$$\pi_{\mathcal{C}_{\mathcal{S}}}^{\infty} : \mathcal{M}_{\mathcal{S}}^{\infty} \rightarrow (\mathbb{R}^m)^{\infty}$$

and

$$\eta_{\mathcal{C}_{\mathcal{S}}}^{\infty} : \mathcal{M}_{\mathcal{S}}^{\infty} \rightarrow (\mathbb{R}^m)^{\infty}$$

(note that the map $\eta_{\mathcal{C}_{\mathcal{S}}}^{\infty}$ is also a homeomorphism).

Now, recall that $(\mathbb{R}^m)^{\infty}$ is also homeomorphic to the unit m -sphere \mathbb{S}^m (the inverse of the stereographic projection, denoted $s : (\mathbb{R}^m)^{\infty} \rightarrow \mathbb{S}^m$, is such a homeomorphism). Then, one can also consider the map $\pi_{\mathbb{S}^m} : \mathbb{S}^m \rightarrow \mathbb{S}^m$ (resp. $\eta_{\mathbb{S}^m} : \mathbb{S}^m \rightarrow \mathbb{S}^m$) such that the right square of the following diagram commutes considering the map $\pi_{\mathcal{C}_{\mathcal{S}}}^{\infty}$ (resp. $\eta_{\mathcal{C}_{\mathcal{S}}}^{\infty}$):

$$\begin{array}{ccccc}
\mathcal{M}_{\mathcal{S}} & \longleftarrow & \mathcal{M}_{\mathcal{S}}^{\infty} & \xrightarrow{s \circ \eta_{\mathcal{C}_{\mathcal{S}}}^{\infty}} & \mathbb{S}^m \\
\pi_{\mathcal{C}_{\mathcal{S}}} \downarrow & \eta_{\mathcal{C}_{\mathcal{S}}} \downarrow & \pi_{\mathcal{C}_{\mathcal{S}}}^{\infty} \downarrow & \eta_{\mathcal{C}_{\mathcal{S}}}^{\infty} \downarrow & \pi_{\mathbb{S}^m} \downarrow \eta_{\mathbb{S}^m} \downarrow \\
\mathbb{R}^m & \longleftarrow & (\mathbb{R}^m)^{\infty} & \xrightarrow{s} & \mathbb{S}^m
\end{array}$$

i.e.

$$\pi_{\mathbb{S}^m} = s \circ \pi_{\mathbf{C}_S}^\infty \circ (s \circ \eta_{\mathbf{C}_S}^\infty)^{-1} = s \circ \pi_{\mathbf{C}_S}^\infty \circ (\eta_{\mathbf{C}_S}^\infty)^{-1} \circ s^{-1}$$

and

$$\eta_{\mathbb{S}^m} = s \circ \eta_{\mathbf{C}_S}^\infty \circ (s \circ \eta_{\mathbf{C}_S}^\infty)^{-1} = s \circ \text{id}_{(\mathbb{R}^m)^\infty} \circ s^{-1} = \text{id}_{\mathbb{S}^m},$$

where $s \circ \eta_{\mathbf{C}_S}^\infty$ is (by composition) a homeomorphism from \mathcal{M}_S^∞ to \mathbb{S}^m .

Step III. $\pi_{\mathbf{C}_S}$ and $\eta_{\mathbf{C}_S}$ are properly homotopic. Moreover, $\pi_{\mathbb{S}^m}$ and $\eta_{\mathbb{S}^m}$ are homotopic.

Recall that from structure theorem (Theorem 1.2.1), there exists a proper homotopy H_S between π_S and η_S . Then, the map

$$\begin{cases} H_{\mathbf{C}_S} : [0, 1] \times \mathcal{M}_S & \rightarrow & \mathbb{R}^m \\ (t, (x, g)) & \mapsto & (\varphi \circ H_S \circ (\text{id}_{[0,1]} \times (\varphi \times \text{id}_{\mathbb{G}})^{-1}))(t, (x, g)) \end{cases}$$

is a proper homotopy between $\pi_{\mathbf{C}_S}$ and $\eta_{\mathbf{C}_S}$, since: (i) $H_{\mathbf{C}_S}(0, (\cdot, \cdot)) = \pi_{\mathbf{C}_S}$ and $H_{\mathbf{C}_S}(1, (\cdot, \cdot)) = \eta_{\mathbf{C}_S}$; (ii) it is a continuous map (by composition); (iii) it is a proper map (both φ and $(\text{id}_{[0,1]} \times (\varphi \times \text{id}_{\mathbb{G}})^{-1})$ are homeomorphisms).

Now, consider the map

$$\begin{cases} H_{\mathbf{C}_S}^\infty : [0, 1] \times \mathcal{M}_S^\infty & \rightarrow & (\mathbb{R}^m)^\infty \\ (t, (x, g)) & \mapsto & \begin{cases} H_{\mathbf{C}_S}(t, (x, g)) & \text{if } (x, g) \neq \infty \\ \infty & \text{otherwise} \end{cases} \end{cases},$$

and remark that for every $(x, g) \in \mathcal{M}_S^\infty$,

$$H_{\mathbf{C}_S}^\infty(0, (x, g)) = \begin{cases} H_{\mathbf{C}_S}(0, (x, g)) = \pi_{\mathbf{C}_S}(x, g) = \pi_{\mathbf{C}_S}^\infty(x, g) & \text{if } (x, g) \neq \infty \\ \infty = \pi_{\mathbf{C}_S}^\infty(\infty) & \text{otherwise} \end{cases}$$

and that

$$H_{\mathbf{C}_S}^\infty(1, (x, g)) = \begin{cases} H_{\mathbf{C}_S}(1, (x, g)) = \eta_{\mathbf{C}_S}(x, g) = \eta_{\mathbf{C}_S}^\infty(x, g) & \text{if } (x, g) \neq \infty \\ \infty = \eta_{\mathbf{C}_S}^\infty(\infty) & \text{otherwise} \end{cases}.$$

Hence, it only remains to show that $H_{\mathbf{C}_S}^\infty$ is continuous in order to obtain that it is a homotopy between $\pi_{\mathbf{C}_S}^\infty$ and $\eta_{\mathbf{C}_S}^\infty$. Once done, observe that the map

$$\begin{cases} H_{\mathbb{S}^m} : [0, 1] \times \mathbb{S}^m & \rightarrow & \mathbb{S}^m \\ (t, x) & \mapsto & (s \circ H_{\mathbf{C}_S}^\infty \circ (\text{id}_{[0,1]} \times (s \circ \eta_{\mathbf{C}_S}^\infty)^{-1}))(t, x) \end{cases}$$

is therefore a homotopy between $\pi_{\mathbb{S}^m}$ and $\eta_{\mathbb{S}^m}$.

We show that $H_{\mathbf{C}_S}^\infty$ is a continuous map. Let O be an open subset of $(\mathbb{R}^m)^\infty$. We show that $(H_{\mathbf{C}_S}^\infty)^{-1}(O)$ is open in $[0, 1] \times \mathcal{M}_S^\infty$.

First, suppose that $\infty \notin O$. By definition, O is open in \mathbb{R}^m . Then, notice that $(H_{\mathbf{C}_S}^\infty)^{-1}(O) = H_{\mathbf{C}_S}^{-1}(O)$ is open in $[0, 1] \times \mathcal{M}_S$ (since $H_{\mathbf{C}_S}$ is continuous), which implies that it is also open in $[0, 1] \times \mathcal{M}_S^\infty$.

Last, suppose that $\infty \in O$. By definition, there exists a compact subspace K of \mathbb{R}^m such that $O = (\mathbb{R}^m)^\infty \setminus K$. Then,

$$(H_{\mathbf{C}_S}^\infty)^{-1}(O) = ([0, 1] \times \{\infty\}) \cup (([0, 1] \times \mathcal{M}_S) \setminus H_{\mathbf{C}_S}^{-1}(K)).$$

To prove that this set is open in $[0, 1] \times \mathcal{M}_S^\infty$, consider $(t, (x, g)) \in (H_{\mathbf{C}_S}^\infty)^{-1}(O)$. We show that there exists $\epsilon > 0$ such that

$$V_\epsilon = ([0, 1] \cap (t - \epsilon, t + \epsilon)) \times (\mathcal{M}_S^\infty \setminus K_\epsilon)$$

is an open neighborhood of $(t, (x, g))$ in $[0, 1] \times \mathcal{M}_S^\infty$ which is included in $(H_{\mathbf{C}_S}^\infty)^{-1}(O)$, where

$$\begin{aligned} K_\epsilon &= \{(x', g') \in \mathcal{M}_S : \exists t' \in [0, 1] \cap [t - \epsilon, t + \epsilon], (t', (x', g')) \in H_{\mathbf{C}_S}^{-1}(K)\} \\ &= \{(x', g') \in \mathcal{M}_S : \exists t' \in [0, 1] \cap [t - \epsilon, t + \epsilon], H_{\mathbf{C}_S}(t', (x', g')) \in K\}. \end{aligned}$$

We proceed in two steps.

Substep III.1. For every $\epsilon > 0$, K_ϵ is a compact subspace of \mathcal{M}_S , and V_ϵ is an open subset of $[0, 1] \times \mathcal{M}_S^\infty$. Furthermore, $V_\epsilon \subset (H_{\mathbf{C}_S}^\infty)^{-1}(O)$.

Let $\epsilon > 0$. First, we show that K_ϵ is a compact subspace of \mathcal{M}_S , and V_ϵ is an open subset of $[0, 1] \times \mathcal{M}_S^\infty$. Consider a sequence $(x^\ell, g^\ell)_{\ell \in \mathbb{N}}$ in K_ϵ . By definition, for every $\ell \in \mathbb{N}$, there exists $t^\ell \in [0, 1] \cap [t - \epsilon, t + \epsilon]$ such that $(t^\ell, (x^\ell, g^\ell)) \in H_{\mathbf{C}_S}^{-1}(K)$. However, since $K \subset \mathbb{R}^m$ is compact and since $H_{\mathbf{C}_S}$ is a proper map, this implies that $H_{\mathbf{C}_S}^{-1}(K)$ is a compact subspace of $[0, 1] \times \mathcal{M}_S$. Thus, the sequence $(t^\ell, (x^\ell, g^\ell))_{\ell \in \mathbb{N}}$ in $H_{\mathbf{C}_S}^{-1}(K)$ admits a subsequence which converges to some element $(t^*, (x^*, g^*))$ of $H_{\mathbf{C}_S}^{-1}(K)$. Finally, since $[0, 1] \cap [t - \epsilon, t + \epsilon]$ is a closed subset of \mathbb{R} , this implies that t^* belongs to $[0, 1] \cap [t - \epsilon, t + \epsilon]$, therefore that $(x^*, g^*) \in K_\epsilon$. In particular, because $K_\epsilon \subset \mathcal{M}_S$ is compact, $V_\epsilon = ([0, 1] \cap (t - \epsilon, t + \epsilon)) \times (\mathcal{M}_S^\infty \setminus K_\epsilon)$ is (by definition) open in $[0, 1] \times \mathcal{M}_S^\infty$. Last, we show that $V_\epsilon \subset (H_{\mathbf{C}_S}^\infty)^{-1}(O)$. Let $(t', (x', g')) \in V_\epsilon$. This property is directly verified when $(x, g) = \infty$, so suppose that $(x, g) \neq \infty$. In that case, by definition of V_ϵ and of K_ϵ , note that for every $t'' \in [0, 1] \cap [t - \epsilon, t + \epsilon]$, $(t'', (x', g')) \notin H_{\mathbf{C}_S}^{-1}(K)$, i.e. $(t'', (x', g')) \in (([0, 1] \times \mathcal{M}_S) \setminus H_{\mathbf{C}_S}^{-1}(K)) \subset (H_{\mathbf{C}_S}^\infty)^{-1}(O)$. Thus, $V_\epsilon \subset (H_{\mathbf{C}_S}^\infty)^{-1}(O)$ (the last property being in particular verified for $t' \in [0, 1] \cap (t - \epsilon, t + \epsilon)$).

Substep III.2. There exists $\epsilon > 0$ such that $(t, (x, g)) \in V_\epsilon$.

Again, this property is clearly verified when $(x, g) = \infty$, so suppose that $(x, g) \neq \infty$, and suppose that for every $\epsilon > 0$, $(t, (x, g)) \notin V_\epsilon$. In that case, since $(t, (x, g)) \in (H_{\mathcal{C}_S}^\infty)^{-1}(O)$, this means that

$$(t, (x, g)) \in (([0, 1] \times \mathcal{M}_S) \setminus H_{\mathcal{C}_S}^{-1}(K))$$

and that for every $\epsilon > 0$, $(x, g) \in K_\epsilon$. In particular, for every integer $k \geq 1$, (x, g) belongs to $K_{\frac{1}{k}}$, which implies that there exists a sequence $(t^\ell)_{\ell \in \mathbb{N}}$ in $[0, 1]$ which converges to t , and such that for every $\ell \in \mathbb{N}$, $H_{\mathcal{C}_S}(t^\ell, (x, g)) \in K$. However, since $H_{\mathcal{C}_S}$ is a continuous map and since K is closed in \mathbb{R}^m (K is a compact subspace of \mathbb{R}^m , which is itself Hausdorff), this implies that $\lim_{t^\ell \rightarrow t} H_{\mathcal{C}_S}(t^\ell, (x, g)) = H_{\mathcal{C}_S}(t, (x, g)) \in K$, which contradicts the fact that $(t, (x, g)) \in (([0, 1] \times \mathcal{M}_S) \setminus H_{\mathcal{C}_S}^{-1}(K))$.

Step IV. $\deg(\pi_{\mathbb{S}^m}) = 1$. Moreover, $\pi_{\mathcal{C}_S}$ is a surjective map. In particular, $\pi_{\mathcal{S}}$ is also surjective (i.e. every society in \mathcal{S} admits a pairwise stable network).

Since $\eta_{\mathbb{S}^m} = \text{id}_{\mathbb{S}^m}$ (from Step II), $\deg(\eta_{\mathbb{S}^m}) = 1$, from Proposition 1.4.2 (in Appendix 1.4.1, first point). Moreover, since $\pi_{\mathbb{S}^m}$ and $\eta_{\mathbb{S}^m}$ are homotopic (from Step III), $\deg(\pi_{\mathbb{S}^m}) = 1$, again from Proposition 1.4.2 (in Appendix 1.4.1, third point). Hence, $\pi_{\mathbb{S}^m}$ is a surjective map, from Proposition 1.4.3 (in Appendix 1.4.1). Now, remark that this implies that $\pi_{\mathcal{C}_S}$ is a surjective map. Indeed, because $\pi_{\mathbb{S}^m}$ is surjective, for every $x \in \mathbb{R}^m$, there exists $y \in \mathbb{S}^m$ such that $\pi_{\mathbb{S}^m}(y) = s(x)$, which implies that $x = (s^{-1} \circ \pi_{\mathbb{S}^m})(y)$. Furthermore, by definition of $\pi_{\mathbb{S}^m}$, this is equivalent to say that

$$\begin{aligned} x &= (s^{-1} \circ (s \circ \pi_{\mathcal{C}_S}^\infty \circ (s \circ \eta_{\mathcal{C}_S}^\infty)^{-1}))(y) \\ &= (\pi_{\mathcal{C}_S}^\infty \circ (s \circ \eta_{\mathcal{C}_S}^\infty)^{-1})(y) \\ &= \pi_{\mathcal{C}_S}^\infty((\eta_{\mathcal{C}_S}^\infty)^{-1}(s^{-1}(y))) \\ &= \pi_{\mathcal{C}_S}((\eta_{\mathcal{C}_S}^\infty)^{-1}(s^{-1}(y))), \end{aligned}$$

where $(\eta_{\mathcal{C}_S}^\infty)^{-1}(s^{-1}(y)) \in \mathcal{M}_S$, since $x \neq \infty$.

Remark that the map $\pi_{\mathcal{S}}$ is also surjective (i.e. every society in \mathcal{S} admits a pairwise stable network). Indeed, for every $v \in \mathcal{S}$, there exists $(x, g) \in \mathcal{M}_S$ such that $\pi_{\mathcal{C}_S}(x, g) = \varphi(v)$ (since $\pi_{\mathcal{C}_S}$ is surjective), i.e. $v = (\varphi^{-1} \circ \varphi)(v) =$

$(\varphi^{-1} \circ \pi_{\mathbf{C}_S})(x, g)$, where, by definition of $\pi_{\mathbf{C}_S}$,

$$\begin{aligned} (\varphi^{-1} \circ \pi_{\mathbf{C}_S})(x, g) &= (\varphi^{-1} \circ (\varphi \circ \pi_S \circ (\varphi \times \text{id}_{\mathbb{G}})^{-1}))(x, g) \\ &= (\pi_S \circ (\varphi \times \text{id}_{\mathbb{G}})^{-1})(x, g) \\ &= \pi_S((\varphi \times \text{id}_{\mathbb{G}})^{-1}(x, g)), \end{aligned}$$

where $(\varphi \times \text{id}_{\mathbb{G}})^{-1}(x, g) = (v^x, g) \in \mathcal{P}_S$. Hence, we recover in particular (but for a “smaller” set of societies) the result of Bich and Morhaim [6], which states that every continuous own-weights quasiconcave society admits a pairwise stable network.

Step V. There exists a generic subset G of \mathbb{R}^m such that for every connected component C of G , there exists a nonempty finite set F_C such that $(\pi_{\mathbf{C}_S}^{-1}(C), C, \pi_{\mathbf{C}_S}, F_C)$ is a covering space. In particular, for every $x \in G$, the society v^x admits a strictly positive finite number of pairwise stable networks.

Note that $\pi_{\mathbf{C}_S} : \mathcal{M}_S \rightarrow \mathbb{R}^m$ is a continuous semi-algebraic map, which is also surjective (from Step IV), and recall that $\dim(\mathcal{M}_S) = m = \dim(\mathbb{R}^m)$ (from Step I). Thus, from Theorem 1.4.5 (in Appendix 1.4.2), this directly implies that there exists a generic subset G of \mathbb{R}^m (which is now fixed) such that for every connected component C of G , there exists a nonempty finite set F_C such that $(\pi_{\mathbf{C}_S}^{-1}(C), C, \pi_{\mathbf{C}_S}, F_C)$ is a covering space.

Now, from Theorem 1.4.2 (in Appendix 1.4.1), observe that for every connected component C of G , the previous property is equivalent to the following one: for every $x \in C$, there exists an open subset $V_{\mathbf{C}_S}^x$ of C which contains x and such that $\pi_{\mathbf{C}_S}^{-1}(V_{\mathbf{C}_S}^x) = \bigcup_{k \in F_C} (V_{\mathbf{C}_S}^x)_k$, where $((V_{\mathbf{C}_S}^x)_k)_{k \in F_C}$ is a family of pairwise disjoint open subsets of \mathcal{M}_S such that for every $k \in F_C$, the map $(x', g) \in (V_{\mathbf{C}_S}^x)_k \mapsto \pi_{\mathbf{C}_S}(x', g) \in V_{\mathbf{C}_S}^x$ is a homeomorphism.

Also, remark that since for every connected component C of G and every $x \in C$, F_C and $\pi_{\mathbf{C}_S}^{-1}(x)$ are homeomorphic, and since $(\varphi \times \text{id}_{\mathbb{G}})^{-1}$ is a homeomorphism, for every $x \in G$,

$$\text{card}(\pi_{\mathbf{C}_S}^{-1}(\varphi^{-1}(x))) = \text{card}((\varphi \times \text{id}_{\mathbb{G}})^{-1}(\pi_{\mathbf{C}_S}^{-1}(x))) = \text{card}(\pi_{\mathbf{C}_S}^{-1}(x)) = \text{card}(F_{C^x})$$

(by definition of $\pi_{\mathbf{C}_S}$), where C^x is the connected component of G which contains x . To put it in another way: the society $v^x = \varphi^{-1}(x)$ admits a strictly positive finite number of pairwise stable networks.

Step VI. For every $x \in G$, the society v^x admits an odd number of pairwise stable networks.

Let $x \in G$ and C^x be the connected component of G which contains x . From Step V, consider an open subset $V_{\mathbf{C}_S}^x$ of C^x which contains x and a family $((V_{\mathbf{C}_S}^x)_k)_{k \in F_{C^x}}$ of pairwise disjoint open subsets of \mathcal{M}_S such that $\pi_{\mathbf{C}_S}^{-1}(V_{\mathbf{C}_S}^x) = \bigcup_{k \in F_{C^x}} (V_{\mathbf{C}_S}^x)_k$, and such that for every $k \in F_{C^x}$, the map $(x', g) \in (V_{\mathbf{C}_S}^x)_k \mapsto \pi_{\mathbf{C}_S}(x', g) \in V_{\mathbf{C}_S}^x$ is a homeomorphism (recall that F_{C^x} is a nonempty finite set). Observe that $V_{\mathbf{C}_S}^x$ is also open in \mathbb{R}^m , since G is open in \mathbb{R}^m (G being generic in \mathbb{R}^m) and $C^x \supset V_{\mathbf{C}_S}^x$ is open in G (as a connected component of G). Furthermore, remark that $\pi_{\mathbf{C}_S}^{-1}(x) = (\pi_{\mathbf{C}_S}^\infty)^{-1}(x)$, that $V_{\mathbf{C}_S}^x$ is also open in $(\mathbb{R}^m)^\infty$, and that for every $k \in F_{C^x}$, $(V_{\mathbf{C}_S}^x)_k$ is also open in \mathcal{M}_S^∞ , since $x \neq \infty$.

Now, let $V_{\mathbb{S}^m}^x$ be the image by the homeomorphism s of $V_{\mathbf{C}_S}^x$, and for every $k \in F_{C^x}$, let $(V_{\mathbb{S}^m}^x)_k$ be the image by the homeomorphism $s \circ \eta_{\mathbf{C}_S}^\infty$ of $(V_{\mathbf{C}_S}^x)_k$. Since, both s and $\eta_{\mathbf{C}_S}^\infty$ are homeomorphisms, note that $V_{\mathbb{S}^m}^x$ is an open subset of \mathbb{S}^m which contains $s(x)$ and such that

$$\begin{aligned} \pi_{\mathbb{S}^m}^{-1}(V_{\mathbb{S}^m}^x) &= \pi_{\mathbb{S}^m}^{-1}(s(V_{\mathbf{C}_S}^x)) \\ &= (\pi_{\mathbb{S}^m}^{-1} \circ s)(V_{\mathbf{C}_S}^x) \\ &= (s \circ \eta_{\mathbf{C}_S}^\infty)(\pi_{\mathbf{C}_S}^{-1}(V_{\mathbf{C}_S}^x)) \quad (\text{by definition of } \pi_{\mathbb{S}^m}) \\ &= (s \circ \eta_{\mathbf{C}_S}^\infty)\left(\bigcup_{k \in F_{C^x}} (V_{\mathbf{C}_S}^x)_k\right) \\ &= \bigcup_{k \in F_{C^x}} (s \circ \eta_{\mathbf{C}_S}^\infty)((V_{\mathbf{C}_S}^x)_k) \\ &= \bigcup_{k \in F_{C^x}} (V_{\mathbb{S}^m}^x)_k, \end{aligned}$$

where $((V_{\mathbb{S}^m}^x)_k)_{k \in F_{C^x}}$ is a family of pairwise disjoint open subsets of \mathbb{S}^m such that for every $k \in F_{C^x}$, the map $x' \in (V_{\mathbb{S}^m}^x)_k \mapsto \pi_{\mathbb{S}^m}(x') \in V_{\mathbb{S}^m}^x$ is a homeomorphism. Observe that for every $k \in F_{C^x}$, there exists a unique $\alpha_k^x \in (V_{\mathbb{S}^m}^x)_k$ such that

$$\pi_{\mathbb{S}^m}^{-1}(s(x)) = \{\alpha_k^x : k \in F_{C^x}\},$$

by definition of the family $((V_{\mathbb{S}^m}^x)_k)_{k \in F_{C^x}}$. Moreover, for every $k \in F_{C^x}$, because the restriction $\pi_{\mathbb{S}^m}|_{(V_{\mathbb{S}^m}^x)_k}$ of the map $\pi_{\mathbb{S}^m}$ from $(V_{\mathbb{S}^m}^x)_k$ to \mathbb{S}^m is a topological embedding, then

$$\deg_{s(x)}(\pi_{\mathbb{S}^m}|_{(V_{\mathbb{S}^m}^x)_k}) = \pm 1,$$

from Proposition 1.4.4 (in Appendix 1.4.1). Finally, since $\deg(\pi_{\mathbb{S}^m}) = 1$ (from Step IV), $\text{card}(F_{C^x})$ is odd, from Theorem 1.4.1 (in Appendix 1.4.1).

Therefore, since $\text{card}(\pi_S^{-1}(\varphi^{-1}(x))) = \text{card}(F_{C^x})$ (from Step V), $v^x = \varphi^{-1}(x)$ admits an odd number of pairwise stable networks.

Step VII. General case.

We now treat the general case, i.e. we do not suppose anymore that the set C_S is equal to \mathbb{R}^m .

Substep VII.1. Direct sum decomposition of \mathbb{R}^{m_i} and cancellation of the “ \mathcal{A}_i part” of C_{S_i} ($i \in N$) - application of the first part of the proof.

For every $i \in N$, let

$$\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i} = \text{Span}\left(\left\{\prod_{j \in L} g_{jl}^{k_{jl}} : k \in \mathbb{N}^L, \deg(k) \leq \delta_i\right\} \setminus (\{g_{il} : l \neq i\} \cup \{1\})\right)$$

be the linear subspace of $\mathbb{R}_{\delta_i}[g]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[g]$, except the ones in \mathcal{A}_i . By definition, observe that $\mathbb{R}_{\delta_i}[g] = \mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i} \oplus \mathcal{A}_i$, thus that

$$\varphi_i(\mathbb{R}_{\delta_i}[g]) = \mathbb{R}^{m_i} = \mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}} \oplus \mathbf{C}_{\mathcal{A}_i} = \varphi_i(\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}) \oplus \varphi_i(\mathcal{A}_i).$$

Moreover, denote by $\Pi_{-\mathcal{A}_i}$ the linear projection from the vector space $\mathbb{R}^{m_i} = \mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}} \oplus \mathbf{C}_{\mathcal{A}_i}$ (of coefficients of polynomial functions in $\mathbb{R}_{\delta_i}[g]$) to the space $\mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}}$ (of coefficients of polynomial functions in $\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}$), and consider the set $\Pi_{-\mathcal{A}_i}(C_{S_i})$.

Substep VII.2. Semi-algebraic decomposition of $\Pi_{-\mathcal{A}_i}(C_{S_i})$ ($i \in N$) - application of the first part of the proof.

For every $i \in N$, since C_{S_i} is a semi-algebraic set (from semi-algebraicity assumption in Definition 1.3.3) and since $\Pi_{-\mathcal{A}_i}$ is a semi-algebraic map, the set $\Pi_{-\mathcal{A}_i}(C_{S_i})$ is also semi-algebraic, from Proposition 1.4.8 (in Appendix 1.4.2). In particular, from Proposition 1.4.9 (in Appendix 1.4.2),

$$\Pi_{-\mathcal{A}_i}(C_{S_i}) = \bigcup_{k=1}^{r_i} T_k^i,$$

where $r_i \in \mathbb{N}$, and where $(T_k^i)_{k=1}^{r_i}$ is a family of pairwise disjoint semi-algebraic sets such that for every $k \in \{1, \dots, r_i\}$, T_k^i is semi-algebraically homeomorphic to an open hypercube $(0, 1)^{d_k^i}$, for some $d_k^i \in \mathbb{N}$. Thus,

$$\prod_{i \in N} \Pi_{-\mathcal{A}_i}(C_{S_i}) = \prod_{i \in N} \bigcup_{k=1}^{r_i} T_k^i = \bigcup_{\lambda \in \Lambda} \prod_{i \in N} T_{\lambda_i}^i, \quad (1.9)$$

where $\Lambda = \prod_{i \in N} \{1, \dots, r_i\}$.

Substep VII.3. Addition of \mathcal{A}_i to $\varphi_i^{-1}(T_{\lambda_i}^i)$ ($i \in N$) - application of the first part of the proof.

For every $\lambda \in \Lambda$, consider the set

$$\mathcal{V}_\lambda = \prod_{i \in N} \varphi_i^{-1}(T_{\lambda_i}^i) + \prod_{i \in N} \mathcal{A}_i = \prod_{i \in N} (\varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i),$$

and define $\mathcal{V}_{\lambda_i}^i = \varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i$, for every $i \in N$.

Substep VII.4. For every $\lambda \in \Lambda$, \mathcal{V}_λ is \mathcal{A} -semi-algebraically regular and $\mathbf{C}_{\mathcal{V}_\lambda}$ is homeomorphic to some Euclidean space of dimension $p_\lambda \leq m$, for some integer p_λ - application of the first part of the proof.

Observe that for every $\lambda \in \Lambda$:

1. \mathcal{V}_λ satisfies concavity assumption (in Definition 1.3.3) because for every $i \in N$, $T_{\lambda_i}^i \subset \Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i}) = \Pi_{-\mathcal{A}_i}(\varphi_i(\mathcal{S}_i))$, which implies that $\varphi_i^{-1}(T_{\lambda_i}^i) \subset (\varphi_i^{-1} \circ \Pi_{-\mathcal{A}_i} \circ \varphi_i)(\mathcal{S}_i) \subset \mathcal{S}_i$, hence that

$$\mathcal{V}_{\lambda_i}^i = \varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i \subset \mathcal{S}_i + \mathcal{A}_i = \mathcal{S}_i$$

(the last equality coming from the fact that \mathcal{S} satisfies \mathcal{A} -invariance assumption), where \mathcal{S} satisfies concavity assumption (i.e. $\mathcal{S}_i \subset \mathcal{F}_i$).

2. \mathcal{V}_λ clearly satisfies \mathcal{A} -invariance assumption (in Definition 1.3.3).
3. \mathcal{V}_λ satisfies semi-algebraicity assumption (in Definition 1.3.3) because for every $i \in N$, $\mathcal{V}_{\lambda_i}^i$,

$$\mathbf{C}_{\mathcal{V}_{\lambda_i}^i} = \varphi_i(\mathcal{V}_{\lambda_i}^i) = \varphi_i(\varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i) = T_{\lambda_i}^i + \varphi_i(\mathcal{A}_i) = T_{\lambda_i}^i + \mathbf{C}_{\mathcal{A}_i},$$

where both $T_{\lambda_i}^i$ and $\mathbf{C}_{\mathcal{A}_i}$ are semi-algebraic sets; the result follows from Corollary 1.4.2 (in Appendix 1.4.2).

4. For every $i \in N$, since $\text{Span}(T_{\lambda_i}^i)$ and $\text{Span}(\mathbf{C}_{\mathcal{A}_i}) = \mathbf{C}_{\mathcal{A}_i}$ are in direct sum (because $\mathbf{C}_{\mathbb{R}_{\delta_i}[g]_{-\mathcal{A}_i}} \supset \text{Span}(T_{\lambda_i}^i)$ and $\mathbf{C}_{\mathcal{A}_i}$ are in direct sum), $\mathbf{C}_{\mathcal{V}_{\lambda_i}^i}$ is homeomorphic to $T_{\lambda_i}^i \times \mathbf{C}_{\mathcal{A}_i}$ (see the proof of Corollary 1.4.5 in Appendix 1.4.2), thus homeomorphic to $(0, 1)^{d_{\lambda_i}^i}$, for some $d_{\lambda_i}^i \in \mathbb{N}$, which implies that $\mathbf{C}_{\mathcal{V}_\lambda} = \prod_{i \in N} \mathbf{C}_{\mathcal{V}_{\lambda_i}^i}$ is itself homeomorphic to some Euclidean space of dimension $p_\lambda \leq m$, for some integer p_λ (n.b.: this condition is important in order to be able to use what has been done from Step I to Step VI).

Therefore, from the first part of the proof (Steps I-VI), for every $\lambda \in \Lambda$, there exists a generic semi-algebraic subset G_λ of $\mathbf{C}_{\mathcal{V}_\lambda}$ such that for every $x \in G_\lambda$, the society v^x has an odd number of pairwise stable networks. This implies that for every x in

$$G = \bigcup_{\lambda \in \Lambda^{\text{op}}} G_\lambda,$$

where

$$\Lambda^{\text{op}} = \{\lambda \in \Lambda : \forall i \in N, T_{\lambda_i}^i \text{ is open in } \Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})\},$$

the society v^x admits an odd number of pairwise stable networks.

Substep VII.5. The family $(\mathbf{C}_{\mathcal{V}_\lambda})_{\lambda \in \Lambda}$ forms a cover of $\mathbf{C}_\mathcal{S}$ - genericity of G in $\mathbf{C}_\mathcal{S}$.

Because \mathcal{S} satisfies \mathcal{A} -invariance assumption, notice first that for every $i \in N$,

$$\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i}) + \mathbf{C}_{\mathcal{A}_i} = \mathbf{C}_{\mathcal{S}_i}. \quad (1.10)$$

Now, observe that

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} \mathbf{C}_{\mathcal{V}_\lambda} &= \bigcup_{\lambda \in \Lambda} \varphi(\mathcal{V}_\lambda) = \bigcup_{\lambda \in \Lambda} \varphi\left(\prod_{i \in N} (\varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i)\right) = \bigcup_{\lambda \in \Lambda} \prod_{i \in N} \varphi_i(\varphi_i^{-1}(T_{\lambda_i}^i) + \mathcal{A}_i) \\ &= \bigcup_{\lambda \in \Lambda} \prod_{i \in N} (T_{\lambda_i}^i + \varphi_i(\mathcal{A}_i)) = \bigcup_{\lambda \in \Lambda} \prod_{i \in N} (T_{\lambda_i}^i + \mathbf{C}_{\mathcal{A}_i}) = \bigcup_{\lambda \in \Lambda} \prod_{i \in N} T_{\lambda_i}^i + \prod_{i \in N} \mathbf{C}_{\mathcal{A}_i} \\ &= \prod_{i \in N} \Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i}) + \prod_{i \in N} \mathbf{C}_{\mathcal{A}_i} \text{ (from Equation (1.9))} \\ &= \prod_{i \in N} (\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i}) + \mathbf{C}_{\mathcal{A}_i}) \\ &= \prod_{i \in N} \mathbf{C}_{\mathcal{S}_i} \text{ (from Equation (1.10))} \\ &= \mathbf{C}_\mathcal{S}, \end{aligned}$$

i.e. $(\mathbf{C}_{\mathcal{V}_\lambda})_{\lambda \in \Lambda}$ forms a cover of $\mathbf{C}_\mathcal{S}$. Furthermore, remark that

$$\mathbf{C}_\mathcal{S} \setminus G \subset \bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda) \cup \mathbf{C},$$

where

$$\mathbf{C} = \bigcup_{\lambda \in \Lambda \setminus \Lambda^{\text{op}}} \mathbf{C}_{\mathcal{V}_\lambda}$$

(for every $\lambda \in \Lambda$, $\lambda \in \Lambda \setminus \Lambda^{\text{op}}$ if and only if there exists $i \in N$ such that $T_{\lambda_i}^i$ is not open in $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{\mathcal{S}_i})$). Indeed, let $x \in \mathbf{C}_\mathcal{S}$ such that $x \notin G$ (i.e.

$x \in \bigcap_{\lambda \in \Lambda^{\text{op}}} \mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda$). Since $(\mathbf{C}_{\mathcal{V}_\lambda})_{\lambda \in \Lambda}$ forms a cover of \mathbf{C}_S , there exists $\ell \in \Lambda$ such that $x \in \mathbf{C}_{\mathcal{V}_\ell}$. Hence: (i) either $\ell \in \Lambda^{\text{op}}$, which implies that $x \in \mathbf{C}_{\mathcal{V}_\ell} \setminus G_\ell \subset \bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda)$ (because $x \notin G$); (ii) or $\ell \in \Lambda \setminus \Lambda^{\text{op}}$, which implies that $x \in \mathbf{C}$.

