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Contraction Analysis of Nonlinear Systems on Riemannian Manifolds

Dongjun Wu

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Analyse de contraction des systèmes non-linéaires sur des variétés Riemanniennes

*Contraction Analysis of Nonlinear Systems on Riemannian
Manifolds*

**Thèse de doctorat de l'université Paris-Saclay et de
Harbin Institute of Technology**

École doctorale n° 580 : sciences et technologies de l'information et de
la communication (STIC)
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Systèmes** (Université Paris-Saclay, CNRS, CentraleSupélec) et **Center for Control
Theory and Guidance Technology** (Harbin Institute of Technology), sous la
direction de **Antoine Chaillet**, professeur, et la co-direction de **Guangren Duan**,
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Abstract

Stability of equilibrium points of nonlinear systems is one of the central issues of nonlinear control theory and applications. Stability analysis often boils down to searching for a Lyapunov candidate that adequately dissipates along the system's solutions. The last two decades have witnessed a growing need to go beyond stability of an equilibrium, by imposing that any two solutions of a system eventually converge to one another. Such an incremental version of Lyapunov stability (contraction) indeed proves useful in observer design, synchronization and trajectory tracking. However, analysis methods to contraction are still far from being standardized, particularly for systems evolving on manifolds such as rotation dynamics in special orthonormal group, Lagrangian systems modeled in non-Euclidean configuration space and quantum systems in density matrix space. The main objective of this thesis is to provide further understanding of contractive systems on manifolds and to propose applicable methods to ensure contraction. More precisely, the contributions of the thesis are:

- C1** Introduce the new tool, based on the complete lift, for contraction analysis. This new tool makes it possible to carry out contraction analysis on manifolds in a coordinate-free manner and to understand the geometric essence of contractive systems.
- C2** Show that Finsler-Lyapunov functions play a similar role for contraction as Lyapunov functions for stability analysis. In particular, we show that a contractive system always admits a Finsler-Lyapunov function.
- C3** Provide new geometric characterizations of contractive systems. First, we show that contraction can be fully characterized on a tubular neighborhood of the base manifold of the tangent bundle, therefore relaxing the main results of **C1**. Second, we establish a connection between Lyapunov stability and contraction. The connection is made by Krasovskii's method. It is shown that a Lyapunov function can be directly constructed using the information of contraction, in which the latter is

concerned with objects in the tangent bundle while the former is an object on the base manifold.

C4 Study local exponential stability of nontrivial solutions of systems on manifolds. It is found that such stability has close relationship with contraction, and is easier to use than contraction in some situations. Necessary and sufficient conditions are obtained. An illustrative example, namely, convergence analysis of observer of Lagrangian system is given to show the effectiveness of the approach.

C5 Study robustness of contraction by converting contraction into transverse stability. It is shown that contraction is robust when the system flow along the horizontal manifold is hyperbolic. The method is then extended to study robustness of hyperbolic flow along compact submanifold.

Keywords: Contraction, Nonlinear Systems, Riemannian Manifolds

Résumé en français

La stabilité des points d'équilibre des systèmes non linéaires est l'un des problèmes centraux de la théorie et des applications du contrôle non linéaire. L'analyse de stabilité se résume souvent à la recherche d'un candidat Lyapunov qui se dissipe de manière adéquate le long des solutions du système. Les deux dernières décennies ont été marquées par un besoin croissant d'aller au-delà de la stabilité des un équilibre, en imposant que deux solutions quelconques d'un système finissent par converger l'une vers l'autre. Une telle version incrémentale de la stabilité de Lyapunov (contraction) s'avère en effet utile dans la conception, la synchronisation et le suivi de trajectoire des observateurs. Cependant, les méthodes d'analyse à la contraction sont encore loin d'être standardisées, en particulier pour les systèmes évoluant sur des variétés telles que la dynamique de rotation dans un groupe orthonormé spécial, les systèmes Lagrangiens modélisés dans un espace de configuration non Euclidien et les systèmes quantiques dans un espace de matrice de densité. L'objectif principal de cette thèse est d'approfondir la compréhension des systèmes contractifs sur les variétés et de proposer des méthodes applicables pour assurer la contraction. Plus précisément, les contributions de la thèse sont :

- C1** Présenter le nouvel outil, basé sur la complete lift, pour l'analyse de la contraction. Ce nouvel outil permet d'effectuer des analyses de contraction sur des variétés sans coordonnées et de comprendre l'essence géométrique des systèmes contractifs.
- C2** Montrer que les fonctions de Finsler-Lyapunov jouent un rôle similaire pour la contraction que les fonctions de Lyapunov pour l'analyse de stabilité. En particulier, on montre qu'un système contractif admet toujours une fonction de Finsler-Lyapunov.
- C3** Proposer de nouvelles caractérisations géométriques des systèmes contractifs. Tout d'abord, on montre que la contraction peut être entièrement caractérisée sur un voisinage tubulaire de la variété de base du fibré tangent, relâchant ainsi les principaux résultats de **C1**. Deuxièmement, on établit un lien entre la stabilité de Lyapunov et la contraction. La connexion est établie par la méthode de Krasovskii.

On montre qu'une fonction de Lyapunov peut être directement construite en utilisant l'information de contraction, dans laquelle cette dernière concerne des objets dans le fibré tangent tandis que la première est un objet sur la variété de base.

- C4** Étudier la stabilité exponentielle locale de solutions non triviales des systèmes sur des variétés. Il s'avère qu'une telle stabilité a une relation étroite avec la contraction, et est plus facile à utiliser que la contraction dans certaines situations. Les conditions nécessaires et suffisantes sont obtenues. Un exemple illustratif, à savoir, l'analyse de convergence de l'observateur du système Lagrangien est donné pour montrer l'efficacité de l'approche.
- C5** Étudier la robustesse de la contraction en convertissant la contraction en stabilité transversale. On montre que la contraction est robuste lorsque l'écoulement du système le long de la variété horizontale est hyperbolique. La méthode est ensuite étendue pour étudier la robustesse du flot hyperbolique le long d'une sous-variété compacte.

Mots clés: Contraction, Systèmes non-linéaire, Variété Riemanniennes

To my motherland and my parents

Contents

Publication List	15
1 Prologue	17
1.1 Research Background and Literature Review	17
1.1.1 The Notion of Incremental Stability	17
1.1.2 Historical Perspective on Incremental Stability	18
1.2 Motivation of the Thesis	21
1.3 Terminologies	24
2 Background Material	27
2.1 Smooth Manifolds	27
2.2 Riemannian-Finsler Geometry	33
2.3 A Tour in Contraction Analysis	37
2.3.1 Extreme Stability and Incremental Stability	37
2.3.2 The Method via matrix measure	39
2.3.3 Differential Contraction Analysis	40
2.3.4 Contraction Analysis on Finsler Manifolds	42
2.3.5 Contraction Analysis on Riemannian Manifolds	44
2.3.6 Transverse Exponential Stability	45
2.3.7 Two Applications	47
2.3.7.1 Observer Design	47
2.3.7.2 Synchronization	47
3 Stability analysis on Riemannian Manifolds	49
3.1 Stability Analysis on Riemannian Manifolds	49
3.1.1 Converse theorem on Riemannian manifold	52
3.1.2 Discussions and applications	59
3.2 Local Exponential Stability of Trajectories	61

3.2.1	Fundamentals	62
3.2.2	LES and complete lift	64
3.2.3	Boundedness and aperiodicity of LES solutions	68
3.3	A Brief Summary	69
4	Contraction Analysis on Riemannian Manifolds	71
4.1	Fundamental Theories	72
4.2	Complete lift and Intrinsic Contraction Analysis	76
4.2.1	Quadratic Finsler-Lyapunov Function on Riemannian Manifolds	81
4.3	Converse Results of Incremental Exponential Stability	83
4.4	Tubular Neighborhood Property	88
4.5	Krasovskii's Theorem and Contraction	91
4.6	Contraction and LES	96
4.7	Volume Shrinking	98
4.8	A Brief Summary	99
5	Some Applications	101
5.1	Optimal Contraction Rate of Equilibrium Searching on Riemannian Manifolds	101
5.1.1	Zero Curvature Case	103
5.1.2	Non-negative Constant Curvature Case	104
5.2	Gradient-based Filter on $SO(3)$	105
5.3	Intrinsic Speed Observer of Lagrangian Systems	108
5.3.1	Jacobi equation and stability	108
5.3.2	Dynamical Systems Along a Trajectory	110
5.3.3	Local Exponential Stability and Contraction of Lagrangian Systems	112
5.3.4	Speed Observer of Lagrangian Systems	115
5.4	Synchronization	116
5.5	A Brief Summary	125
6	Extensions: Robustness of Transverse Stability of Submanifolds	127
6.1	Anosov Flow and Normally Hyperbolic Invariant Manifold	129
6.2	Transverse Input to State Stability	131
6.3	Towards Anosov Flow on a Manifold	136
6.3.1	Projection to a Normal Bundle	136
6.3.2	Locally Exponentially Stable Manifold	140

CONTENTS 13

6.4 A Brief Summary 146

Conclusion and Future Perspective 147

Résumé substantiel en français 151

Bibliography 155

Publication List

The thesis is based on the following articles, some of which have been published or under review.

Journal Articles

- **Dongjun Wu** and Guang-Ren Duan. Further geometric and Lyapunov characterizations of incrementally stable systems on Finsler manifolds. *IEEE Transactions on Automatic Control*, 2021, doi: 10.1109/TAC.2021.3122377.
- **Dongjun Wu** and Guang-ren Duan. Local properties of exponential contractive systems. *Automatica*, 132:109844, 2021.
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- Bowen Yi, Romeo Ortega, **Dongjun Wu**, and Weidong Zhang. Orbital stabilization of nonlinear systems via Mexican sombrero energy shaping and pumping-and-damping injection. *Automatica*, 112:108661, 2020.
- Joel Ferguson, **Dongjun Wu**, and Romeo Ortega. On matched disturbance suppression for port-Hamiltonian systems. *IEEE Control Systems Letters*, 4(4):892–897, 2020.

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- Bowen Yi, Romeo Ortega, **Dongjun Wu**, and Weidong Zhang. Two constructive solutions to orbital stabilization of nonlinear systems via passivity-based control. In 2019 *IEEE 58th Conference on Decision and Control (CDC)*, pages 2027–2032. IEEE, 2019.

Works in Preparation

- **Dongjun Wu**, Bowen Yi, and Antoine Chaillet. Uniform local exponential stability of trajectories on Riemannian manifolds with applications to observer design. In preparation.
- **Dongjun Wu** and Antoine Chaillet. Robust transverse stability of hyperbolic flows on manifolds and applications. In preparation.
- **Dongjun Wu**. Synchronization under Finsler and Riemannian metric. In preparation.

Chapter 1

Prologue

1.1 Research Background and Literature Review

1.1.1 The Notion of Incremental Stability

Stability analysis lies in the core of nonlinear control theory. Amongst the many methods up-to-date to analyze the stability of nonlinear dynamical systems, Lyapunov's second method (or Lyapunov's indirect method) has been the most widely used one. The simplest reason for this fact is that Lyapunov's second method converts a problem which involves solving a differential equation to a simpler task of searching for a scalar Lyapunov function, and the ability to use a scalar function to analyze the stability of a system with higher dimension greatly simplifies the problem. Various developments based on Lyapunov's second method are covered in many standard textbooks of nonlinear control, such as [58, 131, 108]. Needless to say, Lyapunov's second method has become the fundamental tool not only for stability analysis, but also for wide range of control problems including constructive design [109], adaptive control [88], passivity-based control [90]. Therefore, Lyapunov's pioneering work [77] which was published in 1892, is a milestone in course of the development of control theory and control applications.

Despite the dominating role of Lyapunov's stability and Lyapunov's second method in the realm of nonlinear control, in some situations however, an *incremental* form of stability [41] is needed. In these occasions, instead of studying the convergence of solutions to an equilibrium, one is concerned with the convergence of a set of solutions toward one another.

As an illustration, Figure 1.1 shows three trajectories $x_1(t)$, $x_2(t)$ and $x_3(t)$ of a given dynamical system; then incremental asymptotic stability (to be defined rigorously in

Chapter 4, see Definition 4.1.3) implies that the three trajectories converge to a *single trajectory*. Nonetheless, one should take care that this single trajectory is not unique, on the contrary, any one of $x_1(t)$, $x_2(t)$ and $x_3(t)$ is such a trajectory that attracts the other two. At the first glance, this rather peculiar behaviour — compared to the stability of equilibria — seems to be difficult to satisfy, while in reality, there does exist many important examples which are incrementally stable, including (but not limited to) some which have already been dealt with in classical control theory, e.g., exponentially stable linear systems. On the other hand, some control and observation problems can be naturally transformed to or interpreted as problems of making certain systems incrementally stable [2, 107]. Therefore, developing applicable theories and tools for incremental stability analysis has become the aim of many control theorists.

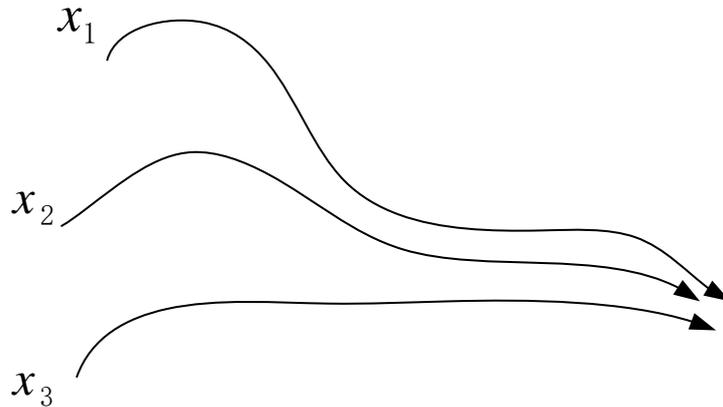


Figure 1.1: Illustration of Incremental stability

1.1.2 Historical Perspective on Incremental Stability

In this section, we give a literature review of contraction analysis. But before we start, it is necessary to make a remark on the terminology “contraction”, since so far we have only talked about incremental stability.

As indicated in the thesis title, the thesis is about contraction analysis. We emphasize that contraction is a loose synonym of incremental stability in our context. The reason why we have chosen to use the terminology “contraction” instead of the more accurate one in the title of the thesis is that most of the results in this thesis regarding incremental stabilities stem from the original works of W. Lohmiller, J. Slotine, F. Forni, and R. Sepulchre, in which incremental stability is usually called contraction. In particular, F.

Forni and R. Sepulchre have referred to “contraction analysis” as incremental stability analysis in [41], which we are going to follow.

Extreme Stability and Incremental Stability

The study of contraction in the control community can date back to the 1950s and 1960s [67, 66]. At the time it was termed as extreme stability, which was well documented in the book of T. Yoshizawa [141]. In [141], a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is defined to be extremely asymptotically stable provided that given any pair of solutions $x(t)$ and $y(t)$ to the system, there holds $x(t) \rightarrow y(t)$ as $t \rightarrow \infty$. It is customary to tackle extreme stability via augmenting the system by its identical copy, denoted as $\dot{y} = f(y)$, and then study the stability (attraction) of the unbounded diagonal set $D = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ of the augmented system. Based on this transformation, set version Lyapunov theorems can be proposed to analyze extreme stability. More recent results in this line of research can be found in the works of D. Angeli [8, 9], M. Zamani *et al.* [142, 143] and V. Fromion *et al.* [44, 45, 46].

Convergent Systems

In addition to extreme stability, another important notion concerning incremental stability, entitled *convergent system*, has also been extensively studied since the 1960s. As pointed out in [94], this subject was pioneered by B. P. Demidovich [36, 35]. As definition, a system $\dot{x} = f(x)$ is called convergent on a positively invariant set if all solutions in this domain converge asymptotically to a single solution. Standard definitions can be found in the works of A. Pavlov *et al.* [93, 94, 95], and the work of B. Rüffer *et al.* [101]. One of the most remarkable results of convergent systems is derived from the so called Demidovich condition — a matrix inequality — which guarantees the exponential convergence of each pair of solutions to each other. Unlike extreme stability analysis, Demidovich condition does not involve an augmentation procedure.

Strong links as well as some subtle differences exist between convergent systems and incrementally stable systems. The reader is referred to [101, 53] for details on this matter.

Differential Contraction Analysis

Between 1996 and 1998, W. Lohmiller and J.J. Slotine proposed the notion of *contraction analysis* of nonlinear systems in a series of articles [71, 69, 72, 70] These results were later summarized in W. Lohmiller’s PhD thesis [76], see also the PhD thesis of J. Jouffroy [54] for a complete list of references. A crucial step of the differential contraction analysis method is the calculation of the *virtual dynamics* of the system, which characterize the evolution of the infinitesimal distance between two sufficiently close trajectories. Then it

was argued that as long as the virtual dynamics converge exponentially, each pair of solutions necessarily converge to each other, at the same rate. It should be mentioned that the state space setting of the contraction analysis mentioned above is the Euclidean space equipped with a Riemannian norm. This Riemannian norm is utilized to characterize the infinitesimal distance, which depends on the state.

An alternative way for contraction analysis is via the matrix measure method proposed by E. Sontag *et al.* [116, 5], which proves to be powerful to handle systems with input, see for example [103], where it is shown that contraction implies entrainment for certain systems with input. This method applies to systems living in normed vector spaces, even infinite dimensional ones [5]. E. Sontag's works have been enriched by many other researchers in various aspects, such as by Z. Aminzare *et al.* for contraction of systems described by PDEs [4, 6, 110], by M. Margaliot *et al.* for some generalizations of contraction [84, 83, 83, 117, 135], and by M. di Bernardo *et al.* for contraction of networked systems [104, 39, 34, 34]. It is worth mentioning that, different from the methods developed by W. Lohmiller and J.J. Slotine, the matrix measure method assigns a *constant metric* to the state space.

Based on previous works, in 2014, F. Forni and R. Sepulchre laid a differential framework for contraction analysis on manifolds [41]. They noticed that it is the metric rather than the linear structure which is essential for contraction analysis. Hence they chose the Riemannian-Finsler manifold as the working space on which a distance is naturally defined. The authors then proposed a sufficient condition for contraction based on a novel concept — the Finsler-Lyapunov function. This condition is a partial differential inequality written in local coordinates. The advantage of F. Forni's and R. Sepulchre's work lies in at least two aspects. First, it provides a possibility to carry out contraction analysis on manifolds. Second, it is a rather general framework which includes a bunch of previous results as special cases, or to put it in another way, many previous known results such as the contraction analysis initiated by W. Lohmiller and J.J. Slotine, and the matrix measure methods developed by E. Sontag *et al.* can be interpreted by this theory [41].

Besides general theory of contraction, some researchers are interested in developing contraction theory for systems with certain structures. J. Simpson-Porco and F. Bullo studied intrinsic contraction on Riemannian manifolds [22, 112], in which they get intrinsic criterion to check contraction. R. Reyes-Báez, R. Ortega and A. Yaghmaei *et al.* studied contraction of port-Hamiltonian systems [99, 137, 16]. Q. Pham *et al.* studied contraction of stochastic systems [97, 98]. We underscore that the idea of contraction of stochastic systems has already been used in a much older work [123] in 1976. A. van der Schaft, Y.

Kawano, F. Forni *et. al* studied differential passivity systems [41, 57, 62, 130, 128], which has close relationships with contraction.

Applications of Contraction Analysis

Applications of contraction theory can be found in various fields, ranging from mechanical systems to system biology. The incremental characteristic of contraction makes itself particularly useful when the control objective is to achieve certain convergence property of a pair, or some pairs of solutions.

To explain the underlying idea of applications of contraction, perhaps the most illustrative example is the two agents synchronization problem. For a system with two agents described by $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_1, x_2)$, $x_1, x_2 \in \mathbb{R}^n$, synchronization is achieved as long as $|X_1(t, x_1, x_2) - X_2(t, x_1, x_2)| \rightarrow 0$ as $t \rightarrow \infty$, where $X_i(t, x_1, x_2)$ corresponds to the solution of the i -th agent with initial condition x_1, x_2 , $i = 1, 2$. Notice that synchronization necessarily implies $f_1(x, x) = f_2(x, x)$ for all $x \in \mathbb{R}^n$. Now consider an auxiliary system $\dot{x} = f_2(x_1, x) - f_1(x_1, x) + f_1(x_1, x_2)$. It is readily checked that $X_i(t, x_1, x_2)$, $i = 1, 2$ are solutions to this system. Therefore, if this auxiliary system is incrementally exponentially stable uniformly in x_1 and x_2 , then the two agents synchronize. This simple observation makes it possible to apply contraction theory to synchronization and consensus problems in various situations [87, 37, 114, 133, 34, 102, 104].

Based on the above analysis of synchronization, one can argue in a similar fashion for observer design. In fact, it can be easily shown that observer design may be viewed as a special case of synchronization. Contraction-based observer has been studied extensively in the literature and it is still an active area [2, 140, 53, 133, 106, 107, 72, 73, 74, 75], among which we want to mention is the observer design for free Lagrangian systems proposed by N. Aghannan and P. Rouchon in [2]. The observer in [2] is intrinsic in the sense that the construction of the observer does not rely on local coordinates. However, the contraction analysis was done in local coordinates.

Other applications include trajectory tracking of mechanical systems [93, 100, 82, 137], stabilization of limit cycle [81, 116], output regulation [82, 93], to name a few. Recently, contraction analysis has also been applied for learning [80, 61, 125, 113, 124].

1.2 Motivation of the Thesis

Most control problems can be dealt with in Euclidean spaces, either because the most common state spaces in application are Euclidean or many of the control problems are local in nature so that analysis in local coordinate chart will suffice. There exist however

some important control problems for which the natural setting of the state spaces are differentiable manifolds, and meanwhile, for these problems, working in local coordinates cannot solve them soundly. Typical examples may include trajectory tracking, observer design and motion planning, and that in these situations, the problems can hardly be transformed into set point stabilization ones in Euclidean spaces. Therefore, it is desirable to develop control methods for systems evolving on manifolds, which is the theme of many popular monographs and textbooks, such as [24] (geometric control of mechanical systems), [33] (control of quantum systems), [3, 55, 105] (control methods for systems on Lie groups).

This thesis is devoted to contraction analysis on manifolds. In the literature review of contraction covered in previous section, we have seen that most of the theories and applications address systems evolving in Euclidean spaces. Only a few are discussed on manifolds, among which the most important ones are [112] and [41]. In [41], a differential framework for contraction analysis was developed, though the analysis was carried out in local coordinates. In [112], J. Simpson-Porco and F. Bullo proposed an intrinsic method to study contraction analysis on Riemannian manifolds. However, this particular case is limited in applications because only a very special class of Finsler-Lyapunov function was considered, i.e., the Riemannian inner product.

In view of the importance of systems on manifolds and the fact that contraction is far less understood for systems of this class, this thesis aims at providing a geometric framework for contraction analysis on manifolds. We expect at the end of this thesis to develop a theory for contraction analysis parallel to Lyapunov stability analysis such that one can carry out contraction analysis on manifolds in a geometric way other than working with local coordinates.

The rest of the thesis is structured as follows, as shown in Figure 1.2.

Chapter 2 recalls some necessary background material and provides a tour for contraction analysis. In particular, a brief introduction to smooth manifolds and Riemannian geometry is given.

The main contributions of the thesis are reported in Chapters 3 to 6. Chapter 3 is concerned with stability analysis on Riemannian manifolds, in particular, local exponential stability analysis of nontrivial solutions. This chapter also includes some important preparations for the next chapter.

Chapter 4 focuses on geometric contraction analysis on manifolds. This chapter contains the essential theoretical results of the thesis. In this chapter, we start by introducing the basic tools for contraction analysis on manifolds, and then provides geometric char-

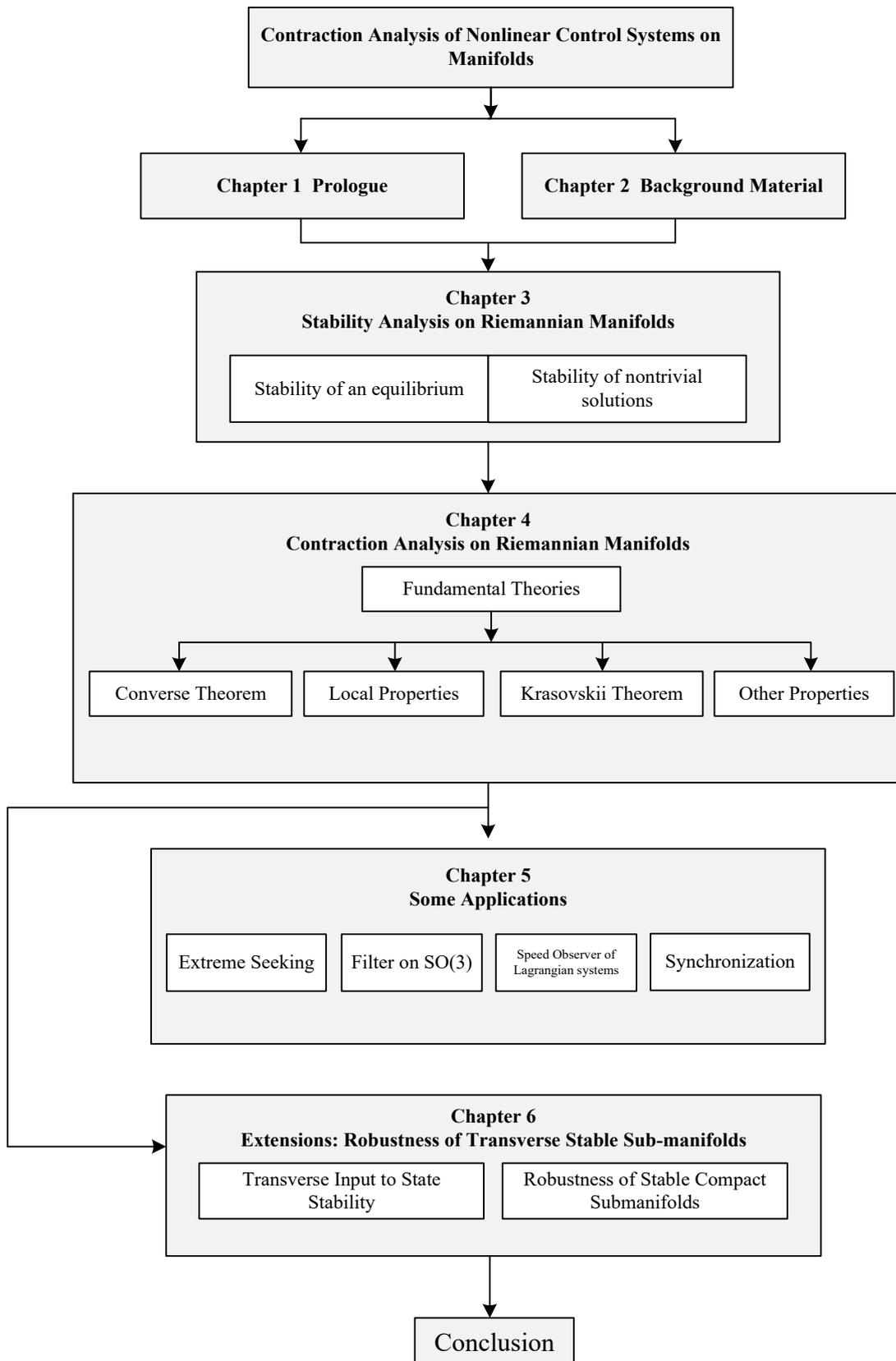


Figure 1.2: Structure of the Thesis

acterizations of contractive systems.

Chapter 5 studies robustness of contraction. In this chapter, first we study robustness of contraction via transverse linearization techniques in Euclidean space. Then we extend the notion of robustness to compact invariant submanifolds.

Chapter 6 covers applications of contraction. In this chapter, we study several application problems, namely, extreme seeking on Riemannian manifolds via Euler method, filter on $SO(3)$, speed observer for Lagrangian systems and synchronization.

1.3 Terminologies

Notation of some particular sets

$B_x(r)$ denotes the closed ball of radius r centered at x , i.e., the set $\{y : d(y, x) \leq r\}$, \bar{S} the closure of set S , \mathbb{R}_+ the set of non-negative real numbers.

Some notations from Riemannian geometry

Throughout the manuscript, M denotes a smooth Riemannian manifold with a metric g or $\langle \cdot, \cdot \rangle$. $|\cdot|$ is the induced Riemannian norm, i.e., $|v| = \sqrt{g(v, v)}$, $T_p M$ the tangent space of M at p , $T_p^* M$ the cotangent space of M at p , $\pi : TM \rightarrow M$ the projection map, $\mathcal{X}(M)$ the set of smooth vector fields on M , $d(x, y)$ the Riemannian distance between x and y , P_a^b the parallel transport from $T_a M$ to $T_b M$, ∇ the Levi-Civita connection, $\frac{D}{dt}$ the Covariant derivative, $\ell(\gamma)$ the arc-length of a rectifiable curve γ .

Miscellaneous

A class \mathcal{K} function is a continuous mapping $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is strictly increasing and which vanishes at the origin. A class \mathcal{K} function α is said to belong to class \mathcal{K}_∞ if $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$. A class \mathcal{KL} function is a continuous mapping $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies: 1) for each fixed s , $r \mapsto \beta(r, s)$ is class \mathcal{K} and 2) for each fixed r , $s \mapsto \beta(r, s)$ is non-increasing and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. $X(t, x_0)$ denotes the solution to the autonomous ordinary differential equation $\dot{x} = f(x)$ with initial condition x_0 at initial time $t = 0$; $\phi(t; t_0, x_0)$ denotes the solution or flow of a nonautonomous ordinary differential equation $\dot{x} = f(t, x)$ with initial condition x_0 at initial time $t = t_0$. C^k will denote continuously differentiable functions of order k , $1 \leq k \leq \infty$. $\mathcal{L}_f V$ denotes the Lie derivative of a function V with respect to a vector field f . The symbol \simeq means isomorphism, e.g., $\mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$.

Abbreviations

GAS	Globally asymptotically stable
GES	Globally exponentially stable
LES	Locally exponentially stable
FLF	Finsler-Lyapunov function
IS	Incrementally stable
IES	Incrementally exponentially stable
GIES	Globally incrementally exponentially stable
IAS	Incrementally asymptotically stable
GIAS	Globally incrementally asymptotically stable

Chapter 2

Background Material

2.1 Smooth Manifolds

In order to make this manuscript self contained, we spend a few pages to provide a brief introduction to the theory of smooth manifolds. The material is rather standard and can be found in textbooks such as [59, 68, 126].

The building blocks for smooth manifolds are topology and calculus.

Definition 2.1.1 (Topological space). A *topology* on a set S is a collection \mathcal{T} of subsets of S with the following properties:

- $\emptyset, S \in \mathcal{T}$;
- Finite intersection and arbitrary union of elements in \mathcal{T} are still in \mathcal{T} .

The set S equipped with a topology \mathcal{T} is called a *topological space* and the elements in \mathcal{T} are called open sets. The notation (S, \mathcal{T}) means “set S equipped with the topology \mathcal{T} .” An open set containing a point is called a neighborhood of that point.

Definition 2.1.2 (Topological base, second countable space). A collection \mathcal{B} of open sets of S is called a (*topological*) *base* if every open set in S is a union of sets in \mathcal{B} . S is called *second countable* if it admits a countable base.

Definition 2.1.3 (Hausdorff Space). A topological space S is *Hausdorff* if given any two distinct points $x, y \in S$, there exist two disjoint open sets U, V , with $x \in U$ and $y \in V$.

Definition 2.1.4 (Continuous mapping). Let (M, \mathcal{T}_1) and (N, \mathcal{T}_2) be two topological spaces. A mapping $f : M \rightarrow N$ is called a *continuous mapping* if $f^{-1}(U) \in \mathcal{T}_1$ for all $U \in \mathcal{T}_2$.

Definition 2.1.5 (Homeomorphism). A continuous mapping $f : M \rightarrow N$ is said to be a *homeomorphism* if it has a continuous inverse. In this case, M and N are said to be *homeomorphic*.

Roughly speaking, a (topological) manifold is a topological space which is locally Euclidean in the sense that each point has a neighborhood which is homeomorphic to an open set of \mathbb{R}^n . The precise definition is provided below.

Definition 2.1.6 (Topological manifold). M is a *topological manifold of dimension n* if the following properties hold:

- M is a second countable Hausdorff topological space;
- for each point of M , there exists an open set U , homeomorphic to an open subset of \mathbb{R}^n via a mapping ϕ .

ϕ is called a *coordinate mapping*, the pair (U, ϕ) is called a (*coordinate*) *chart*, and a collection of charts (U_α, ϕ_α) , $\alpha \in A$ such that $\cup_{\alpha \in A} U_\alpha = M$ is called an *atlas* for M , where the set A is an index set.

The starting point moving from a topological manifold to a smooth manifold is the introduction of a smooth structure.

Definition 2.1.7 (Differentiable structure). A *differentiable structure of class C^k* ($1 \leq k \leq \infty$) on a topological manifold M is a designation of an atlas $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ with the following properties:

- $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ is C^k for all $\alpha, \beta \in A$;
- the collection \mathcal{F} is maximal in the following sense: if (U, ϕ) is a chart such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \phi) \in \mathcal{F}$.

Notice that the two domains $\phi_\beta(U_\alpha \cap U_\beta)$ and $\phi_\alpha(U_\alpha \cap U_\beta)$ are in \mathbb{R}^n and are homeomorphic, and hence class C^k function is well-defined.

Definition 2.1.8 (Differentiable manifold). A topological manifold M with a differentiable structure of class k is called a *differentiable manifold of class C^k* . When $k = \infty$, we call M a smooth manifold.

Since all the smooth manifolds treated in this manuscript are of class C^∞ , for brevity, the term “manifold” will be referred to as smooth manifold unless otherwise stated.

Definition 2.1.9 (Smooth functions). Let M and N be two smooth manifolds. A mapping $f : M \rightarrow N$ is called a C^k if for every $p \in M$, there exist C^k charts $(U, \phi) \ni p$ and $(V, \psi) \ni f(p)$ such that $f(U) \subseteq V$ and $\psi \circ f \circ \phi^{-1}$ is a C^k function from $\phi(U) \subseteq \mathbb{R}^m$ to $\psi(V) \subseteq \mathbb{R}^n$. In particular, when $k = \infty$, f is called a smooth function.

The following diagram is an illustration of the above definition:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{R}^m & \xrightarrow{\tilde{f}} & \mathbb{R}^n \end{array}$$

where $\tilde{f} = \psi \circ f \circ \phi^{-1}$ is called the coordinate representation of f . Denote the set of smooth functions from M to N as $C^\infty(M; N)$. When $N = \mathbb{R}$, we simply write $C^\infty(M; \mathbb{R}) =: C^\infty(M)$. f is called a diffeomorphism when it has a smooth inverse, and in this case M and N are said to be diffeomorphic. The set $C^\infty(M)$ is an algebra over \mathbb{R} under point-wise multiplication.

Next, we define tangent space and tangent bundle on smooth manifolds. There are several standard ways to define a tangent space at a given point $p \in M$. We follow [68]:

Definition 2.1.10 (Tangent space). A linear map $v : C^\infty(M) \rightarrow \mathbb{R}$ is called a *derivation at p* if it satisfies

$$v(fg) = f(p)v(g) + g(p)v(f), \quad f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p , denoted by T_pM is called the *tangent space of M at p* . The linear — since T_pM is a linear space — dual of T_pM is called the cotangent space at p and is denoted T_p^*M .

Definition 2.1.11 (Differential). The *differential* $df_p : T_pM \rightarrow T_{f(p)}N$ of a smooth map $f : M \rightarrow N$ is defined via $df_p(v)(g) = v(g \circ f)$ for all $g \in C^\infty(N)$ and $v \in T_pM$.

The tangent space and cotangent space have the same dimension as vector spaces as the dimension of the manifold M . From T_pM and T_p^*M , one can construct vector bundles.

Definition 2.1.12 (Vector bundle, section). A *vector bundle* of rank m consists of a pair of topological spaces E and M , with a continuous surjective map $\pi : E \rightarrow M$ such that for each $x \in M$, the subset $E_x := \pi^{-1}(x) \subseteq E$ is a vector space isomorphic to \mathbb{R}^m , and for every point $x \in M$ there exists an open neighborhood $x \in U \subseteq M$ and a *local trivialization*

$$\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m.$$

Here Φ is a homeomorphism which restricts to a linear isomorphism $E_y \rightarrow \{y\} \times \mathbb{R}^m$ for each $y \in U$. A *section* of the bundle $\pi : E \rightarrow M$ is a map $s : M \rightarrow E$ such that $\pi \circ s = \text{Id}_M$.

We are in position to define tensor bundles.

Definition 2.1.13 (Multilinear function). Given linear spaces E_i , $i = 1, \dots, m$ and V over \mathbb{R} , a multi-linear function from $E_1 \times \dots \times E_m$ to V is a mapping $f : E_1 \times \dots \times E_m \rightarrow V$ such that $f \circ p_i : E_i \rightarrow V$ is linear for all $i = 1, \dots, m$, where $p_i : E_i \rightarrow E_1 \times \dots \times E_m$ is defined by $p_i(v) = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, v, 0, \dots, 0)$.

Given two multi-linear functions

$$f : E_1 \times \dots \times E_m \rightarrow F \text{ and } g : E_1 \times \dots \times E_m \rightarrow G, \quad (2.1)$$

define a morphism $f \mapsto g$ to be a linear mapping $h : F \rightarrow G$ which makes the following diagram commutative.

$$\begin{array}{ccc} & & F \\ & \nearrow f & \downarrow h \\ E_1 \times \dots \times E_m & & \\ & \searrow g & G \end{array}$$

Figure 2.1: Morphism of multi-linear functions

Definition 2.1.14 (Tensor Product). Given vector spaces E_1, \dots, E_m over \mathbb{R} . The *tensor product* of E_1, \dots, E_m is the universal object (see [65, Chapter 1, §11]) which makes the above diagram commutative.

Thus a tensor product is the “unique” (more precisely, universal) object $(f, E_1 \otimes \dots \otimes E_m)$ (usually f is not specified explicitly), such that for any multi-linear mapping g , there exists a unique *linear* mapping h which makes the following diagram commutative.

Thus, a tensor product “lifts” a multilinear mapping g to a linear mapping h . For the existence and detailed construction of tensor product, we refer to [65].

Definition 2.1.15 (Tensor Bundle). The tensor bundle of type (r, s) (*contravariant of order r and covariant of order s*) is defined as

$$T_s^r(M) = \bigsqcup_{p \in M} \underbrace{T_p M \otimes \dots \otimes T_p M}_{r \text{ times}} \otimes \underbrace{T_p^* M \otimes \dots \otimes T_p^* M}_{s \text{ times}} \quad (2.2)$$

where \bigsqcup represents the disjoint union of sets.

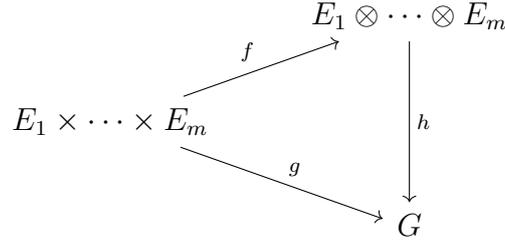


Figure 2.2: Morphism of multilinear functions

In particular, $(1, 0)$ -tensor bundle is called the tangent bundle and is written as TM and $(0, 1)$ -tensor bundle is called the cotangent bundle is written as T^*M . The tensor bundle admits a natural smooth structure induced by the smooth structure on M which makes the projection map $\pi_s^r : T_s^r M \rightarrow M$, $\pi_s^r(p, e) = p$ smooth, see for example [59]. Hereafter, we assume that the tensor bundle is always equipped with this smooth structure. It should be noted that tensor bundle is a special case of vector bundle, and thus we can define sections.

Definition 2.1.16 (Tensor field). A smooth (r, s) -tensor field on M is a smooth map $s : M \rightarrow T_s^r M$ such that $\pi_s^r \circ s(p) = p, \forall p \in M$.

Intuitively, a tensor field is a smooth assignment of tensors at each point $p \in M$. The simplest but the most important tensor field is the $(1, 0)$ -tensor field, i.e., the vector field on M . Denote $\mathcal{X}(M)$ the set of smooth vector fields on M .

Given a chart (U, ϕ) on M , define $\frac{\partial}{\partial x^i}|_p = (d\phi|_p)^{-1}(e_i)$ where $\{e_i\}_{i=1, \dots, n}$ is the standard Euclidean base. Define the dual dx^i via the relation $dx^i|_p(\partial/\partial x^j|_p) = \delta_j^i$. Then an element in $T_s^r M|_p$ can be expressed as

$$t|_p = t_{j_1 \dots j_s}^{i_1 \dots i_r}(p) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \Big|_p$$

using Einstein summation notation. When $t_{j_1 \dots j_s}^{i_1 \dots i_r}(p)$ varies smoothly with respect to p , then $t : p \mapsto t|_p$ is a smooth (r, s) -tensor field. A $(0, 2)$ -tensor field can be written as $g = \sum_{i, j=1}^n g_{ij} dx^i \otimes dx^j$. g is said to be *symmetric* if $g_{ij} = g_{ji}$ and in this case, it is conventional to write $g = g_{ij} dx^i dx^j$.

