Spin chain with A and D-type algebra and Coderivative

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Abstract

In this thesis, our investigation focuses on spin chains exhibiting fundamental symmetries, these spin chains are recognized as the fundamental integrable systems. We delve into analyzing particular integrable quantum spin chains characterized by these symmetries. These spin chains are regarded as the introductory phase in the exploration of certain two-dimensional field theories. Notably, several functional relationships inherent within these spin chains have been extrapolated to field theories through the application of a finite set of equations, with the aim of discerning their spectrum.

This thesis commences with an exposition on the extensively researched rational spin chain endowed with GL(n) symmetry, employing the Coderivative operator to construct a polynomial Q-operator facilitating the diagonalization of the Hamiltonian. We demonstrate its equivalence to an alternative construction rooted in representations that are explicitly formulated in terms of harmonic oscillators.

Subsequently, our focus shifts to the comparatively less explored spin chain characterized by SO(2r) symmetry. Within representations where Q-operators are accessible, we proceed to construct these operators. Subsequently, we employ various methodologies to construct the operators above for general representations. These efforts indicate that the Coderivative alone may not be adequate for describing general representations within auxiliary space. However, we remain optimistic that these endeavors will contribute to identifying additional tools necessary for their characterization.

Résumé

Dans cette thèse, notre investigation se concentre sur les chaînes de spins présentant des symétries fondamentales, ces chaînes de spins étant reconnues comme des systèmes intégrables fondamentaux. Nous plongeons dans l'analyse de chaînes de spins quantiques intégrables particulières caractérisées par ces symétries. Ces chaînes de spins sont considérées comme la phase introductive dans l'exploration de certaines théories de champs bidimensionnelles. Notamment, plusieurs relations fonctionnelles inhérentes à ces chaînes de spins ont été extrapolées aux théories de champs grâce à l'application d'un ensemble fini d'équations, afin de discerner leur spectre.

Cette thèse débute par une exposition sur des chaînes de spins largement étudiées: les chaînes de spins rationnelles dotées de la symétrie GL(n). Pour ces chaînes, l'opérateur codérivée permet construire un opérateur polynomial Q facilitant la diagonalisation de l'hamiltonien. Nous démontrons son équivalence avec une construction alternative basée sur des représentations explicitement formulées en termes d'oscillateurs harmoniques.

Ensuite, notre attention se tourne vers des chaînes de spins caractérisées par la symétrie SO(2r), qui sont comparativement moins explorée . Dans les représentations où les opérateurs Q sont accessibles, nous procédons à la construction de ces opérateurs. Ensuite, nous employons différentes méthodologies pour construire les opérateurs ci-dessus pour des représentations générales. Ces efforts indiquent que le codérivé seul pourrait ne pas être adéquat pour décrire les représentations générales dans l'espace auxiliaire. Cependant, nous restons optimistes quant au fait que ces efforts contribueront à identifier des outils supplémentaires nécessaires à leur caractérisation.

Chapter I

Introduction

In 1925, one of the earliest resolution of a classical one-dimensional many-body interacting systems was made by E. Ising [Isi24]; this will be later known as the *Ising model*. Soon after that, W. Heisenberg was studying the quantum origins of magnetism in material [Hei28]. One can argue that these attempts are the origin of what we know in the field of theoretical physics as *Spin chains*.

The Heisenberg spin chain is a system (lattice) of spin-1/2 particles with a nearest neighbors interactions. A general form of this system (with no external sources) is known in the literature as the XYZ-spin chain where one can write the totally anisotropic quantum Hamiltonian as:

$$H_{XYZ} = -\sum_{i=1}^{N} \left(\gamma_1 \sigma_i^{(1)} \sigma_{i+1}^{(1)} + \gamma_2 \sigma_i^{(2)} \sigma_{i+1}^{(2)} + \gamma_3 \sigma_i^{(3)} \sigma_{i+1}^{(3)} \right)$$

In the last expression, $\sigma_i^{(k)}$ are the local spin operators, which act as Pauli matrices in the i^{th} local subspace of the total Hilbert space, and $k = 1, 2, 3$. The physical interpretation of the coefficients γ_k is known as *Slater's exchange integral*. The last equation describes the simple case of one-dimensional spin chain of length N with periodicity condition $\sigma_{N+1}^{(k)} = \sigma_1^{(k)}$. Know models can be extracted from the previous relation; by setting the longitudinal anisotropy coupling constants $\gamma_1 = \gamma_2 = \gamma$ and $\gamma_3 = \Delta\gamma$, one obtains the famous XXZ-model, whereas by setting the isotropic constants $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, we obtain the famous XXX-model, which will be the studied system in this thesis.

In 1931, Hans Bethe [Bet31] showed that the XXX-model can be solved analytically to obtain exact wavefunctions and their eigenvalues; this method is now known as *coordinate Bethe ansatz*. It parametrizes the Bethe wavefunctions with a complex *spectral parameters* that satisfies a system of equations known as *Bethe equations*, and the roots of these equations are known as *Bethe roots*. Bethes work was generalized to other one-dimensional quantum systems; the most notable examples include the non-linear Schrodinger equations NLS model [LL63] and the XXZ-model [Orb58].

From the exact solutions of the Ising model by Onsager [Ons44] up to that of the hard hexagon model by Baxter [Bax80], researchers have repeatedly observed the remarkable similarity between one-dimensional quantum integrable models and exactly solvable models of

two-dimensional statistical physics. The most notable example of this similarity is the work by Lieb [Lie67], where he used the transfer matrix method to find the six-vertex model wavefunction and verify the similarity with the wavefunction obtained from the Bethe ansatz for the XXZ-spin model. In 1968, the work done by Lax [Lax68]and Gardner, Greene, Kruskal and Miura [GGKM67] on the Korteweg and de Vries (KdV) equation led to the realisation that non-linear problem of the KdV equation can be rephrased in terms of evolution problem of a linear operator, which surprisingly turns out to be the Schrödinger operator in the case of KdV equation. This method of using the Lax operator was further developed by Zakharov, Faddeev and Shabat in [ZS70, FZ71, ZS79] and it came to be known as classical inverse scattering method. These simultaneous developments in the classical integrable systems as well as exactly solvable lattice models played a crucial role towards the development of the *quantum inverse scattering method* by Faddeev, Sklyanin and Takhtadzhyan [FST79].

At the same time, in a different branch of physics, string theory was developed, and a series of major steps paved the way for connecting the two sectors of physics. Historically, string theory was seen as a candidate for a unified theory of everything (starting as a theory that will describe strong interaction [Ven68]), that is, quantum gravity alongside gauge theories that describe all other interactions in nature. One of these major steps was done by 'tHooft in [tH74]. In his work, 'tHooft demonstrated that in a gauge theory with $SU(\mathsf{n})$ symmetry, if we take a large n limit, the expansion in terms of Feynman diagrams turns out to be the string perturbative expansion over Riemann surfaces. Similar work by Wilson [Wil74] shows that the strong coupling expansions can be seen as propagating closed strings.

Another significant milestone was the establishment of Gauge/String Dualities by Maldacena [Mal98]. Soon after that, the work by Gubser, Klebanov, Polyakov [GKP98] and Witten [Wit96] was the elaboration on some of the important aspects of correspondence. In his work, Maldacena showed that a four-dimensional supersymmetric gauge theory $\mathcal{N} = 4$ with $SU(\mathsf{n})$ gauge group is equivalent to type IIB string theory in $AdS_5 \times S^5$ background by proposing a conjecture that relaizes the gauge/string duality. Now, one can foresee a four-dimensional theory as a boundary of AdS_5, while the S^5 represents the geometry that arises due to the internal supersymmetry of the gauge theory. This formulation has laid the groundwork for comprehending the duality between certain gravity theories and quantum field theories, i. e. the AdS/CFT correspondence.

In 2002, significant progress was achieved by Berenstein, Maldacena, and Nastase [BMN02], as well as by Gubser, Klebanov, and Polyakov [GKP02], in the field of AdS/CFT duality. Their work employed the semiclassical approach to string theory, focusing on perturbative corrections around classical solutions of the string equation of motion in the $AdS_5 \times S^5$ background. Specifically, they explored expressing the theory in terms of alternative quantum numbers at both classical and quantum-corrected levels. Subsequently, in 2003, Minahan and Zarembo [MZ03] provided initial evidence of integrability in $\mathcal{N} = 4$ Super Yang-Mills, establishing a connection between the one-loop dilatation operator and a spin chain with $SO(6)$ symmetry. This revelation paved the way for the extension of integrability to all one-loop operators, followed by

subsequent advancements to encompass two-loop and three-loop operators within $\mathcal{N} = 4$ Super Yang-Mills. Ultimately, Beisert et al. [BKS03] conjectured that integrability might extend to all-loop features. The presence of integrability on both sides of the duality is particularly notable, as it manifests through the analogy between the dilatation operator in $\mathcal{N} = 4$ Super Yang-Mills and the Hamiltonian of a one-dimensional spin chain in the weak coupling limit. This alignment suggests promising avenues for further exploration, potentially revealing additional insights into the theory's properties through the utilization of the Yang-Baxter ansatz.

In this thesis, we will explore the construction methods outlined in the literature for a set of commuting transfer matrices (T-operators) of spin chains. These matrices are linked to the simplest representations of quantum groups, which are in turn associated with finite-dimensional representations of classical Lie algebras. Additionally, the functional relations within this theory play a significant role in the study of integrable quantum systems. These operators appear to offer a solution for certain bilinear relationships known as the Hirota equation. This concept was further generalized in [KP92, KNS94], where an infinite system of such equations was derived. This infinite system of equations, coupled with the constraint of polynomiality (analyticity in field theory) in T-operators, is referred to as T-systems. Remarkably, these T-systems have proven to be ubiquitous structures, demonstrating relevance in solving $\mathcal{N} = 4$ Super Yang-Mills equations. The T-system is not the exclusive common structure among many integrable systems; others, including the Y-system and Q-systems, also exist. However, within this study, we will refrain from exploring Y-system. The Q-system, on the other hand, comprises a system of equations, known as QQ-relations, involving a finite number of commuting operators, in contrast to the infinite number of T-functions.

It has been observed that the Bethe equations, tasked with encoding the spectrum, can be reformulated as polynomial equations derived from QQ-relations, which operate upon a family of polynomial functions recognized as Q-functions. Subsequently, it became apparent that when extracting the spectrum from these QQ-relations, no 'unphysical solutions' arise, in contrast to the Bethe equations where such solutions necessitate identification and elimination. This was demonstrated by Huang, Lu, and Mukhin in [HLM19], and Chernyak, Leurent, and Volin in [CLV22], where they proved that the algebraic number of solutions to these relations is equal to the dimension of the spin chain Hilbert space. The establishment of these Q-functions proves challenging, intertwined with the completeness of Bethe Ansatz, thus demanding their explicit construction: they manifest as the eigenvalues of the Q-operators. Consequently, this manuscript will thoroughly examine the construction of these Q-operators. Furthermore, the T-functions can be represented as determinants of shifted Q-functions [KLWZ97], facilitating a compact diagonalization of the Hamiltonian. This compactness arises from the fact that QQ-relations encode the algebraic properties of the system's conserved charges.

Until recently, the study of the Q-system was far less developed compared to the T- and Y-systems. In fact, the first appearance of the Q-system was in the work by Baxter on the eight-vertex model [Bax72]. Fortunately, much progress in this topic was achieved by doing a case-by-case study of various models; some noticeable examples are [Bax72, Bax82, BS90,

GP92, BLZ97, Smi00, BT06]. From these studies, a theory that allows constructing Q-operators was needed. In 2011, the work done by V. Bazhanov, Lukowski, Meneghelli, Staudacher and Frassek[BLMS10, BFL$^+$11] to construct Q-operators by using the theory of Yangians (which is the simplest example of a quantum groups) was introduced for a spin chain with $GL(\mathsf{n})$ symmetry, where they utilized new aspects of the theory of infinite-dimensional representations of Yangians, which led to the construction of Q-operators, more details about the construction of these operators is in Appendix(A.1).

Prior to the construction using oscillator representations, an idea introduced by Kazakov and Vieira [KV08] that allowed the construction of conserved T-operators using a differential operator called *Coderivative*. This method reduced the algebraic understanding needed in constructing T-operators to only knowing the character and the generating series of the representation introduced in the auxiliary space. Additionally, it reduced the construction of Q-operators to a combinatorial diagrammatical description. After that, Kazakov, Leurent, and Tsuboi [KLT12] used the same differential operator to give an operatorial form to the functional *TQ-relations* for the supersymmetric spin chain with $GL(n|m)$ symmetry; this allowed to define all the T- and Q-operators in all levels of nesting.

The main objective behind the work presented in this thesis is to generalize the ideas presented in [KV08, KLT12] to a different symmetry group other than $GL(\mathsf{n})$. We study the case of a spin chain with $SO(2\mathsf{r})$ symmetry in the auxiliary space and introduce a modified version of the Coderivative operator to describe the conserved T-operators and Q-operators for the new symmetry. The other objective would be to show the equivalence between the oscillator and differential constructions of Q-operators.

Outline of the thesis

This thesis is organized into two chapters. The first chapter is dedicated to discussing the construction of spin chain operators with a $GL(\mathsf{n})$ symmetry in the auxiliary space. The second chapter uses the ideas introduced in the first chapter to model a system of spin chains with $SO(2\mathsf{r})$ symmetry.

The first chapter is divided into two sections. The initial section, as detailed in the first chapter, commences with a brief introduction to the transfer matrix (T-operators) of a spin chain with a twist in the auxiliary space and its connection to the Heisenberg spin chain. Then we proceed to formulate the findings established in [BFL$^+$11] concerning the construction of the oscillator description of the$R$-matrix. Finally, rewriting the T-operators using the the Coderivative operator intruduced in [KV08], and as an original result, in Section II.1.5 we show that a Yang-Baxter equation with one auxiliary space is satisfied due to *Schwartz's property* of the differential operator. In the subsequent section, Section II.2, we argue the equivalence between the two construction methodologies discussed in the preceding section of the chapter. This is accomplished by composing the Wronskian determinant of shifted Q-functions, enabling us to express the transfer matrix. Leveraging the concepts introduced in [BLMS10, BFL$^+$11, FKT21], we decode the nested expression of the transfer matrix, utilizing the nested transfer

matrix expression uncovered in [KLT12]. Additionally, we employ the concept of normalized limits, introduced in the notation section of the introduction.

The second chapter, as detailed in Chapter III, also commences by providing a brief overview of the existing literature concerning spin chains with $SO(2\mathsf{r})$ symmetry. Here, we apply the same methodologies introduced in the first chapter to express the transfer matrix T- and Q-operators with the new symmetry. In Section III.1.3, we present a modified definition of the Coderivative operator, enabling us to reformulate the expressions of T-operators for the spinorial and symmetric representations. Subsequently, we commence a discussion on the reformulation of the Q-operators for these representations based on their functional relations. Following this, in Section III.3, we explore various methods to verify the limits of the available solutions and attempt to formulate an expression for T-operators with general rectangular representations in the auxiliary space. Additionally, we allude to the notion that utilizing only the Coderivative operator may be inadequate for describing a general case of the rectangular representation.

I.1 Notations and symbols

Throughout this script, we are going to be using the concept physical space frequently, so it will be very convenient to define the basis of such a system basis for our Hilbert space to be able to introduce all operators needed in the study of this system. As we will see in the first chapter, the Hilbert space is defined as $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$, where every small $\mathcal{H}_i$ defines a physical space in our spin chain (represented as a vertical line in Fig(II.1)). Thus, we need to assign an orthonormal basis for each space in our Hilbert space, i. e. for the space $\mathcal{H}_i$ we have the vectors $|1\rangle_i, |2\rangle_i, \ldots, |d_i\rangle_i$ representing it [1]. With that, our Hilbert space is defined with a set of orthonormal vectors as:

$$|m_1, m_2, \ldots, m_N\rangle \equiv |m_1\rangle_1 \otimes |m_2\rangle_2 \otimes \cdots \otimes |m_N\rangle_N \tag{I.1.1}$$

with each m_i to be finite and each m_i to be smaller than d_i. Now, we are ready to define the necessary operators for our systems. In this study, it's more convenient to introduce a general definition of an arbitrary linear operator $\mathcal{O}$ (an N tensor operator) such that the matrix/tensor coefficients take the form:

$$\mathcal{O}^{i_1,i_2,\ldots,i_N}_{j_1,j_2,\ldots,j_N} = \langle i_1, i_2, \ldots, i_N| \, \mathcal{O} \, |j_1, j_2, \ldots, j_N\rangle \tag{I.1.2}$$

Where (i, j) in the previous relation will denote what we will refer to throughout this text as (outgoing/ingoing) indices, respectively. The number of these indices refers to the length of the chain; each ingoing index can be contracted with an outgoing or an ingoing index.

Among the pivotal operators in this study are $\mathcal{P}$ and $\mathcal{Q}$. Their matrix coefficients play a crucial role in comprehending our systems. Beginning with $\mathcal{P}$, we can elucidate its actions on a basis vector as illustrated in (I.1.1):

$$\mathcal{P}_\sigma : |m_1, m_2, \ldots, m_N\rangle \mapsto \left|m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(N)}\right\rangle \tag{I.1.3a}$$

[1] We are considering the case when $\mathcal{H}_i$'s are finite-dimensional, i. e. d_i is a finite number

$$\text{i. e.} \qquad (\mathcal{P}_\sigma)^{i_1,i_2,\ldots,i_N}_{j_1,j_2,\ldots,j_N} = \prod_{k=1}^{N} \delta^{\sigma(i_k)}_{j_k} \tag{I.1.3b}$$

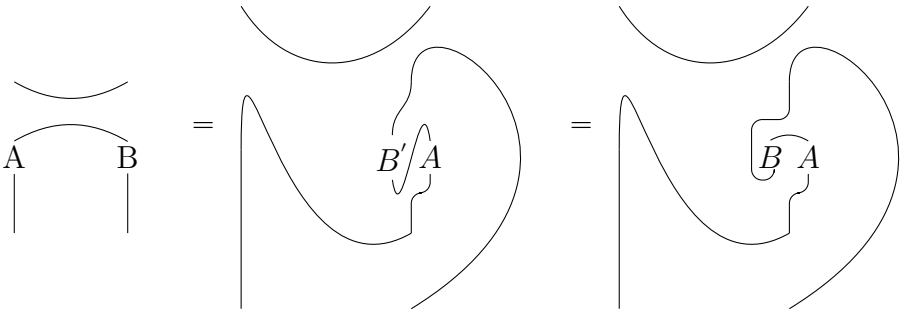
where $\sigma \in \mathcal{S}_N$. This operator arranges the way we link the ingoing index with the outgoing ones. A similar statement can be found for $\mathcal{Q}$, this operator is a bit more complicated to explain, but it is defined as:

$$(\mathcal{Q}_{a,b})^{i_1,i_2,\ldots,i_N}_{j_1,j_2,\ldots,j_N} = \delta^{i_a}_{i_b^{'}} \delta^{j_a^{'}}_{j_b} \prod_{\substack{k=1 \\ k\neq a,b}}^{N} \delta^{i_k}_{j_k} \tag{I.1.4}$$

The prime means that we flipped the ingoing/outgoing labels before connecting them, i. e. $\left(A^i_j\right)^{'} = A^{j^{'}}_{i^{'}}$[2]. A very interesting property of $\mathcal{Q}$ comes from the way it connects operators to one another. To see that, let us take the tensor product of two operators A, B, where the entrys of these operators commute among themselves, and act by a $\mathcal{Q}$ on them. The tensor coefficients of these operators are represented as:

$$\left(\mathcal{Q}\left(A\otimes B\right)\right)^{i_1,i_2}_{j_1,j_2} = \mathcal{Q}^{i_1,i_2}_{k_1,k_2}\left((A\otimes B)\right)^{k_1,k_2}_{j_1,j_2} = \delta^{i_1}_{i_2^{'}}\delta^{k_1}_{k_2^{'}}A^{k_1}_{j_1}B^{k_2}_{j_2} = \delta^{i_1}_{i_2^{'}}A^{k_1}_{j_1}B^{k_1^{'}}_{j_2} = \left(\mathcal{Q}\left(\mathbb{I}\otimes\left(B^{'}A\right)^{'}\right)\right)^{i_1,i_2}_{j_1,j_2} \tag{I.1.5}$$

This question can be diagrammatically represented as:



It is worth noting that the form of (I.1.5) represents a choice in the labeling of spaces. (I.1.5) can equivalently be expressed as:

$$\left(\mathcal{Q}\left(A\otimes B\right)\right)^{i_1,i_2}_{j_1,j_2} = \mathcal{Q}\left(\left(A^{'}B\right)^{'}\otimes\mathbb{I}\right)^{i_1,i_2}_{j_1,j_2} \tag{I.1.6}$$

Index notations

In this thesis, we are going to use a set of unified notations throughout the manuscript unless stated otherwise, these notations will be listed here.

One of the more common notations is the partition of non-negative integers as $\lambda = (\lambda_1, \lambda_2, \dots)$ with the descending property $\lambda_1 \geqslant \lambda_2 \dots$. The number of non-zero integers in the partition is finite when we talk about finite-dimensional representation operators. More of these notations

[2]The meaning of the prime notation for square matrix entry is $j^{'} = \mathsf{n} - i + 1$, where n is the size of the matrix.

are below:

$\mathbb{C}$	set of complex numbers
$\mathbb{I}$	identity matrix
N	length of the spin chain
$\delta_{i,j}$	Kronecker's delta function
$\mathcal{P}$	permutation operator
$\mathcal{Q}$	transposed Permutation operator
g	twist operator

Scaler notation

In this thesis some objects, such as the sets of rapidities or spectral parameters, and the eigenvalues of the twist operator, will be scalers. They are listed as:

z	free parameter
$u = \{u_1, u_2, \ldots, u_N\}$	set of complex parameter spectral parameter
$\{x_1, x_2, \ldots, x_N\}$	eigenvalues of the twist operator
$\chi_\lambda(g)$	character of the twist operator

If a scaler is written with indices, it can be thought of as the scaler multiplied by the identity operator, for example: $(x_c)^i_j = \delta^i_j x_c$, where $c \in [\![1, N]\!]$ is a label to indicate which eigenvalue we have from the twist.

To complement the idea of the character, let us recall the notation of the representation of a group $\mathcal{G}$ to be (V, ρ), where V stands for the vector space of the group and ρ is the morphism $\rho : \mathcal{G} \to GL(V)$, i. e. a map from $\mathcal{G}$ to $GL(V)$ such that:

$$\forall g_1, g_2 \in \mathcal{G}, \qquad \rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2) \tag{I.1.7}$$

It is worth noting that the fundamental representation is the one with a vector space $\mathbb{C}^n$ and a morphism $\rho(g) = g$.

In this manuscript, we will use different notations of the morphism $\rho, \rho_s, \rho_\lambda, etc$ to denote different representations of the algebra, all of which follow the general identity :

$$\chi : \begin{cases} \mathcal{G} \to \mathbb{C} \\ g \to \mathrm{Tr}(\rho(g)) \end{cases} \tag{I.1.8}$$

Whereas, a morphism between two Lie algebras $\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}$ and $\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}$ is a linear map $\pi : \mathfrak{g} \to \mathfrak{h}$ such that it preserves the Lie bracket, meaning for all $x, y \in \mathfrak{g}$, we have:

$$\pi\left([x,y]_\mathfrak{g}\right) = [\pi(x), \pi(y)]_\mathfrak{h} \tag{I.1.9}$$

$GL(\mathsf{n})$ and $SO(2\mathsf{r})$ notations

We have two different algebras to study in the manuscript, thus we have two sets of operators for each algebra. We will differentiate between the operators of these algebra as shown in the table.

For the $GL(\mathsf{n})$ symmetry, we have the notation to be unified for all arbitrary representations of the algebra.

$GL(\mathsf{n})$ notation	
$\hat{D}$	Coderivative operator
J	fundamental algebra generators
$\mathcal{R}$	a solution of the Yang-Baxter equation
$\mathcal{T}$	conserved charges or T-operators
$\mathbb{Q}$	Q-operators for a spin chain

Unlike the the $GL(\mathsf{n})$ case, we have two families of representations of the algebra with different solution that require us to use deferent notation in the manuscript, these representation are know as the spinorial and symmetric representation:

$SO(2\mathsf{r})$ notation	
$\mathbf{D}$	Coderivative operator
F	fundamental algebra generators
$\mathbf{R}$	a solution of the Yang-Baxter equation with the spinorial symmetry
$\mathfrak{R}$	a solution of the Yang-Baxter equation with the symmetric representation
$\mathbf{T}$	conserved charges or T-operators with spinorial symmetry
$\mathtt{T}$	conserved charges or T-operators with symmetric representation
$\mathbf{S}$	Q-operators for of the spinorial symmetry
$\mathbf{Q}$	Q-operators for of the symmetric representation

I.1.1 Shifts and Normalized limits

This part of the introduction will be devoted to introducing two important notations that will be used often in this text.

Shifts

We will introduce a notion of shit that will help us simplify the bulky formulas we will have later in the manuscript. Some functions will be formulated as $f^{[k]}(u)$, where $f(u)$ is an arbitrary

function of the spectral parameter u. The notation we have is a shift in u, which we will define as:

$$f^{[k]}(u) \equiv f\left(u + \frac{k}{2}\right) \tag{I.1.10}$$

This notation will be carried throughout this thesis[3].

Normalized limits

In this thesis, we will consider the leading order of given quantities (for example, the leading order of polynomials) and omit normalizations that make the quantity convergent. To do this, we introduce the notation $\mathcal{N}\lim$, which allows us not to write the normalization factor explicitly. This notion means that we multiply by a normalization which fixes the leading term of the limit.

For instance, let us take the example:

$$\mathcal{N}\lim_{z\to\infty} 7^z(z^3u^4 - z^4u^3 + z^4) = \lim_{z\to\infty} \frac{7^z(z^3u^4 - z^4u^3 + z^4)}{-7^z z^4} = u^3 - 1$$

?The normalization was fixed by requiring that the limit is non-zero but finite, and that the result to be a monic polynomial in u.

Now, let us provide a more precise definition of $\mathcal{N}\lim$:

Definition

Let Λ be a complex parameter, and $P(\Lambda, u)$ be a Λ-dependent polynomial in u, i. e. $P(\Lambda, u) = \sum_{i=0}^{d} f_i(\Lambda)u^i$ for a given degree d. We wish to study the $\Lambda \to \Lambda_0$ limit. We first define

$$\tilde{P}(\Lambda, u) = \lim_{\Lambda\to\Lambda_0} \frac{P(\Lambda, u)}{\max_i |f_i(\Lambda)|} = \sum_{i=0}^{d'} c_i u^i \tag{I.1.11}$$

where the degree d' may be smaller than the initial degree d. Then we define the normalized limit of P (denoted as $\mathcal{N}\lim_{\Lambda\to\Lambda_0} P(\Lambda, u)$) as:

$$\mathcal{N}\lim_{\Lambda\to\Lambda_0} P(u) = \frac{\tilde{P}(\Lambda, u)}{c_{d'}} \tag{I.1.12}$$

which imposes the monicity of $\mathcal{N}\lim_{\Lambda\to\Lambda_0} P(\Lambda, u)$ (as a polynomial in u).

In some situation, the situation will spasify that the result we obtin from $\mathcal{N}\lim_{\Lambda\to\Lambda_0} P(\Lambda, u)$ to be of a specific value v. In that context, we should change (I.1.12) to:

$$\mathcal{N}\lim_{\Lambda\to\Lambda_0} P(u) = \frac{v}{c_{d'}}\tilde{P}(u) \tag{I.1.13}$$

Several properties follow from this definition. An obvious property is that for an arbitrary polynomial $P(u)$ (of degree d), we have:

$$\mathcal{N}\lim_{\Lambda\to\infty} P(u + \Lambda) = 1, \tag{I.1.14}$$

[3]There are other references that also introduce similar notation with different shifts; this is really up to the author.

because the only coefficient of $P(u+\Lambda)$ with magnitude Λ^d is the constant coefficient. In what follow, we will write several properties which are less trivial and will be used in this thesis.

Lemma I.1.1. *If a function $f(\Lambda,u)$ is the product $x^{\Lambda}P(\Lambda,u)$, where $P(\Lambda,u)$ is an arbitrary polynomial in the two variables Λ and u, then:*

$$\boxed{\underset{\Lambda\to\infty}{\mathcal{N}\lim} f(\Lambda,u) = \underset{z\to\frac{1}{x}}{\mathcal{N}\lim} \sum_{\Lambda\geqslant 0} f(\Lambda,u)\, z^{\Lambda}} \tag{I.1.15}$$

Proof. Lemma(I.1.1):
Let us denote $P(\Lambda,u)=\sum_{k=0}^{d}\sum_{l=0}^{m_k} c_{k,l}\Lambda^k u^l$ (where d is the degree of $P(\Lambda,u)$ as a function of Λ). Then, the left hand side of (I.1.15) is equal to the monic polynomial $\frac{1}{c_{d,m_d}}\sum_{l=0}^{m_d} c_{d,l}u^l$. This can be shown as:

$$\underset{\Lambda\to\infty}{\mathcal{N}\lim} f(\Lambda,u) = \frac{\max|c_{d,l}|}{c_{d,m_d}}\tilde{P}(u) = \frac{\max|c_{d,l}|}{c_{d,m_d}} \underbrace{\lim_{\Lambda\to\infty}\frac{f(\Lambda,u)}{x^{\Lambda}\Lambda^d\max|c_{d,l}|}}_{\frac{1}{\max|c_{d,l}|}\sum_{l=0}^{m_d}c_{d,l}u^l} = \frac{1}{c_{d,m_d}}\sum_{l=0}^{m_d} c_{d,l}u^l \tag{I.1.16}$$

On the other hand, the right hand side of (I.1.15) is a combination of infinite sums:

$$\sum_{\Lambda\geqslant 0}\Lambda^n (xz)^{\Lambda} = \frac{\sum_{m=0}^{n-1}\gamma(n,m)(xz)^{m+1}}{(1-xz)^{n+1}} \tag{I.1.17}$$

where the explicit expressions of the coefficients $\gamma(k,\beta)$ in the numerator of these fractions are known, and they are called Eulerian numbers[4]. It is clear that when taking the limit $z\to\frac{1}{x}$, the pole with the highest multiplicity comes from the term with highest degree in Λ, consequently the right hand side is equal to $\frac{1}{c_{d,m_d}}\sum_{l=0}^{m_d} c_{d,l}u^l$. This can be shown as:

$$\underset{z\to\frac{1}{x}}{\mathcal{N}\lim}\sum_{\Lambda\geqslant 0} f(\Lambda,u)\, z^{\Lambda} = \frac{\max|c_{d,l}|}{d!c_{d,m_d}}\tilde{P}(u) = \frac{\max|c_{d,l}|}{d!c_{d,m_d}} \underbrace{\lim_{z\to\frac{1}{x}}\frac{\sum_{\Lambda\geqslant 0}P(\Lambda,u)(xz)^{\Lambda}}{1/(1-xz)^{d+1}\max|c_{d,l}|}}_{\frac{\sum_{m=0}^{d-1}\gamma(d,m)}{\max|c_{d,l}|}\sum_{l=0}^{m_d}c_{d,l}u^l} = \frac{1}{c_{d,m_d}}\sum_{l=0}^{m_d} c_{d,l}u^l \tag{I.1.18}$$

In the second equality, the $d!$ comes from the Eulerian numbers property: $\sum_{k=0}^{n-1}\gamma(n,k) = n!$. □

Now that we have the concept of normalized limits, we will give a more general form of the previous lemma:

Lemma I.1.2. *If a function $f(\Lambda,u)$ is the finite sum $\sum_i x_i^{\Lambda}P_i(\Lambda,u)$, where the non-zero numbers x_i are pairwise distinct, and $P_i(\Lambda,u)$ are an arbitrary non-zero polynomials in two variables Λ and u then:*

$$\boxed{\underset{z\to\frac{1}{x_\ell}}{\mathcal{N}\lim}\sum_{\Lambda\geqslant 0} f(\Lambda,u)\, z^{\Lambda} = \underset{\Lambda\to\infty}{\mathcal{N}\lim} P_\ell(\Lambda,u)} \tag{I.1.19}$$

[4]This sum is ill-defined when $n=0$, and it should be replaced by 1.

In the case when $z \to \frac{1}{x_\ell}$, the only singular term in $\sum_{\lambda \geqslant 0} f(\lambda, u) z^\lambda$ comes from $x_\ell^\lambda P_\ell(\lambda, u)$, where all the other terms are finite functions in the limit. Thus, the proof follow as the proof of Lemma(I.1.2)

There are two distinct cases we need to look at here: If $\forall i \neq \ell, |x_i| > |x_\ell|$, the (I.1.19) holds in the open disk $D\left(0, \frac{1}{\max |x_\ell|}\right)$: in the disk, the sum $\sum_{\Lambda \geqslant 0} f(\Lambda, u) z^\Lambda = \sum_i \sum_{\Lambda \geqslant 0} (z x_i)^\Lambda P_i(\Lambda, u)$ with the only singular term being $(z x_\ell)^\Lambda P_i(\Lambda, u)$ when taking the limit $z \to \frac{1}{x_\ell}$ making all the other terms subleading and (I.1.19) hold.

By contrast if x_ℓ is not smaller than the other x_i, then (I.1.19) holds in the sense of analytic continuation: $\mathcal{N}\lim_{z \to \frac{1}{x_\ell}} \sum_{\Lambda \geqslant 0} f(\Lambda, u) z^\Lambda$ is an abuse of notation to denote a process where one first computes the sum over Λ when $|z| < \min_i |x_i|$, then continues analytically the resulting function of z (it forms a rational function for $z \in \mathbb{C} \backslash \{x_1, x_2, \dots\}$), and then takes the normalized limit where $z \to \frac{1}{x_\ell}$. In this limit, the term $(z x_\ell)^\Lambda P_\ell(\Lambda, u)$ is still the only singular terms, all other terms are subleading, and one gets (I.1.19).

Remark: In practice, we may use the following corollary of lemm(I.1.2):

If a function $f(\Lambda, u)$ is the finite sum $\sum_i x_i^\Lambda P_i(\Lambda, u) Q_i(u + \Lambda)$, where the numbers x_i are pairwise distinct, and $P_i(\Lambda, u)$ are an arbitrary non-zero polynomials in two variables Λ and u whereas each Q_i is a univariate polynomial:

$$\mathcal{N}\lim_{z \to \frac{1}{x_\ell}} \sum_{\Lambda \geqslant 0} f(\Lambda, u) z^\Lambda = \mathcal{N}\lim_{\Lambda \to \infty} P_\ell(\Lambda, u) \tag{I.1.20}$$

This corollary follows from the previous property and from the fact that $\mathcal{N}\lim_{\Lambda \to \infty} P_\ell(\Lambda, u) Q_\ell(u + \Lambda) = \mathcal{N}\lim_{\Lambda \to \infty} P_\ell(\Lambda, u)$, as follows from (I.1.14).

Limits of diagonalizable operators

In this thesis, we will often take the normalized limit of matrix-valued polynomials in the variable u (for instance, the limit of a transfer matrix).

For simplicity, we do not give a general prescription for the normalized limit of an arbitrary matrix (for instance, a non-diagonalizable matrix, or a matrix whose eigenvectors are functions of u and Λ).

It will turn out that in this article, we will always take the limit of a diagonalizable matrix $M = P^{-1} D P$, where $D = \mathrm{diag}(d_1, d_2, \dots)$ is diagonal and where the matrix P depends only on the physical parameters of the model (the twist and the inhomogeneities), and not on the parameters u and Λ. In that case, we define $\mathcal{N}\lim_{\Lambda \to \infty} M = P^{-1} \mathrm{diag}(\mathcal{N}\lim_{\Lambda \to \infty} d_1, \mathcal{N}\lim_{\Lambda \to \infty} d_2, \dots) P$.

The precise discussion of this point is postponed to Sec(II.2.3). It will turn out that the normalization factors are block-wise multiples of identity, for instance

$$\left(\begin{array}{c|cc|c} a & 0 & 0 & 0 \\ \hline 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ \hline 0 & 0 & 0 & c \end{array}\right)$$

where the corresponding blocks correspond to simultaneous eigenspaces of all T- and Q-operators: these blocks correspond to the sectors of the Hilbert space.

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Contents

Chapter II

Q-operators for spin chains with $GL(\mathsf{n})$ symmetry.

II.1 Spin chain and the construction of the T-operators

In this chapter, we are going to explain the changes in constructing Q-operators with $GL(\mathfrak{n})$ symmetry and show that they are equivalent. The two ways in question are the oscillator description (which was formulated in a series of papers [BLMS10, BFL$^+$11, FKT21]), and the differential description Coderivative (which was introduced in [KV08]).

To do this, we will briefly introduce the recipe for a quantum integrable spin chain, give a formal definition of T-operators, and show that they give rise to a family of conserved charges. Then, we will present a way to explicitly manipulate these operators using a differential operator called Coderivative, as shown in [KV08]. We will also elaborate on how we can modify a specific Yang-Baxter equation using a differential operator, where we motivate a proof of the Yang-Baxter equation using a property of the differential operator only. Additionally, in the two descriptions, the functional relations of our integrable system were described using Q-operators. We will show that the construction of T- and Q-operators are equivalent between the oscillator and differential representations.

II.1.1 Introduction to quantum integrable systems

In this section, we will review the definition and primary properties of integrable spin chain. This subject has been extensively researched, and there is a wealthy literature available on the topic. Some of the references that aid in introducing and fully developing this subject include: [Gau83] for a complete general overview of the subject of integrability and spin chains; [Bet31] which introduces the idea of Coordinate Bethe Ansatz and devalobe the one-dimensional Heisenberg spin chain (also [KMGT97, KHM98, KHM00] for a comblite discussion); and[FST79, NW13, Skl82, AV87] where they develop the spectrum of the model using Algebraic Bethe Ansatz (and [MT00, Sla18, Gro17, Sla07, NW14, KMST05] for more detail).

