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Valentin Rapenne

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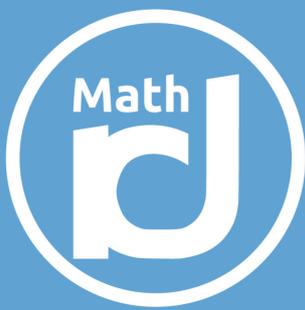
## Étude de quelques problèmes liés aux marches aléatoires branchantes et aux marches aléatoires renforcées

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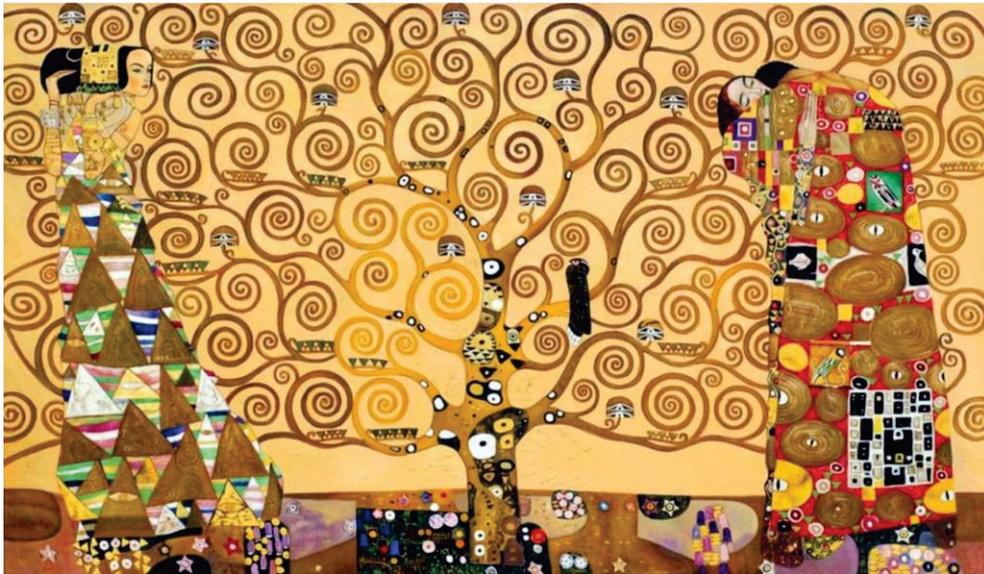




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Étude de quelques problèmes liés  
aux marches aléatoires branchantes et aux  
marches aléatoires renforcées



Valentin Rapenne

Thèse de doctorat

Université Claude Bernard



Lyon 1



*Là où il y a une volonté, il y a un chemin.*

*Vladimir Ilitch Oulianov*

## Remerciements

Cette thèse n'aurait bien sûr jamais pu être accomplie sans les précieux conseils de ma directrice Xinxin Chen et de mon directeur Christophe Sabot. Je tiens donc à les remercier tout particulièrement pour tout ce qu'il m'ont appris durant ces trois années ainsi que pour leur soutien sans faille.

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## Résumé

Cette thèse a pour objectif d'étudier les marches aléatoires branchantes spatiales et les marches aléatoires renforcées. Nous nous intéresserons tout particulièrement au processus de saut renforcé par sommets (communément appelé VRJP en raison de son acronyme anglais). Le VRJP est un processus renforcé qui peut être défini sur n'importe quel graphe localement fini et dont le renforcement dépend d'un paramètre  $W > 0$ . Ce paramètre peut également varier sur chaque arête du graphe mais nous le supposons constant dans ce résumé. Grâce à de précédents travaux (voir [152], [153], [154] et [139]), on sait que le VRJP est presque sûrement récurrent sur  $\mathbb{Z}$  et  $\mathbb{Z}^2$  pour toute valeur de  $W$ . A l'inverse, sur  $\mathbb{Z}^d$  avec  $d \geq 3$ , on sait qu'il existe une unique valeur  $W_c(d)$  telle que le VRJP sur  $\mathbb{Z}^d$  avec paramètre  $W > 0$  est récurrent si  $W < W_c(d)$  et transient si  $W > W_c(d)$ . Quant aux marches branchantes, il s'agit de marches aléatoires qui ont la propriété de se diviser au cours du temps. Leurs propriétés en dimension 1 sont à présent connues de manière très fine. (Voir par exemple [25], [24], [2], [37], [86] et [117].) Par ailleurs, on peut considérer aussi des marches branchantes en dimension supérieure comme dans [106], [72] ou encore [169] et [171].

En premier lieu, nous nous intéresserons à un modèle de branchement critique en temps discret partant d'un processus de Poisson  $\Theta_0$  sur  $\mathbb{R}^d$  avec  $d \in \mathbb{N}^*$ . Cela nous amène à considérer la suite de processus ponctuels  $(\Theta_n)_{n \in \mathbb{N}}$  résultant de cette construction. Si la loi du mouvement de la marche branchante est dans le domaine d'attraction de la Gaussienne, alors on sait (voir [50], [60] et [49]) que  $(\Theta_n)_{n \in \mathbb{N}}$  converge en loi vers le processus ponctuel nul si  $d \in \{1, 2\}$  et, qu'à l'inverse,  $(\Theta_n)_{n \in \mathbb{N}}$  converge en loi vers un processus ponctuel non trivial si  $d \geq 3$ . Cependant, les preuves de ces résultats avaient deux problèmes. Soit elles supposaient des hypothèses très restrictives, soit elles se plaçaient dans le cadre d'un modèle continu en temps ce qui permettait, chose impossible pour nous, d'utiliser des méthodes venant de la théorie des équations aux dérivées partielles. Dans cette thèse, nous prouvons à nouveau le résultat de convergence lorsque  $d \geq 3$  à l'aide d'une preuve unifiant les différents cas abordés par la littérature, le tout dans un formalisme plus moderne. En outre, le processus limite obtenu est invariant en loi par rapport à la marche branchante. Grâce à [32], une caractérisation de ces processus invariants était connue dans le cadre d'un modèle continu en temps. Là encore, la preuve de [32] utilisait des EDPs. Dans cette thèse, nous fournissons une nouvelle preuve de la caractérisation des processus ponctuels invariants en loi pour les processus de branchement critiques sous des hypothèses plus générales et en ayant seulement recours à des outils probabilistes.

Par ailleurs, dans cette thèse, nous utilisons également les marches branchantes comme outil pour étudier le VRJP sur les arbres. Dans le cadre de l'étude du VRJP, Sabot et Zeng ont construit une martingale positive  $(\psi_n)_{n \in \mathbb{N}}$  dans [154]. Cette martingale joue un rôle crucial puisqu'elle converge vers 0 si et seulement si le VRJP est récurrent. Par ailleurs, si cette martingale est bornée dans  $L^p$  pour  $p$  assez grand, on peut l'utiliser pour montrer que le VRJP a un comportement asymptotiquement diffusif sur  $\mathbb{Z}^d$  avec  $d \geq 3$ . Dans cette thèse, nous montrerons que  $(\psi_n)_{n \in \mathbb{N}}$  est uniformément intégrable dès que  $W > W_c(d)$  mais nous ne parvenons pas à étudier les moments dans  $L^p$  dans le cas où le graphe sous-jacent est  $\mathbb{Z}^d$ . Par contre nous prouvons que  $(\psi_n)_{n \in \mathbb{N}}$  est bornée dans  $L^p$  pour tout  $p > 1$  dès que le VRJP est transient sur les arbres. En outre, nous nous intéressons à la vitesse de décroissance vers 0 de  $(\psi_n)_{n \in \mathbb{N}}$  lorsque le VRJP est récurrent sur les arbres.

Par la suite, nous étudions les propriétés spectrales d'un opérateur de Schrödinger aléatoire  $H_\beta$  sur  $\mathbb{Z}^d$  qui a été introduit dans [153] et [154] dans le cadre de l'étude du VRJP. Pour tout paramètre  $W > 0$  associé au VRJP,  $H_\beta$  sur  $\mathbb{Z}^d$  a une densité d'états intégrée  $n_{W,d}$ . Nous montrons que lorsque  $W$  est petit, on a  $n_{W,d}(E) \simeq \sqrt{E}$  lorsque  $E$  tend vers 0. Cela est très différent du comportement bien connu du modèle d'Anderson dont la densité d'états décroît exponentiellement vite au bord du spectre d'après la propriété des "queues de Lifshitz". Au contraire, lorsque  $W$

est grand et  $d \geq 3$ , on va montrer que  $n_{W,d}(E) = O(E)$  lorsque  $E$  tend vers 0. Il y a donc une transition de phase concernant le comportement de la densité d'états.

En outre, dans cette thèse, nous considérerons des limites d'échelle du VRJP sur des graphes unidimensionnels en prolongeant les travaux réalisés dans [113]. Nous utiliserons ces limites d'échelle pour donner une nouvelle preuve des propriétés de Matsumoto et Yor concernant des fonctionnelles exponentielles du mouvement Brownien. De plus, nous construirons une version continue de l'opérateur  $H_\beta$  et de son inverse  $G_\beta$  sur des cercles et nous calculerons la densité d'états de cet opérateur continu.

Enfin, nous poursuivrons les travaux initiés par Sabot et Zeng dans [155] et approfondis ensuite par Gérard dans [76] en prouvant une généralisation multidimensionnelle des propriétés de Matsumoto-Yor.

# Table des matières

<b>1</b>	<b>Introduction</b>	<b>11</b>
1.1	How to read this thesis . . . . .	11
1.2	Reinforced processes . . . . .	11
1.2.1	Exchangeability and De Finetti's theorem . . . . .	11
1.2.2	The edge-reinforced random walk . . . . .	13
1.2.3	The Vertex Reinforced Jump Process . . . . .	15
1.2.4	The supersymmetric hyperbolic sigma model, localisation and delocalisation	18
1.2.5	The $\beta$ -potential and the environment of the VRJP . . . . .	19
1.2.6	The infinite version of $\beta$ , the martingale $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$ and its applications . .	21
1.3	Random Schrödinger operators . . . . .	24
1.3.1	Background on self-adjoint operators . . . . .	24
1.3.2	Physical interpretation . . . . .	26
1.3.3	An important example of random Schrödinger operator : the Anderson model	27
1.3.4	The density of states . . . . .	29
1.3.5	Link with the VRJP . . . . .	30
1.4	Branching random walks . . . . .	31
1.4.1	One-dimensional branching random walks . . . . .	31
1.4.2	Critical spatial branching random walks . . . . .	35
1.4.3	Spatial branching random walks starting from a point process . . . . .	36
1.5	Our results regarding branching random walks . . . . .	39
1.6	Our main results regarding the VRJP . . . . .	41
1.6.1	Main results of Chapter 3 . . . . .	41
1.6.2	Main results of Chapter 4 . . . . .	44
1.6.3	Main results of Chapter 5 . . . . .	48
1.6.3.1	A new approach to Matsumoto-Yor properties . . . . .	48
1.6.3.2	The space-continuous operator on circles . . . . .	50
1.6.3.3	New Dufresne type identities . . . . .	51
1.6.3.4	The density of states on the real line . . . . .	52
1.6.4	Main results of Chapter 6 . . . . .	53
<b>2</b>	<b>Invariant measures of critical branching random walks in high dimension</b>	<b>55</b>
2.1	Introduction . . . . .	55
2.1.1	Definition of the model and first notation . . . . .	55
2.1.2	Previous results . . . . .	57
2.1.3	Main results . . . . .	57
2.1.4	Organisation of the paper . . . . .	59
2.2	Preliminaries . . . . .	59
2.2.1	Local-limit Theorem . . . . .	59
2.2.2	Heat kernel estimate . . . . .	60
2.2.3	A warm-up about point processes . . . . .	60
2.2.4	Spine method and change of measure . . . . .	61

2.2.5	A critical branching random walk conditioned on survival in a given set	63
2.3	Proof of Theorem 2.2	69
2.4	Compatibility of $\Lambda_{\infty}^{d,X}$	72
2.5	Characterization of invariant measures	74
2.5.1	Preliminary Lemmas	75
2.5.2	Proof of Theorem 2.3	79
2.6	Heavy tail generalization	82
2.6.1	New setting	82
2.6.2	Generalized proofs	83
2.7	Remarks and further discussion	87
<b>3</b>	<b>The asymptotic behaviour of the martingale associated with the VRJP</b>	<b>91</b>
3.1	Introduction and first definitions	91
3.2	Context and statement of the results	93
3.2.1	General notation	93
3.2.2	The $\beta$ -potential and the martingale $(\psi_n)_{n \in \mathbb{N}}$	93
3.2.3	Notation associated with the VRJP	94
3.2.3.1	General notation for the VRJP	94
3.2.3.2	Notation for the VRJP on trees	95
3.2.4	The phase transition	95
3.2.5	Statement of the results	95
3.2.5.1	Results on $\mathbb{Z}^d$	95
3.2.5.2	Results on Galton-Watson trees	95
3.3	Background	97
3.3.1	Marginals and conditional laws of the $\beta$ -potential	97
3.3.2	Warm-up about the VRJP	98
3.3.3	Specificities of the tree	99
3.3.4	$\beta$ -potential and path expansions	100
3.3.5	Warm-up about branching random walks	102
3.4	Preliminary lemmas	103
3.4.1	$\psi_n(o)$ as a mixture of Inverse Gaussian distributions and proof of Theorem 3.1	103
3.4.2	Resistance formula on a tree	106
3.4.3	Burkholder-Davis-Gundy inequality	108
3.4.4	Link between $\hat{G}_n$ and $\tilde{G}_n$	109
3.5	The transient phase	110
3.6	The subcritical phase	113
3.6.1	Proof of Theorem 3.3	113
3.6.2	Proof of Theorem 3.4	120
3.6.3	Proof of Proposition 3.5	122
3.6.4	Proof of Proposition 3.6	124
3.7	The critical point	125
3.7.1	Proof of Theorem 3.7	125
3.7.2	Positive recurrence at the critical point	126
<b>4</b>	<b>The density of states of <math>H_{\beta}</math></b>	<b>129</b>
4.1	Introduction	129
4.2	Context and statement of the results	130
4.2.1	Notation	130
4.2.2	The $\beta$ -field and the operator $H_{\beta}$	131
4.2.3	The Green function and the density of states	133
4.2.4	Results about the density of states	133

4.2.5	Localization for high energies . . . . .	134
4.3	Background . . . . .	134
4.3.1	Restriction properties . . . . .	134
4.3.2	The field $\tilde{\beta}$ . . . . .	135
4.4	Preliminaries . . . . .	136
4.4.1	A lemma about the Dirichlet operator associated with $H_\beta$ . . . . .	136
4.4.2	The spectral measure and the density of states . . . . .	138
4.4.3	The iterated resolvent formula . . . . .	139
4.5	Proof of Propositions 4.1 and 4.2 . . . . .	139
4.6	Proof of Theorem 4.3 . . . . .	140
4.7	Proof of Theorem 4.4 . . . . .	141
4.8	Proof of Proposition 4.5 . . . . .	144
4.9	Proof of Theorem 4.6 . . . . .	145
<b>5</b>	<b>One-dimensional scaling limit of the operator <math>H_\beta</math></b>	<b>151</b>
5.1	Introduction . . . . .	151
5.2	Context and statement of the results . . . . .	153
5.2.1	General notation . . . . .	153
5.2.2	The random potential $\beta$ . . . . .	153
5.2.3	A new approach of the Matsumoto-Yor properties in relation with the mixing measure of the VRJP . . . . .	156
5.2.4	Scaling limit and continuous version of $H_\beta$ on the circle . . . . .	157
5.2.4.1	Definition of the discrete operator on the circle . . . . .	157
5.2.4.2	Definition of the continuous limit . . . . .	158
5.2.4.3	Results of convergence . . . . .	158
5.2.4.4	Dufresne's type identities in law . . . . .	158
5.2.4.5	The continuous random operator $\mathcal{H}^{(\lambda)}$ . . . . .	159
5.2.5	Continuous version of $H_\beta$ on the real line . . . . .	160
5.2.6	The asymptotic density of states . . . . .	161
5.2.7	Organisation of the paper . . . . .	161
5.3	Background on the $\beta$ -potential . . . . .	161
5.4	Preliminary lemmas . . . . .	162
5.5	Proof of the results of section 5.2.3 . . . . .	167
5.6	Study of the discrete operator on the circle . . . . .	175
5.6.1	A simple description of the random potential $\beta$ . . . . .	176
5.6.2	Computation of $G_\beta^{(\lambda,n)}$ . . . . .	176
5.7	Convergence of the discretized Green function . . . . .	179
5.8	Study of $\mathcal{G}^{(\lambda)}$ . . . . .	181
5.9	Proof of identities in law . . . . .	182
5.10	Study of $\mathcal{H}^{(\lambda)}$ . . . . .	184
5.11	Proof of asymptotic results on $\mathcal{H}^{(\lambda)}$ . . . . .	187
<b>6</b>	<b>A multi-dimensional version of the Matsumoto-Yor properties</b>	<b>199</b>
6.1	Introduction . . . . .	199
6.2	Context and statement of the results . . . . .	200
6.2.1	Results in dimension one . . . . .	200
6.2.2	Brownian motions with interacting drifts and the random $\beta$ potential . . . . .	202
6.2.3	Main results : A multi-dimensional version of the opposite drift theorem, the conditional Matsumoto-Yor property and consequences . . . . .	204
6.2.4	Two open questions . . . . .	206
6.3	Multi-dimensional time change : Proof of Theorem 6.2 and Theorem 6.3 . . . . .	207
6.3.1	Justification of the Lamperti time change . . . . .	207

6.3.2	Proof of Theorem 6.2 . . . . .	208
6.3.3	First proof of (i) : using the strong Markov property of Theorem H . . . . .	209
6.3.4	Second proof of (i) : using the mixing measure and Girsanov's theorem . . . . .	211
6.3.5	Time change on the conditional process . . . . .	216
6.4	Conditioning in Lamperti time scale : Proof of Theorem 6.4, 6.5 and 6.6 . . . . .	217
6.4.1	Proof of Theorem 6.4 . . . . .	217
6.4.2	Proof of Theorem 6.5 . . . . .	220
6.4.3	Proof of Theorem 6.6 . . . . .	220
<b>A</b>		<b>223</b>
A.1	Simulations . . . . .	223
A.2	Interpretation . . . . .	227

# Chapitre 1

## Introduction

### 1.1 How to read this thesis

This thesis is about branching random walks, reinforced random walks, random operators and the links between these topics. In this introduction, we start by describing the state of the art regarding these subjects. Afterwards, we explain which are the main results of this thesis and we give short insights into their proofs. The next chapters are based on the articles which have been written during this thesis. In particular, each chapter is self-contained and can be read independently. Finally, in the appendix, we present numerical simulations which are related to Chapter 4.

### 1.2 Reinforced processes

#### 1.2.1 Exchangeability and De Finetti's theorem

Let us start with the simplest example of reinforced process we can imagine : Pólya's urn. This model was introduced in [61] and has been studied and generalized in many works including [12], [13] and [90]. We focus here on a very simple case. Let  $a \in \mathbb{N}^*$  and  $b \in \mathbb{N}^*$ . We start with an urn containing  $a$  white balls and  $b$  black balls. At every step, we choose a ball uniformly at random in the urn and we put it back in the urn together with a new ball with the same color as the chosen ball. More precisely, this process defines two sequences  $(U_1(n), U_2(n))_{n \in \mathbb{N}}$  such that  $U_1(0) = a$ ,  $U_2(0) = b$  and for every  $n \in \mathbb{N}$ ,

$$(U_1(n+1), U_2(n+1)) = (U_1(n), U_2(n)) + (\xi_1(n+1), \xi_2(n+1))$$

where conditionally on  $(U_1(k), U_2(k), 0 \leq k \leq n)$ ,

- $(\xi_1(n+1), \xi_2(n+1)) = (1, 0)$  with probability  $\frac{U_1(n)}{a+b+n}$ .
- $(\xi_1(n+1), \xi_2(n+1)) = (0, 1)$  with probability  $\frac{U_2(n)}{a+b+n}$ .

The process  $(U_1(n), U_2(n))_{n \in \mathbb{N}}$  is reinforced in the sense that the more you pick balls of some color, the more likely it is to pick balls of this color. Moreover, a simple computation yields the following result :

**Proposition 1.1.** *Let  $n \in \mathbb{N}^*$ . Let  $(x_1, \dots, x_n) \in \{0, 1\}^n$ . Let  $L = \sum_{i=1}^n x_i$ . It holds that*

$$\mathbb{P}(\xi_1(1) = x_1, \dots, \xi_1(n) = x_n) = \frac{\prod_{k=0}^{L-1} (a+k) \times \prod_{k=0}^{n-L-1} (b+k)}{\prod_{k=0}^{n-1} (a+b+k)}.$$

A remarkable fact which stems directly from Proposition 1.1 is that for every  $n \in \mathbb{N}^*$ , for every  $(x_1, \dots, x_n) \in \{0, 1\}^n$  and for every permutation  $\sigma$  of  $\llbracket 1, n \rrbracket$ ,

$$\mathbb{P}(\xi_1(1) = x_1, \dots, \xi_1(n) = x_n) = \mathbb{P}(\xi_1(1) = x_{\sigma(1)}, \dots, \xi_1(n) = x_{\sigma(n)}). \quad (1.1)$$

For any measure,  $\nu$  on  $[0, 1]$  we can define a random variable  $\zeta^{*,\nu}$  which has distribution  $\nu$  and a sequence of random variables  $(\zeta^\nu(k))_{k \in \mathbb{N}^*}$  such that conditionally on  $\zeta^{*,\nu}$ ,  $(\zeta^\nu(k))_{k \in \mathbb{N}^*}$  is a sequence of i.i.d Bernoulli random variables with parameter  $\zeta^{*,\nu}$ .

The property given by (1.1) is called exchangeability. Actually, it is the main ingredient in order to apply the De Finetti's theorem. (See [65].)

**Theorem 1.2** (De Finetti's theorem). *Let  $(\xi(k))_{k \in \mathbb{N}^*}$  be an exchangeable sequence of random variables whose values are in  $\{0, 1\}$ . Then, there exists a measure  $\nu$  on  $[0, 1]$  such that*

$$(\xi(k))_{k \in \mathbb{N}^*} \stackrel{\text{law}}{=} (\zeta^\nu(k))_{k \in \mathbb{N}^*}.$$

Basically, De Finetti's theorem states that if a sequence of random variables whose values are in  $\{0, 1\}$  is exchangeable, then, it is a sequence of independent coin tossings where the bias of the coin is itself a random variable. Therefore, one can apply De Finetti's Theorem to the case of Pólya's urn which implies there exists a measure  $\nu$  on  $[0, 1]$ , a random variable  $\zeta_1^{*,\nu}$  with distribution  $\nu$  and an i.i.d sequence of Bernoulli random variables  $(\zeta_1^\nu(k))_{k \in \mathbb{N}^*}$  with parameter  $\zeta_1^{*,\nu}$  such that

$$(\xi_1(k))_{k \in \mathbb{N}^*} \stackrel{\text{law}}{=} (\zeta_1^\nu(k))_{k \in \mathbb{N}^*}.$$

The random variable  $\zeta_1^{*,\nu}$  with distribution  $\nu$  is called "the random environment" of Pólya's urn. If we know the distribution  $\nu$ , we will be able to entirely characterize the distribution of the process  $((U_1(k), U_2(k)))_{k \in \mathbb{N}^*}$ . Actually, a simple computation shows that for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \int_0^1 x^k d\nu(x) &= \mathbb{E} \left[ (\zeta_1^{*,\nu})^k \right] \\ &= \mathbb{P}(\zeta_1^\nu(1) = 1, \zeta_1^\nu(2) = 1, \dots, \zeta_1^\nu(k) = 1) \\ &= \mathbb{P}(\xi_1(1) = 1, \xi_1(2) = 1, \dots, \xi_1(k) = 1) \\ &= \prod_{i=0}^{k-1} \frac{(a+i)}{a+b+i}. \end{aligned} \quad (1.2)$$

Then, one can recognise the moments of a Beta distribution with parameters  $(a, b)$ , that is, a distribution whose density is

$$\mathbf{1}\{x \in (0, 1)\} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}.$$

Therefore, by the theorem of moments (see Theorem 30.1 in [27]),  $\nu$  is a Beta distribution with parameters  $(a, b)$ . In this case, the distribution of the random environment was quite easy to compute but it is sometimes much more difficult. Nevertheless, one may wonder in which context the random environment does exist. In [51], Diaconis and Freedman proved a generalization of De Finetti's theorem.

Let  $(V, E)$  be a locally finite countable graph.  $V$  denotes the set of vertices and  $E$  denotes the set of (non-oriented) edges. Furthermore, let us define the set of oriented edges of  $(V, E)$  by

$$\hat{E} := \bigcup_{\{i,j\} \in E} \{(i, j), (j, i)\}.$$

A path  $\sigma$  of length  $n$  in  $V$  is a finite sequence  $(\sigma_0, \sigma_1, \dots, \sigma_n)$  such that for every  $k \in \llbracket 0, n-1 \rrbracket$ ,  $(\sigma_k, \sigma_{k+1}) \in \hat{E}$ . For every oriented edge  $e \in \hat{E}$  and for every path  $\sigma$ , one can define

$$\omega(e, \sigma) = |\{k \in \llbracket 0, n-1 \rrbracket, e = (\sigma_k, \sigma_{k+1})\}|$$

where  $|\cdot|$  denotes the cardinality. Then, let us define an equivalence relation  $\sim$  on the set of paths. Two paths  $\sigma^{(1)}$  and  $\sigma^{(2)}$  are equivalent, that is,  $\sigma^{(1)} \sim \sigma^{(2)}$ , if and only if they start from the same vertex and for every  $e \in \hat{E}$ ,  $\omega(e, \sigma^{(1)}) = \omega(e, \sigma^{(2)})$ . A sequence of random variables  $(X_n)_{n \in \mathbb{N}}$  is said to be partially exchangeable if and only if for every paths  $\sigma^{(1)}$  and  $\sigma^{(2)}$  with the same size  $n$  such that  $\sigma^{(1)} \sim \sigma^{(2)}$ ,

$$\mathbb{P}\left(X_0 = \sigma_0^{(1)}, \dots, X_n = \sigma_n^{(1)}\right) = \mathbb{P}\left(X_0 = \sigma_0^{(2)}, \dots, X_n = \sigma_n^{(2)}\right).$$

A transition kernel on  $(V, E)$  is an element  $K$  of  $[0, 1]^{\hat{E}}$  such that for every  $i \in V$ ,  $\sum_{\{i, j\} \in E} K(i, j) = 1$ .

Let  $\mathcal{K}$  be the set of transition kernels on  $(V, E)$ . For any distribution  $\nu$  on  $\mathcal{K}$  and for any  $i_0 \in V$  one can define a random variable  $K^\nu$  on  $\mathcal{K}$  whose distribution is  $\nu$  and a sequence of random variables  $(Y_n^{i_0, \nu})_{n \in \mathbb{N}}$  which is a Markov chain with transition kernel  $K^\nu$  starting from  $i_0$  conditionally on  $K^\nu$ .  $(Y_n^{i_0, \nu})_{n \in \mathbb{N}}$  is a random walk in the random environment  $K^\nu$ . Sometimes,  $(Y_n^{i_0, \nu})_{n \in \mathbb{N}}$  is also called a "mixture of Markov chains".

**Theorem 1.3** (Section 2 in [51]). *Let  $(V, E)$  be a locally finite countable graph. Let  $i_0 \in V$ . Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of random variables such that  $X_0 = i_0$  and for every  $k \in \mathbb{N}$ ,  $X_k \in V$  and  $\{X_k, X_{k+1}\} \in E$ . Moreover, we assume that  $(X_n)_{n \in \mathbb{N}}$  is partially exchangeable. Further, we assume that  $(X_n)_{n \in \mathbb{N}}$  is recurrent in the sense that almost surely there is an infinite number of  $k$  such that  $X_k = X_0$ . Then, there exists a unique distribution  $\nu$  on  $\mathcal{K}$  such that*

$$(X_n)_{n \in \mathbb{N}} \stackrel{\text{law}}{=} (Y_n^{i_0, \nu})_{n \in \mathbb{N}}.$$

By Theorem 1.3, if  $(X_n)_{n \in \mathbb{N}}$  is partially exchangeable and recurrent, then  $(X_n)_{n \in \mathbb{N}}$  is a Markov chain in random environment. Therefore a relevant way to study the behaviour of  $(X_n)_{n \in \mathbb{N}}$  is to find the distribution  $\nu$  and study its properties. However, in [51], Freedman and Diaconis showed that Theorem 1.3 can fail if we do not assume recurrence of the process. Now, let us focus on a particular case of partially exchangeable process : the Edge-Reinforced Random Walk.

### 1.2.2 The edge-reinforced random walk

The Edge-Reinforced Random Walk (ERRW) was introduced by Coppersmith and Diaconis in [45]. Let  $(V, E)$  be a locally finite graph. Let  $a \in (0, +\infty)^E$ . For every  $e \in E$ ,  $a_e$  is the weight of the non-oriented edge  $e$ . Let  $i_0 \in V$  be the initial point. Then, let us define a stochastic process  $(X_n)_{n \in \mathbb{N}}$  with law  $\mathbb{P}_{a, i_0}$  such that

- $\mathbb{P}_{a, i_0}(X_0 = i_0) = 1$ ,
- for every  $n \in \mathbb{N}$  and for every  $i \in V$ ,

$$\mathbb{P}_{a, i_0}(X_{n+1} = i | X_0, \dots, X_n) = \frac{w_{\{X_n, i\}}(n)}{\sum_{\{X_n, j\} \in E} w_{\{X_n, j\}}(n)}$$

where for every  $e \in E$  and for every  $n \in \mathbb{N}$ ,  $w_e(n) = a_e + \sum_{k=0}^{n-1} \mathbf{1}_{\{\{X_k, X_{k+1}\} = e\}}$ .

**Remark 1.1.** *We insist on the fact that the weights are associated with non-oriented edges in the ERRW. There is also an oriented version of the ERRW but it is a very different model which is known as the random walk in Dirichlet environment. This model has been studied for example in [150], [161], [151] and [140].*

There are a few important questions concerning the ERRW. One wonder whether the ERRW is recurrent or transient? If it is recurrent, is it exponentially localized? If it is transient, does it exhibit a diffusive behaviour? Especially, one would like to know what is the asymptotic

behaviour of the ERRW on classical graphs like trees or  $\mathbb{Z}^d$  and whether it depends on the reinforcement  $a \in (0, +\infty)^E$  or not. Now, let us assume that all weights are equals to some constant  $a$ . Heuristically, when  $a$  is very large, the reinforcement is very small and the same behaviour as for the simple random walk is expected. On the contrary, if  $a$  is very small then the reinforcement is huge and we expect the ERRW to be recurrent. In particular, it is natural to conjecture that the ERRW on  $\mathbb{Z}$  or  $\mathbb{Z}^2$  is always recurrent and that there is a phase transition between recurrence and transience on  $\mathbb{Z}^d$  when  $d \geq 3$ . We will see in the introduction of this thesis that these conjectures are true even if their proofs required three decades of work involving many different authors.

The first step towards this conjecture has been made in [136], where Pemantle proved there is a phase transition in the case of the binary tree.

**Theorem 1.4** (Theorem 1 in [136]). *Let us consider the ERRW on an infinite binary tree such that each edge has the same weight  $a$ . Then, there exists a constant  $a_0 > 0$  such that*

- (i) *If  $a < a_0$ , the ERRW is recurrent,*
- (ii) *If  $a > a_0$ , the ERRW is transient.*

The proof of Theorem 1.4 relies on the fact that a tree does not have cycles. Therefore, it is possible to show that the ERRW on trees is a random walk whose jumps from a vertex to another one are given by i.i.d Pólya's urns on each vertex. Moreover, by De Finetti's theorem, these Pólya's urns are associated with Dirichlet random variables. That is why, the ERRW on trees is a random walk in Dirichlet environment which gives a lot of tools in order to analyse this random walk.

However, when the graph  $(V, E)$  has cycles, everything is much more complicated. Nevertheless, Coppersmith and Diaconis observed the following general fact about the ERRW :

**Proposition 1.5.** *For any locally finite graph  $(V, E)$ , for any  $a \in (0, +\infty)^E$  and for any  $i_0 \in V$ , the ERRW with law  $\mathbb{P}_{a, i_0}$  is partially exchangeable.*

Proposition 1.5 is not difficult to prove but it has huge consequences. Indeed, if the ERRW is recurrent, by Theorem 1.3, the ERRW should be a random walk in random environment. Moreover, in [127], Merkl and Rolles proved that the ERRW is a random walk in random environment without assuming the recurrence of the process. As usual, one would like to know what is the distribution of this random environment. For example, does it have an explicit formula for its density? Actually, Coppersmith and Diaconis found such a formula on any finite graph in [45] and it is so amazing that it was called later the "magic formula".

**Theorem 1.6** (Magic formula). *Let  $(V, E)$  be a finite graph with  $n$  vertices. Let  $a \in (0, +\infty)^E$ . Let  $i_0 \in V$ . Let us choose  $e_0 \in E$ . Let  $\Omega_{e_0} = (0, \infty)^{E \setminus \{e_0\}}$  which is the space of conductances where we imposed that the conductance of the edge  $e_0$  is 1. Every  $x \in \Omega_{e_0}$  can be written as  $(x_e)_{e \in E \setminus \{e_0\}}$ . We define  $x_{e_0} = 1$ . For every  $i \in V$ , we denote*

$$y_i = \sum_{e \in E, i \in e} x_e \text{ and } a_i = \sum_{e \in E, i \in e} a_e.$$

*Then the ERRW  $(X_n)_{n \in \mathbb{N}}$  with distribution  $\mathbb{P}_{i_0, a}$  is distributed as a random walk with a random transition kernel  $K$ . Moreover, for every  $\{i, j\} \in E$ ,*

$$K(i, j) = \frac{x_{\{i, j\}}}{y_i}$$

*where  $(x_e)_{e \in E}$  has the following density on  $\Omega_{e_0}$  :*

$$C(i_0, a) \sqrt{y_{i_0}} \frac{\prod_{e \in E} x_e^{a_e}}{\prod_{i \in V} y_i^{(a_i+1)/2}} \times \sqrt{\sum_{T \in \mathcal{T}} \prod_{e \in T} x_e} \times \prod_{e \in E \setminus \{e_0\}} \frac{dx_e}{x_e}$$

where  $\mathcal{T}$  is the set of spanning trees of  $(V, E)$  and

$$C(i_0, a) = \frac{2^{1-n+\sum_{e \in E} a_e}}{\pi^{(n-1)/2}} \times \frac{\prod_{i \in V} \Gamma\left(\frac{1}{2}(a_i + 1 - \mathbf{1}\{i = i_0\})\right)}{\prod_{e \in E} \Gamma(a_e)}.$$

Theorem 1.6 was originally proved in [45] but this paper was never published. However, one can find two different proofs of the magic formula in [95] and in [126]. This formula is not very easy to use in practical cases. By the matrix-tree theorem (see [168]) the term which is a sum on spanning trees can be interpreted as the determinant of any minor of the matrix  $M^{(x)}$  which is defined by

$$M^{(x)}(i, j) = \mathbf{1}\{i = j\} \left( \sum_{\{k, i\} \in E} x_k \right) - x_{i, j}$$

for every  $i, j \in V$ . This kind of determinantal term creates long-range interactions which are quite difficult to study. However, Theorem 1.6 was used by Merkl and Rolles in order to prove the recurrence of the ERRW on  $\mathbb{Z} \times \llbracket 1, d \rrbracket$  in [129] and on a modified version of  $\mathbb{Z}^2$  in [131] for any initial weights. However, at this point, it was still not possible to prove any result of transience and the recurrence on  $\mathbb{Z}^2$  for any reinforcement was still unproved. We present in the next section a continuous-time process which is related to the ERRW, the VRJP, which was decisive in further investigations of the ERRW.

### 1.2.3 The Vertex Reinforced Jump Process

Let  $(V, E)$  be a locally finite graph. For every  $i, j \in V$ , let  $W_{i, j} = W_{j, i}$  be a non-negative weight. We will always assume that for every  $i, j \in V$ ,  $W_{i, j} > 0$  if and only if  $\{i, j\} \in E$ . For now, we assume that  $W_{i, i} = 0$  for every  $i \in V$  but this restriction shall be removed later in a slightly different context. Let  $i_0 \in V$  be an initial point. The Vertex Reinforced Jump Process (VRJP) is a continuous time process  $(Y_t)_{t \geq 0}$  which starts from  $i_0$  and such that for every  $t \geq 0$ , conditionally on  $(Y_s, s \leq t)$ , if  $Y_t = i$ , then  $Y$  jumps to a neighbour  $j$  of  $i$  at rate  $W_{i, j} L_j(t)$ , where

$$L_j(t) = 1 + \int_0^t \mathbf{1}\{Y_s = j\} ds.$$

The VRJP was originally introduced by Werner but the first results about the VRJP were obtained by Davis and Volkov in [47] where they proved the following result :

**Proposition 1.7** (Theorem 1.1 in [47]). *Let us consider  $(Y_t)_{t \geq 0}$  which is the VRJP on  $\mathbb{Z}$  where weights are all equal to 1. Then for every  $i \in \mathbb{Z}$ , the almost sure limit  $V_i := \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbf{1}\{Y_s = i\} ds$  exists. Moreover there are i.i.d random variables  $(A_i)_{i \in \mathbb{Z}^*}$  with density*

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2x}(x-1)^2\right)$$

such that for every  $i \in \mathbb{Z}$ ,

$$V_i = \frac{\tilde{V}_i}{\sum_{k=-\infty}^{+\infty} \tilde{V}_k}$$

where  $\tilde{V}_0 = 1$ , for every  $i > 0$ ,  $\tilde{V}_i = \prod_{k=1}^i A_k$  and for every  $i < 0$ ,  $\tilde{V}_i = \prod_{k=i}^{-1} A_k$ .

**Remark 1.2.** In Proposition 1.7, the distribution of the random variables  $(A_i)_{i \in \mathbb{Z}^*}$  is a particular case of the Inverse Gaussian distribution with parameters  $(\mu, \lambda)$  whose distribution is

$$\mathbf{1}\{x > 0\} \frac{\sqrt{\lambda}}{\sqrt{2\pi x^3}} \exp\left(-\frac{\lambda}{2\mu^2 x}(x - \mu)^2\right).$$

The Inverse Gaussian distribution with parameters  $(\mu, \lambda)$  is denoted by  $IG(\mu, \lambda)$ . This distribution will play a crucial role in this thesis and more generally in the theory of the VRJP.

Proposition 1.7 implies the recurrence of the VRJP on  $\mathbb{Z}$  when all weights are equal to 1 and describes precisely what is the amount of time where the VRJP occupies each vertex. Moreover, it is not difficult to prove that if the VRJP on  $\mathbb{Z}$  starts with weights which are all equal to  $W > 0$ , then Proposition 1.7 remains true with  $A_i$  having density  $IG(1, W)$  for every  $i \in \mathbb{Z}^*$ . Consequently, there is no phase transition between recurrence and transience for the VRJP on  $\mathbb{Z}$ . On the contrary, on supercritical Galton-Watson trees, there is a phase transition. The case of  $d$ -ary trees has been studied in [48] and a complete phase diagram has been obtained in [15] for Galton-Watson trees.

**Proposition 1.8** (Theorem 1.1 in [15]). *Let  $\mu$  be an offspring law on  $\mathbb{N}$ . We assume that  $\mu$  is supercritical, that is, its mean  $m$  satisfies  $m > 1$ . Let  $(Y_t)_{t \geq 0}$  be the VRJP with constant weights  $W > 0$  on a Galton-Watson tree with offspring  $\mu$ . For every  $c > 0$ , let*

$$r(c) = \frac{\sqrt{c}}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} \exp\left(-\frac{c}{2x}(x-1)^2\right) dx.$$

Then, we have the following phase transition :

- (i) If  $mr(W) \leq 1$ ,  $(Y_t)_{t \geq 0}$  is almost surely recurrent.
- (ii) If  $mr(W) > 1$ ,  $(Y_t)_{t \geq 0}$  is almost surely transient conditionally on the survival of the underlying Galton-Watson tree.

There is unique positive real number  $W$  such that  $mr(W) = 1$ . It is the transition point and it will be denoted by  $W_c(\mu)$ .

However, a decisive step in the understanding of the VRJP has been achieved by Sabot and Tarrès in [152]. It is easy to see that the VRJP goes faster and faster. That is why, Sabot and Tarrès found a time-change which is convenient for the study of the VRJP. More precisely, if  $(V, E)$  is a locally finite graph and  $(Y_t)_{t \geq 0}$  is the VRJP on  $(V, E)$  with symmetric non-negative weights  $(W_{i,j})_{i,j \in V}$  then one can define the strictly increasing random function  $D$  such that for every  $t \geq 0$ ,

$$D(t) = \sum_{i \in V} (L_i^2(t) - 1).$$

Let us define the time-changed VRJP  $\hat{Z}$  by  $\hat{Z}_t = Y_{D^{-1}(t)}$  for every  $t \geq 0$ . For every  $i \in V$ , and for every  $t \geq 0$ , we define  $l_i(t) = \int_0^t \mathbf{1}\{\hat{Z}_s = i\} ds$ . Sabot and Tarrès obtained the following result :

**Theorem 1.9** (Theorem 2 in [152] or Theorem 2 in [153]). *Let  $(V, E)$  be a finite graph with symmetric non-negative weights  $(W_{i,j})_{i,j \in V}$ . Let  $n$  be the cardinality of  $V$ . Let  $i_0 \in V$ . Let  $(\hat{Z}_t)_{t \geq 0}$  be the time-changed VRJP on  $(V, E)$  with weights  $(W_{i,j})_{i,j \in V}$ . Then, for every  $i \in V$ ,*

$$U_i := \lim_{t \rightarrow +\infty} \frac{1}{2} \ln \left( \frac{l_i(t) + 1}{l_{i_0}(t) + 1} \right)$$

exists almost surely. Moreover, conditionally on  $U$ ,  $\hat{Z}$  is a continuous-time Markov process which jumps from  $i$  to  $j$  at rate

$$\frac{1}{2} W_{i,j} e^{U_j - U_i}.$$

Furthermore,  $(U_j)_{j \in V}$  has an explicit distribution  $Q_{i_0, V}^W$  on  $\{(u_j)_{j \in V}, u_{i_0} = 0\}$  which is given by

$$\frac{1}{(2\pi)^{(n-1)/2}} \exp \left( - \sum_{j \in V} u_j - 2 \sum_{\{i, j\} \in E} W_{i, j} \sinh^2 \left( \frac{1}{2}(u_i - u_j) \right) \right) \sqrt{D(W, u)} \prod_{j \in V \setminus \{i_0\}} du_j$$

where  $D(W, u)$  is any diagonal minor of the matrix  $M(W, u)$  on  $V \times V$  such that for every  $i, j \in V$

$$M(W, u)(i, j) = \mathbf{1}\{i = j\} \left( \sum_{\{k, i\} \in E} W_{i, j} e^{u_i + u_k} \right) - W_{i, j} e^{u_i + u_j}.$$

First, one can observe that, as in the case of the ERRW, the time-changed VRJP  $\hat{Z}$  is a Markov process in a random environment. Moreover, the density of this environment is quite similar to the magic formula obtained for the ERRW, especially concerning the determinantal term. Actually, this is not a coincidence because Sabot and Tarrès proved that the ERRW is a VRJP with random weights.

**Theorem 1.10** (Theorem 1 in [152]). *Let  $(V, E)$  be a locally finite graph. Let  $(a_e)_{e \in E}$  be positive weights on the edge set  $E$ . Let  $(X_n)_{n \in \mathbb{N}}$  be the ERRW on  $(V, E)$  with weights  $(a_e)_{e \in E}$ . Let  $(\mathcal{W}_e)_{e \in E}$  be independent random variables such that for every  $e \in E$ ,  $\mathcal{W}_e$  is a Gamma distribution with parameters  $(a_e, 1)$ . Finally, let  $(Y_n)_{n \in \mathbb{N}}$  be the discrete time process associated with the VRJP, that is, the VRJP taken at jump times, on  $(V, E)$  with weights  $(\mathcal{W}_e)_{e \in E}$ . Then, it holds that*

$$(X_n)_{n \in \mathbb{N}} \stackrel{\text{law}}{=} (Y_n)_{n \in \mathbb{N}}.$$

Theorem 1.10 states that the ERRW is basically a mixture of VRJPs. However, it is not always straightforward to deduce properties about the ERRW directly from properties about the VRJP. Nevertheless, it will be possible to apply on the ERRW the proof ideas which work for the VRJP.

Furthermore, thanks to the field  $U$ , one can reprove Proposition 1.8 about the phase transition of the VRJP on trees. Indeed, in [40], Chen and Zeng made a change of variables in the field  $U$  and they get a very explicit description of the environment of the VRJP on trees.

**Proposition 1.11** (Theorem 3 in [40]). *Let  $(V, E)$  be a rooted tree with root  $o$ . Let  $W > 0$ . Let  $(A_x)_{x \in V \setminus \{o\}}$  be a family of i.i.d random variables with Inverse Gaussian distribution  $IG(1, W)$ . We define  $U_o = 1$  and for every  $x \in V \setminus \{o\}$  we define*

$$U_x = \prod_{o < z \leq x} A_z$$

where the order between vertices of  $V$  is the genealogical order associated with the structure of rooted tree. Then the time-changed VRJP  $(\hat{Z}_t)_{t \geq 0}$  on  $(V, E)$  starting from  $o$  with constant weights  $W$  is a mixture of Markov processes which jumps from  $i$  to  $j$  at rate

$$\frac{1}{2} W e^{U_j - U_i}.$$

**Remark 1.3.** *In Proposition 1.7, the field  $\tilde{V}$  is exactly the same as the field  $U$  in Proposition 1.11. However, the authors of [47] (where Proposition 1.7 was originally proved) did not know that  $\tilde{V}$  could give the random environment of the VRJP. They understood  $\tilde{V}$  only as the mean local times of the VRJP.*

However, as for the magic formula, apart from the case of trees, the density of the field  $U$  in Theorem 1.9 is not easy to handle because of the determinantal term. Nonetheless, the density of the field  $U$  is related to the supersymmetric hyperbolic sigma model which has been studied by Disertori, Spencer and Zirnbauer in [54] and [53].

### 1.2.4 The supersymmetric hyperbolic sigma model, localisation and delocalisation

The supersymmetric hyperbolic sigma model, also called the  $\mathbb{H}^{2|2}$  model, has been introduced by Zirnbauer in [56] and [172] for physical motivations. The main advances concerning the  $\mathbb{H}^{2|2}$  model for our purpose were obtained by Disertori, Spencer and Zirnbauer in [54] and [53]. This model takes values in a supermanifold which contains classical variables and anticommuting variables. By using tools coming from the theory of superanalysis (see [20] for an introduction to this theory), they found a non-probabilistic proof of the fact that  $Q_{i_0, V}^W$ , the measure of the field  $U$ , is a probability measure. Moreover, they obtained the following results :

**Theorem 1.12** (Theorem 2 in [53]). *Let  $d \in \mathbb{N}^*$ . Let  $(V, E)$  be  $\mathbb{Z}^d$  endowed with its natural graph structure. Then, there exists  $C_0 > 0$  and  $W_r(d) > 0$  such that for every  $W < W_r(d)$ , there exists  $r_W > 0$  such that for every finite connected subset  $\Lambda \subset \mathbb{Z}^d$  containing 0 and for every  $x \in \Lambda$ ,*

$$\int e^{u_x/2} dQ_{0, \Lambda}^W(u) \leq C_0 e^{-r_W \|x\|}$$

where  $\|\cdot\|$  is the euclidean norm.

**Theorem 1.13** (Theorem 1 in [54]). *Let  $d \geq 3$ . Let  $(V, E)$  be  $\mathbb{Z}^d$  endowed with its natural graph structure. Then, there exists  $W_t(d) > 0$  such that for every  $W > W_t(d)$ , for every finite connected subset  $\Lambda \subset \mathbb{Z}^d$  containing 0 and for every  $x, y \in \Lambda$ ,*

$$\int \cosh^m(u_x - u_y) dQ_{0, \Lambda}^W(u) \leq 2$$

provided that  $m \leq W^{1/8}$ .

**Remark 1.4.** *In the two previous theorems, we made a small abuse of notation. Indeed, when  $W$  is a positive number, we use the notation  $Q_{i_0, V}^W$  to denote the measure of the field  $U$  (see Theorem 1.9) associated with the VRJP starting from  $i_0$  on some graph  $(V, E)$  where all weights are assumed to be equal to  $W$ .*

Theorem 1.12 states that the environment of the VRJP exhibits exponential decay when  $W$  is small. On the contrary, Theorem 1.13 implies that the environment of the VRJP is almost constant when  $W$  is large and  $d \geq 3$ . Consequently, by combining their own results with Theorem 1.12 and Theorem 1.13, Sabot and Tarrès obtained the following result :

**Theorem 1.14** (Corollary 1 and Corollary 3 in [152]). *Let  $d \in \mathbb{N}^*$ . Let  $W > 0$ . Let us consider the discrete time process  $(Y_n)_{n \geq 0}$  associated with the VRJP on  $\mathbb{Z}^d$  starting from 0 with constant weights  $W$ .*

- (i) *If  $W < W_r(d)$ ,  $(Y_n)_{n \geq 0}$  is a mixture of positive recurrent Markov chains.*
- (ii) *If  $d \geq 3$  and  $W > W_t(d)$ ,  $(Y_n)_{n \geq 0}$  is a mixture of transient Markov chains.*

By Theorem 1.10, it is natural to wonder whether one can prove the same result for the ERRW by means of the supersymmetric hyperbolic sigma model. Actually, it is possible but it requires stronger results even if the proof is in the same spirit as Theorems 1.12 and 1.13.

**Theorem 1.15** (Corollary 2 in [152] or Theorem 1 in [8] for (i) and Theorem 1 in [52] for (ii)). *Let  $d \in \mathbb{N}^*$ . Let  $a > 0$ . Let us consider the ERRW  $(X_n)_{n \geq 0}$  on  $\mathbb{Z}^d$  starting from 0 with constant weights  $a$ .*

- (i) *There exists  $a_r(d) > 0$  such that if  $a < a_r(d)$ ,  $(X_n)_{n \geq 0}$  is a mixture of positive recurrent Markov chains.*
- (ii) *If  $d \geq 3$ , there exists  $a_t(d) > 0$  such that if  $a > a_t(d)$ ,  $(X_n)_{n \geq 0}$  is a mixture of transient Markov chains.*

We mention that Theorem 1 in [8] was proved with a method which does not use the VRJP but only the partial exchangeability of the ERRW. Finally, the supersymmetric hyperbolic sigma model was a very powerful tool which allowed to make a major breakthrough in the understanding of the VRJP and the ERRW. However, at this point, many questions remain unsolved :

- Is there a unique transition point for the VRJP and the ERRW on  $\mathbb{Z}^d$  with  $d \geq 3$ , that is, is it true that  $W_r(d) = W_t(d)$  and  $a_r(d) = a_t(d)$  for every  $d \in \mathbb{N} \setminus \{0, 1, 2\}$ ?
- Is the VRJP or the ERRW always recurrent on  $\mathbb{Z}^2$ ?
- In Theorem 1.6 and 1.9, we gave a representation of the environment of the ERRW and the VRJP on finite graphs. Is it possible to get a useful representation on infinite graphs?

In order to answer these questions, we need to introduce a new object : the  $\beta$ -potential.

### 1.2.5 The $\beta$ -potential and the environment of the VRJP

As we explained at the end of the previous subsection, we would like to get a representation of the environment of the VRJP on infinite graphs. To do so, we first need to make a change of variables involving the field  $U$  on any finite graph.

Let  $(V, E)$  be a finite graph with  $n$  vertices. Let  $(W_{i,j})_{i,j \in V}$  be symmetric non-negative weights. Let  $i_0 \in V$ . Recall that  $Q_{i_0, V}^W$  is the measure of the field  $U$  given by Theorem 1.9 which is associated with the VRJP on  $V$  starting from  $i_0$  with weights  $(W_{i,j})_{i,j \in V}$ . Let  $(U_i)_{i \in V}$  be a random field with distribution  $Q_{i_0, V}^W$  and let  $\gamma$  be a Gamma random variable with parameters  $(1/2, 1)$  which is independent of  $(U_i)_{i \in V}$ . Now, for every  $i \in V$ , we define

$$\beta_i = \frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} e^{U_j - U_i} + \mathbf{1}\{i = i_0\} \gamma. \quad (1.3)$$

Then, Sabot, Tarrès and Zeng introduced this random potential  $(\beta_i)_{i \in V}$  and computed its density.

**Proposition 1.16** (Theorem 1 in [153]). *The density of  $(\beta_i)_{i \in V}$  is*

$$\left(\frac{2}{\pi}\right)^{n/2} \mathbf{1}\{H_\beta > 0\} \frac{\exp\left(-\frac{1}{2}\langle 1, H_\beta 1 \rangle\right)}{\sqrt{\det(H_\beta)}}$$

where  $\langle \cdot, \cdot \rangle$  is the usual euclidean scalar product,  $1$  in the scalar product stands for the vector  $(1, 1, \dots, 1)$  and  $H_\beta$  is a symmetric matrix on  $V \times V$  such that for every  $i, j \in V$ ,

$$H_\beta(i, j) = 2\beta_i \mathbf{1}\{i = j\} - W_{i,j}. \quad (1.4)$$

The measure which is associated with this density will be denoted by  $\nu_V^W$ .

**Remark 1.5.** *In Proposition 1.16, we have to clarify that for every  $i \in V$ , it is possible that  $W_{i,i} > 0$ . Even if it does not really make sense for the VRJP, it will be useful to allow that case for the potential  $(\beta_i)_{i \in V}$ .*

**Remark 1.6.** *By means of the  $\beta$ -potential, Sabot, Tarrès and Zeng gave in [153] a purely computational proof of the fact that  $Q_{i_0, V}^W$  has total mass 1. This is the third proof of this fact. The first one was obtained in [54] by means of superanalysis and the second one was obtained in [152] with probabilistic tools.*

One can also compute the Laplace transform of  $(\beta_i)_{i \in V}$ .

**Proposition 1.17** (Proposition 1 in [153]). *Let  $(t_i)_{i \in V} \in \mathbb{R}_+^V$ . Then, it holds that*

$$\int e^{-\langle t, \beta \rangle} \nu_V^W(d\beta) = \exp\left(-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} \left(\sqrt{(t_i + 1)(t_j + 1)} - 1\right)\right) \prod_{i \in V} \frac{1}{\sqrt{1 + t_i}}. \quad (1.5)$$

Thanks to the Laplace transform given by (1.5), we can deduce directly the following proposition.

**Proposition 1.18.** *Let  $(V, E)$  be a finite graph. Let  $(W_{i,j})_{i,j \in V}$  be symmetric non-negative weights. Let  $(\beta_i)_{i \in V}$  be a random vector with distribution  $\nu_V^W$ . Then, it holds that*

(i) *For every  $i \in V$ ,  $\frac{1}{2\beta_i - W_{i,i}}$  is an Inverse Gaussian random variable*

$$IG \left( \frac{1}{\sum_{\{i,j\} \in E, j \neq i} W_{i,j}}, 1 \right).$$

(ii) *Let  $V_1$  and  $V_2$  be two subsets of  $V$  which are not related by any edge. Then  $(\beta_i)_{i \in V_1}$  and  $(\beta_i)_{i \in V_2}$  are independent.*

**Remark 1.7.** *Remark that  $2\beta_i - W_{i,i}$  is a reciprocal Inverse Gaussian distribution. It is an important fact which explains the major role which is played by the Inverse Gaussian distribution in the sequel of this thesis.*

We started from  $(U_i)_{i \in V}$  with distribution  $Q_{i_0, V}^W$  and by a change of variables, we obtained  $(\beta_i)_{i \in V}$  with distribution  $\nu_V^W$ . However, it is possible to go the other way. If  $(\beta_i)_{i \in V} \in \mathbb{R}_+^V$  and if  $H_\beta$  is invertible, then we can define the matrix  $G_\beta$  which is the inverse of  $H_\beta$ . Remark that if  $(\beta_i)_{i \in V} \sim \nu_V^W$ ,  $H_\beta$  is positive definite and thus invertible almost surely.

**Proposition 1.19** (Theorem 3 in [153]). *Let  $(V, E)$  be a finite graph. Let  $(W_{i,j})_{i,j \in V}$  be symmetric non-negative weights. Let  $(\beta_i)_{i \in V}$  be a random vector with distribution  $\nu_V^W$ . Let  $i_0 \in V$ . Then,*

$$\left( \frac{G_\beta(i_0, i)}{G_\beta(i_0, i_0)} \right)_{i \in V}$$

*has distribution  $Q_{i_0, V}^W$ . Moreover,  $G_\beta(i_0, i_0)$  is independent from  $\left( \frac{G_\beta(i_0, i)}{G_\beta(i_0, i_0)} \right)_{i \in V}$  and  $\frac{1}{2G_\beta(i_0, i_0)}$  is distributed like a Gamma random variable with parameters  $(1/2, 1)$ . In particular the time-changed VRJP starting from  $i_0$  with weights  $(W_{i,j})_{i,j \in V}$  is a mixture of Markov processes which jumps from  $i \in V$  to  $j \in V$  at rate*

$$\frac{1}{2} W_{i,j} \frac{G_\beta(i_0, j)}{G_\beta(i_0, i)}.$$

Therefore, thanks to Proposition 1.19, it is possible to recover the environment of the VRJP from the random vector  $(\beta_i)_{i \in V}$  with distribution  $\nu_V^W$ . A remarkable fact is that the measure  $\nu_V^W$  belongs to a more general family of measures.

**Proposition 1.20** (Theorem 2.2 on [110]). *Let  $(V, E)$  be a finite graph with  $n$  vertices. Let  $(W_{i,j})_{i,j \in V}$  be symmetric non-negative weights. Let  $(\eta_i)_{i \in V} \in \mathbb{R}_+^V$ . Let us define the measure  $\nu_V^{W, \eta}$  as*

$$\mathbf{1}_{\{H_\beta > 0\}} \left( \frac{2}{\pi} \right)^{n/2} \frac{\exp \left( -\frac{1}{2} \langle 1, H_\beta 1 \rangle - \frac{1}{2} \langle \eta, G_\beta \eta \rangle + \langle 1, \eta \rangle \right)}{\sqrt{\det(H_\beta)}} d\beta.$$

*Then  $\nu_V^{W, \eta}$  is a probability measure. Moreover, we can compute its Laplace transform. For any  $t \in \mathbb{R}_+^V$ ,*

$$\begin{aligned} & \int e^{-\langle t, \beta \rangle} \nu_V^{W, \eta}(d\beta) \\ &= \exp \left( - \sum_{i \in V} \eta_i (\sqrt{t_i + 1} - 1) - \frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} \left( \sqrt{(1+t_i)(1+t_j)} - 1 \right) \right) \prod_{i \in V} \frac{1}{\sqrt{1+t_i}}. \end{aligned}$$

Observe that if  $\eta = 0$ , then  $\nu_V^W$  and  $\nu_V^{W,\eta}$  are equal. This generalization of  $\nu_V^W$  is very useful because it exhibits stability properties with respect to restriction and conditioning. For any matrix  $M$  and any subsets  $U_1$  and  $U_2$  of  $V$ , we denote the restriction of the matrix  $M$  to the set  $U_1 \times U_2$  by  $M_{U_1, U_2}$ . Moreover, for any subset  $U_1$  of  $V$ , for any vector  $v$  of  $\mathbb{R}^V$  the restriction of  $v$  to  $U_1$  is denoted by  $v_{U_1}$  and  $\mathbf{1}_{U_1}$  is the vector of  $\mathbb{R}^{U_1}$  whose entries are all equal to 1.

**Lemma 1.21** (Lemma 5 in [154] and Proposition 4.3 in [110]). *Let  $(V, E)$  be a finite graph. Let  $(W_{i,j})_{i,j \in V}$  be symmetric non-negative weights. Let  $\eta = (\eta_i)_{i \in V} \in \mathbb{R}_+^V$ . Let  $U$  be a finite subset of  $V$  and let  $U^c = V \setminus U$ . Let  $(\beta_i)_{i \in V}$  be a random vector with distribution  $\nu_V^{W,\eta}$ .*

(i)  $(\beta_i)_{i \in U}$  has distribution  $\nu_U^{W_{U,U}, \hat{\eta}}$  where  $\hat{\eta}$  is a vector on  $U$  such that

$$\hat{\eta} = \eta_U + W_{U, U^c} \mathbf{1}_{U^c}.$$

(ii) Conditionally on  $(\beta_i)_{i \in U}$ ,  $(\beta_i)_{i \in U^c}$  has distribution  $\nu_{U^c}^{\tilde{W}, \check{\eta}}$  where  $\tilde{W}$  is a matrix on  $U^c \times U^c$  and  $\check{\eta}$  is a vector on  $U^c$  such that

$$\hat{W} = W_{U^c, U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} W_{U, U^c}$$

where  $H_\beta|_{U \times U}$  is the restriction and

$$\check{\eta} = \eta_{U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} \eta_U.$$

**Remark 1.8.** *Lemma 1.21 explains why it is natural to consider the case where  $W_{i,i} > 0$  for some  $i \in V$  for the  $\beta$ -potential. Indeed this case arises naturally when we condition  $(\beta_i)_{i \in U^c}$  on  $(\beta_i)_{i \in U}$ .*

In the case of finite graphs, by Proposition 1.19, the time-changed VRJP starting from  $i_0$  jumps from  $i$  to  $j$  at rate

$$\frac{1}{2} W_{i,j} \frac{G_\beta(i_0, j)}{G_\beta(i_0, i)}$$

where  $G_\beta$  is the inverse of the matrix  $H_\beta$ . One may wonder whether it remains true when  $H_\beta$  is an operator on an infinite graph. Does its "inverse" give the environment of the VRJP on infinite graphs? The problem is that  $H_\beta$  is not invertible on infinite graphs. Actually, on particular graphs as  $\mathbb{Z}^d$  or an infinite tree, one can show that the spectrum of  $H_\beta$  is exactly  $\mathbb{R}_+$ . Nonetheless, Sabot and Zeng showed in [154] that it is possible to construct a kind of pseudo-inverse of  $H_\beta$  which can play the role of the environment of the VRJP on infinite graphs. To do so, they developed a new tool which is the martingale  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  discussed below.

### 1.2.6 The infinite version of $\beta$ , the martingale $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$ and its applications

In this subsection, we will work on an infinite locally finite graph  $(V, E)$  with non-negative symmetric weights  $(W_{i,j})_{i,j \in V}$  and a root  $o$ . For example,  $(V, E)$  can be  $\mathbb{Z}^d$  with  $o = 0$  or an infinite rooted tree. Now, we want to construct an infinite version of  $\beta$ . Let us consider an increasing sequence  $(V_n)_{n \in \mathbb{N}}$  of finite subsets of  $V$  which contain  $o$  such that  $\bigcup_{n \in \mathbb{N}} V_n = V$ . For every  $n \in \mathbb{N}$ , we denote by  $E_n$  the set of edges of  $E$  whose ends lie in  $V_n$ . Then, for every  $n \in \mathbb{N}$ , we can construct a wired version  $(\tilde{V}_n, \tilde{E}_n)$  of  $(V_n, E_n)$ . For every  $n \in \mathbb{N}$ , we define  $\tilde{V}_n = V_n \cup \{\delta\}$  and

$$\tilde{E}_n = E_n \cup \{\{i, \delta\}, i \in V_n \text{ such that } \exists j \notin V_n, \{i, j\} \in E\}.$$

Besides, for every  $n \in \mathbb{N}$ ,  $(\tilde{V}_n, \tilde{E}_n)$  is endowed with the symmetric weights  $W^{(n)}$  on  $\tilde{V}_n \times \tilde{V}_n$  such that for every  $i, j \in \tilde{V}_n$ ,  $W_{i,j}^{(n)} = W_{i,j}$  if  $i, j \in V_n$  and  $W_{i,\delta}^{(n)} = \sum_{j \notin V_n, \{i,j\} \in E} W_{i,j}$  if  $\{i, \delta\} \in \tilde{E}_n$ . By applying Lemma 1.21, this leads us to the following result :

**Proposition 1.22** (Lemma 1 in [154]). *The sequence of measures  $(\nu_{V_n}^{W(n)}|_{V_n})_{n \in \mathbb{N}}$  is compatible. In particular, there exists a measure  $\nu_V^W$  on the infinite graph  $V$  such that for every  $n \in \mathbb{N}$*

$$\nu_V^W|_{V_n} = \nu_{V_n}^{W(n)}|_{V_n}.$$

By Proposition 1.22, we can construct  $(\beta_i)_{i \in V}$  even if  $V$  is infinite. Now, let  $\beta$  be a random element of  $\mathbb{R}_+^V$  with distribution  $\nu_V^W$  and let  $H_\beta$  be the infinite volume operator on  $V \times V$  which is associated with  $\beta$  in the same way as in (1.4). For every  $n \in \mathbb{N}$ , we define the sigma-field  $\mathcal{F}_n := \sigma(\beta_i, i \in V_n)$ . Then, following Definition 1 in [154], we define the operator  $(\hat{G}_\beta^{(n)}(i, j))_{i, j \in V}$  by

$$\hat{G}_\beta^{(n)}(i, j) = (H_\beta|_{V_n \times V_n})^{-1}(i, j)$$

if  $i, j \in V_n$  and  $\hat{G}_\beta^{(n)}(i, j) = 0$  otherwise. Moreover, we define the random vector  $(\psi_\beta^{(n)}(i))_{i \in V_n}$  as the unique solution of the equation

$$\begin{aligned} (H_\beta \psi_\beta^{(n)})(i) &= 0, & \text{for every } i \in V_n, \\ \psi_\beta^{(n)}(i) &= 1, & \text{for every } i \in V_n^c. \end{aligned}$$

The idea behind the definition of  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  is to create an eigenstate of  $H_\beta$  for the eigenvalue 0 when  $n$  goes to infinity. However, at first sight,  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  does not have any reason to converge. Nevertheless, miraculously, it is a martingale.

**Theorem 1.23** (Theorem 1 in [154]).

- (i) *For every  $i \in V$ ,  $(\psi_\beta^{(n)}(i))_{n \in \mathbb{N}}$  is a positive martingale with respect to the filtration  $(\mathcal{F}_n)_{n \geq 0}$ . In particular, it converges almost surely towards some finite random variable  $\psi_\beta(i)$ .*
- (ii) *For every  $i, j \in V$ ,  $(\hat{G}_\beta^{(n)}(i, j))_{n \in \mathbb{N}}$  is increasing almost surely and converges almost surely towards some finite random variable  $\hat{G}_\beta(i, j)$ .*

The proof of Theorem 1.23 relies on the stability of the measures of the form  $\nu_V^{W, \eta}$  under conditioning and restriction which is described in Proposition 1.21. Thanks to Theorem 1.23, Sabot and Zeng were able to construct the environment of the VRJP on infinite graphs.

**Theorem 1.24** (Theorem 1 in [154]). *Let  $\beta$  be a random element of  $\mathbb{R}_+^V$  with distribution  $\nu_V^W$ . Let  $\gamma$  be a random variable with Gamma distribution with parameters  $(1/2, 1)$  which is independent of  $\beta$ . Then, for every  $i, j \in V$ , let us define*

$$G_{\beta, \gamma}(i, j) := \hat{G}_\beta(i, j) + \frac{1}{2\gamma} \psi_\beta(i) \psi_\beta(j).$$

*Let  $i_0 \in V$  and let  $(\hat{Z}_t)_{t \geq 0}$  be the time-changed VRJP on  $V$  starting from  $i_0$  with weights  $(W_{i, j})_{i, j \in V}$ . Then,  $\hat{Z}$  has the same distribution as the mixture of Markov processes which starts from  $i_0$  and jumps from  $i$  to one of its neighbour  $j$  at rate*

$$\frac{1}{2} W_{i, j} \frac{G_{\beta, \gamma}(i_0, j)}{G_{\beta, \gamma}(i_0, i)}.$$

**Remark 1.9.** *Thanks to Theorem 1.24, we know a representation of the VRJP on an infinite graph as a mixture of Markov processes. One may wonder whether there is a unique possible distribution for this environment or not. This question has been studied by Gerard in [77]. He proved there is uniqueness of the environment when the VRJP is recurrent. Moreover, he proved the same uniqueness property when the VRJP walks on the lattice  $\mathbb{Z}^d$  with  $d \geq 1$  with small reinforcement, that is, for large  $W$ . Moreover, when the VRJP is transient on trees, Gerard proved there is an infinite number of distinct representations of the VRJP.*

Furthermore, the martingale  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  has been used by Sabot and Zeng in [154] in order to prove the diffusive behaviour of the VRJP (and of the ERRW too) when  $W$  is large and  $d \geq 3$ .

**Theorem 1.25** (Theorem 3 in [154]). *Let  $W > 0$ . Let  $d \geq 3$ . Let  $(Y_n)_{n \in \mathbb{N}}$  be the discrete-time process which is associated with the VRJP on  $\mathbb{Z}^d$  with constant weights  $W > 0$ . For every  $t \geq 0$ , we define*

$$Y_t^{(n)} = \frac{Y_{[tn]}}{\sqrt{n}}.$$

*Then, there exists  $W_b(d) > 0$  such that if  $W > W_b(d)$ , then for every  $T > 0$  the following convergence does hold for the Skorokhod topology :*

$$\left( Y_t^{(n)} \right)_{t \in [0, T]} \xrightarrow[n \rightarrow +\infty]{law} \left( \alpha_t^{(d, W)} \right)_{t \in [0, T]}$$

*where  $\alpha^{(d, W)}$  is a Brownian motion on  $\mathbb{R}^d$  with covariance matrix  $\sigma(W, d)^2 Id$  for some  $\sigma(W, d) \in (0, +\infty)$ .*

**Remark 1.10.** *The proof of Theorem 1.25 requires that  $\psi_\beta(i)$  and  $1/\psi_\beta(i)$  are in  $L^4$  for every  $i \in \mathbb{Z}^d$ . It is true for large  $W$  thanks to the results of [54]. However, we do not know whether the rescaled VRJP converges toward a Brownian motion as soon as the VRJP is transient or not. That is why, the study of the  $L^p$ -boundedness of the martingale  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  will be an important topic in this thesis.*

Besides, the martingale  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  gives a very simple criterion for recurrence or transience.

**Proposition 1.26** (Theorem 1 in [154]). *Let  $(\hat{Z}_t)_{t \geq 0}$  be the time-changed VRJP on  $V$  starting from  $i_0$  with non-negative symmetric weights  $(W_{i,j})_{i,j \in V}$ . Then, it holds that*

- (i)  $\hat{Z}$  is a mixture of transient Markov processes if and only if  $\psi_\beta(i) > 0$  for every  $i \in V$ .
- (ii)  $\hat{Z}$  is a mixture of recurrent Markov processes if and only if  $\psi_\beta(i) = 0$  for every  $i \in V$ .

By means of Proposition 1.26, we get a convenient tool in order to prove the recurrence of the VRJP. Indeed if we have an estimate of the type

$$E \left[ \left( \psi_\beta^{(n)}(i) \right)^s \right] \leq \varepsilon_n \tag{1.6}$$

for all  $i \in V$  where  $s < 1$  and  $\varepsilon_n \xrightarrow[n \rightarrow +\infty]{} 0$ , then it implies that the VRJP is a mixture of recurrent Markov processes. However, in [128], in the case of the ERRW, Merkl and Rolles proved an estimate with polynomial decay of the fractional moments of the environment on  $\mathbb{Z}^2$  which can be related to (1.6). Moreover, such an estimate with polynomial decay for the VRJP was proved simultaneously in [149] and [102]. Together with Proposition 1.26, this implies the following result which answers an ancient question concerning the ERRW and the VRJP :

**Theorem 1.27.** *Let  $a > 0$ . Let  $W > 0$ . Let  $(X_n)_{n \in \mathbb{N}}$  be the ERRW on  $\mathbb{Z}^2$  with constant weights  $a$ . Let  $(\hat{Z}_t)_{t \geq 0}$  be the time-changed VRJP with constant weights  $W > 0$ . Then  $(X_n)_{n \in \mathbb{N}}$  is a mixture of recurrent Markov chains and  $(\hat{Z}_t)_{t \geq 0}$  is a mixture of recurrent Markov processes.*

Another important question concerning the VRJP and the ERRW was to know whether there is a unique phase transition between recurrence and transience on  $\mathbb{Z}^d$  when  $d \geq 3$ . Usually, for example in the case of Bernoulli bond percolation or the Ising model (see [82] and [67]), such a result is proved thanks to a monotonicity property. For example, let us assume that we can prove that for some  $s \in ]0, 1[$  and for any  $n \in \mathbb{N}$

$$W \mapsto \mathbb{E}_W \left[ \left( \psi_\beta^{(n)}(0) \right)^s \right] \tag{1.7}$$

is non-decreasing where  $\mathbb{E}_W$  means that the VRJP starts with constant weights  $W$ . If such a monotonicity property is true, together with Proposition 1.26, it would prove that there is a unique transition between recurrence and transience for the VRJP with constant weights. The basic idea in order to prove that (1.7) is non-decreasing would be to show that for every  $n \in \mathbb{N}$ , the random variable  $\psi_\beta^{(n)}(0)$  is stochastically non-decreasing as a function of  $W$ . However, it is not difficult to check that it is not true on some simple examples. Therefore, it was necessary to be more subtle. By means of a clever coupling, Poudevigne proved a different property of convex monotonicity which implies that (1.7) is non-decreasing.