To end this section, we introduce two important definitions that will be used frequently throughout the thesis, namely, *Lie transport* and *complete lift*. There are several equivalent ways to define complete lift, and we have followed [30].

Definition 2.1.17 (Lie transport [30]). Consider the ODE $\dot{x} = f(t, x)$, and its flow $\phi(s; t, x)$, i.e., the solution to the following ODE:

$$\frac{d}{dt} \phi(t; s, x) = f(t, \phi(t; s, x)), \quad \phi(s; s, x) = x$$

$\forall x \in M, t \geq s \geq 0$. The Lie transport of the vector $W \in T_x M$ along the ODE is defined as the push forward of the vector W along the curve $s \mapsto \phi(s; t, x)$ by the flow $x \mapsto \phi(s, t, x)$ for $s \geq t$. And we denote $\text{Lie}(W)(t, s) = (\phi_{s,t})_* W$, where $\phi_{s,t}(x) = \phi(t; s, x)$. Thus $\text{Lie}(W)$ defines a vector field along the curve $\sigma(t, s) = \phi(t; s, x)$.

Remark 2.1. We remark that $W \mapsto \text{Lie}(W)$ defines a linear operator in the sense that for any constant α , $\text{Lie}(\alpha W)(t, s) = \alpha \text{Lie}(W)(t, s)$. The operator also satisfies the semi-group property in the following sense:

$$\text{Lie}(\text{Lie}(W)(r, s))(t, r) = \text{Lie}(W)(t, s)$$

Having at hand the notion of Lie transport, we can now proceed to define the complete lift of a vector field. But before going into technical details, we provide some intuitions. Loosely speaking, the complete lift is a linearization procedure. To understand this, we recall the linearization in \mathbb{R}^n . Consider an ordinary differential equation $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$, and a solution $q(\cdot)$. The linearization of the system along $q(\cdot)$ should be understood as the linearization of the error dynamics $\dot{e} = f(q+e) - f(q)$, where $e := x - q$. This is obviously $\delta\dot{q} = \frac{\partial f}{\partial x}(q)\delta q$, where we have denoted δq as the linearized state. We show next that the solution to this linearized error dynamics is the Lie transport. By definition, the Lie transport of a (constant) vector v is $(\phi_{t,0})_*(v)$, but

$$\begin{aligned} \frac{d}{dt}(\phi_{t,0})_*(v) &= \frac{d}{dt} \frac{\partial \phi(t; 0, x)}{\partial x}(v) \\ &= \frac{\partial}{\partial x} \frac{d\phi(t; 0, x)}{dt}(v) = \frac{\partial}{\partial x} f(\phi(t; 0, x))(v) \\ &= \frac{\partial f}{\partial x}(\phi(t; 0, x)) \frac{\partial \phi(t; 0, x)}{\partial x}(v) \\ &= \frac{\partial f}{\partial x}(\phi(t; 0, x))(\phi_{t,0})_*(v), \end{aligned}$$

which shows it is indeed the case by replacing x by q and v by $\delta q(0)$. Therefore, the vector field $\frac{\partial f}{\partial x}(q)\delta q$ can be obtained by taking the time derivative of the Lie transport, which however, results in a vector field on the second order tangent bundle when \mathbb{R}^n is replaced by a manifold M , since the Lie transport forms a curve in TM .

Definition 2.1.18 (Complete lift [30]). Consider the time-varying system (3.1). Given a point $v \in TM$, let $(s, t) \mapsto \sigma(t, s)$ be the integral curve of f with $\sigma(s, s) = \pi(v)$. Let V be the vector field along σ obtained by Lie transport of v by f . Then (σ, V) defines a curve in TM through v . For every $t \geq s$, the *complete lift* of f into TTM is defined at v as the tangent vector to the curve (σ, V) at $t = s$. We denote this vector field by $\tilde{f}(v, t)$ for each $v \in TM$ and each $t \geq 0$.

Figure 2.1 shows the lifting procedure for a time-invariant vector field f : σ is an integral curve of the vector field f in M , V a vector field along σ constructed by Lie transport of the flow generated by f . $\tilde{f}(V, t)$ is the velocity of the curve $(\sigma(t), V(t))$.

Remark 2.2. The discussion before Definition 2.1.18 shows that for a vector field f in \mathbb{R}^n , the complete lift of f at a point (x, v_x) is the vector $\text{col}(f(t, x), \frac{\partial f(t, x)}{\partial x} v_x)$.

The above procedure can be written in a more compact form as follows. Given a vector field f on M , f can be viewed as a mapping from M to TM , and we define a mapping $X^T : TM \rightarrow TTM$ (which is again, a vector field) as

$$X^T(v_x) = \frac{\partial}{\partial t} \Big|_{t=0} (d\phi_t^X(v_x)), \tag{2.3}$$

where $\phi_t^X(x) = X(t, x)$ and $d\phi_t^X$ is its differential map. It is straightforward to verify that this definition coincides with Definition 2.1.18. See for example Remark S1.10 of the online supplementary material of the book [24].

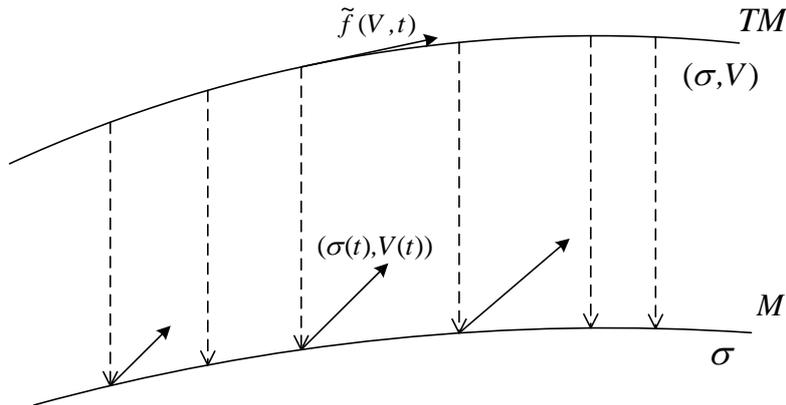


Figure 2.3: Schematic view of the complete lift.

2.2 Riemannian-Finsler Geometry

Riemannian geometry is among the most important tools and objects which will be used in this manuscript. This section does not intend to provide a thorough introduction to Riemannian geometry. Rather, it serves as a reference for some of the basic notions of Riemannian geometry. For standard textbooks, we refer to [96, 27, 47, 60].

The use of Finsler geometry is not essential in this context. However, it helps explain some interesting examples in the manuscript which cannot be covered by Riemannian

geometry. Therefore, the author decides to list some of the fundamental concepts of Finsler geometry which will be used in this manuscript. Interesting readers are referred to [15] for more details.

Definition 2.2.1 (Riemannian metric). A *Riemannian metric* on M is a smooth symmetric covariant $(0, 2)$ -tensor field g on M which is positive definite at each point. The pair (M, g) is called a Riemannian manifold.

The Riemannian metric g defines an inner product on the tangent space and often will be denoted as $\langle \cdot, \cdot \rangle$.

Definition 2.2.2 (Affine connection). An *affine connection* on a differentiable manifold M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M),$$

denoted as $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ and which satisfies the following properties:

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;
- $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;
- $\nabla_X (fY) = f\nabla_X Y + X(f)Y$.

where $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.

It can be easily shown that $\nabla_X Y$ is a pointwise operator with respect to X . Namely, if $X(p) = W(p)$ for some point $p \in M$, then $\nabla_X Y(p) = \nabla_W Y(p)$. Therefore, we can restrict the affine connection to the pullback bundle $\nabla : \Gamma^\infty(\gamma^*TM) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ where $\gamma : \mathbb{R} \rightarrow M$ is a smooth curve. Details about pullback bundle can be found in [127]. Since this case is of special importance, it has a name: *covariant derivative*. More precisely, given a vector field $V \in \mathcal{X}(M)$, and a smooth curve γ , the covariant derivative of V along γ is defined as $\frac{DV}{dt}(t) = \nabla_{\gamma'(t)} V(t)$. A connection is *symmetric* or *torsion free* if $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathcal{X}(M)$.

Theorem 2.1 (Levi-Civita connection). Given a Riemannian manifold (M, g) , there is a unique symmetric connection which is compatible with the Riemannian metric, i.e.,

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \quad X, Y, Z \in \mathcal{X}(M)$$

This connection is called the *Levi-Civita connection* associated with the Riemannian manifold (M, g) .

Definition 2.2.3 (Geodesic). A parametrized curve $\gamma : I \rightarrow M$ is a *geodesic* at t_0 if $\frac{D\gamma'(t)}{dt}(t_0) = 0$; if γ is a geodesic at $t \in I$, we say γ is a geodesic.

A geodesic has constant speed since

$$\frac{d}{dt} \langle \gamma'(t), \gamma'(t) \rangle = 2 \left\langle \frac{D}{dt} \gamma'(t), \gamma'(t) \right\rangle = 0.$$

The geodesic is said to be normalized if its speed is 1.

Theorem 2.2 (Existence of Geodesic). Given any point $p \in M$ and any tangent vector $v \in T_p M$, there is a geodesic $c : (-\epsilon, \epsilon) \rightarrow M$ with initial point $c(0) = p$ and $c'(0) = v$, where $\epsilon > 0$ is some positive constant. Moreover, this geodesic is unique in the sense that any other geodesic satisfying the same initial conditions agrees with c on the intersection of their domains.

Write $\exp_p(v) := c(1)$ when the right hand side is defined. It can be shown that the map $\exp_p : T_p M \rightarrow M$, which we call the *exponential map*, is a local diffeomorphism. Thus geodesic exists between two sufficiently close points. Given $V \subseteq T_p M$ such that $\exp_p|_V$ is a diffeomorphism, we call $U := \exp_p(V)$ a *normal neighborhood* of p .

Definition 2.2.4 (Parallel transport). Given a smooth curve $c : [a, b] \rightarrow M$, V a vector field along c . If $\frac{D}{dt} V(t) = 0$ for $t \in [a, b]$, then we say that $V(b)$ is obtained from $V(a)$ via parallel transport. We denote $V(b) = P_a^b V(a)$.

Given a curve $c : [a, b] \rightarrow M$, the arc-length of c is defined as

$$\ell(c) = \int_a^b \sqrt{\langle c'(t), c'(t) \rangle} dt.$$

Given two points $p, q \in M$, the distance between them is defined as $d(p, q) = \inf_c \ell(c)$ where the infimum is taken over all piecewise smooth curves c joining q to p .

Theorem 2.3 (Minimizing property of geodesic). Let $p \in M$, U a normal neighborhood of p , and $U \supseteq B_r := \{q \in M : d(q, p) \leq r\}$ a normal ball centered at p . Let $\gamma : [0, 1] \rightarrow B_r$ be a geodesic with $\gamma(0) = p$. If $c : [0, 1] \rightarrow M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $\ell(\gamma) \leq \ell(c)$.

Theorem 2.4 (Hopf-Rinow Theorem). Let M be a Riemannian manifold and let $p \in M$. The following assertions are equivalent

- \exp_p is defined on all of $T_p M$;
- M is complete as a metric space;

- M is geodesically complete, i.e., any two points on M can be joined by a piecewise C^1 geodesic.

Definition 2.2.5 (Curvature). The *curvature* on a Riemannian manifold M is a $(1,3)$ -tensor R defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

It can be easily checked that R is indeed a tensor.

Definition 2.2.6 (Sectional curvature). Given two linearly independent tangent vectors $v, w \in T_p M$, the sectional curvature of the two dimensional plane $\text{span}\{v, w\}$ at p is defined as

$$\text{sec}(v, w) := \frac{g(R(w, v)v, v)}{g(v, v)g(w, w) - g(v, w)^2}. \quad (2.4)$$

If $\text{sec}(v, w)$ is constant for all such two dimensional planes at each point, we say that the manifold has constant sectional curvature, or constant curvature for short.

Definition 2.2.7 (Variation of a curve). Given $\gamma : [a, b] \rightarrow M$ a continuously differentiable curve, a *variation of the curve* γ is a continuously differentiable mapping $F : (-\epsilon, \epsilon) \times [a, b] \rightarrow M$, such that $F(0, s) = \gamma(s)$, where $\epsilon > 0$ is some constant.

Let F be a variation of γ , and denote $\ell(c)$ the arc-length of a curve c . Then we have the following formula:

$$\left. \frac{d}{dt} \right|_{t=0} \ell(F(t, \cdot)) = \frac{1}{|\partial F / \partial s|} \left[\left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right\rangle_{s=a}^{s=b} - \int_a^b \left\langle \frac{\partial F}{\partial t}, \nabla_{\partial / \partial s} \frac{\partial F}{\partial s} \right\rangle ds \right]_{t=0} \quad (2.5)$$

(2.5) is called the *first variation of arclength*.

Next we recall some definitions from Finsler geometry. As S. Chern has put it [28], “Finsler geometry is just Riemannian geometry without the quadratic equation”. The theory of Finsler geometry shares a lot in common with Riemannian geometry. For example, one can define geodesic, curvature etc. on a Finsler manifold.

Definition 2.2.8 (Finsler Structure). A *Finsler Structure* of M is a function $F : TM \rightarrow [0, \infty)$ with the following properties:

- regularity: F is smooth on $TM \setminus \{0\}$;
- positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;

- strong convexity: The $n \times n$ Hessian matrix

$$(g_{ij}) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite at every point of $TM \setminus \{0\}$.

Given a smooth curve $\gamma : [a, b] \rightarrow M$, define the Finsler arc-length by

$$\ell(\gamma) = \int_a^b F(\gamma(t), \gamma'(t)) dt$$

and the Finsler distance between two points p, q as $d(p, q) = \inf_{\gamma} \ell(\gamma)$ where the infimum is taken over all piecewise C^1 curves γ joining q to p .

2.3 A Tour in Contraction Analysis

In this section, we list some fundamental results which have been obtained in the literature. It is our aim to provide a tour for the reader in theory and applications of contraction analysis. Some classical definitions related to contraction are recalled, which will then be followed with representative theorems. These theorems are stated without proof in this section, but we will come back to these results along the thesis.

2.3.1 Extreme Stability and Incremental Stability

Definition 2.3.1 (Extreme stability [141]). Consider the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2.6}$$

and its identical copy

$$\dot{y} = f(y), \quad y \in \mathbb{R}^n. \tag{2.7}$$

Then the system (2.6) is called *extremely stable* (*asymptotically stable*, *exponentially stable*) if the diagonal set $D =: \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ is stable (asymptotically stable, exponentially stable) for the augmented system (2.6)-(2.7).

Denote $X(t, x_0)$ the solution to the system (2.6) with initial condition x_0 at $t = 0$. The asymptotic stability of the set D implies contraction since the two sub-systems (2.6) and (2.7) can be viewed as the same one with different initial conditions and that the stability of D characterize the convergence of $X(t, x)$ to $X(t, y)$, where x and y are arbitrary.

Theorem 2.5 ([141]). Consider the system (2.6) and its copy (2.7). If there exists a continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$, and class \mathcal{K} functions a, b, c such that

$$a(|x - y|) \leq V(x, y) \leq b(|x - y|), \quad (2.8)$$

$$\frac{\partial V(x, y)}{\partial x} f(x) + \frac{\partial V(x, y)}{\partial y} f(y) \leq -c(|x - y|) \quad (2.9)$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, then the system is extremely asymptotically stable.

In this theorem, the function V is indeed a set Lyapunov function. In fact, $|x - y| = \text{dist}((x, y), D)$ where “dist” stands for distance and D the diagonal set defined as before.

Similar results concerning extreme stability and extreme exponential stability can be derived.

Incremental stability, a more recent synonym of extreme stability, is defined as follows.

Definition 2.3.2 (δ GAS [8]). We say that the system (2.6) is incrementally globally asymptotically stable (δ GAS) if there exists a function β of class \mathcal{KL} so that for all $x, y \in \mathbb{R}^n$ and all $t \geq 0$, the following holds

$$|X(t, x) - X(t, y)| \leq \beta(|x - y|, t) \quad (2.10)$$

Following D. Angeli, one can define incremental stability (δ S) on $Q \subseteq \mathbb{R}^n$ if (2.10) is replaced by

$$|X(t, x) - X(t, y)| \leq \alpha(|x - y|), \quad \forall x, y \in Q \quad (2.11)$$

for some class \mathcal{K} function α , and incremental exponential stability (δ ES) on $Q \subseteq \mathbb{R}^n$ if (2.10) is replaced by

$$|X(t, x) - X(t, y)| \leq K e^{-\lambda t} |x - y|, \quad \forall x, y \in Q \quad (2.12)$$

for some positive constants K and λ .

Theorem 2.6 ([8]). The system (2.6) is δ GAS if and only if there exist a continuous function $U : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ and \mathcal{K}_∞ functions α_1, α_2 , such that for all $x, y \in \mathbb{R}^n$

$$\alpha_1(|x - y|) \leq U(x, y) \leq \alpha_2(|x - y|), \quad (2.13)$$

and for all $t \geq 0$,

$$U(X(t, x), X(t, y)) - U(x, y) \leq - \int_0^t \alpha(|X(s, x) - X(s, y)|) ds \quad (2.14)$$

where α is a continuous positive definite function.

We remark that Theorem 2.5 and Theorem 2.6 share the same spirit in the following sense. Assume that U is continuously differentiable. Dividing t for $t > 0$ from both sides of (2.14) results in

$$\frac{U(X(t, x), X(t, y)) - U(x, y)}{t} \leq -\frac{1}{t} \int_0^t \alpha(|X(s, x) - X(s, y)|) ds.$$

Now let $t \rightarrow 0+$, one can get

$$\dot{U}(x, y) = \frac{\partial U(x, y)}{\partial x} f(x) + \frac{\partial U(x, y)}{\partial y} f(y) \leq -\alpha(|x - y|)$$

We can see that this inequality has the same form as (2.9), except that the function α here is only a continuous positive definite function instead of class \mathcal{K} , and that Theorem 2.6 provides necessary and sufficient conditions, while Theorem 2.5 is only sufficient. This similarity is not very surprising since both theorems translate extreme stability as the Lyapunov stability of the diagonal set D . Nevertheless, the seeking for the function V in Theorem 2.5 and the function U in Theorem 2.6 is quite difficult. More importantly, such method does not take advantage of the fact that each of the two sub-systems is the copy of each other.

2.3.2 The Method via matrix measure

The matrix measure, or the logarithm norm is a rather old concept [32, 121]. The definition is as follows: let A be a square matrix, and $|\cdot|$ a matrix norm, then the matrix measure associated with this norm is defined as

$$\mu(A) = \lim_{h \rightarrow 0+} \frac{|I + hA| - 1}{h}, \quad (2.15)$$

where I is the identity matrix. Matrix measure is a classical tool for the stability analysis of linear time varying systems [131]. E. Sontag *et al.* introduced this for contraction analysis [116]. Their main result can be stated in the following theorem.

Theorem 2.7 ([116]). Consider the system (2.6) and let $C \subset \mathbb{R}^n$ be a convex set. If there exists a matrix measure μ and a positive constant c , such that

$$\mu(J_f(x)) \leq -c, \quad \forall x \in C, t \geq 0, \quad (2.16)$$

where J_f is the Jacobian matrix of f , i.e.

$$J_f(x) = \frac{\partial f}{\partial x}(x).$$

then given any $x, y \in \mathbb{R}^n$ such that $X(t, x), X(t, y) \in C$ for all $t \geq 0$, there holds

$$|X(t, x) - X(t, y)| \leq e^{-ct} |x - y|, \quad \forall t \geq 0. \quad (2.17)$$

Remark 2.3. Notice that the inequality (2.17) is stronger than (2.12).

Compared to Theorem 2.5 and 2.6, Theorem 2.7 has the advantage that one only needs to calculate the scalar quantity $\mu(J_f)$. Since the logarithm norm can be defined for operators on Hilbert space, the method also owns the advantage of being extendable to larger classes of systems, for example, systems described by partial differential equations [5]. The limitation is that the scalar quantity $\mu(J_f)$ may encode much less information than the Jacobian matrix J_f itself. Another limitation is that for systems not evolving in vector spaces, the matrix measure is not defined.

2.3.3 Differential Contraction Analysis

In 1996, W. Lohmiller and J.J. Slotine proposed the notion of contraction analysis and explored its applications in a series of papers [72, 73, 74, 75]. The intuition is as follows. Consider two copies of the system (2.6), i.e., $\dot{x} = f(x)$ and $\dot{y} = f(y)$. The solutions to these systems can be seen as two different solutions to the system with different initial conditions. When the two solutions are sufficiently close, we can make the following approximation:

$$\dot{x} - \dot{y} = f(x) - f(y) \approx \frac{\partial f(x)}{\partial x}(x - y)$$

Based on this observation, W. Lohmiller and J.J. Slotine defined the following so called virtual dynamics:

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x)\delta x, \quad (2.18)$$

where δx is called the virtual displacement, see Figure 2.4. Now the square distance between x and y is approximately $\delta x^T \delta x$. By (2.18), the time derivative of this quantity is:

$$\frac{d}{dt}(\delta x^T \delta x) = 2\delta x^T \delta\dot{x} = 2\delta x^T \frac{\partial f}{\partial x} \delta x.$$

Let $\lambda_{max}(x)$ be maximum eigenvalue of the symmetric part of the Jacobian $J_f(x)$, then we have

$$\frac{d}{dt}(\delta x^T \delta x) \leq 2\lambda_{max}(x)\delta x^T \delta x.$$

Invoking Gronwall lemma, it follows that

$$|\delta x(t)| \leq |\delta x(0)| e^{\int_0^t \lambda_{max}(x(s)) ds}.$$

If there exists a positive constant β such that $\lambda_{max}(x) \leq -\beta$ for all $x \in \mathbb{R}^n$ then the virtual displacement (the “infinitesimal distance”) converges to 0 exponentially. Using path integral, this implies that all paths converge to each other exponentially. This is summarized in the following definition and theorem.

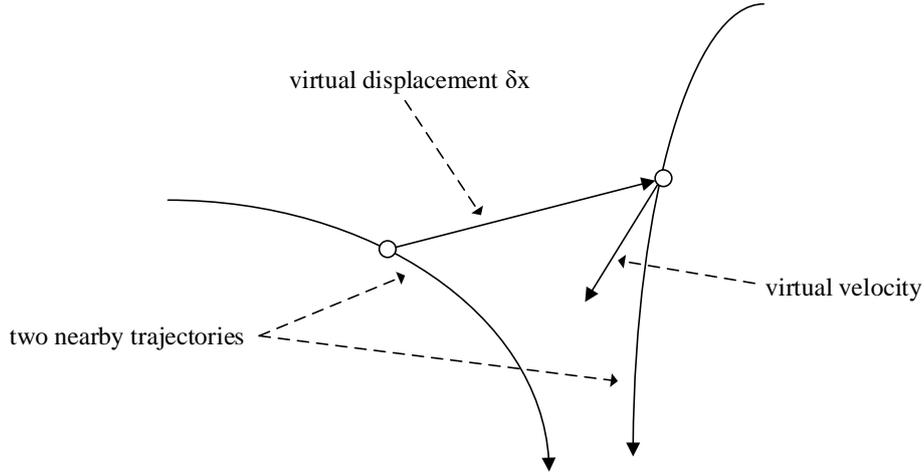


Figure 2.4: The dynamics of the virtual displacement

Definition 2.3.3. Consider the system (2.6) and a set $Q \subset M$. The set Q is called a region of contraction if the symmetric part of the Jacobian of f is negative definite over Q .

Theorem 2.8 ([72]). Given the system (2.6), Q a region of contraction and a trajectory $X(t, x)$ contained in Q . Then any trajectory which starts in Q remains in Q and converges to $X(t, x)$ exponentially.

The authors have also extended this result to systems defined on Euclidean space equipped with a quadratic form (in this case, the Euclidean space becomes a Riemannian manifold, see Section 2.2). To gain some insights, we assume that Q is the whole space \mathbb{R}^n and consider the “coordinate transform” $\delta z = \Theta(x)\delta x$, where $\Theta(x)$ is a square matrix. Then

$$\delta z^T \delta z = \delta x^T T(x) \delta x, \quad (2.19)$$

where $T(x) = \Theta^T(x)\Theta(x)$. The time derivative of δz reads

$$\frac{d}{dt}\delta z = \dot{\Theta}(x)\delta x + \Theta(x)\delta \dot{x} = \left(\dot{\Theta}(x) + \Theta(x)\frac{\partial f}{\partial x} \right) \Theta^{-1}(x)\delta z = F(x)\delta z,$$

where $F(x)$ is

$$F(x) = \left(\dot{\Theta}(x) + \Theta(x)\frac{\partial f}{\partial x} \right) \Theta^{-1}(x). \quad (2.20)$$

We get

$$\frac{d}{dt}(\delta z^T \delta z) = 2\delta z^T F(x)\delta z.$$

The dynamics of the “infinitesimal distance” can be characterized as follows:

$$\frac{d}{dt}(\delta x^T T(x) \delta x) = \delta x^T \left(\frac{\partial f^T}{\partial x} T(x) + \dot{T}(x) + T(x) \frac{\partial f}{\partial x} \right) \delta x.$$

Then the region of contraction is the set on which there holds

$$\frac{\partial f^T}{\partial x} T(x) + T(x) \frac{\partial f}{\partial x} + \dot{T}(x) \leq -\beta T(x), \quad (2.21)$$

where β is a positive constant.

The above analysis leads to the following:

Definition 2.3.4 ([72]). Given the system (2.6), a region $Q \subset M$ is called a contraction region with respect to a positive definite metric $T(x) = \Theta^T(x)\Theta(x)$, if the inequality (2.21) is fulfilled for all $x \in Q$.

Theorem 2.9 ([72]). Given the system (2.6), a contraction region Q with respect to the metric $T(x) = \Theta^T(x)\Theta(x)$ and a solution $X(t, x)$ contained in Q , all the trajectories which start in Q remain in Q and converge to $X(t, x)$ exponentially.

Remark 2.4. This method provides a differential approach for contraction analysis. One no longer considers the copy of the system but the variation of the system. This is obviously a new way of thinking and a turning point for contraction analysis. But we remark that some of the analysis in the paper [72] is not easy to justify mathematically. For example, the term “virtual displacement” and “infinitesimal distance” are not precise mathematical languages and should be further clarified. Faced with this issue, F. Forni and R. Sepulchre proposed a framework in [41] that we are going to introduce in next subsection, which successfully solved the mentioned problems.

2.3.4 Contraction Analysis on Finsler Manifolds

Based on the works of W. Lohmiller and J.J. Slotine [72], F. Forni and R. Sepulchre proposed in 2014 a differential Lyapunov framework for contraction analysis [41]. They suggested to study contraction using Finsler geometry on manifolds. As we have seen in Section 2.2, Finsler geometry is a kind of metric geometry, which includes Riemannian geometry as a special case. It does not cause any trouble in this thesis, however, to view a Finsler structure as a Riemannian norm.

The following definition and theorem are due to F. Forni and R. Sepulchre [41].

Definition 2.3.5 ([41]). Given a manifold M equipped with a Finsler structure F , a candidate Finsler-Lyapunov function is a continuously differentiable function $V : TM \rightarrow \mathbb{R}_+$, such that

$$c_1 F(x, \delta x)^p \leq V(x, \delta x) \leq c_2 F(x, \delta x)^p, \quad (2.22)$$

is verified for all tangent vectors $(x, \delta x) \in TM$, where TM is the tangent bundle of M , and c_1, c_2 and $p \geq 1$ are some positive constants.

Loosely speaking, like a Riemannian metric, a Finsler structure is a characterization of the infinitesimal distance between two sufficiently close points on a manifold. For a rigorous definition, see Definition 2.2.8.

Theorem 2.10 ([41]). Consider the system (2.6) defined on M , in which f is C^2 , a connected and forward invariant set Q , and a function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. Let V be a candidate Finsler-Lyapunov function such that in local coordinates,

$$\frac{\partial V(x, \delta x)}{\partial x} f(x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} \delta x \leq -\alpha(V(x, \delta x)), \quad (2.23)$$

for each $t \in \mathbb{R}$, $x \in Q \subset M$, and $\delta x \in T_x M$. Then the system (2.6) is

(IS) incrementally stable on Q if $\alpha(s) = 0$, $\forall s \geq 0$.

(IAS) incrementally asymptotically stable on Q if α is a class \mathcal{K} function.

(IES) incrementally exponentially stable on Q if $\alpha(s) = \lambda s$, for all $s \geq 0$ and some $\lambda > 0$.

Remark 2.5. In the above definition and theorem, the key object is the newly introduced Finsler-Lyapunov function. This theorem can recover many classical results in contraction theory. But it should be noted that (2.23) is expressed in local coordinates. So on the one hand, its geometric meaning is not straightforward, on the other, if one wants to analyze global contraction further analysis will be needed. Nevertheless, Theorem 2.10 is already rich enough to derive some interesting results.

As an example, recall that in Section 2.3.2, we have discussed the contraction properties using matrix measure. It can be shown that the matrix measure method is a special case of the above theorem. In fact, suppose that (2.16) holds and consider the Finsler-Lyapunov candidate $V(x, \delta x) = |\delta x|$ by setting $F(x, \delta x) = |\delta x|$ and $p = 1$ [41]. Since $V(x, \delta x)$ does not depend on x , by (2.23), we have

$$\frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} = \lim_{h \rightarrow 0^+} \frac{V(x, \delta x + h J_f(x) \delta x) - V(x, \delta x)}{h}$$

$$\begin{aligned}
&\leq \lim_{h \rightarrow 0^+} \frac{|I + hJ_f(x)||\delta x| - |\delta x|}{h} \\
&= \lim_{h \rightarrow 0^+} \frac{|I + hJ_f(x)| - 1}{h} V(x, \delta x) \\
&= \mu(J_f(x))V(x, \delta x) \\
&= -cV(x, \delta x), \quad \forall t \geq 0, x \in C, \delta x \in \mathbb{R}^n
\end{aligned}$$

where $J_f(x)$ is the Jacobian of $f(x)$. Therefore the condition 3 in Theorem 2.10 is satisfied, which implies that the system is IES. Note that the function $V(x, \delta x) = |\delta x|$ is not C^1 . Therefore, strictly speaking, it does not meet all the requirements of a Finsler-Lyapunov candidate. However, Theorem 2.10 can still hold by using Dini derivative in (2.23).

The differential contraction analysis method proposed by W. Lohmiller and J.J. Slotine [72] is also a special case of Theorem 2.10. In fact, consider the Finsler-Lyapunov candidate $V(x, \delta x) = \delta x^T M(x) \delta x$, where $M(x)$ is the same as in Section 2.3.3 and $F(x, \delta x) = \sqrt{\delta x^T M(x) \delta x}$ and $p = 2$ in Definition 2.3.5. Then by (2.23) we know that

$$\dot{V}(x, \delta x) = \delta x^T \left(\frac{\partial f^T}{\partial x} M(x) + M(x) \frac{\partial f}{\partial x} + \dot{M}(x) \right) \delta x.$$

Therefore, if the condition (2.21) is satisfied, we have

$$\dot{V}(x, \delta x) \leq -\lambda \delta x^T M(x) \delta x.$$

Invoking Theorem 2.10, the system is again IES.

2.3.5 Contraction Analysis on Riemannian Manifolds

Riemannian manifold is one of the most important classes of Finsler manifold. In fact, all the manifolds that we will meet in this thesis are Riemannian, therefore, it is worthwhile to see what kind of result can one get when Theorem 2.10 is specified to Riemannian manifold case. Consider the Riemannian manifold M . Let ∇ be the Levi-Civita connection associated with this metric. J. Simpson-Porco and F. Bullo proved the following theorem.

Theorem 2.11 ([112]). Consider the system (2.6), a connected set Q and a positive constant λ . If

$$\langle \nabla_{v_x} f, v_x \rangle \leq -\lambda |v_x|^2, \quad \forall v_x \in T_x Q. \quad (2.24)$$

Then there exists $K \geq 1$, such that for all $x_1, x_2 \in Q$, there holds

$$d(X(t, x_1), X(t, x_2)) \leq K e^{-\lambda t} d(x_1, x_2).$$

Remark 2.6. To gain an insight about Theorem 2.11, simply consider the Euclidean space \mathbb{R}^n equipped with an inner product $\langle v, v \rangle = v^T P v$ where $P \in \mathbb{R}^{n \times n}$ is a constant symmetric positive definite matrix. In this case, the covariant derivative is the directional derivative, and therefore the inequality (2.24) reads

$$v^T \frac{\partial f(x)}{\partial x} P v \leq -\lambda v^T P v, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n$$

or

$$P \frac{\partial f(x)}{\partial x} + \frac{\partial f(x)}{\partial x} P \leq -\lambda P, \quad x \in \mathbb{R}^n.$$

Thus one recovers the inequality (2.21) when T is constant.

It should be noted that this theorem is derived independently of the theory developed by F. Forni and R. Sepulchre [41]. In Chapter 4, we show that it can be obtained from the intrinsic form of Theorem 2.10.

2.3.6 Transverse Exponential Stability

In 2016, V. Andrieu *et al.* studied the so called transverse exponential stability [7]. Such stability has close relationships with contraction properties, and will be used in Chapter 6 for robust analysis, so we state their main results here.

Consider the system

$$\dot{e} = F(e, x), \quad \dot{x} = G(e, x), \quad (2.25)$$

where $e \in \mathbb{R}^{n_e}$, $x \in \mathbb{R}^{n_x}$. For convenience, denote $(E(t, e_0, x_0), X(t, e_0, x_0))$ the solution with initial condition (e_0, x_0) . The authors have given the following definitions.

Definition 2.3.6 ([7]). The system (2.25) is called

- TULES-NL (Transversal uniform local exponential stability): If there exist positive numbers r, k, λ , such that for all $(e_0, x_0, t) \in B_{0_e}(r) \times \mathbb{R}^{n_x} \times \mathbb{R}_+$, there holds

$$|E(t, e_0, x_0)| \leq k |e_0| e^{-\lambda t}$$

- UES-TL (Uniform exponential stability for the transversally linear system): If for the system

$$\dot{\tilde{x}} = \tilde{G}(\tilde{x}) := G(0, \tilde{x}),$$

there exist positive constants $\tilde{k}, \tilde{\lambda}$, such that the solutions $(\tilde{E}(t, \tilde{e}_0, \tilde{x}_0), \tilde{X}(t, \tilde{x}_0))$ of the system

$$\dot{\tilde{e}} = \frac{\partial F}{\partial e}(0, \tilde{x}) \tilde{e}, \quad \dot{\tilde{x}} = \tilde{G}(\tilde{x})$$

satisfy

$$|\tilde{E}(t, \tilde{e}_0, \tilde{x})| \leq \tilde{k}e^{-\tilde{\lambda}t}|\tilde{e}_0|,$$

$$\forall(t, \tilde{e}_0, \tilde{x}_0) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{R}_+.$$

- ULMTE (Uniform Lyapunov Matrix Transversal Equation): If for all positive matrices Q , there exists corresponding $P : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_e \times n_e}$, positive constants p_1, p_2 , such that for all $\tilde{x} \in \mathbb{R}^{n_x}$, there holds

$$\lim_{h \rightarrow 0} \frac{P(\tilde{X}(h, \tilde{x})) - P(\tilde{x})}{h} + P(\tilde{x}) \frac{\partial F}{\partial e}(0, \tilde{x}) + \frac{\partial F}{\partial e}(0, \tilde{x})^T P(\tilde{x}) \leq -Q$$

and

$$p_1 I \leq P(\tilde{x}) \leq p_2 I.$$

In [7], the authors proved that under mild conditions, the three properties in Definition 2.3.6 are equivalent. Since these assumptions are too technical, we do not go into details now.

The relationships between this definition and contraction can be seen from the following arguments: let $F(e, x) = f(x) + f(x - e)$, $G(e, x) = f(x)$, since

$$\begin{aligned} X(t, e_0, x_1) &= x_1 + \int_0^t f(X(s, e_0, x_1)) ds \\ X(t, e_0, x_2) &= x_2 + \int_0^t f(X(s, e_0, x_2)) ds, \end{aligned}$$

then

$$X(t, e_0, x_2) - X(t, e_0, x_1) = x_2 - x_1 + \int_0^t f(X(s, e_0, x_2)) - f(X(s, e_0, x_1)) ds.$$

Denote $\varphi(t) = X(t, e_0, x_2) - X(t, e_0, x_1)$, then one gets

$$\varphi(t) = \varphi(0) + \int_0^t f(X(s, e_0, x_1) + \varphi(s)) - f(X(s, e_0, x_1)) ds.$$

Besides,

$$E(e_0, x_1, t) = e_0 + \int_0^t f(X(s, e_0, x_1) + E(s, e_0, x_1)) - f(X(s, e_0, x_1)) ds.$$

Let $e_0 = x_2 - x_1$. By the uniqueness of the existence of solutions, we get

$$E(t, x_2 - x_1, x_1) = X(t, x_2 - x_1, x_2) - X(t, x_2 - x_1, x_1).$$

Therefore, the dynamics of e characterizes the dynamics of the difference between two solutions of the system (2.6).

2.3.7 Two Applications

We mention two important applications that will be dealt with in this thesis, namely, contraction-based observer and synchronization.

2.3.7.1 Observer Design

Consider a system

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \quad (2.26)$$

where y is the measured output. The target is to design an observer to reconstruct x .

A contraction-based observer is proceeded as follows. If there exists a function g such that, $f(x)$ can be rewritten as $f(x) = g(x, h(x))$ in such a way that the system

$$\dot{z} = g(z, h(x(t)))$$

is contractive, where $h(x(t))$ is viewed as a time varying signal, then an observer can be constructed as follows

$$\dot{\hat{x}} = g(\hat{x}, y). \quad (2.27)$$

The convergence of this observer is obvious since $x(t)$ is a particular solution to (2.27) and by the contraction of this system, all solutions should converge to this particular solution.

2.3.7.2 Synchronization

Consider two coupled systems [133]:

$$\begin{aligned} \dot{x}_1 &= f(x_1) \\ \dot{x}_2 &= f(x_2) + u(x_1) - u(x_2). \end{aligned}$$

where $x_1, x_2 \in \mathbb{R}^m$ is the state, $f(x_i)$ the non-coupled dynamics, and $u(x_1) - u(x_2)$ the coupled force. According to contraction analysis, if the vector field $f(x) - u(x)$ is contracting, then since $x_1(t)$ is a solution to the second equation, we assert that $x_1(t)$ converges to $x_2(t)$ exponentially, meaning that $x_1(t)$ and $x_2(t)$ synchronize.

Obviously, the above reasoning can be also used to analyze the synchronization of the following system with n subsystems.

$$\begin{cases} \dot{x}_1 = f(x_1, t) \\ \dot{x}_2 = f(x_2, t) + u(x_1) - u(x_2) \\ \vdots \\ \dot{x}_n = f(x_n, t) + u(x_{n-1}) - u(x_n). \end{cases}$$

Multi-agent systems with other topologies have also been reported in [\[133\]](#).

Chapter 3

Stability analysis on Riemannian Manifolds

In this chapter, we study stability notions on Riemannian manifolds. Although the thesis is mainly about contraction analysis, it will be seen in later chapters that stability analysis and contraction analysis share much in common. For example, many techniques used in the proof of converse theorem for stability will also appear in the proof of converse contraction theorem. Thus this chapter serves as a preparation for contraction analysis on manifolds, but it is also important in its own right.

We will meet two kinds of stability, 1) the stability of an equilibrium and 2) the stability of a nontrivial trajectory. For 1), the main result is the converse Lyapunov function theorem. For 2), a theory for the analysis of local exponential stability of a trajectory will be developed. Unlike in Euclidean space, on a manifold, problem 2) usually cannot be converted to 1), therefore it should be treated separately.

3.1 Stability Analysis on Riemannian Manifolds

Two kinds of systems will be met in this manuscript, namely, time varying system Σ_1 and time-invariant system Σ_2 .

$$\Sigma_1 : \quad \dot{x} = f(t, x), \quad t \in \mathbb{R}_+, x \in M \quad (3.1)$$

$$\Sigma_2 : \quad \dot{x} = f(x), \quad t \in \mathbb{R}_+, x \in M \quad (3.2)$$

where M is the state space. The vector fields $f(t, x)$ and $f(x)$ are assumed to be C^1 with respect to t, x . An equilibrium is a point x_* such that $f(t, x_*) = 0, \forall t \in \mathbb{R}_+ (f(x_*) = 0)$. Since time invariant systems are strictly included in the set of time varying systems, all

the results obtained for time varying systems directly carry over to the autonomous case, but the converse is not true.

In this section, we prove converse Lyapunov theorem for stability of an equilibrium on Riemannian manifolds. The most relevant results to this section can be found in [122], however, we follow a different approach which does not depend on local coordinates and does not rely on converse results in Euclidean space. To begin with, we set up some necessary backgrounds.