Thus, the definition of an integrable spin chain in the literature: *spin chains are quantum integrable systems if they have a sufficient number of independent commuting T-operators*[1], a description of these systems is provided in Fig(II.1). In the figure, a vertical line stands for the vector space, and a set of these lines corresponds to a tensor product of these vector spaces. The horizontal lines denote the so-called Auxiliary spaces; they take an ellipse shape in the figure, indicating a trace over Auxiliary spaces. Also, at each crossing of a vertical with the horizontal lines, there is a ball representing the so-called R-matrix, where these matrices are solutions of a key identity called the Yang-Baxter equation (for more details on the solutions of the Yang-Baxter equation check II.1.3), and the two (blue, purple) circles represent the so-called Twist operator. In the Figure, we labeled a set of vertical lines with an ellipse as $\mathcal{T}$. Thus, $\mathcal{T}_1$ and $\mathcal{T}_2$ are operators on the Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i = (\mathbb{C}^{\mathfrak{n}})^{\otimes N}$. The key argument in proving the commutation relation proposed in Fig(II.1) is the Yang-Baxter equation:

$$\mathcal{R}_{i,j}(u-v)\,\mathcal{R}_{i,k}(u)\,\mathcal{R}_{j,k}(v) = \mathcal{R}_{j,k}(v)\,\mathcal{R}_{i,k}(u)\,\mathcal{R}_{i,j}(u-v) \tag{II.1.1a}$$

[1] The number of these operators is equal to the dimension of the Hilbert space.

$$\text{i. e.} \boxed{\mathcal{R}_{i,j}(u) = u\mathbb{I} + \mathcal{P}_{i,j}} \tag{II.1.1b}$$

$$\text{Where } \mathcal{P} = \sum_{\alpha,\beta=1}^{n} J_{\alpha,\beta} \otimes J_{\beta,\alpha}, \qquad \text{and } J_{\alpha,\beta} = |\alpha\rangle\langle\beta| \tag{II.1.1c}$$

Where (II.1.1c) is the same as (I.1.3), $\mathcal{R}_{i,j}(u)$ in (II.1.1b) is a solution of the Yang-Baxter

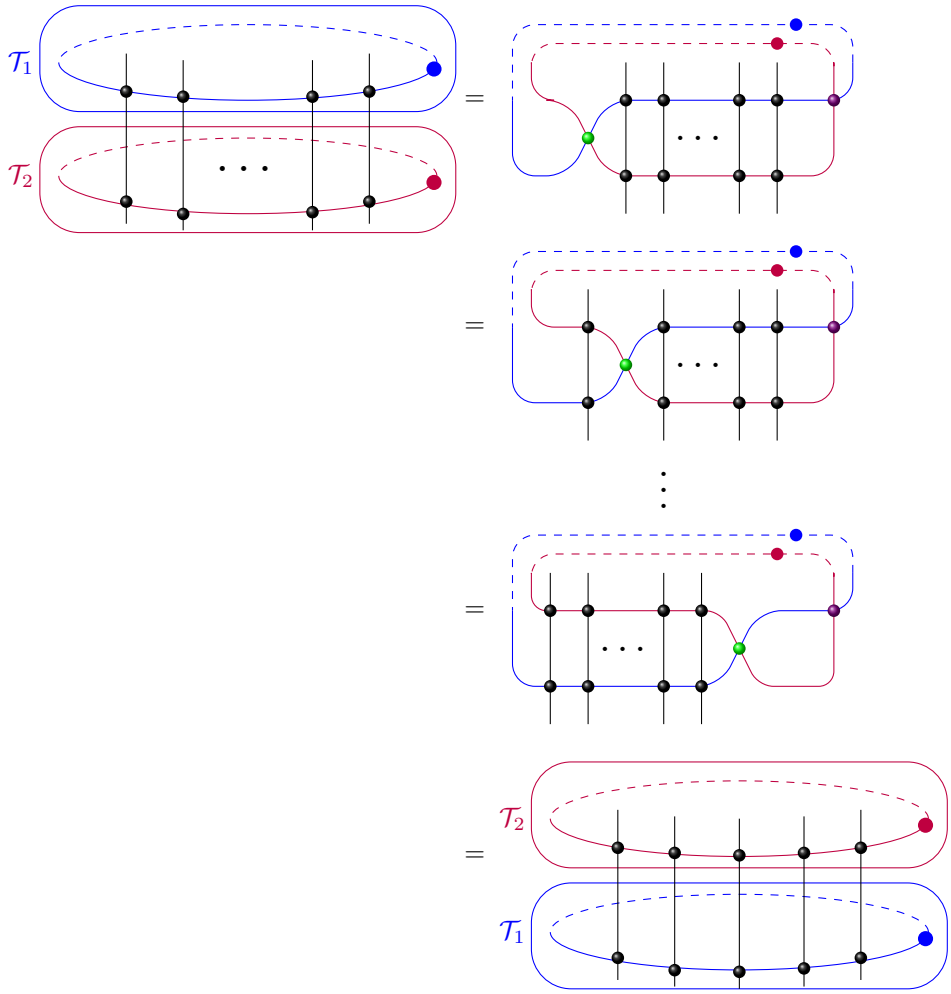


Figure II.1: This is a rough graphical description of the commutation relation between the so-called T-operators. The first part of this diagram represents two operators, $\mathcal{T}_1$ and $\mathcal{T}_2$ represented as two ellipses with vertical lines crossing each ellipse. Then, using the famous Yang-Baxter equation, one can show the commutativity between T-operators.

equation (II.1.1a) and will be referred to as R-matrix, (i, j, k) are labels discribing three different vector spaces V_i,V_j, and V_k (in the figure, these spaces are in the fundamental representation), and $u, v \in \mathcal{C}$ are called spectral parameters. Each R-matrix acts on a tensor product of these spaces [2]. Now, we can show the steps taken in Fig(II.1) using (II.1.1a) with the unitarity condition of the R-matrices $(\mathcal{R}_{i,j}(u)\mathcal{R}_{j,i}(-u) \propto \mathbb{I})$[3] represented in Fig(II.1) as the (green/violet) balls. Using the diagrammatical definition of (II.1.1a) shown in Fig(II.2), we can move the green

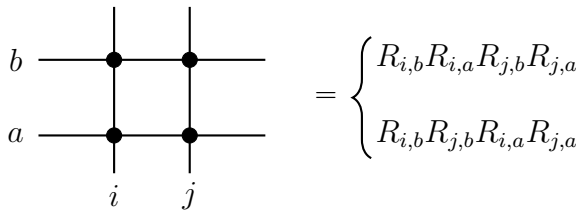
[2]The R-matrix acts trivially on a vector space that it is not assigned to. For example, $\mathcal{R}_{j,k}(u)$ acts trivially on V_i.

[3]The unitarity condition $\mathcal{R}_{i,j}(u)\mathcal{R}_{j,i}(-u) = \zeta(u)\mathcal{I}$, where $\zeta(u)$ is a polynomial of degree two in u.

ball from the left hand side of a vertical line to the right hand side of the line, as proposed in the second diagram in Fig(II.1). This process is repeated for all other vertical lines, and we end up with the last diagram in the figure.

In order for the reader to follow the diagram in Fig(II.1), it is important to introduce the convention used to read the diagram into formulas. Starting with two R-matrices on two different physical spaces (i, j) and an auxiliary space (a), then two R-matrices on one physical space with two auxiliary spaces:

As we can see, the diagram is read from left to right and from bottom to top. Then, one can ask if we have two physical spaces with two auxiliary spaces with four R-matrices i. e.:



There is an ambiguity in the order of the operators; as stated in the last picture. This can be solved by noticing that operators on different spaces do commute, i. e.:

$$[A \otimes \mathbb{I}, \mathbb{I} \otimes B] = 0 \tag{II.1.2}$$

Thus allow us to see that $R_{j,b}R_{i,a} = R_{i,a}R_{j,b}$, which also remove this ambiguity when writing the mathematical expression of Fig(II.1).

We also skipped an important step in showing the commutativity between $\mathcal{T}_1$ and $\mathcal{T}_2$, and that is the commutation between the R-matrix and the twist operator. As stated before, the (blue, purple) circles are what we call twist operators; they are group elements $g \in GL(\mathsf{n})$ and can be in any arbitrary algebraic representation. In this section, we will show the commutation relation between R-matrix and the fundamental twist operators[4], i. e. $[g_{a_1} \otimes g_{a_2}, \mathcal{R}_{a_1,a_2}(u)] = 0$, where a_1, a_2 refers to the two auxiliary spaces in Fig(II.1). The proof is straightforward; taking the definition of the R-matrix from (II.1.1b), one notice that $\mathbb{I}$ commutes with everything, and from the general property of the permutation operator:

$$\mathcal{P}_{1,2}(A \otimes B) = (B \otimes A)\mathcal{P}_{1,2}$$

From the last relation, we can see that the permutation operator exchanges the positions of the operators, indicating an exchange in the physical space position of the twist operators. This

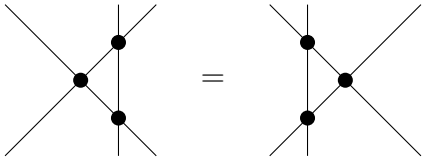


Figure II.2: The diagrammatical representation of the Yang-Baxter equation (II.1.1a)

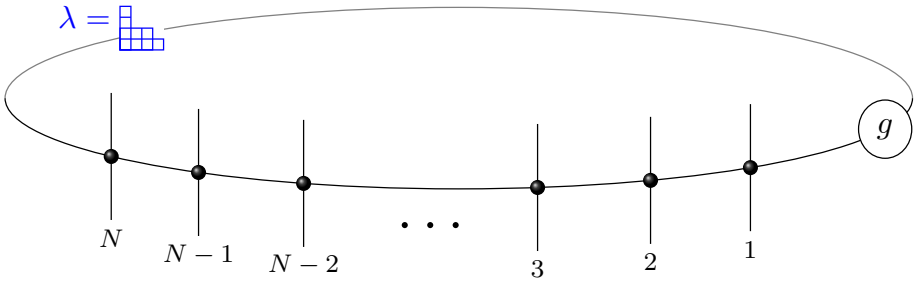


Figure II.3: A graphical description of a T-operators with a general representation λ in the auxiliary space. The R-matrix are represented as small black balls and multiplied along the auxiliary space. whereas the set of vertical lines physical space relabels the Hilbert space, these spaces are defined in the fundamental representation $\lambda = \square$.

property of $\mathcal{P}$ relabels the twist operators $a_1 \leftrightarrow a_2$, making it the last step in proving the commutativity between T-operators impling that $[g_{a_1} \otimes g_{a_2}, \mathcal{R}_{a_1,a_2}(u)] = 0$ holds.

After proving that T-operators commute, we will give the general description of these operators from Fig(II.1). Generally and throughout this thesis, the representation of the auxiliary space can be something more complicated than the fundamental representation. We will refer to a general representation of the symmetry group as λ unless stated otherwise, λ will be denoting a Young diagram. By contrast, we will only have the fundamental ones in our physical spaces. In representation (λ), the twist operator will be referred to as $\rho_\lambda(g)$ and the general definition of T-operators:

$$\boxed{\mathcal{T}^{(\lambda)}(u) = \mathrm{tr}_\lambda\left(\mathcal{R}_{N,\lambda}(u-\theta_N)\cdot\mathcal{R}_{N-1,\lambda}(u-\theta_{N-1})\ldots\mathcal{R}_{1,\lambda}(u-\theta_1)\,\rho_\lambda(g)\right)} \tag{II.1.3a}$$

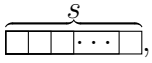
$$\text{Where } \mathcal{R}_{i,\lambda}(u) = u + \overbrace{\sum_{\alpha,\beta} J_{\alpha,\beta}\otimes\rho_\lambda(J_{\beta,\alpha})}^{\rho_{i,\lambda}(\mathcal{P})} \tag{II.1.3b}$$

Equation (II.1.3a) reads as a trace of a product of R-matrix with spectral parameter u and an inhomogeneities θ_i, also a twist operator in a general representation of the algebra (the graphical description of (II.1.3a) is shown in Fig(II.3)). Additionally, (II.1.3b) represents R-matrices that satisfies the Yang-Baxter equation for a given representation λ in the auxiliary space.

[4]For the general representation, check Sec(II.1.5)

A generalization to the commutation relation presented in Fig(II.1) is:

$$[\mathcal{T}^{(\lambda)}(u), \mathcal{T}^{(\mu)}(u)] = 0 \tag{II.1.4}$$

where (II.1.4) is the commutation relation between T-operators with different representations of the algebra in their auxiliary spaces. This relation holds when both T-operators share the same twist g and inhomogeneities θ_i. When restricting the representation in (II.1.4) to be the symmetric one, i. e. $\lambda = (1, s) =$  , the proof of (II.1.4) then depends on writing the projector operator $P_{(1,s)}$; this was explained in [KV08]. (In the paper, they explain that the projector of the symmetric representation is written as a product of fundamental R-matrix).

II.1.2 T-operators and Hamiltonian

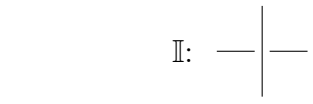
In this thesis, we are interested in a spin chain with nearest-neighbor interactions. This is apparent from the definition of the Hamiltonian operator of the spin chain in the introduction. In the literature, the Hamiltonian can be obtained from T-operators, using the fromula:

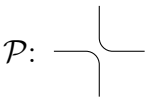
$$\boxed{H = \frac{2}{\mathsf{n}} N - 2 \frac{\partial_u \mathcal{T}(u)}{\mathcal{T}(u)}\bigg|_{u=0}} \tag{II.1.5}$$

where N is the length of the spin chain and n is related to the rank of the algebra. Notice that T-operators in (II.1.5) are in the fundamental representation $\mathcal{T}^{\square}(u)$, and we have dropped the notation $\square$ For simplicity, the derivation of the Hamiltonian involves finding the derivative of [5] revealing it to be a trace of sums of products of permutation operators with a missing site. This arises due to the relations $\mathcal{R}_{i,j}(0) = \mathcal{P}_{i,j}$ and $\dot{\mathcal{R}}_{i,j}(0) = \delta_{i,j}$, where $\dot{\mathcal{R}} = \partial_u \mathcal{R}$. The same procedure applies to $\mathcal{T}(u)^{-1}$. This can be depicted graphically, as shown in Fig. (II.4). Thus, the Hamiltonian of this system reads:

$$\boxed{H = \frac{2}{\mathsf{n}} N - 2 \sum_{k=1}^{N-1} \mathcal{P}_{k,k+1} - 2\mathcal{P}_{1,N} \cdot g_N \cdot g_1^{-1}} \tag{II.1.6}$$

Here g_1, g_N denote the twist operators associated with the first and N^{th} quantum spaces, respectively. The term involving the twist operator arises from the second segment of the diagram, as the two twist operators are not contracted. Additionally, the diagrammatic representation of the operators $\mathbb{I}$ and $\mathcal{P}$, are provided below:

$\mathbb{I}$: 

$\mathcal{P}$: 

[5] One can find an extensive calculation on how to extract the Hamiltonian from (II.1.5) in the literature. For example, the $GL(2)$ was presented in [Sla18].

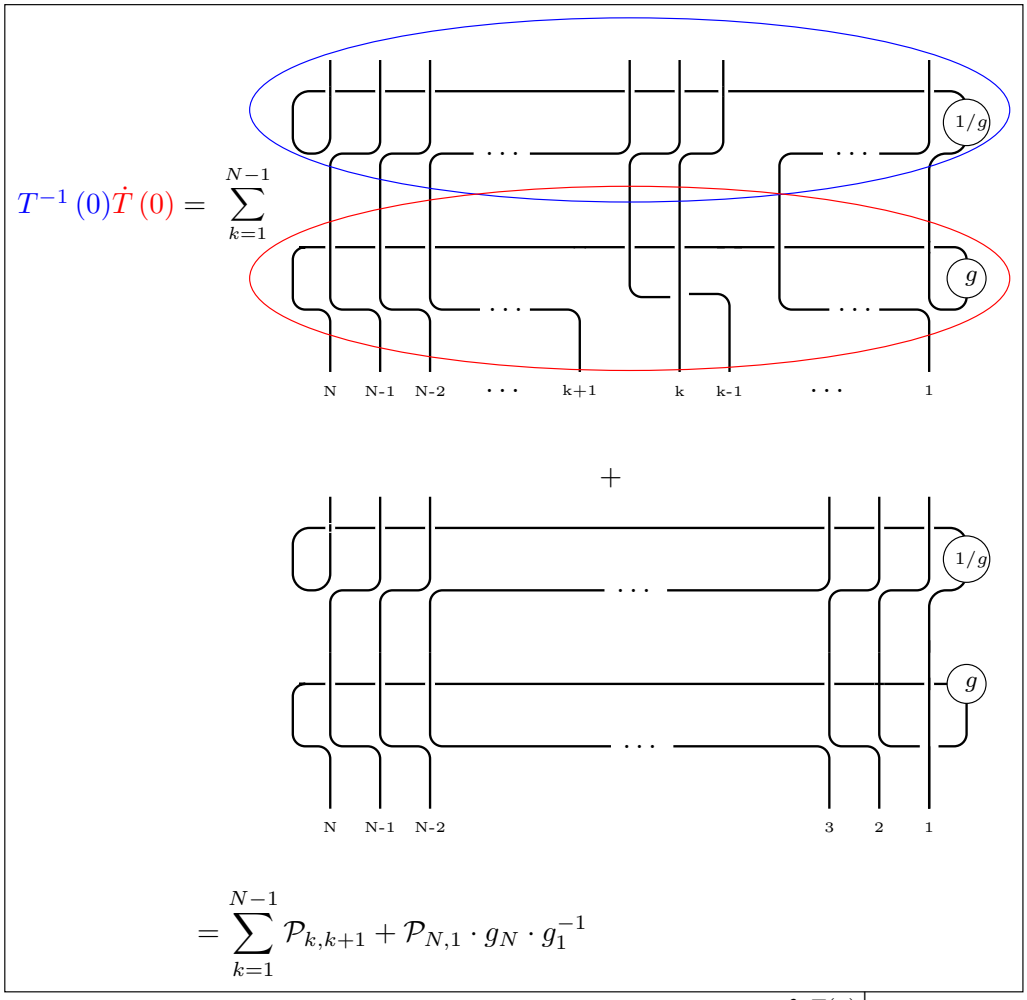


Figure II.4: This is a diagrammatical representation of the object $\frac{\partial_u T(u)}{T(u)}\Big|_{u=0}$. The R-matrices are substituted with permutation operators due to $\mathcal{R}_{i,j}(0) = \mathcal{P}_{i,j}$ except for k and $k-1$ lines in the first segment (and N and 1 lines in the second segment) of the diagram. In these instances, they denote an identity operator due to the derivative of the R-matrix. If a vertical line begins at position k at the bottom and terminates at position l at the top, it signifies the factor $\delta_{k,l}$.

One can see from (II.1.5) that one of the T-operators is the Hamiltonian, thus the family of commuting T-operators are actually conserved charges. In order to show that our T-operators (II.1.3a) describes an integrable system, we will show how it is connected to the well-known Heisenberg spin chain.

Integrablity and the Heisenberg spin chain

The Heisenberg spin chain is a simple integrable system; usually, it is referred to as the $XXX_{1/2}$ spin chain. To show the link between our definition of integrability and this system, we are going to recover the Hamiltonian of the $XXX_{1/2}$ spin chain by restricting our general statement about integrability to $GL(2)$ and the twist to identity.

This will simplify the Hamiltonian formula (II.1.6) to[6]:

$$H_{XXX_{1/2}} = N - 2\sum_{k=1}^{N} \mathcal{P}_{k,k+1} \tag{II.1.7}$$

To recover the well-known Hamiltonian of the Heisenberg spin chain, one must notice that the permutation operator $\mathcal{P}$ defined in (II.1.1c) can be rewritten using the 2×2 matrix representation of the $GL(2)$ generators. We have four generators associated with this algebra: $(J_{1,1}, J_{1,2}, J_{2,1}, J_{2,2})$, and the matrix realization of these operators in the fundamental representation takes the form:

$$J_{i,j} \left|(k)\right\rangle = \delta_{j,k} \left|(i)\right\rangle \tag{II.1.8}$$

This translates to a matrix representation where all of the coefficients are equal to zeros, except the ones at position (i, j):

$$\text{and } J_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad J_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Consequentially, using (II.1.1c), we can define the matrix representation of the permutation operator using the Pauli matrices as ($\sigma^x = \big(J_{1,2} + J_{2,1}\big)$, $\sigma^y = -i\big(J_{1,2} - J_{2,1}\big)$ and $\sigma^z = \big(J_{1,1} - J_{2,2}\big)$):

$$\boxed{\mathcal{P}_{i,j} = \frac{1}{2}\left(\mathbb{I} + \vec{\sigma}_i \cdot \vec{\sigma}_j\right)} \tag{II.1.9a}$$

$$\text{Where } \vec{\sigma}_i \cdot \vec{\sigma}_j = \sigma_i^x \cdot \sigma_j^x + \sigma_i^y \cdot \sigma_j^y + \sigma_i^z \cdot \sigma_j^z \tag{II.1.9b}$$

where the labeling (i, j) in (II.1.9a) and (II.1.9b) refers to the positions on quantum spaces (i) and (j), respectively. Substituting (II.1.9a) in (II.1.7), one obtains:

$$\boxed{H_{XXX_{1/2}} = -\sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}} \tag{II.1.10}$$

The Hamiltonian describes the simplest spin chain with nearest neighbor interactions and corresponds to a quantum version of the Ising model, i. e. the Heisenberg spin chain. Notice that the minus sign in (II.1.10) means that our spin chain describes the ferromagnetic model.

II.1.3 Yang-Baxter equation

In this subsection, we will introduce the solution of the cornerstone equation for an integrable system, the Yang-Baxter equation. We are going to motivate the solution in the fundamental representation of the auxiliary space ($V_i = \mathbb{C}^2$) and demonstrate that one of its non-trivial solutions is the permutation operator $\mathcal{P}$.

[6] In the intruduction, we have an overall factor γ in the Hamiltonian, which can be translated by multiplying (II.1.6) and (II.1.7) with γ.

By construction, the operators $\mathcal{R}(u)$ of the Yang-Baxter equation (II.1.1a) are an $\mathsf{n}^2 \times \mathsf{n}^2$ matrices, such that $\mathcal{R}(u) : \mathbb{C}^{\mathsf{n}} \otimes \mathbb{C}^{n} \to \mathbb{C}^{\mathsf{n}} \otimes \mathbb{C}^{n}$. One can start by looking for a non-trivial solution of a simpler version of (II.1.1a) (by setting the spectral parameters u, v to zeros). An obvious solution would be the identity matrix (which is not interesting). A less obvious solution would be the permutation matrix $\mathcal{P}$. Indeed, by taking the identity:

$$\mathcal{P}_{i,j}\mathcal{P}_{i,k}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\underbrace{\mathcal{P}_{j,k}\mathcal{P}_{i,j}\mathcal{P}_{i,k}\mathcal{P}_{j,k}}_{\mathcal{P}_{i,k}\mathcal{P}_{i,j}} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}\mathcal{P}_{i,j} \tag{II.1.11}$$

where we have used $\mathcal{P}^2 = \mathbb{I}$ and multiplied the last equation with $\mathcal{P}^2_{j,k}$ from the left, giving us the second equality. One can see that (II.1.11) satisfies the Yang-Baxter equation.

Now, we would look for solutions dependent on the spectral parameters by setting $R(u) = \phi(u)\mathbb{I} + c\mathcal{P}$, where $\phi(u)$ is a smooth function of u and c is a constant [7]. By substituting the proposed R-matrix in (II.1.1a) and expanding with respect to powers of the constant c, one finds that the non-trivial equation comes from the coefficients of c^2:

$$\begin{gathered}\phi(u)\mathcal{P}_{i,j}\mathcal{P}_{j,k} + \phi(-v)\mathcal{P}_{i,j}\mathcal{P}_{i,k} + \phi(u-v)\mathcal{P}_{i,k}\mathcal{P}_{j,k} \\ = \\ \phi(u)\mathcal{P}_{j,k}\mathcal{P}_{i,j} + \phi(-v)\mathcal{P}_{i,k}\mathcal{P}_{i,j} + \phi(u-v)\mathcal{P}_{j,k}\mathcal{P}_{i,k}\end{gathered}$$

Multiplying the last equation by $\mathcal{P}_{i,k}$ from the left and $\mathcal{P}_{i,j}$ from the right, we obtain:

$$\phi(u)\mathbb{I} + \phi(-v)\mathbb{I} + \phi(u-v)\mathcal{P}_{j,k}\mathcal{P}_{i,j} = \phi(u)\mathcal{P}_{i,k}\mathcal{P}_{j,k} + \phi(-v)\mathbb{I} + \phi(u-v) \tag{II.1.12}$$

Finally, we multiply by $\mathcal{P}_{j,k}$ from the left and obtain:

$$\Big(\phi(u) + \phi(-v) - \phi(u-v)\Big)\mathcal{P}_{i,j} = \Big(\phi(u) + \phi(-v) - \phi(u-v)\Big)\mathcal{P}_{j,k} \tag{II.1.13}$$

Thus, $R(u) = \phi(u)\mathbb{I} + c\mathcal{P}$ is a solution of the Yang-Baxter equation if and only if:

$$\phi(u-v) = \phi(u) + \phi(-v)$$

and the only well-known linear function that satisfies the last identity is $\phi(u) = au$, where a is a constant. Giving us:

$$\boxed{R(u) = au\mathbb{I} + c\mathcal{P}} \tag{II.1.14}$$

The constant a can be absorbed in c, and by setting $c = 1$ we obtain (II.1.1b). This equation is a solution to the Yang-Baxter equation in the fundamental representation; a more general solution (general representation $V = \mathbb{C}^{\mathsf{n}}$) can be found.

II.1.3.1 Oscillator representation and the R-matrix

In appendix(A.1), the fundamental relations of the Yangian algebra were introduced. We can use these relations to formulate an oscillator description of the R-matrix. Knowing that

[7] If one has a solution $R(u)$ of the Yang-Baxter equation, then $\phi(u)R(u)$ is also a solution.

the R-matrices are $\mathsf{n} \times \mathsf{n}$ matrices whose matrix elements are operators valued functions of the spectral parameter belonging to the Yangian algebra $\mathcal{Y}(gl(n))$. Indeed, the Yang-Baxter equation provide to define the relations of the matrix elements of the R-matrix which are the relations of the Yangian operators $t_{i,j}$ [8].

We will change the notations in this section slightly; the indices $\{i_1, i_2, j_1, j_2\}$ will be the physical labeling of the operators, and they will be referred to as (outgoing, ingoing) for (i, j), respectively. From the left hand side of the Yang-Baxter equation, one can write:

$$\begin{aligned}
&\left(\mathcal{R}(u-v)\left(\tilde{\mathcal{R}}(u)\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes\tilde{\mathcal{R}}(v)\right)\right)^{i_1,i_2}_{j_1,j_2} \\
&\qquad = \left((u-v)\,\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right)\left(\tilde{\mathcal{R}}(u)_{k_1,m_1}\,\delta^{k_2}_{m_2}\right)\left(\delta^{m_1}_{j_1}\tilde{\mathcal{R}}(v)_{m_2,j_2}\right) \\
&\qquad = (u-v)\,\tilde{\mathcal{R}}(u)_{i_1,j_1}\,\tilde{\mathcal{R}}(u)_{i_2,j_2} + \tilde{\mathcal{R}}(u)_{i_2,j_1}\,\tilde{\mathcal{R}}(u)_{i_1,j_2}
\end{aligned} \tag{II.1.15}$$

where $\tilde{\mathcal{R}}$ is in $\mathsf{n} \times \mathsf{n}$ matrix that acts on a single copy of the space $\mathbb{C}^{\mathsf{n}}$. The second equality comes from expanding the product, were we also used:

$$\left(\mathbb{I}^{\otimes(k-1)} \otimes \overbrace{\tilde{\mathcal{R}}}^{k^{th}-position} \otimes \mathbb{I}^{\otimes(\mathsf{n}-k+1)}\right)^{i_1,\ldots,i_{\mathsf{n}}}_{j_1,\ldots,j_{\mathsf{n}}} = \prod_{\substack{1\leqslant m\leqslant n \\ m\neq k}} \delta^{i_m}_{j_m}\tilde{\mathcal{R}}_{i_k,j_k} \tag{II.1.16}$$

The same goes for the right hand side of the Yang-Baxter equation.

$$\left(\left(\mathbb{I}\otimes\tilde{\mathcal{R}}(v)\right)\left(\tilde{\mathcal{R}}(u)\otimes\mathbb{I}\right)\mathcal{R}(u-v)\right)^{i_1,i_2}_{j_1,j_2} = (u-v)\,\tilde{\mathcal{R}}(v)_{i_2,j_2}\,\tilde{\mathcal{R}}(u)_{i_1,j_1} + \tilde{\mathcal{R}}(v)_{i_2,j_1}\,\tilde{\mathcal{R}}(u)_{i_1,j_2} \tag{II.1.17}$$

Now, by substituting the left hand side and right hand side in the Yang-Baxter equation, we find the commutation relation between the matrix entry of $\tilde{\mathcal{R}}$:

$$\left[\tilde{\mathcal{R}}(u)_{i_1,j_1}, \tilde{\mathcal{R}}(v)_{i_2,j_2}\right] = \frac{1}{u-v}\left(\tilde{\mathcal{R}}(v)_{i_2,j_1}\,\tilde{\mathcal{R}}(u)_{i_1,j_2} - \tilde{\mathcal{R}}(u)_{i_2,j_1}\,\tilde{\mathcal{R}}(v)_{i_1,j_2}\right) \tag{II.1.18}$$

which is exactly the same as (A.1.6) up to changing R-matrices into t-operators. One can also show that if we write the R-matrix as generating series of the form (A.1.5), we find:

$$\left[\tilde{\mathcal{R}}^{(r)}_{i,j}, \tilde{\mathcal{R}}^{(s)}_{k,l}\right] = \sum_{a=1}^{min(r,s)}\left(\tilde{\mathcal{R}}^{(a-1)}_{k,j}\tilde{\mathcal{R}}^{(r+s-a)}_{i,l} - \tilde{\mathcal{R}}^{(r+s-a)}_{k,j}\tilde{\mathcal{R}}^{(a-1)}_{i,l}\right) \tag{II.1.19}$$

where $\tilde{\mathcal{R}}^{(a)}$ means that we take the a^{th} coefficient in the Laurent series defined in (A.1.5). Thus, the problem of solving the Yang-Baxter equation reduces to the construction of representations of the infinite-dimensional quadratic algebra (II.1.19). With the evaluation map defined in

[8] This section is a derivation from [BFL+11]

(A.1.8), we see that the infinite series of the R-matrix diminishes after the second term. With a trivial change in normalization, these representations correspond to:

$$\tilde{\mathcal{R}}_{i,j}(u) = u\delta_{i,j} + J_{i,j} \tag{II.1.20}$$

where $J_{i,j}$ denotes the generators of the algebra in the defining representation which they satisfy the commutation relation (A.1.1). In defining the R-matrices, we are going to introduce a generalization to the system by changing (A.1.1) and (A.1.5), i. e.$\tilde{\mathcal{R}}_j^{i,(0)}$ not to be equal to the identity matrix δ_j^i, instead we are going to leave it to be $\tilde{\mathcal{R}}_j^{i,(0)}$. With the fact that the R-matrix is $GL(\mathsf{n})$-invariant (the commutation between R-matrix and the twist operators shown in Sec(II.1.1)), and satisfy the Yang-Baxter relation[9], one can show that:

$$\tilde{\mathcal{R}}^{(0)} = diag\left(\underbrace{1,...,1}_{m},\underbrace{0,...,0}_{\mathsf{n}-m}\right) \qquad m = 1,2,...,\mathsf{n} \tag{II.1.21}$$

It is easy to see that (II.1.21) is equal to the identity matrix if $m = \mathsf{n}$. According to (II.1.19), we only need to find the matrix elements of $\tilde{\mathcal{R}}^{(1)}$ since $\tilde{\mathcal{R}}^{(r)} = 0, \forall r \geqslant 2$. Thus, we write the remaining coefficient of $\tilde{\mathcal{R}}^{(1)}$ in the form:

$$\tilde{\mathcal{R}}^{(1)} = \left(\begin{array}{c|c} A_{a,b} & B_{a,\bar{b}} \\ \hline C_{\bar{a},b} & D_{\bar{a},\bar{b}} \end{array}\right) \tag{II.1.22}$$

where A, B, C and D are operator-valued matrices of dimensions $m \times m, m \times (\mathsf{n}-m), (\mathsf{n}-m) \times m$ and $(\mathsf{n}-m) \times (\mathsf{n}-m)$, respectively. Notice that the undashed and dashed indices run from:

$$1 \leqslant a, b \leqslant m, \qquad m+1 \leqslant \bar{a}, \bar{b} \leqslant n \tag{II.1.23}$$

By inserting (II.1.22) in (II.1.19), we obtain the commutation relations of the matrix elements:

$$[A_{a,b}, A_{c,e}] = \delta_{a,e}A_{c,b} - \delta_{c,b}A_{a,e}, \qquad [A_{a,b}, B_{c,\bar{c}}] = -\delta_{b,c}B_{a,\bar{c}}, \qquad [A_{a,b}, B_{\bar{c},c}] = \delta_{a,c}C_{\bar{c},b}$$

$$\left[B_{a,\bar{b}}, B_{c,\bar{e}}\right] = 0, \qquad \left[B_{a,\bar{b}}, C_{\bar{a},b}\right] = \delta_{a,b}D_{\bar{a},\bar{b}}, \qquad [C_{\bar{a},b}, C_{\bar{c},e}] = 0 \tag{II.1.24}$$

$$\left[D_{\bar{a},\bar{b}}, L_{c,e}\right] = \left[D_{\bar{a},\bar{b}}, L_{c,\bar{c}}\right] = \left[D_{\bar{a},\bar{b}}, L_{\bar{c},\bar{e}}\right] = 0$$

I will show here how to find the first and last commutation relations; the rest follow in the same manner. To obtain the first relation, we set $r = s = 1$ in (II.1.19), then for convenience we rewrite the indices as $i \to a, j \to b, k \to c$ and $l \to e$, notice that we are in the undashed sector of (II.1.22) [10], given us the commutation relation of $\tilde{\mathcal{R}}_{a,b}^{(1)} = A_{a,b}$. On the other hand, if we look for the commutation relations of $D_{\bar{a},\bar{b}}$ in (II.1.19) we notice that the labeling of order zero

[9] If the R-matrix satisfies the Yang-Baxter relation, then $\mathbb{G}\tilde{\mathcal{R}}(u)\mathbb{F}$ where $\mathbb{G}, \mathbb{F} \in gl(n)$ also satisfies the Yang-Baxter.

[10] Notice that (II.1.21) is equal to the delta function in the undashed sector.

are $\tilde{\mathcal{R}}^{(0)}_{k,\bar{l}}, \tilde{\mathcal{R}}^{(0)}_{\bar{k},l}$ or $\tilde{\mathcal{R}}^{(0)}_{\bar{k},\bar{l}}$, these elements are all zeros since we are not in the undashed sector of (II.1.21), giving us the last three commutation relations in (II.1.24). It was realized in [BFL$^+$11] that the resulting algebra in (II.1.24) can be realized as a direct product of the algebra $gl(m)$ with $m(n-m)$ copy of a harmonic oscillator algebra:

$$gl(m) \otimes \mathcal{H}^{\otimes m(n-m)} \tag{II.1.25}$$

By introducing $m(n-m)$ independent oscillator pairs $(b_{\bar{a},b}, b^\dagger_{b,\bar{a}})$, where $\bar{a} = m+1, \ldots, n$ and $b = 1, \ldots, m$, that satisfy the commutation relation:

$$\left[b_{\bar{a},b}, b^\dagger_{c,\bar{e}}\right] = \delta_{\bar{a},\bar{e}}\delta_{b,c} \tag{II.1.26}$$

The connection of the algebra (II.1.24) to the product (II.1.25) was established in [BFL$^+$11] where they defined an evaluation homomorphism map from the infinite-dimensional Yangian algebra (II.1.19) to the finite-dimensional algebra (II.1.25) which reads as:

$$\tilde{\mathcal{R}}_{\{1,2,\ldots,p\}} = \left(\begin{array}{c|c} u\delta_{a,b} + J_{b,a} - \sum_{\bar{c}=p+1}^{n}\left(b^\dagger_{a,\bar{c}}b_{\bar{c},b} + \frac{1}{2}\delta_{a,b}\right) & b^\dagger_{a,\bar{b}} \\ \hline -b_{\bar{a},b} & \delta_{\bar{a},\bar{b}} \end{array}\right) \tag{II.1.27}$$

II.1.4 Coderivative: the operational description of T-operators

The differential method of constructing T-operators was first introduced in the work by Kazakov and Vieira [KV08]. This approach stemmed from how the R-matrices were defined (II.1.14) using the permutation operator $\mathcal{P}$. The concept presented in [KV08], involved rewriting the tensor product from the operational description in (II.1.3a) using a differential operator independent of the representation used in our auxiliary space. The operator is referred to as Coderivative. An aim of this approach is to reduce the terms influenced by the representation, among other reasons that will be detailed later. With that, the definition of T-operators was transformed into a set of operators that act on an object described by the representation.

As a start point, let us introduce the Coderivative operator:

$$\begin{aligned} \hat{D} \otimes f(g) &\equiv \frac{\partial}{\partial \varepsilon^t} \otimes f\left(e^{\langle \varepsilon | J \rangle} g\right)\Big|_{\varepsilon=0} \\ &= \sum_{i,j=1}^{\mathsf{n}} J_{i,j} \otimes \frac{\partial}{\partial \varepsilon_{j,i}} f\left(e^{\sum_{\alpha,\beta=1}^{\mathsf{n}} \varepsilon_{\alpha,\beta} J_{\alpha,\beta}} g\right)\Big|_{\varepsilon\to 0} \end{aligned} \tag{II.1.28}$$

where f is a N-tensor valued function of the twist operator g, i. e. $f(g)^{i_1,i_2,\ldots,i_N}_{j_1,j_2,\ldots,j_N}$, $J_{\alpha,\beta}$ is a $GL(\mathsf{n})$ generators and $\langle \varepsilon | J \rangle = \sum_{\alpha,\beta=1}^{\mathsf{n}} \varepsilon_{\alpha,\beta} J_{\alpha,\beta}$ denote the inner product. To help us understand how one can rewrite T-operators using (II.1.28), we first need to see the action of Coderivative on $\rho_\lambda(g)$. Using (II.1.28), one obtains:

$$\begin{aligned} \hat{D} \otimes \rho_\lambda(g) &= \sum_{i,j} J_{i,j} \otimes \frac{\partial}{\partial \varepsilon_{j,i}} \rho_\lambda\left(e^{\sum_{\alpha,\beta} \varepsilon_{\alpha,\beta} J_{\alpha,\beta}} \cdot g\right)|_{\varepsilon=0} \\ &= \sum_{i,j} J_{i,j} \otimes \left(\frac{\partial}{\partial \varepsilon_{j,i}} e^{\sum_{\alpha,\beta} \varepsilon_{\alpha,\beta} \pi_\lambda\left(J_{\alpha,\beta}\right)} \cdot \rho_\lambda(g)\right)|_{\varepsilon=0} \\ &= \sum_{i,j} J_{i,j} \otimes \pi_\lambda(J_{j,i}) \cdot \rho_\lambda(g) = \mathcal{P}_{1,\lambda}\left(\mathbb{I} \otimes \rho_\lambda(g)\right) \end{aligned} \tag{II.1.29}$$

Notice that the action of Coderivative on the twist with an arbitrary representation (λ) produces a generalized permutation operator, which appears in the definition of the R-matrix with representation (λ) in the auxiliary space (II.1.3b). This allows us to replace the generalized permutation operators in (II.1.3a) with $\hat{D}$.