**Theorem 1.28** (Theorem 6 in [139]). *Let  $(V, E)$  be a finite graph. Let  $(W_{i,j}^-)_{i,j \in V}$  and  $(W_{i,j}^+)_{i,j \in V}$  be two sets of symmetric non-negative weights such that for every  $i, j \in V$ ,*

$$W_{i,j}^- \leq W_{i,j}^+.$$

*Let  $\varphi$  be a concave function. Let  $i, j \in V$ . Then, it holds that*

$$\int \varphi \left( \frac{G_\beta(i, j)}{G_\beta(i, i)} \right) \nu_V^{W^-} (d\beta) \leq \int \varphi \left( \frac{G_\beta(i, j)}{G_\beta(i, i)} \right) \nu_V^{W^+} (d\beta).$$

Moreover, it is possible to interpret  $\psi_\beta^{(n)}(0)$  as a random variable of the form  $\frac{G_\beta(i, j)}{G_\beta(i, i)}$ . Together with Theorem 1.28, it implies that (1.7) is non-decreasing as a function of  $W$ . Consequently, thanks to Theorem 1.28, Poudevigne proved there is a unique phase transition for the VRJP (and actually also for the ERRW).

**Theorem 1.29** (Theorem 2 in [139]). *Let  $d \geq 3$ . There exists  $W_c(d) \in (0, +\infty)$  such that the VRJP on  $\mathbb{Z}^d$  with initial constant weights  $W$  is recurrent if  $W < W_c(d)$  and transient if  $W > W_c(d)$ .*

**Theorem 1.30** (Theorem 3 in [139]). *Let  $d \geq 3$ . There exists  $a_c(d) \in (0, +\infty)$  such that the ERRW on  $\mathbb{Z}^d$  with initial constant weights  $a$  is recurrent if  $a < a_c(d)$  and transient if  $a > a_c(d)$ .*

Finally, thanks to the field  $\beta$  on infinite graphs, some of the most important questions regarding the VRJP and the ERRW were solved. During the developpement of the theory about the VRJP, the infinite volume self-adjoint operator  $H_\beta$  appeared to be a very important object. That is why we would like to understand how the properties of  $H_\beta$  as an operator, especially its spectral properties, are related to the asymptotic behaviour of the VRJP. A part of this thesis is devoted to this kind of questions. Actually,  $H_\beta$  belongs to a vast category of objects which are called random Schrödinger operators. Now, let us look at the theory of random Schrödinger operators.

## 1.3 Random Schrödinger operators

### 1.3.1 Background on self-adjoint operators

In this section, all operators are defined on a separable complex Hilbert space  $\mathcal{H}$  with some scalar product  $(\cdot, \cdot)$ . For example, if  $(V, E)$  is an infinite locally finite graph, we will consider

$$\mathcal{H} = l^2(V) := \left\{ \phi \in \mathbb{C}^V, \sum_{x \in V} |\phi(x)|^2 < +\infty \right\}$$

with scalar product

$$(\phi, \chi) := \sum_{x \in V} \overline{\phi(x)} \chi(x)$$

for every  $\phi, \chi \in l^2(V)$ . We denote by  $\|\cdot\|$  the euclidean norm which is associated with  $(\cdot, \cdot)$ . A linear operator  $A$  on  $\mathcal{H}$  is a linear application from some subset  $\mathcal{D}(A)$  of  $\mathcal{H}$  into  $\mathcal{H}$  such that  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ . Moreover,  $\mathcal{D}(A)$  is called the domain of  $A$ . It is very common that  $\mathcal{D}(A) \subsetneq \mathcal{H}$ .

Indeed it allows to manage with unbounded operators. Now, let us consider a linear operator  $(A, \mathcal{D}(A))$  on  $\mathcal{H}$ . Then, we define the domain of the adjoint of  $A$  by

$$\mathcal{D}(A^*) := \{\phi \in \mathcal{H}, \exists \eta \in \mathcal{H}, \forall \chi \in \mathcal{D}(A) : (\phi, A\chi) = (\eta, \chi)\}.$$

Then  $(A, \mathcal{D}(A))$  is self-adjoint if and only if

(i)  $A$  is symmetric, that is, for every  $\phi, \chi \in \mathcal{D}(A)$ ,

$$(\phi, A\chi) = (A\phi, \chi).$$

(ii)  $\mathcal{D}(A^*) = \mathcal{D}(A)$ .

One crucial example of self-adjoint operator on  $l^2(V)$  is the discrete laplacian  $\Delta$  where  $V$  has bounded degree.

$$\begin{aligned} \Delta &: l^2(V) &\rightarrow & l^2(V) \\ &\phi &\mapsto & \Delta\phi \end{aligned}$$

where for every  $x \in V$ ,

$$(\Delta\phi)(x) := \sum_{\{x,y\} \in E} (\phi(y) - \phi(x)).$$

At present, we consider a self-adjoint operator  $(A, \mathcal{D}(A))$  on  $\mathcal{H}$ . The spectrum of  $A$  is defined as

$$\sigma(A) := \{z \in \mathbb{C}, (A - z) : \mathcal{D}(A) \rightarrow \mathcal{H} \text{ is not bijective.}\}.$$

As  $A$  is self-adjoint,  $\sigma(A)$  is included in  $\mathbb{R}$ . Moreover, it is always a closed subset of  $\mathbb{R}$ . Now, let us introduce the spectral measure which is a crucial object in the study of self-adjoint operators.

**Proposition 1.31** (Section A.2 in [6]). *Let  $\phi \in \mathcal{H}$ . Then there exists a unique real-valued measure  $\mu_\phi$  whose support is included in  $\sigma(A)$  such that for every  $z \in \mathbb{C}$  with  $\text{Im}(z) > 0$ ,*

$$(\phi, (A - z)^{-1}\phi) = \int_{\sigma(A)} \frac{d\mu_\phi(E)}{E - z}.$$

Moreover,  $\mu_\phi$  is a finite measure whose total mass is  $\|\phi\|^2$  and  $\mu_\phi$  is called the spectral measure of  $A$  associated with the vector  $\phi$ . If  $\phi, \chi \in \mathcal{H}$ , then by polarization, there exists a complex measure  $\mu_{\phi,\chi}$  such that for every  $z \in \mathbb{C}$  with  $\text{Im}(z) > 0$ ,

$$(\phi, (A - z)^{-1}\chi) = \int_{\sigma(A)} \frac{d\mu_{\phi,\chi}(E)}{E - z}.$$

The construction of the spectral measure is explained roughly in [6] but one can also look at chapter VII in [143]. Thanks to the spectral measure, one can apply functional calculus to  $A$ . Let  $F$  be a bounded measurable function. Then we can define the linear self-adjoint operator  $F(A)$  such that for every  $\phi, \chi \in \mathcal{H}$ ,

$$(\phi, F(A)\chi) = \int_{\sigma(A)} F(E) d\mu_{\phi,\chi}(E).$$

Moreover, one can also use the spectral measure in order to classify the elements of  $\mathcal{H}$ . It is well known that every measure  $\mu$  can be decomposed as

$$\mu = \mu^{ac} + \mu^{sc} + \mu^{pp}$$

where  $\mu^{ac}$  is absolutely continuous,  $\mu^{sc}$  is singular and  $\mu^{pp}$  is a pure-point measure. Now, we can apply this decomposition to the spectral measure. We define

$$\mathcal{H}^{ac} := \{\phi \in \mathcal{H}, \mu_\phi = \mu_\phi^{ac}\}, \mathcal{H}^{sc} := \{\phi \in \mathcal{H}, \mu_\phi = \mu_\phi^{sc}\} \text{ and } \mathcal{H}^{pp} := \{\phi \in \mathcal{H}, \mu_\phi = \mu_\phi^{pp}\}.$$

Then, these subspaces of  $\mathcal{H}$  are closed vector spaces which give an orthogonal decomposition of  $\mathcal{H}$ .

**Proposition 1.32** (Theorem VII.4 in [143]). *We have the following orthogonal decomposition :*

$$\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}.$$

By means of this classification of vectors, one can also divide the spectrum into three parts.

$$\sigma^{ac}(A) = \sigma(A|_{\mathcal{H}_{ac}}), \sigma^{sc}(A) = \sigma(A|_{\mathcal{H}_{sc}}) \text{ and } \sigma^{pp}(A) = \sigma(A|_{\mathcal{H}_{pp}}).$$

**Remark 1.11.** *There is no reason in general for  $\sigma^{ac}(A)$ ,  $\sigma^{sc}(A)$  and  $\sigma^{pp}(A)$  to be disjoint. However it is true (or conjectured to be true) in all the cases which will be studied in this thesis.*

This classification of vectors and of the spectrum has physical meaning through the notions of localization and delocalization. The connection between the theory of operators and quantum physics is made rigorous in the next subsection thanks to the RAGE Theorem.

### 1.3.2 Physical interpretation

This subsection explains briefly how we can interpret self adjoint operators from the point of view of quantum physics. For more details, one can consult [6] and [84]. Let us fix an infinite locally finite graph with bounded degree  $(V, E)$  and let  $(\zeta(x))_{x \in V} \in \mathbb{R}^V$  be a real-valued function on  $V$  which is called the potential. We can imagine the potential  $(\zeta(x))_{x \in V} \in \mathbb{R}^V$  as impurities in a crystal structure which is represented by the graph  $V$ . For sake of convenience, we will often write  $\zeta$  to denote the diagonal operator such that for every  $\phi \in l^2(V)$  for all  $x \in V$ ,

$$(\zeta\phi)(x) = \zeta(x)\phi(x).$$

Remark that it is possible that  $\zeta\phi$  is not in  $l^2(V)$  anymore. Then, a Schrödinger operator on  $l^2(V)$  is an operator of the form

$$A_\zeta = -\Delta + \zeta.$$

It is not difficult to prove that it is self-adjoint with domain

$$\mathcal{D}(A_\zeta) := \left\{ \phi \in l^2(V), \sum_{x \in V} \zeta(x)^2 |\phi(x)|^2 < +\infty \right\}.$$

Let  $\phi \in V$  be such that

$$\sum_{x \in V} |\phi(x)|^2 = 1.$$

Let us imagine a quantum particule which walks on the graph  $V$  with the potential  $\zeta$ . As it is a quantum particule, we do not know exactly where it is located. Let us say that at time 0, the particle is at site  $x$  with probability  $|\phi(x)|^2$ . One may wonder what is the probability that this particle is at  $x \in V$  at time  $t \in \mathbb{R}_+$ . Actually, this probability equals  $|\phi(t, x)|^2$  where  $\phi(\cdot, \cdot)$  is solution of the famous Schrödinger equation :

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, x) = -iA_\zeta \phi(t, x) \\ \phi(0, \cdot) = \phi. \end{cases}$$

Thanks to functional calculus, we get a simple representation of  $(\phi(t, x))_{t \geq 0, x \in V}$ . Indeed, for every  $t \geq 0$ ,

$$\phi(t, \cdot) = e^{-itA_\zeta} \phi.$$

Remark that  $e^{-itA_\zeta}$  is a unitary operator. Consequently, for every  $t \geq 0$ ,

$$\sum_{x \in V} |\phi(t, x)|^2 = 1$$

which means that  $|\phi(t, \cdot)|^2$  remains a probability measure for every  $t > 0$ . Now, let us consider a self adjoint operator  $\mathcal{R}$  on  $l^2(V)$ . A fundamental point of quantum physics is that  $\mathcal{R}$  can be interpreted as an observation of the particle, for example its position or its impulsions. Let  $I$  be an interval of  $\mathbb{R}$ . Let  $P_I$  be the function which is 1 on  $I$  and 0 elsewhere. Then the probability  $\mathbb{P}_{\phi,t}(\mathcal{R} \in I)$  that at time  $t$ ,  $\mathcal{R}$  lies in  $I$  can be computed by

$$\mathbb{P}_{\phi,t}(\mathcal{R} \in I) = (\phi(t, \cdot), P_I(\mathcal{R})\phi(t, \cdot)) = \|P_I(\mathcal{R})e^{-itA_\zeta}\phi\|^2. \quad (1.8)$$

Then, it is possible to get a physical interpretation of  $\mathcal{H}^{ac}$ ,  $\mathcal{H}^{sc}$  and  $\mathcal{H}^{pp}$  thanks to the following result which is due to Ruelle, Amrein, Georgescu and Enss (RAGE).

**Theorem 1.33** ((RAGE) Theorem 2.6 in [6]). *Let  $\mathcal{H}$  be the Hilbert space  $l^2(V)$  and let  $A_\zeta$  be a Schrödinger operator on  $l^2(V)$ . Let  $(\mathcal{R}_L)_{L \in \mathbb{N}}$  be a sequence of compact self-adjoint operators which converges towards the identity. Then, it holds that*

(i)

$$\mathcal{H}^{ac} \oplus \mathcal{H}^{sc} = \left\{ \phi \in \mathcal{H}, \lim_{L \rightarrow +\infty} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \|\mathcal{R}_L e^{-itA_\zeta} \phi\|^2 dt = 0 \right\},$$

(ii)

$$\mathcal{H}^{pp} = \left\{ \phi \in \mathcal{H}, \lim_{L \rightarrow +\infty} \sup_{t \in \mathbb{R}} \|(Id - \mathcal{R}_L)e^{-itA_\zeta} \phi\| = 0 \right\}.$$

By the interpretation given by (1.8), (i) in Theorem 1.33 tells us that if the particle starts from a state  $\phi \in \mathcal{H}^{ac} \oplus \mathcal{H}^{sc}$ , then the particle spends a fraction of time in finite subsets of  $V$  which is asymptotically zero. Therefore, continuity of the spectrum corresponds to quantum delocalization. On the contrary, (ii) in Theorem 1.33 tells us that if the particle starts from a state  $\phi \in \mathcal{H}^{pp}$ , then the particle will stay in finite subsets of  $V$  with high probability. Thus, pure-point spectrum corresponds to quantum localization. Therefore, a fundamental question about self-adjoint operators is to know whether their spectrum is continuous or pure-point. For example, the opposite of the Laplacian  $-\Delta$  on  $\mathbb{Z}^d$  has only absolutely continuous spectrum and  $\sigma(-\Delta) = [0, 4d]$ . On the contrary, the diagonal operator  $\zeta$  has only pure point spectrum which is

$$\overline{\{\zeta(x), x \in V\}}.$$

### 1.3.3 An important example of random Schrödinger operator : the Anderson model

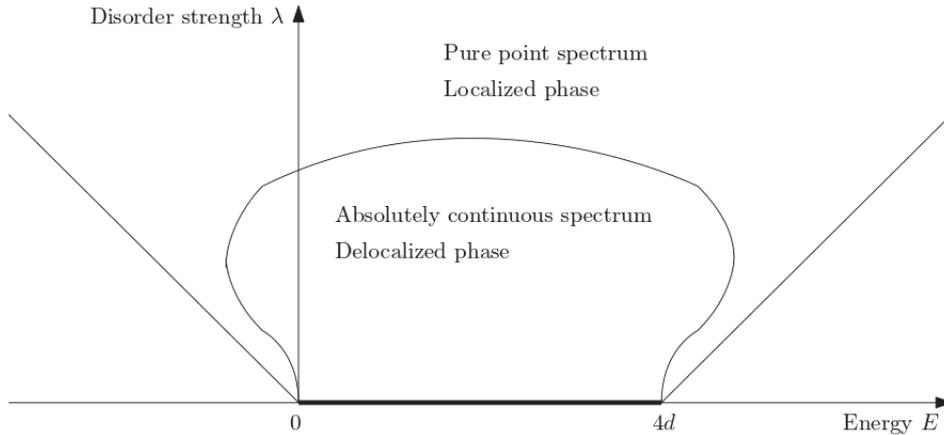
Let  $d \in \mathbb{N}^*$  and  $\lambda > 0$ . In his seminal paper [7], Anderson introduced the following random Schrödinger operator :

$$A_{\lambda\zeta} = -\Delta + \lambda\zeta$$

where

$$\left\{ \zeta(x), x \in \mathbb{Z}^d \right\}$$

is a random i.i.d potential on  $\mathbb{Z}^d$ . This model is known as the Anderson model. The Laplacian term  $-\Delta$  has absolutely continuous spectrum whereas  $\zeta$  is a very disordered operator which has pure point spectrum. At first sight, it is not obvious to know which part of the operator, the continuous one or the pure point one, will prevail in the spectrum of  $A_\zeta$ . By the RAGE Theorem, this question is directly linked to the following one : if an electron walks in a metal with many impurities, is this electron able to move far away from its original position, that is, is the impure metal conductive or not ? In [7], Anderson explained in a non rigorous way that the metal should not be conductive when  $\lambda$  is large. Moreover, it is reasonable to expect that the nature of the spectrum of  $A_{\lambda\zeta}$  depends on the disorder strength  $\lambda$  and the dimension  $d$ . In [1], it was conjectured that there is only pure point spectrum in dimension 1 and 2 for any  $\lambda > 0$  and that a phase transition should occur in dimension  $d \geq 3$ .



**FIGURE 1.1** – Conjectured phase diagram of the Anderson model with uniform potential on  $[-\frac{1}{2}, \frac{1}{2}]$ .

Let us assume that for every  $x \in \mathbb{Z}^d$ , the distribution of the random variable  $\zeta_x$  has a support  $\mathcal{S}$ . Then by Corollary III.3 in [104], the spectrum of  $A_{\lambda\zeta}$  is almost surely

$$[0, 4d] + \lambda\mathcal{S}.$$

Is this spectrum only pure point, only absolutely continuous or a mix of both types? Today, when  $d \geq 3$ , the phase diagram of the Anderson model on  $\mathbb{Z}^d$  is conjectured to look like Figure 1.1.

When  $d \geq 3$ , it is expected that for small  $\lambda$ , that is for small disorder, we should see delocalization (absolutely continuous spectrum) for small energies. This first part of the conjecture remains unproved nowadays. On the contrary at extreme energies or high disorder, the Anderson model is localized (only pure point spectrum). This second part of the conjecture has been proved rigorously by mathematicians but it took a few decades.

The first thing about the Anderson model which has been proved by mathematicians is the localization in dimension 1 for any  $\lambda > 0$ . It has been proved rigorously in [88] and [80]. In [70], by a multiscale analysis, Fröhlich and Spencer proved that for any  $d \in \mathbb{N}^*$ , if  $\lambda$  is large, then the Green function  $(A_{\lambda\zeta} - z)^{-1}(x, y)$  decreases exponentially fast for every  $z \in \mathbb{C} \setminus \mathbb{R}$  with respect to the distance  $d(x, y)$  between the vertices  $x$  and  $y$ . More precisely, they obtained the following result :

**Theorem 1.34** (Theorem 1.2 in [70]). *Let  $d \in \mathbb{N}^*$ . Let  $\zeta$  be an i.i.d potential on  $\mathbb{Z}^d$  such that for every  $x \in \mathbb{Z}^d$ ,  $\zeta_x$  has a bounded density. Then, there exist  $\lambda_0 > 0$  and  $C > 0$  such that for every  $\lambda > \lambda_0$ , for every  $E \in \mathbb{R}$ , for every  $\varepsilon > 0$ , for every  $N > 0$  and for every  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{P} \left( |(A_{\lambda\zeta} - E - i\varepsilon)^{-1}(0, x)| \leq e^{N - \|x\|} \right) \geq 1 - \frac{C}{N}.$$

This was the first result of localization at large disorder which was proved rigorously for any dimension. In [118], Martinelli and Scoppola showed that the result of [70] implies that  $A_{\lambda\zeta}$  has only pure-point spectrum when  $\lambda$  is large for every  $d \in \mathbb{N}^*$ . In [5], Aizenman and Molchanov proved also the exponential decrease of the Green function and the pure-point nature of the spectrum for large disorder or extreme energies. Actually their proof uses a method called "the method of fractional moments" which is different of the method of Fröhlich and Spencer.

Nowadays, we still do not know how to show delocalization on  $\mathbb{Z}^d$ . Such a result seems to be very hard. Consequently, the Anderson model on the Cayley tree (sometimes called the Bethe lattice) sparked off a huge interest because it can be viewed a toy model of the Anderson model on  $\mathbb{Z}^d$ . The localization on the Cayley tree is known in a very detailed way when  $\lambda$  is large. (See for example Chapter 16 in [6].) Moreover, in the particular case of the Cayley tree, delocalization has been proved for small values of  $\lambda$  by Klein in [99]. However, it was not possible to adapt

his proof on  $\mathbb{Z}^d$  with  $d \geq 3$ . Localization and delocalization properties have also been shown for random Schrödinger operators which are not the Anderson operator  $A_{\lambda\zeta}$ . For example, one can look at [98], [79] and [29]. Remark also that we presented the Anderson model in a discrete setting. This discrete model is known as the Anderson tight-binding model. However, it is also possible to consider the continuous Anderson operator on  $\mathbb{R}^d$  :

$$-\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + dB$$

where  $dB$  is a white noise. When  $d = 1$ , this model has been studied first by Halperin in [85]. Fukushima and Nakao gave a rigorous formulation of the related eigenvalue problem in [71]. They found an explicit form for the cumulative distribution function of spectral values. More recently, in [58], Dumaz and Labbé proved very precise results concerning the shape of the localized eigenvectors of the continuous Anderson model in dimension 1. Besides, in [59], considering the continuous Anderson model  $\mathcal{H}_L$  on  $(0, L)$  with Dirichlet boundary condition, they proved the existence of a delocalized phase of  $\mathcal{H}_L$  for eigenvalues corresponding to energies  $E \propto L$  and  $E \gg L$ . Moreover, in [105], Labbé proved it is possible to construct the continuous Anderson model rigorously on  $(-L, L)^d$  for  $d \in \{1, 2, 3\}$ .

### 1.3.4 The density of states

Let  $(\zeta_x)_{x \in \mathbb{Z}^d}$  be a random potential on  $\mathbb{Z}^d$ . For every  $x \in \mathbb{Z}^d$ , we write  $\tau_x$  the translation  $y \mapsto y + x$  on  $\mathbb{Z}^d$ . We say that  $\zeta$  is stationary when for every  $A \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Z}^d}$

$$\mathbb{P}(\zeta \in A) = \mathbb{P}(\zeta \in \tau_x(A)).$$

Furthermore, we assume that  $\zeta$  is ergodic when for every  $A \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Z}^d}$  such that for every  $x \in \mathbb{Z}^d$ ,  $\tau_x(A) = A$ , it holds that

$$\mathbb{P}(\zeta \in A) \in \{0, 1\}.$$

Of course, i.i.d potentials are ergodic. In the case of ergodic stationary potentials, one can define a very important object which is known as the Density of States (DOS). For every  $x \in \mathbb{Z}^d$ , we define  $\delta_x$  which is the vector of  $l^2(\mathbb{Z}^d)$  which is 1 at  $x$  and 0 elsewhere. If  $\Lambda$  is a finite set,  $|\Lambda|$  denotes its cardinality.

**Proposition 1.35** (Theorem 3.14 in [6]). *Let  $A_\zeta = -\Delta + \zeta$  be a random Schrödinger operator such that  $\zeta$  is an ergodic stationary potential on  $\mathbb{Z}^d$  with  $d \in \mathbb{N}^*$ . Let  $(\Lambda_L)_{L \in \mathbb{N}}$  be an increasing sequence of boxes of  $\mathbb{Z}^d$  such that  $\bigcup_{L \in \mathbb{N}} \Lambda_L = \mathbb{Z}^d$ . Then, there exists a measure  $\nu$  on  $\mathbb{R}$  such that for every continuous function  $f$  it holds that*

$$\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} (\delta_x, f(A_\zeta) \delta_x) \xrightarrow[L \rightarrow +\infty]{a.s.} \int_{-\infty}^{\infty} f(E) d\nu(E).$$

*The measure  $\nu$  is called the density of states of  $A_\zeta$  and it does not depend on the sequence of boxes  $(\Lambda_L)_{L \in \mathbb{N}}$ .*

The density of states  $\nu$  can be interpreted as the density of spectral values in the spectrum of  $A_\zeta$ . For sake of convenience, we often prefer to use the integrated density of states  $n$  which is defined by  $n(E) = \nu((-\infty, E))$ . A natural problem consists in finding the properties of  $n$ . A very general result concerning the integrated density of states was found by Wegner in [163].

**Proposition 1.36** (Corollary 5.25 in [97]). *Let  $(\zeta_x)_{x \in \mathbb{Z}^d}$  be a random potential on  $\mathbb{Z}^d$  which is ergodic and stationary. Let us assume that for every  $x \in \mathbb{Z}^d$ ,  $\zeta_x$  has density  $g$  and that  $g$  is bounded. Let  $n$  be the integrated density of states which is associated with  $A_\zeta$ . Then, there exists  $C > 0$  such that for every  $E \in \mathbb{R}$  and every  $\varepsilon > 0$ ,*

$$n(E + \varepsilon) - n(E - \varepsilon) \leq C \|g\|_{\infty} \varepsilon.$$

Therefore, under mild conditions,  $n$  must be a Lipschitz function. Actually, in [112], Lifschitz remarked that at the edges of the spectrum, the density of states decreases exponentially fast when the potential is i.i.d.

**Theorem 1.37** (Theorem 6.1 in [97]). *Let  $\zeta$  be an i.i.d potential on  $\mathbb{Z}^d$ . We assume that the infimum of the support of  $\zeta_x$  is 0 for every  $x \in \mathbb{Z}^d$ . In particular the infimum of the spectrum of  $A_\zeta$  is 0. Let  $n$  be the integrated density of states of  $A_\zeta$ . Then*

$$\frac{\ln(|\ln(n(E))|)}{\ln(E)} \xrightarrow{E \rightarrow 0} -\frac{d}{2}.$$

Therefore, if  $\zeta$  is i.i.d, then at the bottom of the spectrum, let us say 0, the integrated density of states behaves roughly as  $e^{-E^{-d/2+o(1)}}$ . The idea of the proof of Theorem 1.37 is that creating an eigenvalue at the bottom of the spectrum of  $A_\zeta$  requires a huge number of the random variables  $(\zeta(x))_{x \in \mathbb{Z}^d}$  to be small. As the random variables  $(\zeta(x))_{x \in \mathbb{Z}^d}$  are independent, this event has an exponentially small probability.

Now, let us look at the connection between these notions regarding random Schrödinger operators and the VRJP.

### 1.3.5 Link with the VRJP

Let  $d \in \mathbb{N}^*$ . Let  $W > 0$ . Recall that by Proposition 1.22, there exists a potential  $(\beta_x)_{x \in \mathbb{Z}^d}$  on  $\mathbb{Z}^d$  with distribution  $\nu_V^W$  where  $V = \mathbb{Z}^d$  which is linked to the VRJP on  $\mathbb{Z}^d$  with constant weights  $W$ . Remark that

$$\frac{H_\beta}{W} = -\Delta + \left( \frac{2\beta}{W} - 2d \right).$$

Consequently,  $H_\beta/W$  is a random Schrödinger operator with an ergodic potential  $\frac{2\beta}{W} - 2d$ . Actually, it looks like the Anderson model where  $1/W$  would correspond to the disorder strength  $\lambda$  in the Anderson model. Nevertheless, the potential  $(\beta_x)_{x \in \mathbb{Z}^d}$  is not i.i.d. On the contrary, the correlations of  $(\beta_x)_{x \in \mathbb{Z}^d}$  make  $H_\beta$  non-negative. By Theorem 1.24, we know that  $H_\beta$  is strongly related to the VRJP. On  $\mathbb{Z}$  and  $\mathbb{Z}^2$ , by Theorem 1.27, the VRJP is recurrent almost surely for any value of  $W$ . This can be interpreted as a kind of localization property. On the contrary, on  $\mathbb{Z}^d$  with  $d \geq 3$ , the VRJP exhibits a unique phase transition between recurrence and transience according to Theorem 1.29. It can be interpreted as a kind of phase transition between localization and delocalization. Besides, in the transient phase,  $\psi$  can be viewed as a delocalized pseudo-eigenstate of  $H_\beta$  for the spectral value 0. Consequently, as an operator, it is reasonable to expect that  $H_\beta$  exhibits the same kind of phase transition as the Anderson model. The properties of  $H_\beta$  as a random Schrödinger operator were first studied in [154].

**Proposition 1.38** (Theorem 2 in [154]). *Let  $V = \mathbb{Z}^d$  with  $d \in \mathbb{N}^*$  and let  $W > 0$ . Then, under  $\nu_V^W$ ,*

- (i) *The spectrum of  $H_\beta$  is included in  $\mathbb{R}_+$ .*
- (ii) *For every  $i, j \in \mathbb{Z}^d$ , almost surely,*

$$\hat{G}_\beta(i, j) = \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} (H_\beta + \varepsilon)^{-1}(i, j).$$

- (iii)  *$H_\beta \psi = 0$  and  $\psi$  has at most polynomial growth in the sense that for every  $p > d$ , almost surely, there exists a random  $C > 0$  such that for every  $x \in \mathbb{Z}^d$ ,*

$$|\psi_\beta(x)| \leq C \|x\|^p.$$

Actually, we will see in this thesis that the spectrum of  $H_\beta$  is exactly  $\mathbb{R}_+$ . The question of the localization of  $H_\beta$  has been partially studied by Zeng and Collevocchio in [43]. They proved the following result :

**Theorem 1.39** (Theorem 2.4 in [43]). *Let  $V = \mathbb{Z}^d$  with  $d \in \mathbb{N}^*$  and let  $W > 0$ . Then, there exists  $W_1(d) > 0$  such that for every  $W \in (0, W_1(d))$ , under  $\nu_V^W$ ,  $H_\beta$  is localized in the sense that it has almost surely a complete set of orthonormal eigenfunctions which decay exponentially.*

**Remark 1.12.** *Similarly as in Remark 1.4, in this section, we made a small abuse of notation. Indeed, when  $W$  is a positive number, we use the notation  $\nu_V^W$  to denote the probability measure of the potential  $\beta$  associated with the VRJP on some graph  $(V, E)$  where all weights are assumed to be equal to  $W$ .*

The proof of Zeng and Collevocchio relies on the properties of the potential  $\beta$  and the result of Aizenman and Molchanov in [5]. One goal of this thesis is to go further in the understanding of  $H_\beta$  as an operator. In particular, we will focus on its integrated density of states which has many surprising properties.

## 1.4 Branching random walks

In this thesis, branching random walks arise simultaneously as a tool and as an object of study. In this section, we describe the most important results regarding branching random walks which are important in order to understand Chapter 2 and Chapter 3.

### 1.4.1 One-dimensional branching random walks

First, let us define one-dimensional branching random walks. Let  $\chi$  be a point process on  $\mathbb{R}$  with a finite number of points. More precisely  $\chi = \{\varrho_i, 1 \leq i \leq N\}$  where  $N$  is an integer-valued random variable and for every  $i \in \llbracket 1, N \rrbracket$ ,  $\varrho_i \in \mathbb{R}$ . Let  $o$  be a first individual called the root and let  $S(o) = 0$  be the initial position of  $o$ .  $(o, S(o)) = (o, 0)$  constitutes the generation 0 of the branching random walk. The generation 1 of the branching random walk is given by  $\chi$ :  $o$  has  $N$  children with positions  $\{\varrho_1, \dots, \varrho_N\}$ . Then, next generations are defined as follows: Let  $n \in \mathbb{N}$ . We associate a new point process  $\chi_u := \{\varrho_i^u, 1 \leq i \leq N_u\}$  to every individual  $u$  at generation  $n$ . If the position of  $u$  is  $S_u$ , then the positions of the children of  $u$  will be given by the point process  $\{\varrho_i^u + S_u, 1 \leq i \leq N_u\}$ . Moreover, we assume that the point processes  $\{\chi_u, u \text{ individual at generation } n\}$  are i.i.d with distribution  $\chi$ . Further, we assume that these point processes are independent from the previous generations. The point process

$$\{\varrho_i^u + S_u, 1 \leq i \leq N_u, u \text{ individual at generation } n\}$$

represents the  $(n + 1)$ -th generation of the process. This construction gives us an underlying Galton-Watson tree  $V$ . Then, the branching random walk can be represented as  $(u, S_u)_{u \in V}$  where  $S_u$  is the position of the individual  $u$ . For every  $u \in V$ , the generation of  $u$  is denoted by  $|u|$ . The structure of rooted tree gives a natural genealogical order  $\leq$  on  $V$ . The parent of  $u \in V$ , for the genealogical order is denoted by  $\tilde{u}$ . A simple example of branching random walk can be constructed as follows:

- Let  $V$  be a Galton-Watson tree with a root  $o$ .
- Let  $\{\xi_v\}_{v \in V \setminus \{o\}}$  be real valued random variables which are i.i.d conditionally on  $V$ .
- For every  $u \in V$ , we define

$$S_u = \sum_{o < v \leq u} \xi_v.$$

In this case the positions of the branching random walks are independent of the underlying Galton-Watson tree but remark that it is not necessarily true in the definition of a branching random walk.

By Proposition 1.11, one can easily check that, when  $V$  is a Galton Watson tree with root  $o$ , the discrete-time process which is associated with the VRJP on  $V$  with constant weights  $W > 0$

is a random walk in random environment whose conductances  $(c(u, \tilde{u}))_{u \in V \setminus \{o\}}$  are given by

$$c(u, \tilde{u}) = A_u \prod_{o < v \leq \tilde{u}} A_v^2$$

for every  $u \in V \setminus \{o\}$  where  $(A_u)_{u \in V \setminus \{o\}}$  are i.i.d Inverse Gaussian random variables with parameters  $(1, W)$ . Therefore the conductances of the VRJP on a tree form a process which is very similar to

$$(\tilde{c}(u, \tilde{u}))_{u \in V \setminus \{o\}} := \left( \exp \left( \sum_{o < v \leq u} 2 \ln(A_v) \right) \right)_{u \in V \setminus \{o\}}$$

which is the exponential of a branching random walk. That is why one-dimensional branching random walks are a very convenient tool in order to study the VRJP on trees.

A natural problem regarding branching random walks consists in looking at their asymptotic behaviour. Let  $(u, S_u)_{u \in V}$  be a branching random walk. For every  $n \in \mathbb{N}$ , we define

$$I_n := \min_{|u|=n} S_u.$$

Of course, the asymptotic behaviour of  $(I_n)_{n \in \mathbb{N}}$  depends on the point process  $\chi$  which was chosen in order to define the branching random walk. Actually, the behaviour of  $(u, S_u)_{u \in V}$  is mostly linked to the function  $F$  which is defined by

$$F(t) = \ln \left( \mathbb{E} \left[ \sum_{|u|=1} e^{-tS_u} \right] \right).$$

Moreover, we will always assume that the underlying Galton Watson tree  $V$  is supercritical. In particular  $F(0) > 0$ . Besides, the survival event of  $V$  is denoted by  $\mathcal{S}$ . As  $V$  is assumed to be supercritical,  $\mathcal{S}$  has positive probability. In [25], Biggins found the asymptotic behaviour of  $(I_n)_{n \in \mathbb{N}}$  on  $\mathcal{S}$ .

**Proposition 1.40** (Theorem 4 in [25]). *Let us assume that  $F(t) < +\infty$  for some  $t > 0$ . For any  $a \in \mathbb{R}$ , let  $\mu(a) := \inf \{ta + F(t), t \geq 0\}$  and  $\gamma := \inf \{a \in \mathbb{R}, \mu(a) \geq 0\}$ . Then, conditionally on  $\mathcal{S}$  it holds that*

$$\frac{I_n}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \gamma.$$

Moreover,

$$\gamma = \inf_{t > 0} \frac{F(t)}{t}.$$

Therefore, when  $\gamma \neq 0$ , we know the first order of the asymptotic behaviour of  $(I_n)_{n \in \mathbb{N}}$ . However, Proposition 1.40 does not explain what is going on when  $\gamma = 0$ . Actually, it is not very difficult to check that  $\gamma = 0$  when

$$F(1) = 0 \quad \text{and} \quad F'(1) = 0. \tag{1.9}$$

Moreover, by section 7 of chapter 3 of [89], it is always possible to reduce to (1.9) under mild assumptions. Therefore, the main issue now consists in finding the asymptotic behaviour of  $(I_n)_{n \in \mathbb{N}}$  assuming (1.9). The main tool in order to answer this question is the martingale

$$(\mathcal{W}_n)_{n \in \mathbb{N}} = \left( \sum_{|u|=n} e^{-S_u} \right)_{n \in \mathbb{N}}.$$

By Biggins's theorem in [24] (see [114] for a simpler proof), it holds that

$$\mathcal{W}_n \xrightarrow[n \rightarrow +\infty]{a.s.} 0 \tag{1.10}$$

when we assume (1.9). One may wonder what is the decay rate of  $\mathcal{W}_n$ . A partial answer was given by Aïdekon and Shi in [2].

**Theorem 1.41** (Theorem 1.1 in [2]). *Let us assume that the underlying Galton-Watson tree is supercritical and that*

$$\sigma^2 := \mathbb{E} \left[ \sum_{|u|=n} S_u^2 e^{-S_u} \right] < +\infty.$$

Moreover we assume (1.9) and

$$\mathbb{E} [X \ln_+^2(X)] < +\infty \text{ and } \mathbb{E} [\tilde{X} \ln_+(\tilde{X})] < +\infty$$

where  $X = \sum_{|u|=1} e^{-S_u}$  and  $\tilde{X} = \sum_{|u|=1} (S_u \wedge 0) \cdot e^{-S_u}$ . Then, conditionally on  $\mathcal{S}$ , it holds that

$$\sqrt{n} \mathcal{W}_n \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \sqrt{\frac{2}{\pi \sigma^2}} D_\infty$$

where  $D_\infty$  is the limit of the derivative martingale  $\left( \sum_{|u|=n} S_u e^{-S_u} \right)_{n \in \mathbb{N}}$  for whom Chen has found a necessary and sufficient condition of convergence in [37].

Remark that for every  $n \in \mathbb{N}$ ,  $e^{-I_n} \leq \mathcal{W}_n$ . Together with Theorem 1.41, this implies that, in probability,

$$\liminf_{n \rightarrow +\infty} \frac{I_n}{\ln(n)} \geq 1/2.$$

Actually, in [86], Hu and Shi proved a much more precise result :

**Theorem 1.42** (Theorem 1.2 in [86]). *Let us assume (1.9) and that the underlying Galton-Watson tree is supercritical. Moreover, we assume that there exists  $\delta_1 > 0$ ,  $\delta_2 > 0$  and  $\delta_3 > 0$  such that*

$$\mathbb{E} \left[ \left( \sum_{|u|=1} 1 \right)^{1+\delta_1} \right] + \mathbb{E} \left[ \sum_{|u|=1} e^{-(1+\delta_2)S_u} \right] + \mathbb{E} \left[ \sum_{|u|=1} e^{\delta_3 S_u} \right] < +\infty.$$

Then, conditionally on  $\mathcal{S}$ , it holds that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{I_n}{\ln(n)} &= \frac{3}{2} \quad a.s. \\ \liminf_{n \rightarrow +\infty} \frac{I_n}{\ln(n)} &= \frac{1}{2} \quad a.s. \\ \lim_{n \rightarrow +\infty} \frac{I_n}{\ln(n)} &= \frac{3}{2} \quad \text{in probability.} \end{aligned}$$

In order to prove this result, Hu and Shi introduced  $(\mathcal{W}_{n,\beta})_{n \in \mathbb{N}}$  such that for every  $n \in \mathbb{N}$  and every  $\beta > 1$ ,

$$\mathcal{W}_{n,\beta} = \sum_{|u|=n} e^{-\beta S_u}.$$

They proved the following estimate regarding  $\mathcal{W}_{n,\beta}$  which will be very useful in Chapter 3 in order to study the VRJP on trees.

**Proposition 1.43** (Theorem 1.6 in [86]). *Make the same assumptions as in Theorem 1.42. Let  $\beta > 1$ . Let  $r \in (0, 1/\beta)$ . Then, it holds that*

$$\mathbb{E} [\mathcal{W}_{n,\beta}^r] = n^{-\frac{3r\beta}{2} + o_n(1)}.$$

Furthermore, in chapter 3 of this thesis, we will use estimates regarding the asymptotic behaviour of another quantity which is similar to  $(I_n)_{n \in \mathbb{N}}$ .

**Theorem 1.44** (Theorem 1.4 in [64]). *Let  $(u, S_u)_{u \in V}$  be a branching random walk satisfying (1.9). For every  $x \in V$ , let us define*

$$\bar{S}(x) = \max_{o < u \leq x} S(u).$$

*In addition, we define*

$$\sigma^2 = \mathbb{E} \left[ \sum_{|u|=1} S(u)^2 e^{-S(u)} \right].$$

*Then, conditionally on  $\mathcal{S}$ , almost surely, it holds that*

$$\lim_{n \rightarrow +\infty} \frac{\min_{|u|=n} \bar{S}(u)}{n^{1/3}} = \left( \frac{3\pi^2 \sigma^2}{2} \right)^{1/3}.$$

Moreover, branching random walks have a continuous-time counterpart which is called the branching Brownian motion. In this thesis, we focus on discrete-time branching random walks but we often generalize theorems which had been proved previously in the continuous-time setting. Let us give a short insight into the main results concerning the branching Brownian motion.

The binary branching Brownian motion is defined in the following way :

- We start with 1 initial particle  $o$  at the origin. It behaves as a Brownian motion until an exponential time (with parameter 1)  $T$  where  $o$  divides itself into two independent Brownian motions which start from the position of  $o$  at time  $T$ .
- Each particle  $u$  behaves as a Brownian motion and divides into two Brownian motions at rate 1.

For every  $t \in \mathbb{R}_+$ , we denote the number of particles at time  $t$  by  $n(t)$  and the set of particles at time  $t$  by  $N(t)$ . For every  $t \geq 0$  and for every  $u \in N(t)$ , we denote the position of  $u$  at time  $t$  by  $S_u(t)$ . Moreover, for every  $t \geq 0$ , we define  $M(t) = \max \{S_u(t), u \in N(t)\}$ .  $(M(t))_{t \geq 0}$  is the continuous-time analogue of  $(I_n)_{n \in \mathbb{N}}$ . Let  $f$  be a positive measurable function. Then, let us define for every  $t \geq 0$ ,

$$h(t, x) = \mathbb{E} \left[ \prod_{u \in N(t)} f(x + S_u(t)) \right].$$

A remarkable fact is that  $h$  is solution of the famous *FKPP* equation which is

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} + h^2 - h$$

with  $h(0, \cdot) = f$ . This equation has been first studied by Fisher in [66] and by Kolmogorov, Petrovsky and Piskunov in [100]. In [125], McKean found the link between the FKPP equation and the branching Brownian motion. Furthermore, with  $f = \mathbf{1}\{\cdot > 0\}$ , for every  $t \geq 0$  and every  $x \in \mathbb{R}$ , remark that

$$h(t, x) = \mathbb{P}(M(t) \leq x).$$

Therefore, one can use the FKPP equation in order to study the distribution of  $M$ . More precisely, McKean showed in [125] the following result : which was generalized by Bramson in [30]

**Proposition 1.45** ((3) in [125]). *It holds that*

$$M(t) - m(t) \xrightarrow[t \rightarrow +\infty]{law} M^*$$

*where  $m(t)$  is the median of  $M(t)$  for every  $t \geq 0$ . Moreover, for every  $x \in \mathbb{R}$ ,  $\mathbb{P}(M^* \leq x) = w(x)$  with  $w$  solution of the ODE*

$$\frac{1}{2} w'' + \sqrt{2} w' + w^2 - w = 0.$$

Further, in [31] and [30], Bramson proved that, as  $t$  goes to infinity

$$m(t) = \sqrt{2t} - \frac{3}{2\sqrt{2}} \ln(t) + O(1).$$

Nowadays, new progresses have been made regarding the branching Brownian motion. For example, one can consult [38], [19], [10] and [4] for an insight into the recent litterature concerning the branching Brownian motion.

### 1.4.2 Critical spatial branching random walks

Spatial branching random walks are defined in the same way as one-dimensional branching random walks, except that the random variables  $\varrho_i$  in the point process  $\chi = \{\varrho_i, 1 \leq i \leq N\}$  can now take values in a set which is not  $\mathbb{R}$  but  $\mathbb{Z}^d$  or  $\mathbb{R}^d$ . Usually, as in the one-dimensional case, one can consider a simpler case of spatial branching random walks which is defined as follows :

- Let  $\mu$  be a probability measure on  $\mathbb{N}$  and let  $\mathcal{P}$  be a probability measure on  $\mathbb{R}^d$ .  $\mu$  is called the offspring law and  $\mathcal{P}$  is the motion law.
- Let  $V$  be a Galton-Watson tree with a root  $o$ .
- Let  $\{\xi_v\}_{v \in V \setminus \{o\}}$  be i.i.d random variables with distribution  $\mathcal{P}$  which are independent of  $V$ .
- For every  $u \in V$ , we define

$$S_u = \sum_{o < v \leq u} \xi_v.$$

Moreover, in this thesis, we will assume that spatial branching random walks are critical. It means that

$$\sum_{k=0}^{+\infty} k\mu(k) = 1.$$

In other words, we assume that the underlying Galton-Watson tree  $V$  is critical. It is well known that it implies that  $V$  is almost surely finite. Moreover, in [101] (see [75] for a modern proof) Kolmogorov proved the following result :

**Lemma 1.46** (Theorem 3.1 in [75]). *Assume that  $\mu$  is a critical measure on  $\mathbb{N}$  and let  $\sigma^2 = \sum_{k=1}^{+\infty} k(k-1)\mu(k) < +\infty$ . Let  $(Z_n)_{n \in \mathbb{N}}$  be the successive generation sizes of a Galton-Watson tree with offspring law  $\mu$ . Then,*

$$\mathbb{P}(Z_n > 0) \underset{n \rightarrow +\infty}{\sim} \frac{2}{\sigma^2 n}.$$

Thanks to Lemma 1.46, in [166], Yaglom found the limit in law of  $Z_n$  conditionally on  $Z_n > 0$ .

**Proposition 1.47** (Theorem 1 in [74]). *Assume that  $\mu$  is a critical measure on  $\mathbb{N}$  and let  $\sigma^2 = \sum_{k=1}^{+\infty} k(k-1)\mu(k) < +\infty$ . Let  $(Z_n)_{n \in \mathbb{N}}$  be the successive generation sizes of a Galton-Watson tree with offspring law  $\mu$ .*

$$\mathcal{L} \left( \frac{Z_n}{n} \middle| Z_n > 0 \right) \underset{n \rightarrow +\infty}{\xrightarrow{\text{law}}} \mathcal{E}$$

where  $\mathcal{E}$  is an exponential distribution which has density

$$\frac{2}{\sigma^2} \exp(-2x/\sigma^2) dx.$$

For every  $d \in \mathbb{N}^*$ , let  $\mathcal{P}_d$  be a distribution which has probability  $1/(2d+1)$  on every element of the form  $(0, \dots, 0, \pm 1, 0, \dots, 0)$  in  $\mathbb{Z}^d$  and probability  $1/(2d+1)$  on  $0 \in \mathbb{Z}^d$ . In dimension  $d \geq 3$ , in [106], Lalley and Zheng proved the following result which generalizes Proposition 1.47 :

**Theorem 1.48** (Theorem 5 in [106]). *Let  $d \geq 3$ . Let  $\mu$  be a critical measure on  $\mathbb{N}$  such that  $\sigma^2 = \sum_{k=1}^{+\infty} k(k-1)\mu(k) < +\infty$ . Let us consider a spatial branching random walk with offspring law  $\mu$  and motion law  $\mathcal{P}_d$ . For every  $n \in \mathbb{N}$ , let  $Z_n$  be the number of particles in generation  $n$ , let  $\Omega_n$  be the number of occupied sites in generation  $n$  and for every  $j \in \mathbb{N}^*$ , let  $M_n(j)$  be the number of vertices of  $\mathbb{Z}^d$  which contain exactly  $j$  particles in generation  $n$ . Then there exists a sequence  $(\kappa_j)_{j \in \mathbb{N}^*} \in \mathbb{R}_+^{\mathbb{N}^*}$  such that  $\sum_{j=1}^{+\infty} j\kappa_j = 1$  and*

$$\mathcal{L} \left( \frac{Z_n}{n}, \left( \frac{M_n(j)}{n} \right)_{j \geq 1}, \frac{\Omega_n}{n} \middle| Z_n > 0 \right) \xrightarrow[n \rightarrow +\infty]{law} \left( 1, (\kappa_j)_{j \geq 1}, \sum_{j=1}^{+\infty} \kappa_j \right) \cdot \mathcal{E}$$

where  $\mathcal{E}$  is an exponential distribution which has density

$$\frac{2}{\sigma^2} \exp(-2x/\sigma^2) dx.$$

In dimension 1 and 2, the behaviour of the spatial branching random walk is very different. In dimension 1, it is not hard to see that  $\Omega_n$  is of order  $\sqrt{n}$ . (See section 7 in [146].) Moreover, in dimension 2, Lalley and Zheng proved the following result :

**Proposition 1.49** (Theorem 7 in [106]). *Let  $\mu$  be a critical measure on  $\mathbb{N}$  such that  $\sigma^2 = \sum_{k=1}^{+\infty} k(k-1)\mu(k) < +\infty$ . Let us consider a spatial branching random walk with offspring law  $\mu$  and motion law  $\mathcal{P}_2$ . For every  $n \in \mathbb{N}$ , let  $Z_n$  be the number of particles in generation  $n$  and let  $\Omega_n$  be the number of occupied sites in generation  $n$ . Then, conditionally on  $Z_n > 0$ ,  $\ln(n)n^{-1}\Omega_n$  is tight.*

Consequently, according to Theorem 1.48 and Proposition 1.49, spatial branching random walks which are conditioned on survival exhibit a very different behaviour when  $d \geq 3$  compared to lower dimension. Indeed, in dimension 1 and 2, there is not enough space and particles are accumulating. On the contrary, when  $d \geq 3$ , particles spread in an homogeneous way.

Dimension 2 is a kind of critical dimension when we condition on survival at time  $n$ . Let us mention that it is also possible to consider a spatial conditioning. For example, one can condition the Galton-Watson tree to have  $n$  vertices and make  $n$  go to infinity. This approach is explored by Le Gall and Lin in [72] and in [73]. One can also consider a branching random walk starting from  $x$  and condition it to visit 0 and make  $\|x\|$  go to infinity. This type of conditioning is studied by Zhu in [169], [170] and [171]. Some crucial estimates of Zhu have also been generalized by Angel, Hutchcroft and Jarai in [9] and more recently by Asselah and Schapira in [11]. In the spatial conditioning, the critical dimension is 4 whereas it was 2 in the time conditioning.

In their articles, Lalley, Zheng, Le Gall, Lin, Zhu, Asselah and Schapira used a very important tool in the area of branching random walks which is the spine decomposition. The principle of this spine decomposition is that, by the means of a change of measure, one can decompose the branching random walk into a simple random walk which is called the spine on which we attach smaller branching random walks which are independent from the spine. In Chapter 2 we give a precise explanation of the spine decomposition and we use it in order to find properties of spatial branching random walks starting from a point process.

### 1.4.3 Spatial branching random walks starting from a point process

Let  $d \in \mathbb{N}^*$ . Let  $\Theta$  be a point process on  $\mathbb{R}^d$ , that is, a random locally finite integer-valued measure on  $\mathbb{R}^d$ .  $\Theta$  can be represented in the following way :

$$\Theta = \sum_{i \in I} \delta_{x_i}.$$

Let  $\mu$  be an offspring law on  $\mathbb{N}$ . We do not assume criticality of  $\mu$  for now. Let  $\mathcal{P}$  be a motion law on  $\mathbb{R}^d$ . Then, conditionally on  $\Theta$ , for every  $i \in I$ , let us consider a branching random walk  $S^{(i)}$  with offspring law  $\mu$  and motion law  $\mathcal{P}$  starting from  $x_i$ . Besides, we assume that, conditionally on  $\Theta$ , these branching random walks  $(S^{(i)})_{i \in I}$  are independent. Then, for every  $n \in \mathbb{N}$ , we can construct the measure  $\Theta_n$  which is defined by

$$\Theta_n = \sum_{i \in I} \sum_{|u|=n} \delta_{S_u^{(i)}}.$$

Remark that  $\Theta_n$  is not always locally finite for any  $n \in \mathbb{N}$  if we do not make assumptions on  $\Theta_0$ ,  $\mu$  and  $\mathcal{P}$ . Actually, for each choice of  $\Theta_0$ ,  $\mu$  and  $\mathcal{P}$ , it is necessary to verify whether  $\Theta_n$  is locally finite for every  $n \in \mathbb{N}$  or not. For example if  $\Theta_0$  is a Poisson point process with intensity  $e^{\|x\|^3} dx$ ,  $\mu$  is critical and  $\mathcal{P}$  is a standard Gaussian distribution, then  $\Theta_1$  is not locally finite. On the contrary, if  $\Theta_0$  is a homogeneous Poisson point process and  $\mu$  is critical, then  $\Theta_n$  is locally finite for every integer  $n$ . If for every bounded subset  $A$  of  $\mathbb{R}^d$ ,  $\mathbb{E}[\Theta(A)] < +\infty$ , then we say that  $\Theta$  has finite intensity and  $A \mapsto \mathbb{E}[\Theta(A)]$  is called the intensity measure of  $\Theta$ . However, such a measure does not exist for every point process  $\Theta$ .

A natural problem regarding spatial branching random walks starting from a point process is to know whether  $(\Theta_n)_{n \in \mathbb{N}}$  converges towards the null point process or not. This question has been studied a lot in the 70's and the 80's and engendered the theory of cluster fields which is summarized in [124]. We say that a point process  $\Theta$  is cluster-invariant (with respect to the branching random walk with offspring law  $\mu$  and motion law  $\mathcal{P}$ ) if and only if for every  $n \in \mathbb{N}$ ,  $\Theta_n \stackrel{\text{law}}{=} \Theta$ . A branching random walk  $D(\mu, \mathcal{P})$  with offspring law  $\mu$  and motion law  $\mathcal{P}$  is said to be stable if there exists a point process  $\Theta$  which is invariant in law by translation, which is cluster-invariant with respect to  $D(\mu, \mathcal{P})$  and which has finite intensity. In [94], thanks to his backward tree method, Kallenberg proved the following result.

**Theorem 1.50** (Particular case of Theorem 1.2 in [94]). *Let  $d \in \mathbb{N}^*$ . Let  $\Theta$  be a point process on  $\mathbb{R}^d$  which has a finite intensity  $\rho$  and whose law is invariant by translation. Let us assume that for every  $n \in \mathbb{N}$ , the random measure  $\Theta_n$  is locally finite where  $\Theta_n$  is obtained from  $\Theta$  by performing the  $n$ th generation of the branching random walk  $D(\mu, \mathcal{P})$  with critical offspring law  $\mu$  and motion law  $\mathcal{P}$ . Then  $(\Theta_n)_{n \in \mathbb{N}}$  converges in law for the vague topology towards some limiting point process  $\Theta_\infty$ . Moreover  $\Theta_\infty$  is either zero if  $D(\mu, \mathcal{P})$  is unstable or it is non-zero and its intensity measure is  $\rho$  if  $D(\mu, \mathcal{P})$  is stable.*

According to Theorem 1.50, the notion of stability is very useful in order to analyse the asymptotic behaviour of branching random walks starting from a point process. Therefore, it is a big question to know whether a branching random walk is stable or not. Let  $\beta \in ]0, 1]$  and let  $\alpha \in ]0, 2]$ . Then, we can consider the critical offspring law  $\mu_\beta$  on  $\mathbb{N}$  such that for every  $s \in [0, 1]$ ,

$$\sum_{k=0}^{+\infty} \mu_\beta(k) s^k = s + \frac{1}{2}(1-s)^{1+\beta}.$$

Moreover, we define the motion law  $\mathcal{P}_{\alpha,d}$  which is defined by its characteristic function such that for every  $y \in \mathbb{R}^d$ ,

$$\int \exp(i\langle y, x \rangle) \mathcal{P}_{\alpha,d}(dx) = \exp\left(-\left(\sum_{k=1}^d |y_k|^2\right)^{\alpha/2}\right).$$

Then, generalizing [50] and [60], Gorostiza and Wakolbinger proved in [81], in the case of a continuous-time model, that there is a critical dimension between stability and instability.

**Theorem 1.51** (Theorem 2.2 in [81]). *The continuous-time branching random process with offspring law  $\mu_\beta$  and motion law  $\mathcal{P}_{\alpha,d}$  is stable if and only if  $d > \alpha/\beta$ .*

In [49], considering the discrete-time model, Dawson and Fleischmann proved that the critical dimension is also  $\alpha/\beta$  in the lattice case. Another important question concerning spatial branching random walks starting from a point process is to characterize the set of cluster-invariant point processes. Assuming some strong hypothesis, it is possible to find the set of the laws of cluster-invariant point processes.

**Theorem 1.52** (Theorem 12.4.6 in [124]). *Let  $d \in \mathbb{N}^*$ . Let  $\mu$  be a critical offspring law and let  $\mathcal{P}$  be a non lattice motion law on  $\mathbb{R}^d$ . We assume that the branching random walk associated with  $(\mu, \mathcal{P})$  is stable. For every  $l \in \mathbb{R}_+$ , we denote by  $P_\infty^{(l)}$  the non-trivial limit in law of the sequence  $(\Theta_n^{(l)})_{n \in \mathbb{N}}$  starting from the Poisson point process  $\Theta^{(l)}$  with intensity  $l \cdot \text{Leb}$ .  $P_\infty^{(l)}$  is a translation-invariant, cluster-invariant point process with intensity  $l \cdot \text{Leb}$ . Let  $\Theta$  be a point process on  $\mathbb{R}^d$  which is cluster-invariant and whose law is invariant by translation. Then, there exists a probability measure  $\mu$  on  $\mathbb{R}_+$  such that the law  $\mathcal{L}(\Theta)$  of  $\Theta$  satisfies*

$$\mathcal{L}(\Theta) = \int_0^{+\infty} \mathcal{L}\left(P_\infty^{(l)}\right) d\mu(l).$$

Theorem 1.52 characterizes the set of cluster-invariant point processes which are invariant by translation. Actually, this assumption of invariance by translation is very strong. In 11.10.5 in [124], one can find a counter-example where a one-dimensional cluster-invariant point process is not invariant by translation. In [92], in some cases, Kabluchko found a criterion to know when there exists such cluster-invariant processes which are not invariant by translation. However, there are some important cases where all the cluster-invariant point processes are invariant by translation. For example, in [32], Bramson, Cox and Greven studied the case of the critical binary branching Brownian motion by means of estimates on the PDE

$$\frac{\partial h}{\partial t} = \frac{1}{2} \Delta h - h^2.$$

The critical binary branching Brownian motion on  $\mathbb{R}^d$  is defined as follows :

- We start with 1 initial particle  $o$  at the origin. It behaves as a Brownian motion in  $\mathbb{R}^d$  until an exponential time  $T$  with parameter 1. At time  $T$ , the particle  $o$  dies with probability 1/2 or divides into two independent Brownian motions in  $\mathbb{R}^d$  starting from the position of  $o$  at time  $T$ .
- Each particle  $u$  behaves as a Brownian motion in  $\mathbb{R}^d$  and at rate 1, it dies with probability 1/2 or divides into two independent Brownian motions in  $\mathbb{R}^d$  with probability 1/2.

**Theorem 1.53** (Theorem 1 in [32]). *Let  $d \geq 3$ . There exists a family  $(\tilde{P}_\infty^{(l)})_{l \geq 0}$  of point processes which are cluster-invariant with respect to the critical binary branching Brownian motion in  $\mathbb{R}^d$  such that for every  $l \in \mathbb{R}_+$ ,  $\tilde{P}_\infty^{(l)}$  has constant intensity  $l$ . Moreover, for every  $l \in \mathbb{R}_+$ ,  $\tilde{P}_\infty^{(l)}$  is translation-invariant. Let  $\Theta$  be a cluster-invariant point process with respect to the critical binary branching Brownian motion in  $\mathbb{R}^d$ . Then, there exists a probability measure  $\mu$  on  $\mathbb{R}_+$  such that,*

$$\mathcal{L}(\Theta) = \int_0^{+\infty} \mathcal{L}\left(\tilde{P}_\infty^{(l)}\right) d\mu(l).$$

Remark that, in Theorem 1.53, the only assumption which is made is cluster-invariance.  $\Theta$  is not assumed to be translation-invariant anymore. In other words, in that case, all cluster-invariant point processes are invariant by translation. In [39], Chen, Garban and Shekhar proved a similar result for the one-dimensional branching Brownian motion with critical drift. The aim of chapter 2 is to generalize Theorem 1.53 in the discrete-time setting by means of methods which are developed in [39].

## 1.5 Our results regarding branching random walks

This section will briefly describe the results of chapter 2. It is based on my paper [142] which was written under the supervision of my PhD advisor Xinxin Chen. In this chapter we will study spatial branching random walks in  $\mathbb{R}^d$  with critical offspring law  $\mu$  and motion law  $\mathcal{P}$  in three different cases :

1. Hypothesis  $\mathcal{H}_1$  : We assume that  $d \geq 3$ . The distribution  $\mathcal{P}$  is a  $d$ -dimensional Gaussian random variable with mean 0 and covariance matrix  $\Sigma = I_d$  where  $I_d$  is the identity matrix. Moreover  $\sigma^2 := \sum_{k=0}^{+\infty} k^2 \mu(k) < +\infty$ .
2. Hypothesis  $\mathcal{H}_2$  : We assume that  $\mathcal{P}$  can be written as

$$\mathcal{P} = \sum_{x \in \mathcal{R}} p(x) \delta_x$$

where  $\mathcal{R}$  is a finite subset of  $\mathbb{Z}^d$  with  $d \geq 3$ . Moreover  $p$  is symmetrical in the sense that for every  $x \in \mathcal{R}$ ,  $-x \in \mathcal{R}$  and  $p(-x) = p(x)$ .  $\mathcal{P}$  has a positive definite covariance matrix  $\Sigma$ . In addition, we assume that the random walk generated by the motion  $\mathcal{P}$  is aperiodic. Furthermore  $\sigma^2 := \sum_{k=0}^{+\infty} k^2 \mu(k) < +\infty$ .

3. Hypothesis  $\mathcal{H}_3$  : The motion law  $\mathcal{P}$  is given by a spherically symmetric  $\alpha$ -stable law with  $\alpha \in ]0, 2]$ . More precisely for every  $y \in \mathbb{R}^d$ ,

$$\int \exp(i\langle y, x \rangle) \mathcal{P}(dx) = \exp\left(-\left(\sum_{k=1}^d |y_k|^2\right)^{\alpha/2}\right).$$

The critical offspring law  $\mu$  has no second moment anymore. However we assume that there exists  $\beta \in ]0, 1]$  such that for every  $\gamma < \beta$ ,  $\sum_{k=0}^{+\infty} k^{1+\gamma} \mu(k) < +\infty$ . Moreover, we assume that  $d > \alpha/\beta$ .

**Remark 1.13.** Hypothesis  $\mathcal{H}_1$  is a particular case of  $\mathcal{H}_3$  with  $(\alpha, \beta) = (2, 1)$ . However we distinguish the special case of Hypothesis  $\mathcal{H}_1$  because the proofs of the forthcoming results are very similar when we assume Hypotheses  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . On the contrary, if we assume  $\mathcal{H}_3$  with general  $\alpha$  and  $\beta$ , then we need to slightly change the proofs.

Now, let  $X$  be a non-negative random variable. If we assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_3$ , let  $\Lambda_0^{d,X}$  be a Poisson point process with distribution

$$PPP(X\lambda(dx))$$

where  $\lambda$  is the Lebesgue measure. If we assume hypothesis  $\mathcal{H}_2$ , let  $\Lambda_0^{d,X}$  be the discrete Poisson point process

$$PPP\left(X \sum_{x \in \mathbb{Z}^d} \delta_x\right).$$

Let us define the sequence  $(\Lambda_n^{d,X})_{n \in \mathbb{N}}$  which is obtained by attaching independent spatial branching random walks with motion law  $\mathcal{P}$  and offspring law  $\mu$  to the point process  $\Lambda_0^{d,X}$ . Moreover, a "closed ball" designates a euclidean closed ball of  $\mathbb{R}^d$ . Under hypothesis  $\mathcal{H}_2$ , we always assume that a closed ball contains at least one point of  $\mathbb{Z}^d$ .

**Theorem 1.54** (Convergence Theorem). *We assume hypotheses  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  or  $\mathcal{H}_3$ . Then, there exists a non-trivial point process  $\Lambda_\infty^{d,X}$  such that*

$$\Lambda_n^{d,X} \xrightarrow[n \rightarrow +\infty]{law} \Lambda_\infty^{d,X}$$

*in the vague topology.*

*Idea of the proof of Theorem 1.54.*

Let us sketch the proof of Theorem 1.54 under hypothesis  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . If a critical branching process starts from a single particle and if we condition it to visit a closed ball  $A$ , we will prove in Chapter 2 that the limiting point process is some point process  $\mathcal{N}_A$ . However, most of the critical branching processes starting from particles in  $\Lambda_0^{d,X}$  will not reach  $A$  because of transience (in dimension  $d \geq 3$ ) or because of extinction of the branching process. Let us make this intuition more quantitative. By homogeneity of the Poisson point process  $\Lambda_0^{d,X}$ , for every  $M \geq 1$ ,

$$\Lambda_0^{d,X}(B(0, M\sqrt{n})) \simeq \Theta(n^{d/2}).$$

Moreover, any particle of  $\Lambda_0^{d,X}$  located at  $x \in \mathbb{R}^d$  with  $\|x\| \gg \sqrt{n}$  is too far from  $A$  to have descendants in  $A$  at time  $n$ . Indeed, a centered random walk with second moment is at distance  $O(\sqrt{n})$  from 0 at time  $n$ . Moreover, we will prove in Chapter 2 that a particle in  $\Lambda_0^{d,X}$  located at  $x \in B(0, M\sqrt{n})$  has descendants which reach  $A$  at time  $n$  with probability which is of order  $\Theta(n^{-d/2})$ . If  $Z_n^{(y)}(\cdot)$  is the occupation measure of the  $n$ -th generation of the critical branching process starting from a particle  $y$  in the support of  $\Lambda_0^{d,X}$ , combining both previous approximations, we get that

$$\int \mathbf{1}\{Z_n^{(y)}(A) \geq 1\} d\Lambda_0^{d,X}(y) \simeq \Theta(n^{d/2}) \times \Theta(n^{-d/2}) = \Theta(1).$$

Therefore, the number  $P_A$  of particles in  $\Lambda_0^{d,X}$  whose descendants reach  $A$  is of order  $\Theta(1)$ . Moreover, for each of these particles, the positions of their descendants in  $A$  form a point process distributed as  $\mathcal{N}_A$ . Consequently,  $\Lambda_\infty^{d,X}$  is a layering of  $P_A$  independent copies of  $\mathcal{N}_A$ .  $\square$

**Remark 1.14.** *Theorem 1.54 is not totally new. Actually the case of Hypothesis  $\mathcal{H}_3$  was already known in a continuous-time setting in [81]. However our result is stated in discrete time. Therefore, we cannot use PDEs estimates as in [81] which makes necessary to use probabilistic tools. Moreover the lattice case of Hypothesis  $\mathcal{H}_2$  was already treated in the discrete-time setting in [49] with a different formalism. Nonetheless, one advantage of our proof is that we can treat the three different hypotheses with a unified proof in a modern formalism. (However the proof requires slight modifications in the case of  $\mathcal{H}_3$  but the main idea remains the same.)*

It is clear that the limiting point processes obtained in Theorem 1.54 are cluster-invariant. In fact, all cluster-invariant point processes have this form.

**Theorem 1.55** (Characterization of cluster-invariant measures). *Let us assume hypotheses  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  or  $\mathcal{H}_3$ . Let  $\Theta$  be a cluster-invariant point process. Then there exists a non-negative random variable  $X$  such that :*

$$\Theta \stackrel{\text{law}}{=} \Lambda_\infty^{d,X}.$$

Moreover  $\Lambda_\infty^{d,X}$  is cluster-invariant for every non-negative random variable  $X$ .

*Idea of the proof of Theorem 1.55.*

Let us consider a cluster-invariant point process  $\Theta$ . Let  $f$  be a non-negative continuous compactly supported function whose support is included in a closed ball  $A$ . By cluster-invariance, we remark that for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right].$$

Then, we investigate the asymptotics of the right-hand side above in order to describe the distribution of  $\Theta$ . We will show that the particles  $x$  in the support of  $\Theta$  such that  $\|x\| \geq M\sqrt{n}$  can be neglected. That is why,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right] \approx \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right) \right] \right]$$

where under  $\mathbf{P}$ ,  $S$  is a branching random walk with offspring law  $\mu$  and motion law  $\mathcal{P}$  starting from 0. Under  $\mathbf{P}$ , for every set  $\tilde{A}$ ,  $Z_n(\tilde{A})$  denotes the number of particles of the branching random walk which lie in  $\tilde{A}$  in generation  $n$ . Furthermore, we will prove in Chapter 2 that there exists a quantity  $R_{f,A}$  such that, uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$\ln \left( \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right) \approx R_{f,A} \mathbf{P}(Z_n(A-x) \geq 1).$$

Therefore, for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] \approx \mathbb{E} [\exp(R_{f,A} L_n)] \quad (1.11)$$

where

$$L_n := \int_{B(0, M\sqrt{n})} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx).$$

In Chapter 2, we will prove that  $(L_n)_{n \in \mathbb{N}^*}$  is tight. Therefore, by Prokhorov's theorem, up to the extraction of a subsequence,  $(L_n)_{n \in \mathbb{N}^*}$  converges in law toward some random variable  $Y$ . Therefore, by making  $n$  go to infinity along this subsequence in (1.11), we obtain that

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} [\exp(R_{f,A} Y)].$$

Actually, we will see that the right hand-side is exactly the Laplace transform of  $\Lambda_{\infty}^{d,X}$  for some random variable  $X$  which is related to  $Y$ . It concludes the proof.  $\square$

## 1.6 Our main results regarding the VRJP

Most of the results of this thesis concern the VRJP but we focus on different objects including the martingale  $(\psi_{\beta}^{(n)})_{n \in \mathbb{N}}$ , the operator  $H_{\beta}$  on  $\mathbb{Z}^d$ , the scaling limits of  $H_{\beta}$  on one-dimensional continuous topological spaces or a system of SDEs which is associated with the random potential  $\beta$ .

### 1.6.1 Main results of Chapter 3

In chapter 3, we study the asymptotic behaviour of the martingale  $(\psi_{\beta}^{(n)})_{n \in \mathbb{N}}$ . It is based on the paper [141] which has been written under the supervision of my PhD advisors C. Sabot and X. Chen. The proofs of Theorems 1.56, 1.57, 1.58 and 1.59 below use one-dimensional branching random walks. By means of different tricks, we adapt the tools which are known about branching random walks in the particular case of the VRJP on trees.

As explained in Remark 1.10, the  $L^p$ -boundedness of  $(\psi_{\beta}^{(n)})_{n \in \mathbb{N}}$  on  $\mathbb{Z}^d$ , is crucial in order to prove diffusivity of the VRJP. We are not able to prove the  $L^p$ -boundedness of  $(\psi_{\beta}^{(n)})_{n \in \mathbb{N}}$  for the transient VRJP on  $\mathbb{Z}^d$ . However we can always prove uniform integrability.