Substep VII.6. $\dim(\mathbf{C}_S \setminus G) < \dim(\mathbf{C}_S)$ - **genericity of G in \mathbf{C}_S .**

Note that the dimension of $\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda)$ is strictly less than the dimension of \mathbf{C}_S , because for every $\lambda \in \Lambda$, $\dim(\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda) < \dim(\mathbf{C}_{\mathcal{V}_\lambda})$ (from G_λ being generic in $\mathbf{C}_{\mathcal{V}_\lambda}$) and $\dim(\mathbf{C}_{\mathcal{V}_\lambda}) \leq \dim(\mathbf{C}_S)$ (from Corollary 1.4.4 in Appendix 1.4.2); the result follows from Proposition 1.4.11 (in Appendix 1.4.2). Also, the dimension of \mathbf{C} is strictly less than the dimension of \mathbf{C}_S . Indeed, for every $\lambda \in \Lambda \setminus \Lambda^{\text{op}} = \{\lambda \in \Lambda : \exists i \in N, T_{\lambda_i}^i \text{ is not open in } \Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i})\}$, consider $j \in N$ such that $T_{\lambda_j}^j$ is not open in $\Pi_{-\mathcal{A}_j}(\mathbf{C}_{S_j})$, and remark that $\dim(T_{\lambda_j}^j) < \dim(\Pi_{-\mathcal{A}_j}(\mathbf{C}_{S_j}))$ (from Corollary 1.4.3 in Appendix 1.4.2). Thus, for every $\lambda \in \Lambda \setminus \Lambda^{\text{op}}$,

$$\begin{aligned}
\dim(\mathbf{C}_{\mathcal{V}_\lambda}) &= \dim\left(\prod_{i \in N} (T_{\lambda_i}^i + \mathbf{C}_{\mathcal{A}_i})\right) \\
&= \sum_{i \in N} \dim(T_{\lambda_i}^i + \mathbf{C}_{\mathcal{A}_i}) \quad (\text{from Proposition 1.4.11}) \\
&= \sum_{i \in N} \dim(T_{\lambda_i}^i) + \sum_{i \in N} \dim(\mathbf{C}_{\mathcal{A}_i}) \quad (\text{from Corollary 1.4.5}) \\
&< \sum_{i \neq j} \dim(\Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i})) + \dim(\Pi_{-\mathcal{A}_j}(\mathbf{C}_{S_j})) + \sum_{i \in N} \dim(\mathbf{C}_{\mathcal{A}_i}) \\
&= \sum_{i \in N} \dim(\Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i}) + \mathbf{C}_{\mathcal{A}_i}) \quad (\text{from Corollary 1.4.5}) \\
&= \dim\left(\prod_{i \in N} (\Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i}) + \mathbf{C}_{\mathcal{A}_i})\right) \quad (\text{from Proposition 1.4.11}) \\
&= \dim\left(\prod_{i \in N} \mathbf{C}_{S_i}\right) \quad (\text{from Equation (1.10)}) \\
&= \dim(\mathbf{C}_S).
\end{aligned}$$

Therefore, the result follows from Proposition 1.4.11 (in Appendix 1.4.2). Finally, from Step VII.5 and from Corollary 1.4.4 (in Appendix 1.4.2),

$$\dim(\mathbf{C}_S \setminus G) \leq \dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda) \cup \mathbf{C}\right),$$

and from Proposition 1.4.11 (in Appendix 1.4.2),

$$\dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda) \cup \mathbf{C}\right) = \max\left\{\dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{V}_\lambda} \setminus G_\lambda)\right), \dim(\mathbf{C})\right\} < \dim(\mathbf{C}_S).$$

Substep VII.7. G is open in \mathbf{C}_S - genericity of G in \mathbf{C}_S .

Observe that for every $\lambda \in \Lambda^{\text{op}} = \{\lambda \in \Lambda : \forall i \in N, T_{\lambda_i}^i \text{ is open in } \Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i})\}$, G_λ is open in \mathbf{C}_S because G_λ is open in $\mathbf{C}_{\mathcal{V}_\lambda}$ (from G_λ being generic in $\mathbf{C}_{\mathcal{V}_\lambda}$) and because $\mathbf{C}_{\mathcal{V}_\lambda}$ is open in \mathbf{C}_S . Indeed, to understand this last point, remark that since for every $\lambda \in \Lambda^{\text{op}}$ and every $i \in N$, $T_{\lambda_i}^i$ is open in $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i})$, $\mathbf{C}_{\mathcal{V}_{\lambda_i}^i} = T_{\lambda_i}^i + \mathbf{C}_{\mathcal{A}_i}$ is open in $\Pi_{-\mathcal{A}_i}(\mathbf{C}_{S_i}) + \mathbf{C}_{\mathcal{A}_i}$, i.e. is open in \mathbf{C}_{S_i} (from Equation (1.10)). Therefore, $\mathbf{C}_{\mathcal{V}_\lambda} = \prod_{i \in N} \mathbf{C}_{\mathcal{V}_{\lambda_i}^i}$ is open in $\mathbf{C}_S = \prod_{i \in N} \mathbf{C}_{S_i}$. Finally, $G = \bigcup_{\lambda \in \Lambda^{\text{op}}} G_\lambda$ is open in \mathbf{C}_S (as an arbitrary union of open subsets of \mathbf{C}_S).

This step ends the proof of the oddness theorem. \square

Chapter 2

Topological Structure and Generic Oddness of the Graph of Nash Equilibria

$\mathcal{N}_{\mathcal{U}}$	<i>Graph of Nash equilibria associated to the set \mathcal{U} of games</i>
$\pi_{\mathcal{U}}$	<i>Projection from $\mathcal{N}_{\mathcal{U}}$ to \mathcal{U}, where \mathcal{U} is a set of games</i>
\mathcal{F}_i	<i>Set of continuous payoff functions of player i which are concave in x_i and with continuous first-order derivative with respect to the i, ℓ-th variable, for every $\ell \in \{1, \dots, \mu_i\}$ ($\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$)</i>
$\text{Cst}(X, \mathbb{R})$	<i>Set of constant payoff functions ($\mathcal{C} = \text{Cst}(X, \mathbb{R})^N$)</i>
\mathcal{L}_i	<i>Set of payoff functions of player i which are linear in x_i and which only depend on this strategy ($\mathcal{L} = \prod_{i \in N} \mathcal{L}_i$)</i>
\mathcal{A}_i	<i>Set of payoff functions of player i which are affine in x_i and which only depend on this strategy ($\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$)</i>
$\eta_{\mathcal{R}}$	<i>Homeomorphism from $\mathcal{N}_{\mathcal{R}}$ to \mathcal{R} of structure theorem, where \mathcal{R} is a \mathcal{A}-regular set of games</i>
$\mathbb{R}_{\delta_i}[x]$	<i>Set of payoff functions of player i which are polynomial of x and whose degree is less or equal to δ_i ($\mathbb{R}_{\delta}[x] = \prod_{i \in N} \mathbb{R}_{\delta_i}[g]$)</i>
φ_i	<i>Vector space isomorphism which assigns to each payoff function of player i in $\mathbb{R}_{\delta_i}[x]$ its coefficients in \mathbb{R}^{m_i}, where $m_i \in \mathbb{N}$ depends on δ_i ($m = \sum_{i \in N} m_i$, $\varphi = \times_{i \in N} \varphi_i : \mathbb{R}_{\delta}[x] \rightarrow \mathbb{R}^m$)</i>
$\mathbf{C}_{\mathcal{U}}$	<i>Set of coefficients of polynomial games in \mathcal{U} (i.e. $\mathbf{C}_{\mathcal{U}} = \varphi(\mathcal{U})$), where $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$</i>
u^y	<i>Polynomial game in \mathcal{U} whose coefficients correspond to y (i.e. $u^y = \varphi^{-1}(y)$), where $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$ and where $y \in \mathbf{C}_{\mathcal{U}}$</i>
$\mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})}$	<i>Linear subspace of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the constant one ($\mathbb{R}_{\delta}[x]_{-C} = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})}$)</i>
Π_{-C}	<i>Linear projection from $\mathbb{R}^m = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-C}} \oplus \mathbf{C}_C$ to $\mathbf{C}_{\mathbb{R}_{\delta}[x]_{-C}}$</i>
$\mathbb{R}_{\delta_i}[x]_{-\mathcal{A}_i}$	<i>Linear subspace of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the ones in \mathcal{A}_i ($\mathbb{R}_{\delta}[x]_{-A} = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]_{-\mathcal{A}_i}$)</i>
Π_{-A}	<i>Linear projection from $\mathbb{R}^m = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-A}} \oplus \mathbf{C}_A$ to $\mathbf{C}_{\mathbb{R}_{\delta}[x]_{-A}}$</i>
$(T_{\lambda})_{\lambda=1}^r$	<i>Semi-algebraic decomposition of $\Pi_{-A}(\mathbf{C}_{\mathcal{S}})$, where \mathcal{S} is a \mathcal{A}-semi-algebraically regular set of games and where $r \in \mathbb{N}$ depends on \mathcal{S}</i>
\mathcal{U}_{λ}	<i>Minkowski sum of $\varphi^{-1}(T_{\lambda})$ and \mathcal{A}, where \mathcal{S} is a \mathcal{A}-semi-algebraically regular set of games and where $\lambda \in \Lambda = \{1, \dots, r\}$</i>
Λ^{op}	<i>Subset of indices $\lambda \in \Lambda$ such that T_{λ} is open in $\Pi_{-A}(\mathbf{C}_{\mathcal{S}})$, where \mathcal{S} is a \mathcal{A}-semi-algebraically regular set of games</i>

Table 2.1: Table of notations of Chapter 2

This chapter is based on a preprint written by Philippe Bich and Julien Fixary which is to appear in *Games and Economic Behavior* [4].

2.1 Introduction

In this second chapter, we provide our *structure theorem* and our *oddness theorem* in game theory. The chapter is organized as follows: (i) in Subsection 2.2.1 (of Section 2.2), we first recall some basic definitions and notations about game theory and we define the *graph of Nash equilibria associated to any set of games*; (ii) in Subsection 2.2.2, we introduce the notion of \mathcal{A} -regular set of games and we present our structure theorem (Theorem 2.2.2); (iii) in Subsection 2.3.1 (of Section 2.3), we introduce the notion of $\dim(\mathcal{L})$ -semi-algebraically regular set of games and we present our oddness theorem (Theorem 2.3.1); (iv) in Subsection 2.3.2, we provide several examples of applications of oddness theorem; (v) in Section 2.4 (Appendix), we provide first the necessary reminders about topological degree of proper continuous maps between topological oriented m -manifolds (Subsection 2.4.1), and we provide next the proofs of structure theorem (Subsection 2.4.2) and of oddness theorem (Subsection 2.4.3).

2.2 Topological structure of the graph of Nash equilibria

2.2.1 The graph of Nash equilibria

First of all, we recall some elementary definitions and notations from game theory.

Definition 2.2.1. A *set of players* is a finite set N such that $\text{card}(N) \geq 1$. For every set N of players and every $i \in N$, a *set of strategies of player i* is an arbitrary set denoted X_i . For every set N of players, every $i \in N$ and every set X_i of strategies of player i , the set $X = \prod_{i \in N} X_i$ is called the *set of strategy profiles*.

Definition 2.2.2. A (*strategic-form*) *game* is 3-tuple $(N, (X_i)_{i \in N}, (u_i)_{i \in N})$ composed by a set N of players, a family $(X_i)_{i \in N}$ of sets of strategies and a family $(u_i)_{i \in N}$ of maps from X to \mathbb{R} , where for every $i \in N$, the map u_i is called the *payoff function of player i* . For every set N of players and every family $(X_i)_{i \in N}$ of sets of strategies, the set of all games whose the set of players is equal to N and whose sets of strategies correspond to $(X_i)_{i \in N}$ can

be identified to the set

$$\mathbf{Gam} = \{N \rightarrow \{X \rightarrow \mathbb{R}\}\}.$$

Throughout this chapter, we consider a fixed set N of players, and for every $i \in N$, we consider a fixed integer $\mu_i \in \mathbb{N}$ and a fixed nonempty convex compact semi-algebraic subset X_i of \mathbb{R}^{μ_i} such that $\text{int}(X_i) \neq \emptyset$ which corresponds to the set of strategies of player i .

Notations. For every player $i \in N$, every strategy x_i of player i is also denoted $(x_{i,1}, \dots, x_{i,\mu_i})$. For every player $i \in N$, $X_{-i} = \prod_{j \neq i} X_j$. For every player $i \in N$, every $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$ and every $d_i \in X_i$, $x' = (d_i, x_{-i}) \in X$ is the strategy profile defined by $x'_j = x_j$, for every $j \neq i$, and $x'_i = d_i$. For every strategy profile $x \in X$ and every player $i \in N$, $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$.

Definition 2.2.3. Let $u \in \mathbf{Gam}$ be a game. A strategy profile $x \in X$ is a *Nash equilibrium* of u if for every $i \in N$ and every $d_i \in X_i$,

$$u_i(d_i, x_{-i}) \leq u_i(x).$$

Definition 2.2.4. Let $\mathcal{U} \subset \mathbf{Gam}$ be a set of games. The *Nash correspondence associated to \mathcal{U}* is the correspondence

$$\left\{ \begin{array}{l} \Psi_{\mathcal{U}} : \mathcal{U} \rightarrow X \\ u \mapsto \{x \in X : x \text{ is a Nash equilibrium of } u\} \end{array} \right. .$$

The graph of the Nash correspondence associated to \mathcal{U} is called the *graph of Nash equilibria associated to \mathcal{U}* and is denoted $\mathcal{N}_{\mathcal{U}}$, i.e.

$$\mathcal{N}_{\mathcal{U}} = \text{Gr}(\Psi_{\mathcal{U}}) = \{(u, x) \in \mathcal{U} \times X : x \text{ is a Nash equilibrium of } u\}.$$

The projection from $\mathcal{N}_{\mathcal{U}}$ to \mathcal{U} is denoted $\pi_{\mathcal{U}}$, i.e. $\pi_{\mathcal{U}}(u, x) = u$, for every $(u, x) \in \mathcal{N}_{\mathcal{U}}$.

2.2.2 \mathcal{A} -regular sets of games and structure theorem

Payoff functions which are considered in this chapter have to satisfy some differentiability and some concavity properties.

Definition 2.2.5. For every $i \in N$,

$$\mathcal{C}_i = \{u_i \in \mathcal{F}(X, \mathbb{R}) : \forall x_{-i} \in X_{-i}, u_i(\cdot, x_{-i}) \text{ is concave}\},$$

$$\mathcal{D}_i = \{u_i \in \mathcal{C}^0(X, \mathbb{R}) : \forall \ell \in \{1, \dots, \mu_i\}, \partial_{i,\ell} u_i \text{ exists and is continuous}\},$$

where for every $u_i \in \mathcal{C}^0(X, \mathbb{R})$ and every $\ell \in \{1, \dots, \mu_i\}$,

$$\partial_{i,\ell} u_i : x = ((x_{i,1}, \dots, x_{i,\ell}, \dots, x_{i,\mu_i}), x_{-i}) \in X \mapsto \frac{\partial u_i}{\partial x_{i,\ell}}(x) \in \mathbb{R},$$

and

$$\mathcal{F}_i = \mathcal{C}_i \cap \mathcal{D}_i.$$

The set

$$\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$$

is called the *set of own-strategy \mathcal{C}^1 concave games*.

Remark 2.2.1. Since X is a closed convex subspace of \mathbb{R}^{μ_i} such that $\text{int}(X) \neq \emptyset$, for every $i \in N$, every $u_i \in \mathcal{C}^0(X, \mathbb{R})$ and every $\ell \in \{1, \dots, \mu_i\}$, $\partial_{i,\ell} u_i$ is well-defined (see Proposition 1.4.5 and Definition 1.4.5 in Appendix 1.4.1 of Chapter 1).

Remark 2.2.2. For every $i \in N$, every $u_i \in \mathcal{D}_i$ and every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is a \mathcal{C}^1 function (see Definition 1.4.4 in Appendix 1.4.1 of Chapter 1).

Definition 2.2.6. For every $i \in N$, the set \mathcal{D}_i is endowed with the topology generated by all subsets of \mathcal{D}_i of the form

$$\{u_i \in \mathcal{D}_i : \forall x = (x_i, x_{-i}) \in K, u_i(x) \in O \text{ and } \nabla_{x_i} u_i(\cdot, x_{-i}) \in O'\},$$

where K is a compact subspace of X , O is an open subset of \mathbb{R} and O' is an open subset of \mathbb{R}^{μ_i} , and where $\nabla_{x_i} u_i(\cdot, x_{-i})$ denotes the gradient of $u_i(\cdot, x_{-i})$ at x_i . Furthermore, any subset of \mathcal{D}_i is endowed with the induced topology.

Remark 2.2.3. For every $i \in N$, the topology defined on \mathcal{D}_i in Definition 2.2.6 is the one that is used by Predtetchinski [40] in his structure theorem (see Theorem 2.2.1 in what follows).

Let us recall (the first part of) Predtetchinski's structure theorem about the graph of Nash equilibria.

Theorem 2.2.1. (*Predtetchinski [40]*)

The projection $\pi_{\mathcal{F}} : \mathcal{N}_{\mathcal{F}} \rightarrow \mathcal{F}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{F}} : \mathcal{N}_{\mathcal{F}} \rightarrow \mathcal{F}$.

Also, recall that the homeomorphism $\eta_{\mathcal{F}} : \mathcal{N}_{\mathcal{F}} \rightarrow \mathcal{F}$ of Theorem 2.2.1 is defined as follows: for every $(u, x) \in \mathcal{N}_{\mathcal{F}}$, every $i \in N$ and every $z \in X$,

$$\eta_{\mathcal{F}}(u, x)_i(z) = u_i(z) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x_{-i}^0), z_i - x_i \rangle + \langle x_i, z_i \rangle, \quad (2.1)$$

where $x^0 \in X$ is a fixed strategy profile, and where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on \mathbb{R}^{μ_i} .

Now, our structure theorem (see Theorem 2.2.2 below) is a slight extension of Theorem 2.2.1 which states that Predtetchinski's result remains true if we replace \mathcal{F} by any of its subspace which satisfies some assumption of “invariance under affine addition”.¹ For every $i \in N$, we consider the set of payoff functions of player i which are affine in x_i and which only depend on this strategy.

Definition 2.2.7. For every $i \in N$,

$$\mathcal{A}_i = \{x \in X \mapsto \langle \alpha_i, x_i \rangle + c \in \mathbb{R} : \alpha_i \in \mathbb{R}^{\mu_i}, c \in \mathbb{R}\}.$$

Moreover, $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$.

We now introduce the notion of \mathcal{A} -regular set of games which is at the core of our structure theorem.

Definition 2.2.8. A set $\mathcal{R} \subset \text{Gam}$ of games is \mathcal{A} -regular if the two following conditions hold:

1. **(Concavity).** $\mathcal{R} \subset \mathcal{F}$.
2. **(\mathcal{A} -invariance).** $\mathcal{R} + \mathcal{A} = \mathcal{R}$.

Our following result characterizes the topological structure of the graph of Nash equilibria associated to any \mathcal{A} -regular set of games.

Theorem 2.2.2. (*Structure theorem*)

For every \mathcal{A} -regular set \mathcal{R} of games, the projection $\pi_{\mathcal{R}} : \mathcal{N}_{\mathcal{R}} \rightarrow \mathcal{R}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{R}} : \mathcal{N}_{\mathcal{R}} \rightarrow \mathcal{R}$.

The proof is provided in Appendix 2.4.2.

Remark 2.2.4. We emphasize the fact that the assumption of semi-algebraicity of the sets of strategies is not used in the proof of our structure theorem, but only in the proof of our oddness theorem (Theorem 2.3.1 in Section 2.3.1).

Considering an arbitrary \mathcal{A} -regular set \mathcal{R} of games, this theorem provides two important informations:

¹In the second part of his structure theorem, Predtetchinski considers the following subspaces of \mathcal{F} : (i) the space of own-strategy \mathcal{C}^k concave games ($k \in \mathbb{N}$); (ii) the space of games such that for every player $i \in N$, each payoff function of player i is affine in j 's strategy ($j \in N$). Both these cases are particular ones of our structure theorem.

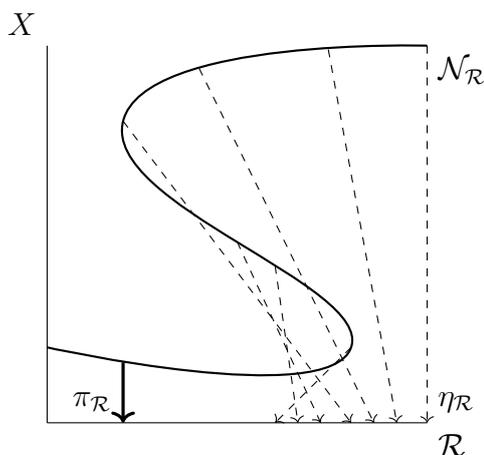


Figure 2.1: $\mathcal{N}_{\mathcal{R}}$, $\pi_{\mathcal{R}}$ (in thick line) and $\eta_{\mathcal{R}}$ (in dashed line)

1. Thanks to $\eta_{\mathcal{R}}$, the graph $\mathcal{N}_{\mathcal{R}}$ of Nash equilibria associated to \mathcal{R} is homeomorphic to \mathcal{R} , which corresponds to the intuition that $\mathcal{N}_{\mathcal{R}}$ can be continuously deformed into the simpler space \mathcal{R} .
2. The projection $\pi_{\mathcal{R}}$ is properly homotopic to the homeomorphism $\eta_{\mathcal{R}}$, which corresponds to the intuition that $\eta_{\mathcal{R}}$ itself can be continuously deformed into the simpler map $\pi_{\mathcal{R}}$.

Structure theorem is a key ingredient in the proof of our oddness theorem (Theorem 2.3.1 in Section 2.3.1), which is itself very important for applications (see Section 2.3.2). Indeed, Theorem 2.3.1 relies partly on some properties of topological degree (see Appendix 2.4.1 for some reminders) which can be derived from Theorem 2.2.2. Figure 2.1 provides a simple representation of the graph $\mathcal{N}_{\mathcal{R}}$ of Nash equilibria associated to any \mathcal{A} -regular set \mathcal{R} of games, of the projection $\pi_{\mathcal{R}}$ and of the homeomorphism $\eta_{\mathcal{R}}$.

2.3 Generic oddness of the graph of Nash equilibria

2.3.1 $\dim(\mathcal{L})$ -strongly semi-algebraically regular sets of games and oddness theorem

In this section, we are interested by sets of games whose payoff functions are polynomial functions of x (with coefficients in \mathbb{R}), and by their corresponding sets of coefficients.

Definition 2.3.1. Let

$$\mathbb{R}[x] = \{x \in X \mapsto \sum_{k \in \mathbb{N}^{\mathbf{L}}} (\alpha_k \prod_{i, \ell \in \mathbf{L}} x_{i, \ell}^{k_{i, \ell}}) \in \mathbb{R} : \forall k \in \mathbb{N}^{\mathbf{L}}, \alpha_k \in \mathbb{R}\},$$

where $\mathbf{L} = \{(i, \ell) : i \in N, \ell \in \{1, \dots, \mu_i\}\}$ (every $(i, \ell) \in \mathbf{L}$ is denoted i, ℓ). For every $i \in N$ and every $\delta_i \in \mathbb{N}$,

$$\mathbb{R}_{\delta_i}[x] = \{u_i \in \mathbb{R}[x] : \deg(u_i) \leq \delta_i\},$$

where for every $u_i \in \mathbb{R}[x]$, $\deg(u_i) = \max\{\deg(k) : \alpha_k \neq 0\}$, with $\deg(k) = \sum_{i, \ell \in \mathbf{L}} k_{i, \ell}$, for every $k \in \mathbb{N}^{\mathbf{L}}$. For every $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, the set

$$\mathbb{R}_{\delta}[x] = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]$$

is called the *set of (δ) -polynomial games*.

Remark 2.3.1. For every $i \in N$ and every $\delta_i \in \mathbb{N}$, $\mathbb{R}_{\delta_i}[x]$ (with its usual operations) is a finite-dimensional vector space. Thus, $\mathbb{R}_{\delta_i}[x]$ is endowed with the unique topology which makes it a Hausdorff topological vector space. Since \mathcal{D}_i is Hausdorff (see Definition 2.2.6), this topology corresponds also to the one induced by \mathcal{D}_i on $\mathbb{R}_{\delta_i}[x]$. Furthermore, with this topology, note that that every linear map from $\mathbb{R}_{\delta_i}[x]$ to any other topological vector space is also continuous.

Definition 2.3.2. Let $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, and consider an order on the set \mathbf{L} and an order on the set $\mathbb{N}^{\mathbf{L}}$. For every $i \in N$, there exists a unique $m_i \in \mathbb{N}$ such that the map

$$\varphi_i : u_i \in \mathbb{R}_{\delta_i}[x] \mapsto (\alpha_k)_{k \in \mathbb{N}^{\mathbf{L}}} \in \mathbb{R}^{m_i}$$

is a well-defined vector space isomorphism. Furthermore, the map

$$\varphi : u \in \mathbb{R}_{\delta}[x] \mapsto \times_{i \in N} \varphi_i(u_i) \in \mathbb{R}^m,$$

where $m = \sum_{i \in N} m_i$, is also a well-defined vector space isomorphism.

Throughout the rest of this chapter, we consider a fixed $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, a fixed order on the set \mathbf{L} and a fixed order on the set $\mathbb{N}^{\mathbf{L}}$ (in particular, we consider also the vector space isomorphism φ_i of Definition 2.3.2, for every $i \in N$).

Notations. Consider a subset $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$ of polynomial games. The set $\varphi(\mathcal{U})$ of coefficients of polynomial games in \mathcal{U} is denoted $\mathbf{C}_{\mathcal{U}}$. By abuse of notation,

both the restriction of φ from \mathcal{U} to \mathbb{R}^m and the restriction of φ from \mathcal{U} to $\mathbf{C}_{\mathcal{U}}$ are denoted φ (however, note that $\varphi : \mathcal{U} \rightarrow \mathbf{C}_{\mathcal{U}}$ is a homeomorphism). For every $y \in \mathbf{C}_{\mathcal{U}}$, the polynomial game in \mathcal{U} whose coefficients correspond to y is denoted u^y , i.e. $u^y = \varphi^{-1}(y)$.

On the other hand, if we consider the set $\text{Cst}(X, \mathbb{R})$ (of constant payoff functions), we can remark that for every game $u \in \mathbf{Gam}$ and every constant game

$$u' \in \mathcal{C} = \{N \rightarrow \text{Cst}(X, \mathbb{R})\},$$

the set of Nash equilibria of u is equal to the set of Nash equilibria of $u + u'$. Thus, we want to focus only on games without constant part.

Definition 2.3.3. For every $i \in N$,

$$\mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})} = \text{Span}\left(\left\{\prod_{j, \ell \in \mathbf{L}} x_{j, \ell}^{k_{j, \ell}} : k \in \mathbb{N}^{\mathbf{L}}, \deg(k) \leq \delta_i\right\} \setminus \{1\}\right).$$

Moreover, $\mathbb{R}_{\delta}[x]_{-\mathcal{C}} = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})}$.

For every player $i \in N$, $\mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})}$ corresponds to the linear subspace of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the one in $\text{Cst}(X, \mathbb{R})$ (i.e. the constant monomial).

Also, for every $i \in N$, we consider the set of payoff functions of player i which are linear in x_i and which only depend on this strategy.

Definition 2.3.4. For every $i \in N$,

$$\mathcal{L}_i = \{x \in X \mapsto \langle \alpha_i, x_i \rangle \in \mathbb{R} : \alpha_i \in \mathbb{R}^{\mu_i}\}.$$

Moreover, $\mathcal{L} = \prod_{i \in N} \mathcal{L}_i$.

Remark 2.3.2. Notice that $\mathcal{A} = \mathcal{L} + \mathcal{C}$.

In the following, we introduce the notions of \mathcal{A} -semi-algebraically regular set of games, of $\dim(\mathcal{A})$ -semi-algebraically regular set of games, and of $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games.

Definition 2.3.5. Let $\mathcal{S} \subset \mathbf{Gam}$ be a set of games:

- \mathcal{S} is said to be \mathcal{A} -semi-algebraically regular if the three following conditions hold:
 1. **(Concavity).** $\mathcal{S} \subset \mathcal{F}$.
 2. **(Semi-algebraicity).** $\mathcal{S} \subset \mathbb{R}_{\delta}[x]$ and $\mathbf{C}_{\mathcal{S}}$ is a semi-algebraic set.

3. (**\mathcal{A} -invariance**). $\mathcal{S} + \mathcal{A} = \mathcal{S}$.
- \mathcal{S} is said to be $\dim(\mathcal{A})$ -semi-algebraically regular if the three following conditions hold:
 1. (**Concavity**). $\mathcal{S} \subset \mathcal{F}$.
 2. (**Semi-algebraicity**). $\mathcal{S} \subset \mathbb{R}_\delta[x]$ and $\mathbf{C}_\mathcal{S}$ is a semi-algebraic set.
 3. (**$\dim(\mathcal{A})$ -invariance**). $\dim(\mathbf{C}_{\mathcal{S}+\mathcal{A}}) = \dim(\mathbf{C}_\mathcal{S})$.
 - \mathcal{S} is said to be $\dim(\mathcal{L})$ -strongly semi-algebraically regular if the three following conditions hold:
 1. (**Concavity**). $\mathcal{S} \subset \mathcal{F}$.
 2. (**Strong semi-algebraicity**). $\mathcal{S} \subset \mathbb{R}_\delta[x]_{-\mathcal{C}}$ and $\mathbf{C}_\mathcal{S}$ is a semi-algebraic set.
 3. (**$\dim(\mathcal{L})$ -invariance**). $\dim(\mathbf{C}_{\mathcal{S}+\mathcal{L}}) = \dim(\mathbf{C}_\mathcal{S})$.

\mathcal{A} -semi-algebraically regular sets of games are particular cases of \mathcal{A} -regular set of games whose payoff functions are polynomial, with additional assumptions on their associated sets of coefficients. However, $\dim(\mathcal{A})$ -semi-algebraically regular sets of games and $\dim(\mathcal{L})$ -strongly semi-algebraically regular sets of games are not \mathcal{A} -regular in general since \mathcal{A} -invariance assumption might not be verified. Hence, $\dim(\mathcal{A})$ -semi-algebraic regularity is weaker than \mathcal{A} -semi-algebraic regularity. On the other hand, recall that even if $\dim(\mathcal{L})$ -strongly semi-algebraic regularity is not weaker than \mathcal{A} -semi-algebraic regularity (from a logical point of view), it has the interest to not take into account the constant part of any game, which is irrelevant regarding its set of Nash equilibria (as discussed above).

Also, to say that a set $\mathcal{U} \subset \mathbb{R}_\delta[x]$ of games satisfies semi-algebraicity assumption means that the set $\mathbf{C}_\mathcal{U}$ of coefficients of polynomial games in \mathcal{U} can be defined using a finite number of polynomial equalities or inequalities (see Appendix 1.4.2 in Chapter 1 for some reminders about real semi-algebraic geometry). Furthermore, to say that \mathcal{U} satisfies strong semi-algebraicity assumption means that \mathcal{U} satisfies the previous condition and that games in \mathcal{U} have no constant part.

Now, we present our oddness theorem which is in the spirit of Wilson's oddness theorem [43], but for polynomial payoff functions satisfying the last regularity condition in Definition 2.3.5 (recall that a semi-algebraic subset G of a semi-algebraic set S is said to be a *generic subset of S* if $\dim(S \setminus G) < \dim(S)$, and if G is open in S).

Theorem 2.3.1. (*Oddness theorem*)

For every $\dim(\mathcal{L})$ -strongly semi-algebraically regular set \mathcal{S} of games, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.

Analogously to Theorem 1.3.1 in Chapter 1, our result states that “most” games in game theory should have an odd number of Nash equilibria, when payoff functions are polynomial and satisfy the “standard” concavity assumption. However, Theorem 2.3.1 is less assumptions demanding: (i) we do not require anymore \mathcal{S} to be a cartesian product of spaces $(\mathcal{S}_i)_{i \in N}$; (ii) we now drop the constant parts (which are not relevant when one deals with Nash equilibria); (iii) $\dim(\mathcal{A})$ -invariance assumption (see the second point of Definition 2.3.5) is weaker than \mathcal{A} -invariance assumption (see the first point of Definition 2.3.5). Furthermore, this oddness result can be obtained in a very similar way in the framework of network formation theory.²

Remark that $\dim(\mathcal{L})$ -strong semi-algebraic regularity seems to be tight, since we can find simple polynomial own-strategy concave games without constant part for which oddness theorem is false when we remove $\dim(\mathcal{L})$ -invariance assumption.

Example 2.3.1. Suppose that $N = \{1, 2\}$, and that $X_1 = X_2 = [0, 1]$. Moreover, suppose that $\delta_1 = 1$ and that $\delta_2 = 2$. Consider the subset

$$\mathcal{U} = \{(x \in X \mapsto ax_1 \in \mathbb{R}, x \in X \mapsto b(1 - x_1)x_2 \in \mathbb{R}) : (a, b) \in \mathbb{R} \times \mathbb{R}\}$$

of polynomial games. The set \mathcal{U} satisfies concavity assumption and strong semi-algebraicity assumption. Indeed, $\mathbf{C}_{\mathcal{U}}$ is of the form

$$\{(a, 0, 0) : a \in \mathbb{R}\} \times \{(0, 0, b, b, 0, 0) : b \in \mathbb{R}\},$$

which is a linear subspace of \mathbb{R}^9 of dimension 2 (in particular, it is a semi-algebraic set of dimension 2). However, it does not satisfies $\dim(\mathcal{L})$ -invariance assumption because $\mathbf{C}_{\mathcal{U}+\mathcal{L}}$ is of the form

$$\{(a, 0, 0) : a \in \mathbb{R}\} \times \{(0, 0, b, c, 0, 0) : (b, c) \in \mathbb{R} \times \mathbb{R}\},$$

which is a linear subspace of \mathbb{R}^9 of dimension 3. As a matter of fact, Theorem 2.3.1 does not hold since for every $x_2 \in [0, 1]$, $(1, x_2)$ is a Nash equilibrium of the game whose coefficients correspond to $((a, 0, 0), (0, 0, b, b, 0, 0))$, for every $(a, b) \in [0, +\infty) \times \mathbb{R}$. The set

$$S = \{(a, 0, 0) : a \in [0, +\infty)\} \times \{(0, 0, b, b, 0, 0) : b \in \mathbb{R}\}$$

²This justifies Remark 1.3.2 in Section 1.3.2 of Chapter 1.

is a full-dimensional semi-algebraic subset of $\mathbf{C}_{\mathcal{U}}$ such that for every $y \in S$, the game u^y has an infinite number of Nash equilibria (i.e. we cannot find a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{U}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria).

One could argue that $\dim(\mathcal{L})$ -invariance assumption seems not easy to be verified, and that an assumption of \mathcal{L} -invariance would be simpler to approach (a set $\mathcal{S} \subset \mathbf{Gam}$ of games would satisfy \mathcal{L} -invariance assumption if $\mathcal{S} + \mathcal{L} = \mathcal{S}$; analogously to \mathcal{A} -invariance assumption). However, the following simple example highlights the practical use of $\dim(\mathcal{L})$ -invariance assumption when one wants to model some economical problematics (more interesting examples are provided in Section 2.3.2). The idea is that $\dim(\mathcal{L})$ -invariance assumption allows to have a larger degree of freedom on the possible values of the coefficients of the payoff functions that we consider.

Example 2.3.2. Suppose that $N = \{1, 2\}$, and that $X_1 = X_2 = [0, 1]$. Moreover, suppose that $\delta_1 = \delta_2 = 1$. Consider the subset

$$\mathcal{U} = \{(x \in X \mapsto ax_1 \in \mathbb{R}, x \in X \mapsto bx_2 \in \mathbb{R}) : (a, b) \in [0, +\infty) \times [0, +\infty)\}$$

of polynomial games. The set \mathcal{U} satisfies concavity assumption and strong semi-algebraicity assumption. Indeed, $\mathbf{C}_{\mathcal{U}}$ is of the form

$$\{(a, 0, 0) : a \in [0, +\infty)\} \times \{(0, b, 0) : b \in [0, +\infty)\},$$

which is a semi-algebraic set of dimension 2. This set does not satisfy \mathcal{L} -invariance assumption because $\mathbf{C}_{\mathcal{U}+\mathcal{L}}$ is of the form

$$\{(a, 0, 0) : a \in \mathbb{R}\} \times \{(0, b, 0) : b \in \mathbb{R}\}.$$

However, the set $\mathbf{C}_{\mathcal{U}}$ satisfies $\dim(\mathcal{L})$ -invariance assumption since the dimension of $\mathbf{C}_{\mathcal{U}+\mathcal{L}}$ is also equal to 2. In fact, observe that

$$G = \{(a, 0, 0) : a \in (0, +\infty)\} \times \{(0, b, 0) : b \in (0, +\infty)\}$$

is a generic semi-algebraic subset of $\mathbf{C}_{\mathcal{U}}$ such that for every $y \in G$, the game u^y has a unique Nash equilibrium which corresponds to $(1, 1)$.

Sketch of proof

The full proof is provided in Appendix 2.4.3.

First of all, remark that \mathcal{A} -invariance assumption is not necessarily verified for a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set \mathcal{S} of games. Hence, the proof of Theorem 2.3.1 cannot be exactly the same as the one of Theorem 1.3.1 (in Chapter 1). The idea is to proceed as follows:

1. In the first part of the proof (Step I), we show that for every \mathcal{A} -semi-algebraically regular set \mathcal{S} of games, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria. The demonstration is quite similar to the one of Theorem 1.3.1 (in Chapter 1). However, it has been slightly revisited and modified; the main difference lies in the use of topological degree between topological oriented manifolds of the same dimension (instead of topological degree between unit spheres of the same dimension), which avoids to use compactifications, thus simplifies the demonstration.
2. In Step II, we first prove that for every $\dim(\mathcal{A})$ -semi-algebraically regular set \mathcal{S} of games, the set $\mathcal{S} + \mathcal{A}$ is \mathcal{A} -semi-algebraically regular. This allows to use the previous result in order to obtain a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}+\mathcal{A}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria, and to show that the set $\mathbf{C}_{\mathcal{S}} \cap G$ is a generic semi-algebraic subset of $\mathbf{C}_{\mathcal{S}}$ (which satisfies the same oddness property).
3. Finally, in the last step (Step III), we demonstrate our oddness theorem. First, we show that for every $\dim(\mathcal{L})$ -strongly semi-algebraically regular set \mathcal{S} of games, the set $\mathcal{S} + \mathcal{C}$ is $\dim(\mathcal{A})$ -semi-algebraically regular. Then, using the previous step, we obtain a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}+\mathcal{C}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria, and we consider the projection $\Pi_{-\mathcal{C}}(G)$ of G to the vector space $\mathbf{C}_{\mathbb{R}_{\delta}[x]_{-\mathcal{C}}}$ of coefficients of polynomial games without constant part. This set satisfies the oddness property since constant parts are not relevant to determine Nash equilibria of any game, and is generic in $\mathbf{C}_{\mathcal{S}}$.