The next logical step is to check the action of two Coderivatives on $\rho_\lambda(g)$. Thankfully, the Coderivative operator obeys the Leibniz rule:

$$\hat{D} \otimes (f(g) \cdot h(g)) = \left(\hat{D} \otimes f(g)\right) \cdot (\mathbb{I} \otimes h(g)) + (\mathbb{I} \otimes f(g)) \cdot \left(\hat{D} \otimes h(g)\right) \tag{II.1.30}$$

Thus, by acting with a second Coderivative on (II.1.29), one obtain:

$$\begin{aligned} \hat{D} \otimes \hat{D} \otimes \rho_\lambda(g) = \hat{D} \otimes (\mathcal{P}_{2,\lambda} \cdot (\mathbb{I} \otimes \rho_\lambda(g))) = \mathcal{P}_{2,\lambda} \cdot \left(\hat{D} \otimes (\mathbb{I} \otimes \rho_\lambda(g))\right) \\ = \mathcal{P}_{2,\lambda}\mathcal{P}_{1,\lambda} \cdot (\mathbb{I} \otimes \mathbb{I} \otimes \rho_\lambda(g)) \end{aligned} \tag{II.1.31}$$

In this relation, it is evident from the right hand side that we have a reversed labeling of our physical spaces when using $\hat{D}$. By induction, one can show that an arbitrary number of Coderivatives acting on $\rho_\lambda(g)$ will produce a set of permutation operators:

$$\hat{D}^{\otimes N} \otimes \rho_\lambda(g) = \mathcal{P}_{N,\lambda}\mathcal{P}_{N-1,\lambda} \ldots \mathcal{P}_{1,\lambda} \cdot \left(\mathbb{I}^{\otimes N} \otimes \rho_\lambda(g)\right) \tag{II.1.32}$$

Thus, one can think of reformulating the T-operators in terms of Coderivative by replacing the $\mathcal{P}$-operators with a Coderivative operator[11]. The T-operators takes the form:

$$\begin{aligned} \mathcal{T}^{(\lambda)}(u) &= \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_N + \hat{D}\right) \chi_\lambda(g) \\ &= \bigotimes_{i=1}^{N} \left(u_i + \hat{D}\right) \otimes \chi_\lambda(g) \end{aligned} \tag{II.1.33a}$$

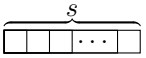
$$\text{Where } u_i = u - \theta_i, \qquad \text{And } \chi_\lambda(g) = \mathrm{tr}_\lambda(\rho_\lambda(g)) \tag{II.1.33b}$$

As mentioned at the beginning of the section, (II.1.33a) is a description of T-operators with the differential operator $\hat{D}$ that acts on the character of the representation (II.1.33b). The two definitions of T-operators provided in (II.1.3a) and (II.1.33a) have a conceptual difference in constructing T-operators, which are built using R-matrices which correspond to spin in our spin chain and then take a trace on the auxiliary space, whereas the latter starts with the character of our system, and spins are added up by the action of the Coderivative on the character [12].

With the help of $\hat{D}$, one can: use the fact that we have a Wronskian expression of symmetric T-operators $\mathcal{T}^{(1,s)}(u)$[13] to describe a more general representation of a T-operators $\mathcal{T}^{(\lambda)}(u)$, this

[11] When replacing the $\mathcal{P}$-operators with Coderivatives, one should not forget about the flipping in the Hilbert space, as we have shown in (II.1.31).

[12] Also, from (II.1.33a) or (II.1.3a), one notice that these operators are polynomials in u

[13] $\mathcal{T}^{(1,s)}(u)$ is a T-operators with a symmetric representation in the auxiliary space; $(1,s) =$ 

refers to the number of (rows, columns) respectively of the representation (as shown in Fig. II.3).

was well documented in [KV08] (check [Leu12] for a more detailed discussion). Also, it allows us to describe fundamental objects from the generating series of the symmetry character called Q-operators, these operators were described in [KLT12] using Coderivative (they have managed to show the computations of Q-operators reduced to a combinatorial description of permutation diagrams). In this PhD, we will use this operator to provide a proof of the Yang-Baxter equation with a general function of the twist $f(g)$ in the auxiliary space (see Sec(II.1.5)). Then, in Sec(II.2), we will show how to obtain a nested T- and Q-operators from the two descriptions (oscillator and differential) and prove that the two results are equivalent.

II.1.5 Coderivative and the Yang-Baxter equation.

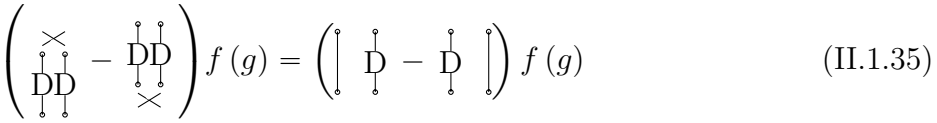
Now that we have (II.1.33a), one can ask about the family of functions that satisfies the modified Yang-Baxter equation using the Coderivative operator. This question was answered in [KLT12], where it was obtained by the action of Coderivative on an arbitrary collection of the character (that was a consequence of the properties of the R-matrix). In this section, we will provide a formal proof of the question using Schwartz's property of the differential operator.

This can be proven by taking the first step of rewriting the Coderivative version of the particular Yang-Baxter equation (a Yang-Baxter equation with one auxiliary space and two physical ones):

$$\mathcal{R}^{i_1,i_2}_{k_1,k_2}(u-v)\left(u+\hat{D}\right)^{k_1}_{j_1}\left(v+\hat{D}\right)^{k_2}_{j_2}f(g)=\left(v+\hat{D}\right)^{i_2}_{k_2}\left(u+\hat{D}\right)^{i_1}_{k_1}\mathcal{R}^{k_1,k_2}_{j_1,j_2}(u-v)\,f(g) \quad \text{(II.1.34)}$$

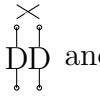
where the indices $\{i_1,i_2,j_1,j_2\}$ are the physical labeling of the operators, and they will be referred to as (outgoing, ingoing) for (i,j) , respectively, whereas the indices $\{k_1,k_2\}$ will refer to a contraction between the operators. An expression like $\left(u+\hat{D}\right)^{\alpha}_{\beta}$ read as $u\delta_{\alpha,\beta}+\hat{D}^{\alpha}_{\beta}$

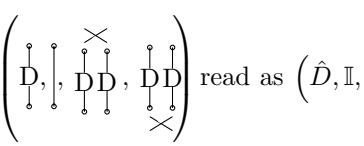
The proof of (II.1.34) boils down to the nontrivial identity we obtained in Appendix(B.1.1). In the appendix, we collected the terms of all the powers of the spectral parameters in (II.1.34) and found out that the only nontrivial one to be [14]:



(II.1.35)

Proof. of (II.1.34)*:*

First, notice that the matrix coefficients of the operators  and has a permutation that exchanges the outgoing and ingoing labeling of its physical spaces respectively. For generality,

[14]The diagrammatic representation of operators  read as $\left(\hat{D},\mathbb{I},\mathcal{P}\left(\hat{D}\otimes\hat{D}\right),\left(\hat{D}\otimes\hat{D}\right)\mathcal{P}\right)$, respectively.

we assumed a function of the twist $f(g)$ to be on the auxiliary space, using (II.1.28) the left hand side of (II.1.35) reads:

$$\left(\begin{array}{c} \times \\ \text{DD} \end{array} - \begin{array}{c} \text{DD} \\ \times \end{array} \right)^{i_1,i_2}_{j_1,j_2} f(g) = \frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\phi}\left(f\left(e^{\phi J_{j_2,i_1}} e^{\varepsilon J_{j_1,i_2}}\right) - f\left(e^{\phi J_{j_1,i_2}} e^{\varepsilon J_{j_2,i_1}}\right) \right)|_{\varepsilon=\phi=0}$$

$$= \frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\phi}\Bigg(f\Big(g + \left(\varepsilon J_{j_1,i_2} + \phi J_{j_2,i_1} + \varepsilon\phi\delta_{i_1,j_1} J_{j_2,i_2}\right) g + \mathcal{O}\left(\phi^2\right) + \mathcal{O}\left(\varepsilon^2\right)\Big) \tag{II.1.36}$$

$$- f\Big(g + \left(\varepsilon J_{j_2,i_1} + \phi J_{j_1,i_2} + \varepsilon\phi\delta_{i_2,j_2} J_{j_1,i_1}\right) g + \mathcal{O}\left(\phi^2\right) + \mathcal{O}\left(\varepsilon^2\right)\Big)\Bigg)|_{\varepsilon=\phi=0}$$

To elaborate, the second equality was obtained by taking the multiplication of the Taylor expansion of the exponential functions and leaving only the terms with degree one in ε and ϕ. Assuming that $f \in \mathcal{C}^2$, this will help us in writing (II.1.36) as a differential function using directional derivatives. For example, the matrix coefficient of the term $(J_{j_1,i_2} g)$ from (II.1.36) reads:

$$(J_{j_1,i_2} g)^{k_1}_{k_2} = \overbrace{(J_{j_1,i_2})^{k_1}_{m}}^{\delta_{j_1,k_1}\delta_{i_2,m}} g^{m}_{k_2} = \delta_{j_1,k_1} g^{i_2}_{k_2}$$

Using that, we can write $\begin{array}{c} \times \\ \text{DD} \end{array}$ as:

$$\begin{array}{c} \times \\ \text{DD} \end{array} f(g) = \Bigg(f(g) + \sum_{k_1,k_2} \frac{\partial f}{\partial g_{k_1,k_2}} \left[\varepsilon\delta_{k_1,j_1} g^{i_2}_{k_2} + \phi\delta_{k_1,j_2} g^{i_1}_{k_2} + \varepsilon\phi\delta_{i_1,j_1}\delta_{i_1,k_1} g^{i_2}_{k_2} \right] +$$

$$\sum_{k_1,k_2,k_3,k_4} \frac{\partial^2 f}{\partial g_{k_1,k_2}\partial g_{k_3,k_4}} \left[\varepsilon\phi \left(\delta_{k_1,j_1} g^{i_2}_{k_2}\right)\left(\delta_{k_3,j_2} g^{i_1}_{k_4}\right) \right] + \mathcal{O}\left(\phi^2\right) + \mathcal{O}\left(\varepsilon^2\right) \Bigg)|_{\varepsilon=\phi=0} \tag{II.1.37}$$

The same goes for $\begin{array}{c} \text{DD} \\ \times \end{array}$ up to changing the indices $(i_1 \leftrightarrow i_2), (j_1 \leftrightarrow j_2)$ and $(\varepsilon \leftrightarrow \phi)$, the last exchange is permitted; scene ε, ϕ are dummy variables. After collecting the terms together, one finds:

$$\left(\begin{array}{c} \times \\ \text{DD} \end{array} - \begin{array}{c} \text{DD} \\ \times \end{array} \right) f(g) = \sum_{k_1,k_2} \frac{\partial f}{\partial g_{k_1,k_2}} \left(\delta_{i_1,j_1}\delta_{k_1,j_2} g^{i_2}_{k_2} - \delta_{i_2,j_2}\delta_{k_1,j_1} g^{i_1}_{k_2} \right) \tag{II.1.38}$$

In the last expression, we notice that we have a free Kronecker delta (a Kronecker delta with no index k to contract with another function), and they are labeled with the first or second quantum indices $(i_1, j_1), (i_2, j_2)$, respectively. To prove the equality in (II.1.35), one can easily do the same steps as before for the right hand side of (II.1.35) and obtain the same result as (II.1.38).

□

II.2 Q-operators

In the study of spin chains, one key object is the Q-operators, a family of commuting T-operators that commute with T-operators (and hence with the Hamiltonian), these operators were heavily studied in the literature, some of the references about the subject are: [KLWZ97, GKLT11, Tsu13] where they generate the Q-operators from the Bethe algebra, and [BLZ99, Tsu10, KLV16] showing the various functional relations these operators satisfy and there connection to Bethe Ansatz. This section will be a review to two different constructions of Q-operators, but let us first mention that they allow us to express T-operators via the following *Wronskian* determinant:

$$\mathcal{T}^{(\lambda)}(u) = \chi_\lambda(g)\, u^N + \dots \tag{II.2.1a}$$

$$\propto \begin{vmatrix} x_1^{\lambda_1}\mathbb{Q}^{[2\lambda_1]}_{\{1\}}(u) & x_2^{\lambda_1}\mathbb{Q}^{[2\lambda_1]}_{\{2\}}(u) & \dots & x_{\mathsf{n}}^{\lambda_1}\mathbb{Q}^{[2\lambda_1]}_{\{\mathsf{n}\}}(u) \\ x_1^{\lambda_2-1}\mathbb{Q}^{[2\lambda_2-2]}_{\{1\}}(u) & x_2^{\lambda_2-1}\mathbb{Q}^{[2\lambda_2-2]}_{\{2\}}(u) & \dots & x_{\mathsf{n}}^{\lambda_2-1}\mathbb{Q}^{[2\lambda_2-2]}_{\{\mathsf{n}\}}(u) \\ \vdots & \vdots & \dots & \vdots \\ x_1^{\lambda_{\mathsf{n}}-\mathsf{n}+1}\mathbb{Q}^{[2\lambda_{\mathsf{n}}-2\mathsf{n}+2]}_{\{1\}}(u) & x_2^{\lambda_{\mathsf{n}}-\mathsf{n}+1}\mathbb{Q}^{[2\lambda_{\mathsf{n}}-2\mathsf{n}+2]}_{\{2\}}(u) & \dots & x_{\mathsf{n}}^{\lambda_{\mathsf{n}}-\mathsf{n}+1}\mathbb{Q}^{[2\lambda_{\mathsf{n}}-2\mathsf{n}+2]}_{\{\mathsf{n}\}}(u) \end{vmatrix}$$

$$\propto \left| \left(x_j^{\lambda_k-k+1}\mathbb{Q}^{[2\lambda_k-2k+2]}_{\{j\}}(u) \right)_{j,k\in[\![1,\mathsf{n}]\!]} \right|, \tag{II.2.1b}$$

where the $\propto$ symbol is a normalization factor that is fixed by (II.2.1a) (which sets the coefficient of u^N to $\chi_\lambda(g)$, as follows from the definition (II.1.33a)), and $x_1, \dots, x_{\mathsf{n}}$ are the eigenvalues of the twist operator g [15]. The Q-operators are hence directly related to the Hamiltonian, and they provide a way to diagonalize the Hamiltonian, known as *Wronskian Bethe equations* [CLV22].

The operators $\mathbb{Q}_{\{j\}}(u)$ that appear in this expression are the *single-indexed* Q-operators, as opposed to *multi-indexed* Q-operators $\mathbb{Q}_{\{i_1,i_2,\dots\}}(u)$ that we will also introduce, and which will be labeled by arbitrary subsets $\{i_1, i_2, \dots\} \subset \{1, 2, \dots, \mathsf{n}\}$.

It turns out that the minors of the determinant play an important role (which will be used in Sec(II.2.2.3)). To exemplify this idea, let us take an arbitrary subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, \mathsf{n}\}$ and some weights $\mu_1 \geqslant \mu_2 \cdots \geqslant \mu_k$ (associated to a representation of $GL(\mathsf{n})$), one can define the minor

$$\mathcal{T}^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}(u) = \chi_\mu(\tilde{g})\; u^d + \dots \qquad \text{where} \qquad \tilde{g} = \mathrm{diag}(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \tag{II.2.2a}$$

$$\propto \left| \left(x_{i_j}^{\mu_\ell-\ell+1}\mathbb{Q}^{[2\mu_\ell-2\ell+2]}_{\{i_j\}}(u) \right)_{j,\ell\in[\![1,k]\!]} \right|, \tag{II.2.2b}$$

where d is the degree of $\mathcal{T}^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}(u)$ as a polynomial in u[16], and we see that the normalization factor is now given by the character of a $GL(k)$ element $\tilde{g}$ with eigenvalues $x_{i_1}, x_{i_2}, \dots, x_{i_k}$,

[15] In this thesis, we will address the case of the twist with distinct eigenvalues, i. e. $x_i \neq x_j$ if $i \neq j$.

[16] Notice that d itself is an operator, in the sense that the degree of the polynomial varies from one sector to another in the Hilbert space. One can view all T-operators Q-operators, *etc.* as block-diagonal (where the blocks correspond to the number of spins pointing in each direction), and in each block d is an integer multiple of identity. Hence it is conceptually easier to restrict equation (II.2.2) to any of these blocks, and see d as an integer. For more details, check out the discussion on conserved charges in this section.

acting in representation $(\mu_1, \dots \mu_k)$. In (II.2.2a), the number k of elements of the subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, \mathsf{n}\}$ is sometimes referred to as the *nesting level*[17].

A case of particular importance is when $(\mu_1, \dots, \mu_k) = (0, \dots, 0)$, and then it defines the Q-operators:

$$\mathbb{Q}_{\{i_1,i_2,\dots,i_k\}}(u) = \mathcal{T}^{(0,\dots,0)}_{\{i_1,i_2,\dots,i_k\}}(u) \ . \tag{II.2.3}$$

II.2.1 Constructions of the nested T-operators

Let us now hint at two of the constructions that can be found in the literature, that shows these Q-operators do exist and are polynomial, by explicitly constructing them. One construction [KLT12] was obtained from the Coderivative, by constructing explicitly an operator as (we will motivate this expression in the subsection II.2.2.3):

$$\mathcal{T}^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}(u) = \underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \ \bigotimes_{i=1}^{N}\left(u_i + \mathsf{n} - k + \hat{D}\right)\left[\prod_{j\notin\{i_1,i_2,\dots,i_k\}} w(z_j)\chi_\mu(\tilde{g})\right] \tag{II.2.4}$$

(where we remind that $\tilde{g} = \mathrm{diag}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$) and $w(z) = \sum_{s\geqslant 0}\chi_{(s)}z^s$ is the generating series of the symmetric character. Finding out that this operator obeys (II.2.2), and the leading coefficients in u is fixed by (II.2.1).

Another construction relies on finding an explicit infinite-dimensional representation and an R-matrix such that the nested T-operators of (II.2.2) is equal to the trace of the product of these R-matrix (times the action of twist in the representation). This construction was written in a series of papers (including [BLMS10],[BFL$^+$11] and [FKT21]) and we show below how to rewrite it in terms of Coderivatives. It follows the three following steps:

- The determinant in (II.2.1) reads $\sum_\sigma (-1)^\sigma \prod_{\ell\in[\![1,\mathsf{n}]\!]} x_{\sigma(\ell)}^{\lambda_\ell-\ell+1}\mathbb{Q}^{[2\lambda_\ell-2\ell+2]}_{\{\sigma(\ell)\}}(u)$. In this sum over permutations σ, it turns out that each term can be written as a trace of product of R-matrix (times twist), and the authors explicit this R-matrix in terms of oscillator representations.

  This step can easily be rephrased in terms of Coderivatives: the character $\chi_\lambda(g)$ of (II.1.33a) turns out to be equal to $\left|\left(x_j^{\lambda_i+\mathsf{n}-i}\right)_{i,j\in[\![1,\mathsf{n}]\!]}\right| \Big/ \prod_{i<j}(x_i - x_j)$. One can expand this determinant, and we denote $\chi^{(\sigma)}_\lambda(g) = \prod_{\ell\in[\![1,\mathsf{n}]\!]} x_{\sigma(\ell)}^{\lambda_\ell+\mathsf{n}-\ell} \Big/ \prod_{i<j}(x_i - x_j)$ so that we obtain $\chi_\lambda(g) = \sum_\sigma(-1)^\sigma\chi^{(\sigma)}_\lambda(g)$, hence $\mathcal{T}(u) = \sum_\sigma(-1)^\sigma\bigotimes_{i=1}^N\left(u_i+\hat{D}\right)\chi^{(\sigma)}_\lambda(g)$. We will not need it in the present discussion, but it turns out that an infinite-dimensional representation and an R-matrix can be found, which are associated with the character $\chi^{(\sigma)}_\lambda(g)$.

[17] By contrast, some authors refer to $\mathsf{n}-k$ as the nesting level.

- The second step is to take a limit where only a subset of factors survive in the product $\prod_{\ell\in[\![1,\mathsf{n}]\!]} x_{\sigma(\ell)}^{\lambda_\ell-\ell+1}\mathbb{Q}_{\{\sigma(\ell)\}}^{[2\lambda_\ell-2\ell+2]}(u)$ (which is the product exhibited in the previous step for a fixed permutation σ).

  More precisely, we set $\lambda_i = \mu_{i-\mathsf{n}+k} + \mathsf{n} - k$ when $i > \mathsf{n} - k$, which means that in the last lines of (II.2.1) the powers of x_j (and the shift of $\mathbb{Q}$-functions) are the same as in (II.2.2), namely $(\lambda_{\mathsf{n}-k+1} - \mathsf{n} + k, \dots, \lambda_{\mathsf{n}} - \mathsf{n} + 1) = (\mu_1, \dots \mu_k - k + 1)$.

  We moreover focus on specific permutations σ in this determinant: if we denote the set $[\![1,\mathsf{n}]\!]\backslash\{i_1, i_2, \dots, i_k\}$ as $\{j_1, \dots j_{\mathsf{n}-k}\}$, we chose a permutation sigma such that $\sigma(1) = j_1$, $\sigma(2) = j_2$, …, $\sigma(\mathsf{n} - k) = j_{\mathsf{n}-k}$ whereas the last terms $(\sigma(\mathsf{n} - k + 1), \dots, \sigma(\mathsf{n}))$ form an arbitrary permutation $\tilde{\sigma}$ of $\{i_1, i_2, \dots, i_k\}$.

  Then, when we take the limit $\lambda_i \to \infty$ for each $i \in [\![1, \mathsf{n} - k]\!]$, we see that the product $\prod_{\ell\in[\![1,\mathsf{n}]\!]} x_{\sigma(\ell)}^{\lambda_\ell-\ell+1}\mathbb{Q}_{\{\sigma(\ell)\}}^{[2\lambda_\ell-2\ell+2]}(u)$ reduces to $\prod_{\ell\in[\![1,k]\!]} x_{i_{\tilde{\sigma}(\ell)}}^{\mu_\ell-\ell+1}\mathbb{Q}_{\{i_{\tilde{\sigma}(\ell)}\}}^{[2\mu_\ell-2\ell+2]}(u)$.

  Indeed, the rest of the product is $\lim_{\lambda_a\to\infty}\prod_{a\in[\![1,\mathsf{n}-k]\!]} x_{j_a}^{\lambda_a-a+1}\mathbb{Q}_{\{j_a\}}^{[2\lambda_a+2a+2]}(u)$ which is an overall $\tilde{\sigma}$ independent constant, which we normalize out[18].

  It turns out that in this limit, the R-matrices have a well-defined limit; hence, the result can be expressed as a trace of the product of R-matrix (times a twist). This expression in terms of R-matrices was expressed in [BLMS10] for $GL(2)$, where it is introduced as a $\lambda \to \infty$ limit as in the above paragraph. By contrast, at higher rank, it was introduced in [BFL$^+$11] and [FKT21] without a $\lambda \to \infty$ limit, Appendix A.1.2 gives slightly more details on this construction and shows that it coincides indeed with the limit where λ_1, λ_2, …, $\lambda_{\mathsf{n}-k}$ are infinite.

- The third step is to perform the sum over the permutation $\tilde{\sigma}$, which allows to recover the right hand side of (II.2.2b).

  If we rewrite this step in terms of Coderivatives, it expresses $\mathcal{T}_{\{i_1,i_2,\dots,i_k\}}^{(\mu_1,\dots,\mu_k)}(u)$ as

$$\mathcal{T}_{\{i_1,i_2,\dots,i_k\}}^{(\mu_1,\dots,\mu_k)}(u) = \sum_{\tilde{\sigma}}(-1)^{\tilde{\sigma}} \operatorname*{\mathcal{M}lim}_{\substack{\lambda_1\to\infty\\ \vdots \\ \lambda_{\mathsf{n}-k}\to\infty}} \bigotimes_{i=1}^{N}\left(u_i + \hat{D}\right)\chi_\lambda^{(\sigma)}(g) \tag{II.2.5}$$

  But if we recall the definition of $\chi_\lambda^{(\sigma)}(g)$, and the above relations between λ and μ and between σ and $\tilde{\sigma}$, we obtain that[19]

$$\chi_\lambda^{(\sigma)}(g) = \frac{(\det g)^{\mathsf{n}-k}\, x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}}{\prod_{1\leqslant a<b\leqslant\mathsf{n}-k}(x_{j_a}-x_{j_b})\prod_{\substack{1\leqslant a\leqslant\mathsf{n}-k\\ 1\leqslant\ell\leqslant k}}(x_{j_a}-x_{i_\ell})}\chi_\mu^{(\tilde{\sigma})}(\tilde{g}) \tag{II.2.6}$$

[18] The normalization follows exactly as described in Sec(I.1.1) or Sec(II.2.2.3) for detailed calculation.

[19] Equation (II.2.6) follows from the definition of the χ^σ and of λ and of σ, which give

$$\chi_\lambda^\sigma(g) = \frac{x_{j_1}^{\lambda_1+n-1}\dots x_{j_{n-k}}^{\lambda_{n-k}+k} x_{i_{\tilde{\sigma}(1)}}^{\mu_1+n-1}\dots x_{i_{\tilde{\sigma}(k)}}^{\mu_k+\mathsf{n}-k}}{\prod_{1\leqslant a<b\leqslant\mathsf{n}-k}(x_{j_a}-x_{j_b})\prod_{\substack{1\leqslant a\leqslant\mathsf{n}-k\\ 1\leqslant\ell\leqslant k}}(x_{j_a}-x_{i_\ell})\prod_{1\leqslant\ell<m\leqslant k}(x_{i_\ell}-x_{i_m})},$$

and $\chi_\mu^{\tilde{\sigma}}(\tilde{g}) = \dfrac{x_{i_{\tilde{\sigma}(1)}}^{\mu_1+k-1}\dots x_{i_{\tilde{\sigma}(k)}}^{\mu_k}}{\prod_{1\leqslant\ell<m\leqslant k}(x_{i_\ell}-x_{i_m})}$.

hence

$$\mathcal{T}^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}(u) = \underset{\substack{\lambda_1\to\infty\\ \vdots \\ \lambda_{\mathsf{n}-k}\to\infty}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N}\left(u_i+\hat{D}\right)\frac{(\det g)^{\mathsf{n}-k}\, x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}}{\prod_{1\leqslant a<b\leqslant \mathsf{n}-k}(x_{j_a}-x_{j_b})\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}(x_{j_a}-x_{i_\ell})}\chi_\mu(\tilde{g}) \tag{II.2.7}$$

The way of writing the character as in (II.2.6) will show to be a key step in the argument of equivalence between the two constructions of Q-operators.

II.2.2 Relation between these two constructions

In the present subsection, we will show that the two constructions hinted in the previous subsection II.2.1 actually match.

To this end, we will first intruduce a new diagramalcal notation to help us show that (II.2.4) found in one of the two constructions:

II.2.2.1 Diagrammatic expressions from the Leibniz rule

Before we go any further, we notice that (II.2.4) and (II.2.7) involve Coderivatives of products of multiple factors. We will now see that such an expression can be expressed via some diagrams:

To start with, let us consider a product of two scalar factors $\mathcal{A}$ and $\mathcal{B}$, and let us compute $\bigotimes_{i=1}^{N}\left(u_i+\hat{D}\right)\mathcal{A}\mathcal{B}$ at lengths $N=1$ and $N=2$:

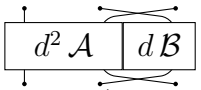
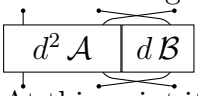
At length one, the Leibniz rule (II.1.30) gives $\left(u_1+\hat{D}\right)\mathcal{A}\mathcal{B} = \left(u_1+\frac{\hat{D}\mathcal{A}}{\mathcal{A}}+\frac{\hat{D}\mathcal{B}}{\mathcal{B}}\right)\mathcal{A}\mathcal{B}$, which we shall write as $\left(\boxed{u}+\boxed{d\,\mathcal{A}}+\boxed{d\,\mathcal{B}}\right)\mathcal{A}\mathcal{B}$ where $\boxed{u}$ (resp $\boxed{d\,\mathcal{A}}$, resp $\boxed{d\,\mathcal{B}}$) denotes a factor u_i (resp $\frac{\hat{D}\mathcal{A}}{\mathcal{A}}$ resp $\frac{\hat{D}\mathcal{B}}{\mathcal{B}}$). At length two, $\left(u_1+\hat{D}\right)\left(u_2\mathcal{A}\mathcal{B}+\mathcal{B}\hat{D}\mathcal{A}+\mathcal{A}\hat{D}\mathcal{B}\right)$ involves a Leibniz rule where the Coderivative can act on either of the two factors of each term. We therefore obtain:

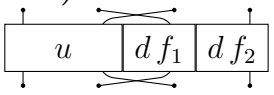
$$\bigotimes_{i=1}^{2}\left(u_i+\hat{D}\right)\mathcal{A}\mathcal{B} = \Big(\boxed{u}\boxed{u} + \boxed{u}\boxed{d\,\mathcal{A}} + \boxed{u}\boxed{d\,\mathcal{B}} + \boxed{d\,\mathcal{A}}\boxed{u} + \boxed{d\,\mathcal{B}}\boxed{u} + \boxed{d\,\mathcal{B}}\boxed{d\,\mathcal{A}} + \boxed{d^2\,\mathcal{A}} + \boxed{d\,\mathcal{A}}\boxed{d\,\mathcal{B}} + \boxed{d^2\,\mathcal{B}}\Big)\mathcal{A}\mathcal{B}, \tag{II.2.8}$$

where for instance $\boxed{u}\boxed{d\,\mathcal{A}}$ denotes $u_1\mathbb{I}\otimes\frac{\hat{D}\mathcal{A}}{\mathcal{A}}$, whereas $\boxed{d\,\mathcal{B}}\boxed{u}$ denotes $\frac{\hat{D}\mathcal{B}}{\mathcal{B}}\otimes u_2\mathbb{I}$, and $\boxed{d^2\,\mathcal{A}}$ denotes $\frac{\hat{D}^2\mathcal{A}}{\mathcal{A}}$.

Before we give the general statement for the outcome of Leibniz rule, let us also consider length $N=3$, but now for simplicity we omit the u_i:

$$\hat{D}^{\otimes 3}\mathcal{A}\mathcal{B} = \Big(\boxed{d^2\,\mathcal{B}}\boxed{d\,\mathcal{A}} + \boxed{d^2\,\mathcal{A}}\boxed{d\,\mathcal{B}} + \boxed{d^3\,\mathcal{A}} + \boxed{d\,\mathcal{B}}\boxed{d^2\,\mathcal{A}} + \boxed{d^2\,\mathcal{A}}\boxed{d\,\mathcal{B}} + \boxed{d^2\,\mathcal{B}}\boxed{d\,\mathcal{A}} + \boxed{d^3\,\mathcal{B}} + \boxed{d\,\mathcal{A}}\boxed{d^2\,\mathcal{B}}\Big)\mathcal{A}\mathcal{B}, \tag{II.2.9}$$

where $d^3\,\mathcal{B}$ denotes $\frac{\hat{D}^3\mathcal{B}}{\mathcal{B}}$, and where the crossings of external legs in  keep track of the ingoing and outgoing indices: the matrix coefficient at position $(i_1 i_2 i_3, j_1 j_2 j_3)$ in  is $(d^2\,\mathcal{A})^{i_1 i_3}_{j_1 j_3}\,(d\,\mathcal{B})^{i_2}_{j_2}$.

At this point it becomes clear that for an arbitrary product of factors $f_1 f_2 \dots f_k$, the Leibniz rule allows to expand $\bigotimes_{i=1}^{N}\left(u_i + \hat{D}\right) f_1 f_2 \dots f_k$ as the sum of $(k+1)^N$ diagrams: each diagram is characterized by a partition of $[\![1, N]\!]$ into $k+1$ sets: $A_0 \sqcup A_1 \sqcup \dots \sqcup A_k = [\![1, N]\!]$ and we associate the legs with position in A_1 to $d\,f_1$, the legs with position in A_2 to $d\,f_2$, etc. and the legs with position in A_0 to factors $u_i \mathbb{I}$. For instance, in $\bigotimes_{i=1}^{4}\left(u_i + \hat{D}\right) f_1 f_2$, the term associated to the partition $\{1,3\} \sqcup \{2\} \sqcup \{4\}$ corresponds to the diagram  , which means $u_1 u_3 \mathbb{I} \otimes \frac{\hat{D}\,f_1}{f_1} \otimes \mathbb{I} \otimes \frac{\hat{D}\,f_2}{f_2}$.

And we have to remember that the sum of these diagrams is multiplied by $f_1 f_2 \dots f_k$ (in the same way that there were overall factors $\mathcal{A}\mathcal{B}$ in (II.2.8) and (II.2.9)).

II.2.2.2 Equivalence of the two constructions

Following the previous subsection, the equivalence of these two constructions would amount to the equality

$$\begin{aligned}
&\underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N}\left(u_i + \mathsf{n} - k + \hat{D}\right)\left[\prod_{a=1}^{\mathsf{n}-k}\prod_{i=1}^{\mathsf{n}}\frac{1}{1 - x_i z_{j_a}}\chi_\mu\left(\tilde{g}\right)\right] \\
&\qquad = \underset{\substack{\lambda_1\to\infty\\ \vdots \\ \lambda_{\mathsf{n}-k}\to\infty}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N}\left(u_i + \hat{D}\right)\frac{(\det g)^{\mathsf{n}-k}\, x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}}{\prod_{1\leqslant a<b\leqslant \mathsf{n}-k}\left(x_{j_a} - x_{j_b}\right)\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}\left(x_{j_a} - x_{i_\ell}\right)}\chi_\mu(\tilde{g})
\end{aligned} \tag{II.2.10}$$

This equailty will follow from Lemma(I.1.2). In the previous equation one can see that the polynomial produced is not monic in u. Thus, the normalization in Sec(I.1.1) can be fixed according to (I.1.13) using the highest coefficient in u from (II.2.2a).

II.2.2.2.1 Polynomiality in λ

Let us now conclude, from the way that limits are defined up to a normalisation, and from the above expressions of the Leibniz rule, that in the right hand side of (II.2.10), we actually take the limit of a polynomial function of the λ.

When we express the right hand side of (II.2.10) from the Leibniz rule as above, it gives a sum of diagrams, multiplied by the factor $\frac{(\det g)^{\mathsf{n}-k} x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}}{\prod_{1\leqslant a<b\leqslant \mathsf{n}-k}\left(x_{j_a} - x_{j_b}\right)\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}}\chi_\mu(\tilde{g})$ (which is the product $f_1 \dots f_k$ of the previous paragraph). In this product, the factors $x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}$ are the only ones that contain λ, whereas in the diagrams, we only have polynomials in λ.

Indeed, it turns out that there exists polynomial[20] $P_{n,i}(\lambda)$ such that $\hat{D}^n x_i^\lambda = P_{n,i}(\lambda) x_i^\lambda$, and this polynomial is expressed by recurrence as $P_{1,i}(\lambda) = \lambda p_i$ and $P_{n,i}(\lambda) = \lambda P_{n-1,i}(\lambda) \otimes p_i + \hat{D} P_{n-1,i}(\lambda)$[21] (this last equality follows from the Leibnitz rule when we differentiate $P_{n-1,i}(\lambda) x_i^\lambda$). It follows that in each diagram, the factors $d^k x_j^\lambda = P_{k,j}(\lambda)$ are polynomials in λ, and hence we are in a condition that allows to apply the therom.

It follows that the right hand side of (II.2.10) is equal to

$$\underset{\substack{z_{j_1}\to 1/x_{j_a}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \quad \bigotimes_{i=1}^{N} \left(u_i + \hat{D}\right) \frac{(\det g)^{\mathsf{n}-k} \prod_{a=1}^{\mathsf{n}-k} \frac{1}{1-x_{j_a} z_{j_a}}}{\prod_{1\leqslant a<b\leqslant \mathsf{n}-k} (x_{j_a} - x_{j_b}) \prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k \\ 1\leqslant \ell\leqslant k}} (x_{j_a} - x_{i_\ell})} \chi_\mu(\tilde{g}) \tag{II.2.11}$$

Terms that do not contribute to the $z_j \to 1/x_j$ limit

Let us show now that II.2.11 is also equal to

$$\underset{\substack{z_{j_1}\to 1/x_{j_a}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \quad \bigotimes_{i=1}^{N} \left(u_i + \hat{D}\right) \frac{(\det g)^{\mathsf{n}-k} \prod_{a=1}^{\mathsf{n}-k} \frac{1}{1-x_{j_a} z_{j_a}}}{\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k \\ 1\leqslant \ell\leqslant k}} (x_{j_a} - x_{i_\ell})} \chi_\mu(\tilde{g}) \tag{II.2.12}$$

i. e. that the factors $\frac{1}{(x_{j_a} - x_{j_b})}$ are completely irrelevant:

To this end, we should first notice that in (II.2.11), (II.2.12) the action of Coderivative gives a rational function of all z_j. The limit $z_j \to 1/x_j$ simply picks the coefficient of the most singular term in its Laurent series. To identify this coefficient, the first step is to understand the multiplicity of the pole at $z_j \to 1/x_j$.