Definition 3.1.1. Suppose x_* is an equilibrium for the system (3.1), then x_* is said to be

1. (locally) uniformly stable (LUS) if there exists a class \mathcal{K} function α and a positive constant c , independent of t_0 , such that for all $t \geq t_0 \geq 0$ and $x_0 \in B_{x_*}(c)$:

$$d(\phi(t; t_0, x_0), x_*) \leq \alpha(d(x_0, x_*)), \quad \forall t \geq t_0 \geq 0;$$

2. (locally) uniformly asymptotically stable (LUAS) if there exists a class \mathcal{KL} function β and a positive constant c , independent of t_0 , such that for all $t \geq t_0 \geq 0$ and $x_0 \in B_{x_*}(c)$:

$$d(\phi(t; t_0, x_0), x_*) \leq \beta(d(x_0, x_*), t - t_0); \quad (3.3)$$

3. (locally) exponentially stable (LES) if there exists three positive constants K , λ and c such that for all $t \geq t_0 \geq 0$ and $x_0 \in B_{x_*}(c)$:

$$d(\phi(t; t_0, x_0), x_*) \leq K e^{-\lambda(t-t_0)} d(x_0, x_*); \quad (3.4)$$

4. uniformly globally asymptotically stable (UGAS) if (3.3) is satisfied for all $x_0 \in M$;
globally exponentially asymptotically stable (GES) if (3.4) is satisfied for all $x_0 \in M$.

Remark 3.1. In [24], [122], stability definitions are given via the ε - δ language. However, it is not hard to show that the two ways are equivalent, see for example [122]. Similar to the Euclidean case, the comparison functions are introduced to simplify the stability analysis, especially for time varying systems.

In Euclidean space, one needs to calculate the partial derivate of the Lyapunov candidate V . However, on a manifold, the partial derivative of a function is normally not a coordinate-free notion. To handle this, Lie derivate will be used instead. On the other hand, we will deal with time varying systems, therefore we introduce the concept of *timed Lie derivative*.

Definition 3.1.2 (Timed Lie derivative). Rewrite the system (3.1) as follows:

$$\begin{cases} \frac{dx(t)}{dt} = f(s(t), x(t)) \\ \frac{ds(t)}{dt} = 1, \end{cases} \quad (3.5)$$

with initial condition

$$x(t_0) = x_0, \quad s(t_0) = t_0.$$

The defined system (3.5) is now time-invariant, thus the Lie derivative of a function along this system is well-defined. The Lie derivative of V with respect to (3.1) is defined as the Lie derivative of V with respect to the system (3.5), and is denoted $\mathcal{L}_f V$. More precisely,

$$\mathcal{L}_f V(t, x) := L_{\bar{f}} V(t, x) \quad (3.6)$$

where \bar{f} is the augmented vector field $\text{col}(f(s, x), 1)$.

Remark 3.2. $L_{\bar{f}} V$ is well defined since it is the usual Lie derivative of a time-invariant function with respect to a time-invariant vector field. Since the flow of \bar{f} is $(\phi_f(t; t_0, x_0), t - t_0)$. In coordinates, at point (t_0, x_0) , it reads

$$\begin{aligned} L_{\bar{f}} V(t_0, x_0) &= \lim_{t \rightarrow t_0} \frac{V(t, \phi_f(t; t_0, x_0)) - V(t_0, x_0)}{t - t_0} \\ &= \frac{\partial V}{\partial t}(t_0, x_0) + \frac{\partial V}{\partial x}(t_0, x_0) f(t_0, x_0), \end{aligned}$$

which coincides with the time derivative of V along (3.1) and is indeed the correct definition we need.

We are now in position to state the Lyapunov stability theorem on Riemannian manifolds.

Theorem 3.1. Let x_* be an equilibrium point of the system (3.1) and D be an open connected neighborhood of x_* . Let $V : \mathbb{R}_+ \times D \rightarrow \mathbb{R}_+$ be a Lyapunov candidate such that

$$W_1(d(x, x_*)) \leq V(t, x) \leq W_2(d(x, x_*)), \quad \forall t \geq 0, \quad x \in D, \quad (3.7)$$

then x_* is uniformly stable if

$$\mathcal{L}_f V(t, x) \leq 0;$$

it is uniformly asymptotically stable if

$$\mathcal{L}_f V(t, x) \leq -W_3(d(x, x_*)), \quad (3.8)$$

where W_i are class \mathcal{K} functions. If $W_i(r) = c_i r^p$, where $c_i > 0$, for $i = 1, 2, 3$ and $p > 0$, then (3.7) and 3.8) together imply exponential stability.

This theorem can be proved by repeating the procedures used in Euclidean space [58] by noticing that

$$\frac{d}{dt}V(t, \phi(t; t_0, x_0)) = \mathcal{L}_f V(t, \phi(t; t_0, x_0)).$$

3.1.1 Converse theorem on Riemannian manifold

In this subsection, we prove the converse theorem of Theorem 3.1. To streamline our idea, in the sequel, we only prove the global version, which can be easily extended to local case.

Recall that, in the proof of converse theorems for GES, there is a key assumption: the global Lipschitz condition. In \mathbb{R}^n , $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be globally Lipschitz continuous if there exists a constant L such that

$$|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^n$$

where $|\cdot|$ is the Euclidean norm. On Riemannian manifold, if f is a vector field, $f(x)$ and $f(y)$ will live in different tangent spaces, so it is not possible to compare them directly. In [122], the authors considered the tangent map

$$Tf : TM \rightarrow TTM.$$

At every point $x \in M$, $T_x f$ is a linear operator. The authors assume this operator to be uniformly bounded and claim that [122, (3.61)]

$$|T_x f(t, x)(X)|_e \leq c_2 |T_x f(t, x)(X)|_g, \quad (3.9)$$

when

$$|X|_e \leq c_2 |X|_g, \quad \forall X \in T_x M \quad (3.10)$$

where $|\cdot|_e$ and $|\cdot|_g$ stand for the Euclidean and Riemannian metric respectively, and condition (3.9) will be used as the Lipschitz continuity condition on Riemannian manifolds. However, $T_x f(t, x)(X)$ lives in $T_{f(x)}TM$ so its Riemannian norm needs to be defined. There exist canonical Riemannian metrics on the second order tangent bundle, such as the Sasaki metric, we remark that however, even if $|T_x f(t, x)(X)|_g$ is replaced by a Riemannian metric on TTM , the implication from (3.10) to (3.9) is not straightforward.

Instead of defining a metric on TTM and studying the tangent map, we consider the Riemannian version of Lipschitz continuity. This definition can be found for example in [26, Chapter II.3]. Intuitively, we transport two tangent vectors into a same tangent space so that we can compare them.

For a complete Riemannian manifold M , given $x, y \in M$, there exists a minimizing geodesic curve $\gamma : [0, 1] \rightarrow M$ joining x to y . Given $W \in T_x M$, let $W(t)$ be the parallel

transport of W along γ , i.e., $W(t) \in T_{\gamma(t)}M$ and $DW(t)/dt = 0$, for all $t \in [0, 1]$. Then we denote $P_x^y W =: W(1) \in T_y M$, i.e. we transport the vector W in $T_x M$ to $T_y M$.

Definition 3.1.3 (Lipschitz continuity). A vector field f on M is said to be globally Lipschitz continuous on M , if there exists a constant $L > 0$ such that for all $p, q \in M$ and all γ geodesic joining p to q , there holds

$$\left| P_p^q f(p) - f(q) \right| \leq Ld(p, q)$$

where $|\cdot|$ is the norm induced by the Riemannian metric, d the Riemannian distance and L is called the Lipschitz constant.

Assume that f is C^1 . Then it can be easily shown that if $|\nabla_{c'(0)} f| \leq L$ for all $c(t)$ with $|c'(0)| = 1$, then, f is Lipschitz continuous with constant L , and vice versa. Since the Levi-Civita connection ∇ is an affine connection, we have $\nabla_v V = |v| \nabla_{v/|v|} V$, consequently, $|\nabla_{c'(0)} f| \leq L$ is equivalent to $|\nabla_v f| \leq L|v|$ for all $v \in TM$.

Lemma 3.1. Given a C^1 vector field f on a complete Riemannian manifold M , then f is Lipschitz with constant L on M if and only if $|\nabla_v f| \leq L|v|$ for $v \in TM$.

Proof. Necessity: Suppose that f is Lipschitz continuous with constant L . Given $v \in TM$, with $|v| = 1$, there exists a normalized minimizing geodesic $\gamma : [0, T] \rightarrow M$ with $\gamma(0) = \pi(v)$ and $\gamma'(0) = v$. Then

$$\begin{aligned} |\nabla_v f| &= |\nabla_{\gamma'(0)} f| = \left| \lim_{t \rightarrow 0^+} \frac{P_{\gamma(t)}^{\gamma(0)} f(\gamma(t)) - f(\gamma(0))}{t} \right| \\ &= \lim_{t \rightarrow 0^+} \left| \frac{P_{\gamma(t)}^{\gamma(0)} f(\gamma(t)) - f(\gamma(0))}{t} \right| \\ &\leq \lim_{t \rightarrow 0^+} \left| \frac{Ld(\gamma(0), \gamma(t))}{t} \right| \\ &= L \end{aligned}$$

since γ is normalized. Since ∇ is affine connection, the necessity follows.

Sufficiency: Suppose that $|\nabla_v f| \leq L|v|$ for all $v \in TM$. Given $p, q \in M$ and a normalized minimizing geodesic γ joining p to q with $\gamma(0) = p$ and $\gamma(t) = q$. Then

$$\begin{aligned} |P_p^q f(p) - f(q)| &= |P_{\gamma(t)}^{\gamma(0)} f(\gamma(t)) - f(\gamma(0))| \\ &= \left| \int_0^t \frac{d}{ds} P_{\gamma(s)}^{\gamma(0)} f(\gamma(s)) ds \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \int_0^t -\nabla_{\gamma'(s)} f(\gamma(s)) \right| ds \\
&\leq Lt \\
&= Ld(p, q)
\end{aligned}$$

This completes the proof. \square

Assume that f is C^1 . In Euclidean space, the covariant derivative is simply the directional derivative, i.e., $\nabla_v f = \frac{\partial f}{\partial x}$, by the above lemma we see that f is Lipschitz continuous with constant L if and only if $|\nabla_v f = \frac{\partial f}{\partial x}| \leq L$. This indeed corresponds to the usual definition of Lipschitz continuity since it is equivalent to saying $|f(x) - f(y)| \leq L|x - y|$. We will see in the following that on a Riemannian manifold, it is the covariant derivative rather than the tangent map which comes into play.

The following lemma is key to the proof of the converse theorem.

Lemma 3.2. Assume that the vector field of (3.1) globally Lipschitz continuous with constant L , then there holds the following estimation

$$d(x_1, x_2)e^{-L(\tau-t)} \leq d(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \leq d(x_1, x_2)e^{L(\tau-t)}, \quad (3.11)$$

for all $\tau \geq t$, $x_1, x_2 \in M$.

Proof. Suppose that x_1 and x_2 are joined by a normalized geodesic $\gamma : [0, \hat{s}] \rightarrow M$, with $\gamma(0) = x_1$ and $\gamma(\hat{s}) = x_2$, where \hat{s} is the arc-length of γ . Then the map $F(t, s) = \phi(t; t_0, \gamma(s))$ defines a variation of γ . By the first variation formula of arc length (2.5), we have

$$\begin{aligned}
\left. \frac{d}{d\tau} d(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \right|_{\tau=t} &= \left\langle \frac{\partial \phi(t; t_0, \gamma(s))}{\partial s}, \frac{\partial \phi(t; t_0, \gamma(s))}{\partial t} \right\rangle \Bigg|_{s=0, \tau=t}^{s=\hat{s}, \tau=t} \\
&= \langle \gamma'(\hat{s}), f(x_2) \rangle - \langle \gamma'(0), f(x_1) \rangle \\
&= \langle P_{x_2}^{x_1} \gamma'(\hat{s}), P_{x_2}^{x_1} f(x_2) \rangle - \langle \gamma'(0), f(x_1) \rangle \\
&= \langle \gamma'(0), P_{x_2}^{x_1} f(x_2) - f(x_1) \rangle, \quad (3.12)
\end{aligned}$$

where the third equality follows from the inner product preserving property of the parallel transport operator. Since γ is normalized, by the Lipschitz continuity, we have

$$\left| \frac{d}{d\tau} d(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \right|_{\tau=t} \leq Ld(x_1, x_2).$$

Using the semi-group property of the flow, for any $s > t$, we have

$$\left. \frac{d}{d\tau} d(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \right|_{\tau=s}$$

$$= \frac{d}{d\tau} d(\phi(\tau; s, \phi(s; t, x_1)), \phi(\tau; s, \phi(s; t, x_2))) \Big|_{\tau=s},$$

hence

$$\left| \frac{d}{d\tau} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2) \right|_{\tau=s} \leq Ld(\phi(s; t, x_1), \phi(s; t, x_2)).$$

Or equivalently,

$$-Ld(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \leq \frac{d}{d\tau} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2) \leq Ld(\phi(\tau; t, x_1), \phi(\tau; t, x_2)),$$

from which we get (3.11) invoking Gronwall lemma. \square

Remark 3.3. As we have remarked earlier, when working in Euclidean space, the Lipschitz condition in the Definition 3.1.3 is simply $|f(t, x) - f(t, y)| \leq L|x - y|$. And Lemma 3.2 amounts to

$$|x_1 - x_2|e^{-L(\tau-t)} \leq |\phi(\tau; t, x_1) - \phi(\tau; t, x_2)| \leq |x_1 - x_2|e^{L(\tau-t)}. \quad (3.13)$$

This is well-known, see for example in [58].

Our main result in this section is the following converse result for GES on Riemannian manifolds.

Theorem 3.2. Assume that $f(\cdot, x)$ is globally Lipschitz (with constant L). Let x_* be a GES equilibrium point of the system (3.1) on M . Then there exists a continuous Lyapunov candidate V satisfying the following properties:

1. There exist two positive constants c_1 and c_2 , such that

$$c_1d(x, x_*) \leq V(t, x) \leq c_2d(x, x_*), \quad \forall x \in M, t \geq 0. \quad (3.14)$$

2. The Lie derivative of $V(t, x)$ in the sense of Definition 3.1.2 along the system satisfies

$$\mathcal{L}_f V(t, x) \leq -c_3V(t, x), \quad \forall x \in M, t \geq 0 \quad (3.15)$$

where c_3 is a positive constant.

3. If $d(\cdot, x_*) : M \rightarrow \mathbb{R}$ is class C^1 . Then for every t , the differential of $V(t, x)$, $dV(t, x) \in T^*M$ is uniformly bounded on T^*M :

$$|dV(t, x)| \leq c_4, \quad \forall x \in M, t \geq 0 \quad (3.16)$$

where c_4 is a positive constant independent of t and x .

Remark 3.4. For $\omega \in T^*M$, the norm of ω is defined as

$$|\omega| = \sup_{\langle v, v \rangle = 1} \langle \omega | v \rangle$$

where $\langle \cdot | \cdot \rangle$ stands for the pairing between ω and v .

Proof. Item 1: Consider the function

$$V(t, x) = \int_t^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau. \quad (3.17)$$

Setting $x_2 = x_*$ in Lemma 3.2, we get the following estimate by the fact that x_* is an equilibrium point:

$$d(x, x_*)e^{-L(\tau-t)} \leq d(\phi(\tau; t, x), x_*) \leq d(x, x_*)e^{L(\tau-t)}, \quad (3.18)$$

$\forall \tau \geq t, \forall x \in M$. Thus the defined function (3.17) admits the following bounds:

$$\begin{aligned} V(t, x) &= \int_t^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau \\ &\geq \int_t^{t+\delta} d(x, x_*)e^{-L(\tau-t)} d\tau \\ &= \frac{1 - e^{-L\delta}}{L} d(x, x_*), \end{aligned}$$

and

$$\begin{aligned} V(t, x) &= \int_t^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau \\ &\leq \int_t^{t+\delta} K e^{-\lambda(\tau-t)} d(x, x_*) d\tau \\ &= \frac{K(1 - e^{-\lambda\delta})}{L} d(x, x_*). \end{aligned}$$

So we can find two positive constants c_1, c_2 such that

$$c_1 d(x, x_*) \leq V(t, x) \leq c_2 d(x, x_*), \quad \forall x \in M, t \geq 0. \quad (3.19)$$

Item 2: In order to estimate the evolution of V along the system's solutions, we again utilize the semi-group property:

$$V(s, \phi(s; t, x)) = \int_s^{s+\delta} d(\phi(\tau; s, \phi(s; t, x)), x_*) d\tau = \int_s^{s+\delta} d(\phi(\tau; t, x), x_*) d\tau.$$

Therefore

$$\mathcal{L}_f V(t, x) = \left. \frac{d}{ds} V(s, \phi(s; t, x)) \right|_{s=t} = d(\phi(t + \delta; t, x), x_*) - d(x, x_*)$$

$$\begin{aligned}
&\leq -(1 - Ke^{-\lambda\delta})d(x, x_*) \\
&= -K'd(x, x_*), \quad \forall t, \forall x \in M,
\end{aligned} \tag{3.20}$$

where δ is chosen such that $K' > 0$. By (3.19),

$$\mathcal{L}_f V(t, x) \leq -K'd(x, x_*) \leq -\frac{K'}{c_2}V(t, x).$$

Now c_3 can be set as $c_3 = K'/c_2$.

Item 3: Denote $h_t(x) = V(t, x)$, then for any $v \in T_x M$,

$$dh_t(v) = \left. \frac{d}{ds} \right|_{s=0} h_t(c(s))$$

where $c : [-\varepsilon, \varepsilon] \rightarrow M$ is a smooth curve with $c'(0) = v$, $\varepsilon > 0$ a constant. Hence

$$dh_t(v) = \int_t^{t+\delta} \left. \frac{d}{ds} d(\phi(\tau; t, c(s)), x_*) \right|_{s=0} d\tau \tag{3.21}$$

Since $(\tau, s) \mapsto \phi(\tau; t, c(s))$ is a variation of the curve c (see Definition 2.2.7), then by the first variation formula,

$$\left. \frac{d}{ds} d(\phi(\tau; t, c(s)), x_*) \right|_{s=0} = \langle \phi(\tau; t, x)_* v, \gamma'(1) \rangle,$$

where $\phi(\tau; t, x)_* v$ is the push forward of the vector $c'(0)$ by the map $x \mapsto \phi(\tau; t, x)$ and γ is the normalized geodesic joining x_* to $\phi(\tau; t, x)$. Hence

$$\left| \left. \frac{d}{ds} d(\phi(\tau; t, c(s)), x_*) \right|_{s=0} \right|^2 \leq \langle \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \rangle. \tag{3.22}$$

Now we estimate the term on the right hand side.

$$\begin{aligned}
\frac{d}{d\tau} \frac{1}{2} \langle \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \rangle &= \left\langle \frac{D}{d\tau} \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \right\rangle \\
&= \left\langle \nabla_{f(\phi(\tau; t, x))} \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \right\rangle \\
&= \left\langle \nabla_{\phi(\tau; t, x)_* v} f(\phi(\tau; t, x)), \phi(\tau; t, x)_* v \right\rangle \\
&\quad + \langle [f(\phi(\tau; t, x)), \phi(\tau; t, x)_* v], \phi(\tau; t, x)_* v \rangle,
\end{aligned}$$

where we have used the symmetry of the Levi-Civita connection, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

However,

$$[f(\phi(\tau; t, x)), \phi(\tau; t, x)_* v] = L_{f(\phi(\tau; t, x))} \phi(\tau; t, x)_* v$$

$$= \frac{d}{d\tau} \phi^* \phi_* v = \frac{d}{d\tau} v = 0.$$

Therefore

$$\begin{aligned} \frac{d}{d\tau} \frac{1}{2} \langle \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \rangle &= \langle \nabla_{\phi(\tau; t, x)_* v} f(\phi(\tau; t, x)), \phi(\tau; t, x)_* v \rangle \\ &\leq L \langle \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \rangle \end{aligned}$$

where we have used that fact that $|\nabla_v f| \leq L|v|$. So

$$\langle \phi(\tau; t, x)_* v, \phi(\tau; t, x)_* v \rangle \leq e^{2L(\tau-t)} |v|^2$$

or

$$\begin{aligned} |\phi(\tau; t, x)_* v| &\leq e^{L(\tau-t)} |v|, \quad \forall \tau \geq t. \\ dh_t(v) &\leq \int_t^{t+\delta} e^{L(\tau-t)} |v| d\tau = \frac{e^{L\delta} - 1}{L} |v|. \end{aligned}$$

So we have obtained

$$|d_x V(t, x)(v)| \leq c_4 |v|$$

or

$$|d_x V(t, x)| \leq c_4, \quad \forall x \in M,$$

where $c_4 = (e^{L\gamma} - 1)/L$. □

Remark 3.5. In Item 3, the distance function $d(x, x_*)$ is required to be C^1 . This is however, not guaranteed in general. For example, in Euclidean space, $d(x, x_*) = |x - x_*|$, which is not differentiable at the point x_* . To remedy this, it suffices to consider the following Lyapunov candidate:

$$V(t, x) = \int_t^{t+\gamma} d(\phi(\tau; t, x), x_*)^2 d\tau \quad (3.23)$$

The proof can be carried out in exactly the same way as Theorem 3.2, except that the claims of Item 1 and 3 should change accordingly: the bound of $V(t, x)$ should be

$$c_1 d(x, x_*)^p \leq V(t, x) \leq c_2 d(x, x_*)^2$$

and dV satisfies

$$|dV(t, x)| \leq c_4 d(x, x_*).$$

Remark 3.6. In contrast to the proof in [122], all the proof here is coordinate-free. So if the system has an invariant set U as region of attraction, then a Lyapunov function can be naturally defined everywhere on U .

Remark 3.7. In Euclidean space, the Lyapunov candidate becomes

$$V(t, x) = \int_t^{t+\delta} |\phi(\tau; t, x)|^2 d\tau$$

this reduces to the standard construction, see for example [58].

3.1.2 Discussions and applications

Theorem 3.2 requires the system to be globally stable. However, such requirement is not essential. In fact, all the procedures of the proof can be done locally in the same manner. Hence we can obtain local version of converse theorems.

The extension to asymptotically stability is also not difficult. Following [58], we just need to modify the Lyapunov candidate to

$$V(t, x) = \int_t^\infty G(d(\phi(\tau; t, x), x_*))d\tau$$

where G is constructed from the following Massera's lemma.

Lemma 3.3 (Massera). Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a positive, continuous, strictly decreasing function with $g(t) \rightarrow 0$ as $t \rightarrow \infty$. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a positive, continuous, non decreasing function. Then, there exists a function $G(t)$ such that

1. G and its derivative G' are class \mathcal{K} functions;
2. for any continuous function $u(t)$ that satisfies $0 \leq u(t) \leq g(t)$ for all $t \geq 0$, there exist positive constants k_1 and k_2 , independent of u , such that

$$\int_0^\infty G(u(t))dt \leq k_1; \quad \int_0^\infty G'(u(t))h(t)dt \leq k_2.$$

The rest of the proof can be done similarly as that of Theorem 3.2. That is to say, we have the following theorem.

Theorem 3.3. Let x_* be a UGAS equilibrium point of the system (3.1) on the M , i.e.,

$$d(\phi(t; t_0, x_0), x_*) \leq \beta(d(x_0, x_*), t - t_0), \quad \forall t \geq t_0, \quad x_0 \in M$$

for a class \mathcal{KL} function β . Then there exists a Lyapunov candidate V such that for all $t \geq t_0 \geq 0$ and all $x \in M$, the following three properties hold

1. V is C^1 and satisfies

$$\alpha_1(d(x, x_*)) \leq V(t, x) \leq \alpha_2(d(x, x_*)),$$

2. The timed Lie derivative of $V(t, x)$ along the system satisfies

$$\mathcal{L}_f V(t, x) \leq -\alpha_3(V(t, x))$$

3. If $d(\cdot, x_*) : M \rightarrow \mathbb{R}$ is class C^1 . Then for every t , the differential of $V(t, x)$, $dV(t, x) \in T^*M$ satisfies

$$|dV(t, x)| \leq \alpha_4(V(t, x)) \quad (3.24)$$

where α_i , $i = 1, 2, 3, 4$ are class \mathcal{K}_∞ functions.

As an application, we show that Theorem 3.2 can be applied to prove the input-to-state stability (ISS) of a class of systems. The classical form of this theorem can be found in [58].

Corollary 3.1. Consider the control system

$$\dot{x} = f(t, x, u) \quad (3.25)$$

on Riemannian manifold M , where f is C^1 and globally Lipschitz in x and u is bounded. Additionally, assume f is globally Lipschitz in u with constant L , i.e.,

$$|f(t, x, u) - f(t, x, 0)| \leq L|u|.$$

If the unforced system $\dot{x} = f(t, x, 0)$ is GES with respect to equilibrium point $x = 0$, then the system (3.25) is ISS.

Proof. By Theorem 3.2, a Lyapunov function V verifying the three conditions can be constructed for the unforced system $\dot{x} = f(t, x, 0)$. Rewrite

$$f(t, x, u) = f_1 + f_2$$

where

$$\begin{aligned} f_1 &= f(t, x, 0) \\ f_2 &= f(t, x, u) - f(t, x, 0). \end{aligned}$$

By assumption, $\mathcal{L}_{f_1}V \leq -c_3V$. The Lie derivative of $V(t, x)$ with respect to (3.25) reads

$$\begin{aligned} \mathcal{L}_fV &= \frac{\partial V}{\partial t} + L_fV = \frac{\partial V}{\partial t} + dV(f_1 + f_2) \\ &= \mathcal{L}_{f_1}V + dV[f(t, x, u) - f(t, x, 0)] \\ &\leq -c_3V + c_4|f(t, x, u) - f(t, x, 0)| \\ &\leq -c_3V + c_4L|u|_\infty. \end{aligned}$$

Now invoking Gronwall's lemma, we conclude that the system (3.25) is ISS. \square

3.2 Local Exponential Stability of Trajectories

In this section, we study another important kind of stability on Riemannian manifolds: the stability of a particular solution which is not an equilibrium. We call such solution a nontrivial solution.

It is well known that local stability of an equilibrium can be analyzed via linearization in local coordinates (Lyapunov indirect method), just like in Euclidean spaces. Many control objectives request to go beyond stability of a fixed point and to assess stability of a *particular solution* $X(\cdot)$ of the system. This problem classically arises in observer design [19, 18], trajectory tracking [52, 90, 25], orbital stabilization [139]. In Euclidean spaces, this may be solved by introducing an error variable between the target trajectory and the actual state and by studying its dynamics, which is a nonlinear time-varying system. In particular, local exponential stability (LES) of $X(\cdot)$ can be characterized by the linearization of this error dynamics near the origin.

For systems evolving on Riemannian manifolds, stability analysis of a given solution $X(\cdot)$ is more challenging. The difficulty arises from two aspects. On the one hand, the “error dynamics” is more involved than in the Euclidean case, as the induced Riemannian distance on manifolds can hardly be used to derive error dynamics directly: there are, indeed, no generally preferred definition of tracking (or observation, synchronization) errors for such systems. In practice, one has to choose an error vector according to the structure of the manifold [25, 64, 79]. On the other hand, the alternative method (via first-order approximation or partial contraction) is nontrivial when applied to Riemannian manifolds, since it is usually a daunting task to calculate the differential dynamics which involves complicated computations of parallel transport. Overcoming these two major challenges is the main motivation of this section.

We provide an alternative way to study LES of trajectories on Riemannian manifolds. More precisely, we show that LES of a given trajectory is equivalent to exponential stability of the origin of the *complete lift* of the system along the trajectory. In this way, we remove the need of obtaining error dynamics and simplify the problem of stability of a trajectory to that of an equilibrium. Complete lift (sometimes called tangent lift or prolongation) has already been used to study various control problems: see for instance [29, 129, 24, 23]. In particular, [23] has already remarked that the complete lift can be seen as a linearization procedure.

For systems evolving in Euclidean spaces, studying stability of a nontrivial solution via first-order approximation has already been used and is known as partial (or virtual) contraction [133, 41]. A successful application may be found in [21] for the stability of

extended Kalman filter.

Stability of a particular solution is tightly linked to contraction analysis or incremental stabilities [72, 41, 8] that we have introduced in Chapter 2. In contraction analysis, stability is imposed on every trajectory since it is required that any pair of solutions converge to one another. On the one hand, this allows to conduct the analysis or control design without any prior knowledge about the target trajectory $X(\cdot)$. On the other hand, this requirement limits the domain of applicability of this approach. For instance, observer design for non-uniformly observable systems can be achieved (although the requested uniformity does not hold) by imposing persistency of excitation (PE) [92] or by ensuring uniform complete observability along a specific class of trajectories [17]. Another illustration is that of trajectory tracking and formation control of non-holonomic systems, which require PE conditions to achieve asymptotic stability [78]. Taking account of these issues, we study stability of nontrivial solutions on Riemannian manifolds in this section.

3.2.1 Fundamentals

In this section, we assume the system is of the form (3.1) and that the system is forward complete. Let X denote a particular solution of this system, namely $\dot{X}(t) = f(t, X(t))$ for all $t \geq 0$. Local exponential stability of the particular solution X is defined as follows.

Definition 3.2.1 (LES of a solution). The solution X of the system (3.1) is *locally exponentially stable* (LES) if there exist positive constants c, K and λ such that, for all $t_0 \geq 0$ and all $x_0 \in M$ satisfying $d(x_0, X(t_0)) < c$, it holds that

$$d(\phi(t; t_0, x_0), X(t)) \leq Kd(x_0, X(t_0))e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0. \quad (3.26)$$

Remark 3.8. Intuitively, an LES solution “attracts” all neighboring trajectories at an exponential rate that is independent of the initial time t_0 . This definition should not be confused with exponential stability of a path, in which solutions are requested to exponentially converge to the set $\{X(t) : t \geq 0\}$, and which thus constitutes a weaker property than that of Definition 3.2.1. In particular, a periodic orbit of an *autonomous system* can never be LES: for two different initial points x, y on the periodic orbit, $\limsup_{t \rightarrow \infty} d(\phi(t; t_0, x), \phi(t; t_0, y)) \geq d(x, y)$. However, the periodic solution to a time-varying system can be LES. A simple example can be $\dot{x} = -x + \cos(t) + \sin(t)$ to which $\sin(t)$ is an LES solution as can be easily checked from the definition.

In the case when X is constant (meaning $X(t) = X_0$ for all $t \geq t_0$), Definition 3.2.1 coincides with the definition of LES of an equilibrium, see Definition 3.1.1, and also

[24, 122]. It is worth stressing that LES solutions may form a dense set in M : for instance, every solution of $\dot{x} = Ax$ is LES when A is a Hurwitz matrix.

Remark 3.9. It is well-known that if the autonomous system (3.2) is LES at an equilibrium point x_* , then there exists a neighborhood U of x_* such that U is forward invariant. This is easily shown by applying the Theorem 3.2. For example, let V be a Lyapunov function constructed via the converse theorem. Let $D = \{x : V(x) \leq c\}$ where $c > 0$ is a sufficiently small constant. Obviously D invariant since $\dot{V} \leq 0$. A natural question is, if a non-trivial solution X is LES, does there exist an invariant set which contains X ? In fact, this question can be answered in a similar fashion by constructing a ‘‘Lyapunov function’’.

Proposition 3.1. Suppose that X is an LES solution to the system (3.1). Then there exists a function $V : \mathbb{R}_+ \times U \rightarrow \mathbb{R}_+$, such that

- U is an open neighborhood of X ;
- there exist positive constants c_1, c_2, c_3 satisfying

$$c_1 d(x, X(t))^2 \leq V(t, x) \leq c_2 d(x, X(t))^2,$$

and

$$\mathcal{L}_f V(t, x) \leq -c_3 d(x, X(t))^2$$

for all $(t, x) \in \mathbb{R}_+ \times U$.

Proof. Let

$$V(t, x) = \int_t^\infty d(\phi(\tau; t, x), X(\tau))^2 d\tau. \quad (3.27)$$

Then the proposition is proved by repeating verbatim the procedure in the proof of Theorem 3.2. \square

Now we are able to construct an invariant open neighborhood of X . For this, let

$$D_c := \bigcup_{t \geq 0} \{x : V(t, x) < c\}.$$

where c is a sufficiently small positive constant. Then

- D_c is nonempty and open since it is the union of open sets;
- $\{X(t) : t \geq 0\} \subseteq D_c$ since $V(t, X(t)) = 0$ for all $t \geq 0$. Thus D_c is an open neighborhood of X ;

- D_c is invariant, since if $x \in D_c$, by definition, there exists a $t \geq 0$, such that $V(t, x) < c$. Since $\dot{V} \leq 0$, we have

$$V(s, \phi(s; t, x)) \leq V(t, x) < c, \quad \forall s \geq t$$

Therefore, $\phi(s; t, x) \in D_c$ for all $s \geq t$.

3.2.2 LES and complete lift

The results presented in this thesis make an extensive use of the complete lift of a vector field which we have introduced in Section 2.1. Complete lift is a term widely used in differential geometry [138, 30], see Definition 2.1.18. Its application can also be found in control theory. For example, A. van der Schaft *et al.* used this concept to study prolonged system and differential passivity [29, 129, 128] in a coordinate free manner. F. Bullo and A. D. Lewis used it to study the linearization of nonlinear mechanical systems, see [24] and its online supplementary materials. Complete lift is also referred to as *prolongation* [31].

Definition 3.2.2 (Complete lift along a solution). The *complete lift of the system (3.1) along a given solution X* is defined as

$$\dot{v} = \tilde{f}(v, t), \quad v(t) \in T_{X(t)}M, \quad (3.28)$$

where \tilde{f} denotes the complete lift of f .

The most important property of the complete lift system is its *linearity* at a fixed fibre, namely, if v_1 and v_2 are two solutions to (3.28), then so is $\alpha_1 v_1 + \alpha_2 v_2$, where α_1, α_2 are two arbitrary real constants.

From the above definition, one can easily verify that any solution v of (3.28) has the property that $\pi(v(t)) = X(t)$ for all $t \geq 0$, where π is the projection map. Hence we say that (3.28) defines a dynamical system along the particular solution X .

Definition 3.2.3. The system (3.28) is exponentially stable if there exist two positive constants k, λ such that

$$|v(t)| \leq k e^{-\lambda t} |v(0)|, \quad \forall t \geq 0 \quad (3.29)$$

Our main result in this section is the following, which can be viewed as an analogue of the Lyapunov direct method on Riemannian manifolds.

Theorem 3.4. If a solution X of (3.1) is LES, then the complete lift of the system (3.1) along X is exponentially stable. If the system (3.1) is periodic in t and X is bounded, the converse is also true.

The above statement thus allows to transform the stability analysis of a solution into the stability analysis of a point (the “origin” of the complete lift, or more precisely, the zero section of the pullback bundle X^*TM). The proof of this theorem relies on the following lemma.

Lemma 3.4. Let γ_1, γ_2 be two continuously differentiable curves in M , where M is a Riemannian manifold. If $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$, then $d(\gamma_1(s), \gamma_2(s)) = O(s^2)$ when $s > 0$ is sufficiently small.

Proof. Fixing $s > 0$, let $t \mapsto F(t, s)$, $t \in [0, T]$ where $T = d(\gamma_1(s), \gamma_2(s))$ is the normalized geodesic joining $\gamma_1(s)$ to $\gamma_2(s)$. Then $F(t, s)$ forms a smooth variation along the geodesic. By the the first variational formula of arc-length (see Section 2.2), we have

$$\frac{d}{ds}d(\gamma_1(s), \gamma_2(s)) = \langle \gamma_2'(s), v_2'(s) \rangle - \langle \gamma_1'(s), v_1'(s) \rangle$$

where

$$v_1'(s) = \frac{\partial F}{\partial t}(0, s), \quad v_2'(s) = \frac{\partial F}{\partial t}(T, s).$$

Clearly $v_1'(s) \rightarrow v_2'(s)$ as $s \rightarrow 0$ since $T \rightarrow 0$. This together with the fact $\gamma_1'(s) \rightarrow \gamma_2'(s)$ implies

$$\left. \frac{d}{ds} \right|_{s=0+} d(\gamma_1(s), \gamma_2(s)) = 0$$

Hence we can conclude that $d(\gamma_1(s), \gamma_2(s)) = O(s^2)$. \square

Proof of Theorem 3.4. (\implies) Assume that the solution X is LES. Then there exist $c, K, \lambda > 0$ such that (3.26) is satisfied. Given any $t_0 \geq 0$ and any $x_0 \in M$ such that $d(x_0, X(t_0)) < c$, denote the minimizing normalized (*i.e.*, with unit speed) geodesic joining $X(t_0)$ to x_0 as $\gamma : [0, \hat{s}] \rightarrow M$, where $\hat{s} := d(X(t_0), x_0) \geq 0$. It holds in particular that

$$\gamma(0) = X(t_0), \quad \gamma(\hat{s}) = x_0.$$

Let $v_0 \in TM$ be such that $\pi(v_0) = X(t_0)$ and $v_0 = \gamma'(0)$, and let $v(\cdot)$ be the solution to the complete lift system (3.28) starting from v_0 at t_0 . Then it holds that

$$\hat{s} |v(t)| = d\left(\exp_{X(t)}(\hat{s}v(t)), X(t)\right), \quad \forall t \geq t_0, \quad (3.30)$$

where $\exp_x : TM \rightarrow M$ is the exponential map, by choosing \hat{s} sufficiently small that \exp is defined. Using the metric property of d , we have that

$$\begin{aligned} & d\left(\exp_{X(t)}\left(\hat{s}v(t)\right), X(t)\right) \\ & \leq d\left(\exp_{X(t)}\left(\hat{s}v(t)\right), \phi(t; t_0, x_0)\right) + d\left(\phi(t; t_0, x_0), X(t)\right) \\ & \leq d\left(\exp_{X(t)}\left(\hat{s}v(t)\right), \phi(t; t_0, x_0)\right) + K\hat{s}e^{-\lambda(t-t_0)}. \end{aligned} \quad (3.31)$$

It follows from (3.30) and (3.31) that

$$|v(t)| \leq \kappa(\hat{s}) + Ke^{-\lambda(t-t_0)} \quad (3.32)$$

where, for each fixed $t \geq t_0$,

$$\kappa(\hat{s}) := \frac{d\left(\exp_{X(t)}\left(\hat{s}v(t)\right), \phi(t; t_0, x_0)\right)}{\hat{s}}. \quad (3.33)$$

Note that κ is a function of both t and \hat{s} , but omitting the t argument does not affect the following analysis. We next show that the term $\kappa(\hat{s})$ is of order $O(\hat{s})$. To this end first notice that, since $x_0 = \gamma(\hat{s})$, this term can be rewritten as

$$\kappa(\hat{s}) = \frac{d\left(\exp_{X(t)}\left(\hat{s}v(t)\right), \phi(t; t_0, \gamma(\hat{s}))\right)}{\hat{s}}$$

Consider the functions $\alpha_1 : \hat{s} \mapsto \exp_{X(t)}(\hat{s}v(t))$ and $\alpha_2 : \hat{s} \mapsto \phi(t; t_0, \gamma(\hat{s}))$ (here again, we have omitted the t argument). Then it holds that $\alpha_1(0) = \alpha_2(0) = X(t)$ and $\alpha_1'(0) = \alpha_2'(0) = v(t)$. Thus by Lemma 3.4, we have $\kappa(\hat{s}) = O(\hat{s})$, or

$$\lim_{\hat{s} \rightarrow 0+} \kappa(\hat{s}) = 0, \quad \text{as } \hat{s} \rightarrow 0+$$

Now letting $\hat{s} \rightarrow 0+$ in (4.25) and recalling that the geodesic has unit speed, we conclude from (3.32) that

$$|v(t)| \leq K|v(t_0)|e^{-\lambda(t-t_0)},$$

for any $v(t_0) \in T_{X(t_0)}M$, meaning that the origin of the complete lift is LES.

The proof of the converse is postponed to Chapter 4, Section 4.6. \square

A direct consequence of Theorem 3.4 is the following sufficient condition for LES of a solution.

Corollary 3.2. Consider a particular solution X of system (3.1). If there exists a constant $k > 0$ such that

$$\langle \nabla_v f(t, x), v \rangle|_{x=X(t)} \leq -k\langle v, v \rangle, \quad \forall v \in T_{X(t)}M, \quad t \geq 0, \quad (3.34)$$

then the solution X is LES.

Proof. Let $v(\cdot)$ denote the solution of the complete lift system (3.28) along the solution X with initial condition v_0 and consider the Lyapunov function V defined as $V(v) := \frac{1}{2}\langle v, v \rangle$. Then, for all $t \geq t_0 \geq 0$, it holds that

$$\begin{aligned} \frac{dV(v(t))}{dt} &= \langle \nabla_{\dot{X}(t)} v(t), v(t) \rangle \\ &= \langle \nabla_{v(t)} \dot{X}(t), v(t) \rangle \\ &= \langle \nabla_{v(t)} f(X(t), t), v(t) \rangle \\ &\leq -k \langle v(t), v(t) \rangle \\ &\leq -2kV(v(t)), \end{aligned}$$

where we have used the fact that $\nabla_{\dot{X}(t)} v(t) = \nabla_{v(t)} \dot{X}(t)$ since $[\dot{X}(t), v(t)] = 0$. It follows that $|v(t)| \leq |v(t_0)|e^{-k(t-t_0)}$, meaning that the origin of the complete lift is LES. The conclusion follows by Theorem 3.4. \square

Remark 3.10. In [112], a similar result to Corollary 3.2 is obtained. The only difference is that the left hand side term of (3.34) is evaluated for all $v \in TM$ in [112], which is exactly the meaning of contraction that we will study in the next chapters. For this reason, we postpone the discussion of the details to the next chapter. The advantage of Corollary 3.2 is that it does not involve the calculation of the complete lift system along X .