2.3.2 Some applications of oddness theorem

Multiaffine games

Here, suppose that $N = \{1, \dots, n\}$. Moreover, suppose that for every $i \in N$, the set X_i of strategies of player i corresponds to the unit $\mu_i - 1$ -simplex $\Delta^{\mu_i-1} \subset \mathbb{R}^{\mu_i}$, and that δ_i is “large enough”. Consider the set

$$\mathcal{A}^n(X, \mathbb{R}) = \{N \rightarrow A_n(X_1, \dots, X_n, \mathbb{R})\}$$

of *multiaffine games*, where $A_n(X_1, \dots, X_n, \mathbb{R})$ is the set of all multiaffine maps from $X = X_1 \times \dots \times X_n$ to \mathbb{R} .³

Proposition 2.3.1. *The set $\mathcal{A}^n(X, \mathbb{R})$ of all multiaffine games is a \mathcal{A} -semi-algebraically regular set of games. Therefore, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{A}^n(X, \mathbb{R})}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.*

Remark 2.3.3. The previous proposition is closely related to Kohlberg-Mertens' structure theorem [33] (see also Predtetchinski [40], Corollary 1).

From now on, and until the end of this section, we suppose that for every $i \in N$, the set X_i of strategies of player i is a nonempty compact interval of \mathbb{R} ; when $X_i \subset [0, +\infty)$, a strategy of player i can be interpreted as an amount of time or effort to exert some activity.

Linear perturbations

In this section, we prove that oddness theorem holds for generic linear perturbations of a given game $\bar{u} = (\bar{u}_i)_{i \in N}$, where for every $i \in N$, $\bar{u}_i \in \mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})} \cap \mathcal{F}_i$, i.e. \bar{u}_i is polynomial with a degree less or equal to δ_i and without constant part, and is concave in x_i .

Consider a semi-algebraic subset A of \mathbb{R}^N such that $\dim(A) = \text{card}(N)$ (a typical case is when A is a product of intervals of nonempty interior), and for every $\alpha = (\alpha_i)_{i \in N} \in A$, consider the game

$$u^\alpha = (x \in X \mapsto \bar{u}_i(x) + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

parameterized by α .

The following proposition states that polynomial games of the form u^α ($\alpha \in A$) admit generically an odd number of Nash equilibria.

Proposition 2.3.2. *The set*

$$\mathcal{S} = \{(x \in X \mapsto \bar{u}_i(x) + \alpha_i x_i \in \mathbb{R})_{i \in N} : \alpha \in A\},$$

³Recall that: (i) an *affine space* is a 3-tuple $(E, \vec{E}, +)$, where E is a set, \vec{E} is a vector space and $+$ is a transitive free action of the additive group of \vec{E} on E (in the case where $E = \vec{E}$, $+$ can be taken as the vector sum of \vec{E}); (ii) a map $f : E \rightarrow F$ between two affine spaces is *affine* if there exists a linear map $\vec{f} : \vec{E} \rightarrow \vec{F}$ such that for every $x, y \in \vec{E}$, $\vec{f}(x - y) = f(x) - f(y)$; (iii) a map $f : \prod_{i=1}^n E_i \rightarrow F$ between a product of $n \in \mathbb{N}$ affine spaces and an affine space is *multiaffine* if for every $i = 1, \dots, n$ and every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \prod_{j \neq i} E_j$, the map $f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$ is affine (recall that the set of all multiaffine maps from $E_1 \times \dots \times E_n$ to F is denoted $A_n(E_1, \dots, E_n, F)$).

is a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games. Furthermore, there exists a generic semi-algebraic subset G of A such that for every $\alpha \in G$, the game u^α has an odd number of Nash equilibria.

Proof. By definition, \mathcal{S} satisfies concavity assumption (i.e. $\mathcal{S} \subset \mathcal{F}$). Now, remark that the set $\mathbf{C}_\mathcal{S} \subset \mathbb{R}^m$ of coefficients associated to \mathcal{S} is of the form

$$\prod_{i \in N} E_i^1 \times \left(\prod_{i \in N} E_i^2 + A \right),$$

where for every player $i \in N$, E_i^1 is a finite product of singletons (corresponding to the coefficients of the payoff function of player i in the game \bar{u} associated to the monomials other than x_i) and E_i^2 is a singleton (corresponding to the coefficient of the payoff function of player i in the game \bar{u} associated to the monomial x_i). Hence, \mathcal{S} satisfies strong semi-algebraicity assumption, since A is semi-algebraic (by assumption). Furthermore, \mathcal{S} satisfies also $\dim(\mathcal{L})$ -invariance assumption, since

$$\dim(\mathbf{C}_\mathcal{S}) = \dim(A) = \text{card}(N) = \dim(\mathbf{C}_{\mathcal{S}+\mathcal{L}})$$

(the dimension of each singleton being equal to 0 and by assumption, $\dim(A) = \text{card}(N)$). Thus, \mathcal{S} is a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games, and from Theorem 2.3.1, there exists a generic semi-algebraic subset $\mathbf{C}_\mathcal{S}^0$ of $\mathbf{C}_\mathcal{S}$ such that for every $y \in \mathbf{C}_\mathcal{S}^0$, the game u^y has an odd number of Nash equilibria. Now, since the trivial map f which associates to any $\alpha \in A$ the “same” element in $\mathbf{C}_\mathcal{S}$ (up to singletons) is a semi-algebraic homeomorphism, for every $\alpha \in G = f^{-1}(\mathbf{C}_\mathcal{S}^0)$, the game u^α has an odd number of Nash equilibria, where G is a generic subset of A . \square

Quadratic perturbations

Similarly to the previous section, we now prove that oddness theorem holds for generic quadratic perturbations of a given game $\bar{u} = (\bar{u}_i)_{i \in N}$, where for every $i \in N$, $\bar{u}_i \in \mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})} \cap \mathcal{F}_i$, i.e. \bar{u}_i is polynomial with a degree less or equal to δ_i and without constant part, and is concave in x_i .

Consider a semi-algebraic subset A of \mathbb{R}^N such that $\dim(A) = \text{card}(N)$, a semi-algebraic subset B of \mathbb{R}^{L_d} and a semi-algebraic subset C of $[0, +\infty)^N$, where $L_d = \{(i, j) \in N^2 : i \neq j\}$ (every $(i, j) \in L_d$ is denoted i, j). Now, for every $(\alpha, \beta, \gamma) \in A \times B \times C$, consider the game

$$u^{\alpha, \beta, \gamma} = \left(x \in X \mapsto \bar{u}_i(x) - \gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i \in \mathbb{R} \right)_{i \in N}$$

parameterized by α , β and γ .

The following proposition states that polynomial games of the form $u^{\alpha,\beta,\gamma}$ ($(\alpha, \beta, \gamma) \in A \times B \times C$) admit generically an odd number of Nash equilibria.

Proposition 2.3.3. *The set*

$$\mathcal{S} = \{(x \in X \mapsto \bar{u}_i(x) - \gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i \in \mathbb{R})_{i \in N} : (\alpha, \beta, \gamma) \in A \times B \times C\},$$

is a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games. Furthermore, there exists a generic semi-algebraic subset G of $A \times B \times C$ such that for every $(\alpha, \beta, \gamma) \in G$, the game $u^{\alpha,\beta,\gamma}$ has an odd number of Nash equilibria.

Proof. By definition, \mathcal{S} satisfies concavity assumption (i.e. $\mathcal{S} \subset \mathcal{F}$). Now, remark that the set $\mathbf{C}_{\mathcal{S}} \subset \mathbb{R}^m$ of coefficients associated to \mathcal{S} is of the form

$$\prod_{i \in N} E_i^1 \times \left(\prod_{i \in N} E_i^2 - C \right) \times \left(\prod_{i \in N} E_i^3 + B \right) \times \left(\prod_{i \in N} E_i^4 + A \right),$$

where for every player i , E_i^1 is a finite product of singletons (corresponding to the coefficients of the payoff function of player i in the game \bar{u} associated to the monomials other than x_i^2 , $(x_i x_j)_{j \neq i}$, and x_i), E_i^2 is a singleton (corresponding to the coefficient of the payoff function of player i in the game \bar{u} associated to the monomial x_i^2), E_i^3 is a finite product of singletons (corresponding to the coefficients of the payoff function of player i in the game \bar{u} associated to the monomials $(x_i x_j)_{j \neq i}$), and E_i^4 is a singleton (corresponding to the coefficient of the payoff function of player i in the game \bar{u} associated to the monomial x_i). Hence, \mathcal{S} satisfies strong semi-algebraicity assumption, since A , B and C are semi-algebraic (by assumption). Furthermore, \mathcal{S} satisfies also $\dim(\mathcal{L})$ -invariance assumption, since

$$\begin{aligned} \dim(\mathbf{C}_{\mathcal{S}}) &= \dim(A \times B \times C) \\ &= \dim(A) + \dim(B) + \dim(C) \\ &= \text{card}(N) + \dim(B) + \dim(C) \\ &= \dim(\mathbf{C}_{\mathcal{S}+\mathcal{L}}) \end{aligned}$$

(the dimension of each singleton being equal to 0 and by assumption, $\dim(A) = \text{card}(N)$). Thus, \mathcal{S} is a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games, and from Theorem 2.3.1, there exists a generic semi-algebraic subset $\mathbf{C}_{\mathcal{S}}^0$ of $\mathbf{C}_{\mathcal{S}}$ such that for every $y \in \mathbf{C}_{\mathcal{S}}^0$, the game u^y has an odd number of Nash equilibria. Now, since the trivial map f which associates to any

$(\alpha, \beta, \gamma) \in A \times B \times C$ the “same” element in \mathbf{C}_S (up to singletons) is a semi-algebraic homeomorphism, for every $(\alpha, \beta, \gamma) \in G = f^{-1}(\mathbf{C}_S^0)$, the game $u^{\alpha, \beta, \gamma}$ has an odd number of Nash equilibria, where G is a generic subset of $A \times B \times C$. \square

As an application of the model with linear perturbations and of the model with quadratic perturbations, we prove that there exists generically an odd number of Nash equilibria for several models of *games on networks* (introduced in network formation theory): Patacchini-Zenou’s model [38] about juvenile delinquency and conformism, Calvó-Armengol-Patacchini-Zenou’s model [12] about social networks in education, Konig-Liu-Zenou’s model [34] about R&D networks, Helsley-Zenou’s model [28] about social networks and interactions in cities, etc. These models are in fact particular cases of Jackson-Zenou’s benchmark quadratic model [44]. In the following, we consider the set $L = \{\{i, j\} : (i, j) \in N^2, i \neq j\}$ of links (on N) (every link $\{i, j\} \in L$ is denoted ij), and the set $\mathbb{G} = [0, 1]^L$ of (weighted) networks (on N). For every network $g \in \mathbb{G}$ and every link $ij \in L$, $g(\{i, j\})$ is denoted g_{ij} and is called the *weight associated to ij (in g)*; this quantity measures the strength of link ij in the network g .

Jackson-Zenou’s benchmark quadratic model with ex ante heterogeneity [44]

Let $c \in (0, +\infty)$, and suppose that for every $i \in N$, payoff function of player i is defined by

$$x \in X \mapsto -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij} x_i x_j + \alpha_i x_i,$$

where $g \in \mathbb{G}$ and $\alpha_i \in [0, +\infty)$. For every $i \in N$, we can rewrite payoff function of player i in the following way:

$$x \in X \mapsto -\gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i,$$

where for every player $j \neq i$, $\beta_{i,j} = \beta_{j,i} = c g_{ij} \in [0, c]$, and where $\gamma_i = \frac{1}{2}$.

We can obtain two different generic existence results (which are not comparable), depending on which parameters of the model are fixed:

- First, as an application of the model with linear perturbations, we can consider the set

$$\mathcal{S} = \{(x \in X \mapsto \bar{u}_i(x) + \alpha_i x_i \in \mathbb{R})_{i \in N} : \alpha \in A\},$$

where for every player $i \in N$, and every $x \in X$,

$$\bar{u}_i(x) = -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij}x_i x_j,$$

and where $A = [0, +\infty)^N$. From Proposition 2.3.2, we obtain a generic subset G^{lin} of A such that for every $\alpha \in G^{\text{lin}}$, the game

$$u^\alpha = (x \in X \mapsto \bar{u}_i(x) + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

has an odd number of Nash equilibria. This application corresponds to the case where the network g is supposed to be fixed.

- Second, as an application of the model with quadratic perturbations, we can consider the set

$$\mathcal{S} = \{(x \in X \mapsto \bar{u}_i(x) - \gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i \in \mathbb{R})_{i \in N} : (\alpha, \beta, \gamma) \in A \times B \times C\},$$

where for every player $i \in N$, and every $x \in X$, $\bar{u}_i(x) = 0$, and where $A = [0, +\infty)^N$, $B = \{\beta = (\beta_{i,j})_{i,j \in L_d} : \forall i \in N, \forall j \neq i, \beta_{i,j} = \beta_{j,i} \in [0, c]\}$ and $C = \{\frac{1}{2}\}^N$. From Proposition 2.3.3, we obtain a generic subset G of $A \times B \times C$ such that for every $(\alpha, \beta, \gamma) \in G$, the game

$$u^{\alpha, \beta, \gamma} = (x \in X \mapsto \bar{u}_i(x) - \gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

has an odd number of Nash equilibria. Moreover, we can even find a generic subset G^{quad} of $A \times \mathbb{G} \times C$ such that for every $(\alpha, g, \gamma) \in G^{\text{quad}}$, the game

$$u^{\alpha, g, \gamma} = (x \in X \mapsto -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij} x_i x_j + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

has an odd number of Nash equilibria. Indeed, the map

$$(\alpha, (\beta_{i,j})_{i,j \in L_d}, \gamma) \in A \times B \times C \mapsto (\alpha, (\frac{\beta_{ij}}{c})_{ij \in L}, \gamma) \in A \times \mathbb{G} \times C,$$

with $\beta_{ij} = \beta_{i,j}$ ($ij \in L$), is a well-defined semi-algebraic homeomorphism, since for every player $i \in N$ and every player $j \neq i$, $\beta_{i,j} = \beta_{j,i}$, and since $c \neq 0$. This application corresponds to the case where the network g is not supposed to be fixed.

Jackson-Zenou's benchmark quadratic model with ex ante heterogeneity and global congestion [44]

Consider the benchmark quadratic model with ex ante heterogeneity with the following modification: let $c' \in [0, +\infty)$, and suppose that for every $i \in N$, payoff function of player i is defined by

$$x \in X \mapsto -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij}x_i x_j - c' \sum_{j \neq i} x_i x_j + \alpha_i x_i.$$

Again, for every $i \in N$, payoff function of player i can be written in the following way:

$$x \in X \mapsto -\gamma_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \alpha_i x_i,$$

where for every player $j \neq i$, $\beta_{i,j} = \beta_{j,i} = c g_{ij} - c' \in [-c', c - c']$, and where $\gamma_i = \frac{1}{2}$.

Similarly to the previous model (without global congestion):

- If we apply Proposition 2.3.2 (i.e. if we consider the case where the network g is supposed to be fixed), then we obtain a generic subset G^{lin} of $[0, +\infty)^N$ such that for every $\alpha \in G^{\text{lin}}$, the game

$$u^\alpha = (x \in X \mapsto \bar{u}_i(x) + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

has an odd number of Nash equilibria, where

$$\bar{u}_i(x) = -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij}x_i x_j - c' \sum_{j \neq i} x_i x_j$$

($i \in N, x \in X$).

- If we apply Proposition 2.3.3 (i.e. if we consider the case where the network g is not supposed to be fixed), then we obtain a generic subset G^{quad} of $[0, +\infty)^N \times \mathbb{G} \times \{\frac{1}{2}\}^N$ such that for every $(\alpha, g, \gamma) \in G^{\text{quad}}$, the game

$$u^{\alpha, g, \gamma} = (x \in X \mapsto -\frac{1}{2}x_i^2 + c \sum_{j \neq i} g_{ij}x_i x_j - c' \sum_{j \neq i} x_i x_j + \alpha_i x_i \in \mathbb{R})_{i \in N}$$

has an odd number of Nash equilibria, where $\bar{u}_i(x) = 0$ ($i \in N, x \in X$).

2.4 Appendix

2.4.1 Reminders about topological degree of a proper continuous map between topological oriented m -manifolds

Let X be an oriented topological m -manifold and Y be an oriented connected topological m -manifold. For every proper continuous map $f : X \rightarrow Y$, one can associate to f an integer $\deg(f) \in \mathbb{Z}$ called the *degree of f* (Dold [19], Proposition and Definition 4.5, p. 268).

Proposition 2.4.1. (Dold [19], Exercises 4.10, 3., p. 271)

Let $f, g : X \rightarrow Y$ be two proper continuous maps. If f and g are homotopic, then $\deg(f) = \deg(g)$.

Proposition 2.4.2. *Let $f : X \rightarrow Y$ be a proper continuous map. If f is a topological embedding onto an open subset of Y , then $\deg(f) = \pm 1$.*

Proof. See Dold's comment [19] after Definition 4.2, p. 267. □

Proposition 2.4.3. *Let $f : X \rightarrow Y$ be a proper continuous map. If $\deg(f) \neq 0$, then f is surjective.*

Proof. See Dold's comment [19] after Definition 4.2, p. 267. □

Theorem 2.4.1. *Let $f : X \rightarrow Y$ be a proper continuous map and $y \in Y$ such that $f^{-1}(y) = \{x_1, \dots, x_n\}$, where $n > 0$. Moreover, let $V = \bigcup_{i=1}^n V_i$, where $(V_i)_{i=1}^n$ is a family of pairwise disjoint open subsets of X such that for every $i \in \{1, \dots, n\}$, $x_i \in V_i$. Then,*

$$\deg(f) = \sum_{i=1}^n \deg(f|_{V_i}).$$

Furthermore, if $\deg(f) = \pm 1$, and if for every $i \in \{1, \dots, n\}$, $f|_{V_i}$ is a topological embedding, then n is odd.

Proof. The proof follows from Dold [19], Proposition 4.7, p. 269 and is similar to the one of Theorem 1.4.1 in Section 1.4.1 of Chapter 1. □

2.4.2 Proof of structure theorem

Let $x^0 \in X$ be a fixed strategy profile and \mathcal{R} be a \mathcal{A} -regular set of games (i.e. \mathcal{R} satisfies concavity assumption and \mathcal{A} -invariance assumption in Definition 2.2.8).

Consider the restriction $\eta_{\mathcal{R}}$ of the homeomorphism $\eta_{\mathcal{F}}$ (of Theorem 2.2.1) from $\mathcal{N}_{\mathcal{R}}$ to \mathcal{F} , and the restriction $\rho_{\mathcal{R}}$ of the map $\eta_{\mathcal{F}}^{-1}$ from \mathcal{R} to $\mathcal{N}_{\mathcal{F}}$ (which are well-defined, from concavity assumption). It is sufficient to prove that $\eta_{\mathcal{R}}(\mathcal{N}_{\mathcal{R}}) \subset \mathcal{R}$ and that $\rho_{\mathcal{R}}(\mathcal{R}) \subset \mathcal{N}_{\mathcal{R}}$ in order to obtain that $\mathcal{N}_{\mathcal{R}}$ and \mathcal{R} are homeomorphic (in that case, the restriction of $\eta_{\mathcal{R}}$ from $\mathcal{N}_{\mathcal{R}}$ to \mathcal{R} is such a homeomorphism and its inverse is the restriction of $\rho_{\mathcal{R}}$ from \mathcal{R} to $\mathcal{N}_{\mathcal{R}}$):

- Recall that for every $(u, x) \in \mathcal{N}_{\mathcal{R}}$, every $i \in N$ and every $z \in X$,

$$\eta_{\mathcal{R}}(u, x)_i(z) = u_i(z) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x_{-i}^0), z_i - x_i \rangle + \langle x_i, z_i \rangle$$

(see Equation (2.1) in Section 2.2.2). Remark that $\eta_{\mathcal{R}}(u, x)_i$ is equal to u_i up to an element of \mathcal{A}_i . Thus, \mathcal{A} -invariance assumption directly implies that $\eta_{\mathcal{R}}(\mathcal{N}_{\mathcal{R}}) \subset \mathcal{R}$.

From now on, by abuse of notation, the map $(u, x) \in \mathcal{N}_{\mathcal{R}} \mapsto \eta_{\mathcal{F}}(u, x) \in \mathcal{R}$ is also denoted $\eta_{\mathcal{R}}$.

- Recall that the inverse of the homeomorphism $\eta_{\mathcal{F}} : \mathcal{N}_{\mathcal{F}} \rightarrow \mathcal{F}$ of Theorem 2.2.1 is defined as follows: for every $u \in \mathcal{F}$, $\eta_{\mathcal{F}}^{-1}(u) = (\underline{v}^u, x^u)$, where for every $i \in N$ and every $z \in X$,

$$\underline{v}_i^u(z) = u_i(z) - \langle \nabla_{x_i^u} u_i(\cdot, x_{-i}^u) - \nabla_{x_i^u} u_i(\cdot, x_{-i}^0), z_i - x_i^u \rangle - \langle x_i^u, z_i \rangle,$$

and where x_i^u is the unique maximizer of the strictly concave map $x_i \in X_i \mapsto u_i(x_i, x_{-i}^0) - \frac{1}{2} \langle x_i, x_i \rangle$ (see Predtetchinski [40]). Similarly as before, observe that \underline{v}_i^u is equal to u_i up to an element of \mathcal{A}_i . Thus, \mathcal{A} -invariance assumption directly implies that $\rho_{\mathcal{R}}(\mathcal{R}) \subset \mathcal{N}_{\mathcal{R}}$.

From now on, by abuse of notation, the map $u \in \mathcal{R} \mapsto \rho_{\mathcal{F}}(u) \in \mathcal{N}_{\mathcal{R}}$ is also denoted $\rho_{\mathcal{R}}$.

Now, consider the map

$$\begin{cases} H_{\mathcal{R}} : [0, 1] \times \mathcal{N}_{\mathcal{R}} & \rightarrow & \mathcal{F} \\ (t, (x, u)) & \mapsto & (1-t)\pi_{\mathcal{R}}(x, u) + t\eta_{\mathcal{R}}(x, u) \end{cases} .$$

For every $(t, (x, u)) \in [0, 1] \times \mathcal{N}_{\mathcal{R}}$, there exists $a \in \mathcal{A}$ such that

$$H_{\mathcal{R}}(t(x, u)) = (1-t)\pi_{\mathcal{R}}(x, u) + t\eta_{\mathcal{R}}(x, u) = (1-t)u + t(u+a) = u + ta \in \mathcal{R} + \mathcal{A},$$

since $\eta_{\mathcal{R}}(u, x)_i$ is equal to u_i up to an element of \mathcal{A}_i . Hence, \mathcal{A} -invariance assumption implies that $H_{\mathcal{R}}([0, 1] \times \mathcal{N}_{\mathcal{R}}) \subset \mathcal{R}$. Finally, the map $(t, (x, u)) \in [0, 1] \times \mathcal{N}_{\mathcal{R}} \mapsto (1 - t)\pi_{\mathcal{R}}(u, x) + t\eta_{\mathcal{R}}(u, x) \in \mathcal{R}$ (which is also denoted $H_{\mathcal{R}}$, by abuse of notation) is a proper homotopy between $\pi_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ (the proof is similar to the one of Theorem 1.2.1, Step VII in Section 1.4.3 of Chapter 1, knowing that for every $i \in N$, X_i is compact).

This step ends the proof of the structure theorem. \square

2.4.3 Proof of oddness theorem

From now on, consider a fixed strategy profile $x^0 \in X$. Recall that for every \mathcal{A} -regular set \mathcal{R} of games, $\eta_{\mathcal{R}}$ is the homeomorphism from $\mathcal{N}_{\mathcal{R}}$ to \mathcal{R} of structure theorem (Theorem 2.2.2): for every $(u, x) \in \mathcal{N}_{\mathcal{R}}$, every $i \in N$ and every $z \in X$,

$$\eta_{\mathcal{R}}(u, x)_i(z) = u_i(z) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x_{-i}^0), z_i - x_i \rangle + \langle x_i, z_i \rangle$$

(see Equation (2.1) in Section 2.2.2). Also, for every subset $\mathcal{U} \subset \mathbb{R}_{\delta}[x]$ of polynomial games, consider the set

$$\mathcal{M}_{\mathcal{U}} = \{(\varphi(u), x) \in \mathbf{C}_{\mathcal{U}} \times X : (u, x) \in \mathcal{N}_{\mathcal{U}}\} \subset \mathbb{R}^m \times X.$$

Step I. For every \mathcal{A} -semi-algebraically regular set \mathcal{S} of games, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.

Let \mathcal{S} be a \mathcal{A} -semi-algebraically regular set of games (i.e. \mathcal{S} satisfies concavity assumption, \mathcal{A} -invariance assumption and semi-algebraicity assumption in Definition 2.3.5). In particular, note that \mathcal{S} is also \mathcal{A} -regular (by definition).

Substep I.1. A decomposition result.

For every $i \in N$, let

$$\begin{aligned} & \mathbb{R}_{\delta_i}[x]_{-\mathcal{A}_i} \\ &= \text{Span}(\{ \prod_{j, \ell \in \mathbf{L}} x_{j, \ell}^{k_{j, \ell}} : k \in \mathbb{N}^{\mathbf{L}}, \deg(k) \leq \delta_i \} \setminus (\{x_{i, \ell} : \ell \in \{1, \dots, \mu_i\}\} \cup \{1\})) \end{aligned}$$

be the linear subspace of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the ones in \mathcal{A}_i , and let $\mathbb{R}_{\delta}[x]_{-\mathcal{A}} = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]_{-\mathcal{A}_i}$. By definition, observe that $\mathbb{R}_{\delta}[x] = \mathbb{R}_{\delta}[x]_{-\mathcal{A}} \oplus \mathcal{A}$, thus that

$$\varphi(\mathbb{R}_{\delta}[x]) = \mathbb{R}^m = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-\mathcal{A}}} \oplus \mathbf{C}_{\mathcal{A}} = \varphi(\mathbb{R}_{\delta}[x]_{-\mathcal{A}}) \oplus \varphi(\mathcal{A}).$$

Moreover, denote by $\Pi_{-\mathcal{A}}$ the linear projection from the vector space $\mathbb{R}^m = \mathbf{C}_{\mathbb{R}_\delta[x]_{-\mathcal{A}}} \oplus \mathbf{C}_{\mathcal{A}}$ (of coefficients of polynomial games in $\mathbb{R}_\delta[x]$) to the space $\mathbf{C}_{\mathbb{R}_\delta[x]_{-\mathcal{A}}}$ (of coefficients of polynomial games in $\mathbb{R}_\delta[x]_{-\mathcal{A}}$), and consider the set $\Pi_{-\mathcal{A}}(\mathbf{C}_{\mathcal{S}})$.

Since $\mathbf{C}_{\mathcal{S}}$ is a semi-algebraic set (from semi-algebraicity assumption in Definition 2.3.5) and since $\Pi_{-\mathcal{A}}$ is a semi-algebraic map, the set $\Pi_{-\mathcal{A}}(\mathbf{C}_{\mathcal{S}})$ is also semi-algebraic, from Proposition 1.4.8 (in Appendix 1.4.2 of Chapter 1). In particular, from Proposition 1.4.9 (in Appendix 1.4.2 of Chapter 1),

$$\Pi_{-\mathcal{A}}(\mathbf{C}_{\mathcal{S}}) = \bigcup_{\lambda=1}^r T_\lambda,$$

where $r \in \mathbb{N}$, and where $(T_\lambda)_{\lambda=1}^r$ is a family of pairwise disjoint semi-algebraic sets such that for every $\lambda \in \Lambda = \{1, \dots, r\}$, T_λ is semi-algebraically homeomorphic to an open hypercube $(0, 1)^{d_\lambda}$, for some $d_\lambda \in \mathbb{N}$. For every $\lambda \in \Lambda$, consider the set

$$\mathcal{U}_\lambda = \varphi^{-1}(T_\lambda) + \mathcal{A},$$

Observe that for every $\lambda \in \Lambda$:

1. \mathcal{U}_λ satisfies concavity assumption (in Definition 2.3.5) because $T_\lambda \subset \Pi_{-\mathcal{A}}(\mathbf{C}_{\mathcal{S}}) = \Pi_{-\mathcal{A}}(\varphi(\mathcal{S}))$, which implies that $\varphi^{-1}(T_\lambda) \subset (\varphi^{-1} \circ \Pi_{-\mathcal{A}} \circ \varphi)(\mathcal{S}) \subset \mathcal{S}$, hence that

$$\mathcal{U}_\lambda = \varphi^{-1}(T_\lambda) + \mathcal{A} \subset \mathcal{S} + \mathcal{A} = \mathcal{S}$$

(the last equality coming from the fact that \mathcal{S} satisfies \mathcal{A} -invariance assumption), where \mathcal{S} satisfies concavity assumption (i.e. $\mathcal{S} \subset \mathcal{F}$).

2. \mathcal{U}_λ clearly satisfies \mathcal{A} -invariance assumption (in Definition 2.3.5).
3. \mathcal{U}_λ satisfies semi-algebraicity assumption (in Definition 2.3.5) because

$$\mathbf{C}_{\mathcal{U}_\lambda} = \varphi(\mathcal{U}_\lambda) = \varphi(\varphi^{-1}(T_\lambda) + \mathcal{A}) = T_\lambda + \varphi(\mathcal{A}) = T_\lambda + \mathbf{C}_{\mathcal{A}},$$

where both T_λ and $\mathbf{C}_{\mathcal{A}}$ are semi-algebraic sets; the result follows from Corollary 1.4.2 (in Appendix 1.4.2 of Chapter 1).

4. Since $\text{Span}(T_\lambda)$ and $\text{Span}(\mathbf{C}_{\mathcal{A}}) = \mathbf{C}_{\mathcal{A}}$ are in direct sum (because $\mathbf{C}_{\mathbb{R}_\delta[x]_{-\mathcal{A}}}$ - which contains $\text{Span}(T_\lambda)$ - and $\mathbf{C}_{\mathcal{A}}$ are in direct sum), $\mathbf{C}_{\mathcal{U}_\lambda}$ is homeomorphic to $T_\lambda \times \mathbf{C}_{\mathcal{A}}$ (see the proof of Corollary 1.4.5 in Appendix 1.4.2 of Chapter 1), thus homeomorphic to $(0, 1)^{e_\lambda}$, for some $e_\lambda \in \mathbb{N}$.

Hence, for every $\lambda \in \Lambda$, \mathcal{U}_λ is a \mathcal{A} -semi-algebraically regular set of games such that $\mathbf{C}_{\mathcal{U}_\lambda}$ is homeomorphic to \mathbb{R}^{e_λ} .

From now on, and until Step I.6, consider a fixed $\lambda \in \Lambda$.

Substep I.2. The set $\mathcal{M}_{\mathcal{U}_\lambda}$ is semi-algebraic and $\dim(\mathcal{M}_{\mathcal{U}_\lambda}) = \dim(\mathbf{C}_{\mathcal{U}_\lambda})$. Moreover, both $\mathcal{M}_{\mathcal{U}_\lambda}$ and $\mathbf{C}_{\mathcal{U}_\lambda}$ are oriented connected topological e_λ -manifolds.

First, we show that $\mathcal{M}_{\mathcal{U}_\lambda}$ is a semi-algebraic set. From concavity assumption, remark that for every $(y, x) \in \mathbb{R}^m \times X$, $(y, x) \in \mathcal{M}_{\mathcal{U}_\lambda}$ if and only if for every $i \in N$ and every $d_i \in X_i$,

$$\langle \nabla_{x_i} u_i^y(\cdot, x_{-i}), d_i - x_i \rangle \leq 0.$$

These conditions involve a finite number of equalities and of inequalities with semi-algebraic maps, thus $\mathcal{M}_{\mathcal{U}_\lambda}$ is a semi-algebraic set.

Last, we show that the map

$$\begin{cases} \eta_{\mathcal{U}_\lambda} : \mathcal{M}_{\mathcal{U}_\lambda} & \rightarrow & \mathbf{C}_{\mathcal{U}_\lambda} \\ (y, x) & \mapsto & (\varphi \circ \eta_{\mathcal{U}_\lambda})(u^y, x) \end{cases}$$

is a semi-algebraic homeomorphism, which implies that $\dim(\mathcal{M}_{\mathcal{U}_\lambda}) = \dim(\mathbf{C}_{\mathcal{U}_\lambda})$, from Proposition 1.4.12 (in Appendix 1.4.2 of Chapter 1). From its definition and from the definition of $\eta_{\mathcal{U}_\lambda}$, the map $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is semi-algebraic if and only if for every $i \in N$, the map

$$(y, x) \in \mathcal{M}_{\mathcal{U}_\lambda} \mapsto \varphi_i(z \mapsto u_i^y(z) + \langle \nabla_{x_i} u_i^y(\cdot, x_{-i}) - \nabla_{x_i} u_i^y(\cdot, x_{-i}^0), z_i - x_i \rangle + \langle x_i, z_i \rangle) \in \mathbb{R}^{m_i}$$

is semi-algebraic. Observe that this is the case, since each coefficient of the polynomial function

$$z \in X \mapsto u_i^y(z) + \langle \nabla_{x_i} u_i^y(\cdot, x_{-i}) - \nabla_{x_i} u_i^y(\cdot, x_{-i}^0), z_i - x_i \rangle + \langle x_i, z_i \rangle \in \mathbb{R}$$

is itself a polynomial function of (y, x) , for every $(y, x) \in \mathcal{M}_{\mathcal{U}_\lambda}$. In order to understand why the map $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is a homeomorphism, consider the following diagram:

$$\begin{array}{ccccc} \mathcal{N}_{\mathcal{F}} & \longleftarrow & \mathcal{N}_{\mathcal{U}_\lambda} & \xrightarrow{\varphi \times \text{id}_X} & \mathcal{M}_{\mathcal{U}_\lambda} \\ \pi_{\mathcal{F}} \downarrow \downarrow \eta_{\mathcal{F}} & & \pi_{\mathcal{U}_\lambda} \downarrow \downarrow \eta_{\mathcal{U}_\lambda} & & \pi_{\mathbf{C}_{\mathcal{U}_\lambda}} \downarrow \downarrow \eta_{\mathbf{C}_{\mathcal{U}_\lambda}} \\ \mathcal{F} & \longleftarrow & \mathcal{U}_\lambda & \xrightarrow{\varphi} & \mathbf{C}_{\mathcal{U}_\lambda} \end{array}$$

where $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is such that the right square commutes considering the map $\pi_{\mathcal{U}_\lambda}$, i.e.

$$\pi_{\mathbf{C}_{\mathcal{U}_\lambda}} = \varphi \circ \pi_{\mathcal{U}_\lambda} \circ (\varphi \times \text{id}_X)^{-1}$$

(which is the restriction of the canonical projection $\mathbb{R}^m \times \mathbb{R}^{\sum_{i \in N} \mu_i} \rightarrow \mathbb{R}^m$ from $\mathcal{M}_{\mathcal{U}_\lambda}$ to $\mathbf{C}_{\mathcal{U}_\lambda}$). Now, notice that by definition, $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ makes commute the same square considering the map $\eta_{\mathcal{U}_\lambda}$, i.e.

$$\eta_{\mathbf{C}_{\mathcal{U}_\lambda}} = \varphi \circ \eta_{\mathcal{U}_\lambda} \circ (\varphi \times \text{id}_X)^{-1}.$$

Therefore, $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is a homeomorphism as the composition of the three homeomorphisms φ , $\eta_{\mathcal{U}_\lambda}$ and $(\varphi \times \text{id}_X)^{-1}$. Finally, since $\mathbf{C}_{\mathcal{U}_\lambda}$ is homeomorphic to \mathbb{R}^{e_λ} (from Step I.1), this implies that both $\mathcal{M}_{\mathcal{U}_\lambda}$ and $\mathbf{C}_{\mathcal{U}_\lambda}$ are oriented connected topological e_λ -manifolds.

Substep I.3. $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ and $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ are properly homotopic and $\deg(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}) = \pm 1$. In particular, $\pi_{\mathcal{U}_\lambda}$ is a surjective map (i.e. every game in \mathcal{U}_λ admits a Nash equilibrium).