As one can see from the right hand side of (II.2.4), (II.2.11) and (II.2.12) there are poles that come from the limit $z_j \to \frac{1}{x_j}$. These poles come solely from the factors $\frac{1}{1-x_{j_\ell} z_{j_\ell}}$ and their derivatives. Whereas the rest of the factors and their derivatives are $\mathcal{C}^\infty$ in the vicinity of $\frac{1}{x_j}$, they do not contribute to the singularity.

The degree of a pole is sector dependant, i. e. it varies from one sector in the Hilbert space to another. We will call "sectors" the subspaces spanned by basis vectors $|V_N\rangle \equiv |v_{k_1}, v_{k_2}, \ldots, v_{k_N}\rangle$ where a fixed number of factors v_{k_ℓ} is equal to each v_k. Such that, in the eigenvectors of basis, $g_1, g_2, \ldots, g_N$ read as $g_l |V_N\rangle = x_{k_l} |V_N\rangle$[22]. Then, a singularity at $z \to \frac{1}{x_j}$ comes from a factor $d^k \frac{1}{1-x_j z}$ (the notation of derivative d is introduced in Sec(II.2.2.1)). This can be explained by viewing $\frac{1}{1-x_j z_j}$ as a matrix element of the operator $\frac{1}{det(1-g z_j)}$, such that $\frac{1}{1-x_j z_\ell} = \langle v_j | \left(\frac{1}{1-g z_\ell}\right) |v_j\rangle$.

[20]Note that $P_{n,i}(\lambda)$ is an operator valued polynomial, ie the coefficient of each power of λ is a complicated function of g.

[21]The parameter p_i is the projector on the eigenspace associated with the eigenvalue x_i and is described in Appendix(B.1).

[22]For a chain with N spins in it, $g_\ell = \mathbb{I}^{\otimes(\ell-1)} \otimes g \otimes \mathbb{I}^{\otimes(N-\ell+1)}$ refare to how it act on a vector $|V_N\rangle$ of the Hilbert space.

Then, we write the matrix coefficients of an opeartor $d^N \frac{1}{det(1-gz_\ell)}$ as:

$$\langle v_{i_1} \dots, v_{i_N}| \, d^N \frac{1}{det\,(1-gz_\ell)} \, |v_{j_1} \dots, v_{j_N}\rangle = \sum_{\sigma \in \mathcal{S}_N} \prod_{k=1}^{N} \left(\frac{(x_k z_\ell)^{\theta(\sigma(k)-k)}}{1-x_k z_\ell} \right) \delta_{j_k}^{(i_{\sigma k})} \tag{II.2.13}$$

The expression on the right hand side of $d^N \frac{1}{det(1-gz_\ell)}$ was found in [KLT12], $\theta(k) = \begin{cases} 1, & \text{if } k \geqslant 0 \\ 0, & \text{if } k < 0 \end{cases}$ is the Heaviside step function, and the sum is taken over all permutations in $\mathcal{S}_N$. This relation tells us that when you act by $d^N \frac{1}{det(1-gz_\ell)}$ on a vector $|V_N\rangle$, the highest multiplicity of a pole when $z \to \frac{1}{x_j}$ comes from all of the factors of $d^N \frac{1}{det(1-gz_\ell)}$ when acted on $|v_j, \dots, v_j\rangle$, that being the sector with the highest multiplicity. Other secotrs have lower degrees due to some of the factors acting on different vectors.

To a non-negative integers $m_1, m_2, \dots, m_{\mathsf{n}}$ such that $\sum_\ell m_\ell = N$, we associate the sector $\mathcal{S}_{(m_1,m_2,\dots,m_{\mathsf{n}})}$ spanned by all the basis vectors $|v_{i_1} \dots v_{i_N}\rangle$ such that $\forall k \in \{1, 2, \dots, \mathsf{n}\} : m_k = \#\{p | i_p = k\}$. As pointed out in [KLT12], this is a conserved number, which can be produced by an operator M_j such that $M_j \, |V_N\rangle = m_j \, |V_N\rangle$. This operator is a conserved quantity of the system, where:

$$[M_j, \mathcal{P}] \, = [M_j, \bigotimes_{i=1}^{N} \left(u_i + \hat{D} \right) f\,(g)] \, = 0 \tag{II.2.14}$$

The first commutation is straightforward due to the fact that $\mathcal{P}$ does not change the number of charges in the sector, as defined in (I.1.3a). Also, $\bigotimes_{i=1}^{N} \left(u_i + \hat{D} \right) f\,(g)$ is made of permutation operators and operators which are diagonal on the same basis as the twist g. That is why we have $[M_j, \bigotimes_{i=1}^{N} \left(u_i + \hat{D} \right) f\,(g)] \, = 0$.

Now we are ready to compare (II.2.11) to (II.2.12): When we expand the Leibniz rule of both expressions, as in (II.2.8) and (II.2.9), they differ by the factors $\frac{1}{x_{j_a} - x_{j_b}}$ which appear both as some factors $d^{\cdots} \frac{1}{x_{j_a} - x_{j_b}}$ is several diagrams, and as a global prefactor (in the factor $f_1 f_2 \dots f_k$ that multiplies the sum of diagrams). The contribution as a global prefactor disappears when we take the limit, because this limit involves a global normalisation which forces the leading order term to be given by (II.2.2a). Therefore, expressions (II.2.11) and (II.2.12) differ only by some factors $d^{\cdots} \frac{1}{x_{j_a} - x_{j_b}}$ is several diagrams. The expression of $d^{\cdots} \frac{1}{x_{j_a} - x_{j_b}}$ is itself made of several terms, and each term contains a p_{j_a} or a p_{j_b} or the (potentially iterated) derivative of the p_{j_a} or p_{j_b}. It follows that such terms give zero on vectors $|v_{k_1} v_{k_2} \dots v_{k_N}\rangle$ unless there is at least one ingoing leg that is attached to a v_{j_a} (or v_{j_b}). But if this is the case, then the corresponding ingoing leg is not attached to a diagram $d^{\cdots} \frac{1}{1 - x_{j_a} z_{j_a}}$ (or $d^{\cdots} \frac{1}{1 - x_{j_b} z_{j_b}}$) hence the corresponding term is subleading in the $z_{j_a} \to 1/x_{j_a}$ limit (or in the $z_{j_b} \to 1/x_{j_b}$ limit) and such term does not contribute in the limit.

Therefore, when we take the limit, we obtain the equality between (II.2.11) and (II.2.12). The exact same argument can be applied to the left hand side of (II.2.10), where it allows to drop the factors $\frac{1}{1 - x_i z_{j_a}}$ when $i = j_b$ (ie an index of $[\![1, \mathsf{n}]\!] \backslash \{i_1, i_2, \dots, i_k\}$) for a $b \neq a$. Then the left hand side of (II.2.10) is equal to $\lim \bigotimes_{i=1}^{N} \left(u_i + \hat{D} \right) \left[\prod_{a=1}^{\mathsf{n}-k} \frac{1}{1 - x_{j_a} z_{j_a}} \prod_{\ell=1}^{k} \frac{1}{1 - x_{i_\ell} z_{j_a}} \chi_\mu\,(\tilde{g}) \right]$.

Therefore, the proof of equality reduces to proving the equality

$$\underset{\substack{z_{j_1}\to 1/x_{j_a}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N}\left(u_i+\mathsf{n}-k+\hat{D}\right)\left[\prod_{a=1}^{\mathsf{n}-k}\frac{1}{1-x_{j_a}z_{j_a}}\prod_{\ell=1}^{k}\frac{1}{1-x_{i_\ell}z_{j_a}}\chi_\mu\left(\tilde{g}\right)\right]$$
$$= \underset{\substack{z_{j_1}\to 1/x_{j_a}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N}\left(u_i+\hat{D}\right)\frac{(\det g)^{\mathsf{n}-k}\prod_{a=1}^{\mathsf{n}-k}\frac{1}{1-x_{j_a}z_{j_a}}}{\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}\left(x_{j_a}-x_{i_\ell}\right)}\chi_\mu(\tilde{g}) \quad \text{(II.2.15)}$$

At this point we have to use the relation $\hat{D}\det g=\det g$, which allows to rewrite the right hand side as $\underset{\substack{z_{j_1}\to 1/x_{j_a}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\lim} \bigotimes_{i=1}^{N}\left(u_i+\mathsf{n}-k+\hat{D}\right)\frac{\prod_{a=1}^{\mathsf{n}-k}\frac{1}{1-x_{j_a}z_{j_a}}}{\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}\left(x_{j_a}-x_{i_\ell}\right)}\chi_\mu(\tilde{g})$.

Next, the expressions only differ by factors $\frac{1}{1-x_{i_\ell}z_{j_a}}$ (in the left hand side) compared to $\frac{1}{x_{j_a}-x_{i_\ell}}$ (in the right hand side). Therefore it only remains to show that factors $\frac{1}{x_{j_a}-x_{i_\ell}}$ can be replaced with $\frac{1}{1-x_{i_\ell}z_{j_a}}$ without changing the limit, exactly like we showed previously that the factors $\frac{1}{x_{j_a}-x_{j_b}}$ can be dropped to obtain the equality of (II.2.11) and (II.2.12).

To this end, we have to consider the quantity $d^r\frac{1}{x_{j_a}-x_{i_\ell}}$, which itself contains several terms: if all Coderivatives successively act on the x_{i_ℓ} (and its derivatives) and never on the x_{j_a} then we obtain the same as $\lim_{z_{j_a}\to\frac{1}{x_{j_a}}} d^r\frac{1}{1/z_{j_a}-x_{i_\ell}}$, because $\hat{D}z_{j_a}=0$. Whereas if at least one derivative acts on x_{j_a} (and the next Coderivatives potentially act on the result), then we get terms that contain p_{j_a} in at least one of the outer legs. As discussed above, such terms are subleading and should be dropped. Hence we can replace each factor $\frac{1}{x_{j_a}-x_{i_\ell}}$ with $\frac{1}{1/z_{j_a}-x_{i_\ell}}$ which can be written as $\frac{1}{z_{j_a}}\frac{1}{1-x_{i_\ell}z_{j_a}}$, and the factor $\frac{1}{z_{j_a}}$ is not impacted by Coderivatives hence it is absorbed into the normalisation when we take the limit.

This concludes the proof that the two expressions of $T^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}$ coincide, i. e. that the two above constructions of nested T-operators (and of nested Q-operators via (II.2.3)) do coincide.

II.2.2.3 Derivation of (II.2.4)

This discussion was motivated by Appendix C of [CLV22] for a *single-indexed* Q-operators. We are going to capitalize on that idea and deduce the nested T-operatorsin (II.2.2) and *multi-indexed* Q-operatorsin (II.2.3). To be more precise, we will show that nested T- and Q-operators are only dependant on the nested character and the generating series of the removed eigenvalues.

To start with, if we use Lemma (I.1.2) to isolate terms with factor $x_1^{\lambda_1}$ in (II.2.1), we obtain that:

$$\underset{z_1\to 1/x_1}{\mathcal{N}\lim}\sum_{\lambda_1\geqslant 0}\mathcal{T}^{(\lambda)}(u)\,z_1^{\lambda_1}=\begin{vmatrix} x_2^{\lambda_2-1}\mathbb{Q}^{[2\lambda_2-2]}_{\{2\}}(u) & \dots & x_{\mathsf{n}}^{\lambda_2-1}\mathbb{Q}^{[2\lambda_2-2]}_{\{\mathsf{n}\}}(u)\\ \vdots & \dots & \vdots \\ x_2^{\lambda_{\mathsf{n}}-\mathsf{n}+1}\mathbb{Q}^{[2\lambda_{\mathsf{n}}-2\mathsf{n}+2]}_{\{2\}}(u) & \dots & x_{\mathsf{n}}^{\lambda_{\mathsf{n}}-\mathsf{n}+1}\mathbb{Q}^{[2\lambda_{\mathsf{n}}-2\mathsf{n}+2]}_{\{\mathsf{n}\}}(u)\end{vmatrix} \quad \text{(II.2.16)}$$

where the right hand side differs from (II.2.1) by the removal of the first line and column of the determinant.

If we iterate the process and denote $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\mathsf{n}-k}, \mu_1, \dots, \mu_k)$ similarly to section II.2.1[23], then we obtain:

$$\begin{aligned}\mathcal{T}^{(\mu_1,\dots,\mu_k)}_{\{i_1,i_2,\dots,i_k\}}(u-\mathsf{n}+k) &= \underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \sum_{\lambda_{j_1}\geqslant 0}\dots\sum_{\lambda_{j_{\mathsf{n}-k}}\geqslant 0} \mathcal{T}^{(\lambda)}(u)\, z_{j_1}^{\lambda_{j_1}}\dots z_{j_{\mathsf{n}-k}}^{\lambda_{j_{\mathsf{n}-k}}} \qquad \text{(II.2.17)}\\ &= \underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim} \bigotimes_{m=1}^{N}\left(u_m+\hat{D}\right)\sum_{\substack{\lambda_{j_1}\geqslant 0\\ \vdots \\ \lambda_{j_{\mathsf{n}-k}}\geqslant 0}} z_{j_1}^{\lambda_{j_1}}\dots z_{j_{\mathsf{n}-k}}^{\lambda_{j_{\mathsf{n}-k}}}\det\left(\chi_{(\lambda_j-j+i)}(g)\right)_{1\leqslant i,j\leqslant n}\end{aligned}$$

where $\chi_{(\lambda_j-j+i)}(g)$ is the character in the symmetric representation representation $(s)=(s,0,\dots)$, and where we used the Weyl formula express $\chi_\lambda(g)$ as $\det\left(\chi_{(\lambda_j-j+i)}(g)\right)_{1\leqslant i,j\leqslant n}$.

The right hand side of (II.2.17) is made of coderivatives of the following determinant:

$$\begin{aligned}&\sum_{\substack{\lambda_{j_1}\geqslant 0\\ \vdots \\ \lambda_{j_{\mathsf{n}-k}}\geqslant 0}} z_{j_1}^{\lambda_{j_1}}\dots z_{j_{\mathsf{n}-k}}^{\lambda_{j_{\mathsf{n}-k}}}\det\left(\chi_{(\lambda_j-j+i)}(g)\right)_{1\leqslant i,j\leqslant n}\\ &= \begin{vmatrix} \sum_{\lambda_1\geqslant 0} z_{j_1}^{\lambda_1}\chi_{\lambda_1}(g) & \dots & \sum_{\lambda_1\geqslant 0} z_{j_1}^{\lambda_1}\chi_{\lambda_1+\mathsf{n}-1}(g)\\ \vdots & & \vdots \\ \sum_{\lambda_{\mathsf{n}-k}\geqslant 0} z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}}\chi_{\lambda_{\mathsf{n}-k}-\mathsf{n}+k+1}(g) & \dots & \sum_{\lambda_{\mathsf{n}-k}\geqslant 0} z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}}\chi_{\lambda_{\mathsf{n}-k}+k}(g)\\ \chi_{\mu_1-\mathsf{n}+k}(g) & \dots & \chi_{\mu_1+k-1}(g)\\ \vdots & & \vdots \\ \chi_{\mu_k-\mathsf{n}+1}(g) & \dots & \chi_{\mu_k}(g)\end{vmatrix}\\ &= \begin{vmatrix} w(z_{j_1}) & \dots & w(z_{j_1})/z_{j_1}^{\mathsf{n}-1}\\ \vdots & & \vdots \\ z_{j_{\mathsf{n}-k}}^{\mathsf{n}-k-1}w\left(z_{j_{\mathsf{n}-k}}\right) & \dots & w\left(z_{j_{\mathsf{n}-k}}\right)/z_{j_{\mathsf{n}-k}}^{k}\\ \chi_{\mu_1-\mathsf{n}+k}(g) & \dots & \chi_{\mu_1+k-1}(g)\\ \vdots & & \vdots \\ \chi_{\mu_k-\mathsf{n}+1}(g) & \dots & \chi_{\mu_k}(g)\end{vmatrix} + \text{regular terms}\end{aligned}$$

where the *regular terms* are boundary terms for the sum over λ. For instance, for the top-right coefficient of the matrix, one has

$$\begin{aligned}\sum_{\lambda_1\geqslant 0} z_{j_1}^{\lambda_1}\chi_{(\lambda_1+\mathsf{n}-1)}(g) &= \left(\sum_{\lambda_1\geqslant 1-\mathsf{n}} - \sum_{\lambda_1=1-\mathsf{n}}^{-1}\right)\left(z_{j_1}^{\lambda_1}\chi_{(\lambda_1+\mathsf{n}-1)}(g)\right) \qquad \text{(II.2.18)}\\ &= \left(w(z_{j_1}) - \sum_{s=0}^{\mathsf{n}-2} z_{j_1}^{s}\chi_s(g)\right)/z_{j_1}^{\mathsf{n}-1} \qquad \text{(II.2.19)}\end{aligned}$$

and $\sum_{\lambda_1}^{\mathsf{n}-2} z_{j_1}^{\lambda_1}\chi_{\lambda_1}(g)/z_{j_1}^{\lambda_1}$ gives one of the regular terms. We call them "regular terms" because in the limit $z_j\to 1/x_j$ these terms (and their coderivative) have no pole (unlike $w(z_j)$), hence these terms do not contribute to the $\mathcal{N}\lim$ in (II.2.17).

[23]The reader may notice a slight change of convention with respect to section II.2.1 where the convention was $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{\mathsf{n}-k},\mu_1+\mathsf{n}-k,\dots,\mu_k+\mathsf{n}-k)$.

The previous discussion compresses (II.2.17) to:

$$\mathcal{T}^{(\mu_1,\ldots,\mu_k)}_{\{i_1,i_2,\ldots,i_k\}}(u-\mathsf{n}+k)=\underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\mathcal{N}\lim}\ \bigotimes_{m=1}^{N}\left(u_m+\hat{D}\right)\begin{vmatrix}\left(z_{z_{j_a}}^{a-b}w\left(z_{j_a}\right)\right)_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ b\in[\![1,\mathsf{n}]\!]}}\\ \left(\chi_{(\mu_{a-k}-a+b)}(g)\right)_{\substack{\mathsf{n}-k+1\leqslant a\leqslant \mathsf{n}\\ b\in[\![1,\mathsf{n}]\!]}}\end{vmatrix} \tag{II.2.20}$$

where the generating series factors out from the determinant, and due to the properties of the determinant, one can deduce that:

$$\begin{vmatrix}\left(x_{j_a}^{b-a}\right)_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ b\in[\![1,\mathsf{n}]\!]}}\\ \left(\chi_{(\lambda_a-a+b)}(g)\right)_{\substack{\mathsf{n}-k+1\leqslant a\leqslant \mathsf{n}\\ b\in[\![1,\mathsf{n}]\!]}}\end{vmatrix}=\prod_{a=1}^{\mathsf{n}-k}\left(x_{j_a}^{1-a}\right)\chi_\mu\left(\tilde{g}\right) \tag{II.2.21}$$

where $\prod_{a=1}^{\mathsf{n}-k}\left(x_{j_a}^{1-a}\right)$ is a prefactor of the removed eigenvalues, and $\chi_\mu(\tilde{g})$ is the nested character. From the discussion in Sec(II.2.2.2) for the non-contributing factors of the limit, we know $\hat{D}^{\otimes N}\cdot\prod_{a=1}^{\mathsf{n}-k}\left(x_{j_a}^{1-a}\right)$ is subleading when taking the normalized limit. After shifting the spectral parameter $u\rightsquigarrow u+\mathsf{n}-k$, we end up with (II.2.4).

Proof. of (II.2.21):
The equality in (II.2.21) stems from the fact that we have two equivalent definitions of the character: the CBR and 1^{st} Weyl formula:

$$\chi_\lambda^{CBR}(g)=\left|\left(\chi_{\lambda_i+j-i}\right)_{1\leqslant i,j\leqslant \mathsf{n}}\right| \tag{II.2.22a}$$

$$\chi_\lambda^{Weyl}(g)=\frac{\left|\left(x_j^{\lambda_i+\mathsf{n}-i}\right)_{1\leqslant i,j\leqslant \mathsf{n}}\right|}{\Delta\left(x_1,\ldots,x_{\mathsf{n}}\right)} \tag{II.2.22b}$$

The steps here have already been done for (II.2.17), but we will do the steps precisely in order to find the prefactor in (II.2.21). Starting with taking the infinite sums[24] of (II.2.22a):

$$\sum_{\substack{\lambda_1\geqslant 0\\ \vdots\\ \lambda_{\mathsf{n}-k}\geqslant 0}}\chi_\lambda^{CBR}(g)\,z_{j_1}^{\lambda_1}\ldots z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}}=\begin{vmatrix}\left(z_{z_{j_a}}^{a-b}w\left(z_{j_a}\right)\right)_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ b\in[\![1,\mathsf{n}]\!]}}\\ \left(\chi_{(\mu_{a-k}-a+b)}(g)\right)_{\substack{\mathsf{n}-k+1\leqslant a\leqslant \mathsf{n}\\ b\in[\![1,\mathsf{n}]\!]}}\end{vmatrix} \tag{II.2.23}$$

The next step is to eliminate the singularities of the expression when taking the limit. We propose the normalization factor $(1-xz)$, which will [25] read as:

$$\underset{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots\\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}}{\lim}\ \prod_{a=1}^{\mathsf{n}-k}(1-x_{j_a}z_{j_a})\sum_{\substack{\lambda_1\geqslant 0\\ \vdots\\ \lambda_{\mathsf{n}-k}\geqslant 0}}\chi_\lambda^{CBR}(g)\,z_{j_1}^{\lambda_1}\ldots z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}}=\prod_{a=1}^{\mathsf{n}-k}x_{j_a}^{\mathsf{n}-1}\frac{\begin{vmatrix}\left(x_{j_a}^{b-a}\right)_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ b\in[\![1,\mathsf{n}]\!]}}\\ \left(\chi_{(\mu_{a-k}-a+b)}(g)\right)_{\substack{\mathsf{n}-k+1\leqslant a\leqslant \mathsf{n}\\ b\in[\![1,\mathsf{n}]\!]}}\end{vmatrix}}{\Delta\left(x_{j_1},\ldots,x_{j_{\mathsf{n}-k}}\right)} \tag{II.2.24}$$

[24] The CBR formula (II.2.22a) is equivalent to (II.2.1) when $N=0$.

[25] $\Delta(x_1,\ldots,x_m)$ is defined as $\left|x_a^{a-b}\right|_{1\leqslant a,b\leqslant m}$.

The prefactor infrote of the determinant comes from the normalized limit on the generating series of the symmetric characters:

$$\boxed{\lim_{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}} \prod_{a=1}^{\mathsf{n}-k}\left(1-x_{j_a}z_{j_a}\right)w\left(z_{j_a}\right)=\frac{\prod_{a=1}^{\mathsf{n}-k}x_{j_a}^{\mathsf{n}-1}}{\Delta\left(x_{j_1},\ldots,x_{j_{\mathsf{n}-k}}\right)}} \tag{II.2.25}$$

Using the same steps for the 1^{st} Weyl formula of the character, we obtain:

$$\begin{aligned}\lim_{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}} \prod_{a=1}^{\mathsf{n}-k}\left(1-x_{j_a}z_{j_a}\right)\sum_{\substack{\lambda_1\geqslant 0\\ \vdots \\ \lambda_{\mathsf{n}-k}\geqslant 0}}\chi_\lambda^{Weyl}(g)\,z_{j_1}^{\lambda_1}\ldots z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}} &= \frac{1}{\Delta\left(x_{j_1},\ldots,x_{j_{\mathsf{n}}}\right)}\begin{vmatrix}\left(\delta_{a,b}x_{j_a}^{(\mathsf{n}-a)}\right)_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ b\in[\![1,\mathsf{n}]\!]}}\\ \left(x_{j_a}^{\lambda_a+\mathsf{n}-a}\right)_{\substack{\mathsf{n}-k+1\leqslant a\leqslant \mathsf{n}\\ b\in[\![1,\mathsf{n}]\!]}}\end{vmatrix}\\ &= \frac{\prod_{a=1}^{\mathsf{n}-k}\left(x_{j_a}^{\mathsf{n}-a}\right)}{\Delta\left(x_{j_1},\ldots,x_{j_{\mathsf{n}}}\right)}\left|\left(x_{j_a}^{\lambda_a+\mathsf{n}-a}\right)_{\mathsf{n}-k+1\leqslant a,b\leqslant \mathsf{n}}\right|\end{aligned} \tag{II.2.26}$$

One can also write the numerator of the right hand side in the previous equation as:

$$\left|\left(x_{j_a}^{\lambda_a+\mathsf{n}-a}\right)_{k+1\leqslant a,b\leqslant \mathsf{n}}\right| = \Delta\left(x_{j_{k+1}},\ldots,x_{j_{\mathsf{n}}}\right)\chi_\mu\left(\tilde{g}\right), \qquad \tilde{g}=\mathrm{diag}\left(x_{j_{k+1}},\ldots,x_{j_{\mathsf{n}}}\right) \tag{II.2.27}$$

This will allow us to rewrite (II.2.26):

$$\lim_{\substack{z_{j_1}\to 1/x_{j_1}\\ \vdots \\ z_{j_{\mathsf{n}-k}}\to 1/x_{j_{\mathsf{n}-k}}}} \prod_{a=1}^{\mathsf{n}-k}\left(1-x_{j_a}z_{j_a}\right)\sum_{\substack{\lambda_1\geqslant 0\\ \vdots \\ \lambda_{\mathsf{n}-k}\geqslant 0}}\chi_\lambda^{Weyl}(g)\,z_{j_1}^{\lambda_1}\ldots z_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}} = \prod_{a=1}^{\mathsf{n}-k}x_{j_a}^{\mathsf{n}-a}\frac{\chi_\mu\left(\tilde{g}\right)}{\Delta\left(x_{j_1},\ldots,x_{j_{\mathsf{n}-k}}\right)} \tag{II.2.28}$$

Finally, with the equivalence between (II.2.24) and (II.2.28), one obtain (II.2.21). □

II.2.3 Normalized limit of operators

In the introduction, we manipulated normalized limits of operator-valued polynomials without thoroughly discussing whether they have the same definition and can be manipulated in the same manner as normalized limits of complex-valued polynomials.

II.2.3.1 Small size example

To clarify the question, let us start with a simple explicit example: In a $gl(2)$-spin chain of length $N=2$, the operator $\mathbb{Q}_1(u)$ defined by equation (II.2.3) and (II.2.4), is given by $\mathbb{Q}_1(u) = \mathcal{N}\lim_{z_2\to 1/x_2}\bigotimes_{i=1}^{N}\left(u_i+1+\hat{D}\right)w(z_2)$. In the basis ($|11\rangle$, $|12\rangle$, $|21\rangle$, $|22\rangle$), the expression (found in [KV08]):

$$\hat{D}_{j_1}^{i_1}\hat{D}_{j_2}^{i_2}\ldots\hat{D}_{j_N}^{i_N}w(z) = \sum_{\sigma\in\mathcal{S}_N}\prod_{k=1}^{N}\left(\frac{(g\,z)^{\Theta(k-\sigma(k))}}{1-g\,z}\right)_{j_k}^{i_{\sigma k}}w(z), \tag{II.2.29}$$

allows to write it as:

$$Q_1(u) = \underset{z_2\to 1/x_2}{\mathcal{N}\lim} \left(\begin{array}{c|cc|c} A & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 \\ 0 & & & 0 \\ \hline 0 & 0 & 0 & C \end{array}\right) w(z_2) \tag{II.2.30a}$$

$$\text{where } A = (u_1 + \tfrac{1}{1-x_1z_2})(u_2 + \tfrac{1}{1-x_1z_2}) + \tfrac{x_1z_2}{(1-x_1z_2)^2}\,, \tag{II.2.30b}$$

$$B = \begin{pmatrix} (u_1+\frac{1}{1-x_1z_2})(u_2+\frac{1}{1-x_2z_2}) & \frac{x_1z_2}{(1-x_1z_2)(1-x_2z_2)} \\ \frac{x_2z_2}{(1-x_1z_2)(1-x_2z_2)} & (u_1+\frac{1}{1-x_2z_2})(u_2+\frac{1}{1-x_1z_2}) \end{pmatrix}, \tag{II.2.30c}$$

$$\text{and } C = (u_1 + \tfrac{1}{1-x_2z_2})(u_2 + \tfrac{1}{1-x_2z_2}) + \tfrac{x_2z_2}{(1-x_2z_2)^2}\,. \tag{II.2.30d}$$

As expected, the block structure is given by the sectors of the Hilbert space[26], which were defined in section II.2.2.1.

The Normalized limit was introduced in section I.1.1 by saying that *we multiply by a normalisation which makes the limit finite and fixes the leading coefficient of the limit, as a polynomial in u.*

Following this recipe, we would compute the normalized limit as:

$$Q_1(u) = \lim_{z_2\to 1/x2} \frac{\left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$(1-x_2z_2)\mathbb{I}$}} & 0 \\ 0 & & & 0 \\ \hline 0 & 0 & 0 & \frac{(1-x_2z_2)^2}{2} \end{array}\right)}{w(z_2)} \left(\begin{array}{c|cc|c} A & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 \\ 0 & & & 0 \\ \hline 0 & 0 & 0 & C \end{array}\right) w(z_2) \tag{II.2.31}$$

$$= \left(\begin{array}{c|cc|c} (u_1 + \frac{x_2}{x_2-x_1})(u_2 + \frac{x_2}{x_2-x_1}) + \frac{x_2^2}{(x_2-x_1)^2} & 0 & 0 & 0 \\ \hline 0 & u_1 + \frac{x_2}{x_2-x_1} & \frac{x_1}{x_2-x_1} & 0 \\ 0 & \frac{x_2}{x_2-x_1} & u_2 + \frac{x_2}{x_2-x_1} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \tag{II.2.32}$$

We notice for instance that the matrix coefficients of this normalized limit are not the normalized limit of the matrix coefficients: for instance the normalized limit of the off-diagonal coefficients of B (in (II.2.30c)) is equal to one (because they are u-independent), whereas the corresponding coefficients in (II.2.32) are $\frac{x_1}{x_2-x_1}$ (resp $\frac{x_2}{x_2-x_1}$).

In what follows we precise a posteriori the definition of normalized limit for these precise operators, so that it gives indeed (II.2.32) and that the manipulations of normalized limit in sections II.2.2.3 and II.2.1 are legitimate.

II.2.3.2 Sector-wise definition

The simplest way to defined normalized limits for the present article is to notice that we are manipulating very specific operators: On the one hand, all of them are block-diagonal, where

[26]This block structure is a consequence of the diagrammatic expression which express T- and Q-operators as sums of products of permutation operators and diagonal matrices.

the blocks correspond to the sectors of the Hilbert space, introduced in Sec(II.2.2.1); and on the other hand, the normalisation factor which we introduce to make the limit finite is block-wise proportional to identity.

The block structure (i. e. the statement that the sectors of the Hilbert space are stable spaces of all considered operators) follows simply from the fact that we always considers operators which are sums of products of permutation operators and diagonal matrices. This property is for instance very manifest in (II.2.29), and more generally it follows from the Leibnitz rule and from (B.1.6) and (B.1.10).

Due to this block-diagonality, we will only generalise equations (I.1.11-I.1.12) inside each sector. In each sector, we generalize "$\max_i |f_i(\Lambda)|$" (in (I.1.11)) as $\max_i \|f_i(\Lambda)\|$, where $\|\dots\|$ denotes an arbitrary norm. Indeed, we remind that all norms are equivalent in finite dimensional spaces, and we will only use this norm to capture the scaling (*e.g.* the multiplicity of a pole).

II.2.3.2.1 Explicit example

For instance in the example (II.2.30), if we focus on the sector $\mathcal{S}_{(1,1)}$, we use the norm $\|M\| = \sum |m_{i,j}|$, and we denote by f_i the coefficients of the polynomial $P(u) = \sum_{i=0}^{d} f_i u^i$ as in (I.1.11)[27]), then we have:

$$f_0 = \begin{pmatrix} (\frac{1}{1-x_1z_2}-\theta_1)(\frac{1}{1-x_2z_2}-\theta_2) & \frac{x_1z_2}{(1-x_1z_2)(1-x_2z_2)} \\ \frac{x_2z_2}{(1-x_1z_2)(1-x_2z_2)} & (\frac{1}{1-x_2z_2}-\theta_1)(\frac{1}{1-x_1z_2}-\theta_2) \end{pmatrix} w(z_2) \tag{II.2.33}$$

$$f_1 = \begin{pmatrix} \frac{1}{1-x_1z_2}+\frac{1}{1-x_2z_2}-\theta_1-\theta_2 & 0 \\ 0 & \frac{1}{1-x_2z_2}+\frac{1}{1-x_1z_2}-\theta_1-\theta_2 \end{pmatrix} w(z_2) \tag{II.2.34}$$

$$f_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) w(z_2) \tag{II.2.35}$$

$$\text{hence } \|f_2\| = 2|w(z_2)|, \qquad \|f_1\| = \frac{2}{|1-x_2z_2|}|w(z_2)| + \mathcal{O}(1) \tag{II.2.36}$$

$$\text{and } \|f_0\| = \left(\left|\tfrac{1}{1-x_1z_2} - \theta_1\right| + \tfrac{|x_1|+|x_2|}{|1-x_1z_2|} + \left|\tfrac{1}{1-x_1z_2} - \theta_2\right|\right) \frac{|w(z_2)|}{|1-x_2z_2|} + \mathcal{O}(1) \tag{II.2.37}$$

Therefore, $\max_i \|f_i\|$ scales like $\frac{|w(z_2)|}{|1-x_2z_2|}$. In order to follow precisely (I.1.11-I.1.12) we can introduce an α such $\max_i \|f_i\| = \alpha \frac{w(z_2)}{1-x_2z_2}$ – this function α is a maximum of several functions of the eigenvalues and the inhomogeneities[28] – which has a finite limit α_0 when $z_2 \to 1/x_2$. Then in equations (I.1.11-I.1.12), $\tilde{P}(u)$ is given by

$$c_0 = \begin{pmatrix} \frac{1}{1-x_1/x_2}-\theta_1 & \frac{x_1/x_2}{1-x_1/x_2} \\ \frac{1}{1-x_1/x_2} & \frac{1}{1-x_1/x_2}-\theta_2 \end{pmatrix} \Big/ \alpha_0, \quad c_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) / \alpha_0 \quad \text{and} \quad d' = 1 \tag{II.2.38}$$

Then c_1 is proportional to identity, which allows to manipulate it as if it was a scalar in (I.1.12), and we get the normalized limit announced in (II.2.32), with an overall normalisation factor $\frac{1}{\max_i \|f_i(\Lambda)\| c_{d'}} = \frac{(1-x_2z_2)}{w(z_2)}$.

[27] Notice that compared to (I.1.11), the limit $z_2 \to 1/x_2$ plays the role of the limit $\Lambda \to \Lambda_0$.

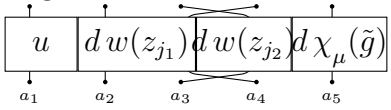
[28] Notice that if $\max_i \|f_i\|$ was replaced by another (necessarily equivalent) norm – for instance $\sum_i \|f_i\|$, then the function α would be modified but the discussion would remain exactly the same.

II.2.3.2.2 General justification

The main propoerty which enabled to generalize the definitions (I.1.11-I.1.12) to operators, is that on each sector the coefficient $c_{d'}$ used in (I.1.12) is a scalar (a multiple of identity).

Let us show that in sector $\mathcal{S}_{(m_1,m_2,\dots,m_{\mathsf{n}})}$ (as defined in Sec(II.2.2.1)), the normalized limit (II.2.4) involves the renormalisation by a factor:

$$\frac{1}{\max_i \|f_i(\Lambda)\| c_{d'}} = \prod_{a=1}^{\mathsf{n}-k} \frac{(1-x_{j_a} z_{j_a})^{m_{j_a}}}{m_{j_a}!} \Bigg/ \left(\chi_\mu(\tilde{g}) \prod_{a=1}^{\mathsf{n}-k} w(z_{j_a}) \right)$$

First let us explain the factor $\prod_{a=1}^{\mathsf{n}-k} \frac{(1-x_{j_a} z_{j_a})^{m_{j_a}}}{m_{j_a}!}$. We remember that the factor $(1-x_{j_a} z_{j_a})^{m_{j_a}}$ was identified in Sec(II.2.2.2.1) where we identified the multiplicity of the poles in each sector of the Hilbert space. The factor $\frac{1}{m_{j_a}!}$ follows from the expression of the terms with leading degree in u: when we act on a given basis vector $|a_1, \dots a_N\rangle$, we get diagrams such as 

(in the notations of Sec(II.2.2.1)), where the leading diagrams (see Sec(II.2.2.2.1) are the diagrams such that *for each* $p \in \{1, 2, \dots N\}$*, such that* $a_p \in \{j_1, j_2, \dots, j_{\mathsf{n}-k}\}$*, the index* a_p *is connected to the* $d\, w(z_{a_p})$. Among those, one single diagram has maximal degree in u: the diagram such that *for each* $p \in \{1, 2, \dots N\}$*, such that* $a_p \in \{i_1, i_2, \dots, i_{\mathsf{n}-k}\}$*, the index* a_p *is connected to the* u. From the expression (II.2.29) of $d\, w(z_{j_b})$, we see that this diagram has a $\prod_{a=1}^{\mathsf{n}-k} m_{j_a}!$ which comes from the sum over permutations in each $d\, w(z_{j_a})$.

Now than we have explained the factor $\prod_{a=1}^{\mathsf{n}-k} \frac{(1-x_{j_a} z_{j_a})^{m_{j_a}}}{m_{j_a}!}$, we should remember that in the sum of diagrams given in Sec(II.2.2.1), all diagrams are multiplied by $\chi_\mu(\tilde{g}) \prod_{a=1}^{\mathsf{n}-k} w(z_{j_a})$, and in order to absorb this factor we need a normalisation:

$$\frac{1}{\max_i \|f_i(\Lambda)\| c_{d'}} = \prod_{a=1}^{\mathsf{n}-k} \frac{(1-x_{j_a} z_{j_a})^{m_{j_a}}}{m_{j_a}!} \Bigg/ \left(\chi_\mu(\tilde{g}) \prod_{a=1}^{\mathsf{n}-k} w(z_{j_a}) \right)$$

II.2.3.3 Diagonalizability

In the intruduction, another claim was given to say that the normalized limit would reduce to normalized limits of complex-valued polynomial:

The claim is that we only consider the normalized limit of diagonalizable matrices $M(u, \Lambda) = P^{-1} D P$, where the change of basis P is indenpendant of u and of the parameter Λ (or z_j) with respect to which we take the limit. Denoting the diagonal part as $D = \mathrm{diag}(d_1, d_2, \dots)$, the statement is then that $\mathcal{N}\lim P^{-1} D P = P^{-1} \mathrm{diag}(\mathcal{N}\lim d_1, \mathcal{N}\lim d_2, \dots) P$.