**Theorem 1.56.** *We assume that  $V = \mathbb{Z}^d$  with  $d \geq 3$  and that the weights on the edges of  $\mathbb{Z}^d$  are constant and equal to some  $W > 0$ . Moreover we assume that  $W > W_c(d)$ . Then the martingale  $(\psi_{\beta}^{(n)}(0))_{n \in \mathbb{N}}$  is uniformly integrable with respect to  $\nu_V^W$ .*

*Idea of the proof of Theorem 1.56.*

Thanks to the stability of the distributions of the form  $\nu_V^{W,\eta}$  under restriction and conditioning

which is given by Lemma 1.21, we will prove in Chapter 3 that for every  $n \in \mathbb{N}$ , the law of  $\psi_\beta^{(n)}(0)$  satisfies

$$\mathcal{L}\left(\psi_\beta^{(n)}(0) | \mathcal{G}_n\right) = IG\left(1, \frac{\psi_\beta^{(n)}(0)}{\hat{G}_\beta^{(n)}(0,0)}\right) \quad (1.12)$$

for some  $\sigma$ -field  $\mathcal{G}_\beta^{(n)}$  such that  $\frac{\psi_\beta^{(n)}(0)}{\hat{G}_\beta^{(n)}(0,0)}$  is measurable with respect to  $\mathcal{G}_n$ . As the VRJP is transient because  $W > W_c(d)$ , we know that  $\frac{\psi_\beta^{(n)}(0)}{\hat{G}_\beta^{(n)}(0,0)}$  converges almost surely towards  $\frac{\psi_\beta(0)}{\hat{G}_\beta(0,0)}$  which is strictly positive. Therefore, by making  $n$  go to infinity in (1.12), we obtain

$$\mathcal{L}\left(\psi_\beta(0) | \mathcal{G}_\infty\right) = IG\left(1, \frac{\psi_\beta(0)}{\hat{G}_\beta(0,0)}\right)$$

for some  $\sigma$ -field  $\mathcal{G}_\infty$  such that  $\frac{\psi_\beta(0)}{\hat{G}_\beta(0,0)}$  is measurable with respect to  $\mathcal{G}_\infty$ . In particular,  $\mathbb{E}[\psi_\beta(0)] = 1 = \mathbb{E}\left[\psi_\beta^{(n)}(0)\right]$  for every  $n \in \mathbb{N}$ . By Scheffé's lemma, it implies that  $(\psi_\beta^{(n)}(0))_{n \in \mathbb{N}}$  converges in  $L^1$ . In particular,  $(\psi_\beta^{(n)}(0))_{n \in \mathbb{N}}$  is a uniformly integrable martingale.  $\square$

Moreover, in the case of Galton-Watson trees, when the VRJP is transient, we are able to show  $L^p$ -boundedness of  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$ . In the next three theorems  $o$  designates the root of the tree.

**Theorem 1.57.** *Let  $V$  be a Galton-Watson tree with root  $o$  with offspring law  $\mu$  such that  $\mu(0) = 0$  and the mean  $m$  of  $\mu$  satisfies  $m > 1$ . Let  $W > W_c(\mu)$ . Then, for every  $p \in ]1, +\infty[$ , the martingale  $(\psi_\beta^{(n)}(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  with respect to  $\nu_V^W$ .*

On the contrary, when  $W \leq W_c(\mu)$ , we know that the VRJP is recurrent which means that  $(\psi_\beta^{(n)}(o))_{n \in \mathbb{N}}$  converges to 0 and therefore is not uniformly integrable. In Chapter 3, we compute the rate of convergence of  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  toward 0.

**Theorem 1.58.** *Let  $V$  be a Galton-Watson tree with a supercritical offspring law  $\mu$  satisfying mild hypotheses. Let  $W < W_c(\mu)$ . Then, it holds that,  $\nu_V^W$ -almost surely,*

$$\lim_{n \rightarrow +\infty} \frac{\ln(\psi_\beta^{(n)}(o))}{n} = -\tau(m, W)$$

with  $\tau(m, W) > 0$ .

Moreover, at the transition point  $W_c(\mu)$ , we found a different behaviour.

**Theorem 1.59.** *Let  $V$  be a Galton-Watson tree with a supercritical offspring law  $\mu$  satisfying mild hypotheses. Let  $m$  be the mean of  $\mu$ . We assume that  $W = W_c(\mu)$ . Then, it holds that*

$$\frac{\ln(\psi_\beta^{(n)}(o))}{n^{1/3}} \xrightarrow[n \rightarrow +\infty]{a.s.} -\varrho_c$$

where  $\varrho_c = \frac{1}{2} \left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3}$  with  $\sigma^2 = 16m \int_0^{+\infty} \frac{\sqrt{W_c(\mu) \ln(x)^2}}{\sqrt{2\pi x}} e^{-\frac{W_c(\mu)}{2}(x+1/x-2)} dx$ .

The proof of these three theorems relies on the link between the square of  $\psi_\beta^{(n)}$  and the effective resistance of the system. Therefore, controlling the  $\psi_\beta^{(n)}$  is tantamount to controlling this effective resistance. To do so, we will use tools coming from the theory of branching random walks.

*Idea of the proof of Theorem 1.57.*

Let  $W > W_c(\mu)$ . We aim to show that  $(\psi_\beta^{(n)}(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  for every  $p > 1$ . First,  $(\hat{G}_\beta^{(n)}(o, o))_{n \in \mathbb{N}}$  is the bracket of  $(\psi_\beta^{(n)}(o))_{n \in \mathbb{N}}$  in the sense that  $(\psi_\beta^{(n)}(o)^2 - \hat{G}_\beta^{(n)}(o, o))_{n \in \mathbb{N}}$  is a martingale. Therefore, by Burkholder-Davis-Gundy inequality, for every  $p > 1$ , it is enough to show that there exists  $C > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\mathbb{E} \left[ \left( \hat{G}_\beta^{(n)}(o, o) \right)^{p/2} \right] \leq C$$

in order to prove that  $\left( \mathbb{E} \left[ \left( \psi_\beta^{(n)}(o) \right)^p \right] \right)_{n \in \mathbb{N}}$  is bounded. Now, how can we control  $\hat{G}_\beta^{(n)}(o, o)$ ? Surprisingly, everything is easier if we make a rank-one perturbation. More precisely, there exists a potential  $(\beta_i)_{i \in V}$  on the tree  $V$  and a random variable  $\gamma$  with Gamma distribution with parameter  $(1/2, 1)$  which is independent of  $\beta$  such that for every  $i \in V$ ,

$$\beta_i = \tilde{\beta}_i + \mathbf{1}\{i = o\}\gamma.$$

Moreover, if  $(A_i)_{i \in V \setminus \{o\}}$  are independent Inverse Gaussian random variables with parameters  $(1, W)$  which are independent of  $\gamma$ , we can construct  $\tilde{\beta}$  in the following way : for every  $i \in V$ ,

$$\tilde{\beta}_i = \frac{W}{2} \left( \sum_{\tilde{u}=i} A_u + \mathbf{1}\{i \neq o\} \frac{1}{A_i} \right).$$

Now, let us consider the operator  $\tilde{H}_\beta$  such that for every  $i, j \in V$ ,

$$\tilde{H}_\beta(i, j) = \mathbf{1}\{i = j\}2\tilde{\beta}_i - \mathbf{1}\{\{i, j\} \in E\}W.$$

Remark that  $\tilde{H}_\beta$  and  $H_\beta$  differ only at point  $(o, o)$ . We will show in Chapter 3 that for every  $n \in \mathbb{N}$ ,  $\tilde{H}_\beta|_{V_n \times V_n}$  is positive definite almost surely. In particular, we are allowed to define  $\tilde{G}_\beta^{(n)}$  which is the inverse of  $\tilde{H}_\beta|_{V_n \times V_n}$ . By Cramer's formula, for every  $n \in \mathbb{N}$ , it holds that

$$\hat{G}_\beta^{(n)}(o, o) = \frac{\tilde{G}_\beta^{(n)}(o, o)}{1 + 2\gamma\tilde{G}_\beta^{(n)}(o, o)}. \quad (1.13)$$

Moreover, by a direct computation, one can use (1.13) to prove that for every  $p > 1$ , there exists  $a_p > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\mathbb{E} \left[ \left( \hat{G}_\beta^{(n)}(o, o) \right)^{p/2} \right] \simeq a_p \mathbb{E} \left[ \left( \tilde{G}_\beta^{(n)}(o, o) \right)^{(p-1)/2} \right].$$

Thus, in order to conclude the proof, it is enough to prove that for every  $\alpha > 0$ ,

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \left[ \left( \tilde{G}_\beta^{(n)}(o, o) \right)^\alpha \right] < +\infty.$$

Moreover,  $\left( \tilde{G}_\beta^{(n)}(o, o) \right)_{n \in \mathbb{N}}$  is increasing. Therefore, it has almost surely a limit  $\tilde{G}_\beta^{(\infty)}(o, o)$ . Consequently, we need that for every  $\alpha > 0$ ,

$$\mathbb{E} \left[ \left( \tilde{G}_\beta^{(\infty)}(o, o) \right)^\alpha \right] < +\infty. \quad (1.14)$$

Then, in Chapter 3, we will prove the following remarkable fact :

$$\tilde{G}_\beta^{(\infty)}(o, o) = \mathcal{R}(o \longleftrightarrow \infty)$$

where  $\mathcal{R}(o \longleftrightarrow \infty)$  is the effective resistance from 0 to infinity on  $V$  with the random conductances  $(c(x, y))_{\{x, y\} \in E}$  such that for every  $x \in V$ ,

$$c(x, \tilde{x}) = A_x \prod_{o < z \leq \tilde{x}} A_z^2.$$

By Proposition 1.11, these conductances are the conductances of the discrete time VRJP. Thus, as  $W > W_c(\mu)$ , we know that  $\tilde{G}_\beta^{(\infty)}(o, o) = \mathcal{R}(o \longleftrightarrow \infty) < +\infty$  almost surely. However, we need (1.14) which is a stronger result than the finiteness  $\mathcal{R}(o \longleftrightarrow \infty)$ . Actually, by means of the theory of electrical networks (see [116]), one can prove that  $\mathcal{R}(o \longleftrightarrow \infty)$  is solution of the following equation in law :

$$\mathcal{R}(o \longleftrightarrow \infty) = \frac{1}{\sum_{i=x}^{\tau} \frac{A_i^2 W}{A_i + W \mathcal{R}(i)}} \quad (1.15)$$

where  $(\mathcal{R}(i))_{i=x}^{\tau}$  are i.i.d copies of  $\mathcal{R}(o \longleftrightarrow \infty)$  which are independent of  $(A_i)_{i=x}^{\tau}$ . Thanks to the study of the equation (1.15), we will prove (1.14).  $\square$

*Idea of the proof of Theorem 1.58 and Theorem 1.59.*

Let  $(A_z)_{z \in V \setminus \{o\}}$  be i.i.d Inverse Gaussian random variables with parameters  $(1, W)$ . For every  $x \in V \setminus \{o\}$ , let us define again the local conductance

$$c(x, \tilde{x}) = A_x \prod_{o < z \leq \tilde{x}} A_z^2.$$

For every  $n \in \mathbb{N}$ , let  $\mathcal{R}(0 \longleftrightarrow \delta_n)$  be the effective conductance between the root  $o$  and the point  $\delta_n$  with local conductances  $c$ .  $\delta_n$  is obtained by a wired contraction of  $V \setminus V_n$ . The details of the construction are given in Chapter 3. Besides, when  $W \leq W_c(\mu)$ , if  $\gamma = \frac{1}{2\tilde{G}_\beta(o, o)}$ , we will prove that for every  $n \in \mathbb{N}$  the following identity in law does hold

$$\psi_\beta^{(n)}(o)^2 \times 2\gamma \times (1 + 2\gamma \mathcal{R}(o \longleftrightarrow \delta_n)) \stackrel{law}{=} 2\Gamma(1/2, 1) \quad a.s. \quad (1.16)$$

Therefore, by means of (1.16) and Borel-Cantelli lemma, it is not difficult to show that, up to some negligible terms,

$$\ln(\psi_\beta^{(n)}(o)) \simeq \frac{1}{2} \ln(\mathcal{R}(o \longleftrightarrow \delta_n)) \quad a.s.$$

Then, it is possible to obtain almost-sure estimates for  $\ln(\mathcal{R}(o \longleftrightarrow \delta_n))$  thanks to the theory of branching random walks. In the non-critical case, we will use results of Hu and Shi in [87]. In the critical case, we have to apply finer estimates. In particular, we will use the results of Faraud, Hu and Shi in [64].  $\square$

## 1.6.2 Main results of Chapter 4

Let  $W > 0$  and let  $d \in \mathbb{N}^*$ . Let us denote by  $\nu_d^W$  the distribution of the random potential  $\beta$  which is associated with the VRJP on  $\mathbb{Z}^d$  with constant weights  $W > 0$ . In chapter 4, we study the integrated density of states  $n_{W,d}$  of the operator  $H_\beta$  on  $\mathbb{Z}^d$  where  $\beta$  has distribution  $\nu_d^W$ . We will show that the spectrum of  $H_\beta$  is  $\mathbb{R}_+$ . Therefore  $n_{W,d}$  is supported on  $\mathbb{R}_+$ . By the property of Lifschitz tails (see Theorem 1.37), in the case of the Anderson model, the density of states decreases exponentially fast at the bottom of the spectrum. However, Sabot and Zeng suspected that Lifschitz tails should not occur for  $H_\beta$  and that we should have

$$n_{W,d}(E) \simeq \sqrt{E}$$

when  $E$  goes to 0, at least for certain values of  $W$ . During my thesis, I found some partial answers to this conjecture. Nevertheless, simultaneously, Disertori, Rojas-Molina and Zeng proved results which are very similar to mine. When we discovered this situation we decided to publish an article together which yielded the paper [55]. However the original version of my proofs of the results in [55] was slightly different. In Chapter 4, I present my original proofs of the results of [55]. First, I proved a universal bound for the density of states of the operator  $H_\beta$ .

**Theorem 1.60.** *Let  $d \in \mathbb{N}^*$ . There exists a positive constant  $K_1$  which does not depend on  $d$  such that for every  $W > 0$  and for every  $E > 0$ ,*

$$n_{W,d}(E) \leq K_1 E^{1/2}.$$

Moreover, when  $W > 0$  is small, it holds that the upper bound of Theorem 1.60 is the good asymptotic order up to some logarithmic correction.

**Theorem 1.61.** *Let  $d \in \mathbb{N}^*$ . There exists  $W_-^{(d)} > 0$  such that for every  $W < W_-^{(d)}$ , there exists a positive constant  $K_{2,d,W}$  such that for every  $E \in (0, 1)$ ,*

$$n_{W,d}(E) \geq K_{2,d,W} \frac{\sqrt{E}}{|\ln(E)|^d}.$$

Moreover, in the particular case of the dimension 1, we have  $W_-^{(1)} = +\infty$ .

By Theorems 1.60 and 1.61, when  $W$  is small, we have  $n_{W,d}(E) \simeq \sqrt{E}$ . Nevertheless, it is not true anymore when  $W$  is large.

**Proposition 1.62.** *Let  $d \geq 3$ . Then, there exists  $W_+^{(d)} > 0$  such that for every  $W > W_+^{(d)}$ , there exists a positive constant  $K_{3,d,W}$  such that for every  $E > 0$ ,*

$$n_{W,d}(E) \leq K_{3,d,W} E.$$

Thus, according to Theorem 1.61 and Proposition 1.62, there is a phase transition regarding the behaviour of the density of states at the bottom of the spectrum. We conjecture that this phase transition for the density of states may coincide with the phase transition between recurrence and transience for the VRJP but we do not know how to prove it for now.

*Idea of the proof of Theorem 1.60.*

Let  $\varrho^*$  be the distribution of  $\beta_0$  conditionally on  $\sigma(\beta_x, x \neq 0)$ . By Wegner's bound (see Proposition 1.36), if  $\varrho^*$  had a bounded density then it would imply  $n_{W,d}(E) \leq CE$  for some constant  $C > 0$ . However, by the conditioning property in Lemma 1.21, one can compute the density of  $\varrho^*$  which is

$$\mathbf{1}\{\beta > D\} \frac{1}{\sqrt{\pi(\beta - D)}} e^{-(\beta - D)} e^{-\frac{1}{4(\beta - D)} \frac{\psi_\beta(0)^2}{\tilde{G}_\beta(0,0)^2}} e^{\frac{\psi_\beta(0)}{\tilde{G}_\beta(0,0)}} \quad (1.17)$$

where  $D$  is a positive random variable which is measurable with respect to  $\sigma(\beta_x, x \neq 0)$ . This density function is not bounded. Therefore, we cannot apply Proposition 1.36. Nonetheless, according to a generalization of Wegner's bound, we can obtain a bound of the type  $n_{W,d}(E) \leq C\sqrt{E}$  for some constant  $C > 0$  if there exists  $\tilde{C} > 0$  such that for every  $t \geq 0$  and for every  $x \in \mathbb{R}$ ,

$$\varrho^*([x - t, x + t]) \leq \tilde{C}\sqrt{t}. \quad (1.18)$$

Using (1.17), we can show an estimate of the form of (1.18). Indeed, for every  $x \in \mathbb{R}$  and every  $t \geq 0$ ,

$$\begin{aligned} \varrho^*([x-t, x+t]) &\leq \int_{x-t}^{x+t} \mathbf{1}\{\beta > D\} \frac{1}{\sqrt{\pi(\beta-D)}} e^{-(\beta-D)} e^{-\frac{1}{4(\beta-D)} \frac{\psi_\beta(0)^2}{\hat{G}_\beta(0,0)^2}} e^{\frac{\psi_\beta(0)}{\hat{G}_\beta(0,0)}} d\beta \\ &= \int_{x-t}^{x+t} \mathbf{1}\{\beta > D\} \frac{1}{\sqrt{\pi(\beta-D)}} e^{-\frac{1}{4(\beta-D)} \left(\frac{\psi_\beta(0)}{\hat{G}_\beta(0,0)} - 2(\beta-D)\right)^2} d\beta \\ &\leq \int_{x-t-D}^{x+t-D} \mathbf{1}\{\beta > 0\} \frac{1}{\sqrt{\pi\beta}} d\beta \\ &\leq \int_0^{2t} \frac{1}{\sqrt{\pi\beta}} d\beta \\ &= \frac{2\sqrt{2t}}{\sqrt{\pi}} \end{aligned}$$

which concludes the proof.  $\square$

*Idea of the proof of Theorem 1.61.*

For every  $L \in \mathbb{N}^*$ , let  $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ . Let us denote by  $H_L^D$  the Dirichlet restriction of  $H_\beta$  in the box  $\Lambda_L$ . More precisely, for every  $x, y \in \Lambda_L$ ,

- If  $x$  and  $y$  are neighbours in  $\mathbb{Z}^d$ ,  $H_L^D(x, y) = -W$
- If  $x$  and  $y$  are not equal and not neighbours in  $\mathbb{Z}^d$ ,  $H_L^D(x, y) = 0$ .
- If  $x = y$ ,  $H_L^D(x, y) = 2\beta_x + \sum_{\{x,k\} \in E(\mathbb{Z}^d), k \notin \Lambda_L} W$

where  $E(\mathbb{Z}^d)$  is the edge set of  $\mathbb{Z}^d$ . Moreover, the inverse of  $H_L^D$  is denoted by  $G^{(L),D}$ . In order to prove Theorem 1.61, we use the Dirichlet-Neumann bracketing which tells us that for any  $L \in \mathbb{N}^*$ ,

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\nu_d^W} [N(H_L^D, E)]$$

where  $N(H_L^D, E)$  is the number of eigenvalues of  $H_L^D$  which are lower than  $E$ . However, we cannot control the eigenvalues of  $H_L^D$  except the first one. Indeed the smallest eigenvalue of  $H_L^D$  is lower than  $G^{(L),D}(0,0)^{-1}$  by the min-max principle. Consequently,

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \nu_d^W \left( \frac{1}{G^{(L),D}(0,0)} \leq E \right).$$

Furthermore, if  $L$  is large,  $G^{(L),D}(0,0)$  is almost  $\hat{G}_\beta^{(L)}(0,0)$  which converges towards  $\hat{G}_\beta(0,0)$ . Therefore, approximately,

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \nu_d^W \left( \frac{1}{\hat{G}_\beta(0,0)} \leq E \right).$$

Besides, if  $W$  is small enough, the VRJP is recurrent. By Proposition 1.26,  $\psi_\beta(0) = 0$  almost surely. By Theorem 1.24, it implies that

$$\hat{G}(0,0) = G_{\beta,\gamma}(0,0) - \frac{1}{2\gamma} \psi_\beta(0)^2 = G_{\beta,\gamma}(0,0).$$

However, by Proposition 1.19,  $1/(2G_{\beta,\gamma}(0,0))$  has  $\Gamma(1/2, 1)$  distribution. Actually, in Proposition 1.19, the result is stated on finite graphs but one can show that the distribution of diagonal terms of  $G_{\beta,\gamma}$  remains the same on infinite graphs. Moreover, the cumulative distribution function of the  $\Gamma(1/2, 1)$  distribution behaves as  $\sqrt{E}$  when  $E$  goes to 0. It yields the lower bound in Theorem 1.61. Some work still needs to be done in order to justify that  $G^{(L),D}(0,0) \simeq \hat{G}_\beta^{(L)}(0,0)$  for a good choice of  $L$  with  $L \simeq |\ln(E)|$ . This is the main difficulty of this proof.  $\square$

*Proof of Theorem 1.62.*

We write  $\mathbb{E}_{\nu_d^W}$  when we integrate with respect to  $\nu_d^W$ . Let us denote by  $\alpha_{W,d}$  the non-integrated density of states of  $H_\beta$ . For every  $\eta > 0$ , we have  $\nu_d^W$ -a.s.,

$$\int_0^{+\infty} \frac{1}{\eta + u} d\mu_{\delta_0}(u) = (H_\beta + \eta)^{-1}(0, 0) \quad (1.19)$$

where  $\mu_{\delta_0}$  is the spectral measure of  $H_\beta$  which is associated with the vector  $\delta_0$ . By Proposition 1.38, we know that  $(H_\beta + \eta)^{-1}(0, 0) \xrightarrow{\eta \rightarrow 0} \hat{G}_\beta(0, 0)$ ,  $\nu_d^W$ -a.s. Moreover, by monotone convergence theorem,

$$\int_0^{+\infty} \frac{1}{\eta + u} d\mu_{\delta_0}(u) \xrightarrow{\eta \rightarrow 0} \int_0^{+\infty} \frac{1}{u} d\mu_{\delta_0}(u).$$

Together with (1.19), this implies that,  $\nu_d^W$ -a.s.,

$$\int_0^{+\infty} \frac{1}{u} d\mu_{\delta_0}(u) = \hat{G}_\beta(0, 0). \quad (1.20)$$

Taking the expectation we obtain,

$$\int_0^{+\infty} \frac{1}{u} d\alpha_{W,d}(u) = \mathbb{E}_{\nu_d^W} \left[ \hat{G}_\beta(0, 0) \right].$$

However, recall that  $(\hat{G}_\beta^{(n)}(0, 0))_{n \in \mathbb{N}}$  is the bracket of the martingale  $(\psi_\beta^{(n)}(0))_{n \in \mathbb{N}}$ . Moreover, by Theorem 1 in [54] (see Lemma 9 in [154] for a more precise explanation.), if  $d \geq 3$ , there exists  $W_+^{(d)} > 0$  such that for every  $W > W_+^{(d)}$ ,  $(\psi_\beta^{(n)}(0))_{n \in \mathbb{N}}$  is bounded in  $L^2$  and thus

$$\mathbb{E}_{\nu_d^W} \left[ \hat{G}_\beta(0, 0) \right] < +\infty.$$

Consequently, for every  $W > W_+^{(d)}$ ,

$$\begin{aligned} n_{W,d}(E) &= \int_0^E d\alpha_{W,d}(u) \\ &\leq E \int_0^E \frac{1}{u} d\alpha_{W,d}(u) \\ &\leq E \int_0^{+\infty} \frac{1}{u} d\alpha_{W,d}(u) \\ &= E \times \mathbb{E}_{\nu_d^W} \left[ \hat{G}_\beta(0, 0) \right]. \end{aligned}$$

This concludes the proof. □

In Chapter 4, we also prove the localization for high energies of the Green function which is associated with the operator  $H_\beta$  :

**Theorem 1.63.** *Let  $d \in \mathbb{N}^*$ . For every  $W > 0$ , there exist  $K_{4,d,W} > 0$ ,  $\mu_{d,W} > 0$  and  $E_{d,W} > 0$  such that for every  $z \in \mathbb{C} \setminus \mathbb{R}$  such that  $|z| \geq E_{d,W}$ , for every  $x, y \in \mathbb{Z}^d$ ,*

$$\mathbb{E}_{\nu_d^W} \left[ |(H_\beta - z)^{-1}(x, y)|^{1/4} \right] \leq K_{4,d,W} \exp(-\mu_{d,W} \|x - y\|).$$

*Proof.*

By the finite volume criterion given by (11.5) in Theorem 11.2 of [6] applied just on 1 point, we

only need to check that for every  $\varepsilon > 0$ , there exists  $c > 0$  such that for every  $z \in \mathbb{C}$  such that  $|z| \geq c$ ,

$$\mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - z|^{1/4}} \right] \leq \varepsilon. \quad (1.21)$$

In Chapter 4, we will prove something stronger than (1.21). Indeed, there exists  $C > 0$  such that for every  $E \geq 1$ ,

$$\mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - E|^{1/4}} \right] \leq CE^{-1/4}.$$

Actually, the finite volume criterion (11.5) in [6] is stated for an i.i.d potential. However,  $(\beta_x)_{x \in \mathbb{Z}^d}$  is not i.i.d. but 1-dependent. Therefore, a part of Chapter 4 is devoted to an adaptation of the proof of Theorem 11.2 in [6].

□

### 1.6.3 Main results of Chapter 5

Chapter 5 is devoted to the study of the scaling limits of the operator  $H_\beta$  on continuous topological objects as the line or circles and some related topics. It is based on a work which has been achieved with my PhD advisor Christophe Sabot. First, we want to understand the continuous-space version of  $H_\beta$  and  $G_\beta$  by considering a fine-mesh limit on discrete graphs. In [113], Lupu, Sabot and Tarrès managed to find the scaling limit of  $G_\beta/G_\beta(0,0)$  on  $\mathbb{R}$  but they mostly focused in this paper on the dynamics of the continuous-space VRJP. Here, we will not look at the dynamics of the VRJP but we will carry on with the study of the continuous-space version of  $G_\beta$ . We will do that on the simplest graph which is not a tree, that is, the circle and we will make the size of the circle go to infinity. By inverting the continuous-space version of  $G_\beta$ , we will obtain a continuous-space version of  $H_\beta$ . Several other results stems from this analysis :

- We find a discrete-time version of the so-called Matsumoto-Yor properties and we will obtain a new proof of continuous-time Matsumoto-Yor properties by taking the limit. This result is not directly linked to the continuous-space version of  $H_\beta$  but the main tools of the proof are very similar. In particular, it uses the same scaling-limit of the  $\beta$  potential.
- We prove generalizations of the famous Dufresne's identity (see [57]) which states that

$$\int_0^{+\infty} e^{2\alpha_s - s} ds \stackrel{\text{law}}{=} \frac{1}{2\gamma}$$

where  $\gamma$  is a Gamma random variable with parameters  $(1/2, 1)$  and  $\alpha$  is a Brownian motion.

- We compute the density of states of the continuous-space version of  $H_\beta$  on  $\mathbb{R}$  which has a remarkably simple expression.

#### 1.6.3.1 A new approach to Matsumoto-Yor properties

First, we use this scaling limit in order to give a new proof of the Matsumoto-Yor properties. Matsumoto-Yor properties are beautiful results concerning exponential functionals of the Brownian motion which were proved in [121] and [122]. Let  $\alpha$  be a standard Brownian motion on  $\mathbb{R}_+$ . Then we can define the associated geometric Brownian motion  $e$  as  $(e_t)_{t \geq 0} = (\exp(\alpha_t - t/2))_{t \geq 0}$ . Moreover, let us define the related exponential functionals  $T$  and  $Z$  such that for every  $t > 0$ ,

$$T_t = \int_0^t e_s^2 ds \text{ and } Z_t = \frac{T_t}{e_t}.$$

For every  $t \geq 0$ , we define two sigma-fields  $\mathcal{A}_t = \sigma(\alpha_s, s \leq t)$  and  $\mathcal{Z}_t = \sigma(Z_s, s \leq t)$ . Then, Matsumoto and Yor proved the following results :

**Theorem 1.64** (Theorem 1.6 and Proposition 1.7 in [122]).

(i) For every  $t > 0$ ,  $\mathcal{Z}_t \subsetneq \mathcal{A}_t$ .

(ii)  $Z$  is a diffusion process whose infinitesimal generator is

$$\frac{1}{2}z^2 \frac{d^2}{dz^2} + (1+z) \frac{d}{dz}.$$

(iii) For every  $t > 0$ , the conditional distribution of  $e_t$  knowing  $\mathcal{Z}_t$  is an Inverse Gaussian distribution with parameter  $(1, 1/Z_t)$ . More precisely, for every  $t > 0$ , the conditional density of  $e_t$  knowing  $\mathcal{Z}_t$  is

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi Z_t x^3}} e^{-\frac{1}{Z_t} \frac{(x-1)^2}{2x}} dx.$$

In chapter 5, we show that some functionals of the  $\beta$  potential on  $\mathbb{N}^*$  furnish a discrete-time analogous version of Theorem 1.64. Moreover, by considering a relevant scaling limit of the  $\beta$  potential we give a new proof of Theorem 1.64.

*Idea of the new proof of Theorem 1.64.*

Actually, Theorem 1.64 has a discrete counterpart which is given by the VRJP on  $\mathbb{N}^*$ . Let  $m > 0$  and let  $K_m$  be a weight operator on the line graph  $\mathbb{N}^*$  such that for every  $i \in \mathbb{N}^*$ ,  $K_m(i, i+1) = K_m(i+1, i) = m$ . All other entries of  $K_m$  are zero. Then we can define the random operator  $\mathbf{H}_\beta^{(m)}$  on the discrete half-line  $\mathbb{N}^*$  associated with the random field  $\beta \sim \nu_{\mathbb{N}^*}^{K_m}$ . We write  $\mathbf{H}_\beta^{(m)}$  in bold letters in order to avoid the confusion with  $H_\beta^{(\lambda, n)}$  on the discrete circle which shall be introduced later. Now, for every  $n \in \mathbb{N}^*$ , let us define  $\hat{\mathbf{G}}_\beta^{(n, m)} = \left( (\mathbf{H}_\beta^{(m)})|_{[[1, n]] \times [[1, n]]} \right)^{-1}$ . For every  $n \in \mathbb{N}^*$ , we define also

$$\psi_\beta^{(n, m)} = \hat{\mathbf{G}}_\beta^{(n, m)}(1, n)m \text{ and } Z_\beta^{(n, m)} = \frac{\hat{\mathbf{G}}_\beta^{(n, m)}(1, 1)}{\psi_\beta^{(n, m)}}.$$

$\psi_\beta^{(n, m)}$  and  $Z_\beta^{(n, m)}$  will play the same role as  $e_t$  and  $Z_t$  respectively in the discrete-time setting. Furthermore, for every  $n \in \mathbb{N}^*$ , we define

$$\mathcal{A}_{n, m} = \sigma(\psi_\beta^{(k, m)}, 1 \leq k \leq n) \text{ and } \mathcal{Z}_{n, m} = \sigma(Z_\beta^{(k, m)}, 1 \leq k \leq n).$$

The interest of these discrete objects is that they give a discrete version of the results of Matsumoto and Yor :

**Lemma 1.65.** *Let  $m \in \mathbb{N}^*$  be fixed.*

(i) For every  $n \in \mathbb{N}^*$ ,  $\mathcal{Z}_{n, m} \subsetneq \mathcal{A}_{n, m}$ .

(ii)  $(Z_\beta^{(n, m)})_{n \in \mathbb{N}^*}$  is a Markov process. More precisely, for every  $n \in \mathbb{N}^*$ , the law of  $Z_\beta^{(n+1, m)}$  conditionally on  $\mathcal{Z}_{n, m}$  is

$$\frac{Z_\beta^{(n, m)}}{m} \times \frac{1}{IG \left( \frac{1}{m + \frac{1}{Z_\beta^{(n, m)}}}, 1 \right)}.$$

(iii) For every  $n \geq 1$ , the conditional distribution of  $\psi_\beta^{(n, m)}$  knowing  $\mathcal{Z}_{n, m}$  is an Inverse Gaussian distribution with parameter  $(1, 1/Z_\beta^{(n, m)})$ . More precisely, for every  $n \geq 1$  the conditional density of  $\psi_\beta^{(n, m)}$  knowing  $\mathcal{Z}_{n, m}$  is

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi Z_\beta^{(n, m)} x^3}} e^{-\frac{1}{Z_\beta^{(n, m)}} \frac{(x-1)^2}{2x}} dx.$$

A remarkable fact is that the space in the context of the VRJP corresponds to the time in the context of Matsumoto-Yor properties. Then, the main idea of the proof is to take the limit in (i), (ii) and (iii) of Lemma 1.65. It is possible because  $\psi_\beta^{(n,m)}$  converges towards the geometric Brownian motion thanks to a proper renormalisation. More precisely, let  $(\tilde{\psi}^{(m)}(t))_{t \geq 0}$  and  $(\tilde{Z}^{(m)}(t))_{t \geq 0}$  be the continuous linear interpolations of  $(\psi_\beta^{[mt],m})_{t \geq 0}$  and  $(Z_\beta^{[mt],m})_{t \geq 0}$ . Then, we will prove that the following convergence does hold for the topology of uniform convergence on compact sets :

$$\left( \tilde{\psi}^{(m)}(t), \tilde{Z}^{(m)}(t) \right)_{t \geq 0} \xrightarrow[m \rightarrow +\infty]{law} (e_t, Z_t)_{t \geq 0}.$$

Together with Lemma 1.65, it implies directly (iii) and (i) in Theorem 1.64. However, it still requires some work in order to deduce (ii) of Theorem 1.64.  $\square$

### 1.6.3.2 The space-continuous operator on circles

Another part of Chapter 5 consists in constructing a continuous-space version of  $H_\beta$  on some one-dimensional topological spaces as circles. Let  $n \in \mathbb{N}^*$ . Let  $\lambda > 0$ . We define the discretized circle  $\mathcal{C}_{[\lambda n]} = \{-[\lambda n], \dots, [\lambda n]\}$  such that  $-[\lambda n]$  and  $[\lambda n] + 1$  are identified. Let  $W_n^{(\lambda)}$  be a matrix on  $\mathcal{C}_{[\lambda n]}$  such that  $(W_n^{(\lambda)})_{i,j}$  is 0 if  $i$  and  $j$  are not connected and is  $n$  otherwise. Let us denote  $H_\beta^{(\lambda,n)}$  the matrix associated with the random potential  $\beta$  with distribution  $\nu_{\mathcal{C}_{[\lambda n]}}^{W_n^{(\lambda)}}$ . Let  $\mathcal{C}^{(\lambda)}$  be the circle  $[-\lambda, \lambda]$  where the points  $-\lambda$  and  $\lambda$  are topologically identified. We denote by  $G_\beta^{(\lambda,n)}$  the inverse of  $H_\beta^{(\lambda,n)}$ . We define also a continuous bilinear interpolation  $(\tilde{G}_\beta^{(\lambda,n)}(t, t'))_{t, t' \in \mathcal{C}^{(\lambda)}}$  of  $(G_\beta^{(\lambda,n)}([nt], [nt']))_{t, t' \in \mathcal{C}^{(\lambda)}}$ .

Moreover, we introduce a limiting continuous version  $\mathcal{G}^{(\lambda)}$  of  $\tilde{G}_\beta^{(\lambda,n)}$  as follows : Let  $B$  be a Brownian motion on  $\mathbb{R}$  such that  $B(0) = 0$  almost surely. We define the geometric Brownian motion  $M$  by

$$(M_t)_{t \in \mathbb{R}} = (e^{Bt - t/2})_{t \in \mathbb{R}}.$$

Then, the symmetric random kernel  $\mathcal{G}^{(\lambda)}$  on  $\mathcal{C}^{(\lambda)}$  is defined by

$$\mathcal{G}^{(\lambda)}(t, t') = \frac{M_{t'} M_t}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_{t'}^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^{t'} \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right)$$

for every  $t \leq t' \in \mathcal{C}^{(\lambda)}$ . Then, we proved the following convergence result :

**Theorem 1.66.** *Let  $\lambda > 0$ . Then*

$$\tilde{G}_\beta^{(\lambda,n)} \xrightarrow[n \rightarrow +\infty]{law} \mathcal{G}^{(\lambda)}$$

*for the topology of uniform convergence on  $(\mathcal{C}^{(\lambda)})^2$ .*

*Idea of the proof of Theorem 1.66.*

Let  $(A_i)_{i \in \mathcal{C}_{[\lambda n]}}$  be i.i.d Inverse Gaussian random variables with parameters  $(1, n)$ . We will prove that the random potential  $\beta$  with distribution  $\nu_{\mathcal{C}_{[\lambda n]}}^{W_n^{(\lambda)}}$  can be constructed in the following way : for every  $i \in \mathcal{C}_{[\lambda n]}$ ,

$$\beta_i = \frac{n}{2} \left( A_{i+1} + \frac{1}{A_i} \right).$$

Remark that this representation is not valid on any graph. Here, the unidimensional structure of the underlying graph is crucial in order to make work this proof. By means of tedious computations involving this representation of the random potential  $\beta$ , one can get an explicit form for the inverse  $G_\beta^{(\lambda,n)}$  of  $H_\beta^{(\lambda,n)}$ . Thus, up to some negligible term, we can write  $\tilde{G}_\beta^{(\lambda,n)}$  as  $\Phi^{(\lambda)}(Y^{(n)})$

where  $\Phi^{(\lambda)}$  is an explicit functional from the space of continuous functions on  $\mathcal{C}^{(\lambda)}$  into the space of continuous functions on  $\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)}$  and  $t \mapsto Y_t^{(n)}$  is a random continuous function such that if  $j/n \leq t < (j+1)/n$ ,

$$Y_t^{(n)} = \prod_{i=-[\lambda n]}^j A_i + n(t - j/n) \left( \prod_{i=-[\lambda n]}^{j+1} A_i - \prod_{i=-[\lambda n]}^j A_i \right).$$

Moreover, we will prove that  $(Y^{(n)})_{n \in \mathbb{N}}$  converges in law for the topology of uniform convergence on compact sets toward a geometric Brownian motion  $X$  starting from  $-\lambda$ . Therefore,  $(\tilde{G}_\beta^{(\lambda, n)})_{n \in \mathbb{N}^*}$  converges in law for the topology of uniform convergence on  $\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)}$  toward  $\Phi^{(\lambda)}(X)$ . Moreover, in Chapter 5, the functional  $\Phi^{(\lambda)}$  is computed explicitly. By means of this explicit form, we can show that

$$\Phi^{(\lambda)}(X) \stackrel{\text{law}}{=} \mathcal{G}^{(\lambda)}.$$

□

Besides,  $\mathcal{G}^{(\lambda)}$  can be viewed as an operator on  $L^2([-\lambda, \lambda])$  such that for every  $f \in L^2([-\lambda, \lambda])$  and every  $t \in \mathcal{C}^{(\lambda)}$ ,

$$\mathcal{G}^{(\lambda)} f(t) = \int_{\mathcal{C}^{(\lambda)}} \mathcal{G}^{(\lambda)}(t, t') f(t') dt'.$$

We will show that  $\mathcal{G}^{(\lambda)}$  is positive definite and thanks to the following theorem, we can inverse it explicitly. Let us define the domain

$$\mathcal{D}(\mathcal{H}^{(\lambda)}) = \left\{ g \in L^2([-\lambda, \lambda]), \quad \left(\frac{g}{M}\right)' \in L^2([-\lambda, \lambda]), \quad \left(M^2 \left(\frac{g}{M}\right)'\right)' \in L^2([-\lambda, \lambda]), \right. \\ \left. g(-\lambda) = g(\lambda), \quad M_{-\lambda} \left(\frac{g}{M}\right)'(-\lambda) = M_\lambda \left(\frac{g}{M}\right)'(\lambda) \right\}.$$

In the definition above the derivation  $'$  is used in the sense of distributions.

**Theorem 1.67.** *Let  $\lambda > 0$ . Then the image of  $\mathcal{G}^{(\lambda)}$  is  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Moreover,  $\mathcal{G}^{(\lambda)}$  has a bijective inverse  $\mathcal{H}^{(\lambda)}$  from  $\mathcal{D}(\mathcal{H}^{(\lambda)})$  onto  $L^2([-\lambda, \lambda])$ . For every  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ ,*

$$\mathcal{H}^{(\lambda)} g = -\frac{1}{M} \left( M^2 \left( \frac{g}{M} \right)' \right)'.$$

Furthermore,  $\mathcal{H}^{(\lambda)}$  is a positive self-adjoint operator (for the classical inner-product on  $L^2([-\lambda, \lambda])$ ) with domain  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ .

*Idea of the proof of Theorem 1.67.*

We will just verify that

$$\mathcal{G}^{(\lambda)} \mathcal{H}^{(\lambda)} = Id_{\mathcal{D}(\mathcal{H}^{(\lambda)})} \text{ and } \mathcal{H}^{(\lambda)} \mathcal{G}^{(\lambda)} = Id_{L^2([-\lambda, \lambda])}$$

through very heavy computations. □

### 1.6.3.3 New Dufresne type identities

Theorem 1.66 implies also a lot of identities in law involving the geometric Brownian motion which generalize Dufresne's identity.

**Proposition 1.68.** *Let  $f$  be a deterministic continuous non-negative function on  $\mathcal{C}^{(\lambda)}$ . Then, the following identity in law does hold :*

$$\frac{1}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{1}{M_u^2} \left( M_{-\lambda} \int_u^\lambda f(t) M_t dt + M_\lambda \int_{-\lambda}^u M_t f(t) dt \right)^2 du \stackrel{\text{law}}{=} \frac{\left( \int_{-\lambda}^\lambda f(t) dt \right)^2}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameter  $(1/2, 1)$ .

*Idea of the proof of Proposition 1.68.*

Let  $f$  be a deterministic continuous non-negative function on  $\mathcal{C}^{(\lambda)}$ . By Lemma 8.1 in [77], for every  $n \in \mathbb{N}^*$ , for every  $\eta \in \mathbb{R}_+^{\mathcal{C}_{[\lambda n]}}$ ,

$$\sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_{\beta}^{(\lambda, n)}(i, j) \eta_i \eta_j \stackrel{\text{law}}{=} \frac{\left( \sum_{i \in \mathcal{C}_{[\lambda n]}} \eta_i \right)^2}{2\gamma}$$

where  $\gamma$  is distributed like a gamma distribution with parameter  $(1/2, 1)$ . Now, let us apply this fact with  $\eta_i = n^{-1}f(i/n)$ . Then we obtain that

$$\left( \frac{1}{n} \sum_{i \in \mathcal{C}_{[\lambda n]}} f(i/n) \right)^2 \frac{1}{2\gamma} \stackrel{\text{law}}{=} \frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_{\beta}^{(\lambda, n)}(i, j) f(i/n) f(j/n).$$

The left-hand side converges in law toward  $\left( \int_{-\lambda}^{\lambda} f(x) dx \right)^2 \frac{1}{2\gamma}$  because  $f$  is assumed to be continuous. Now, let us focus on the right-hand side. By Theorem 1.66, in the topology of uniform convergence on  $\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)}$  it holds that :

$$\tilde{\mathcal{G}}_{\beta}^{(\lambda, n)} \xrightarrow[n \rightarrow +\infty]{\text{law}} \mathcal{G}^{(\lambda)}$$

Therefore, it is not difficult to prove that

$$\frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_{\beta}^{(\lambda, n)}(i, j) f(i/n) f(j/n) \xrightarrow[n \rightarrow +\infty]{\text{law}} \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \mathcal{G}^{(\lambda)}(t, t') f(t) f(t') dt dt'.$$

Consequently,

$$\int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \mathcal{G}^{(\lambda)}(t, t') f(t) f(t') dt dt' \stackrel{\text{law}}{=} \left( \int_{-\lambda}^{\lambda} f(x) dx \right)^2 \frac{1}{2\gamma}.$$

Moreover, using the explicit form of  $\mathcal{G}^{(\lambda)}$ , one can show that

$$\int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \mathcal{G}^{(\lambda)}(t, t') f(t) f(t') dt dt'$$

equals the left hand-side of Proposition 1.68. □

#### 1.6.3.4 The density of states on the real line

As  $\mathcal{G}^{(\lambda)}$  is a compact self-adjoint operator, one can show that the spectrum of  $\mathcal{H}^{(\lambda)}$  is almost surely a random increasing infinite sequence of eigenvalues which is denoted by  $(E_k(\lambda))_{k \in \mathbb{N}}$ . Moreover, one can compute the asymptotic density of states of  $\mathcal{H}^{(\lambda)}$  when  $\lambda$  goes to infinity. For every  $E \in \mathbb{R}_+^*$ , let us define the random variable  $N_{\lambda}(E)$  which is the number of eigenvalues of  $\mathcal{H}^{(\lambda)}$  which are lower than  $E$ . Then, we have the following result :

**Theorem 1.69.** *For every  $E > 0$ ,*

$$\frac{N_{\lambda}(E)}{2\lambda} \xrightarrow[\lambda \rightarrow +\infty]{\mathbb{P}} \frac{\sqrt{E}}{\pi}.$$

*Idea of the proof of Theorem 1.69.*

The equation  $\mathcal{H}^{(\lambda)}\varphi_E = E\varphi_E$  is equivalent to the Sturm-Liouville equation

$$(M^2\varphi_E')' + EM^2\varphi_E = 0$$

with  $\varphi(-\lambda) \neq 0$ . Some classical results on Sturm-Liouville equations imply that  $\varphi_E$  can be written as  $\varphi_E = R_E \sin(\theta_E)$  where  $R_E$  never vanishes. Therefore,  $\varphi_E$  vanishes  $k$  times in  $[-\lambda, \lambda]$  if and only if  $\theta_E(\lambda) \in [k\pi, (k+1)\pi]$ . Moreover, Sturm-Liouville oscillation theorem states that  $\varphi_{E_k(\lambda)}$  vanishes approximately  $k$  times. Consequently, if  $N_\lambda(E) = k$ , then  $E_k(\lambda) \leq E < E_{k+1}(\lambda)$  which implies that

$$N_\lambda(E) = k \simeq \frac{\theta_{E_k(\lambda)}(\lambda)}{\pi} \simeq \frac{\theta_E(\lambda)}{\pi}.$$

Therefore, for every fixed  $E > 0$ , we only have to study the asymptotic behaviour of  $\theta_E(\lambda)$  when  $\lambda$  goes to infinity. However, according to the theory of Sturm-Liouville equations,  $\theta_E$  is solution of the following ODE with random coefficients :

$$\theta'_E = M_t^{-2} \cos(\theta_E)^2 + EM_t^2 \sin(\theta_E)^2.$$

Moreover a very surprising fact is that  $\zeta_E := -\frac{\cot(\theta_E)}{M^2}$  is a Markov process with explosions. Actually explosions of  $\zeta_E$  occur precisely when  $\theta_E$  is a multiple of  $\pi$ . Therefore, we only have to count the explosions of  $\zeta_E$ . However, by the Markov property, the explosion times of  $\zeta_E$  are i.i.d random variables. As a consequence, the number of explosions of  $\zeta_E$  before time  $t$ , that is the number of times where  $\theta_E$  is a multiple of  $\pi$  before  $t$  is a renewal process which can be studied thanks to classical tools. However, one problem of the Sturm-Liouville equation is that  $M$  is not differentiable. Consequently, we cannot apply directly the Sturm-Liouville theory. Nevertheless, we can apply it with a regularized version of  $M$ . That is why,  $M$  can be replaced by  $M^{(n)}$  which is a  $C^2$  random function which converges uniformly almost surely to  $M$  as  $n$  goes to infinity. (For example,  $M^{(n)}$  can be taken as a polynomial interpolation of  $M$ .) The theory of Sturm-Liouville equation is robust enough to allow us to make  $n$  goes to infinity.  $\square$

One can remark that this asymptotic density of states is the same as the density of states of  $-\Delta$  on  $\mathbb{R}$ . However the spectrum of  $-\Delta$  and  $\mathcal{H}^{(\lambda)}$  is very different at a microscopic scale.

#### 1.6.4 Main results of Chapter 6

Chapter 6 comes from a joint paper which has been written with Gerard, Sabot and Zeng. This paper will be published very soon. Actually, at the beginning, this paper was written only by Gerard, Sabot and Zeng and was posted on arxiv as a preprint. It consisted in applying a Lamperti time-scale to a system of SDEs which was found by Sabot and Zeng in [155] and which is related to the  $\beta$  potential. Thanks to this Lamperti time-scale, they proved a multidimensional generalization of the Matsumoto-Yor opposite drift theorem which was originally proved in dimension 1 in [119]. However, the authors were not totally satisfied of their preprint because they did not manage to relate their results to the Matsumoto-Yor properties given by Theorem 1.64 even if such a link was strongly suspected. My contribution consists in having made this link rigorous. More precisely, under the supervision of C. Sabot, I proved Theorems 6.4, 6.5 and 6.6 in Chapter 6.

Let  $(V, E)$  be a finite graph. Let  $(W_{i,j})_{i,j \in V}$  be non-negative symmetric weights on  $V \times V$  such that  $W_{i,j} > 0$  if and only if  $\{i, j\} \in E$ . For every  $t \in (\mathbb{R}_+)^V$ , we denote by  $K_t$  the matrix  $K_t = Id - tW$  where  $t$  stands for the diagonal matrix whose diagonal entries are given by  $(t_i)_{i \in V}$ . Let  $\theta, \eta \in (0, +\infty)^V$ . Let  $(\varrho, T)$  be solution of the following system of SDEs :

$$\begin{cases} \varrho_i(v) = \log(\theta_i) + \tilde{B}_i(v) + \int_0^v \left( -\frac{1}{2} - e^{\varrho_i(u)} \left( \tilde{W}^{(u)}(e^{\varrho(u)} + T(u)\eta) + \eta \right)_i \right) du, \\ T_i(v) = \int_0^v e^{2\varrho_i(u)} du, \end{cases} \quad (1.22)$$

for  $i \in V$  and  $v \geq 0$ , where  $(\tilde{B}_i)_{i \in V}$  is a  $|V|$ -dimensional standard Brownian motion,  $e^{\varrho(u)}$  denotes the vector  $(e^{\varrho_i(u)})_{i \in V}$ , and

$$\tilde{W}^{(u)} = WK_{T(u)}^{-1} = W(\text{Id} - T(u)W)^{-1}.$$

Gerard, Sabot and Zeng had already proved the following result which makes a link with the random potential  $\beta$  :

**Theorem 1.70.** *Let  $(\varrho, T)$  be the solution of the previous SDE.*

(i) *For all  $i \in V$ , we have*

$$T_i(u) = \int_0^u e^{2\varrho_i(v)} dv \xrightarrow[u \rightarrow \infty]{a.s.} T_i^\infty,$$

where  $\left(\frac{1}{2T_i^\infty}\right)_{i \in V}$  is distributed according to  $\nu_V^{W, \theta, \eta}$  which is a generalization of  $\nu_V^{W, \eta}$ .

(ii) *There exists a standard  $|V|$ -dimensional Brownian motion  $B^*$  which is independent of  $T^\infty$  such that for  $i \in V$  and  $u \geq 0$ ,*

$$\varrho_i(u) = \log(\theta_i) + B_i^*(u) + \frac{1}{2}u + \log\left(\frac{T_i^\infty - T_i(u)}{T_i^\infty}\right).$$

*In particular, the processes  $(\varrho_i, T_i)_{i \in V}$  are independent site by site conditionally on  $T^\infty$ .*

Theorem 1.70 gives a multidimensional generalization of the Matsumoto-Yor opposite drift theorem. Indeed the drift  $-\frac{1}{2}$  in (1.22) becomes  $\frac{1}{2}$  in (ii) of Theorem 1.70. Thanks to Theorem 1.70, we proved the following multidimensional generalization of Theorem 1.64 :

**Theorem 1.71.** *For every  $u \geq 0$  and for every  $i \in V$ , we define  $Z_i(u) = T_i(u) \exp(-\varrho_i(u))$ . Let us denote by  $(\mathcal{Z}_u)_{u \geq 0}$  the natural filtration associated with  $(Z(u))_{u \geq 0}$ . Then, it holds that*

(i) *For every  $u > 0$ ,*

$$\mathcal{Z}_u \not\subseteq \sigma(\varrho_v, v \leq u).$$

(ii) *The process  $(Z(u))_{u \geq 0}$  has independent components and for every  $i \in V$ , there exists a Brownian motion  $\hat{B}_i$  such that  $(Z_i(u))_{u \geq 0}$  is solution of the SDE*

$$dZ_i(u) = Z_i(u)d\hat{B}_i(u) + (\theta_i + Z_i(u))du.$$

(iii)  *$(Z(u))_{u \geq 0}$  is independent of  $T^\infty$ .*

(iv) *For every  $u > 0$ , let us define  $\beta(u) = (\beta_i(u))_{i \in V} = (1/(2T_i(u)))_{i \in V}$ . Then, for every  $u > 0$ , the conditional law of  $\beta(u)$  given  $\mathcal{Z}_u$  is*

$$\mathcal{L}\left(\beta(u) \middle| \mathcal{Z}_u, Z(u) = z\right) = \nu_V^{W, \theta, \eta + 1/z}$$

where  $\eta + 1/z$  is the vector  $(\eta_i + 1/z_i)_{i \in V}$ .

# Chapitre 2

## Invariant measures of critical branching random walks in high dimension

### Abstract

This chapter is based on the article [142] which has been written under the supervision of Xinxin Chen. In this chapter, we characterize cluster-invariant point processes for critical branching spatial processes on  $\mathbb{R}^d$  for all large enough  $d$  when the motion law is  $\alpha$ -stable or has a finite discrete range. More precisely, when the motion is  $\alpha$ -stable with  $\alpha \leq 2$  and the offspring law  $\mu$  of the branching process has an heavy tail such that  $\mu(k) \sim k^{-2-\beta}$ , then we need the dimension  $d$  to be strictly larger than the critical dimension  $\alpha/\beta$ . In particular, when the motion is Brownian and the offspring law  $\mu$  has a second moment, this critical dimension is 2. Contrary to the previous work of Bramson, Cox and Greven in [32] whose proof used PDE techniques, our proof uses probabilistic tools only.

### 2.1 Introduction

#### 2.1.1 Definition of the model and first notation

Let  $\mu$  be a probability distribution on  $\mathbb{N}$  called the "offspring law". We assume that  $\mu$  is critical, that is,  $\sum_{k=0}^{+\infty} k\mu(k) = 1$  and non-trivial, that is,  $\mu(1) < 1$ . Let  $\mathcal{P}$  be a probability distribution on  $\mathbb{R}^d$  with  $d \in \mathbb{N}^*$ .  $\mathcal{P}$  is called the "motion law". We define a discrete-time critical branching process on  $\mathbb{R}^d$  in the following way :

We start with one particle at an initial position  $x \in \mathbb{R}^d$ . It is generation 0 of the process. Let  $\mathcal{Z}_n$  be the set of particles at generation  $n$ . Every particle  $u \in \mathcal{Z}_n$  gives birth independently to  $N_u$  offsprings where  $N_u$  has law  $\mu$ . The offsprings of  $u$  jump independently of each others from the position of  $u$  according to the motion law  $\mathcal{P}$ . All the offsprings of  $\mathcal{Z}_n$  with their new positions form the  $(n+1)$ -th generation of the branching process. We remark that the underlying genealogical tree  $\mathbf{T}_{gen}$  of the model is a Galton-Watson tree with offspring law  $\mu$ . As  $\mu$  is critical,  $\mathbf{T}_{gen}$  will die out a.s. For every branching process defined in this article, we use the following notation :

$$\begin{aligned} |u| &:= \text{the generation of some particle } u \\ \mathcal{Z}_n &:= |\mathcal{Z}_n| = \text{the total number of particles in generation } n \\ \{S_u : |u| = n\} &:= \text{the set of particle positions in generation } n \\ \mathcal{Z}_n(A) &:= \text{the number of particles in the set } A \text{ in generation } n \end{aligned}$$

A special case of this model is when  $\mathcal{P}$  is a standard Gaussian distribution. In this situation the model is called the critical Wiener branching process. One can refer to [144], [147] and [145] for more properties of the critical Wiener branching process. Another particular case is when  $\mathcal{P}$  corresponds to the simple random walk on  $\mathbb{Z}^d$ . In [106], Lalley and Zheng studied this critical

branching simple random walk conditioned on long survival when  $d \geq 2$ . In particular, they prove a spatial generalization of the famous Yaglom's theorem when  $d \geq 3$ .

In the previous paragraph, we defined a critical branching process starting from a single particle. However we can also start from a point process in  $\mathbb{R}^d$ . Let us consider a point process, that is, a random locally finite integer valued measure :

$$\Theta = \sum_{i \in I} \delta_{x_i}$$

where  $I$  is finite or countable. For every point  $x_i$ , it is possible to define a critical branching process with critical offspring law  $\mu$  and motion law  $\mathcal{P}$  starting from an ancestor located at  $x_i$ . For every  $i \in I$  and every  $n \in \mathbb{N}$ , we denote by  $\{S_u^{(i)}, |u| = n\}$  the set of positions of the  $n$ -th generation of the branching process starting from an ancestor located at  $x_i$ . We assume that these branching processes are independent of each other. This collection of critical branching processes gives us a sequence  $(\Theta_n)_{n \in \mathbb{N}}$  of random integer valued measures on  $\mathbb{R}^d$ . More precisely  $\Theta_0 = \Theta$  and at each time  $n \geq 1$ ,

$$\Theta_n = \sum_{i \in I} \sum_{|u|=n} \delta_{S_u^{(i)}}.$$

We have to be careful here because it is possible that for some  $n$ ,  $\Theta_n$  is not locally finite anymore. For example, let us consider a critical Wiener branching process. If we start with a Poisson point process  $\Theta_0$  of intensity  $e^{\|x\|^3} dx$ , even  $\Theta_1$  is not a point process anymore. However  $(\Theta_n)_{n \in \mathbb{N}}$  is a sequence of locally finite point processes when  $\Theta$  is a Poisson point process of constant intensity. Indeed in this situation, if  $A$  is a closed ball of  $\mathbb{R}^d$ ,  $\mathbb{E}[\Theta_n(A)]$  remains constant and proportional to the volume of  $A$  because  $\mu$  is critical. One can refer to [93] or [124] for more information on point processes. Moreover, we offer a short warm-up about point processes in subsection 2.2.3 for this article to be self-contained.

A critical branching process starting from a point process is a special case of a more general theory known as the theory of cluster fields. This theory was originally developed by Liemant, Kerstan, Matthes and Prehn in several papers (in German), e.g [111] and [96]. The book [124] summarizes most of important facts about this theory. (Especially chapters 11 and 12.) A very important issue about point processes is to know whether  $(\Theta_n)_{n \in \mathbb{N}}$  converges toward a non-trivial point process or not. This is strongly linked with the concept of stability which is studied in [94]. It is well-known that in dimension 1 or 2, if the motion  $\mathcal{P}$  is a Gaussian random variable with mean 0 and variance 1,  $(\Theta_n)_{n \in \mathbb{N}}$  converges in law in the vague sense toward the trivial null measure. (See for example sections 7 and 8 of [60] in a continuous time setting.) On the contrary, if  $\Theta_0$  is a Poisson point process with constant intensity, in dimension  $d \geq 3$ ,

$$\Theta_n \xrightarrow[n \rightarrow +\infty]{law} \Lambda$$

where  $\Lambda$  is non-trivial. (See [50] in a continuous-time setting.) This limiting point process  $\Lambda$  is cluster-invariant, that is, if  $\Theta_0 = \Lambda$ , then  $\Theta_n$  is distributed as  $\Lambda$  for every  $n \in \mathbb{N}^*$ . This raises the natural question of classification of all cluster-invariant point processes. If we assume spatial stationarity (that is,  $\Theta_0(\cdot + x) \stackrel{law}{=} \Theta_0$  for every  $x \in \mathbb{R}^d$ ) or boundedness of the intensity (that is, there exists  $C > 0$  such that  $\mathbb{E}[\Theta_0(A)] \leq CVol(A)$  for every borel set  $A$  in  $\mathbb{R}^d$ ) of the initial process  $\Theta_0$ , the classification of cluster-invariant point processes is a well-known fact in a very general setting. For example, one can refer to Theorem 12.4.6 in [124]. However, spatial stationarity is a very strong hypothesis. The classification of cluster-invariant point processes for critical branching Brownian motion when  $d \geq 3$  is treated without any extra assumption like spatial stationarity in [32]. Their proof is given in a continuous time setting and is based on PDE's techniques. Our article aims to characterize all cluster-invariant point processes in a more general setting by using only probabilistic tools. We also mention that the method which is used in this article is inspired by [39] which treats 1-dimensional binary branching Brownian motion with critical drift.

### 2.1.2 Previous results

Let us recall the known result on critical branching Brownian motion in continuous time which is defined as follows :

- It starts with one particle.
- Each particle dies at rate 1.
- At the end of its life, a particle is replaced by 0 particle with probability 1/2 and by 2 particles with probability 1/2.
- During its lifetime, a particle moves like a  $d$ -dimensional standard Brownian motion.

Let us summarize the following results of [60], [50] and [32]. For every random variable  $X$ ,  $\mathcal{L}(X)$  stands for the law of  $X$ .

**Theorem 2.1** ([60], [50] and [32]). *If we consider a critical branching Brownian motion starting from a Poisson point process  $\Theta_0$  of constant intensity  $\theta$ , it induces a continuous time family of point processes  $(\Theta_t)_{t \geq 0}$  by collecting the positions of the alive particles at time  $t$ . Then we have the following results :*

- (i) *If  $d \in \{1, 2\}$ , then  $(\Theta_t)_{t \geq 0}$  converges vaguely toward the null point process.*
- (ii) *If  $d \geq 3$ , then  $(\Theta_t)_{t \geq 0}$  converges vaguely toward a non-trivial point process  $\tilde{\Lambda}_\infty^{d, \theta}$ .*

Furthermore, it characterizes the set of cluster-invariant point processes, that is, point processes  $\Theta$  such that  $\Theta_t \stackrel{\text{law}}{=} \Theta$  for every  $t \in \mathbb{R}_+$  if  $\Theta_0 = \Theta$ . Indeed,

$$\{\mathcal{L}(\Theta), \Theta \text{ is cluster-invariant}\}$$

is a convex set whose extreme points are

$$\{\mathcal{L}(\tilde{\Lambda}_\infty^{d, \theta}), \theta \in \mathbb{R}_+\}.$$

**Remark 2.1.** *The first point (i) stems from sections 7 and 8 of [60]. The second point (ii) is Theorem 7.1 in [50]. Moreover, characterization of cluster-invariant point processes is Theorem 1 in [32].*

### 2.1.3 Main results

Now let us state our results which generalize Theorem 2.1 in the discrete time setting. Indeed we work with a general critical offspring law  $\mu$  and a quite general motion law  $\mathcal{P}$ . We consider the following three cases for  $\mathcal{P}$  and  $\mu$  in this paper.

1. Hypothesis  $\mathcal{H}_1$  : We assume that  $d \geq 3$ . The distribution  $\mathcal{P}$  is a  $d$ -dimensional Gaussian random variable with mean 0 and covariance matrix  $\Sigma = I_d$  where  $I_d$  is the identity matrix. Moreover  $\sigma^2 := \sum_{k=0}^{+\infty} k^2 \mu(k) < +\infty$ .
2. Hypothesis  $\mathcal{H}_2$  : We assume that  $\mathcal{P}$  can be written as

$$\mathcal{P} = \sum_{x \in \mathcal{R}} p(x) \delta_x$$

where  $\mathcal{R}$  is a finite subset of  $\mathbb{Z}^d$  with  $d \geq 3$ . Moreover  $p$  is symmetrical in the sense that for every  $x \in \mathcal{R}$ ,  $-x \in \mathcal{R}$  and  $p(-x) = p(x)$ .  $\mathcal{P}$  has a positive definite covariance matrix  $\Sigma$ . In addition, we assume that the random walk generated by the motion  $\mathcal{P}$  is aperiodic. Furthermore  $\sigma^2 := \sum_{k=0}^{+\infty} k^2 \mu(k) < +\infty$ .

3. Hypothesis  $\mathcal{H}_3$  : The motion law  $\mathcal{P}$  is given by a spherically symmetric  $\alpha$ -stable law with  $\alpha \in ]0, 2]$ . More precisely for every  $y \in \mathbb{R}^d$ ,

$$\int \exp(i \langle y, x \rangle) \mathcal{P}(dx) = \exp \left( - \left( \sum_{k=1}^d |y_k|^2 \right)^{\alpha/2} \right).$$

The critical offspring law  $\mu$  has no second moment anymore. However we assume that there exists  $\beta \in ]0, 1]$  such that for every  $\gamma < \beta$ ,  $\sum_{k=0}^{+\infty} k^{1+\gamma} \mu(k) < +\infty$ . Moreover, we assume that  $d > \alpha/\beta$ .

Now, let  $X$  be a non-negative random variable. If we assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_3$ , let  $\Lambda_0^{d,X}$  be a Poisson point process with distribution

$$PPP(X\lambda(dx))$$

where  $\lambda$  is the Lebesgue measure. If we assume hypothesis  $\mathcal{H}_2$ , let  $\Lambda_0^{d,X}$  be the discrete Poisson point process

$$PPP\left(X \sum_{x \in \mathbb{Z}^d} \delta_x\right).$$

As in subsection 2.1.1, we can define the sequence  $(\Lambda_n^{d,X})_{n \in \mathbb{N}}$  by considering the critical branching process starting from  $\Lambda_0^{d,X}$ .  $(\Lambda_n^{d,X})_{n \in \mathbb{N}}$  is a sequence of locally finite measures because the underlying Galton-Watson tree is critical and we started from an homogeneous Poisson point process. In particular, for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E}\left[\Lambda_n^{d,X}(A) | \sigma(X)\right] = \mathbb{E}\left[\Lambda_0^{d,X}(A) | \sigma(X)\right] = X \text{Vol}(A) < +\infty$$

where  $\text{Vol}(A)$  is the Lebesgue measure of  $A$  if we assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_3$  and the cardinality of  $A$  if we assume hypothesis  $\mathcal{H}_2$ . In this paper, a "closed ball" designates a euclidean closed ball of  $\mathbb{R}^d$ . Under hypothesis  $\mathcal{H}_2$ , we always assume that a closed ball contains at least one point of  $\mathbb{Z}^d$ .

**Theorem 2.2** (Convergence Theorem). *We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Then, there exists a non-trivial point process  $\Lambda_\infty^{d,X}$  such that*

$$\Lambda_n^{d,X} \xrightarrow[n \rightarrow +\infty]{law} \Lambda_\infty^{d,X}$$

*in the vague topology. In addition, the point process  $\Lambda_\infty^{d,X}$  can be described in the following way on every closed ball  $A$  :*

$$\Lambda_\infty^{d,X}(\cdot \cap A) = \sum_{i=1}^{P_A} \mathcal{N}_A^{(i)}$$

*where  $(\mathcal{N}_A^{(i)})_{i \geq 1}$  are i.i.d copies of a point process  $\mathcal{N}_A$  defined in Proposition 2.16 and  $P_A$  is a Poisson random variable of parameter  $X I_A$  where  $I_A$  is a constant defined in definition 2.14.  $P_A$  and  $(\mathcal{N}_A^{(i)})_{i \geq 1}$  are assumed to be independent.*

A point process  $\Theta$  is said to be cluster-invariant if the sequence  $(\Theta_n)_{n \in \mathbb{N}}$  obtained by the critical branching process starting from  $\Theta_0 = \Theta$  as in subsection 2.1.1 satisfies

$$\Theta_n \stackrel{law}{=} \Theta$$

for every  $n \in \mathbb{N}$ . One sees easily that the limiting point processes obtained in Theorems 2.1 and 2.2 are cluster-invariant. In fact, all cluster-invariant point processes are given by the following theorem :

**Theorem 2.3** (Characterization of cluster-invariant measures). *Let us assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Let  $\Theta$  be a cluster-invariant point process. Then there exists a non-negative random variable  $X$  such that :*

$$\Theta \stackrel{law}{=} \Lambda_\infty^{d,X}.$$

*Moreover  $\Lambda_\infty^{d,X}$  is cluster-invariant for every non-negative random variable  $X$ .*

Theorems 2.2 and 2.3 assumed hypothesis  $\mathcal{H}_1$  or hypothesis  $\mathcal{H}_2$ . Under these hypotheses, the proofs are quite similar. Under the hypothesis  $\mathcal{H}_3$ , the results of Theorems 2.2 and 2.3 remain true. However proofs require some slight modifications. We focus on these modifications in section 2.6.

**Theorem 2.4.** *Let us assume hypothesis  $\mathcal{H}_3$ . Then the conclusions of Theorems 2.2 and 2.3 remain true.*

**Remark 2.2.** *Actually, assuming hypothesis  $\mathcal{H}_2$  the convergence of  $(\Lambda_n^{d,X})_{n \in \mathbb{N}}$  toward  $\Lambda_\infty^{d,X}$  was already known by combining Theorem 5.1 in [49] and the stability criterion which is given by Kallenberg in [94]. Moreover, assuming hypotheses  $\mathcal{H}_1$  and  $\mathcal{H}_3$ , stability of the cluster field was proved in [81] in a continuous-time setting but not in our discrete setting. It changes a lot of things because the continuous-time setting enables to use methods involving PDEs. Moreover, some of these proofs used the notation of the book [124] which are quite difficult to handle. However our proof unifies all these cases in a unique proof involving probabilistic tools only and which uses the modern notation of branching random walks.*

### 2.1.4 Organisation of the paper

- In section 2.2, we begin by recalling some results about local limit theorems, heat kernel estimates and point processes. In subsection 2.2.4, we quickly explain the classical "spine method" which is a fundamental tool in this article. Finally, in subsection 2.2.5, we study the branching process starting from a single particle conditionally on survival in a given set.
- In section 2.3, we prove Theorem 2.2.
- Theorem 2.2 gives us the point process  $\Lambda_\infty^{d,X}$  as a limiting object. We give an independent proof of the compatibility of  $\Lambda_\infty^{d,X}$  in section 2.4.
- In section 2.5, we prove Theorem 2.3.
- Then, we briefly explain how to prove analogous versions of Theorems 2.2 and 2.3 under the hypothesis  $\mathcal{H}_3$  in section 2.6.
- We finish in section 2.7 by a short discussion about our results.

## 2.2 Preliminaries

### 2.2.1 Local-limit Theorem

The motion law  $\mathcal{P}$  is not always a Gaussian random variable. However all our computations are easier in this case. The local-limit Theorem is a fundamental tool in order to make a link between general random walks and Gaussian random walks.

**Proposition 2.5** (Local-limit Theorem, Theorem 2.1.1 of [108]). *Let us assume that  $\mathcal{P}$  satisfies hypothesis  $\mathcal{H}_2$ . Let  $(\hat{S}_n)_{n \in \mathbb{N}}$  be a random walk with motion law  $\mathcal{P}$  starting from 0. Then we have the following uniform convergence :*

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{Z}^d} n^{d/2} \left| \mathbb{P}(\hat{S}_n = x) - \frac{1}{(2\pi)^{d/2} \det(\Sigma)^{1/2}} \times \exp\left(-\frac{1}{2n} \langle x, \Sigma^{-1} x \rangle\right) \right| = 0.$$

Remark that hypothesis  $\mathcal{H}_2$  is really strong. Actually we only need the finiteness of the second moment of  $\mathcal{P}$  for the local-limit Theorem to be true. (See Theorem 2.3.9 in [108].) Moreover, this theorem implies the following useful corollary.

**Corollary 2.6.** *Let  $(\hat{S}_n)_{n \in \mathbb{N}}$  be a random walk whose motion law satisfies hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Let  $A$  be a closed ball. Then there exists a constant  $c_d$ , depending only on the dimension  $d$  and*

the motion law  $\mathcal{P}$ , such that for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{P}(\hat{S}_n \in A) \leq c_d \frac{|A|}{n^{d/2}}$$

where  $|A|$  is the Lebesgue measure of  $A$ .