Recall that from structure theorem (Theorem 2.2.2), there exists a proper homotopy $H_{\mathcal{U}_\lambda}$ between $\pi_{\mathcal{U}_\lambda}$ and $\eta_{\mathcal{U}_\lambda}$. Then, the map

$$\left\{ \begin{array}{l} H_{\mathbf{C}_{\mathcal{U}_\lambda}} : [0, 1] \times \mathcal{M}_{\mathcal{U}_\lambda} \rightarrow \mathbf{C}_{\mathcal{U}_\lambda} \\ (t, (y, x)) \mapsto (\varphi \circ H_{\mathcal{U}_\lambda} \circ (\text{id}_{[0,1]} \times (\varphi \times \text{id}_X)^{-1}))(t, (y, x)) \end{array} \right.$$

is a proper homotopy between $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ and $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$, since: (i) $H_{\mathbf{C}_{\mathcal{U}_\lambda}}(0, (\cdot, \cdot)) = \pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ and $H_{\mathbf{C}_{\mathcal{U}_\lambda}}(1, (\cdot, \cdot)) = \eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$; (ii) it is a continuous map (by composition); (iii) it is a proper map (both φ and $(\text{id}_{[0,1]} \times (\varphi \times \text{id}_X)^{-1})$ are homeomorphisms).

Now, since $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is a homeomorphism (from Step I.2), $\deg(\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}) = \pm 1$, from Proposition 2.4.2 (in Appendix 2.4.1). Moreover, since $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ and $\eta_{\mathbf{C}_{\mathcal{U}_\lambda}}$ are properly homotopic, $\deg(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}) = \pm 1$, from Proposition 2.4.1 (in Appendix 2.4.1). Hence, $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is a surjective map, from Proposition 2.4.3 (in Appendix 2.4.1). Observe that this implies that $\pi_{\mathcal{U}_\lambda}$ is also surjective (i.e. every game in \mathcal{U}_λ admits a Nash equilibrium). Indeed, for every $u \in \mathcal{U}_\lambda$, there exists $(y, x) \in \mathcal{M}_{\mathcal{U}_\lambda}$ such that $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}(y, x) = \varphi(u)$ (since $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ is surjective), i.e. $u = (\varphi^{-1} \circ \varphi)(u) = (\varphi^{-1} \circ \pi_{\mathbf{C}_{\mathcal{U}_\lambda}})(y, x)$, where, by definition of $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$,

$$\begin{aligned} (\varphi^{-1} \circ \pi_{\mathbf{C}_{\mathcal{U}_\lambda}})(y, x) &= (\varphi^{-1} \circ (\varphi \circ \pi_{\mathcal{U}_\lambda} \circ (\varphi \times \text{id}_X)^{-1}))(y, x) \\ &= (\pi_{\mathcal{U}_\lambda} \circ (\varphi \times \text{id}_X)^{-1})(y, x) \\ &= \pi_{\mathcal{U}_\lambda}((\varphi \times \text{id}_X)^{-1}(y, x)), \end{aligned}$$

where $(\varphi \times \text{id}_X)^{-1}(y, x) = (u^y, x) \in \mathcal{N}_{\mathcal{U}_\lambda}$.

Substep I.4. There exists a generic subset G_λ of $\mathbf{C}_{\mathcal{U}_\lambda}$ such that for every connected component C of G_λ , there exists a nonempty finite set F_C such that $(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(C), C, \pi_{\mathbf{C}_{\mathcal{U}_\lambda}}, F_C)$ is a covering space. In particular, for every $y \in G_\lambda$, the game u^y admits a strictly positive finite number of Nash equilibria.

Note that $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}} : \mathcal{M}_{\mathcal{U}_\lambda} \rightarrow \mathbf{C}_{\mathcal{U}_\lambda}$ is a continuous semi-algebraic map, which is also surjective (from Step I.3), and recall that $\dim(\mathcal{M}_{\mathcal{U}_\lambda}) = \dim(\mathbf{C}_{\mathcal{U}_\lambda})$ (from

Step I.2). Thus, from Theorem 1.4.5 (in Appendix 1.4.2 of Chapter 1), this directly implies that there exists a generic subset G_λ of $\mathbf{C}_{\mathcal{U}_\lambda}$ (which is now fixed until Step I.6) such that for every connected component C of G_λ , there exists a nonempty finite set F_C such that $(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(C), C, \pi_{\mathbf{C}_{\mathcal{U}_\lambda}}, F_C)$ is a covering space.

Now, from Theorem 1.4.2 (in Appendix 1.4.1 of Chapter 1), observe that for every connected component C of G_λ , the previous property is equivalent to the following one: for every $y \in C$, there exists an open subset $V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ of C which contains y and such that $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y) = \bigcup_{k \in F_C} (V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k$, where $((V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k)_{k \in F_C}$ is a family of pairwise disjoint open subsets of $\mathcal{M}_{\mathcal{U}_\lambda}$ such that for every $k \in F_C$, the map $(y', x) \in (V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k \mapsto \pi_{\mathbf{C}_{\mathcal{U}_\lambda}}(y', x) \in V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ is a homeomorphism.

Also, remark that since for every connected component C of G_λ and every $y \in C$, F_C and $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(y)$ are homeomorphic, and since $(\varphi \times \text{id}_X)^{-1}$ is a homeomorphism, for every $y \in G_\lambda$,

$$\text{card}(\pi_{\mathcal{U}_\lambda}^{-1}(\varphi^{-1}(y))) = \text{card}((\varphi \times \text{id}_X)^{-1}(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(y))) = \text{card}(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(y)) = \text{card}(F_{C^y})$$

(by definition of $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$), where C^y is the connected component of G_λ which contains y . To put it in another way: the society $u^y = \varphi^{-1}(y)$ admits a strictly positive finite number of Nash equilibria.

Substep I.5. For every $y \in G_\lambda$, the game u^y admits an odd number of Nash equilibria.

Let $y \in G_\lambda$ and C^y be the connected component of G_λ which contains y . From Step I.4, consider an open subset $V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ of C^y which contains y and a family $((V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k)_{k \in F_{C^y}}$ of pairwise disjoint open subsets of $\mathcal{M}_{\mathcal{U}_\lambda}$ such that $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}^{-1}(V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y) = \bigcup_{k \in F_{C^y}} (V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k$, and such that for every $k \in F_{C^y}$, the map $(y', x) \in (V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k \mapsto \pi_{\mathbf{C}_{\mathcal{U}_\lambda}}(y', x) \in V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ is a homeomorphism (recall that F_{C^y} is a nonempty finite set). Observe that $V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ is also open in $\mathbf{C}_{\mathcal{U}_\lambda}$, since G_λ is open in $\mathbf{C}_{\mathcal{U}_\lambda}$ (G_λ being generic in $\mathbf{C}_{\mathcal{U}_\lambda}$) and $C^y \supset V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ is open in G_λ (as a connected component of G_λ).

Now, observe that for every $k \in F_{C^y}$, there exists a unique $\alpha_k^y \in (V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k$ such that

$$\pi_{\mathcal{U}_\lambda}^{-1}(y) = \{\alpha_k^y : k \in F_{C^y}\},$$

by definition of the family $((V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k)_{k \in F_{C^y}}$. Moreover, for every $k \in F_{C^y}$, because the restriction $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}|_{(V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k}$ of the map $\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}$ from $(V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k$ to $\mathbf{C}_{\mathcal{U}_\lambda}$ is

a topological embedding onto $V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y$ (which is open in $\mathbf{C}_{\mathcal{U}_\lambda}$), then

$$\deg(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}|_{(V_{\mathbf{C}_{\mathcal{U}_\lambda}}^y)_k}}) = \pm 1,$$

from Proposition 2.4.2 (in Appendix 2.4.1). Finally, since $\deg(\pi_{\mathbf{C}_{\mathcal{U}_\lambda}}) = \pm 1$ (from Step I.3), $\text{card}(F_{C^y})$ is odd, from Theorem 2.4.1 (in Appendix 2.4.1). Therefore, since $\text{card}(\pi_{\mathcal{U}_\lambda}^{-1}(\varphi^{-1}(y))) = \text{card}(F_{C^y})$ (from Step I.4), $v^y = \varphi^{-1}(y)$ admits an odd number of Nash equilibria.

Substep I.6. An amalgamation result.

From the first part of the proof (Substeps I.1-I.5), for every $\lambda \in \Lambda$, there exists a generic semi-algebraic subset G_λ of $\mathbf{C}_{\mathcal{U}_\lambda}$ such that for every $y \in G_\lambda$, the game u^y has an odd number of Nash equilibria. This implies that for every y in

$$G = \bigcup_{\lambda \in \Lambda^{\text{op}}} G_\lambda,$$

where

$$\Lambda^{\text{op}} = \{\lambda \in \Lambda : T_\lambda \text{ is open in } \Pi_{-\mathcal{A}}(\mathbf{C}_S)\},$$

the game u^y admits an odd number of Nash equilibria (recall that $\Pi_{-\mathcal{A}}(\mathbf{C}_S) = \bigcup_{\lambda=1}^r T_\lambda$; see Substep I.1).

Because \mathcal{S} satisfies \mathcal{A} -invariance assumption, notice first that

$$\Pi_{-\mathcal{A}}(\mathbf{C}_S) + \mathbf{C}_\mathcal{A} = \mathbf{C}_S. \quad (2.2)$$

Now, observe that

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} \mathbf{C}_{\mathcal{U}_\lambda} &= \bigcup_{\lambda \in \Lambda} \varphi(\mathcal{U}_\lambda) = \bigcup_{\lambda \in \Lambda} \varphi(\varphi^{-1}(T_\lambda) + \mathcal{A}) \\ &= \bigcup_{\lambda \in \Lambda} (T_\lambda + \varphi(\mathcal{A})) = \bigcup_{\lambda \in \Lambda} (T_\lambda + \mathbf{C}_\mathcal{A}) = \bigcup_{\lambda \in \Lambda} T_\lambda + \mathbf{C}_\mathcal{A} \\ &= \Pi_{-\mathcal{A}}(\mathbf{C}_S) + \mathbf{C}_\mathcal{A} \\ &= \mathbf{C}_S \text{ (from Equation (2.2))} \end{aligned}$$

i.e. $(\mathbf{C}_{\mathcal{U}_\lambda})_{\lambda \in \Lambda}$ forms a cover of \mathbf{C}_S . Furthermore, remark that

$$\mathbf{C}_S \setminus G \subset \bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda) \cup \mathbf{C},$$

where

$$\mathbf{C} = \bigcup_{\lambda \in \Lambda \setminus \Lambda^{\text{op}}} \mathbf{C}_{\mathcal{U}_\lambda}$$

(for every $\lambda \in \Lambda$, $\lambda \in \Lambda \setminus \Lambda^{\text{op}}$ if and only if T_λ is not open in $\Pi_{-\mathcal{A}}(\mathbf{C}_S)$). Indeed, let $y \in \mathbf{C}_S$ such that $y \notin G$ (i.e. $y \in \bigcap_{\lambda \in \Lambda^{\text{op}}} \mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda$). Since $(\mathbf{C}_{\mathcal{U}_\lambda})_{\lambda \in \Lambda}$ forms a cover of \mathbf{C}_S , there exists $\ell \in \Lambda$ such that $y \in \mathbf{C}_{\mathcal{U}_\ell}$. Hence: (i) either $\ell \in \Lambda^{\text{op}}$, which implies that $y \in \mathbf{C}_{\mathcal{U}_\ell} \setminus G_\ell \subset \bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda)$ (because $y \notin G$); (ii) or $\ell \in \Lambda \setminus \Lambda^{\text{op}}$, which implies that $y \in \mathbf{C}$.

We now show that G is generic in \mathbf{C}_S , which will end Step I:

- First, we show that $\dim(\mathbf{C}_S \setminus G) < \dim(\mathbf{C}_S)$.

Note that the dimension of $\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda)$ is strictly less than the dimension of \mathbf{C}_S , because for every $\lambda \in \Lambda$, $\dim(\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda) < \dim(\mathbf{C}_{\mathcal{U}_\lambda})$ (from G_λ being generic in $\mathbf{C}_{\mathcal{U}_\lambda}$) and $\dim(\mathbf{C}_{\mathcal{U}_\lambda}) \leq \dim(\mathbf{C}_S)$ (from Corollary 1.4.4 in Appendix 1.4.2 of Chapter 1); the result follows from Proposition 1.4.11 (in Appendix 1.4.2 of Chapter 1). Also, the dimension of \mathbf{C} is strictly less than the dimension of \mathbf{C}_S . Indeed, for every $\lambda \in \Lambda \setminus \Lambda^{\text{op}} = \{\lambda \in \Lambda : T_\lambda \text{ is not open in } \Pi_{-\mathcal{A}}(\mathbf{C}_S)\}$, remark that $\dim(T_\lambda) < \dim(\Pi_{-\mathcal{A}}(\mathbf{C}_S))$ (from Corollary 1.4.3 in Appendix 1.4.2 of Chapter 1). Thus, for every $\lambda \in \Lambda \setminus \Lambda^{\text{op}}$,

$$\begin{aligned} \dim(\mathbf{C}_{\mathcal{U}_\lambda}) &= \dim(T_\lambda + \mathbf{C}_A) \\ &= \dim(T_\lambda) + \dim(\mathbf{C}_A) \text{ (from Corollary 1.4.5 of Chapter 1)} \\ &< \dim(\Pi_{-\mathcal{A}}(\mathbf{C}_S)) + \dim(\mathbf{C}_A) \\ &= \dim(\Pi_{-\mathcal{A}}(\mathbf{C}_S) + \mathbf{C}_A) \text{ (from Corollary 1.4.5 of Chapter 1)} \\ &= \dim(\mathbf{C}_S) \text{ (from Equation (2.2)).} \end{aligned}$$

Therefore, the result follows from Proposition 1.4.11 (in Appendix 1.4.2 of Chapter 1). Finally, from Corollary 1.4.4 (in Appendix 1.4.2 of Chapter 1),

$$\dim(\mathbf{C}_S \setminus G) \leq \dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda) \cup \mathbf{C}\right),$$

and from Proposition 1.4.11 (in Appendix 1.4.2 of Chapter 1),

$$\dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda) \cup \mathbf{C}\right) = \max\left\{\dim\left(\bigcup_{\lambda \in \Lambda} (\mathbf{C}_{\mathcal{U}_\lambda} \setminus G_\lambda)\right), \dim(\mathbf{C})\right\} < \dim(\mathbf{C}_S).$$

- Last, we show that G is open in \mathbf{C}_S .

Observe that for every $\lambda \in \Lambda^{\text{op}} = \{\lambda \in \Lambda : T_\lambda \text{ is open in } \Pi_{-\mathcal{A}}(\mathbf{C}_S)\}$, G_λ is open in \mathbf{C}_S because G_λ is open in $\mathbf{C}_{\mathcal{U}_\lambda}$ (from G_λ being generic in $\mathbf{C}_{\mathcal{U}_\lambda}$) and because $\mathbf{C}_{\mathcal{U}_\lambda}$ is open in \mathbf{C}_S . Indeed, to understand this last

point, remark that since for every $\lambda \in \Lambda^{\text{op}}$, T_λ is open in $\Pi_{-\mathcal{A}}(\mathbf{C}_S)$, $\mathbf{C}_{u_\lambda} = T_\lambda + \mathbf{C}_\mathcal{A}$ is open in $\Pi_{-\mathcal{A}}(\mathbf{C}_S) + \mathbf{C}_\mathcal{A}$, i.e. is open in \mathbf{C}_S (from Equation (2.2)). Finally, $G = \bigcup_{\lambda \in \Lambda^{\text{op}}} G_\lambda$ is open in \mathbf{C}_S (as an arbitrary union of open subsets of \mathbf{C}_S).

Step II. For every $\dim(\mathcal{A})$ -semi-algebraically regular set \mathcal{S} of games, there exists a generic semi-algebraic subset G of \mathbf{C}_S such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.

Let \mathcal{S} be a $\dim(\mathcal{A})$ -semi-algebraically regular set of games (i.e. \mathcal{S} satisfies concavity assumption, $\dim(\mathcal{A})$ -invariance assumption and semi-algebraicity assumption in Definition 2.3.5).

Consider the set $\mathcal{S} + \mathcal{A}$ of games, and observe that it is \mathcal{A} -semi-algebraically regular: (i) concavity assumption is directly verified; (ii) \mathcal{A} -invariance assumption is verified since $(\mathcal{S} + \mathcal{A}) + \mathcal{A} = \mathcal{S} + \mathcal{A}$; (iii) semi-algebraicity assumption is verified since both \mathbf{C}_S and $\mathbf{C}_\mathcal{A}$ are semi-algebraic sets, and since $\mathbf{C}_{\mathcal{S}+\mathcal{A}} = \mathbf{C}_S + \mathbf{C}_\mathcal{A}$; the result follows from Corollary 1.4.2 in Appendix 1.4.2 of Chapter 1. Hence, from Step I, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S}+\mathcal{A}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.

Now, consider such a subset G of $\mathbf{C}_{\mathcal{S}+\mathcal{A}}$ and define $G' = \mathbf{C}_S \cap G$. In particular, note that for every $y \in G'$, the game u^y has an odd number of Nash equilibria. Finally, observe that G' is generic in \mathbf{C}_S , from the following lemma applied to $S = \mathbf{C}_S$, $T = \mathbf{C}_{\mathcal{S}+\mathcal{A}}$, and $T_0 = G$.

Lemma 2.4.1. *Let S, T be two semi-algebraic sets such that $S \subset T$ and $\dim(S) = \dim(T)$. If T_0 is a generic subset of T , then $S \cap T_0$ is a generic subset of S .*

Proof. Remark that $S \setminus (S \cap T_0) \subset T \setminus T_0$, which implies that $\dim(S \setminus (S \cap T_0)) \leq \dim(T \setminus T_0)$, from Corollary 1.4.4 in Appendix 1.4.2 of Chapter 1. Now, since $\dim(T \setminus T_0) < \dim(T)$ (from T_0 being generic in T) and since $\dim(S) = \dim(T)$ (by assumption), note that $\dim(S \setminus (S \cap T_0)) < \dim(S)$. Moreover, $S \cap T_0$ is open in S : T_0 is open in T (from T_0 being generic in T) and $S \subset T$, so the result follows from the definition of the induced topology on S . \square

Step III. Oddness theorem.

Let \mathcal{S} be a $\dim(\mathcal{L})$ -strongly semi-algebraically regular set of games (i.e. \mathcal{S} satisfies concavity assumption, strong semi-algebraicity assumption and $\dim(\mathcal{L})$ -

invariance assumption in Definition 2.3.5).

Recall that for every $i \in N$,

$$\mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})} = \text{Span}\left(\left\{\prod_{j, \ell \in \mathbf{L}} x_{j, \ell}^{k_{j, \ell}} : k \in \mathbb{N}^{\mathbf{L}}, \deg(k) \leq \delta_i\right\} \setminus \{1\}\right)$$

corresponds to the linear subspace of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the constant monomial, and that $\mathbb{R}_{\delta}[x]_{-c} = \prod_{i \in N} \mathbb{R}_{\delta_i}[x]_{-\text{Cst}(X, \mathbb{R})}$. By definition, observe that $\mathbb{R}_{\delta}[x] = \mathbb{R}_{\delta}[x]_{-c} \oplus \mathcal{C}$, thus that

$$\varphi(\mathbb{R}_{\delta}[x]) = \mathbb{R}^m = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-c}} \oplus \mathbf{C}_{\mathcal{C}} = \varphi(\mathbb{R}_{\delta}[x]_{-c}) \oplus \varphi(\mathcal{C}).$$

Moreover, denote by Π_{-c} the linear projection from the vector space $\mathbb{R}^m = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-c}} \oplus \mathbf{C}_{\mathcal{C}}$ (of coefficients of polynomial games in $\mathbb{R}_{\delta}[x]$) to the space $\mathbf{C}_{\mathbb{R}_{\delta}[x]_{-c}}$ (of coefficients of polynomial games in $\mathbb{R}_{\delta}[x]_{-c}$).

Consider the set $\mathcal{S} + \mathcal{C}$ of games, and observe that it is $\dim(\mathcal{A})$ -semi-algebraically regular: (i) concavity assumption is directly verified; (ii) $\dim(\mathcal{A})$ -invariance assumption is verified since

$$\begin{aligned} \dim(\mathbf{C}_{(\mathcal{S} + \mathcal{C}) + \mathcal{A}}) &= \dim(\mathbf{C}_{(\mathcal{S} + \mathcal{C}) + (\mathcal{L} + \mathcal{C})}) \\ &= \dim(\mathbf{C}_{(\mathcal{S} + \mathcal{L}) + \mathcal{C}}) \\ &= \dim(\mathbf{C}_{\mathcal{S} + \mathcal{L}}) + \dim(\mathbf{C}_{\mathcal{C}}) \text{ (from Corollary 1.4.5 of Chapter 1)} \\ &= \dim(\mathbf{C}_{\mathcal{S}}) + \dim(\mathbf{C}_{\mathcal{C}}) \\ &\text{(from } \dim(\mathcal{L})\text{-invariance assumption on } \mathcal{S}\text{)} \\ &= \dim(\mathbf{C}_{\mathcal{S} + \mathcal{C}}) \text{ (from Corollary 1.4.5 of Chapter 1);} \end{aligned}$$

(iii) semi-algebraicity assumption is verified since both $\mathbf{C}_{\mathcal{S}}$ and $\mathbf{C}_{\mathcal{C}}$ are semi-algebraic sets, and since $\mathbf{C}_{\mathcal{S} + \mathcal{C}} = \mathbf{C}_{\mathcal{S}} + \mathbf{C}_{\mathcal{C}}$; the result follows from Corollary 1.4.2 in Appendix 1.4.2 of Chapter 1. Hence, from Step II, there exists a generic semi-algebraic subset G of $\mathbf{C}_{\mathcal{S} + \mathcal{C}}$ such that for every $y \in G$, the game u^y has an odd number of Nash equilibria.

Now, consider such a subset G of $\mathbf{C}_{\mathcal{S} + \mathcal{C}}$ and define $G' = \Pi_{-c}(G)$. In particular, note that for every $y \in G'$, the game u^y has an odd number of Nash equilibria (constant parts are not relevant when one deals with Nash equilibria). Finally, observe that G' is generic in $\mathbf{C}_{\mathcal{S}}$, from the following lemma applied to $p = m$, $E_1 = \mathbf{C}_{\mathbb{R}_{\delta}[x]_{-c}}$, $E_2 = S_2 = \mathbf{C}_{\mathcal{C}}$, $S_1 = \mathbf{C}_{\mathcal{S}}$, and $T_0 = G$.

Lemma 2.4.2. *Let E_1, E_2 be two linear subspaces of \mathbb{R}^p ($p \in \mathbb{N}$) such that E_1 and E_2 are in direct sum, and let Π_{E_1} be the linear projection from \mathbb{R}^p to E_1 . Moreover, let S_1 (resp. S_2) be a semi-algebraic subset of E_1 (resp. E_2). If T_0 is a generic subset of $S_1 + S_2$, then $\Pi_{E_1}(T_0)$ is a generic subset of S_1 .*

Proof. Recall that from Proposition 1.4.8 (in Appendix 1.4.2 of Chapter 1), $\Pi_{E_1}(T_0)$ is semi-algebraic.

Now, consider a generic subset T_0 of $S_1 + S_2$, and suppose that $\Pi_{E_1}(T_0)$ is not generic in S_1 .

First, we prove that $\Pi_{E_1}(T_0)$ is open in S_1 (which will imply that the dimension of $S_1 \setminus \Pi_{E_1}(T_0)$ is equal to the dimension of S_1). To do so, consider $x_1 \in \Pi_{E_1}(T_0)$. We show that there exists a neighborhood of x_1 in S_1 which is included in $\Pi_{E_1}(T_0)$. By definition of $\Pi_{E_1}(T_0)$, there exists $t \in T_0$ such that $x_1 = \Pi_{E_1}(t)$, and by definition of T_0 , there exists $x_2 \in S_2$ such that $t = x_1 + x_2$ (fix such elements $t \in T_0$ and $x_2 \in S_2$). Since T_0 is open in $S_1 + S_2$ (from T_0 being generic in $S_1 + S_2$), there exists a neighborhood V^t of t in $S_1 + S_2$ (now fixed) which is included in T_0 . Now, consider the map $f : (y_1, y_2) \in S_1 \times S_2 \mapsto y_1 + y_2 \in S_1 + S_2$. Since f is a continuous map, for every neighborhood V of $t = f(x_1, x_2)$ in $S_1 + S_2$, there exists a neighborhood U^t of (x_1, x_2) in $S_1 \times S_2$ such that $f(U^t) \subset V$ (by definition); consider $V^t \ni t$, and consider such a neighborhood U^t of (x_1, x_2) in $S_1 \times S_2$. Because for every $(y_1, y_2) \in S_1 \times S_2$, the set of all subsets of $S_1 \times S_2$ of the form $V^{y_1} \times V^{y_2}$ (where V^{y_1} is a neighborhood of y_1 in S_1 and V^{y_2} is a neighborhood of y_2 in S_2) forms a neighborhood basis of (y_1, y_2) for the topology on $S_1 \times S_2 \subset E_1 \times E_2$, there exists some neighborhoods V^{x_1} of x_1 in S_1 and V^{x_2} of x_2 in S_2 such that $V^{x_1} \times V^{x_2} \subset U^t$, which implies that $f(V^{x_1} \times V^{x_2}) \subset f(U^t) \subset V^t$ (i.e. $V^{x_1} + V^{x_2} \subset V^t$). In particular, note that V^{x_1} is a neighborhood of x_1 in S_1 which is included in $\Pi_{E_1}(T_0)$ (for every $x'_1 \in V^{x_1}$, $x'_1 + x_2 \in V^{x_1} + V^{x_2} \subset V^t \subset T_0$, which implies that $x'_1 \in \Pi_{E_1}(T_0)$).

Now, since $\Pi_{E_1}(T_0)$ is not generic in S_1 and since $\Pi_{E_1}(T_0)$ is open in S_1 , one obtains that $\dim(S_1 \setminus \Pi_{E_1}(T_0)) = \dim(S_1)$, thus that

$$\dim((S_1 \setminus \Pi_{E_1}(T_0)) + S_2) = \dim((S_1 \setminus \Pi_{E_1}(T_0)) \times S_2) = \dim(S_1 \times S_2) = \dim(S_1 + S_2),$$

since, by assumption, E_1 and E_2 are in direct sum, and from Corollary 1.4.5 (in Appendix 1.4.2 of Chapter 1). However, remark that

$$(S_1 \setminus \Pi_{E_1}(T_0)) + S_2 \subset (S_1 + S_2) \setminus T_0$$

(for every $x_1 + x_2 \in (S_1 \setminus \Pi_{E_1}(T_0)) + S_2$, $x_1 + x_2 \in (S_1 + S_2) \setminus T_0$, otherwise $x_1 \in \Pi_{E_1}(T_0)$, which directly leads to a contradiction), which implies that

$$\dim(S_1 + S_2) = \dim((S_1 \setminus \Pi_{E_1}(T_0)) + S_2) \leq \dim((S_1 + S_2) \setminus T_0) \leq \dim(S_1 + S_2),$$

from Corollary 1.4.4 (in Appendix 1.4.2 of Chapter 1). This contradicts the genericity of T_0 in $S_1 + S_2$. \square

This step ends the proof of the oddness theorem. \square

Chapter 3

Unknottedness of the Graph of Pairwise Stable Networks and Dynamics in Networks

\mathbb{G}	Set of weighted networks on N
$\mathcal{P}_{\mathcal{V}}$	Graph of pairwise stable networks associated to the set \mathcal{V} of societies
$\pi_{\mathbb{G}}$	Projection from $\mathcal{P}_{\mathcal{V}}$ to \mathbb{G} , where \mathcal{V} is a set of societies
\mathcal{F}_i	Set of continuous payoff functions of agent i which are concave in g_{ij} and with continuous first-order derivative with respect to the ij -th variable, for every $j \neq i$ ($\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$)
\mathcal{A}_i	Set of payoff functions of agent i which are affine in $(g_{ij})_{j \neq i}$ and which only depend on those weights ($\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$)
$\eta_{\mathcal{R}}$	Homeomorphism from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} of structure theorem, where \mathcal{R} is a \mathcal{A} -regular set of societies (for every $(v, g) \in \mathcal{P}_{\mathcal{R}}$, $\eta_{\mathcal{R}}(v, g) = (\bar{u}_i^{v, g})_{i \in N}$, where for every $i \in N$, $\bar{u}_i^{v, g} = v_i + h_i^g[v] + l_i^g$)
$\rho_{\mathcal{R}}$	Inverse of $\eta_{\mathcal{R}}$, where \mathcal{R} is a \mathcal{A} -regular set of societies (for every $v \in \mathcal{R}$, $\rho_{\mathcal{R}}(v) = ((\underline{u}_i^v)_{i \in N}, (g_{ij}^v)_{ij \in L})$, where for every $i \in N$, $\underline{u}_i^v = v_i - h_i^g[v] - l_i^g$)
$\mathcal{E}_{\mathcal{V}}$	Initial ambient space associated to \mathcal{V} : cartesian product of \mathcal{V} and \mathbb{G} , where $\mathcal{V} \subset \mathcal{F}$ is a subset of own-weights \mathcal{C}^1 concave societies
\mathcal{V}^0	Trivial copy of \mathcal{V} : cartesian product of \mathcal{V} and $\{g^0\}$, where $\mathcal{V} \subset \mathcal{F}$ is a subset of own-weights \mathcal{C}^1 concave societies and $g^0 \in \mathbb{G}$
$\mathcal{E}_{\mathcal{V}}^{\varepsilon}$	(ε) -ambient space associated to a subset \mathcal{V} of own-weights \mathcal{C}^1 concave societies, where $\varepsilon \in (0, +\infty)$ ($\mathcal{E}_{\mathcal{V}}^{\varepsilon} = \mathcal{V} \times \mathbb{G}^{\varepsilon}$, where $\mathbb{G}^{\varepsilon} = [-\varepsilon, 1 + \varepsilon]^L$); if there is no ambiguity, then $\tilde{\mathcal{E}}_{\mathcal{V}} = \mathcal{E}_{\mathcal{V}}^{\varepsilon}$ and $\tilde{\mathbb{G}} = \mathbb{G}^{\varepsilon}$
$\kappa_{\mathcal{R}}^0$	Unknot associated to a \mathcal{A} -regular set \mathcal{R} of societies and g^0 , where $g^0 \in \mathbb{G}$ ($\kappa_{\mathcal{R}}^0 : v \in \mathcal{R} \mapsto (v, g^0) \in \tilde{\mathcal{E}}_{\mathcal{R}}$, $\text{im}(\kappa_{\mathcal{R}}^0) = \mathcal{R}^0$)
$\kappa_{\rho_{\mathcal{R}}}$	Knot of pairwise stable networks associated to a \mathcal{A} -regular set \mathcal{R} of societies and $\rho_{\mathcal{R}}$, ($\kappa_{\rho_{\mathcal{R}}} : v \in \mathcal{R} \mapsto \rho_{\mathcal{R}}(v) \in \tilde{\mathcal{E}}_{\mathcal{R}}$, $\text{im}(\kappa_{\rho_{\mathcal{R}}}) = \mathcal{P}_{\mathcal{R}}$)
$\theta_{\mathcal{R}}$	Ambient isotopy between $\kappa_{\rho_{\mathcal{R}}}$ and $\kappa_{\mathcal{R}}^0$ of unknottedness theorem, where \mathcal{R} is a \mathcal{A} -regular set of societies
$\mathbf{Z}(D_v), \mathbf{Z}(D)$	$\mathbf{Z}(D) = \{(v, g) \in \mathcal{E}_{\mathcal{V}} : g \in \mathbf{Z}(D_v)\}$, with $\mathbf{Z}(D_v)$ being the set of zeros of the vector field $D_v = D(v, \cdot)$, where D is a network dynamic on some subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies

Table 3.1: Table of notations of Chapter 3

This chapter is based on a preprint written by Julien Fixary [20].

3.1 Introduction

In this last chapter, we provide our *unknottedness theorem* and our *dynamics equivalence theorem* in network formation theory. The chapter is organized as follows: (i) in Subsection 3.2.1 (of Section 3.2), we first recall some definitions and notations introduced in Chapter 1 - in particular, we provide some reminders about the graph of pairwise stable networks (associated to any set of societies), about \mathcal{A} -regularity, and we recall our structure theorem (Theorem 3.2.1); (ii) in Subsection 3.2.2, we present our unknottedness theorem (Theorem 3.2.2); (iii) in Subsection 3.3.1 (of Section 3.3), we introduce the notions of *network dynamic* and of *extended network dynamic*; (iv) in Subsection 3.3.2, we present our dynamics equivalence theorem (Theorem 3.3.1); (v) in Subsection 3.3.3, we provide some consequences of the dynamics equivalence theorem - in particular, we present our indices equality theorem (Corollary 3.3.1); (vi) in Section 3.4 (Appendix), we provide first the necessary reminders about elementary notions of knot theory, about vector bundles and about differential geometry (Subsection 3.4.1 and Subsection 3.4.2), and we provide next the proofs of unknottedness theorem (Subsection 3.4.3), of dynamics equivalence theorem (Subsection 3.4.4) and of indices equality theorem (Subsection 3.4.5).

3.2 Unknottedness of the graph of pairwise stable networks

3.2.1 The graph of pairwise stable networks and \mathcal{A} -regular sets of societies

First, we recall some definitions and notations from network formation theory introduced in Chapter 1.

Definition 3.2.1. A *set of agents* is a finite set N such that $\text{card}(N) \geq 2$. For every set N of agents, the set $L = \{\{i, j\} : (i, j) \in N \times N, i \neq j\}$ is called the *set of links (on N)* and the set $\mathbb{G} = [0, 1]^L$ is called the *set of (weighted) networks (on N)*. Furthermore, the vector space \mathbb{R}^L (with its usual operations) is endowed with the Euclidean norm, i.e.

$$\|\cdot\| : g \in \mathbb{R}^L \mapsto \sqrt{\sum_{ij \in L} g_{ij}^2} \in \mathbb{R},$$

and \mathbb{G} is endowed with the induced topology. A (*weighted*) *society* is a couple $(N, (v_i)_{i \in N})$, where for every $i \in N$, the map v_i is called the *payoff function of agent i* . For every set N of agents, the set of all societies whose the set of agents is equal to N can be identified to the set

$$\text{Soc} = \{N \rightarrow \{\mathbb{G} \rightarrow \mathbb{R}\}\}.$$

Throughout this chapter, we consider a fixed set N of agents.

Notations. Every link $\{i, j\} \in L$ is denoted ij . For every network $g \in \mathbb{G}$ and every link $ij \in L$, $g(\{i, j\})$ is denoted g_{ij} and is called the *weight associated to ij (in g)*. For every link $ij \in L$, $L_{-ij} = L \setminus ij$ and $\mathbb{G}_{-ij} = [0, 1]^{L_{-ij}}$. For every link $ij \in L$, every $g_{-ij} = (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$ and every $w \in [0, 1]$, $g' = (w, g_{-ij}) \in \mathbb{G}$ is the network defined by $g'_{kl} = g_{kl}$, for every $kl \neq ij$, and $g'_{ij} = w$. For every network $g \in \mathbb{G}$ and every link $ij \in L$, $g_{-ij} = (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$.

Definition 3.2.2. Let $v \in \text{Soc}$ be a society. A network $g \in \mathbb{G}$ is *pairwise stable (with respect to v)* if for every $ij \in L$, the two following conditions hold:

1. For every $w \in [0, g_{ij})$, $v_i(w, g_{-ij}) \leq v_i(g)$ and $v_j(w, g_{-ij}) \leq v_j(g)$.
2. For every $w \in (g_{ij}, 1]$, $v_i(w, g_{-ij}) \leq v_i(g)$ or $v_j(w, g_{-ij}) \leq v_j(g)$.

For every set $\mathcal{V} \subset \text{Soc}$ of societies, the *pairwise stable networks correspondence associated to \mathcal{V}* is the correspondence

$$\left\{ \begin{array}{l} \Psi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbb{G} \\ v \mapsto \{g \in \mathbb{G} : g \text{ is pairwise stable with respect to } v\} \end{array} \right. .$$

The graph of the pairwise stable networks correspondence associated to \mathcal{V} is called the *graph of pairwise stable networks associated to \mathcal{V}* and is denoted $\mathcal{P}_{\mathcal{V}}$, i.e.

$$\mathcal{P}_{\mathcal{V}} = \text{Gr}(\Psi_{\mathcal{V}}) = \{(v, g) \in \mathcal{V} \times \mathbb{G} : g \text{ is pairwise stable with respect to } v\}.$$

The projection from $\mathcal{P}_{\mathcal{V}}$ to \mathcal{V} is denoted $\pi_{\mathcal{V}}$ and the projection from $\mathcal{P}_{\mathcal{V}}$ to \mathbb{G} is denoted $\pi_{\mathbb{G}}$ (by abuse of notation).