Although this argument looks rather simple in order to understand what is meant by normalized limit of an operator, we will only sketch its justification, which requires two claims:

1. For each u and Λ (or z_j), the operator $M(u)$ that we consider is diagonalisable.

2. This operator comutes in itself, in the sense that arbitrary u and u_0, and Λ and Λ_0, we have $[M(u,\Lambda), M(u_0,\Lambda_0)] = 0$. This implies that they are co-diagonalisable, i. e. that the matrix P of change of basis can be chosen independent of u and Λ.

The property 2 follows from the fact that all operators of the form $\bigotimes_{i=1}^{N}\left(u_i\mathbb{I}+\hat{D}\right)F(x_1,x_2,\dots,x_{\mathsf{n}})$ (where F is a scalar-valued, rationnal function) belong to the Bethe Algebra, hence they commute with each other. This property was justified in [KLT12] when F is a symmetric function of the eigenvalues x_i. For arbitrary functions, one has to take a limit of class functions, as was discussed for instance in Appendix C of [CLV22] to show that it still belongs to the Bethe Algebra.

The property 1 is more subtle: for generic values of the inhomogeities θ_i it is known that the elements of Bethe algebra do not have Jordan blocks. But there are exceptionnal values of inhomogeneities for which that is not the case anymore. Therefore, the *diagonalisability* argument, works only for generic values of inhomogeneities.

For all expressions of the form $\bigotimes_{m=1}^{N}\left(u_m+\hat{D}\right)\left[F\prod_{\ell=1}^{\mathsf{n}-k}w(z_\ell)\right]$ – where the rational function F has no pole when $z_j\to 1/x_j$ – the normalisation factor is $\prod_{a=1}^{\mathsf{n}-k}\frac{(1-x_{j_a}z_{j_a})^{m_{j_a}}}{m_{j_a}!}\Big/\left[F\prod_{\ell=1}^{\mathsf{n}-k}w(z_\ell)\right]$, as follows by the very same arguments.

Additionally, if we compute T- and Q-operators using (II.2.7) instead of (II.2.4), then we identify the normalisation factor – in the sector $\mathcal{S}_{(m_1,m_2,\dots,m_{\mathsf{n}})}$ – as $\prod_{a=1}^{\mathsf{n}-k}\lambda_a^{m_{j_a}}\Big/\frac{(\det g)^{\mathsf{n}-k}x_{j_1}^{\lambda_1+k-1}\dots x_{j_{\mathsf{n}-k}}^{\lambda_{\mathsf{n}-k}+2k-\mathsf{n}}\ \chi_{\mu}(\tilde{g})}{\prod_{1\leqslant a<b\leqslant \mathsf{n}-k}\left(x_{j_a}-x_{j_b}\right)\prod_{\substack{1\leqslant a\leqslant \mathsf{n}-k\\ 1\leqslant \ell\leqslant k}}\left(x_{j_a}-x_{i_\ell}\right)}$.

In all cases, what see if that when we restrict to a sector in the Hilbert space, the definition (II.2.3-II.2.4) make sense with a scalar normalisation factor, which justifies the use of normalised limits in Sec(II.2.2.3) and Sec(II.2.1).

Chapter III

Spin chains with $SO(2\mathsf{r})$ symmetry.

III.1 Spin chain and Conserved charges

Similar to what we did in the previous chapter, we are going to study integrable spin chain with a different symmetry. In doing so, we will assume the integrable system in the previous chapter and change the symmetry of our system to $SO(2\mathsf{r})$. With the new symmetry, we will give a new formal definition to a differential operator, which allows us to explicitly manipulate the operators of the system, where these operators are defined in the oscillator description [ZZ79, Fra20].

Unlike the $GL(\mathsf{n})$ case, the new symmetry will prove to be more complicated in constructing T- and Q-operators with an arbitrary representation in the auxiliary space. It is possible to write T-operators with rectangular representations in the auxiliary space from a Wronskian determinant as presented in [FFK21, ESV20], but it will turn out that finding a differential discripting (an expression of the T-operators with only Coderivatives, spectral parameters, and the twist) of these operators is unfeasible.

III.1.1 The Yang-Baxter equation and T-operators

As stated, we are going to assume the same integrable system with the $SO(2\mathsf{r})$ symmetry, which indicate the existance of T-operators. We saw in Sec(II.1) that the Yang-Baxter equation is the fundamental idea in describing an integrable spin chain, and the R-matrices to be its solutions. The definition of these operators was found in [ZZ79] for the fundamental representation and it reads:

$$\text{Where: } \boxed{R_{a_1,a_2}(u) = \left(u+\frac{\kappa}{2}\right)\left(u-\frac{\kappa}{2}\right)\mathbb{I} + \left(u+\frac{\kappa}{2}\right)\mathcal{P}_{a_1,a_2} - \left(u-\frac{\kappa}{2}\right)\mathcal{Q}_{a_1,a_2}} \tag{III.1.1a}$$

$$\text{Where, } \kappa = \mathsf{r}-1, \quad \mathcal{P} = \sum_{\alpha,\beta=1}^{2\mathsf{r}} J_{\alpha,\beta}\otimes F_{\beta,\alpha}, \quad \mathcal{Q} = \sum_{\alpha,\beta=1}^{2\mathsf{r}} J_{\alpha,\beta}\otimes F_{\alpha',\beta'} \tag{III.1.1b}$$

The parameter κ is a number related to the rank of the Lie algebra; $(\mathcal{P},\mathcal{Q})$ are defined in the intruduction and $(J_{i,j},F_{i,j})$ are the $gl(n)$ and the $so(2\mathsf{r})$ fundamental generators, respectively. The matrix representation of these operators is of size $2\mathsf{r}\times 2\mathsf{r}$ where $gl(n)$ generators obey the standard relations $J_{i,j}J_{k,l}=\delta_{j,k}J_{i,l}$ and the $so(2\mathsf{r})$ [1] reads $F_{i,j}=J_{i,j}-J_{j',i'}$.

The construction of the T-operators follows exactly as in Sec(II.1.1) and they have the following definition/relations:

$$[\mathcal{T}(u),\mathcal{T}(v)]=0, \qquad \mathcal{T}(u)\equiv \operatorname{tr}\Big(R_N(u)\cdot R_{N-1}(u)\cdots R_1(u)\Big) \tag{III.1.2}$$

The only notable difference is the building blocks (R-matrix), which will be the reason for the limited freedom in the form of the allowed functions in the auxiliary space. As it is the case in Sec(II.1.2), the Hamiltonian of our system is defined through Fig(III.1), this is the

[1] In this notation, the genreator $F_{j,i}$ obeys $F_{i',j'}=-F_{j,i}$, which is defined in [Fra20]. As matrices, the prime filps the matrix entries $X_{i,j}$ as follows: $X'_{i,j}=X_{\mathsf{r}-i+1,\mathsf{r}-i+1}$; this also means $X''_{i,j}=X_{i,j}$.

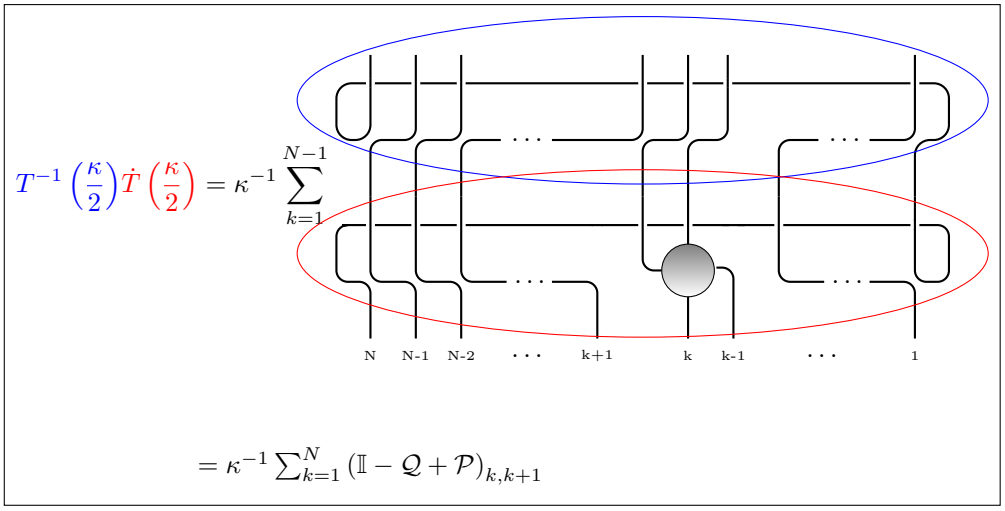


Figure III.1: This is a diagrammatic description of the non-twisted Hamiltonian of a spin chain with $SO(2\mathsf{r})$ symmetry. This follows exactly like the non-twisted diagram description of the Hamiltonian of the $GL(\mathsf{n})$ case up to a difference in the derivative of $R^{'}(\kappa/2) = (\kappa\mathbb{I} + \mathcal{P} - Q)$, which is represented as the shaded circle, and for the rest of the lines, $R(\kappa/2) = \mathcal{P}$.

diagrammatical description of the non-twisted Hamiltonian. Notice that we are taking the derivative in Fig(III.1) at $u = \frac{\kappa}{2}$ due to (III.1.1a). The diagramalcal representation of $\mathbb{I}$, $\mathcal{P}$ is in Sec(II.1.2), and the $\mathcal{Q}$ is represented as:

$\mathcal{Q}$: 

Each vector space has a permutation operator due to $R(\kappa/2) = \mathcal{P}$ exipt the k^{th} one (where we have the shaded ball), which has $R^{'}(\kappa/2) = (\kappa\mathbb{I} + \mathcal{P} - Q)$. From Fig(III.1) and (II.1.5), we can write the Hamiltonian as:

$$\boxed{H = \frac{\partial}{\partial u}\log\left(T(u)\right)\Big|_{u=\frac{\kappa}{2}} = \kappa^{-1}\sum_{i=1}^{N}(\mathbb{I} - \mathcal{Q} + \kappa\mathcal{P})_{i,i+1}} \tag{III.1.3}$$

III.1.2 Generalizing T-operators

The process of finding a bigger family of commuting T-operators will be labeled as the generalization process of family of T-operators. When taking $SO(2\mathsf{r})$ symmetry in the auxiliary space is not similar to the case with $GL(\mathsf{n})$ symmetry (described in [KLT12] or section 1 of chapter 1 in [Leu12]). As specified in Fig(III.2), the R-matrices that are related to the symmetric and spinorial representations are generated by the algebra generators only, whereas the other R-matrices related to the nodes labeled with unknowns are constructed with more than the generators of the algebra; check [KK21a, KOW22, Oka07].

This process of generalizing T-operators in the $GL(\mathsf{n})$ case was possible due to (II.1.3b). That equation describes the R-matrix in any arbitrary representation of the algebra in the auxiliary space by the generators of the algebra in that representation. The same process can

be done for the spinorial and symmetric representations of $SO(2\mathsf{r})$, due to the fact that the R-matrices in these representations are described by the generators of the algebra $so(2\mathsf{r})$ only.

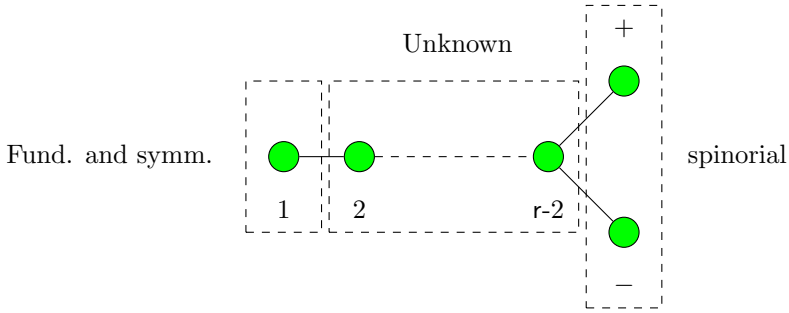
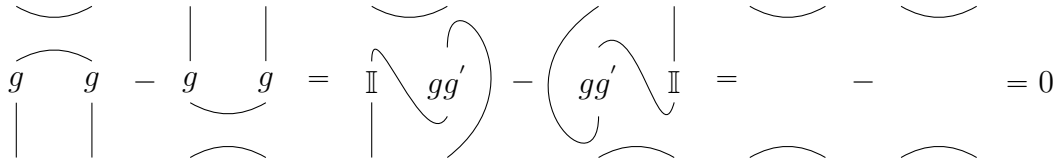


Figure III.2: $SO(2\mathsf{r})$ Dynkin diagram. We are distinguishing the diagonal into three parts if we know the R-matrix of the symmetry proposed by the nodes of the diagram.

A good starting point would be to introduce T-operators with a twist in the fundamental representation ($\lambda = \square$), then discuss the symmetries separately.

Introducing the twist and inhomogeneities.

In this case, we are following the same steps presented in Sec(III.1.1), the extra thing to checking $[\mathcal{Q}, g \otimes g] = 0$, which shows that the twist is a group element $g \in SO(2\mathsf{r})$. This can be done using (I.1.5) and (I.1.6). In the interest of not writing too many indices, let us use the diagrammatical version of it:



In the first equality, we use (I.1.5) to move one of the twist operators from one physical space to the other (we stress that $\mathcal{Q}$ work as a contraction between these spaces) up to priming one of the twist operators, then using $gg^{'} = \mathbb{I}$ (which is the orthogonality condition of $SO(2\mathsf{r})$ elements) in the third equality. Thus, T-operators with a twist in the auxiliary space takes the form:

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0, \qquad \mathcal{T}(u) \equiv \mathrm{tr}\Big(\mathcal{R}_N(u-\theta_N) \cdot \mathcal{R}_{N-1}(u-\theta_{N-1}) \cdots \mathcal{R}_1(u-\theta_1)\, g\Big) \quad \text{(III.1.4)}$$

We have introduced two extra elements to our T-operators, the twist g and the inhomogeneities θ. The introduction of the twist will generalize the description of the Hamiltonian (III.1.3) to a new operator with terms depending on the twist.

The twisted Hamiltonian is described in Fig(III.3), where one can see that the sum splits into two factors; the first is similar to Fig(III.1) (up to a change in the sum), giving us the same expression as (III.1.3), whereas the last part of the figure gives rise to the terms that are twist dependent, i. e. $\left(\kappa^{-1} g_N \left(\mathbb{I} - \mathcal{Q} + \kappa\mathcal{P}\right)_{N,1} g_1^{-1}\right)$. Thus, the twisted Hamiltonian takes the form:

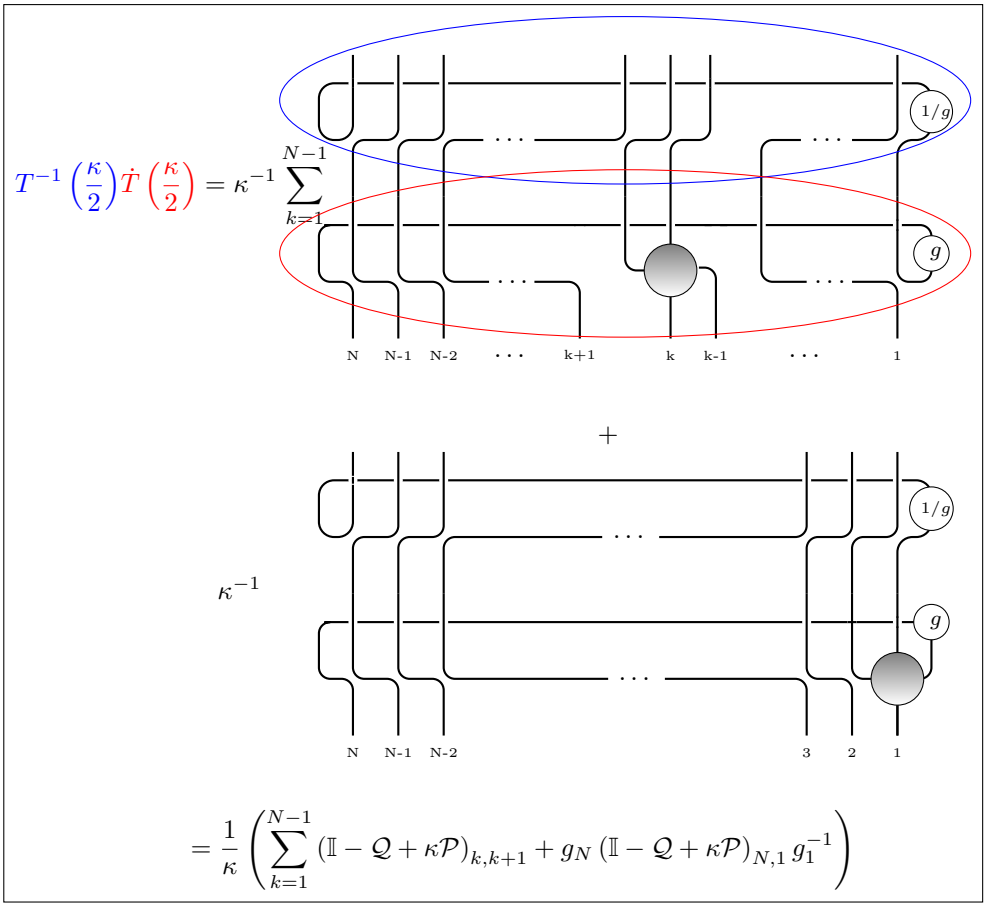


Figure III.3: This is a diagrammatic description of the twisted Hamiltonian of $SO(2\mathsf{r})$ spin chains. The first part of the diagram follows as Fig(III.1), which means the lines connecting the twists cancel out. For the second part of the diagram, we have the twist connected to $R^{'}(u)\Big|_{u=\frac{\kappa}{2}} = (\kappa\mathbb{I}+\mathcal{P}-\mathcal{Q})$ which gives rise to the twist term in the expression.

$$\boxed{H_{twist} = \kappa^{-1}\left(\sum_{i=1}^{N-1}\left(\mathbb{I}-\mathcal{Q}+\kappa\mathcal{P}\right)_{i,i+1} + g_N\left(\mathbb{I}-\mathcal{Q}+\kappa\mathcal{P}\right)_{N,1} g_1^{-1}\right)} \tag{III.1.5}$$

One can notice that the twist Hamiltonian (III.1.5) is equal to the non-twisted one (III.1.3) when taking the limit $(g \to 1)$, i. e. $H = H_{twist}\Big|_{g=1}$.

From here on, we are going to distinguish between the T-operators depending on the symmetry they have in the auxiliary space.

III.1.2.0.1 General representations.

After introducing the twist into the auxiliary space, we can now change the representation into the spinorial or symmetric ones, and then write the corresponding T-operators of these representations. Since we have two different representations, we are going to treat each case separately.

Spinorial T-operators

The R-matrices in the spinorial representation are realized to be linear in the spectral parameter u (check out [Res85, Bes85], also [KK21b] for a discussion on representation construction, [CDI13] for a complete discussion on the spinorial solutions of the Yang-Baxter equation and [MRV16, FKSZ20, ESV20, EV21, KS95] for the spinorial QQ-relations), which takes the form:

$$\mathbf{R}\left(u\right)=u+\sum_{\alpha,\beta=1}^{2\mathsf{r}}J_{\alpha,\beta}\otimes\pi_{\pm}\left(F_{\beta,\alpha}\right) \tag{III.1.6}$$

This definition is very similar to the definition of an R-matrix in the $gl(n)$ case (up to exchanging $GL(\mathsf{n})$ generators for $SO(2\mathsf{r})$ ones). With that, the T-operators are defined in a similar manner as in (II.1.3), and they read as:

$$\left[\mathbf{T}_{(\pm)}\left(u\right),\mathbf{T}_{(\pm)}\left(v\right)\right]=0,\qquad\mathbf{T}_{(\pm)}\left(u\right)\equiv\mathrm{tr}\left(\mathbf{R}_{N}\left(u-\theta_{N}\right)\cdot\mathbf{R}_{N-1}\left(u-\theta_{N-1}\right)\cdots\mathbf{R}_{1}\left(u-\theta_{1}\right)\rho_{\pm}\left(g\right)\right) \tag{III.1.7}$$

where the trace is taken over the auxiliary space, $(\pm)$ refers to the two spinorial nodes in Fig(III.2) and $\rho_{\pm}$ is the morphism of the spinorial representations.

Symmetric T-operators

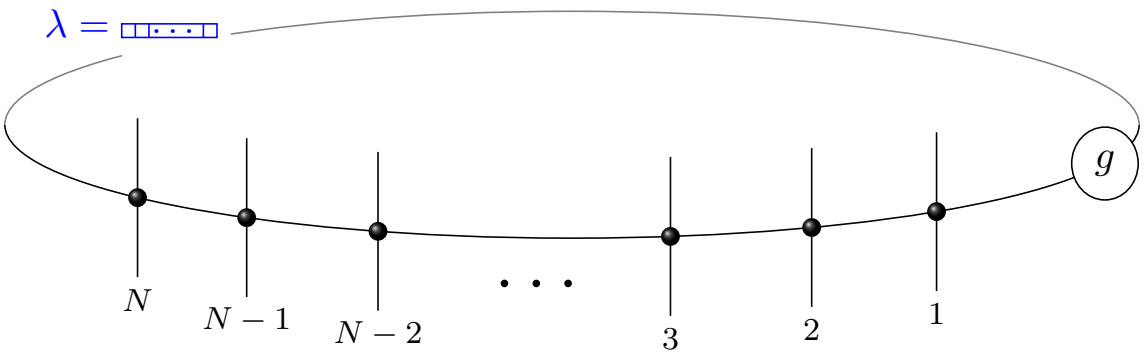


Figure III.4: A graphical description to T-operators with a symmetric representation in the auxiliary space ($\lambda=(s,0,0,...)$). This graph follows the same description as Fig(II.3).

The fundamental R-matrix defined in (III.1.1a) is the special case when taking the representation $\lambda=(1,0,0,\dots)$ of the symmetric R-matrix $\mathfrak{R}\left(u\right)$. For the symmetric representation, the R-matrix was introduced in [Fra20, Res85] (also, check [KK21b] for the symmetric representation description) as:

$$\begin{aligned}\mathfrak{R}\left(u\right)=\left(u^{2}-\frac{1}{4}\left(\left(\kappa-1\right)^{2}+2\kappa s+s^{2}\right)\right)\mathbb{I}+\left(u+\frac{\kappa}{2}\right)\sum_{\alpha,\beta=1}^{2\mathsf{r}}J_{\alpha,\beta}\otimes\pi_{s}\left(F_{\beta,\alpha}\right)+\\ \frac{1}{2}\sum_{\alpha,\beta=1}^{2\mathsf{r}}\sum_{k=1}^{2\mathsf{r}}\left(J_{\alpha,\beta}\otimes\pi_{s}\left(F_{k,\beta}\right)\pi_{s}\left(F_{\alpha,k}\right)\right)\end{aligned} \tag{III.1.8}$$

Thus, the definition of the T-operators reads (check out Fig(III.4)):

$$\left[\mathtt{T}_{(1,s)}(u),\mathtt{T}_{(1,s')}(v)\right]=0,\qquad \mathtt{T}_{(1,s)}(u)\equiv \mathrm{tr}\Big(\mathfrak{R}_N(u-\theta_N)\cdot\mathfrak{R}_{N-1}(u-\theta_{N-1})\cdots\mathfrak{R}_1(u-\theta_1)\,\rho_{(s)}(g)\Big) \tag{III.1.9}$$

where the trace is taken over the auxiliary space, (s,s') are two symmetric representations on the auxiliary space and $\rho_{(s)}$ is the morphism of the symmetric representation.

It is worth noting that the Casimir operator of the symmetric representations is equal to the representation-dependent part of (III.1.8), and it is defined as:

$$C=\mathrm{Tr}\left(\sum_{i,j=1}^{2\mathsf{r}}\pi_s(F_{k,j})\,\pi_s(F_{i,k})\right)=2\left(2\kappa s+s^2\right) \tag{III.1.10}$$

This will be useful for the next section when we formulate symmetric T-operators with the differential opeartor.

III.1.3 Coderivative: for $so(2\mathsf{r})$ case

Motivated by Sec(II.1.4), we are going to introduce a new differential operator (modified version of the $gl(n)$ case (II.1.28)), which will be referred to as $\mathbf{D}$. As is the case in Sec(II.1.4), constriction T-operators using $\mathbf{D}$ will be conceptually similar to $GL(\mathsf{n})$ case (which will be governed by the action of $\mathbf{D}$ on the character). The definition of the modified operator $\mathbf{D}$:

$$\boxed{\mathbf{D}\otimes f(g)\equiv\sum_{i,j}J_{i,j}\otimes\frac{\partial}{\partial\varepsilon_{j,i}}f\left(e^{\sum_{\alpha,\beta=1}^{2\mathsf{r}}\varepsilon_{\alpha,\beta}F_{\alpha,\beta}}g\right)\Big|_{\varepsilon=0}} \tag{III.1.11}$$

where f is a tensor-valued function of the twist $g\in SO(2\mathsf{r})$, i. e. $f^{i_1,i_2,\ldots,i_N}_{j_1,j_2,\ldots,j_N}(g)$, ε is an $2\mathsf{r}\times 2\mathsf{r}$ matrix, and $J_{\alpha,\beta}$ are the generators of $GL(\mathsf{n})$, whereas $F_{\alpha,\beta}$ are the generators of $SO(2\mathsf{r})$. The action of $\mathbf{D}$ does indeed add a spin to the chain, as is the case in $GL(\mathsf{n})$. In order to rewrite T-operators using the new differential operator $\mathbf{D}$, we need to study the action of $\mathbf{D}$ on the general twist operator $\rho_\lambda(g)$:

$$\begin{aligned}\mathbf{D}\otimes\rho_\lambda(g)&=\sum_{i,j}J_{i,j}\otimes\frac{\partial}{\partial\varepsilon_{j,i}}\rho_\lambda\left(e^{\sum_{\alpha,\beta}\varepsilon_{\alpha,\beta}F_{\alpha,\beta}}\cdot g\right)|_{\varepsilon=0}\\&=\sum_{i,j}\left(J_{i,j}\otimes\pi_\lambda(F_{j,i})\right)\cdot\left(\mathbb{I}\otimes\rho_\lambda(g)\right)\end{aligned} \tag{III.1.12}$$

When restricting the representation of the last equation to the fundamental ($\lambda=\square$), the expression $\sum_{i,j}\left(J_{i,j}\otimes\pi_\lambda(F_{j,i})\right)$ becomes $(\mathcal{P}_{1,\lambda}-\mathcal{Q}_{1,\lambda})$, which are the permutation $\mathcal{P}$ and the transposed permutation $\mathcal{Q}$. Fortunately, $\mathbf{D}$ also obey the Leibniz rule:

$$\mathbf{D}\otimes\left(f(g)\cdot h(g)\right)=\left(\mathbf{D}\otimes f(g)\right)\cdot\left(\mathbb{I}\otimes h(g)\right)+\left(\mathbb{I}\otimes f(g)\right)\cdot\left(\mathbf{D}\otimes h(g)\right) \tag{III.1.13}$$

With that, we can then check the action of two $\mathbf{D}$ operators on the fundamental twist:

$$\begin{aligned}\mathbf{D}\otimes\mathbf{D}\otimes\rho_\lambda(g)&=(\mathcal{P}-\mathcal{Q})_{2,\lambda}\cdot\left(\mathbf{D}\otimes\left(\mathbb{I}\otimes\rho_\lambda(g)\right)\right)\\&=(\mathcal{P}-\mathcal{Q})_{2,\lambda}\,(\mathcal{P}-\mathcal{Q})_{1,\lambda}\cdot\left(\mathbb{I}\otimes\mathbb{I}\otimes\rho_\lambda(g)\right)\end{aligned} \tag{III.1.14}$$

Notice that the two quantum labelings are swapped (which is the same as the $GL(\mathsf{n})$ case). With the definition (III.1.11) and properties of $\mathbf{D}$, we will be able to redefine the T-operators in the two known representations according to the definitions of their R-matrices.

Remark 1. *One can think of the $SO(2\mathsf{r})$ operator $\mathbf{D}$ as two $GL(\mathsf{n})$ operators $\hat{D}$ up to priming one of them. In our chain, the physical spaces are defined with a fundamental representation of the algebra $\lambda = \square$, thus in* (III.1.11) *the $so(2\mathsf{r})$ generators $F_{\alpha,\beta}$ can be replaced with $gl(n)$ generators, i. e. $F_{\alpha,\beta} = J_{\alpha,\beta} - J_{\beta',\alpha'}$, which in terns implize that:*

$$\boxed{\mathbf{D} \equiv \hat{D} - \hat{D}'} \tag{III.1.15}$$

From (III.1.15)*, we can see that the $SO(2\mathsf{r})$ operator $\mathbf{D}$ have the following property: $\mathbf{D}' = -\mathbf{D}$.*

Spinorial representation

Reformulating the T-operators with $SO(2\mathsf{r})$ symmetry will take the same steps as the $GL(\mathsf{n})$ case. In the spinorial case, the definition of R-matrix provided in (III.1.6) is extremely similar to the $GL(\mathsf{n})$ one. Following the same logic as in Sec(II.1.4), it is straightforward to write the spinorial T-operators as:

$$\boxed{\mathbf{T}_{(\pm)}(u) = \bigotimes_{i=1}^{N} (u_i + \mathbf{D})\, \chi_{\pm}} \tag{III.1.16}$$

Where $\chi_{\pm}$ is the finite-dimensional character of this representation ($\pm$ is the labeling of the two spinorial nodes in Fig. (III.2)). The exact form of this character is given in Sec(III.2.2).

Symmetric (Fundamental) representation

For the symmetric case, finding the operatorial description of T-operators is trickier than the $GL(\mathsf{n})$ or the spinorial cases due to representation dependence in the R-matrix, and the term with products of $SO(2\mathsf{r})$ generates in (III.1.8). For this particular case, the reformulated expression of T-operators takes on the form:

$$\boxed{\mathtt{T}_{(1,s)}(u) = \bigotimes_{i=1}^{N} \left[u_i^2 - \frac{1}{4}(k-1)^2 - \frac{1}{8}\operatorname{Tr}\left(\mathbf{D}^2\right) + \left(u_i + \frac{\kappa}{2}\right)\mathbf{D} + \frac{1}{2}\left(\mathbf{D}^2\right)' \right] \chi_s} \tag{III.1.17}$$

Here, χ_s represents the finite symmetric character. To comprehend how we arrived at the final formula, we must determine the expression of the Casimir and the second-degree coefficient in the generators $so(2\mathsf{r})$ in (III.1.8) as Coderivative operators. The other terms in (III.1.8) follow straightforwardly, as in the case of $GL(\mathsf{n})$. To achieve this, let us expand the following expression and demonstrate that it is equivalent to the second-degree expression in (III.1.8):

$$\begin{aligned} \sum_{\alpha,\beta=1}^{2\mathsf{r}} \left(J_{\alpha,\beta} \otimes \pi_s\left(F_{\beta,\alpha}\right)\right)^2 &= \sum_{\alpha,\beta=1}^{2\mathsf{r}} \sum_{k=1}^{2\mathsf{r}} \left(J_{\alpha,k} \otimes \pi_s\left(F_{k,\alpha}\right)\right)\left(J_{k,\beta} \otimes \pi_s\left(F_{\beta,k}\right)\right) \\ &= \sum_{\alpha,\beta=1}^{2\mathsf{r}} \sum_{k=1}^{2\mathsf{r}} \left(J_{\alpha,\beta} \otimes \pi_s\left(F_{k,\beta}\right)\pi_s\left(F_{\alpha,k}\right)\right) \end{aligned} \tag{III.1.18}$$

This observation leads us to consider two Coderivative operator with a contraction between them, although caution must be exercised when dealing with indices. Taking $\mathbf{D}^2$ and using (III.1.11) and ommiting $\rho_s(g)$ for simplicity, we find:

$$\mathbf{D}^{\alpha_1}_{\beta_1}\mathbf{D}^{\alpha_2}_{\beta_2} = \sum_{\alpha_1,\beta_1}\sum_{\alpha_2,\beta_2}\left(J_{\alpha_2,\beta_2}\otimes\pi_s\left(F_{\beta_2,\alpha_2}\right)\right)\left(J_{\alpha_1,\beta_1}\otimes\pi_s\left(F_{\beta_1,\alpha_1}\right)\right) \tag{III.1.19}$$

To ensure that the expression in (III.1.19) matches with (III.1.18), we need to replace $\alpha_1 = \beta_2$ with k and α_2 with α, β_1 with β. This is expressed in terms of Coderivatives as $\mathbf{D}^k_\beta\mathbf{D}^\alpha_k = \left(\mathbf{D}^2\right)'^2$. The illustration of $\left(\mathbf{D}^2\right)'$ diagrammatically take the form  . The same idea applies to the description of the Casimir operation with Coderivatives.

III.2 Symmetric and Spinorial Q-operators.

Similar to what has been done in Sec(II.2), we have a family of commuting Q-operators that also commute with T-operators. These operators allow us to express T-operators via a *Wronskian* determinant obtained in [FFK21] for both representations separately.

In the spinorial case, the *Wronskian* relation describing T-operators was defined in [FFK21] with weights $\lambda = \left(\frac{s}{2},...,\frac{s}{2},\pm\frac{s}{2}\right)$ as:

$$\mathbf{T}_{(\pm)}(u) \propto \sum_{\{\alpha_i\}_\pm}\chi^+_{\vec{\alpha}}\left(-\frac{\kappa}{2}\right)\mathbf{S}^{[\kappa+s]}_{\vec{\alpha}}(u)\,\mathbf{S}^{[-\kappa-s]}_{-\vec{\alpha}}(u) \tag{III.2.1a}$$

$$\chi^+_{\vec{\alpha}}(z) = \prod_{i=1}^{\mathsf{r}} x_i^{\alpha_i z}\prod_{1\leqslant j<k\leqslant\mathsf{r}}\frac{1}{1-1/(x_j^{\alpha_j}x_k^{\alpha_k})} \tag{III.2.1b}$$

where $\mathbf{S}(u)$ are the spinorial Q-operators and $\chi^+_\alpha(z)$ is the infinite-dimensional characters. We define $\vec{\alpha} = (\alpha_1,\ldots,\alpha_\mathsf{r})$ with $\alpha_i = \pm 1$, $\forall i \in [\![1,\mathsf{r}]\!]$, and $\{\alpha_i\}_\pm$ denotes all permissible configurations where $\prod_j \alpha_j = \pm 1$. In [FFK21], the definition of $\mathbf{S}(u)$ was given as a non-monic polynomial in u, a polynomial with a normalization that will allow the case $N = 0$ in (III.2.1a) to give the finite dimensional character. We are not spasifing the normalization factor, which is why we are using the symbol $\propto$.

Similar to what has been done for the spinorial case, the *Wronskian* describing the symmetric T-operators takes the form:

$$\mathsf{T}_{(1,s)}(u) \propto \left(\mathbf{Q}^{[\kappa+s]}_{\{i\}}(u)\,\mathbf{Q}^{[-\kappa-s]}_{\{i'\}}(u) + \mathbf{Q}^{[-\kappa-s]}_{\{i\}}(u)\,\mathbf{Q}^{[\kappa+s]}_{\{i'\}}(u)\right) \tag{III.2.2}$$

where $\mathbf{Q}_{\{i\}}(u)$ are the *single-indexed* Q-functions[3]. In what follows, we will show how to construct T- and Q-operators using the same ideas as in Sec(II.2).

A good starting point would be to give a brief description of the infinite-dimensional T-operators in the spinorial case.

[2]The operator $\left(\mathbf{D}^2\right)'$ does not follow the same Leibniz rule as $\mathbf{D}$. The proof is in Appendix(B.2)

[3]The diagonalization of T-operators has been done in [DK87] using the algebraic Bethe ansatz, and [GR20] for the trigonometric case.