An alternative proof

We now provide an alternative proof of Corollary 3.2, as the technique will be adopted to study the stability of Lagrangian systems in later chapters. Choose a curve $\gamma : [0, 1] \rightarrow M$ passing through $X(t_0)$ at $s = 0$. Then, given any $t_0 \geq 0$, $(t, s) \mapsto \phi(t; t_0, \gamma(s))$ forms a variation of the curve X . Let $q : (s, t) \mapsto \phi(t; t_0, \gamma(s))$, $q' : s \mapsto \frac{\partial q}{\partial s}(s, t)$, and $\dot{q} : t \mapsto \frac{\partial q}{\partial t}(s, t)$. It can be verified that q' is the solution to the complete lift system (3.2.2) with the initial condition $\gamma'(0)$. Taking the covariant derivative on both sides of (3.1) along X , we get

$$\nabla_{q'} \dot{q} = \nabla_{\dot{q}} q' = \nabla_{q'} f.$$

Hence

$$\frac{1}{2} \frac{d}{dt} \langle q', q' \rangle|_{s=0} = \langle \nabla_{q'} f, q' \rangle|_{s=0} \leq -k \langle q', q' \rangle|_{s=0}.$$

Note that the curve $q'|_{s=0}$ is the Lie transport of the vector $q'|_{s=0, t=0}$ along the curve q and hence the solution to the complete lift system. Since γ is arbitrary, the conclusion follows from Theorem 3.4.

Revisit of LES of an equilibrium

Suppose that the system (3.1) has an equilibrium $x_* = 0$. We characterize local exponential stability of the equilibrium using Theorem 3.4. Since we are concerned with LES, without loss of generality, assume that the system (3.1) evolves in \mathbb{R}^n with a Riemannian metric given by a constant symmetric positive definite matrix P , and that $X(t) = 0$ for all $t \geq 0$. Then the solution to the complete lift system along $X(t) \equiv 0$ is

$$\dot{V}(t) = \frac{\partial f(t, 0)}{\partial x} V. \quad (3.35)$$

That is, (3.35) shares the same form of the linearized system of (3.1) near the origin. Theorem 3.4 says that if the system (3.35) is exponentially stable, so is the system (3.1), which is what the well-known Lyapunov indirect method says. There is however, a difference between (3.35) and the linearization of (3.1): in (3.35), the vector $V(t) \in T_0\mathbb{R}^n$ here is a tangent vector. The reason why they share the same form in Euclidean space is that the tangent space at the origin can be naturally identified with the state space \mathbb{R}^n . However, on a Riemannian manifold, such identification does not hold along a nontrivial solution.

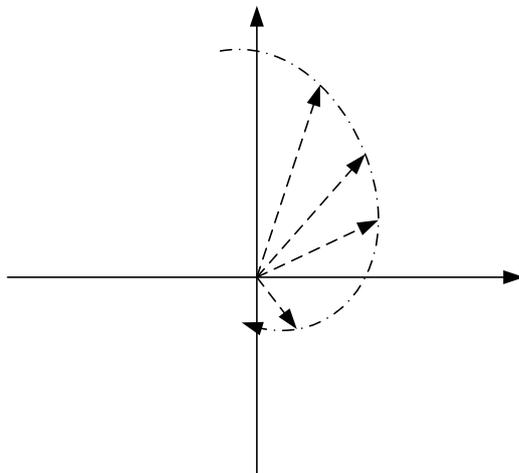


Figure 3.1: Complete lift in Euclidean space

3.2.3 Boundedness and aperiodicity of LES solutions

When restricting to time-invariant systems, namely the system (3.2), some interesting features can be derived from Theorem 3.4. For example, any LES solution of (3.2) is necessarily bounded and it cannot be periodic unless it is constant.

Corollary 3.3. Let X be an LES solution of the time-invariant system (3.2). Then it is bounded and, if it is periodic, it is necessarily constant.

Note: This result does not hold for time-varying systems. For instance, as remarked in [101], the solution $X(t) = t$ is LES for the system $\dot{x} = t - x$, and it is clearly unbounded. Similarly, the solution $X(t) = \sin(t)$ is LES for $\dot{x} = \sin(t) - x$ and it is clearly periodic.

Proof. For time-invariant system, $v = \dot{X}$ is a solution of the complete lift system of (3.2). Then it follows from Theorem 3.4 that there exist $K, \lambda > 0$ such that

$$|\dot{X}(t)| \leq K|\dot{X}(0)|e^{-\lambda t}, \quad \forall t \geq 0. \quad (3.36)$$

Hence X cannot be periodic unless it is constant. Furthermore, $d(X(t), X(0)) \leq \ell(\gamma)$ where γ is the curve $s \mapsto X(s)$, $s \in [0, t]$. Therefore, it holds from (3.36) that

$$\begin{aligned} d(X(t), X(0)) &\leq \int_0^t |\dot{X}(s)| ds \\ &\leq \int_0^t k|\dot{X}(0)|e^{-\lambda s} ds \\ &= \frac{k|\dot{X}(0)|}{\lambda}(1 - e^{-\lambda t}) \\ &\leq \frac{k|\dot{X}(0)|}{\lambda}, \end{aligned}$$

which shows that X is bounded. □

In [48, Lemma 1], the authors recently obtained a similar result for *autonomous* systems, namely, there is a unique attractive equilibrium – limit cycle will never appear – in a forward invariant set in which the system is incrementally exponentially stable.

As we will see in the next chapter, incremental exponential stability has close relationship with LES.

Applications of the results in this chapter will be reported in Chapter 5.

3.3 A Brief Summary

In this chapter, we have studied stability analysis on Riemannian manifolds. The main contributions are now reviewed.

First, we proved converse Lyapunov theorem on Riemannian manifolds in a coordinate free manner. In the proof, a key step is to use a proper definition of Lipschitz continuity on Riemannian manifolds.

Second, we studied local exponential stability of nontrivial solutions on Riemannian manifolds by lifting the system along the target trajectory into the pullback tangent bundle. Since the lifted system is fibre-wise linear and it is therefore easier to study.

Another important task of this chapter is to lay some foundations for next chapter. In particular, some estimations will prove to be useful. Furthermore, at the end of next chapter, we will find out that there exist strong relationships between LES of nontrivial solutions and contraction.

Chapter 4

Contraction Analysis on Riemannian Manifolds

In this chapter, we propose a *geometric* (or *intrinsic*) framework for contraction analysis on Riemannian manifolds. By geometric or intrinsic, we mean that the approach does not depend on the choice of local coordinates. As we have pointed out in Chapter 1, although contraction theory has been well established for systems evolving in Euclidean spaces, it is generally less understood for systems on manifolds. It is true that when we are concerned with local contraction, working in local coordinates is possible, see for example [41, 136]. But there are at least two drawbacks of such method: first, calculation in local coordinates may be quite involved; second, when expressed in local coordinates, it may be difficult to see how the geometry (e.g., the curvature) of the manifold affects the contraction of the system. In view of these issues, we rely on the work proposed by F. Forni and R. Sepulchre [41] to develop an intrinsic framework for contraction analysis on manifolds.

In [41], F. Forni and R. Sepulchre introduced two essential objects, namely, the Finsler structure and the Finsler-Lyapunov function. Then sufficient conditions of incremental stability are deduced by utilizing these two notions. The advantage of this framework was justified by showing that numerous previous works in the literature can be unified utilizing this formalism. Nevertheless, there still remain several important issues and interesting questions in this framework that need to be addressed:

- Q1** Most of the results in [41] as well as their proofs are handled in local coordinates. Therefore, the geometric interpretations of the differential conditions obtained therein need to be clarified. This leads to the following question: what is the geometric essence of this framework?

Q2 The main theorem in [41] (Theorem 2.10) gives a sufficient condition for incremental stability. A natural question is: is it also necessary? Equivalently, can we prove converse theorems?

Q3 When a system has an equilibrium, incremental stability implies stability of the equilibrium. In this case, what is the connection between incremental stability and stability of the equilibrium?

We provide answers to the questions above in this chapter:

- We give an intrinsic form condition expressed in the tangent bundle, which guarantees the incremental stability of the system. This is achieved by studying the behaviour of the complete lift of the system. The result easily recovers one of the main results in [41] and gives new geometric insights to it.
- We prove converse theorems for exponential incremental stability in a coordinate-free way. The results are expressed in the tangent bundle involving no copy of the original system (cf. [8]).
- We show that contraction can be fully characterized in a tubular neighborhood of the tangent bundle.
- We reveal the relationship between incremental stability and stability, which are linked by the so called Krasovskii's theorem. More precisely, we prove that a Lyapunov function can be directly constructed from the information of contraction.
- The connection between local exponential stability of nontrivial solutions and contraction will also be established.

4.1 Fundamental Theories

It is shown in [41] that a natural setting for contraction analysis is the Finsler geometry. This work introduced the concept Finsler-Lyapunov function (FLF) which is crucial for the characterization of incremental stability. We propose a somewhat relaxed definition of FLF, which will be sufficient for geometric contraction analysis.

Definition 4.1.1. Given a Finsler structure F (refer to Chapter 2 for the definition of Finsler structure) on the manifold M , a *candidate Finsler-Lyapunov function on $U \subseteq M$* is a C^1 function $V : \mathbb{R}_+ \times TM|_U \rightarrow \mathbb{R}_+$ satisfying

$$\alpha_1(F(x, \delta x)) \leq V(t, x, \delta x) \leq \alpha_2(F(x, \delta x)), \quad (4.1)$$

for all $(t, x, \delta x) \in \mathbb{R}_+ \times TM|_U$, where α_1, α_2 are class \mathcal{K}_∞ functions.

Remark 4.1. In [41], a candidate Finsler-Lyapunov function is defined as a function V satisfying

$$c_1 F(x, \delta x)^p \leq V(t, x, \delta x) \leq c_2 F(x, \delta x)^p, \quad (4.2)$$

for some $p \geq 1$ and some $c_1, c_2 > 0$. This is a special case of (4.1) by taking $\alpha_1(s) = c_1|s|^p$ and $\alpha_2(s) = c_2|s|^p$. Hence Definition 4.1.1 is a relaxed version, and we always refer to (4.11) as a Finsler-Lyapunov function in this thesis.

Remark 4.2. To the best of our knowledge, it is J. Mierczyński who firstly used Finsler structure to define Lyapunov functions [86, 85]. But we should underscore that the functions in [85, 86] are defined quite differently from the Finsler-Lyapunov function here in that the former ones are defined on a cone field of the tangent bundle while the later is defined on the whole tangent bundle. The functions in [85, 86] are defined to study cooperative systems. We thus believe that they may have some close relationships with differential positivity studied by F. Forni *et al.* in [40].

Remark 4.3. In Definition 4.1.1, the conditions imposed on α_1 and α_2 can be relaxed to class \mathcal{K} functions, once we have shown the local properties of contraction in Section 4.4.

Since a Riemannian manifold is easier to handle compared to a Finsler one, we focus on Riemannian manifold, and later we discuss how to extend to the Finsler case. In the setting of Riemannian manifold, a candidate Finsler-Lyapunov function satisfies the following condition,

$$\alpha_1(|\delta x|) \leq V(t, x, \delta x) \leq \alpha_2(|\delta x|) \quad (4.3)$$

since $F(x, \delta x) = |\delta x|$, where $|\cdot|$ denotes the induced norm of the Riemannian metric, i.e. $|\delta x| = \sqrt{\langle \delta x, \delta x \rangle}$. The Riemannian distance induced by g is

$$d(x_1, x_2) = \inf_{\gamma \in \Gamma(x_1, x_2)} \int_0^1 |\gamma'(s)| ds$$

where $\Gamma(x_1, x_2)$ is the set of continuous piecewise C^1 curves joining x_1 to x_2 . For any $\gamma \in \Gamma(x_1, x_2)$, there are finitely many points at which γ is not differentiable, therefore when evaluating the above integration, γ' can be taken arbitrarily at these points.

In order to be able to handle local and global contraction at the same time, we give the following definition.

Definition 4.1.2. Let D be a connected subset of the manifold M . Then D is said to have *GC property* if one of the following conditions is satisfied

A1 $D \subseteq M$ is compact with C^1 boundary.

A2 $D \subseteq M$ is closed and geodesically convex.¹

A3 $D = M$.

We are now in position to give the definition of incremental stabilities.

Definition 4.1.3. Given a forward invariant set D of the system (3.1), satisfying the GC property, then the system is called

1. *incrementally stable (IS) on D* if there exists a class \mathcal{K} function α , such that for all $t \geq t_0 \geq 0$ and $x_1, x_2 \in D$,

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \alpha(d(x_1, x_2)); \quad (4.4)$$

In particular, it is called *globally incrementally stable (GIS)* if $D = M$.

2. *incrementally asymptotically stable (IAS) on D* if there exists a class \mathcal{KL} function β , such that for all $t \geq t_0 \geq 0$ and $x_1, x_2 \in D$,

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \beta(d(x_1, x_2), t - t_0); \quad (4.5)$$

In particular, it is called *globally incrementally asymptotically stable (GIAS)* if $D = M$.

3. *incrementally exponentially stable (IES) on D* if there exists $K \geq 1$, $\lambda > 0$, such that for all $t \geq t_0 \geq 0$ and $x_1, x_2 \in D$,

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq K e^{-\lambda(t-t_0)} d(x_1, x_2); \quad (4.6)$$

In particular, it is called *globally incrementally exponentially stable (GIES)* if $D = M$.

Remark 4.4. Incremental stability may be better called *uniform* incremental stability, as the function α is independent of the initial state t_0 . But to keep consistency with the literature, we have chosen not to do so (c.f. [41, Definition 1]). Likewise, the above defined incremental asymptotic stability is indeed *uniform* incremental asymptotic stability.

¹In this thesis, a geodesically convex subset $U \subseteq M$ is such that for any two arbitrary points in U , there exists a C^1 minimizing geodesic joining the two points and lying entirely in U .

Remark 4.5. In the third item of Definition 4.1.3, when $K = 1$, the system is sometimes called *contraction*, see for example [5, 83]. Other researchers refer to contraction as certain incremental stabilities. For example, F. Forni and R. Sepulchre call incremental stability analysis as contraction analysis [72, 41]. In this thesis, we adopt the latter convention.

Remark 4.6. In the above definition, the conditions from (4.4) to (4.6) need to be satisfied for all pairs x_1, x_2 in the domains in concern. This can be relaxed: it is sufficient that these conditions are satisfied for all pairs (x_1, x_2) with $d(x_1, x_2) < c$, where c is some fixed positive constant, see [8]. For example, the system (3.1) is IAS on D if (4.5) is verified for all $\{(x_1, x_2) : d(x_1, x_2) < c\} \cap D$ for some positive constant c .

This remark leads to the following proposition which will be used in next subsection.

Proposition 4.1. Suppose that the system (3.1) is IAS, *i.e.*, (4.5) is satisfied. Additionally, assume that there exist class \mathcal{K} function α , which is differentiable at $r = 0$, and a decreasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $\eta(s) \rightarrow 0$ as $s \rightarrow \infty$, such that

$$\beta(r, s) \leq \alpha(r)\eta(s), \quad \forall r, s \geq 0.$$

Then the system (3.1) is IES.

Proof. Since $r \mapsto \alpha(r)$ is differentiable at $r = 0$, there exist positive constants a, b such that $\alpha(r) \leq ar$, $\forall r \in [0, b]$. Moreover, there exists a constant $T > 0$, such that for all $r \in [0, b]$, there holds $\beta(r, T) \leq cr$, where $c \in (0, 1)$ is a constant. Now given x_1, x_2 , with $d(x_1, x_2) \leq b$, by assumption, we have

$$d(\phi(t_0 + T, t_0, x_1), \phi(t_0 + T, t_0, x_2)) \leq \beta(d(x_1, x_2), T) \leq cd(x_1, x_2),$$

for all $t_0 \geq 0$. Using the semi-group property of ϕ , we can obtain

$$d(\phi(t_0 + nT, t_0, x_1), \phi(t_0 + nT, t_0, x_2)) \leq c^n d(x_1, x_2) = e^{n \ln c} d(x_1, x_2).$$

Now for arbitrary $t \geq t_0$, there exists $n \in \mathbb{N}$, such that $t = nT + t_0 + r$ for $r \in (0, T]$, and therefore

$$\begin{aligned} d(\phi(t, t_0, x_1), \phi(t, t_0, x_2)) &\leq e^{n \ln c} d(\phi(t_0 + r, t_0, x_1), \phi(t_0 + r, t_0, x_2)) \\ &\leq c' e^{n \ln c} d(x_1, x_2) \\ &= K e^{-\lambda(t-t_0)} d(x_1, x_2) \end{aligned}$$

where

$$K = c' e^{-r \ln c / T}, \quad \lambda = -\ln c / T$$

and c' is a positive constant. Invoking Remark 4.6, the proposition follows. \square

4.2 Complete lift and Intrinsic Contraction Analysis

Theorem 2.10 revealed the significance of Finsler-Lyapunov function in contraction analysis. In this Chapter, we explore further the relationship between Finsler-Lyapunov function and contraction. Notice that the results in [41] are local since the conditions are represented in local coordinates. This poses the following problem: in Theorem 2.10, the set D needs to be invariant; assume now that the manifold M is covered by three coordinate charts U_1, U_2, U_3 , and $U_1 \cap U_2 = \emptyset$. If we have already known that the system is GIAS, then neither U_1 nor U_2 can be invariant (otherwise $\phi(t; t_0, x_1)$ and $\phi(t; t_0, x_2)$ with $x_1 \in U_1, x_2 \in U_2$ cannot converge to each other). Now that U_1 and U_2 are not invariant sets, Theorem 2.10 cannot be applied on the two sets simultaneously, hence to analyze GIAS of the system, further analysis will be needed.

To overcome this difficulty, we provide an intrinsic proof of Theorem 2.10 [41]. In particular, an intrinsic form of (2.23) will be given. The key ingredients we need to achieve this goal are two concepts from differential geometry that we have already used in the Chapter 3: the Lie transport of a vector (Definition 2.1.17) and the complete lift of a vector field (Definition 2.1.18). In order to study contraction, Definition 3.2.2 will be modified by just replacing $v(t) \in T_{X(t)}M$ with $v \in TM$ and keeping all the rest unchanged. With these ingredients at hand, coordinate free form of Theorem 2.10 can be proved. For this, we need a technical lemma.

Lemma 4.1. Let $D \subseteq M$ be a compact set with C^1 boundary and suppose that there exists a geodesically convex subset U containing D (this is the case when (M, g) is complete). Then there exists a positive constant $K \geq 1$, such that

$$d(x_1, x_2) \leq \inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma) \leq Kd(x_1, x_2), \quad \forall x_1, x_2 \in D \quad (4.7)$$

where $\ell(\gamma)$ stands for the length of the curve γ and $\Gamma(x_1, x_2)$ the set of continuous piecewise C^1 curves joining x_1 to x_2 .

Proof. Define the function $\sigma : D \times D \rightarrow \mathbb{R}$ as

$$\sigma(x_1, x_2) = \begin{cases} \frac{\inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma)}{d(x_1, x_2)}, & x_1 \neq x_2 \\ 1, & x_1 = x_2 \end{cases}$$

Let $D_1 = \{(x_1, x_2) | x_1 = x_2 \in D\}$ be the diagonal set of $D \times D$. We now show that σ is continuous on $D \times D$. Since the boundary of D is C^1 , it can be shown that the minimizing curve (if it exists) in D between any two points is C^1 and is thus rectifiable.

Case 1, $x_1 \neq x_2$: Let $S_{i,\delta} := B_\delta(x_i) \cap D$, $i = 1, 2$ and $\delta > 0$ small enough such that $S_{1,\delta} \cap S_{2,\delta} = \emptyset$ and smaller than the injectivity radius.² Suppose that $\gamma : [0, 1] \rightarrow D$ is the minimizing curve when it ranges over $\Gamma(x_1, x_2)$ (when the infimum $\inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma)$ is not attained, the procedure is similar by replacing γ with an arbitrary close curve). Now consider two arbitrary points $y_i \in S_{i,\delta}$. Since δ is smaller than the injectivity radius, there exist two minimizing geodesics η_1 joining y_1 to x_1 and η_2 joining x_2 to y_2 such that $\ell(\eta_i) \leq \delta$. Let the minimizing curve joining y_2 to y_1 be $\bar{\gamma}$, see Figure 4.1. Then by the minimizing property, we have

$$\ell(\gamma) \leq \ell(\eta_1 \cup \bar{\gamma} \cup \eta_2) = \ell(\bar{\gamma}) + 2\delta$$

and

$$\ell(\bar{\gamma}) \leq \ell(\eta_1 \cup \gamma \cup \eta_2) = \ell(\gamma) + 2\delta$$

Combining the two inequalities, we get

$$|\ell(\gamma) - \ell(\bar{\gamma})| \leq 2\delta$$

Since δ is arbitrary, $\inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma)$ is continuous on $D \setminus D_1$. On the other hand, the Riemannian distance function d is continuous, hence σ is continuous on $D \setminus D_1$.

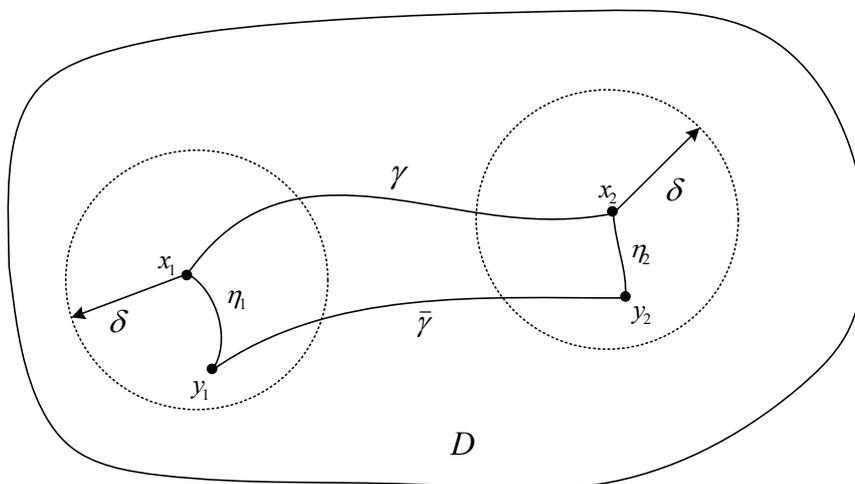


Figure 4.1: Proof Illustration

Case 2, $x_1 = x_2$: It suffices to show that

$$\lim_{d(x_1, x_2) \rightarrow 0^+} \frac{\inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma)}{d(x_1, x_2)} = 1. \quad (4.8)$$

²Recall that $B_\delta(x)$ is the geodesic ball centered at x with radius δ .

When x_1, x_2 are both in the interior of D , then (4.8) is trivially satisfied, therefore it suffices to prove (4.8) when x_1 and x_2 are near the boundary. Without loss of generality, assume $x_1 \in \partial D$. Let $s = d(x_1, x_2)$, and $\gamma : [0, s] \rightarrow M$ be the normalized geodesic joining x_1 to x_2 . Let $\eta : [0, s] \rightarrow D$ be any curve passing through x_1 with $\eta'(0) = \gamma'(0)$. Such curve exists thanks to the differentiability of the boundary ∂D . Then $|\ell(\eta) - d(x_1, x_2)| = O(s^2)$. Therefore, for sufficiently small $s > 0$,

$$\frac{\inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma)}{d(x_1, x_2)} \leq \frac{\ell(\eta)}{d(x_1, x_2)} = 1 + O(d(x_1, x_2)).$$

Letting $s \rightarrow 0+$, we immediately get (4.8).

Now that σ is a continuous function on $D \times D$, which is compact, σ attains a maximum on $D \times D$, say K . This implies (4.7). \square

Lemma 4.2. If there exists a closed subset $U \supseteq D$, such that U is geodesically convex and D has the GC property, then, the following defines a metric on D which makes D a complete metric space:

$$d_K(x, y) := \inf_{\gamma \in \Gamma(x_1, x_2)} \ell(\gamma). \quad (4.9)$$

Proof. If $D = M$, then by assumption M is geodesically complete. By Hopf-Rinow theorem, M is a complete metric space under the Riemannian distance $d = d_K$.

If D is a closed geodesically convex subset. Then again by Hopf-Rinow theorem, D is a complete metric space under the Riemannian distance $d = d_K$.

If D is a compact space of M with C^1 boundary, it suffices to show that d_K defines a metric on D . Then by (4.7), d_K is equivalent to d and the lemma follows. But clearly we have

- $d_K(x, y) = d_K(y, x)$ by reversing the parameterization of the curve;
- $d_K(x, z) \leq d_K(x, y) + d_K(y, z)$ because the curve joining x to y and the curve joining y to z can be joined together to form into a curve joining x to z .
- $d_K(x, y) \geq 0$ and $d_K(x, y) = 0$ if and only if $x = y$.

This completes the proof. \square

Remark 4.7. The inequality (4.7) implies that d_K and d are equivalent norms and induce the same topology on D .

The above lemma allows us to treat the invariant set D as if it is geodesically convex when either one of the assumptions from A1 to A3 of Definition 4.1.2 is satisfied. In particular, when D satisfies assumption A1 or A3, the constant K can be set to 1.

Theorem 4.1. Consider the system (3.1) defined on a Riemannian manifold M , a forward invariant set D satisfying the GC property defined in Definition 4.1.2, a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and the dynamical system defined by the complete lift of f ,

$$\dot{v} = \tilde{f}(v, t), \quad v \in TM|_D. \quad (4.10)$$

Let V be a candidate Finsler-Lyapunov function on D , i.e., there exist two class \mathcal{K}_∞ functions α_1 and α_2 , such that for all $(t, v) \in \mathbb{R}_+ \times TM|_D$,

$$\alpha_1(|\delta x|) \leq V(t, x, \delta x) \leq \alpha_2(|\delta x|). \quad (4.11)$$

and

$$\mathcal{L}_{\tilde{f}}V(t, v) \leq -\alpha(V(t, v)) \quad (4.12)$$

for all $(t, v) \in \mathbb{R}_+ \times TM|_D$, where $\mathcal{L}_{\tilde{f}}$ is the timed Lie derivative along the flow of \tilde{f} , see Definition 3.1.2. Then the system (3.1) is

C1 IS on D if $\alpha(s) = 0$ for each $s \geq 0$;

C2 IAS on D if α is a class \mathcal{K} function;

C3 IES on D if $\alpha_i(s) = c_i|s|^p$ for $i = 1, 2$ and $\alpha(s) = \lambda s$ for some $\lambda > 0$.

All the incremental stabilities above are global when $D = M$.

Proof. By Definition 2.1.18, the trajectory of the system (4.10) started from v is the Lie transport of the vector v along $\phi(t; t_0, \pi(v))$. Given two points $x_1, x_2 \in D$, there is a normalized curve $\gamma : [0, \ell] \rightarrow M$ joining x_1 to x_2 such that γ is a minimizer of $\inf_{\eta \in \Gamma(x_1, x_2)} \ell(\eta)$ by the GC property of D . Then there exists a constant $k \geq 1$ such that $\ell(\gamma) = \ell \leq kd(x_1, x_2)$ and k depends only on the geometry of D .

The following expression defines a curve in TM :

$$t \mapsto \left(\phi(t; t_0, \gamma(s)), \frac{\partial}{\partial s} \phi(t; t_0, \gamma(s)) \right) \in TM|_D$$

which is the Lie transport of the vector $\gamma'(s)$ along the curve $\phi(t; t_0, \gamma(s))$ for $s \in [0, \ell]$ and hence is the solution to (4.10). The following estimate is then obvious:

$$\begin{aligned} d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) &\leq \int_0^\ell \left| \frac{\partial}{\partial s} \phi(t; t_0, \gamma(s)) \right| ds \\ &\leq \int_0^\ell \alpha_1^{-1} \left(V \left(t, \phi(t; t_0, \gamma(s)), \frac{\partial}{\partial s} \phi(t; t_0, \gamma(s)) \right) \right) ds \end{aligned}$$

$$= \int_0^\ell \alpha_1^{-1}(V(t, \text{Lie}(\gamma'(s))(t; t_0))) ds \quad (4.13)$$

On the other hand, (4.12) implies that

$$\frac{d}{dt}V(t, \text{Lie}(\gamma'(s))(t; t_0)) \leq -\alpha(V(t, \text{Lie}(\gamma'(s))(t; t_0))) \quad (4.14)$$

(IS) If $\alpha(s) = 0$ for all $s \in \mathbb{R}_+$, then (4.14) implies that

$$V(t, \text{Lie}(\gamma'(s))(t; t_0)) \leq V(t_0, \text{Lie}(\gamma'(s))(t_0; t_0)) \leq \alpha_2(|\gamma'(s)|) = \alpha_2(1)$$

for all $t \geq t_0$. Hence it follows from (4.13) that

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \int_0^\ell \alpha_1^{-1} \circ \alpha_2(1) ds \leq b \cdot d(x_1, x_2)$$

where $b = k\alpha_1^{-1} \circ \alpha_2(1)$. Therefore, the system (3.1) is IS on D .

(IAS) If α is a class \mathcal{KL} function, then by [115, Lemma 6.1], there exists a class \mathcal{KL} function β such that

$$V(t, \text{Lie}(\gamma'(s))(t; t_0)) \leq \beta(V(t_0, \gamma'(s)), t - t_0) \leq \beta(\alpha_2(1), t - t_0),$$

and it follows from (4.13) that

$$\begin{aligned} d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) &\leq \int_0^\ell \alpha_1^{-1}(\beta(\alpha_2(1), t - t_0)) ds \\ &\leq k\alpha_1^{-1}(\beta(\alpha_2(1), t - t_0)) \cdot d(x_1, x_2) \\ &=: \bar{\beta}(d(x_1, x_2), t - t_0) \end{aligned}$$

where $\bar{\beta}$ is clearly a class \mathcal{KL} function. Thus the system (3.1) is IAS on D .

(IES) If $\alpha(s) = \lambda s$, then (4.14) implies

$$V(t, \text{Lie}(\gamma'(s))(t; t_0)) \leq V(t_0, \gamma'(s))e^{-\lambda(t-t_0)} \leq \alpha_2(1)e^{-\lambda(t-t_0)} \quad (4.15)$$

and it follows from (4.13) that

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \int_0^\ell \alpha_1^{-1}(\alpha_2(1)e^{-\lambda(t-t_0)}) ds \leq Ke^{-\lambda'(t-t_0)} \cdot d(x_1, x_2)$$

for some $K, \lambda' > 0$, where we have used Proposition 4.1.

Thus the system is IES on D . This completes the proof. \square

Remark 4.8. We remark that the above proof is an intrinsic version of the proof of [41, Theorem 1]. It can be easily adapted to prove other intrinsic version results of [41]. The complete lift technique will be used throughout this manuscript, in particular, to prove converse theorems and reveal the connection between incremental stability and Lyapunov stability of an equilibrium. Thus we underscore that this subsection is crucial for the rest of the thesis.

To see that Theorem 4.1 is indeed the intrinsic form of Theorem 2.10 [41], we only need the following lemma [30].

Lemma 4.3. Suppose that TM has local coordinates $\{x, v\}$ and TTM is locally spanned by $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v_i}\right\}_{i=1, \dots, n}$, where n is the dimension of the manifold M . Then in these coordinates, \tilde{f} reads

$$\tilde{f}(t, v) = \begin{bmatrix} f(t, \pi(v)) \\ \frac{\partial f}{\partial x}(t, \pi(v))v \end{bmatrix}.$$

Remark 4.9. In [112], J. Simpson-Porco and F. Bullo gave a coordinate free proof of a contraction theorem on Riemannian manifold. But we should notice that there are several differences between our result and theirs. Firstly, the function $\langle\langle v_x, v_x \rangle\rangle$ considered in [112] is a special case of the more general Finsler-Lyapunov function considered here. Second, the proof in [112] relies on the Levi-Civita connection defined on the Riemannian manifold. In the whole thesis however, we do not use any connection on the manifold. Because of this, the proof can be easily extended to Finsler manifold without considering any connections, thus simplifying the analysis.

Next, we prove a more striking result: in Theorem 4.1, Condition C2 is sufficient to guarantee IES.

Corollary 4.1. Assume in Theorem 4.1 that all the conditions above C1 are met. Then Condition C2 is sufficient to imply that the system is IES.

Proof. In the proof of Theorem 4.1, we have shown that if C2 is satisfied, then there exists a class \mathcal{KL} function β , such that

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq k\alpha_1^{-1}(\beta(\alpha_2(1), t - t_0)) \cdot d(x_1, x_2)$$

for all $x_1, x_2 \in D$ and all $t \geq t_0 \geq 0$. Thus the newly defined class \mathcal{KL} function $\bar{\beta}(r, s) := \psi(s) \cdot r$ where $\psi(s) := k\alpha_1^{-1}(\beta(\alpha_2(1), s))$ decreases monotonically to 0 as $s \rightarrow \infty$, where k is as in the proof of Theorem 4.1. Invoking Proposition 4.1, the corollary follows. \square

4.2.1 Quadratic Finsler-Lyapunov Function on Riemannian Manifolds

There exists a special class of Finsler-Lyapunov functions on Riemannian manifolds, namely, quadratic Finsler-Lyapunov function, which is of particular importance in applications. Therefore, we study in detail this matter in this subsection.

Definition 4.2.1 (Positive Definite Operator). A positive definite operator P on the Riemannian manifold M is a $(1, 1)$ -tensor field, expressed as

$$P(x) = P_i^j(x) dx^i \otimes \frac{\partial}{\partial x^j} \quad 3$$

in local coordinate, such that

- $P_i^j(x) = P_j^i(x)$ for all $i, j \in \{1, \dots, n\}$ and $x \in M$; (This is equivalent to saying that P is self adjoint)
- there exist two positive constants k_1 and k_2 , such that $k_1|v|^2 \leq \langle Pv, v \rangle \leq k_2|v|^2$ for all $v \in TM$. ⁴

Corollary 4.2. Consider the system (3.2) defined on (M, g) . If there exists a positive operator P on M and a constant $k > 0$, such that

$$\langle (\nabla_f P)v, v \rangle + \langle P\nabla_v f, v \rangle + \langle Pv, \nabla_v f \rangle \leq -k|v|^2, \quad v \in TM \quad (4.16)$$

then the system is IES.

Proof. Consider the Finsler-Lyapunov function

$$V(v) = \langle Pv, v \rangle,$$

then

$$\mathcal{L}_f V(v_0) = \left. \frac{d}{dt} \langle P(\pi(v(t)))v(t), v(t) \rangle \right|_{t=t_0}$$

where $v(t)$ is the solution to the complete lift system with initial condition v_0 . Since both $P(\pi(v(t)))v(t)$ and $v(t)$ are vector fields along the integral curve of f , then by the compatibility of the Levi-Civita connection, we have

$$\begin{aligned} \mathcal{L}_f V(v_0) &= \langle \nabla_f(Pv(t))|_{t=t_0}, v_0 \rangle + \langle Pv_0, \nabla_f v(t)|_{t=t_0} \rangle \\ &= \langle (\nabla_f P)v_0 + P(\nabla_f v(t))|_{t=t_0}, v_0 \rangle + \langle Pv_0, \nabla_f v(t)|_{t=t_0} \rangle \end{aligned}$$

Since $[f(\pi(v(t))), v(t)] = 0$, there holds

$$\nabla_f v(t)|_{t=t_0} = \nabla_{v(t)} f(\pi(v(t)))|_{t=t_0} + [f(\pi(v(t))), v(t)] = \nabla_{v_0} f(\pi(v_0)), \quad (4.17)$$

hence

$$\mathcal{L}_f V(v_0) = \langle (\nabla_f P)v_0, v_0 \rangle + \langle P(\nabla_{v_0} f), v_0 \rangle + \langle Pv_0, \nabla_{v_0} f \rangle \quad (4.18)$$

Since v_0 is arbitrary, invoking Theorem 4.1 and (4.16), the corollary follows. \square

³In order that P is a tensor, $\sum_{i,j} \frac{\partial \tilde{x}^p}{\partial x^j} \frac{\partial x^i}{\partial \tilde{x}^q} P_i^j(x(\tilde{x})) = P_p^q(\tilde{x})$, $\forall \tilde{x} \in M$ should be satisfied for coordinate transform $\tilde{x} = \tilde{x}(x)$. For example, when $P_i^j = \delta_i^j$, P is a $(1, 1)$ -tensor.

⁴ Pv is defined pointwisely via the natural isomorphism $\text{Hom}(V, W) \simeq V^* \otimes W$.

Remark 4.10. $\nabla_f P$ is the covariant derivative of the tensor P in the direction of f , which is again a $(1, 1)$ -tensor. See [96, Section 2.2.2] for the definition of covariant derivative of an arbitrary tensor.

Remark 4.11. When P is the identity operator, (4.16) becomes

$$\langle \nabla_v f, v \rangle \leq -\frac{k}{2}|v|^2, \quad \forall v \in TM.$$

This has been proven in [112].

Remark 4.12. Compared to Theorem 4.1, Corollary 4.2 does not involve the calculation of the complete lift of the system. Thus it is easier to implement.

Remark 4.13. When specified to \mathbb{R}^n with the standard Euclidean inner product as the Riemannian metric and noticing that $\nabla_f P = \mathcal{L}_f P$, (4.16) reads

$$\mathcal{L}_f P(x) + P(x) \frac{\partial f(x)}{\partial x} + \frac{\partial f(x)^T}{\partial x} P(x) \leq -kI.$$

This formula has been used in for example [106] and [7]. However, the matrix P was interpreted as a Riemannian metric in \mathbb{R}^n in the two mentioned articles. The above result suggests that on Riemannian manifolds, it is naturally interpreted P as a positive operator, which is a $(1, 1)$ -tensor instead of a $(0, 2)$ one.

4.3 Converse Results of Incremental Exponential Stability

In this section, we prove that the conditions in Theorem 4.1 to ensure IES are not only sufficient but also necessary. That is, if the system is IES, then we will be able to find a Finsler-Lyapunov function.

In [8], D. Angeli gave a necessary and sufficient conditions of GIAS using incremental Lyapunov functions, which is a set version of Lyapunov functions and involves augmenting the original system with its copy. In comparison, what we are going to prove is a differential version and augmentation is not needed. In [7], V. Andrieu *et al.* proved a converse theorem for IES systems defined on \mathbb{R}^n . See also [56] for converse theorems for monotone systems.

In order to streamline the idea, we assume the system to be GIES. Extension to local version is straightforward.

Theorem 4.2. Consider the system (3.1) defined on a Riemannian manifold (M, g) with $f \in C^2$ globally Lipschitz continuous with constant L in the sense of Definition 3.1.3. Then the system is GIES if and only if there exists a (possibly time dependent) C^1 Finsler-Lyapunov function $V : \mathbb{R}_+ \times TM \rightarrow \mathbb{R}$ such that the following conditions are satisfied

1. there exists two positive constants c_1, c_2 , such that

$$c_1|v|^2 \leq V(t, v) \leq c_2|v|^2, \quad \forall (t, v) \in \mathbb{R}_+ \times TM$$

where $|\cdot|$ is the Riemannian norm.

2. the timed Lie derivative of V along the system (4.10) satisfies

$$\mathcal{L}_{\tilde{f}}V(t, v) \leq -kV(t, v), \quad \forall (t, v) \in \mathbb{R}_+ \times TM \quad (4.19)$$

for some positive constant $k > 0$, where \tilde{f} is the complete lift of f .

To prove the theorem, we need a few lemmas.

Lemma 4.4. If the system (3.1) is GIES, i.e.