Definition 3.2.3. For every $i \in N$,

$$\mathcal{C}_i = \{v_i \in \mathcal{F}(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \text{ is concave}\},$$

$$\mathcal{D}_i = \{v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \partial_{ij} v_i \text{ exists and is continuous}\},$$

where for every $v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R})$ and every $j \neq i$,

$$\partial_{ij}v_i : g = (g_{ij}, g_{-ij}) \in \mathbb{G} \mapsto \frac{\partial v_i}{\partial g_{ij}}(g) \in \mathbb{R},$$

and

$$\mathcal{F}_i = \mathcal{C}_i \cap \mathcal{D}_i.$$

The set

$$\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$$

is called the *set of own-weights \mathcal{C}^1 concave societies*. For every $i \in N$, the vector space \mathcal{D}_i (with its usual operations) is endowed with the following norm:

$$\|\cdot\|_i : v_i \in \mathcal{D}_i \mapsto \max\{\max\{\|v_i\|_\infty, \|\partial_{ij}v_i\|_\infty\} : j \neq i\} \in \mathbb{R},$$

where for every $v_i \in \mathcal{D}_i$ and every $j \neq i$, $\|v_i\|_\infty = \sup_{g \in \mathbb{G}} |v_i(g)|$ and $\|\partial_{ij}v_i\|_\infty = \sup_{g \in \mathbb{G}} |\partial_{ij}v_i(g)|$. Furthermore, any subset of \mathcal{D}_i is endowed with the induced topology.

Definition 3.2.4. For every $i \in N$,

$$\mathcal{A}_i = \{g \in \mathbb{G} \mapsto \sum_{j \neq i} \alpha_{ij} g_{ij} + c \in \mathbb{R} : \forall j \neq i, \alpha_{ij} \in \mathbb{R}, c \in \mathbb{R}\}.$$

Moreover, $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$.

We now recall the notion of \mathcal{A} -regular set of societies which plays also an important role in this chapter.

Definition 3.2.5. A set $\mathcal{R} \subset \text{Soc}$ of societies is \mathcal{A} -regular if the two following conditions hold:

1. (**Concavity**). $\mathcal{R} \subset \mathcal{F}$.
2. (**\mathcal{A} -invariance**). $\mathcal{R} + \mathcal{A} = \mathcal{R}$.

We also recall our structure theorem (see Theorem 1.2.1 in Chapter 1).¹

Theorem 3.2.1. *For every \mathcal{A} -regular set \mathcal{R} of societies, the projection $\pi_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ is properly homotopic to some homeomorphism $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$.*

¹To be precise, this result was stated for \mathcal{A} -regular set of societies of the form $\prod_{i \in N} \mathcal{R}_i$, where for every $i \in N$, $\mathcal{R}_i \subset \mathcal{F}(\mathbb{G}, \mathbb{R})$. Nevertheless, Theorem 1.2.1 holds in fact for any \mathcal{A} -regular set of societies.

Recall that for every \mathcal{A} -regular set \mathcal{R} of societies, the homeomorphism $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ of Theorem 3.2.1 is built in the following way: consider a network $g^0 \in \mathbb{G}$ and define

$$\begin{cases} \eta_{\mathcal{R}} : & \mathcal{P}_{\mathcal{R}} & \rightarrow & \mathcal{R} \\ & (v, g) & \mapsto & \bar{u}^{v,g} \end{cases} ,$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\bar{u}_i^{v,g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}. \quad (3.1)$$

The inverse of $\eta_{\mathcal{R}}$ is defined as

$$\begin{cases} \rho_{\mathcal{R}} : & \mathcal{R} & \rightarrow & \mathcal{P}_{\mathcal{R}} \\ & v & \mapsto & (\underline{u}^v, g^v) \end{cases} ,$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\underline{u}_i^v(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0)) (\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} g_{ij}^v \gamma_{ij}, \quad (3.2)$$

and where for every link $ij \in L$, $g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\}$, with $w_{i,j}^v \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_i[v] : & [0, 1] & \rightarrow & \mathbb{R} \\ & w & \mapsto & v_i(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases}$$

and $w_{j,i}^v \in [0, 1]$ being the unique maximizer of the strictly concave function

$$\begin{cases} q_j[v] : & [0, 1] & \rightarrow & \mathbb{R} \\ & w & \mapsto & v_j(w, g_{-ij}^0) - \frac{w^2}{2} \end{cases} .$$

In order to simplify the rest of this chapter, for every agent $i \in N$, every network $g^0 \in \mathbb{G}$, and every $(v, g) \in \mathcal{R} \times \mathbb{G}$, define the maps

$$\begin{cases} h_i^g[v] : & \mathbb{G} & \rightarrow & \mathbb{R} \\ & \gamma & \mapsto & \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) \end{cases}$$

and

$$\begin{cases} l_i^g : & \mathbb{G} & \rightarrow & \mathbb{R} \\ & \gamma & \mapsto & \sum_{j \neq i} \gamma_{ij} g_{ij} \end{cases} ,$$

so that

$$\bar{u}_i^{v,g} = v_i + h_i^g[v] + l_i^g \quad \text{and} \quad \underline{u}_i^v = v_i - h_i^g[v] - l_i^g.$$

3.2.2 Unknottedness theorem

Notations. Let $\mathcal{V} \subset \mathcal{F}$ be a subset of own-weights \mathcal{C}^1 concave societies. $\mathcal{E}_{\mathcal{V}} = \mathcal{V} \times \mathbb{G}$. For every network $g^0 \in \mathbb{G}$, $\mathcal{V}^0 = \mathcal{V} \times \{g^0\}$.

In this section, for every subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies, we consider an “enlargement” of the space $\mathcal{E}_{\mathcal{V}} = \mathcal{V} \times \mathbb{G}$ which we call *ambient space associated to \mathcal{V}* .

Definition 3.2.6. Let $\varepsilon \in (0, +\infty)$ and $\mathcal{V} \subset \mathcal{F}$ be a subset of own-weights \mathcal{C}^1 concave societies. The (ε) -ambient space (associated to \mathcal{V}) is the topological space $\mathcal{E}_{\mathcal{V}}^{\varepsilon} = \mathcal{V} \times \mathbb{G}^{\varepsilon}$, where $\mathbb{G}^{\varepsilon} = [-\varepsilon, 1 + \varepsilon]^L$. If there is no ambiguity, $\mathcal{E}_{\mathcal{V}}^{\varepsilon}$ (resp. \mathbb{G}^{ε}) is also denoted $\tilde{\mathcal{E}}_{\mathcal{V}}$ (resp. $\tilde{\mathbb{G}}$).

Throughout the rest of this chapter, consider a fixed real number $\varepsilon > 0$.

Definition 3.2.7. Let $g^0 \in \mathbb{G}$ and \mathcal{R} be a \mathcal{A} -regular set of societies. The *unknot associated to \mathcal{R}* (and g^0) is the topological embedding

$$\left\{ \begin{array}{l} \kappa_{\mathcal{R}}^0 : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}} \\ v \mapsto (v, g^0) \end{array} \right. .$$

Moreover, the *knot of pairwise stable networks associated to \mathcal{R}* and $\rho_{\mathcal{R}}$ is the topological embedding

$$\left\{ \begin{array}{l} \kappa_{\rho_{\mathcal{R}}} : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}} \\ v \mapsto \rho_{\mathcal{R}}(v) \end{array} \right. .$$

By analogy to knot theory (a branch of topology which studies topological embeddings of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ into \mathbb{R}^3 or \mathbb{S}^3 , called *knots*), the term “unknot” (in the previous definition) comes from the trivial knot $(x, y) \in \mathbb{S}^1 \mapsto (x, y, 0) \in \mathbb{R}^3$ which is itself originally called the *unknot*. In our case, the idea is that for every network $g^0 \in \mathbb{G}$ and every \mathcal{A} -regular set \mathcal{R} of societies, if we consider the set of all topological embeddings of \mathcal{R} into the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$ (instead of those mentioned before), then \mathcal{R} can be seen as trivially embedded by $\kappa_{\mathcal{R}}^0$ into $\tilde{\mathcal{E}}_{\mathcal{R}}$. Figure 3.1 provides a simple representation of the unknot $\kappa_{\mathcal{R}}^0$ associated to \mathcal{R} (for $g^0 \in \mathbb{G}$), and of the knot $\kappa_{\rho_{\mathcal{R}}}$ of pairwise stable networks associated to \mathcal{R} and $\rho_{\mathcal{R}}$, where \mathcal{R} is an arbitrary \mathcal{A} -regular set of societies.

Recall that for every topological spaces X, Y and every topological embeddings $e^1, e^2 : X \rightarrow Y$, an *ambient isotopy between e^1 and e^2* is a continuous map $\theta : [0, 1] \times Y \rightarrow Y$ such that $\theta(0, \cdot) = \text{id}_Y$, $\theta(1, \cdot) \circ e^1 = e^2$ and for every $t \in [0, 1]$, $\theta(t, \cdot)$ is a homeomorphism (see Appendix 3.4.1 for some

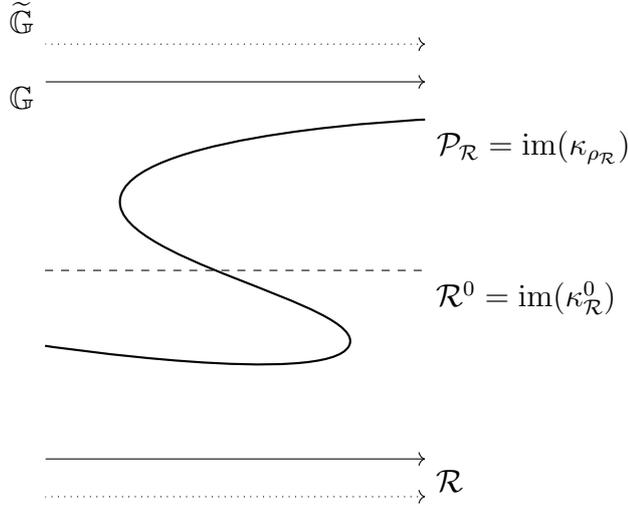


Figure 3.1: $\mathcal{P}_{\mathcal{R}} = \text{im}(\kappa_{\rho_{\mathcal{R}}})$ (in thick line) and $\mathcal{R}^0 = \text{im}(\kappa_{\mathcal{R}}^0)$ (in dashed line)

reminders). With reference to Theorem 3.2.1 (structure theorem), the following result provides more insights on the topological structure of the graph of pairwise stable networks associated to any \mathcal{A} -regular set of societies.

Theorem 3.2.2. (*Unknottedness theorem*)

For every \mathcal{A} -regular set \mathcal{R} of societies, the knot $\kappa_{\rho_{\mathcal{R}}} : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} and $\rho_{\mathcal{R}}$ is ambient isotopic to the unknot $\kappa_{\mathcal{R}}^0 : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ associated to \mathcal{R} within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$, through an ambient isotopy $\theta_{\mathcal{R}}$ which does not deform the boundary of $\tilde{\mathcal{E}}_{\mathcal{R}}$.

Consider a \mathcal{A} -regular set \mathcal{R} of societies. Again, by analogy to knot theory, the term “unknottedness” comes from the fact that $\kappa_{\rho_{\mathcal{R}}} : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ can be seen as “equivalent” to the unknot $\kappa_{\mathcal{R}}^0 : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$. Intuitively, unknottedness theorem states that the graph $\mathcal{P}_{\mathcal{R}} = \text{im}(\kappa_{\rho_{\mathcal{R}}})$ of pairwise stable networks associated to \mathcal{R} can be continuously deformed (using $\theta_{\mathcal{R}}$) into a trivial copy $\mathcal{R}^0 = \text{im}(\kappa_{\mathcal{R}}^0)$ of the space \mathcal{R} of societies within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$; the key idea being that $\theta_{\mathcal{R}}$ does not deform only $\mathcal{P}_{\mathcal{R}}$, but also the entire space $\tilde{\mathcal{E}}_{\mathcal{R}}$.² Note that unknottedness theorem is stronger than the first part of structure theorem (Theorem 3.2.1): for every \mathcal{A} -regular set \mathcal{R} of societies, since $\kappa_{\rho_{\mathcal{R}}} : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ is ambient isotopic to $\kappa_{\mathcal{R}}^0 : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ within $\tilde{\mathcal{E}}_{\mathcal{R}}$, then $\kappa_{\rho_{\mathcal{R}}}(\mathcal{R}) = \mathcal{P}_{\mathcal{R}}$ and

²In knot theory, the notion of homeomorphism is not sufficient in order to classify knots since by definition, the image of \mathbb{S}^1 by any knot is homeomorphic to \mathbb{S}^1 itself. The (stronger) notion of ambient isotopy is more useful since it allows to distinguish in a sharper way a knot from the others (see Appendix 3.4.1 for some reminders).

$\kappa_{\mathcal{R}}^0(\mathcal{R}) = \mathcal{R}^0$ are homeomorphic, where \mathcal{R}^0 is itself homeomorphic to \mathcal{R} .

Before providing a sketch of proof of Theorem 3.2.2, let us come back in a slightly more precise way on two particular works in game theory that we already mentioned in the Introduction (of this thesis) and at the beginning of Chapter 1: Demichelis and Germano's paper [17], and Predtetchinski's paper [40]. These works are indeed very linked to the results of this chapter: Demichelis and Germano provided an unknottedness theorem in the case of mixed Nash equilibria of finite strategic-form games, and Predtetchinski provided (in addition to his structure theorem) a generalization of Demichelis-Germano's unknottedness theorem in the case of Nash equilibria of own-strategy \mathcal{C}^1 concave games. Furthermore, in the same paper, Demichelis and Germano proved several important results about the notion of *Nash dynamic* (these results will be very useful for our next section).

Sketch of proof

The full proof is provided in Appendix 3.4.3.

Considering a fixed network $g^0 \in \mathbb{G}$ and a fixed \mathcal{A} -regular set \mathcal{R} of societies, the demonstration goes as follows:

1. In Step I, we show that the knot $\kappa_{\rho_{\mathcal{R}}} : v \in \mathcal{R} \mapsto \rho_{\mathcal{R}}(v) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} and $\rho_{\mathcal{R}}$, and the topological embedding $e^{\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}} : v \in \mathcal{R} \mapsto (v, (\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}})(v)) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ are ambient isotopic within $\tilde{\mathcal{E}}_{\mathcal{R}}$, through an ambient isotopy $\theta_{\mathcal{R}}^1$. The idea is to continuously deform the graph $\mathcal{P}_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} into the graph of the continuous map $\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}} : \mathcal{R} \rightarrow \mathbb{G}$, within the ambient space. Figure 3.2 provides an illustration of this proof step.
2. In Step II, for every continuous map $f : \mathcal{R} \rightarrow \mathbb{G}$, we show that the topological embedding $e^f : v \in \mathcal{R} \mapsto (v, f(v)) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ and the unknot $\kappa_{\mathcal{R}}^0 : v \in \mathcal{R} \mapsto (v, g^0) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ are ambient isotopic within $\tilde{\mathcal{E}}_{\mathcal{R}}$, through an ambient isotopy $\theta_{\mathcal{R}}^{2,f}$. The idea is to continuously deform the graph of any continuous map from \mathcal{R} to \mathbb{G} into the trivial copy \mathcal{R}^0 of \mathcal{R} , within the ambient space. Figure 3.3 provides an illustration of this proof step.
3. In Step III, using ambient isotopies $\theta_{\mathcal{R}}^1$ of Step I and $\theta_{\mathcal{R}}^{2,f}$ of Step II (for $f = \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}$), we construct, within $\tilde{\mathcal{E}}_{\mathcal{R}}$, an ambient isotopy $\theta_{\mathcal{R}}$ between the knot $\kappa_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} and $\rho_{\mathcal{R}}$, and the unknot $\kappa_{\mathcal{R}}^0$, which ends the proof of this theorem. A major feature of $\theta_{\mathcal{R}}$ is that it does not deform the boundary of the ambient

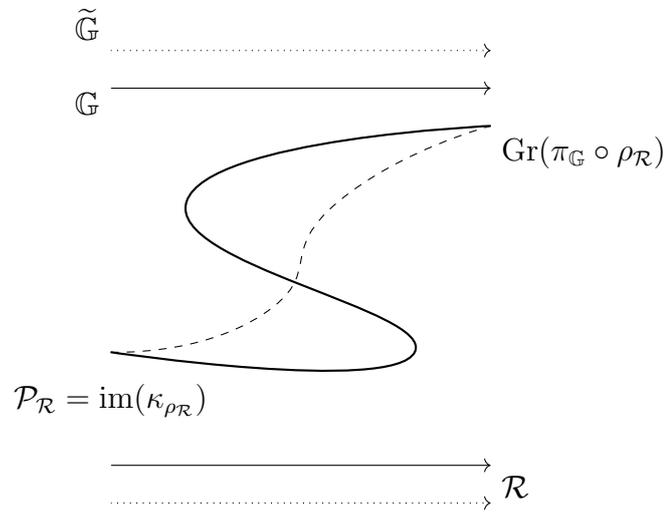


Figure 3.2: Deformation of $\mathcal{P}_{\mathcal{R}}$ (in thick line) into $\text{Gr}(\pi_{\mathcal{G}} \circ \rho_{\mathcal{R}})$ (in dashed line) within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}} = \mathcal{R} \times \tilde{\mathcal{G}}$

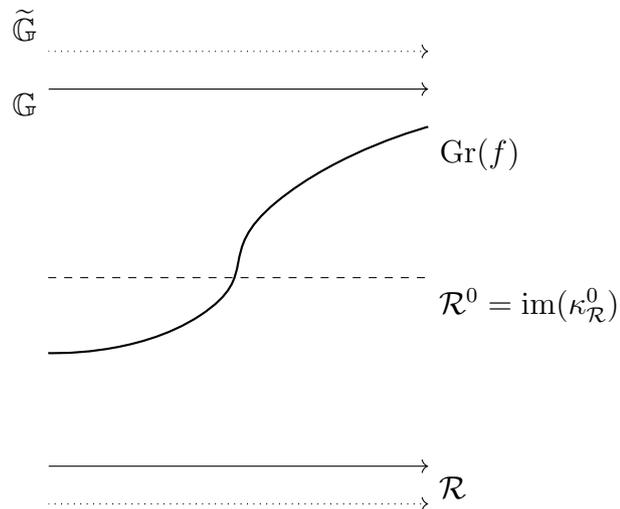


Figure 3.3: Deformation of $\text{Gr}(f)$ (in thick line) into \mathcal{R}^0 (in dashed line) within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}} = \mathcal{R} \times \tilde{\mathcal{G}}$

space $\tilde{\mathcal{E}}_{\mathcal{R}}$; this will play an important role in the proof of our dynamics equivalence theorem (Theorem 3.3.1 in Section 3.3.2).

3.3 Dynamics in networks

3.3.1 Network dynamics and extended network dynamics

In this section, we introduce the notions of *network dynamic* and of *extended network dynamic* on an arbitrary subset \mathcal{V} of own-weights \mathcal{C}^1 concave societies (i.e. an arbitrary subset of \mathcal{F}). Briefly, a network dynamic (resp. an extended network dynamic) on \mathcal{V} is a family of vector fields $(D_v)_{v \in \mathcal{V}}$ on \mathbb{G} (resp. $(\tilde{D}_v)_{v \in \mathcal{V}}$ on $\tilde{\mathbb{G}}$) such that for every $v \in \mathcal{V}$, the set of zeros of D_v (resp. of \tilde{D}_v) coincide with the set of pairwise stable networks of v .³

Definition 3.3.1. Let $\mathcal{V} \subset \mathcal{F}$ be a subset of own-weights \mathcal{C}^1 concave societies:

- A *network dynamic on \mathcal{V}* is a continuous map $D : \mathcal{E}_{\mathcal{V}} \rightarrow \text{T}\mathbb{G}$ such that the two following conditions hold:
 1. For every $v \in \mathcal{V}$, $D_v = D(v, \cdot)$ is a vector field on \mathbb{G} .
 2. $\mathbf{Z}(D) = \mathcal{P}_{\mathcal{V}}$, where $\mathbf{Z}(D) = \{(v, g) \in \mathcal{E}_{\mathcal{V}} : g \in \mathbf{Z}(D_v)\}$, with $\mathbf{Z}(D_v)$ being the set of zeros of the vector field D_v .
- An *extended network dynamic on \mathcal{V}* is a continuous map $\tilde{D} : \tilde{\mathcal{E}}_{\mathcal{V}} \rightarrow \text{T}\tilde{\mathbb{G}}$ such that the two following conditions hold:
 1. For every $v \in \mathcal{V}$, $\tilde{D}_v = \tilde{D}(v, \cdot)$ is a vector field on $\tilde{\mathbb{G}}$.
 2. $\mathbf{Z}(\tilde{D}) = \mathcal{P}_{\mathcal{V}}$, where $\mathbf{Z}(\tilde{D}) = \{(v, g) \in \tilde{\mathcal{E}}_{\mathcal{V}} : g \in \mathbf{Z}(\tilde{D}_v)\}$, with $\mathbf{Z}(\tilde{D}_v)$ being the set of zeros of the vector field \tilde{D}_v .

Remark 3.3.1. Remark that \mathbb{G} (resp. $\tilde{\mathbb{G}}$) is a convex subset of \mathbb{R}^L , which implies that it is contractible in \mathbb{R}^L , thus that the tangent bundle $\text{T}\mathbb{G}$ of \mathbb{G} (resp. $\text{T}\tilde{\mathbb{G}}$ of $\tilde{\mathbb{G}}$) is trivial (see Proposition 3.4.1 in Appendix 3.4.1). Consequently, Theorem 3.4.1 (see Appendix 3.4.1) states in particular that any network dynamic D on a subset \mathcal{V} of own-weights \mathcal{C}^1 concave societies can be

³Recall that the tangent bundle $\text{T}\mathbb{G}$ of \mathbb{G} is the disjoint union of all tangent spaces $\text{T}_g\mathbb{G}$ to \mathbb{G} at g ($g \in \mathbb{G}$), and that a vector field on \mathbb{G} is an assignment of a tangent vector in $\text{T}_g\mathbb{G}$ to each network $g \in \mathbb{G}$ (see Appendix 3.4.2 for more details). Moreover, each tangent space $\text{T}_g\mathbb{G}$ ($g \in \mathbb{G}$) can be identified to \mathbb{R}^L , and we will see in what follows that each vector field on \mathbb{G} can be seen as a continuous map from \mathbb{G} to \mathbb{R}^L .

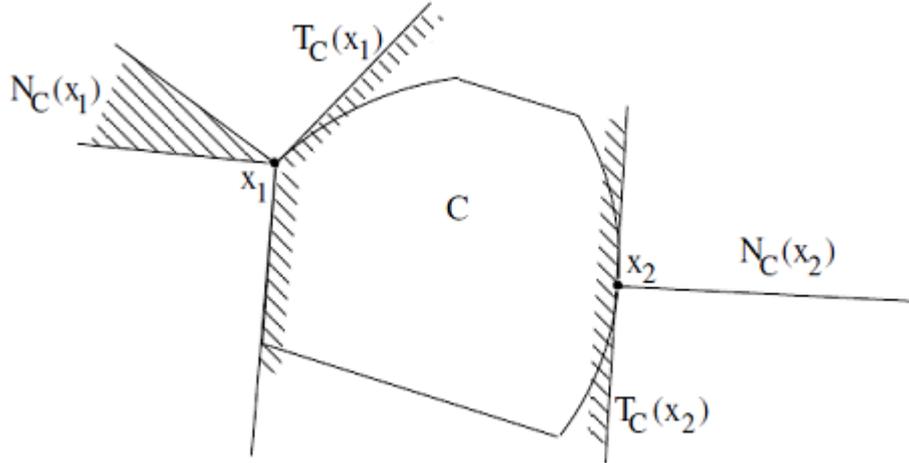


Figure 3.4: *Variational geometry of a convex set; examples of normal cones (and of tangent cones) (Rockafellar-Wets [42], p. 204)*

treated as a continuous map from $\mathcal{E}_{\mathcal{V}}$ to \mathbb{R}^L such that $D^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{V}}$, and that any extended network dynamic \tilde{D} on \mathcal{V} can be treated as a continuous map from $\tilde{\mathcal{E}}_{\mathcal{V}}$ to \mathbb{R}^L such that $\tilde{D}^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{V}}$. Sometimes, this identification will be implicitly or explicitly used in some definitions or in some proofs in order to avoid complex details.

In the following, we introduce the notions of *inward-pointing network dynamic* and of *strongly inward-pointing extended network dynamic*. Before that, recall that for every convex subset C of \mathbb{R}^m ($m \in \mathbb{N}$) and every $x \in C$, the *normal cone to C at x* is defined as

$$N_C(x) = \{y \in \mathbb{R}^m : \forall x' \in C, \langle y, x' - x \rangle \leq 0\}$$

(see Rockafellar-Wets [42], pp. 203-204).

Definition 3.3.2. Let $\mathcal{V} \subset \mathcal{F}$ be a subset of own-weights \mathcal{C}^1 concave societies. A network dynamic D on \mathcal{V} is said to be *inward-pointing* if for every $v \in \mathcal{V}$, every $g \in \partial\mathbb{G}$, and every $x \in N_{\mathbb{G}}(g)$,

$$\langle D_v(g), x \rangle \leq 0.$$

Figure 3.5 provides a “slice” of an inward-pointing network dynamic D for $\text{card}(L) = 2$, and for some subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies and some $v \in \mathcal{V}$.

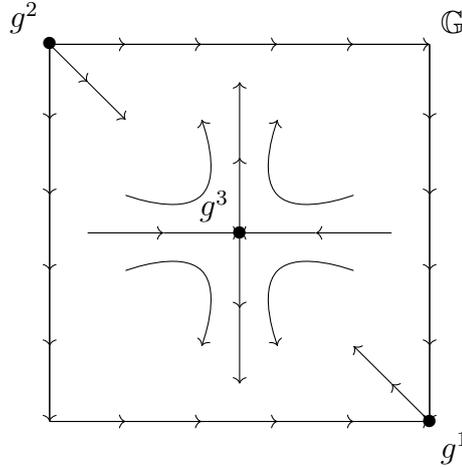


Figure 3.5: “Slice” of D , with $\mathbf{Z}(D_v) = \{g^1, g^2, g^3\}$

Definition 3.3.3. Let $\mathcal{V} \subset \mathcal{F}$ be a subset of own-weights \mathcal{C}^1 concave societies. An extended network dynamic \tilde{D} on \mathcal{V} is said to be *strongly inward-pointing* if for every $v \in \mathcal{V}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\mathbb{G}^{\varepsilon'}}(g) \setminus \{0_{\mathbb{R}^L}\}$,

$$\langle \tilde{D}_v(g), x \rangle < 0.$$

Similarly as above, Figure 3.6 provides a “slice” of a strongly inward-pointing extended network dynamic \tilde{D} for $\text{card}(L) = 2$, and for some subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies and some $v \in \mathcal{V}$.

The two notions of inward-pointing network dynamic and of strongly inward-pointing extended network dynamic are in fact quite natural. If one wants to deal with flows associated to vector fields on \mathbb{G} , then the first notion is worth to consider; at the boundary of \mathbb{G} , we do not want to move “outside”. The second notion aims to precise how an extended network dynamic should “normally” behave on the ambient space $\tilde{\mathcal{E}}_{\mathcal{V}}$ ($\mathcal{V} \subset \mathcal{F}$), which corresponds to an “artificial enlargement” of the space $\mathcal{E}_{\mathcal{V}}$. Indeed, the practical feature of $\tilde{\mathcal{E}}_{\mathcal{V}}$ lies in the fact that it allows us to be able to describe what happens outside \mathbb{G} (especially around isolated zeros on $\partial\mathbb{G}$). However, starting from any element in $\tilde{\mathbb{G}} \setminus \mathbb{G}$, we also want to be sure to move back to \mathbb{G} (by following the path induced by the underlying extended network dynamic); this will be the case if the considered extended network dynamic is strongly inward-pointing.

Example 3.3.1. (Pairwise best-response dynamic)

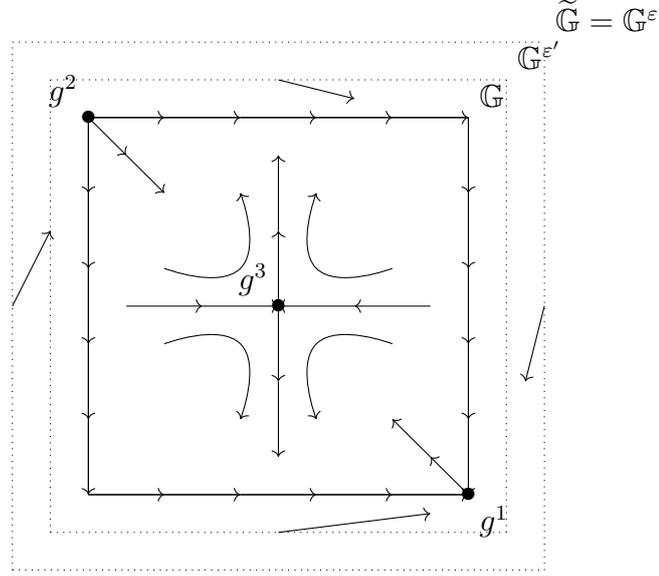


Figure 3.6: “Slice” of \tilde{D} , with $\mathbf{Z}(\tilde{D}_v) = \{g^1, g^2, g^3\}$

For every $i \in N$, let

$$\mathcal{V}_i = \{v_i \in \mathcal{F}_i : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \text{ is strictly concave}\}.$$

Moreover, consider the map

$$D : (v, g) \in \mathcal{V} \times \mathbb{G} \mapsto (\min\{\omega_i(v_i, g_{-ij}), \omega_j(v_j, g_{-ij})\} - g_{ij})_{ij \in L} \in \mathbb{R}^L,$$

where $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$, and where for every $(v, g) \in \mathcal{V} \times \mathbb{G}$, and every $ij \in L$,

$$\omega_i(v_i, g_{-ij}) = \operatorname{argmax}_{w \in [0,1]} v_i(w, g_{-ij})$$

and

$$\omega_j(v_j, g_{-ij}) = \operatorname{argmax}_{w \in [0,1]} v_j(w, g_{-ij}).$$

We show that D is an inward-pointing network dynamic.

First, observe that $D^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{V}}$. Indeed, for every $(v, g) \in \mathcal{V} \times \mathbb{G}$, g is pairwise stable with respect to v if and only if for every $ij \in L$, $g_{ij} = \min\{\omega_i(v_i, g_{-ij}), \omega_j(v_j, g_{-ij})\}$ (from strict concavity of the maps $v_i(\cdot, g_{-ij})$ and $v_j(\cdot, g_{-ij})$). Equivalently, g is pairwise stable with respect to v if and only if for every $ij \in L$, $\min\{\omega_i(v_i, g_{-ij}), \omega_j(v_j, g_{-ij})\} - g_{ij} = 0$, i.e. if and only if $D(v, g) = 0_{\mathbb{R}^L}$.

Now, we show that D is inward-pointing. Let $v \in \mathcal{V}$, $g \in \partial\mathbb{G}$, and $x \in N_{\mathbb{G}}(g) = \{y \in \mathbb{R}^L : \forall \gamma \in \mathbb{G}, \langle y, \gamma - g \rangle \leq 0\}$. By definition,

$$\langle x, \gamma - g \rangle \leq 0$$

for every $\gamma \in \mathbb{G}$. In particular, remark that

$$\langle \hat{\gamma} - g, x \rangle \leq 0,$$

for $\hat{\gamma} = (\min\{\omega_i(v_i, g_{-ij}), \omega_j(v_j, g_{-ij})\})_{ij \in L} \in \mathbb{G}$. Finally, observe that

$$D_v(g) = (\min\{\omega_i(v_i, g_{-ij}), \omega_j(v_j, g_{-ij})\} - g_{ij})_{ij \in L} = (\hat{\gamma}_{ij} - g_{ij})_{ij \in L} = \hat{\gamma} - g.$$

Hence,

$$\langle D_v(g), x \rangle \leq 0.$$

3.3.2 Dynamics equivalence theorem

The second main result of this chapter, called *dynamics equivalence theorem*, states that any two strongly inward-pointing extended network dynamics on an arbitrary \mathcal{A} -regular set \mathcal{R} of societies are homotopic within the set of all extended network dynamics on \mathcal{R} , i.e. through a homotopy $\tilde{H}_{\mathcal{R}}$ such that for every $t \in [0, 1]$, $\tilde{H}_{\mathcal{R}}(t, \cdot, \cdot) : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \text{T}\mathbb{G}$ is also an extended network dynamic on \mathcal{R} .

Theorem 3.3.1. (*Dynamics equivalence theorem*)

For every \mathcal{A} -regular set \mathcal{R} of societies, any two strongly inward-pointing extended network dynamics on \mathcal{R} are homotopic, through a homotopy $\tilde{H}_{\mathcal{R}}$ such that for every $t \in [0, 1]$, $\tilde{H}_{\mathcal{R}}(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} .

Dynamics equivalence theorem is a result which is in the spirit of Demichelis and Germano's previously mentioned paper [17], who provide a similar theorem in game theory, in the case of extended Nash dynamics.⁴

Sketch of proof

The full proof is provided in Appendix 3.4.4.

Consider a fixed network $g^0 \in \mathbb{G}$ and a fixed \mathcal{A} -regular set \mathcal{R} of societies.

⁴Consider a set $I = \{1, \dots, n\}$ of players (for some $n \in \mathbb{N}$) and the family $(X_i)_{i \in I}$ of sets of strategies such that for every $i \in I$, X_i corresponds to the unit $\mu_i - 1$ simplex $\Delta^{\mu_i - 1} \subset \mathbb{R}^{\mu_i}$ (for some $\mu_i \in \mathbb{N}$). Moreover, consider the space $\mathcal{L}^n(X, \mathbb{R}) = \{I \rightarrow L_n(X_1, \dots, X_n, \mathbb{R})\}$ of multilinear games. Observe that the notion of *Nash dynamic* (resp. *extended Nash dynamic*) is analogous to the one of inward-pointing network dynamic on $\mathcal{L}^n(X, \mathbb{R})$ (resp. strongly inward-pointing extended network dynamic on $\mathcal{L}^n(X, \mathbb{R})$). For example, recall that a Nash dynamic is a continuous map $D : \mathcal{L}^n(X, \mathbb{R}) \times X \rightarrow \text{TX}$ (with $X = \prod_{i \in I} X_i$) such that: (i) for every $u \in \mathcal{L}^n(X, \mathbb{R})$, $D(u, \cdot)$ is a vector field on X ; (ii) $\mathbf{Z}(D) = \mathcal{N}_{\mathcal{L}^n(X, \mathbb{R})}$; (iii) for every $u \in \mathcal{L}^n(X, \mathbb{R})$, every $x \in \partial X$, and every $z \in N_X x$, $\langle D_u(x), z \rangle \leq 0$.

Observe that $\tilde{\mathbb{G}}$ is a convex subset of \mathbb{R}^L , which implies that it is contractible. Hence, from Proposition 3.4.1 (see Appendix 3.4.1), $\mathrm{T}\tilde{\mathbb{G}}$ is trivial. Consequently, using Theorem 3.4.1 (see Appendix 3.4.2), any extended network dynamic on \mathcal{R} is treated throughout all this proof as a continuous map $\tilde{D} : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L$ such that $\tilde{D}^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{R}}$, in order to simplify the demonstration.

Now, consider two extended network dynamics \tilde{D}, \tilde{D}' on \mathcal{R} . The required homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D} and \tilde{D}' of Theorem 3.3.1 is built in a two-step process:

$$\left\{ \begin{array}{l} \tilde{H}_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} \tilde{H}_{\mathcal{R}}^1(2t, (v, g)) & \text{if } t \in [0, \frac{1}{2}] \\ \tilde{H}_{\mathcal{R}}^2(2t - 1, (v, g)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array} \right. ,$$

where the maps $\tilde{H}_{\mathcal{R}}^1, \tilde{H}_{\mathcal{R}}^2 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L$ are explained in what follows:

1. The map $\tilde{H}_{\mathcal{R}}^1$ is built in Step I. The idea is to deform the map \tilde{D} into a continuous map $\tilde{D}_{\partial} : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L$ which is equal to \tilde{D} on $\mathcal{E}_{\mathcal{R}} = \mathcal{R} \times \mathbb{G}$, and which is equal to \tilde{D}' on $\mathcal{R} \times \partial\tilde{\mathbb{G}}$, i.e.

$$\tilde{D}_{\partial}|_{\mathcal{E}_{\mathcal{R}}} = \tilde{D}|_{\mathcal{E}_{\mathcal{R}}} \text{ and } \tilde{D}_{\partial}|_{\mathcal{R} \times \partial\tilde{\mathbb{G}}} = \tilde{D}'|_{\mathcal{R} \times \partial\tilde{\mathbb{G}}}$$

(this property will be crucial for the construction of the map $\tilde{H}_{\mathcal{R}}^2$). Then, $\tilde{H}_{\mathcal{R}}^1$ is defined as the straight-line homotopy between \tilde{D} and \tilde{D}_{∂} ; this homotopy is such that for every $t \in [0, 1]$, $\tilde{H}_{\mathcal{R}}^1(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} .