III.2.1 Infinite dimensional spinorial T-operators

The spinorial oscillator representation was introduced in a series of papers [FFK21, FKT21], and the fusion relation of this representation reads:

$$\mathbf{R}_{\vec{\alpha}}\left(u+y_{\vec{\alpha}}\right)\mathbf{R}_{-\vec{\alpha}}\left(u-y_{-\vec{\alpha}}-\kappa\right)\propto\mathbf{R}^{+}\left(u\right) \tag{III.2.3a}$$

$$\mathbf{R}^{+}\left(u\right)=u+\sum_{\alpha,\beta}J_{\alpha,\beta}\otimes F_{\beta,\alpha} \tag{III.2.3b}$$

In the above expression, the symbol $\propto$ signifies that we have omitted certain factors that do not play a role in the present discussion, and the parameter y_α plays the role of the representation label. In [FKT21], there was a discussion on how to extract $\mathbf{R}_{\vec{\alpha}}\left(u\right)$ from the fusion relation. We will briefly describe the discussion and show how one can write the result using $\mathbf{D}$. The authors of [FKT21] started by restricting $\vec{\alpha}$ to $(+,\ldots,+)$, thus (III.2.3a) reads as (we are going to drop α in y):

$$\mathbf{R}_{(+,\ldots,+)}\left(u+y\right)\mathbf{R}_{(-,\ldots,-)}\left(u-y-\kappa\right)\propto\mathbf{R}^{+}\left(u\right) \tag{III.2.4}$$

This fusion relation is quite similar to the $GL(2)$ fusion relation defined in [BLMS10] (In this case, $\mathbf{R}_{(+,\ldots,+)}$ and $\mathbf{R}_{(-,\ldots,-)}$ are represented as two distinct diagonal blocks in $\mathbf{R}^{+}$), and by following the same ideas as in Appendix(A.1.2), they were able to write the two opeartors:

$$\begin{cases} \mathbf{R}_{(+,\ldots,+)}\left(u\right)=\lim\limits_{y\to\infty}\left(\overbrace{\begin{pmatrix}\mathbb{I}_{\mathsf{r}\times\mathsf{r}} & 0\\ 0 & \frac{1}{-2y}\mathbb{I}_{\mathsf{r}\times\mathsf{r}}\end{pmatrix}}^{\mathcal{N}_{(+,\ldots,+)}}\cdot\mathbf{R}^{+}\left(u-y\right)\right) \\ \mathbf{R}_{(-,\ldots,-)}\left(u\right)=\lim\limits_{y\to\infty}\left(\mathbf{R}^{+}\left(u+y+\kappa\right)\cdot\overbrace{\begin{pmatrix}\frac{1}{2y}\mathbb{I}_{\mathsf{r}\times\mathsf{r}} & 0\\ 0 & \mathbb{I}_{\mathsf{r}\times\mathsf{r}}\end{pmatrix}}^{\mathcal{N}_{(-,\ldots,-)}}\right) \end{cases} \tag{III.2.5}$$

With that, they can define two of the infinite-dimensional T-operators $(\mathbf{T}^{+}_{(+,\ldots,+)}\left(u\right),\mathbf{T}^{+}_{(-,\ldots,-)}\left(u\right))$. Instead of doing the same for different $\vec{\alpha}$, they introduced a transformation that allows us to find $\mathbf{T}^{+}_{(\vec{\alpha})}\left(u\right)$ from $\mathbf{T}^{+}_{(+,\ldots,+)}\left(u\right)$. This is due to the fact that different $\mathbf{T}^{+}_{(\vec{\alpha})}\left(u\right)$ are distinct by their diagonal blocs and their eigenvalues. They refer to the transformation as particle-hole transformation, which takes care of both criteria and is defined as:

$$\mathbf{T}^{+}_{(\vec{\alpha})}\left(u\right)=\left(\mathbb{S}_{\vec{\alpha}}\otimes\cdots\otimes\mathbb{S}_{\vec{\alpha}}\right)\mathbf{T}^{+}_{(+,\ldots,+)}\left(u\right)\left(\mathbb{S}_{\vec{\alpha}}\otimes\cdots\otimes\mathbb{S}_{\vec{\alpha}}\right)^{-1}\Big|_{\left\{x_i\to\frac{1}{x_i}|\alpha_i=-1\right\}} \tag{III.2.6a}$$

$$\mathbb{S}_{\vec{\alpha}}=\prod_{1\leqslant j\leqslant r}^{\alpha_j=-1}\left(J_{j,j'}-J_{j',j}+\sum_{\substack{1\leqslant k\leqslant\mathsf{r}\\ k\neq j}}\left(J_{k,k}+J_{k',k'}\right)\right) \tag{III.2.6b}$$

where $J_{a,b}$ satisfy (II.1.8). Using the transformation, one obtain the general relation for the spinorial T-operators:

$$\mathbf{T}^{+}_{(\vec{\alpha})}(u) = \lim_{\substack{y_\alpha \to \infty \\ \forall \alpha \in \vec{\alpha}}} \mathrm{Tr}\left(\check{\mathbf{R}}_{N,\vec{\alpha}}(u) \cdot \check{\mathbf{R}}_{N-1,\vec{\alpha}}(u) \cdots \check{\mathbf{R}}_{1,\vec{\alpha}}(u)\, \rho^{+}_{\vec{\alpha}}(g)\right) \tag{III.2.7a}$$

$$\text{Where } \check{\mathbf{R}}_{\vec{\alpha}}(u) = \mathcal{N}_{\vec{\alpha}} \mathbf{R}^{+}(u), \qquad \mathcal{N}_{\vec{\alpha}} = \mathbb{S}_{\vec{\alpha}} \mathcal{N}_{(+,\ldots,+)} \mathbb{S}^{-1}_{\vec{\alpha}} \tag{III.2.7b}$$

where $\left(\rho^{+}_{\vec{\alpha}}(g)\right)$ is the infinite-dimensional twist operator. Notice that (III.2.6a),(III.2.7a) define the rest of the 2^r infinite-dimensional T-operators.

By using the definition of $\mathbf{R}^{+}(u)$ in (III.2.3b), we replace the sum of generators $(\sum_{i,j} J_{i,j} \otimes F_{j,i})$ with $\mathbf{D}$ using (III.1.12). Then, the trace and the transformation act solely on the twist $(\rho^{+}(g))$, which gives us the infinite-dimensional character, and (III.2.7a) becomes:

$$\boxed{\mathbf{T}^{+}_{(\vec{\alpha})}(u) = \bigotimes_{i=1}^{N} (u_i + \mathbf{D}) \cdot \chi^{+}_{\vec{\alpha}}(z)} \tag{III.2.8}$$

where the infinite-dimensional character χ^{+}_{α} is defined as the trace of $\left(\mathbb{S}_{\vec{\alpha}} \rho^{+}_{\vec{\alpha}}(g)\, \mathbb{S}_{\vec{\alpha}}\right)$ and given in (III.2.1b).

III.2.2 Spinorial Q-operators.

Using (III.2.1a) and the ideas in Sec(II.2.2.3), we will motivate a Coderivative description for the spinorial Q-operators, a description where the Coderivative acts on a somewhat generating series of the spinorial character. Starting by shifting the spectral parameter $u \rightsquigarrow u - \frac{1}{2}(\kappa + s)$ in (III.2.1a), then taking the infinite sum $\sum_{s \geqslant 0}$:

$$\sum_{s \geqslant 0} \mathbf{T}^{[-\kappa-s]}_{(\pm)}(u)\, z^s \propto \sum_{\{\alpha_i\}_{\pm}} \chi^{+}_{\vec{\alpha}}\left(-\kappa - \frac{s}{2}\right) \mathbf{S}_{\vec{\alpha}}(u) \sum_{s \geqslant 0} \left(\mathbf{S}^{[-2\kappa-2s]}_{-\vec{\alpha}}(u)\, z^s\right) \tag{III.2.9}$$

The infinite sum on the right hand side has a radius of conferences $|z| < \min\left|\frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i}}\right|$. Indeed, by denoting $\frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i s}} \mathbf{S}_{-\vec{\alpha}}(u) = \frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i s}} \sum_{k=0}^{\mathcal{M}_d} \mathcal{C}_{-\alpha, \mathcal{M}_d - k} u^k$, and using the normalized limit defined in Lemma(I.1.1). As discussed in [CLV22] and in Sec(I.1.1), the limit $z \to \frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i s}}$ of the infinite sum on the right hand side of the previous equation gives us singularities with different orders depending on the sector (the infinite sum gives us the Eulerian number formula (I.1.17)). Thus, one finds the relation between the spinorial T- and Q-operators to be:

$$\mathbf{S}_{\vec{\alpha}}(u) = \underset{z \to \frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i}}}{\mathcal{N}\lim} \sum_{s \geqslant 0} \mathbf{T}^{[-\kappa-s]}_{(\pm)}(u)\, z^s \tag{III.2.10}$$

Finally, substituting (III.1.16) in the last formula, we can write $\mathbf{S}_{\vec{\alpha}}(u)$ using $\mathbf{D}$ as:

$$\boxed{\mathbf{S}_{\vec{\alpha}}(u) = \underset{z \to \frac{1}{\prod_{i=1}^{r} x_i^{\alpha_i}}}{\mathcal{N}\lim} \bigotimes_{i=1}^{N} \left(u_i - \frac{\kappa}{2} - z\frac{\partial}{\partial z} + \mathbf{D}\right) \sum_{s \geqslant 0} \chi_{\vec{\alpha},s} z^s} \tag{III.2.11}$$

The factor s is replaced by $z\frac{\partial}{\partial z}$ [4]. The action of $\mathbf{D}$ on the character is well-understood due to Appendix(B.1) [5]. The normalization in (III.2.11) takes the form:

$$\mathcal{N} \propto \left(1 - z\prod_{i=1}^{\mathsf{r}} x_i^{\alpha_i}\right)^{\mathcal{M}_d+1} \tag{III.2.12}$$

Where $\propto$ is used to emphasize the missing factors that will make our polynomial monic. One can argue that the normalization in (III.2.11) nullifies all components that are not in the $\vec{\alpha}$ direction, which is a similar statement to the proof of Lemma(I.1.2).

One can also describe the finite-dimensional character from the infinite-dimensional ones (III.2.1a) using the fact that T- and Q-operators are polynomials in u with a coefficient of the leading order in u to be $\chi_{\pm}$ and χ^{+}, respectively. By collecting the coefficients of the leading order in u (or by taking a chain of length $N = 0$) of both sides of (III.2.1a), one obtains:

$$\boxed{\chi_{\pm} = \sum_{\{\alpha_i\}_{\pm}} \chi^{+}_{\vec{\alpha}}\left(\frac{s}{2}\right)} \tag{III.2.13}$$

III.2.3 Symmetric Q-operators

With the same idea as the previous subsection, we can define a Coderivative description of the *single-indexed* Q-operators using (III.2.2). This description is an action of Coderivative on the generating series of the symmetric character. Starting with shifting the spectral parameter $u \rightsquigarrow u - \frac{1}{2}(\kappa + s)$, then taking the infinite sum $\sum_{s\geqslant 0}$:

$$\sum_{s\geqslant 0} \mathsf{T}^{[-\kappa-s]}_{(1,s)}(u) \propto \sum_{i=1}^{\mathsf{r}} \left(\mathbf{Q}_{\{i\}}(u)\left(\sum_{s\geqslant 0}\mathbf{Q}^{[-2\kappa-2s]}_{\{i'\}}(u)\right) + \left(\sum_{s\geqslant 0}\mathbf{Q}^{[-2\kappa-2s]}_{\{i\}}(u)\right)\mathbf{Q}_{\{i'\}}(u)\right) \tag{III.2.14}$$

Where $\propto$ is used to remove factors that have no impact on our present argument. The infinite sum of the last expression has a radius of convergence $|z| \leqslant \min\left|\frac{1}{x_{i'}}\right|$ for $\mathbf{Q}_{\{i'\}}(u)$ and $|z| \leqslant \min|\frac{1}{x_i}|$ for $\mathbf{Q}_{\{i\}}(u)$. Indeed, if we take $\mathbf{Q}_{\{i'\}}(u) = \frac{1}{x_i^u}\sum_{k=0}^{\mathcal{M}'_d}\mathcal{C}_{i',\mathcal{M}'_d-k}u^k$, and $\mathbf{Q}_{\{i\}}(u) = \frac{1}{x_{i'}^u}\sum_{k=0}^{\mathcal{M}_d}\mathcal{C}_{i,\mathcal{M}_d-k}u^k$, then using the same arguments as the previous chapter, we find the relation between the symmetric T-operators and single-indexed Q-operators:

$$\mathbf{Q}_{\{i\}}(u) = \mathcal{N}\lim_{z\to\frac{1}{x_{i'}}}\sum_{s\geqslant 0}\mathsf{T}^{[-\kappa-s]}_{(1,s)}(u)\,z^s \tag{III.2.15a}$$

$$\mathbf{Q}_{\{i'\}}(u) = \mathcal{N}\lim_{z\to\frac{1}{x_i}}\sum_{s\geqslant 0}\mathbf{T}^{[-\kappa-s]}_{(1,s)}(u)\,z^s \tag{III.2.15b}$$

[4] The sum $\sum_{s\geqslant 0}s^j z^s = z^j\frac{\partial^j}{\partial z^j}\sum_{s\geqslant 0}z^s$.

[5] We are unaware of a generating series for the spinorial representation. Therefore, we have kept the infinite sum in the right hand side of (III.2.11).

Using (III.1.17), we can formulate the symmetric Q-operators as:

$$\mathbf{Q}_{\{i\}}(u) = \underset{z\to\frac{1}{x_i'}}{\mathcal{N}\lim} \bigotimes_{k=1}^{N} \left[u_k u_k^{[2\kappa]} + u_k z\frac{\partial}{\partial z} + \frac{1}{4}(2\kappa-1) + \left(u_k^{[-\frac{\kappa}{2}]} - \frac{z}{2}\frac{\partial}{\partial z}\right)\mathbf{D} + \frac{1}{2}\left(\mathbf{D}^2\right)' \right] \left(1-z^2\right) w(z) \tag{III.2.16}$$

where a formnula for $\mathbf{Q}_{\{i'\}}(u)$ is exactly the same as the one for $\mathbf{Q}_{\{i\}}(u)$ up changing the limit to $z\to\frac{1}{x_i}$ and the normalization. The normalization expression of Q-operators in the previous relation take the form: $(1-x_{i'}z)^{\mathcal{M}_d+1}$ (resp. $(1-x_iz)^{\mathcal{M}'_d+1}$) for $\mathbf{Q}_{\{i\}}(u)$ (resp. $\mathbf{Q}_{\{i'\}}(u)$).

Conserved numbers

In the case of the $so(2\mathsf{r})$ symmetry group, the basis vectors will be labelled as

$$v_{\mathsf{r}'}, v_{(\mathsf{r}-1)'}, \dots v_{2'1'}, v_{1'}, v_1, v_2, \dots, v_{\mathsf{r}}$$

Then the sectors will not be defined by fixing the number of occurrences of each factor v_k as in $GL(\mathsf{n})$, but by fixing the difference between the number of occurrences of v_k and the number of occurrences of $v_{k'}$:

To an N-tuple of integers $m_1, m_2, \dots, m_{\mathsf{r}}$ such that $N - \sum_\ell |m_\ell| \in 2\mathbb{Z}_{\geqslant 0}$, we associate the sector spanned by all the basis vectors $|v_{i_1}\dots v_{i_N}\rangle$ such that $\forall k \in \{1,2,\dots,\mathsf{r}\}$:

$$m_k = \#\{p|i_p = k\} - \#\{p|i_p = k'\}$$

This is a conserved number, which means we have an operator M_j such that $M_j|V_N\rangle = m_j|V_N\rangle$. This operator has the same relations as (II.2.14) plus $[\mathcal{Q}, M_j] = 0$.

To understand $[\mathcal{Q}, M_j]$, and without loss of generality, let us take the action of $\mathcal{Q}_{1,2}$ on a state, $\mathcal{Q}_{1,2}|v_{k_1}, v_{k_2}, v_{k_3}, \dots\rangle = \delta_{v_{k_1'}, v_{k_2}} \sum_k |k, k', v_{k_3}, \dots\rangle$. Now, by acting on that state with M_j^{so}, we find:

$$M_j^{so}\mathcal{Q}_{1,2}|v_{k_1}, v_{k_2}, v_{k_3}, \dots\rangle = \begin{cases} 0 = \mathcal{Q}_{1,2}M_j^{so}|v_{k_1}, v_{k_2}, v_{k_3}, \dots\rangle, & \text{if } v_{k_1} \neq v_{k_2'} \\ \underbrace{\mathcal{Q}_{1,2}M_j^{so}|v_{k_1}, v_{k_2}, v_{k_3}, \dots\rangle}_{m_j^{so}\mathcal{Q}_{1,2}|v_{k_1}, v_{k_2}, v_{k_3}, \dots\rangle} & , \text{ if } v_{k_1} = v_{k_2'} \end{cases} \tag{III.2.17}$$

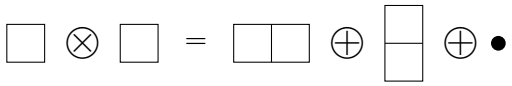
Thus showing $[\mathcal{Q}, M_j^{so}] = 0$.

III.3 Coderivatives are not sufficient to describe other representations of $SO(2\mathsf{r})$

As argued below, we expect that Coderivatives are not sufficient to describe other representations of $SO(2\mathsf{r})$. We have tested several methods in building T- and Q-operators for a general

rectangular representation in the auxiliary space; these attempts included changing the Casimir or other terms that depend on the number of rows in a Young diagram. As we will see later, all these attempts fail to describe a more general representation due to the fact that Coderivatives (powers of Coderivative, contractions, and other complicated coefficients) cannot produce the R-matrix of a generic rectangular representation. This is even apparent for the antisymmetric representation; if we interpret the leading coefficient of the spectral parameter in T-operators as the character of the $SO(2\mathsf{r})$ representation associated with this Yangian representation, then it does not correspond to an irreducible representation of $SO(2\mathsf{r})$. Therefore, we could expect a Yangian representation where some generators are independent from the so(2r) generators. Such terms cannot be produced with a Coderivative, which only produces the so(2r) generators (see (III.1.12)).

This can be inferred from examining the Yangian representation theory introduced in [FH91] which enables the construction of R-matrices for general rectangular representations, along with the representation theory of $SO(2\mathsf{r})$. As an example, are aware from the representation theory of $SO(2\mathsf{r})$ that the space $V = (\mathcal{C}^{2\mathsf{r}})^{\otimes 2}$ can be decomposed into:



(III.3.1)

On the right hand side, the representations  ,  are defined from the space of symmetric and antisymmetric tensors respectively, and • is known as the dot representation. It turns out that the dimension of symmetric representation on the right hand side is the same when studying the symmetric Yangian algebra, whereas this is not true for the antisymmetric one. It seems that for this specific example, the combination of the last two representations on the right hand side would produce the irreducible representation of the Yangian algebra.

The spinorial and symmetric representations possess certain algebraic properties that render them particularly suitable for the construction of Q-operators, notably due to the fact that all operators in the expression of the R-matrices are generators of the Lie algebra $so(2\mathsf{r})$. These families of irreducible representations are simple to investigate with Coderivative operator. However, there exist other representations for which an R-matrix is known [KK21a, KOW22, Oka07, FH91], but their expressions are less explicit and not solely in terms of $so(2\mathsf{r})$ generators. These R-matrices are referred to as Kirillov-Reshetikhin R-matrices and present a natural extension for further investigation beyond the scope of this manuscript. The exploration of these representations remains a task for future research.

In the following sections, we will share insights gained from our exploration, hoping they will prove beneficial for future research endeavors. We begin by mimicking Sec(II.1.5) of the $GL(\mathsf{n})$ case, wherein we consider a general function of the twist operator $f(g)$ in the auxiliary space and examine the non-trivial coefficients obtained. Then, inspired by the discussion in [KV08], we attempt to seek T-operators with general rectangular representation from the CBR formula (III.3.4), derived in [Che86, BR90]. When considering only the rectangular representations, the CBR formula is equivalent to the bilinear identity Hirota identity, appearing in [KNS11,

Res83, KP92], and also in [KNS94] where it was shown that the symmetric T-operators are a solution of the Hirota identity for the nodes on the tail of Fig(III.2). This identity reads:

$$\mathtt{T}_{(a+1,s)}(u)\,\mathtt{T}_{(a-1,s)}(u)=\mathtt{T}_{(a,s)}\left(u+\frac{1}{2}\right)\mathtt{T}_{(a,s)}\left(u-\frac{1}{2}\right)-\mathtt{T}_{(a,s+1)}(u)\,\mathtt{T}_{(a,s-1)}(u) \tag{III.3.2}$$

We have emphasized that our desired representations on the auxiliary space are general rectangular representations. In the $GL(\mathsf{n})$ case, this statement can be generalized to arbitrary representations on the auxiliary space due to $\mathcal{T}_{(0,s)}(u)=\mathcal{T}_{(a,0)}(u)=\mathcal{T}_{(0,0)}(u)=\prod_{i=1}^{N}u_i$, which is not the case in $SO(2\mathsf{r})$:

$$\mathtt{T}_{(0,s)}(u)=\bigotimes_{i=1}^{N}\left(u_i^2-\frac{1}{4}\left((\kappa-1)^2+2\kappa s+s^2\right)\right) \tag{III.3.3}$$

This is not equal to $\mathtt{T}_{0,0}(u)=\prod_{i=1}^{N}\left(u_i^2-\frac{1}{4}(\kappa-1)^2\right)$, and no gauge transformation will achieve such equality. Know that, the CBR identity for genearic rectangular representations take the form:

$$\boxed{\mathtt{T}_{(a,s)}(u)=\frac{\left|\left(\mathtt{T}_{(1,s+i-j)}^{[a-i-j+1]}(u)\right)_{1\leqslant i,j\leqslant a}\right|}{\prod_{n=1}^{N}\prod_{k=1}^{a-1}\mathtt{T}_{(0,s)}^{[a-2k]}(u)}} \tag{III.3.4}$$

This relation can be formulated by taking the cases $a=2,3,\ldots$ in (III.3.2) and writing $\mathtt{T}_{(2,s)}(u),\mathtt{T}_{(3,s)}(u),etc$ using $\mathtt{T}_{(1,s)}(u)$. So as an example, when $a=2$, (III.3.2) takes the form:

$$\mathtt{T}_{(2,s)}(u)=\frac{\mathtt{T}_{(1,s)}^{[1]}(u)\,\mathtt{T}_{(1,s)}^{[-1]}(u)-\mathtt{T}_{(1,s+1)}(u)\,\mathtt{T}_{(1,s-1)}(u)}{\mathtt{T}_{(0,s)}(u)} \tag{III.3.5}$$

Which can be seen as a 2×2 determinant:

$$\mathtt{T}_{(2,s)}(u)=\frac{\begin{vmatrix}\mathtt{T}_{(1,s)}^{[1]}(u) & \mathtt{T}_{(1,s-1)}(u)\\ \mathtt{T}_{(1,s+1)}(u) & \mathtt{T}_{(1,s)}^{[-1]}(u)\end{vmatrix}}{\mathtt{T}_{(0,s)}(u)} \tag{III.3.6}$$

Which does coincide with (III.3.4). Next, we can do the same for $a=3$:

$$\mathtt{T}_{(3,s)}(u)=\frac{\mathtt{T}_{(2,s)}^{[1]}(u)\,\mathtt{T}_{(2,s)}^{[-1]}(u)-\mathtt{T}_{(2,s+1)}(u)\,\mathtt{T}_{(2,s-1)}(u)}{\mathtt{T}_{(1,s)}(u)} \tag{III.3.7}$$

If $\mathtt{T}_{1,s}(u)$ is non-zero, then we can plug (III.3.6) in (III.3.7) to get:

$$\begin{aligned}
\mathtt{T}_{3,s}(u) &= \frac{1}{\mathtt{T}_{(1,s)}(u)}\left(\frac{\begin{vmatrix}\mathtt{T}^{[2]}_{(1,s)}(u) & \mathtt{T}^{[1]}_{(1,s-1)}(u)\\ \mathtt{T}^{[1]}_{(1,s+1)}(u) & \mathtt{T}_{(1,s)}(u)\end{vmatrix}}{\mathtt{T}^{[1]}_{(0,s)}(u)}\frac{\begin{vmatrix}\mathtt{T}_{(1,s)}(u) & \mathtt{T}^{[-1]}_{(1,s-1)}(u)\\ \mathtt{T}^{[-1]}_{(1,s+1)}(u) & \mathtt{T}^{[-2]}_{(1,s)}(u)\end{vmatrix}}{\mathtt{T}^{[-1]}_{(0,s)}(u)}\right)\\
&\quad-\frac{1}{\mathtt{T}_{(1,s)}(u)}\left(\frac{\begin{vmatrix}\mathtt{T}^{[1]}_{(1,s+1)}(u) & \mathtt{T}_{(1,s)}(u)\\ \mathtt{T}_{(1,s+2)}(u) & \mathtt{T}^{[-1]}_{(1,s+1)}(u)\end{vmatrix}}{\mathtt{T}_{(0,s+1)}(u)}\frac{\begin{vmatrix}\mathtt{T}^{[1]}_{(1,s-1)}(u) & \mathtt{T}_{(1,s-2)}(u)\\ \mathtt{T}_{(1,s)}(u) & \mathtt{T}^{[-1]}_{(1,s-1)}(u)\end{vmatrix}}{\mathtt{T}_{(0,s-1)}(u)}\right)\\
&= \mathtt{T}^{[2]}_{(1,s)}(u)\left(\frac{\mathtt{T}_{(1,s)}(u)\,\mathtt{T}^{[-2]}_{(1,s)}(u)-\mathtt{T}^{[-1]}_{(1,s-1)}(u)\,\mathtt{T}^{[-1]}_{(1,s+1)}(u)}{\mathtt{T}^{[1]}_{(0,s)}(u)\,\mathtt{T}^{[-1]}_{(0,s)}(u)}\right)\\
&\quad+\mathtt{T}^{[1]}_{(1,s-1)}(u)\left(\frac{\mathtt{T}^{[-1]}_{(1,s-1)}(u)\,\mathtt{T}_{(1,s+2)}(u)-\mathtt{T}^{[1]}_{(1,s+1)}(u)\,\mathtt{T}^{[-2]}_{(1,s)}(u)}{\mathtt{T}^{[1]}_{(0,s)}(u)\,\mathtt{T}^{[-1]}_{(0,s)}(u)}\right)\\
&\quad+\mathtt{T}_{(1,s-2)}(u)\left(\frac{\mathtt{T}^{[1]}_{(1,s+1)}(u)\,\mathtt{T}^{[-1]}_{(1,s+1)}(u)-\mathtt{T}_{(1,s)}(u)\,\mathtt{T}_{(1,s+2)}(u)}{\mathtt{T}^{[1]}_{(0,s)}(u)\,\mathtt{T}^{[-1]}_{(0,s)}(u)}\right)\\
&= \frac{\begin{vmatrix}\mathtt{T}^{[2]}_{(1,s)}(u) & \mathtt{T}^{[1]}_{(1,s-1)}(u) & \mathtt{T}_{(1,s-2)}(u)\\ \mathtt{T}^{[1]}_{(1,s+1)}(u) & \mathtt{T}_{(1,s)}(u) & \mathtt{T}^{[-1]}_{(1,s-1)}(u)\\ \mathtt{T}_{(1,s+2)}(u) & \mathtt{T}^{[-1]}_{(1,s+1)}(u) & \mathtt{T}^{[-2]}_{(1,s)}(u)\end{vmatrix}}{\mathtt{T}^{[1]}_{(0,s)}(u)\,\mathtt{T}^{[-1]}_{(0,s)}(u)}
\end{aligned}\tag{III.3.8}$$

Again, this coincides with (III.3.4), and the reason why is simply the fact that (III.3.4) satisfies the Hirota equation. Then a simple recurrence shows that if the Hirota equation holds, then one gets iteratively the rectangular CBR formula (III.3.4).

III.3.1 Coderivative and the Yang-Baxter equation

Knowing that the naively constructed Yang-Baxter equation is not satisfied with a general rectangular representation in the auxiliary space, it might be interesting to see the coefficients that break down the statement. By Replicating the approach in Sec(II.1.5) for the spinorial and symmetric cases, the Yang-Baxter equation is depicted diagrammatically in Fig(III.5), where  is the graphical representation of $\mathbf{D}$. Also, for simplicity, we will denote the factors with different powers in the spectral parameters as:

$$\begin{cases}\tilde{R}(u)=u^2+\mathbf{D}u, \quad \tilde{R}(0)=-\dfrac{1}{4}\left((\kappa-1)^2+\dfrac{1}{2}\mathbf{D}\mathbf{D}\right)+\dfrac{\kappa}{2}\mathbf{D}+\dfrac{1}{2}\mathbf{D}\mathbf{D}, & \text{if } \lambda \text{ is symmetric}\\ \tilde{R}(u)=u, \qquad\qquad \tilde{R}(0)=\mathbf{D}, & \text{if } \lambda \text{ is spinorial}\end{cases}$$

In doing so, Fig(III.5) is the diagrammatically description of the Yang-Baxter equation:

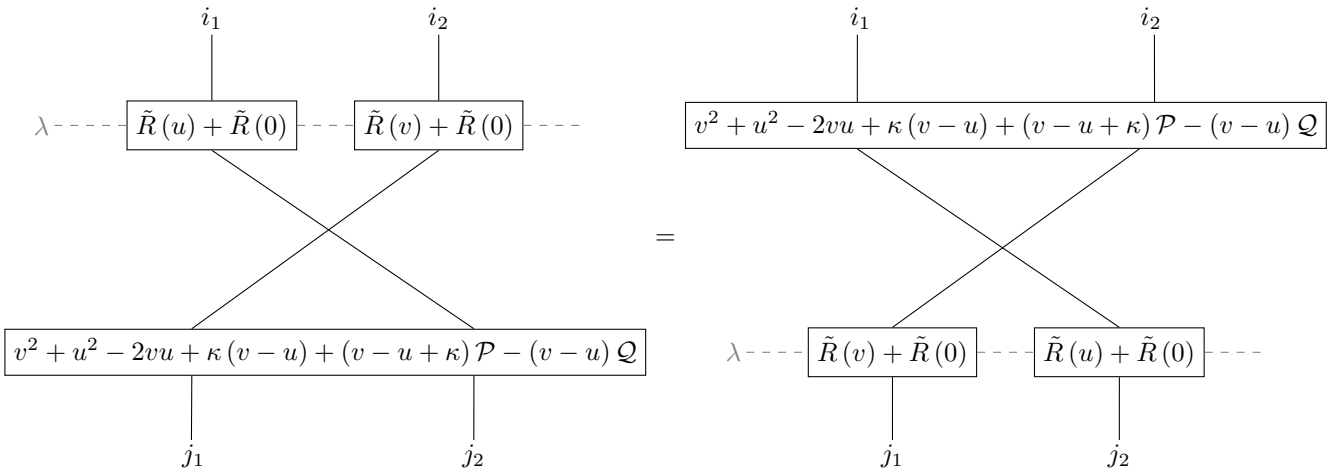
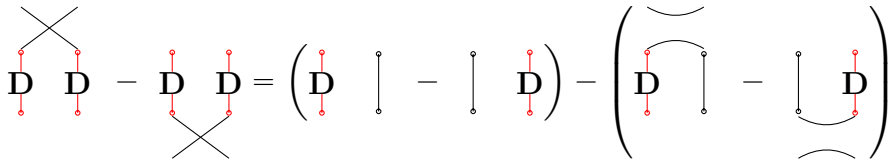


Figure III.5: A diagrammatical representation of Yang-Baxter equation with $SO(2\mathsf{r})$ symmetry. It read exactly like Fig(B.3) up to the definition of the R-matrix (III.1.8).

Spinorial Yang-Baxter equation

For the spinorial case, the process of determining the coefficients of the various spectral parameter powers is straightforward, as we only encounter 2 nontrivial factors:

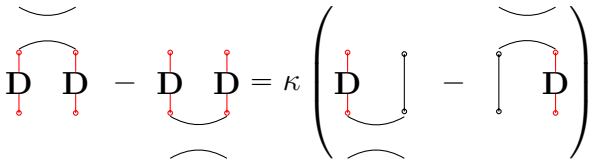
- Factors of the spectral power u^2v^0 and uv:



(III.3.9)

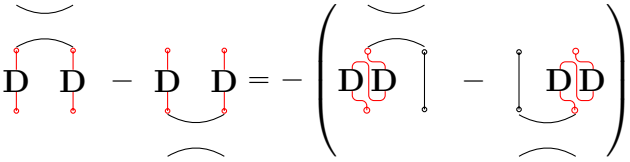
  This is the $so(2\mathsf{r})$ virsion of (B.1.15), and can be proven as in Sec(II.1.5).

- Factors of the spectral power u^1v^0:



(III.3.10)

  This identity immediately fails when $f(g) \neq \chi_{\pm}$. Additionally, one can verify the equality of the left hand side and the right hand side using only (I.1.5) and Remark(1) to shift the Coderivatives on the left hand side. Upon doing so, we obtain:

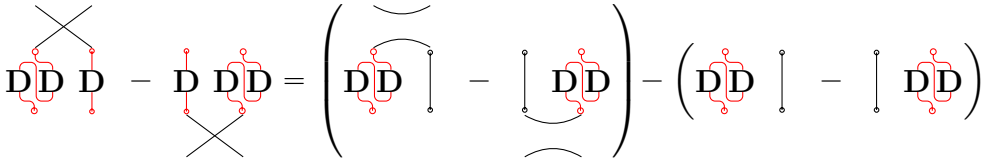


(III.3.11)

  This expression differs from (III.3.10). However, it does not provide clear guidance on what additions or modifications are required in our R-matrix o ensure a general solution of the YB equation in the spinorial representation.

Symmetric Yang-Baxter equation

This case proves to be significantly more intricate than the spinorial or the $GL(\mathsf{n})$ cases. Upon following a similar procedure as before, we have identified several non-trivial factors that fail to hold when generalizing our representation, i. e. $f(g) \neq \chi_s$. Surprisingly, among those that are successful for general representations, we rediscovered again (A.1.7) and (III.3.11). One of the most notable identities we found arises from collecting the factors of the spectral parameter u^3 or v^3, which is expressed as:



(III.3.12)

Although we were unable to prove this equation in a manner similar to that in Sec(II.1.5), we have verified its validity on the computer for $f(g) = g^{\otimes 3}$. Using the ideas in Sec(II.2.2.1), we can confidently assert that it holds ture for $f(g) = g^{\otimes k}, \forall k \in \mathbb{N}$.

III.3.2 Fusion and symmetric T-operators

Another idea that we had was to try and describe the symmetric T-operators using two $GL(\mathsf{n})$ R-matrices with different contractions between them. By virtue of (III.1.15), we can rewrite (III.1.17) using $gl(n)$ Coderivatives $\hat{D}$ with varying contractions. It's worth noting that the representation-dependent part s or the Casimir operator C can also be written using a traced $\hat{D}$ or $\textcircled{\mathbf{D}}$. Still, it is subtle to introduce $gl(n)$ operators to act on $so(2\mathsf{r})$ characters; in the next paragraph, we will discuss the setup difference between the two cases and how to modify it.

Setup of $\textcircled{\mathbf{D}}$ in $GL(\mathsf{n})$.

let us define a function $f(g)$ which has a degree d, i. e. $f(\zeta g) = \zeta^d f(g)$. Then $\textcircled{\mathbf{D}} f(g) = deg(f(g))\, f(g)$.

Proof. It is easy to see the connection between the degree of a function of the twist $f(g)$ and $\textcircled{\mathbf{D}}$. By defining the function $\Gamma(g) = deg(f(g))\, f(g)$, and asking what is the degree of the function:

$$\begin{aligned} \Gamma(g) &= deg(f(\zeta g)) f(\zeta g) \\ &= \zeta^d deg(\zeta^d f(g)) f(g) \\ &= \zeta^{2d} deg(f(g)) f(g) \end{aligned} \tag{III.3.13}$$

which is $\textcircled{\mathbf{D}}^2 f(g)$, and by a simple recurrence we can show that $\textcircled{\mathbf{D}}^m f(g) = d^m f(g)$. It is also true for x_k and for g (which have degree 1), for p_j (which has degree zero and $\textcircled{\mathbf{D}} p_j = 0$), and product/tensor product go nicely through thanks to the leibnitz rule (and degree of a product is the sum of the degrees). It also goes through for sums obviously. □

Because $\hat{D}$ preserve the degree in $f(g)$, it also follows that for instance $\hat{D}\mathbf{D}\mathbf{D}\hat{D}\hat{D}f(g) = d^2\hat{D}\hat{D}\hat{D}f(g)$.

Setup of $\mathbf{D}$ in $SO(2\mathsf{r})$.

The definition of $\mathbf{D}f(g)$ is ambiguous in the $so(2\mathsf{r})$ setup. A priori, if a function $f(g)$ defined for $g \in SO(2\mathsf{r})$, then only $\mathbf{D}f(g)$ is well defined (not $\hat{D}f(g)$).

For instance $\chi_s^{(gl)}(g) - \chi_{s-2}^{(gl)}(g)$ and $\chi_s^{(gl)}(g) - (\det g)^{1/r}\,\chi_{s-2}^{(gl)}(g)$ are two functions in $GL(\mathsf{n})$ which both coincide with $\chi_s^{(so)}(g)$ when $g \in SO(2\mathsf{r})$. These two functions do not have the same $\hat{D}$, but they have the same $\mathbf{D}$. This reflects the fact that $\mathbf{D}\chi_s^{(so)}$ is well defined, but not $\hat{D}\chi_s^{(so)}$ (hence not $\mathbf{D}\chi_s^{(so)}$ either).

Statement

The function $f_s(g) = \chi_s^{(gl)}(g) - (\det g)^{1/r}\,\chi_{s-2}^{(gl)}(g)$ is defined in $gl(n)$ and has degree s in g. Hence $\mathbf{D}f_s = sf_s$.

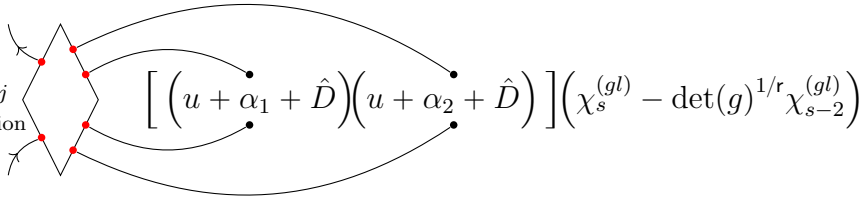
Thus $\mathbf{D}\sum_{s\geqslant 0} z^s f_s = \sum_{s\geqslant 0} sz^s f_s$ (and it still goes through after inserting arbitrarily many $\hat{D}$ (or even $\mathbf{D}$)). Thus, if we redefine the $so(2\mathsf{r})$ character as $\left(\chi_s^{(gl)}(g) - (\det g)^{1/r}\,\chi_{s-2}^{(gl)}(g)\right)$, then we obtain that for this character $\mathbf{D}$ does exactly the same as s.