### 2.2.2 Heat kernel estimate

In addition to the local-limit Theorem, we also need the following so-called heat kernel estimate :

**Proposition 2.7** (Heat kernel estimate). *Let us assume that  $\mathcal{P}$  satisfies hypothesis  $\mathcal{H}_2$ . Let  $(\hat{S}_n)_{n \in \mathbb{N}}$  be a random walk with motion law  $\mathcal{P}$  starting from 0. Then there exists a positive constant  $C_1$  such that for every  $x \in \mathbb{Z}^d$  and for every  $n \in \mathbb{N}^*$  :*

$$\mathbb{P}(\hat{S}_n = x) \leq C_1 n^{-d/2} \exp\left(-\frac{\|x\|^2}{C_1 n}\right).$$

Moreover there exist positive constants  $\tau$  and  $C_2$  such that for every  $n \in \mathbb{N}^*$  and for every  $x \in \mathbb{Z}^d$  satisfying  $\|x\| \leq \tau n$  :

$$\mathbb{P}(\hat{S}_n = x) \geq C_2 n^{-d/2} \exp\left(-\frac{\|x\|^2}{C_2 n}\right).$$

*Proof of Proposition 2.7.*

A random walk  $(\hat{S}_n)_{n \in \mathbb{N}}$  with motion law  $\mathcal{P}$  under the hypothesis  $\mathcal{H}_2$  can be interpreted as a random walk with conductances  $(c(x, y))_{x, y \in G}$  where  $G$  is a graph whose vertices are given by  $\mathbb{Z}^d$  and  $(x, y)$  is an edge of  $G$  iff  $x - y \in \mathcal{R}$ . We recall that  $\mathcal{R}$  is the finite support of  $\mathcal{P}$ . Moreover, for every edge  $(x, y)$  in the graph  $G$ ,  $c(x, y) = p(y - x) = p(x - y)$ . For every  $R \geq 1$ , we define  $\mathcal{B}(R) = \{x \in G, d(0, x) \leq R\}$  and  $V(R) = |\mathcal{B}(R)|$  where  $d$  is the graph distance in  $G$ . According to Theorem 3.3.5 and Proposition 3.3.2 of [103], two conditions have to be checked in order to satisfy heat kernel estimate :

- Condition  $(VD)$  : There exists a positive constant  $C$  such that for every  $R \geq 1$ ,

$$V(2R) \leq CV(R).$$

- Condition  $(WPI(2))$  : There exists a positive constant  $C'$  such that for every  $R \geq 1$  and for every  $f : \mathcal{B}(R) \mapsto \mathbb{R}$ ,

$$\sum_{x \in \mathcal{B}(R)} (f(x) - \bar{f}_R)^2 \leq C' R^2 \sum_{x, y \in \mathcal{B}(2R)} c(x, y) (f(x) - f(y))^2$$

$$\text{where } \bar{f}_R = \frac{1}{V(R)} \sum_{y \in \mathcal{B}(R)} f(y).$$

Verifying condition  $(VD)$  is straightforward in our case. Moreover Condition  $(WPI(2))$  is a consequence of Theorem 4.1 in [156] applied with  $p = 2$ .  $\square$

### 2.2.3 A warm-up about point processes

For reader's convenience, we recall a few facts about point processes. One can refer to [93] and [124] for more details. A point measure  $m$  is an integer-valued measure on  $\mathbb{R}^d$  such that  $m$  is finite on all compact sets of  $\mathbb{R}^d$ . We denote by  $\mathbf{N}$ , the set of point measures. We can equip  $\mathbf{N}$  with a  $\sigma$ -field  $\mathfrak{N}$  which is the smallest  $\sigma$ -field that makes measurable the applications  $m \mapsto \int f dm$  for every  $f$  in  $\mathcal{F}_c(\mathbb{R}^d)$ , the set of compactly supported continuous functions on  $\mathbb{R}^d$ . A point process  $\Theta$  is then defined as a random variable from a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  into  $(\mathbf{N}, \mathfrak{N})$ . It is a random collection of positions which is locally finite. Sometimes, we use the following abuse of

notation : we write  $x \in \Theta$  as  $x$  runs through the support of  $\Theta$  with multiplicity. For example, if  $\Theta(\{x\}) = 3$ ,  $x$  is counted three times. A natural question is to know how to characterize the distribution of a point process. Let us state the following useful criterion.

**Lemma 2.8** (Corollary 2.3 in [93]). *Let  $\xi_1$  and  $\xi_2$  be two point processes. Then the following are equivalent :*

1.  $\xi_1$  and  $\xi_2$  have the same distribution.
2. For every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$ , the set of non-negative compactly supported continuous functions,

$$\mathbb{E} \left[ \exp \left( - \int f(x) d\xi_1(x) \right) \right] = \mathbb{E} \left[ \exp \left( - \int f(x) d\xi_2(x) \right) \right].$$

Further, a sequence of point processes  $(\xi_n)_{n \in \mathbb{N}}$  is said to converge in law in the vague topology toward a point process  $\xi$  if and only if  $\int f d\xi_n \xrightarrow[n \rightarrow +\infty]{law} \int f d\xi$  for every  $f \in \mathcal{F}_c(\mathbb{R}^d)$ . One can also say that  $\xi_n$  converges vaguely toward  $\xi$ . Actually, the following criterion characterizes convergence in law in the vague topology.

**Lemma 2.9** (Theorem 4.11 in [93]). *A sequence of point processes  $(\xi_n)_{n \in \mathbb{N}}$  converges in law toward a point process  $\xi$  in the vague topology if and only if*

$$\mathbb{E} \left[ \exp \left( - \int f(x) d\xi_n(x) \right) \right] \xrightarrow[n \rightarrow +\infty]{} \mathbb{E} \left[ \exp \left( - \int f(x) d\xi(x) \right) \right]$$

for every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$ , the set of non-negative compactly supported continuous functions.

We also need a criterion for the existence of a limit point process :

**Lemma 2.10.** *Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of point processes. Let us assume that for every  $f \in \mathcal{F}_c(\mathbb{R}^d)$  and for every  $\eta \in \mathbb{R}$ ,*

$$\mathbb{E} \left[ \exp \left( i\eta \int f(x) d\xi_n(x) \right) \right] \xrightarrow[n \rightarrow +\infty]{} \Phi_f(\eta)$$

and that  $\eta \mapsto \Phi_f(\eta)$  is a continuous function, then  $(\xi_n)_{n \in \mathbb{N}}$  converges in law in the vague topology toward some point process  $\xi$ .

*Proof of Lemma 2.10.*

We combine Corollary 4.14 in [93] and strong Lévy's continuity Theorem. (See [69].) □

## 2.2.4 Spine method and change of measure

A key ingredient in our proof is the following spine method which was developed largely in the study of branching processes. (See [115] or section 2 in [87].) Let us begin with some useful notation. First, we introduce  $\mathcal{U} := \{\emptyset\} \cup \bigcup_{k=1}^{+\infty} (\mathbb{N}^*)^k$  which is called Neveu's space. Every  $u = i_1 \cdots i_r$  represents the labelling of a particle at generation  $r$ . Indeed  $u$  is the  $i_r$ -th offspring of the  $i_{r-1}$ -th offspring of  $\cdots$  of the  $i_1$ -th offspring of the root  $\emptyset$ . The length  $r$  of  $u$  is denoted by  $|u|$ . The parent of  $u$  is denoted by  $\tilde{u}$ . Moreover, we introduce the partial order  $\leq$  on  $\mathcal{U}$  by  $u \leq v$  if  $u$  is an ancestor of  $v$ . Observe that the notion of parent gives a natural (not locally finite) graph structure on  $\mathcal{U}$  : we put an edge between  $u$  and  $\tilde{u}$  for every  $u \in \mathcal{U} \setminus \{\emptyset\}$ . Every subset  $\mathcal{X}$  of  $\mathcal{U}$  inherits the graph structure of  $\mathcal{U}$ . In particular, for every  $\mathcal{X} \subset \mathcal{U}$  and for every  $x \in \mathcal{X}$ , we denote the connected component of  $x$  in  $\mathcal{X}$  by  $C(x, \mathcal{X})$ . If  $u, v \in \mathcal{U}$ , we denote by  $uv$  the concatenation of  $u$  and  $v$ . Furthermore, we need the notion of plane tree. A subgraph  $t$  of  $\mathcal{U}$  is a plane tree if and only if

- $\emptyset \in t$ ,
- $t$  is locally finite,

— If there exists  $j \in \mathbb{N}^*$  such that  $uj \in t$ , then  $u \in t$  and  $ui \in t$  for every  $i \in \{1, \dots, j\}$ . Then let us define  $\mathcal{V} := \{(u, s) : u \in \mathcal{U}, s \in \mathbb{R}^d\}$ . If  $\mathcal{T}$  is a subset of  $\mathcal{V}$ , let us define

$$\mathcal{T}_{gen} := \{u \in \mathcal{U} : \exists s \in \mathbb{R}^d, (u, s) \in \mathcal{T}\}.$$

Finally we define the set of plane marked trees

$$\mathcal{E} = \{\mathcal{T} \subset \mathcal{V} : C(\emptyset, \mathcal{T}_{gen}) \text{ is a plane tree, } \forall u \in \mathcal{T}_{gen}, \exists! s_u \in \mathbb{R}^d \text{ such that } (u, s_u) \in \mathcal{T}\}.$$

For every  $n \in \mathbb{N}$ , we define the  $\sigma$ -field  $\mathcal{F}_n$  on  $\mathcal{E}$  by  $\mathcal{F}_n := \sigma(\{(u, s_u) : |u| \leq n\})$  and we denote by  $\mathcal{F}_\infty$  the  $\sigma$ -field  $\sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)$ . Let  $\mathbf{T} : \mathcal{E} \rightarrow \mathcal{E}$  be the identity map. If we have a probability measure  $\mathbf{M}$  on  $\mathcal{E}$ ,  $\mathbf{T}$  can be seen as a random object of  $\mathcal{E}$  with distribution  $\mathbf{M}$ . We use the notation  $\mathbb{E}_{\mathbf{M}}$  to mean that we integrate functionals of  $\mathbf{T}$  with respect to  $\mathbf{M}$ . For every  $\mathcal{T} \in \mathcal{E}$  with underlying tree  $\mathcal{T}_{gen}$ , we define a new element  $\mathcal{T}^u$  of  $\mathcal{E}$  starting from  $u$  and its underlying tree as

$$\mathcal{T}_{gen}^u := \{v \in \mathcal{U} : uv \in \mathcal{T}_{gen}\}$$

and

$$\mathcal{T}^u = \{(v, s_{uv} - s_u) : v \in \mathcal{T}_{gen}^u\}.$$

Moreover, for every  $u \in \mathcal{U}$ , we define the measurable function  $\mathbf{T}^u$  from  $\mathcal{E}$  into  $\mathcal{E}$  by  $\mathbf{T}^u(\mathcal{T}) = \mathcal{T}^u$  for every  $\mathcal{T} \in \mathcal{E}$ . In the same way, all random variables  $S_u, Z_n, Z_n(A)$  which are introduced in subsection 2.1.1 can be seen as measurable functions from  $\mathcal{E}$  into  $\mathbb{R}$ . Let us introduce the measure  $\mathbf{P}$  on  $\mathcal{E}$  which is the probability distribution of the critical branching process starting from an ancestor located at 0 defined in subsection 2.1.1. Then, we introduce a new measure  $\mathbf{Q}$  thanks to the following change of measure :

$$\mathbf{Q}|_{\mathcal{F}_n} := Z_n \cdot \mathbf{P}|_{\mathcal{F}_n}$$

where  $Z_n$  is the number of particles at generation  $n$ . One can remark that  $(Z_n)_{n \in \mathbb{N}}$  is a martingale with respect to  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  under  $\mathbf{P}$ .

Now, let us introduce a "size-biased" branching process. We define a new size-biased law  $\nu$  on  $\mathbb{N}$  by  $\nu(k) = k\mu(k)$  for every  $k \in \mathbb{N}$ . This is a probability measure because  $\mu$  has mean 1. Motions still have distribution  $\mathcal{P}$ . Then, we proceed recursively to construct the size-biased version of our branching process :

We start with one particle called  $w_0$  located at 0 in  $\mathbb{R}^d$ . It defines the generation 0 of the process. Now let  $\mathcal{Z}_n$  be the set of particles in generation  $n$ . We also have a marked particle  $w_n$  among  $\mathcal{Z}_n$ . The particle  $w_n$  gives birth to  $\hat{N}_{w_n}$  children with  $\hat{N}_{w_n} \sim \nu$ . The children of  $w_n$  jump independently from the position of  $w_n$  according to the motion law  $\mathcal{P}$ . Among the children of  $w_n$ , we choose uniformly at random a special particle called  $w_{n+1}$ . Moreover, every particle  $u \in \mathcal{Z}_n \setminus \{w_n\}$  gives birth independently to  $N_u$  offsprings where  $N_u$  has law  $\mu$  as in the classical branching process. The offsprings of  $u \in \mathcal{Z}_n \setminus \{w_n\}$  jump from the position of  $u$  according to the motion law  $\mathcal{P}$ . The offsprings of  $\mathcal{Z}_n$  with their new positions (including the marked particle  $w_{n+1}$ ) form the  $(n+1)$ -th generation of the branching process.

With this construction we get a distinguished ray of particles  $(w_n)_{n \in \mathbb{N}}$  called the spine. Let us define the set  $\mathcal{E}^*$  which consists in elements  $\mathcal{T}$  of  $\mathcal{E}$  with an infinite distinguished ray. The size-biased critical branching process defined above gives us a probability distribution  $\mathbf{Q}^*$  on  $\mathcal{E}^*$ . Then we have the following proposition establishing a link between  $\mathbf{Q}$  and  $\mathbf{Q}^*$ .

**Proposition 2.11** ([115]). *The marginal of  $\mathbf{Q}^*$  with respect to  $\mathcal{E}$  (that is, we forget the distinguished ray) is  $\mathbf{Q}$ . Furthermore, for every  $n \in \mathbb{N}^*$  and for every particle  $u$  at generation  $n$ ,*

$$\mathbf{Q}^*(w_n = u | \mathcal{F}_n) = \frac{1}{Z_n}.$$

We must insist on the fact that, under  $\mathbf{Q}^*$ , the spine  $(S_{w_n})_{n \in \mathbb{N}}$  is a random walk starting from 0 with motion law  $\mathcal{P}$ . We denote by  $\mathcal{S}$  the  $\sigma$ -field generated by the spine and  $\mathcal{G}$  the  $\sigma$ -field generated by the spine and the number of children of particles in the spine and the positions of the brothers of the spine. Let us denote by  $\mathfrak{B}$  the set of brothers of particles in the spine. It is clear from our construction that we get the following proposition.

**Proposition 2.12.** *Under  $\mathbf{Q}^*$ , conditionally on  $\mathcal{G}$  (or  $\mathcal{S}$ ),  $(\mathbf{T}^u)_{u \in \mathfrak{B}}$  are independent and  $\mathbf{T}^u$  has distribution  $\mathbf{P}$  for every  $u \in \mathfrak{B}$ .*

**Remark 2.3.** *In the sequel of this paper, we use a lot the measures  $\mathbf{P}, \mathbf{Q}$  and  $\mathbf{Q}^*$  and the associated expectations  $\mathbb{E}_{\mathbf{P}}, \mathbb{E}_{\mathbf{Q}}$  and  $\mathbb{E}_{\mathbf{Q}^*}$ . When we work with one branching random walk which has been constructed in a normal way (that is, as in the introduction), we use  $\mathbf{P}$  and when we work with one branching random walk which has been constructed in the size-biased way, we use  $\mathbf{Q}^*$  or  $\mathbf{Q}$ . Sometimes we use the probability measure  $\mathbb{P}$  or the associated expectation  $\mathbb{E}$ . It means that we integrate with respect to a point process or sometimes with respect to a simple random walk. (We do not distinguish these two different cases in the notation for sake of convenience but we always specify it when we consider a simple random walk.)*

### 2.2.5 A critical branching random walk conditioned on survival in a given set

We are now going to prove a key lemma for this article. Let us recall some notation. In every generation  $n$ ,  $\{S_u, |u| = n\}$  is the collection of positions of all particles in generation  $n$ .  $Z_n$  is the total number of particles in generation  $n$  and for any closed ball  $A$ , we denote by  $Z_n(A)$  the number of particles lying in  $A$  in generation  $n$ . Moreover the set of continuous functions with compact support is denoted by  $\mathcal{F}_c(\mathbb{R}^d)$ .

**Lemma 2.13** (Key Lemma). *Let  $A$  be a closed ball. We assume  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Let  $f \in \mathcal{F}_c(\mathbb{R}^d)$  whose values are in  $\mathbb{R}_-$  or  $i\mathbb{R}$  and such that  $\text{supp}(f) \subset A$ . Let  $M > 0$ . Then, there exists a constant  $I_{A,f}$  such that, uniformly in  $x \in B(0, M\sqrt{n})$ , as  $n$  goes to infinity,*

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= (1 + o_n(1)) \times (2\pi n)^{-d/2} \det(\Sigma)^{-1/2} \exp \left( -\frac{1}{2n} \langle x, \Sigma^{-1} x \rangle \right) I_{A,f}. \end{aligned}$$

*Proof of Lemma 2.13.*

We do the proof only under the hypothesis  $\mathcal{H}_2$ . Assuming the hypothesis  $\mathcal{H}_1$ , the proof follows exactly the same lines but with a few notation changes. We start by using the spine decomposition and the change of measure described in subsection 2.2.4. This yields that for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{P}} \left[ \frac{\mathbf{1}\{Z_n(A-x) \geq 1\} Z_n(A-x)}{Z_n(A-x)} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}} \left[ \frac{Z_n(A-x)}{Z_n} \frac{\mathbf{1}\{Z_n(A-x) \geq 1\}}{Z_n(A-x)} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in A-x\}}{Z_n(A-x)} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right]. \end{aligned} \tag{2.1}$$

In the second equality, we used the definition of  $\mathbf{Q}$  and in the last one we used Proposition 2.11. Now, let us introduce some notation. For every  $k \in \mathbb{N}$ , let  $B(w_{k+1})$  be the set of brothers of  $w_{k+1}$ , that is, the children of  $w_k$  which are not  $w_{k+1}$ .  $|B(w_{k+1})|$  stands for the number of individuals in  $B(w_{k+1})$ . If  $\mathcal{Z}$  is a set of particles and  $g$  a function, let us define :

$$\langle \mathcal{Z}, g \rangle := \sum_{u \in \mathcal{Z}} g(S_u).$$

Further, for every particle  $u$ , we denote by  $\mathcal{Z}_k^u$  the  $k$ -th generation of the branching process  $\mathbf{T}^u$  starting from  $u$ . Moreover, the positions in  $\mathbf{T}^u$  are shifted by  $S_u$ , that is, for every  $v \in \mathbf{T}_{gen}^u$ , the position of  $v$  is given by  $S_v^u := S_{uv} - S_u$ . For every set  $\tilde{A}$ , for every  $k \geq 0$  and for every particle  $u$ , we define  $Z_k^u(\tilde{A})$  by

$$Z_k^u(\tilde{A}) = \sum_{v \in \mathcal{Z}_k^u} \mathbf{1}\{S_v^u \in \tilde{A}\}.$$

Further, for every particle  $u$ , we denote by  $\varrho_u$  the random variable  $S_u - S_{\tilde{u}}$ . Then, by spinal decomposition we know that,

$$\sum_{|u|=n} f(S_u + x) = f(S_{w_n} + x) + \sum_{k=0}^{n-1} \sum_{u \in B(w_{k+1})} \langle \mathcal{Z}_{n-k-1}^u, f(S_{w_k} + \varrho_u + x + \cdot) \rangle.$$

In the same way,

$$Z_n(A - x) = \mathbf{1}\{S_{w_n} \in A - x\} + \sum_{k=0}^{n-1} \sum_{u \in B(w_{k+1})} Z_{n-k-1}^u(A - x - S_{w_k} - \varrho_u).$$

For sake of clarity, let us introduce for every  $k \in \{0, \dots, n-1\}$  and for every  $z \in \mathbb{R}^d$  :

$$\begin{aligned} Y_{f,n-k}(z) &:= \sum_{u \in B(w_{k+1})} \langle \mathcal{Z}_{n-k-1}^u, f(z + \varrho_u + \cdot) \rangle \\ Y_{n-k}(z) &:= \sum_{u \in B(w_{k+1})} Z_{n-k-1}^u(A - z - \varrho_u). \end{aligned}$$

Moreover, by convention we define for every  $z \in \mathbb{R}^d$  :

$$Y_{f,0}(z) := f(z).$$

With this notation, (2.1) becomes

$$\begin{aligned} &\mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A - x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in A - x\}}{1 + \sum_{k=0}^{n-1} Y_{n-k}(S_{w_k} + x)} \exp \left( \sum_{k=0}^n Y_{f,n-k}(S_{w_k} + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in A - x\}}{1 + \sum_{k=1}^n Y_k(S_{w_{n-k}} + x)} \exp \left( \sum_{k=0}^n Y_{f,k}(S_{w_{n-k}} + x) \right) \right]. \end{aligned} \quad (2.2)$$

Let  $K > 0$ . We write  $\mathbb{E}_{\mathbf{Q}^*}[\cdot | \mathcal{S}]$  when we mean that we condition on the spine. By conditional Markov inequality we deduce that

$$\begin{aligned} & \mathbf{Q}^* \left( S_{w_n} \in A - x, \sum_{k=K+1}^n Y_k(S_{w_{n-k}} + x) \geq 1 \right) \\ & \leq \mathbb{E}_{\mathbf{Q}^*} \left[ \mathbf{1}\{S_{w_n} \in A - x\} \mathbb{E}_{\mathbf{Q}^*} \left[ \sum_{k=K+1}^n Y_k(S_{w_{n-k}} + x) \middle| \mathcal{S} \right] \right]. \end{aligned} \quad (2.3)$$

However by definition of  $\mathbf{Q}^*$  and  $Y_k(\cdot)$  and by Proposition 2.12, for any  $k \in \{K+1, \dots, n\}$ ,

$$\begin{aligned} \mathbb{E}_{\mathbf{Q}^*} \left[ Y_k(S_{w_{n-k}} + x) \middle| \mathcal{S} \right] &= \mathbb{E}_{\mathbf{Q}^*} \left[ |B(w_{n-k+1})| \middle| \mathcal{S} \right] \times \mathbb{E}_{\mathbf{P}} [Z_{k-1}(A - x - y - \varrho)] \big|_{y=S_{w_{n-k}}} \\ &= \sigma^2 \times \mathbb{E}_{\mathbf{P}} [Z_{k-1}(A - x - y - \varrho)] \big|_{y=S_{w_{n-k}}} \\ &= \sigma^2 \times \mathbb{P} \left( \hat{S}_{k-1} \in A - x - y - \varrho \right) \big|_{y=S_{w_{n-k}}} \end{aligned} \quad (2.4)$$

where  $(\hat{S}_k)_{k \in \mathbb{N}}$  is a random walk whose i.i.d increments have distribution  $\mathcal{P}$  and  $\varrho$  is a random variable with distribution  $\mathcal{P}$  which is independent of  $Z_{k-1}(\cdot)$  under  $\mathbf{P}$  and independent of  $\hat{S}$  under  $\mathbb{P}$ . In the second equality we used the fact that  $1 + |B(w_{n-k+1})|$  has distribution  $\nu$  under  $\mathbf{Q}^*$  and  $1 + \sigma^2$  is the mean of  $\nu$ . To obtain the third equality, we used the fact that the branching process is critical under  $\mathbf{P}$ . Consequently, thanks to Corollary 2.6 and identity (2.4), we obtain for any  $k \in \{K+1, \dots, n\}$ ,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ Y_k(S_{w_{n-k}} + x) \middle| \mathcal{S} \right] \leq \sigma^2 |A| c_d k^{-d/2}. \quad (2.5)$$

Therefore, combining identities (2.5) and (2.3) and Corollary 2.6 again, there exists a constant  $C > 0$  such that

$$\begin{aligned} \mathbf{Q}^* \left( S_{w_n} \in A - x, \sum_{k=K+1}^n Y_k(S_{w_{n-k}} + x) \geq 1 \right) &\leq \sigma^2 |A|^2 c_d^2 \times n^{-d/2} \sum_{k=K+1}^{+\infty} k^{-d/2} \\ &\leq C \times n^{-d/2} K^{1-d/2}. \end{aligned}$$

We remark that  $K^{1-d/2} = o_K(1)$  if and only if  $d \geq 3$ . Thus, together with (2.2), it yields

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A - x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in (A - x)\}}{1 + \sum_{k=1}^K Y_k(S_{w_{n-k}} + x)} \exp \left( \sum_{k=0}^K Y_{f,k}(S_{w_{n-k}} + x) \right) \right] + o_K(1) n^{-d/2}. \end{aligned} \quad (2.6)$$

Now let us introduce some new notation. For every  $k \in \mathbb{N}^*$  and for every  $z \in \mathbb{R}^d$ ,

$$\begin{aligned} \tilde{Y}_{f,k}(z) &:= \sum_{u \in B(w_{k+1})} \langle Z_{k-1}^u, f(z + \varrho_u + \cdot) \rangle \\ \tilde{Y}_k(z) &:= \sum_{u \in B(w_{k+1})} Z_{k-1}^u(A - z - \varrho_u). \end{aligned}$$

Moreover, by convention, we define for every  $z \in \mathbb{R}^d$  :

$$\tilde{Y}_{f,0}(z) := f(z).$$

With this new notation, by reversing time in the spine, (2.6) yields,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} - S_{w_K} \in (A-x - S_{w_K})\}}{1 + \sum_{k=1}^K \tilde{Y}_k((S_{w_n} - S_{w_K}) + x + (S_{w_K} - S_{w_k}))} \exp \left( \sum_{k=0}^K \tilde{Y}_{f,k}((S_{w_n} - S_{w_k}) + x) \right) \right] \\ & \quad + o_K(1)n^{-d/2}. \end{aligned} \quad (2.7)$$

For every  $z \in \mathbb{Z}^d$ , let us denote  $p_{n-K}(z) = \mathbf{Q}^*(S_{w_n} - S_{w_K} = z)$ . Let us define also the function

$$F_{K,A} : \quad \mathbb{R}^{K+1} \quad \longrightarrow \quad \mathbb{C} \\ (y, y_1, \dots, y_K) \quad \longmapsto \quad \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{1}{1 + \sum_{k=1}^K \tilde{Y}_k(y - y_k)} \exp \left( \sum_{k=0}^K \tilde{Y}_{f,k}(y - y_k) \right) \right].$$

Therefore, by (2.7) and the independence under  $\mathbf{Q}^*$  between  $S_{w_n} - S_{w_K}$ ,  $(S_{w_k})_{1 \leq k \leq K}$  and  $(\tilde{Y}_{f,k}(\cdot))_{1 \leq k \leq K}$ , it holds that

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \sum_{y \in A \cap \mathbb{Z}^d} p_{n-K}(y-x - S_{w_K}) F_{K,A}(y, S_{w_k}, 1 \leq k \leq K) \right] + o_K(1)n^{-d/2}. \end{aligned} \quad (2.8)$$

Using Proposition 2.5, it holds that, uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= (2\pi(n-K))^{-d/2} \det(\Sigma)^{-1/2} \\ & \quad \times \sum_{y \in A \cap \mathbb{Z}^d} \mathbb{E}_{\mathbf{Q}^*} \left[ \exp \left( -\frac{1}{2(n-K)} \langle V_{x,y,K}, \Sigma^{-1} V_{x,y,K} \rangle \right) F_{K,A}(y, S_{w_k}, 1 \leq k \leq K) \right] \\ & \quad + o_K(1)n^{-d/2} + o_{n,K}(1)n^{-d/2} \end{aligned} \quad (2.9)$$

where for every  $K \in \mathbb{N}$ ,  $o_{n,K}(1)$  tends to zero as  $n$  goes to infinity and  $V_{x,y,K} = y - x - S_{w_K}$ . However, we remark that,

$$\begin{aligned} & \left| \frac{1}{2(n-K)} \langle V_{x,y,K}, \Sigma^{-1} V_{x,y,K} \rangle - \frac{1}{2n} \langle x, \Sigma^{-1} x \rangle \right| \\ & \leq \left( \frac{1}{2(n-K)} - \frac{1}{2n} \right) |\langle x, \Sigma^{-1} x \rangle| + \frac{1}{2(n-K)} \left| \langle V_{x,y,K}, \Sigma^{-1} V_{x,y,K} \rangle - \langle x, \Sigma^{-1} x \rangle \right| \\ & \leq \frac{\|\Sigma^{-1}\|}{2(n-K)} \left( \frac{K\|x\|^2}{n} + \|y\|^2 + \|S_{w_K}\|^2 + 2\|x\|(\|y\| + \|S_{w_K}\|) + 2\|y\| \cdot \|S_{w_K}\| \right) \\ & \leq \frac{\|\Sigma^{-1}\|}{2(n-K)} \left( \frac{K\|x\|^2}{n} + 2\|y\|^2 + 2\|S_{w_K}\|^2 + 2\|x\|(\|y\| + \|S_{w_K}\|) \right). \end{aligned} \quad (2.10)$$

Therefore, on the event  $\{\|S_{w_K}\| \leq K\}$ , we can use (2.10) and the inequality  $|1 - e^{-t}| \leq 2|t|$  in a neighborhood of 0 to show that there is  $N_K \in \mathbb{N}^*$  such that for every  $n \geq N_K$  and for every  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \left| \exp\left(-\frac{1}{2(n-K)}\langle V_{x,y,K}, \Sigma^{-1}V_{x,y,K}\rangle\right) - \exp\left(-\frac{1}{2n}\langle x, \Sigma^{-1}x\rangle\right) \right| \\ & \leq \exp\left(-\frac{1}{2n}\langle x, \Sigma^{-1}x\rangle\right) \times \frac{2\|\Sigma^{-1}\|}{(n-K)} \left( \frac{K\|x\|^2}{n} + \|y\|^2 + \|S_{w_K}\|^2 + \|x\|(\|y\| + \|S_{w_K}\|) \right) \\ & \leq \frac{2\|\Sigma^{-1}\|}{(n-K)} \left( \frac{KM^2}{2} + C_A^2 + K^2 + M\sqrt{n}(C_A + K) \right) \end{aligned} \quad (2.11)$$

where  $C_A = \max_{y \in A} \|y\|$ . Furthermore, by Proposition 2.7,

$$\mathbf{Q}^*(\|S_{w_K}\| \geq K) = o_K(1). \quad (2.12)$$

Finally, using inequalities (2.11) and (2.12) in identity (2.9), we know that uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp\left(\sum_{|u|=n} f(S_u + x)\right) \right] \\ & = (2\pi n)^{-d/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2n}\langle x, \Sigma^{-1}x\rangle\right) \sum_{y \in A \cap \mathbb{Z}^d} \mathbb{E}_{\mathbf{Q}^*} [F_{K,A}(y, S_{w_k}, 1 \leq k \leq K)] \\ & \quad + o_K(1)n^{-d/2} + o_{n,K}(1)n^{-d/2}. \end{aligned} \quad (2.13)$$

Now, let us introduce the function  $G_{A,f}$  defined by

$$G_{A,f} : \mathbb{R}^d \rightarrow \mathbb{C} \\ y \mapsto \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{1}{1 + \sum_{k=1}^{+\infty} \tilde{Y}_k(y - S_{w_k})} \exp\left(\sum_{k=0}^{+\infty} \tilde{Y}_{f,k}(y - S_{w_k})\right) \right].$$

$G_{A,f}$  is well-defined because the infinite sums in its expression are actually finite sums almost surely. Indeed, we can prove exactly as in (2.5) that for every  $k \in \mathbb{N}^*$ ,

$$\mathbf{Q}^* \left( \tilde{Y}_k(y - S_{w_k}) \geq 1 \right) \leq \mathbb{E}_{\mathbf{Q}^*} \left[ \tilde{Y}_k(y - S_{w_k}) \right] \leq \sigma^2 |A| c_d k^{-d/2}.$$

As  $d \geq 3$ , this is summable. Thus, by Borel-Cantelli Lemma, almost surely, there is only a finite number of integers  $k \in \mathbb{N}^*$  such that  $\tilde{Y}_k(y - S_{w_k}) \geq 1$ . As  $\text{supp}(f) \subset A$ , there is also only a finite number of integers  $k \in \mathbb{N}^*$  such that  $\tilde{Y}_{f,k}(y - S_{w_k}) \neq 0$ . Consequently, the random variable inside the expectation in the definition of  $G_{A,f}$  is well-defined. Moreover the modulus of this quantity is lower than 1. Therefore,  $G_{A,f}$  is well-defined. We define also

$$I_{A,f} := \sum_{y \in A \cap \mathbb{Z}^d} G_{A,f}(y).$$

Besides, by the dominated convergence theorem,

$$\sum_{y \in A \cap \mathbb{Z}^d} \mathbb{E}_{\mathbf{Q}^*} [F_{K,A}(y, S_{w_k}, 1 \leq k \leq K)] = I_{A,f} + o_K(1).$$

Finally, using this in (2.13), we proved that uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u+x) \right) \right] \\ &= (2\pi n)^{-d/2} \det(\Sigma)^{-1/2} \exp \left( -\frac{1}{2n} \langle x, \Sigma^{-1}x \rangle \right) \times I_{A,f} \\ &+ o_K(1)n^{-d/2} + o_{n,K}(1)n^{-d/2}. \end{aligned} \quad (2.14)$$

It concludes the proof.  $\square$

In the proof of Lemma 2.13, the quantity  $I_{A,f}$  naturally arises. It is a very important quantity in the sequel of this article. We often refer to its definition.

**Definition 2.14.** *Let  $A$  be a closed ball and let  $f$  be a continuous function whose support is in  $A$  and whose values are in  $-\mathbb{R}$  or in  $i\mathbb{R}$ . With the notation which is introduced in the proof of Lemma 2.13, we recall that :*

$$\begin{aligned} G_{A,f} &: \mathbb{R}^d \rightarrow \mathbb{C} \\ y &\mapsto \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{1}{1 + \sum_{k=1}^{+\infty} \tilde{Y}_k(y - S_{w_k})} \exp \left( \sum_{k=0}^{+\infty} \tilde{Y}_{f,k}(y - S_{w_k}) \right) \right]. \end{aligned}$$

- Under the hypothesis  $\mathcal{H}_1$ , we define  $I_{A,f} := \int_A G_{A,f}(y) dy$ .
- Under the hypothesis  $\mathcal{H}_2$ , we define  $I_{A,f} := \sum_{y \in A \cap \mathbb{Z}^d} G_{A,f}(y)$ .

Moreover, we often use the notation  $I_A := I_{A,0}$ .

We remark that the definition 2.14 is reminiscent of the structure of backward tree introduced by Kallenberg in [94].

As a particular case of the Lemma 2.13, we can estimate the probability for the branching process to survive in a specified area of  $\mathbb{R}^d$ .

**Proposition 2.15.** *Let  $A$  be a closed ball. We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . For every  $n \in \mathbb{N}^*$ , let us define  $a_n = a\sqrt{n}$  for some  $a \in \mathbb{R}^d$ . Then we have the following asymptotic equivalence*

$$\mathbf{P}(Z_n(A - a_n) \geq 1) \underset{n \rightarrow +\infty}{\sim} I_A \times (2\pi)^{-d/2} \det(\Sigma)^{-1/2} e^{-\langle a, \Sigma^{-1}a \rangle / 2} n^{-d/2}$$

with  $I_A$  defined in definition 2.14.

*Proof of Proposition 2.15.*

Apply Lemma 2.13 with  $f \equiv 0$  and  $M$  larger than  $\|a\|$ .  $\square$

We are now able to prove our first proposition concerning convergence of point processes.

**Proposition 2.16.** *Let  $A$  be a closed ball. We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . For every  $n \in \mathbb{N}$ , let us define  $a_n = a\sqrt{n}$  for some  $a \in \mathbb{R}^d$ . Then, under probability measure  $\mathbf{P}$ , we have the following convergence in law*

$$\mathcal{L} \left( \sum_{|u|=n} \delta_{S_u+a_n} \mid Z_n(A - a_n) \geq 1 \right) \xrightarrow[n \rightarrow +\infty]{law} \mathcal{N}_A$$

where  $\mathcal{N}_A$  is a point process on  $A$  which does not depend on  $a$ . Moreover for every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ ,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] = I_A^{-1} I_{A,-f}$$

where the terms of the right-hand side were defined in definition 2.14.

*Proof of Proposition 2.16.*

By Lemma 2.10, it is enough to prove that for every  $f \in \mathcal{F}_c(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ , there exists a continuous function  $\eta \mapsto \Phi_f(\eta)$  such that for every  $\eta \in \mathbb{R}$ ,

$$\mathbb{E}_{\mathbf{P}} \left[ \exp \left( i\eta \sum_{|u|=n} f(S_u + a_n) \right) \middle| Z_n(A - a_n) \geq 1 \right] \xrightarrow{n \rightarrow +\infty} \Phi_f(\eta).$$

By Lemma 2.13,

$$\mathbb{E}_{\mathbf{P}} \left( \exp \left( i\eta \sum_{|u|=n} f(S_u + a_n) \right) \mathbf{1}_{\{Z_n(A - a_n) \geq 1\}} \right)$$

is asymptotically

$$(1 + o_n(1)) \frac{I_{A,inf}}{(2\pi)^{d/2}} \det(\Sigma)^{-1/2} e^{-\langle a, \Sigma^{-1} a \rangle / 2} \frac{1}{n^{d/2}}.$$

Combining this with Proposition 2.15 we get that

$$\mathbb{E}_{\mathbf{P}} \left( \exp \left( i\eta \sum_{|u|=n} f(S_u) \right) \middle| Z_n(A - a_n) \geq 1 \right) \xrightarrow{n \rightarrow +\infty} I_A^{-1} I_{A,inf}.$$

Looking at the expression of  $I_{A,inf}$  which is given in definition 2.14, we deduce from dominated convergence theorem that  $\eta \mapsto I_{A,inf}$  is a continuous function. It concludes the proof of the first part of Proposition 2.16 concerning convergence in law toward a point process  $\mathcal{N}_A$ . The second part of Proposition 2.16 concerning the Laplace transform of  $\mathcal{N}_A$  is obtained by exactly the same computations. We just have to replace  $i\eta f$  by  $-f$ .  $\square$

## 2.3 Proof of Theorem 2.2

### Strategy of the proof of Theorem 2.2

If a critical branching process starts from a single particle and if we condition it to visit a closed ball  $A$ , we proved in the previous section that the limiting point process is  $\mathcal{N}_A$ . However, most of the critical branching processes starting from particles in  $\Lambda_0^{d,\theta}$  will not reach  $A$  because of transience (in dimension  $d \geq 3$ ) or because of extinction of the branching process. Let us make this intuition more quantitative. By homogeneity of the Poisson point process  $\Lambda_0^{d,\theta}$ , for every  $M \geq 1$ ,

$$\Lambda_0^{d,\theta}(B(0, M\sqrt{n})) \simeq \Theta(n^{d/2}).$$

Moreover, any particle of  $\Lambda_0^{d,\theta}$  located at  $x \in \mathbb{R}^d$  with  $\|x\| \gg \sqrt{n}$  is too far from  $A$  to have descendants in  $A$  at time  $n$ . Indeed, a centered random walk with second moment is at distance  $O(\sqrt{n})$  from 0 at time  $n$ . Moreover, we proved in the previous section that a particle in  $\Lambda_0^{d,\theta}$  located at  $x \in B(0, M\sqrt{n})$  has descendants which reach  $A$  at time  $n$  with probability

$$\mathbf{P}(Z_n(A - x) \geq 1) \simeq \Theta(n^{-d/2}).$$

If  $Z_n^{(y)}(\cdot)$  is the occupation measure of the  $n$ -th generation of the critical branching process starting from a particle  $y$  in the support of  $\Lambda_0^{d,\theta}$ , combining both previous approximations, we get that

$$\int \mathbf{1}\{Z_n^{(y)}(A) \geq 1\} d\Lambda_0^{d,\theta}(y) \simeq \Theta(n^{d/2}) \times \Theta(n^{-d/2}) = \Theta(1).$$

Therefore, the number  $P_A$  of particles in  $\Lambda_0^{d,\theta}$  whose descendants reach  $A$  is of order  $\Theta(1)$ . Moreover, for each of these particles, the positions of the descendants in  $A$  form a point process distributed as  $\mathcal{N}_A$ . That is why, we will get an independent layering of  $P_A$  copies of  $\mathcal{N}_A$ .

Now, let us prove the following lemma.

**Lemma 2.17.** *Let  $A$  be a closed ball. Let  $f \in \mathcal{F}_c(\mathbb{R}^d)$  whose values are in  $\mathbb{R}_-$  or in  $i\mathbb{R}$  and such that  $\text{supp}(f) \subset A$ . Then, assuming hypothesis  $\mathcal{H}_1$ ,*

$$\mathbb{E}_{\mathbf{P}} \left[ \int_{\mathbb{R}^d} \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u+x) \right) dx \right] \underset{n \rightarrow +\infty}{\sim} I_{A,f}$$

where the definition of  $I_{A,f}$  is given in definition 2.14. Moreover, assuming hypothesis  $\mathcal{H}_2$ ,

$$\mathbb{E}_{\mathbf{P}} \left[ \sum_{x \in \mathbb{Z}^d} \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u+x) \right) \right] \underset{n \rightarrow +\infty}{\sim} I_{A,f}.$$

*Proof of Lemma 2.17.*

Let us prove this lemma under the hypothesis  $\mathcal{H}_2$ . Assuming hypothesis  $\mathcal{H}_1$ , the proof is similar. For sake of clarity, we do the proof only with  $f \equiv 0$ . Let  $M > 0$ . Let us use the notation  $B_{\mathbb{Z}^d}(0, t)$  for the set of elements of  $\mathbb{Z}^d$  whose euclidean norm is less than  $t$ .  $B(0, t)$  denotes the standard euclidean ball. By Markov inequality and criticality of the branching process under  $\mathbf{P}$ , we get for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) &\leq \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \mathbb{E}_{\mathbf{P}} [Z_n(A-x)] \\ &= \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \mathbb{P}(\hat{S}_n \in A-x). \end{aligned} \quad (2.15)$$

where  $(\hat{S}_k)_{k \in \mathbb{N}}$  is a random walk with motion law  $\mathcal{P}$ . Therefore, by Proposition 2.7, there exists  $C > 0$  such that for every  $n \in \mathbb{N}$  :

$$\sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) \leq Cn^{-d/2} \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \exp \left( -\frac{\|x\|^2}{Cn} \right). \quad (2.16)$$

Thus, by standard inequalities, there exists a constant  $C' > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) &\leq C'n^{-d/2} \int_{B(0, M\sqrt{n})^c} \exp \left( -\frac{\|x\|^2}{C'n} \right) dx \\ &= C' \int_{B(0, M)^c} \exp \left( -\frac{\|x\|^2}{C'} \right) dx \\ &= o_M(1). \end{aligned} \quad (2.17)$$

Moreover, by Lemma 2.13,

$$\begin{aligned} &\sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})} \mathbf{P}(Z_n(A-x) \geq 1) \\ &= (2\pi n)^{-d/2} \det(\Sigma)^{-1/2} I_A \times \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})} \exp \left( -\frac{1}{2n} \langle x, \Sigma^{-1}x \rangle \right) + M^d o_n(1). \end{aligned} \quad (2.18)$$

However, one can observe that,

$$\begin{aligned} & (2\pi n)^{-d/2} \det(\Sigma)^{-1/2} \times \sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})} \exp\left(-\frac{1}{2n} \langle x, \Sigma^{-1}x \rangle\right) \\ &= (2\pi)^{-d/2} \det(\Sigma)^{-1/2} \int_{B(0, M)} \exp\left(-\frac{1}{2} \langle x, \Sigma^{-1}x \rangle\right) dx + M^d o_n(1) \\ &= 1 + o_M(1) + M^d o_n(1). \end{aligned}$$

By substituting this into identity (2.18), we get

$$\sum_{x \in B_{\mathbb{Z}^d}(0, M\sqrt{n})} \mathbf{P}(Z_n(A - x) \geq 1) = I_A + o_M(1) + M^d o_n(1). \quad (2.19)$$

Then, combining identities (2.19) and (2.17) yields

$$\sum_{x \in \mathbb{Z}^d} \mathbf{P}(Z_n(A - x) \geq 1) = I_A + o_M(1) + M^d o_n(1) \quad (2.20)$$

which concludes the proof.  $\square$

Now we are ready to prove Theorem 2.2.

*Proof of Theorem 2.2.*

We only work on the case where  $X$  is a constant  $\theta$ . The general case is obtained by integrating the constant case with respect to the law of  $X$ , that is,  $\mathbb{P}_X$ , and by using the dominated convergence theorem. First we prove Theorem 2.2 under the hypothesis  $\mathcal{H}_2$ . We recall that under this hypothesis,  $\Lambda_0^{d, \theta}$  is a point process on  $\mathbb{Z}^d$  such that each site contains a number of particles following a Poisson distribution of parameter  $\theta$ . Let  $f \in \mathcal{F}_c(\mathbb{R}^d)$ . There exists a closed ball  $A$  such that  $\text{supp}(f) \subset A$ . Let  $\eta$  be a real number. Recall that, by a small abuse of notation, for any point process  $\Theta$ , " $x \in \Theta$ " takes into account the multiplicity of  $x$  in  $\Theta$ . Then, standard computations yield

$$\begin{aligned} \mathbb{E} \left[ \exp \left( i\eta \int f(x) \Lambda_n^{d, \theta}(dx) \right) \right] &= \mathbb{E} \left[ \prod_{x \in \Lambda_0^{d, \theta}} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( \sum_{|u|=n} i\eta f(S_u + x) \right) \right] \right] \\ &= \exp \left( \theta \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( \sum_{|u|=n} i\eta f(S_u + x) \right) - 1 \right] \right). \quad (2.21) \end{aligned}$$

Thus, we only need to investigate the asymptotical behaviour of

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( \sum_{|u|=n} i\eta f(S_u + x) \right) - 1 \right]$$

However, as  $\text{supp}(f) \subset A$ , it holds that

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( \sum_{|u|=n} i\eta f(S_u + x) \right) - 1 \right] \\ &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A - x) \geq 1\} \left( \exp \left( \sum_{|u|=n} i\eta f(S_u + x) \right) - 1 \right) \right]. \quad (2.22) \end{aligned}$$

By Lemma 2.17, this quantity (2.22) converges toward  $I_{A,inf} - I_A$ . Therefore, together with identity (2.21), we deduce

$$\mathbb{E} \left[ \exp \left( i\eta \int f(x) \Lambda_n^{d,\theta}(dx) \right) \right] = (1 + o_n(1)) \exp(\theta(I_{A,inf} - I_A)).$$

By the description of  $I_{A,inf}$  given in definition 2.14, the function  $\eta \mapsto \exp(\theta(I_{A,inf} - I_A))$  is continuous. That is why, by Proposition 2.10, there exists a point process  $\Lambda_\infty^{d,\theta}$  such that

$$\Lambda_n^{d,\theta} \xrightarrow[n \rightarrow +\infty]{law} \Lambda_\infty^{d,\theta}.$$

In order to do the proof under the hypothesis  $\mathcal{H}_1$ , we just need an identity which is similar to (2.21). This can be obtained thanks to the exponential formula for Poisson Point Process. (See [93].) If we replace sums by integrals, the rest of the proof follows the same lines. Now, let  $g \in \mathcal{F}_c(\mathbb{R}^d)^+$  such that  $supp(g) \subset A$ . Following exactly the same computations than above, we get that,

$$\mathbb{E} \left[ \exp \left( - \int g(x) \Lambda_\infty^{d,\theta}(dx) \right) \right] = \exp(\theta(I_{A,-g} - I_A)).$$

Consequently, by Proposition 2.16,

$$\mathbb{E} \left[ \exp \left( - \int g(x) \Lambda_\infty^{d,\theta}(dx) \right) \right] = \exp \left( \theta I_A \left( \mathbb{E} \left[ \exp \left( - \int g(x) \mathcal{N}_A(dx) \right) \right] - 1 \right) \right).$$

Therefore,  $\Lambda_\infty^{d,\theta}$  is a Poissonian sum of i.i.d copies of  $\mathcal{N}_A$ , as stated in Theorem 2.2.  $\square$

In the proof of Theorem 2.2, we obtained the Laplace transform of  $\Lambda_\infty^{d,X}$ .

**Proposition 2.18** (Laplace transform). *For every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  whose support is included in some closed ball  $A$ , we have*

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Lambda_\infty^{d,X}(dx) \right) \right] = \mathbb{E} [\exp(X \times (I_{A,-f} - I_A))]$$

where  $I_A$  and  $I_{A,-f}$  are defined in definition 2.14. Another reformulation is

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Lambda_\infty^{d,X}(dx) \right) \right] = \mathbb{E} [\exp(X I_A R_{f,A})]$$

where

$$R_{f,A} = \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] - 1.$$

**Remark 2.4.** *Let  $A$  and  $B$  be two closed balls such that  $A \subset B$ . By the previous Proposition 2.18 we get two formulas to compute the Laplace transform of  $\Lambda_\infty^{d,X}$ , one with respect to  $A$ , the other one with respect to  $B$ . These formulas must be equal. This is discussed in the following section.*

## 2.4 Compatibility of $\Lambda_\infty^{d,X}$

Theorem 2.2 shows that  $\Lambda_\infty^{d,X}$ , as a limit of  $\left( \Lambda_n^{d,X} \right)_{n \in \mathbb{N}}$ , is a point process. In particular, it must be compatible. Here, we give an independent proof of compatibility of the limiting point processes of the form  $\Lambda_\infty^{d,X}$  obtained in Theorem 2.2. In all this section,  $X$  is a non-negative random variable.

**Lemma 2.19.** *Let  $A_1$  and  $A_2$  be two closed balls such that  $A_1 \subset A_2$ . We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Then*

$$\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1) = \frac{I_{A_1}}{I_{A_2}}$$

where  $I_{A_1}$  and  $I_{A_2}$  are defined in definition 2.14.

*Proof of Lemma 2.19.*

By Proposition 2.16, we know that under  $\mathbf{P}$ ,

$$\mathcal{L} \left( \sum_{|u|=n} \delta_{S_u} \middle| Z_n(A_2) \geq 1 \right) \xrightarrow[n \rightarrow +\infty]{law} \mathcal{N}_{A_2}.$$

Then, by Lemma 4.1 in [93], we get

$$\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1) = \lim_{n \rightarrow +\infty} \mathbf{P}(Z_n(A_1) \geq 1 | Z_n(A_2) \geq 1) = \lim_{n \rightarrow +\infty} \frac{\mathbf{P}(Z_n(A_1) \geq 1)}{\mathbf{P}(Z_n(A_2) \geq 1)}$$

The estimate given by Proposition 2.15 concludes the proof.  $\square$

**Lemma 2.20.** *Let  $A_1$  and  $A_2$  be two closed balls such that  $A_1 \subset A_2$ . We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ .*

$$\mathcal{L} \left( \mathcal{N}_{A_2}(\cdot \cap A_1) \middle| \mathcal{N}_{A_2}(A_1) \geq 1 \right) \stackrel{law}{=} \mathcal{N}_{A_1}.$$

*Proof of Lemma 2.20.*

Let  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A_1$ . Let us observe that,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_2}(dx) \right) \mathbf{1}\{\mathcal{N}_{A_2}(A_1) \geq 1\} \right] \\ &= \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_2}(dx) \right) \right] - \mathbb{P}(\mathcal{N}_{A_2}(A_1) = 0). \end{aligned} \quad (2.23)$$

However by Proposition 2.16,

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u) \right) \mathbf{1}\{Z_n(A_2) \geq 1\} \right]}{\mathbf{P}(Z_n(A_2) \geq 1)} = \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_2}(dx) \right) \right]. \quad (2.24)$$

The left-hand side in (2.24) can be rewritten as

$$\frac{\mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u) \right) \mathbf{1}\{Z_n(A_1) \geq 1\} \right] + \mathbf{P}(Z_n(A_2) \geq 1, Z_n(A_1) = 0)}{\mathbf{P}(Z_n(A_1) \geq 1)} \times \frac{\mathbf{P}(Z_n(A_1) \geq 1)}{\mathbf{P}(Z_n(A_2) \geq 1)}.$$

By Lemma 2.19 and Propositions 2.15 and 2.16, this converges toward

$$\left( \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_1}(dx) \right) \right] + \frac{1}{\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)} - 1 \right) \mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1).$$

Consequently, together with (2.24), this implies that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_2}(dx) \right) \right] \\ &= \left( \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_1}(dx) \right) \right] + \frac{1}{\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)} - 1 \right) \mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1). \end{aligned} \quad (2.25)$$

Therefore, using identities (2.23) and (2.25), we know that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_2}(dx) \right) \middle| \mathcal{N}_{A_2}(A_1) \geq 1 \right] \\ &= \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_1}(dx) \right) \right] + \frac{1}{\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)} - 1 - \frac{\mathbb{P}(\mathcal{N}_{A_2}(A_1) = 0)}{\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)}. \end{aligned}$$

This is exactly  $\mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_{A_1}(dx) \right) \right]$ . Thus, by Lemma 2.8, this finishes the proof.  $\square$

Thanks to the two previous lemmas, we are now able to deduce the compatibility of  $\Lambda_\infty^{d,X}$ .

**Proposition 2.21.** *Let  $A_1$  and  $A_2$  be two closed balls such that  $A_1 \subset A_2$ . We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . In Theorem 2.2, we get a Poissonian way to define  $\xi_{A_1} := \Lambda_\infty^{d,X}(\cdot \cap A_1)$  and  $\xi_{A_2} := \Lambda_\infty^{d,X}(\cdot \cap A_2)$ . Then,*

$$\xi_{A_2}(\cdot \cap A_1) \stackrel{law}{=} \xi_{A_1}.$$

*Proof of Proposition 2.21.*

Let us first construct  $\xi_{A_2}$  with the construction given in Theorem 2.2. Consider  $P_{A_2}$  a Poisson Random variable of parameter  $XI_{A_2}$  and  $(\mathcal{N}_{A_2}^{(k)})_{k \in \mathbb{N}}$  a family of i.i.d copies of  $\mathcal{N}_{A_2}$  which is independent of  $P_{A_2}$ . Then,

$$\xi_{A_2} = \sum_{k=1}^{P_{A_2}} \mathcal{N}_{A_2}^{(k)}.$$

With this construction,

$$\xi_{A_2}(\cdot \cap A_1) = \sum_{k=1}^{P_{A_2}} \mathbf{1}\{\mathcal{N}_{A_2}^{(k)}(A_1) \geq 1\} \mathcal{N}_{A_2}^{(k)}(\cdot \cap A_1).$$

By the previous Lemma 2.20, we only have to check that  $\sum_{k=1}^{P_{A_2}} \mathbf{1}\{\mathcal{N}_{A_2}^{(k)}(A_1) \geq 1\}$  has a Poisson law with parameter  $XI_{A_1}$ . It is a Poissonian sum of independent Bernoulli random variables of parameter  $\mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)$ . It is well known that this is still a Poisson random variable with parameter  $XI_{A_2} \times \mathbb{P}(\mathcal{N}_{A_2}(A_1) \geq 1)$ . This parameter is equal to  $XI_{A_1}$  by Lemma 2.19.  $\square$

## 2.5 Characterization of invariant measures

**Strategy of the proof of Theorem 2.3** Let us consider a cluster-invariant point process  $\Theta$ . Let  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  whose support is included in a closed ball  $A$ . By cluster-invariance, we remark that for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right].$$

Then, we are looking for the asymptotics of the right-hand side above in order to describe the distribution of  $\Theta$ . Remark that particles  $x$  of  $\Theta$  such that  $\|x\| \geq M\sqrt{n}$  can be neglected. That is why,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right] \approx \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right].$$

However, uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$\begin{aligned} & \ln \left( \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right) \\ &= \ln \left( 1 + \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \right) \\ &\approx \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \\ &\approx R_{f,A} \mathbf{P}(Z_n(A-x) \geq 1) \end{aligned}$$

where the last approximation stems from Lemma 2.13 and

$$R_{f,A} = \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] - 1.$$

Finally, we get that for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right] \approx \mathbb{E} \left[ \exp \left( R_{f,A} \int_{B(0, M\sqrt{n})} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx) \right) \right].$$

In order to conclude the proof, we only need to find a subsequence along which

$$L_n := \int_{B(0, M\sqrt{n})} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx)$$

converges in distribution. Therefore, it would be enough to show that  $(L_n)_{n \in \mathbb{N}^*}$  is tight and use Prokhorov's theorem.

### 2.5.1 Preliminary Lemmas

In order to make the strategy above more rigorous, we need a few lemmas. The proof of the following lemma is very similar to the proof of proposition 4.(b) in [32].

**Lemma 2.22.** *Let  $\Theta$  be a cluster-invariant point process. Let  $A$  be a closed ball. Then, for every  $n \in \mathbb{N}^*$ ,*

$$\int \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx)$$

*is finite almost surely. Moreover,*

$$\left( \int \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx) \right)_{n \in \mathbb{N}}$$

*is a tight sequence of random variables.*

*Proof of Lemma 2.22.*

Let  $t > 0$ . By cluster-invariance of  $\Theta$ , it holds that

$$\mathbb{E}[\exp(-t\Theta(A))] = \mathbb{E}[\exp(-t\Theta_n(A))]. \quad (2.26)$$

By definition of  $\Theta_n$ , (2.26) yields

$$\begin{aligned} \mathbb{E}[\exp(-t\Theta(A))] &= \mathbb{E}\left[\prod_{x \in \Theta} \mathbb{E}_{\mathbf{P}}[\exp(-tZ_n(A-x))]\right] \\ &= \mathbb{E}\left[\exp\left(\int \ln\left(\mathbb{E}_{\mathbf{P}}[\exp(-tZ_n(A-x))]\right)\Theta(dx)\right)\right]. \end{aligned} \quad (2.27)$$

However, using inequality  $\ln(1+t) \leq t$  for every  $t > -1$ , one gets that for any  $x \in \mathbb{R}^d$

$$\begin{aligned} &\ln\left(\mathbb{E}_{\mathbf{P}}[\exp(-tZ_n(A-x))]\right) \\ &= \ln\left(1 + \mathbb{E}_{\mathbf{P}}\left[\mathbf{1}\{Z_n(A-x) \geq 1\}\left(\exp(-tZ_n(A-x)) - 1\right)\right]\right) \\ &\leq \mathbb{E}_{\mathbf{P}}\left[\mathbf{1}\{Z_n(A-x) \geq 1\}\left(\exp(-tZ_n(A-x)) - 1\right)\right] \\ &\leq (e^{-t} - 1)\mathbf{P}(Z_n(A-x) \geq 1). \end{aligned} \quad (2.28)$$

Combining (2.27) and (2.28), one gets that

$$\begin{aligned} \mathbb{E}[\exp(-t\Theta(A))] &\leq \mathbb{E}\left[\exp\left((e^{-t} - 1) \int \mathbf{P}(Z_n(A-x) \geq 1)\Theta(dx)\right)\right] \\ &= \mathbb{E}[\exp((e^{-t} - 1)W_n)] \end{aligned} \quad (2.29)$$

where

$$W_n := \int \mathbf{P}(Z_n(A-x) \geq 1)\Theta(dx).$$

From (2.29), we deduce that for every  $L > 0$ ,

$$\mathbb{E}[\exp(-t\Theta(A))] \leq 1 - \mathbb{P}(W_n \geq L) + \mathbb{P}(W_n \geq L)\exp((e^{-t} - 1)L). \quad (2.30)$$

Therefore, for every  $L > 0$ ,

$$\mathbb{P}(W_n \geq L) \leq \frac{1 - \mathbb{E}[\exp(-t\Theta(A))]}{1 - \exp((e^{-t} - 1)L)}. \quad (2.31)$$

By letting  $L$  go toward infinity and  $t$  go to 0 in (2.31), we get that

$$\mathbb{P}(W_n = +\infty) = 0.$$

Moreover inequality (2.31) is clearly sufficient to prove tightness.  $\square$

**Lemma 2.23.** *Let  $A$  be a closed ball. We assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Then there exist positive constants  $\kappa$  and  $\lambda$  such that for every  $M \geq 1$ , there exists an integer  $N_M$  such that for every  $n \geq N_M$  and for every  $x \in B(0, M\sqrt{n})^c$ ,*

$$\mathbf{P}(Z_n(A-x) \geq 1) \leq \kappa e^{-M^2/\kappa} \mathbf{P}(Z_{\lambda n}(A-x) \geq 1).$$

*Proof of Lemma 2.23.*

Let  $(b_1, b_2, \dots)$  be an infinite sequence of real numbers. Let  $n \in \mathbb{N}^*$ . Obviously,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \left( 1 + \sum_{k=1}^n Y_k(b_k) \right)^{-1} \right] \leq 1 \quad (2.32)$$

where  $Y_k$  refers to the notation given in the proof of Lemma 2.13. Moreover, by Jensen's inequality,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \left( 1 + \sum_{k=1}^n Y_k(b_k) \right)^{-1} \right] \geq \left( \mathbb{E}_{\mathbf{Q}^*} \left[ 1 + \sum_{k=1}^n Y_k(b_k) \right] \right)^{-1}.$$

However, in a similar manner as in the proof of identity (2.5), it holds that

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \sum_{k=1}^n Y_k(b_k) \right] \leq |A| c_d \sigma^2 \sum_{k=1}^{+\infty} \frac{1}{k^{d/2}} := C_{A,d,\sigma} < +\infty.$$

Therefore, we get that for every sequence  $(b_1, b_2, \dots)$  and for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \left( 1 + \sum_{k=1}^n Y_k(b_k) \right)^{-1} \right] \geq \frac{1}{1 + C_{A,d,\sigma}}. \quad (2.33)$$

However we can recall from the proof of Lemma 2.13 that for every  $n \in \mathbb{N}^*$  and for any  $x \in B(0, M\sqrt{n})^c$ ,

$$\mathbf{P}(Z_n(A-x) \geq 1) = \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}_{\{S_{w_n} \in (A-x)\}}}{1 + \sum_{k=1}^n Y_k(S_{w_{n-k}+x})} \right]$$

where the spine  $(S_{w_k})_{k \in \mathbb{N}^*}$  is a random walk with motion law  $\mathcal{P}$ . Moreover, by definition of  $\mathbf{Q}^*$ ,  $(S_{w_k})_{k \in \mathbb{N}^*}$  is independent of the random variables  $(Y_k(\cdot))_{k \in \mathbb{N}^*}$  under  $\mathbf{Q}^*$ . Therefore, for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, M\sqrt{n})^c$ ,

$$\mathbf{P}(Z_n(A-x) \geq 1) = \mathbb{E}_{\mathbf{Q}^*} \left[ \mathbf{1}_{\{S_{w_n} \in (A-x)\}} \mathbb{E}_{\mathbf{Q}^*} \left[ \left( 1 + \sum_{k=1}^n Y_k(b_k) \right)^{-1} \right]_{b_k = S_{w_{n-k}+x}} \right]. \quad (2.34)$$

Combining identities (2.32), (2.33) and (2.34) yields

$$(1 + C_{A,d,\sigma})^{-1} \mathbf{Q}^*(S_{w_n} \in A-x) \leq \mathbf{P}(Z_n(A-x) \geq 1) \leq \mathbf{Q}^*(S_{w_n} \in A-x). \quad (2.35)$$

Consequently, under the hypothesis  $\mathcal{H}_1$ , for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, M\sqrt{n})^c$ ,

$$\frac{\mathbf{P}(Z_n(A-x) \geq 1)}{\mathbf{P}(Z_{2n}(A-x) \geq 1)} \leq \frac{(1 + C_{A,d,\sigma}) \int_A e^{-\|y-x\|^2/(2n)} dy}{\int_A e^{-\|y-x\|^2/(4n)} dy}.$$

Therefore, under the hypothesis  $\mathcal{H}_1$ , there exists a positive constant  $C'_{A,d,\sigma}$  such that for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, M\sqrt{n})^c$ ,

$$\begin{aligned} \frac{\mathbf{P}(Z_n(A-x) \geq 1)}{\mathbf{P}(Z_{2n}(A-x) \geq 1)} &\leq C'_{A,d,\sigma} e^{-\|x\|^2/(4n)} \\ &\leq C'_{A,d,\sigma} e^{-M^2/4}. \end{aligned} \quad (2.36)$$

Thus, (2.36) proves Lemma 2.23 under the hypothesis  $\mathcal{H}_1$ . Under the hypothesis  $\mathcal{H}_2$ , we will use Proposition 2.7 to give a similar proof. Let us use constants  $\tau$ ,  $C_1$  and  $C_2$  introduced in Proposition 2.7. Moreover, under hypothesis  $\mathcal{H}_2$ ,  $\mathcal{P}$  has a finite range. Therefore there exists a constant  $C_3$  such that  $\mathbf{P}(Z_n(A-x) \geq 1) = 0$  for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, C_3 n)^c$ . Now let us define  $r = \max(C_2 C_3 / (\tau C_1), 2)$ . Let  $n$  be an integer larger than  $(M/C_3)^2$  and let  $x$  be such that  $M\sqrt{n} < \|x\| \leq C_3 n$ . Then, we get

$$\begin{aligned} \mathbf{P}(Z_n(A-x) \geq 1) &\leq \sum_{y \in A \cap \mathbb{Z}^d} \mathbf{Q}^*(S_{w_n} = y-x) \\ &\leq C_1 n^{-d/2} \sum_{y \in A \cap \mathbb{Z}^d} \exp\left(-\frac{\|x-y\|^2}{C_1 n}\right) \end{aligned} \quad (2.37)$$

where we used identity (2.35) in the first inequality and Proposition 2.7 in the second one. Then, as  $\|x\| \geq M\sqrt{n}$ , we get

$$\mathbf{P}(Z_n(A-x) \geq 1) \leq C \times n^{-d/2} \exp\left(-\frac{(r-1)}{C_1 r} M^2\right) \sum_{y \in A \cap \mathbb{Z}^d} \exp\left(-\frac{\|x-y\|^2}{C_2 (r C_1 C_2^{-1} n)}\right) \quad (2.38)$$

where  $C$  depends only on  $A$  and  $C_1$ . However, by definition of  $r$ ,  $\|x\| \leq C_3 n \leq \tau r C_1 C_2^{-1} n$ . Therefore, by the lower bound in Proposition 2.7, there exists  $C' > 0$  such that for every  $x$  such that  $M\sqrt{n} < \|x\| \leq C_3 n$ ,

$$\begin{aligned} \mathbf{P}(Z_n(A-x) \geq 1) &\leq C' \times \exp\left(-\frac{(r-1)}{C_1 r} M^2\right) \sum_{y \in A \cap \mathbb{Z}^d} \mathbf{Q}^*(S_{w_{r C_1 C_2^{-1} n}} = y-x) \\ &\leq C' \times (1 + C_{A,d,\sigma}) \times \exp\left(-\frac{(r-1)}{C_1 r} M^2\right) \mathbf{P}\left(Z_{r C_1 C_2^{-1} n}(A-x) \geq 1\right) \end{aligned} \quad (2.39)$$

where we used identity (2.35) in the second inequality. If  $\|x\| \geq C_3 n$ , then the left-hand side in (2.39) is zero. Thus (2.39) is also true in this case. This concludes the proof.  $\square$

**Lemma 2.24.** *Let  $A$  be a closed ball. Let us assume hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . Let  $\Theta$  be a cluster-invariant point process. Then,*

$$\lim_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}\left(\int_{B(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx) \geq e^{-M^2/(2\kappa)}\right) = 0.$$

*Proof of Lemma 2.24.*

By Lemma 2.23, for every  $M \geq 1$  and for every  $n \geq N_M$ ,

$$\begin{aligned} \mathbb{P}\left(\int_{B(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx) \geq e^{-M^2/(2\kappa)}\right) \\ \leq \mathbb{P}\left(\int_{B(0, M\sqrt{n})^c} \mathbf{P}(Z_{\lambda n}(A-x) \geq 1) \Theta(dx) \geq \kappa^{-1} e^{M^2/(2\kappa)}\right) \\ \leq \mathbb{P}\left(\int \mathbf{P}(Z_{\lambda n}(A-x) \geq 1) \Theta(dx) \geq \kappa^{-1} e^{M^2/(2\kappa)}\right). \end{aligned}$$

However, by Lemma 2.22,

$$\left(\int \mathbf{P}(Z_{\lambda n}(A-x) \geq 1) \Theta(dx)\right)_{n \in \mathbb{N}}$$

is tight. This concludes the proof.  $\square$

### 2.5.2 Proof of Theorem 2.3

*Proof of Theorem 2.3.*

Let  $A$  be a closed ball. Let  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ . Let  $M > 0$ . By cluster-invariance of  $\Theta$ , for any  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] &= \mathbb{E} \left[ \exp \left( - \int f(x) \Theta_n(dx) \right) \right] \\ &= \mathbb{E} \left[ \prod_{x \in \Theta} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right] \\ &= \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right. \\ &\quad \left. \times \prod_{x \in \Theta \cap B(0, M\sqrt{n})^c} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right]. \end{aligned} \quad (2.40)$$

First, let us look at the random variable

$$J_{n,M} := \int_{B(0, M\sqrt{n})^c} \ln \left( \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right) \Theta(dx). \quad (2.41)$$

As  $\text{supp}(f) \subset A$ ,  $J_{n,M}$  is equal to

$$\int_{B(0, M\sqrt{n})^c} \ln \left( 1 + \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \right) \Theta(dx). \quad (2.42)$$

Moreover, we remark that for every  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \left| \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \right| &\leq \mathbf{P}(Z_n(A-x) \geq 1) \\ &\leq \mathbb{E}_{\mathbf{P}}[Z_n(A-x)] \\ &= \mathbb{P}(\hat{S}_n \in A-x) \end{aligned}$$

where  $(\hat{S}_n)_{n \in \mathbb{N}}$  is a random walk with motion law  $\mathcal{P}$ . Together with Lemma 2.6, this implies :

$$\sup_{x \in \mathbb{R}^d} \left| \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \right| \leq c_d \frac{|A|}{n^{d/2}} \xrightarrow{n \rightarrow +\infty} 0. \quad (2.43)$$

Therefore, as  $|\ln(1+t)| \leq 2|t|$  for every  $t$  small enough, the combination of identities (2.42) and (2.43) implies that there exists an integer  $\tilde{N}$  such that for every  $n \geq \tilde{N}$ ,

$$|J_{n,M}| \leq 2 \int_{B(0, M\sqrt{n})^c} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx)$$

almost surely. Therefore, using Lemma 2.24,

$$\mathbb{P} \left( |J_{n,M}| \geq 2e^{-M^2/(2\kappa)} \right) = o_{n,M}(1) \quad (2.44)$$

where  $o_{n,M}(1)$  is a function  $(\varepsilon_{n,M})_{n \in \mathbb{N}^*, M \in \mathbb{R}_+^*}$  such that  $\lim_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \varepsilon_{n,M} = 0$ . Furthermore, by (2.40), for every  $n \geq \tilde{N}$ ,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] \\ &= \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \times \exp(J_{n,M}) \right] \\ &= \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right] \\ &+ \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \times (\exp(J_{n,M}) - 1) \right]. \end{aligned} \quad (2.45)$$

However,

$$\left| \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \times (\exp(J_{n,M}) - 1) \right] \right|$$

is lower than,

$$\begin{aligned} \mathbb{E} [|\exp(J_{n,M}) - 1|] &= \mathbb{E} [|\exp(J_{n,M}) - 1| \mathbf{1}\{|J_{n,M}| < 2e^{-M^2/(2\kappa)}\}] \\ &+ \mathbb{E} [|\exp(J_{n,M}) - 1| \mathbf{1}\{|J_{n,M}| \geq 2e^{-M^2/(2\kappa)}\}] \\ &\leq o_{n,M}(1) + \mathbb{P}(|J_{n,M}| \geq 2e^{-M^2/(2\kappa)}) \\ &= o_{n,M}(1) \end{aligned} \quad (2.46)$$

where the second inequality comes from the fact that  $J_{n,M}$  is non-positive and the last equality comes from identity (2.44). Consequently, combining (2.45) and (2.46), it holds that,

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right] + o_{n,M}(1). \quad (2.47)$$

Furthermore, as  $\text{supp}(f) \subset A$ ,

$$\mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right] = \mathbb{E} \left[ \exp \left( \int_{B(0, M\sqrt{n})} h_n(x) \Theta(dx) \right) \right] \quad (2.48)$$

where  $h_n(x) = \ln \left( 1 + \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \left( \exp \left( - \sum_{|u|=n} f(S_u + x) \right) - 1 \right) \right] \right)$ .