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq ke^{-\lambda(t-t_0)}d(x_1, x_2), \quad \forall t \geq t_0 \geq 0 \quad (4.20)$$

for some positive constants k and λ , and all $x_1, x_2 \in M$, then the Lie transport of each vector v along a trajectory of the system (3.1) satisfies

$$|\text{Lie}(v)(t, t_0)| \leq ke^{-\lambda(t-t_0)}|v|, \quad \forall v \in TM. \quad (4.21)$$

Proof. Since Lie is a linear operator as remarked after Definition 2.1.17, it suffices to prove that

$$|\text{Lie}(v)(t, t_0)| \leq ke^{-\lambda(t-t_0)}, \quad \forall v \in TM$$

with $|v| = 1$. Given $x_1, x_2 \in M$, denote the normalized geodesic joining x_1 to x_2 as $\gamma : [0, \ell] \rightarrow M$ with $0 \leq \ell = d(x_1, x_2)$. Let $v \in TM$, $|v| = 1$, $\pi_{TM}(v) = x_1$ and $v = \gamma'(0)$. Denoting $v_t = \text{Lie}(v)(t, t_0)$, we have

$$\ell|v_t| = d\left(\exp_{\phi(t; t_0, x_1)}(lv_t), \phi(t; t_0, x_1)\right), \quad (4.22)$$

by the property of the exponential map, see Theorem 2.2. Since we have assumed that the Riemannian manifold is complete, \exp_x is defined on TM for all $x \in M$. Using the metric property of d , we have

$$d\left(\exp_{\phi(t; t_0, x_1)}(lv_t), \phi(t; t_0, x_1)\right) \leq d\left(\exp_{\phi(t; t_0, x_1)}(lv_t), \phi(t; t_0, x_2)\right)$$

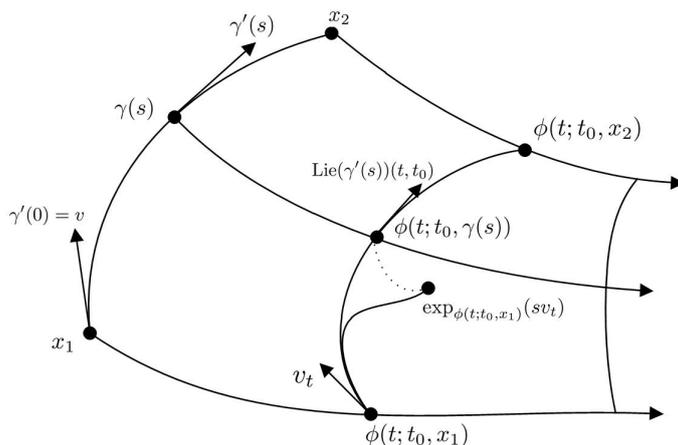


Figure 4.2: Illustration of the proof

$$+ d(\phi(t; t_0, x_2), \phi(t; t_0, x_1)) \quad (4.23)$$

$$\leq d(\exp_{\phi(t; t_0, x_1)}(\ell v_t), \phi(t; t_0, x_2)) + k\ell e^{-\lambda(t-t_0)}, \quad (4.24)$$

where the second inequality holds due to (4.20). From (4.22) and (4.24) we get

$$|v_t| \leq \frac{d(\exp_{\phi(t; t_0, x_1)}(\ell v_t), \phi(t; t_0, x_2))}{\ell} + k e^{-\lambda(t-t_0)}. \quad (4.25)$$

See Figure. 4.2 for an illustration. Now we want to show that the first term on the right hand side is of order $O(\ell)$ and hence converges to 0 as $\ell \rightarrow 0$. Since $x_2 = \gamma(\ell)$, the above formula can also be written as

$$\kappa(\ell) = \frac{d(\exp_{\phi(t; t_0, x_1)}(\ell v_t), \phi(t; t_0, \gamma(\ell)))}{\ell}$$

To this end, we consider the two functions $\alpha_1(\ell) = \exp_{\phi(t; t_0, x_1)}(\ell v_t)$ and $\alpha_2(\ell) = \phi(t; t_0, \gamma(\ell))$. We have $\alpha_1(0) = \alpha_2(0) = x_1$ and $\alpha_1'(0) = \alpha_2'(0) = v_t$. Thus

$$\kappa(\ell) = \frac{d(\alpha_1(\ell), \alpha_2(\ell))}{\ell} = O(\ell)$$

invoking Lemma 3.4. Now letting $\ell \rightarrow 0$ in (4.25), we obtain (4.21), which completes the proof. \square

The lower bound of $\text{Lie}(v)(t; t_0)$ is also needed.

Lemma 4.5. Suppose that f in (3.1) is Lipschitz continuous with constant L in the sense of Definition 3.1.3, then the Lie transport satisfies $|\text{Lie}(v)(t; t_0)| \geq |v|e^{-L(t-t_0)}$, $\forall v \in TM$.

Proof. Let $\gamma(s) = \exp_x(sv)$, so $\gamma'(0) = v$. From Lemma 3.2, we have the following inequality for $s > 0$:

$$\frac{d(x, \gamma(s))e^{-L(\tau-t)}}{s} \leq \frac{d(\phi(t; t_0, x), \phi(t; t_0, \gamma(s)))}{s},$$

in which the left hand side is nothing but $|v|e^{-L(\tau-t)}$. Letting $s \rightarrow 0+$,

$$\begin{aligned} \lim_{s \rightarrow 0+} \frac{d(\phi(t; t_0, x), \phi(t; t_0, \gamma(s)))}{s} &= \frac{d}{ds} \Big|_{s=0} d(\phi(t; t_0, x), \phi(t; t_0, \gamma(s))) \\ &= \lim_{s \rightarrow 0} \left\langle \frac{\partial \phi(t; t_0, \gamma(s))}{\partial s}, \bar{\gamma}'(\ell) \right\rangle \\ &\leq \lim_{s \rightarrow 0} \left| \frac{\partial \phi(t; t_0, \gamma(s))}{\partial s} \right| \\ &= |\text{Lie}(v)(t; t_0)| \end{aligned}$$

where $\bar{\gamma} : [0, \ell] \rightarrow M$ is the normalized geodesic joining $\phi(t; t_0, x)$ to $\phi(t; t_0, \gamma(s))$. Thus the proof is completed. \square

Now we are in position to prove Theorem 4.2.

Proof of Theorem 4.2. Necessity has already been proven in Section 4.2. It remains to prove the converse.

Step 1: We consider the following candidate Finsler-Lyapunov function:

$$V(t, v) = \int_t^{t+\delta} |\text{Lie}(v)(\tau; t)|^2 d\tau \quad (4.26)$$

where $\delta > 0$. From Lemma 3.2 and Lemma 4.4, we can estimate the lower and upper bound of $V(t, v)$:

$$\begin{aligned} V(t, v) &\geq |v|^2 \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau = \frac{1 - e^{-2L\delta}}{2L} |v|^2 \\ V(t, v) &\leq |v|^2 \int_t^{t+\delta} e^{-2\lambda(\tau-t)} d\tau = \frac{1 - e^{-2\lambda\delta}}{2\lambda} |v|^2. \end{aligned}$$

Thus there exists two positive constants c_1, c_2 such that

$$c_1|v|^2 \leq V(t, v) \leq c_2|v|^2.$$

Step 2: By the property of Lie transport, we know that

$$\text{Lie}(\text{Lie}(v)(t; s))(\tau; t) = \text{Lie}(v)(\tau; s), \quad \forall v \in TM, \tau \geq t \geq s$$

hence

$$V(s, \text{Lie}(v)(s, t)) = \int_s^{s+\gamma} |\text{Lie}(\text{Lie}(v)(s, t))(\tau; s)|^2 d\tau$$

$$= \int_s^{s+\delta} |\text{Lie}(v)(\tau; t)|^2 d\tau$$

The timed Lie derivative satisfies

$$\begin{aligned} \mathcal{L}_{\tilde{f}}V(t, v) &= \left. \frac{d}{ds} V(s, \text{Lie}(v)(s, t)) \right|_{s=t} \\ &= |\text{Lie}(v)(t + \delta; t)|^2 - |v|^2 \\ &\leq -(1 - K^2 e^{-2\lambda\delta})|v|^2 \\ &\leq -\frac{1 - K^2 e^{-2\lambda\delta}}{c_2} V(t, v). \end{aligned}$$

By choosing δ large enough such that $1 - K^2 e^{-2\lambda\delta} > 0$, we obtain (4.19) with $k = (1 - K^2 e^{-2\lambda\delta})/c_2$. \square

Remark 4.14. In the proof of Theorem 4.2, the Finsler-Lyapunov function (4.26) is constructed as a time dependent function, even if the system (3.1) is time-invariant. In order to construct a time-invariant one for autonomous system, one may consider the following Finsler-Lyapunov candidate,

$$V(v) = \int_0^\infty |\text{Lie}(v)(t; 0)|^2 dt \quad (4.27)$$

and it can be shown that (4.27) is a *bona fide* Finsler-Lyapunov function.

Remark 4.15. When TM is equipped with Sasaki metric, then similar to Theorem 3.2, one can easily show that the differential of the function V constructed above has following bound,

$$|dV(t, v)|_s \leq c|v|, \quad v \in TM$$

where $|\cdot|_s$ is induced by the Sasaki norm.

Remark 4.16. Like Theorem 4.1, the above proof can be extended to the Finsler case, by replacing the Riemannian metric g_{ij} with $\partial^2(F^2)/\partial x_i \partial x_j$.

Remark 4.17. In [7], the authors obtained similar results of Theorem 4.2 in Euclidean space, see Proposition 1, 2, 3 [7]. More precisely, they proved the equivalence of TLES-NL, UES-TL and ULMTE defined in Definition 2.3.6, see also [7]. We clarify their differences with our results. First, in [7], the state space is \mathbb{R}^n with a metric described by positive definite matrices. Compared to Finsler manifolds, it is easier to deal with and excludes some interesting examples, see for example [41]. In contrast, Finsler structure is the key object in the theory that we have developed in this thesis, it is more general and admits more complex structures. More importantly, it helps us single out what are the more

essential conditions needed to guarantee contraction properties. For example, in [7], it is required that the second order partial derivatives of f are uniformly bounded. On Finsler manifold, this condition is no longer sufficient; instead, conditions imposed on the covariant derivative is needed. Second, the Lyapunov function constructed in [7] is quite different from the Finsler-Lyapunov function constructed in (4.26).

The following table exhibits a correspondence between stability and incremental stability. The most remarkable difference comes to the function V : for incremental stability, the argument $|\delta x|$ of the functions α_i is part of the state $(x, \delta x)$, while for Lyapunov stability, one needs to take into consideration of the whole state.

Table 4.1: A comparison between incremental stability and Lyapunov stability

Incremental Stability	Lyapunov Stability
State space: TM	State space: M
Finsler-Lyapunov function	Lyapunov function
$\alpha_1(\delta x) \leq V(t, x, \delta x) \leq \alpha_2(\delta x)$	$\alpha_1(x) \leq V(t, x) \leq \alpha_2(x)$
$\mathcal{L}_{\tilde{f}}V(t, x, \delta x) \leq -\alpha_3(\delta x)$	$\mathcal{L}_fV(t, x) \leq -\alpha_3(x)$

4.4 Tubular Neighborhood Property

In this section, we prove a technical result concerning the local property of contraction. We show that IES can be fully characterized on a tubular neighborhood of the base manifold in the tangent bundle. To this end, we first give the following definition.

Definition 4.4.1 (Local Finsler-Lyapunov function). Given a forward invariant (under the flow of the system (3.1)) set D satisfying the GC property, a function V is called a local Finsler-Lyapunov function on D if (4.11) is satisfied for all $x \in D$, $t \geq 0$ and all $|\delta x| < c$ for some positive constant c .

Remark 4.18. A local Finsler-Lyapunov function is thus defined on some set $S = \{(x, \delta x) \in TM|_D : |\delta x| < c\}$, which is an open neighbourhood of the base set D in the tangent bundle.

Theorem 4.3. Consider the system (3.1). Suppose that f is globally Lipschitz. Let V be a local Finsler-Lyapunov function defined on the set S as in (4.11), such that the following inequality holds for all $t \geq 0$ and all $(x, \delta x) \in S$, i.e.,

$$\frac{\partial V(x, \delta x)}{\partial x} f(t, x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(t, x)}{\partial x} \delta x \leq -\alpha(V(x, \delta x)),$$

then the system is IES.

Proof. Without loss of generality, assume that D is geodesically convex. Let $\phi(t; t_0, x_0)$ be the solution of the system (3.1) at time $t \geq t_0$ with initial condition (t_0, x_0) . Given any two different points $x_1, x_2 \in D$. (When $x_1 = x_2$, the proof is trivial) Consider the minimizing geodesic $l(s)$, $s \in [0, 1]$ joining x_1 to x_2 . We can estimate the norm of $\delta\phi(t; t_0, s) := \frac{\partial}{\partial s}\phi(t; t_0, l(s))$. In fact

$$\frac{d}{dt} \frac{\partial \phi(t; t_0, l(s))}{\partial s} = \frac{\partial f(t, \phi(t; t_0, l(s)))}{\partial x} \frac{\partial \phi(t; t_0, l(s))}{\partial s},$$

hence

$$|\delta\phi(t; t_0, s)| \leq e^{L(t-t_0)/2} |\delta\phi(t_0; t_0, s)| = e^{L(t-t_0)/2} d(x_1, x_2),$$

for all $t \geq t_0$, $s \in [0, 1]$ where L stands for the Lipschitz constant of f on D .

Let $\bar{V}(t, s) = V(\phi(t; t_0, s), \delta\phi(t; t_0, s))$ for $t \geq t_0$, $s \in [0, 1]$. Its time derivative reads

$$\frac{d\bar{V}(t, s)}{dt} = \frac{\partial V(x, \delta x)}{\partial x} f(x, t) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x, t)}{\partial x} \delta x \Big|_{x=\phi(t; t_0, l(s)), \delta x=\delta\phi(t; t_0, s)} \quad (4.28)$$

Denote

$$T_0 = \frac{2}{L} \ln \left(\frac{c}{d(x_1, x_2)} \right), \quad T_1 = \frac{1}{\lambda} \ln \frac{c_2}{c_1}.$$

Choose γ sufficiently small such that when $0 < d(x_1, x_2) < \gamma$, $T_0 > T_1 \geq t_0$. Given that $t \in [t_0, t_0 + T_0]$, one has $|\delta\phi(t; t_0, s)| \leq e^{L(t-t_0)/2} d(x_1, x_2) < c$, consequently, by (2.23),

$$\frac{d\bar{V}(t, s)}{dt} \leq -\lambda \bar{V}(t, s), \quad \forall s \in [0, 1],$$

or

$$\bar{V}(t, s) \leq e^{-\lambda(t-t_0)} \bar{V}(t_0, s), \quad \forall s \in [0, 1].$$

The distance between x_1, x_2 thus satisfies

$$\begin{aligned} d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) &\leq \int_0^1 |\delta\phi(t; t_0, s)| ds \\ &\leq c_1^{-1/p} \int_0^1 \bar{V}(t, s)^{1/p} ds \\ &\leq c_1^{-1/p} \int_0^1 e^{-\lambda(t-t_0)/p} \bar{V}(t_0, s)^{1/p} ds \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{c_2}{c_1}\right)^{1/p} \int_0^1 e^{-\lambda(t-t_0)/p} |\delta\phi(t_0; t_0, s)| ds \\
&= \left(\frac{c_2}{c_1}\right)^{1/p} \exp\left(-\frac{\lambda}{p}(t-t_0)\right) d(x_1, x_2)
\end{aligned} \tag{4.29}$$

for $t \in [t_0, t_0 + T_0]$, and all $t_0 \geq 0$.

From the definition of T_0 and T_1 , we know that $c_3 := (c_2/c_1)^{1/p} \exp\left(-\frac{\lambda}{p}T_0\right) < 1$. In particular,

$$d(\phi(t_0 + T_0; t_0, x_1), \phi(t_0 + T_0; t_0, x_2)) \leq \left(\frac{c_2}{c_1}\right)^{1/p} e^{-\frac{\lambda}{p}T_0} d(x_1, x_2) = c_3 d(x_1, x_2),$$

for all $t_0, T_0 \geq 0$ and $x_1, x_2 \in D$. By the semi-group property of the flow of the ordinary differential equation (3.1), we have

$$\begin{aligned}
&d(\phi(t_0 + 2T_0; t_0, x_1), \phi(t_0 + 2T_0; t_0, x_2)) \\
&= d(\phi(t_0 + 2T_0; t_0 + T_0, \phi(t_0 + T_0; t_0, x_1)), \phi(t_0 + 2T_0; t_0 + T_0, \phi(t_0 + T_0; t_0, x_2))) \\
&\leq c_3 d(\phi(t_0 + T_0; t_0, x_1), \phi(t_0 + T_0; t_0, x_2)) \\
&\leq c_3^2 d(x_1, x_2).
\end{aligned}$$

By induction, one can easily obtain

$$d(\phi(t_0 + nT_0; t_0, x_1), \phi(t_0 + nT_0; t_0, x_2)) \leq c_3^n d(x_1, x_2) = e^{-\eta n T} d(x_1, x_2),$$

where

$$\eta = -\frac{\ln c_3}{T} > 0.$$

Now for any $t \geq t_0 \geq 0$, there exist an integer $n \geq 0$, and a constant $0 \leq T_t < T_0$ such that $t = nT + T_t + t_0$. Therefore

$$\begin{aligned}
&d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \\
&= d(\phi(t_0 + T_t + nT_0; t_0 + T_t, \phi(t_0 + T_t; t_0, x_1)), \phi(t_0 + T_t + nT_0; t_0 + T_t, \phi(t_0 + T_t; t_0, x_2))) \\
&\leq e^{-\eta n T_0} d(\phi(t_0 + T_t; t_0, x_1), \phi(t_0 + T_t; t_0, x_2)) \\
&\leq e^{-\eta n T_0} d(x_1, x_2) \\
&= e^{-\eta(t-t_0-T_t)} d(x_1, x_2) \\
&\leq c_4 e^{-\eta(t-t_0)} d(x_1, x_2).
\end{aligned}$$

where η and c_4 are independent of t .

We conclude that there exist three positive constants γ, η, c_4 , such that whenever $d(x_1, x_2) < \gamma$,

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq c_4 e^{-\eta(t-t_0)} d(x_1, x_2),$$

for all $t \geq t_0 \geq 0$. Invoking Remark 4.6, the theorem follows. \square

Remark 4.19. In the above proof, we start by choosing two points x_1, x_2 which are sufficiently close. Alternatively, we can assume that the two points are arbitrary, and choose the minimizing geodesic whose velocity is smaller than a given constant. After that, similar procedures as above can be used to arrive at the same conclusion.

Example 4.1. Consider the system $\dot{x} = -2x - \cos x + u(t)$, where $u(t)$ is a time-varying signal. Let $V(x, \delta x) = 1 - \cos \delta x$. Then we have $\dot{V} = -(2 - \sin x)\delta x \sin(\delta x)$. By Taylor expansion, when δx is sufficiently small, $V(x, \delta x) = \frac{1}{2}(\delta x)^2 + O(\delta^4 x)$. Then V is only a local FLF. Furthermore, $\dot{V} \leq -(\delta x)^2 + O(\delta^4 x)$, hence $\dot{V} \leq -cV$ for δx sufficiently small with some constant $1 < c < 2$. Thus by Theorem 4.3, we can conclude that the system $\dot{x} = -2x - \cos x + u(t)$ is IES.

4.5 Krasovskii's Theorem and Contraction

When a system is IES on D and has $x_* \in D$ as an equilibrium point, i.e., $f(t, x_*) = 0$, for all $t \geq 0$. It is obvious that x_* is exponentially stable and D is a region of attraction.

For time-invariant or periodic systems, more can be said.

Theorem 4.4. Let the system (3.1) be T -periodic, i.e., $f(t, x) = f(t + T, x)$ for all $t \geq 0$ and all $x \in M$. Suppose that $D \subseteq M$ satisfies the GC property and that there exists a continuous function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$, and

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \gamma(t - t_0)d(x_1, x_2), \quad \forall x_1, x_2 \in D, t \geq t_0 \geq 0, \quad (4.30)$$

Assume furthermore that f is globally Lipschitz on D . Then, there exist a unique periodic solution $\alpha(t)$, whose period is kT for some $k \in \mathbb{N}$, constants $C > 0$ and $\lambda > 0$, such that

$$d(\phi(t; t_0, x_1), \alpha(t)) \leq Ce^{-\lambda(t-t_0)}d(x_1, \alpha(0)), \quad \forall t \geq t_0 \geq 0, x_1 \in D.$$

Proof. Define a mapping P as

$$P : x \mapsto \phi(kT; 0, x) \quad (4.31)$$

where $k \in \mathbb{N}$ is big enough that $\gamma(kT) = c < 1/K$. It follows that $d(Px, Py) = d(\phi(kT; 0, x), \phi(kT; 0, y)) \leq cd(x, y)$. Since D satisfies the GC property, P is a contraction mapping on the complete metric space D . Invoking Banach contraction mapping theorem, there exists a unique x such that $Px = x$. Let $\alpha(t) = \phi(t; 0, x)$, which is a solution to the system (3.1). Then

$$\alpha(t + kT) = \phi(t + kT; 0, x)$$

$$\begin{aligned}
&= \phi(t + kT; kT, \phi(kT, 0, x)) \\
&= \phi(t; 0, \phi(kT, 0, x)) \\
&= \phi(t; 0, x) \\
&= \alpha(t), \quad \forall t \geq 0
\end{aligned}$$

meaning that α is a periodic function with period kT . Clearly, (4.30) implies that there exist two positive constants C, λ , independent of t_0 , such that

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq Ce^{-\lambda(t-t_0)}d(x_1, x_2), \quad \forall x_1, x_2 \in D, t \geq t_0 \geq 0. \quad (4.32)$$

Hence

$$\begin{aligned}
d(\phi(t; t_0, y), \alpha(t)) &= d(\phi(t; t_0, y), \phi(t; 0, x)) \\
&= d(\phi(t; t_0, y), \phi(t; t_0, \phi(t_0; 0, x))) \\
&\leq Ce^{-\lambda(t-t_0)}d(y, \phi(t_0; 0, x)) \\
&= Ce^{-\lambda(t-t_0)}d(\phi(0; 0, y), \phi(t_0; 0, x)) \\
&\leq Ce^{Lt_0}e^{-\lambda(t-t_0)}d(x, y)
\end{aligned} \quad (4.33)$$

where L is the Lipschitz constant associated with the set D . Now each $t_0 \in \mathbb{R}_+$ can be written as $t_0 = mkT + s_0$ for some $m \in \mathbb{N}$ and $s_0 \in [0, kT)$. Therefore, $\phi(t; t_0, y) = \phi(t - mkT; t_0 - mkT, y) = \phi(t - mkT; s_0, y)$. Therefore, in (4.33), we can assume that $t_0 < kT$, and consequently, $Ce^{Lt_0} < Ce^{kLT}$. By setting $C' = Ce^{kLT}$, we immediately get

$$d(\phi(t; t_0, y), \alpha(t)) \leq C'e^{-\lambda(t-t_0)}d(\alpha(0), y), \quad \forall t \geq t_0 \geq 0, y \in D$$

which completes the proof. \square

In particular, when the system is autonomous, it admits a unique equilibrium which is exponentially stable.

Corollary 4.3. Consider the autonomous system (3.2). Suppose that $D \subseteq M$ satisfies the GC property and that there exists a continuous function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$, and

$$d(\phi(t; t_0, x_1), \phi(t; t_0, x_2)) \leq \gamma(t - t_0)d(x_1, x_2), \quad \forall x_1, x_2 \in D, t \geq t_0 \geq 0, \quad (4.34)$$

Then, the system has a unique equilibrium point $x \in D$, and there exist constants $C > 0$ and $\lambda > 0$, such that

$$d(\phi(t; t_0, x_1), x) \leq Ce^{-\lambda(t-t_0)}d(x_1, x), \quad \forall t \geq t_0 \geq 0, x_1 \in D.$$

Remark 4.20. We make a remark on the difference between Theorem 4.4 and Theorem 2 in [116]. In [116], the domain D is assumed to be a convex set in Euclidean space and the system is assumed to have the property $\mu(J_f(t, x)) \leq -c$ for all $x \in D$, $t \geq 0$. (Recall that $J_f(t, x)$ is the Jacobian matrix of f and c is a positive constant.) In the theorem above, the domain D is a submanifold with GC property which is more general than convexity, and the condition (4.30) is strictly weaker than restricting a uniform negative bound on the matrix measure of the Jacobian of the system.

Corollary 4.3 is of special interest to us. If an autonomous system is incrementally exponentially stable on D , then by Corollary 4.3, there exists a unique exponentially stable equilibrium point in D . Then two questions can be asked.

LF (Lyapunov function construction): The converse Lyapunov theorem (see e.g. [58]) tells us that there should exist a Lyapunov function $W(t, x)$ (not a Finsler-Lyapunov function) for the system (3.1) along which, the time derivative of the Lyapunov function is negative definite. Now, having the IES property at hand, by Theorem 4.2, a Finsler-Lyapunov function can be constructed. A natural question is, can we construct a Lyapunov function based on the information of this Finsler-Lyapunov function?

ES (Equilibrium searching): How to find the equilibrium point by numerical methods?

We answer the first question here and leave the second to Chapter 5.

Construction of Lyapunov Function

The following proposition gives an answer to the question **LF**. As we will see, it is a rediscovery and extension of the classical Krasovskii's method used for the construction of Lyapunov function [58]. We highlight that the following theorem is valid even for time varying system.

Theorem 4.5. Suppose the system (3.1) is IES with a Finsler-Lyapunov function $V(t, v)$ such that $\mathcal{L}_{\tilde{f}}V \leq -kV$ for a positive constant k . Assume furthermore the system has an equilibrium point x_* . Then

- the system is GES;
- given a smooth time invariant vector field h on M such that

$$\eta_1(d(x, x_*)) \leq |h(x)| \leq \eta_2(d(x, x_*))$$

for two class \mathcal{K} functions η_1, η_2 and that $[f, h] = 0$, then the function $W(t, x) = V(t, x, h(x))$ is a Lyapunov function for the system.

We need the following lemma to prove the theorem, which is interesting in its own right.

Lemma 4.6. Consider the system (3.1). If there exists a vector field $h(x)$ on M such that $[f, h] = 0$, then $h(\phi(t; t_0, x_0))$ is the unique solution to the system $\dot{v} = \tilde{f}(t, v)$ with initial condition $(x_0, h(x_0))$. In particular, the solution to the complete lift system of the system (3.2) started from $(x_0, f(x_0))$ is $(\phi(t; t_0, x_0), f(\phi(t; t_0, x_0)))$.

Proof. The Lie bracket of f and h can be calculated as

$$[f, h](\phi(t; t_0, x_0)) = \left. \frac{d(\phi^*h)}{ds}(\phi(s; t_0, x_0)) \right|_{s=t} = 0,$$

where ϕ^* is the pullback of ϕ . Thus $\phi^*h(\phi(t; t_0, x_0)) = \text{constant} = (x_0, h(x_0))$, or

$$h(\phi(t; t_0, x_0)) = \phi(t; t_0, x_0)_*(x_0, h(x_0)) = \text{Lie}(h(x_0))(t, t_0).$$

which completes the first half of the lemma. Since $[f, f] = 0$ is always true, the last claim also follows. \square

Proof of Theorem 4.5. It can be readily checked that $W(t, x)$ is a positive definite Lyapunov candidate. Using the above lemma, we have

$$\begin{aligned} \mathcal{L}_f W(t, x) &= \left. \frac{d}{d\tau} V(\tau, h(\phi(\tau; t, x))) \right|_{\tau=t} \\ &= \mathcal{L}_{\tilde{f}} V(t, h(x)) \\ &\leq -kV(t, h(x)) \\ &= -kW(t, x), \end{aligned}$$

showing that $W(t, x)$ is indeed a Lyapunov function. \square

Corollary 4.4. Consider the system $\dot{x} = f(t, x)$, where $x \in \mathbb{R}^n$, with $f(0, t) = 0$. If the system is IES with a Finsler-Lyapunov function $V(t, x, \delta x)$ and there exists a smooth vector field $h(x)$ on \mathbb{R}^n such that $[f, h] = 0$, where $h = 0$ if and only if $x = 0$, then the function $W(t, x) = V(t, x, h(x))$ is a Lyapunov function and the system is exponentially stable. In particular, $W(t, x)$ can be chosen as $V(t, x, f(x))$ when the system is time invariant.

Proof. The time derivative of $W(t, x)$ reads

$$\dot{W}(t, x) = \dot{V}(t, x, h(x))$$

$$\begin{aligned}
&= \frac{\partial V(t, x, h(x))}{\partial t} + \frac{\partial V(t, x, h(x))}{\partial x} f(x, t) \\
&\quad + \frac{\partial V(t, x, h(x))}{\partial \delta x} \frac{\partial h(x)}{\partial x} f(x, t) \\
&= \frac{\partial V(t, x, h(x))}{\partial t} + \frac{\partial V(t, x, h(x))}{\partial x} f(x, t) \\
&\quad + \frac{\partial V(t, x, h(x))}{\partial \delta x} \frac{\partial f(x, t)}{\partial x} h(x) \\
&\leq -kV(t, x, h(x)) \\
&= -kW(t, x),
\end{aligned}$$

where the third equality follows from the fact that in Euclidean space,

$$[f, h] = \frac{\partial f}{\partial x} h - \frac{\partial h}{\partial x} f.$$

Thus we see that the system is exponentially stable with Lyapunov function $W(t, x)$. \square

Theorem 4.5 recovers and extends the so called Krasovskii's method [58]: if there exist two constant positive definite matrices P and Q such that

$$P \frac{\partial f(x)}{\partial x} + \left[\frac{\partial f(x)}{\partial x} \right]^T P \leq -Q, \quad (4.35)$$

then $V(x) = f^T(x)Pf(x)$ can serve as a Lyapunov function for the system since h can be taken as f . Clearly, if (4.35) is satisfied, $\delta^T x P \delta x$ is a Finsler-Lyapunov function for the system. Then the Krasovskii's method is a direct consequence of Corollary 4.4. We consider two examples.

Example 4.2. Consider the linear system $\dot{x} = Ax$. Suppose there exists a Finsler-Lyapunov function $V = \delta x^T P \delta x$, such that $A^T P + PA = -I$. Then since $[Ax, x] = 0$, Corollary 4.4 tells us that when replacing δx with x , V becomes a Lyapunov function, i.e. $W(x) = x^T P x$. Furthermore, $x^T B^T P B x$ is also a Lyapunov function as long as B is invertible and commutes with A since in this case $[Ax, Bx] = (BA - AB)x = 0$.

Example 4.3. We consider the case when the matrix measure of the Jacobian $J_f(t, x) = \partial f(x, t)/\partial x$ satisfies

$$\mu(J(x)) \leq -c, \quad \forall x \in \mathbb{R}^n$$

for some positive constant c . This is considered in [5] for example. The Finsler-Lyapunov function can be chosen as $V(x, \delta x) = |\delta x|$, and the Lyapunov function $W(x) = |f(x)|$. Indeed, it can be readily checked that

$$\dot{W}(x(t)) = \lim_{h \rightarrow 0^+} \frac{|f(x + hf(x))| - |f(x)|}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0^+} \frac{1}{h} \left(\left| f(x) + h \frac{\partial f(\xi)}{\partial x} f(x) \right| - |f(x)| \right) \\
&\leq \lim_{h \rightarrow 0^+} \frac{|I + hJ(t, \xi)| - 1}{h} |f(x)| \\
&= \mu(J(\xi)) |f(x)| \\
&\leq -cW(x(t)).
\end{aligned}$$

Remark 4.21. We remark that the results obtained by F. Bullo [24] and K. Kosaraju [63] (when the input u is 0) regarding Krasovskii's method are special cases of Corollary 4.4.

4.6 Contraction and LES

In Section 3.2, we studied local exponential stability of nontrivial solutions of the system (3.1) and in Section 4.2 of this chapter, we studied contraction analysis via geometric methods. This section is to establish connections between the two different subjects, namely, LES of particular solutions and the contraction property. The reader may refer to [101, 8] for the this issue in Euclidean spaces.

In Section 3.2, it has been shown in Theorem 3.4 that LES of a solution X implies exponential stability of the complete lift along X . It remains to show that for a bounded X , the converse is also true.

Proof of the converse of Theorem 3.4. Suppose that the system (3.1) is periodic and the complete lift system (3.28) along X is exponentially stable. By the remarks below Proposition 3.1, there exists an open forward invariant set D_c containing X . Since X is bounded, the closure of D_c is compact. As a consequence, \bar{D}_c verifies the GC property.

Thanks to the (fibre) linear structure of the complete lift, one can construct a C^1 quadratic Lyapunov function $V : \mathbb{R}_+ \times X^*TM$ ⁵ such that the following inequalities are satisfied for some positive constants c_1, c_2 and c_3

$$\begin{aligned}
c_1|v|^2 &\leq V(t, v) \leq c_2|v|^2 \\
\frac{\partial V}{\partial t}(t, v) + \frac{\partial V}{\partial v}(t, v)\tilde{f}(v, t) &\leq -c_3|v|^2.
\end{aligned} \tag{4.36}$$

for all $v \in X^*TM$. Due to the quadratic nature of $V(t, v)$ and the linearity of \tilde{f} with respect to v , the function

$$F(t, v) = \frac{\partial V}{\partial t}(t, v) + \frac{\partial V}{\partial v}(t, v)\tilde{f}(v, t)$$

⁵ X^*TM is the pullback bundle.

is again quadratic in v . Assume $T > 0$ to be the period of X and denote

$$S(TM|_{\bar{D}_c}) := \{v \in TM|_{\bar{D}_c} : |v| = 1\}.$$

Equip TM with the Sasaki metric, then for each $v \in S(TM|_{\bar{D}_c})$, the distance from v to the set $\{v \in X^*TM : |v| = 1\}$ is sufficiently small by choosing D_c sufficiently small.

By the continuity of V and F and the inequalities (4.36), it follows that there exist some positive constants c'_1, c'_2 and c'_3 , such that

$$\begin{aligned} \min_{[0,T] \times S(TM|_{\bar{D}_c})} V(t, v) &\geq c'_1 \\ \max_{[0,T] \times S(TM|_{\bar{D}_c})} V(t, v) &\leq c'_2 \\ \max_{[0,T] \times S(TM|_{\bar{D}_c})} F(t, v) &\leq -c'_3 \end{aligned}$$

for D_c sufficiently small. Recalling the quadratic nature of V and F with respect to v , (4.36) is valid for all $(t, v) \in [0, T] \times TM|_{\bar{D}_c}$ by replacing c_i by c'_i . Hence the system (3.1) is incrementally exponentially stable on D_c . Thus the particular solution X must be LES with D_c being a region of attraction. \square

In the proof of the above theorem, we have also proved the following:

Proposition 4.2. Suppose that the system (3.1) is periodic and X is a bounded solution to it. Then X is LES if and only if there exists an open neighborhood of X on which the system (3.1) is IES.

A straightforward consequence of Proposition 4.2 is stated in the following corollary.

Corollary 4.5. Assume that the system (3.1) has an equilibrium point $x_* \in M$. Then x_* is LES if and only if there exists a forward invariant open neighborhood of x_* on which the system is IES.

In fact, for this special case, one can proceed much more easily. Since LES of an equilibrium is a local property, it is sufficient to work in an open neighborhood of $0 \in \mathbb{R}^n$ and assume $x_* = 0$. It is well-known that, 0 is LES if and only if the linearized system

$$\dot{x} = A(t)x \tag{4.37}$$

is exponentially stable, where

$$A(t) = \frac{\partial f}{\partial x}(t, 0).$$

From linear system theory, we know that the system (4.37) is exponentially stable if and only if there exist two C^1 positive definite matrix functions $P, Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ verifying the differential Lyapunov matrix equation

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) = -Q(t), \quad \forall t \geq 0.$$

and $P(t)$ and $Q(t)$ satisfies the following bound

$$\begin{aligned} c_1 I &\leq P(t) \leq c_2 I, \\ b_1 I &\leq Q(t) \leq c_2 I \end{aligned}$$

for all $t \geq 0$ and some positive constants c_1, c_2, b_1, b_2 . When the system is periodic, obviously

$$\left| \frac{\partial f}{\partial x}(t, x) - \frac{\partial f}{\partial x}(0, x) \right| < c_4, \quad \forall t \geq 0, |x| < \delta$$

where $\delta > 0$. Now consider the following candidate Finsler-Lyapunov function for the system (3.1),

$$V(t, v) = v^T P(t)v.$$

Then

$$\begin{aligned} \mathcal{L}_{\tilde{f}} V(t, v) &= v^T \dot{P}(t) + P(t)A(t) + A(t)^T P(t)v + 2v^T P(t) \left(\frac{\partial f}{\partial x}(t, x) - \frac{\partial f}{\partial x}(t, 0) \right) v \\ &\leq (-b_1 + c_2 c_4) |v|^2. \end{aligned}$$

Choosing δ and then c_4 sufficiently small, the constant $k := -b_1 + c_2 c_4$ can be made negative. Thus the system is IES on a forward invariant open neighborhood of the origin.

Remark 4.22. In [40], the authors proved similar result to this corollary for *autonomous* systems in Euclidean space. The paper [101] focuses on asymptotic stability and asymptotic contraction, also in Euclidean space.

4.7 Volume Shrinking

Given a Riemannian density V_g on the manifold M , the volume of an measurable open set D is defined as

$$\text{vol}(D) = \int_D dV_g.$$

Consider the time-invariant system (3.2). Define the flow of the set D as

$$D_t = \phi(t; t_0, D).$$

Lemma 4.7. If the vector field f of the system (3.2) satisfies

$$\langle \nabla_v f, v \rangle \leq -k|v|^2, \quad v \in TM, \quad (4.38)$$

then $\operatorname{div} f(x) \leq -kn$ for all $x \in M$, where n is the dimension of M .

Proof. Recall that

$$\operatorname{div} = \operatorname{tr} \circ \nabla$$

where ∇ is the covariant derivative operator. The vector field f can be viewed as a $(1, 0)$ -tensor field, and thus ∇f is a $(1, 1)$ -tensor field defined as $\langle \nabla f(v)|w \rangle = \langle \nabla_v f, w \rangle$, for $v, w \in TM$. Equivalently, recall the isomorphism $V \otimes W^* = \operatorname{Hom}(V, W)$, then $\nabla f(x)$ can also be seen as a linear operator from $T_x M$ to $T_x M$ defined as $v \mapsto \nabla_v f$. Hence by (4.38),

$$\operatorname{div} f(x) = \operatorname{tr}(\nabla f) \leq -kn,$$

which completes the proof. □

Theorem 4.6. If the system (3.2) is contractive and condition (4.38) is verified, then the system is exponentially volume shrinking in the sense that the volume of the flow of a measurable set D shrinks exponentially.

Proof. By the transport formula, we have

$$\begin{aligned} \frac{d}{dt} \operatorname{vol}(D_t) &= \int_{D_t} \operatorname{div} f dV_g \\ &\leq -kn \operatorname{vol}(D_t) \end{aligned}$$

by the above lemma. This justifies the conclusion. □

4.8 A Brief Summary

In this chapter, we have obtained some new results of contraction analysis on Riemannian manifolds.

First, based on the tool “complete lift”, we have given an intrinsic condition for incremental stabilities. This condition is simple but fairly general which covers many important results in the literature.

Second, we have proven the converse theorem of contraction on Riemannian manifolds. This converse theorem, for the first time, completely justifies the legitimacy of the introduction of Finsler-Lyapunov function for contraction analysis.

Third, we have obtained several other characterizations of contractive systems: (1) The tubular neighborhood property shows that IES can be characterized fully on a tubular neighborhood of the base manifold; (2) Krasovskii's method reveals the intimate connections between incremental stability and Lyapunov stability on manifolds. Meanwhile, it gives a geometric interpretation of the classical Krasovskii's theorem.

To summarize, a geometric framework for contraction analysis on Riemannian manifolds has been established.

Chapter 5

Some Applications

This chapter is concerned with several applications of the theory developed so far. We study three examples:

- Equilibrium seeking on Riemannian manifold;
- Gradient based filter on $SO(3)$;
- Speed observer of Lagrangian systems on Riemannian manifold;
- Synchronization on manifolds.

5.1 Optimal Contraction Rate of Equilibrium Searching on Riemannian Manifolds

We now come back to the **ES** problem that we have mentioned in Section 4.5. That is, suppose that the time-invariant system (3.2) is IES on an invariant set $D \subseteq M$ with the GC property (Definition 4.1.2), then by Corollary 4.3, there exists an equilibrium point $x_* \in D$ which is exponentially stable. The problem is to search this equilibrium via numerical methods.

In [22], F. Bullo *et al.* have considered the Euler method on a Riemannian manifold:

$$x_{k+1} = \exp_{x_k}(\alpha f(x_k)) \tag{5.1}$$

where \exp is the exponential map, and α a positive constant, see also [132].

It has been pointed out in [22] that when the system (3.2) is IES, then the following mapping

$$F : x \mapsto \exp_x(\alpha f(x))$$

is a Banach contraction mapping and an “optimal contraction rate” can be derived when working in Euclidean space, but it is an open conjecture [22] whether it is still true for Riemannian manifold. In this subsection, we show that under certain conditions, F is indeed a Banach contraction mapping when the system (3.2) is IES.

Consider two points $x, y \in M$, and the normalized minimizing geodesic $\gamma : [0, \hat{s}] \rightarrow M$. Obviously,

$$\begin{aligned} d(\exp_x(\alpha f(x)), \exp_y(\alpha f(y))) &\leq \int_0^{\hat{s}} \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s))) \right| ds \\ &\leq \sqrt{\hat{s} \int_0^{\hat{s}} \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s))) \right|^2 ds} \end{aligned} \quad (5.2)$$

See Figure 5.1 for an illustration. Since $s \mapsto \exp_{\gamma(s)}(\alpha f(\gamma(s)))$ is a geodesic variation

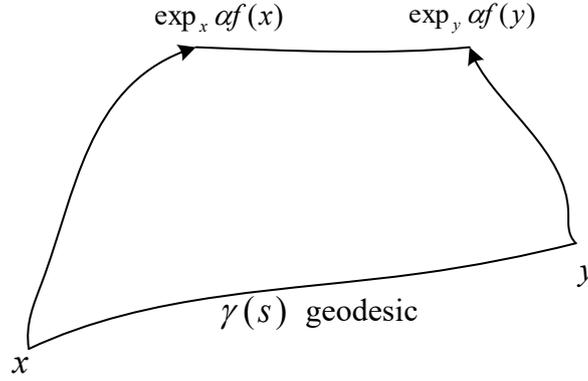


Figure 5.1: Equilibrium Seeking

along the geodesic $r \mapsto \exp_{\gamma(s)}(rf(\gamma(s)))$ at $s > 0$, it holds that

$$\frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s))) = J_s(\alpha)$$

where $J_s(r)$ is the Jacobi field along the geodesic $r \mapsto \exp_{\gamma(s)}(rf(\gamma(s)))$, $r \in [0, \alpha]$ with $J_s(0) = \gamma'(s)$ and $J'_s(0) = \nabla_{\gamma'(s)} f(\gamma(s))$.