With that, we can write the symmetric T-operators using the modified character and the $gl(n)$ coderivatives as:

$$\mathsf{T}_{1,s}(u) = \bigotimes_{i=1}^{N}\Bigg[\left(u_i^2 - \frac{1}{4}\left((\kappa-1)^2 + 2\kappa\mathbf{D} + \mathbf{D}^2\right)\right)\mathbb{I} + \left(u_i + \frac{\kappa}{2}\right)\hat{D} - \left(u_i + \frac{\kappa}{2}\right)\mathbf{D} + \frac{1}{2}\underbrace{\left(\hat{D}\hat{D} - \hat{D}\mathbf{D} - \mathbf{D}\hat{D} + \mathbf{D}\mathbf{D}\right)}_{\mathbf{D}\mathbf{D}}\Bigg]\cdot\left(\chi_s^{(gl)} - \det(g)^{1/\mathsf{r}}\chi_{s-2}^{(gl)}\right) \tag{III.3.14}$$

The last equation can be described with two operators $\left(u + \hat{D}\right)$ with varying contractions, this can be represented as:

$$\mathsf{T}_{1,s}(u) = \sum_{j\in\text{all combination}} \beta_j$$

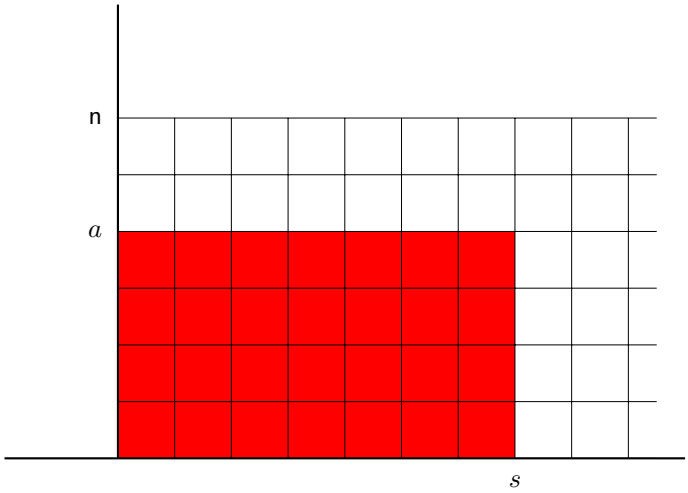


In the expression above, the two lines on left hand side of the diamond represent the ingoing and outgoing, while the other lines represent all the possible ways the two operators can be contracted together. The factors (α_1, α_2) denote a shift in the spectral parameter, and β_j represents a prefactor. There are fifteen different ways of contracting in the last expression, and the objective is to solve all the resulting linear equations to determine the prefactors and

the shifts of the spectral parameter. Regrettably, a rapid collection of the coefficients reveals that no feasible solution can be attained.

III.3.3 CBR determinant

This section aims to establish a relation for general rectangular T-operators. Utilizing (III.3.4), we will replicate the methodology employed in [KV08] to describe T-operators with arbitrary



A rectangular Young diagram
$(\lambda = (7, 7, 7, 7, 0, 0, \dots))$.

Figure III.6: Young diagram and (a, s) Lattice for $SO(2\mathsf{r})$.

representations for $GL(\mathsf{n})$. We commence by introducing rectangular T-operators which will be denoted as $\mathsf{T}_{(a,s)}(u)$, where (a, s) denotes the number of (rows, column) in a Young diagram, respactivly (as in Fig(III.6)).

We are aware that rectangular T-operators are non-zero polynomials in u due to (III.3.2). Furthermore, rectangular T-operators can only be discribe with other rectangular T-operators, this can be demonstrated through the fat hook relation:

$$\mathbb{L} = \left\{ (a, s) \in \mathbb{N} \times \mathbb{Z} \quad \middle| \quad s \geqslant 0, \quad \textit{and } 0 \leqslant a \leqslant \mathsf{r} - 3 \right\} \tag{III.3.15}$$

The explicit computations for this process are exceedingly immense, and they are detailed in Appendix(B.2). To provide a brief overview of our approach, we start with expanding the determinant in (III.3.4) and extracting the coefficients of each individual power of the spectral parameter. Subsequently, we endeavor to condense these coefficients, expressing them as determinants of T-operators of lengths (N-1), denoted as $\overline{\mathsf{T}_{(1,s)}}(u)$. However, We have observed that not all coefficients will simplify to our naive guess, resulting in our expression for general rectangular T-operators becoming exceedingly bulky.

The rationale behind this approach was to emulate [KV08], where they define a Coderivative expression of T-operators with a general representation of $GL(\mathsf{n})$ in the auxiliary space. In

[KV08], the expansion of the rectangle T-operators truncated after the u^{-1} and the Coderivative operator adheres to the Leibniz rule. However, in $SO(2\mathsf{r})$ expansion is truncated after u^{-2}, yielding more terms to attempt to compactify, and the operator $\left(\mathbf{D}^2\right)'$ does not adhere to the Leibniz rule (which is also detailed in Appendix(B.2)).

After a careful computation, detailed in Appendix(B.2), the description of general rectangular T-operators takes the form:

$$
\begin{aligned}
\mathbf{T}_{a,s}(u) = \bigotimes_{i=1}^{N} & \left(u_i^{[\kappa-a]} u_i^{[-\kappa+a]} - \frac{1}{8}\operatorname{Tr}\left(\mathbf{D}^2\right) + u_i^{[\kappa]}\mathbf{D} + \frac{1}{2}\left(\mathbf{D}^2\right)' \right) \det_{1\leqslant i,j\leqslant a}\left(\chi_{1,s+i-j}\right) \\
& + \frac{1}{2\prod_{n>1}^{N}\prod_{k=1}^{a-1} u_n^{[\kappa+s-a+2k]} u_n^{[-\kappa-s+a-2k]}} \sum_{\sigma\in S_a} (-1)^{\sigma} \Bigg[\\
& \sum_{k=1}^{a}\prod_{i_1=1}^{k-1} \overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u) \left((k+\sigma(k)-a-1)\,\mathbf{D}\overline{\mathsf{T}^{[a-k-\sigma(k)+1]}_{1,s+k-\sigma(k)}}(u) \right) \prod_{i_2=k+1}^{a} \overline{\mathsf{T}^{[a-i_2-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u) \times \\
& + \sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a} \left(\prod_{i_1=1}^{k_1-1} \overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u) \left(\mathbf{D}\overline{\mathsf{T}^{[a-k_1-\sigma(k_1)+1]}_{1,s+k_1-\sigma(k_1)}}(u) \right) \prod_{i_2=k_1+1}^{k_2-1} \overline{\mathsf{T}^{[a-i_2-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u) \right. \\
& + \sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a} \left(\prod_{i_1=1}^{k_1-1} \overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u) \left(\mathbf{D}\overline{\mathsf{T}^{[a-k_1-\sigma(k_1)+1]}_{1,s+k_1-\sigma(k_1)}}(u) \right) \prod_{i_2=k_1+1}^{k_2-1} \overline{\mathsf{T}^{[a-i_2-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u) \times \right. \\
& \qquad \left. \left(\mathbf{D}\overline{\mathsf{T}^{[a-k_2-\sigma(k_2)+1]}_{1,s+k_2-\sigma(k_2)}}(u) \right) \prod_{i_3=k_2+1}^{a} \overline{\mathsf{T}^{[a-i_3-\sigma(i_3)+1]}_{1,s+i_3-\sigma(i_3)}}(u) \right. \\
& - \sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a} \left(\prod_{i_1=1}^{k_1-1} \overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u) \mathbf{D}\overline{\mathsf{T}^{[a-k_1-\sigma(k_1)+1]}_{1,s+k_1-\sigma(k_1)}}(u) \prod_{i_2=k_1+1}^{k_2-1} \overline{\mathsf{T}^{[a-i_2-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u) \times \right. \\
& \qquad \left. \left. \left(\mathbf{D}\overline{\mathsf{T}^{[a-k_2-\sigma(k_2)+1]}_{1,s+k_2-\sigma(k_2)}}(u) \right) \prod_{i_3=k_2+1}^{a} \overline{\mathsf{T}^{[a-i_3-\sigma(i_3)+1]}_{1,s+i_3-\sigma(i_3)}}(u) \right)^{p_1} \right]
\end{aligned}
\tag{III.3.16}
$$

Where p_1 denotes the priming of the first quantum space entries of the matrix coefficients. As previously mentioned, for a general rectangular T-operators, the coefficient of the leading order in u is not equivalent to the character of the rectangular representation in $so(2\mathsf{r})$, but rather to the character of the Yangian irreducible representations which are composed of the characters of the symmetric representation of the $so(2\mathsf{r})$ algebra. This can be observed from the expression $\det_{1\leqslant i,j\leqslant a}\left(\chi_{1,s+i-j}\right)$. It is noteworthy that when considering a spin chain with only one particle, the non-compact terms cancel out, as evident from $\overline{\mathsf{T}_{(1,s)}}(u) = \chi_s$. This enables our general T-operators to be sufficiently compact for expression solely in terms of the Coderivative operator.

III.4 Conclusion

The methods described in the manuscript allow to generalize to $so(2\mathsf{r})$ spin chains a construction that had been introduced for $gl(n)$ spin chains in [KV08] and [KLT12]. This construction uses Coderivatives, and the derivation that we provide for the expressions of Q-operators and nested T-operators is suitable for generalizations from $gl(n)$ to $so(2\mathsf{r})$. We also showed that it is equivalent to the construction by [BLMS10, BFL$^+$11, FKT21] who defined them as traces over infinite-dimensional representations.

When we apply this construction to the T- and Q-operators of $so(2\mathsf{r})$ spin chain, they explicitly read as combinatoric expressions written as sums of diagrams, but explicit computations (by "brute force") turned out to be much more complicated than in the $GL(\mathsf{n})$ case.

Moreover, there are some representations (in the auxiliary space), for which the R-matrix is not explicited in the literature and for which Coderivatives have so far failed to express the T-operators, except as determinants of other T- or Q-operators. We expect that in order to express these T-operators (without refering to other T- or Q-operators), other ingredients than the Coderivative are necessary. A key piece of evidence lies in the leading coefficient of the T-operators, which correspond to a reducible representation of $so(2\mathsf{r})$.

It would be interesting to explicit the R-matrix of rectangular representations, following the steps of [KK21b], in order to find out what generators allow to build the R-matrix. This may allow to find another operator than the present Coderivative, out of which the R-matrix of rectangular representations could be expressed. Interestingly enough, the method used to deduce Q-operators from T-operators (in particular Sec(II.2.1) and Sec(II.2.2.3)) uses very few properties of the Coderivatives (mostly its linearity) and should still apply if another operator is added to (or replaces) the Coderivative.

Appendix A

Notations2

A.1 Yangian Algebra

I want in this section to discuss the relationship between the classical Lie algebra and the Yangian algebra. The Yangian forms a remarkable family of quantum groups related to the rational solutions of the classical Yang-Baxter equation, where the defining relation of the Yangian can be written in the form of an RTT-relation. This was done for the Lie algebra $\mathfrak{g}$, and $\mathcal{Y}(\mathfrak{g})$ is the Yangian for that Lie algebra, which was developed in the frame work of the quantum inverse scattering method by L.D. Faddeev and St. Ptersburg.

We will show the defining relation of the Yangian and how they allow a special algebraic technique The R-matrix formalism to be used in studying and describing the structure of the representation of these algebras[1].

A.1.1 Yangian for $gl(n)$

The definition of the Yangian $\mathcal{Y}(gl(n))$ can be motivated by a generalization of the defining relation of the Lie algebra $gl(n)$. Thus, by taking the generators $J_{i,j}$ of $gl(n)$ in their defining representation, they obey the commutation relation:

$$[J_{i,j}, J_{k,l}] = \delta_{k,j} J_{i,l} - \delta_{i,l} J_{k,j} \tag{A.1.1}$$

It is also known that the powers of the generators J also satisfy the commutation relation:

$$[J_{i,j}, (J^s)_{k,l}] = \delta_{k,j} (J^s)_{i,l} - \delta_{i,l} (J^s)_{k,j} \tag{A.1.2}$$

Proof. (A.1.2) By induction.

Case $s = 1$ is satisfied due to (A.1.1). By the induction hypothesis, we have (A.1.2), and we show that the case $(s = s + 1)$ is also satisfied:

$$\begin{aligned}
\left[J_{i,j}, (e^{s+1})_{k,l}\right] &= [J_{i,j}, (e^s)_{k,m}] J_{m,l} + (e^s)_{k,m} [J_{i,j}, J_{m,l}] \\
&= \left(\delta_{k,j} (e^s)_{i,l} - \delta_{i,l} (e^s)_{k,j}\right) J_{m,l} + (e^s)_{k,m} \left(\delta_{k,j} J_{i,l} - \delta_{i,l} J_{k,j}\right) \\
&= \delta_{k,j} (e^{s+1})_{i,l} - \delta_{i,l} (e^{s+1})_{k,j}
\end{aligned}$$

[1] For a more detailed discussion about Yangian algebra, check out [Mol07, KS95]

□

A generalization of (A.1.2), which is less known and can be proven by induction, takes the form:

$$\left[(J^{r+1})_{i,j},(J^s)_{k,l}\right]-\left[(J^r)_{i,j},(J^{s+1})_{k,l}\right]=(J^r)_{k,j}(J^s)_{i,l}-(J^s)_{k,j}(J^r)_{i,l} \tag{A.1.3}$$

Now we are ready to introduce the definition of the Yangian $\mathcal{Y}(gl(n))$. Replacing J^r in (A.1.3) with an abstract generator $t_{i,j}^{(r)}$, we obtain the Yangian defining relation.

Definition A.1.1. *The Yangian $\mathcal{Y}(gl(n))$ is a unital associative algebra over $\mathbb{C}$ with countably many generators $t_{i,j}^{(1)},t_{i,j}^{(2)},\ldots$ where $i,j=1,2,\ldots,\mathsf{n}$ and the defining relation:*

$$\left[t_{i,j}^{(r+1)},t_{k,l}^{(s)}\right]-\left[t_{i,j}^{(r)},t_{k,l}^{(s+1)}\right]=t_{k,j}^{(r)}t_{i,l}^{(s)}-t_{k,j}^{(s)}t_{i,l}^{(r)} \tag{A.1.4}$$

where $r,s=0,1,\ldots$ and $t_{i,j}^{(0)}=\delta_{i,j}$

With this definition and the generating series:

$$t_{i,j}(u)=\sum_{r=0}^{\infty}t_{i,j}^{(r)}u^{-r} \tag{A.1.5}$$

we can write (A.1.4) in the form:

$$\left[t_{i,j}(u),t_{k,l}(v)\right]=\frac{1}{(u-v)}\left(t_{k,j}(u)\,t_{i,l}(v)-t_{k,j}(v)\,t_{i,l}(u)\right) \tag{A.1.6}$$

The parameters u,v are assumed to commute with each other and the generators $t_{i,j}$.

Proposition 1. *The following relation is equivalent to* (A.1.4).

$$\left[t_{i,j}^{(r)},t_{k,l}^{(s)}\right]=\sum_{a=1}^{min(r,s)}\left(t_{k,j}^{(a-1)}t_{i,l}^{(r+s-a)}-t_{k,j}^{(r+s-a)}t_{i,l}^{a-1}\right) \tag{A.1.7}$$

Proof. By substituting the generating series of the Yangian (A.1.5) and $\frac{1}{u-v}=\sum_{k\geqslant 0}u^{-1-k}v^k$ in (A.1.7), then collecting the coefficient of $u^{-r}v^{-s}$ with $r\leqslant s$, we find:

$$\left[t_{i,j}^{(r)},t_{k,l}^{(s)}\right]=\sum_{a=1}^{r}\left(t_{k,j}^{(a-1)}t_{i,l}^{(r+s-a)}-t_{k,j}^{(r+s-a)}t_{i,l}^{a-1}\right)$$

and for the case $r>s$, we find:

$$\left[t_{i,j}^{(r)},t_{k,l}^{(s)}\right]=\sum_{a=s+1}^{r}\left(t_{k,j}^{(a-1)}t_{i,l}^{(r+s-a)}-t_{k,j}^{(r+s-a)}t_{i,l}^{a-1}\right)$$

The last two expressions agree with (A.1.7) in their respective cases, thus completing the proof. □

Finally, we will define the homomorphism map ρ_{n} that allows us to connect the Yangian algebra to what we have done so far in the previous section.

Definition A.1.2. *The evaluation homomorphism:*

$$\rho_{\mathsf{n}} : t_{i,j}(u) \hookrightarrow \delta_{i,j} + J_{i,j} u^{-1} \tag{A.1.8}$$

is a surjective homomorphism $\mathcal{Y}(gl(n)) \hookleftarrow \mathcal{U}(gl(n))$. *Moreover, the map*

$$J_{i,j} \hookrightarrow t_{i,j}^{(1)} \tag{A.1.9}$$

define an embedding $\mathcal{U}(gl(n)) \hookrightarrow \mathcal{Y}(gl(n))$

It's easy to see that if one takes the case $r = s = 1$ in (A.1.7), we get:

$$\left[t_{i,j}^{(1)}, t_{k,l}^{(1)}\right] = \delta_{k,j} t_{i,l}^{(1)} - \delta_{i,l} t_{k,j}^{(1)}$$

and the evaluation homomorphism we recover (A.1.1).

A.1.2 Oscillators description of Q-operators

This section will be dedicated to discussing the construction of T- and Q-operators from a series of papers [BLMS10, BFL$^+$11, FKT21] that uses the oscillator representation (creation/annihilation operators) to build $\tilde{\mathcal{R}}$. This was described using the fusion relation found in [BFL$^+$11]:

$$\tilde{\mathcal{R}}_{\{1\}}\left(u + \lambda_1^{'}\right) \cdot \tilde{\mathcal{R}}_{\{2\}}\left(u + \lambda_2^{'}\right) \cdots \tilde{\mathcal{R}}_{\{\mathsf{n}\}}\left(u + \lambda_{\mathsf{n}}^{'}\right) \propto \tilde{\mathcal{R}}^{+}(u) \tag{A.1.10a}$$

$$\tilde{\mathcal{R}}^{+}(u) = u\mathbb{I} + \sum_{i,j} J_{i,j} \otimes \rho_{\lambda}^{+}(J_{j,i}) \tag{A.1.10b}$$

where $\lambda_i^{'} = \lambda_i + \frac{\mathsf{n}+2i-1}{2}$ are the shifted weights of the represent $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\mathsf{n}})$. The idea is to extract the building blocks R-matrix and define the nested T- and Q-operators as:

$$\mathcal{T}_{\{j\}}^{+,(\lambda)}(u) = \mathrm{tr}_{\lambda}\left(\tilde{\mathcal{R}}_{\{j\}}(u) \otimes \tilde{\mathcal{R}}_{\{j\}}(u) \otimes \cdots \otimes \tilde{\mathcal{R}}_{\{j\}}(u) \cdot \rho_{\lambda}^{+}(g)\right) \tag{A.1.11a}$$

$$\mathbb{Q}_{\{j\}}(u) = \mathcal{T}_{\{j\}}^{(0,0,\ldots)}(u) \tag{A.1.11b}$$

This approach was explained precisely for $GL(2)$ in [BFL$^+$11]. Where, extracting $\tilde{\mathcal{R}}$-operators from (A.1.10a) (when restricted to n=2) took the form:

$$\begin{cases} \tilde{\mathcal{R}}_{\{1\}}(u) \propto \lim\limits_{\lambda_2 \to \infty} \left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix} \cdot \tilde{\mathcal{R}}^{+}(u - \lambda_1) \right) \\ \tilde{\mathcal{R}}_{\{2\}}(u) \propto \lim\limits_{\lambda_1 \to \infty} \left(\tilde{\mathcal{R}}^{+}(u - \lambda_2) \cdot \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & 1 \end{pmatrix} \right) \end{cases} \tag{A.1.12}$$

As you can see, we need to take a limit of a normalized $\tilde{\mathcal{R}}^{+}(u)$, where the position of the normalization multiplication changed between $\tilde{\mathcal{R}}_1(u)$ and $\tilde{\mathcal{R}}_2(u)$. This shows that finding a similar prescription for an algebra with $\mathsf{n} \geqslant 3$ is not possible (due to the non-commutativity between the normalization and the $\tilde{\mathcal{R}}_{\{j\}}(u)$). Fortunately, when taking the trace of (A.1.10a) and

implementing the same process as described in [BFL$^+$11], one obtains the infinite-dimensional T-operators $\mathcal{T}^{+}_{\{j\}}(u)$ (due to the trace, there is no commutativity problem), allowing us to write:

$$\mathcal{T}^{+,(\lambda_i)}_{\{i\}}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\dots,\mathsf{n}]\!]\setminus\{i\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{\{i\}}\,\mathrm{tr}_\lambda\left(\tilde{\mathcal{R}}^{+}(u)\otimes\tilde{\mathcal{R}}^{+}(u)\otimes\cdots\otimes\tilde{\mathcal{R}}^{+}(u)\cdot\rho^{+}_{\lambda}(g)\right) \tag{A.1.13}$$

Now, we can use the same logic introduced in Sec(II.1.4) in rewriting the finite-dimensional T-operators using $\hat{D}$ on the infinite-dimensional ones defined in (A.1.13), one obtain:

$$\boxed{\mathcal{T}^{+,(\lambda_i)}_{\{i\}}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\dots,\mathsf{n}]\!]\setminus\{i\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{\{i\}}\bigotimes_{k=1}^{N}\left(u_k+\hat{D}\right)\chi^{+}_{\lambda}(g)} \tag{A.1.14}$$

In (A.1.14), we defined the infinite-dimensional nested T-operators acting on the full infinite-dimensional character, then removing all of the directions which are not i. For now we will assume a normalization that satisfies this description (details of the normalization will be in Sec(A.1.3)). Now, using the BGG relation $\left(\mathcal{T}^{(\lambda)}(u)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{\sigma}\,\mathcal{T}^{+,(\lambda)}_{\sigma(\lambda_{\mathsf{n}}+\rho_{\mathsf{n}})-\rho_{\mathsf{n}}}(u)\right)$ introduced in [BFL$^+$11], where where ρ_{n} is a constant n-component vector $\left(\rho_{\mathsf{n}}=\left(\frac{\mathsf{n}-1}{2},\frac{\mathsf{n}-3}{2},\dots,\frac{1-\mathsf{n}}{2}\right)\right)$ and (A.1.13) to define the nested finite-dimensional T-operators and (A.1.11b) to define the nested Q-operators:

$$\begin{aligned}\mathcal{T}^{(\lambda_i)}_{\{i\}}(u) &= \lim_{\substack{\mathcal{J}\in[\![1,\dots,\mathsf{n}]\!]\setminus\{i\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{\{i\}}\sum_{\sigma\in\mathcal{S}_n}(-1)^{\sigma}\,\mathcal{T}^{+,(\lambda)}_{\sigma(\lambda_{\mathsf{n}}+\rho_{\mathsf{n}})-\rho_{\mathsf{n}}}(u)\\ &= \lim_{\substack{\mathcal{J}\in[\![1,\dots,\mathsf{n}]\!]\setminus\{i\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{\{i\}}\bigotimes_{k=1}^{N}\left(u_k+\hat{D}\right)\overbrace{\sum_{\sigma\in\mathcal{S}_n}(-1)^{\sigma}\,\chi^{+}_{\sigma(\lambda_{\mathsf{n}}+\rho_{\mathsf{n}})-\rho_{\mathsf{n}}}(g)}^{\chi_\lambda(g)}\end{aligned} \tag{A.1.15a}$$

$$\mathbb{Q}_{\{i\}}(u) = \mathcal{T}^{(0,0,\dots)}_{\{i\}}(u)\Big|_{\lambda=(0,0,\dots)} \tag{A.1.15b}$$

A similar (but more interesting for later discussion) formula can be written for these operators by reformulating the character (writing the part with direction $\{i\}$ independently from the rest of the character):

$$\mathcal{T}^{+,(\lambda_i)}_{\{i\}}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\dots,\mathsf{n}]\!]\setminus\{i\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{\{i\}}\bigotimes_{k=1}^{N}\left(u_k+\hat{D}\right)\cdot\left[\left(x_i^{\lambda_i}\prod_{k<i}\frac{1}{1-x_i/x_k}\right)\chi^{+}_{\{\bar{i}\}}(g)\right] \tag{A.1.16a}$$

$$\text{Where, }\chi^{+}_{\{\bar{i}\}}(g) = \prod_{j\neq i}x_j^{\lambda_j}\prod_{\substack{1\leqslant k<j\leqslant\mathsf{n}\\ j,k\neq i}}\frac{1}{1-x_j/x_k} \tag{A.1.16b}$$

where (A.1.16b) is the nested character (removing a direction $\{i\}$ from the full character $\chi^{+}_{\lambda}(g)$). At this point, we can generalize our expressions of T-operators and Q-operators to an arbitrary nesting level I:

$$\mathcal{T}^{+,(\lambda_I)}_{I}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\mathsf{n}]\!]\setminus I\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_{I}\bigotimes_{k=1}^{N}\left(u_k+\hat{D}\right)\left[\left(\prod_{i\in I}x_i^{\lambda_i}\prod_{\substack{i\in I\\ k\notin I}}^{k<i}\frac{1}{1-x_i/x_k}\right)\chi^{+}_{I}(g)\right] \tag{A.1.17a}$$

$$\text{Where, } \chi_I^+(g) = \prod_{i\notin I} x_i^{\lambda_i} \prod_{\substack{1\leqslant k<i\leqslant \mathsf{n}\\ i,k\notin I}} \frac{1}{1-x_i/x_k} \tag{A.1.17b}$$

And similar to (A.1.15a) and (A.1.15b), the finite-dimensional nested T- and Q-operators take the form:

$$\boxed{\mathcal{T}_I^{(\lambda_I)}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\ldots,\mathsf{n}]\!]\setminus\{I\}\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_I \bigotimes_{k=1}^{N}\left(u_k+\hat{D}\right) \sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma \left[\left(\prod_{i\in I} x_j^{\lambda_i} \prod_{\substack{i\in I\\ k\notin I}}^{k<i} \frac{1}{1-x_i/x_k}\right)\chi_I^+(g)\right]} \tag{A.1.18a}$$

$$\boxed{\mathbb{Q}_I(u)\,\mathcal{T}_I^{(0,0,\ldots)}(u)} \tag{A.1.18b}$$

A.1.3 Normalization

In the discussion about the construction of T- and Q-operators from the oscillators representation, we have left out the definition of the normalizations that will allow our argument in Sec(A.1.2) to be satisfied. To define the normalization, we need to study the object $\mathcal{T}_I^{+,(\lambda)}(u)$ in (A.1.17) for finite lengths ($N=0,1$ for example) to build an idea about the normalization. Starting with ($N=0$), equation (A.1.17) becomes:

$$\mathcal{T}_I^{+,(\lambda)}(u) = \lim_{\substack{\mathcal{J}\in[\![1,\mathsf{n}]\!]\setminus I\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_I \quad \chi_\lambda^+(g) \tag{A.1.19}$$

We know that the last expression should give us the infinite-dimensional nested character, which means that our normalization should nullify the other parts of the full character:

$$\boxed{\mathcal{N}_I^{(N=0)} = \frac{1}{\chi_I^+(g)}} \tag{A.1.20}$$

The last equation ensures the non-existence of any directions other than the j^{th} one in the nested character[2]. Next, let us check (A.1.14) for ($N=1$):

$$\lim_{\substack{\mathcal{J}\in[\![1,\mathsf{n}]\!]\setminus I\\ \forall k\in\mathcal{J},\lambda_k\to\infty}} \mathcal{N}_I \begin{pmatrix} u+\lambda_1+* & & \\ & \ddots & \\ & & u+\lambda_{\mathsf{n}}+* \end{pmatrix} \chi_\lambda^+(g) \tag{A.1.21}$$

where ($*$) denotes factors that are not essential to our discussion. The limit will produce a ($\mathsf{n}-|I|$) infinites (in this limit, all ($\lambda_k\in[\![1,\mathsf{n}]\!]\setminus I$) are infinites). Thus, the normalization should nullify all these infinities and remove the non-desirable factors from the character as follows:

$$\mathcal{N}_I^{(N=1)} = \frac{1}{\chi_I^+(g)}\left(\prod_{k\in I}\lambda_k^{-1}\right) \tag{A.1.22}$$

For higher lengths ($N\geqslant 2$), the degree of λ is sector-dependent. To clarify, each entry in the matrix will have a mixture of different powers of λ_i's and other factors ($*$), and each power of

[2]Equation (A.1.20) is length-independent (this factor will appear in the normalization for any length).

λ_i will be equivalent to the total spin number in the i^{th}'s directions. To nullify these infinites, we multiply with a normalization:

$$\mathcal{N}_I^{(N\geqslant 2)} = \frac{1}{\chi_I^+(g)} \left(\prod_{k\in I} \lambda_k^{-m_k} \right) \tag{A.1.23}$$

where m_k is the total spin number in the direction k (See Sec(II.2.2.3) for more details on m_k).

Appendix B

Coderivative and its properties

B.1 Coderivative $\hat{D}$ and the eigenvalues

In this section, we are going to define the action of the Coderivative on the eigenvalues and the eigenvectors (x_i, p_i), respectively. The eigenvalues are the elements that describe the character of a representation, whereas the eigenvectors are the projectors into the eigenspace of the twist associated with the eigenvalues.

The Coderivative was introduced in [KV08], and in [KLT12] they showed how one can use it on a combination of the generating series of the symmetric character, from which they defined a differential expression for T- and Q-operators, then from the Wronskian relation describing T-operators with arbitrary representation from the symmetric ones, they obtain an expression using Coderivative on the character of these representation. In this section, we will motivate the action of the Coderivative on the character when built as a combination of eigenvalues x_i's; this will be particularly important for the results obtained in Sec(II.2) and Sec(III.2).

To answer this question, one needs to define $\hat{D}x_i$. Since the Coderivative acts on the function of the twist g (II.1.28), one should try and write the twist in terms of the eigenvalues. To do that, first, we are going to assume a diagonal twist for simplicity, then let us take the trace of the general twist (g^m):

$$\mathrm{Tr}(g^m) = \sum_{i=1}^{\mathsf{n}} x_i^m \tag{B.1.1}$$

Using the previous equation and the fact that $\left(\hat{D} \otimes \mathrm{Tr}\left(g^k\right) = kg^k\right)$ [1], one obtains:

$$mg^m = \sum_i m\left(\hat{D}x_i\right) x_i^{m-1} \tag{B.1.2}$$

[1] Check [KV08] for the action of $\hat{D}$ on the twist.

The matrix representation of the last equation with different powers n takes the form:

$$\begin{pmatrix} g \\ g^2 \\ \vdots \\ g^{\mathsf{n}} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & ... & 1 & 1 \\ x_1 & x_2 & ... & x_{\mathsf{n}-1} & x_{\mathsf{n}} \\ & & \vdots & & \\ x_1^{\mathsf{n}} & x_2^{\mathsf{n}} & ... & x_{\mathsf{n}-1}^{\mathsf{n}} & x_{\mathsf{n}}^{\mathsf{n}} \end{pmatrix}}_{\text{Vandermonde matrix}} \begin{pmatrix} \hat{D}x_1 \\ \hat{D}x_2 \\ \vdots \\ \hat{D}x_{\mathsf{n}} \end{pmatrix} \tag{B.1.3}$$

Finally, using the inverse of the Vandermonde determinant [2], one finds the action of Coderivative on the eigenvalue x_i to be:

$$\hat{D}x_i = g \prod_{\substack{1 \leqslant k \leqslant \mathsf{n} \\ k \neq i}} \frac{(g - x_k)}{(x_i - x_k)} \tag{B.1.4}$$

The previous relation tells us that all of the directions in the twist are zeros except the i^{th}-direction. For example, let us take the simple case of a diagonal twist of $GL(2)$. Then (B.1.4) reads:

$$\begin{cases} \hat{D}x_1 = \begin{pmatrix} x_1 \left(\frac{x_1 - x_2}{x_1 - x_2}\right) & 0 \\ 0 & x_2 \left(\frac{x_2 - x_2}{x_1 - x_2}\right) \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix} \\ \hat{D}x_2 = \begin{pmatrix} -x_1 \left(\frac{x_1 - x_1}{x_1 - x_2}\right) & 0 \\ 0 & -x_2 \left(\frac{x_2 - x_1}{x_1 - x_2}\right) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x_2 \end{pmatrix} \end{cases} \tag{B.1.5}$$

This can be generalized to $GL(\mathsf{n})$, and it reads:

$$\boxed{\hat{D}x_i = p_i x_i} \tag{B.1.6}$$

This is a unique solution, where p_i is the projector into the eigenspace associated with the eigenvalue x_i. Now, to have successive actions of Coderivative on the characters, one should define the $\hat{D} \otimes p_i$. Following the same idea we used in defining $\hat{D}x_i$, we look for the action of Coderivative on the general twist g^m:

$$\hat{D} \otimes g^m = \sum_i^{\mathsf{n}} \left(m \left(p_i \otimes p_i \right) x_i^m + \left(\hat{D} \otimes p_i \right) x_i^m \right) \tag{B.1.7}$$

Using (II.1.28), we can show that $\hat{D} \otimes g^m = \mathcal{P}\left(\sum_{k=0}^{m-1} \left(g^k \otimes g^{m-k} \right) \right)$. The matrix representation of the previous relationship has the form:

$$\begin{aligned} \begin{pmatrix} \mathcal{P}\left(1 \otimes g\right) \\ \mathcal{P}\left(1 \otimes g^2 + g \otimes g\right) \\ \vdots \\ \mathcal{P}\left(\sum_{k=0}^{m-1} \left(g^k \otimes g^{m-k} \right) \right) \end{pmatrix} &= \begin{pmatrix} x_1 & x_2 & \dots & x_{\mathsf{n}} \\ 2x_1^2 & 2x_2^2 & \dots & 2x_{\mathsf{n}}^2 \\ & & \vdots & \\ mx_1^m & mx_2^m & \dots & mx_{\mathsf{n}}^m \end{pmatrix} \begin{pmatrix} p_1 \otimes p_1 \\ p_2 \otimes p_2 \\ \vdots \\ p_{\mathsf{n}} \otimes p_{\mathsf{n}} \end{pmatrix} \\ &+ \begin{pmatrix} x_1 & x_2 & \dots & x_{\mathsf{n}} \\ x_1^2 & x_2^2 & \dots & x_{\mathsf{n}}^2 \\ & & \vdots & \\ x_1^m & x_2^m & \dots & x_{\mathsf{n}}^m \end{pmatrix} \begin{pmatrix} \hat{D} \otimes p_1 \\ \hat{D} \otimes p_2 \\ \vdots \\ \hat{D} \otimes p_{\mathsf{n}} \end{pmatrix} \end{aligned} \tag{B.1.8}$$

[2] This is true when considering the twist to have pairwise distinct eigenvalues.

Taking the inverse of the two Vandermondes, one obtains the complicated expression [3]:

$$\hat{D}\otimes p_i=\sum_{m=1}^{\mathsf{n}}(-1)^{n-m}\sum_{\substack{1\leqslant j_1<\cdots<j_m\leqslant m\\ j_1,j_2,\ldots,j_m!=i}}\frac{x_{j_1}\ldots x_{j_m}}{x_i\prod_{\substack{k=1\\k\neq i}}^{n}(x_i-x_k)}\Bigg(\mathcal{P}\sum_{k=0}^{m-1}g^k\otimes g^{m-k}- (n-m)\sum_{l=1}^{n}(p_j\otimes p_j)x_j^m\Bigg) \tag{B.1.9}$$

As stated, this expression is very bulky. On the other hand, we have a more compact guessed expression of the same object, and it reads:

$$\hat{D}\otimes p_i=\mathcal{P}\left(\mathbb{I}\otimes g\right)\cdot\left[\frac{1-p_i}{x_i-g}\otimes p_i+p_i\otimes\frac{1-p_i}{x_i-g}\right] \tag{B.1.10}$$

This expression can be obtained by applying $\hat{D}$ on the projectors p_i in every direction i, but it is an extremely complicated process to do for high-ranked algebra. We will show that this guessed statement is compatible with our solution for $\hat{D}p_i$ by replacing $\hat{D}\otimes p_i$ in (B.1.7) with (B.1.10) and then asking if the following equation:

$$\sum_{i=1}^{\mathsf{n}}\Bigg(\mathcal{P}\left(\mathbb{I}\otimes g\right)\cdot\left[\frac{1-p_i}{x_i-g}\otimes p_i+p_i\otimes\frac{1-p_i}{x_i-g}\right]x_i^m+m\left(p_i\otimes p_i\right)x_i^m\Bigg)- \mathcal{P}_{1,2}\left(\sum_{k=0}^{m-1}\left(g^k\otimes g^{m-k}\right)\right)=0 \tag{B.1.11}$$

is equal to zero on the general basis vectors $(|a,a\rangle,|a,b\rangle)$:

- When (B.1.11) acts on the vector $|a,a\rangle$:

$$\left(0+mx_a^m\right)|a,a\rangle-\sum_{k=0}^{m-1}\overbrace{x_a^kx_a^{m-k}}^{x_a^{\mathsf{n}}}|a,a\rangle=0 \tag{B.1.12}$$

- When (B.1.11) acts on the vector $|a,b\rangle$

$$\left(x_b\left(\frac{x_b^m}{x_b-x_a}+\frac{x_a^m}{x_a-x_b}\right)+0\right)|b,a\rangle-\sum_{k=0}^{m-1}\left(x_a^kx_b^{m-k}\right)|b,a\rangle\stackrel{?}{=}0 \tag{B.1.13}$$

  where $\mathcal{P}\,|a,b\rangle=|b,a\rangle$, and the projector operators p assign a direction to the expression; for example, when the expression $\frac{1-p_i}{x_i-g}\otimes p_i$ acts on the sate $|a,b\rangle$, it means that p_i takes only one term from the sum, and that is when $i=b$ (more or less like a δ-function). One also knows that $\sum_{k=0}^{m-1}\left(x_a^kx_b^{m-k}\right)=x_b\left(\frac{x_a^m-x_b^m}{x_a-x_b}\right)$, proving that (B.1.13) is equal to zero.

[3] There is an even more complicated way of defining this relation, by dividing (B.1.6) with an x_i and then taking the Coderivative.

This proves that our guessed formula (B.1.10) is indeed an expression that is compatible with (B.1.9).

Notice that each factor $\frac{1-p_i}{x_i-g}$ may look singular because $\frac{1}{x_i-g}$ is infinite on the eigenspace associated to x_i. But the numerator $1-p_i$ is also zero on this eigenspace, so that we get $\frac{0}{0}$. In fact, expression (B.1.10) is valid if we take the convention $\frac{0}{0}=0$, i. e. the abusive notation $\frac{1-p_i}{x_i-g}$ is defined by $\frac{1-p_i}{x_i-g}p_j = \begin{cases} 0, \text{ if } i=j \\ \frac{1}{x_i-x_j}p_j, \text{ else.} \end{cases}$

B.1.1 Yang-Baxter equation and Coderivative.

This section will be dedicated to explaining and finding the coefficients of (II.1.34) in Sec(II.1.5). This will be done by writing (II.1.34) with the auxiliary indices, then drawing the equation for simplicity.