2. The map $\tilde{H}_{\mathcal{R}}^2$ is built from Step II to Step VIII. Now, the idea is to deform the map \tilde{D}_{∂} into the map \tilde{D}' . Briefly:

- In Step II, we want to simplify this construction by using the ambient isotopy $\theta_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ of the unknottedness theorem (Theorem 3.2.2). More precisely, instead of directly deform \tilde{D}_{∂} into \tilde{D}' , our aim is to deform the map $\tilde{\Delta}_{\partial} = \tilde{D}_{\partial} \circ (\theta_{\mathcal{R}}(1, \cdot, \cdot))^{-1}$ into the map $\tilde{\Delta}' = \tilde{D}' \circ (\theta_{\mathcal{R}}(1, \cdot, \cdot))^{-1}$. By doing so, once we find a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$ such that for every $t \in [0, 1]$, $(\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0 (= \mathcal{R} \times \{g^0\})$, then

$$\left\{ \begin{array}{l} \tilde{H}_{\mathcal{R}}^2 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}(1, \cdot, \cdot))(v, g) \end{array} \right.$$

is a homotopy between \tilde{D}_∂ and \tilde{D}' such that for every $t \in [0, 1]$, the map $\tilde{H}_\mathcal{R}^2(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} (by definition of $\theta_\mathcal{R}$).

- From Step III to Step VII, we proceed to a kind of "backand-forth motion" in order to built a homotopy $\tilde{\Phi}_\mathcal{R}$ between $\tilde{\Delta}_\partial$ and $\tilde{\Delta}'$ as mentioned above. The following diagram summarizes this construction:

$$\begin{array}{ccccc} \tilde{\Delta}_\partial & \xrightarrow{\tilde{\Phi}_\mathcal{R}^1} & \tilde{\Delta}^1 & \xrightarrow{\tilde{\Phi}_\mathcal{R}^2} & \tilde{\Delta}^2 \\ \downarrow & & & & \parallel \\ \tilde{\Delta}' & \xleftarrow{\tilde{\Phi}_\mathcal{R}^4} & \hat{\Delta}^1 & \xleftarrow{\tilde{\Phi}_\mathcal{R}^3} & \hat{\Delta}^2 \end{array}$$

Each map of this diagram is detailed in the proof. However, the key point is the equality in the last column of the diagram between

$$\left\{ \begin{array}{l} \tilde{\Delta}^2 : \quad \tilde{\mathcal{E}}_\mathcal{R} \quad \rightarrow \quad \mathbb{R}^L \\ (v, g) \quad \mapsto \quad \begin{cases} \frac{\tilde{\Delta}_\partial(v, \mathbf{g}_\partial^g)}{\|\tilde{\Delta}_\partial(v, \mathbf{g}_\partial^g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \hat{\Delta}^2 : \quad \tilde{\mathcal{E}}_\mathcal{R} \quad \rightarrow \quad \mathbb{R}^L \\ (v, g) \quad \mapsto \quad \begin{cases} \frac{\tilde{\Delta}'(v, \mathbf{g}_\partial^g)}{\|\tilde{\Delta}'(v, \mathbf{g}_\partial^g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right. ,$$

where for every $g \in \tilde{\mathbb{G}} \setminus \{g^0\}$, $\mathbf{g}_\partial^g \in \partial\tilde{\mathbb{G}}$ is detailed in Step IV. Indeed, this equality comes from: (i) the fact that the ambient isotopy $\theta_\mathcal{R}$ does not deform the boundary of the ambient space $\tilde{\mathcal{E}}_\mathcal{R}$; (ii) the property of the map $\tilde{H}_\mathcal{R}^1$ (built in Step I) mentioned before. The map $\tilde{\Phi}_\mathcal{R}$ is then built using the homotopies $(\tilde{\Phi}_\mathcal{R}^k)_{k=1}^4$ (details about these maps are provided in the proof).

3. We conclude about the construction of $\tilde{H}_\mathcal{R}$ in Step VIII.

3.3.3 Some consequences of dynamics equivalence theorem

Indices equality theorem

Recall that for a given vector field V on an arbitrary smooth manifold, and a given isolated zero z of V , the *index of V at z* is an indicator which helps

to quantify the behavior of V around z (i.e. V may circulate around z , it may have a source, a sink, a saddle, etc.). It's formal definition is recalled and illustrated in Appendix 3.4.2. The following corollary of Theorem 3.3.1 establishes a link between the indices of strongly inward-pointing extended network dynamics on \mathcal{R} at any isolated zero (i.e. at any isolated pairwise stable network), where \mathcal{R} is an arbitrary \mathcal{A} -regular set of societies.

Corollary 3.3.1. (*Indices equality theorem*)

Let \mathcal{R} be a \mathcal{A} -regular set of societies. For every strongly inward-pointing extended network dynamics \tilde{D} and \tilde{D}' on \mathcal{R} and every $v \in \mathcal{R}$, the index of \tilde{D}_v at g is equal to the index of \tilde{D}'_v at g , for every isolated point g of $\mathbf{Z}(\tilde{D}_v) = \mathbf{Z}(\tilde{D}'_v)$.

The proof is provided in Appendix 3.4.5.

Index of a zero of a network dynamic

Definition 3.3.4. Let \mathcal{V} be a subset of \mathcal{F} , and D be an inward-pointing network dynamic on \mathcal{V} . A strongly inward-pointing extended network dynamic D^ε on \mathcal{V} is called a (ε -)extension of D to $\tilde{\mathcal{E}}_\mathcal{V}$ if $D^\varepsilon|_{\mathcal{E}_\mathcal{V}} = D$.

The next proposition establishes a link between inward-pointing network dynamics and strongly inward-pointing extended network dynamics.

Proposition 3.3.1. *For every subset $\mathcal{V} \subset \mathcal{F}$ of own-weights \mathcal{C}^1 concave societies, any inward-pointing network dynamic on \mathcal{V} admits an extension to $\tilde{\mathcal{E}}_\mathcal{V}$.*

Proof. Let r be the projection from $\tilde{\mathbb{G}}$ to the closest point in \mathbb{G} (this map is a retraction of $\tilde{\mathbb{G}}$ on \mathbb{G} , i.e. a continuous map such that its composition with the inclusion $\mathbb{G} \hookrightarrow \tilde{\mathbb{G}}$ is the identity map $\text{id}_\mathbb{G}$ on \mathbb{G}), $b : \tilde{\mathbb{G}} \rightarrow [0, 1]$ be a continuous map such that $b(\mathbb{G}) = 1$ and $b(\partial\tilde{\mathbb{G}}) = 0$, and $V : \mathbb{G}^\varepsilon = \tilde{\mathbb{G}} \rightarrow \mathbb{R}^L$ be a (continuous) vector field on \mathbb{G}^ε such that $V(\mathbb{G}) = 0_{\mathbb{R}^L}$ and for every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\mathbb{G}^{\varepsilon'}}(g)$, $\langle V(g), x \rangle < 0$. Consider the map

$$\begin{cases} D^\varepsilon : & \mathcal{E}_\mathcal{V}^\varepsilon & \rightarrow & \mathbb{R}^L \\ & (v, g) & \mapsto & b(g)D(v, r(g)) + V(g) \end{cases} .$$

First, one obtains directly that for every $(v, g) \in \mathcal{E}_\mathcal{V}$,

$$D^\varepsilon(v, g) = b(g)D(v, r(g)) + V(g) = D(v, g).$$

Indeed, since $g \in \mathbb{G}$, then $b(g) = 1$, $r(g) = g$ and $V(g) = 0$, which implies that

$$\mathcal{P}_{\mathcal{V}} \subset (D^\varepsilon)^{-1}(0_{\mathbb{R}^L}). \quad (3.3)$$

Second, observe that $D^\varepsilon = b(D \circ (\text{id}_{\mathcal{V}} \times r)) + V$ is a continuous map since D , b , r and V are continuous. Last, observe that for every $v \in \mathcal{V}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$ and every $x \in \mathbb{N}_{\mathbb{G}^{\varepsilon'}}(g)$,

$$\begin{aligned} \langle D_v^\varepsilon(g), x \rangle &= \langle b(g)D(v, r(g)) + V(g), x \rangle \\ &= b(g)\langle D(v, r(g)), x \rangle + \langle V(g), x \rangle < 0. \end{aligned}$$

Indeed: (i) by definition, $b(g) \in [0, 1]$; (ii) $\langle D(v, r(g)), x \rangle \leq 0$ because $r(g) \in \partial\mathbb{G}$ (by definition) and because D is inward-pointing (by assumption); (iii) by definition, $\langle V(g), x \rangle < 0$. Therefore, for every $v \in \mathcal{V}$, D_v^ε has no zeros on $\mathbb{G}^\varepsilon \setminus \mathbb{G}$, which finally implies that $(D^\varepsilon)^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{V}}$ (using Equation (3.3)). \square

Considering a regular set \mathcal{R} of societies, an inward-pointing network dynamic D on \mathcal{R} and any extension D^ε of D to $\tilde{\mathcal{E}}_{\mathcal{R}}$ (from Proposition 3.3.1), Corollary 3.3.1 states that for every $v \in \mathcal{R}$, and every isolated zero g of D_v^ε , the index of D_v^ε at g does not depend on the choice of this extension. Thus, even if we technically cannot talk about the index of D_v at an isolated zero g on $\partial\mathbb{G}$ ($v \in \mathcal{R}$), we can still describe how D_v behaves around g using any extension D^ε (because such an extension is strongly inward-pointing, no zero is added outside \mathbb{G} , which makes it a “good representation” of D on $\mathcal{E}_{\mathcal{R}}^\varepsilon$). In particular, we can see here why the ambient space $\mathcal{E}_{\mathcal{R}}^\varepsilon$ is useful to consider.

Indices equality theorem with semi-algebraic regularity

In this section, we are interested by sets of societies whose payoff functions are polynomial functions of g (with coefficients in \mathbb{R}). Hence, we recall first some definitions introduced in Chapter 1.

Definition 3.3.5. Let

$$\mathbb{R}[g] = \left\{ g \in \mathbb{G} \mapsto \sum_{k \in \mathbb{N}^L} (\alpha_k \prod_{ij \in L} g_{ij}^{k_{ij}}) \in \mathbb{R} : \forall k \in \mathbb{N}^L, \alpha_k \in \mathbb{R} \right\}.$$

For every $i \in N$ and every $\delta_i \in \mathbb{N}$,

$$\mathbb{R}_{\delta_i}[g] = \{v_i \in \mathbb{R}[g] : \deg(v_i) \leq \delta_i\},$$

where for every $v_i \in \mathbb{R}[g]$, $\deg(v_i) = \max\{\deg(k) : \alpha_k \neq 0\}$, with $\deg(k) = \sum_{ij \in L} k_{ij}$, for every $k \in \mathbb{N}^L$. For every $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, the set

$$\mathbb{R}_\delta[g] = \prod_{i \in N} \mathbb{R}_{\delta_i}[g]$$

is called the *set of (δ) -polynomial societies*.

Definition 3.3.6. Let $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, and consider some order on the set L and some order on the set \mathbb{N}^L . For every $i \in N$, there exists a unique $m_i \in \mathbb{N}$ such that the map

$$\varphi_i : v_i \in \mathbb{R}_{\delta_i}[g] \mapsto (\alpha_k)_{k \in \mathbb{N}^L} \in \mathbb{R}^{m_i}$$

is a well-defined vector space isomorphism. Furthermore, the map

$$\varphi : v \in \mathbb{R}_\delta[g] \mapsto \times_{i \in N} \varphi_i(v_i) \in \mathbb{R}^m,$$

where $m = \sum_{i \in N} m_i$, is also a well-defined vector space isomorphism.

Throughout the rest of this section, we consider a fixed $\delta = (\delta_i)_{i \in N} \in \mathbb{N}^N$, a fixed order on the set L and a fixed order on the set \mathbb{N}^L (in particular, we consider also the vector space isomorphism φ_i of Definition 1.3.2, for every $i \in N$).

We now recall the notion of \mathcal{A} -semi-algebraically set of societies

Definition 3.3.7. A set $\mathcal{S} \subset \text{Soc}$ of societies is \mathcal{A} -semi-algebraically regular if the three following conditions hold:

1. **(Concavity).** $\mathcal{S} \subset \mathcal{F}$.
2. **(\mathcal{A} -invariance).** $\mathcal{S} + \mathcal{A} = \mathcal{S}$.
3. **(Semi-algebraicity).** $\mathcal{S} \subset \mathbb{R}_\delta[g]$ and $\varphi(\mathcal{S})$ is a semi-algebraic set.

We also recall our oddness theorem (see Theorem 1.3.1 in Chapter 1).⁵

Theorem 3.3.2. *Let \mathcal{S} be a \mathcal{A} -semi-algebraically regular set of societies. There exists a generic semi-algebraic subset G of the set $\varphi(\mathcal{S})$ (of coefficients of polynomial payoff functions in \mathcal{S}) such that for every $x \in G$, the society $v^x = \varphi^{-1}(x)$ has an odd number of pairwise stable networks.*

⁵Similarly to a previous footnote: this result was stated for \mathcal{A} -semi-algebraically regular set of societies of the form $\prod_{i \in N} \mathcal{S}_i$ (for every $i \in N$, $\mathcal{S}_i \subset \mathcal{F}(\mathbb{G}, \mathbb{R})$), but we can demonstrate that the result holds for any \mathcal{A} -semi-algebraically regular set of societies.

Now, observe that in the framework of \mathcal{A} -semi-algebraically regular sets of societies, one can mix indices equality theorem (Corollary 3.3.1) with Theorem 3.3.2 in order to obtain another interesting version of this result.

Corollary 3.3.2. (*Indices equality theorem with semi-algebraic regularity*)

Let \mathcal{S} be a \mathcal{A} -semi-algebraically regular set of societies. There exists a generic semi-algebraic subset G of $\varphi(\mathcal{S})$ such that for every strongly inward-pointing extended network dynamics \tilde{D} and \tilde{D}' on \mathcal{S} and every $x \in G$, the index of \tilde{D}_{v^x} at g is equal to the index of \tilde{D}'_{v^x} at g , for every point g of $\mathbf{Z}(\tilde{D}_{v^x}) = \mathbf{Z}(\tilde{D}'_{v^x})$.

The proof follows directly from the ones of Corollary 3.3.1 and Theorem 3.3.2.

The difference between Corollary 3.3.2 and Corollary 3.3.1 lies in the fact that if we consider a \mathcal{A} -semi-algebraically regular set \mathcal{S} of societies, and strongly inward-pointing extended network dynamics \tilde{D}, \tilde{D}' on \mathcal{S} , then for “almost every” society $v \in \mathcal{S}$, we are now able to talk about the equality of the indices of \tilde{D}_v and \tilde{D}'_v at any zero, since in that case, each zero is by definition isolated.

3.4 Appendix

3.4.1 Reminders about topology and vector bundles

Elementary concepts of topology (homeomorphisms, topological embeddings, homotopies, etc.) have been introduced in Section 1.4.1 of Chapter 1. The aim of this section is to present other notions that are used in this chapter.

Knots and ambient isotopies

Definition 3.4.1. Let X, Y be two topological spaces and $e^1, e^2 : X \rightarrow Y$ be two topological embeddings. An *ambient isotopy between e^1 and e^2* is a continuous map $\theta : [0, 1] \times Y \rightarrow Y$ such that $\theta(0, \cdot) = \text{id}_Y$, $\theta(1, \cdot) \circ e^1 = e^2$ and for every $t \in [0, 1]$, $\theta(t, \cdot)$ is a homeomorphism. If such a map exists, then e^1 and e^2 are said to be *ambient isotopic within Y* .

Remark 3.4.1. The previous definition find its roots in knot theory, which is a branch of topology which studies topological embeddings of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ into \mathbb{R}^3 (or sometimes, into \mathbb{S}^3), called *knots*. Figure 3.7 provides two examples of knots: the *clover-leaf knot* and the *figure-eight knot* (n.b.: often, the term “knot” refers both to the topological embedding that is considered and to the image of \mathbb{S}^1 by this embedding). The notion of homeomorphism

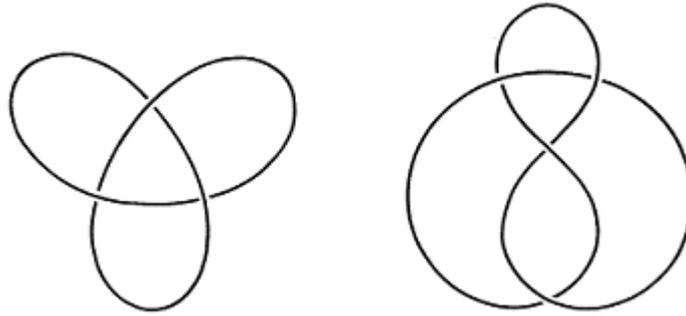


Figure 3.7: *Clover-leaf knot (left) and figure-eight knot (right), embedded into \mathbb{R}^3 (Crowell-Fox [13], p. 4)*

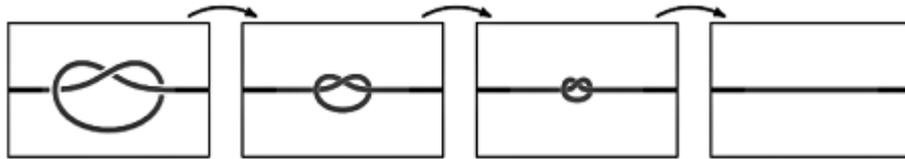


Figure 3.8: *The clover-leaf knot is not ambient isotopic to the unknot (ref.: a lecture course on knot theory from University of Moscow)*

is not sufficient in order to classify knots since by definition, any knot is homeomorphic to \mathbb{S}^1 (i.e. homeomorphic to the *unknot*). In fact, it is quite natural to remark that, for example, the clover-leaf knot is not “similar” to the figure-eight knot, nor to the unknot. Ambient isotopy concept (which is stronger than homeomorphism concept) aims to distinguish knots in a better manner. Informally, in order to illustrate this notion, consider the example of the clover-leaf knot. Figure 3.8 illustrates how this knot can be homeomorphically deformed into the unknot. However, the clover-leaf knot is not ambient isotopic to the unknot. Indeed, observe that if such an ambient isotopy would exist, then the whole ambient space (here, \mathbb{R}^3) would be deformed, which would imply an overlap problem and a lack of injectivity (imagine a tubular neighborhood of the clover-leaf knot: the deformation of Figure 3.8 would be problematic, passing from the third picture to the last one).

Vector bundles

Definition 3.4.2. Let E be a topological space, B be a connected topological space, $p : E \rightarrow B$ be a continuous map and $m \in \mathbb{N}$. The tuple (E, B, p, m) is a (real) vector bundle (of total space E , of base space B , of projection p and

of rank m) if the two following conditions hold:

1. For every $b \in B$, $p^{-1}(b)$ is endowed with the structure of a m -dimensional real vector space.
2. For every $b \in B$, there exists an open subset V of B which contains b and a homeomorphism ψ making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow{\psi} & V \times \mathbb{R}^m \\ \downarrow p & \swarrow \text{pr}_V & \\ V & & \end{array}$$

where $\text{pr}_V : (x, y) \in V \times \mathbb{R}^m \mapsto x \in V$, and such that for every $b' \in V$, the restricted map $x \in p^{-1}(b') \mapsto \psi(x) \in \{b'\} \times \mathbb{R}^m$ is a vector space isomorphism.

A *section* of (E, B, p, m) is a continuous map $s : B \rightarrow E$ such that $p \circ s = \text{id}_B$.

Example 3.4.1.

- Let B be a connected topological space and $m \in \mathbb{N}$. The *trivial bundle of rank m over B* is the vector bundle

$$(B \times \mathbb{R}^m, B, \text{pr}_B, m),$$

where $\text{pr}_B : (b, y) \in B \times \mathbb{R}^m \mapsto b \in B$.

- Let M be a smooth m -manifold (with or without boundary, with or without corners). The *tangent bundle of M* is the vector bundle

$$(TM, M, p, m),$$

where TM is the disjoint union of the tangent spaces $(T_a M)_{a \in M}$ to M and $p : (a, v) \in TM \mapsto a \in M$ (for more details, see Section 3.4.2).

Definition 3.4.3. Let $(E, B, p, m), (E', B, p', m')$ be two vector bundles over the same base space. A continuous map $F : E \rightarrow E'$ is a *vector bundle homomorphism* if the two following conditions hold:

1. The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow p & \swarrow p' & \\ B & & \end{array}$$

2. For every $b \in B$, the restricted map $x \in p^{-1}(b) \mapsto F(x) \in (p')^{-1}(b)$ is a linear map.

Moreover, the map F is a *vector bundle isomorphism* if F is a bijection, and if F and F^{-1} are vector bundle homomorphisms. In that case, (E, B, p, m) and (E', B, p', m') are said to be *isomorphic*.

Definition 3.4.4. A vector bundle (E, B, p, m) is *trivial* if it is isomorphic to the trivial bundle $(B \times \mathbb{R}^m, B, \text{pr}_B, m)$ of rank m over B .

Definition 3.4.5. A topological space X is *contractible* if there exists $x^* \in X$ and a homotopy between the (continuous) map $r_{x^*} : x \in X \mapsto x^* \in X$ and the identity map id_X on X .

Example 3.4.2.

- \mathbb{R}^m and the unit m -hypercube $[0, 1]^m$ are contractible.
- The unit m -sphere \mathbb{S}^m is not contractible.

Definition 3.4.6. A topological space X is *paracompact* if every open cover of X admits a locally finite open subcover.

Example 3.4.3.

- Every compact space is paracompact.
- Any subspace of \mathbb{R}^m is paracompact (since it is metrizable; cf. *Stone's theorem*).

Proposition 3.4.1. *Any vector bundle over a contractible paracompact base space is trivial.*

Proof. The proof is similar to Bott-Tu [9], Corollary 6.9, p. 59. □

3.4.2 Reminders about differential geometry

Remark 3.4.2. In the following, the term “smooth m -manifold” (without further qualification) means a smooth m -manifold with or without boundary, with or without corners (e.g. see Lee [36], pp. 10-15, p.25 and p. 415 for some reminders about these definitions).

Tangent spaces and vector fields

Definition 3.4.7. Let M be a smooth m -manifold and $a \in M$. A linear map $d : C^\infty(M, \mathbb{R}) \rightarrow \mathbb{R}$ is a *derivation at a* if it satisfies the Leibniz identity:

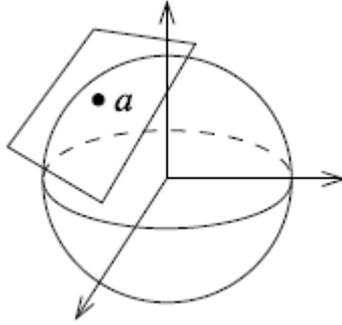


Figure 3.9: *Tangent space to S^2 at a (Lee [36], p. 52)*

for every $f, g \in C^\infty(M, \mathbb{R})$,

$$d(fg) = d(f)g(a) + f(a)d(g).$$

The vector space of all derivations at a is denoted T_aM and is called the *tangent space to M at a* . Furthermore, the tuple (TM, M, p, m) , where

$$TM = \coprod_{a \in M} T_aM$$

and $p : (a, v) \in TM \mapsto a \in M$, is a vector bundle of rank m which is called the *tangent bundle of M* .⁶

Definition 3.4.8. Let M be a smooth m -manifold. A section $V : M \rightarrow TM$ of the tangent bundle of M is called a *vector field on M* . The vector space of all vector fields on M is denoted $\mathfrak{X}(M)$. Moreover, for every vector field $V \in \mathfrak{X}(M)$ on M , the set

$$\mathbf{Z}(V) = \{a \in M : V(a) = (a, \vec{0}_a)\},$$

where $\vec{0}_a$ is the derivation at a constantly equal to 0, is called the *set of zeros of V* . A zero $z \in \mathbf{Z}(V)$ of V is *isolated* if there exists an open neighborhood of z in M which does not intersect the set $\mathbf{Z}(V) \setminus \{z\}$. A zero $z \in \mathbf{Z}(V)$ of V is (*manifold*) *interior* if there exists a chart (U, ϕ) of M such that $\phi(U)$ is an open subset of \mathbb{R}^m (such a chart is called an *interior chart of M*) and $z \in U$.

Theorem 3.4.1. *Let M be a smooth m -manifold with trivial tangent bundle and X be a topological space. There exists a bijection Ψ from $C^0(X \times M, \mathbb{R}^m)$*

⁶See Lee [36], Proposition 3.18, p. 66 and Proposition 10.4, p. 252 for some reminders about the natural topology on TM .

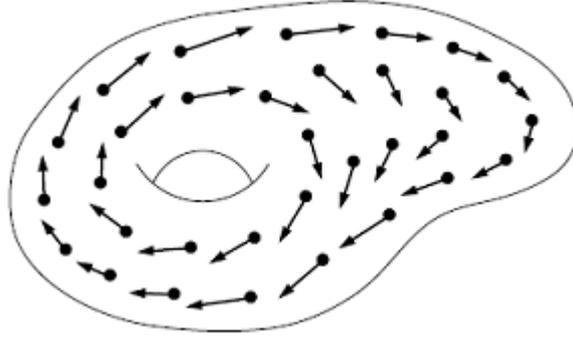


Figure 3.10: Illustration of a vector field on a smooth manifold (Lee [36], p. 175)

to the set

$$\mathfrak{D} = \{D \in \mathcal{C}^0(X \times M, \text{TM}) : \forall x \in X, D(x, \cdot) \in \mathfrak{X}(M)\}$$

with the following properties:

1. For every $f, f' \in \mathcal{C}^0(X \times M, \mathbb{R}^m)$: there exists a homotopy H between f and f' if and only if there exists a homotopy H' between $\Psi(f)$ and $\Psi(f')$ such that for every $t \in [0, 1]$, $H'(t, \cdot, \cdot) \in \mathfrak{D}$.
2. For every $f \in \mathcal{C}^0(X \times M, \mathbb{R}^m)$, $f^{-1}(0_{\mathbb{R}^m}) = \mathbf{Z}(\Psi(f))$, where

$$\mathbf{Z}(\Psi(f)) = \{(x, a) \in X \times M : a \in \mathbf{Z}(\Psi(f)(x, \cdot))\}.$$

Proof. By definition, since TM is trivial, there exists a vector bundle isomorphism from (TM, M, p, m) to $(M \times \mathbb{R}^m, M, \text{pr}_M, m)$. Let $F : \text{TM} \rightarrow M \times \mathbb{R}^m$ be such an isomorphism. Observe that the map

$$\left\{ \begin{array}{ll} \Psi : \mathcal{C}^0(X \times M, \mathbb{R}^m) & \rightarrow \mathfrak{D} \\ f & \mapsto \Psi(f) \end{array} \right. ,$$

with

$$\left\{ \begin{array}{ll} \Psi(f) : X \times M & \rightarrow \text{TM} \\ (x, a) & \mapsto (F^{-1} \circ (\text{pr}'_M \times f) \circ \Delta_{X \times M})(x, a) \end{array} \right.$$

(where $\text{pr}'_M : (x, a) \in X \times M \rightarrow a \in M$ and $\Delta_{X \times M} : (x, a) \in X \times M \rightarrow ((x, a), (x, a)) \in (X \times M) \times (X \times M)$ is the diagonal function on $X \times M$) is a bijection whose inverse is the map

$$\left\{ \begin{array}{ll} \Psi^{-1} : \mathfrak{D} & \rightarrow \mathcal{C}^0(X \times M, \mathbb{R}^m) \\ D & \mapsto \Psi^{-1}(D) \end{array} \right. ,$$

with

$$\left\{ \begin{array}{l} \Psi^{-1}(D) : X \times M \rightarrow \mathbb{R}^m \\ (x, a) \mapsto (\text{pr}_{\mathbb{R}^m} \circ F \circ D)(x, a) \end{array} \right. .$$

Now, we prove the first property. Let $f, f' \in \mathcal{C}^0(X \times M, \mathbb{R}^m)$. First, let H be a homotopy between f and f' . Observe that the map

$$\left\{ \begin{array}{l} H' : [0, 1] \times X \times M \rightarrow \text{TM} \\ (t, x, a) \mapsto \Psi(H(t, \cdot, \cdot))(x, a) \end{array} \right.$$

is continuous (by composition), and that $H'(0, \cdot, \cdot) = \Psi(f)$ and $H'(1, \cdot, \cdot) = \Psi(f')$. Furthermore, for every $t \in [0, 1]$ and every $(x, a) \in X \times M$, remark that

$$\begin{aligned} (p \circ H'(t, x, \cdot))(a) &= (p \circ F^{-1})(a, H(t, x, a)) \\ &= \text{pr}_M(a, H(t, x, a)) \text{ (by definition of } F) \\ &= a, \end{aligned}$$

i.e. that for every $t \in [0, 1]$ and every $x \in X$, $H'(t, x, \cdot) \in \mathfrak{X}(M)$, hence that $H'(t, \cdot, \cdot) \in \mathfrak{D}$. Conversely, let H' be a homotopy between $\Psi(f)$ and $\Psi(f')$ such that for every $t \in [0, 1]$, $H'(t, \cdot, \cdot) \in \mathfrak{D}$. Then, the map

$$\left\{ \begin{array}{l} H : [0, 1] \times X \times M \rightarrow \mathbb{R}^m \\ (t, x, a) \mapsto (\text{pr}_{\mathbb{R}^m} \circ F \circ H')(t, x, a) \end{array} \right.$$

is the required homotopy between f and f' , since it is continuous (by composition), and since $H(0, \cdot, \cdot) = f$ and $H(1, \cdot, \cdot) = f'$.

Finally, we prove the second property. Let $f \in \mathcal{C}^0(X \times M, \mathbb{R}^m)$. By definition,

$$\begin{aligned} \mathbf{Z}(\Psi(f)) &= \{(x, a) \in X \times M : a \in \mathbf{Z}(\Psi(f)(x, \cdot))\} \\ &= \{(x, a) \in X \times M : \Psi(f)(x, a) = (a, \vec{0}_a)\} \\ &= \{(x, a) \in X \times M : F^{-1}(a, f(x, a)) = (a, \vec{0}_a)\} \end{aligned}$$

(where $\vec{0}_a$ is the derivation at a constantly equal to 0, for every $a \in M$). First, let $(x, a) \in f^{-1}(0_{\mathbb{R}^m})$. Since F^{-1} is a vector bundle homomorphism, the restricted map $(a', y) \in \{a\} \times \mathbb{R}^m \mapsto F^{-1}(a', y) \in \{a\} \times T_a M$ is a linear map, which implies that

$$F^{-1}(a, f(x, a)) = F^{-1}(a, 0) = (a, \vec{0}_a),$$

thus that $(x, a) \in \mathbf{Z}(\Psi(f))$. Last, let $(x, a) \in X \times M$ such that $F^{-1}(a, f(x, a)) = (a, \vec{0}_a)$. Observe that $F(F^{-1}(a, f(x, a))) = (a, f(x, a)) = F(a, \vec{0}_a)$. Since F

is a vector bundle homomorphism, the restricted map $(a', v) \in \{a\} \times T_a M \rightarrow F(a', v) \in \{a\} \times \mathbb{R}^m$ is a linear map, which implies that

$$(a, f(x, a)) = F(a, \vec{0}_a) = (a, 0),$$

thus that $(x, a) \in f^{-1}(0_{\mathbb{R}^m})$. \square

Remark 3.4.3. A particular case of the previous theorem will be useful in this chapter: suppose that $X = \{\star\}$ is a singleton. For every vector bundle isomorphism $F : TM \rightarrow M \times \mathbb{R}^m$, following the constructions of the previous proof, the map

$$\left\{ \begin{array}{l} \Psi_\star : \mathcal{C}^0(M, \mathbb{R}^m) \rightarrow \mathfrak{X}(M) \\ f \mapsto \left\{ \begin{array}{l} \Psi_\star(f) : M \rightarrow TM \\ a \mapsto (F^{-1} \circ (\text{id}_M \times f) \circ \Delta_M)(a) \end{array} \right. \end{array} \right.$$

(sometimes denoted $\Psi_{\star, F}$) is a bijection whose inverse is the map

$$\left\{ \begin{array}{l} \Psi_\star^{-1} : \mathfrak{X}(M) \rightarrow \mathcal{C}^0(M, \mathbb{R}^m) \\ V \mapsto \left\{ \begin{array}{l} \Psi_\star^{-1}(V) : M \rightarrow \mathbb{R}^m \\ a \mapsto (\text{pr}_{\mathbb{R}^m} \circ F \circ V)(a) \end{array} \right. \end{array} \right.$$

and it satisfies the following properties:

1. For every $f, f' \in \mathcal{C}^0(M, \mathbb{R}^m)$: there exists a homotopy H between f and f' if and only if there exists a homotopy H' between $\Psi_\star(f)$ and $\Psi_\star(f')$ such that for every $t \in [0, 1]$, $H'(t, \cdot) \in \mathfrak{X}(M)$.
2. For every $f \in \mathcal{C}^0(M, \mathbb{R}^m)$, $f^{-1}(0_{\mathbb{R}^m}) = \mathbf{Z}(\Psi_\star(f))$.

A particular vector bundle isomorphism, denoted here F_c (“**c**” for “canonical”), is usually considered in the case where M is an open subset of \mathbb{R}^m .⁷ In such a case, one can define

$$\left\{ \begin{array}{l} F_c : TM \rightarrow M \times \mathbb{R}^m \\ \left(a, \frac{\partial}{\partial x_1} \Big|_a \right) \mapsto (a, e_1) = (a, (1, 0, 0, \dots)) \\ \left(a, \frac{\partial}{\partial x_2} \Big|_a \right) \mapsto (a, e_2) = (a, (0, 1, 0, \dots)) \\ \dots \\ \left(a, \frac{\partial}{\partial x_m} \Big|_a \right) \mapsto (a, e_m) = (a, (\dots, 0, 0, 1)) \end{array} \right. ,$$

⁷This notion will be useful in order to be able to define the notion of index of a zero of a vector field, on an arbitrary smooth m -manifold.

where for every $a \in M$, $\{\frac{\partial}{\partial x_1}\big|_a, \frac{\partial}{\partial x_2}\big|_a, \dots, \frac{\partial}{\partial x_m}\big|_a\}$ is the canonical basis of T_aM , with for every $i \in \{1, \dots, m\}$,

$$\frac{\partial}{\partial x_i}\bigg|_a : f \in C^\infty(M, \mathbb{R}) \mapsto \frac{\partial f}{\partial x_i}(a) \in \mathbb{R}.$$

In that case, the map $\Psi_{*,F,c}$ defined above is simply denoted $\Psi_{*,c}$.

Index of a zero of a vector field

Definition 3.4.9. Let O be an open subset of \mathbb{R}^m and $f : O \rightarrow \mathbb{R}^m$ be a continuous map. Moreover, let $z \in f^{-1}(0_{\mathbb{R}^m})$ be an isolated zero of f , i.e. there exists $\epsilon > 0$ such that $\overline{B}(z, \epsilon) \cap (f^{-1}(0_{\mathbb{R}^m}) \setminus \{z\}) = \emptyset$, where $\overline{B}(z, \epsilon) = \{y \in \mathbb{R}^m : \|z - y\|_2 \leq \epsilon\}$. The *index of f at z* is defined as the topological degree of the map

$$\begin{cases} f^z : \partial\overline{B}(z, \epsilon) & \rightarrow \mathbb{S}^{m-1} \\ x & \mapsto \frac{f(x)}{\|f(x)\|_2} \end{cases}$$

and is denoted $\text{ind}_z(f)$.⁸

Remark 3.4.4. The index $\text{ind}_z(f)$ of f at z does not depend on ϵ (e.g. see Guillemin-Pollack [25], p. 133).

Definition 3.4.10. Let O be an open subset⁹ of \mathbb{R}^m , $V : O \rightarrow TO$ be a vector field on O and $z \in \mathbf{Z}(V)$ be an isolated zero of V . The *index of V at z* is defined as the index of $\Psi_{*,c}^{-1}(V)$ at z and is denoted $\text{ind}_z(V)$, i.e.

$$\text{ind}_z(V) = \text{ind}_z(\Psi_{*,c}^{-1}(V)),$$

where $\Psi_{*,c}^{-1}(V) : O \rightarrow \mathbb{R}^m$ is defined in Remark 3.4.3.

Definition 3.4.11. Let M, N be two smooth manifolds, $V \in \mathfrak{X}(M)$ be a vector field on M , $V' \in \mathfrak{X}(N)$ be a vector field on N , and $f : M \rightarrow N$ be a smooth map. The maps V and V' are said to be *f -related* if the following diagram commutes (in the category of sets):

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ V \uparrow & & V' \uparrow \\ M & \xrightarrow{f} & N \end{array}$$

⁸See Appendix 2.4.1 in Chapter 2 for some reminders about topological degree of a proper continuous map. Note that the (continuous) map f^z is proper since it goes from the compact space $\partial\overline{B}(z, \epsilon)$ to the Hausdorff space \mathbb{S}^{m-1} .

⁹Here, O is seen as an embedded m -submanifold of \mathbb{R}^m (see Lee [36], Proposition 5.1, p. 99). Also, the tangent bundle of O is trivial (see Lee [36], Proposition 3.20, p. 67).