Hence

$$\int_0^{\hat{s}} \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s))) \right|^2 ds = \int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds \quad (5.3)$$

where $J_s(r)$ is a solution to

$$\begin{aligned} J''_s(r) + R(\varphi'_s(r), J_s(r))\varphi'_s(r) &= 0 \\ J_s(0) = \gamma'(s), \quad J'_s(0) &= \nabla_{\gamma'} f(\gamma(s)) \end{aligned} \quad (5.4)$$

with $r \in [0, \hat{s}]$ and $\varphi_s(r) = \exp_{\gamma(s)} rf(\gamma(s))$, $\varphi'_s(0) = f(\gamma(s))$.

Based on the property of Jacobi fields, we can obtain some useful estimations. We consider two particular cases, namely, the zero curvature case and non-negative constant curvature case.

5.1.1 Zero Curvature Case

Let us consider the zero curvature case, i.e., when $M = \mathbb{R}^n$. In this case, $J_s(\alpha) = \gamma'(s) + \alpha \frac{df(\gamma(s))}{ds}$, where $\gamma : [0, \hat{s}] \rightarrow \mathbb{R}^n$ is the line segment with unit speed joining x to y , with $|x - y| = \hat{s}$. Explicitly, $\gamma(s) = x - \frac{x-y}{\hat{s}}s$. Thus $J_s(\alpha) = -\frac{x-y}{\hat{s}} + \alpha \frac{df(\gamma(s))}{ds}$. Denote $f'_s = \frac{df(\gamma(s))}{ds}$, then we have

$$\begin{aligned} \int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds &= \int_0^{\hat{s}} \left[1 + 2\alpha \left\langle \frac{y-x}{\hat{s}}, f'_s \right\rangle + \alpha^2 |f'_s|^2 \right] ds \\ &= |x-y| + 2\alpha \left\langle \frac{y-x}{|x-y|}, f(y) - f(x) \right\rangle + \alpha^2 \int_0^{\hat{s}} |f'_s|^2 ds \end{aligned}$$

Assume the simplest condition of IES is imposed on the system (3.2), namely,

$$\frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} \leq -cI$$

where c is a positive constant (Demidovich condition with $P = I$). This is equivalent to saying

$$(y-x)^T(f(y) - f(x)) \leq -c|x-y|^2$$

for all $x, y \in \mathbb{R}^n$, see [22] for details. (Alternatively, one may rewrite $\int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds$

$$\frac{1}{2} \int_0^{\hat{s}} \int_0^{\alpha} \langle J'_s(\alpha), J_s(\alpha) \rangle ds + \int_0^{\hat{s}} \langle J_s(0), J_s(0) \rangle ds$$

and this equivalence will not be needed explicitly.) Additionally, assume that f is globally ℓ -Lipschitz continuous. Then we get from above that

$$\begin{aligned} \int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds &\leq |x-y| - 2\alpha c|x-y| + \alpha^2 \int_0^{\hat{s}} \left| \frac{\partial f}{\partial x} \right|^2 ds \\ &\leq |x-y|(1 - 2\alpha c + \alpha^2 \ell^2) \end{aligned}$$

Then from (5.2), one can obtain

$$\begin{aligned} d(F(x), F(y)) &\leq |x-y| \sqrt{1 - 2\alpha c + \alpha^2 \ell^2} \\ &\leq \sqrt{1 - 2\alpha c + \alpha^2 \ell^2} d(x, y) \end{aligned}$$

Then the “best contraction rate” is attained when $\kappa(\alpha) = 1 - 2\alpha c + \alpha^2 \ell^2$ is minimized. Obviously, this is achieved at $\alpha = c/\ell^2$ and the corresponding contraction rate is $\sqrt{1 - c^2/\ell^2}$. (Notice that $c < \ell$ is always true since the Lipschitz constant is always larger than the contraction rate.) This is indeed the result obtained in [22].

5.1.2 Non-negative Constant Curvature Case

Suppose that the manifold M has non-negative curvature K , see Definition 2.2.6. The system (3.2) is ℓ -Lipschitz in the sense of Definition 3.1.3, and that the system is IES, namely, there holds

$$\langle \nabla_v f, v \rangle \leq -c|v|^2, \quad v \in TM \quad (5.5)$$

for some positive constant c . Let

$$U(t, s) = \langle J'_s(t), J'_s(t) \rangle + \langle R(\varphi'_s(t), J_s(t))\varphi'_s(t), J_s(t) \rangle$$

Then since the manifold has constant curvature, $\nabla R = 0$, therefore,

$$\begin{aligned} \frac{\partial U}{\partial t} &= 2\langle J''_s(t), J'_s(t) \rangle + 2\langle R(\varphi'_s(t), J_s(t))\varphi'_s(t), J'_s(t) \rangle \\ &\quad + \langle (\nabla_{\partial_t} R)(\varphi'_s(t), J_s(t))\varphi'_s(t), J_s(t) \rangle \\ &= 2\langle J''_s(t), J'_s(t) \rangle + 2\langle R(\varphi'_s(t), J_s(t))\varphi'_s(t), J'_s(t) \rangle \\ &= -2\langle R(\varphi'_s(t), J_s(t))\varphi'_s(t), J'_s(t) \rangle + 2\langle R(\varphi'_s(t), J_s(t))\varphi'_s(t), J'_s(t) \rangle \\ &= 0 \end{aligned}$$

Thus $U(t, s) = U(0, s)$ for all $t \geq 0$, $s \in [0, \hat{s}]$.

And we have

$$\begin{aligned} \int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds &= 2 \int_0^{\hat{s}} \int_0^\alpha \langle J'_s(r), J_s(r) \rangle dr ds + \int_0^{\hat{s}} |J_s(0)|^2 ds \\ &= 2L + \int_0^{\hat{s}} |\gamma'(s)|^2 ds \\ &= 2L + \hat{s} \end{aligned}$$

where

$$\begin{aligned} L &= \int_0^{\hat{s}} \int_0^\alpha \langle J'_s(r), J_s(r) \rangle dr ds \\ &= \int_0^{\hat{s}} \int_0^\alpha \left(\int_0^r \frac{d}{dt} \langle J'_s(t), J_s(t) \rangle dt + \langle J'_s(0), J_s(0) \rangle \right) dr ds \\ &= \int_0^{\hat{s}} \int_0^\alpha \left(\int_0^r \langle J''_s(t), J_s(t) \rangle + \langle J'_s(t), J'_s(t) \rangle dt + \langle J'_s(0), J_s(0) \rangle \right) dr ds \\ &= \int_0^{\hat{s}} \int_0^\alpha \left(\int_0^r \langle -R(\varphi'_s(t), J_s(t))\varphi'_s(t), J_s(t) \rangle + \langle J'_s(t), J'_s(t) \rangle dt + \langle J'_s(0), J_s(0) \rangle \right) dr ds \\ &= \int_0^{\hat{s}} \int_0^\alpha \int_0^r (\langle -2R(\varphi'_s(t), J_s(t))\varphi'_s(t), J_s(t) \rangle + U(t, s) + \langle J'_s(0), J_s(0) \rangle) dt dr ds \\ &\leq \int_0^{\hat{s}} \int_0^\alpha \left(\int_0^r U(0, s) dt \right) dr ds + \int_0^{\hat{s}} \int_0^\alpha \langle J'_s(0), J_s(0) \rangle dr ds \\ &\leq \frac{1}{2}\alpha^2 \int_0^{\hat{s}} U(0, s) ds - c \int_0^{\hat{s}} \int_0^\alpha dr ds \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \int_0^{\hat{s}} U(0, s) ds &= \int_0^{\hat{s}} |\nabla_{\gamma'} f|^2 + \langle R(\nabla_{\gamma'} f, \gamma') \nabla_{\gamma'} f, \gamma' \rangle ds \\ &\leq \int_0^{\hat{s}} ((1 + K)\ell^2) ds \\ &= (1 + K)\hat{s}\ell^2 \end{aligned}$$

Hence

$$\int_0^{\hat{s}} \langle J_s(\alpha), J_s(\alpha) \rangle ds \leq \hat{s}(1 - 2c\alpha + \alpha^2(1 + K)\ell^2) \quad (5.7)$$

Thus the best contraction rate is achieved at $\alpha_* = \frac{c}{(1+K)\ell^2}$, and the corresponding contraction rate is

$$\sqrt{1 - \frac{c^2}{(1+K)\ell^2}}$$

Notice that when $K = 0$, we immediately recover the zero curvature case.

5.2 Gradient-based Filter on $SO(3)$

Gradient based methods are extensively used when it comes to systems on manifolds. See for example, [50, 64, 112] and the references therein. The following statement, which results from Theorem 5.1, focuses on gradient systems evolving on a manifold.

Proposition 5.1. Let $V : M \times M \rightarrow \mathbb{R}_+$ be a twice continuously differentiable function satisfying: $V(y, y) = 0$, $V(x, y) > 0$ whenever $x \neq y$, and $\text{Hess } V(v, v) \geq c_1|v|^2$. Consider the system

$$\dot{x} = -k \text{grad } V(x, X(t)) + \eta(x, t), \quad (5.8)$$

where $k \in \mathbb{R}$ and $\eta : M \times \mathbb{R}_+ \rightarrow TM$ is a smooth vector field in a neighborhood of $X : \mathbb{R}_+ \rightarrow M$ with the following properties:

P1 $\eta(X(t), t) = \dot{X}(t)$, for all $t \geq 0$;

P2 there exists a constant $c_2 > 0$ such that

$$|\nabla_v \eta(x, t)|_{x=X(t)} \leq c_2|v|, \quad \forall v \in T_{X(t)}M. \quad (5.9)$$

Then, X is a solution of (5.8) and, for all $k > \frac{c_2}{c_1}$, it is LES for (5.8).

Proof. First observe that, since V attains a global minimum on the diagonal, it holds that $\text{grad} V(X(t), X(t)) = 0$ for all $t \geq 0$. It follows from **P1** that X is a solution of (5.8). Moreover, it holds that

$$\begin{aligned} & \left\langle \nabla_v [-k \text{grad} V(x, X(t)) + \eta(x, t)], v \right\rangle \Big|_{x=X(t)} \\ &= -k(\text{Hess} V)(v, v)|_{T_{X(t)}M} + \langle \nabla_v \eta(x, t), v \rangle \Big|_{x=X(t)} \\ &\leq -(kc_1 - c_2)|v|^2. \end{aligned}$$

The conclusion follows by Corollary 3.2. □

Remark 5.1. A natural choice for η is $\eta(x, t) := P_{X(t)}^x \dot{X}(t)$, where the operator $P_{X(t)}^x$ is the parallel transport from $X(t)$ to x . Then, at each time $t > 0$, $P_{X(t)}^x \dot{X}(t)$ defines a vector field in a neighborhood of $X(t)$. It follows that $\nabla_v P_{X(t)}^x \dot{X}(t) = 0$, $\forall v \in T_{X(t)}M$. In an Euclidean space, $P_{X(t)}^x \dot{X}(t)$ is simply $\dot{X}(t)$, and clearly $\nabla_v \dot{X}(t) = 0$ for all $v \in T_x M$ and all $x \in \mathbb{R}^n$.

Remark 5.2. If the solution X is bounded, then η can be chosen as any smooth extension of X since the formula (5.9) is always valid in this case thanks to the linear dependence of v on both sides.

Consider the attitude dynamics

$$\dot{R} = R\Omega(t) \tag{5.10}$$

where $R \in SO(3)$ is the attitude and $\Omega(t) \in \mathfrak{so}(3)$ corresponds to the angular velocity. The Lie group $SO(3)$ is a Riemannian manifold with the bi-invariant metric $\langle X, Y \rangle = \text{tr}(X^T Y)$. The corresponding Riemannian norm $|X|_F = \sqrt{\text{tr}(X^T X)}$ is the Frobenius norm.

It is tempting to design a low pass filter for the dynamics (5.10), see for example [79]. To this end, we apply Proposition 5.1. Define a function

$$V(\hat{R}, R) = \frac{1}{2} |\hat{R} - R|_F^2, \quad \hat{R}, R \in SO(3).$$

The gradient of V at \hat{R} is the unique vector $\text{grad} V$ satisfying

$$dV(X) = \langle \text{grad} V, X \rangle, \quad \forall X \in T_{\hat{R}} SO(3),$$

from which it can be readily checked that

$$\text{grad} V = \frac{1}{2} \hat{R}(R^T \hat{R} - \hat{R}^T R). \tag{5.11}$$

By Proposition 5.1 (with $\eta(\hat{R}, t) = \hat{R}\Omega(t)$), $R(t)$ is an exponentially stable solution to

$$\begin{aligned}\dot{\hat{R}} &= -k \operatorname{grad} V(\hat{R}, R) + \hat{R}\Omega(t) \\ &= -\frac{k}{2} \hat{R}(R^T(t)\hat{R} - \hat{R}^T R(t)) + \hat{R}\Omega(t)\end{aligned}$$

by noticing that $\eta(\hat{R}, t) = \hat{R}\Omega(t)$ is a smooth extension of \dot{R} to $SO(3)$ and that $SO(3)$ is compact. Therefore, the following is a locally exponential filter for the system (5.10)

$$\dot{\hat{R}} = -\frac{k}{2} \hat{R}(R^T \hat{R} - R^T \hat{R}) + \hat{R}\Omega \quad (5.12)$$

More precisely, there exist two positive constants c, k , such that

$$d(\hat{R}(t), R(t)) \leq ke^{-\lambda t} d(\hat{R}(0), R(0)), \quad \forall d(\hat{R}(0), R(0)) < c, \quad t \geq 0$$

Remark 5.3. The filter (5.12) has also been obtained in [79].

The above analysis is based on the theory developed in Chapter 3, namely, the local exponential stability of a particular solution. This problem can also be tackled via the contraction analysis method developed in Chapter 4.

Consider the following dynamics

$$\dot{R} = R\Omega, \quad (5.13)$$

where Ω is now a function of R (c.f. 5.10). In order to apply contraction analysis, we need first derive the complete lift system of (5.13). To this end, recall that the tangent bundle of a Lie group G is trivial, i.e., $TG \simeq G \times T_e G$, we have

$$TSO(3) \simeq SO(3) \times \mathfrak{so}(3)$$

via the mapping $(R, R\Omega) \mapsto (R, \Omega)$. Hence the second order tangent bundle has the corresponding isomorphism

$$TTSO(3) \simeq TSO(3) \times T\mathfrak{so}(3) \simeq TSO(3) \times \mathfrak{so}(3)^2$$

Therefore, the complete lift system of (5.13) is

$$\frac{d}{dt} \begin{bmatrix} R \\ V \end{bmatrix} = \begin{bmatrix} R\Omega(R) \\ d\Omega_R(RV) \end{bmatrix}$$

For example, when $\Omega(R) = -Z^T R + R^T Z$, the complete lift system reads

$$\dot{V} = -Z^T R V + V^T R^T Z.$$

Choose the Finsler-Lyapunov function $W(V) = \frac{1}{2}|V|_F^2$, it follows that

$$\dot{W} = -2\text{tr}(V^T Z^T R V)$$

When $R = Z$, we have $\dot{W} = -4W$, this suggests that the system (5.13) is locally IES near the solution near Z , where Z can be any solution to (5.13).

5.3 Intrinsic Speed Observer of Lagrangian Systems

Speed observer design for Lagrangian systems is of great practical and theoretical interest. See for example [14, 89, 91] and the references therein. What we are interested in this section is the intrinsic speed observer for Lagrangian system

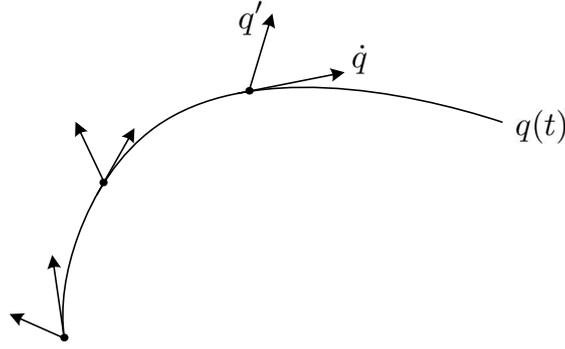
$$\nabla_{\dot{q}} \dot{q} = u, \quad q(t) \in M, \quad u(t) \in T_{q(t)}M \quad (5.14)$$

for all $t \geq 0$, where M is the configuration space equipped with a kinematic metric, see [24, Chapter 4]. Our results are mainly motivated by the work of P. Rouchon [2]. Similar results are also found in [20, 11]. Before observation, we first need to develop an intrinsic stability theory for Lagrangian systems on Riemannian manifolds. After that we combine this with the tools developed in previous chapters to provide an alternative convergence analysis method other than the one in [2]. Compared to the three mentioned works, the proposed method will greatly simplify the analysis procedure. In particular, the analysis is intrinsic and does not involve any calculations in local coordinates. To achieve this goal, some preparations will be covered in this section, which are also interesting in their own rights.

5.3.1 Jacobi equation and stability

Jacobi equation is a second order differential equation which characterizes geodesic variation along a given geodesic. The solutions to the Jacobi equation form into a vector field on the Riemannian manifold, called the Jacobi field. The most remarkable fact it reveals is that the Jacobi field is related to the curvature of the manifold. In this section, we study the Jacobi field from the stability analysis point of view.

Consider the system (5.14) with zero input, i.e., $u = 0$. Then the solutions to the system are geodesics. Choose a geodesic q and find a smooth geodesic variation $(t, s) \mapsto q(t, s)$ such that $q(t, 0) = q(t)$. For convenience, denote $\dot{q} = \frac{\partial q}{\partial t}|_{s=0}$ and $q' = \frac{\partial q}{\partial s}|_{s=0}$. Then both \dot{q} and q' are vector fields along the geodesic q , but \dot{q} is tangent to the geodesic while q' is generally transverse to it. See Figure 5.2.

Figure 5.2: \dot{q} and q' along a geodesic $t \mapsto q(t)$

Taking the covariant derivative of the system (5.14) with zero input along the curve $s \mapsto q(t, s)$ results in $\nabla_{q'} \nabla_{\dot{q}} \dot{q} = 0$, and using the relation

$$\begin{cases} \nabla_{q'} \nabla_{\dot{q}} - \nabla_{\dot{q}} \nabla_{q'} = R(\dot{q}, q') \\ \nabla_{\dot{q}} q' = \nabla_{q'} \dot{q} \end{cases} \quad (5.15)$$

where R is the curvature tensor, we obtain

$$\begin{aligned} \nabla_{\dot{q}} \nabla_{\dot{q}} q' &= \nabla_{\dot{q}} \nabla_{q'} \dot{q} \\ &= \nabla_{q'} \nabla_{\dot{q}} \dot{q} - R(\dot{q}, q') \dot{q} \\ &= -R(\dot{q}, q') \dot{q} \end{aligned}$$

or

$$\frac{D^2 q'}{dt^2} = -R(\dot{q}, q') \dot{q}, \quad (5.16)$$

which is the well-known Jacobi equation. For (5.16), choose the ‘‘Lyapunov function’’

$$V(\dot{q}, q') = \left\langle \frac{Dq'}{dt}, \frac{Dq'}{dt} \right\rangle + \langle R(\dot{q}, q') \dot{q}, q' \rangle.$$

Assume that (M, g) is a constant curvature manifold, that is, there exists a constant $K \in \mathbb{R}$, such that

$$\langle R(\dot{q}, q') \dot{q}, q' \rangle = K \langle \dot{q}, \dot{q} \rangle \langle q', q' \rangle$$

for all \dot{q}, q' . The time derivative of V reads

$$\begin{aligned} \dot{V} &= 2 \left\langle \frac{D^2 q'}{dt^2}, \frac{Dq'}{dt} \right\rangle + \left\langle R(\dot{q}, \frac{Dq'}{dt}) \dot{q}, q' \right\rangle + \left\langle R(\dot{q}, q') \dot{q}, \frac{Dq'}{dt} \right\rangle \\ &= 2 \left\langle -R(\dot{q}, q') \dot{q}, \frac{Dq'}{dt} \right\rangle + 2 \left\langle R(\dot{q}, q') \dot{q}, \frac{Dq'}{dt} \right\rangle \end{aligned}$$

$$= 0,$$

where we have used the fact that $\frac{D\dot{q}}{dt} = 0$. Thus we have

$$V(\dot{q}, q') = |\dot{q}'|^2 + K|q'|^2 = \text{constant}.$$

When $K > 0$, the phase plot of (\dot{q}', q') is a series of ellipses, so one geodesic oscillates along the other, as shown in Figure 5.3, in which the two drawing curves are geodesics and the arrows represent the Jacobi field. When $K < 0$, the geodesic flow diverges. This may be better understood if we rewrite (5.16) as

$$\frac{D^2 q'}{dt^2} = -\text{grad}_{q'} V$$

and this shares the same form of the motion of a particle in a potential field with potential force V , see also [13].

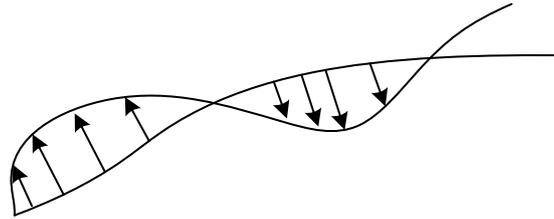


Figure 5.3: Jacobi field on positive curvature manifold

5.3.2 Dynamical Systems Along a Trajectory

To handle Lagrangian systems, we need to introduce some necessary concepts. Consider the vector bundle

$$T^k M = \bigoplus_{i=1}^k TM \text{ and } \Omega^k = T^k M|_X,$$

i.e., Ω^k the restriction of $T^k M$ to the curve $t \mapsto X(t)$, which is still a vector bundle. Given a curve c in M , the set of smooth k -vector fields along c is the smooth section of Ω^k . Explicitly,

$$\Sigma^k = \{V \in \Omega^k | V^i(t), t \in \mathbb{R}_{\geq 0} \text{ are smooth vector fields along } c(t), \forall i = 1, \dots, k\}, \quad (5.17)$$

with $k \in \mathbb{N}_+$ and $V^i(t)$ the i -th component of the vector $V(t)$, or equivalently

$$V(t) = \begin{bmatrix} V^1(t) \\ V^2(t) \\ \vdots \\ V^k(t) \end{bmatrix}, \quad V^i(t) \in T_{c(t)}M.$$

Denote also

$$\frac{DV(t)}{dt} = \begin{bmatrix} \frac{DV^1(t)}{dt} \\ \frac{DV^2(t)}{dt} \\ \vdots \\ \frac{DV^k(t)}{dt} \end{bmatrix}.$$

We equip Ω^k with the inner product (and hence a norm)

$$\langle W(t), V(t) \rangle = \langle W^1(t), V^1(t) \rangle + \cdots + \langle W^k(t), V^k(t) \rangle,$$

where \langle, \rangle on the right hand side are the Riemannian metrics on M . The inner product defined above has an important property, namely, for $V, W \in \Sigma^k$,

$$\begin{aligned} \frac{d}{dt} \langle V, (P \otimes I_n)W \rangle &= L_{\dot{c}(t)} \langle V, (P \otimes I_n)W \rangle \\ &= \left\langle \frac{DV}{dt}, (P \otimes I_n)W \right\rangle + \left\langle V, (P \otimes I_n) \frac{DW}{dt} \right\rangle, \end{aligned}$$

where P is a constant matrix of dimension $k \times k$, I_n the identity matrix of dimension $n \times n$, and \otimes stands for Kronecker product of matrices.

Using the above notations, one can define dynamical systems on Σ^k

$$\frac{DV(t)}{dt} = f(t, V(t)), \quad V \in \Sigma^k, \quad (5.18)$$

where f is smooth and $f(t, 0) = 0$ for $t \geq 0$.

Definition 5.3.1. The system (5.18) is said to be *locally exponentially stable* if there exist positive constants K, λ and b such that

$$|V(t)| \leq K|V(0)|e^{-\lambda t}$$

for all $|V(0)| < b$. If the above is satisfied for all $V(0)$, then the system (5.18) is said to be *globally exponentially stable*.

Proposition 5.2. Let $A \in \mathbb{R}^{k \times k}$ be a constant Hurwitz matrix. Then the following system

$$\frac{DV}{dt} = (A \otimes I_n)V, \quad V \in \Sigma^k \quad (5.19)$$

is globally exponentially stable, where Σ^k is defined as (5.17).

Proof. Since A is Hurwitz, there exists a positive definite matrix $P \in \mathbb{R}^{k \times k}$ such that

$$PA + A^T P = -I_k.$$

Consider now the Lyapunov function $W = \langle V, (P \otimes I)V \rangle$. Differentiating W along c (see (5.17)) yields

$$\begin{aligned} \frac{dW}{dt} &= \left\langle \frac{DV}{dt}, (P \otimes I)V \right\rangle + \left\langle V, (P \otimes I) \frac{DV}{dt} \right\rangle \\ &= \langle (A \otimes I)V, (P \otimes I)V \rangle + \langle V, (P \otimes I)(A \otimes I)V \rangle \\ &= \langle V, (A^T \otimes I)(P \otimes I)V \rangle + \langle V, (P \otimes I)(A \otimes I)V \rangle \\ &= \langle V, [(A^T P + PA) \otimes I]V \rangle \\ &= -\langle V, V \rangle \\ &\leq -kW, \end{aligned}$$

for some $k > 0$. This completes the proof. \square

5.3.3 Local Exponential Stability and Contraction of Lagrangian Systems

Clearly, the reason that trajectories of the Jacobi equation (5.16) oscillate around a given geodesic is due to the curvature term on the right hand side and the lack of damping. Therefore, analogous to the second order system

$$\ddot{x} + \alpha \dot{x} + \beta x = 0, \quad x \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R}$$

in Euclidean space, intuitively, to achieve local exponential stability, the Jacobi equation (5.16) should be modified by adding a damping term:

$$\frac{D^2 q'}{dt^2} + \alpha \frac{Dq'}{dt} + \beta q' = 0 \quad (5.20)$$

with positive constants α and β where the covariant derivatives are evaluated along the prescribed solution.

Stability notions rely on metric (or distance in the case of Riemannian manifold). So, the choice of the metric is crucial. For Lagrangian systems, the state space TM admits the natural Sasaki metric [138] induced from the Riemannian metric on M , which we recall now.

Let $(p, v) \in TM$ and V, W tangent vectors in TM at (p, v) . Choose curves in TM

$$\alpha : t \mapsto (p(t), v(t)), \beta : t \mapsto (q(t), w(t)),$$

with $p(0) = q(0) = p$, $v(0) = w(0) = v$, $v'(0) = V$, $w'(0) = W$. Define the inner product on TM by

$$\langle V, W \rangle_{(p,v)} = \langle d\pi(V), d\pi(W) \rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{dt}(0) \right\rangle_p. \quad (5.21)$$

We call this metric the Sasaki metric on TM .

Having this metric at hand, one can calculate the length of a curve $w(s) = (c(s), v(s))$ lying in TM , which is

$$\begin{aligned} \ell(w) &= \int \sqrt{\langle w'(s), w'(s) \rangle} ds \\ &= \int \sqrt{\langle d\pi(w'(s)), d\pi(w'(s)) \rangle + \left\langle \frac{Dv(s)}{ds}, \frac{Dv(s)}{ds} \right\rangle} ds \\ &= \int \sqrt{\langle c'(s), c'(s) \rangle + \left\langle \frac{Dv(s)}{ds}, \frac{Dv(s)}{ds} \right\rangle} ds \end{aligned}$$

in which the third equality is due to the simple fact that

$$\begin{aligned} d\pi(w'(s)) &= d\pi(dw(\partial/\partial s)) \\ &= d(\pi \circ w)(\partial/\partial s) = dc(\partial/\partial s) = c'(s) \end{aligned}$$

where $\pi : TM \rightarrow M$ is the natural projection.

Assumption 5.1. For each pair of points (q, V) and (p, W) in TM , the minimizing geodesic that joins (q, v) to (p, W) always exists.

This assumption is reasonable since we work always locally.

Theorem 5.1. Consider a Lagrangian system (5.14) and its complete lift $\dot{w} = W$ along the solution $(q_*(t), \dot{q}_*(t))$. Assume that Assumption 5.1 holds. If there exist positive constants K and λ such that

$$|d\pi(w(t; w_0))|^2 + \left| \frac{Dd\pi(w(t; w_0))}{dt} \right|^2 \leq K e^{-\lambda t} \left(|d\pi(w_0)|^2 + \left| \frac{Dd\pi(w(t; w_0))}{dt} \right|_{t=0}^2 \right) \quad (5.22)$$

for all $w_0 \in T_{(q_*(0), \dot{q}_*(0))}TM$, where $w(t; w_0)$ is the solution to the complete lift of the Lagrangian system with initial condition w_0 and the covariant derivative is evaluated along $q_*(t)$. Then the solution $(q_*(t), \dot{q}_*(t))$ is locally exponentially stable. Namely,

$$d_{TM}((X(t), \dot{q}(t)), (q_*(t), \dot{q}_*(t))) \leq K' e^{-\lambda' t} d_{TM}((q(0), \dot{q}(0)), (q_*(0), \dot{q}_*(0))),$$

for $(q(0), \dot{q}(0)) \in B_\epsilon(q_*(0), \dot{q}_*(0))$ with $\epsilon > 0$ small enough and K', λ' two positive constants.

Proof. Consider a point $(q_1, v_1) \in TM$, and the integral curve $\eta_1(t) = (q_1(t), \dot{q}_1(t))$ passing through it at time $t = 0$. Denote $\eta_0(t) = (X(t), \dot{q}(t))$. By Assumption 5.1, there exists a unique minimizing geodesic $\gamma(s) = (q(s), v(s))$, $s \in [0, 1]$ joining (q_0, v_0) to (q_1, v_1) , that is, $\gamma(0) = (q_0, v_0)$, $\gamma(1) = (q_1, v_1)$. Now the family of integral curves $t \mapsto (q(s, t), \frac{\partial q(s, t)}{\partial t})$, $s \in [0, 1]$ passing through $\gamma(s)$ at time $t = 0$ forms a variation of the curve η_0 . The curve $s \mapsto (q(s, t), \frac{\partial q(s, t)}{\partial t})$ joins $\eta_0(t)$ to $\eta_1(t)$. Therefore, we have the estimation of the distance between the two points $\eta_0(t)$ and $\eta_1(t)$:

$$d_{TM}(\eta_0(t), \eta_1(t)) \leq \int_0^1 \sqrt{\left| \frac{\partial q}{\partial s}(s, t) \right|^2 + \left| \nabla_{\frac{\partial}{\partial s}} \frac{\partial q}{\partial t} \right|^2} ds. \quad (5.23)$$

By the definition of complete lift, we know that

$$\frac{\partial q(s, t)}{\partial s} = d\pi(w(t; \frac{\partial \gamma(s)}{\partial s})).$$

Hence by (5.22),

$$\begin{aligned} d_{TM}(\eta_0(t), \eta_1(t)) &\leq K e^{-\lambda t} \int_0^1 \sqrt{\left| \frac{\partial q(s)}{\partial s} \right|^2 + \left| \nabla_{\frac{\partial}{\partial s}} v(s) \right|^2} ds \\ &= K e^{-\lambda t} d_{TM}((q_0, v_0), (q_1, v_1)), \end{aligned}$$

which completes the proof. \square

Corollary 5.1. The solution X to the Lagrangian system (5.14) is LES if there exist positive constants α, β , $i = 1, 2$, such that

$$\frac{D^2 d\pi(w)}{dt^2} + \alpha \frac{D d\pi(w)}{dt} + \beta d\pi(w) = 0, \quad (5.24)$$

for all w_0 where $w = w(t; w_0)$ is as in Theorem 5.1 and the covariant derivative is taken along X .

Proof. Let $V = \begin{bmatrix} \pi(w) \\ \frac{Dd\pi(w)}{dt} \end{bmatrix}$, then the system (5.24) can be rewritten as $\frac{DV}{dt} = (A \otimes I_n)V$,

where

$$A = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix},$$

which is Hurwitz. The corollary is then proved by invoking Proposition 5.2. \square

This corollary will serve as a powerful tool to analyse the convergence of the speed observer in the next section.

5.3.4 Speed Observer of Lagrangian Systems

Consider the Lagrangian system (5.14). The objective is to design a speed observer for $\dot{q}(t)$ knowing the information of $q(t)$ and $u(t)$. The following intrinsic speed observer in proposed in [2] for the system (5.14) when $u = 0$:

$$\begin{cases} \dot{\hat{q}} = \hat{v} - \alpha \nabla F(\hat{q}, q) \\ \nabla_{\hat{q}} \hat{v} = -\beta \nabla F(\hat{q}, q) + R(\hat{v}, \nabla F) \hat{v} + P_q^{\hat{q}} u(q, t). \end{cases} \quad (5.25)$$

where α, β are positive constants and $F(x, y) = \frac{1}{2}d^2(x, y)$ with d the Riemannian distance function. In [2], the authors used contraction analysis in local coordinates to study the local exponential convergence of the observer.

We provide an alternative way for this procedure based on the techniques we have developed in this thesis, which will largely simplify the analysis in [2]. Moreover, unlike that in [2], the analysis will be coordinate free.

Theorem 5.2. Consider the system (5.14), where M is a Riemannian manifold. Then (5.25) is a local exponential speed observer for the system in the sense that there exist some positive constants c, k, λ , such that

$$d(\hat{q}(t), q(t)) \leq ke^{-\lambda t} d(\hat{q}(0), q(0)), \quad \forall t \geq 0, d(\hat{q}(0), q(0)) < c \quad (5.26)$$

Proof. It suffices to show that each solution $(s(t), \dot{s}(t))$ to the following system is LES:

$$\begin{cases} \dot{q} = v - \alpha \nabla F(q, s(t)) \\ \nabla_{\dot{q}} v = -\beta \nabla F(q, s(t)) + R(v, \nabla F)v + P_{s(t)}^q u(s(t), t), \end{cases} \quad (5.27)$$

The system (5.25) is equivalent to the following Lagrangian system:

$$\nabla_{\dot{q}}(\dot{q} + \alpha \nabla F) = -\beta \nabla F + R(\dot{q} + \alpha \nabla F, \nabla F)(\dot{q} + \alpha \nabla F) + P_{s(t)}^q u(s(t), t),$$

or

$$\begin{aligned}\nabla_{\dot{q}}\dot{q} &= -\alpha\nabla_{\dot{q}}\nabla F - \beta\nabla F + R(\dot{q}, \nabla F)(\dot{q} + \alpha\nabla F) \\ &\quad + P_{s(t)}^q u(s(t), t).\end{aligned}$$

Taking the covariant derivative in the direction of q' on the left hand side of the above formula leads to

$$\nabla_{q'}\nabla_{\dot{q}}\dot{q} = \frac{D^2q'}{dt^2} + R(\dot{q}, q')\dot{q}, \quad (5.28)$$

and on the right hand side

$$\begin{aligned}& -\alpha\nabla_{q'}\nabla_{\dot{q}}\nabla F - \beta\nabla_{q'}\nabla F + \nabla_{q'}[R(\dot{q}, \nabla F)(\dot{q} + \alpha\nabla F)] \\ &= -\alpha\nabla_{\dot{q}}\nabla_{q'}\nabla F - \alpha R(\dot{q}, q')\nabla F - \beta\nabla_{q'}\nabla F \\ &\quad + \nabla_{q'}[R(\dot{q}, \nabla F)(\dot{q} + \alpha\nabla F)] \\ &= -\alpha\nabla_{\dot{q}}q' - \beta q' + R(\dot{q}, \nabla_{q'}\nabla F)\dot{q} \quad (\text{since } \nabla F(s(t), s(t)) = 0) \\ &= -\alpha\nabla_{\dot{q}}q' - \beta q' + R(\dot{q}, q')\dot{q},\end{aligned}$$

where we have used the relations $\nabla_{q'}\nabla F|_{q=s(t)} = q'$ and $\nabla_{q'}P_{s(t)}^q u(s(t), t) = 0$ for all $t \geq 0$. Combining this with (5.28) yields

$$\frac{D^2q'}{dt^2} + \alpha \frac{Dq'}{dt} + \beta q' = 0. \quad (5.29)$$

Now invoking Corollary 5.1, the theorem is proved. \square

In [2], the authors considered speed observer for Lagrangian system in the form of (5.25). Note that by setting $u = 0$, we immediately get that result.

5.4 Synchronization

Synchronization is a broad research subject which finds applications in a variety of areas ranging from manufacturing industry to systems biology. It is impossible for us to provide a complete reference list for this subject; however, there are many excellent survey articles, see for example [120, 38, 1] and the many references therein.

In this thesis, we are interested in studying synchronization via contraction analysis. To the best of the author's knowledge, contraction theory was first introduced to study synchronization in [114]. This work was then followed by a more detailed and systematic research article [133], in which W. Wang and J. Slotine proposed the so called *partial contraction* as the basic tool to study synchronization. More recent works on this topic

can be found in [34, 87, 41]. We try to establish a geometric framework and develop a general theory for synchronization in this section. It can partly serve as a mathematical interpretation for the ideas proposed in [133].

Consider a system with $n + 1$ sub-systems

$$\begin{cases} \dot{x}_0 = f_0(x_0, x_1, \dots, x_n) \\ \dot{x}_1 = f_1(x_0, x_1, \dots, x_n) \\ \dot{x}_2 = f_2(x_0, x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_0, x_1, \dots, x_n) \end{cases} \quad (5.30)$$

where $x_i \in M$, and $f_i(x_0, x_1, \dots, x_n) \in T_{x_i}M$ for all $i \in \{0, 1, \dots, n\}$

Definition 5.4.1. The system (5.30) is said to achieve *local exponential synchronization* (LE-sync) if there exist two positive constants K, λ , such that the following holds for some positive constant c :

$$d(x_i(t), x_j(t)) \leq K e^{-\lambda t} d(x_i(0), x_j(0)), \quad (5.31)$$

for all $x_i(0), x_j(0) \in M$ with $d(x_i(0), x_j(0)) < c$, and all $t \geq 0$, $i, j \in \{0, 1, \dots, n\}$. If c can be chosen arbitrarily, the system is said to achieve *global exponential synchronization* (GE-sync).

Definition 5.4.2. We say that the system (5.30) *synchronize to the leader x_0 locally* if there exist two positive constant K, λ such that the following holds for some positive constant c :

$$d(x_i(t), x_0(t)) \leq K e^{-\lambda t} d(x_i(0), x_0(0)) \quad (5.32)$$

for all $x_i(0), x_0(0) \in M$ with $d(x_i(0), x_0(0)) < c$, and all $t \geq 0$, $i \in \{1, \dots, n\}$. If the constant c can be chosen arbitrarily, we say that the system *synchronize to the leader x_0 globally*.

To begin with, consider two coupled nonlinear systems

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \quad (5.33)$$

where x_1, x_2 lie in M . We call the first line of (5.33) *sub-system x_1* and the second line *sub-system x_2* . The aim is to study the exponential stability of the diagonal set $D_M = \{(x_1, x_2) \in M \times M : x_1 = x_2\}$. Clearly, this is achieved only when D_M is an invariant set, which is equivalent to saying that

$$f_1(x, x) = f_2(x, x), \quad \forall x \in M. \quad (5.34)$$

Case 1: $M = \mathbb{R}^n$ with Euclidean metric

When $M = \mathbb{R}^n$, introduce the error variable $e = x_1 - x_2$ and we get

$$\begin{cases} \dot{e} = f_1(x_1, x_1 - e) - f_2(x_1, x_1 - e) \\ \dot{x}_1 = f_1(x_1, x_1 - e) \end{cases} \quad (5.35)$$

Let $F(e, x) = f_1(x, x - e) - f_2(x, x - e)$ and $G(e, x) = f_1(x_1, x_1 - e)$. Invoking condition (5.34), we see that the system can be written in the form of (2.25), i.e., $\dot{e} = F(e, x)$, $\dot{x} = G(e, x)$ with $F(0, x) = 0$ for all $x \in \mathbb{R}^n$.

Based on the error dynamics (5.35) and Definition 2.3.6, it is obvious that the system (5.33) achieves LE-sync if the system (5.35) is TULES-NL (transversally uniformly locally exponentially stable).

Therefore, we can easily prove the following result.

Proposition 5.3. Consider the system (5.33).

1. If there exist two constants $r, c > 0$ and a matrix function C^1 function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ such that

- (a) there exists positive constants p_1, p_2 , such that

$$p_1 I \leq P(x) \leq p_2 I, \quad \forall x \in \mathbb{R}^n, \quad (5.36)$$

- (b)

$$\left| \frac{\partial P}{\partial x}(x) \right| \leq c, \quad \forall x \in \mathbb{R}^n, \quad (5.37)$$

- (c) there exists a positive definite matrix Q , such that

$$L_{f_1(x,x)} P(x) - P(x) \left(\frac{\partial(f_1 - f_2)}{\partial x_2}(x, x) \right) - \left(\frac{\partial(f_1 - f_2)}{\partial x_2}(x, x) \right)^T P(x) \leq -Q, \quad (5.38)$$

for all $x \in \mathbb{R}^n$.