The formula with auxiliary indices take the form:

$$\mathcal{R}^{i_1,i_2}_{k_1,k_2}(u-v)\,\tilde{\mathcal{R}}(u)^{k_1,\alpha_1}_{j_1,\alpha_2}\tilde{\mathcal{R}}(v)^{k_2,\alpha_2}_{j_2,\alpha_3} = \tilde{\mathcal{R}}(v)^{i_2,\alpha_1}_{k_2,\alpha_2}\tilde{\mathcal{R}}(u)^{i_1,\alpha_2}_{k_1,\alpha_3}\mathcal{R}^{k_1,k_2}_{j_1,j_2}(u-v) \tag{B.1.14}$$

similar to the notation in Sec(II.1.5), we denoted the indices (i,j) to refer to the (outgoing, ingoing), respectively; (k)'s denote a contraction of operators on the physical space. Whereas the (α) labels the auxiliary indices, notice that in the previous equation, we have a contraction on the auxiliary space represented by (α_2). The diagrammatical representation of (B.1.14) is shown in Fig(B.1).

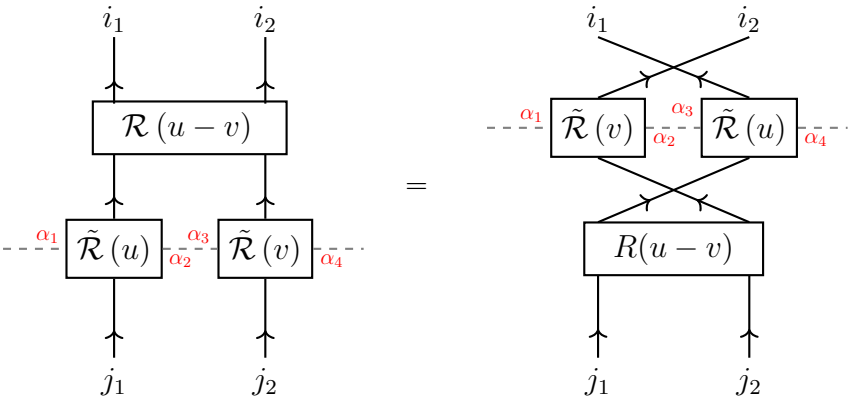


Figure B.1: This is the diagrammatical representation of (B.1.14). The gray dashed line represents the auxiliary space, and (α)'s govern the operator contraction with the auxiliary space. The black straight lines are the representation of the physical spaces, which are labeled by (i,j), and (k)'s govern the contraction of the operators with the physical space.

In (B.1.14) and from Fig(B.1), we notice that the left hand side has a swapping between its outgoing indices and the contracted indices as well, which means the existance of a permutation operator[4]. We can swap the positions of the two operators $\tilde{\mathcal{R}}(u)$ and $\tilde{\mathcal{R}}(v)$ in the physical space

[4]Permutations operators are represented diagrammatically as a cross.

using the identity:

$$\mathcal{P}\,(A\otimes B)\,\mathcal{P} = B\otimes A$$

Now, we need to take into account the way these operators contract with the auxiliary space after the swapping. The same operators are contracted to the same positions in the auxiliary space, making the unpleasant figure:

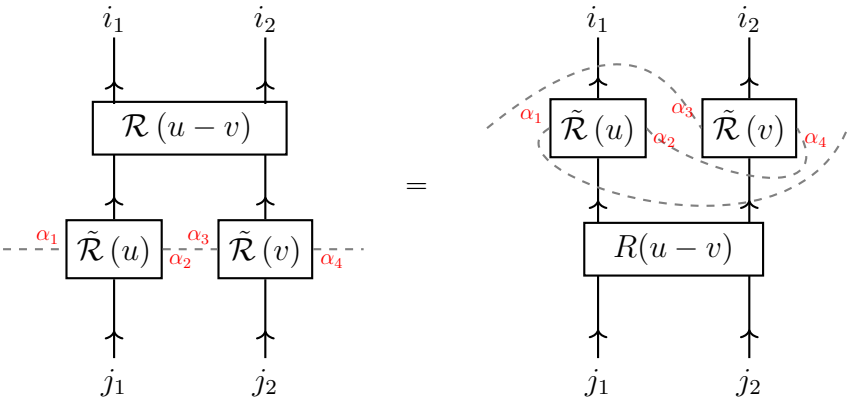


Figure B.2: This is the diagrammatical result of swapping the operators on left hand side of Fig(B.1). The difference between the auxiliary space on both sides of the diagram is due to the swapping of the two operators in the physical space and keeping the same contraction with the auxiliary space.

In order to find the relations imposed by Yang-Baxter equation on the Coderivative, we will write $\tilde{\mathcal{R}}\,(u)$ and $\tilde{\mathcal{R}}\,(v)$ using (II.1.33a). This will impose a swapping in the physical spaces that come from (II.1.31), and using the permutation identity, one will end up with Fig(B.3).

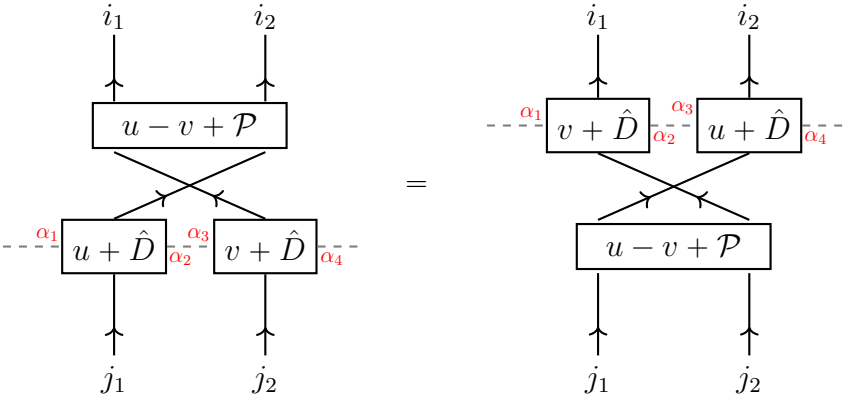
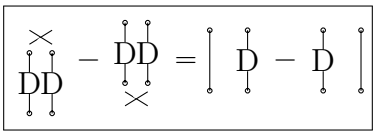


Figure B.3: A diagrammatical representation of (B.1.14) after replaceing the operators with their $\hat{D}$. Notice that we are ommiting the identity operator $\mathbb{I}$

Now, we are ready to find all non-trivial coefficient from Fig(B.3). Fortunately, there is only one non-trivial coefficient that can be found, and it comes from the spectral parameters of powers u^1 or v^1:



(B.1.15)

The proof of (B.1.15) will be discussed in Sec(II.1.5).

B.2 Comutations of rectangular T-operators in Sec(III.3.3)

This section will detail the mathematical steps needed to derive (III.3.16). Start by writing the shifted symmetry T-operators in the numerator and denominator of (III.3.4) using the convention introduced in Sec(II.2), the read as:

$$\mathtt{T}_{1,s}\left(u\right)=\bigotimes_{i=1}^{N}\left[u_{i}^{[\kappa+s-1]}u_{i}^{[-\kappa-s+1]}-\frac{s}{2}+u_{i}^{[\kappa]}\mathbf{D}+\frac{1}{2}\left(\mathbf{D}^{2}\right)^{'}\right]\cdot\chi_{s} \tag{B.2.1a}$$

$$\prod_{n=1}^{N}\prod_{k=1}^{a-1}\mathtt{T}_{0,s}^{[a-2k]}\left(u\right)=\prod_{i=1}^{N}\prod_{k=1}^{a-1}\left(u_{i}^{[\kappa+s-a+2k]}\right)\left(u_{i}^{[-\kappa-s+a-2k]}\right) \tag{B.2.1b}$$

As stated in Sec(III.3.3), we will write T-operators of length N as a polynomial of degree one in the spectral parameter $\left(\left(u_{1}^{[\kappa+s-1]}\right)\left(u_{1}^{[-\kappa-s+1]}\right)-\frac{s}{2}+u_{1}^{[\kappa]}\mathbf{D}+\frac{1}{2}\left(\mathbf{D}^{2}\right)^{'}\right)$ that acts on T-operators of length $(N-1)$, where the latter will be referred to as $\overline{\mathtt{T}_{1,s}}\left(u\right)$:

$$\mathtt{T}_{1,s}\left(u\right)=\left[u_{1}^{[\kappa+s-1]}u_{1}^{[-\kappa-s+1]}-\frac{s}{2}+u_{1}^{[\kappa]}\mathbf{D}+\frac{1}{2}\left(\mathbf{D}^{2}\right)^{'}\right]\cdot\overline{\mathtt{T}_{1,s}}\left(u\right) \tag{B.2.2}$$

With that, we can factor out $u_{1}^{[\kappa+s-a]}u_{1}^{[-\kappa-s+a]}$ from the determinant, then expand the determinant, which will give:

$$\begin{aligned}\mathtt{T}_{a,s}\left(u\right)=&\frac{u_{1}^{[\kappa+s-a]}u_{1}^{[-\kappa-s+a]}}{\prod_{n>1}^{N}\prod_{k=1}^{a-1}u_{n}^{[\kappa+s-a+2k]}u_{n}^{[-\kappa-s+a-2k]}}\sum_{\sigma\in S_{a}}\left(-1\right)^{\sigma}\prod_{i=1}^{a}\Bigg(\Bigg\{1+\\&\frac{u_{1}^{[a-i-\sigma(i)+r]}\mathtt{D}+\frac{1}{2}\left(\mathtt{D}^{2}\right)^{'}-\frac{s+i-\sigma(i)}{2}}{u_{1}^{[\kappa+s+a-2\sigma(i)]}u_{1}^{[-\kappa-s+a-2i+2]}}\Bigg\}\overline{\mathtt{T}_{1,s+i-j}^{[a-i-j+1]}}\left(u\right)\Bigg)\end{aligned} \tag{B.2.3}$$

We are going to drop the notation $\otimes$ between $\mathbf{D}$ and T-operators for simplicity. Due to the commutativity between T-operators, the last expression can be written as:

$$\begin{aligned}\mathtt{T}_{a,s}\left(u\right)=&\frac{u_{1}^{[\kappa+s-a]}u_{1}^{[-\kappa-s+a]}}{\prod_{n>1}^{N}\prod_{k=1}^{a-1}u_{n}^{[\kappa+s-a+2k]}u_{n}^{[-\kappa-s+a-2k]}}\Bigg\{\det_{1\leqslant i,j\leqslant a}\left(\overline{\mathtt{T}_{1,s+i-j}^{[a-i-j+1]}}\left(u\right)\right)\\&+\frac{1}{u_{1}^{2}}\sum_{\sigma\in S_{a}}\left(-1\right)^{\sigma}\sum_{k=1}^{a}\prod_{i_{1}\neq k}^{k-1}\overline{\mathtt{T}_{1,s+i_{1}-\sigma(i_{1})}^{[a-i_{1}-\sigma(i_{1})+1]}}\left(u\right)\times\\&\frac{\left(u_{1}^{[a-k-\sigma(k)+r]}\mathbf{D}+\frac{1}{2}\left(\mathbf{D}^{2}\right)^{'}-\frac{1}{2}\left(s+k-\sigma(k)\right)\right)\overline{\mathtt{T}_{1,s+k-\sigma(k)}^{[a-k-\sigma(k)+1]}}\left(u\right)}{\left(1+\frac{1}{2u_{1}}\left(\kappa+s+a-2\sigma(k)\right)\right)\left(1-\frac{1}{2u_{1}}\left(\kappa+s-a+2k-2\right)\right)}\prod_{i_{2}=k+1}^{a}\overline{\mathtt{T}_{1,s+i_{2}-\sigma(i_{2})}^{[a-i_{1}-\sigma(i_{2})+1]}}\left(u\right)\\&+\frac{1}{u_{1}^{2}}\sum_{\sigma\in S_{a}}\left(-1\right)^{\sigma}\sum_{k_{1}=1}^{a}\sum_{k_{2}=k_{1}+1}^{a}\Bigg(\prod_{i_{1}=1}^{k_{1}-1}\overline{\mathtt{T}_{1,s+i_{1}-\sigma(i_{1})}^{[a-i_{1}-\sigma(i_{1})+1]}}\left(u\right)\mathbf{D}\overline{\mathtt{T}_{1,s+k_{1}-\sigma(k_{1})}^{[a-k_{1}-\sigma(k_{1})+1]}}\left(u\right)\prod_{i_{2}=k_{1}+1}^{k_{2}-1}\times\\&\overline{\mathtt{T}_{1,s+i_{2}-\sigma(i_{2})}^{[a-i_{1}-\sigma(i_{2})+1]}}\left(u\right)\mathbf{D}\overline{\mathtt{T}_{1,s+k_{2}-\sigma(k_{2})}^{[a-k_{2}-\sigma(k_{2})+1]}}\left(u\right)\prod_{i_{3}=k_{2}+1}^{a}\overline{\mathtt{T}_{1,s+i_{3}-\sigma(i_{3})}^{[a-i_{3}-\sigma(i_{3})+1]}}\left(u\right)\Bigg)+\mathcal{O}\left(\frac{1}{u_{1}^{3}}\right)\Bigg\}\end{aligned} \tag{B.2.4}$$

This expression is different than what the authors of [KV08] found, because our R-matrix in the $so(2\mathsf{r})$ case is of order two in the spectral parameter u, which gives us more terms in the expansion to look through. We will deal with each power of u_1 in the last expression individually.

For now, let us focus on the second term in the expansion and collect the coefficients of (u_1^{-1}, u_1^{-2}):

- For u_1^{-1}:

$$\begin{aligned}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k=1}^{a}\prod_{i_1=1}^{k-1}\overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u)\left(\mathbf{D}\overline{\mathsf{T}^{[a-k-\sigma(k)+1]}_{1,s+k-\sigma(k)}}(u)\right)\prod_{i_2=k+1}^{a}\overline{\mathsf{T}^{[a-i_1-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u)\\ =\mathbf{D}\det_{1\leqslant i,j\leqslant a}\left(\overline{\mathsf{T}^{[a-i-j+1]}_{1,s+i-j}}(u)\right)\end{aligned} \tag{B.2.5}$$

  This is a Coderivative acting on a determinant of shifted symmetric T-operators.

- For u_1^{-2}:

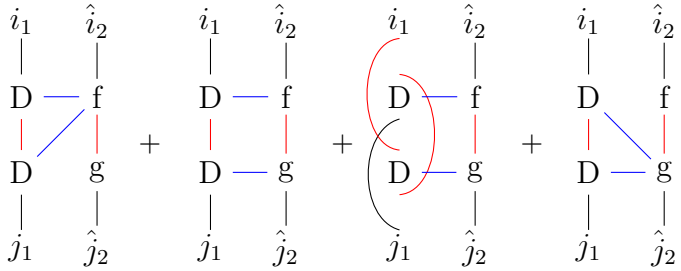
$$\begin{aligned}&\frac{1}{2}(\kappa\mathbf{D}-as)\det_{1\leqslant i,j\leqslant a}\left(\overline{\mathsf{T}^{[a-i-j+1]}_{1,s+i-j}}(u)\right)+\frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k=1}^{a}\prod_{i_1=1}^{k-1}\overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u)\times\\ &\left(\left(\mathbf{D}^2\right)'\overline{\mathsf{T}^{[a-k-\sigma(k)+1]}_{1,s+k-\sigma(k)}}(u)\right)\prod_{i_2=k+1}^{a}\overline{\mathsf{T}^{[a-i_1-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u)\\ &+\frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k=1}^{a}\prod_{i_1=1}^{k-1}\overline{\mathsf{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u)\left((k+\sigma(k)-a-1)\,\mathbf{D}\overline{\mathsf{T}^{[a-k-\sigma(k)+1]}_{1,s+k-\sigma(k)}}(u)\right)\times\\ &\qquad\prod_{i_2=k+1}^{a}\overline{\mathsf{T}^{[a-i_1-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u)\end{aligned} \tag{B.2.6}$$

  Unfortunately, this expression could not be completely expressed as a Coderivative acting on some expression of T-operators.

Notice that the factor of $(\mathbf{D}^2)'$ was not written as $\left[(\mathbf{D}^2)'\det_{1\leqslant i,j\leqslant a}\left(\overline{\mathsf{T}^{[a-i-j+1]}_{1,s+i-j}}(u)\right)\right]$, this is due to the fact that $(\mathbf{D}^2)'$ does not follow the usual Leibniz rule (III.1.13). Instead, the action of this operator on a product of N-tensor valued functions takes the form:

$$\begin{aligned}\left((\mathbf{D}^2)'\cdot(f\times g)\right)^{i_1,\dots,i_N}_{j_1,\dots,j_N}&=\mathbf{D}^{k_1}_{j_1}\mathbf{D}^{i_1}_{k_1}\left(f^{\overbrace{i_2,\dots,i_n}^{\hat{i}_2}}_{\underbrace{k_2,\dots,k_n}_{\hat{k}_2}}g^{\overbrace{k_2,\dots,k_n}^{\hat{k}_2}}_{\underbrace{j_2,\dots,j_n}_{\hat{j}_2}}\right)\\ &=\mathbf{D}^{k_1}_{j_1}\mathbf{D}^{i_1}_{k_1}f^{\hat{i}_2}_{\hat{k}_2}g^{\hat{k}_2}_{\hat{j}_2}+\mathbf{D}^{i_1}_{k_1}f^{\hat{i}_2}_{\hat{k}_2}\mathbf{D}^{k_1}_{j_1}g^{\hat{k}_2}_{\hat{j}_2}+\mathbf{D}^{k_1}_{j_1}f^{\hat{i}_2}_{\hat{k}_2}\mathbf{D}^{i_1}_{k_1}g^{\hat{k}_2}_{\hat{j}_2}+f^{\hat{i}_2}_{\hat{k}_2}\mathbf{D}^{k_1}_{j_1}\mathbf{D}^{i_1}_{k_1}g^{\hat{k}_2}_{\hat{j}_2}\end{aligned} \tag{B.2.7}$$

Which can be representationed diagrammatically as: Notice that we have referred to $l_2,\dots,l_n$ in (B.2.7) as $\hat{l}_2$ for simplicity where $l=i,j$ and k. Whereas, in the diagram, the black lines



show how an object is connected to the external points i, j, the red ones show how we contract these objects together, and the blue ones show which Coderivative acting on which object f, g. With that, we can act by $(\mathbf{D}^2)'$ on $\det\limits_{1\leqslant i,j\leqslant a}\left(\overline{\mathrm{T}^{[a-i-j+1]}_{1,s+i-j}}(u)\right)$ and write:

$$
\begin{aligned}
(\mathbf{D}^2)' \det_{1\leqslant i,j\leqslant a}\left(\overline{\mathrm{T}^{[a-i-j+1]}_{1,s+i-j}}(u)\right) &= \sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k=1}^{a}(\mathbf{D}^2)'\overline{\mathrm{T}^{[a-k-\sigma(k)+1]}_{1,s+k-\sigma(k)}}(u)\prod_{i\neq k}^{a}\overline{\mathbf{T}^{[a-i-\sigma(i)+1]}_{(1,s+i-\sigma(i))}}(u)\\
&+\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\left(\prod_{i_1=1}^{k_1-1}\overline{\mathrm{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u)\left(\mathbf{D}\overline{\mathrm{T}^{[a-k_1-\sigma(k_1)+1]}_{1,s+k_1-\sigma(k_1)}}(u)\right)\prod_{i_2=k_1+1}^{k_2-1}\overline{\mathrm{T}^{[a-i_1-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u)\times\right.\\
&\left.\left(\mathbf{D}\overline{\mathrm{T}^{[a-k_2-\sigma(k_2)+1]}_{1,s+k_2-\sigma(k_2)}}(u)\right)\prod_{i_3=k_2+1}^{a}\overline{\mathrm{T}^{[a-i_3-\sigma(i_3)+1]}_{1,s+i_3-\sigma(i_3)}}(u)\right)\\
&+\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\left(\prod_{i_1=1}^{k_1-1}\overline{\mathrm{T}^{[a-i_1-\sigma(i_1)+1]}_{1,s+i_1-\sigma(i_1)}}(u)\left(\mathbf{D}'\overline{\mathrm{T}^{[a-k_1-\sigma(k_1)+1]}_{1,s+k_1-\sigma(k_1)}}(u)\right)\prod_{i_2=k_1+1}^{k_2-1}\overline{\mathrm{T}^{[a-i_1-\sigma(i_2)+1]}_{1,s+i_2-\sigma(i_2)}}(u)\times\right.\\
&\left.\left(\mathbf{D}'\overline{\mathrm{T}^{[a-k_2-\sigma(k_2)+1]}_{1,s+k_2-\sigma(k_2)}}(u)\right)\prod_{i_3=k_2+1}^{a}\overline{\mathrm{T}^{[a-i_3-\sigma(i_3)+1]}_{1,s+i_3-\sigma(i_3)}}(u)\right)^{p_1}
\end{aligned}
\tag{B.2.8}
$$

After a tedious calculation and noticing the similarity between (B.2.8) and (B.2.4), we fine:

$$\mathbf{T}_{a,s}(u) = \frac{1}{\prod_{n>1}^{N}\prod_{k=1}^{a-1} u_n^{[\kappa+s-a+2k]}u_n^{[-\kappa-s+a-2k]}}\Bigg[\Bigg(u_1^{[\kappa-a]}u_1^{[-\kappa+a]} - \frac{1}{8}\operatorname{Tr}\left(\mathbf{D}^2\right) + u_1^{[\kappa]}\mathbf{D}$$
$$+ \frac{1}{2}\left(\mathtt{D}^2\right)'\Bigg)\det_{1\leqslant i,j\leqslant a}\left(\overline{\mathtt{T}_{1,s+i-j}^{[a-i-j+1]}}(u)\right)\Bigg]$$
$$+ \frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k=1}^{a}\prod_{i_1=1}^{k-1}\overline{\mathtt{T}_{1,s+i_1-\sigma(i_1)}^{[a-i_1-\sigma(i_1)+1]}}(u)\left((k+\sigma(k)-a-1)\,\mathbf{D}\overline{\mathtt{T}_{1,s+k-\sigma(k)}^{[a-k-\sigma(k)+1]}}(u)\right)\times$$
$$\prod_{i_2=k+1}^{a}\overline{\mathtt{T}_{1,s+i_2-\sigma(i_2)}^{[a-i_2-\sigma(i_2)+1]}}(u) + \frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\Bigg(\prod_{i_1=1}^{k_1-1}\overline{\mathtt{T}_{1,s+i_1-\sigma(i_1)}^{[a-i_1-\sigma(i_1)+1]}}(u)\times$$
$$\left(\mathbf{D}\overline{\mathtt{T}_{1,s+k_1-\sigma(k_1)}^{[a-k_1-\sigma(k_1)+1]}}(u)\right)\prod_{i_2=k_1+1}^{k_2-1}\overline{\mathtt{T}_{1,s+i_2-\sigma(i_2)}^{[a-i_2-\sigma(i_2)+1]}}(u)$$
$$+ \frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\Bigg(\prod_{i_1=1}^{k_1-1}\overline{\mathtt{T}_{1,s+i_1-\sigma(i_1)}^{[a-i_1-\sigma(i_1)+1]}}(u)\left(\mathbf{D}\overline{\mathtt{T}_{1,s+k_1-\sigma(k_1)}^{[a-k_1-\sigma(k_1)+1]}}(u)\right)\prod_{i_2=k_1+1}^{k_2-1}\overline{\mathtt{T}_{1,s+i_2-\sigma(i_2)}^{[a-i_2-\sigma(i_2)+1]}}(u)\times$$
$$\left(\mathbf{D}\overline{\mathtt{T}_{1,s+k_2-\sigma(k_2)}^{[a-k_2-\sigma(k_2)+1]}}(u)\right)\prod_{i_3=k_2+1}^{a}\overline{\mathtt{T}_{1,s+i_3-\sigma(i_3)}^{[a-i_3-\sigma(i_3)+1]}}(u)$$
$$- \frac{1}{2}\sum_{\sigma\in S_a}(-1)^{\sigma}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\Bigg(\prod_{i_1=1}^{k_1-1}\overline{\mathtt{T}_{1,s+i_1-\sigma(i_1)}^{[a-i_1-\sigma(i_1)+1]}}(u)\mathbf{D}\overline{\mathtt{T}_{1,s+k_1-\sigma(k_1)}^{[a-k_1-\sigma(k_1)+1]}}(u)\prod_{i_2=k_1+1}^{k_2-1}\overline{\mathtt{T}_{1,s+i_2-\sigma(i_2)}^{[a-i_2-\sigma(i_2)+1]}}(u)\times$$
$$\left(\mathbf{D}\overline{\mathtt{T}_{1,s+k_2-\sigma(k_2)}^{[a-k_2-\sigma(k_2)+1]}}(u)\right)\prod_{i_3=k_2+1}^{a}\overline{\mathtt{T}_{1,s+i_3-\sigma(i_3)}^{[a-i_3-\sigma(i_3)+1]}}(u)\Bigg)^{p_1} \tag{B.2.9}$$

Finally, we should do the same process for all other special parameters $(u_i)_{i\in[\![2,N]\!]}$, one find out (III.3.16).

Bibliography

[AV87] L. V. Avdeev and A. A. Vladimirov. Exceptional solutions to the bethe ansatz equations. *Theoretical and Mathematical Physics*, 69(2):1071–1079, 1987.

[Bax72] Rodney J Baxter. Partition function of the eight-vertex lattice model. *Annals of Physics*, 70(1):193–228, 1972.

[Bax80] R J Baxter. Hard hexagons: exact solution. *Journal of Physics A: Mathematical and General*, 13(3):L61, mar 1980.

[Bax82] R. J. Baxter. *Exactly solved models in statistical mechanics*. 1982.

[Bes85] N Y Beshetikhin. Hamiltonian structures for integrable field theory models. ii. models with o(n) and sp(2k) symmetry on a one-dimensional lattice. *Theor. Math. Phys.; (United States)*, 11 1985.

[Bet31] H. A. Bethe. Zur Theorie der Metalle. i. Eigenwerte und Eigenfunktionen der linearen Atomkette. *Zeit. für Physik*, 71:205, 1931.

[BFL+11] Vladimir V. Bazhanov, Rouven Frassek, Tomasz Lukowski, Carlo Meneghelli, and Matthias Staudacher. Baxter Q-Operators and Representations of Yangians. *Nucl. Phys. B*, 850:148–174, 2011.

[BKS03] N. Beisert, C. Kristjansen, and M. Staudacher. The Dilatation operator of conformal N=4 superYang-Mills theory. *Nucl. Phys. B*, 664:131–184, 2003.

[BLMS10] Vladimir V. Bazhanov, Tomasz Lukowski, Carlo Meneghelli, and Matthias Staudacher. A Shortcut to the Q-Operator. *J. Stat. Mech.*, 1011:P11002, 2010.

[BLZ97] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Integrable structure of conformal field theory. 2. Q operator and DDV equation. *Commun. Math. Phys.*, 190:247–278, 1997.

[BLZ99] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Integrable structure of conformal field theory. 3. The Yang-Baxter relation. *Commun. Math. Phys.*, 200:297–324, 1999.

[BMN02] David Eliecer Berenstein, Juan Martin Maldacena, and Horatiu Stefan Nastase. Strings in flat space and pp waves from N=4 superYang-Mills. *JHEP*, 04:013, 2002.

[BR90] V. Bazhanov and N. Reshetikhin. Restricted Solid on Solid Models Connected With Simply Based Algebras and Conformal Field Theory. *J. Phys. A*, 23:1477, 1990.

[BS90] VV Bazhanov and Yu G Stroganov. Chiral potts model as a descendant of the six-vertex model. *Journal of Statistical Physics*, 59:799–817, 1990.

[BT06] Andrei G. Bytsko and Jorg Teschner. Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model. *J. Phys. A*, 39:12927–12981, 2006.

[CDI13] D. Chicherin, S. Derkachov, and A. P. Isaev. The spinorial R-matrix. *J. Phys. A*, 46:485201, 2013.

[Che86] Ivan Cherednik. Special bases of irreducible representations of a degenerate affine hecke algebra. *Functional Analysis and Its Applications*, 20:76–78, 1986.

[CLV22] Dmitry Chernyak, Sébastien Leurent, and Dmytro Volin. Completeness of Wronskian Bethe Equations for Rational $\mathfrak{gl}_{m|n}$ Spin Chains. *Commun. Math. Phys.*, 391(3):969–1045, 2022.

[DK87] H.J. De Vega and M. Karowski. Exact bethe ansatz solution of o(2n) symmetric theories. *Nuclear Physics B*, 280:225–254, 1987.

[ESV20] Simon Ekhammar, Hongfei Shu, and Dmytro Volin. Extended systems of Baxter Q-functions and fused flags I: simply-laced case. 8 2020.

[EV21] Simon Ekhammar and Dmytro Volin. Bethe Algebra using Pure Spinors. 4 2021.

[FFK21] Gwenaël Ferrando, Rouven Frassek, and Vladimir Kazakov. QQ-system and Weyl-type transfer matrices in integrable SO(2r) spin chains. *JHEP*, 02:193, 2021.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[FKSZ20] Edward Frenkel, Peter Koroteev, Daniel S. Sage, and Anton M. Zeitlin. q-Opers, QQ-Systems, and Bethe Ansatz. 2 2020.

[FKT21] Rouven Frassek, Ivan Karpov, and Alexander Tsymbaliuk. Transfer matrices of rational spin chains via novel BGG-type resolutions. 12 2021.

[Fra20] Rouven Frassek. Oscillator realisations associated to the D-type Yangian: Towards the operatorial Q-system of orthogonal spin chains. *Nucl. Phys. B*, 956:115063, 2020.

[FST79] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan. The Quantum Inverse Problem Method. 1. *Teor. Mat. Fiz.*, 40:194–220, 1979.

[FZ71] L. D. Faddeev and V. E. Zakharov. Korteweg-de Vries equation: A Completely integrable Hamiltonian system. *Funct. Anal. Appl.*, 5:280–287, 1971.

[Gau83] M. Gaudin. *La fonction d'onde de Bethe*. Collection CEA : Série scientifique. Masson, 1983.

[GGKM67] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Method for solving the korteweg-devries equation. *Phys. Rev. Lett.*, 19:1095–1097, Nov 1967.

[GKLT11] Nikolay Gromov, Vladimir Kazakov, Sebastien Leurent, and Zengo Tsuboi. Wronskian Solution for AdS/CFT Y-system. *JHEP*, 01:155, 2011.

[GKP98] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998.

[GKP02] S. S. Gubser, I. R. Klebanov, and Alexander M. Polyakov. A Semiclassical limit of the gauge / string correspondence. *Nucl. Phys. B*, 636:99–114, 2002.

[GP92] M. Gaudin and V. Pasquier. The periodic Toda chain and a matrix generalization of the bessel function's recursion relations. *J. Phys. A*, 25:5243–5252, 1992.

[GR20] Allan Gerrard and Vidas Regelskis. Nested algebraic bethe ansatz for deformed orthogonal and symplectic spin chains. *Nuclear Physics B*, 956:115021, 2020.

[Gro17] Nikolay Gromov. Introduction to the Spectrum of $N = 4$ SYM and the Quantum Spectral Curve. 8 2017.

[Hei28] W. Heisenberg. Zur Theorie des Ferromagnetismus. *Z. Phys.*, 49(9-10):619–636, 1928.

[HLM19] Chenliang Huang, Kang Lu, and Evgeny Mukhin. Solutions of $\mathfrak{gl}_{m|n}$ XXX Bethe ansatz equation and rational difference operators. *J. Phys. A*, 52(37):375204, 2019.

[Isi24] Ernst Ising. *Beitrag zur theorie des ferro-und paramagnetismus*. PhD thesis, Grefe & Tiedemann Hamburg, Germany, 1924.

[KHM98] Michael Karbach, Kun Hu, and Gerhard Müller. Introduction to the bethe ansatz ii. *Computers in Physics*, 12(6):565–573, 1998.

[KHM00] Michael Karbach, Kun Hu, and Gerhard Muller. Introduction to the bethe ansatz iii. *arXiv preprint cond-mat/0008018*, 2000.

[KK21a] D. Karakhanyan and R. Kirschner. Representations of orthogonal and symplectic Yangians. *Nucl. Phys. B*, 967:115402, 2021.

[KK21b] D. Karakhanyan and R. Kirschner. Representations of orthogonal and symplectic yangians. *Nuclear Physics B*, 967:115402, 2021.

[KLT12] Vladimir Kazakov, Sebastien Leurent, and Zengo Tsuboi. Baxter's Q-operators and operatorial Backlund flow for quantum (super)-spin chains. *Commun. Math. Phys.*, 311:787–814, 2012.

[KLV16] Vladimir Kazakov, Sebastien Leurent, and Dmytro Volin. T-system on T-hook: Grassmannian Solution and Twisted Quantum Spectral Curve. *JHEP*, 12:044, 2016.

[KLWZ97] I. Krichever, O. Lipan, P. Wiegmann, and A. Zabrodin. Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations. *Commun. Math. Phys.*, 188:267–304, 1997.

[KMGT97] Michael Karabach, Gerhard Müller, Harvey Gould, and Jan Tobochnik. Introduction to the bethe ansatz i. *Computers in Physics*, 11(1):36–43, 1997.

[KMST05] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras. Master equation for spin-spin correlation functions of the XXZ chain. *Nucl. Phys. B*, 712:600–622, 2005.

[KNS94] Atsuo Kuniba, Tomoki Nakanishi, and Junji Suzuki. Functional relations in solvable lattice models. 1: Functional relations and representation theory. *Int. J. Mod. Phys. A*, 9:5215–5266, 1994.

[KNS11] Atsuo Kuniba, Tomoki Nakanishi, and Junji Suzuki. T-systems and Y-systems in integrable systems. *J. Phys. A*, 44:103001, 2011.

[KOW22] Hiroto Kusano, Masato Okado, and Hideya Watanabe. Kirillov-reshetikhin modules and quantum k-matrices. *arXiv preprint arXiv:2209.10325*, 2022.

[KP92] A. Klumper and P. A. Pearce. Conformal weights of RSOS lattice models and their fusion hierarchies. *Physica A*, 183:304, 1992.

[KS95] Atsuo Kuniba and Junji Suzuki. Analytic Bethe Ansatz for fundamental representations of Yangians. *Commun. Math. Phys.*, 173:225–264, 1995.

[KV08] Vladimir Kazakov and Pedro Vieira. From characters to quantum (super)spin chains via fusion. *JHEP*, 10:050, 2008.

[Lax68] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on Pure and Applied Mathematics*, 21(5):467–490, 1968.

[Leu12] Sebastien Leurent. *Integrable systems and AdS/CFT duality*. PhD thesis, Paris U., IV, 2012.

[Lie67] Elliott H. Lieb. Exact solution of the problem of the entropy of two-dimensional ice. *Phys. Rev. Lett.*, 18:692–694, Apr 1967.

[LL63] Elliott H. Lieb and Werner Liniger. Exact analysis of an interacting bose gas. i. the general solution and the ground state. *Phys. Rev.*, 130:1605–1616, May 1963.

[Mal98] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[Mol07] A. Molev. *Yangians and Classical Lie Algebras*. Mathematical surveys and monographs. American Mathematical Society, 2007.

[MRV16] Davide Masoero, Andrea Raimondo, and Daniele Valeri. Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case. *Commun. Math. Phys.*, 344(3):719–750, 2016.

[MT00] J. M. Maillet and V. Terras. On the quantum inverse scattering problem. *Nucl. Phys. B*, 575:627–644, 2000.

[MZ03] J. A. Minahan and K. Zarembo. The Bethe ansatz for N=4 superYang-Mills. *JHEP*, 03:013, 2003.

[NW13] Rafael I Nepomechie and Chunguang Wang. Algebraic bethe ansatz for singular solutions. *Journal of Physics A: Mathematical and Theoretical*, 46(32):325002, jul 2013.

[NW14] Rafael I. Nepomechie and Chunguang Wang. Twisting singular solutions of Bethe's equations. *J. Phys. A*, 47(50):505004, 2014.

[Oka07] Masato Okado. Existence of Crystal Bases for Kirillov-Reshetikhin Modules of Type D. *Publ. Res. Inst. Math. Sci. Kyoto*, 43:977–1004, 2007.

[Ons44] Lars Onsager. Crystal statistics. i. a two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117–149, Feb 1944.

[Orb58] R. Orbach. Linear antiferromagnetic chain with anisotropic coupling. *Phys. Rev.*, 112:309–316, Oct 1958.

[Res83] N. Yu. Reshetikhin. A Method Of Functional Equations In The Theory Of Exactly Solvable Quantum Systems. *Lett. Math. Phys.*, 7:205–213, 1983.

[Res85] N. Yu. Reshetikhin. Integrable Models of Quantum One-dimensional Magnets With O(N) and Sp(2k) Symmetry. *Theor. Math. Phys.*, 63:555–569, 1985.

[Skl82] E. K. Sklyanin. Quantum version of the method of inverse scattering problem. *Journal of Soviet Mathematics*, 19(5):1546–1596, Jul 1982.

[Sla07] Nikita A Slavnov. The algebraic bethe ansatz and quantum integrable systems. *Russian Mathematical Surveys*, 62(4):727, 2007.

[Sla18] Nikita Andreevich Slavnov. Algebraic bethe ansatz. *arXiv preprint arXiv:1804.07350*, 2018.

[Smi00] F A Smirnov. Dual baxter equations and quantization of the affine jacobian. *Journal of Physics A: Mathematical and General*, 33(16):3385, apr 2000.

[tH74] Gerard 't Hooft. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B*, 72:461, 1974.

[Tsu10] Zengo Tsuboi. Solutions of the T-system and Baxter equations for supersymmetric spin chains. *Nucl. Phys. B*, 826:399–455, 2010.

[Tsu13] Zengo Tsuboi. Wronskian solutions of the T, Q and Y-systems related to infinite dimensional unitarizable modules of the general linear superalgebra gl(M—N). *Nucl. Phys. B*, 870:92–137, 2013.

[Ven68] G. Veneziano. Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories. *Nuovo Cim. A*, 57:190–197, 1968.

[Wil74] Kenneth G. Wilson. Confinement of Quarks. *Phys. Rev. D*, 10:2445–2459, 1974.

[Wit96] Edward Witten. Small instantons in string theory. *Nucl. Phys. B*, 460:541–559, 1996.

[ZS70] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Journal of Experimental and Theoretical Physics*, 34:62–69, 1970.

[ZS79] V. E. Zakharov and A. B. Shabat. Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. *Funct. Anal. Appl.*, 13:166–174, 1979.

[ZZ79] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. *Annals Phys.*, 120:253–291, 1979.