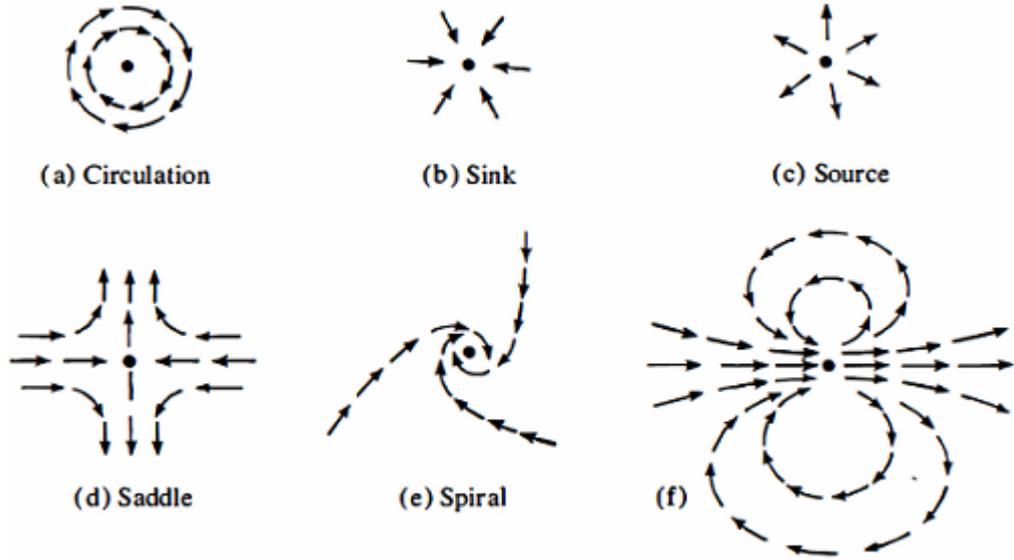


Figure 3.11: *Examples of vector fields' zeros on some open subset of \mathbb{R}^2 : (a) $\text{ind}_z(V) = 1$; (b) $\text{ind}_z(V) = 1$; (c) $\text{ind}_z(V) = 1$; (d) $\text{ind}_z(V) = -1$; (e) $\text{ind}_z(V) = 1$; (f) $\text{ind}_z(V) = 2$ (Guillemin-Pollack [25], p. 133)*

where $Tf : (a, d) \in TM \mapsto (f(a), T_a f(d)) \in TN$ is the global tangent map of f , with $T_a f(d)(g) = d(g \circ f) \in \mathbb{R}$, for every $g \in \mathcal{C}^\infty(N, \mathbb{R})$ ($T_a f(d) \in T_{f(a)}N$).

Definition 3.4.12. (Lee [36], Proposition 8.19, p. 183)

Let M, N be two smooth manifolds, $V \in \mathfrak{X}(M)$ be a vector field on M and $f : M \rightarrow N$ be a diffeomorphism. There exists a unique vector field $f_*V \in \mathfrak{X}(N)$ on N , called the *pushforward of V by f* , such that V and f_*V are f -related, which is defined as follows: for every $a \in N$,

$$f_*V(a) = (T_{f^{-1}(a)}f \circ V \circ f^{-1})(a).$$

Definition 3.4.13. Let M be a smooth m -manifold, $V \in \mathfrak{X}(M)$ be a vector field on M , $z \in \mathbf{Z}(V)$ be an interior isolated zero of V and (U, ϕ) be an interior chart of M such that $z \in U$. The *index of V at z* is defined as the index of the pushforward ϕ_*V of V by ϕ at $\phi(z)$, i.e. the index of the vector field

$$\begin{cases} \phi_*V : \phi(U) & \rightarrow & T\phi(U) \\ & a & \mapsto & (T_{\phi^{-1}(a)}\phi \circ V \circ \phi^{-1})(a) \end{cases}$$

(on the open subset $\phi(U)$ of \mathbb{R}^m) at $\phi(z)$, and is denoted $\text{ind}_z(V)$.

Remark 3.4.5. The quantity $\text{ind}_z(V)$ does not depend on the chart (U, ϕ) (e.g. see Guillemin-Pollack [25], p. 134).

3.4.3 Proof of unknottedness theorem

From now on, consider a fixed network $g^0 \in \mathbb{G}$ and a fixed \mathcal{A} -regular set \mathcal{R} of societies. Recall that $\eta_{\mathcal{R}} : (v, g) \in \mathcal{P}_{\mathcal{R}} \mapsto \bar{u}^{v,g} \in \mathcal{R}$ is the homeomorphism of structure theorem (Theorem 3.2.1), defined as follows: for every $(v, g) \in \mathcal{P}_{\mathcal{R}}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\bar{u}_i^{v,g}(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij} \gamma_{ij}$$

(see Equation (3.1)), i.e.

$$\bar{u}_i^{v,g} = v_i + h_i^g[v] + l_i^g,$$

where

$$h_i^g[v] : \gamma \in \mathbb{G} \mapsto \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) \in \mathbb{R}$$

and

$$l_i^g : \gamma \in \mathbb{G} \rightarrow \sum_{j \neq i} \gamma_{ij} g_{ij} \in \mathbb{R}.$$

Moreover, recall that the inverse of $\eta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}} \rightarrow \mathcal{R}$ is the map $\rho_{\mathcal{R}} : v \in \mathcal{R} \mapsto (\underline{u}^v, g^v) \in \mathcal{P}_{\mathcal{R}}$ which is defined as follows:

- For every $v \in \mathcal{R}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\underline{u}_i^v(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0)) (\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} g_{ij}^v \gamma_{ij},$$

(see Equation (3.2)), i.e.

$$\underline{u}_i^v = v_i - h_i^{g^v}[v] - l_i^{g^v}.$$

- For every link $ij \in L$, $g_{ij}^v = \min\{w_{i,j}^v, w_{j,i}^v\}$, with $w_{i,j}^v \in [0, 1]$ (resp. $w_{j,i}^v \in [0, 1]$) being the unique maximizer of the strictly concave function $q_i[v] : w \in [0, 1] \rightarrow v_i(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$ (resp. $q_j[v] : w \in [0, 1] \rightarrow v_j(w, g_{-ij}^0) - \frac{w^2}{2} \in \mathbb{R}$).

Step I. The knot $\kappa_{\rho_{\mathcal{R}}} : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} and $\rho_{\mathcal{R}}$ is ambient isotopic to the topological embedding $e^{\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}} : v \in \mathcal{R} \mapsto (v, (\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}})(v)) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ within $\tilde{\mathcal{E}}_{\mathcal{R}}$, through an ambient isotopy $\theta_{\mathcal{R}}^1$.

Let r be the projection from $\tilde{\mathbb{G}}$ to the closest point in \mathbb{G} (this map is a retraction of $\tilde{\mathbb{G}}$ on \mathbb{G} , i.e. a continuous map such that its composition with the inclusion $\mathbb{G} \hookrightarrow \tilde{\mathbb{G}}$ is the identity map $\text{id}_{\mathbb{G}}$ on \mathbb{G}), and $b : \tilde{\mathbb{G}} \rightarrow [0, 1]$ be a continuous map such that $b(\mathbb{G}) = 1$ and $b(\partial\tilde{\mathbb{G}}) = 0$.

Now, let

$$\begin{cases} \theta_{\mathcal{R}}^1 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \tilde{\mathcal{E}}_{\mathcal{R}} \\ (t, (v, g)) & \mapsto & (\bar{u}^{t,v,g}, g) \end{cases} ,$$

where for every $i \in N$,

$$\bar{u}_i^{t,v,g} = v_i + tb(g)(h_i^{r(g)}[v] + l_i^{r(g)}),$$

and let

$$\begin{cases} \xi_{\mathcal{R}}^1 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \tilde{\mathcal{E}}_{\mathcal{R}} \\ (t, (v, g)) & \mapsto & (\underline{u}^{t,v,g}, g) \end{cases} ,$$

where for every $i \in N$,

$$\underline{u}_i^{t,v,g} = v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)}).$$

Notice that from \mathcal{A} -invariance assumption, $\theta_{\mathcal{R}}^1$ and $\xi_{\mathcal{R}}^1$ are well-defined maps.

First, $\theta_{\mathcal{R}}^1$ is a continuous map since both r and b are continuous maps (for more details, see Step V of the proof of Theorem 1.2.1 in Appendix 1.4.3 of Chapter 1, whose demonstration is quite similar).

Second, for every $i \in N$ and every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\bar{u}_i^{0,v,g} = v_i + 0b(g)(h_i^{r(g)}[v] + l_i^{r(g)}) = v_i.$$

Hence, $\theta_{\mathcal{R}}^1(0, \cdot, \cdot) = \text{id}_{\tilde{\mathcal{E}}_{\mathcal{R}}}$.

Third, for every $v \in \mathcal{R}$,

$$(\theta_{\mathcal{R}}^1(1, \cdot, \cdot) \circ \kappa_{\rho_{\mathcal{R}}})(v) = \theta_{\mathcal{R}}^1(1, \rho_{\mathcal{R}}(v)) = \theta_{\mathcal{R}}^1(1, (\underline{u}^v, g^v)) = (\bar{u}^{1, \underline{u}^v, g^v}, g^v),$$

where for every $i \in N$,

$$\begin{aligned}
\bar{u}_i^{1, \underline{u}^v, g^v} &= \underline{u}_i^v + 1b(g^v)(h_i^{r(g^v)}[\underline{u}^v] + l_i^{r(g^v)}) \\
&= (v_i - h_i^{g^v}[v] - l_i^{g^v}) + b(g^v)(h_i^{r(g^v)}[\underline{u}^v] + l_i^{r(g^v)}) \\
&= (v_i - h_i^{g^v}[v] - l_i^{g^v}) + (h_i^{r(g^v)}[\underline{u}^v] + l_i^{r(g^v)}) \quad (\text{since } g^v \in \mathbb{G}, b(g^v) = 1) \\
&= (v_i - h_i^{g^v}[v] - l_i^{g^v}) + (h_i^{g^v}[\underline{u}^v] + l_i^{g^v}) \quad (\text{since } g^v \in \mathbb{G}, r(g^v) = g^v) \\
&= v_i - h_i^{g^v}[v] + h_i^{g^v}[\underline{u}^v].
\end{aligned}$$

Observe that for every $\gamma \in \mathbb{G}$,

$$\begin{aligned}
h_i^{g^v}[\underline{u}^v](\gamma) &= \sum_{j \neq i} (\partial_{ij} \underline{u}_i^v(g_{ij}^v, g_{-ij}^v) - \partial_{ij} \underline{u}_i^v(g_{ij}^v, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) \\
&= \sum_{j \neq i} (\partial_{ij} (v_i - h_i^{g^v}[v] - l_i^{g^v})(g_{ij}^v, g_{-ij}^v) \\
&\quad - \partial_{ij} (v_i - h_i^{g^v}[v] - l_i^{g^v})(g_{ij}^v, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) \\
&= \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0))(\gamma_{ij} - g_{ij}^v) \\
&\quad (\text{since both } \partial_{ij} h_i^{g^v}[v] \text{ and } \partial_{ij} l_i^{g^v} \text{ are constant maps}) \\
&= h_i^{g^v}[v](\gamma).
\end{aligned}$$

Thus, $\theta_{\mathcal{R}}^1(1, \cdot, \cdot) \circ \kappa_{\rho_{\mathcal{R}}} = e^{\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}$.

Last, it remains to show that for every $t \in [0, 1]$, $\theta_{\mathcal{R}}^1(t, \cdot, \cdot)$ is a homeomorphism with inverse $\xi_{\mathcal{R}}^1(t, \cdot, \cdot)$. Let $t \in [0, 1]$. First, both $\theta_{\mathcal{R}}^1(t, \cdot, \cdot)$ and $\xi_{\mathcal{R}}^1(t, \cdot, \cdot)$ are continuous maps (as both $\theta_{\mathcal{R}}^1$ and $\xi_{\mathcal{R}}^1$ are continuous maps). Second, for every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$(\theta_{\mathcal{R}}^1(t, \cdot, \cdot) \circ \xi_{\mathcal{R}}^1(t, \cdot, \cdot))(v, g) = \theta_{\mathcal{R}}^1(t, (\underline{u}^{t, v, g}, g)) = (\bar{u}^{t, \underline{u}^{t, v, g}, g}, g),$$

where for every $i \in N$,

$$\begin{aligned}
\bar{u}_i^{t, \underline{u}^{t, v, g}, g} &= \underline{u}_i^{t, v, g} + tb(g)(h_i^{r(g)}[\underline{u}^{t, v, g}] + l_i^{r(g)}) \\
&= v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)}) + tb(g)(h_i^{r(g)}[\underline{u}^{t, v, g}] + l_i^{r(g)}) \\
&= v_i + tb(g)(h_i^{r(g)}[\underline{u}^{t, v, g}] - h_i^{r(g)}[v]).
\end{aligned}$$

Observe that for every $\gamma \in \mathbb{G}$,

$$\begin{aligned}
& h_i^{r(g)}[\underline{u}^{t,v,g}](\gamma) \\
&= \sum_{j \neq i} (\partial_{ij} \underline{u}_i^{t,v,g}(r(g)_{ij}, r(g)_{-ij})) - \partial_{ij} \underline{u}_i^{t,v,g}(r(g)_{ij}, g_{-ij}^0)(\gamma_{ij} - r(g)_{ij}) \\
&= \sum_{j \neq i} (\partial_{ij}(v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)}))(r(g)_{ij}, r(g)_{-ij})) \\
&\quad - \partial_{ij}(v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)}))(r(g)_{ij}, g_{-ij}^0)(\gamma_{ij} - r(g)_{ij}) \\
&= \sum_{j \neq i} (\partial_{ij} v_i(r(g)_{ij}, r(g)_{-ij}) - \partial_{ij} v_i(r(g)_{ij}, g_{-ij}^0))(\gamma_{ij} - r(g)_{ij}) \\
&\quad (\text{since both } \partial_{ij} h_i^{r(g)}[v] \text{ and } \partial_{ij} l_i^{r(g)} \text{ are constant maps}) \\
&= h_i^{r(g)}[v](\gamma),
\end{aligned}$$

which implies that $\bar{u}^{t, \underline{u}^{t,v,g}, g} = v$. Similarly, one can verify that for every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$(\xi_{\mathcal{R}}^1(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}^1(t, \cdot, \cdot))(v, g) = \xi_{\mathcal{R}}^1(t, (\bar{u}^{t,v,g}, g)) = (\underline{u}^{t, \bar{u}^{t,v,g}, g}, g) = (v, g).$$

Finally, from the previous points, $\theta_{\mathcal{R}}^1$ is by definition an ambient isotopy between $\kappa_{\mathcal{R}}$ and the topological embedding $e^{\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}$ within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$.

Remark that for every $t \in [0, 1]$,

$$\theta_{\mathcal{R}}^1(t, \cdot, \cdot)|_{\mathcal{R} \times \partial \tilde{\mathbb{G}}} = \text{id}_{\mathcal{R} \times \partial \tilde{\mathbb{G}}},$$

i.e. the ambient isotopy $\theta_{\mathcal{R}}^1$ does not deform the boundary of the ambient space.

Step II. For every continuous map $f : \mathcal{R} \rightarrow \mathbb{G}$, the topological embedding $e^f : v \in \mathcal{R} \mapsto (v, f(v)) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ is ambient isotopic to the unknot $\kappa_{\mathcal{R}}^0 : \mathcal{R} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ within $\tilde{\mathcal{E}}_{\mathcal{R}}$, through an ambient isotopy $\theta_{\mathcal{R}}^{2,f}$.

For every link $ij \in L$ and every $w \in [0, 1]$, consider the piecewise linear maps χ_{ij}^w and ζ_{ij}^w defined by

$$\left\{ \begin{array}{lcl} \chi_{ij}^w : & [-\varepsilon, 1 + \varepsilon] & \rightarrow [-\varepsilon, 1 + \varepsilon] \\ & -\varepsilon & \mapsto -\varepsilon \\ & g_{ij}^0 & \mapsto w \\ & 1 + \varepsilon & \mapsto 1 + \varepsilon \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} \zeta_{ij}^w : [-\varepsilon, 1 + \varepsilon] & \rightarrow & [-\varepsilon, 1 + \varepsilon] \\ & -\varepsilon & \mapsto -\varepsilon \\ & w & \mapsto g_{ij}^0 \\ & 1 + \varepsilon & \mapsto 1 + \varepsilon \end{array} \right. .$$

Furthermore, for every $ij \in L$, consider the maps

$$\chi_{ij} : (w, \tilde{w}) \in [0, 1] \times [-\varepsilon, 1 + \varepsilon] \mapsto \chi_{ij}^w(\tilde{w}) \in [-\varepsilon, 1 + \varepsilon]$$

and

$$\zeta_{ij} : (w, \tilde{w}) \in [0, 1] \times [-\varepsilon, 1 + \varepsilon] \mapsto \zeta_{ij}^w(\tilde{w}) \in [-\varepsilon, 1 + \varepsilon].$$

Lemma 3.4.1. *Let $ij \in L$:*

1. *For every $w \in [0, 1]$, χ_{ij}^w is a homeomorphism with inverse ζ_{ij}^w .*
2. *$\chi_{ij}^{g_{ij}^0} = \zeta_{ij}^{g_{ij}^0} = \text{id}_{[-\varepsilon, 1 + \varepsilon]}$.*

Now, let

$$\left\{ \begin{array}{lcl} \theta_{\mathcal{R}}^{2,f} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \tilde{\mathcal{E}}_{\mathcal{R}} \\ & (t, (v, g)) & \mapsto (v, \bar{\gamma}^{t,v,g}) \end{array} \right. ,$$

where for every $ij \in L$,

$$\bar{\gamma}_{ij}^{t,v,g} = \zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, g_{ij}),$$

and let

$$\left\{ \begin{array}{lcl} \xi_{\mathcal{R}}^{2,f} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \tilde{\mathcal{E}}_{\mathcal{R}} \\ & (t, (v, g)) & \mapsto (v, \underline{\gamma}^{t,v,g}) \end{array} \right. ,$$

where for every $ij \in L$,

$$\underline{\gamma}_{ij}^{t,v,g} = \chi_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, g_{ij}).$$

First, $\theta_{\mathcal{R}}^{2,f}$ is continuous as the composition of continuous maps (recall that by assumption, the map $f : \mathcal{R} \rightarrow \mathbb{G}$ is continuous).

Second, for every $ij \in L$ and every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\bar{\gamma}_{ij}^{0,v,g} = \zeta_{ij}((1-0)g_{ij}^0 + 0f(v)_{ij}, g_{ij}) = \zeta_{ij}(g_{ij}^0, g_{ij}).$$

From Lemma 3.4.1 (property 2), $\zeta_{ij}^{g_{ij}^0} = \zeta_{ij}(g_{ij}^0, \cdot)$ is the identity map on $[-\varepsilon, 1 + \varepsilon]$. Hence, $\theta_{\mathcal{R}}^{2,f}(0, \cdot, \cdot) = \text{id}_{\tilde{\mathcal{E}}_{\mathcal{R}}}$.

Third, for every $v \in \mathcal{R}$,

$$(\theta_{\mathcal{R}}^{2,f}(1, \cdot, \cdot) \circ e^f)(v) = \theta_{\mathcal{R}}^{2,f}(1, (v, f(v))) = (v, \bar{\gamma}^{1,v,f(v)}),$$

where for every $ij \in L$,

$$\begin{aligned} \bar{\gamma}_{ij}^{1,v,f(v)} &= \zeta_{ij}((1-1)g_{ij}^0 + 1f(v)_{ij}, f(v)_{ij}) \\ &= \zeta_{ij}^{f(v)_{ij}}(f(v)_{ij}) \\ &= g_{ij}^0 \text{ (by definition of } \zeta_{ij}) \end{aligned}$$

Thus, $\theta_{\mathcal{R}}^{2,f}(1, \cdot, \cdot) \circ e^f = \kappa_{\mathcal{R}}^0$.

Last, it remains to show that for every $t \in [0, 1]$, $\theta_{\mathcal{R}}^{2,f}(t, \cdot, \cdot)$ is a homeomorphism with inverse $\xi_{\mathcal{R}}^{2,f}(t, \cdot, \cdot)$. Let $t \in [0, 1]$. First, both $\theta_{\mathcal{R}}^{2,f}(t, \cdot, \cdot)$ and $\xi_{\mathcal{R}}^{2,f}(t, \cdot, \cdot)$ are continuous maps (as both $\theta_{\mathcal{R}}^{2,f}$ and $\xi_{\mathcal{R}}^{2,f}$ are continuous maps). Second, for every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$(\theta_{\mathcal{R}}^{2,f}(t, \cdot, \cdot) \circ \xi_{\mathcal{R}}^{2,f}(t, \cdot, \cdot))(v, g) = \theta_{\mathcal{R}}^{2,f}(t, (v, \underline{\gamma}^{t,v,g})) = (v, \bar{\gamma}^{t,v,\underline{\gamma}^{t,v,g}}),$$

where for every $ij \in L$,

$$\begin{aligned} \bar{\gamma}_{ij}^{t,v,\underline{\gamma}^{t,v,g}} &= \zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, \underline{\gamma}_{ij}^{t,v,g}) \\ &= \zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, \chi_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, g_{ij})) \\ &= (\zeta_{ij}^{(1-t)g_{ij}^0 + tf(v)_{ij}} \circ \chi_{ij}^{(1-t)g_{ij}^0 + tf(v)_{ij}})(g_{ij}) \\ &= g_{ij} \text{ (from Lemma 3.4.1, property 1)}. \end{aligned}$$

Similarly, one can verify that for every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$(\xi_{\mathcal{R}}^{2,f}(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}^{2,f}(t, \cdot, \cdot))(v, g) = \xi_{\mathcal{R}}^{2,f}(t, (v, \bar{\gamma}^{t,v,g})) = (v, \underline{\gamma}^{t,v,\bar{\gamma}^{t,v,g}}) = (v, g).$$

Finally, from the previous points, $\theta_{\mathcal{R}}^{2,f}$ is by definition an ambient isotopy between e^f and the unknot $\kappa_{\mathcal{R}}^0$ within the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$.

Remark that for every $t \in [0, 1]$,

$$\theta_{\mathcal{R}}^{2,f}(t, \cdot, \cdot)|_{\mathcal{R} \times \partial \tilde{\mathcal{G}}} = \text{id}_{\mathcal{R} \times \partial \tilde{\mathcal{G}}},$$

i.e. the ambient isotopy $\theta_{\mathcal{R}}^{2,f}$ does not deform the boundary of the ambient space.

Step III. Unknottedness theorem.

To finish, from Step I and Step II, notice that the map

$$\begin{cases} \theta_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \tilde{\mathcal{E}}_{\mathcal{R}} \\ (t, (v, g)) & \mapsto & (\theta_{\mathcal{R}}^{2, \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}^1(t, \cdot, \cdot))(v, g) \end{cases}$$

is an ambient isotopy between $\kappa_{\rho_{\mathcal{R}}}$ and the unknot $\kappa_{\mathcal{R}}^0$ within $\tilde{\mathcal{E}}_{\mathcal{R}}$. Indeed:

- (i) $\theta_{\mathcal{R}}$ is continuous as the composition of continuous maps;
- (ii) $\theta_{\mathcal{R}}^{2, \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}(0, \cdot, \cdot) \circ \theta_{\mathcal{R}}^1(0, \cdot, \cdot) = \text{id}_{\tilde{\mathcal{E}}_{\mathcal{R}}} \circ \text{id}_{\tilde{\mathcal{E}}_{\mathcal{R}}} = \text{id}_{\tilde{\mathcal{E}}_{\mathcal{R}}}$;
- (iii) $(\theta_{\mathcal{R}}^{2, \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}(1, \cdot, \cdot) \circ \theta_{\mathcal{R}}^1(1, \cdot, \cdot)) \circ \kappa_{\rho_{\mathcal{R}}} = \theta_{\mathcal{R}}^{2, \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}(1, \cdot, \cdot) \circ e^{\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}} = \kappa_{\mathcal{R}}^0$;
- (iv) for every $t \in [0, 1]$, $\theta_{\mathcal{R}}(t, \cdot, \cdot)$ is a homeomorphism with inverse $\xi_{\mathcal{R}}^1(t, \cdot, \cdot) \circ \xi_{\mathcal{R}}^{2, \pi_{\mathbb{G}} \circ \rho_{\mathcal{R}}}(t, \cdot, \cdot)$.

Also, again from Step I and Step II, remark that for every $t \in [0, 1]$,

$$\theta_{\mathcal{R}}(t, \cdot, \cdot)|_{\mathcal{R} \times \partial \tilde{\mathbb{G}}} = \text{id}_{\mathcal{R} \times \partial \tilde{\mathbb{G}}},$$

i.e. the ambient isotopy $\theta_{\mathcal{R}}$ does not deform the boundary of the ambient space.

This step ends the proof of the unknottedness theorem. \square

3.4.4 Proof of dynamics equivalence theorem

From now on, consider a fixed network $g^0 \in \mathbb{G}$ and a fixed \mathcal{A} -regular set \mathcal{R} of societies. Recall that $\tilde{\mathbb{G}}$ is a convex subset of \mathbb{R}^L , which implies that it is a contractible subspace of \mathbb{R}^L (since $\tilde{\mathbb{G}}$ is a convex, for every $g^* \in \tilde{\mathbb{G}}$, the map $(t, g) \in [0, 1] \times \tilde{\mathbb{G}} \mapsto tg + (1-t)g^* \in \tilde{\mathbb{G}}$ is a well-defined homotopy between the constant map $g \in \tilde{\mathbb{G}} \mapsto g^* \in \tilde{\mathbb{G}}$ and the identity map $\text{id}_{\tilde{\mathbb{G}}}$ on $\tilde{\mathbb{G}}$). Thus, from Proposition 3.4.1 (see Section 3.4.1), the tangent bundle $T\tilde{\mathbb{G}}$ of $\tilde{\mathbb{G}}$ is trivial. Consequently, using Theorem 3.4.1 (see Section 3.4.2), any extended network dynamic on \mathcal{R} is now treated throughout all this proof as a continuous map \tilde{D} from $\tilde{\mathcal{E}}_{\mathcal{R}}$ to \mathbb{R}^L such that $\tilde{D}^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{R}}$ (for the sake of simplicity).

Also, the following lemma will be used several times in this proof:

Lemma 3.4.2. *Let X, Y be two metrizable topological spaces, A be a subspace of X , $f_1 : A \rightarrow Y$ and $f_2 : X \setminus A \rightarrow Y$. Suppose that:*

- (i) both f_1 and f_2 are continuous;

- (ii) for every $x^* \in \partial A$ and every sequence $(x^\ell)_{\ell \in \mathbb{N}}$ in A such that $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, $f_1(x^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*)$;
- (iii) for every $x^* \in \partial(X \setminus A)$ and every sequence $(x^\ell)_{\ell \in \mathbb{N}}$ in $X \setminus A$ such that $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, $f_2(x^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*)$.

Then, the map

$$\begin{cases} f: X \rightarrow Y \\ x \mapsto \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in X \setminus A \end{cases} \end{cases}$$

is continuous.

Proof. Let $x^* \in X$ and $(x^\ell)_{\ell \in \mathbb{N}}$ be a sequence in X such that $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$.

First, suppose that $x^* \in \text{int}(A)$ (in that case, $f(x^*) = f_1(x^*)$). Since $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, there exists $n \in \mathbb{N}$ such that for every $\ell \geq n$, $x^\ell \in \text{int}(A)$ (which implies that $f(x^\ell) = f_1(x^\ell)$, for every $\ell \geq n$). Hence, $f(x^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*)$, i.e. f is continuous at x^* .

Second, suppose that $x^* \in \text{int}(X \setminus A)$ (in that case, $f(x^*) = f_2(x^*)$). Since $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, there exists $n \in \mathbb{N}$ such that for every $\ell \geq n$, $x^\ell \in \text{int}(X \setminus A)$ (which implies that $f(x^\ell) = f_2(x^\ell)$, for every $\ell \geq n$). Hence, $f(x^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*)$, i.e. f is continuous at x^* .

Last, suppose that $x^* \in \partial A = \partial(X \setminus A)$. Consider the sets of indices

$$I_A = \{\ell \in \mathbb{N} : x^\ell \in A\} \text{ and } I_{X \setminus A} = \{\ell \in \mathbb{N} : x^\ell \in X \setminus A\},$$

and consider the two subsequences $(x_A^\ell)_{\ell \in \mathbb{N}} = (x^\ell)_{\ell \in I_A}$ of $(x^\ell)_{\ell \in \mathbb{N}}$ in A , and $(x_{X \setminus A}^\ell)_{\ell \in \mathbb{N}} = (x^\ell)_{\ell \in I_{X \setminus A}}$ of $(x^\ell)_{\ell \in \mathbb{N}}$ in $X \setminus A$. Since $x^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, one obtains that $x_A^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$ and $x_{X \setminus A}^\ell \xrightarrow{\ell \rightarrow +\infty} x^*$, thus that

$$f_1(x_A^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*) \text{ and } f_2(x_{X \setminus A}^\ell) \xrightarrow{\ell \rightarrow +\infty} f(x^*),$$

using second and third assumptions in the statement of the lemma. In other words, considering an open subset U of Y which contains $f(x^*)$: (i) there exists $n_A \in \mathbb{N}$ (now fixed) such that for every $\ell \geq n_A$, $f_1(x_A^\ell) \in U$; (ii) there exists $n_{X \setminus A} \in \mathbb{N}$ (now fixed) such that for every $\ell \geq n_{X \setminus A}$, $f_2(x_{X \setminus A}^\ell) \in U$. Finally, by definition of f and since $\mathbb{N} = I_A \cup I_{X \setminus A}$ with $I_A \cap I_{X \setminus A} = \emptyset$, observe that for every $\ell \in \mathbb{N}$, $f(x^\ell)$ is either equal to $f_1(x_A^\ell)$, or to $f_2(x_{X \setminus A}^\ell)$, for some $\ell' \in \mathbb{N}$. Therefore, setting $n = n_A + n_{X \setminus A}$, one obtains that for every $\ell \geq n$, $f(x^\ell) \in U$, i.e. f is continuous at x^* . \square

Let $\tilde{D}, \tilde{D}' : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L$ be two strongly inward-pointing extended network dynamics on \mathcal{R} .

Step I. First step in the construction of a homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D} and \tilde{D}' .

First, define

$$\left\{ \begin{array}{l} \tilde{D}_{\partial} : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} \tau^g \tilde{D}(v, g) + (1 - \tau^g) \tilde{D}'(v, g) & \text{if } g \in \tilde{\mathbb{G}} \setminus \mathbb{G} \\ \tilde{D}(v, g) & \text{otherwise} \end{cases} \end{array} \right. ,$$

where for every $g \in \tilde{\mathbb{G}}$,

$$\tau^g = \frac{\|g - g^{1/2}\|_{\infty} - \frac{1}{2} - \varepsilon}{-\varepsilon},$$

with $g^{1/2} \in \mathbb{G}$ being the network such that for every $ij \in L$, $g_{ij}^{1/2} = \frac{1}{2}$. The map \tilde{D}_{∂} is continuous since:

- (i) the map $(v, g) \in \mathcal{R} \times \tilde{\mathbb{G}} \setminus \mathbb{G} \mapsto \tau^g \tilde{D}(v, g) + (1 - \tau^g) \tilde{D}'(v, g) \in \mathbb{R}^L$ is continuous (in particular, the map $g \in \tilde{\mathbb{G}} \setminus \mathbb{G} \mapsto \tau^g \in \mathbb{R}$ is itself continuous);
- (ii) the map $(v, g) \in \mathcal{R} \times \text{int}(\mathbb{G}) \mapsto \tilde{D}(v, g) \in \mathbb{R}^L$ is continuous as the restriction of the continuous map \tilde{D} from $\mathcal{R} \times \text{int}(\mathbb{G}) \subset \tilde{\mathcal{E}}_{\mathcal{R}}$ to \mathbb{R}^L ;
- (iii) for every $(v, g) \in \mathcal{R} \times \partial\mathbb{G}$, $\tau^g \tilde{D}(v, g) + (1 - \tau^g) \tilde{D}'(v, g) = \tilde{D}(v, g)$.

Moreover, notice that

$$\tilde{D}_{\partial}|_{\mathcal{E}_{\mathcal{R}}} = \tilde{D}|_{\mathcal{E}_{\mathcal{R}}} \text{ and } \tilde{D}_{\partial}|_{\mathcal{R} \times \partial\tilde{\mathbb{G}}} = \tilde{D}'|_{\mathcal{R} \times \partial\tilde{\mathbb{G}}}. \quad (3.4)$$

Second, define

$$\left\{ \begin{array}{l} \tilde{H}_{\mathcal{R}}^1 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (1 - t) \tilde{D}(v, g) + t \tilde{D}_{\partial}(v, g) \end{array} \right.$$

The map $\tilde{H}_{\mathcal{R}}^1$ is a homotopy between \tilde{D} and \tilde{D}_{∂} ; more precisely, it corresponds to the straight-line homotopy between \tilde{D} and \tilde{D}_{∂} . Furthermore, for every $t \in [0, 1]$, the map $\tilde{H}_{\mathcal{R}}^1(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} . Indeed, for every $t \in [0, 1]$:

- (i) $\tilde{H}_{\mathcal{R}}^1(t, \cdot, \cdot) : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L$ is continuous since $\tilde{H}_{\mathcal{R}}^1$ is itself a continuous map;
- (ii) $(\tilde{H}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{R}}$.

To understand the previous last point, remark first that for every $v \in \mathcal{R}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\mathbb{G}^{\varepsilon'}}(g) \setminus \{0_{\mathbb{R}^L}\}$,

$$\begin{aligned} \langle \tilde{D}_{\partial}(v, g), x \rangle &= \langle \tau^g \tilde{D}(v, g) + (1 - \tau^g) \tilde{D}'(v, g), x \rangle \\ &\quad (\text{since } \partial\mathbb{G}^{\varepsilon'} \subset \tilde{\mathbb{G}} \setminus \mathbb{G}, \text{ for every } \varepsilon' \in (0, \varepsilon)) \\ &= \tau^g \langle \tilde{D}(v, g), x \rangle + (1 - \tau^g) \langle \tilde{D}'(v, g), x \rangle \\ &< 0 \end{aligned}$$

since τ^g belongs to $[0, 1]$ (g belonging to $\tilde{\mathbb{G}} \setminus \mathbb{G}$), and since both \tilde{D} and \tilde{D}' are strongly inward-pointing (by assumption). Hence, for every $v \in \mathcal{R}$, $(\tilde{D}_{\partial})_v$ has no zeros on $\tilde{\mathbb{G}} \setminus \mathbb{G}$. Moreover, knowing from Equation (3.4) that $\tilde{D}_{\partial}|_{\mathcal{E}_{\mathcal{R}}} = \tilde{D}|_{\mathcal{E}_{\mathcal{R}}}$, one obtains directly that $(\tilde{D}_{\partial})^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{R}}$. Finally, observe that for every $v \in \mathcal{R}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, every $x \in N_{\mathbb{G}^{\varepsilon'}}(g) \setminus \{0_{\mathbb{R}^L}\}$, and every $t \in [0, 1]$,

$$\begin{aligned} \langle \tilde{H}_{\mathcal{R}}^1(t, v, g), x \rangle &= \langle (1 - t) \tilde{D}(v, g) + t \tilde{D}_{\partial}(v, g), x \rangle \\ &= (1 - t) \langle \tilde{D}(v, g), x \rangle + t \langle \tilde{D}_{\partial}(v, g), x \rangle \\ &< 0 \end{aligned}$$

since by assumption, \tilde{D} is strongly inward-pointing, and since from the above calculation, \tilde{D}_{∂} is also strongly inward-pointing. Hence, for every $v \in \mathcal{R}$, $\tilde{H}_{\mathcal{R}}^1(t, v, \cdot)$ has no zeros on $\tilde{\mathbb{G}} \setminus \mathbb{G}$. Therefore, one obtains that for every $t \in [0, 1]$, $(\tilde{H}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{P}_{\mathcal{R}}$ since for every $(v, g) \in \mathcal{E}_{\mathcal{R}}$,

$$\begin{aligned} \tilde{H}_{\mathcal{R}}^1(t, v, g) &= (1 - t) \tilde{D}(v, g) + t \tilde{D}_{\partial}(v, g) \\ &= (1 - t) \tilde{D}(v, g) + t \tilde{D}(v, g) \quad (\text{from Equation (3.4)}) \\ &= \tilde{D}(v, g). \end{aligned}$$

Step II. Use of the ambient isotopy $\theta_{\mathcal{R}}$ of unknottedness theorem.