- (d)

$$\begin{aligned} \left| \frac{\partial^2 f_1}{\partial x_2 \partial x_2}(x_1, x_2) \right| \leq c, & \quad \left| \frac{\partial^2 f_2}{\partial x_2 \partial x_2}(x_1, x_2) \right| \leq c, \\ \left| \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(x_1, x_2) \right| \leq c, & \quad \left| \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right| \leq c \end{aligned} \quad (5.39)$$

for all $x_1, x_2 \in \mathbb{R}^n$ with $|x_1 - x_2| \leq r$.

then the system (5.33) achieves LE-sync.

2. Conversely, if the system (5.33) achieves LE-sync and that (5.39) is satisfied, then there exists a continuous matrix function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ such that 1(a) and 1(c) are satisfied as above, where $L_{f_1}P$ should be understood in the Dini derivative sense.

Proof. See [7]. □

Therefore, the following matrix provides characterization of LE-sync of system (5.33).

$$\frac{\partial f_2}{\partial x_2}(x_1, x_2) - \frac{\partial f_1}{\partial x_2}(x_1, x_2),$$

or

$$\frac{\partial f_1}{\partial x_1}(x_1, x_2) - \frac{\partial f_2}{\partial x_1}(x_1, x_2),$$

by swapping the subscripts “1” and “2” in the analysis.

Example 5.1. Consider the simplest case: two coupled scalar linear systems

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases}$$

By (5.34), necessary condition of synchronization is

$$a_{11}x + a_{12}x = a_{21}x + a_{22}x, \quad \forall x \in \mathbb{R}$$

or equivalently, $a_{11} + a_{12} = a_{21} + a_{22}$. (This implies that the matrix $A = [a_{ij}]$ has a real eigenvalue whose corresponding eigenvector is $[1, 1]^T$.) So if $a_{11} - a_{21} < 0$ (or $a_{22} - a_{12} < 0$), then the above analysis implies that x_1 should converge exponentially to x_2 . This is satisfied when A is a Metzler matrix.

Example 5.2. In [133], the authors studied the one-way coupling configuration

$$\begin{cases} \dot{x}_1 = f(x_1, t) \\ \dot{x}_2 = f(x_2, t) + u(x_1) - u(x_2) \end{cases} \quad (5.40)$$

where $x_1, x_2 \in \mathbb{R}^m$. It is claimed that if the function $f - u$ is contracting, i.e., the system $\dot{x} = f(x, t) - u(x)$ is IES, then x_1 and x_2 will reach synchrony exponentially. This is obvious by noticing that

$$\frac{\partial f(x_1, t)}{\partial x_1} - \frac{\partial [f(x_2, t) + u(x_1) - u(x_2)]}{\partial x_1} = \frac{\partial [f(x_1, t) - u(x_1)]}{\partial x_1}.$$

Case 2: $M = \mathbb{R}^n$ with a Riemannian metric

When \mathbb{R}^n is equipped with a Riemannian metric g , or equivalently there exists a C^1 positive definite function $P : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$, such that $g(v_x, v_x) = v_x^T P(x) v_x$. The distance between two given points x_1, x_2 is no longer $|x_1 - x_2|$. In this case, instead of constructing an error dynamics, we construct a one parameter set of auxiliary systems:

$$\dot{x}_s = F(x_1, x_2, x_s) \quad (5.41)$$

where $F(x_1, x_2, x_s) = f_2(x_1, x_s) - f_1(x_1, x_s) + f_1(x_1, x_2)$, $x_s \in \mathbb{R}^n$, $s \in [1, 2]$, and (x_1, x_2) the solution to (5.33). Alternatively, one can also consider

$$\dot{x}_s = f_1(x_s, x_2) - f_2(x_s, x_2) + f_2(x_1, x_2) \quad x_s \in \mathbb{R}^n \quad (5.42)$$

See Figure 5.4.

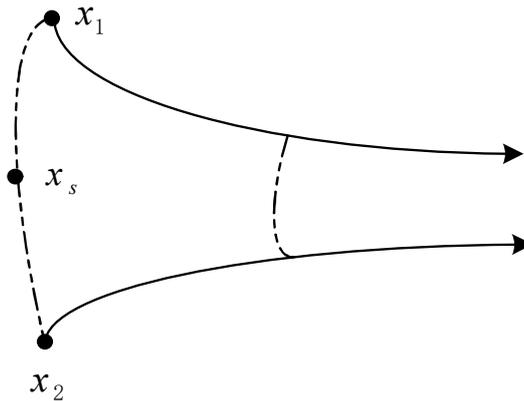


Figure 5.4: A Parameterized Auxiliary Systems

Remark 5.4. The auxiliary system (5.41) has the following important property: when $s = 1$, it reduces to the sub-system x_1 and the sub-system x_2 when $s = 2$. Similar techniques have been used in observer design [106, 7], but the construction of an auxiliary system for the purpose of synchronization is new, and we will show the construction can be easily extended to the Riemannian manifold case.

We adopt the following notations: $\phi_1(t; x_1, x_2)$, $\phi_2(t; x_1, x_2)$ the solution to (5.33) with initial condition (x_1, x_2) , and $\phi_s(t; x_1, x_2, x_s)$ the solution to (5.41) with initial condition (x_1, x_2, x_s) . For simplicity, we sometimes omit the initial conditions and write $x_1(t)$, $x_2(t)$ and $x_s(t)$ when clear from the context.

Fixing two points $x_1, x_2 \in \mathbb{R}^n$, let $\gamma : [1, 2] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be the line segment joining x_1 to x_2 , i.e., $\gamma(1) = x_1$, $\gamma(2) = x_2$. Invoking Remark 5.4, $s \mapsto \phi_s(t; x_1, x_2, \gamma(s))$ is a smooth curve joining $\phi_1(t; x_1, x_2)$ to $\phi_2(t; x_1, x_2)$.

Meanwhile,

$$\begin{aligned} & \frac{d}{dt} \frac{\partial \phi_s(t; x_1, x_2, \gamma(s))}{\partial s} \\ &= \frac{\partial}{\partial s} \frac{d \phi_s(t; x_1, x_2, \gamma(s))}{dt} \\ &= (D_{x_2} f_2 - D_{x_2} f_1)(\phi_1(t; x_1, x_2), \phi_s(t; x_1, x_2, \gamma(s))) \frac{\partial \phi_s(t; x_1, x_2, \gamma(s))}{\partial s} \end{aligned}$$

in which $D_{x_2} f_2$ is the differential of the mapping $x_2 \mapsto f_2(x_1, x_2)$. Thus $\frac{\partial \phi_s(t; x_1, x_2, \gamma(s))}{\partial s}$ is the solution to:

$$\dot{v}_s = \left(\frac{\partial f_2}{\partial x_2}(x_1, x_s) - \frac{\partial f_1}{\partial x_2}(x_1, x_s) \right) v_s, \quad v_s(0) = \gamma'(s). \quad (5.43)$$

where x_s is as in (5.41).

Proposition 5.4. Consider the system (5.33) in which f_1, f_2 are assumed to be C^1 , and $M = \mathbb{R}^n$ is equipped with a Riemannian metric. If there exists a constant $c > 0$, such that

$$P(y) \left(\frac{\partial f_2}{\partial x_2}(x_1, y) - \frac{\partial f_1}{\partial x_2}(x_1, y) \right) + \left(\frac{\partial f_2}{\partial x_2}(x_1, y) - \frac{\partial f_1}{\partial x_2}(x_1, y) \right)^T P(y) \quad (5.44)$$

$$+ L_{F(x_1, x_2, y)} P(y) \leq -cI \quad (5.45)$$

for all $x_1, x_2, y \in \mathbb{R}^n$, then the system (5.33) achieves GE-sync.

Noticing the relation

$$d(\phi_1(t; x_1, x_2), \phi_2(t; x_1, x_2)) \leq \int_1^2 \left| \frac{\partial \phi_s(t; x_1, x_2, \gamma(s))}{\partial s} \right| ds, \quad (5.46)$$

we see immediately that the distance between $\phi_1(t; x_1, x_2)$ and $\phi_2(t; x_1, x_2)$ can be characterized by (5.43). Thus proof of this proposition is straightforward by imitating the procedure of the proof of Theorem 4.1 and hence is omitted.

Remark 5.5. In [7], the authors used transverse linearization techniques to study LE-sync for systems in Euclidean space. In comparison, Proposition 5.4 is concerned with GE-sync on Riemannian manifolds.

Case 3: M a general Riemannian manifold

For general Riemannian manifold, the methods used in the above two cases no longer work. For example, the auxiliary system

$$\dot{x}_s = f_2(x_1, x_s) - f_1(x_1, x_s) + f_1(x_1, x_2) \quad (5.47)$$

does not make sense any more in that $f_1(x_1, x_s)$ and $f_1(x_1, x_2)$ are tangent vectors in $T_{x_1}M$ but $f_2(x_1, x_s) - f_1(x_1, x_s) + f_1(x_1, x_2)$ needs to be in $T_{x_s}M$. To handle this, we modify the auxiliary system (5.41) into the following form

$$\dot{x}_s = f_2(x_1, x_s) - P_{x_1}^{x_s} f_1(x_1, x_s) + P_{x_1}^{x_s} f_1(x_1, x_2). \quad (5.48)$$

Again it can be easily verified that when s varies from 1 to 2, the vector field $\bar{F}(x_1, x_2, x_s) := f_2(x_1, x_s) - P_{x_1}^{x_s} f_1(x_1, x_s) + P_{x_1}^{x_s} f_1(x_1, x_2)$ varies from $f_1(x_1, x_2)$ to $f_2(x_1, x_2)$.

Thus to estimate the distance between $\phi_1(t; x_1, x_2)$ and $\phi_2(t; x_1, x_2)$, we use

$$d(\phi_1(t; x_1, x_2), \phi_2(t; x_1, x_2)) \leq \int_1^2 \left| \frac{\partial \phi_s(t; x_1, x_2, \gamma(s))}{\partial s} \right| ds \quad (5.49)$$

But

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial \phi_s}{\partial s}, \frac{\partial \phi_s}{\partial s} \right\rangle &= \left\langle \frac{D}{\partial t} \frac{\partial \phi_s}{\partial s}, \frac{\partial \phi_s}{\partial s} \right\rangle \\ &= \left\langle \frac{D}{\partial s} \frac{\partial \phi_s}{\partial t}, \frac{\partial \phi_s}{\partial s} \right\rangle = \left\langle \nabla_{\frac{\partial \phi_s}{\partial s}} \bar{F}(\phi_1, \phi_2, \phi_s), \frac{\partial \phi_s}{\partial s} \right\rangle \end{aligned}$$

therefore, it remains to study the quadratic form

$$Q_{x_1, x_2}(v_y) = \left\langle \nabla_{v_y} \bar{F}(x_1, x_2, y), v_y \right\rangle, v_y \in T_y M. \quad (5.50)$$

We have the following proposition.

Proposition 5.5. Consider the system (5.33) in which f_1, f_2 are assumed to be C^1 , and M is a Riemannian manifold with metric \langle, \rangle . If there exists a constant $c > 0$, such that

$$Q_{x_1, x_2}(v_y) \leq -c|v_y|^2, \quad \forall x_1, x_2, y \in M, v_y \in T_y M \quad (5.51)$$

where Q_{x_1, x_2} is the quadratic form defined in (5.50), then the system (5.33) achieves GE-sync.

Remark 5.6. This proposition is a generalization of Proposition 5.4.

For a system with two sub-systems, there is no difference between Definition 5.4.1 (leaderless synchronization) and Definition 5.4.2 (synchronization to a leader). This is not true for a system with more than two sub-systems, as $d(x_i(t), x_0(t)) \leq ke^{-\lambda t} d(x_i(0), x_0(0))$

for all $i \in \{1, \dots, n\}$, $n \geq 2$ and some positive constants k, λ does not necessarily imply $d(x_i(t), x_j(0)) \leq k'e^{-\lambda t}d(x_i(0), x_j(0))$ for all $i, j \in \{0, 1, \dots, n\}$ for some positive constants k', λ' . However, one can always treat a leaderless synchronization problem with n agents as n leader following problems by setting x_i as the leader at the i -th step.

In this section, we show how to extend the results obtained in previous subsection to synchronization of multi-agent systems. We start with leader following synchronization.

In analogy with the two coupled sub-system, we construct a parameterized system:

$$\begin{aligned} \dot{x}_i^s &= \sum_{j=1}^n P_i(f_i - f_0)(\underbrace{x_0, \dots, x_0}_{j \text{ times}}, x_j^s, x_{j+1}, \dots, x_n) \\ &\quad - \sum_{j=1}^n P_i(f_i - f_0)(\underbrace{x_0, \dots, x_0}_{j \text{ times}}, x_j, x_{j+1}, \dots, x_n) \\ &\quad + P_i f_i(x_0, \dots, x_n), \quad i = 1, \dots, n \end{aligned} \quad (5.52)$$

where $x_i^s(0) = \gamma_i(s)$, $s \in [0, 1]$, in which γ_i is a smooth curve joining $x_0(0)$ to $x_i(0)$, i.e., $\gamma_i(0) = x_0(0)$, $\gamma_i(1) = x_i(0)$; P_i is an operator which takes a vector $f(u_1, \dots, u_i, \dots, u_n)$ to $P_{u_i}^{x_i^s} f(u_1, \dots, u_i, \dots, u_n)$. This is illustrated in Figure 5.5.

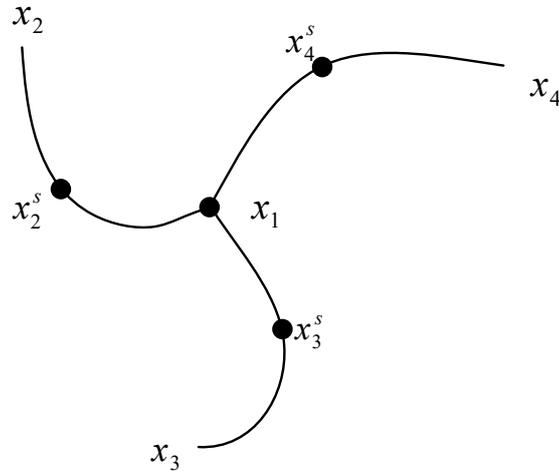


Figure 5.5: A Parameterized Auxiliary Systems of n Subsystems

Rewrite (5.52) as

$$\dot{x}_i^s = F_i(x_0, \dots, x_n; x_1^s, \dots, x_n^s). \quad (5.53)$$

We have the following key observation:

$$\dot{x}_0 = F_i(x_0, \dots, x_n; x_0, \dots, x_0), \quad \forall i = 1, \dots, n \quad (5.54)$$

$$\dot{x}_i = F_i(x_0, \dots, x_n; x_1, \dots, x_n). \quad \forall i = 1, \dots, n \quad (5.55)$$

Lemma 5.1. Given two ordinary differential equations: $\dot{y}_1 = f(t, y_1, y_2)$ and $\dot{y}_2 = f(t, y_2, y_2)$ with f smooth. If $y_1(0) = y_2(0)$, then $y_1(t) = y_2(t)$ for all $t \geq 0$.

Proof. By assumption, $y_2(t)$ is a particular solution to $\dot{y}_1 = f(t, y_1, y_2)$ hence by uniqueness of solution, $y_1(t) = y_2(t)$ for all $t \geq 0$. \square

Lemma 5.2. The map $[0, 1] \ni s \mapsto x_i^s(t)$ forms a smooth curve joining $x_0(t)$ to $x_i(t)$ for all $i = 1, \dots, n$.

Proof. Denote

$$X_1 = \underbrace{(x_0, \dots, x_0)}_{n \text{ times}}, \quad X_2 = (x_1, \dots, x_n), \quad X^s = (x_1^s, \dots, x_n^s),$$

then equations (5.53) to (5.55) can be rewritten as

$$\dot{X}^s = F(x_0, X_2, X^s), \quad (5.56)$$

$$\dot{X}_1 = F(x_0, X_1, X_1) \quad (5.57)$$

$$\dot{X}_2 = F(x_0, X_1, X_2) \quad (5.58)$$

for a smooth F . By construction, $X^0(0) = X_1(0)$ and $X^1(0) = X_2(0)$, therefore by Lemma 5.1, we have $X^0(t) = X_1(t)$ and $X^1(t) = X_2(t)$ for all $t \geq 0$. Equivalently,

$$x_i^0(t) = x_0(t), \quad x_i^1(t) = x_i(t), \quad \forall i = 1, \dots, n, \quad t \geq 0. \quad (5.59)$$

Thus the lemma follows. \square

Lemma 5.2 can now help us estimate the distance between $x_i(t)$ and $x_0(t)$. In effect,

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial x_s^i}{\partial s}, \frac{\partial x_s^i}{\partial s} \right\rangle &= \left\langle \frac{D}{\partial t} \frac{\partial x_s^i}{\partial s}, \frac{\partial x_s^i}{\partial s} \right\rangle \\ &= \left\langle \frac{D}{\partial s} \frac{\partial x_s^i}{\partial t}, \frac{\partial x_s^i}{\partial s} \right\rangle = \left\langle \nabla_{\frac{\partial x_s^i}{\partial s}} F_i(x_0, \dots, x_n; x_1^s, \dots, x_n^s), \frac{\partial x_s^i}{\partial s} \right\rangle \end{aligned}$$

therefore, it remains to study the n quadratic forms

$$Q_{x_0, \dots, x_n, y_1, \dots, y_n}^i(v_{y_i}) = \left\langle \nabla_{v_{y_i}} F_i(x_0, \dots, x_n; y_1, \dots, y_n), v_{y_i} \right\rangle, \quad v_{y_i} \in T_{v_{y_i}} M \quad (5.60)$$

with $i = 1, \dots, n$, and we can conclude without difficulty that if there exist n positive constants $c_i, i = 1, \dots, n$, such that

$$Q_{x_0, \dots, x_n, y_1, \dots, y_n}^i(v_{y_i}) \leq -c_i |v_{y_i}|^2, \quad v_{y_i} \in T_{y_i} M, \quad i = 1, \dots, n \quad (5.61)$$

for all $x_0, \dots, x_n, y_1, \dots, y_n \in M$ and $v_{y_i} \in T_{y_i} M$, then the system (5.30) achieves GE-sync to the leader x_0 .

5.5 A Brief Summary

In this chapter, we have studied some applications of the theory that we have developed in previous chapters.

First, equilibrium seeking on Riemannian manifold was studied. In particular, we proved a conjecture posed in [22] under the assumption of non-negative constant curvature manifold. Notably, we obtained the optimal step size for the Euler method on Riemannian manifolds.

Second, we studied gradient filter on $SO(3)$ using contraction method.

Third, speed observer of Lagrangian system on manifolds was studied. We started by establishing contraction theory for Lagrangian system, based on which the convergence of the intrinsic speed observer was analyzed in a coordinate free manner. The analysis is greatly simplified compared to existing results in the literature.

At last, we studied the synchronization problem from a contraction point of view.

These examples shows the advantage of geometric contraction analysis developed in previous chapters.

Chapter 6

Extensions: Robustness of Transverse Stability of Submanifolds

In this chapter, we study robustness properties of IES systems, and then extend the results to the robustness of stable submanifolds. At the first stage, we want to know how contraction is affected when a disturbance is introduced into the system. This question is valuable for example, when designing a contraction based observer [111].

For systems evolving in an Euclidean space, this question can be well understood by making use of the transverse stability theory developed in [7]. For example, consider a nonlinear system with an unknown disturbance $d_1 : \mathbb{R}_+ \rightarrow \mathbb{R}^m$,

$$\dot{x} = f(x, d_1), \quad x \in \mathbb{R}^m$$

and an observer in the following form

$$\dot{\hat{x}} = f(\hat{x}, 0) + B(y - h(\hat{x}, 0))$$

where $y = h(x, d_2) \in \mathbb{R}^m$ is the measured output, $B \in \mathbb{R}^{n \times m}$ and d_2 the measure noise. If $d_1 = 0$, $d_2 = 0$ and the system $\dot{z} = f(z, 0) - Bh(z, 0)$ is contractive, then the observer converges exponentially. To study the robustness, introduce the error term $e = \hat{x} - x$, then

$$\dot{e} = f(e + x, 0) - f(x, d_1) + B(h(x, d_2) - h(x + e, 0))$$

Let $F(e, x, d) = f(e + x, 0) - f(x, d_1) + B(h(x, d_2) - h(x + e, 0))$, $d = (d_1, d_2)$ and $G(x, d) = f(x, d_1)$. Then the error system, together with the dynamics of x can be written as

$$\begin{cases} \dot{e} = F(e, x, d) \\ \dot{x} = G(x, d) \end{cases} \quad (6.1)$$

where F satisfies $F(0, x, 0) = 0, \forall x \in \mathbb{R}^n$. Observe that when $d = 0$, the system (6.1) is in the form of (2.25). Thus, the robustness of contraction is equivalent to the robustness of the transverse (exponential) stability of the system (2.25). Since it has been proven in [7] that transverse exponential stability implies the existence of a “Lyapunov function”, one can therefore use this function to analyze the robustness.

In fact, we can go one step further. Loosely speaking, transverse exponential stability is equivalent to the exponential stability of the set $W = \{(e, x) | e = 0\} \subseteq \mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$ of the system (2.25) (the assumption that $F(0, x, 0) = 0$ for all x guarantees the invariance of W in the absence of d). Therefore, robust transverse exponential stability is equivalent to the robust exponential stability of the set W . Notice that W is a plane in $\mathbb{R}^{n_e \times n_x}$, a submanifold with simple structure. One may ask if it is possible to derive a parallel theory for more general class of submanifolds. More precisely, the uniform local exponential stability of W , or TULES-NL defined in Definition 2.3.6 can be characterized by the uniform exponential stability (UES-TL) of the linearization of the system (2.25) along W , and this linearization can aid the robustness analysis of the set W . The question is whether we can study the robust stability of more general classes of submanifolds via the linearization along the submanifold once “linearization” of is properly defined.

We are going to study an important class of submanifolds, namely, compact submanifolds, and dynamics along them. Unlike in the Euclidean case, in which the transverse linearization system can be easily defined, for systems along a compact submanifold, however, this is much more involved. One important problem which will be met is that a compact manifold generally cannot be covered by a single coordinate chart, and therefore how to define a global linearization system along the manifold is not so obvious.

This chapter is organized as follows: we introduce some basic notions regarding invariant dynamics along a submanifold, notably Anosov flow and normal hyperbolic invariant manifold theory. Then we show in Euclidean space that hyperbolicity plays a crucial role for robustness. Next, to extend the result to a more general class of submanifolds, we develop a technique to project the complete lift system along a submanifold to Euclidean space and thus a workable global linearization system can be obtained. Having this linearized system at hand, robustness analysis will be carried out.

6.1 Anosov Flow and Normally Hyperbolic Invariant Manifold

In this chapter, we focus on nonlinear time invariant system (3.2), i.e., $\dot{x} = f(x)$, $x \in M$. For convenience, define $\phi_t : M \rightarrow M$ as $\phi_t(x) = \phi(t; 0, x)$, and we call ϕ_t the flow of the system (3.2).

Definition 6.1.1 ([12]). Consider the system (3.2). The flow ϕ_t is called an Anosov flow if there exists a splitting of the tangent bundle

$$T_x M = E^s(x) \oplus E^u(x) \oplus \mathbb{R}E^0(x),$$

such that the following properties hold:

1. the splitting is invariant under the flow $D\phi_t : TM \rightarrow TM$;
2. $E^0(x)$ coincides with the direction of $f(x)$;
3. there exist constants $k, \lambda > 0$ such that for all $t > 0$,

$$\|D\phi_t(v)\| \leq ke^{-\lambda t}\|v\|, \quad \forall v \in E^s$$

and

$$\|D\phi_t(v)\| \geq ke^{\lambda t}\|v\|, \quad \forall v \in E^u.$$

We can see that if the dimension of E^u is everywhere zero along a closed curve γ , then by Theorem 3.4, γ is a locally exponentially stable (LES) curve. More generally, if there exists a compact invariant submanifold ¹ $U \subseteq M$, such that $E^u(x) \subseteq T_x U$ for all $x \in U$, then is the submanifold U LES?

Example 6.1. When M is \mathbb{R}^n and U is an embedded smooth submanifold, we now try to find criteria to check whether the flow is Anosov. To this end, we consider the linearization of the system (3.2):

$$\dot{v} = \frac{\partial f}{\partial x}(x)v. \quad (6.2)$$

If $df(T_x U) = T_x U$ for all $x \in \mathbb{R}^n$ and we can find the following decomposition of \mathbb{R}^n :

$$\mathbb{R}^n = E^s(x) \oplus E^u(x) \oplus T_x U$$

¹Recall that an invariant manifold is such that the vector field defining the system is tangent to this submanifold.

such that $E^s(x)$ and $E^u(x)$ are invariant subspaces of $\partial f(x)/\partial x$. In particular, if $E^s(x) = \text{span}\{v_1(x), \dots, v_k(x)\}$, $E^u(x) = \text{span}\{v_{k+1}, \dots, v_{k+l}\}$ and

$$\frac{\partial f(x)}{\partial x} v_i(x) = \lambda_i(x) v_i(x), \quad \forall 1 \leq i \leq k+l$$

where λ_i are smooth, $\lambda_i(x) \leq -c$, $\forall 1 \leq i \leq k$, $\lambda_i(x) \geq c$, $\forall k+1 \leq i \leq k+l$ for a positive constant c , then the flow along the submanifold U is Anosov. Clearly, a linear system $\dot{x} = Ax$ where A is diagonalizable is Anosov along the subspace $\{x \in \mathbb{R}^n : Ax = 0\}$.

It is generally hard to check whether a flow is Anosov or not. However, a flow along an LES manifold is automatically Anosov with $\dim E^u = 0$.

We now introduce a more important class of flow along submanifolds, namely, flow along a normally hyperbolic invariant manifold (NHIM). It will be shown later in this chapter that NHIM will play a central role for the study of robust transverse exponential stability.

Definition 6.1.2 ([134]). Let $U \subseteq M$ be a smooth invariant submanifold. If there exists a splitting of the tangent bundle

$$T_x M = E^s(x) \oplus E^u(x) \oplus T_x U$$

such that each of the three components is invariant under the flow ϕ_t and there exist some positive constants c, λ, γ , such that

1. $|D\phi_t(v_s)| \leq ce^{-\lambda(t-t_0)}|v_s|$, $\forall v_s \in E^s$, $t \geq 0$;
2. $|D\phi_t(v_u)| \geq ce^{\lambda(t-t_0)}|v_u|$, $\forall v_u \in E^u$, $t \geq 0$;
3. $|D\phi_t(v_h)| \leq ce^{\gamma(t-t_0)}|v_h|$, $\forall v_h \in TU$, $t \geq 0$;
4. $\lambda > \gamma > 0$.

then the manifold U is called an NHIM.

Remark 6.1. We underscore that the requirement $\gamma < \lambda$ is crucial for hyperbolicity, which is the only difference to an Anosov flow. Later, we will see that an NHIM has some robust properties which an Anosov flow does not have.

Example 6.2. Let us consider a simple example of an NHIM. Consider the system (2.25). If there exist two positive constants $\alpha > \beta > 0$, such that

$$\left(\frac{\partial F}{\partial e}\right)^T(0, x) + \frac{\partial F}{\partial e}(0, x) \leq -2\alpha I \quad (6.3)$$

$$\left| \frac{\partial G(0, x)}{\partial x} \right| < \beta, \quad \forall x \in \mathbb{R}^{n_x} \quad (6.4)$$

then the invariant manifold $W = \{(e, x) | e = 0\}$ is an NHIM.

Proof. Observe that

$$\begin{bmatrix} v_e(t) \\ v_x(t) \end{bmatrix} = D\phi_t \begin{bmatrix} v_e(0) \\ v_x(0) \end{bmatrix}$$

Then the complete lift system of (2.25) along W is

$$\begin{bmatrix} \dot{v}_e \\ \dot{v}_x \end{bmatrix} = \begin{pmatrix} \frac{\partial F}{\partial e}(0, x) & 0 \\ \frac{\partial G}{\partial e}(0, x) & \frac{\partial G}{\partial x}(0, x) \end{pmatrix} \begin{bmatrix} v_e \\ v_x \end{bmatrix} \quad (6.5)$$

By identifying $T_{(e,x)}(\mathbb{R}^{n_e} \times \mathbb{R}^{n_x})$ with $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$, the above equation shows clearly that $\mathbb{R}^{n_e} \times \{0_x\}$ is invariant since \dot{v}_e is independent of v_x . Similarly, $\{0_e\} \times \mathbb{R}^{n_x}$ is also invariant since when $v_e = 0$, we have $\dot{v}_x = \frac{\partial G}{\partial x}(0, x)v_x$.

Furthermore, the condition (6.3) implies that $|v_e(t)| \leq e^{-\alpha t}|v_e(0)|$ for all $t \geq 0$, and the condition (6.4) implies that $|v_x(t)| \leq e^{\beta t}|v_x(0)|$ for all $t \geq 0$. Therefore, $E_{(0,x)}^s = \mathbb{R}^{n_e} \times \{0\}$, $E_{(0,x)}^u = 0$ and $T_{(0,x)}U = T_{(e,x)}W = \{0_e\} \times \mathbb{R}^{n_x}$. This makes W an NHIM according to Definition 6.1.2. \square

6.2 Transverse Input to State Stability

We define transverse input to state stability as follows.

Definition 6.2.1. Consider the system

$$\begin{cases} \dot{e} = F(e, x, d) \\ \dot{x} = G(e, x, d) \end{cases} \quad (6.6)$$

where $(e, x, d) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$, and F and G are assumed to be C^2 with respect to (e, x, d) . The time varying signal $d(t)$ is a continuous and bounded disturbance. The system is called e -transverse input to state stable (e -tISS for short) if there exist a class \mathcal{K}_∞ function γ , a class \mathcal{KL} function β and a positive constant r , such that

$$|E(t, e_0, x_0)| \leq \beta(|e_0|, t) + \gamma(\|d\|_\infty).$$

for all $(t, e_0, x_0) \in \mathbb{R}_+ \times B_e(r) \times \mathbb{R}^{n_x}$.

Remark 6.2. It should be emphasized that on the right hand side of the above inequality, β and γ are independent of x_0 . Clearly, e -tISS is a special case of input to state stability with respect to a set, see for example [119].

Assumption 6.1. The system (6.6) with $d = 0$ is TULES-NL, namely, there exist positive real numbers r, k and λ such that we have for all $(e_0, x_0, t) \in B_e(r) \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$

$$|E(t, e_0, x_0)| \leq k|e_0|e^{-\lambda t}.$$

Assumption 6.2. There exists a positive constant ρ such that $\left| \frac{\partial G}{\partial x}(0, x, 0) \right| < \rho < \lambda$ where λ is the decay rate of the linearized e -subsystem.

Invoking Example 6.2, this assumption makes the manifold $W = \{(e, x) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} : e = 0\}$ a normally hyperbolic invariant manifold.

The following two propositions are from [7], which we are going to recall for further use.

Proposition 6.1 ([7]). Consider the system (6.6). If Assumption 6.1 holds, then there exists a continuous function $P : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_e \times n_e}$ and strictly positive real numbers p_1 and p_2 such that P has a derivative $L_{G(0, \tilde{x})}P(\tilde{x}) := \lim_{h \rightarrow 0} (P(\tilde{X}(h, \tilde{x})) - P(\tilde{x}))/h$ and we have, for all $\tilde{x} \in \mathbb{R}^{n_x}$,

$$L_{G(0, \tilde{x}, 0)}P(\tilde{x}) + P(\tilde{x}) \frac{\partial F}{\partial e}(0, \tilde{x}, 0) + \left(\frac{\partial F}{\partial e} \right)^T(0, \tilde{x}, 0)P(\tilde{x}) \leq -I \quad (6.7)$$

$$p_1 I \leq P(\tilde{x}) \leq p_2 I; \quad (6.8)$$

furthermore, the matrix $P(\tilde{x})$ can be constructed explicitly as

$$P(\tilde{x}) = \int_0^\infty \left(\frac{\partial \tilde{E}}{\partial \tilde{e}} \right)^T(0, \tilde{x}, t) \frac{\partial \tilde{E}}{\partial \tilde{e}}(0, \tilde{x}, t) dt. \quad (6.9)$$

where $(\tilde{E}(t, \tilde{e}, \tilde{x}), \tilde{X}(t, \tilde{x}))$ is the solution to the system

$$\begin{cases} \dot{\tilde{e}} = \frac{\partial F}{\partial e}(0, \tilde{x}, 0)\tilde{e}, \\ \dot{\tilde{x}} = G(0, \tilde{x}, 0). \end{cases} \quad (6.10)$$

The converse of Proposition 6.1 was also proved in [7]:

Proposition 6.2 ([7]). Suppose that there exists a continuous matrix function $P : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_e \times n_e}$ and strictly positive real numbers p_1 and p_2 such that (6.7) and (6.8) are satisfied. Additionally, if

$$\left| \frac{\partial P(\tilde{x})}{\partial \tilde{x}} \right| < C, \quad \forall \tilde{x} \in \mathbb{R}^{n_x} \quad (6.11)$$

for some positive constant C , then Assumption 6.1 holds, i.e. the system (6.6) is TULES-NL when for $d = 0$.

In Proposition 6.2, a key assumption is (6.11), we show this is automatically satisfied when the system flow of (2.25) along $W = \{(e, x) : e = 0\}$ is normally hyperbolic.

Proposition 6.3. Under Assumption 6.2 and 6.1, the matrix function $P(\tilde{x})$ defined in 6.9 is globally Lipschitz and hence is differentiable almost everywhere. In particular, there exists a constant $C > 0$ such that

$$\left| \frac{\partial P(\tilde{x})}{\partial \tilde{x}} \right| < C,$$

for almost all $\tilde{x} \in \mathbb{R}^{n_x}$.

Lemma 6.1. Consider the following ordinary differential equation,

$$\dot{b}(t) = A(t)b(t) + c(t).$$

Assume that

- 1) $\dot{b} = A(t)b$ is exponentially stable;
- 2) there exist some positive constants a, δ , such that $|c(t)| \leq ae^{-\delta t}$ for all $t \geq 0$.

Then there exist some positive constants c_1 and γ such that $|b(t)| \leq c_1(|b(0)| + a)e^{-\gamma t}$ for all $t \geq 0$.

The proof of this Lemma is quite standard and hence omitted.

Proof of Proposition 6.3. Let $\alpha(\tilde{e}, \tilde{x}, \tilde{y}, t) = \tilde{E}(t, \tilde{e}, \tilde{x}) - \tilde{E}(t, \tilde{e}, \tilde{y})$, then

$$\frac{d}{dt}\alpha(\tilde{e}, \tilde{x}, \tilde{y}, t) = \frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{y}))\alpha(\tilde{e}, \tilde{x}, \tilde{y}, t) + b(\tilde{e}, \tilde{x}, \tilde{y}, t)$$

where

$$b(\tilde{e}, \tilde{x}, \tilde{y}, t) = \left(\frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{x})) - \frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{y})) \right) \tilde{E}(t, \tilde{e}, \tilde{x}),$$

and

$$\begin{aligned} |b(\tilde{e}, \tilde{x}, \tilde{y}, t)| &= \left| \left(\frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{x})) - \frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{y})) \right) \tilde{E}(t, \tilde{e}, \tilde{x}) \right| \\ &\leq c_1 \max \left| \frac{\partial^2 F}{\partial e \partial x} \right| |\tilde{X}(t, \tilde{x}) - \tilde{X}(t, \tilde{y})| \cdot e^{-\lambda t} |\tilde{e}| \\ &\leq c_2 e^{-(\lambda-\rho)t} |\tilde{x} - \tilde{y}| \cdot |\tilde{e}|. \end{aligned}$$

By assumption, the system $\dot{\alpha}(\tilde{e}, \tilde{x}, \tilde{y}, t) = \frac{\partial F}{\partial e}(0, \tilde{X}(t, \tilde{y}))\alpha(\tilde{e}, \tilde{x}, \tilde{y}, t)$ is exponentially stable (uniformly in \tilde{y}), then by Lemma 6.1, there holds

$$|\tilde{E}(t, \tilde{e}, \tilde{x}) - \tilde{E}(t, \tilde{e}, \tilde{y})| \leq k|\tilde{e}| \cdot |\tilde{x} - \tilde{y}|e^{-\gamma t}, \quad \forall t \geq 0 \quad (6.12)$$

hence

$$\left| \frac{\partial \tilde{E}}{\partial \tilde{x}}(t, \tilde{e}, \tilde{x}) \right| \leq c_1 |\tilde{e}| e^{-c_2 t}, \quad \forall \tilde{x} \in \mathbb{R}^{n_x}, t \geq 0,$$

and

$$\left| \frac{\partial^2 \tilde{E}}{\partial \tilde{x} \partial e}(t, \tilde{e}, \tilde{x}) \right| \leq c_1 e^{-c_2 t}, \quad \forall \tilde{x} \in \mathbb{R}^{n_x}, t \geq 0.$$

due to the linearity of \tilde{E} with respect to \tilde{e} . Thus $P(\tilde{x})$ is globally Lipschitz. By Rademacher's theorem, the proposition follows and further more, we have

$$\left| \frac{\partial P(\tilde{x})}{\partial \tilde{x}} \right| = \left| \int_0^\infty \frac{\partial}{\partial \tilde{x}} \left[\frac{\partial^T \tilde{E}}{\partial \tilde{e}}(t, 0, \tilde{x}) Q \frac{\partial \tilde{E}}{\partial e}(t, 0, \tilde{x}) \right] dt \right| \leq C.$$

□

Now that $P(\tilde{x})$ is differentiable almost everywhere, the term $L_{G(0, \tilde{x}, 0)}P(\tilde{x})$ can be safely written as

$$L_{G(0, \tilde{x}, 0)}P(\tilde{x}) = \sum_i \frac{\partial P(\tilde{x})}{\partial \tilde{x}_i} G_i(0, \tilde{x}, 0).$$

for almost all x .

The following proposition will be useful for us.

Proposition 6.4 ([7]). If the system (2.25) is TULES-NL, and there exist positive real numbers ρ, μ and c such that

$$\left| \frac{\partial F}{\partial e}(0, x, 0) \right| \leq \mu, \quad \left| \frac{\partial G}{\partial x}(0, x, 0) \right| \leq \rho, \quad \forall x \in \mathbb{R}^{n_x}$$

and for all $(e, x) \in B_e(kr) \times \mathbb{R}^{n_x}$,

$$\left| \frac{\partial^2 F}{\partial e \partial e}(e, x, 0) \right| \leq c, \quad \left| \frac{\partial^2 F}{\partial x \partial e}(0, x, 0) \right| \leq c, \quad \left| \frac{\partial G}{\partial e}(e, x, 0) \right| \leq c.$$

then the property UES-TL holds.

We have the following theorem.

Theorem 6.1. Under Assumptions 6.2 and 6.1, suppose that F and G are globally Lipschitz with respect to d . Then the system (2.25) is e -tISS.

Proof. Let $V(e, x) = e^T P(x)e$. Using characterization of ISS with respect to a set [119], it suffices to prove that there exist positive constants a, b such that $\dot{V}(e, x) \leq -a|e|^2 + b|d|^2$ for all $|d| < c$ for some positive constant c and small e . The time derivative of V along the system (6.6) reads

$$\begin{aligned}
\dot{V}(e, x) &= 2e^T P(x)F(e, x, d) + e^T L_{G(e,x,d)}P(x)e \\
&= 2e^T P(x)(F(e, x, d) - F(e, x, 0)) + 2e^T P(x)F(e, x, 0) \\
&\quad + e^T L_{G(e,x,d)-G(e,x,0)}P(x)e + e^T L_{G(e,x,0)-G(0,x,0)}P(x)e + e^T L_{G(0,x,0)}P(x)e \\
&\leq 2c|e| \cdot |d| + 2c|e|^2|d| + C|e|^3 \\
&\quad + 2e^T P(x)F(e, x, 0) + e^T L_{G(0,x,0)}P(x)e \\
&= 2c|e| \cdot |d| + 2c|e|^2|d| + C|e|^3 \\
&\quad + e^T \left(\int_0^1 L_{G(0,x,0)}P(x) + P(x) \frac{\partial F}{\partial e}(se, x, 0) + \frac{\partial^T F}{\partial e}(se, x, 0)P(x) ds \right) e \\
&\leq 2c|e| \cdot |d| + 2c|e|^2|d| + c|e|^3 - c|e|^2 + c|e|^3 \\
&\leq c\gamma|e|^2 + \frac{c|d|^2}{\gamma} - c|e|^2 \\
&\leq -a|e|^2 + b|d|^2
\end{aligned}$$

with $0 < \gamma < 1$; we have assumed e small and used the following inequalities and the inequalities in Proposition 6.4

$$\begin{aligned}
|F(e, x, d) - F(e, x, 0)| &\leq c|d| \\
|L_{G(e,x,d)-G(e,x,0)}P(x)| &\leq c|d| \\
|L_{G(e,x,0)-G(0,x,0)}P(x)| &\leq c|e| \\
2|e| \cdot |d| &\leq \gamma|e|^2 + \frac{|d|^2}{\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left[P(x) \frac{\partial F}{\partial e}(se, x, 0) + \frac{\partial^T F}{\partial e}(se, x, 0)P(x) \right] - \left[P(x) \frac{\partial F}{\partial e}(0, x, 0) + \frac{\partial^T F}{\partial e}(0, x, 0)P(x) \right] \right| \\
&\leq \left| P(x) \left[\frac{\partial F}{\partial e}(se, x, 0) - \frac{\partial F}{\partial e}(0, x, 0) \right] - \left[\frac{\partial^T F}{\partial e}(se, x, 0) - \frac{\partial^T F}{\partial e}(0, x, 0) \right] P(x) \right| \\
&\leq c|P(x)| \cdot |e| \leq c|e|,
\end{aligned}$$

for all $0 \leq s \leq 1$. □

6.3 Towards Anosov Flow on a Manifold

For general submanifold of \mathbb{R}^n , it is not possible to represent a submanifold as $\{e = 0\}$ for some e , especially for compact submanifolds. Therefore, the results of the previous section are not applicable. We develop an alternative method to handle the problem. Since most of the systems of interest in control theory evolve in Euclidean space or on a submanifold of it by appropriate embedding, in what follows we restrict our attention to this case.