Nevertheless, by Lemma 2.13, uniformly in  $x \in B(0, M\sqrt{n})$ ,

$$h_n(x) = (1 + o_n(1)) R_{f,A} \mathbf{P}(Z_n(A-x) \geq 1) \quad (2.49)$$

where we recall that

$$R_{f,A} = \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] - 1 = \frac{I_{A,-f}}{I_A} - 1.$$

Consequently, using (2.49) in (2.48) yields

$$\begin{aligned} & \mathbb{E} \left[ \prod_{x \in \Theta \cap B(0, M\sqrt{n})} \mathbb{E}_{\mathbf{P}} \left[ \exp \left( - \sum_{|u|=n} f(S_u + x) \right) \right] \right] \\ &= \mathbb{E} \left[ \exp \left( (1 + o_n(1)) R_{f,A} \int_{B(0, M\sqrt{n})} \mathbf{P}(Z_n(A - x) \geq 1) \Theta(dx) \right) \right]. \end{aligned} \quad (2.50)$$

Moreover, by Lemma 2.22, we know that

$$\int_{B(0, M\sqrt{n})} \mathbf{P}(Z_n(A - x) \geq 1) \Theta(dx)$$

is tight. Therefore, by Prokhorov's Theorem (see for example [26]), there exists a subsequence  $r_n = r_n(M, A)$  such that

$$\int_{B(0, M\sqrt{r_n})} \mathbf{P}(Z_{r_n}(A - x) \geq 1) \Theta(dx)$$

converges in law toward some random variable  $Y_{M,A}$ . Combining this with (2.47), (2.50) and letting  $n$  go toward  $+\infty$ , we deduce that

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} [\exp(Y_{M,A} R_{f,A})] + o_M(1). \quad (2.51)$$

Besides, thanks to Lemma 2.22, the tightness of  $(Y_{M,A})_{M \in \mathbb{R}_+^*}$  is easily checked. Thus a subsequence of  $(Y_{M,A})_{M \in \mathbb{N}^*}$  converges in law toward some finite, non-negative random variable  $Y_A$ . Letting  $M$  go to infinity in (2.51), we deduce that for every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ ,

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] &= \mathbb{E} [\exp(Y_A R_{f,A})] \\ &= \mathbb{E} [\exp(X_A I_A R_{f,A})] \end{aligned} \quad (2.52)$$

with  $X_A := Y_A \times I_A^{-1}$  where  $I_A$  is defined in definition 2.14. Then, we can compare the expression of the Laplace transform given in Proposition 2.18 and the expression given by identity (2.52) and we get that for every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$  :

$$\mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} \left[ \exp \left( - \int f(x) \Lambda_\infty^{d, X_A}(dx) \right) \right].$$

Consequently, by Lemma 2.8,  $\Theta(\cdot \cap A)$  has the same law as  $\Lambda_\infty^{d, X_A}(\cdot \cap A)$ . At this point, the proof of Theorem 2.3 is almost complete. The last subtlety is that we only proved our result in  $A$ . Of course we could consider larger and larger balls  $A$  but we have to verify that  $A \subset B$  implies  $X_A \stackrel{\text{law}}{=} X_B$ . Thus, let us consider two closed balls  $A$  and  $B$  such that  $A \subset B$ . Let  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ . The two formulas for the Laplace transform given by (2.52) with  $A$  and  $B$  are equals :

$$\mathbb{E} [\exp(X_A I_A R_{f,A})] = \mathbb{E} \left[ \exp \left( - \int f(x) \Theta(dx) \right) \right] = \mathbb{E} [\exp(X_B I_B R_{f,B})]. \quad (2.53)$$

As  $\text{supp}(f) \subset A$ , we know that

$$\begin{aligned} R_{f,B} &= \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_B(dx) \right) - 1 \right] \\ &= \mathbb{E} \left[ \mathbf{1}\{\mathcal{N}_B(A) \geq 1\} \left( \exp \left( - \int f(x) \mathcal{N}_B(dx) \right) - 1 \right) \right]. \end{aligned} \quad (2.54)$$

Therefore, using (2.54) and Lemmas 2.19 and 2.20, we obtain that

$$\begin{aligned} \frac{I_B}{I_A} R_{f,B} &= \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_B(dx) \right) - 1 \middle| \mathcal{N}_B(A) \geq 1 \right] \\ &= \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] - 1 \\ &= R_{f,A}. \end{aligned} \tag{2.55}$$

Thus  $I_A R_{f,A} = I_B R_{f,B}$ . Together with (2.53), for every  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ , this implies that

$$\mathbb{E} [\exp(X_A I_A R_{f,A})] = \mathbb{E} [\exp(X_B I_A R_{f,A})].$$

Moreover

$$I_A R_{f,A} = I_A \times \left( \mathbb{E} \left[ \exp \left( - \int f(x) \mathcal{N}_A(dx) \right) \right] - 1 \right).$$

It is equal to 0 for  $f = 0$  and  $-I_A R_{f,A}$  can be made as close as we want from  $I_A$  by choosing  $f$  of larger and larger infinite norm. Consequently, for every  $t \in [0, I_A[$ ,

$$\mathbb{E} [\exp(-tX_A)] = \mathbb{E} [\exp(-tX_B)].$$

It is sufficient in order to say that  $X_A$  and  $X_B$  have the same distribution because  $I_A > 0$ . It concludes the proof.  $\square$

## 2.6 Heavy tail generalization

### 2.6.1 New setting

In this section, we prove Theorem 2.4 which states that Theorems 2.2 and 2.3 remain true under the hypothesis  $\mathcal{H}_3$ . We recall that hypothesis  $\mathcal{H}_3$  consists in the following assumptions : The motion law  $\mathcal{P}$  is given by spherically symmetric  $\alpha$ -stable laws with  $\alpha \in ]0, 2]$ . More precisely, for every  $y \in \mathbb{R}^d$ ,

$$\int \exp(i\langle y, x \rangle) \mathcal{P}(dx) = \exp \left( - \left( \sum_{k=1}^d |y_k|^2 \right)^{\alpha/2} \right).$$

The critical offspring law  $\mu$  does not have second moment anymore. However we assume that there exists  $\beta \in ]0, 1]$  such that for every  $\gamma < \beta$ ,  $\sum_{k=0}^{+\infty} k^{1+\gamma} \mu(k) < +\infty$ . Moreover we assume that  $d > \alpha/\beta$ .

In this new setting,  $\alpha \in ]0, 2]$ . Of course,  $\alpha = 2$  is the Brownian case which has been treated under the hypothesis  $\mathcal{H}_1$  when  $\beta = 1$ . Assuming hypothesis  $\mathcal{H}_3$ , any particle is following a spherically symmetric  $\alpha$ -stable Lévy process until its death. One can refer to [157] for more informations about general Lévy processes.

**Remark 2.5.** *The typical example of an offspring law  $\mu$  satisfying hypothesis  $\mathcal{H}_3$  is given by  $f(s) = s + \frac{1}{2}(1-s)^{1+\beta}$  with  $\beta \in ]0, 1]$ , where  $f(s) = \sum_{k=0}^{+\infty} \mu(k)s^k$  for every  $s \in [0, 1]$ .*

Of course, it would be much too long to modify the entirety of our proofs. In this section, we just indicate to the reader how to change the key steps in the proofs of Theorem 2.2 and 2.3. In the sequel, we often use a random variable  $\chi$  such that for every  $y \in \mathbb{R}^d$ ,

$$\mathbb{E} \left[ e^{i\langle y, \chi \rangle} \right] = \exp \left( - \left( \sum_{k=1}^d |y_k|^2 \right)^{\alpha/2} \right).$$

Let us call such a random variable  $\chi$  "a standard ( $d$ -dimensional)  $\alpha$ -stable law". We denote by  $L_{\alpha,d}$  the density of  $\chi$ . Except for a few values of  $\alpha$ ,  $L_{\alpha,d}$  has not any closed form.

### 2.6.2 Generalized proofs

In order to prove generalized versions of Theorem 2.2 we have to prove a new version of Lemma 2.13. Under the hypothesis  $\mathcal{H}_3$ ,  $I_A$  and  $I_{A,f}$  are defined as in definition 2.14.

**Lemma 2.25** (Key Lemma - generalized version). *We assume hypothesis  $\mathcal{H}_3$ . Let  $A$  be a closed ball. Let  $f \in \mathcal{F}_c(\mathbb{R}^d)$  whose values are in  $i\mathbb{R}$  or in  $\mathbb{R}_-$  and such that  $\text{supp}(f) \subset A$ . Let  $M > 0$ . Uniformly in  $x \in B(0, Mn^{1/\alpha})$ , as  $n$  goes to infinity,*

$$\mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] = (1 + o_n(1)) \frac{L_{\alpha,d}(x/n^{1/\alpha})}{n^{d/\alpha}} I_{A,f}.$$

**Remark 2.6.**  $I_{A,f}$  is defined as in definition 2.14 but the motion is now of Levy type. The well-definiteness of  $I_{A,f}$  can be proved exactly as in the following proof of Lemma 2.25.

Now, let us prove Lemma 2.25.

*Proof of Lemma 2.25.*

With the same notation as in the proof of Lemma 2.13 we know that

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in A-x\}}{1 + \sum_{k=1}^n Y_k(S_{w_{n-k}} + x)} \exp \left( \sum_{k=0}^n Y_{f,k}(S_{w_{n-k}} + x) \right) \right]. \end{aligned}$$

Now let us prove that we can restrict the sum above to a finite number of terms, as in the proof of Lemma 2.13. Let  $K > 0$ . Let us consider  $\gamma > 0$  such that  $\gamma < \beta \leq 1$ . Recall the notation  $\mathbb{E}_{\mathbf{Q}^*} [\cdot | \mathcal{G}]$  when we condition on the spine and on the number of brothers of every particle in the spine and the positions of the brothers of the spine. Then, by conditional Markov inequality,

$$\begin{aligned} & \mathbf{Q}^* \left( S_{w_n} \in A-x, \sum_{k=K+1}^n Y_k(x + S_{w_{n-k}}) \geq 1 \right) \\ & \leq \mathbb{E}_{\mathbf{Q}^*} \left[ \mathbf{1}\{S_{w_n} \in A-x\} \mathbb{E}_{\mathbf{Q}^*} \left[ \left( \sum_{k=K+1}^n Y_k(x + S_{w_{n-k}}) \right)^\gamma \middle| \mathcal{G} \right] \right]. \end{aligned}$$

We know that for any non negative numbers  $(\lambda_i)_{1 \leq i \leq n}$ ,  $\left( \sum_{i=1}^n \lambda_i \right)^\gamma \leq \sum_{i=1}^n \lambda_i^\gamma$  because  $0 < \gamma < 1$ . Consequently,

$$\begin{aligned} & \mathbf{Q}^* \left( S_{w_n} \in A-x, \sum_{k=K+1}^n Y_k(x + S_{w_{n-k}}) \geq 1 \right) \\ & \leq \mathbb{E}_{\mathbf{Q}^*} \left[ \mathbf{1}\{S_{w_n} \in A-x\} \sum_{k=K+1}^n \mathbb{E}_{\mathbf{Q}^*} \left[ Y_k(x + S_{w_{n-k}})^\gamma \middle| \mathcal{G} \right] \right]. \end{aligned} \quad (2.56)$$

However, for every  $k \in \{K+1, \dots, n\}$ , by definition of  $Y_k(\cdot)$ ,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ Y_k(x + S_{w_{n-k}})^\gamma \middle| \mathcal{G} \right] = \mathbb{E}_{\mathbf{Q}^*} \left[ \left( \sum_{u \in B(w_{n-k+1})} Z_{k-1}^u(A-x - S_{w_{n-k}} - \varrho_u) \right)^\gamma \middle| \mathcal{G} \right]$$

where for every  $u \in B(w_{n-k+1})$ , recall that  $\varrho_u = S_u - S_{w_{n-k}}$ . Then, as  $0 < \gamma < 1$ , by Jensen's inequality,

$$\begin{aligned} \mathbb{E}_{\mathbf{Q}^*} \left[ Y_k(x + S_{w_{n-k}})^\gamma \middle| \mathcal{G} \right] &\leq \mathbb{E}_{\mathbf{Q}^*} \left[ \sum_{u \in B(w_{n-k+1})} Z_{k-1}^u(A - x - S_{w_{n-k}} - \varrho_u) \middle| \mathcal{G} \right]^\gamma \\ &= \left( \sum_{u \in B(w_{n-k+1})} \mathbb{E}_{\mathbf{Q}^*} \left[ Z_{k-1}^u(A - x - S_{w_{n-k}} - \varrho_u) \middle| \mathcal{G} \right] \right)^\gamma. \end{aligned} \quad (2.57)$$

However, by construction of  $\mathbf{Q}^*$ , for every  $k \in \{K+1, \dots, n\}$  and for every  $u \in B(w_{n-k+1})$ , it holds that

$$\mathbb{E}_{\mathbf{Q}^*} \left[ Z_{k-1}^u(A - x - S_{w_{n-k}} - \varrho_u) \middle| \mathcal{G} \right] = \mathbb{E}_{\mathbf{P}} [Z_{k-1}(A - z - \varrho)] \Big|_{z=x+S_{w_{n-k}}, \varrho=\varrho_u}.$$

Then, by criticality of the branching process under  $\mathbf{P}$ , for every  $k \in \{K+1, \dots, n\}$  and for every  $u \in B(w_{n-k+1})$ ,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ Z_{k-1}^u(A - x - S_{w_{n-k}} - \varrho_u) \middle| \mathcal{G} \right] = \mathbb{P} \left( (k-1)^{1/\alpha} \chi \in A - z - \varrho \right) \Big|_{z=x+S_{w_{n-k}}, \varrho=\varrho_u}$$

where  $\chi$  is a standard  $d$ -dimensional  $\alpha$ -stable law. Then, as the density  $L_{\alpha,d}$  is bounded, there exists a constant  $c_\alpha > 0$  such that for every  $k \in \{K+1, \dots, n\}$  and for every  $u \in B(w_{n-k+1})$ ,

$$\begin{aligned} \mathbb{E}_{\mathbf{Q}^*} \left[ Z_{k-1}^u(A - x - S_{w_{n-k}} - \varrho_u) \middle| \mathcal{G} \right] &= \int_{A/(k-1)^{1/\alpha}} L_{\alpha,d} \left( y - \frac{x + S_{w_{n-k}} + \varrho_u}{(k-1)^{1/\alpha}} \right) dy \\ &\leq c_\alpha |A| k^{-d/\alpha}. \end{aligned} \quad (2.58)$$

Combining, (2.56), (2.57) and (2.58), we get

$$\begin{aligned} \mathbf{Q}^* \left( S_{w_n} \in A - x, \sum_{k=K+1}^n Y_k(x + S_{w_{n-k}}) \geq 1 \right) \\ \leq c_\alpha^\gamma |A|^\gamma \sum_{k=K+1}^n k^{-d\gamma/\alpha} \mathbb{E}_{\mathbf{Q}^*} [\mathbf{1}\{S_{w_n} \in A - x\} | B(w_{n-k+1}) |^\gamma]. \end{aligned} \quad (2.59)$$

However for every  $k \in \{1, \dots, n\}$ ,  $1 + |B(w_{n-k+1})|$ , that is, the number of children of  $w_{n-k}$ , has the distribution  $\nu$  and is independent of the spine under  $\mathbf{Q}^*$ . We recall that  $\nu$  is defined by  $\nu(k) = k\mu(k)$ . We assumed that  $\int x^{1+\gamma} \mu(dx) < +\infty$ . Thus,  $\hat{c}_\gamma := \int x^\gamma \nu(dx) < +\infty$ . Therefore, together with (2.59), this yields

$$\begin{aligned} \mathbf{Q}^* \left( S_{w_n} \in A - x, \sum_{k=K+1}^n Y_k(x + S_{w_{n-k}}) \geq 1 \right) &\leq c_\alpha^\gamma |A|^\gamma \hat{c}_\gamma \sum_{k=K+1}^n k^{-d\gamma/\alpha} \times \mathbf{Q}^* (S_{w_n} \in A - x) \\ &\leq c_\alpha^{1+\gamma} |A|^{1+\gamma} \hat{c}_\gamma n^{-d/\alpha} \sum_{k=K+1}^{+\infty} k^{-d\gamma/\alpha}. \end{aligned} \quad (2.60)$$

Now, we need the sum to be convergent. For this, we need  $d\gamma/\alpha > 1$ , that is,  $d > \frac{\alpha}{\gamma}$ . However we assumed that  $d > \frac{\alpha}{\beta}$ . Thus, this is possible to find such a  $\gamma < \beta$ . Therefore, uniformly in

$x \in B(0, Mn^{1/\alpha})$ , it holds that

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{\mathbf{1}\{S_{w_n} \in A-x\}}{1 + \sum_{k=1}^K Y_k(S_{w_{n-k}} + x)} \exp \left( \sum_{k=0}^K Y_{f,k}(S_{w_{n-k}} + x) \right) \right] + o_K(1)n^{-d/\alpha}. \end{aligned} \quad (2.61)$$

Recall that for every  $m \in \mathbb{N}^*$  the density of  $S_{w_m}$  is given by  $z \mapsto m^{-d/\alpha} L_{\alpha,d}(m^{-1/\alpha}z)$ . Therefore, using (2.61) and following the same lines as in Lemma 2.13, we get

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= (n-K)^{-d/\alpha} \mathbb{E}_{\mathbf{Q}^*} \left[ \int_A L_{\alpha,d}((n-K)^{-1/\alpha}(y-x-S_{w_K})) F_{K,A}(y, S_{w_k}, 1 \leq k \leq K) dy \right] \\ &+ o_K(1)n^{-d/\alpha}. \end{aligned} \quad (2.62)$$

The function  $F_{K,A}$  in (2.62) is defined exactly as in the proof of Lemma 2.13. Remark that

$$(n-K)^{-1/\alpha}(y-x-S_{w_K}) = -n^{-1/\alpha}x + \underbrace{\frac{1-(1-K/n)^{1/\alpha}}{(n-K)^{1/\alpha}}x}_{\varepsilon_{n,K,x,y}} + (n-K)^{-1/\alpha}(y-S_{w_K}). \quad (2.63)$$

On the event  $\{\|S_{w_k}\| \leq K^{2/\alpha}\}$ , uniformly in  $x \in B(0, Mn^{1/\alpha})$ , the error term  $\varepsilon_{n,K,x,y}$  can be majorized in the following way

$$\begin{aligned} \|\varepsilon_{n,K,x,y}\| &\leq M \times o_{n,K}(1) + (\sup_{z \in A} \|z\|) \times (n-K)^{-1/\alpha} + K^{2/\alpha}(n-K)^{-1/\alpha} \\ &= o_{n,K}(1) \end{aligned} \quad (2.64)$$

where for every  $K \in \mathbb{N}^*$ ,  $o_{n,K}(1)$  tends to zero as  $n$  goes to infinity. Moreover,  $L_{\alpha,d}$  is continuous and goes to 0 at infinity. Therefore, this function is uniformly continuous. This implies that, uniformly in  $x \in B(0, Mn^{1/\alpha})$ , on the event  $\{\|S_{w_k}\| \leq K^{2/\alpha}\}$ ,

$$|L_{\alpha,d}(n^{-1/\alpha}x) - L_{\alpha,d}((n-K)^{-1/\alpha}(y-x-S_{w_K}))| = o_{n,K}(1) \quad (2.65)$$

where  $L_{\alpha,d}(-n^{-1/\alpha}x)$  was replaced by  $L_{\alpha,d}(n^{-1/\alpha}x)$  by symmetry of  $L_{\alpha,d}$ . Furthermore, the event  $\{\|S_{w_K}\| \geq K^{2/\alpha}\}$  can be neglected because

$$\mathbf{Q}^*(\|S_{w_K}\| \geq K^{2/\alpha}) = o_K(1). \quad (2.66)$$

Therefore, combining (2.62), (2.65) and (2.66), we get that, uniformly in  $x \in B(0, Mn^{1/\alpha})$ ,

$$\begin{aligned} & \mathbb{E}_{\mathbf{P}} \left[ \mathbf{1}\{Z_n(A-x) \geq 1\} \exp \left( \sum_{|u|=n} f(S_u + x) \right) \right] \\ &= n^{-d/\alpha} L_{\alpha,d}(n^{-1/\alpha}x) \mathbb{E}_{\mathbf{Q}^*} \left[ \int_A F_{K,A}(y, S_{w_k}, 1 \leq k \leq K) dy \right] \\ &+ o_K(1)n^{-d/\alpha} + o_{n,K}(1)n^{-d/\alpha}. \end{aligned} \quad (2.67)$$

Thanks to (2.67), the end of the proof goes along the same lines as in the proof of Lemma 2.13.  $\square$

Thanks to this Lemma 2.25, we can derive a generalized version of Theorem 2.2 in the same way as before. All the proof is the same. We only have to replace Lemma 2.13 by Lemma 2.25. Moreover, proving Theorem 2.3 under the hypothesis  $\mathcal{H}_3$  is very similar with proving the same theorem under hypotheses  $\mathcal{H}_1$  or  $\mathcal{H}_2$ . We only need slight modifications for the generalization of Lemma 2.24. Let us adapt this lemma under the hypothesis  $\mathcal{H}_3$ .

**Lemma 2.26.** *Let  $A$  be a closed ball. Let us assume hypothesis  $\mathcal{H}_3$ . Let  $\Theta$  be a cluster-invariant point process. Then,*

$$\lim_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P} \left( \int_{B(0, Mn^{1/\alpha})^c} \mathbf{P}(Z_n(A-x) \geq 1) \Theta(dx) \geq M^{-\alpha/4} \right) = 0.$$

By a careful reading, one can convince oneself that one can prove a generalized version of Theorem 2.3 thanks to this generalized Lemma 2.26. This Lemma 2.26 is a direct consequence of Lemma 2.22 and the following Lemma 2.27 which is a generalized version of Lemma 2.23.

**Lemma 2.27.** *We assume hypothesis  $\mathcal{H}_3$ . Let  $A$  be a closed ball. There exists  $C > 0$  such that for every  $M \geq 1$ , for every  $n$  large enough and for every  $x \in B(0, Mn^{1/\alpha})^c$ ,*

$$\mathbf{P}(Z_n(A-x) \geq 1) \leq C \frac{\mathbf{P}(Z_{M^{\alpha/2}n}(A-x) \geq 1)}{M^{\alpha/2}}.$$

*Proof of Lemma 2.27.*

As  $d > \alpha/\beta$ , following the same lines as in the proof of Lemma 2.23, there exists a positive constant  $C_{A,d,\alpha,\beta}$  such that for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, Mn^{1/\alpha})^c$ ,

$$\begin{aligned} \frac{\mathbf{P}(Z_n(A-x) \geq 1)}{\mathbf{P}(Z_{M^{\alpha/2}n}(A-x) \geq 1)} &\leq \frac{C_{A,d,\alpha,\beta} \mathbb{P}(n^{1/\alpha} \chi \in A-x)}{\mathbb{P}((M^{\alpha/2}n)^{1/\alpha} \chi \in A-x)} \\ &= C_{A,d,\alpha,\beta} \frac{\int_{(A-x)/n^{1/\alpha}} L_{\alpha,d}(y) dy}{\int_{(A-x)/(M^{\alpha/2}n)^{1/\alpha}} L_{\alpha,d}(y) dy} \end{aligned} \quad (2.68)$$

where we recall that  $\chi$  is a standard  $d$ -dimensional  $\alpha$ -stable law and  $L_{\alpha,d}$  is its density. According to [28], there exists a constant  $c_{\alpha,d}$  such that

$$L_{\alpha,d}(z) \underset{\|z\| \rightarrow +\infty}{\sim} c_{\alpha,d} \|z\|^{-d-\alpha}.$$

In particular, there exist  $c_{\alpha,d}^{(1)} > 0$  and  $c_{\alpha,d}^{(2)} > 0$  such that for every  $z \in \mathbb{R}^d$  such that  $\|z\| \geq 1$ ,

$$c_{\alpha,d}^{(1)} \|z\|^{-d-\alpha} \leq L_{\alpha,d}(z) \leq c_{\alpha,d}^{(2)} \|z\|^{-d-\alpha}. \quad (2.69)$$

However for  $x \in B(0, Mn^{1/\alpha})^c$ ,  $\|n^{-1/\alpha} M^{-1/2} x\| \geq \sqrt{M}$ . Consequently, if  $M \geq 1$ , by (2.69), for every  $x \in B(0, Mn^{1/\alpha})^c$ ,

$$c_{\alpha,d}^{(1)} \|n^{-1/\alpha} M^{-1/2} x\|^{-d-\alpha} \leq L_{\alpha,d}(n^{-1/\alpha} M^{-1/2} x) \leq c_{\alpha,d}^{(2)} \|n^{-1/\alpha} M^{-1/2} x\|^{-d-\alpha}$$

and

$$c_{\alpha,d}^{(1)} \|n^{-1/\alpha} x\|^{-d-\alpha} \leq L_{\alpha,d}(n^{-1/\alpha} x) \leq c_{\alpha,d}^{(2)} \|n^{-1/\alpha} x\|^{-d-\alpha}.$$

Together with (2.68) it implies that there exists a constant  $C > 0$  such that for every  $n \in \mathbb{N}^*$  and for every  $x \in B(0, Mn^{1/\alpha})^c$ ,

$$\begin{aligned} \frac{\mathbf{P}(Z_n(A-x) \geq 1)}{\mathbf{P}(Z_{M^{\alpha/2}n}(A-x) \geq 1)} &\leq C \frac{\int_{(A-x)/n^{1/\alpha}}^{\infty} \|y\|^{-d-\alpha} dy}{\int_{(A-x)/(M^{\alpha/2}n)^{1/\alpha}}^{\infty} \|y\|^{-d-\alpha} dy} \\ &= C \times M^{-\alpha/2} \end{aligned}$$

where the last equality is obtained by the changes of variable  $y' = n^{1/\alpha}y$  in the integral above and  $y'' = \sqrt{M}n^{1/\alpha}y$  in the integral below. It concludes the proof of Lemma 2.27.  $\square$

## 2.7 Remarks and further discussion

Let us begin with a few remarks concerning our results.

**Remark 2.7.** *In this paper, we discussed three different hypotheses. Observe that, hypothesis  $\mathcal{H}_2$  is a generalization of hypothesis  $\mathcal{H}_1$ . Indeed, in both cases,  $\mathcal{P}$  has a second moment. Consequently it is in the domain of attraction of a Gaussian distribution. However, having a second moment is not enough for us. Indeed, in our arguments, we need the random walk associated to  $\mathcal{P}$  to meet the heat-kernel estimate stated in Proposition 2.7. Our results hold for any distribution  $\mathcal{P}$  which has a finite second moment and satisfies this heat-kernel estimate. For now, we do not know how to remove this technical assumption. Under hypotheses  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , as  $\mathcal{P}$  is in the domain of attraction of a Gaussian distribution, the critical branching process with motion law  $\mathcal{P}$  is analogous to a binary branching Brownian motion. In particular, assuming  $\mathcal{H}_1$  or  $\mathcal{H}_2$ , the critical dimension is always 2 as in Theorem 2.1. On the contrary, under hypothesis  $\mathcal{H}_3$ ,  $\mathcal{P}$  is no more in the Brownian domain. This leads to a change of the critical dimension, among other consequences. Indeed, under hypothesis  $\mathcal{H}_3$ , we can prove our results for  $d > \alpha/\beta$ . Moreover, assuming hypothesis  $\mathcal{H}_3$ , Gorostiza and Wakolbinger showed in [81] (in a continuous setting but the qualitative behaviour must be the same.) that if  $d \leq \alpha/\beta$ , then the sequence  $(\Lambda_n^{d,X})_{n \in \mathbb{N}}$  converges to the null point process. Therefore,  $\alpha/\beta$  is the critical dimension if we assume hypothesis  $\mathcal{H}_3$ .*

**Remark 2.8.** *Actually, the heat kernel estimate in  $\mathcal{H}_2$  is not really necessary for Theorem 2.2. We see in the proof of Lemma 2.17 that we only need the existence of a positive integrable function  $g$  such that for every  $x \in \mathbb{Z}^d$  and for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}(\hat{S}_n = x) \leq n^{-d/2}g(n^{-1/2}x)$  where  $\hat{S}$  is the random walk associated to  $\mathcal{P}$ . This is possible as soon as  $\mathcal{P}$  has a moment of order  $d+1$  thanks to the asymptotic expansion of the local-limit theorem. (See Theorem 22.1 in [21].) However, the heat kernel estimate in assumption  $\mathcal{H}_2$  is crucial in the proof of Theorem 2.3 for technical reasons.*

Now, let us try to get another insight about our results. Thanks to Theorem 2.2, we know a local description of  $\Lambda_\infty^{d,X}$  on every closed ball  $A$ . Indeed, if  $P_A$  is a Poisson random variable of parameter  $XI_A$ , then  $\Lambda_\infty^{d,X}(\cdot \cap A)$  can be constructed as a layering of  $P_A$  independent copies of  $\mathcal{N}_A$ . A natural question is to know whether this is possible to describe  $\mathcal{N}_A$  and  $I_A$  in terms of  $\Lambda_\infty^{d,X}$ . We prove that there exists another interesting description of  $\Lambda_\infty^{d,X}$  which explains the meaning of  $\mathcal{N}_A$  and  $I_A$ . Let us define  $\mathbf{W}$  the set of measures  $W$  on the set  $\mathbf{N}$  of point measures on  $\mathbb{R}^d$  which satisfy :

- a) For any closed ball  $A$  of  $\mathbb{R}^d$ ,  $W(\{\mu \in \mathbf{N} : \mu(A) > 0\}) < +\infty$ .
- b)  $W(\{\emptyset\}) = 0$  where  $\emptyset$  is the null point measure.

If  $W \in \mathbf{W}$ , we can define a Poisson point process  $\Xi$  on  $\mathbf{N}$  with intensity measure  $W$ .  $\Xi$  can be written as

$$\Xi = \sum_{i \in I} \delta_{N_i}$$

where  $\{N_i, i \in I\}$  is a countable collection of point measures. Then we are able to define a new point process  $\mathcal{E}_W$  on  $\mathbb{R}^d$  as

$$\mathcal{E}_W = \sum_{i \in I} N_i$$

where the sum of point measures means that we overlay them.  $\mathcal{E}_W$  is still a locally finite point process thanks to assumption *a*). We mention that the point process  $\mathcal{E}_W$  is introduced in subsection 2.1 of [124] in a slightly different (but equivalent) way.

**Proposition 2.28.** *We assume hypothesis  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  or  $\mathcal{H}_3$ . Then for every  $\theta \in \mathbb{R}_+^*$ , there exists a unique measure  $W^{d,\theta} \in \mathbf{W}$  such that :*

(i)

$$\mathcal{E}_{W^{d,\theta}} \stackrel{\text{law}}{=} \Lambda_\infty^{d,\theta}.$$

(ii) For any closed ball  $A$ ,  $W^{d,\theta}(\{\mu \in \mathbf{N} : \mu(A) \geq 1\}) = \theta I_A$ .

(iii) For any closed ball  $A$ , for any borel set  $U$  of the set of point measures on  $A$ ,

$$W^{d,\theta} \left( \left\{ \mu : \mu(\cdot \cap A) \in U \right\} \middle| \left\{ \mu \in \mathbf{N} : \mu(A) \geq 1 \right\} \right) = \mathbb{P}(\mathcal{N}_A \in U).$$

(iv) Let  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  whose support is a closed ball  $A$ . Then for every interval  $I \subset \mathbb{R}_+^*$ ,

$$W^{d,\theta} \left( \left\{ \mu \in \mathbf{N} : \int f(x) d\mu(x) \in I \right\} \right) = \theta I_A \mathbb{P} \left( \int f(x) d\mathcal{N}_A(x) \in I \right).$$

*Proof of Proposition 2.28.*

$\Lambda_\infty^{d,\theta}$  is infinitely divisible. Indeed, it is obtained through a branching process starting from a Poisson point process which is infinitely divisible. Then, Theorem 2.1.10 in [124] gives (i). By the local description of  $\Lambda_\infty^{d,\theta}$  given in Theorem 2.2, for any closed ball  $A$ , it holds that

$$\begin{aligned} e^{-\theta I_A} &= \mathbb{P} \left( \Lambda_\infty^{d,\theta}(A) = 0 \right) \\ &= \mathbb{P} \left( \mathcal{E}_{W^{d,\theta}}(A) = 0 \right). \end{aligned} \tag{2.70}$$

Now, let us recall that  $\mathcal{E}_{W^{d,\theta}}$  is defined through a Poisson point process  $\Xi = \sum_{i \in I} \delta_{N_i}$  on  $\mathbf{N}$  with intensity  $W^{d,\theta}$ . Then, from (2.70), we deduce

$$\begin{aligned} e^{-\theta I_A} &= \mathbb{P} (N_i(A) = 0, \forall i \in I) \\ &= \mathbb{P} (\Xi(\{\mu : \mu(A) \geq 1\}) = 0). \end{aligned}$$

Therefore, as  $\Xi$  is a Poisson point process on  $\mathbf{N}$  with intensity  $W^{d,\theta}$ ,

$$e^{-\theta I_A} = \mathbb{P} (\Xi(\{\mu : \mu(A) \geq 1\}) = 0) = e^{-W^{d,\theta}(\{\mu : \mu(A) \geq 1\})}.$$

This proves (ii). Now, let us consider a closed ball  $A$  and  $f \in \mathcal{F}_c^+(\mathbb{R}^d)$  such that  $\text{supp}(f) \subset A$ . For every point measure  $\mu$ , let us define  $H_f(\mu) = \int f(x) d\mu(x)$ . Then, by definition of  $\mathcal{E}_{W^{d,\theta}}$ ,

$$\mathbb{E} \left( \exp \left( - \int f(x) d\mathcal{E}_{W^{d,\theta}}(x) \right) \right) = \mathbb{E} \left[ \exp \left( - \int H_f(\mu) d\Xi(\mu) \right) \right]. \tag{2.71}$$

However,  $\Xi$  is a Poisson point process with intensity  $W^{d,\theta}$ . Therefore, by Lemma 3.1 in [93], (2.71) yields

$$\begin{aligned} & \mathbb{E} \left( \exp \left( - \int f(x) d\mathcal{E}_{W^{d,\theta}}(x) \right) \right) \\ &= \exp \left( - \int (1 - \exp(-H_f(\mu))) dW^{d,\theta}(\mu) \right) \\ &= \exp \left( \int \exp \left( - \int f(x) d\mu(x) \right) - 1 dW^{d,\theta}(\mu) \right) \\ &= \exp \left( \int \left( \exp \left( - \int f(x) d\mu(x) \right) - 1 \right) \mathbf{1}_{\{\mu(A) \geq 1\}} dW^{d,\theta}(\mu) \right) \end{aligned}$$

where the last equality stems from the fact that  $\text{supp}(f) \subset A$ . By (ii), this implies that

$$\begin{aligned} & \mathbb{E} \left( \exp \left( - \int f(x) d\mathcal{E}_{W^{d,\theta}}(x) \right) \right) \\ &= \exp \left( \theta I_A \left( \int \exp \left( - \int f(x) d\mu(x) \right) dW^{d,\theta}(\mu | \mu(A) \geq 1) - 1 \right) \right). \end{aligned} \quad (2.72)$$

However, by (i), we know that

$$\mathbb{E} \left( \exp \left( - \int f(x) d\mathcal{E}_{W^{d,\theta}}(x) \right) \right) = \mathbb{E} \left( \exp \left( - \int f(x) d\Lambda_\infty^{d,\theta}(x) \right) \right).$$

Consequently, by Proposition 2.18, we get

$$\mathbb{E} \left( \exp \left( - \int f(x) d\mathcal{E}_{W^{d,\theta}}(x) \right) \right) = \exp \left( \theta I_A \left( \mathbb{E} \left[ \exp \left( - \int f(x) d\mathcal{N}_A(x) \right) \right] - 1 \right) \right). \quad (2.73)$$

Moreover, the left-hand sides in (2.72) and (2.73) are equals. This implies that

$$\int \exp \left( - \int f(x) d\mu(x) \right) dW^{d,\theta}(\mu | \mu(A) \geq 1) = \mathbb{E} \left[ \exp \left( - \int f(x) d\mathcal{N}_A(x) \right) \right]. \quad (2.74)$$

This proves (iii). Moreover, (iv) is a direct consequence of (ii) and (iii).  $\square$

An interesting property of the measure  $W^{d,\theta}$  is that it is  $\sigma$ -finite but not finite. This is proved by the following proposition. We assume  $\mathcal{H}_1$  for clarity of the discussion.

**Proposition 2.29.** *Assuming hypothesis  $\mathcal{H}_1$ ,  $I_{B(0,r)}$  converges toward infinity when  $r$  is going to infinity.*

*Proof of Proposition 2.29.*

Let  $A$  be a closed ball. Using the notation introduced in definition 2.14, we recall that

$$I_A = \int_A \mathbb{E}_{\mathbf{Q}^*} \left[ \frac{1}{1 + \sum_{k=1}^{+\infty} \tilde{Y}_k(y - S_{w_k})} \right] dy.$$

Therefore, by Jensen's inequality,

$$I_A \geq \int_A \frac{1}{\mathbb{E}_{\mathbf{Q}^*} \left[ 1 + \sum_{k=1}^{+\infty} \tilde{Y}_k(y - S_{w_k}) \right]} dy. \quad (2.75)$$

However, with the notation of the proof of Lemma 2.13, for every  $k \in \mathbb{N}^*$ ,

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \tilde{Y}_k(y - S_{w_k}) \right] = \mathbb{E}_{\mathbf{Q}^*} \left[ \sum_{u \in B(w_{k+1})} Z_{k-1}^u(A - y + S_{w_k} - \varrho_u) \right].$$

Then, by construction of  $\mathbf{Q}^*$ , it holds that

$$\mathbb{E}_{\mathbf{Q}^*} \left[ \tilde{Y}_k(y - S_{w_k}) \right] = \mathbb{E}_{\mathbf{Q}^*} [|B(w_{k+1})|] \mathbb{E}_{\mathbf{Q}^*} \left[ \mathbb{E}_{\mathbf{P}} [Z_{k-1}(A - y + z)] |_{z=S_{w_k} - \varrho} \right] \quad (2.76)$$

where  $\varrho$  is independent of  $S_{w_k}$ . Moreover  $\varrho$  is a standard Gaussian random variable because we assume hypothesis  $\mathcal{H}_1$ . We know that, under  $\mathbf{Q}^*$ ,  $1 + |B(w_{k+1})|$  has law  $\nu$  which has expectation  $1 + \sigma^2$ . Moreover, the branching process is critical under  $\mathbf{P}$ . Thus,  $\mathbb{E}_{\mathbf{P}} [Z_{k-1}(A - y + z)] = \mathbb{P}(\hat{S}_{k-1} \in A - y + z)$  where  $\hat{S}$  is a brownian motion. Therefore, together with (2.76), this yields

$$\begin{aligned} \mathbb{E}_{\mathbf{Q}^*} \left[ \tilde{Y}_k(y - S_{w_k}) \right] &= \sigma^2 \mathbb{P}(\hat{S}_{2k} \in A - y) \\ &= \sigma^2 (4\pi k)^{-d/2} \int_A \exp(-\|x - y\|^2/4k) dx. \end{aligned} \quad (2.77)$$

Consequently, combining (2.75) and (2.77), we get

$$I_A \geq \int_A \left( 1 + \sigma^2 \sum_{k=1}^{+\infty} (4\pi k)^{-d/2} \int_A \exp(-\|x - y\|^2/(4k)) dx \right)^{-1} dy. \quad (2.78)$$

Now, if  $A = B(0, r)$ , a change of variable in (2.78) yields

$$\begin{aligned} I_{B(0,r)} &\geq r^d \int_{B(0,1)} \left( 1 + \sigma^2 \sum_{k=1}^{+\infty} (4\pi k)^{-d/2} r^d \int_{B(0,1)} \exp(-r^2\|x - y\|^2/(4k)) dx \right)^{-1} dy \\ &= \int_{B(0,1)} \left( r^{-d} + \sigma^2 \sum_{k=1}^{+\infty} (4\pi k)^{-d/2} \int_{B(0,1)} \exp(-r^2\|x - y\|^2/(4k)) dx \right)^{-1} dy. \end{aligned} \quad (2.79)$$

However, by the dominated convergence theorem, as  $d \geq 3$ , for every  $y \in B(0, 1)$ ,

$$\lim_{r \rightarrow +\infty} r^{-d} + \sigma^2 \sum_{k=1}^{+\infty} (4\pi k)^{-d/2} \int_{B(0,1)} \exp(-r^2\|x - y\|^2/(4k)) dx = 0.$$

Consequently, using Fatou's Lemma in (2.79),

$$\lim_{r \rightarrow +\infty} I_{B(0,r)} = +\infty.$$

□

## Chapitre 3

# The asymptotic behaviour of the martingale associated with the VRJP

### Abstract

This chapter is based on the article [141] which has been written under the supervision of Christophe Sabot and Xinxin Chen. We study the asymptotic behaviour of the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  associated with the Vertex Reinforced Jump Process (VRJP). We show that it is bounded in  $L^p$  for every  $p > 1$  on trees and uniformly integrable on  $\mathbb{Z}^d$  in all the transient phase of the VRJP. Moreover, when the VRJP is recurrent on trees, we have good estimates on the moments of  $\psi_n(o)$  and we can compute the exact decreasing rate  $\tau$  such that  $n^{-1} \ln(\psi_n(o)) \sim -\tau$  almost surely where  $\tau$  is related to standard quantities for branching random walks. Besides, on trees, at the critical point, we show that  $n^{-1/3} \ln(\psi_n(o)) \sim -\varrho_c$  almost surely where  $\varrho_c$  can be computed explicitly. Furthermore, at the critical point, we prove that the discrete-time process associated with the VRJP is a mixture of positive recurrent Markov chains. Our proofs use properties of the  $\beta$ -potential associated with the VRJP and techniques coming from the domain of branching random walks.

### 3.1 Introduction and first definitions

Let  $(V, E)$  be a locally finite graph. Let  $W > 0$ . In [48], Davis and Volkov introduced a continuous self-reinforced random walk  $(Y_s)_{s \geq 0}$  known as the Vertex Reinforced Jump Process (VRJP) which is defined as follows : the VRJP starts from some vertex  $i_0 \in V$  and conditionally on the past before time  $s$ , it jumps from a vertex  $i$  to one of its neighbour  $j$  at rate  $WL_j(s)$  where

$$L_j(s) = 1 + \int_0^s \mathbf{1}\{Y_u = s\} du.$$

In [152], Sabot and Tarrès defined the time-change  $D$  such that for every  $s \geq 0$ ,

$$D(s) = \sum_{i \in V} (L_i(s)^2 - 1).$$

Then, they introduced the time-changed process  $(Z_t)_{t \geq 0} = (Y_{D^{-1}(t)})_{t \geq 0}$ . If  $V$  is finite, this process is easier to analyse than  $Y$  because it is a mixture of Markov processes whose mixing field has a density which is known explicitly. The density of the mixing field of  $Z$  was already known as a hyperbolic supersymmetric sigma model. This supersymmetric model was first studied in [54] and

[53] and Sabot and Tarrès combined these previous works with their own results in order to make some important progress in the knowledge of the VRJP. However, their formula for the density of the environment of the VRJP was true only on finite graphs. This difficulty has been solved in [153] and [154] where Sabot, Tarrès and Zeng introduced a  $\beta$ -potential with some distribution  $\nu_V^W$  which allows to have a representation of the environment of the VRJP on infinite graphs. Thanks to this  $\beta$ -potential, Sabot and Zeng introduced a positive martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  which converges toward some random variable  $\psi(o)$ . A remarkable fact is that  $\psi(o) = 0$  if and only if the VRJP is recurrent. Moreover, they proved a 0-1 law for transitive graphs. On these graphs, the VRJP is either almost surely recurrent or almost surely transient.

We can study the VRJP on any locally finite graph  $V$ . However, in this paper, we will focus only on the two most important cases :

- First, we can consider the case where  $V = \mathbb{Z}^d$ . In this case, when  $d \in \{1, 2\}$ , the VRJP is always recurrent. (See [154], [149] and [102].) On the contrary, when  $d \geq 3$ , Sabot and Tarrès proved in [152] that the time-changed VRJP is recurrent for small  $W$  and that it is transient for large  $W$ . Further, in [139], thanks to a clever coupling of  $\psi_n(o)$  for different weights, Poudevigne proved there is a unique transition point  $W_c(d)$  between recurrence and transience on  $\mathbb{Z}^d$  if  $d \geq 3$ .
- Another interesting case for the VRJP is when  $V$  is a tree. In this case, the environment of the VRJP is easy to describe thanks to independent Inverse Gaussian random variables. Using this representation of the environment, in [40], Chen and Zeng proved there is a unique phase transition between recurrence and transience on supercritical Galton-Watson trees for the time-changed VRJP. (This result was already proved in [15] but the proof of [15] was very different and did not use the representation of the VRJP as a mixture of Markov processes.) Furthermore the transition point  $W_c(\mu)$  can be computed explicitly and depends only on the mean of the offspring law  $\mu$  of the Galton-Watson tree.

Therefore, if  $V$  is a Galton-Watson tree or  $\mathbb{Z}^d$  with  $d \geq 3$ , the following dichotomy is known : there exists  $W_c \in \mathbb{R}_+^*$  (depending on  $V$ ) such that

If  $W < W_c$ , then a.s,  $\psi(o) = 0$ , i.e the VRJP is recurrent.

If  $W > W_c$ , then a.s,  $\psi(o) > 0$ , i.e the VRJP is transient.

The recurrence of the VRJP can be regarded as a form of "strong disorder". Indeed, if  $W$  is small, the reinforcement, i.e the disorder of the system compared to a simple random walk, is very strong. Therefore, the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  associated with the system vanishes only when there is strong disorder. This situation is reminiscent of directed polymers in random environment. One can refer to [44] for more information on this topic. In the case of directed polymers, there is a positive martingale  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  which converges toward a random variable  $\mathcal{M}_\infty$ .  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  and  $(\psi_n(o))_{n \in \mathbb{N}}$  play analogous roles in different contexts. Indeed,  $\mathcal{M}_\infty > 0$  a.s if and only if the system exhibits "weak disorder", exactly as for  $\psi(o)$ . However, on  $\mathbb{Z}^d$  or on trees, this is possible that  $\mathcal{M}_\infty > 0$  a.s but  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is not bounded in  $L^2$ . (See [35] and [33].) Therefore, a natural question regarding  $(\psi_n(o))_{n \in \mathbb{N}}$  is to know when it is bounded in  $L^p$  for a fixed value of  $p > 1$ . Moreover, as shown in the proof of Theorem 3 in [154],  $L^p$  boundedness of the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  on  $\mathbb{Z}^d$  for sufficiently large  $p$  implies the existence of a diffusive regime for the VRJP, i.e the VRJP satisfies a central-limit theorem. We would like to know whether this diffusive regime coincides with the transient regime or not. This gives another good reason to study the moments of  $(\psi_n(o))_{n \in \mathbb{N}}$ . Using [54], [154] and [139], one can prove that, on  $\mathbb{Z}^d$  with  $d \geq 3$ , for any  $p > 1$ , there exists a threshold  $W^{(p)}(d)$  such that  $(\psi_n(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  for every  $W > W^{(p)}(d)$ . However, we do not know whether  $W^{(p)}(d) = W_c(d)$  for every  $p > 1$  or not. In this paper, we will prove that  $(\psi_n(o))_{n \in \mathbb{N}}$  is uniformly integrable on  $\mathbb{Z}^d$  as soon as the VRJP is transient. Moreover, we will prove that  $(\psi_n(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  for any  $p > 1$  as soon as  $W > W_c(\mu)$  on trees.

Furthermore, we will also look at the rate of convergence toward 0 of  $(\psi_n(o))_{n \in \mathbb{N}}$  on trees when  $W < W_c(\mu)$  under mild assumptions. We have a  $L^p$  version and an almost sure version of

the estimate of the decay of  $(\psi_n(o))_{n \in \mathbb{N}}$  toward 0.

Finally a natural question consists in finding the behaviour of the VRJP at the critical point  $W_c$ . On Galton-Watson trees, it was proved in [40] or [15] that the time-changed VRJP is a mixture of recurrent Markov processes at the critical point. In this paper, we prove that it is even a mixture of positive recurrent Markov processes. However the asymptotic behaviour of the VRJP at the critical point on  $\mathbb{Z}^d$  remains unknown. We will also compute the rate of convergence of  $(\psi_n(o))_{n \in \mathbb{N}}$  on trees when  $W = W_c(\mu)$ .

## 3.2 Context and statement of the results

### 3.2.1 General notation

Let  $(V, E)$  be a locally finite countable graph with non oriented edges. We assume that  $V$  has a root  $o$ . We write  $i \sim j$  when  $\{i, j\} \in E$ . For every  $n \in \mathbb{N}$ , we define  $V_n := \{x \in V, d(o, x) \leq n\}$  where  $d$  is the graph distance on  $(V, E)$ . For every  $n \in \mathbb{N}^*$ , we denote the boundary of  $V_n$ , that is  $\{i \in V_n, \exists j \in V_n^c \text{ such that } \{i, j\} \in E\}$ , by  $\partial V_n$ . Let us denote by  $E_n$  the set of edges of  $V_n$ . If  $M$  is a matrix (or possibly an operator) with indices in a set  $A \times B$ , then for every  $A' \subset A$  and  $B' \subset B$ , the restriction of  $M$  to  $A' \times B'$  is denoted by  $M_{A', B'} = (M(i, j))_{(i, j) \in A' \times B'}$ . If  $M$  is a symmetric matrix, we write  $M > 0$  when  $M$  is positive definite.

In this article, we use a lot the Inverse Gaussian distribution. For every  $(a, \lambda)$ , recall that an Inverse Gaussian random variable with parameters  $(a, \lambda) \in \mathbb{R}_+^{*2}$  has density :

$$\mathbf{1}\{x > 0\} \left( \frac{\lambda}{2\pi x^3} \right)^{1/2} \exp \left( -\frac{\lambda(x-a)^2}{2a^2x} \right) dx. \quad (3.1)$$

The law of the Inverse Gaussian distribution with parameters  $(a, \lambda) \in \mathbb{R}_+^{*2}$  is denoted by  $IG(a, \lambda)$ . For  $W > 0$  and  $t \in \mathbb{R}$ , if  $A \sim IG(1, W)$ , we write  $Q(W, t) = \mathbb{E}[A^t]$ . A well-known property of the Inverse Gaussian distribution states that  $Q(W, t) = Q(W, 1-t)$ .

### 3.2.2 The $\beta$ -potential and the martingale $(\psi_n)_{n \in \mathbb{N}}$

Let  $(V, E)$  be an infinite countable graph with non-oriented edges. In this paper, the graph  $(V, E)$  will always have a special vertex  $o$  called the root. Actually, in our results,  $V$  is a rooted tree or  $\mathbb{Z}^d$  with root 0. Let  $W > 0$ . In [154], the authors introduced a random potential  $(\beta_i)_{i \in V}$  on  $V$  with distribution  $\nu_V^W$  such that for every finite subset  $U \subset V$ , for every  $(\lambda_i)_{i \in U} \in \mathbb{R}_+^U$ ,

$$\begin{aligned} & \int \exp \left( -\sum_{i \in U} \lambda_i \beta_i \right) \nu_V^W(d\beta) \\ &= \exp \left( -\frac{1}{2} \sum_{\substack{i \sim j \\ i, j \in U}} W \left( \sqrt{1 + \lambda_i} \sqrt{1 + \lambda_j} - 1 \right) - \sum_{\substack{i \sim j \\ i \in U, j \notin U}} W \left( \sqrt{1 + \lambda_i} - 1 \right) \right) \frac{1}{\prod_{i \in U} \sqrt{1 + \lambda_i}}. \end{aligned} \quad (3.2)$$

Looking at the Laplace transform in (3.2), we see that  $(\beta_i)_{i \in V}$  is 1-dependent, that is, if  $U_1$  and  $U_2$  are finite subsets of  $V$  which are not connected by an edge, then  $(\beta_i)_{i \in U_1}$  and  $(\beta_i)_{i \in U_2}$  are independent under  $\nu_V^W$ . Moreover, the restriction of this potential on finite subsets has a density which is known explicitly. We give the expression of this density in subsection 3.3.1. Furthermore, for every  $(\beta_i)_{i \in V}$ , let us introduce the operator  $H_\beta$  on  $V$  which satisfies :

$$\forall (i, j) \in V^2, H_\beta(i, j) = 2\beta_i \mathbf{1}\{i = j\} - W \mathbf{1}\{i \sim j\}.$$

By proposition 1 in [154], the support of  $\nu_V^W$  is

$$\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, (H_\beta)_{U, U} \text{ is positive definite for all finite subsets } U \subset V\}.$$

Therefore, under  $\nu_V^W$ , for every  $n \in \mathbb{N}$ ,  $(H_\beta)_{V_n, V_n}$  is positive definite. In particular, it is invertible. We denote by  $\hat{G}_n$  the inverse of  $(H_\beta)_{V_n, V_n}$ . Moreover, for  $n \in \mathbb{N}$  and  $\beta \in \mathcal{D}_V^W$ , let us define  $(\psi_n(i))_{i \in V}$  as the unique solution of the equation :

$$\begin{cases} (H_\beta \psi_n)(i) = 0 & \forall i \in V_n \\ \psi_n(i) = 1 & \forall i \in V_n^c \end{cases} \quad (3.3)$$

The idea behind the definition of  $(\psi_n)_{n \in \mathbb{N}}$  is to create an eigenstate of  $H_\beta$  when  $n$  goes to infinity. We can make  $n$  go to infinity thanks to the following proposition :

**Proposition A** (Theorem 1 in [154]). *For any  $i, j \in V$ ,  $(\hat{G}_n(i, j))_{n \in \mathbb{N}^*}$  is increasing  $\nu_V^W$ -a.s. In particular there exists a random variable  $\hat{G}(i, j)$  such that*

$$\hat{G}_n(i, j) \xrightarrow[n \rightarrow +\infty]{} \hat{G}(i, j), \quad \nu_V^W \text{-a.s.}$$

Further, for any  $i, j \in V$ ,

$$\hat{G}(i, j) < +\infty, \quad \nu_V^W \text{-a.s.}$$

Moreover,  $(\psi_n)_{n \in \mathbb{N}}$  is a vectorial martingale with positive components. In particular, for every  $i \in V$  the martingale  $(\psi_n(i))_{n \in \mathbb{N}}$  has an almost sure limit which is denoted by  $\psi(i)$ . Besides,  $(\hat{G}_n)_{n \in \mathbb{N}}$  is the bracket of  $(\psi_n)_{n \in \mathbb{N}}$  in the sense that for every  $i, j \in V$ ,  $(\psi_n(i)\psi_n(j) - \hat{G}_n(i, j))_{n \in \mathbb{N}}$  is a martingale.

This martingale  $(\psi_n)_{n \in \mathbb{N}}$  is crucial in order to study the asymptotic behaviour of the VRJP. One reason for this is that a representation of the environment of the discrete random walk associated with the VRJP starting from  $i_0$  is given by  $(WG(i_0, j)G(i_0, i))_{\{i, j\} \in E}$  where for every  $(i, j) \in V^2$ ,

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma} \psi(i)\psi(j)$$

where  $\gamma$  is random variable with distribution  $\Gamma(1/2, 1)$  which is independent of the random potential  $\beta$ . We will say more about the link between the VRJP and  $(\psi_n)_{n \in \mathbb{N}}$  in Proposition B. Before this, let us give some notation.

### 3.2.3 Notation associated with the VRJP

#### 3.2.3.1 General notation for the VRJP

In the previous section, for every deterministic graph  $(V, E)$ , we introduced the measure  $\nu_V^W$  associated with the  $\beta$ -potential. We write  $\mathbb{E}_{\nu_V^W}$  when we integrate with respect to this measure  $\nu_V^W$ . Moreover, we defined a martingale  $(\psi_n(o))_{n \in \mathbb{N}}$ . For a fixed graph  $V$ , we say that  $(\psi_n(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  if  $\sup_{n \in \mathbb{N}} \mathbb{E}_{\nu_V^W} [\psi_n(o)^p] < +\infty$ . We say that  $(\psi_n(o))_{n \in \mathbb{N}}$  is uniformly integrable if

$$\lim_{K \rightarrow +\infty} \sup_{n \in \mathbb{N}} \mathbb{E}_{\nu_V^W} [\psi_n(o) \mathbf{1}\{\psi_n(o) \geq K\}] = 0.$$

We denote by  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  the discrete time process associated with the VRJP, that is, the VRJP taken at jump times. We will see that it is a mixture of discrete random walks. Let us introduce the probability measure  $\mathbf{P}_{V, W}^{VRJP}$  under which  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  is the discrete time process associated with the VRJP on a graph  $V$  with constant weights  $W$  starting from  $o$ .

### 3.2.3.2 Notation for the VRJP on trees

If  $V$  is a rooted tree, there is a natural genealogical order  $\leq$  on  $V$ . For  $u \in V$ , the parent of  $u$  is denoted by  $\tilde{u}$  and the generation of  $u$  is denoted by  $|u|$ . If  $(x, u) \in V^2$  such that  $x \leq u$ , then  $|u|_x = |u| - |x|$ . If  $V$  is a Galton-Watson tree with offspring law  $\mu$ , let us denote by  $GW^\mu$  the law of  $V$ . Then, let us define the probability measure  $\mathbb{P}_{\mu, W}$  under which we first choose randomly the graph  $V$  with distribution  $GW^\mu$  and then we choose randomly the potential  $(\beta_i)_{i \in V}$  with distribution  $\nu_V^W$ . Moreover, we define  $\mathbf{P}_{\mu, W}^{VRJP}$  under which we first choose randomly the graph  $V$  with distribution  $GW^\mu$  and then we choose randomly a trajectory on  $V$  with distribution  $\mathbf{P}_{V, W}^{VRJP}$ . We write  $\mathbb{E}_{\mu, W}(\cdot)$  and  $\mathbf{E}_{\mu, W}^{VRJP}(\cdot)$  when we integrate with respect to  $\mathbb{P}_{\mu, W}$  and  $\mathbf{P}_{V, W}^{VRJP}$  respectively.

### 3.2.4 The phase transition

The martingale  $\psi$  is very important in order to understand the recurrence or transience of the VRJP as explained by the following proposition :

**Proposition B** ([152], [154], [139] and [40]). *Let us assume that  $(V, E)$  is  $\mathbb{Z}^d$ . Then there exists  $W_c(d) > 0$  depending only on  $d$  such that :*

- *If  $W < W_c(d)$ ,  $\nu_d^W$ -a.s, for every  $i \in \mathbb{Z}^d$ ,  $\psi(i) = 0$  and the VRJP is recurrent.*
- *If  $W > W_c(d)$ ,  $\nu_d^W$ -a.s, for every  $i \in \mathbb{Z}^d$ ,  $\psi(i) > 0$  and the VRJP is transient.*

*Moreover,  $W_c(d) < +\infty$  if and only if  $d \geq 3$ . Now let us assume that  $(V, E)$  is a supercritical Galton Watson tree with offspring law  $\mu$  such that  $\mu(0) = 0$ . Then there exists  $W_c(\mu) \in \mathbb{R}_+^*$  depending only on the mean of  $\mu$  such that :*

- *If  $W \leq W_c(\mu)$ ,  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,  $\psi(i) = 0$  and the VRJP is recurrent.*
- *If  $W > W_c(\mu)$ ,  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,  $\psi(i) > 0$  and the VRJP is transient.*

### 3.2.5 Statement of the results

#### 3.2.5.1 Results on $\mathbb{Z}^d$

For now, on  $\mathbb{Z}^d$ , we are not able to estimate the moments of the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  in the transient phase. However, when  $d \geq 3$ , we can prove uniform integrability of this martingale in the transient phase.

**Theorem 3.1.** *We assume that  $V = \mathbb{Z}^d$  with  $d \geq 3$  and that  $W > W_c(d)$ . Then the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  is uniformly integrable.*

#### 3.2.5.2 Results on Galton-Watson trees

Let  $\mu$  be a probability measure on  $\mathbb{N}$ . In this paper, we use the following hypotheses for Galton-Watson trees :

- Hypothesis  $A_1$  :  $\mu(0) = 0$  and  $m := \sum_{k=1}^{+\infty} k\mu(k) > 1$ .
- Hypothesis  $A_2$  :  $\mu(1) = 0$ .
- Hypothesis  $A_3$  : There exists  $\delta > 0$  such that  $\sum_{k=1}^{+\infty} k^{1+\delta}\mu(k) < +\infty$ .

Our first theorem on trees states that, if  $V$  is a Galton-Watson tree,  $(\psi_n(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$  as soon as the VRJP is transient.

**Theorem 3.2.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypothesis  $A_1$ . Let  $W > W_c(\mu)$ . Then, for every  $p \in ]1, +\infty[$ , the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  is bounded in  $L^p$ ,  $GW^\mu$ -a.s.*

In the recurrent phase, we already know that  $\psi_n(o) \xrightarrow{a.s} 0$  on any graph as  $n$  goes to infinity. Thanks to the theory of branching random walks and the representation of the VRJP with the

$\beta$ -potential, we are able to be much more accurate on trees. Let us introduce some notation related to branching random walks in order to give the precise asymptotics of  $(\psi_n(o))_{n \in \mathbb{N}}$ .

For every  $m > 1$ ,  $W > 0$ , we define

$$\begin{aligned} f_{m,W} &: \mathbb{R} \rightarrow \mathbb{R} \\ t &\mapsto \ln(mQ(W, t)). \end{aligned}$$

Moreover, we will prove in the step 1 of the proof of Theorem 3.3 that there exists a unique  $t^*(m, W) > 0$  such that

$$f'_{m,W}(t^*(m, W)) = \frac{f_{m,W}(t^*(m, W))}{t^*(m, W)}. \quad (3.4)$$

Then, we define  $\tau(m, W) = -f'_{m,W}(t^*(m, W))$ . Thanks to these quantities, we are able to describe the asymptotics of  $(\psi_n(o))_{n \in \mathbb{N}}$  in the two following results. First, we can estimate the moments of  $(\psi_n(o))_{n \in \mathbb{N}}$ .

**Theorem 3.3.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypotheses  $A_1$ ,  $A_2$  and  $A_3$ . Let  $W < W_c(\mu)$ . Then we have the following moment estimates :*

- (i)  $\forall p > 0$ ,  $\mathbb{E}_{\mu,W}[\psi_n(o)^{-p}] = \mathbb{E}_{\mu,W}[\psi_n(o)^{1+p}] = e^{np\tau(m,W)+o(n)}$ .
- (ii)  $\forall p \in ]1 - t^*(m, W), 1[$ ,  $\mathbb{E}_{\mu,W}[\psi_n(o)^p] = \mathbb{E}_{\mu,W}[\psi_n(o)^{1-p}] \leq e^{-n(1-p)\tau(m,W)+o(n)}$

with  $\tau(m, W) > 0$  and  $0 < t^*(m, W) < 1/2$ .

**Remark 3.1.** *In Theorem 3.3, remark that we can not estimate all the moments of  $(\psi_n(o))_{n \in \mathbb{N}}$ . This is due to the non-integrability of high moments of some quantities related to branching random walks. We will be more precise in Proposition K.*

The previous theorem gives good estimates of the moments of  $(\psi_n(o))_{n \in \mathbb{N}}$ . Moreover, it is also possible to give the exact almost sure decreasing rate of  $(\psi_n(o))_{n \in \mathbb{N}}$  if  $W < W_c(\mu)$ .

**Theorem 3.4.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypotheses  $A_1$  and  $A_3$ . Let  $W < W_c(\mu)$ . Then, it holds that,  $\mathbb{P}_{\mu,W}$ -a.s,*

$$\lim_{n \rightarrow +\infty} \frac{\ln(\psi_n(o))}{n} = -\tau(m, W)$$

with  $\tau(m, W) > 0$ .

The following proposition gives an estimate of the behaviour of the decreasing rate  $\tau(m, W)$  near the critical point  $W_c(\mu)$ .

**Proposition 3.5.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypothesis  $A_1$ . In the neighborhood of the critical point  $W_c(\mu)$ ,*

$$\tau(m, W) \underset{W \rightarrow W_c(\mu)}{\sim} \alpha(m)(W_c(\mu) - W)$$

where  $\alpha(m) = 2 + \frac{1}{W_c(\mu)} - 2m \frac{K_1(W_c(\mu))}{K_{1/2}(W_c(\mu))} > 0$  where  $K_\alpha$  is the modified Bessel function of the second kind with index  $\alpha$ .

Following basically the same lines as in the proofs of the previous estimates on  $(\psi_n(o))_{n \in \mathbb{N}}$ , we deduce information on the asymptotic behaviour of the VRJP when  $W < W_c(\mu)$ . More precisely, we can estimate the probability for the VRJP to touch the generation  $n$  before coming back to the root  $o$  when  $W < W_c(\mu)$ . Remind that  $(\tilde{Z}_k)_{k \in \mathbb{N}}$  is the discrete-time process associated with the VRJP on the rooted tree  $V$  starting from  $o$ . We define  $\tau_o^+ = \inf\{k \in \mathbb{N}^*, \tilde{Z}_k = o\}$  and for every  $n \in \mathbb{N}^*$ , we define  $\tau_n = \inf\{k \in \mathbb{N}^*, |\tilde{Z}_k| = n\}$ . Recall that the probability measure  $\mathbf{P}_{\mu,W}^{VRJP}$  is defined in the paragraph 3.2.3.2.

**Proposition 3.6.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypotheses  $A_1$ ,  $A_2$  and  $A_3$ . Let  $W < W_c(\mu)$ . Then we have the following estimate :*

$$-2\tau(m, W) \leq \liminf_{n \rightarrow +\infty} \frac{\ln \left( \mathbf{P}_{\mu, W}^{VRJP}(\tau_o^+ > \tau_n) \right)}{n}$$

and

$$\limsup_{n \rightarrow +\infty} \frac{\ln \left( \mathbf{P}_{\mu, W}^{VRJP}(\tau_o^+ > \tau_n) \right)}{n} \leq -\tau(m, W) \times t^*(m, W)$$

where  $0 < t^*(m, W) < 1/2$ .

**Remark 3.2.** *We suspect that the real decreasing rate in the proposition above is  $-2\tau(m, W)$ . Indeed, we only have a problem of integrability of some functionals related to branching random walks. Up to this technical detail, the upper bound in Proposition 3.6 would be  $-2\tau(m, W)$  too.*

Now, let us look at the behaviour of the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  at the critical point  $W_c(\mu)$ .

**Theorem 3.7.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypothesis  $A_1$  and  $A_3$ . We assume that  $W = W_c(\mu)$ . Then, under  $\mathbb{P}_{\mu, W}$ ,*

$$\frac{\ln(\psi_n(o))}{n^{1/3}} \xrightarrow[n \rightarrow +\infty]{a.s.} -\varrho_c$$

where  $\varrho_c = \frac{1}{2} \left( \frac{3\pi^2 \sigma^2}{2} \right)^{1/3}$  with  $\sigma^2 = 16m \int_0^{+\infty} \frac{\sqrt{W_c(\mu)} \ln(x)^2}{\sqrt{2\pi x}} e^{-\frac{W_c(\mu)}{2}(x+1/x-2)} dx$ .

**Remark 3.3.** *At the critical point, we are not able to have precise  $L^p$  bounds for  $\psi_n(o)$ . Indeed, in the subcritical phase, we have subexponential bounds for some functionals associated with branching random walks. At the critical point, we would need to be more accurate.*

The recurrence of the VRJP on trees at the critical point  $W_c(\mu)$  was already known. The following theorem states that the VRJP on trees is even positive recurrent at the critical point. This result is of a different kind than the previous ones. However, the proof requires the same tools as before.

**Theorem 3.8.** *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypothesis  $A_1$  and  $A_3$ . We assume that  $W = W_c(\mu)$ . Then, the discrete-time VRJP  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  associated with  $(Z_t)_{t \geq 0}$  is a mixture of positive recurrent Markov chains.*

## 3.3 Background

### 3.3.1 Marginals and conditional laws of the $\beta$ -potential

The law  $\nu_V^W$  introduced in section 3.1 was originally defined on finite graphs in [153] with general weights. More precisely, on a finite set  $S$ , we can define a  $\beta$ -potential with some law  $\tilde{\nu}_S^{P, \eta}$  for every  $(\eta_i)_{i \in S} \in \mathbb{R}_+^S$  and every  $P = (W_{i,j})_{i,j \in S^2} \in \mathbb{R}_+^{S^2}$ . One can remark that the weights in the matrix  $P$  are not assumed to be constants anymore. Moreover we allow loops, that is,  $W_{i,i}$  can be non-zero for every  $i \in S$ . The term  $\eta$  is a boundary term which represents the weights of some edges relating  $S$  to some virtual vertices which are out of  $S$ . The probability measure  $\tilde{\nu}_S^{P, \eta}$  is defined in the following way : by Lemma 4 in [154] the function

$$\beta \mapsto \mathbf{1}\{H_\beta^{(S)} > 0\} \left( \frac{2}{\pi} \right)^{|S|/2} e^{-\frac{1}{2}\langle 1, H_\beta^{(S)} 1 \rangle - \frac{1}{2}\langle \eta, (H_\beta^{(S)})^{-1} \eta \rangle + \langle \eta, 1 \rangle} \frac{1}{\sqrt{\det H_\beta^{(S)}}} \quad (3.5)$$

is a density.  $H_\beta^{(S)}$  is a matrix on  $S \times S$  defined by

$$H_\beta^{(S)}(i, j) = 2\beta_i \mathbf{1}\{i = j\} - W_{i,j} \mathbf{1}\{i \sim j\}$$

and  $\mathbf{1}$  stands for the vector  $(1, \dots, 1)$  in  $\mathbb{R}^S$  in the expression (3.5). Then, we can define a probability measure with the density (3.5) and we denote it by  $\tilde{\nu}_S^{P,\eta}(d\beta)$ . Besides, the Laplace transform of  $\tilde{\nu}_S^{P,\eta}$  can be computed and it is very similar to the Laplace transform of  $\nu_V^W$ . Indeed, for any  $\lambda \in \mathbb{R}_+^S$ ,

$$\int e^{-\langle \lambda, \beta \rangle} \tilde{\nu}_S^{P,\eta}(d\beta) = e^{-\langle \eta, \sqrt{\lambda+1} - 1 \rangle - \frac{1}{2} \sum_{i \sim j} W_{i,j} (\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1)} \prod_{i \in S} (1 + \lambda_i)^{-1/2}$$

where  $\sqrt{1 + \lambda}$  is the vector  $(\sqrt{1 + \lambda_i})_{i \in S}$ . Further, the family of distributions of the form  $\tilde{\nu}_S^{P,\eta}$  have a very useful behaviour regarding its marginals and conditional laws. Indeed, marginals and conditional laws are still of the form  $\tilde{\nu}_S^{P,\eta}$ . The following lemma gives a formula for the law of the marginals and the conditional laws :

**Lemma C** (Lemma 5 in [154]). *Let  $S$  be a finite set. Let  $U \subset S$  be a subset of  $S$ . Let  $(\eta_i)_{i \in S} \in \mathbb{R}_+^S$  and  $P = (W_{i,j})_{i,j \in S^2} \in \mathbb{R}_+^{S^2}$ . Under  $\tilde{\nu}_S^{P,\eta}$ ,*

(i)  $\beta_U$  has law  $\tilde{\nu}_U^{P_{U,U}, \hat{\eta}}$ , where for every  $i \in U$ ,  $\hat{\eta}_i = \eta_i + \sum_{j \in U^c} W_{i,j}$ .

(ii) Conditionally on  $\beta_U$ ,  $\beta_{U^c}$  has distribution  $\tilde{\nu}_{U^c}^{\check{P}, \check{\eta}}$  where  $\check{P}$  and  $\check{\eta}$  are defined in the following way : For every  $(i, j) \in U^c \times U^c$ ,

$$\check{P}(i, j) = \check{W}_{i,j} = W_{i,j} + \sum_{k \sim i, k \in U} \sum_{l \sim j, l \in U} W_{i,k} W_{j,l} (H_\beta)_{U,U}^{-1}(k, l).$$

For every  $i \in U^c$ ,

$$\check{\eta}_i = \eta_i + \sum_{k \sim i, k \in U} \sum_{l \in U} W_{i,k} (H_\beta)_{U,U}^{-1}(k, l) \eta_l.$$

In [154], the infinite potential  $\nu_V^W$  is defined thanks to a sequence of potentials of the form  $\tilde{\nu}_{V_n}^{P,\eta}$  on the exhausting sequence  $(V_n)_{n \in \mathbb{N}}$  which is shown to be compatible. More, precisely, the restrictions of  $\nu_V^W$  are given by the following lemma :

**Lemma D.** *Let  $n \in \mathbb{N}^*$ . Let  $(\beta_i)_{i \in V}$  be a random potential following  $\nu_V^W$ . Then  $(\beta_i)_{i \in V_n}$  is distributed as  $\tilde{\nu}_{V_n}^{\hat{P}^{(n)}, \hat{\eta}^{(n)}}$  where*

- For every  $i, j \in V_n$ ,  $\hat{P}^{(n)}(i, j) = W \mathbf{1}\{i \sim j\}$ .
- For every  $i \in V_n$ ,  $\hat{\eta}_i^{(n)} = \sum_{j \sim i, j \notin V_n} W$ .