Consider the ambient isotopy $\theta_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \tilde{\mathcal{E}}_{\mathcal{R}}$ between the knot $\kappa_{\rho_{\mathcal{R}}} : v \in \mathcal{R} \mapsto \rho_{\mathcal{R}}(v) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ of pairwise stable networks associated to \mathcal{R} and the unknot $\kappa_{\mathcal{R}}^0 : v \in \mathcal{R} \mapsto (v, g^0) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ of Theorem 3.2.2, and define

$$\tilde{\Delta}_{\partial} = \tilde{D}_{\partial} \circ (\theta_{\mathcal{R}}(1, \cdot, \cdot))^{-1} \quad \text{and} \quad \tilde{\Delta}' = \tilde{D}' \circ (\theta_{\mathcal{R}}(1, \cdot, \cdot))^{-1}.$$

The map $\tilde{\Delta}_\partial$ (resp. $\tilde{\Delta}'$) is continuous (as the composition of two continuous maps) and is such that

$$\begin{aligned}\tilde{\Delta}_\partial^{-1}(0_{\mathbb{R}^L}) &= (\tilde{D}_\partial \circ (\theta_{\mathcal{R}}(1, \cdot, \cdot))^{-1})^{-1}(0_{\mathbb{R}^L}) \\ &= \theta_{\mathcal{R}}(1, \cdot, \cdot)(\tilde{D}_\partial^{-1}(0_{\mathbb{R}^L})) \\ &= \theta_{\mathcal{R}}(1, \cdot, \cdot)(\mathcal{P}_{\mathcal{R}}) \\ &= \mathcal{R}^0 (= \mathcal{R} \times \{g^0\})\end{aligned}$$

(resp. $(\tilde{\Delta}')^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$). Thus, observe that if $\tilde{\Phi}_{\mathcal{R}}$ is a homotopy between $\tilde{\Delta}_\partial$ and $\tilde{\Delta}'$ such that for every $t \in [0, 1]$, $(\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, then

$$\left\{ \begin{array}{l} \tilde{H}_{\mathcal{R}}^2 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}(1, \cdot, \cdot))(v, g) \end{array} \right.$$

is a homotopy between \tilde{D}_∂ and \tilde{D}' such that for every $t \in [0, 1]$, the map $\tilde{H}_{\mathcal{R}}^2(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} .

Last, recall that the ambient isotopy $\theta_{\mathcal{R}}$ does not deform the boundary of the ambient space $\tilde{\mathcal{E}}_{\mathcal{R}}$, i.e. for every $t \in [0, 1]$,

$$\theta_{\mathcal{R}}(t, \cdot, \cdot)|_{\mathcal{R} \times \partial \tilde{\mathbb{G}}} = \text{id}_{\mathcal{R} \times \partial \tilde{\mathbb{G}}}.$$

Hence, from Equation (3.4), remark that

$$\tilde{\Delta}_\partial|_{\mathcal{R} \times \partial \tilde{\mathbb{G}}} = \tilde{\Delta}'|_{\mathcal{R} \times \partial \tilde{\mathbb{G}}}. \quad (3.5)$$

Step III. First step in the construction of a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_\partial$ and $\tilde{\Delta}'$.

First, define

$$\left\{ \begin{array}{l} \tilde{\Delta}^1 : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} \frac{\tilde{\Delta}_\partial(v, g)}{\|\tilde{\Delta}_\partial(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right. .$$

Continuity of this map is a consequence of Lemma 3.4.2 applied to $X = \tilde{\mathcal{E}}_{\mathcal{R}}$, $Y = \mathbb{R}^L$, $A = \mathcal{R} \times (\tilde{\mathbb{G}} \setminus \{g^0\})$,

$$f_1 : (v, g) \in A \mapsto \frac{\tilde{\Delta}_\partial(v, g)}{\|\tilde{\Delta}_\partial(v, g)\|_2} \|g - g^0\|_2 \in Y$$

and

$$f_2 : (v, g) \in X \setminus A \mapsto 0_{\mathbb{R}^L} \in Y.$$

Assumptions (i) and (iii) of Lemma 3.4.2 are clearly verified. Now, in order to check that assumption (ii) is also satisfied, let $(v^*, g^*) \in \partial A = \mathcal{R}^0$ and let $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ be a sequence in A such that $(v^\ell, g^\ell) \xrightarrow{\ell \rightarrow +\infty} (v^*, g^*)$. Then, for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \|f_1(v^\ell, g^\ell) - \tilde{\Delta}^1(v^*, g^*)\|_2 &= \left\| \left(\frac{\tilde{\Delta}_\partial(v^\ell, g^\ell)}{\|\tilde{\Delta}_\partial(v^\ell, g^\ell)\|_2} \|g^\ell - g^0\|_2 \right) - 0_{\mathbb{R}^L} \right\|_2 \quad (\text{since } g^* = g^0) \\ &= \left\| \frac{\tilde{\Delta}_\partial(v^\ell, g^\ell)}{\|\tilde{\Delta}_\partial(v^\ell, g^\ell)\|_2} \|g^\ell - g^0\|_2 \right\|_2 \\ &= \frac{\|\tilde{\Delta}_\partial(v^\ell, g^\ell)\|_2}{\|\tilde{\Delta}_\partial(v^\ell, g^\ell)\|_2} \|g^\ell - g^0\|_2 \\ &= \|g^\ell - g^0\|_2. \end{aligned}$$

Since $g^\ell \xrightarrow{\ell \rightarrow +\infty} g^*$ (by assumption), and since $g^* = g^0$, one obtains that $\|f_1(v^\ell, g^\ell) - \tilde{\Delta}^1(v^*, g^*)\|_2 \xrightarrow{\ell \rightarrow +\infty} 0$, thus that

$$f_1(v^\ell, g^\ell) \xrightarrow{\ell \rightarrow +\infty} \tilde{\Delta}^1(v^*, g^*).$$

Second, define

$$\begin{cases} \tilde{\Phi}_{\mathcal{R}}^1 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & (1-t)\tilde{\Delta}_\partial(v, g) + t\tilde{\Delta}^1(v, g) \end{cases}.$$

This map is the straight-line homotopy between $\tilde{\Delta}_\partial$ and $\tilde{\Delta}^1$. Furthermore, for every $t \in [0, 1]$, notice that $(\tilde{\Phi}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$. Indeed, for every $t \in [0, 1]$, $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ belongs to $(\tilde{\Phi}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L})$ if and only if: (i) either $g = g^0$; (ii) or $g \neq g^0$ and

$$(1-t)\tilde{\Delta}_\partial(v, g) + t \left(\frac{\tilde{\Delta}_\partial(v, g)}{\|\tilde{\Delta}_\partial(v, g)\|_2} \|g - g^0\|_2 \right) = 0_{\mathbb{R}^L}. \quad (3.6)$$

However, Equation (3.6) has no solution if $g \neq g^0$ (recall that $\tilde{\Delta}_\partial^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$), which implies that $(\tilde{\Phi}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, for every $t \in [0, 1]$.

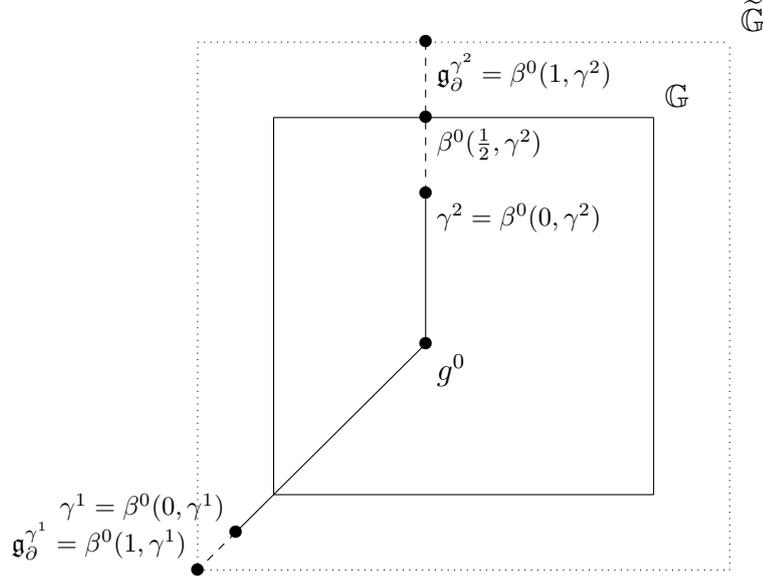


Figure 3.12: *Illustration of β^0*

Step IV. Second step in the construction of a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$.

For every $g \in \tilde{\mathbb{G}} \setminus \{g^0\}$, consider the unique element $\mathfrak{g}_{\partial}^g \in \partial\tilde{\mathbb{G}}$ such that $\mathfrak{g}_{\partial}^g \in \{g' \in \tilde{\mathbb{G}} : \exists t \in [0, +\infty), g' = (1-t)g^0 + tg'\}$, i.e. such that $\mathfrak{g}_{\partial}^g$ belongs to the half-line starting at g^0 and passing through g . Moreover, consider the map

$$\begin{cases} \beta^0 : [0, 1] \times (\tilde{\mathbb{G}} \setminus \{g^0\}) & \rightarrow & \tilde{\mathbb{G}} \\ (t, g) & \mapsto & (1-t)g^0 + t\mathfrak{g}_{\partial}^g \end{cases} .$$

This map is continuous, since the map $g \in \tilde{\mathbb{G}} \setminus \{g^0\} \mapsto \mathfrak{g}_{\partial}^g \in \partial\tilde{\mathbb{G}}$ is itself continuous.

Now, define

$$\begin{cases} \tilde{\Delta}^2 : \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (v, g) & \mapsto & \begin{cases} \frac{\tilde{\Delta}_{\partial}(v, \mathfrak{g}_{\partial}^g)}{\|\tilde{\Delta}_{\partial}(v, \mathfrak{g}_{\partial}^g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{cases} .$$

Continuity of this map is a consequence of Lemma 3.4.2 applied to $X = \tilde{\mathcal{E}}_{\mathcal{R}}$, $Y = \mathbb{R}^L$, $A = \mathcal{R} \times (\tilde{\mathbb{G}} \setminus \{g^0\})$,

$$f_1 : (v, g) \in A \mapsto \frac{\tilde{\Delta}_{\partial}(v, \mathfrak{g}_{\partial}^g)}{\|\tilde{\Delta}_{\partial}(v, \mathfrak{g}_{\partial}^g)\|_2} \|g - g^0\|_2 \in Y$$

and

$$f_2 : (v, g) \in X \setminus A \mapsto 0_{\mathbb{R}^L} \in Y.$$

Assumptions (i) and (iii) of Lemma 3.4.2 are clearly verified. Now, in order to check that assumption (ii) is also satisfied, let $(v^*, g^*) \in \partial A = \mathcal{R}^0$ and let $(v^\ell, g^\ell)_{\ell \in \mathbb{N}}$ be a sequence in A such that $(v^\ell, g^\ell) \xrightarrow{\ell \rightarrow +\infty} (v^*, g^*)$. Then, for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \|f_1(v^\ell, g^\ell) - \tilde{\Delta}^2(v^*, g^*)\|_2 &= \left\| \left(\frac{\tilde{\Delta}_\partial(v^\ell, \mathfrak{g}_\partial^{g^\ell})}{\|\tilde{\Delta}_\partial(v, \mathfrak{g}_\partial^{g^\ell})\|_2} \|g^\ell - g^0\|_2 \right) - 0_{\mathbb{R}^L} \right\|_2 \quad (\text{since } g^* = g^0) \\ &= \left\| \frac{\tilde{\Delta}_\partial(v^\ell, \mathfrak{g}_\partial^{g^\ell})}{\|\tilde{\Delta}_\partial(v^\ell, \mathfrak{g}_\partial^{g^\ell})\|_2} \|g^\ell - g^0\|_2 \right\|_2 \\ &= \frac{\|\tilde{\Delta}_\partial(v^\ell, \mathfrak{g}_\partial^{g^\ell})\|_2}{\|\tilde{\Delta}_\partial(v^\ell, \mathfrak{g}_\partial^{g^\ell})\|_2} \|g^\ell - g^0\|_2 \\ &= \|g^\ell - g^0\|_2. \end{aligned}$$

Since $g^\ell \xrightarrow{\ell \rightarrow +\infty} g^*$ (by assumption), and since $g^* = g^0$, one obtains that $\|f_1(v^\ell, g^\ell) - \tilde{\Delta}^2(v^*, g^*)\|_2 \xrightarrow{\ell \rightarrow +\infty} 0$, thus that

$$f_1(v^\ell, g^\ell) \xrightarrow{\ell \rightarrow +\infty} \tilde{\Delta}^2(v^*, g^*).$$

Last, define

$$\left\{ \begin{array}{l} \tilde{\Phi}_{\mathcal{R}}^2 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} \frac{\tilde{\Delta}_\partial(v, \beta^0(t, g))}{\|\tilde{\Delta}_\partial(v, \beta^0(t, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right. .$$

This map is a homotopy between $\tilde{\Delta}^1$ and $\tilde{\Delta}^2$:

- Again, continuity of this map is a consequence of Lemma 3.4.2 applied to $X = [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}}$, $Y = \mathbb{R}^L$, $A = [0, 1] \times (\mathcal{R} \times (\tilde{\mathcal{G}} \setminus \{g^0\}))$,

$$f_1 : (t, (v, g)) \in A \mapsto \frac{\tilde{\Delta}_\partial(v, \beta^0(t, g))}{\|\tilde{\Delta}_\partial(v, \beta^0(t, g))\|_2} \|g - g^0\|_2 \in Y$$

and

$$f_2 : (t, (v, g)) \in X \setminus A \mapsto 0_{\mathbb{R}^L} \in Y.$$

- For every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\begin{aligned}\tilde{\Phi}_{\mathcal{R}}^2(0, v, g) &= \begin{cases} \frac{\tilde{\Delta}_{\partial}(v, \beta^0(0, g))}{\|\tilde{\Delta}_{\partial}(v, \beta^0(0, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\tilde{\Delta}_{\partial}(v, g)}{\|\tilde{\Delta}_{\partial}(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \tilde{\Delta}^1(v, g).\end{aligned}$$

- For every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\begin{aligned}\tilde{\Phi}_{\mathcal{R}}^2(1, v, g) &= \begin{cases} \frac{\tilde{\Delta}_{\partial}(v, \beta^0(1, g))}{\|\tilde{\Delta}_{\partial}(v, \beta^0(1, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\tilde{\Delta}_{\partial}(v, g)}{\|\tilde{\Delta}_{\partial}(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \tilde{\Delta}^2(v, g).\end{aligned}$$

Furthermore, for every $t \in [0, 1]$, notice that $(\tilde{\Phi}_{\mathcal{R}}^2(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$. Indeed, for every $t \in [0, 1]$, $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$ belongs to $(\tilde{\Phi}_{\mathcal{R}}^2(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L})$ if and only if: (i) either $g = g^0$; (ii) or $g \neq g^0$ and

$$\frac{\tilde{\Delta}_{\partial}(v, \beta^0(t, g))}{\|\tilde{\Delta}_{\partial}(v, \beta^0(t, g))\|_2} \|g - g^0\|_2 = 0_{\mathbb{R}^L} \quad (3.7)$$

However, Equation (3.7) has no solution if $g \neq g^0$ (recall that $(\tilde{\Delta}_{\partial})^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$), which implies that $(\tilde{\Phi}_{\mathcal{R}}^2(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, for every $t \in [0, 1]$.

Step V. Third step in the construction of a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$.

First, remark that the maps $\tilde{\Delta}^2$ (defined in Step IV) and

$$\left\{ \begin{array}{l} \hat{\Delta}^2 : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} \frac{\tilde{\Delta}'(v, g)}{\|\tilde{\Delta}'(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right.$$

are equal. Indeed, recall that from Equation (3.5) of Step II,

$$\tilde{\Delta}_{\partial}|_{\mathcal{R} \times \partial\tilde{\mathcal{G}}} = \tilde{\Delta}'|_{\mathcal{R} \times \partial\tilde{\mathcal{G}}}$$

and that for every $g \in \mathbb{G} \setminus \{g^0\}$, \mathfrak{g}_∂^g belongs to $\partial\tilde{\mathbb{G}}$.

Second, define

$$\left\{ \begin{array}{l} \tilde{\Delta}^1 : \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} \frac{\tilde{\Delta}'(v, g)}{\|\tilde{\Delta}'(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right. .$$

This map is continuous (the proof is similar to the one of the continuity of $\tilde{\Delta}^1$, in Step III).

Last, define

$$\left\{ \begin{array}{l} \tilde{\Phi}_{\mathcal{R}}^3 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} \frac{\tilde{\Delta}'(v, \beta^0(1-t, g))}{\|\tilde{\Delta}'(v, \beta^0(1-t, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \end{array} \right.$$

(recall that the map β^0 is defined at the beginning of Step IV). This map is a homotopy between $\hat{\Delta}^2$ and $\hat{\Delta}^1$:

- The proof of the continuity of $\tilde{\Phi}_{\mathcal{R}}^3$ is similar to the one of $\tilde{\Phi}_{\mathcal{R}}^2$, in Step IV.
- For every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\begin{aligned} \tilde{\Phi}_{\mathcal{R}}^3(0, v, g) &= \begin{cases} \frac{\tilde{\Delta}'(v, \beta^0(1, g))}{\|\tilde{\Delta}'(v, \beta^0(1, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\tilde{\Delta}'(v, \mathfrak{g}_\partial^g)}{\|\tilde{\Delta}'(v, \mathfrak{g}_\partial^g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \hat{\Delta}^2(v, g). \end{aligned}$$

- For every $(v, g) \in \tilde{\mathcal{E}}_{\mathcal{R}}$,

$$\begin{aligned} \tilde{\Phi}_{\mathcal{R}}^3(1, v, g) &= \begin{cases} \frac{\tilde{\Delta}'(v, \beta^0(0, g))}{\|\tilde{\Delta}'(v, \beta^0(0, g))\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\tilde{\Delta}'(v, g)}{\|\tilde{\Delta}'(v, g)\|_2} \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0_{\mathbb{R}^L} & \text{otherwise} \end{cases} \\ &= \hat{\Delta}^1(v, g). \end{aligned}$$

Furthermore, for every $t \in [0, 1]$, notice that $(\tilde{\Phi}_{\mathcal{R}}^3(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$ (the proof is similar to the one of $(\tilde{\Phi}_{\mathcal{R}}^2(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, in Step IV).

Step VI. Fourth step in the construction of a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$.

Define

$$\begin{cases} \tilde{\Phi}_{\mathcal{R}}^4 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & (1-t)\hat{\Delta}^1(v, g) + t\tilde{\Delta}'(v, g) \end{cases}.$$

This map is the straight-line homotopy between $\hat{\Delta}^1$ and $\tilde{\Delta}'$. Furthermore, for every $t \in [0, 1]$, notice that $(\tilde{\Phi}_{\mathcal{R}}^4(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$ (the proof is similar to the one of $(\tilde{\Phi}_{\mathcal{R}}^1(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, in Step III).

Step VII. Last step in the construction of a homotopy $\tilde{\Phi}_{\mathcal{R}}$ between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$.

From Step III to Step VI, the map

$$\begin{cases} \tilde{\Phi}_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & \begin{cases} \tilde{\Phi}_{\mathcal{R}}^1(4t, (v, g)) & \text{if } t \in [0, \frac{1}{4}] \\ \tilde{\Phi}_{\mathcal{R}}^2(4t-1, (v, g)) & \text{if } t \in [\frac{1}{4}, \frac{1}{2}] \\ \tilde{\Phi}_{\mathcal{R}}^3(4t-2, (v, g)) & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ \tilde{\Phi}_{\mathcal{R}}^4(4t-3, (v, g)) & \text{if } t \in [\frac{3}{4}, 1] \end{cases} \end{cases}$$

is a homotopy between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$ such that for every $t \in [0, 1]$,

$$(\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0.$$

Step VIII. Last step in the construction of a homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D} and \tilde{D}' .

Following the remark made in Step II, since $\tilde{\Phi}_{\mathcal{R}}$ is a homotopy between $\tilde{\Delta}_{\partial}$ and $\tilde{\Delta}'$ such that for every $t \in [0, 1]$, $(\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot))^{-1}(0_{\mathbb{R}^L}) = \mathcal{R}^0$, then

$$\begin{cases} \tilde{H}_{\mathcal{R}}^2 : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & (\tilde{\Phi}_{\mathcal{R}}(t, \cdot, \cdot) \circ \theta_{\mathcal{R}}(1, \cdot, \cdot))(v, g) \end{cases}$$

is a homotopy between \tilde{D}_{∂} and \tilde{D}' such that for every $t \in [0, 1]$, the map $\tilde{H}_{\mathcal{R}}^2(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} . Finally, from Step I, the map

$$\begin{cases} \tilde{H}_{\mathcal{R}} : [0, 1] \times \tilde{\mathcal{E}}_{\mathcal{R}} & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & \begin{cases} \tilde{H}_{\mathcal{R}}^1(2t, (v, g)) & \text{if } t \in [0, \frac{1}{2}] \\ \tilde{H}_{\mathcal{R}}^2(2t-1, (v, g)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{cases}$$

is the required homotopy between \tilde{D} and \tilde{D}' such that for every $t \in [0, 1]$, the map $\tilde{H}_{\mathcal{R}}(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} .

This step ends the proof of the dynamics equivalence theorem. \square

3.4.5 Proof of indices equality theorem

Before to start the proof, let us precise that we do not use here the identification of Theorem 3.4.1 (see Appendix 3.4.2), unlike in the proof of Theorem 3.3.1.

First, consider the homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D} and \tilde{D}' defined in Theorem 3.3.1 (in the following, $\tilde{H}_{\mathcal{R}}$ is simply denoted \tilde{H}). Observe that the map

$$\tilde{H}_v^1 : (t, \gamma) \in [0, 1] \times \tilde{\mathbb{G}} \mapsto \tilde{H}(t, (v, \gamma)) \in T\tilde{\mathbb{G}}$$

is a homotopy between \tilde{D}_v and \tilde{D}'_v such that for every $t \in [0, 1]$,

$$\mathbf{Z}(\tilde{H}_v^1(t, \cdot)) = \mathbf{Z}(\tilde{D}_v) = \mathbf{Z}(\tilde{D}'_v) = \text{PS}(v), \quad (3.8)$$

where $\text{PS}(v) = \{\gamma \in \mathbb{G} : \gamma \text{ is pairwise stable with respect to } v\}$, since by definition, for every $t \in [0, 1]$, $\tilde{H}(t, \cdot, \cdot)$ is an extended network dynamic on \mathcal{R} .

Also, let g be an isolated zero of \tilde{D}_v and (U, ϕ) be an interior chart of $\tilde{\mathbb{G}}$ such that $g \in U$ (since $g \in \mathbb{G}$ and $\mathbb{G} \subset \text{int}(\tilde{\mathbb{G}})$, g is necessarily an interior zero of \tilde{D}_v).¹⁰ The index of \tilde{D}_v at g corresponds to the index of the pushforward

$$\phi_* \tilde{D}_v : x \in \phi(U) \mapsto (T_{\phi^{-1}(x)} \phi \circ \tilde{D}_v \circ \phi^{-1})(x) \in T\phi(U)$$

of \tilde{D}_v by ϕ at $\phi(g)$ (see Definition 3.4.13 in Appendix 3.4.2), which corresponds to the index of the map $\Psi_{*,c}^{-1}(\phi_* \tilde{D}_v) : \phi(U) \rightarrow \mathbb{R}^L$ at $\phi(g)$ (see Definition 3.4.10 in Appendix 3.4.2), which is finally equal to the topological degree of the map

$$\begin{cases} [\Psi_{*,c}^{-1}(\phi_* \tilde{D}_v)]^{\phi(g)} : \partial \bar{B}(\phi(g), \epsilon) & \rightarrow \mathbb{S}^{|L|-1} \\ x & \mapsto \frac{\Psi_{*,c}^{-1}(\phi_* \tilde{D}_v)(x)}{\|\Psi_{*,c}^{-1}(\phi_* \tilde{D}_v)(x)\|_2} \end{cases}$$

(see Definition 3.4.9 in Appendix 3.4.2), where $\epsilon > 0$ is such that

$$\bar{B}(\phi(g), \epsilon) \cap ([\Psi_{*,c}^{-1}(\phi_* \tilde{D}_v)]^{-1}(0_{\mathbb{R}^L}) \setminus \{\phi(g)\}) = \emptyset \quad (3.9)$$

¹⁰From Equation (3.8), notice that g is also an interior isolated zero of \tilde{D}'_v , and of $\tilde{H}_v^1(t, \cdot)$, for every $t \in [0, 1]$.

(because g is isolated, such a ϵ is well-defined, and fixed from now on).

First of all, observe that from Equation (3.8), ϵ is also such that

$$\overline{B}(\phi(g), \epsilon) \cap ([\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}'_v)]^{-1}(0_{\mathbb{R}^L}) \setminus \{\phi(g)\}) = \emptyset,$$

and even such that

$$\overline{B}(\phi(g), \epsilon) \cap ([\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{H}_v^1(t, \cdot))]^{-1}(0_{\mathbb{R}^L}) \setminus \{\phi(g)\}) = \emptyset,$$

for every $t \in [0, 1]$. Indeed, observe that for every $t \in [0, 1]$,

$$\mathbf{Z}(\phi_* \tilde{H}_v^1(t, \cdot)) = \mathbf{Z}(\phi_* \tilde{D}_v) = \mathbf{Z}(\phi_* \tilde{D}'_v) = \phi(\text{PS}(v) \cap U); \quad (3.10)$$

for every $x \in \phi(U)$, the tangent map $\mathbb{T}_{\phi^{-1}(x)}\phi$ of ϕ at $\phi^{-1}(x)$ being a vector space isomorphism (because $\phi : U \rightarrow \phi(U)$ is by definition a diffeomorphism), $\ker(\mathbb{T}_{\phi^{-1}(x)}\phi)$ is reduced to the set containing only the zero vector $\overrightarrow{0}_{\phi^{-1}(x)}$ of $\mathbb{T}_{\phi^{-1}(x)}\tilde{\mathbb{G}}$. Last, from Remark 3.4.3, for every $t \in [0, 1]$,

$$[\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{H}_v^1(t, \cdot))]^{-1}(0_{\mathbb{R}^L}) = \mathbf{Z}(\phi_* \tilde{H}_v^1(t, \cdot)). \quad (3.11)$$

One obtains the result using together Equation (3.9), Equation (3.10) and Equation (3.11). In particular (for $t = 1$), this implies that the index of \tilde{D}'_v at g is equal to the topological degree of the map

$$\begin{cases} [\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}'_v)]^{\phi(g)} : \partial \overline{B}(\phi(g), \epsilon) & \rightarrow \mathbb{S}^{|\mathbb{R}^L|-1} \\ x & \mapsto \frac{\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}'_v)(x)}{\|\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}'_v)(x)\|_2} \end{cases} \cdot$$

Now, remark that it is sufficient to show that the maps $[\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}_v)]^{\phi(g)}$ and $[\Psi_{\star, \mathfrak{c}}^{-1}(\phi_* \tilde{D}'_v)]^{\phi(g)}$ are homotopic in order to obtain that $\text{ind}_g(\tilde{D}_v) = \text{ind}_g(\tilde{D}'_v)$. Define the maps

$$\tilde{H}_v^2 : (t, x) \in [0, 1] \times \phi(U) \mapsto \phi_* \tilde{H}_v^1(t, x) = (\mathbb{T}_{\phi^{-1}(x)}\phi \circ \tilde{H}_v^1(t, \cdot) \circ \phi^{-1})(x) \in \mathbb{T}\phi(U)$$

and

$$\tilde{H}_v^3 : (t, x) \in [0, 1] \times \phi(U) \mapsto (\text{pr}_{\mathbb{R}^L} \circ F_{\mathfrak{c}} \circ \tilde{H}_v^2(t, \cdot))(x) \in \mathbb{R}^L,$$

where $F_{\mathfrak{c}}$ is defined in Remark 3.4.3.

$$\begin{array}{ccccc} \mathbb{T}\tilde{\mathbb{G}} & \xrightarrow{\mathbb{T}\phi} & \mathbb{T}\phi(U) & \xrightarrow{F_{\mathfrak{c}}} & \phi(U) \times \mathbb{R}^L \xrightarrow{\text{pr}_{\mathbb{R}^L}} \mathbb{R}^L \\ \tilde{H}_v^1 \uparrow & & \tilde{H}_v^2 \uparrow & & \tilde{H}_v^3 \uparrow \\ [0, 1] \times \tilde{\mathbb{G}} & \xleftarrow{\text{id}_{[0,1]} \times \phi^{-1}} & [0, 1] \times \phi(U) & \xrightarrow{\tilde{H}_v^3} & \mathbb{R}^L \end{array}$$

Observe that \tilde{H}_v^2 is a homotopy between the pushforward $\phi_*\tilde{D}_v$ of \tilde{D}_v by ϕ and the pushforward $\phi_*\tilde{D}'_v$ of \tilde{D}'_v by ϕ : it is continuous (as the composition of continuous maps), and for every $x \in \phi(U)$, $\tilde{H}_v^2(0, x) = \phi_*\tilde{D}_v(x)$ and $\tilde{H}_v^2(1, x) = \phi_*\tilde{D}'_v(x)$. In a similar way, observe that \tilde{H}_v^3 is a homotopy between $\Psi_{*,c}^{-1}(\phi_*\tilde{D}_v) = \text{pr}_{\mathbb{R}^L} \circ F_c \circ \phi_*\tilde{D}_v$ and $\Psi_{*,c}^{-1}(\phi_*\tilde{D}'_v) = \text{pr}_{\mathbb{R}^L} \circ F_c \circ \phi_*\tilde{D}'_v$.

Now, notice that the map

$$\begin{cases} \tilde{H}_v : [0, 1] \times \partial\bar{B}(\phi(g), \epsilon) & \rightarrow \mathbb{S}^{|L|-1} \\ (t, x) & \mapsto \frac{\tilde{H}_v^3(t, x)}{\|\tilde{H}_v^3(t, x)\|_2} \end{cases}$$

is a well-defined homotopy between $[\Psi_{*,c}^{-1}(\phi_*\tilde{D}_v)]^{\phi(g)}$ and $[\Psi_{*,c}^{-1}(\phi_*\tilde{D}'_v)]^{\phi(g)}$. Indeed, let us demonstrate that

$$([0, 1] \times \bar{B}(\phi(g), \epsilon)) \cap ((\tilde{H}_v^3)^{-1}(0_{\mathbb{R}^L}) \setminus ([0, 1] \times \{\phi(g)\}))$$

is empty. By contradiction, suppose that there exists $(t, x) \in [0, 1] \times \bar{B}(\phi(g), \epsilon)$ which belongs also to $(\tilde{H}_v^3)^{-1}(0_{\mathbb{R}^L}) \setminus ([0, 1] \times \{\phi(g)\})$. By definition,

$$\begin{aligned} & (\tilde{H}_v^3)^{-1}(0_{\mathbb{R}^L}) \\ &= \{(t, x) \in [0, 1] \times \phi(U) : (\text{pr}_{\mathbb{R}^L} \circ F_c \circ \tilde{H}_v^2(t, \cdot))(x) = 0_{\mathbb{R}^L}\} \\ &= \{(t, x) \in [0, 1] \times \phi(U) : (\text{pr}_{\mathbb{R}^L} \circ F_c \circ T_{\phi^{-1}(x)}\phi \circ \tilde{H}_v^1(t, \cdot) \circ \phi^{-1})(x) = 0_{\mathbb{R}^L}\}. \end{aligned}$$

Hence, by definition of F_c (cf. Remark 3.4.3) and of the projection $\text{pr}_{\mathbb{R}^L}$, (t, x) belongs to $(\tilde{H}_v^3)^{-1}(0_{\mathbb{R}^L}) \setminus ([0, 1] \times \{\phi(g)\})$ if and only if $x \neq \phi(g)$ and

$$(T_{\phi^{-1}(x)}\phi \circ \tilde{H}_v^1(t, \cdot) \circ \phi^{-1})(x) = \phi_*\tilde{H}_v^1(t, x) = \vec{0}_x,$$

where $\vec{0}_x$ denotes the zero vector of $T_x\phi(U)$. However, recall from Equation (3.9), Equation (3.10) and Equation (3.11), that for every $t' \in [0, 1]$,

$$\mathbf{Z}(\phi_*\tilde{H}_v^1(t', \cdot)) = \mathbf{Z}(T_{\phi^{-1}(x)}\phi \circ \tilde{H}_v^1(t', \cdot) \circ \phi^{-1}) = \phi(\mathbf{PS}(v) \cap U) = \mathbf{Z}(\phi_*\tilde{D}_v).$$

Furthermore, as explained above and by definition of ϵ ,

$$\bar{B}(\phi(g), \epsilon) \cap (\mathbf{Z}(\phi_*\tilde{D}_v) \setminus \{\phi(g)\}) = \emptyset,$$

which implies that for every $t' \in [0, 1]$,

$$\bar{B}(\phi(g), \epsilon) \cap (\mathbf{Z}(\phi_*\tilde{H}_v^1(t', \cdot)) \setminus \{\phi(g)\}) = \emptyset.$$

In particular, one obtains that $\bar{B}(\phi(g), \epsilon) \cap (\mathbf{Z}(\phi_*\tilde{H}_v^1(t, \cdot)) \setminus \{\phi(g)\})$ is empty, which contradicts the fact that (t, x) belongs to the set $(\tilde{H}_v^3)^{-1}(0_{\mathbb{R}^L}) \setminus ([0, 1] \times \{\phi(g)\})$.

Therefore,

$$\begin{cases} \tilde{H}_v : [0, 1] \times \partial\bar{B}(\phi(g), \epsilon) & \rightarrow \mathbb{S}^{|L|-1} \\ (t, x) & \mapsto \frac{\tilde{H}_v^3(t, x)}{\|\tilde{H}_v^3(t, x)\|_2} \end{cases}$$

is a well-defined homotopy between $[\Psi_{\star, \mathfrak{c}}^{-1}(\phi_*\tilde{D}_v)]^{\phi(g)}$ and $[\Psi_{\star, \mathfrak{c}}^{-1}(\phi_*\tilde{D}'_v)]^{\phi(g)}$, which implies that $\text{ind}_g(\tilde{D}_v) = \text{ind}_g(\tilde{D}'_v)$. \square

Part III

Conclusion

This thesis aims to provide elements for the study of the topological structure as well as the generic oddness of both the graph of pairwise stable networks (in network formation theory) and the graph of Nash equilibria (in game theory), in the semi-algebraic case.

In the first chapter, we first show that the graph of pairwise stable networks associated to any \mathcal{A} -regular set \mathcal{R} of societies can be continuously deformed into the set \mathcal{R} itself. A rather natural question is then the following: can we obtain a topological structure result based on a generalization of \mathcal{A} -regularity? On one hand, the second chapter of this thesis provides a beginning of an answer to this question by considering sets of games which do not have necessarily a product form $\prod_{i \in N} \mathcal{U}_i$ (contrary to Chapter 1, where sets of societies do have a product form by assumption). On the other hand, one can observe that \mathcal{A} -invariance assumption is linked to the form of the homeomorphism of the structure theorem (see the proof of Theorem 1.2.1). If we would consider another homeomorphism, then we could maybe weaken \mathcal{A} -invariance assumption, thus weaken \mathcal{A} -regularity. Also, we prove in this first chapter that generically, each society belonging to any \mathcal{A} -semi-algebraically regular fixed set of societies has an odd number of pairwise stable networks. In a somewhat analogous way, one can then ask whether we can obtain a generalization of this result by assuming a regularity hypothesis a little weaker than the one considered, i.e. than \mathcal{A} -semi-algebraic regularity (also in this case, chapter two provides some answers).

In the second chapter, we prove in a rather similar way that the graph of Nash equilibria associated to any \mathcal{A} -regular set \mathcal{R} of games can be continuously deformed into the set \mathcal{R} in question, but also that generically, each game belonging to any fixed \mathcal{A} -semi-algebraically regular set of games has an odd number of Nash equilibria. One can first of all wonder if it is possible to generalize the two previous results by weakening the hypotheses of \mathcal{A} -regularity and of \mathcal{A} -semi-algebraic regularity (in a quite similar way to the case of network formation theory). Also, one can wonder if it is possible to extend the previous results beyond the semi-algebraic case, i.e. assuming more general sets of strategies (for example, o-minimal structures which can be seen as an axiomatic treatment of semi-algebraic geometry). However, a much more fundamental question arises from the following observation: one can note a certain parallelism between game theory (with the concept of Nash equilibrium) and network formation theory (with pairwise stability). The question would then be to ask whether we can formulate this similarity mathematically. More precisely, would there be a “nice” way to associate to any society a game whose Nash equilibria are related to the pairwise stable networks of the underlying society, and conversely, to associate to any game a society

whose pairwise stable networks would be related to the Nash equilibria of the underlying game.

In the last third chapter, we first prove that the graph of pairwise stable networks associated to any \mathcal{A} -regular set \mathcal{R} of societies has no knots (i.e. there is an ambient isotopy between the graph of pairwise stable networks associated to \mathcal{R} and a trivial copy of \mathcal{R} itself). This result improves the one about the topological structure of the graph of pairwise stable networks associated to any \mathcal{A} -regular set of societies which is established in the first chapter. Secondly, we introduce the notion of network dynamic, then we show that any two network dynamics D and D' on any arbitrary \mathcal{A} -regular set \mathcal{R} of societies are homotopic within the set of all network dynamics on \mathcal{R} . An example of problem relating to network dynamics is the following: is it possible to highlight different types of pairwise stable networks by calculating their indices (e.g. a pairwise stable unweighted network, whose weights are either equal to zero or to one, can perhaps be characterized by a particular value of its index). Finally, another question would be to ask whether it would be possible to obtain network dynamics analogous to well-known Nash dynamics in game theory such as for example replicator dynamic or Gul-Pearce-Stacchetti dynamic (for best-response dynamic, see Example 3.3.1 in Chapter 3).

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