6.3.1 Projection to a Normal Bundle

Given a k dimensional submanifold $U^k \subseteq \mathbb{R}^n$, which is invariant under the flow of (3.2), assume that there exist $n - k$ smooth vector fields $\{E^i\}_{i=k+1, \dots, n}$ along U such that $L_{f(x)}E^i(x) = 0$ for all $x \in U$ and all $i = k + 1, \dots, n$. Assign a Riemannian metric near U such that $\{E^i\}_{i=k+1, \dots, n}$ is an orthonormal basis of the normal bundle of U . Assume, for simplicity, that the metric coincides with the Euclidean metric. Denote

$$E_H = [E^1, \dots, E^k] \in \mathbb{R}^{n \times k}, \quad E_N = [E^{k+1}, \dots, E^n] \in \mathbb{R}^{n \times (n-k)}.$$

The above assumption makes the normal bundle of U invariant in the following sense: $L_{f(x)}E^i(x) = 0$ implies

$$\dot{E}^i(x) = \frac{\partial E^i(x)}{\partial x} f(x) = \frac{\partial f(x)}{\partial x} E^i(x).$$

Thus the $n - k$ vector fields $\{E^i\}_{i=k+1, \dots, n}$ along U are $n - k$ linearly independent solutions to the complete lift system $\dot{v} = \frac{\partial f(x)}{\partial x} v$. And we have $E^i(\phi_t(x_0)) = D\phi_t(E^i(x_0))$.

Define the operator $P_N : T\mathbb{R}^n|_U \rightarrow \mathbb{R}^{n-k}$ by

$$P_N(v) = \begin{bmatrix} \langle v, E^{k+1} \rangle \\ \vdots \\ \langle v, E^n \rangle \end{bmatrix}$$

whose restriction to the normal bundle is a linear invertible operator. (We underscore that P_N is defined on the restriction of the tangent bundle on U , which is not equal to the normal bundle of U) The operator can also be written as $P_N(v) = E_N^T(x)v$ where

$$E_N^T(x) = \begin{bmatrix} (E^{k+1}(x))^T \\ \vdots \\ (E^n(x))^T \end{bmatrix} \in \mathbb{R}^{(n-k) \times n}$$

for $x \in U$, which is a full row rank matrix function defined on U . Viewing $y = P_N(v)$ as the output of the complete lift system of the system (3.2) along U , we get the following system

$$\begin{cases} \dot{v} = \frac{\partial f(x)}{\partial x} v \\ y = E_N^T(x)v \end{cases} \quad (6.13)$$

It should be noted that for a linear system $\dot{x} = A(t)x$, the output $y = C(t)x$ is in general not invariant in the following sense: the set $\{x | C(t)x = 0\}$ is not invariant. However, for the above system (6.13), the output y is indeed invariant: if $y(t_0) = P_N(v(t_0)) = 0$ at some moment t_0 , then $v(t_0)$ has no orthogonal component, which means that it is tangent to the submanifold. Since the manifold is invariant, this vector under the flow should always be tangent to the submanifold for all $t \geq t_0$. This implies that $y(t) = 0$ for all $t \geq t_0$.

For invariant output, $C(t_0)x(t_0) = 0$ implies $\dot{C}(t_0)x(t_0) + C(t_0)A(t_0)x(t_0) = 0$, therefore there exists a unique matrix function $D(t)$ such that $\dot{C}(t) + C(t)A(t) = D(t)C(t)$. Consequently, $\dot{y} = (\dot{C}(t) + C(t)A(t))x = D(t)C(t)x = D(t)y$.

Based on these discussions, we conclude that there exists a matrix function $D : U \rightarrow \mathbb{R}^{(n-k) \times (n-k)}$ such that

$$\frac{d}{dt} P_N v = D(x) P_N v. \quad (6.14)$$

Notice that $D(x)$ is a function on U taking values in $\mathbb{R}^{(n-k) \times (n-k)}$. More specifically,

$$D(x) E_N^T(x) = E_N^T(x) \left[\left(\frac{\partial f(x)}{\partial x} \right)^T + \frac{\partial f(x)}{\partial x} \right]$$

By choosing an output, we have in effect partially shifted the complete dynamics along the manifold U to a system in Euclidean space.

When the normal bundle admits a splitting, U can be an Anosov flow.

Proposition 6.5. Suppose that U is a compact invariant submanifold of the system (3.2) in \mathbb{R}^n and that the normal bundle of U is spanned by $\{E^i\}_{i=k+1, \dots, n}$ with $L_f E^i = 0$ for all $i = k+1, \dots, n$ and there exists a C^1 matrix function $R : \mathbb{R}^n \rightarrow \mathbb{R}^{(n-k) \times (n-k)}$ such that

- $R(x)$ is invertible for each $x \in U$, and $R(\cdot)$, $R(\cdot)^{-1}$ and $\frac{\partial R}{\partial x}(\cdot)$ are continuous on U ;
- there exists C^1 matrix functions $S : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times p}$, $T : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times q}$ with $p + q = n - k$ such that

$$\dot{R}(x) + R(x)D(x) = \begin{bmatrix} S(x) & 0_{p \times q} \\ 0_{q \times p} & T(x) \end{bmatrix} R(x)$$

- the systems $\dot{z}_1 = S(x)z_1$, and $\dot{z}_2 = -T(x)z_2$ are exponentially stable.

then the system (3.2) along the manifold U is an Anosov flow, and

$$\begin{aligned} E^s(x) &= (P_N|_{NU})^{-1} \circ R(x)^{-1}(\mathbb{R}^p \oplus \{0_q\}) \\ E^u(x) &= (P_N|_{NU})^{-1} \circ R(x)^{-1}(\{0_p\} \oplus \mathbb{R}^q) \end{aligned}$$

where $P_N|_{NU}$ is the restriction to the normal bundle.

Proof. Let $z = R(x)y$, $x \in U$. Then

$$\begin{aligned} \dot{z} &= \dot{R}(x)y + R(x)D(x)y \\ &= \begin{bmatrix} S(x) & 0 \\ 0 & T(x) \end{bmatrix} z \end{aligned}$$

Therefore by definition, U is Anosov. The rest is obvious. \square

We consider some examples to illustrate the above proposition.

Example 6.3. We revisit the example studied by V. Andrieu et al. mentioned earlier. That is, LES of the invariant set $U = \{(e, x) | e = 0\} \simeq \mathbb{R}^{n_x}$ of the system (2.25). In this example, the state space $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$ is equipped with the usual Euclidean inner product as the Riemannian metric. On U , there is a trivial normal bundle $U \times \mathbb{R}^{n_e}$. Then $P_N : \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_e}$ is $P_N(v_e, v_x) = v_e$. Notice that the complete lift of the system (2.25) on U is

$$\begin{bmatrix} \dot{v}_e \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial e}(0, x) & 0 \\ 0 & \frac{\partial G}{\partial x}(0, x) \end{bmatrix} \begin{bmatrix} v_e \\ v_x \end{bmatrix},$$

therefore $\dot{v}_e = \frac{\partial F}{\partial e}(0, x)v_e$, and $D(x) = I_{n_e}$. If

$$|v_e(t)| \leq ke^{-\lambda(t-t_0)}|v_e(0)|, \quad \forall v_e(0) \in \mathbb{R}^{n_e}$$

Then $E^s(x) = \mathbb{R}^{n_e} \oplus \{0_{n_x}\}$.

Notice the submanifold U is just a linear subspace, thus the theorem applies trivially. We study some more nontrivial examples.

Example 6.4. Consider the speed regulation of current-fed induced motors model taken from [139],

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R\lambda_1 + \frac{\beta R\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} - \omega\lambda_2 + \frac{kR\lambda_2(\omega - \omega_*)}{\beta\sqrt{\lambda_1^2 + \lambda_2^2}} \\ -R\lambda_2 + \frac{\beta R\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} + \omega\lambda_1 - \frac{kR\lambda_1(\omega - \omega_*)}{\beta\sqrt{\lambda_1^2 + \lambda_2^2}} \\ -\frac{k}{\beta}\sqrt{\lambda_1^2 + \lambda_2^2}(\omega - \omega_*) \end{bmatrix}$$

where $(\lambda_1, \lambda_2)^T$ represents the rotor flux and ω the rotor angular velocity. $R > 0$ is the resistance, β and ω_* two positive constants. The complete lift of the system along $U = \{(\lambda_1, \lambda_2) | \lambda_1^2 + \lambda_2^2 = 1\} \times \{\omega_*\}$ reads

$$\begin{aligned}\delta \dot{\lambda}_1 &= \left(-R + \frac{R\lambda_2^2}{\beta^2}\right) \delta \lambda_1 + \left(-\omega - \frac{R\lambda_1\lambda_2}{\beta^2}\right) \delta \lambda_2 + \left(-\lambda_2 + \frac{kR\lambda_2}{\beta^2}\right) \delta \omega \\ \delta \dot{\lambda}_2 &= \left(\omega - \frac{R\lambda_1\lambda_2}{\beta^2}\right) \delta \lambda_1 + \left(-R + \frac{R\lambda_1^2}{\beta^2}\right) \delta \lambda_2 + \left(\lambda_1 - \frac{kR\lambda_1}{\beta^2}\right) \delta \omega.\end{aligned}$$

Choose a normal bundle in a neighborhood of U as

$$\text{span} \left\{ \frac{\lambda_1}{\beta} \frac{\partial}{\partial \lambda_1} + \frac{\lambda_2}{\beta} \frac{\partial}{\partial \lambda_2}, \frac{\partial}{\partial \omega} \right\}$$

Then

$$\begin{aligned}P_N^1(\delta \lambda_1, \delta \lambda_2, \delta \omega) &= \frac{\lambda_1}{\beta} \delta \lambda_1 + \frac{\lambda_2}{\beta} \delta \lambda_2 \\ P_N^2(\delta \lambda_1, \delta \lambda_2, \delta \omega) &= \delta \omega\end{aligned}$$

Calculations show that

$$\begin{aligned}\frac{d}{dt} P_N^1(\delta \lambda_1, \delta \lambda_2, \delta \omega) &= -\frac{\lambda_2 \omega_*}{\beta} \delta \lambda_1 + \frac{\lambda_1}{\beta} \delta \dot{\lambda}_1 + \frac{\lambda_1 \omega_*}{\beta} \delta \lambda_2 + \frac{\lambda_2}{\beta} \delta \dot{\lambda}_2 \\ &= -R P_N^1(\delta \lambda_1, \delta \lambda_2, \delta \omega) \\ \frac{d}{dt} P_N^2(\delta \lambda_1, \delta \lambda_2, \delta \omega) &= -k P_N^2(\delta \lambda_1, \delta \lambda_2, \delta \omega)\end{aligned}$$

or in matrix form

$$\frac{d}{dt} \begin{bmatrix} P_N^1 v \\ P_N^2 v \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 0 & -k \end{bmatrix} \begin{bmatrix} P_N^1 v \\ P_N^2 v \end{bmatrix}.$$

Thus $E^s(\lambda_1, \lambda_2, \omega) = \text{span} \left\{ \frac{\lambda_1}{\beta} \frac{\partial}{\partial \lambda_1} + \frac{\lambda_2}{\beta} \frac{\partial}{\partial \lambda_2}, \frac{\partial}{\partial \omega} \right\} \simeq \mathbb{R}^2$ for all $(\lambda_1, \lambda_2, \omega)$.

The next example is to show that an asymptotically stable submanifold need not be Anosov or an NHIM using the normal projection technique.

Example 6.5. Consider the following planar system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

This system has an invariant set $U = \{(x_1, x_2) | x_1^2 + x_2^2 = 1\}$, i.e., the unit circle. The normal bundle along U can be chosen as the linear space spanned by the tangent vector $v_N = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$. Now $P_N : T\mathbb{R}^2|_{S^1} \rightarrow N$ is

$$P_N \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) = x_1 v_1 + x_2 v_2$$

The complete lift can be written as

$$\begin{aligned}\dot{v}_1 &= -v_2 \\ \dot{v}_2 &= (1 - 2x_1x_2)v_1 - (x_1^2 + 3x_2^2 - 1)v_2\end{aligned}$$

Easy calculations show that

$$\frac{d}{dt}P_N(v) = -2x_2^2P_N(v).$$

On U , the dynamics now reads

$$\begin{aligned}\frac{d}{dt}P_N(v) &= -2x_2^2P_N(v) \\ \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1.\end{aligned}$$

Using LaSalle invariance principle, one can conclude that the $P_N(v)$ subsystem is uniformly asymptotically stable, but it is not exponentially stable. Therefore, the circle is not an Anosov flow.

More generally, if the compact invariant manifold U is a hypersurface of codimension 1 and is given by a level set $U = \{x \in M : h(x) = 0\}$ where $h : M \rightarrow \mathbb{R}$ is smooth. The invariance of U should be guaranteed by $\frac{\partial h}{\partial x}f(x) = 0, \forall x \in U$. Now the one dimensional normal bundle is spanned by the gradient of h . Thus if

$$\left\langle v(t), \frac{\text{grad } h(x(t))}{|\text{grad } h(x(t))|} \right\rangle \leq ke^{-\lambda(t-t_0)} \left\langle v(t_0), \frac{\text{grad } h(x(t_0))}{|\text{grad } h(x(t_0))|} \right\rangle \quad (6.15)$$

the manifold U is Anosov. In this example, $h(x) = \frac{1}{2}(x_1^2 + x_2^2 - 1)$, and $\left\langle v, \frac{\text{grad } h(x)}{|\text{grad } h(x)|} \right\rangle = x_1v_1 + x_2v_2$. But equation (6.15) is not satisfied.

6.3.2 Locally Exponentially Stable Manifold

In this subsection, we consider a special class of Anosov manifold, namely, locally exponentially stable compact submanifold.

We line up the states x, y and v that we mentioned in the previous section to form the following system

$$\begin{cases} \dot{x} = f(x) \\ \dot{v} = \frac{\partial f(x)}{\partial x}v \\ \dot{y} = D(x)y \end{cases} \quad (6.16)$$

where $D(x)$ is defined by (6.14). Let us denote

- $X(t, x_0)$ the solution to the first line with initial condition x_0 at $t = 0$.
- $Y(t, x_0, y_0)$ the solution to the y component with initial condition (x_0, y_0) by viewing the first and third line as an autonomous system.
- $v(t, x_0, v_0)$ the solution to the v component with initial condition (x_0, v_0) by viewing the first and second line as an autonomous system, i.e., the solution to the complete lift system.

Now by the definition of y , there holds

$$Y(x_0, E_N^T(x_0)v_0, t) = E_N^T(X(x_0, t))v(x_0, v_0, t)$$

for all $(x_0, v_0) \in TU$. Noticing also that $E_N^T(x_0)[E_N(x_0)y_0] = y_0$, there holds

$$Y(x_0, y_0, t) = E_N^T(X(x_0, t))v(x_0, E_N(x_0)y_0, t)$$

for all $(x_0, y_0) \in U \times \mathbb{R}^{n-k}$.

Denote

$$N_\epsilon := \{v \in NU : |v| < \epsilon\},$$

for small ϵ , which is diffeomorphic to a tubular neighborhood of U . Identify N_ϵ with the tubular neighborhood via $(\pi(v), v) \mapsto \pi(v) + v$, see [51, Chapter 5, Section 4]. By this identification, a point in the tubular neighborhood of U is written as $x + v$ where $x \in U$ with $|v| < \epsilon$ attached to x .

We give the following definition.

Definition 6.3.1. Consider the time-invariant system (3.2) and a compact invariant manifold U . If there exists a positive constant L , such that

$$f(x)^T \frac{\partial f(x)}{\partial x} f(x) \leq L f(x)^T f(x)$$

for all $x \in U$, then we call L the Lipschitz constant of the system on U .

Definition 6.3.2. Suppose that U is a compact invariant manifold of the system (3.2). We say that U has property

- (P1) Locally exponentially stable (LES), if there exists positive real numbers ε, k and λ such that we have, for all $(x_0, t) \in N_\varepsilon \times \mathbb{R}_{\geq 0}$,

$$d(X(x_0, t), U) \leq k e^{-\lambda t} d(x_0, U). \quad (6.17)$$

(P2) Transversally exponentially stable (TES), if there exist positive real numbers k, λ such that for all $(x_0, y_0, t) \in U \times \mathbb{R}^{n-k} \times \mathbb{R}_{\geq 0}$,

$$|Y(x_0, y_0, t)| \leq ke^{-\lambda t}|y_0|$$

(P3) if for all positive definite matrix Q , there exists a matrix value function $P : U \rightarrow \mathbb{R}^{(n-k) \times (n-k)}$ satisfying the following conditions:

1. $\exists p_1, p_2 > 0$, such that $p_1 I \leq P(x) \leq p_2 I, \forall x \in U$
2. $L_f P + PD + D^T P = -Q$ where $L_f P$ is understood as

$$L_f P(x) = \limsup_{h \rightarrow 0} \frac{P(X(x, h)) - P(x)}{h}$$

(P4) If there exists a C^1 function $V : N_\varepsilon \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} k_1 d(x, U)^2 &\leq V(x) \leq k_2 d(x, U)^2 \\ L_f V(x) &\leq -k_3 V \\ \left| \frac{\partial V}{\partial x} \right| &\leq k_3 d(x, U) \end{aligned}$$

Then we have the following link between (P1) and (P2).

Proposition 6.6. If the compact invariant submanifold U is locally exponentially stable (LES), then U is also transversally exponentially stable (TES). That is, (P1) \Rightarrow (P2).

Proof. Assume that $|y_0|$ is sufficiently small and let $\gamma : [0, 1] \rightarrow N_\varepsilon$ be a segment satisfying $\gamma(0) = x_0$ and $\gamma(s) = x_0 + sE_N y_0$ for $s \in [0, 1]$. For all $s \in (0, 1)$,

$$\begin{aligned} s|E_N Y(x_0, y_0, t)| &= d(X(x_0, t) + sE_N Y(x_0, y_0, t), X(x_0, t)) \\ &\leq d(X(x_0, t) + sE_N Y(x_0, y_0, t), X(x_0 + sy_0, t)) \\ &\quad + d(X(x_0 + sy_0, t), X(x_0, t)) \end{aligned}$$

Multiply by $\frac{1}{s}$ on both sides, it follows that

$$\begin{aligned} |E_N Y(x_0, y_0, t)| &\leq \frac{d(X(x_0, t) + sE_N Y(x_0, y_0, t), X(x_0 + sy_0, t))}{s} \\ &\quad + \frac{d(X(x_0 + sy_0, t), X(x_0, t))}{s} \\ &\leq \frac{\kappa(s)}{s} + \frac{d(X(x_0 + sy_0, t), U)}{s} \\ &\leq \frac{\kappa(s)}{s} + \frac{ke^{-\lambda t} d(x_0 + sy_0, U)}{s} \end{aligned}$$

$$\begin{aligned}
&= \frac{\kappa(s)}{s} + \frac{ke^{-\lambda t}|sy_0|}{s} \\
&= \frac{\kappa(s)}{s} + ke^{-\lambda t}|y_0|,
\end{aligned}$$

where

$$\kappa(s) = d(X(x_0, t) + sE_N Y(x_0, y_0, t), X(x_0 + sE_N y_0, t)).$$

Clearly $\kappa(0) = 0$. We show $\kappa'(0) = 0$. Consider the curve $\gamma_t(s) = X(x_0 + sE_N y_0, t)$, then $\gamma_t'(0)$ is a tangent vector at the base point $X(x_0, t)$ and it is the solution of the complete lift of the system $\dot{x} = f(x)$ along the curve $X(x_0, t)$ with initial condition $(x_0, sE_N y_0)$. Since the initial condition $\gamma_t'(0) = E_N y_0$ is normal to U , then $\gamma_t'(0) \perp U$ for all $t \geq 0$. By uniqueness of solution, $E_N Y(x_0, y_0, t) = \gamma_t'(0)$. Hence we have

$$\left. \frac{\partial [X(x_0, t) + sE_N Y(x_0, y_0, t)]}{\partial s} \right|_{s=0} = E_N Y(x_0, y_0, t) = \left. \frac{\partial X(x_0 + sE_N y_0, t)}{\partial s} \right|_{s=0}$$

Therefore $\kappa(s) = O(s^2)$. Let $s \rightarrow 0+$, we get $|E_N Y(x_0, y_0, t)| \leq ke^{-\lambda t}|y_0|$, or $|Y(x_0, y_0, t)| \leq ke^{-\lambda t}|y_0|$. Since the complete lift system is linear fibre-wise, it holds for all $y_0 \in \mathbb{R}^{n-k}$. \square

Unfortunately we do not know if the converse of Proposition 6.6 is true. However, for NHIM case, it is indeed the case.

Proposition 6.7. If the compact invariant submanifold U is transversally exponentially stable with rate $\lambda > L$, where L is the Lipschitz constant on U , then U is locally exponentially stable.

Proof. The condition $\lambda > L$ implies that U is an NHIM with $\dim E^u = 0$. Therefore, there exists a smooth asymptotic phase $m : N_\varepsilon \rightarrow U$ [134, Theorem 5.6.1]. See Figure 6.1. Let

$$V(x) = [E_N^T(m(x))(x - m(x))]^T P(m(x)) [E_N^T(m(x))(x - m(x))] \quad (6.18)$$

Since x lies in the foliation passing through $m(x)$ whose tangent space is spanned by $E_N^T(m(x))$, we can choose x sufficiently close to U such that

$$|E_N^T(m(x))(x - m(x))| = |x - m(x)| \leq 2d(x, U)$$

On the other hand $|E_N^T(m(x))(x - m(x))| = |x - m(x)| \geq d(x, U)$. Hence $\exists c_1, c_2 > 0$, such that for all $x \in N_\varepsilon$,

$$c_1 d(x, U)^2 \leq V(x) \leq c_2 d(x, U)^2 \quad (6.19)$$

Clearly, E_N and m are both smooth. The time derivative of V reads

$$\dot{V}(x) = 2[E_N^T(m(x))(x - m(x))]^T P(m(x))$$

$$\begin{aligned}
 & \cdot \left[E_N^T(m(x)) \frac{\partial^T f}{\partial x}(m(x))(x - m(x)) + E_N^T(m(x))(f(x) - f(m(x))) \right] \\
 & + [E_N^T(m(x))(x - m(x))]^T L_f P(m(x)) [E_N^T(m(x))(x - m(x))] \\
 & = 2[E_N^T(m(x))(x - m(x))]^T P(m(x)) \left[E_N^T(m(x)) \left(\frac{\partial^T f}{\partial x} + \frac{\partial f}{\partial x} \right) (m(x))(x - m(x)) \right] \\
 & + [E_N^T(m(x))(x - m(x))]^T L_f P(m(x)) [E_N^T(m(x))(x - m(x))] \\
 & + O(|x - m(x)|^3) \\
 & = [E_N^T(m(x))(x - m(x))]^T [L_f P(m(x)) + P(m(x))D(m(x)) + D(m(x))^T P(m(x))] \\
 & \quad \cdot [E_N^T(m(x))(x - m(x))] + O(|x - m(x)|^3) \\
 & \leq -\varepsilon [E_N^T(m(x))(x - m(x))]^T Q(m(x)) [E_N^T(m(x))(x - m(x))] \\
 & \leq -cV(x)
 \end{aligned}$$

Invoking (6.19), we get the inequality (6.17). \square

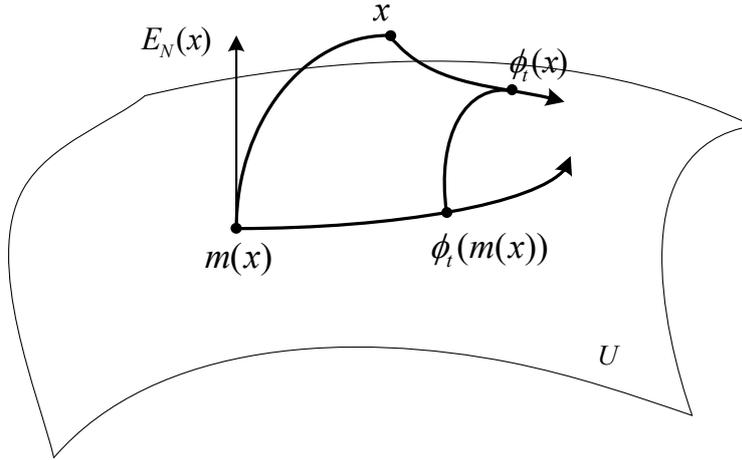


Figure 6.1: Asymptotic phase of NHIM

Furthermore, we can prove the following proposition.

Proposition 6.8. In (P2), if $\lambda > L$ where L is the Lipschitz constant on U , then (P1), (P2) and (P4) are equivalent and (P2) implies (P3).

Proof. The proof consists in showing that (P2) \Rightarrow (P3) and that (P2) \Rightarrow (P4) \Rightarrow (P1) \Rightarrow (P2).

(P2) \Rightarrow (P3): It suffices to prove for $Q = I$. We construct a matrix value function $P(x)$ for $x \in U$ as follows:

$$V(x, y) = y^T P(x) y = \int_0^\infty Y(t, x, y)^T Y(t, x, y) dt$$

Property (P2) makes the integration converge and hence $V(x, y)$ is well defined. Meanwhile, the existence of $P(x)$ is guaranteed by the linearity of $Y(x, y, t)$ with respect to y . We calculate the Lie derivative of $V(x, y)$

$$\begin{aligned}\mathcal{L}_{\bar{f}}V(x, y) &= \left. \frac{d}{ds} V(X(s, x), Y(s, x, y)) \right|_{s=0} \\ &= \left. \frac{d}{ds} \int_0^\infty Y(t, X(s, x), Y(s, x, y))^T Y(t, X(s, x), Y(s, x, y)) dt \right|_{s=0} \\ &= \left. \frac{d}{ds} \int_0^\infty Y(s+t, x, y)^T Y(s+t, x, y) dt \right|_{s=0} \\ &= \left. \frac{d}{ds} \int_s^\infty Y(t, x, y)^T Y(t, x, y) dt \right|_{s=0} \\ &= -y^T y\end{aligned}$$

or equivalently, $L_f P(x) + P(x)D(x) + D(x)^T P(x) = -I$.² The bounds of $P(x)$ can be estimated easily by noticing that

$$V(x, y) \leq \int_0^\infty k^2 e^{-2\lambda t} |y|^2 dt = \frac{k^2}{2\lambda} |y|^2$$

On the other hand $\left| \frac{\partial f}{\partial x} \right| < C$ on U for a large $C \geq L$, therefore $|Y(x, y, t)| = |E_N Y(x, y, t)| \geq e^{-Ct} |y|$ for all $t \geq 0$. Hence

$$V(x, y) \geq \int_0^\infty e^{-2Lt} |y|^2 dt = \frac{1}{2L} |y|^2$$

Take

$$p_1 = \frac{1}{2L}, \quad p_2 = \frac{k^2}{2\lambda},$$

then (P2) implies (P3).

(P2) \Rightarrow (P4): Take $V(x)$ as (6.18). Let $Z(x) = E_N^T(m(x))(x - m(x))$. Then $V(x) = \sum_{i,j} Z_i(x) Z_j(x) P_{ij}(m(x))$. Since $V(x)$ is C^1 ,

$$\begin{aligned}\frac{\partial V}{\partial x_k} &= \sum_{i,j} Z_i(x) Z_j(x) \frac{\partial P_{ij}(m(x))}{\partial x_k} + 2Z^T(x) P(x) \frac{\partial Z(x)}{\partial x_k} \\ &\leq c(|Z(x)|^2 + |Z(x)|)\end{aligned}$$

by noticing that $P(x)$ and $\frac{\partial Z(x)}{\partial x_k}$ are bounded on compact sets. Therefore $\left| \frac{\partial V}{\partial x}(x) \right| \leq 2c|Z(x)| \leq c'd(x, U)$, when $|Z(x)| < 1$.

(P4) \Rightarrow (P1): Trivial. □

Using the results in the previous subsection, the robustness of NHIM can be obtained:

²Like in Section 6.2, we can show that P is differential almost everywhere.

Theorem 6.2. Assume that the submanifold U is an NHIM of the system $\dot{x} = f(x, 0)$. Then is locally input to state stable in the following sense: $\exists \varepsilon > 0$, such that

$$d(\phi_t(x), U) \leq \beta(d(x, U), t) + \gamma(\|d\|_\infty), \quad \forall x \in N_\varepsilon, \quad t \geq 0$$

where β is a class \mathcal{KL} function and γ class \mathcal{K} .

Proof. By Theorem 6.2, there exists a function V as in (P3) for the input-free system $\dot{x} = f(x, 0)$. Hence, for all $x \in N_\varepsilon$,

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V(x)}{\partial x} f(x, 0) + \frac{\partial V(x)}{\partial x} (f(x, d) - f(x, 0)) \\ &\leq -cd(x, U)^2 + c \left| \frac{\partial V(x)}{\partial x} \right| \cdot |d(t)| \\ &\leq -cd(x, U)^2 + cd(x, U) \cdot |d(t)| \\ &\leq -k_1 V + k_2 \|d\|_\infty^2 \end{aligned}$$

where k_1 and k_2 are some positive constants. This completes the proof. \square

Remark 6.3. Results concerning input to state stability of compact submanifolds has been well established, see for example [118]. The novelty of the above theorem is to characterize the input to state stability of compact submanifolds via the “linearized system” along the manifold.

6.4 A Brief Summary

In this chapter, we have mainly studied two problems. First, the robustness of contraction. Second, the robustness of stable compact invariant submanifold.

For the first problem, we studied the robustness of transverse exponential stability in Euclidean space. On the one hand, robustness of contraction can be restated as robustness of transverse exponential stability; on the other, for transverse exponential stability, Lyapunov functions have been constructed in the literature which then can be utilized to analyze the robustness. It turns out that that hyperbolicity condition plays the crucial role for the robustness of contraction. The first problem serves also as a preparation for the second.

For the second problem, we studied the robustness of stable normally hyperbolic invariant manifold. This is an extension of the first problem, but it is generally more difficult to deal with. Motivated by transverse linearization, we constructed Lyapunov functions based on the complete lift of the system along the invariant manifold. Finally, we found that hyperbolicity again plays an important role for robustness.

Conclusion and Future Research Perspective

This thesis has studied contraction in several different aspects, with an emphasis on geometric and Lyapunov characterizations. The main contributions are now summarized as follows.

- A general theory for stability analysis on Riemannian manifolds is formulated. In particular, converse Lyapunov theorem is proved in a coordinate free manner. Then local exponential stability of nontrivial solutions is studied, which proves to have strong connections with contraction.
- Geometric criteria have been obtained to verify contraction. These conditions are derived via the tool “complete lift” of vector fields. Based on the criteria, converse result is obtained, which justifies the important role of Finsler-Lyapunov function.
- The geometric analysis approach has revealed some interesting properties of contraction. The tubular neighborhood property shows that contraction can be characterized locally in the tangent bundle. The Krasovskii’s method proves to be a bridge between stability and contraction, this is well understood by making use of the Finsler-Lyapunov function.
- The developed theories are accompanied with some applications. Three main examples are considered, namely, filter on the special orthogonal group, speed observer of Lagrangian systems and contraction based synchronization. These applications have shown the efficiency of geometric contraction analysis.
- Some robustness properties relating to contraction has been studied. In Euclidean space, this is performed by utilizing the transverse linearization theory. And it has been discovered that robustness is guaranteed by hyperbolicity. As an extension,

robustness of compact submanifolds have also been studied, in which the normal hyperbolicity is crucial to achieve robustness.

Future research directions may be based on the following observations.

- LES of unbounded trajectories. In Chapter 3, we mainly studied LES of a bounded trajectory. More precisely, it was shown that the complete lift system characterizes the LES of a bounded trajectory. But our proof does not carry over directly to unbounded trajectories, which constitutes an interesting direction to be investigated.
- Converse theorem of IAS. We proved converse theorem for IES in Chapter 4. Unfortunately, we do not know how to prove the converse theorem for IAS. For non-incremental stability notions, once the converse theorem for ES is proven, the converse theorem for AS is rather straightforward invoking Massera's lemma. As L. Grüne *et al.* have put it, "Asymptotic stability equals exponential stability, if you twist your eyes" [49]. The incremental counterpart of this is still unclear to us.
- Equilibrium seeking for non-negative curvature manifold. We have studied in Section 5.1 the equilibrium seeking problem for non-positive constant curvature manifold. Notably, we obtained the optimal convergence rate of equilibrium seeking for contractive systems on Riemannian manifolds. Unfortunately, we have not been able to extend the result to more general classes of manifolds. In particular, extending this result to non-constant non-negative curvature manifolds (with a priori upper bound on the curvature) would be worth investigating.
- Differential positivity and contraction. In Chapter 5, we studied synchronization on Riemannian manifolds. This is an active research area and has some close relationships with contraction, [87, 40, 42]. A natural question in that direction is how to relate the results obtained in Section 5.4 to differential positivity.
- Almost global ISS. It was our initial intention to apply the theory developed in Chapter 6 to study almost global input to state stability on manifolds. The problem can be formulated as follows: when a system has multiple isolated ω -limit sets, among which some are invariant submanifolds (see for example [79]) and only one is LES equilibrium point, we would like to know the almost global ISS of this equilibrium point. To answer this, one is forced to study the ISS of each ω -limit set. ISS of hyperbolic equilibrium point has been established via linearization, see for example [10], see also P. Forni's PhD thesis regarding ISS issues of multistable

systems [43]. But to study ISS of nontrivial ω -limit sets rather than equilibrium point via linearization is much more difficult. Unfortunately, we did not have enough time to finish this task.

- Contraction based design. The thesis is mainly about analysis. Developing design techniques based on the geometric methods is our future research task. In [82], I. Manchester and J.J. Slotine proposed the notion *control contraction metric* to design trajectory tracking controllers based on contraction theory. They showed that via the control contraction metric method, trajectory tracking can be solved by a feasible convex optimization procedure. But still, the computational burden is very high. Future research direction may focus on developing efficient contraction based algorithms.

Résumé substantiel en français

Un bref résumé du chapitre 3

Dans ce chapitre, nous avons étudié la stabilité sur des variétés Riemanniennes. Tout d'abord, nous avons prouvé le théorème de Lyapunov inverse sur les variétés Riemanniennes sans recourir à des coordonnées locales. Dans la preuve, une étape clé consiste à utiliser une définition appropriée de la continuité de Lipschitz sur des variétés Riemanniennes. Nous avons également étudié la stabilité exponentielle locale des solutions non triviales sur les variétés Riemanniennes en soulevant le système le long de la trajectoire cible dans le fibré tangent de retrait. Le système soulevé étant linéaire dans le sens des fibres, l'analyse s'en retrouve significativement simplifiée. Une autre tâche importante de ce chapitre est de jeter les bases du chapitre suivant, en fournissant notamment certaines estimations utiles pour la suite.

Un bref résumé du chapitre 4

Dans ce chapitre, nous avons proposé de nouveaux résultats pour l'analyse de la contraction sur des variétés Riemanniennes. Tout d'abord, sur la base de l'outil "complete lift", nous avons donné une condition intrinsèque pour la stabilité incrémentale. Cette condition est simple mais suffisamment générale pour couvrir de nombreux résultats importants de la littérature. Deuxièmement, nous avons établi le théorème inverse de la contraction sur les variétés Riemanniennes. Ce théorème inverse, pour la première fois, justifie complètement la légitimité de l'introduction de la fonction de Finsler-Lyapunov pour l'analyse de la contraction. Troisièmement, nous avons obtenu plusieurs autres caractérisations des systèmes contractifs : (1) La propriété de voisinage tubulaire montre que la stabilité incrémentale exponentielle (IES) peut être entièrement caractérisée sur un voisinage tubulaire de la variété de base ; (2) La méthode de Krasovskii révèle les liens intimes entre la stabilité incrémentale et la stabilité de Lyapunov sur des variétés.

En même temps, l'approche suivie donne une interprétation géométrique du théorème classique de Krasovskii. (3) Le comportement des systèmes IES sous perturbations a été étudié, ce qui montre que le comportement des systèmes IES partage de grandes similarités avec celui des systèmes linéaires. Pour résumer, un cadre géométrique pour l'analyse de la contraction sur les variétés Riemanniennes a été établi.

Un bref résumé du chapitre 5

Dans ce chapitre, nous avons étudié quelques applications de la théorie développée dans les chapitres précédents. Tout d'abord, la recherche d'équilibre des systèmes non-linéaire sur une variété Riemannienne a été étudiée. En particulier, nous avons prouvé une conjecture sous l'hypothèse d'une variété à courbure constante non négative. Notamment, nous avons obtenu la taille de pas « optimale » pour la méthode d'Euler sur les variétés de Riemann. Deuxièmement, nous avons étudié le filtre gradient sur $SO(3)$ en utilisant la méthode de contraction. Troisièmement, l'observateur de vitesse du système Lagrangien sur les variétés a été étudié. On a commencé par établir la théorie de la contraction pour les systèmes Lagrangiens, sur la base de laquelle la convergence de l'observateur de vitesse a été analysée de manière sans utilisation de coordonnées locales. L'analyse est grandement simplifiée par rapport aux résultats existants dans la littérature. Enfin, nous avons étudié le problème de synchronisation du point de vue de la contraction. Ces exemples montrent l'intérêt de l'analyse de contraction géométrique développée dans les chapitres précédents.

Un bref résumé du chapitre 6

Dans ce chapitre, nous avons principalement étudié deux problèmes. Tout d'abord, la robustesse de la contraction. Deuxièmement, la robustesse d'une sous-variété invariante compacte stable.

Pour le premier problème, nous avons étudié la robustesse de la stabilité exponentielle transverse dans un espace Euclidien. D'une part, la robustesse de la contraction peut être reformulée comme robustesse de la stabilité exponentielle transverse ; d'autre part, pour la stabilité exponentielle transversale, des fonctions de Lyapunov ont été construites dans la littérature qui peuvent ensuite être utilisées pour analyser la robustesse. Il s'avère que la condition d'hyperbolicité joue le rôle crucial pour la robustesse de la contraction. Le premier problème sert aussi de préparation au second. Pour le deuxième problème, on a étudié la robustesse d'une variété invariante normalement hyperbolique stable. C'est une

extension du premier problème, mais il est généralement plus difficile à traiter. Motivés par la linéarisation transverse, nous avons construit des fonctions de Lyapunov basées sur le « complete lift » du système le long de la variété invariante. Enfin, nous avons constaté que l'hyperbolicité joue à nouveau un rôle important pour la robustesse.

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Titre : Analyse de contraction des systèmes non-linéaires sur des variétés Riemanniennes

Mots clés : contraction, systèmes non-linéaires, variétés Riemanniennes

Résumé : La stabilité des points d'équilibre des systèmes non linéaires est l'un des problèmes centraux de la théorie du contrôle non linéaire et de ses applications. L'analyse de stabilité se résume souvent à la recherche d'une fonction de Lyapunov candidate qui dissipe de manière adéquate le long des solutions du système. Les deux dernières décennies ont vu un besoin croissant d'aller au-delà de la stabilité d'un équilibre, en imposant que deux solutions d'un système finissent par converger l'une vers l'autre. Une telle version incrémentale de la stabilité de Lyapunov (contraction) s'avère en effet utile dans la conception d'observateurs, la

synchronisation et le suivi de trajectoire. Cependant, les méthodes d'analyse pour la contraction sont encore loin d'être standardisées, en particulier pour les systèmes évoluant sur des variétés telles que les dynamiques de rotation en groupe spécial orthonormé, les systèmes Lagrangiens modélisés en espace de configuration non-Euclidien et les systèmes quantiques en espace matriciel de densité. L'objectif principal de cette thèse est d'approfondir la compréhension de la contraction sur les variétés et de proposer des méthodes applicables pour assurer la contraction.

Title : Contraction Analysis of Nonlinear Systems on Riemannian Manifolds

Keywords : contraction, nonlinear systems, Riemannian manifolds

Abstract : Stability of equilibrium points of nonlinear systems is one of the central issues of nonlinear control theory and applications. Stability analysis often boils down to searching for a Lyapunov candidate that adequately dissipates along the system's solutions. The last two decades have witnessed a growing need to go beyond stability of an equilibrium, by imposing that any two solutions of a system eventually converge to one another. Such an incremental version of Lyapunov stability (contraction) indeed proves useful in observer de-

sign, synchronization and trajectory tracking. However, analysis methods to contraction are still far from being standardized, particularly for systems evolving on manifolds such as rotation dynamics in special orthonormal group, Lagrangian systems modeled in non-Euclidean configuration space and quantum systems in density matrix space. The main objective of this thesis is to provide further understanding of contractive systems on manifolds and to propose applicable methods to ensure contraction.