### 3.3.2 Warm-up about the VRJP

Recall that  $(Z_t)_{t \geq 0} := (Y_{D^{-1}(t)})_{t \geq 0}$  is a time-changed version of the VRJP with constant weights  $W$  on the graph  $V$ . As explained before,  $(Z_t)_{t \geq 0}$  is easier to analyse than  $(Y_t)_{t \geq 0}$  because it is a mixture of Markov processes. In the particular case of finite graphs, Sabot and Tarrès gave an explicit description of the density of a random field associated with the environment.

**Proposition E** (Theorem 2 in [152]). *Let  $(V, E)$  be a finite graph. Let  $W > 0$ . Then, the time-changed VRJP  $(Z_t)_{t \geq 0}$  on  $V$  with constant weights  $W > 0$  starting from  $i_0 \in V$  is a mixture of Markov processes. Moreover, it jumps from  $i$  to  $j$  at rate  $W e^{U_j - U_i}$  where the field  $(U_i)_{i \in V}$  has the following density on the set  $\{(u_i)_{i \in V} \in \mathbb{R}^V, u_{i_0} = 0\}$  :*

$$\frac{1}{\sqrt{2\pi}^{|V|-1}} \exp \left( - \sum_{i \in V} u_i - W \sum_{\{i,j\} \in E} ((\cosh(u_i - u_j) - 1)) \right) \sqrt{D(W, u)} \prod_{i \in V \setminus \{i_0\}} du_i$$

with  $D(W, u) = \sum_{T \in \mathcal{T}} \prod_{\{i,j\} \in T} W e^{u_i + u_j}$  where  $\mathcal{T}$  is the set of spanning trees of  $(V, E)$ .

This density was originally studied in [54] in order to study random band matrices. Remark that the distribution of  $U$  does not have any obvious property of compatibility. Therefore, this was not possible to extend the field  $U$  on a general infinite graph. However, in [153], Sabot, Tarrès and Zeng introduced a smart change of variable which relates the field  $U$  and the  $\beta$ -potential. More precisely, if  $(V, E)$  is a finite graph, then the field  $U$  of Proposition E rooted at  $i_0$  is distributed as  $(G^{(V)}(i_0, i)/G^{(V)}(i_0, i_0))_{i \in V}$  where  $G^{(V)}$  is the inverse of  $H_\beta^{(V)}$  which is the operator associated with the potential  $\beta$  with distribution  $\tilde{\nu}_V^{P,0}$  where  $P(i, j) = W\mathbf{1}\{i \sim j\}$ . In order to have a representation of the environment of the VRJP on infinite graph, Sabot and Zeng extended the  $\beta$ -potential on infinite graphs thanks to the measure  $\nu_V^W$  and they proved the following result :

**Proposition F** (Theorem 1 in [154]). *If  $V$  is  $\mathbb{Z}^d$  with  $d \geq 1$  or an infinite tree, then the time-changed VRJP  $(Z_t)_{t \geq 0}$  on  $V$  with constant weights  $W > 0$  is a mixture of Markov processes. Moreover, the associated random environment can be described in the following way : if the VRJP started from  $i_0$ , it jumps from  $i$  to  $j$  at rate  $(1/2)WG(i_0, j)/G(i_0, i)$  where for every  $i, j \in V$ ,*

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma}\psi(i)\psi(j)$$

where  $\gamma$  is a random variable with law  $\Gamma(1/2, 1)$  which is independent from the the  $\beta$ -potential with distribution  $\nu_V^W$ .

In [77], Gerard proved that, in the case of trees, in the transient phase, there are infinitely many different representations of the environment of the VRJP. In this paper, we will often use a representation which is not the same as the one which is given in Proposition F. Now, let us describe this other representation.

### 3.3.3 Specificities of the tree

In the density given in Proposition E, if the graph is a tree, one can observe that the random variables  $U_i - U_{\tilde{i}}$  are i.i.d and distributed as the logarithm of an Inverse Gaussian random variable. It comes from the fact that the determinant term in the density becomes a product. Therefore, when the graph  $(V, E)$  is an infinite tree with a root  $o$ , this is natural to define an infinite version of the field  $U$  in the following way : for every  $i \in V$ ,

$$e^{U_i} := \prod_{o < u \leq i} A_u$$

where  $(A_i)_{i \in V \setminus \{o\}}$  is a family of independent Inverse Gaussian random variables with parameters  $(1, W)$ . This representation implies directly the following result :

**Proposition G** (Theorem 3 in [40]). *If  $V$  is a tree with root  $o$ , the discrete-time VRJP  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  which is associated with  $(Z_t)_{t \geq 0}$  is a random walk in random environment whose random conductances are given by*

$$c(i, \tilde{i}) = W e^{U_i + U_{\tilde{i}}} = W A_i \prod_{o < u \leq \tilde{i}} A_u^2$$

for every  $i \in V \setminus \{o\}$ .

This representation of the environment of the VRJP on trees is particularly useful because the conductances are almost products of i.i.d random variables along a branch of the tree. This situation is very close from branching random walks. This observation is crucial for the proofs in this paper. In particular, thanks to this representation and its link with branching random walks, this is much easier to compute the critical point on Galton-Watson trees.

**Proposition H** (Theorem 1 in [40] or Theorem 1 in [15]). *Let  $V$  be a Galton-Watson tree with offspring law  $\mu$  satisfying hypothesis  $A_1$ . Then the VRJP on  $V$  with constant weights  $W$  is recurrent if and only if*

$$mQ(W, 1/2) \leq 1$$

where  $m$  is the mean of  $\mu$ . In particular, the critical point  $W_c(\mu)$  is the only solution of the equation

$$mQ(W, 1/2) = 1.$$

Now, remind that our goal is to study the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$ . This martingale is defined through the potential  $\beta$ . If  $V$  is an infinite tree with a special vertex  $o$  called the root, we can couple the field  $U$  and the potential  $\beta$  in the following way : for every  $i \in V$ , we define

$$\tilde{\beta}_i := \frac{W}{2} \sum_{i \sim j} e^{U_j - U_i} = \frac{W}{2} \left( \sum_{\tilde{u}=i} A_u + \mathbf{1}\{i \neq o\} \frac{1}{A_i} \right). \quad (3.6)$$

For every  $i \in V$ ,  $\tilde{\beta}_i$  can be interpreted as the total jump rate of the VRJP at  $i$ . The potential  $\tilde{\beta}$  is very important for our purposes. One reason for that is Lemma 3.15 which makes a link between the effective resistance associated with the VRJP and some quantity defined through  $(\tilde{\beta}_i)_{i \in V}$ . Now, let  $\gamma$  be a Gamma distribution with parameter  $(1/2, 1)$  which is independent of  $(A_i)_{i \in V \setminus \{o\}}$ . Then, let us define

$$\beta = \tilde{\beta} + \mathbf{1}\{\cdot = o\}\gamma. \quad (3.7)$$

**Lemma 3.9.** *Let us assume that  $V$  is a tree. Let  $W > 0$ . Then, the potential  $(\beta_i)_{i \in V}$  defined by (3.7) has law  $\nu_V^W$ .*

*Proof of Lemma 3.9.*

This is a direct consequence of Theorem 3 in [40] and Corollary 2 in [153].  $\square$

From now on, when we work on a tree  $V$ , we always assume that, under  $\nu_V^W$ , the potential  $(\beta_i)_{i \in V}$  is defined by (3.6) and (3.7). This coupling between the field  $U$  and the potential  $(\beta_i)_{i \in V}$  is very important in order to relate our questions regarding the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  to tractable questions about branching random walks. This allows us to apply techniques coming from the area of branching random walks in order to study  $(\psi_n(o))_{n \in \mathbb{N}}$ .

### 3.3.4 $\beta$ -potential and path expansions

In this subsection, we explain how  $\hat{G}$  can be interpreted as a sum over a set of paths. This representation of  $\hat{G}$  will be very useful in the sequel of this paper. A path from  $i$  to  $j$  in the graph  $(V, E)$  is a finite sequence  $\sigma = (\sigma_0, \dots, \sigma_m)$  in  $V$  such that  $\sigma_0 = i$  and  $\sigma_m = j$  and  $\sigma_k \sim \sigma_{k+1}$  for every  $k \in \{0, \dots, m-1\}$ . Let us denote by  $P_{i,j}^V$  the set of paths from  $i$  to  $j$  in  $V$ . Let us also introduce  $\bar{P}_{i,j}^V$  the set of paths from  $i$  to  $j$  which never hit  $j$  before the end of the path. More precisely, it is the set of paths  $\sigma = (\sigma_0, \dots, \sigma_m)$  such that  $\sigma_0 = i$ ,  $\sigma_m = j$  and  $\sigma_k \neq j$  for every  $k \in \{0, \dots, m-1\}$ . For any path  $\sigma = (\sigma_0, \dots, \sigma_m)$ , we denote its length by  $|\sigma| = m$ . For any path  $\sigma$  in  $V$  and for any  $\beta \in \mathcal{D}_V^W$ , let us write,

$$(2\beta)_\sigma = \prod_{k=0}^{|\sigma|-1} (2\beta_{\sigma_k}), \quad (2\beta)_\sigma^- = \prod_{k=0}^{|\sigma|-1} (2\beta_{\sigma_k}).$$

Then, the following lemma stems directly from Proposition 6 in [154] :

**Lemma I** (Proposition 6 in [154]). *Let  $(V, E)$  be any locally finite graph. Let  $W > 0$ . Let  $i, j \in V$ . For any  $\beta \in \mathcal{D}_V^W$ ,*

$$\hat{G}(i, j) = \sum_{\sigma \in P_{i,j}^V} \frac{W^{|\sigma|}}{(2\beta)_\sigma}, \quad \frac{\hat{G}(i, j)}{\hat{G}(i, i)} = \sum_{\sigma \in \bar{P}_{j,i}^V} \frac{W^{|\sigma|}}{(2\beta)_\sigma}.$$

In the special case of trees, we can mix this property with the construction given in subsection 3.3.3 in order to obtain the following lemma.

**Lemma 3.10.** *Let  $V$  be a Galton-Watson tree with a root  $o$  and an offspring law  $\mu$  satisfying hypothesis  $A_1$ . Let us assume that  $W \leq W_c(\mu)$ . Then,  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,*

$$\frac{\hat{G}(o, i)}{\hat{G}(o, o)} = e^{U_i}.$$

*Proof of Lemma 3.10.*

Let us assume that the  $\beta$ -potential is constructed as in subsection 3.3.3. Let us consider the Markov chain  $(\tilde{Z}_k)_{k \in \mathbb{N}^*}$  on  $V$  with conductances given by

$$c(i, \tilde{i}) = W A_i^{-1} \prod_{o < u \leq i} A_u^2 = W e^{U_i + U_{\tilde{i}}}$$

for every  $i \in V$ . Actually, by Proposition G,  $\tilde{Z}$  is the discrete-time process associated with the VRJP. Let us remark that for every  $i \in V$ ,

$$\pi_i := \sum_{j \sim i} c(i, j) = e^{2U_i} 2\tilde{\beta}_i.$$

We denote by  $P_{c,i}$  the probability measure associated with this Markov chain  $\tilde{Z}$  starting from  $i$  with random conductances  $c$ . Let us introduce the stopping time

$$\tau_o = \inf \{n \in \mathbb{N}, \tilde{Z}_n = o\}.$$

If  $\sigma$  is a path, we write  $\{\tilde{Z} \sim \sigma\}$  to mean that  $\tilde{Z}_0 = \sigma_0, \tilde{Z}_1 = \sigma_1$ , etc. Then, it holds that  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,

$$\begin{aligned} P_{c,i}(\tau_o < +\infty) &= \sum_{\sigma \in \bar{P}_{i,o}^V} P_{c,i}(\tilde{Z} \sim \sigma) \\ &= \sum_{\sigma \in \bar{P}_{i,o}^V} \prod_{k=0}^{|\sigma|-1} \frac{W e^{U_{\sigma_k} + U_{\sigma_{k+1}}}}{\pi_{\sigma_k}} \\ &= \sum_{\sigma \in \bar{P}_{i,o}^V} \prod_{k=0}^{|\sigma|-1} \frac{W e^{U_{\sigma_{k+1}} - U_{\sigma_k}}}{2\tilde{\beta}_{\sigma_k}}. \end{aligned} \quad (3.8)$$

There is a telescoping product in (3.8). Consequently, we deduce that  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,

$$P_{c,i}(\tau_o < +\infty) = e^{-U_i} \sum_{\sigma \in \bar{P}_{i,o}^V} \prod_{k=0}^{|\sigma|-1} \frac{W}{2\tilde{\beta}_{\sigma_k}}. \quad (3.9)$$

In identity (3.9), remark that  $\sigma_k$  is always different from  $o$ . Therefore,  $\tilde{\beta}$  can be replaced by  $\beta$  and we obtain that  $\mathbb{P}_{\mu, W}$ -a.s, for every  $i \in V$ ,

$$P_{c,i}(\tau_o < +\infty) = e^{-U_i} \sum_{\sigma \in \bar{P}_{i,o}^V} \prod_{k=0}^{|\sigma|-1} \frac{W}{2\beta_{\sigma_k}}. \quad (3.10)$$

In (3.10), one can observe the same quantity as in Lemma I. Therefore,  $P_{\mu,W}$ -a.s, for every  $i \in V$ ,

$$P_{c,i}(\tau_o < +\infty) = e^{-U_i} \frac{\hat{G}(o, i)}{\hat{G}(o, o)}. \quad (3.11)$$

However, we assumed  $W \leq W_c(\mu)$ . Thus, by Propositions G and B, we know that  $P_{c,i}(\tau_o < +\infty) = 1$ ,  $\mathbb{P}_{\mu,W}$ -a.s. Together with (3.11), this concludes the proof.  $\square$

### 3.3.5 Warm-up about branching random walks

In this subsection, we recall the most important facts about one-dimensionnal branching random walks. Indeed, it is a very important tool in this article. One can refer to [158] for more information on this topic. We consider a point process  $\mathcal{L} := \{\varrho_i, 1 \leq i \leq N\}$  such that  $N$  takes values in  $\mathbb{N}$  and each point  $\varrho_i$  is in  $\mathbb{R}$ . At time 0, there is a unique ancestor called the root  $o$ . We define  $S(o) = 0$ . At time  $n$ , each individual  $u$  generates independently a point process  $\mathcal{L}_u := \{\varrho_i^u, 1 \leq i \leq N_u\}$  with the same law as  $\mathcal{L}$ . Each point in  $\mathcal{L}_u$  stands for a child of  $u$ . The positions of the children of  $u$  are given by the point process  $\{\varrho_i^u + S(u), 1 \leq i \leq N_u\}$ . The children of individuals of the  $n$ -th generation form the  $n + 1$ -th generation. In this way, we get an underlying genealogical Galton-Watson tree  $V$  with  $o$  as a root. For every  $u \in V$ , we denote the position of  $u$  by  $S(u)$ . The set  $\{(u, S(u)), u \in V\}$  is called a branching random walk. Recall that  $|u|$  stands for the generation of  $u \in V$ .

Throughout this subsection, we assume there exists  $\delta > 0$  such that

$$\mathbb{E} \left[ \left( \sum_{|u|=1} 1 \right)^{1+\delta} \right] < +\infty. \quad (3.12)$$

Moreover, we assume that for every  $t \in \mathbb{R}$ ,

$$\mathbb{E} \left[ \sum_{|u|=1} e^{tS(u)} \right] < +\infty. \quad (3.13)$$

Let us introduce the Laplace transform of  $\mathcal{L}$  which is defined as

$$f : \mathbb{R} \rightarrow \mathbb{R} \\ t \mapsto \ln \left( \mathbb{E} \left[ \sum_{|u|=1} e^{-tS(u)} \right] \right).$$

Let us also assume that

$$f(0) > 0, \quad f(1) = f'(1) = 0. \quad (3.14)$$

For every  $n \in \mathbb{N}$  and for every  $\beta > 1$ , let us define,

$$\mathcal{W}_n := \sum_{|u|=n} e^{-S(u)}, \quad \mathcal{W}_{n,\beta} = \sum_{|u|=n} e^{-\beta S(u)}.$$

In [86], Hu and Shi proved the following results :

**Proposition J** (Theorem 1.4 of [86]). *Assume hypotheses (3.12), (3.13) and (3.14) and let  $\beta > 1$ . Conditionally on the system's survival, we have*

$$\limsup_{n \rightarrow +\infty} \frac{\ln(\mathcal{W}_{n,\beta})}{\ln(n)} = -\frac{\beta}{2} \quad a.s., \quad (3.15)$$

$$\liminf_{n \rightarrow +\infty} \frac{\ln(\mathcal{W}_{n,\beta})}{\ln(n)} = -\frac{3\beta}{2} \quad a.s. \quad (3.16)$$

**Proposition K** (Theorem 1.6 in [86]). *Assume hypotheses (3.12), (3.13) and (3.14) and let  $\beta > 1$ . For any  $r \in ]0, 1/\beta[$ ,*

$$\mathbb{E} [\mathcal{W}_{n,\beta}^r] = n^{-3r\beta/2+o(1)}.$$

In many situations, hypothesis (3.14) is not satisfied. However, in most cases, we can transform the branching random walk in order to be reduced to hypothesis (3.14). Indeed, if there exists  $t^* > 0$  such that  $t^* f'(t^*) = f(t^*)$ , then  $(\tilde{S}(u))_{u \in V} := (t^* S(u) + f(t^*)|u|)_{u \in V}$  is a branching random walk satisfying (3.14). However, one still has to check that such a  $t^* > 0$  does exist.

**Proposition L** (Proposition 7.2, Chapter 3 in [89]). *Let us assume that for every  $M \in \mathbb{R}$ ,*

$$\mathbb{P}(\mathcal{L}(] - \infty, -M]) \neq \emptyset) > 0.$$

*Then, there exists  $t^* > 0$  such that  $t^* f'(t^*) = f(t^*)$ .*

**Remark 3.4.** *Be careful when you look at reference [89]. The result is wrongly stated but the proof (of the corrected statement) is correct.*

Moreover, this is possible to know the sign of  $f(t^*)$  and whether  $t^*$  is unique or not.

**Proposition 3.11.** *Let us assume that  $f(0) > 0$  and that there exists  $t^* > 0$  such that  $t^* f'(t^*) = f(t^*)$ . We assume also that  $f$  is strictly convex and that there exists a point  $t_{min}$  such that  $f$  is strictly decreasing on  $[0, t_{min}]$  and strictly increasing on  $[t_{min}, +\infty[$ . Then  $t^*$  is the unique solution in  $\mathbb{R}_+^*$  of the equation  $tf'(t) = f(t)$  and*

$$\text{sgn}(f(t^*)) = \text{sgn}(f(t_{min})).$$

*Moreover,  $t^* < t_{min}$  if  $f(t_{min}) < 0$  and  $t^* > t_{min}$  if  $f(t_{min}) > 0$ .*

*Proof of Proposition 3.11.*

Let us introduce the function  $\Phi : t \mapsto tf'(t) - f(t)$ . As  $f$  is strictly convex, for every  $t \in \mathbb{R}_+^*$ ,  $\Phi'(t) = tf''(t) > 0$ . Therefore,  $\Phi$  is strictly increasing on  $\mathbb{R}_+$ . Thus,  $t^*$  must be unique. Moreover,  $\Phi(t_{min}) = tf'(t_{min}) - f(t_{min}) = -f(t_{min})$ . Thus, if  $f(t_{min}) < 0$ , then  $\Phi(t_{min}) > 0$ . Furthermore,  $\Phi(0) = -f(0) < 0$ . Therefore, as  $t^*$  is the unique zero of  $\Phi$ ,  $t^*$  must be in  $]0, t_{min}[$ . In particular,  $f(t^*) = t^* f'(t^*) < 0$  because  $f$  is strictly decreasing on  $[0, t_{min}]$ . The case where  $f(t_{min}) > 0$  can be treated in the same way.  $\square$

## 3.4 Preliminary lemmas

### 3.4.1 $\psi_n(o)$ as a mixture of Inverse Gaussian distributions and proof of Theorem 3.1

In this subsection,  $V$  is a deterministic countable graph with constant weights  $W > 0$ . For every  $n \in \mathbb{N}$ , we introduce the sigma-field  $\mathcal{G}_n := \sigma((\beta_i)_{i \in V_n \setminus \{o\}})$ . (Recall that  $V_n = \{x \in V, d(o, x) \leq n\}$ .) Moreover, for every  $n \in \mathbb{N}$ , let us introduce

$$D_n := \frac{1}{2} \sum_{o \sim j} W \frac{\hat{G}_n(o, j)}{\hat{G}_n(o, o)}.$$

Then, it is remarkable that  $\psi_n(o)$  has an Inverse Gaussian distribution conditionally on  $\mathcal{G}_n$ .

**Lemma 3.12.** *For every  $n \in \mathbb{N}$ , under  $\nu_V^W$ ,*

(i)

$$\mathcal{L}(\beta_o | \mathcal{G}_n) = D_n + \frac{1}{2 \times IG\left(\frac{\hat{G}_n(o, o)}{\psi_n(o)}, 1\right)}$$

(ii)

$$\mathcal{L}(\psi_n(o)|\mathcal{G}_n) = IG\left(1, \frac{\psi_n(o)}{\hat{G}_n(o, o)}\right)$$

where we recall that  $IG(a, \lambda)$  stands for an Inverse Gaussian distribution with parameters  $a$  and  $\lambda$ .

The computation achieved in the following proof is basically the same as Proposition 3.4 in [43] but we use it in a different way.

*Proof of Lemma 3.12.*

By Lemma D,  $(\beta_i)_{i \in V_n}$  has law  $\tilde{\nu}_{V_n}^{\hat{P}^{(n)}, \hat{\eta}^{(n)}}$  where

$$\hat{\eta}_i^{(n)} = \sum_{j \in V_n^c, i \sim j} W$$

for every  $i \in V_n$  and

$$\hat{P}^{(n)}(i, j) = W \mathbf{1}\{i \sim j\}$$

for every  $i, j \in V_n$ . Further, by Lemma C, the law of  $\beta_o$  conditionally on  $\mathcal{G}_n$  is  $\tilde{\nu}_{\{o\}}^{W_{o,o}, \check{\eta}}$  with :

$$- W_{o,o} = \sum_{o \sim j} \sum_{o \sim k} W^2 \hat{G}_{V_n \setminus \{o\}}(j, k) \text{ where } \hat{G}_{V_n \setminus \{o\}} \text{ is the inverse of } (H_\beta)_{V_n \setminus \{o\}, V_n \setminus \{o\}}.$$

$$- \check{\eta} = \sum_{o \sim j} \sum_{k \in V_n \setminus \{o\}} W \hat{G}_{V_n \setminus \{o\}}(j, k) \hat{\eta}_k^{(n)}.$$

Nevertheless, reasoning on path-expansions (see Lemma I), one remarks that for every  $k \in V_n \setminus \{o\}$ ,

$$\sum_{o \sim j} W \hat{G}_{V_n \setminus \{o\}}(j, k) = \frac{\hat{G}_n(o, k)}{\hat{G}_n(o, o)}. \quad (3.17)$$

Consequently, by definition of  $D_n$  and  $\psi_n(o)$ , it holds that

$$- W_{o,o} = \sum_{o \sim k} W \frac{\hat{G}_n(o, k)}{\hat{G}_n(o, o)} = 2D_n.$$

$$- \check{\eta} = \sum_{k \in V_n \setminus \{o\}} \frac{\hat{G}_n(o, k)}{\hat{G}_n(o, o)} \hat{\eta}_k^{(n)} = \frac{1}{\hat{G}_n(o, o)} \times \sum_{k \in \partial V_n} \hat{G}_n(o, k) \hat{\eta}_k^{(n)} = \frac{\psi_n(o)}{\hat{G}_n(o, o)}.$$

Moreover  $D_n$  and  $\frac{\psi_n(o)}{\hat{G}_n(o, o)}$  are  $\mathcal{G}_n$  measurable. Indeed

$$D_n = \frac{1}{2} \sum_{o \sim k} W \frac{\hat{G}_n(o, k)}{\hat{G}_n(o, o)} \text{ and } \frac{\psi_n(o)}{\hat{G}_n(o, o)} = \sum_{k \in \partial V_n} \frac{\hat{G}_n(o, k) \hat{\eta}_k^{(n)}}{\hat{G}_n(o, o)}.$$

Further, for every  $k \in V_n$ ,  $\frac{\hat{G}_n(o, k)}{\hat{G}_n(o, o)}$  does not depend on  $\beta_o$  by (3.17) and, thus, it is  $\mathcal{G}_n$  measurable. Therefore, by (3.5), conditionally on  $\mathcal{G}_n$ , the law of  $\beta_o$  is given by the density

$$\mathbf{1}\{\beta > D_n\} \frac{1}{\sqrt{\pi(\beta - D_n)}} e^{-(\beta - D_n)} e^{-\frac{1}{4(\beta - D_n)} \frac{\psi_n(o)^2}{\hat{G}_n(o, o)^2}} e^{\frac{\psi_n(o)}{\hat{G}_n(o, o)}}.$$

We can recognise the reciprocal of an Inverse Gaussian distribution. More precisely,

$$\mathcal{L}(\beta_o | \mathcal{G}_n) = D_n + \frac{1}{2 \times IG\left(\frac{\hat{G}_n(o, o)}{\psi_n(o)}, 1\right)}.$$

Besides, as  $\hat{G}_n$  is the inverse of  $(H_\beta)_{V_n, V_n}$ ,  $\beta_o - D_n = \frac{1}{2\hat{G}_n(o, o)}$ . Consequently, as  $D_n$  is  $\mathcal{G}_n$  measurable, this yields

$$\mathcal{L}\left(\hat{G}_n(o, o) | \mathcal{G}_n\right) = IG\left(\frac{\hat{G}_n(o, o)}{\psi_n(o)}, 1\right).$$

Moreover for every positive numbers  $(t, a, b)$ , one can check that  $tIG(a, b) \stackrel{law}{=} IG(ta, tb)$ . Furthermore  $\frac{\hat{G}_n(o, o)}{\psi_n(o)}$  is  $\mathcal{G}_n$  measurable. Thus, it holds that

$$\mathcal{L}(\psi_n(o)|\mathcal{G}_n) = IG\left(1, \frac{\psi_n(o)}{\hat{G}_n(o, o)}\right).$$

□

Moreover, we can pass to the limit in Lemma 3.12. Let us define  $\mathcal{G}_\infty := \sigma\left((\beta_i)_{i \in \mathbb{Z}^d \setminus \{o\}}\right)$ . Let us recall that  $(\hat{G}_n(i, j))_{n \in \mathbb{N}}$  converges toward some finite limit  $\hat{G}(i, j)$  for every  $(i, j) \in V^2$ . Then, we introduce  $D = \frac{1}{2} \sum_{o \sim j} W \frac{\hat{G}(o, j)}{\hat{G}(o, o)}$ .

**Lemma 3.13.** *We assume that  $\psi(o) > 0$ ,  $\nu_V^W$ -a.s. Then, under  $\nu_V^W$ ,*

(i)

$$\mathcal{L}(\beta_o|\mathcal{G}_\infty) = D + \frac{1}{2 \times IG\left(\frac{\hat{G}(o, o)}{\psi(o)}, 1\right)}.$$

(ii)

$$\mathcal{L}(\psi(o)|\mathcal{G}_\infty) = IG\left(1, \frac{\psi(o)}{\hat{G}(o, o)}\right).$$

*Proof of Lemma 3.13.*

Let  $\Lambda$  be a finite subset of  $V$  including  $o$ . Let us define  $\tilde{\Lambda} = \Lambda \setminus \{o\}$ . Let  $A$  be a borelian set of  $\mathbb{R}^{\tilde{\Lambda}}$ . Let  $F$  be a bounded continuous function of  $\mathbb{R}^d$ . Then, by Lemma 3.12, for every  $n$  large enough,

$$\begin{aligned} & \mathbb{E}_{\nu_V^W} [F(\beta_o) \mathbf{1}\{(\beta_i)_{i \in \tilde{\Lambda}} \in A\}] \\ &= \mathbb{E}_{\nu_V^W} \left[ \int_0^{+\infty} F(\beta + D_n) \frac{1}{\sqrt{\pi\beta}} e^{-\frac{1}{4\beta} \left(\frac{\psi_n(o)}{\hat{G}_n(o, o)} - 2\beta\right)^2} d\beta \mathbf{1}\{(\beta_i)_{i \in \tilde{\Lambda}} \in A\} \right]. \end{aligned} \quad (3.18)$$

Moreover, the function

$$(x, y) \mapsto \int_0^{+\infty} F(\beta + x) \frac{1}{\sqrt{\pi\beta}} e^{-\frac{1}{4\beta} (y - 2\beta)^2} d\beta$$

is clearly continuous and uniformly bounded on  $(\mathbb{R}_+^*)^2$ . Therefore, as

$$\left(D_n, \frac{\psi_n(o)}{\hat{G}_n(o, o)}\right) \xrightarrow[n \rightarrow +\infty]{a.s.} \left(D, \frac{\psi(o)}{\hat{G}(o, o)}\right),$$

by means of the dominated convergence theorem, we can take the limit in (3.18) which implies the first point of our lemma. Then, the second point of Lemma 3.13 stems from the first point, exactly in the same way as in the proof of Lemma 3.12.

□

Now we are able to prove Theorem 3.1.

*Proof of Theorem 3.1.*

By Lemma 3.13, we know that

$$\mathcal{L}(\psi(o)|\mathcal{G}_\infty) = IG\left(1, \frac{\psi(o)}{\hat{G}(o, o)}\right).$$

In particular,

$$\mathbb{E}_{\nu_V^W} [\psi(o)] = \mathbb{E}_{\nu_V^W} \left[ IG \left( 1, \frac{\psi(o)}{\hat{G}(o, o)} \right) \right] = 1 \quad (3.19)$$

Thus for every  $n \in \mathbb{N}^*$ ,  $\mathbb{E}_{\nu_V^W} [\psi_n(o)] = \mathbb{E}_{\nu_V^W} [\psi(o)] = 1$ . Moreover,  $\psi_n(o) \xrightarrow[n \rightarrow +\infty]{a.s.} \psi(o)$ . Thus, by Scheffé's lemma,

$$\psi_n(o) \xrightarrow[n \rightarrow +\infty]{L_1} \psi(o).$$

Therefore  $(\psi_n(o))_{n \in \mathbb{N}}$  is uniformly integrable.  $\square$

Besides, Lemma 3.12 implies the following useful result :

**Lemma 3.14.** *Let  $p \in \mathbb{R}$ . For every  $n \in \mathbb{N}$ ,*

$$\mathbb{E}_{\nu_V^W} [\psi_n(o)^p] = \mathbb{E}_{\nu_V^W} [\psi_n(o)^{1-p}].$$

*Proof of Lemma 3.14.*

Let us define  $Y_n = \frac{\psi_n(o)}{\hat{G}_n(o, o)}$ . Then, by Lemma 3.12,

$$\begin{aligned} \mathbb{E}_{\nu_V^W} [\psi_n(o)^p] &= \mathbb{E}_{\nu_V^W} \left[ \int Y_n^{1/2} (2\pi)^{-1/2} x^{p-3/2} \exp \left( -Y_n(x-1)^2/(2x) \right) dx \right] \\ &= \mathbb{E}_{\nu_V^W} \left[ \int Y_n^{1/2} (2\pi)^{-1/2} x^{-p+3/2} x^{-2} \exp \left( -Y_n x(1/x-1)^2/2 \right) dx \right] \\ &= \mathbb{E}_{\nu_V^W} \left[ \int Y_n^{1/2} (2\pi)^{-1/2} x^{(-p+1)-3/2} \exp \left( -Y_n(x-1)^2/(2x) \right) dx \right] \\ &= \mathbb{E}_{\nu_V^W} [\psi_n(o)^{1-p}]. \end{aligned}$$

$\square$

### 3.4.2 Resistance formula on a tree

In this subsection we assume that  $V$  is a tree. Let  $n \in \mathbb{N}$ . Let us define the matrix  $\tilde{H}_n$  on  $V_n \times V_n$  such that for every  $(i, j) \in V_n \times V_n$ ,  $\tilde{H}_n(i, j) = 2\tilde{\beta}_i \mathbf{1}\{i = j\} - W \mathbf{1}\{i \sim j\}$ . We assume that the potentials  $\tilde{\beta}$  and  $\beta$  are constructed as in (3.6) and (3.7). We also introduce  $D_U^{(n)}$  which is the diagonal matrix on  $V_n \times V_n$  with diagonal entries  $D_U^{(n)}(i, i) = e^{U_i}$  for every  $i \in V_n$ . We can observe that  $D_U^{(n)} \tilde{H}_n D_U^{(n)} = M_n$  where for every  $(i, j) \in V_n \times V_n$ ,

$$M_n(i, j) = \sum_{k \sim i} W e^{U_i + U_k} \mathbf{1}\{i = j\} - W e^{U_i + U_j} \mathbf{1}\{i \sim j\}.$$

$M_n$  is almost a conductance matrix with conductances  $W e^{U_i + U_j}$  between two neighbouring vertices  $i$  and  $j$ . However, if  $i \in \partial V_n$ ,

$$M_n(i, i) = \sum_{k \sim i} W e^{U_i + U_k} > \sum_{k \sim i, k \in V_n} W e^{U_i + U_k}.$$

Therefore,  $M_n$  is strictly larger than a conductance matrix (for the order between symmetric matrices). Moreover conductance matrices are non-negative. Thus,  $M_n$  and  $\tilde{H}_n$  are symmetric positive definite matrices. Then, we are allowed to define the inverse  $\tilde{G}_n$  of  $\tilde{H}_n$ . Moreover, for every  $n \in \mathbb{N}$ , we construct a wired version  $(\tilde{V}_n, \tilde{E}_n)$  of  $(V_n, E_n)$  in the following way :

$$\begin{cases} \tilde{V}_n &= V_n \cup \{\delta_n\} \\ \tilde{E}_n &= E_n \cup \{(\delta_n, i), i \in \partial V_n\} \end{cases}$$

where  $\delta_n$  is a new vertex. For every  $(i, j) \in E$ , recall from the notation of Proposition G that  $c(i, j) = We^{U_i+U_j}$ . The conductances  $c$  are the environment of the VRJP. Now, let us introduce a family of conductances  $c_n$  on  $\tilde{E}_n$ .

$$\begin{cases} \forall (i, j) \in E_n, & c_n(i, j) = c(i, j) \\ \forall i \in \partial V_n, & c_n(\delta_n, i) = \sum_{j \sim i, j \in V_n^c} c(i, j) \end{cases}$$

We denote by  $\mathcal{R}(o \longleftrightarrow \delta_n)$  the effective resistance between  $o$  and  $\delta_n$  in  $(\tilde{V}_n, \tilde{E}_n, c_n)$ . Then, we have the following key identity :

**Lemma 3.15.** *If  $V$  is a tree, then, for every  $n \in \mathbb{N}^*$ ,  $\tilde{G}_n(o, o) = \mathcal{R}(o \longleftrightarrow \delta_n)$ .*

*Proof of Lemma 3.15.*

For every  $i \in V_n$ , one defines  $h(i) = \frac{\tilde{G}_n(o, i)e^{-U_i}}{\tilde{G}_n(o, o)}$  and  $h(\delta_n) = 0$ . We are going to prove that  $h$  is harmonic everywhere excepted at  $o$  and  $\delta_n$  where  $h(o) = 1$  and  $h(\delta_n) = 0$ . Let  $i \in V_n \setminus \{o\}$ . Then, it holds that,

$$\begin{aligned} \sum_{i \sim j} c_n(i, j)h(j) &= \sum_{i \sim j, j \in V_n} We^{U_i+U_j} \times \frac{\tilde{G}_n(o, j)e^{-U_j}}{\tilde{G}_n(o, o)} \\ &= \frac{e^{U_i}}{\tilde{G}_n(o, o)} \sum_{i \sim j, j \in V_n} W\tilde{G}_n(o, j). \end{aligned} \quad (3.20)$$

By definition  $\tilde{G}_n = \tilde{H}_n^{-1}$ . Together with (3.20), this yields

$$\sum_{i \sim j} c_n(i, j)h(j) = \frac{e^{U_i}}{\tilde{G}_n(o, o)} \times 2\tilde{\beta}_i\tilde{G}_n(o, i). \quad (3.21)$$

Then, by definition of  $U_i$  and  $\tilde{\beta}_i$ , we infer that

$$\begin{aligned} \sum_{i \sim j} c_n(i, j)h(j) &= \frac{\tilde{G}_n(o, i)}{\tilde{G}_n(o, o)} \times \sum_{i \sim j} We^{U_j} \\ &= \frac{\tilde{G}_n(o, i)e^{-U_i}}{\tilde{G}_n(o, o)} \times \left( c_n(i, \delta_n) + \sum_{i \sim j, j \in V_n} c_n(i, j) \right) \\ &= h(i) \times \sum_{i \sim j} c_n(i, j). \end{aligned}$$

Consequently,  $h$  is harmonic. Therefore, by identity (2.3) in [116],

$$\mathcal{R}(o \longleftrightarrow \delta_n) = \frac{1}{\sum_{o \sim j} c_n(o, j)(1 - h(j))}. \quad (3.22)$$

Besides, it holds that,

$$\begin{aligned} \sum_{o \sim j} c_n(o, j)(1 - h(j)) &= \sum_{o \sim j} We^{U_j} \times \left( 1 - \frac{\tilde{G}_n(o, j)e^{-U_j}}{\tilde{G}_n(o, o)} \right) \\ &= \tilde{G}_n(o, o)^{-1} \sum_{o \sim j} W \left( e^{U_j}\tilde{G}_n(o, o) - \tilde{G}_n(o, j) \right) \end{aligned} \quad (3.23)$$

However  $\tilde{G}_n$  is the inverse of  $\tilde{H}_n$ . Therefore,  $\sum_{o \sim j} W \tilde{G}_n(o, j) = -1 + 2\tilde{\beta}_0 \tilde{G}_n(o, o)$ . Moreover,  $\sum_{o \sim j} W e^{U_j} = 2\tilde{\beta}_0$ . Together with (3.23), this yields

$$\begin{aligned} \sum_{o \sim j} c_n(o, j)(1 - h(j)) &= \tilde{G}_n(o, o)^{-1} \left( 2\tilde{\beta}_0 \tilde{G}_n(o, o) - \left( -1 + 2\tilde{\beta}_0 \tilde{G}_n(0, 0) \right) \right) \\ &= \tilde{G}_n(o, o)^{-1}. \end{aligned} \quad (3.24)$$

Combining (3.22) and (3.24) concludes the proof.  $\square$

By means of Lemma 3.15, one can prove the following lemma which shall be useful later in this paper.

**Lemma 3.16.** *Let  $V$  be a Galton-Watson tree whose offspring law satisfies hypothesis  $A_1$ .*

$$(i) \quad \forall W \in ]0, W_c(\mu)], \quad \lim_{n \rightarrow +\infty} \tilde{G}_n(o, o) = +\infty, \quad \mathbb{P}_{\mu, W} - a.s.$$

$$(ii) \quad \forall W \in ]W_c(\mu), +\infty[, \quad \lim_{n \rightarrow +\infty} \tilde{G}_n(o, o) := \tilde{G}(o, o) < +\infty, \quad \mathbb{P}_{\mu, W} - a.s.$$

*Proof of Lemma 3.16.*

By Propositions G and H,  $W \leq W_c(\mu)$  if and only if the random walk with conductances  $(c_{i,j})_{(i,j) \in E}$  is recurrent almost surely. By Theorem 2.3 in [116], this is equivalent to say that

$$\lim_{n \rightarrow +\infty} \mathcal{R}(o \longleftrightarrow \delta_n) = +\infty.$$

Therefore, Lemma 3.15 concludes the proof.  $\square$

### 3.4.3 Burkholder-Davis-Gundy inequality

As  $(\psi_n(o))_{n \in \mathbb{N}}$  is a martingale, there is a relation between its moments and the moments of its bracket  $(\hat{G}_n(o, o))_{n \in \mathbb{N}}$  under mild assumptions. This relation is known as the BDG inequality. This inequality is not always true for discrete martingales. (See [34].) However, this is always true for continuous martingales. Fortunately, by [155], for every  $n \in \mathbb{N}$ ,  $\psi_n(o)$  can be obtained as the limit of some continuous martingale. That is why we can prove the following lemma :

**Lemma 3.17.** *Let  $V$  be a locally finite graph. Let  $W > 0$ . Let  $p > 1$ . Then, there exist positive constants  $C_{1,p}$  and  $C_{2,p}$  which do not depend on  $V$  and  $W$  such that for every  $n \in \mathbb{N}$ ,*

$$C_{1,p} \mathbb{E}_{\nu_V^W} \left[ \hat{G}_n(o, o)^{p/2} \right] \leq \mathbb{E}_{\nu_V^W} [ |\psi_n(o) - 1|^p ] \leq C_{2,p} \mathbb{E}_{\nu_V^W} \left[ \hat{G}_n(o, o)^{p/2} \right].$$

*Proof of Lemma 3.17.*

By [155], for every  $n \in \mathbb{N}$ , there exists a continuous non-negative martingale  $(\psi_n(o, t))_{t \geq 0}$  such that,

$$\psi_n(o, t) \xrightarrow[t \rightarrow +\infty]{a.s.} \psi_n(o) \quad \text{and} \quad \langle \psi_n(o, t), \psi_n(o, t) \rangle \xrightarrow[t \rightarrow +\infty]{a.s.} \hat{G}_n(o, o) \quad (3.25)$$

where  $\langle \dots, \dots \rangle$  is the bracket for semimartingales. For  $t \geq 0$ , let us introduce  $\psi_n^*(o, t) = \sup_{s \leq t} |\psi_n(o, s) - 1|$ . Then, if  $p > 1$ , by BDG inequality for continuous martingales (see Theorem 4.1 in [148]), there exist positive constants  $\kappa_{1,p}$  and  $\kappa_{2,p}$  such that for every  $n \in \mathbb{N}$ , for every  $t \geq 0$ ,

$$\kappa_{1,p} \mathbb{E}_{\nu_V^W} \left[ \langle \psi_n(o, t), \psi_n(o, t) \rangle^{p/2} \right] \leq \mathbb{E}_{\nu_V^W} [ \psi_n^*(o, t)^p ] \leq \kappa_{2,p} \mathbb{E}_{\nu_V^W} \left[ \langle \psi_n(o, t), \psi_n(o, t) \rangle^{p/2} \right]. \quad (3.26)$$

As  $p > 1$ , by Doob's martingale inequality, there exist  $C_{1,p} > 0$  and  $C_{2,p} > 0$  such that for every  $n \in \mathbb{N}$ , for every  $t \geq 0$ ,

$$C_{1,p} \mathbb{E}_{\nu_V^W} \left[ \langle \psi_n(o, t), \psi_n(o, t) \rangle^{p/2} \right] \leq \mathbb{E}_{\nu_V^W} [|\psi_n(o, t) - 1|^p] \leq C_{2,p} \mathbb{E}_{\nu_V^W} \left[ \langle \psi_n(o, t), \psi_n(o, t) \rangle^{p/2} \right]. \quad (3.27)$$

Let us define  $\psi_n^*(o)$  as the increasing limit of  $\psi_n^*(o, t)$  when  $t$  goes toward infinity. By monotone convergence theorem in (3.26), for every  $n \in \mathbb{N}$ ,

$$\mathbb{E}_{\nu_V^W} [\psi_n^*(o)^p] \leq \kappa_{2,p} \mathbb{E}_{\nu_V^W} [\hat{G}_n(o, o)^p] < +\infty. \quad (3.28)$$

Moreover, for any fixed value of  $n$ ,  $(|\psi_n(o, t) - 1|^p)_{t \geq 0}$  is dominated by  $\psi_n^*(o)^p$  which is integrable by (3.28). Therefore, by dominated convergence theorem, we can make  $t$  go to infinity in (3.27) which concludes the proof.  $\square$

### 3.4.4 Link between $\hat{G}_n$ and $\tilde{G}_n$

Let us recall that  $(\hat{G}_n(o, o))_{n \in \mathbb{N}}$  is the bracket of the martingale  $(\psi_n(o))_{n \in \mathbb{N}}$  whose moments we are seeking an upper bound for. Therefore, it would be very interesting for our purpose to be able to control the moments of  $\hat{G}_n(o, o)$  for  $n \in \mathbb{N}$ . The following lemma shows there is a relation between the moments of  $\hat{G}_n(o, o)$  and the moments of  $\tilde{G}_n(o, o)$  for  $n \in \mathbb{N}$ . Remind that  $\tilde{G}_n(o, o)$  has been defined in subsection 3.4.2. For every  $x > 0$ , let us define

$$F_p(x) = \int_0^{+\infty} \frac{x^p}{(1+2yx)^p} \frac{e^{-y}}{\sqrt{\pi y}} dy.$$

**Lemma 3.18.** *We assume that  $V$  is a deterministic graph. Then, for every  $n \in \mathbb{N}^*$  and for every  $p > 1/2$ ,*

$$\mathbb{E}_{\nu_V^W} [\hat{G}_n(o, o)^p] = \mathbb{E}_{\nu_V^W} [F_p(\tilde{G}_n(o, o))].$$

Moreover,

$$F_p(x) \underset{x \rightarrow +\infty}{\sim} a_p x^{p-1/2} \text{ with } a_p = \int_0^{+\infty} \frac{dy}{(\pi y)^{1/2} (1+2y)^p}.$$

*Proof of Lemma 3.18.*

Let  $n \in \mathbb{N}$ . Recall that  $(H_\beta)_{V_n, V_n} = \tilde{H}_n + 2\gamma E_{o,o}$  where  $E_{o,o}$  is the matrix which has only null coefficients, excepted at  $(o, o)$  where it has coefficient 1. Then, by Cramer's formula, we have the following key-equality :

$$\hat{G}_n(o, o) = \frac{\tilde{G}_n(o, o)}{1 + 2\gamma \tilde{G}_n(o, o)}. \quad (3.29)$$

Remind that  $\gamma$  is a Gamma random variable with parameters  $(1/2, 1)$  which is independent of  $\tilde{\beta}$ . Together with (3.29), this implies directly the link between the moments of  $\hat{G}_n(o, o)$  and  $\tilde{G}_n(o, o)$ . We only have to look at the asymptotic behaviour of  $F_p$ . By a change of variable, for every  $x > 0$ ,

$$F_p(x) = x^{p-1/2} \int_0^{+\infty} \frac{e^{-y/x}}{(1+2y)^p (\pi y)^{1/2}} dy. \quad (3.30)$$

Then, by dominated convergence theorem, if  $p > 1/2$ ,

$$\int_0^{+\infty} \frac{e^{-y/x}}{(1+2y)^p (\pi y)^{1/2}} dy \xrightarrow{x \rightarrow +\infty} a_p. \quad (3.31)$$

$\square$

### 3.5 The transient phase

We are now ready to prove Theorem 3.2. Let us explain quickly the strategy of the proof.

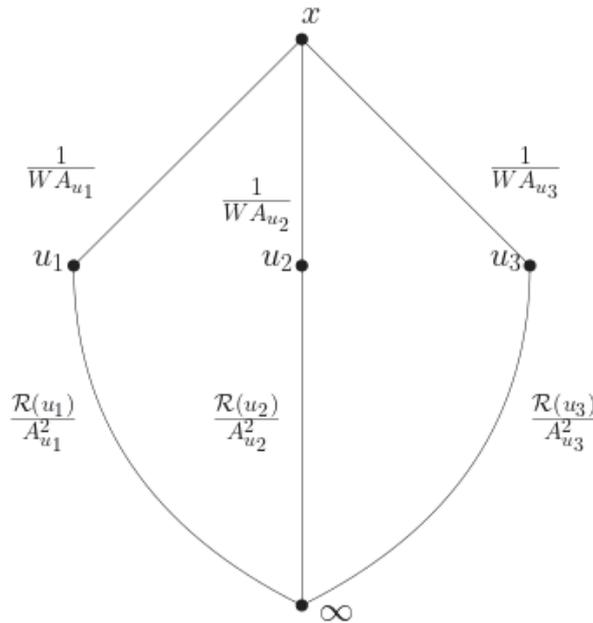
**Strategy of the proof :** The idea is to find an upper bound for the moments of  $\hat{G}_n(o, o)$ . Indeed, it is enough for us because  $(\hat{G}_n(o, o))_{n \in \mathbb{N}}$  is the bracket of  $(\psi_n(o))_{n \in \mathbb{N}}$ . Consequently, by Lemma 3.18, this is enough to find an upper bound for  $\tilde{G}_n(o, o)$  which is also the effective resistance until level  $n$  associated with the environment of the VRJP according to Lemma 3.15. Thus, we only need to show that the global effective resistance  $\mathcal{R}(o \longleftrightarrow \infty)$  has moments of order  $p$  for every  $p > 0$ . By standard computations, the effective resistance of the VRJP on a tree satisfies the equation in law

$$\mathcal{R}(x) = \frac{1}{\sum_{\tilde{i}=x} \frac{A_i^2 W}{A_i + W \mathcal{R}(i)}}$$

where the random variables  $\mathcal{R}(i)$  for  $\tilde{i} = x$  are i.i.d copies of  $\mathcal{R}(x)$ . We will analyse this equation in law in order to bound the moments of the effective resistance.

*Proof of Theorem 3.2.*

**Step 1 :** The potential  $(\beta_i)_{i \in V}$  on  $V$  is constructed as in (3.6). For every  $x \in V$ , recall that  $e^{U_x} = \prod_{o < u \leq x} A_u$ . For every  $x \in V$ , let us define the subtree  $V^x := \{u \in V, x \leq u\}$ . Moreover, for any neighbouring  $i, j \in V^x$ , let us define  $c_x(i, j) = W e^{U_i + U_j - 2U_x}$ . Then, for every  $x \in V$ , let  $\mathcal{R}(x)$  be the electrical resistance between  $x$  and  $\infty$  in the tree  $V^x$  with conductances  $c_x$ . Remark that, under  $\mathbb{P}_{\mu, W}$ ,  $(\mathcal{R}(x))_{x \in V}$  is a family of identically distributed random variables. Furthermore, by Proposition G, as  $W > W_c(\mu)$ ,  $\mathcal{R}(x)$  is finite for every  $x \in V$ ,  $\mathbb{P}_{\mu, W}$ -a.s. The figure 3.1 bellow explains the situation from an electrical point of view.



**FIGURE 3.1** – Electrical network on a subtree. In this situation, the vertex  $x$  has three children,  $u_1, u_2, u_3$ . On each edge the resistance in  $V^x$  is written.

By standard computations on electrical networks we infer that for every  $x \in V$ ,

$$\mathcal{R}(x) = \frac{1}{\sum_{\tilde{i}=x} \frac{A_i^2 W}{A_i + W \mathcal{R}(i)}}.$$

For sake of convenience, we define  $\tilde{\mathcal{R}}(x) = W\mathcal{R}(x)$  for every  $x \in V$ . Therefore, it holds that for every  $x \in V$ ,

$$\tilde{\mathcal{R}}(x) = \frac{1}{\sum_{i=x} \frac{A_i^2}{A_i + \tilde{\mathcal{R}}(i)}}. \quad (3.32)$$

**Step 2 :** The following lines are inspired by the proof of Lemma 2.2 in [3]. For every  $n \in \mathbb{N}$ , the leftest vertex in generation  $n$  of  $V$  is denoted by  $v_n$ . We denote by  $B(v_n)$  the set of "brothers" of  $v_n$ . Remark that this set is possibly empty if  $\mu(1) \neq 0$ . Let  $C > 0$ . Let  $\alpha > 0$ . We define  $c_\alpha = 1$  if  $\alpha \leq 1$  and  $c_\alpha = 2^{\alpha-1}$  otherwise. For every  $n \in \mathbb{N}^*$ , let us introduce the event  $E_n = \{\forall k \in \{1, \dots, n\}, \forall u \in B(v_k), \frac{c_\alpha}{A_u^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(u)^\alpha}{A_u^{2\alpha}} > C\}$ . By convention we write  $\mathbf{1}\{E_0\} := 1$ . Now, let us prove the following key-inequality : for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s,

$$\tilde{\mathcal{R}}(o)^\alpha \leq C \sum_{k=0}^{n-1} \mathbf{1}\{E_k\} \prod_{i=1}^k \left( \frac{c_\alpha}{A_{v_i}^{2\alpha}} \right) + \sum_{k=1}^n \mathbf{1}\{E_k\} A_{v_k}^\alpha \prod_{i=1}^k \left( \frac{c_\alpha}{A_{v_i}^{2\alpha}} \right) + \mathbf{1}\{E_n\} \prod_{i=1}^n \left( \frac{c_\alpha}{A_{v_i}^{2\alpha}} \right) \tilde{\mathcal{R}}(v_n)^\alpha. \quad (3.33)$$

Let us prove it for  $n = 1$ . By (3.32), we can observe that for every child  $u$  of  $o$ ,

$$\tilde{\mathcal{R}}(o)^\alpha \leq \left( \frac{1}{A_u} + \frac{\tilde{\mathcal{R}}(u)}{A_u^2} \right)^\alpha \leq \frac{c_\alpha}{A_u^\alpha} + \frac{c_\alpha}{A_u^{2\alpha}} \tilde{\mathcal{R}}(u)^\alpha. \quad (3.34)$$

If  $E_1$  is satisfied, then we can apply (3.34) with  $u = v_1$  which implies

$$\tilde{\mathcal{R}}(o)^\alpha \leq \mathbf{1}\{E_1\} \left( \frac{c_\alpha}{A_{v_1}^\alpha} + \frac{c_\alpha}{A_{v_1}^{2\alpha}} \tilde{\mathcal{R}}(v_1)^\alpha \right). \quad (3.35)$$

If  $E_1$  is not satisfied, then we can apply (3.34) with a brother of  $v_1$  which implies

$$\tilde{\mathcal{R}}(o)^\alpha \leq C. \quad (3.36)$$

Therefore, combining (3.35) and (3.36), we infer

$$\tilde{\mathcal{R}}(o)^\alpha \leq C + \mathbf{1}\{E_1\} \left( \frac{c_\alpha}{A_{v_1}^\alpha} + \frac{c_\alpha}{A_{v_1}^{2\alpha}} \tilde{\mathcal{R}}(v_1)^\alpha \right) \quad (3.37)$$

which is inequality (3.33) with  $n = 1$ . Remark, that the inequality (3.37) is true even if  $v_1$  is the only child of  $o$ . The proof of (3.33) for any  $n$  is obtained by induction by iterating the inequality (3.37). Moreover, by construction, the events

$$\left( \left\{ \forall u \in B(v_k), \frac{c_\alpha}{A_u^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(u)^\alpha}{A_u^{2\alpha}} > C \right\} \right)_{k \in \mathbb{N}^*}$$

are  $\mathbb{P}_{\mu, W}$ -independent. In addition, the probability of each of these events is the same and it is strictly less than 1 because  $\tilde{\mathcal{R}}(u) < +\infty$  for every  $u \in V$  as  $W > W_c(\mu)$ . Therefore,  $\mathbb{P}_{\mu, W}$ -a.s, there exists  $N \in \mathbb{N}^*$  such that  $\mathbf{1}\{E_n\} = 0$  for every  $n \geq N$ . That is why we can make  $n$  go to infinity in (3.33) which implies,  $\mathbb{P}_{\mu, W}$ -a.s,

$$\tilde{\mathcal{R}}(o)^\alpha \leq C \sum_{k=0}^{+\infty} \mathbf{1}\{E_k\} \prod_{i=1}^k \left( \frac{c_\alpha}{A_{v_i}^{2\alpha}} \right) + \sum_{k=1}^{\infty} \mathbf{1}\{E_k\} A_{v_k}^\alpha \prod_{i=1}^k \left( \frac{c_\alpha}{A_{v_i}^{2\alpha}} \right). \quad (3.38)$$

Now, let us introduce the random set  $\mathcal{A} = \{i \in \mathbb{N}^*, B(v_i) \neq \emptyset\}$  and for every  $k \in \mathbb{N}^*$  the random variable  $\Gamma_k = |\mathcal{A} \cap \{1, \dots, k\}|$ . Under  $GW^\mu$ , the sequence  $(\Gamma_k)_{k \in \mathbb{N}}$  is a random walk whose

increments are independent Bernoulli random variables with parameter  $1 - \mu(1)$ . Further,  $\mathcal{A}$  can be written as  $\{J_1 \leq J_2 \leq J_3 \leq \dots\}$ . For every  $i \in \mathbb{N}^*$ , there exists a brother  $L_i$  of  $v_{J_i}$ . The situation is summarized by the figure 3.2 bellow.

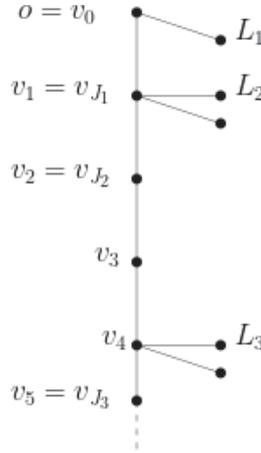


FIGURE 3.2

By construction, conditionally on the underlying Galton-Watson tree, the random variables  $\left(\mathbf{1}\{\forall u \in B(v_k), \frac{c_\alpha}{A_u^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(u)^\alpha}{A_u^{2\alpha}} > C\}\right)_{k \in \mathbb{N}^*}$  and  $(A_{v_k})_{k \in \mathbb{N}^*}$  are mutually independent. Therefore, together with (3.38), this implies that,  $GW^\mu$ -a.s,

$$\mathbb{E}_{\nu_V^W} \left[ \tilde{\mathcal{R}}(o)^\alpha \right] \leq C + \left( C + \frac{Q(W, -\alpha)}{Q(W, -2\alpha)} \right) \sum_{k=1}^{+\infty} (c_\alpha Q(W, -2\alpha))^k \prod_{i=1}^{\Gamma_k} \nu_V^W \left( \frac{c_\alpha}{A_{L_i}^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(L_i)^\alpha}{A_{L_i}^{2\alpha}} > C \right) \quad (3.39)$$

where we recall that  $Q(W, t)$  is the moment of order  $t$  of an Inverse Gaussian random variable with parameters  $(1, W)$ . Remark that, under  $GW^\mu$ , conditionally on  $(\Gamma_k)_{k \in \mathbb{N}^*}$ ,

$$(P_k)_{k \in \mathbb{N}^*} := \left( \nu_V^W \left( \frac{c_\alpha}{A_{L_k}^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(L_k)^\alpha}{A_{L_k}^{2\alpha}} > C \right) \right)_{k \in \mathbb{N}^*}$$

is an *i.i.d* sequence. Therefore, by the strong law of large numbers,  $GW^\mu$ -a.s,

$$\prod_{i=1}^{\Gamma_k} P_i = \exp \left( (\Gamma_k + o(\Gamma_k)) \mathbb{E}_{GW^\mu} [\ln(P_1)] \right).$$

Moreover, by the strong law of large numbers applied with  $(\Gamma_k)_{k \in \mathbb{N}^*}$ ,  $GW^\mu$ -a.s,

$$\prod_{i=1}^{\Gamma_k} P_i = \exp \left( (1 - \mu(1))(k + o(k)) \mathbb{E}_{GW^\mu} [\ln(P_1)] \right). \quad (3.40)$$

Besides, as  $W > W_c(\mu)$ , we know that  $\tilde{\mathcal{R}}(u) < +\infty$  for every  $u \in V$ ,  $\mathbb{P}_{\mu, W}$  a.s. Consequently, by monotone convergence theorem,

$$-\mathbb{E}_{GW^\mu} [\ln(P_1)] = -\mathbb{E}_{GW^\mu} \left[ \ln \left( \nu_V^W \left( \frac{c_\alpha}{A_{L_1}^\alpha} + \frac{c_\alpha \tilde{\mathcal{R}}(L_1)^\alpha}{A_{L_1}^{2\alpha}} > C \right) \right) \right]$$

can be made as large as we want by making  $C$  go toward infinity. Therefore, there exists  $C(\alpha) > 0$  such that

$$\ln(c_\alpha Q(W, -2\alpha)) + (1 - \mu(1)) \mathbb{E}_{GW^\mu} [\ln(P_1)] < 0. \quad (3.41)$$

Hence, for every  $\alpha > 0$ , using (3.41) and (3.40) in (3.39) with  $C = C(\alpha)$  implies that,  $GW^\mu$ -a.s.,

$$I_\alpha := \mathbb{E}_{\nu_W} \left[ \tilde{\mathcal{R}}(o)^\alpha \right] < +\infty. \quad (3.42)$$

**Step 3 :** By (3.42), we can control any moment of  $\tilde{\mathcal{R}}(o)$ . Together with Lemma 3.15, this implies that for every  $\alpha > 0$ , for every  $n \in \mathbb{N}^*$ ,  $GW^\mu$ -a.s.,

$$\mathbb{E}_{\nu_W} \left[ \tilde{G}_n(o, o)^\alpha \right] = \mathbb{E}_{\nu_W} \left[ \mathcal{R}(0 \longleftrightarrow \delta_n)^\alpha \right] \leq W^\alpha \mathbb{E}_{\nu_W} \left[ \tilde{\mathcal{R}}(o)^\alpha \right] = W^\alpha I_\alpha < +\infty. \quad (3.43)$$

Let  $p > 1$ . By Lemma 3.18, for every  $n \in \mathbb{N}^*$ ,  $GW^\mu$ -a.s.,

$$\mathbb{E}_{\nu_W} \left[ \hat{G}_n(o, o)^{p/2} \right] = \mathbb{E}_{\nu_W} \left[ F_{p/2}(\tilde{G}_n(o, o)) \right]$$

where  $F_{p/2}(x) \sim a_{p/2} x^{p/2-1/2}$ . Therefore, together with (3.43), this shows there exists positive constants  $K_1$  and  $K_2$  such that for every  $n \in \mathbb{N}^*$ ,  $GW^\mu$ -a.s.,

$$\begin{aligned} \mathbb{E}_{\nu_W} \left[ \hat{G}_n(o, o)^{p/2} \right] &\leq K_1 + K_2 \mathbb{E}_{\nu_W} \left[ \tilde{G}_n(o, o)^{(p-1)/2} \right] \\ &\leq K_1 + K_2 W I_{(p-1)/2}. \end{aligned} \quad (3.44)$$

By Lemma 3.17, it implies that,  $GW^\mu$ -a.s.,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}_{\nu_W} \left[ \psi_n(o)^p \right] < +\infty.$$

□

**Remark 3.5.** In the proof of Theorem 3.2, identity (3.32) shows that the distribution of  $\hat{G}(o, o)$  is directly linked to the solution of the equation in law

$$\tilde{\mathcal{R}}(o) = \frac{1}{\sum_{\tilde{i}=o} \frac{A_{\tilde{i}}^2}{A_{\tilde{i}} + \tilde{\mathcal{R}}(\tilde{i})}}.$$

A non-trivial solution to this equation must exist in the transient phase. However, we do not know how to express this solution with standard distributions and if it is even possible.

## 3.6 The subcritical phase

### 3.6.1 Proof of Theorem 3.3

In the study of the transient phase, we used the fact that the asymptotic behaviour of  $(\psi_n(o))_{n \in \mathbb{N}}$  is related to the effective resistance associated with the environment of the VRJP. We will also use this crucial property in the recurrent phase. In order to study the effective resistance of the VRJP between  $o$  and the level  $n$ , we will use techniques coming from the area of branching random walks. Indeed the fact that the environment of the VRJP on trees can be expressed as products of independent Inverse Gaussian random variables along branches of the tree makes our situation very similar to branching random walks.

*Proof of Theorem 3.3.*

**Step 1 :** For every vertex  $x$  in the Galton-Watson tree  $V$ , let us define

$$S(x) = - \sum_{o < u \leq x} \ln(A_u).$$

We recall that  $f_{m,W}(t) = \ln(mQ(W, t))$  for every  $t \in \mathbb{R}$ .  $f_{m,W}$  is the Laplace transform associated with the branching random walk  $\{(x, S(x)), x \in V\}$ . In particular, remark that  $\{(x, S(x)), x \in V\}$

satisfies (3.13). By assumption  $A_3$ , it satisfies also (3.12). Remark that  $f_{m,W}(0) = \ln(m) > 0$  because  $m > 1$  by assumption  $A_1$ . Moreover, this is easy to check that  $f_{m,W}$  is strictly convex, strictly decreasing on  $[0, 1/2]$  and strictly increasing on  $[1/2, +\infty[$ . In addition, the support of the point process  $\mathcal{L}$  which is associated with  $\{(x, S(x)), x \in V\}$  is  $\mathbb{R}$  because the support of an Inverse Gaussian distribution is  $\mathbb{R}_+^*$ . Therefore, by Lemma L and Lemma 3.11, there exists a unique  $t^*(m, W) > 0$  such that

$$-\tau(m, W) := f'_{m,W}(t^*(m, W)) = \frac{f_{m,W}(t^*(m, W))}{t^*(m, W)}.$$

For every  $x \in V$ , we define

$$\tilde{S}(x) := t^*(m, W)S(x) + f_{m,W}(t^*(m, W))|x| = t^*(m, W) \left( S(x) - \tau(m, W)|x| \right).$$

By definition of  $t^*(m, W)$ , the branching random walk  $\{(x, \tilde{S}(x)), x \in V\}$  satisfies (3.14). Consequently, with the branching random walk  $\tilde{S}$ , we are allowed to use the results of Hu and Shi, that is, Propositions J and K. Moreover  $W < W_c(\mu)$ . By Proposition H, this is equivalent to say that  $Q(W, 1/2) < 1/m$ . Therefore,  $f_{m,W}(1/2) < 0$ . Thus, by Proposition 3.11,  $t^*(m, W) < 1/2$  and  $\tau(m, W) > 0$ . Now, we are ready to estimate the moments of  $(\psi_n(o))_{n \in \mathbb{N}}$ . By Lemma 3.14, we only have to control  $\mathbb{E}_{\mu,W}[\psi_n(o)^p]$  when  $p > 1$  or  $p \in ]0, \tau(m, W)[$ .

**Step 2 : lower bound in (i).** By Lemma 3.15, we know that for every  $n \in \mathbb{N}$ ,

$$\tilde{G}_n(o, o) = \mathcal{R}(o \longleftrightarrow \delta_n)$$

where  $\mathcal{R}(o \longleftrightarrow \delta_n)$  is the effective resistance between  $o$  and  $\delta_n$  with conductances  $c$ . Recall that if  $i \in V \setminus \{o\}$ , then

$$c(i, \tilde{i}) = W A_i^{-1} \prod_{o < u \leq i} A_u^2.$$

By the Nash-Williams inequality (see 2.15 in [116]), for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu,W}$ -a.s.,

$$\tilde{G}_n(o, o) \geq \frac{1}{W \sum_{|x|=n} A_x^{-1} \prod_{o < y \leq x} A_y^2}. \quad (3.45)$$

Let  $p > 0$ . It holds that, for every  $n \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{E}_{\mu,W} \left[ \tilde{G}_n(o, o)^{p/2} \right] &\geq \frac{1}{W^{p/2}} \mathbb{E}_{\mu,W} \left[ \left( \sum_{|x|=n} A_x^{-1} \prod_{o < y \leq x} A_y^2 \right)^{-p/2} \right] \\ &\geq \frac{1}{W^{p/2}} \mathbb{E}_{\mu,W} \left[ \min_{|x|=n} A_x^{p/2} \times \left( \sum_{|x|=n} \prod_{o < y \leq x} A_y^2 \right)^{-p/2} \right] \\ &= \frac{1}{W^{p/2}} \mathbb{E}_{\mu,W} \left[ \min_{|x|=n} A_x^{p/2} \times \left( \sum_{|x|=n} e^{-2S(x)} \right)^{-p/2} \right] \\ &= \frac{1}{W^{p/2}} e^{p\tau(m,W)n} \mathbb{E}_{\mu,W} \left[ \min_{|x|=n} A_x^{p/2} \times \mathcal{W}_{n,2/t^*(m,W)}^{-p/2} \right] \end{aligned} \quad (3.46)$$

where for every  $\beta > 1$ ,

$$\mathcal{W}_{n,\beta} = \sum_{|x|=n} e^{-\beta \tilde{S}(x)}.$$

By (3.16) in Lemma J, as  $2/t^*(m, W) > 4 > 1$ , we know that,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\limsup_{n \rightarrow +\infty} \frac{\ln(\mathcal{W}_{n, 2/t^*(m, W)})}{\ln(n)} = -1/t^*(m, W).$$

Therefore,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\mathcal{W}_{n, 2/t^*(m, W)}^{-p/2} \geq n^{p/(2t^*(m, W)) + o(1)}. \quad (3.47)$$

Moreover, for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{P}_{\mu, W} \left( \min_{|x|=n} A_x < n^{-2} \right) &= \mathbb{P}_{\mu, W} \left( \bigcup_{|x|=n} \{A_x < n^{-2}\} \right) \\ &\leq \mathbb{E}_{GW\mu} [Z_n \nu_V^W (A < n^{-2})] \end{aligned}$$

where  $A$  has an Inverse Gaussian distribution with parameter  $(1, W)$  and  $Z_n = \sum_{|x|=n} 1$ . In addition, the cumulative distribution function of an Inverse Gaussian random variable decreases exponentially fast at 0. Therefore there exists  $\lambda > 0$  such that for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{P}_{\mu, W} \left( \min_{|x|=n} A_x < n^{-2} \right) &\leq e^{-\lambda n^2} \mathbb{E}_{GW\mu} [Z_n] \\ &\leq m^n e^{-\lambda n^2} \end{aligned} \quad (3.48)$$

which is summable. Therefore, by Borel-Cantelli lemma,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\min_{|x|=n} A_x^{p/2} \geq n^{-p+o(1)}. \quad (3.49)$$

Consequently, using (3.49) and (3.47) and Fatou's lemma, we infer that

$$\mathbb{E}_{\mu, W} \left[ \min_{|x|=n} A_x^{p/2} \times \mathcal{W}_{n, 2/t^*(m, W)}^{-p/2} \right] \geq n^{p/(2t^*(m, W)) - p + o(1)}. \quad (3.50)$$

Then (3.50) and (3.46) imply that,

$$\mathbb{E}_{\mu, W} \left[ \tilde{G}_n(o, o)^{p/2} \right] \geq e^{p\tau(m, W)n + o(n)}. \quad (3.51)$$

Together with Lemma 3.17 and Lemma 3.18, this yields

$$\mathbb{E}_{\mu, W} [\psi_n(o)^{1+p}] \geq e^{p\tau(m, W)n + o(n)}. \quad (3.52)$$

**Step 3 : upper bound in (i).** This part of the proof is partially inspired from [64]. For every  $n \in \mathbb{N}^*$ , let us denote by  $\mathcal{C}(o \longleftrightarrow \delta_n)$  the effective conductance between  $o$  and  $\delta_n$  with respect to conductances  $c_n$ . (See subsection 3.4.2 for the definition of the conductances  $c$  and  $c_n$ .) By Lemma 3.15, for every  $n \in \mathbb{N}^*$ ,

$$\mathcal{C}(o \longleftrightarrow \delta_n) = \tilde{G}_n(o, o)^{-1}. \quad (3.53)$$

Now, we introduce  $(\tilde{Z}_k)_{k \in \mathbb{N}^*}$  a Markov chain on  $V$  with conductances  $c$  starting from  $o$  (which is actually the discrete-time process associated with the VRJP). When we want to integrate only with respect to this Markov chain, we use the notations  $P_{c, o}$  and  $E_{c, o}$ . By definition of the effective conductance, we know that

$$\mathcal{C}(o \longleftrightarrow \delta_n) = W \sum_{\tilde{i}=o} A_i \times P_{c, o}(\tau_n < \tau_o^+) \geq W \sum_{\tilde{i}=o} A_i \times \max_{|x|=n} P_{c, o}(\tau_x < \tau_o^+) \quad (3.54)$$

where  $\tau_n = \inf\{k \in \mathbb{N}, |\tilde{Z}_k| = n\}$ ,  $\tau_x = \inf\{k \in \mathbb{N}, \tilde{Z}_k = x\}$  and  $\tau_o^+ = \inf\{k \in \mathbb{N}^*, \tilde{Z}_k = o\}$ . For every  $x \in V \setminus \{o\}$ , we define  $x_1$  the unique child of  $o$  which is an ancestor of  $x$ . By standard computations, for every  $n \in \mathbb{N}^*$ , for every  $x$  such that  $|x| = n$ ,

$$\begin{aligned} W \sum_{\tilde{i}=o} A_i \times P_{c,o}(\tau_x < \tau_o^+) &= \frac{\sum_{\tilde{i}=o} A_i A_{x_1}^{-1}}{\sum_{o < u \leq x} c(u, \tilde{u})^{-1}} \\ &\geq \frac{1}{\sum_{o < u \leq x} c(u, \tilde{u})^{-1}}. \end{aligned} \quad (3.55)$$

By (3.55) and the expression of  $c$ , we infer that

$$\begin{aligned} W \sum_{\tilde{i}=o} A_i \times P_{c,o}(\tau_x < \tau_o^+) &\geq \frac{W}{\sum_{o < u \leq x} A_u \prod_{o < v \leq u} A_v^{-2}} \\ &\geq \frac{W}{\sum_{o < u \leq x} A_u e^{2S(u)}} \\ &\geq W \frac{e^{-2S_m(x)}}{n} \times \min_{|z| \leq n} A_z^{-1} \end{aligned} \quad (3.56)$$

where  $S_m(x) = \max_{o < u \leq x} S(u)$ . Therefore, combining identities (3.56), (3.54) and (3.53), we get for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\tilde{G}_n(o, o) \leq \frac{n}{W} \times \max_{|z| \leq n} A_z \times e^{2 \min_{|x|=n} S_m(x)}. \quad (3.57)$$

Moreover, as  $\tau(m, W) > 0$ , it holds that for every  $x \in V$ ,

$$\begin{aligned} S_m(x) &= \max_{o < u \leq x} S(u) \\ &= \max_{o < u \leq x} \tilde{S}(u)/t^*(m, W) + \tau(m, W)|u| \\ &\leq \tau(m, W)|x| + (1/t^*(m, W)) \max_{o < u \leq x} \tilde{S}(u) \\ &= \tau(m, W)|x| + (1/t^*(m, W)) \tilde{S}_m(x) \end{aligned} \quad (3.58)$$

where  $\tilde{S}_m(x) = \max_{o < u \leq x} \tilde{S}(u)$ . Combining (3.57) and (3.58), it holds that for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\tilde{G}_n(o, o) \leq \frac{n}{W} \times \max_{|z| \leq n} A_z \times e^{2\tau(m, W)n} \times e^{2/t^*(m, W) \min_{|x|=n} \tilde{S}_m(x)}. \quad (3.59)$$

Let  $p > 0$ . By (3.59) and Cauchy-Schwarz inequality, for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{E}_{\mu, W} \left[ \tilde{G}_n(o, o)^{p/2} \right] &\leq \frac{n^{p/2}}{W^{p/2}} e^{p\tau(m, W)n} \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} A_z^{p/2} \times e^{p/t^*(m, W) \min_{|x|=n} \tilde{S}_m(x)} \right] \\ &\leq \underbrace{\frac{n^{p/2}}{W^{p/2}} e^{p\tau(m, W)n} \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} A_z^p \right]^{1/2}}_{(a)} \underbrace{\mathbb{E}_{\mu, W} \left[ e^{2p/t^*(m, W) \min_{|x|=n} \tilde{S}_m(x)} \right]^{1/2}}_{(b)}. \end{aligned} \quad (3.60)$$

If we show that (a) and (b) have a subexponential growth, it gives the good upper bound for  $\mathbb{E}_{\mu, W} \left[ \tilde{G}_n(o, o)^{p/2} \right]$ . In order to majorize (a), let us introduce a function  $h_p$  on  $\mathbb{R}_+$  which is increasing, convex, bijective and such that there exists  $\gamma_p > 0$  such that  $h_p(x) = e^{(W/4)x^{1/p}}$  for

every  $x > \gamma_p$ . Such a function does clearly exist. By Jensen's inequality, for every  $n \in \mathbb{N}^*$ , it holds that

$$\begin{aligned} h_p \left( \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} A_z^p \right] \right) &\leq \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} h_p(A_z^p) \right] \\ &\leq h_p(\gamma_p) + \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} e^{(W/4)A_z} \right] \\ &\leq h_p(\gamma_p) + \mathbb{E}_{\mu, W} \left[ \sum_{|z| \leq n} e^{(W/4)A_z} \right] \\ &\leq h_p(\gamma_p) + (m-1)^{-1} m^{n+1} \mathbb{E}_{\mu, W} \left[ e^{(W/4)A} \right] \end{aligned}$$

where  $A$  is an Inverse Gaussian distribution with parameters  $(1, W)$ . Remark that  $\mathbb{E}_{\mu, W} [e^{(W/4)A}] < +\infty$ . Thus, there exist positive constants  $C_1$  and  $C_2$  such that for every  $n$  big enough,

$$\begin{aligned} \mathbb{E}_{\mu, W} \left[ \max_{|z| \leq n} A_z^p \right] &\leq h_p^{-1} (C_1 + C_2 m^n) \\ &\leq \left( \frac{4}{W} \ln (C_1 + C_2 m^n) \right)^p. \end{aligned} \quad (3.61)$$

Consequently, (a) in (3.60) has a subexponential growth. Now, let us look at (b) in (3.60). Let us define  $a^* := 2p/t^*(m, W)$ . Let  $\varepsilon > 0$ . Then, remark that for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} (b) &\leq e^{na^*\varepsilon} + \mathbb{E}_{\mu, W} \left[ e^{a^* \min_{|x|=n} \max_{o < u \leq x} \tilde{S}(u)} \mathbf{1} \left\{ \min_{|x|=n} \max_{o < u \leq x} \tilde{S}(u) \geq \varepsilon n \right\} \right] \\ &\leq e^{na^*\varepsilon} + \underbrace{\mathbb{P}_{\mu, W} \left( \min_{|x|=n} \max_{o < u \leq x} \tilde{S}(u) \geq \varepsilon n \right)}_{(c)}^{1/2} \mathbb{E}_{\mu, W} \left[ \sum_{k=1}^n \sum_{|x|=k} e^{2a^* \tilde{S}(x)} \right]^{1/2}. \end{aligned} \quad (3.62)$$

However the term

$$\mathbb{E}_{\mu, W} \left[ \sum_{k=1}^n \sum_{|x|=k} e^{2a^* \tilde{S}(x)} \right] = \sum_{k=1}^n \mathbb{E}_{\mu, W} \left[ \sum_{|x|=1} e^{2a^* \tilde{S}(x)} \right]^k$$

grows exponentially fast when  $n$  goes toward infinity. Therefore we only have to prove that (c) decreases faster than any exponential function. Let  $\delta > 0$ . The crucial point is to remark that for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} &\mathbb{P}_{\mu, W} \left( \min_{|x|=n} \max_{o < u \leq x} \tilde{S}(u) \geq \varepsilon n \right) \\ &\leq \mathbb{P}_{\mu, W} \left( \max_{|z|=\lfloor \delta n \rfloor} \max_{o < u \leq z} \tilde{S}(u) \geq \varepsilon n/2 \right) \\ &\quad + \mathbb{P}_{\mu, W} \left( \forall z, |z| = \lfloor \delta n \rfloor, \min_{|x|= \lfloor (1-\delta)n \rfloor} \max_{z < u \leq x} \tilde{S}_z(u) + \tilde{S}(z) \geq \varepsilon n \cap \tilde{S}(z) \leq \varepsilon n/2 \right) \end{aligned}$$

where  $\tilde{S}_z(u) = \tilde{S}(u) - \tilde{S}(z)$ . Therefore, for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{P}_{\mu, W} \left( \min_{|x|=n} \max_{o < u \leq x} \tilde{S}(u) \geq \varepsilon n \right) &\leq \mathbb{P}_{\mu, W} \left( \max_{|z|=\lfloor \delta n \rfloor} \max_{o < u \leq z} \tilde{S}(u) \geq \varepsilon n/2 \right) \\ &\quad + \mathbb{P}_{\mu, W} \left( \forall z, |z| = \lfloor \delta n \rfloor, \min_{|x|= \lfloor (1-\delta)n \rfloor} \max_{z < u \leq x} \tilde{S}_z(u) \geq \varepsilon n/2 \right). \end{aligned} \quad (3.63)$$

By the branching property, for every  $n \in \mathbb{N}^*$  and hypothesis  $A_2$ ,

$$\begin{aligned} & \mathbb{P}_{\mu,W} \left( \forall z, |z| = \lfloor \delta n \rfloor, \min_{|x|_z = \lfloor (1-\delta)n \rfloor} \max_{z < u \leq x} \tilde{S}_z(u) \geq \varepsilon n/2 \right) \\ & \leq \mathbb{P}_{\mu,W} \left( \min_{|x| = \lfloor (1-\delta)n \rfloor} \max_{o < u \leq x} \tilde{S}(u) \geq \varepsilon n/2 \right)^{2^{\lfloor \delta n \rfloor}}. \end{aligned}$$

Therefore, using inequality (2.12) in [64], there exists  $\eta > 0$  such that for every integer  $n$  which is large enough,

$$\mathbb{P}_{\mu,W} \left( \forall z, |z| = \lfloor \delta n \rfloor, \min_{|x|_z = \lfloor (1-\delta)n \rfloor} \max_{z < u \leq x} \tilde{S}_z(u) \geq \varepsilon n/2 \right) \leq \left( 1 - e^{-\eta n^{1/3}} \right)^{2^{\lfloor \delta n \rfloor}} \quad (3.64)$$

which decreases faster than any exponential function. Now, let  $t > 0$ . By Markov inequality, for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{P}_{\mu,W} \left( \max_{|z| = \lfloor \delta n \rfloor} \max_{o < u \leq z} \tilde{S}(u) \geq \varepsilon n/2 \right) & \leq e^{-n\varepsilon t/2} \sum_{k=1}^{\delta n} \mathbb{E}_{\mu,W} \left[ \sum_{|x|=k} e^{t\tilde{S}(x)} \right] \\ & = e^{-n\varepsilon t/2} \sum_{k=1}^{\delta n} r(t)^k \end{aligned}$$

where  $r(t) = \mathbb{E}_{\mu,W} \left[ \sum_{|x|=1} e^{t\tilde{S}(x)} \right]$ . Consequently, there exists a constant  $C > 0$  such that for every  $n \in \mathbb{N}^*$ ,

$$\mathbb{P}_{\mu,W} \left( \max_{|z| = \lfloor \delta n \rfloor} \max_{o < u \leq z} \tilde{S}(u) \geq \varepsilon n/2 \right) \leq C \exp(n(\delta \ln(r(t)) - t\varepsilon/2)). \quad (3.65)$$

If we take  $t$  large enough and  $\delta$  small enough, we get an exponential decay with a decreasing rate which is as large as we want. Therefore, combining (3.65), (3.64) and (3.63), we know that (c) in (3.62) decreases faster than any exponential function. Consequently, by (3.62), (b) has a subexponential growth. Moreover, we also proved that (a) has subexponential growth. By (3.60), this yields

$$\mathbb{E} \left[ \tilde{G}_n(o, o)^{p/2} \right] \leq e^{p\tau(m,W)n+o(n)}. \quad (3.66)$$

Together with Lemma 3.17 and Lemma 3.18, this yields

$$\mathbb{E}_{\mu,W} \left[ \psi_n(o)^{1+p} \right] \leq e^{p\tau(m,W)n+o(n)}. \quad (3.67)$$

**Step 4 : upper bound in (ii).** For every  $x \in V$ , let us denote by  $\nu_x$  the number of children of  $x$ . For every  $n \in \mathbb{N}^*$ , by definition of  $\psi_n(o)$  we know that

$$\psi_n(o) = W \sum_{|x|=n} \hat{G}_n(o, x) \nu_x.$$

Moreover, for every  $x \in V$ , for every  $n \in \mathbb{N}^*$ ,  $\hat{G}_n(o, x) \leq \hat{G}(o, x)$ . This can be proved thanks to path expansions. (See Lemma I.) Consequently, for every  $n \in \mathbb{N}^*$ ,

$$\psi_n(o) \leq W \sum_{|x|=n} \hat{G}(o, x) \nu_x. \quad (3.68)$$

As  $W < W_c(\mu)$ , by Lemma 3.10, for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s, it holds that

$$\begin{aligned} \psi_n(o) &\leq W \hat{G}(o, o) \sum_{|x|=n} e^{U_x} \nu_x \\ &= W \hat{G}(o, o) \sum_{|x|=n} \prod_{o < u \leq x} A_u \nu_x. \end{aligned} \quad (3.69)$$

Together with the notation introduced in step 1 of this proof, we get that for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s,

$$\psi_n(o) \leq W \hat{G}(o, o) e^{-\tau(m, W)n} \sum_{|x|=n} e^{-\tilde{S}(x)/t^*(m, W)} \nu_x \quad (3.70)$$

By identity (3.29) and Lemma 3.16, as  $W < W_c(\mu)$ , it holds that  $\hat{G}(o, o) = \frac{1}{2\gamma}$ . Together with (3.70) this implies that for every  $n \in \mathbb{N}^*$ ,  $\mathbb{P}_{\mu, W}$ -a.s,

$$\psi_n(o) \leq W \frac{1}{2\gamma} e^{-\tau(m, W)n} \sum_{|x|=n} e^{-\tilde{S}(x)/t^*(m, W)} \nu_x. \quad (3.71)$$

Nevertheless, by the construction of the  $\beta$ -potential introduced in subsection 3.9, we know that  $\gamma$ ,  $(\tilde{S}(x))_{|x|=n}$  and  $(\nu_x)_{|x|=n}$  are independent and  $\gamma$  has a Gamma distribution with parameters  $(1/2, 1)$ . Consequently, for every  $p \in ]0, t^*(m, W)[$ , for every  $n \in \mathbb{N}^*$ , it holds that

$$\mathbb{E}_{\mu, W} [\psi_n(o)^p] \leq W^p e^{-p\tau(m, W)n} \int_0^{+\infty} \frac{x^{-p-1/2}}{\sqrt{4^p \pi}} e^{-x} dx \times \mathbb{E}_{\mu, W} \left[ \left( \sum_{|x|=n} e^{-\tilde{S}(x)/t^*(m, W)} \nu_x \right)^p \right]. \quad (3.72)$$

For every  $p \in ]0, 1/2[$ , we denote

$$\kappa_p = W^p \int_0^{+\infty} \frac{x^{-p-1/2}}{\sqrt{4^p \pi}} e^{-x} dx < +\infty.$$

As  $t^*(m, W) < 1/2 < 1$ , we are allowed to use concavity in (3.72) which implies that for every  $p \in ]0, t^*(m, W)[$ , for every  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{E}_{\mu, W} [\psi_n(o)^p] &\leq \kappa_p e^{-p\tau(m, W)n} \times \mathbb{E}_{\mu, W} \left[ \left( \sum_{|x|=n} e^{-\tilde{S}(x)/t^*(m, W)} \nu_x \right)^{t^*(m, W)} \right]^{p/t^*(m, W)} \\ &\leq \kappa_p e^{-p\tau(m, W)n} \times \mathbb{E}_{\mu, W} \left[ \sum_{|x|=n} e^{-\tilde{S}(x)} \nu_x^{t^*(m, W)} \right]^{p/t^*(m, W)}. \end{aligned} \quad (3.73)$$

However  $(\tilde{S}(x))_{|x|=n}$  and  $(\nu_x)_{|x|=n}$  are independent. Therefore, for every  $n \in \mathbb{N}^*$  and for every  $p \in ]0, t^*(m, W)[$ ,

$$\mathbb{E}_{\mu, W} [\psi_n(o)^p] \leq \kappa_p e^{-p\tau(m, W)n} \times \mathbb{E}_{\mu, W} [\mathcal{W}_n]^{p/t^*(m, W)} \times \mathbb{E}_{\mu, W} [\nu^{t^*(m, W)}]^{p/t^*(m, W)} \quad (3.74)$$

where  $\nu$  has distribution  $\mu$  and  $\mathcal{W}_n = \sum_{|x|=n} e^{-\tilde{S}(x)}$ . Therefore, as  $\mathcal{W}_n$  is a martingale with mean 1, we get that for every  $n \in \mathbb{N}^*$  and for every  $p \in ]0, t^*(m, W)[$ ,

$$\mathbb{E}_{\mu, W} [\psi_n(o)^p] \leq \kappa_p \times \mathbb{E}_{\mu, W} [\nu^{t^*(m, W)}]^{p/t^*(m, W)} \times e^{-p\tau(m, W)n}$$

In order to conclude the proof, we need the same estimate for  $p \in ]1 - t^*(m, W), 1[$ . This stems from Lemma 3.14.  $\square$

### 3.6.2 Proof of Theorem 3.4

First, we need the following lemma which establishes a link "in law" between  $\psi_n(o)$  and the effective resistance associated with the VRJP.

**Lemma 3.19.** *Let  $V$  be a rooted tree with root  $o$ . Let  $W > 0$ . Then, under  $\nu_V^W$ , it holds that for every  $n \in \mathbb{N}^*$ ,*

$$\psi_n(o)^2 \times 2\gamma \times (1 + 2\gamma\mathcal{R}(o \longleftrightarrow \delta_n)) \stackrel{\text{law}}{=} 2\Gamma(1/2, 1)$$

where  $\gamma$  is the  $\Gamma(1/2, 1)$  random variable which was used to define the potential  $\beta$  on a tree (see identity (3.7)) and  $\mathcal{R}(o \longleftrightarrow \delta_n)$  is the effective resistance from  $o$  to  $\delta_n$  associated with the conductances  $c$  defined in Proposition G.

*Proof of Lemma 3.19.*

Let  $n \in \mathbb{N}$ . The proof is based on a coupling with a potential on the wired graph  $\tilde{V}_n$ . (See subsection 3.4.2 for the definition of the wired graph.) Recall that, under  $\nu_V^W$ , thanks to (3.7), the potential  $\beta$  can be decomposed as  $\beta = \tilde{\beta} + \mathbf{1}\{\cdot = o\}\gamma$  where  $\gamma$  and  $\tilde{\beta}$  are independent. For every  $i \in V_n$ , we write  $\hat{\eta}_i^{(n)} = \sum_{j \sim i, j \notin V_n} W$ . Then, recall that  $\psi_n(o) = \hat{G}_n \hat{\eta}^{(n)}$ . In particular, there exists a deterministic function  $F_n$  from  $\mathbb{R}^{|V_n|+1}$  into  $\mathbb{R}^3$  such that

$$(\psi_n(o), \tilde{G}_n(o), 2\gamma) = F_n(\tilde{\beta}_{V_n}, \gamma). \quad (3.75)$$

Now, let us define a potential  $\beta'$  on the wired graph  $\tilde{V}_n$  with distribution  $\tilde{\nu}_{\tilde{V}_n}^{\tilde{P}_n, 0}$  where  $\tilde{P}_n$  is the adjacency matrix of the weighted graph  $\tilde{V}_n$ . We can associate a matrix  $H_{\beta'}$  with the potential  $\beta'$  in the usual way and the inverse of  $H_{\beta'}$  is denoted by  $G'$ . We define  $\gamma' = 1/(2G'(o, o))$  and  $\tilde{\beta}' = \beta' - \mathbf{1}\{\cdot = o\}\gamma'$ . By Theorem 3 in [153],  $\gamma'$  is distributed as  $\Gamma(1/2, 1)$  and is independent of  $\tilde{\beta}'$ . Let us define the matrix  $\tilde{H}_{\beta'}$  in the same way as  $H_{\beta'}$  but we replace  $2\beta'_o$  by  $2\tilde{\beta}'_o$ . Moreover, we define  $\hat{G}'_n$  and  $\tilde{G}'_n$  as the inverse of  $(H_{\beta'})_{V_n, V_n}$  and  $(\tilde{H}_{\beta'})_{V_n, V_n}$  respectively. Further, let us write  $\psi'_n = \hat{G}'_n \hat{\eta}^{(n)}$ . Then, by Proposition 8 in [154], it holds that

$$\frac{1}{2\gamma'} = G'(o, o) = \hat{G}'_n(o, o) + G'(\delta_n, \delta_n)\psi'_n(o)^2. \quad (3.76)$$

The equality (3.76) can be proved by means of the results about path expansions given by Lemma I. By (3.76), we get

$$\frac{\psi'_n(o)^2}{1/(2\gamma') - \hat{G}'_n(o, o)} = \frac{1}{G'(\delta_n, \delta_n)}. \quad (3.77)$$

Besides, by Cramer's formula,

$$\frac{1}{2\gamma'} - \hat{G}'_n(o, o) = \frac{1}{2\gamma'} - \frac{\tilde{G}'_n(o, o)}{1 + 2\gamma'\tilde{G}'_n(o, o)} = \frac{1}{2\gamma'(1 + 2\gamma'\tilde{G}'_n(o, o))}.$$

Together with (3.77), this yields

$$\psi'_n(o)^2 \times 2\gamma' \times (1 + 2\gamma'\tilde{G}'_n(o, o)) = \frac{1}{G'(\delta_n, \delta_n)}. \quad (3.78)$$

Further, with the same function  $F_n$  as in (3.75), it holds that

$$(\psi'_n(o), \tilde{G}'_n(o), 2\gamma') = F_n(\tilde{\beta}'_{V_n}, \gamma'). \quad (3.79)$$

Moreover, the joint law of  $(\tilde{\beta}'_{V_n}, \gamma')$  is the same as the joint law of  $(\tilde{\beta}_{V_n}, \gamma)$ . It stems from the restriction properties in Lemma C and Lemma D. Therefore, combining this with (3.75), (3.79) and (3.78), we obtain that

$$\psi_n(o)^2 \times 2\gamma \times (1 + 2\gamma\tilde{G}_n(o, o)) \stackrel{\text{law}}{=} \psi'_n(o)^2 \times 2\gamma' \times (1 + 2\gamma'\tilde{G}'_n(o, o)) = \frac{1}{G'(\delta_n, \delta_n)}.$$

By Theorem 3 in [153],  $1/G'(\delta_n, \delta_n) \stackrel{law}{=} 2\Gamma(1/2, 1)$  and by Proposition 3.15,  $\tilde{G}_n(o, o) = \mathcal{R}(o \longleftrightarrow \delta_n)$ . This concludes the proof.  $\square$

Now, we are ready to prove Theorem 3.4.

*Proof of Theorem 3.4.*

For every  $n \in \mathbb{N}$ , it holds that

$$\psi_n(o)^2 = \frac{1}{2\gamma(1 + 2\gamma\mathcal{R}(o \longleftrightarrow \delta_n))} \times \Phi_n \quad (3.80)$$

where  $\Phi_n = \psi_n(o)^2 \times 2\gamma(1 + 2\gamma\mathcal{R}(o \longleftrightarrow \delta_n))$ . By Lemma 3.19, we know that for every  $n \in \mathbb{N}$ ,  $\Phi_n \stackrel{law}{=} 2\Gamma(1/2, 1)$ . Therefore for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{P}_{\mu, W}(\Phi_n < 2/n^4) &= \int_0^{1/n^4} \frac{e^{-y}}{\sqrt{\pi y}} dy \\ &\leq \frac{1}{\sqrt{\pi}} \int_0^{1/n^4} \frac{dy}{\sqrt{y}} \\ &= \frac{2}{\sqrt{\pi} n^2} \end{aligned}$$

which is summable. Moreover, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{P}_{\mu, W}(\Phi_n > 2n) &= \int_n^{+\infty} \frac{e^{-y}}{\sqrt{\pi y}} dy \\ &\leq \frac{1}{\sqrt{\pi n}} e^{-n} \end{aligned}$$

which is summable. Consequently, by Borel-Cantelli lemma,  $\mathbb{P}_{\mu, W}$ -a.s, for  $n$  large enough,

$$\frac{2}{n^4} \leq \Phi_n \leq 2n. \quad (3.81)$$

That is why, in order to conclude, we only have to prove that,  $\mathbb{P}_{\mu, W}$ -a.s,

$$\mathcal{R}(o \longleftrightarrow \delta_n) = e^{2\tau(m, W)n + o(n)}.$$

Remark that the identity (3.46) is also true without the expectation and remember from Lemma 3.15 that  $\mathcal{R}(o \longleftrightarrow \delta_n) = \tilde{G}_n(0, 0)$ . Therefore, for every  $n \in \mathbb{N}$ ,

$$\mathcal{R}(o \longleftrightarrow \delta_n) \geq \frac{1}{W} e^{2\tau(m, W)n} \times \min_{|x|=n} A_x \times \mathcal{W}_{n, 2/t^*(m, W)}^{-1}. \quad (3.82)$$

First,  $\min_{|x|=n} A_x$  has at most polynomial decay  $\mathbb{P}_{\mu, W}$ -a.s. This can be shown exactly as in (3.49).

Furthermore, by Proposition J,  $\mathcal{W}_{n, 2/t^*(m, W)}^{-1}$  has also polynomial asymptotics. Consequently, this proves the lower bound of  $\mathcal{R}(o \longleftrightarrow \delta_n)$ . More precisely,  $\mathbb{P}_{\mu, W}$  almost surely,

$$\mathcal{R}(o \longleftrightarrow \delta_n) \geq e^{2\tau(m, W)n + o(n)}.$$

Now, let us prove the upper bound. By (3.59), it holds that

$$\mathcal{R}(0 \longleftrightarrow \delta_n) \leq \frac{n}{W} \times \max_{|z| \leq n} A_z \times e^{2\tau(m, W)n} \times e^{2/t^*(m, W) \min_{|x|=n} \tilde{S}_m(x)}. \quad (3.83)$$

In the same way as in (3.49),  $\max \{A_z : |z| \leq n\}$  has at most polynomial growth  $\mathbb{P}_{\mu, W}$ -a.s. Moreover, by Theorem 1.4 in [64], there exists some constant  $c > 0$  such that  $\min \{\tilde{S}_m(x) : |x| = n\} \sim cn^{1/3}$   $\mathbb{P}_{\mu, W}$ -a.s. This concludes the proof.  $\square$

### 3.6.3 Proof of Proposition 3.5

*Proof of Proposition 3.5.*

Let  $m > 1$ . For every  $W > 0$  and for every  $t > 0$ , let us define

$$F(W, t) = \ln(mQ(W, t)).$$

Obviously,  $F \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}_+^*)$ . We introduce another function  $G$  defined by

$$G(W, t) = F(W, t) - t \frac{\partial F}{\partial t}(W, t)$$

for every  $(t, W) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ . Moreover, by step 1 in the proof of Theorem 3.3, we know that for every  $W > 0$ , there exists a unique  $t^*(m, W) > 0$  such that  $G(W, t^*(m, W)) = 0$ . Further, for every  $(t, W) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ ,

$$\frac{\partial G}{\partial t}(W, t) = -t \frac{\partial^2 F}{\partial t^2}(W, t) = -t \frac{\mathbb{E}_{\mu, W} [A^t] \mathbb{E}_{\mu, W} [\ln(A)^2 A^t] - \mathbb{E}_{\mu, W} [\ln(A) A^t]^2}{\mathbb{E}_{\mu, W} [A^t]^2} \quad (3.84)$$

where  $A$  is an Inverse Gaussian distribution with parameters  $(1, W)$ . From (3.84) and Cauchy-Schwarz inequality, we deduce that for every  $(t, W) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ ,

$$\frac{\partial G}{\partial t}(W, t) < 0. \quad (3.85)$$

Therefore, we can apply the implicit function theorem which implies that  $W \mapsto t^*(m, W)$  is smooth. By Proposition H,  $W_c(\mu)$  is the unique  $W > 0$  such that  $mQ(W, 1/2) = 1$ . Moreover, for every  $W \in \mathbb{R}_+^*$ ,

$$\frac{\partial F}{\partial t}(W, 1/2) = 0 \quad (3.86)$$

because the minimum of  $t \mapsto Q(W, t)$  is achieved for  $t = 1/2$ . Consequently,

$$\begin{aligned} G(W_c(\mu), 1/2) &= F(W_c(\mu), 1/2) - (1/2) \frac{\partial F}{\partial t}(W_c(\mu), 1/2) \\ &= \ln(mQ(W_c(\mu), 1/2)) \\ &= 0. \end{aligned}$$

Therefore,

$$t^*(m, W_c(\mu)) = 1/2. \quad (3.87)$$

Thus, by Taylor expansion in a neighborhood of  $W_c(\mu)$ , it holds that,

$$\begin{aligned} F(W, t^*(m, W)) &= F(W_c(\mu), 1/2) + (W - W_c(\mu)) \frac{\partial F}{\partial W}(W_c(\mu), 1/2) \\ &\quad + (t^*(m, W) - 1/2) \frac{\partial F}{\partial t}(W_c(\mu), 1/2) + o\left(W_c(\mu) - W, t^*(m, W) - 1/2\right) \\ &= (W - W_c(\mu)) \frac{\partial F}{\partial W}(W_c(\mu), 1/2) + o(W_c(\mu) - W) \end{aligned} \quad (3.88)$$

where in the last equality, we used the fact that  $F(W_c(\mu), 1/2) = 0$  and (3.86). Moreover  $o(W_c(\mu) - W, t^*(m, W) - 1/2)$  becomes  $o(W_c(\mu) - W)$  in the last equality because

$$t^*(m, W) - 1/2 = t^*(m, W) - t^*(m, W_c(\mu)) = O(W_c(\mu) - W)$$

as  $t^*(m, \cdot)$  is a smooth function. Besides,

$$\tau(m, W) = -F(W, t^*(m, W))/t^*(m, W) \sim -2F(W, t^*(m, W))$$

in the neighborhood of  $W_c(\mu)$  because  $t^*(m, W_c(\mu)) = 1/2$ . Together with (3.88), it yields

$$\tau(m, W) \underset{W \rightarrow W_c(\mu)}{\sim} 2 \left( \frac{\partial F}{\partial W}(W_c(\mu), 1/2) \right) (W_c(\mu) - W) \quad (3.89)$$

Therefore, we only have to compute  $\frac{\partial F}{\partial W}(W_c(\mu), 1/2)$  in order to conclude the proof. Let us recall that for every  $W > 0$ ,

$$F(W, 1/2) = \ln(m) + \frac{1}{2} \ln(W) + \ln \left( \int_0^{+\infty} \frac{e^{-(W/2)(x+1/x-2)}}{\sqrt{2\pi x}} dx \right). \quad (3.90)$$

Differentiating (3.90), we get

$$\begin{aligned} \frac{\partial F}{\partial W}(W, 1/2) &= \frac{1}{2W} - \frac{1}{2} \frac{\int_0^{+\infty} (x + 1/x - 2)(2\pi)^{-1/2} x^{-1} e^{-(W/2)(x+1/x-2)} dx}{\int_0^{+\infty} (2\pi)^{-1/2} x^{-1} e^{-(W/2)(x+1/x-2)} dx} \\ &= \frac{1}{2W} - \frac{1}{2} \frac{Q(W, 3/2) + Q(W, -1/2) - 2Q(W, 1/2)}{Q(W, 1/2)} \\ &= 1 + \frac{1}{2W} - \frac{Q(W, 3/2)}{Q(W, 1/2)}. \end{aligned} \quad (3.91)$$

In the last equality, we used the fact that  $Q(W, 3/2) = Q(W, -1/2)$ . Moreover, remark that for every  $W > 0$ ,

$$\begin{aligned} Q(W, 3/2) &= \int_1^{+\infty} \sqrt{\frac{W}{2\pi}} \frac{(x + 1/x)}{x} e^{-(W/2)(x+1/x-2)} dx \\ &= \sqrt{\frac{2W}{\pi}} \int_0^{+\infty} \cosh(u) e^{-W(\cosh(u)-1)} du \\ &= \sqrt{\frac{2W}{\pi}} e^W K_1(W) \\ &= \frac{K_1(W)}{K_{1/2}(W)} \end{aligned} \quad (3.92)$$

where  $K_\alpha$  is the modified Bessel function of the second kind with index  $\alpha$ . Besides, recall that  $mQ(W_c(\mu), 1/2) = 1$ . Now, let us evaluate (3.91) at  $W = W_c(\mu)$ . Together with (3.92), this implies

$$\frac{\partial F}{\partial W}(W_c(\mu), 1/2) = 1 + \frac{1}{2W_c(\mu)} - m \frac{K_1(W_c(\mu))}{K_{1/2}(W_c(\mu))}. \quad (3.93)$$

Moreover, we still have to prove that  $\frac{\partial F}{\partial W}(W_c(\mu), 1/2) > 0$ . Actually, it is enough to prove that for every  $W > 0$ ,

$$1 + \frac{1}{2W} - \frac{Q(W, 3/2)}{Q(W, 1/2)} > 0.$$

Exactly as in (3.92), one can prove that

$$Q(W, 1/2) = \frac{K_0(W)}{K_{1/2}(W)}.$$

Therefore, we have to prove that for every  $W > 0$ ,

$$1 + \frac{1}{2W} > \frac{K_1(W)}{K_0(W)}.$$

Nevertheless, it is exactly Corollary 3.3 in [41].  $\square$

### 3.6.4 Proof of Proposition 3.6

*Proof of Proposition 3.6.*

Recall from Proposition G that the measure  $\mathbf{P}_{\mu,W}^{VRJP}$  is defined as follows :

- First, under measure  $\mathbb{P}_{\mu,W}$ , we choose randomly a Galton-Watson tree  $V$  and the random conductances  $c$  on  $V$  which are given by Proposition G.
- Secondly, we choose randomly a trajectory on  $V$  for the discrete-time process  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  with distribution  $P_{c,o}$  where  $P_{c,o}$  is the law of a random walk on the tree  $(V, E)$  starting from  $o$  with conductances  $c$ .

**Step 1 : proof of the lower bound.** Let  $n \in \mathbb{N}^*$ . By Jensen's inequality, it holds that

$$\begin{aligned} \frac{1}{\mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n)} &= \frac{1}{\mathbb{E}_{\mu,W} [P_{c,o}(\tau_o^+ > \tau_n)]} \\ &\leq \mathbb{E}_{\mu,W} \left[ \frac{1}{P_{c,o}(\tau_o^+ > \tau_n)} \right]. \end{aligned} \quad (3.94)$$

However, by definition of the effective resistance, we know that

$$\frac{1}{P_{c,o}(\tau_o^+ > \tau_n)} = W \left( \sum_{\tilde{i}=o} A_i \right) \times \mathcal{R}(o \longleftrightarrow \delta_n).$$

Therefore, by Proposition 3.15

$$\frac{1}{P_{c,o}(\tau_o^+ > \tau_n)} = W \left( \sum_{\tilde{i}=o} A_i \right) \times \tilde{G}_n(o, o).$$

Combining this with (3.94) and Cauchy-Schwarz inequality, there exists a positive constant  $C$  such that

$$\frac{1}{\mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n)} \leq C \sqrt{\mathbb{E}_{\mu,W} [\tilde{G}_n(o, o)^2]}. \quad (3.95)$$

Combining (3.66) and (3.95), we obtain

$$\frac{1}{\mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n)} \leq e^{2\tau(m,W)n+o(n)}.$$

This is exactly the lower bound in Proposition 3.6.

**Step 2 : proof of the upper bound.** Let  $\alpha \in ]0, t^*(m, W)/2[$ . Remark that  $t^*(m, W)/2 < 1/4$  because  $W < W_c(\mu)$ . Let  $n \in \mathbb{N}^*$ . It holds that

$$\begin{aligned} \mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n) &= \mathbb{E}_{\mu,W} [P_{c,o}(\tau_o^+ > \tau_n)] \\ &\leq \mathbb{E}_{\mu,W} [P_{c,o}(\tau_o^+ > \tau_n)^\alpha]. \end{aligned} \quad (3.96)$$

Furthermore, by definition of the effective conductance  $\mathcal{C}(o \longleftrightarrow \delta_n)$  between  $o$  and level  $n$  of the tree, we know that

$$P_{c,o}(\tau_o^+ > \tau_n) = \frac{\mathcal{C}(o \longleftrightarrow \delta_n)}{W \sum_{\tilde{i}=o} A_i}. \quad (3.97)$$

Let  $\varepsilon > 0$  such that  $(1 + 2\varepsilon)\alpha < t^*(m, W)/2$ . Combining Hölder inequality, (3.96) and (3.97), there exists  $C > 0$  such that

$$\mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n) \leq C \mathbb{E}_{\mu,W} \left[ \mathcal{C}(o \longleftrightarrow \delta_n)^{(1+\varepsilon)\alpha} \right]^{1/(1+\varepsilon)}. \quad (3.98)$$

However,  $\tilde{G}_n(o, o)^{-1} = \mathcal{C}(o \longleftrightarrow \delta_n)$ . Consequently, following exactly the same lines as in (3.46), we get

$$\mathcal{C}(o \longleftrightarrow \delta_n) \leq W e^{-2\tau(m,W)n} \times \max_{|x|=n} A_x^{-1} \times \mathcal{W}_{n,2/t^*(m,W)}.$$

Combining this with (3.98), it yields

$$\mathbf{P}_{\mu,W}^{VRJP}(\tau_o^+ > \tau_n) \leq C e^{-2\alpha\tau(m,W)n} \mathbb{E}_{\mu,W} \left[ \max_{|x|=n} A_x^{-(1+\varepsilon)\alpha} \times \mathcal{W}_{n,2/t^*(m,W)}^{(1+\varepsilon)\alpha} \right]^{1/(1+\varepsilon)}. \quad (3.99)$$

Moreover, by Hölder inequality, we get

$$\begin{aligned} & \mathbb{E}_{\mu,W} \left[ \max_{|x|=n} A_x^{-(1+\varepsilon)\alpha} \times \mathcal{W}_{n,2/t^*(m,W)}^{(1+\varepsilon)\alpha} \right] \\ & \leq \mathbb{E}_{\mu,W} \left[ \max_{|x|=n} A_x^{-\alpha(1+\varepsilon)(1+2\varepsilon)/\varepsilon} \right]^{\varepsilon/(1+2\varepsilon)} \times \mathbb{E}_{\mu,W} \left[ \mathcal{W}_{n,2/t^*(m,W)}^{(1+2\varepsilon)\alpha} \right]^{1/(1+2\varepsilon)} \end{aligned} \quad (3.100)$$

One can prove that the first term in (3.100) has at most polynomial growth by following exactly the same lines as for the proof of (3.61). Moreover, the second term in (3.100) decreases with a polynomial decay by Proposition K because  $\alpha(1 + 2\varepsilon) < t^*(m, W)/2$ . Together with (3.99), as  $\alpha$  can be taken as close from  $t^*(m, W)/2$  as we want, this concludes the proof.  $\square$

### 3.7 The critical point

#### 3.7.1 Proof of Theorem 3.7

Now, we are going to prove Theorem 3.7 which describes the asymptotic behaviour of  $(\psi_n(o))_{n \in \mathbb{N}}$  at the critical point.

*Proof of Theorem 3.7.*

For simplicity of notation, we write  $W = W_c(\mu)$  in the entirety of this proof. Exactly as in the proof of Theorem 3.4, by using Lemma 3.19, we only need to find the almost sure behaviour of  $\mathcal{C}(o \longleftrightarrow \delta_n)$ , the effective conductance associated with the VRJP, in order to get the asymptotics of  $\psi_n(o)^2$ . Remember that the local conductance from any vertex  $x \in V \setminus \{o\}$  to  $\tilde{x}$  is

$$W A_x^{-1} \left( \prod_{o < u \leq x} A_u^2 \right)$$

which is not exactly the effective conductance associated with a branching random walk. Remark that for every  $n \in \mathbb{N}$ ,

$$W \min_{|z| \leq n} A_z^{-1} \varrho_n \leq \mathcal{C}(o \longleftrightarrow \delta_n) \leq W \max_{|z| \leq n} A_z^{-1} \varrho_n \quad (3.101)$$

where  $\varrho_n$  is the effective conductance from  $o$  to level  $n$  when the local conductance from any vertex  $x \in V \setminus \{o\}$  to  $\tilde{x}$  is given by

$$\left( \prod_{o < u \leq x} A_u^2 \right).$$

As usual,  $\min_{|z| \leq n} A_z^{-1}$  and  $\max_{|z| \leq n} A_z^{-1}$  have polynomial asymptotics almost surely. Thus, we only need to focus on the behaviour of  $(\varrho_n)_{n \in \mathbb{N}}$ . For every  $x \in V$ , let us denote

$$\hat{S}(x) = -2 \sum_{o < u \leq x} \ln(A_u).$$

We write  $\hat{\psi}(t) = \ln \left( \mathbb{E}_{\mu, W} \left[ \sum_{|x|=1} e^{-t\hat{S}(x)} \right] \right) = \ln \left( \mathbb{E}_{\mu, W} \left[ \sum_{|x|=1} A_x^{2t} \right] \right)$ .

As we are at the critical point and thanks to Proposition H,  $\hat{\psi}$  strictly decreases on  $[0, 1/4]$  and increases strictly on  $[1/4, 1]$ ,  $\hat{\psi}(1/4) = 0$  and  $\hat{\psi}'(1/4) = 0$ . Our  $\varrho_n$  is exactly the same as the one defined in [64] with the branching random walk  $\hat{S}$ . By the proof of Theorem 1.2 in [64], we get that,  $\mathbb{P}_{\mu, W}$ -a.s.,

$$\lim_{n \rightarrow +\infty} \frac{\ln(\varrho_n)}{n^{1/3}} = - \left( \frac{3\pi^2}{2} \times 4 \times \hat{\psi}'(1/4) \right)^{1/3} = - \left( 24\pi^2 \mathbb{E}_{\mu, W} \left[ \sum_{|x|=1} A_x^{1/2} \ln(A_x)^2 \right] \right)^{1/3}.$$

This concludes the proof.  $\square$

### 3.7.2 Positive recurrence at the critical point

Now, let us prove Theorem 3.8.

*Proof of Theorem 3.8.*

We want to prove the positive recurrence of the discrete process  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  associated with  $(Z_t)_{t \geq 0}$ . By Proposition G,  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  is a Markov chain in random conductances with conductances given by

$$c(x, \tilde{x}) = W e^{U_x + U_{\tilde{x}}} = W A_x \prod_{o < u \leq \tilde{x}} A_u^2$$

for every  $x \in V \setminus \{o\}$ . For every  $x \in V$ , let us define

$$\tilde{S}(x) = -\frac{1}{2} \sum_{o < y \leq x} \ln(A_u).$$

We assumed that  $W = W_c(\mu)$ , that is,  $mQ(W, 1/2) = 1$  by Proposition H. Therefore,  $\{(x, \tilde{S}(x)), x \in V\}$  is a branching random walk which satisfies hypothesis (3.14). This is easily checked that it satisfies also (3.13). Moreover it satisfies hypothesis (3.12) by hypothesis  $A_3$ . Therefore, we are allowed to use the results of Hu and Shi (Propositions K and J.) with this branching random walk. Following the notations of Hu and Shi, we define

$$\mathcal{W}_{n,4} := \sum_{|x|=n} e^{-4\tilde{S}(x)} = \sum_{|x|=n} \prod_{o < u \leq x} A_u^2$$

and

$$\mathcal{W}_n := \sum_{|x|=n} e^{-\tilde{S}(x)} = \sum_{|x|=n} \prod_{o < u \leq x} A_u^{1/2}.$$

Further, for every  $n \in \mathbb{N}^*$ , let us define

$$\Lambda_n := \sum_{|x|=n} c(x, \tilde{x}).$$

In order to prove Theorem 3.8, this is enough to prove that for some  $r \in ]0, 1[$ ,

$$\sum_{n=1}^{+\infty} \mathbb{E}_{\mu, W} [\Lambda_n^r] < +\infty. \quad (3.102)$$

Let  $n \in \mathbb{N}^*$  and  $r \in ]0, 1[$ .  $r$  shall be made precise later in the proof. First, let us remark that,

$$\begin{aligned} \mathbb{E}_{\mu, W} [\Lambda_n^r] &\leq \mathbb{E}_{\mu, W} \left[ \left( \sum_{|x|=n} \left( \prod_{o < u \leq x} A_u^2 \right) A_x^{-1} \mathbf{1}_{A_x \geq 1} \right)^r \right] \\ &\quad + \mathbb{E}_{\mu, W} \left[ \left( \sum_{|x|=n} \left( \prod_{o < u \leq x} A_u^2 \right) A_x^{-1} \mathbf{1}_{A_x \leq 1} \right)^r \right] \\ &\leq \mathbb{E}_{\mu, W} [\mathcal{W}_{n,4}^r] + \underbrace{\mathbb{E}_{\mu, W} \left[ \left( \sum_{|x|=n} \left( \prod_{o < u \leq x} A_u^2 \right) A_x^{-1} \mathbf{1}_{A_x \leq 1} \right)^r \right]}_{(a)}. \end{aligned} \quad (3.103)$$

For every  $y \in V$ , let us define the random variable

$$\nu_y = \sum_{\tilde{x}=y} 1$$

which is the number of children of  $y$ . Then, it holds that,

$$\begin{aligned} (a) &= \mathbb{E}_{\mu, W} \left[ \left( \sum_{|y|=n-1} \left( \prod_{o < u \leq y} A_u^2 \right) \sum_{\tilde{x}=y} A_x \mathbf{1}_{A_x \leq 1} \right)^r \right] \\ &\leq \mathbb{E}_{\mu, W} \left[ \left( \sum_{|y|=n-1} \left( \prod_{o < u \leq y} A_u^2 \right) \nu_y \right)^r \right] \\ &\leq n^{3r/2} \mathbb{E}_{\mu, W} [\mathcal{W}_{n-1,4}^r] + \underbrace{\mathbb{E}_{\mu, W} \left[ \left( \sum_{|y|=n-1} \left( \prod_{o < u \leq y} A_u^2 \right) \nu_y \mathbf{1}_{\nu_y \geq n^{3/2}} \right)^r \right]}_{(b)}. \end{aligned} \quad (3.104)$$

Moreover, by Jensen's inequality, if  $r < 1/4$ , we get,

$$\begin{aligned} (b) &\leq \mathbb{E}_{\mu, W} \left[ \left( \sum_{|y|=n-1} \left( \prod_{o < u \leq y} A_u^2 \right) \nu_y \mathbf{1}_{\nu_y \geq n^{3/2}} \right)^{1/4} \right]^{4r} \\ &\leq \mathbb{E}_{\mu, W} \left[ \sum_{|y|=n-1} \left( \prod_{o < u \leq y} A_u^{1/2} \right) \right]^{4r} \mathbb{E}_{\mu, W} \left[ \nu^{1/4} \mathbf{1}_{\nu \geq n^{3/2}} \right]^{4r} \\ &= \mathbb{E}_{\mu, W} [\mathcal{W}_{n-1}]^{4r} \mathbb{E}_{\mu, W} \left[ \nu^{1/4} \mathbf{1}_{\nu \geq n^{3/2}} \right]^{4r} \\ &= \mathbb{E}_{\mu, W} \left[ \nu^{1/4} \mathbf{1}_{\nu \geq n^{3/2}} \right]^{4r} \end{aligned} \quad (3.105)$$

where  $\nu$  has the same distribution as  $\nu_y$  for any  $y \in V$ . The last equality comes from the fact that  $(\mathcal{W}_n)_{n \in \mathbb{N}}$  is a martingale because the branching random walk  $\tilde{S}$  satisfies hypothesis (3.14). Combining identities (3.103), (3.104) and (3.105), in order to make  $\mathbb{E}_{\mu, W} [\Lambda_n^r]$  summable, we need

$$n^{3r/2} \mathbb{E}_{\mu, W} [\mathcal{W}_{n,4}^r] \text{ and } \mathbb{E}_{\mu, W} \left[ \nu^{1/4} \mathbf{1}_{\nu \geq n^{3/2}} \right]^{4r}$$

to be summable. Moreover, recall we assumed that  $r < 1/4$ . By Proposition K, we know that

$$n^{3r/2} \mathbb{E}_{\mu, W} [\mathcal{W}_{n,4}^r] = n^{3r/2} \times n^{-6r+o(1)} = n^{-9r/2+o(1)}.$$

Moreover by Hölder's inequality with  $p = 4$ ,

$$\mathbb{E}_{\mu,W} \left[ \nu^{1/4} \mathbf{1}_{\nu \geq n^{3/2}} \right]^{4r} \leq n^{-9r/2}.$$

In order to conclude, we only need to choose  $r$  between  $2/9$  and  $1/4$  which is possible because  $2/9 < 1/4$ .  $\square$

# Chapitre 4

## The density of states of $H_\beta$

### Abstract

This chapter is based on a work which has been achieved under the supervision of Christophe Sabot. Simultaneously, Disertori, Rojas-Molina and Zeng proved results which were very similar to mine. Therefore, we decided to work together and we published a joint paper. (See [55].) This chapter presents my original proofs of the results of [55]. We study properties of the random Schrödinger operator  $H_\beta$  on  $\mathbb{Z}^d$  where  $\beta$  is the random potential introduced in [153] and [154]. This random potential is strongly linked to the asymptotic behaviour of the Vertex Reinforced Jump Process (VRJP). In this paper we show that when the reinforcement of the VRJP is very large, then the integrated density of states  $n(E)$  associated with  $H_\beta$  is of order  $\sqrt{E}$  up to some log-correction. In particular, contrary to the case of the Anderson model, there is no Lifschitz tail in this regime. Moreover, for  $d \geq 3$ , when the reinforcement is small enough, we show that  $n(E) = O(E)$ . This means that the density of states of the operator  $H_\beta$  exhibits a phase transition. Moreover, it was proved in [43] that the Green function  $(H_\beta - z)^{-1}$  is exponentially localized for every  $z \in \mathbb{C} \setminus \mathbb{R}$  if the reinforcement is large enough. In this paper, we prove that, whatever the reinforcement,  $(H_\beta - z)^{-1}$  is always exponentially localized if  $|z|$  is large enough.

### 4.1 Introduction

This paper concerns the random Schrödinger operator  $H_\beta$  introduced in [153] and [154]. This operator arises in the context of self-reinforced random walks such as the Edge Reinforced Random Walk (ERRW) and the Vertex Reinforced Jump Process (VRJP). The ERRW was introduced by Coppersmith and Diaconis in 1986. (See [45].) In this paper, we focus on the VRJP which was first studied in [47] by Davis and Volkov. The VRJP consists in a continuous self-reinforced walk  $(Y_t)_{t \geq 0}$  which is defined on any locally finite graph  $(V, E)$  as follows : Let  $W > 0$ . The VRJP starts from some vertex  $i_0$  and conditionally on the past at time  $t$ , if  $Y_t = i$ , then the VRJP jumps to a neighbour  $j$  of  $i$  at rate

$$W \left( 1 + \int_0^t \mathbf{1}_{\{Y_s = j\}} ds \right).$$

A natural question is to know whether the ERRW and the VRJP are recurrent or transient. In [152], Sabot and Tarres showed that the ERRW can be interpreted as a VRJP with independent random conductances. Moreover, they proved that a time-changed version of the VRJP is a random walk in random environment. This random environment was already known as a supersymmetric hyperbolic sigma model and was studied in [54]. Thanks to this relation between the mixing field

of the VRJP and the supersymmetric sigma model, Sabot and Tarrès proved the recurrence of the VRJP on  $\mathbb{Z}^d$  for high reinforcement (small  $W$ ) and transience for small reinforcement (large  $W$ ) when  $d \geq 3$ . Actually, there is a unique phase transition between recurrence and transience if  $d \geq 3$ . This was proven by Poudevigne in [139] thanks to a clever coupling argument. In [153] and [154], the authors showed that the random environment of the VRJP is strongly linked to another random real field  $\beta = (\beta_x)_{x \in \mathbb{Z}^d}$  which is ergodic and 1-dependent. The distribution  $\nu_d^W$  of  $\beta$  is totally determined by the dimension  $d$  of the underlying working space  $\mathbb{Z}^d$  and the disorder parameter  $W$ . Then, let us introduce the operator  $H_\beta$  on  $\mathbb{Z}^d$  which satisfies :

$$\forall (x, y) \in (\mathbb{Z}^d)^2, H_\beta(x, y) = 2\beta_x \mathbf{1}\{x = y\} - W \mathbf{1}\{x \sim y\}.$$

Thanks to properties of the  $\beta$ -field, this is a non-negative operator. Further, remark that the operator  $H_\beta$  is of the form  $-W\Delta + V$  where  $V_x = 2\beta_x - 2dW$  and  $\Delta$  is the discrete laplacian, that is,

$$\forall (x, y) \in (\mathbb{Z}^d)^2, -\Delta(x, y) = 2d \mathbf{1}\{x = y\} - \mathbf{1}\{x \sim y\}.$$

Therefore,  $H_\beta$  is a random Schrödinger operator. The aim of this paper is to analyse the spectral properties of  $H_\beta$  and to compare  $H_\beta$  with the classical Anderson model which is  $-\Delta + \lambda V'$  where  $V'$  is a potential with i.i.d coordinates. The Anderson model has been largely studied since its introduction in 1958 by P. W. Anderson in his seminal article [7]. When  $\lambda$  is small and  $d \geq 3$ , it is conjectured that the Anderson model is delocalized in the bulk of the spectrum, that is, the existence of continuous spectrum is expected. Further, in any dimension, when  $\lambda$  is large, it has been proved in a sequence of papers including [70] and [118] that the Anderson model is localized, that is, there is only pure point spectrum. (See [5] for a short proof.) Moreover in [5], it is also shown that the Anderson model is localized for extreme energies. In particular, there is localization for energies which are close of the edges of the spectrum. This is strongly linked to a phenomenon called "Lifschitz tails" which states that the integrated density of states  $n(E)$  decreases exponentially fast at the edges of the spectrum. In this paper, we aim to see whether these properties are true for the operator  $H_\beta$  or not.

When  $W$  is very small, it has been already proved in [43] that the operator  $H_\beta$  is totally localized. In this work, we show that  $H_\beta$  is also localized for high energies. Besides, the property of Lifschitz tails is not true anymore for  $H_\beta$ . Indeed, one result of this paper states that, for  $W$  small enough, we have  $n(E) \simeq \sqrt{E}$ . Moreover, when  $W$  is large enough and  $d \geq 3$ ,  $n(E) = O(E)$ . This means there is a phase transition for the behaviour of the density of states of  $H_\beta$  with respect to the parameter  $W$ .

## 4.2 Context and statement of the results

### 4.2.1 Notation

Let us denote by  $l^2(\mathbb{Z}^d)$  the set of families  $(u_x)_{x \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$  such that  $\sum_{x \in \mathbb{Z}^d} |u_x|^2 < +\infty$ . Moreover, we introduce the related scalar product  $\langle \cdot, \cdot \rangle$  such that for every  $u, v \in l^2(\mathbb{Z}^d)$ ,

$$\langle u, v \rangle = \sum_{x \in \mathbb{Z}^d} \bar{u}_x v_x.$$

We denote by  $\|\cdot\|$  the euclidean norm associated with  $\langle \cdot, \cdot \rangle$ . If  $\Lambda$  is a subset of  $\mathbb{Z}^d$ , we denote by  $\partial\Lambda$  the boundary of  $\Lambda$ , that is the set  $\{x \in \Lambda | \exists y \in \Lambda^c, y \sim x\}$ . The size of a finite subset  $\Lambda$  of  $\mathbb{Z}^d$  is denoted by  $|\Lambda|$ . If  $A$  is a subset of  $\mathbb{Z}^d$ , we write  $\vec{E}(A)$  for the set of oriented edges from  $A$  into  $A^c$ . Moreover,  $A^+ := \{x \in \mathbb{Z}^d | \exists y \in A, \|x - y\|_1 \leq 1\}$ . For any  $L \in \mathbb{N}^*$ , we write  $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ . For every  $x \in \mathbb{Z}^d$ , we denote by  $\delta_x$  the element of  $l^2(\mathbb{Z}^d)$  whose value is 1 at  $x$  and 0 elsewhere.

For every matrix  $M$ , we write  $M(i, j)$  for the coefficient of  $M$  at indices  $i, j \in \mathbb{Z}^d$ . We also use this notation if  $M$  is an operator on  $\mathbb{Z}^d$ . The restriction of  $M$  (a matrix or an operator) on a

finite set  $U$  is denoted by  $M_{U,U}$ . For any vector  $v$ , the restriction of  $v$  to a subset  $U$  is denoted by  $v_U$ . If  $M_1$  and  $M_2$  are two symmetric matrices, we write  $M_1 \geq M_2$  (resp.  $M_1 > M_2$ ) if  $M_1 - M_2$  is a non-negative (positive definite) symmetric matrix.

### 4.2.2 The $\beta$ -field and the operator $H_\beta$

Let  $d \in \mathbb{N}^*$ . Two points  $x$  and  $y$  in  $\mathbb{Z}^d$  are linked by an edge if  $\|x - y\|_1 = 1$ . We denote by  $E_d$ , the set of edges of  $\mathbb{Z}^d$ . We write  $x \sim y$  when  $\{x, y\} \in E_d$ . Let  $W > 0$ . In [154], in order to study the VRJP, Sabot and Zeng introduced a random real field  $(\beta_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}_+^{\mathbb{Z}^d}$  with distribution  $\nu_d^W$  such that for every finite subset  $U \subset \mathbb{Z}^d$ , for every  $(\lambda_x)_{x \in U} \in \mathbb{R}_+^U$ , such that  $\lambda_y = 0, \forall y \notin U$ ,

$$\int \exp \left( - \sum_{x \in U} \lambda_x \beta_x \right) \nu_d^W(d\beta) = \exp \left( - \sum_{\{x,y\} \in E_d, x \in U} W \left( \sqrt{1 + \lambda_x} \sqrt{1 + \lambda_y} - 1 \right) \right) \frac{1}{\prod_{x \in U} \sqrt{1 + \lambda_x}}.$$

We write  $\mathbb{E}_{\nu_d^W}$  when we want to integrate with respect to the probability measure  $\nu_d^W$ . From the expression of the Laplace transform above, we deduce that the field  $(\beta_x)_{x \in \mathbb{Z}^d}$  is 1-dependent. It means that if two subsets  $U$  and  $V$  of  $\mathbb{Z}^d$  are disjoint and are not linked by an edge, then  $\beta_U$  and  $\beta_V$  are independent. Moreover, for every  $x \in \mathbb{Z}^d$ ,  $\frac{1}{2\beta_x}$  is distributed as an inverse Gaussian distribution with parameter  $(1/(2dW), 1)$ . More generally, the density of  $\beta_U$  is known explicitly. (See subsection 4.3.1.) Now, for every  $(\beta_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}_+^{\mathbb{Z}^d}$ , let us define the operator  $H_\beta$  on  $\mathbb{Z}^d$  as in the introduction :

$$\forall (x, y) \in (\mathbb{Z}^d)^2, H_\beta(x, y) = 2\beta_x \mathbf{1}\{x = y\} - W \mathbf{1}\{x \sim y\}.$$

By Proposition 1 in [154], the support of  $\nu_d^W$  is

$$\mathcal{D}_d^W = \{ \beta \in \mathbb{R}_+^{\mathbb{Z}^d}, (H_\beta)_{U,U} \text{ is positive definite for all finite subsets } U \subset \mathbb{Z}^d \}.$$

In order to apply the general theory of operators, we observe that the operator  $H_\beta$  with the right domain is a self-adjoint operator.

**Proposition 4.1.** *For every  $W > 0$ , the operator  $H_\beta$  with domain*

$$\mathcal{D}(H_\beta) = \left\{ \phi \in l^2(\mathbb{Z}^d), \sum_{x \in \mathbb{Z}^d} \beta_x^2 |\phi_x|^2 < +\infty \right\}$$

is  $\nu_d^W$ -a.s a self-adjoint operator for the the scalar product  $\langle \cdot, \cdot \rangle$ . Moreover,  $\nu_d^W$ -a.s, we have

$$\mathcal{D}(H_\beta) \neq l^2(\mathbb{Z}^d).$$

By definition of  $\mathcal{D}_d^W$ ,  $H_\beta$  is a non-negative operator. However, we can be more accurate : we can compute its spectrum. We recall that the spectrum  $\sigma(A)$  of an operator  $A$  is the set  $\{ \lambda \in \mathbb{C}, A - \lambda Id \text{ is not invertible} \}$ .

**Proposition 4.2.** *For every  $W > 0$ ,  $\nu_d^W$ -a.s, the spectrum  $\sigma(H_\beta)$  of  $H_\beta$  is  $\mathbb{R}_+$ .*

By definition of  $\mathcal{D}_d^W$ , for every  $L \in \mathbb{N}^*$ ,  $(H_\beta)_{\Lambda_L, \Lambda_L}$  is invertible  $\nu_d^W$ -a.s. Therefore, we can define  $\hat{G}^{(L)} = (H_\beta)_{\Lambda_L, \Lambda_L}^{-1}$ . (More generally, for any finite set  $U \subset \mathbb{Z}^d$ , we can define  $\hat{G}_U$  as the inverse of  $(H_\beta)_{U,U}$ .) Moreover let us introduce  $\psi^{(L)} \in \mathbb{R}_+^{\mathbb{Z}^d}$  as the unique solution of the equation

$$\begin{cases} (H_\beta \psi^{(L)})(i) = 0 & \forall i \in \Lambda_L \\ \psi^{(L)}(i) = 1 & \forall i \in \Lambda_L^c. \end{cases} \quad (4.1)$$

The definition of  $(\psi^{(L)})_{L \in \mathbb{N}^*}$  is motivated by the creation of an eigenstate of  $H_\beta$  when  $L$  goes to infinity. The following proposition states that such a limit does exist.

**Proposition A** (Theorem 1 in [154]). *For any  $x, y \in \mathbb{Z}^d$ ,  $(\hat{G}^{(L)}(x, y))_{L \in \mathbb{N}^*}$  is positive and increasing  $\nu_d^W$ -a.s. In particular there exists a random variable  $\hat{G}(x, y)$  such that*

$$\hat{G}^{(L)}(x, y) \xrightarrow{L \rightarrow +\infty} \hat{G}(x, y), \quad \nu_d^W \text{-a.s.}$$

Further, for any  $x, y \in \mathbb{Z}^d$ ,

$$\hat{G}(x, y) < +\infty, \quad \nu_d^W \text{-a.s.}$$

Moreover, for every  $x \in \mathbb{Z}^d$ ,  $(\psi^{(L)}(x))_{L \in \mathbb{N}^*}$  is a positive martingale whose almost sure limit is denoted by  $\psi(x)$ . Further  $(\hat{G}^{(L)})_{L \in \mathbb{N}^*}$  is the bracket of  $(\psi^{(L)})_{L \in \mathbb{N}^*}$  in the sense that for every  $x, y \in \mathbb{Z}^d$ ,  $(\psi^{(L)}(x)\psi^{(L)}(y) - \hat{G}^{(L)}(x, y))_{L \in \mathbb{N}^*}$  is a martingale.

By Theorem 2 in [154],  $\hat{G}$  is the inverse of  $H_\beta$  in the sense that for every  $(x, y) \in \mathbb{Z}^d$ ,  $(H_\beta + \eta)^{-1}(x, y)$  goes toward  $\hat{G}(x, y)$  as  $\eta$  goes to zero. Besides, the martingale  $(\psi^{(L)})_{L \in \mathbb{N}^*}$  is a very important tool in order to understand the asymptotic behaviour of the VRJP as explained by the following proposition :

**Proposition B** ([154] and [139]). *Let  $d \in \mathbb{N}^*$ . Then, there exists  $W_c(d) \in (0, +\infty]$  depending only on  $d$  such that :*

- (i) *If  $W < W_c(d)$ ,  $\nu_d^W$ -a.s, for every  $x \in \mathbb{Z}^d$ ,  $\psi(x) = 0$  and the VRJP is recurrent.*
- (ii) *If  $W > W_c(d)$ ,  $\nu_d^W$ -a.s, for every  $x \in \mathbb{Z}^d$ ,  $\psi(x) > 0$  and the VRJP is transient.*
- (iii)  *$W_c(d) < +\infty$  if and only if  $d \geq 3$ .*

Proposition B states there is a unique transition point  $W_c(d)$  between recurrence and transience of the VRJP when  $d \geq 3$ . Furthermore, the positivity of  $\psi$  enables us to determine whether  $W < W_c(d)$  or  $W > W_c(d)$ . When  $W < W_c(d)$ , we say that the reinforcement is large or that the system is highly disordered. Conversely, when  $W > W_c(d)$ , we say that the reinforcement is small or that the system is weakly disordered. Remark that  $(\psi(x))_{x \in \mathbb{Z}^d}$  satisfies  $H_\beta \psi = 0$ . Moreover  $(\psi(x))_{x \in \mathbb{Z}^d}$  is ergodic and stationary. Consequently, by Proposition B, for  $W > W_c(d)$  and  $d \geq 3$ ,  $(\psi(x))_{x \in \mathbb{Z}^d}$  can be interpreted as a delocalized generalized eigenstate (which is not square summable) of  $H_\beta$ . Therefore, the phase transition for the VRJP is also a phase transition for the operator  $H_\beta$  with respect to this eigenstate  $\psi$  associated with  $H_\beta$  at the ground state  $E = 0$ . The goal of this article is to enter the bulk of the spectrum of  $H_\beta$  in order to get some information about the density of states of  $H_\beta$  and the Green function  $(H_\beta - z)^{-1}$  for  $z \in \mathbb{C} \setminus \mathbb{R}$ .

Further,  $(\psi(x))_{x \in \mathbb{Z}^d}$  and  $(\hat{G}(0, x))_{x \in \mathbb{Z}^d}$  also give a representation of the environment of the VRJP. Let us consider a  $\Gamma(1/2, 1)$  random variable  $\gamma_1$  which is independent of  $(\beta_x)_{x \in \mathbb{Z}^d}$ . By a small abuse of notation, we still denote by  $\nu_d^W$  the joint measure  $\nu_d^W \otimes \mathcal{L}(\gamma_1)$ . Then for every  $x, y \in \mathbb{Z}^d$ , let us define

$$\Phi(x, y) = \hat{G}(x, y) + \frac{\psi(x)\psi(y)}{2\gamma_1}. \quad (4.2)$$

This operator  $(\Phi(x, y))_{x, y \in \mathbb{Z}^d}$  is fundamental in order to define the random environment of the VRJP. Indeed, if the VRJP started at 0, then the discrete time process associated with the VRJP has the law of a random walk in random environment with conductance  $W\Phi(0, x)\Phi(0, y)$  between two neighboring vertices  $x$  and  $y$ . Remark that  $\Phi = \hat{G}$  if and only if the VRJP is recurrent. Remark also that  $\Phi$  is not the standard notation for the environment of the VRJP. For example, in [154],  $\Phi$  is denoted by  $G_{\beta, \gamma}$ . However, we do not use the notation  $G_{\beta, \gamma}$  in order to avoid confusion with the Green function.

### 4.2.3 The Green function and the density of states

As  $\sigma(H_\beta) = \mathbb{R}_+$ ,  $\nu_d^W$ -a.s, we can define the Green function  $G_z = (H_\beta - z)^{-1}$  for every  $z \in \mathbb{C} \setminus \mathbb{R}_+$ . In particular, for every  $x, y \in \mathbb{Z}^d$  it defines  $G_z(x, y)$ . By Proposition B.3 in [6], for almost every  $E \in \mathbb{R}$ ,  $\nu_d^W$  - a.s, for every  $x, y \in \mathbb{Z}^d$ ,

$$G_{E+i0}(x, y) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} G_{E+i\varepsilon}(x, y)$$

exists and is finite. The Green function  $G_z$  is the classical inverse of  $H_\beta - z$ . One can ask ourselves how this is related to  $\hat{G}$  and  $\Phi$ . Actually, by Theorem 2 in [154], for every  $x, y \in \mathbb{Z}^d$ ,  $\nu_d^W$ -a.s,

$$\lim_{\eta \rightarrow 0} (H_\beta + \eta)^{-1}(x, y) = \hat{G}(x, y).$$

Therefore,  $\hat{G}$  is directly related to the Green function. It can be seen as the Green function associated with the ground state. However,  $\Phi$  is more unusual. It is not directly linked to the Green function but it is a convenient tool in order to study the VRJP. That is why we will also use  $\Phi$  a lot in this paper.

Besides, for every  $E \geq 0$ , let us define  $N((H_\beta)_{\Lambda_L, \Lambda_L}, E)$  the number of eigenvalues of  $(H_\beta)_{\Lambda_L, \Lambda_L}$  which are smaller than  $E$ . Let  $W > 0$ . Then, by Corollary 3.16 in [6], there exists a measure  $\alpha_{W,d}$  with cumulative distribution function  $n_{W,d}$  such that  $\nu_d^W$ -a.s, for all  $E \geq 0$  except the discontinuity points of  $n_{W,d}$ ,

$$\lim_{L \rightarrow +\infty} \frac{1}{|\Lambda_L|} N((H_\beta)_{\Lambda_L, \Lambda_L}, E) = n_{W,d}(E).$$

The measure  $\alpha_{W,d}$  is called the density of states of  $H_\beta$  and  $n_{W,d}$  is called the integrated density of states of  $H_\beta$ .

### 4.2.4 Results about the density of states

First, we state a general upper-bound on the density of states of  $H_\beta$ .

**Theorem 4.3.** *Let  $d \in \mathbb{N}^*$ . There exists a positive constant  $K_1$  which does not depend on  $d$  such that for every  $W > 0$  and for every  $E > 0$ ,*

$$n_{W,d}(E) \leq K_1 E^{1/2}.$$

The upper bound above stems from the classical Wegner bound. We will have to check only the 1/2-regularity. Besides, in the next theorem we show that at strong disorder, i.e  $W$  small, the previous upper bound gives the good order of the density of states up to some logarithmical corrections. In particular, in this regime it shows that there is no Lifschitz tails for the operator  $H_\beta$ . Moreover the following theorem is true only when  $W$  is smaller than some positive constant  $W_-^{(d)}$ . Actually, for  $d \geq 2$ , this constant  $W_-^{(d)}$  is the same as the one obtained in [43] in order to prove exponential decrease of the Green function  $\hat{G}$  associated with the ground state  $E = 0$  of  $H_\beta$ .

**Theorem 4.4.** *Let  $d \in \mathbb{N}^*$ . There exists  $W_-^{(d)} > 0$  such that for every  $W < W_-^{(d)}$ , there exists a positive constant  $K_{2,d,W}$  such that for every  $E \in (0, 1)$ ,*

$$n_{W,d}(E) \geq K_{2,d,W} \frac{\sqrt{E}}{|\ln(E)|^d}.$$

Moreover, in the particular case of the dimension 1, we have  $W_-^{(1)} = +\infty$ .

In the weak disorder regime, i.e for large  $W$ , if  $d \geq 3$ , the Proposition 4.5 below shows that the behaviour of  $n_{W,d}(E)$  is not  $\sqrt{E}$ . Indeed, the behaviour of  $n_{W,d}(E)$  is linked to the integrability of the ground state  $\hat{G}(0,0)$ . If  $W$  is large enough and  $d \geq 3$ , in [54], Disertori, Spencer and Zirnbauer proved an estimate which enables to control  $\hat{G}(0,0)$ . Combining the proposition below with Theorem 4.4, we observe a phase transition for the density of states of  $H_\beta$ .

**Proposition 4.5.** *Let  $d \geq 3$ . Then, there exists  $W_+^{(d)} > 0$  such that for every  $W > W_+^{(d)}$ , there exists a positive constant  $K_{3,d,W}$  such that for every  $E > 0$ ,*

$$n_{W,d}(E) \leq K_{3,d,W} E.$$

### 4.2.5 Localization for high energies

We shall also prove in this article that the Green function of the operator  $H_\beta$  is exponentially localized when the energy  $E$  is large enough. This completes the result of [43] which states there is exponential localization of the Green function for small  $W$ .

**Theorem 4.6.** *Let  $d \in \mathbb{N}^*$ . For every  $W > 0$ , there exist  $K_{4,d,W} > 0$ ,  $\mu_{d,W} > 0$  and  $E_{d,W} > 0$  such that for every  $z \in \mathbb{C} \setminus \mathbb{R}$  such that  $|z| \geq E_{d,W}$ , for every  $x, y \in \mathbb{Z}^d$ ,*

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] \leq K_{4,d,W} \exp(-\mu_{d,W} \|x - y\|).$$

*In particular, for almost every  $E \geq E_{d,W}$ , for every  $x, y \in \mathbb{Z}^d$ ,*

$$\mathbb{E}_{\nu_d^W} \left[ |G_{E+i0}(x, y)|^{1/4} \right] \leq K_{4,d,W} \exp(-\mu_{d,W} \|x - y\|).$$

**Remark 4.1.** *By the Simon-Wolff criterion (see [159]), Theorem 4.6 implies that, for every  $W > 0$ ,  $\nu_d^W$ -a.s, the spectral measure  $\mu_{\delta_0}$  associated with  $H_\beta$  has empty continuous component in  $[E_W, +\infty[$ . By [43], this property was already known for small  $W$  and for almost every  $E$ . We suspect that this is false for large  $W$  and small  $E$  when  $d \geq 3$ .*

## 4.3 Background

### 4.3.1 Restriction properties

An important fact about the distribution  $\nu_d^W$  is that a lot of information is known about the restrictions of  $\nu_d^W$  on finite sets. Actually the restrictions of  $\nu_d^W$  are in a large family of distributions which are similar to  $\nu_d^W$ . These distributions were originally defined in [153] and generalized in [110]. More precisely, on a finite set  $S$ , we can define a generalized family of  $\beta$  potentials with law  $\tilde{\nu}_S^{P,\eta}$  for every  $(\eta_x)_{x \in S} \in \mathbb{R}_+^S$  and every  $P = (W_{x,y})_{(x,y) \in S^2} \in \mathbb{R}_+^{S^2}$ . The distribution  $\tilde{\nu}_S^{P,\eta}$  is defined in Lemma 4 of [154] through its density which is

$$\tilde{\nu}_S^{P,\eta}(d\beta) := \mathbf{1}\{H_{\beta,S} > 0\} \left( \frac{2}{\pi} \right)^{|S|/2} e^{-\frac{1}{2}\langle 1, H_{\beta,S} 1 \rangle - \frac{1}{2}\langle \eta, H_{\beta,S}^{-1} \eta \rangle + \langle \eta, 1 \rangle} \frac{1}{\sqrt{\det H_{\beta,S}}} d\beta_S \quad (4.3)$$

where  $H_{\beta,S}$  is a matrix on  $S \times S$  defined by

$$H_{\beta,S}(x, y) = 2\beta_x \mathbf{1}\{x = y\} - W_{x,y} \mathbf{1}\{x \sim y\}.$$

Moreover, in the density given by (4.3),  $1$  stands for the vector  $(1, \dots, 1)$  in  $\mathbb{R}^S$ . Besides, the Laplace transform of  $\tilde{\nu}_S^{P,\eta}$  is for any  $\lambda \in \mathbb{R}_+^S$

$$\int e^{-\langle \lambda, \beta \rangle} \tilde{\nu}_S^{P,\eta}(d\beta) = e^{-\langle \eta, \sqrt{\lambda+1} - 1 \rangle - \frac{1}{2} \sum_{x \sim y} W_{x,y} \left( \sqrt{(1+\lambda_x)(1+\lambda_y)} - 1 \right)} \prod_{x \in S} (1 + \lambda_x)^{-1/2}$$

where  $\sqrt{1+\lambda}$  is the vector  $(\sqrt{1+\lambda_x})_{x \in S}$ . Remark this Laplace transform is very similar to the one given in subsection 4.2.2 for  $\nu_d^W$ . However, we do not assume that the weights are constants anymore and there is a "boundary term"  $\eta$ . Moreover we allow loops, that is,  $W_{x,x} \neq 0$  is possible for any  $x \in S$ . An important property of these distributions is that marginals and conditional laws of  $\tilde{\nu}_S^P$  can be computed thanks to the following lemma :

**Lemma C** (Lemma 5 in [154]). *Let  $S$  be a finite set. Let  $U \subset S$  be a subset of  $S$ . Let  $(\eta_x)_{x \in S} \in \mathbb{R}_+^S$  and  $P = (W_{x,y})_{x,y \in S^2} \in \mathbb{R}_+^{S^2}$ . Let  $(\beta_x)_{x \in S}$  be a random potential with distribution  $\tilde{\nu}_S^{P,\eta}$ . Then, it holds that*

$$(i) \beta_U \text{ has law } \tilde{\nu}_U^{P_{U,U},\hat{\eta}}, \text{ where for every } x \in U, \hat{\eta}_x = \eta_x + \sum_{y \in U^c} W_{x,y}.$$

(ii) *Conditionally on  $\beta_U$ ,  $\beta_{U^c}$  has distribution  $\tilde{\nu}_{U^c}^{\check{P},\check{\eta}}$  where  $\check{P}$  and  $\check{\eta}$  are defined in the following way : For every  $(x,y) \in U^c \times U^c$ ,*

$$\check{P}(x,y) = \check{W}_{x,y} = W_{x,y} + \sum_{k \sim x, k \in U} \sum_{l \sim y, l \in U} W_{x,k} W_{y,l} (H_{\beta,S})_{U,U}^{-1}(k,l).$$

For every  $x \in U^c$ ,

$$\check{\eta}_x = \eta_x + \sum_{k \sim x, k \in U} \sum_{l \in U} W_{x,k} (H_{\beta,S})_{U,U}^{-1}(k,l) \eta_l.$$

In [154], the infinite-volume measure  $\nu_d^W$  was obtained by Kolmogorov extension theorem thanks to a compatible sequence of measures of type  $\tilde{\nu}_S^{P,\eta}$ . In particular the restriction of  $\nu_d^W$  on finite sets is given by the following proposition.

**Proposition D.** *Let  $L \in \mathbb{N}^*$ . Let  $(\beta_x)_{x \in \mathbb{Z}^d}$  be a random potential following  $\nu_d^W$ . Then  $(\beta_x)_{x \in \Lambda_L}$  is distributed as  $\tilde{\nu}_{\Lambda_L}^{\hat{P}_L, \hat{\eta}_L}$  where*

- For every  $x, y \in \Lambda_L$ ,  $\hat{P}_L(x, y) = W \mathbf{1}\{x \sim y\}$ .
- For every  $x \in \Lambda_L$ ,  $(\hat{\eta}_L)_x = \sum_{y \sim x, y \notin \Lambda_L} W$ .

### 4.3.2 The field $\tilde{\beta}$

Let us come back to the case where  $\beta$  is defined on the whole set  $\mathbb{Z}^d$ . We denote  $\gamma_2 = \frac{1}{2\Phi(0,0)} = \frac{1}{2\hat{G}(0,0) + \psi(0)^2/\gamma_1}$ . For every,  $x \in \mathbb{Z}^d$ , let us define  $\tilde{\beta}_x = \beta_x - \mathbf{1}\{x = 0\}\gamma_2$ . Then, the law of  $\gamma_2$  can be computed explicitly and is independent of the field  $\tilde{\beta}$ .

**Proposition E** (Proposition 2.4 in [77]). *Under  $\nu_d^W$ ,  $\gamma_2$  has distribution  $\Gamma(1/2, 1)$  and is independent of  $(\tilde{\beta}_x)_{x \in \mathbb{Z}^d}$ .*

In this paper, the field  $\tilde{\beta}$  will be used frequently. Indeed, thanks to  $\tilde{\beta}$ , we can divide  $(\beta_x)_{x \in \mathbb{Z}^d}$  into  $(\tilde{\beta}_x)_{x \in \mathbb{Z}^d}$  and  $\gamma_2$ . We write  $\tilde{H}_\beta := H_{\tilde{\beta}} = H_\beta - 2\gamma_2 \delta_{0,0}$  where  $\delta_{0,0}$  is the operator on  $l^2(\mathbb{Z}^d)$  which is 1 at  $(0,0)$  and zero elsewhere. Surprisingly, the operator  $\tilde{H}_\beta$  is more useful than  $H_\beta$  because the inverse of the restriction to finite boxes of  $\tilde{H}_\beta$  has an interpretation in terms of effective resistance. This interpretation will be made more clear in subsection 4.4.1 for a "Dirichlet" restriction of  $\tilde{H}_\beta$ . In order to consider the inverse of restrictions of  $\tilde{H}_\beta$  we need the following lemma.

**Lemma 4.7.** *For every finite subset  $U$  of  $\mathbb{Z}^d$ ,  $(\tilde{H}_\beta)_{U,U}$  is positive definite  $\nu_d^W$ -a.s.*

*Proof.*

First, let us recall that by (4.2), for every  $x, y \in \mathbb{Z}^d$ ,

$$\Phi(x, y) = \hat{G}(x, y) + \frac{\psi(x)\psi(y)}{2\gamma_1}.$$

By the definition of  $\hat{G}$  and  $\psi$ , for all  $\varphi_1, \varphi_2 \in l^2(\mathbb{Z}^d)$  with finite support, we have

$$\langle \varphi_1, H_\beta \Phi \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle.$$

In particular, for every  $x \in \mathbb{Z}^d$ ,

$$\langle \delta_x, H_\beta \Phi \delta_0 \rangle = \delta_{x,0}. \quad (4.4)$$

If  $x = 0$ , (4.4) implies that

$$2\beta_0 \Phi(0,0) = W \sum_{y \sim 0} \Phi(0,y) + 1$$

which implies that

$$2\tilde{\beta}_0 = 2\beta_0 - \frac{1}{\Phi(0,0)} = W \sum_{y \sim 0} \frac{\Phi(0,y)}{\Phi(0,0)}.$$

If  $x \neq 0$ , (4.4) implies that

$$2\tilde{\beta}_x \Phi(0,x) = 2\beta_x \Phi(0,x) = W \sum_{y \sim x} \Phi(0,y).$$

Therefore, for every  $x \in \mathbb{Z}^d$ ,

$$2\tilde{\beta}_x = W \sum_{y \sim x} \frac{\Phi(0,y)}{\Phi(0,x)}. \quad (4.5)$$

Now, let  $U$  be a finite subset of  $\mathbb{Z}^d$  and let  $D_U$  be the diagonal matrix whose diagonal entries are  $(\Phi(0,x))_{x \in U}$ . Then, by (4.5) we can observe that  $D_U(\tilde{H}_\beta)_{U,U}D_U = W \cdot M$  where for every  $x, y \in U \times U$ ,

$$M(x,y) = \mathbf{1}\{x=y\} \sum_{k \sim x} \Phi(0,x)\Phi(0,k) - \mathbf{1}\{x \sim y\} \Phi(0,x)\Phi(0,y).$$

However, it holds that  $M > M'$  where for every  $x, y \in U \times U$ ,

$$M'(x,y) = \mathbf{1}\{x=y\} \sum_{\substack{k \sim x \\ k \in U}} \Phi(0,x)\Phi(0,k) - \mathbf{1}\{x \sim y\} \Phi(0,x)\Phi(0,y).$$

Remark that  $M'$  is a conductance matrix. Therefore,  $M' \geq 0$  and  $M > M' \geq 0$  which implies  $(\tilde{H}_\beta)_{U,U} > 0$ .  $\square$

## 4.4 Preliminaries

### 4.4.1 A lemma about the Dirichlet operator associated with $H_\beta$

Let  $L \in \mathbb{N}^*$ . Let us denote by  $H_L^D$  the squared matrix of size  $|\Lambda_L|$  defined in the following way :

- If  $x \neq y$ ,  $H_L^D(x,y) = -W\mathbf{1}\{x \sim y\}$
- If  $x = y$ ,  $H_L^D(x,y) = 2\beta_x + \sum_{k \sim x, k \notin \Lambda_L} W$ .

We can also define  $\tilde{H}_L^D$  which is defined as  $H_L^D$  but we replace  $\beta$  by  $\tilde{\beta}$ . Those matrices are "Dirichlet" restrictions of  $H_\beta$  and  $\tilde{H}_\beta$ . They are used many times in this paper because the Neumann-Dirichlet bracketing gives a comparison between the integrated density of states and functionals of Dirichlet restrictions. Remark that  $H_L^D \geq (H_\beta)_{\Lambda_L, \Lambda_L}$  and  $\tilde{H}_L^D \geq (\tilde{H}_\beta)_{\Lambda_L, \Lambda_L}$ . However,  $\nu_d^W$ -a.s.,  $(H_\beta)_{\Lambda_L, \Lambda_L} > 0$  by construction of  $\nu_d^W$  and  $(\tilde{H}_\beta)_{\Lambda_L, \Lambda_L} > 0$  by Lemma 4.7. Therefore, we are able to define  $G^{(L),D}$  and  $\tilde{G}^{(L),D}$  the inverse of  $H_L^D$  and  $\tilde{H}_L^D$  respectively. Now

let us consider a graph  $\Lambda'_L$  which is defined by the vertex set  $\Lambda_L \cup \{\delta_L\}$  where  $\delta_L$  is an arbitrary new vertex and an edge set  $E'_L$ . If  $x, y \in \Lambda_L$ , then  $\{x, y\} \in E'_L$  if and only if  $x$  and  $y$  are connected in  $\mathbb{Z}^d$ . Moreover, for every  $x \in \Lambda_L$ ,  $x$  and  $\delta_L$  are connected if and only if  $x \in \partial\Lambda_L$ . Furthermore, for every  $x, y \in \Lambda_L$ , we define

$$c(x, y) = W \frac{\Phi(0, x)\Phi(0, y)}{\Phi(0, 0)^2}.$$

Besides, for every  $x \in \partial\Lambda_L$ , we define

$$c(x, \delta_L) = \sum_{\substack{y \sim x \\ y \notin \Lambda_L}} W \left( \frac{\Phi(0, x)\Phi(0, y)}{\Phi(0, 0)^2} + \frac{\Phi(0, x)^2}{\Phi(0, 0)^2} \right).$$

Let us denote by  $\mathcal{R}_L(0 \longleftrightarrow \delta_L)$  the effective resistance of the random walk associated with the network  $(\Lambda_L \cup \{\delta_L\}, E'_L, c)$ . Then, we have the following lemma :

**Lemma 4.8.** *For every  $L \in \mathbb{N}^*$ ,*

$$\mathcal{R}_L(0 \longleftrightarrow \delta_L) = \tilde{G}^{(L), D}(0, 0).$$

*Proof.*

Let  $L \in \mathbb{N}^*$ . For every  $x \in \Lambda_L$ , we define  $h(x) = \frac{\Phi(0, 0)\tilde{G}^{(L), D}(0, x)}{\tilde{G}^{(L), D}(0, 0)\Phi(0, x)}$  and  $h(\delta_L) = 0$ . We are going to prove that  $h$  is harmonic for the conductances  $c$  on  $\Lambda_L \setminus \{0\}$ . Let  $x \in \Lambda_L \setminus \{0\}$ . Then, it holds that

$$\begin{aligned} \sum_{y, \{x, y\} \in E'_L} c(x, y)h(y) &= \sum_{\substack{y \sim x \\ y \in \Lambda_L}} W \frac{\Phi(0, x)\tilde{G}^{(L), D}(0, y)}{\Phi(0, 0)\tilde{G}^{(L), D}(0, 0)} \\ &= \frac{\Phi(0, x)}{\Phi(0, 0)\tilde{G}^{(L), D}(0, 0)} \sum_{\substack{y \sim x \\ y \in \Lambda_L}} W \tilde{G}^{(L), D}(0, y). \end{aligned} \quad (4.6)$$

However, by definition,  $\tilde{G}^{(L), D}$  is the inverse of  $\tilde{H}_L^D$ . Together with (4.6), this yields

$$\sum_{y, \{x, y\} \in E'_L} c(x, y)h(y) = \frac{\Phi(0, x)\tilde{G}^{(L), D}(0, x)}{\Phi(0, 0)\tilde{G}^{(L), D}(0, 0)} \left( 2\tilde{\beta}_x + \sum_{\substack{y \sim x \\ y \notin \Lambda_L}} W \right). \quad (4.7)$$

Then, as  $x \neq 0$ ,  $\tilde{\beta}_x = \beta_x$ . Moreover, for every  $x \in \mathbb{Z}^d \setminus \{0\}$ ,

$$2\beta_x \Phi(0, x) = \sum_{y \sim x} W \Phi(0, y).$$

Together with (4.7), this implies that

$$\begin{aligned} \sum_{y, \{x, y\} \in E'_L} c(x, y)h(y) &= \frac{\tilde{G}^{(L), D}(0, x)}{\Phi(0, 0)\tilde{G}^{(L), D}(0, 0)} \left( \sum_{y \sim x} W \Phi(0, y) + \Phi(0, x) \sum_{\substack{y \sim x \\ y \notin \Lambda_L}} W \right) \\ &= \frac{\Phi(0, 0)\tilde{G}^{(L), D}(0, x)}{\tilde{G}^{(L), D}(0, 0)\Phi(0, x)} \left( \sum_{y \sim x} W \frac{\Phi(0, y)\Phi(0, x)}{\Phi(0, 0)^2} + \frac{\Phi(0, x)^2}{\Phi(0, 0)^2} \sum_{\substack{y \sim x \\ y \notin \Lambda_L}} W \right) \\ &= h(x) \left( \sum_{\substack{y \sim x \\ y \in \Lambda_L}} c(x, y) + c(x, \delta_L) \right). \end{aligned} \quad (4.8)$$

Consequently,  $h$  is harmonic, excepted at 0 and  $\delta_L$  where it satisfies  $h(0) = 1$  and  $h(\delta_L) = 0$ . Thus, by (2.3) in section 2.2 in [116], it holds that

$$\mathcal{R}_L(0 \longleftrightarrow \delta_L) = \frac{1}{\sum_{y \sim 0} c(0, y)(1 - h(y))}. \quad (4.9)$$

However,

$$\begin{aligned} \sum_{y \sim 0} c(0, y)(1 - h(y)) &= \sum_{y \sim 0} W \frac{\Phi(0, y)}{\Phi(0, 0)} \left( 1 - \frac{\Phi(0, 0) \tilde{G}^{(L), D}(0, y)}{\tilde{G}^{(L), D}(0, 0) \Phi(0, y)} \right) \\ &= \sum_{y \sim 0} W \frac{\Phi(0, y)}{\Phi(0, 0)} - \sum_{y \sim 0} W \frac{\tilde{G}^{(L), D}(0, y)}{\tilde{G}^{(L), D}(0, 0)}. \end{aligned} \quad (4.10)$$

Nevertheless,  $2\beta_0 \Phi(0, 0) - \sum_{y \sim 0} W \Phi(0, y) = 1$  and  $2\tilde{\beta}_0 \tilde{G}^{(L), D}(0, 0) - \sum_{y \sim 0} W \tilde{G}^{(L), D}(0, y) = 1$ . Together with (4.10), this implies

$$\sum_{y \sim 0} c(0, y)(1 - h(y)) = 2\beta_0 - \frac{1}{\Phi(0, 0)} - \left( 2\tilde{\beta}_0 - \frac{1}{\tilde{G}^{(L), D}(0, 0)} \right). \quad (4.11)$$

However, by Proposition E,  $2\beta_0 - \frac{1}{\Phi(0, 0)} = 2\tilde{\beta}_0$ . Together with (4.11), this implies

$$\sum_{y \sim 0} c(0, y)(1 - h(y)) = \frac{1}{\tilde{G}^{(L), D}(0, 0)}.$$

Therefore, thanks to (4.9), this concludes the proof.  $\square$

#### 4.4.2 The spectral measure and the density of states

In subsection 4.2.3, we defined the density of states as the limit of an empirical distribution of eigenvalues of  $H_\beta$ . However, it is also possible to define the density of states through the spectral measure. This equivalent definition is useful for the needs of this paper.

For any self-adjoint operator  $A$  with domain included in  $l^2(\mathbb{Z}^d)$ , and for every  $\phi_1, \phi_2 \in l^2(\mathbb{Z}^d)$ , there exists a complex measure  $\mu_{\phi_1, \phi_2}$  such that for every  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$\int_{-\infty}^{+\infty} \frac{1}{E - z} d\mu_{\phi_1, \phi_2}(E) = \langle \phi_1, (A - z)^{-1} \phi_2 \rangle.$$

$A - z$  is invertible because the spectrum of a self-adjoint operator is included in  $\mathbb{R}$ .  $\mu_{\phi_1, \phi_2}$  is called the spectral measure of  $A$  at vectors  $\phi_1$  and  $\phi_2$ . We write  $\mu_{\phi_1}$  if  $\phi_1 = \phi_2$ . The support of  $\mu_{\phi_1, \phi_2}$  is always included in the spectrum of  $A$ . Now, let  $f$  be a bounded function which is defined on the spectrum  $\sigma(A)$  of  $A$ . Then we can define the operator  $f(A)$  through the following formula :

$$\langle \phi_1, f(A) \phi_2 \rangle = \int_{\sigma(A)} f(E) d\mu_{\phi_1, \phi_2}(E).$$

The reader can refer to Appendices A and B in [6] for more information on this topic. If we apply these ideas in our context, this allows us to define  $\nu_d^W$ -a.s the operator  $f(H_\beta)$  thanks to the associated (random) spectral measures  $\mu_{\phi_1, \phi_2}$ . Let us define by  $\delta_0$  the element of  $l^2(\mathbb{Z}^d)$  which is 1 at 0 and null elsewhere. Then, the density of states  $\alpha_{W, d}$  can be interpreted as follows :

**Proposition F** (Density of states). *For every  $W > 0$ , for every continuous bounded function  $f$  defined on  $\mathbb{R}_+$ , the density of states  $\alpha_{W, d}$  satisfies*

$$\int_0^{+\infty} f(E) d\alpha_{W, d}(E) = \mathbb{E}_{\nu_d^W} [\langle \delta_0, f(H_\beta) \delta_0 \rangle].$$

Morally,  $\alpha_{W, d}$  can be interpreted as the "expectation" of the spectral measure  $\mu_{\delta_0}$ .

### 4.4.3 The iterated resolvent formula

If  $B$  is a (possibly infinite) subset of  $\mathbb{Z}^d$ , then we can define the operator  $H_\beta|_B$  by  $(H_\beta)_{B,B}$  on  $B \times B$  and 0 elsewhere.  $H_\beta|_B$  is self-adjoint. Therefore, for every  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $H_\beta|_B - zId$  is invertible. Let us denote by  $G_{B,z}$  its inverse which is the partial Green function associated with  $B$ . Let  $B$  be a subset of  $\mathbb{Z}^d$ . Let  $A$  be a subset of  $B$  such that  $A^+ \not\subseteq B$ . Recall that  $A_+ := \{x \in \mathbb{Z}^d | \exists y \in A, \|x - y\|_1 \leq 1\}$ . The following formula is also (11.12) in [6].

**Proposition G.** *Let  $x \in A$  and let  $y \in B \setminus A^+$ . Then, for every  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $\nu_d^W$ -a.s.*

$$G_{B,z}(x, y) = W^2 \sum_{(u, u') \in \tilde{E}(A)} \sum_{(v', v) \in \tilde{E}(A^+)} G_{A,z}(x, u) G_{B,z}(u', v') G_{B \setminus A^+, z}(v, y).$$

*Proof.*

Just apply the resolvent formula twice. □

## 4.5 Proof of Propositions 4.1 and 4.2

The proof of Proposition 4.1 follows classical lines :

*Proof.*

The self-adjointness of  $H_\beta$  is straightforward by definition of the adjoint of an operator. Moreover,  $H_\beta$  is self-adjoint and densely defined. Thus, it is closed by Theorem VIII.1.a) in [143]. Furthermore, let us assume by contradiction there exists a set  $\Omega_0$  of positive  $\nu_d^W$ -measure such that  $\mathcal{D}(H_\beta) = l^2(\mathbb{Z}^d)$  on  $\Omega_0$ . Then, by the closed graph theorem, as  $H_\beta$  is closed, this would imply that  $H_\beta$  is continuous on the event  $\Omega_0$ . Then, this would imply that  $\sup \{\beta_x | x \in \mathbb{Z}^d\} < +\infty$  on  $\Omega_0$ . However, this is false  $\nu_d^W$ -a.s because  $(\beta_x)_{x \in \mathbb{Z}^d}$  is 1-dependent. This concludes the proof. □

The proof of Proposition 4.2 is very similar to Theorem 3.12 in [6] with some slight modifications :

*Proof.*

For every  $x \in \mathbb{Z}^d$ , we denote by  $\tau_x$  the operator on  $\mathbb{C}^{\mathbb{Z}^d}$  defined by  $\tau_x(f) = f(\cdot + x)$  for every  $f \in \mathbb{C}^{\mathbb{Z}^d}$ . By Proposition 3 in [154],  $(\beta_x)_{x \in \mathbb{Z}^d}$  is ergodic for the group of transformations  $(\tau_x)_{x \in \mathbb{Z}^d}$ . This implies straightforwardly that  $H_\beta$  is ergodic as operator (see definition 3.4 in [6]). Recall that  $H_\beta = -W\Delta + 2\beta - 2dW$ . Therefore, we can apply Theorem 3.12 in [6] which implies that,  $\nu_d^W$ -a.s.

$$[0, 4dW] + \mathcal{S} \subset \sigma(H_\beta) \tag{4.12}$$

where

$$\mathcal{S} = \left\{ u \in \mathbb{R} | \forall \varepsilon > 0, \forall L \in \mathbb{N}^*, \nu_d^W \left( \sup_{x \in \Lambda_L} |2\beta_x - 2dW - u| < \varepsilon \right) > 0 \right\}.$$

Remark that

$$\mathcal{S} = -2dW + \mathcal{S}'$$

where

$$\mathcal{S}' = \left\{ u \in \mathbb{R} | \forall \varepsilon > 0, \forall L \in \mathbb{N}^*, \nu_d^W \left( \sup_{x \in \Lambda_L} |2\beta_x - u| < \varepsilon \right) > 0 \right\}.$$

For every  $L \in \mathbb{N}^*$ , by Proposition D,  $(\beta_x)_{x \in \Lambda_L}$  has density :

$$\mathbf{1}\{(H_\beta)_{\Lambda_L, \Lambda_L} > 0\} \left(\frac{2}{\pi}\right)^{|\Lambda_L|/2} e^{-\frac{1}{2}\langle 1, (H_\beta)_{\Lambda_L, \Lambda_L} 1 \rangle - \frac{1}{2}\langle \hat{\eta}_L, (H_\beta)_{\Lambda_L, \Lambda_L}^{-1} \hat{\eta}_L \rangle + \langle \hat{\eta}_L, 1 \rangle} \frac{1}{\det((H_\beta)_{\Lambda_L, \Lambda_L})^{1/2}} d\beta_{\Lambda_L}$$

where  $\mathbf{1}$  is the vector with  $|\Lambda_L|$  entries which are equal to one. Therefore,  $u \in \mathcal{S}'$  if and only if  $u - 2dW + W(-\Delta)_{\Lambda_L, \Lambda_L} > 0$  for every  $L \in \mathbb{N}^*$ . However,  $(-\Delta)_{\Lambda_L, \Lambda_L} > 0$  for every  $L \in \mathbb{N}^*$ . That is why  $[2dW, +\infty[ \subset \mathcal{S}'$ . By (4.12), this implies that  $\mathbb{R}_+ \subset \sigma(H_\beta)$ ,  $\nu_d^W$ -a.s. The other inclusion stems from the fact that  $H_\beta$  is a non-negative operator  $\nu_d^W$ -a.s.  $\square$

## 4.6 Proof of Theorem 4.3

The proof of Theorem 4.3 is an application of Wegner bound. We only have to prove single-site 1/2-regularity. This was already proved in [43] if  $W$  is small. In this section, we give a proof of 1/2-regularity for every  $W > 0$ . Let us define  $D = \frac{1}{2} \sum_{0 \sim j} W \frac{\hat{G}(0, j)}{\hat{G}(0, 0)}$ . Let  $\mathcal{F}_{0,0}$  be the sigma-field  $\sigma(\beta_y, y \neq 0)$ .

**Lemma 4.9.** *Under  $\nu_d^W$ , it holds that*

— *If  $\psi(0) > 0$ , then*

$$\mathcal{L}(\beta_0 | \mathcal{F}_{0,0}) = D + \frac{1}{2 \times IG\left(\frac{\hat{G}(0,0)}{\psi(0)}, 1\right)}$$

*where we recall that  $IG(a, \lambda)$  stands for an inverse Gaussian distribution with parameters  $a$  and  $\lambda$ .*

— *If  $\psi(0) = 0$ , then*

$$\mathcal{L}(\beta_0 | \mathcal{F}_{0,0}) = D + \gamma$$

*where  $\gamma$  is a Gamma random variable with parameters  $(1/2, 1)$  which is independent of  $\mathcal{F}_{0,0}$ .*

*In any of the two cases, conditionally on  $\mathcal{F}_{0,0}$ , the density of  $\beta_0$  is given by*

$$\mathbf{1}\{\beta > D\} \frac{1}{\sqrt{\pi(\beta - D)}} e^{-\frac{1}{4(\beta - D)} \left(\frac{\psi(0)}{\hat{G}(0,0)} - 2(\beta - D)\right)^2} d\beta.$$

*Proof.*

Lemma 4.9 was already proved in Lemma 3.13 in Chapter 3 with a slightly different notation.  $\square$

Now, we are ready to prove Theorem 4.3.

*Proof of Theorem 4.3.*

Let us denote by  $\varrho^*$  the distribution of  $\beta_0$  conditionally on  $\mathcal{F}_{0,0}$ . Let  $x \in \mathbb{R}$ . Let  $t > 0$ . By Lemma 3.13,

$$\begin{aligned} \varrho^*([x - t, x + t]) &= \int_{x-t}^{x+t} \mathbf{1}\{\beta > D\} \frac{1}{\sqrt{\pi(\beta - D)}} e^{-\frac{1}{4(\beta - D)} \left(\frac{\psi(0)}{\hat{G}(0,0)} - 2(\beta - D)\right)^2} d\beta \\ &\leq \int_{x-D-t}^{x-D+t} \mathbf{1}\{\beta > 0\} \frac{1}{\sqrt{\pi\beta}} d\beta. \end{aligned} \quad (4.13)$$

However, remark that for every  $y \in \mathbb{R}$ ,  $\int_{y-t}^{y+t} \mathbf{1}\{\beta > 0\} \beta^{-1/2} d\beta \leq \int_0^{2t} \beta^{-1/2} d\beta$ . Together with 4.13, this implies that

$$\begin{aligned} \varrho^*([x - t, x + t]) &\leq \frac{1}{\sqrt{\pi}} \int_0^{2t} \frac{1}{\sqrt{\beta}} d\beta \\ &= \frac{2\sqrt{2t}}{\sqrt{\pi}}. \end{aligned} \quad (4.14)$$

This proves single-site 1/2-regularity of the potential  $\beta$ . Then, using the Wegner bound concludes the proof. One can look at Theorem 4.6 in [6] for a statement and a proof of the Wegner bound. (The proof in [6] is given on a finite graph but the generalization is straightforward.)  $\square$

## 4.7 Proof of Theorem 4.4

**Strategy of the proof :** In order to prove Theorem 4.4, we use the Dirichlet-Neumann bracketing which tells us that

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\nu_d^W} [N(H_L^D, E)]$$

where  $N(H_L^D, E)$  is the number of eigenvalues of  $H_L^D$  (The Dirichlet restriction of  $H_\beta$  in the box  $\Lambda_L$ .) which are lower than  $E$ . However, we can not control the eigenvalues of  $H_L^D$  excepted the first one. Indeed the smallest eigenvalue of  $H_L^D$  is lower than  $G^{(L),D}(0,0)^{-1}$  by the min-max principle. Consequently,

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \nu_d^W \left( G^{(L),D}(0,0)^{-1} \leq E \right).$$

Furthermore, if  $L$  is large,  $G^{(L),D}(0,0)$  is almost  $\hat{G}^{(L)}(0,0)$  which converges toward  $\hat{G}(0,0)$ . Besides, if  $W$  is small enough, the VRJP is recurrent and thus  $\hat{G}(0,0) = \Phi(0,0)$ . Therefore, approximately,

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \nu_d^W \left( \frac{1}{\Phi(0,0)} \leq E \right).$$

However, according to Proposition E,  $1/(2\Phi(0,0))$  has  $\Gamma(1/2, 1)$  distribution whose cumulative distribution function behaves as  $\sqrt{E}$  when  $E$  goes to 0 which explains the lower bound in Theorem 4.4. Some technical details still need to be checked in order to justify that  $G^{(L),D}(0,0) \simeq \hat{G}^{(L)}(0,0)$  for a good choice of  $L$ . This is the main difficulty of the following proof.

*Proof.*

Let  $E \in (0, 1)$ . Let  $L \in \mathbb{N}^*$ .  $L$  will be fixed later as a function of  $E$ .  $H_L^D$  is diagonalizable because it is symmetric. Now, let us consider  $\lambda_0$  the smallest (random) eigenvalue of  $H_L^D$ . Then, as  $H_L^D$  is positive,  $\lambda_0^{-1}$  is the biggest eigenvalue of  $G^{(L),D}$ . Therefore,  $\nu_d^W$ -a.s,

$$\lambda_0^{-1} = \max_{\|x\|=1} \langle x, G^{(L),D} x \rangle \geq G^{(L),D}(0,0) \quad (4.15)$$

where the maximum is taken on the euclidean unit ball of  $\mathbb{C}^{\Lambda_L}$ . Besides, let us denote by  $N(H_L^D, E)$  the number of eigenvalues of  $H_L^D$  which are smaller than  $E$ . By Dirichlet-Neumann bracketing (see Lemma 4.12 in [6])

$$\begin{aligned} n_{W,d}(E) &\geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\nu_d^W} [N(H_L^D, E)] \\ &\geq \frac{1}{|\Lambda_L|} \nu_d^W (\lambda_0 \leq E). \end{aligned} \quad (4.16)$$

Combining (4.15) and (4.16), we get

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \nu_d^W \left( G^{(L),D}(0,0) \geq \frac{1}{E} \right). \quad (4.17)$$

Recall that  $\beta = \tilde{\beta} + \mathbf{1}\{\cdot = 0\}\gamma_2$ . Then, remark that

$$\begin{aligned} \nu_d^W \left( \gamma_2 \leq \frac{E}{4} \right) &= \nu_d^W \left( \frac{1}{2\gamma_2} \geq \frac{2}{E} \right) \\ &\leq \nu_d^W \left( \frac{1}{2\gamma_2} - G^{(L),D}(0,0) \geq \frac{1}{E} \right) + \nu_d^W \left( G^{(L),D}(0,0) \geq \frac{1}{E} \right). \end{aligned} \quad (4.18)$$

Then, using (4.18) in (4.17), we obtain

$$n_{W,d}(E) \geq \frac{1}{|\Lambda_L|} \left( \nu_d^W \left( \gamma_2 \leq \frac{E}{4} \right) - \nu_d^W \left( \frac{1}{2\gamma_2} - G^{(L),D}(0,0) \geq \frac{1}{E} \right) \right). \quad (4.19)$$

First, let us focus on the term  $\nu_d^W \left( \frac{1}{2\gamma_2} - G^{(L),D}(0,0) \geq \frac{1}{E} \right)$ . We shall prove that this is negligible. By Cramer's formula,  $\nu_d^W$ -a.s it holds that

$$G^{(L),D}(0,0) = \frac{\tilde{G}^{(L),D}(0,0)}{1 + 2\gamma_2 \tilde{G}^{(L),D}(0,0)}.$$

Therefore,  $\nu_d^W$ -a.s

$$\frac{1}{2\gamma_2} - G^{(L),D}(0,0) = \frac{1}{2\gamma_2 \left( 1 + 2\gamma_2 \tilde{G}^{(L),D}(0,0) \right)} \leq \frac{1}{4\gamma_2^2 \tilde{G}^{(L),D}(0,0)}. \quad (4.20)$$

Consequently, by (4.20) and as  $\gamma_2$  is independent of  $\tilde{G}^{(L),D}(0,0)$ , we get

$$\begin{aligned} \nu_d^W \left( \frac{1}{2\gamma_2} - G^{(L),D}(0,0) \geq \frac{1}{E} \right) &\leq \nu_d^W \left( 4\gamma_2^2 \tilde{G}^{(L),D}(0,0) \leq E \right) \\ &= \int_0^{+\infty} \nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right) \frac{e^{-x}}{\sqrt{\pi x}} dx. \end{aligned} \quad (4.21)$$

Let  $x \in \mathbb{R}_+^*$ . Now, we have to control  $\nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right)$ . Recall from Lemma 4.8 that  $\tilde{G}^{(L),D}(0,0)$  can be interpreted as an effective resistance with conductances  $c$  on  $\Lambda_L \cup \{\delta_L\}$ . For every  $k \in \mathbb{N}^*$ , we denote by  $\Pi_k$  the set of edges between  $\Lambda_k$  and  $\Lambda_{k+1}$ .  $\Pi_k$  is what we call a cutset. Indeed, it separates  $\Lambda_k$  and  $\Lambda_{k+1}$ . Then, by the Nash-Williams inequality (see (2.15) in [116]),

$$\frac{1}{\mathcal{R}_L(0 \longleftrightarrow \delta_L)} \leq \sum_{e \in \Pi_{L-1}} c(e).$$

Consequently, by Lemma 4.8,  $\nu_d^W$ -a.s,

$$\frac{1}{\tilde{G}^{(L),D}(0,0)} \leq \sum_{e \in \Pi_{L-1}} c(e). \quad (4.22)$$

By inequality (4.22), we deduce that

$$\nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right) \leq \nu_d^W \left( \left( \sum_{e \in \Pi_{L-1}} c(e) \right)^{1/16} \geq \frac{(2x)^{1/8}}{E^{1/16}} \right). \quad (4.23)$$

However, for  $s \in (0, 1)$ , for any  $n \in \mathbb{N}^*$  and for any  $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}_+^n$ ,

$$\left( \sum_{i=1}^n \lambda_i \right)^s \leq \sum_{i=1}^n \lambda_i^s.$$

Combining this with (4.23), we obtain

$$\begin{aligned} \nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right) &\leq \nu_d^W \left( \sum_{e \in \Pi_{L-1}} c(e)^{1/16} \geq \frac{(2x)^{1/8}}{E^{1/16}} \right) \\ &\leq \frac{E^{1/16}}{(2x)^{1/8}} \times \sum_{e \in \Pi_{L-1}} \mathbb{E}_{\nu_d^W} \left[ c(e)^{1/16} \right]. \end{aligned} \quad (4.24)$$

However, by definition, for  $e = \{x, y\} \in \Pi_{L-1}$ ,  $c(e) = \frac{W\Phi(0,x)\Phi(0,y)}{\Phi(0,0)^2}$ . By (5.2) in [43], there exists  $W_-^{(d)} > 0$  such that for every  $W < W_-^{(d)}$ , there exist positive constants  $C_W$  and  $\tilde{\tau}_W$  such that for every  $x \in \mathbb{Z}^d$ ,

$$\mathbb{E}_{\nu_d^W} \left[ \Phi(0, x)^{1/4} \right] \leq C_W e^{-\tilde{\tau}_W \|x\|}. \quad (4.25)$$

In the particular case of the dimension 1, by [40], for every  $x \in \mathbb{Z}$ , it holds that

$$\Phi(0, x) \stackrel{\text{law}}{=} \prod_{k=0}^{\|x\|} A_k$$

where  $(A_k)_{k \in \mathbb{N}}$  are i.i.d random variables with inverse Gaussian distribution with parameters  $(1, W)$ . Consequently, for every  $x \in \mathbb{Z}$ ,

$$\mathbb{E}_{\nu_d^W} \left[ \Phi(0, x)^{1/4} \right] = \left( \frac{K_{1/4}(W)}{K_{1/2}(W)} \right)^{\|x\|}$$

where for every  $s \in \mathbb{R}$ ,  $K_s$  is the modified Bessel function of the second type with index  $s$ . Moreover,  $\frac{K_{1/4}(W)}{K_{1/2}(W)} < 1$  for any  $W$ . Thus,  $W_-^{(1)} = +\infty$ .

For now, let us assume that  $W < W_-^{(d)}$ . By (4.25) and Cauchy-Schwarz inequality, there exist  $C > 0$  and  $\tau > 0$  which depend on  $W$  but do not depend on  $L$  such that for any  $e \in \Pi_{L-1}$ ,

$$\mathbb{E}_{\nu_d^W} \left[ c(e)^{1/16} \right] \leq C e^{-\tau L}. \quad (4.26)$$

Combining, (4.24) and (4.26), there exists  $C' > 0$  such that

$$\begin{aligned} \nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right) &\leq \frac{E^{1/16}}{(2x)^{1/8}} dCL^{d-1} e^{-\tau L} \\ &\leq C' \frac{E^{1/16}}{(2x)^{1/8}} e^{-(\tau/2)L}. \end{aligned} \quad (4.27)$$

Now, we choose  $L = 1 + \lfloor \frac{19}{8\tau} |\ln(E)| \rfloor$ . Then, for any  $x \geq E^2/2$ ,

$$\frac{E^{1/16}}{(2x)^{1/8}} e^{-(\tau/2)L} \leq \frac{E^{1/16}}{E^{1/4}} E^{19/16} = E. \quad (4.28)$$

Therefore, if  $L = 1 + \lfloor \frac{19}{8\tau} |\ln(E)| \rfloor$ , by (4.27) and (4.28), for any  $x \geq E^2/2$ ,

$$\nu_d^W \left( \tilde{G}^{(L),D}(0,0) \leq \frac{E}{4x^2} \right) \leq C' E. \quad (4.29)$$

Consequently, using (4.29) in (4.21), we deduce there exists a constant  $C'' > 0$  such that

$$\nu_d^W \left( \frac{1}{2\gamma_2} - G^{(L),D}(0,0) \geq \frac{1}{E} \right) \leq C' E + \int_0^{E^2/2} \frac{e^{-x}}{\sqrt{\pi x}} dx \leq C'' E. \quad (4.30)$$

Recall we chose  $L = 1 + \lfloor \frac{19}{8\tau} |\ln(E)| \rfloor$ . Then, using (4.30) in (4.19), we get

$$n_{W,d}(E) \geq \frac{1}{\left(\frac{19}{8\tau} |\ln(E)|\right)^d} \left( \nu_d^W(\gamma_2 \leq E/4) - C'' E \right). \quad (4.31)$$

Moreover, as  $E \leq 1$ ,

$$\begin{aligned} \nu_d^W(\gamma_2 \leq E/4) &= \int_0^{E/4} \frac{e^{-x}}{\sqrt{\pi x}} dx \\ &\geq e^{-1/4} \int_0^{E/4} \frac{1}{\sqrt{\pi x}} dx \\ &= e^{-1/4} \frac{\sqrt{E}}{\sqrt{\pi}}. \end{aligned} \tag{4.32}$$

Together with (4.31), this concludes the proof.  $\square$

## 4.8 Proof of Proposition 4.5

**Strategy of the proof :** The proof of Proposition 4.5 is based on the fact that we can control the Green function  $\hat{G}$  associated with the ground state  $E = 0$  of  $H_\beta$ . This is always true that  $\hat{G}(0, 0) \leq \Phi(0, 0)$ . Recall from Proposition E that  $1/(2\Phi(0, 0))$  has a  $\Gamma(1/2, 1)$  distribution. This random variable has moments  $1/2 - \delta$  for every  $\delta > 0$ . This explains why we have the general bound  $\sqrt{E}$  for the density of states. However, when  $W$  is very large, by [54], this is possible to have better upper bounds for  $\hat{G}(0, 0)$ . This enables us to improve the bound on the density of states.

*Proof.*

For every  $\eta > 0$ , we have  $\nu_d^W$ -a.s.,

$$\int_0^{+\infty} \frac{1}{\eta + u} d\mu_{\delta_0}(u) = (H_\beta + \eta)^{-1}(0, 0). \tag{4.33}$$

By Theorem 2 in [154], we know that  $(H_\beta + \eta)^{-1}(0, 0) \xrightarrow{\eta \rightarrow 0} \hat{G}(0, 0)$ ,  $\nu_d^W$ -a.s. Moreover, by monotone convergence theorem,

$$\int_0^{+\infty} \frac{1}{\eta + u} d\mu_{\delta_0}(u) \xrightarrow{\eta \rightarrow 0} \int_0^{+\infty} \frac{1}{u} d\mu_{\delta_0}(u).$$

Together with (4.33), this implies that,  $\nu_d^W$ -a.s.,

$$\int_0^{+\infty} \frac{1}{u} d\mu_{\delta_0}(u) = \hat{G}(0, 0). \tag{4.34}$$

Taking the expectation we obtain,

$$\int_0^{+\infty} \frac{1}{u} d\alpha_{W,d}(u) = \mathbb{E}_{\nu_d^W} [\hat{G}(0, 0)].$$

However, recall from Theorem A that  $(\hat{G}^{(L)}(0, 0))_{L \in \mathbb{N}}$  is the bracket of the martingale  $(\psi^{(L)}(0))_{L \in \mathbb{N}}$ . Moreover, by Theorem 1 in [54] (see Lemma 9 in [154] for a more precise explanation.), if  $d \geq 3$ , there exists  $W_+^{(d)} > 0$  such that for every  $W > W_+^{(d)}$ ,  $(\psi^{(L)}(0))_{L \in \mathbb{N}}$  is bounded in  $L^2$  and thus  $\mathbb{E}_{\nu_d^W} [\hat{G}(0, 0)] < +\infty$ . Moreover, as  $\alpha_{W,d}$  is a probability measure, by Markov inequality, for every  $W > W_+^{(d)}$ ,

$$\begin{aligned} n_{W,d}(E) &\leq E \int_0^{+\infty} \frac{1}{u} d\alpha_{W,d}(u) \\ &= E \times \mathbb{E}_{\nu_d^W} [\hat{G}(0, 0)]. \end{aligned}$$

This concludes the proof.  $\square$

## 4.9 Proof of Theorem 4.6

In this section, we fix  $W \in \mathbb{R}_+^*$ . The proof of Theorem 4.6 follows the classical method of Aizenman which is explained in details in [6]. Thanks to the two first following lemmas, we prove a property of bidimensional 1/2-regularity for the potential  $\beta$ . Then we can use Theorem 8.3 of [6] and use a finite volume method which is very similar to the proof of Theorem 11.1 in [6].

For any  $x, y \in \mathbb{Z}^d$  with  $x \neq y$ , let us define  $\mathcal{F}_{x,y}$  the sigma field  $\sigma(\beta_l, l \notin \{x, y\})$ . Then we have the following lemma giving the form of the two-sites conditional density of the  $\beta$ -field. The proof follows the ideas of the one-site case but it is a little more involved.

**Lemma 4.10.** *Let us assume that  $(\beta_l)_{l \in \mathbb{Z}^d}$  has distribution  $\nu_d^W$ . Let  $x, y \in \mathbb{Z}^d$  such that  $x \neq y$ . Then, there exists random variables  $\check{W}_x, \check{W}_y, \check{W}_{x,y}$  and  $\check{\eta} = (\check{\eta}_x, \check{\eta}_y)$  which are  $\mathcal{F}_{x,y}$ -measurable such that, conditionally on  $\mathcal{F}_{x,y}$ ,  $(\zeta_x, \zeta_y) = (\beta_x - \check{W}_x/2, \beta_y - \check{W}_y/2)$  has density :*

$$\mathbf{1}\{4v_x v_y > \check{W}_{x,y}^2\} \frac{2}{\pi} \exp\left(- (v_x + v_y - \check{W}_{x,y}) - \frac{\langle \check{\eta}, R_v \check{\eta} \rangle}{8v_x v_y - 2\check{W}_{x,y}^2}\right) \frac{1}{\sqrt{4v_x v_y - \check{W}_{x,y}^2}} e^{\langle \check{\eta}, \mathbf{1} \rangle} dv_x dv_y$$

where  $\mathbf{1}$  stands for the vector  $(1, 1)$  and  $R_v$  stands for the matrix  $\begin{pmatrix} 2v_y & \check{W}_{x,y} \\ \check{W}_{x,y} & 2v_x \end{pmatrix}$ .

*Proof.*

Let  $L \in \mathbb{N}^*$  such that  $\Lambda_L$  includes  $x$  and  $y$ . For every  $l \in \Lambda_L$ , let us define

$$\hat{\eta}(l) = \sum_{k \sim l, k \in \Lambda_L^c} W.$$

Let us write  $\mathcal{F}_{x,y}^L$  the sigma-field  $\sigma(\beta_k, k \in \Lambda_L, k \notin \{x, y\})$ . By Lemma C, conditionally on  $\mathcal{F}_{x,y}^L$ ,  $(\beta_x, \beta_y)$  has distribution  $\tilde{\nu}_{\{x,y\}}^{\check{W}^{(L)}, \check{\eta}_L}$  where

$$\check{W}_{x,y}^{(L)} = W \mathbf{1}\{x \sim y\} + W^2 \sum_{\substack{l \sim x \\ l \neq y}} \sum_{\substack{k \sim y \\ k \neq x}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)$$

and if  $a \in \{x, y\}$ , then

$$\check{W}_a^{(L)} = W^2 \sum_{\substack{l \sim a \\ l \notin \{x,y\}}} \sum_{\substack{k \sim a \\ k \notin \{x,y\}}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l).$$

Moreover , if  $a \in \{x, y\}$ ,

$$\check{\eta}_L(a) = W \sum_{\substack{k \sim a \\ k \notin \{x,y\}}} \sum_{l \in \Lambda_L \setminus \{x,y\}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l) \hat{\eta}(l).$$

Recall that  $\hat{G}_{\Lambda_L \setminus \{x,y\}}$  is the inverse of  $H_\beta$  restricted to  $\Lambda_L \setminus \{x, y\}$ . By definition of  $\tilde{\nu}_{\{x,y\}}^{\check{W}^{(L)}, \check{\eta}_L}$ , the density of  $(\zeta_x, \zeta_y)$  has the right form. We only have to justify that  $\check{W}^{(L)}$  and  $\check{\eta}_L$  converge  $\nu_d^W$ -a.s when  $L$  goes to infinity. First, let us remark that  $(\check{W}^{(L)})_{L \in \mathbb{N}^*}$  has increasing components. (This can be proved thanks to path expansions. See Proposition 6 in [154].) Moreover, the components of  $(\check{W}^{(L)})_{L \in \mathbb{N}^*}$  are bounded  $\nu_d^W$ -a.s. For example, regarding the component of  $\check{W}$  at point  $x$ , for any  $L \in \mathbb{N}^*$ ,

$$\check{W}_x^{(L)} \leq W + W^2 \sum_{\substack{l \sim x \\ l \neq y}} \sum_{\substack{k \sim x \\ k \neq y}} \hat{G}(l, k) < +\infty, \nu_d^W - a.s.$$

Of course, we can do the same thing for the component of  $\check{W}$  at point  $y$ . Therefore,  $(\check{W}^{(L)})_{L \in \mathbb{N}^*}$  converges  $\nu_d^W$ -a.s. The case of  $(\check{\eta}_L)_{L \in \mathbb{N}^*}$  is more complicated. For every  $L \in \mathbb{N}^*$  and for every neighbour  $k$  of  $x$  or  $y$  which is not in  $\{x, y\}$ , let us introduce

$$\psi_{\{x,y\}}^{(L)}(k) = \sum_{\substack{l \sim x \\ l \neq y}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)W + \sum_{\substack{l \sim y \\ l \neq x}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)W + \sum_{l \in \Lambda_L \setminus \{x,y\}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)\hat{\eta}_L.$$

By Lemma 6 in [154], for every neighbour  $k$  of  $x$  or  $y$  which is not in  $\{x, y\}$ ,  $(\psi_{\{x,y\}}^{(L)}(k))_{L \in \mathbb{N}^*}$  is a positive martingale. Therefore, it converges. Moreover, remark that

$$\left( \sum_{\substack{l \sim x \\ l \neq y}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)W + \sum_{\substack{l \sim y \\ l \neq x}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)W \right)_{L \in \mathbb{N}^*}$$

is almost surely increasing and bounded. (Again, this can be proved thanks to path-expansions.) Thus it converges. Combining this with the fact that  $(\psi_{\{x,y\}}^{(L)}(k))_{L \in \mathbb{N}^*}$  converges  $\nu_d^W$ -a.s, we get that for every neighbour  $k$  of  $x$  or  $y$  which is not in  $\{x, y\}$ ,

$$\left( \sum_{l \in \Lambda_L \setminus \{x,y\}} \hat{G}_{\Lambda_L \setminus \{x,y\}}(k, l)\hat{\eta}_L \right)_{L \in \mathbb{N}^*}$$

converges  $\nu_d^W$ -a.s. This implies directly the convergence of  $(\check{\eta}_L)_{L \in \mathbb{N}^*}$ .  $\square$

Let us denote by  $\varrho_2$  the conditional distribution obtained in Lemma 4.10. If  $v_y \in \mathbb{R}$  (resp.  $v_x \in \mathbb{R}$ ), we denote by  $\varrho_2(\cdot | v_y)$  (resp.  $\hat{\varrho}_2(\cdot | v_x)$ ) the distribution  $\varrho_2$  conditionally on the fact that the second coordinate equals  $v_y$  (resp. the first coordinate equals  $v_x$ ). The following lemma states a property of bidimensional 1/2-regularity for the potential  $\beta$ . Unidimensional 1/2-regularity was already proved in Lemma 3.13.

**Lemma 4.11.** *There exists  $C > 0$  such that for every  $t \geq 0$ , for all  $v_x, v_y \in \mathbb{R}$ ,*

$$\varrho_2([v_x - t, v_x + t] | v_y) \leq Ct^{1/2} \text{ and } \hat{\varrho}_2([v_y - t, v_y + t] | v_x) \leq Ct^{1/2}.$$

*Proof.*

By symmetry, we only need to show one of the inequalities. Let  $(v_x, v_y) \in \mathbb{R}^2$ . Let  $t \geq 0$ . Let us denote by  $\tilde{\varrho}_2$  the density associated with the measure  $\varrho_2$  which has been obtained in Lemma 4.10. Then, by a change of variable,

$$\int_{v_x - t}^{v_x + t} \tilde{\varrho}_2(v, v_y) dv = \int_{v_x - \frac{\check{W}_{x,y}^2}{4v_y} - t}^{v_x - \frac{\check{W}_{x,y}^2}{4v_y} + t} \tilde{\varrho}_2(v + \check{W}_{x,y}^2/(4v_y), v_y) dv. \quad (4.35)$$

However, by Lemma 4.10,  $\tilde{\varrho}_2(v + \check{W}_{x,y}^2/(4v_y), v_y)$  is equal to

$$\mathbf{1}\{vv_y > 0\} \frac{2}{\pi\sqrt{4vv_y}} \exp \left( -v - \frac{\check{W}_{x,y}^2}{4v_y} - v_y + \check{W}_{x,y} - \frac{v_y \check{\eta}_x^2 + \left(v + \frac{\check{W}_{x,y}^2}{4v_y}\right) \check{\eta}_y^2 + \check{W}_{x,y} \check{\eta}_x \check{\eta}_y}{4vv_y} + \check{\eta}_x + \check{\eta}_y \right).$$

Remark that the expression inside the exponential above can be rewritten as

$$-\frac{1}{v} \left( v - \frac{1}{2} \left( \check{\eta}_x + \frac{\check{W}_{x,y} \check{\eta}_y}{2v_y} \right) \right)^2 - \frac{1}{v_y} \left( v_y - \frac{1}{2} (\check{\eta}_y + \check{W}_{x,y}) \right)^2. \quad (4.36)$$

Consequently, by (4.36), for every  $v \in \mathbb{R}$ ,

$$\begin{aligned} & \tilde{\varrho}_2(v + \check{W}_{x,y}^2/(4v_y), v_y) \\ &= \mathbf{1}\{vv_y > 0\} \frac{1}{\pi\sqrt{vv_y}} \exp\left(-\frac{1}{v} \left(v - \frac{1}{2} \left(\check{\eta}_x + \frac{\check{W}_{x,y}\check{\eta}_y}{2v_y}\right)\right)^2 - \frac{1}{v_y} \left(v_y - \frac{1}{2} (\check{\eta}_y + \check{W}_{x,y})\right)^2\right). \end{aligned} \quad (4.37)$$

However,

$$\mathbf{1}\{v_y > 0\} \frac{1}{\sqrt{\pi v_y}} \exp\left(-\frac{1}{v_y} \left(v_y - \frac{1}{2} (\check{\eta}_y + \check{W}_{x,y})\right)^2\right)$$

is the density of

$$\frac{1}{2 \times IG\left(\frac{1}{\check{W}_{x,y} + \check{\eta}_y}, 1\right)}$$

where  $IG(a, b)$  is an inverse Gaussian distribution with parameter  $(a, b)$ . Further, by Lemma C, this is exactly the density  $\tilde{\varrho}_1$  of  $\zeta_y$  conditionally on  $\mathcal{F}_{x,y}$ . Therefore, combining (4.37) and (4.35), we get

$$\begin{aligned} \int_{v_x-t}^{v_x+t} \tilde{\varrho}_2(v, v_y) dv &= \tilde{\varrho}_1(v_y) \times \int_{v_x - \frac{\check{W}_{x,y}^2}{4v_y} - t}^{v_x - \frac{\check{W}_{x,y}^2}{4v_y} + t} \mathbf{1}\{v > 0\} \frac{1}{\sqrt{\pi v}} \exp\left(-\frac{1}{v} \left(v - \frac{1}{2} \left(\check{\eta}_x + \frac{\check{W}_{x,y}\check{\eta}_y}{2v_y}\right)\right)^2\right) dv \\ &\leq \tilde{\varrho}_1(v_y) \times \int_{v_x - \frac{\check{W}_{x,y}^2}{4v_y} - t}^{v_x - \frac{\check{W}_{x,y}^2}{4v_y} + t} \mathbf{1}\{v > 0\} \frac{1}{\sqrt{\pi v}} dv. \end{aligned} \quad (4.38)$$

Moreover, the density of  $\varrho_2(\cdot|v_y)$  is  $v \mapsto \tilde{\varrho}_2(v, v_y)/\tilde{\varrho}_1(v_y)$ . Therefore, (4.38) implies directly that

$$\varrho_2(v_x - t, v_x + t|v_y) \leq \int_{v_x - \frac{\check{W}_{x,y}^2}{4v_y} - t}^{v_x - \frac{\check{W}_{x,y}^2}{4v_y} + t} \mathbf{1}\{v > 0\} \frac{1}{\sqrt{\pi v}} dv. \quad (4.39)$$

Further, remark that  $a \mapsto \int_{a-t}^{a+t} \mathbf{1}\{v > 0\} \frac{1}{\sqrt{\pi v}} dv$  is bounded by

$$\int_0^{2t} \frac{1}{\sqrt{\pi v}} dv = \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{t}.$$

Together with (4.39), this concludes the proof.  $\square$

For every  $x \in \mathbb{Z}^d$ , let us denote by  $\mathcal{F}_{x,x}$ , the sigma field  $\sigma(\beta_l, l \in \mathbb{Z}^d, l \neq x)$ . Lemma 4.11 implies the following lemma :

**Lemma 4.12.** *There exists a constant  $\kappa > 0$  such that for any  $W > 0$ , for any  $z \in \mathbb{C} \setminus \mathbb{R}$ , for any  $x, y \in \mathbb{Z}^d$ , for every subset  $B \subset \mathbb{Z}^d$ ,  $\nu_d^W$ -a.s.,*

$$\mathbb{E}_{\nu_d^W} \left[ |G_{B,z}(x, y)|^{1/4} | \mathcal{F}_{x,y} \right] \leq \kappa.$$

*Proof.*

This is a direct consequence of our Lemma 4.11 and Corollary 8.4 in [6].  $\square$

The following lemma states a finite-volume condition (on only one site) which will be necessary in the proof of Theorem 4.6.

**Lemma 4.13.** *For every  $W > 0$  and for every  $E > 0$ ,*

$$\mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - E|^{1/4}} \right] < +\infty.$$

Moreover, for every  $W > 0$ ,

$$\lim_{E \rightarrow +\infty} \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - E|^{1/4}} \right] = 0.$$

*Proof.*

Recall that  $2\beta_0$  is distributed as  $\frac{1}{IG(1/(2dW), 1)}$ . (By Lemma C.) This means that  $2\beta_0$  has density

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi x}} e^{-\frac{(x-2dW)^2}{2x}}. \quad (4.40)$$

Let  $E > 0$ . The finiteness of  $\mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - E|^{1/4}} \right]$  is straightforward from (4.40). Now, we can write

$$\begin{aligned} \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - E|^{1/4}} \right] &= \int_0^{+\infty} \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{(x-2dW)^2}{2x}} dx \\ &\leq e^{2dW} \times \int_0^{+\infty} \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{x}{2}} dx \\ &= e^{2dW} \times (I_1 + I_2 + I_3) \end{aligned} \quad (4.41)$$

where  $I_1 = \int_0^{E/2} f(x) dx$ ,  $I_2 = \int_{E/2}^E f(x) dx$  and  $I_3 = \int_E^{+\infty} f(x) dx$  with  $f(x) = \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{x}{2}}$ . First, let us majorize  $I_1$ .

$$\begin{aligned} I_1 &= \int_0^{E/2} \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{x}{2}} dx \\ &\leq \frac{1}{(E/2)^{1/4}} \times \int_0^{+\infty} \frac{1}{\sqrt{x}} e^{-x/2} dx. \end{aligned} \quad (4.42)$$

The bound (4.42) tends to 0 as  $E$  goes to  $+\infty$ . Now, let us look at  $I_2$ .

$$\begin{aligned} I_2 &= \int_{E/2}^E \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{x}{2}} dx \\ &\leq \frac{1}{\sqrt{2\pi}} (E/2)^{-1/2} e^{-E/4} \int_0^{E/2} \frac{1}{x^{1/4}} dx \\ &\leq \frac{4}{3\sqrt{\pi}} E^{1/4} e^{-E/4}. \end{aligned} \quad (4.43)$$

The bound (4.43) tends to 0 as  $E$  goes to  $+\infty$ . Now, let us look at  $I_3$ .

$$\begin{aligned} I_3 &= \int_E^{+\infty} \frac{1}{\sqrt{2\pi x}} \frac{1}{|x - E|^{1/4}} e^{-\frac{x}{2}} dx \\ &= \int_0^{+\infty} \frac{1}{\sqrt{2\pi(x+E)}} \frac{1}{x^{1/4}} e^{-\frac{(x+E)}{2}} dx \\ &\leq \frac{1}{\sqrt{2\pi}} E^{-1/2} e^{-E/2} \int_0^{+\infty} \frac{1}{x^{1/4}} e^{-x/2} dx. \end{aligned} \quad (4.44)$$

The upper bound (4.44) also goes to 0. Thus, this concludes the proof.  $\square$

Now, we are able to prove Theorem 4.6. The idea of the proof is not new and follows basically the lines of Theorem 11.1 in [6]. However, our setting is slightly different. Indeed, under  $\nu_d^W$ ,  $(\beta_x)_{x \in \mathbb{Z}^d}$  is not an independent potential as in [6]. For every  $z \in \mathbb{C} \setminus \mathbb{R}$ , let us introduce

$$t(|z|) = 2d\kappa W^{1/2} |\vec{E}(\{0\}^+)| \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_0 - |z||^{1/4}} \right]$$

where  $\kappa$  is the constant introduced in Lemma 4.12 and  $|\vec{E}(\{0\}^+)|$  is the number of oriented edges going from  $\{0\}^+$  to its complement.

*Proof.*

Let  $W > 0$ . By Lemma 4.13, there exists  $E_{d,W} > 0$  such that for every  $E \geq E_{d,W}$ ,  $t(E) \leq 1/2$ . Let  $z \in \mathbb{C} \setminus \mathbb{R}$  with modulus  $|z| \geq E_{d,W}$ . Let  $x$  and  $y$  be two points of  $\mathbb{R}^d$ . We assume that  $\|x - y\|_1 \geq 3$ . (Otherwise, the following procedure will not make any sense.) Now, let us apply Proposition G with  $B = \mathbb{Z}^d$  and  $A = \{x\}$  and the fact that for  $s \in (0, 1)$ , for any  $n \in \mathbb{N}^*$  and for any  $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}_+^n$ ,

$$\left( \sum_{i=1}^n \lambda_i \right)^s \leq \sum_{i=1}^n \lambda_i^s.$$

We obtain

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] \leq W^{1/2} \sum_{\substack{u' \sim x \\ (v', v) \in \vec{E}(\{x\}^+)}} \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_x - z|^{1/4}} |G_z(u', v')|^{1/4} |G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)|^{1/4} \right]. \quad (4.45)$$

However, by Lemma 4.12, for every  $u', v' \in \mathbb{Z}^d$ ,  $\nu_d^W$ -a.s.,

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(u', v')|^{1/4} |F_{u', v'}| \right] \leq \kappa. \quad (4.46)$$

Moreover, for every  $(u, u') \in \vec{E}(\{x\})$  and for every  $(v', v) \in \vec{E}(\{x\}^+)$ ,  $2\beta_x$  and  $G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)$  are measurable with respect to  $F_{u', v'}$ . Consequently, (4.45) and (4.46) yield

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] \leq W^{1/2} \kappa \sum_{\substack{u' \sim x \\ (v', v) \in \vec{E}(\{x\}^+)}} \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_x - z|^{1/4}} |G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)|^{1/4} \right]. \quad (4.47)$$

Moreover, under  $\nu_d^W$ , the  $\beta$ -field is 1-dependent. Further, remark that the sets  $\{x\}$  and  $\mathbb{Z}^d \setminus \{x\}^+$  are not related by any edge of  $\mathbb{Z}^d$ . Therefore, by (4.47), we get,

$$\begin{aligned} \mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] &\leq 2dW^{1/2} \kappa \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_x - z|^{1/4}} \right] \sum_{(v', v) \in \vec{E}(\{x\}^+)} \mathbb{E}_{\nu_d^W} \left[ |G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)|^{1/4} \right] \\ &\leq 2dW^{1/2} \kappa \mathbb{E}_{\nu_d^W} \left[ \frac{1}{|2\beta_x - |z||^{1/4}} \right] \sum_{(v', v) \in \vec{E}(\{x\}^+)} \mathbb{E}_{\nu_d^W} \left[ |G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)|^{1/4} \right] \\ &\leq t(|z|) \frac{\sum_{(v', v) \in \vec{E}(\{x\}^+)} \mathbb{E}_{\nu_d^W} \left[ |G_{\mathbb{Z}^d \setminus \{x\}^+, z}(v, y)|^{1/4} \right]}{|\vec{E}(\{x\}^+)|}. \end{aligned} \quad (4.48)$$

Remark that we can iterate (4.48) at least  $\frac{\|x-y\|_1}{2}$  times. (We can not iterated this method an infinite number of times because it is possible that  $\|v - y\|_1 < 3$ .) Then, there exists a finite set  $I$ , a family  $(v_i)_{i \in I} \in (\mathbb{Z}^d)^I$  and a family  $(A_i)_{i \in I}$  of subsets of  $\mathbb{Z}^d$  such that

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] \leq t(|z|)^{\|x-y\|_1/2} \times \frac{\sum_{i \in I} \mathbb{E}_{\nu_d^W} \left[ |G_{A_i, z}(v_i, y)|^{1/4} \right]}{|I|}. \quad (4.49)$$

Then, combining (4.46) and (4.49), we get

$$\mathbb{E}_{\nu_d^W} \left[ |G_z(x, y)|^{1/4} \right] \leq \kappa \times t(|z|)^{\|x-y\|_1/2}. \quad (4.50)$$

Moreover, recall that  $t(|z|) \leq 1/2$  because  $|z| \geq E_{d,W}$ . This proves that  $G_z$  is exponentially localized if  $|z| \geq E_{d,W}$ . This is exactly what we wanted to prove. Then, the analogous result where  $z$  is replaced by  $E + i0$  stems from Fatou's lemma.  $\square$

## Chapitre 5

# One-dimensional scaling limit of the operator $H_\beta$

### Abstract

This chapter is based on a joint work with Christophe Sabot which should be published in a forthcoming paper. Following the previous approach of Lupu, Sabot and Tarrès in [113], we focus on the scaling-limit of the potential  $\beta$  associated with the VRJP on one-dimensional graphs as the line or circles. By considering a relevant scaling of  $\beta$ , we obtain a new surprising proof of the Matsumoto-Yor properties which were proved in [122]. We use also the scaling-limit of  $\beta$  in order to prove new identities in law involving exponential functionals of the Brownian motion. Moreover, by means of the same scaling, we can construct a continuous-space version of  $H_\beta$  on circles and on  $\mathbb{R}$  whereas in [113] they constructed only the continuous-space version of the random environment of the VRJP. Furthermore, we compute the integrated density of states of this operator on  $\mathbb{R}$ .

### 5.1 Introduction

This paper is concerned with the construction and investigation of scaling limits of the random Schrödinger operator associated with the Vertex Reinforced Jump Process (VRJP) on some one-dimensional sets (the one-dimensional torus and the real line). We start by briefly recalling the definition of the VRJP : let  $(V, E)$  be a non-directed locally finite graph and  $(W_{i,j})_{i,j \in V}$  be a family of non-negative conductances such that  $W_{i,j} > 0$  if and only if  $\{i, j\} \in E$ . The VRJP is the continuous self-reinforced random walk  $(Y_s)_{s \geq 0}$  which is defined as follows : the VRJP starts from some vertex  $i_0 \in V$  and conditionally on the past before time  $s$ , it jumps from a vertex  $i$  to one of its neighbour  $j$  at rate  $W_{i,j}L_j(s)$  where

$$L_j(s) = 1 + \int_0^s \mathbf{1}\{Y_u = s\} du.$$

In [152], it was shown, firstly, that the VRJP is closely related to the Edge Reinforced Random Walk, a famous reinforced process introduced by Diaconis and Coppersmith in the 80's, and secondly, that after some time change the VRJP can be represented as a mixture of Markov Jump Processes with a mixing measure given by a marginal of a supersymmetric sigma-field called the  $\mathbb{H}^{2|2}$  model which was introduced by Zirnbauer in [56, 172] and investigated by Disertori, Spencer and Zirnbauer in [54, 53]. A complementary, but closely related, representation of the VRJP in terms of a random Schrödinger operator was provided in [153] on finite graphs. More precisely, if

$(V, E)$  is a non-directed finite graph and  $(W_{i,j})_{i,j \in V}$  some conductances on the edges, then for any potential  $\beta = (\beta_i)_{i \in V}$  on the vertices, we define the discrete Schrödinger operator  $(H_\beta(i, j))_{i,j \in V}$  by

$$H_\beta(i, j) = \mathbf{1}\{i = j\}2\beta_i - \mathbf{1}\{i \sim j\}W_{i,j}.$$

In [153], an explicit probability measure on the set of potentials  $\beta$  is defined (see Proposition 5.1 below). This measure lives on the set where  $H_\beta$  is positive definite, it has several remarkable properties which we recall in section 5.2.2, and it gives a representation of the VRJP in the following sense. After some time-change, the VRJP starting at a vertex  $i_0$  is a mixture of Markov jump process with jump rates from  $i$  to  $j$  given by :

$$\frac{1}{2}W_{i,j} \frac{G_\beta(i_0, j)}{G_\beta(i_0, i)},$$

where  $\beta$  is the random potential and  $G_\beta = (H_\beta)^{-1}$ . That representation, and its generalization to infinite graphs (see [154]), has played an important role in order to understand the asymptotic behavior of the VRJP (see in particular [154, 139]).

In this paper, we will be mainly concerned by that representation and its scaling limits. The question of the scaling limits of the VRJP and its representation is rather natural, but remains still quite mysterious. In [113], the scaling limit of the VRJP itself was analysed. With this goal, the scaling limit of its mixing field, i.e. of the limit of the function  $(G_\beta(0, [nt])/G_\beta(0, 0))_{t \in \mathbb{R}}$  was described on the real line in terms of the geometric Brownian motion. Here, we investigate the scaling limit of  $H_\beta$  and  $G_\beta$  as random operators, both on the one-dimensional torus and the real line. More precisely, the main results of the paper are summarized below :

- By scaling limits, we construct an explicit continuous-space version of the operator  $H_\beta$  and its inverse  $G_\beta$  on circles, and on the real line. We describe the domains of these operators on the circle, while we still face some technical difficulties to describe the domains in the case of the real line. However, we hope to solve this problem in the near future. Note that the one-dimensional torus is not a tree, which induces specific difficulties (the representation of the VRJP on trees is considerably simpler), and that in the case of the real line, we get the full description of the operators, while in [113], only the scaling limit of  $(G_\beta(0, [nt])/G_\beta(0, 0))_{t \geq 0}$  was considered in order to analyse the continuous-space VRJP.
- A natural problem regarding  $H_\beta$  consists in investigating its spectral properties as a self-adjoint operator. In particular, spectral properties of self-adjoint operators are very important, for example if we want to understand the dynamical properties of a quantum particle whose wave-function follows the Schrödinger equation. In this paper, we compute the exact density of states of the limiting operator  $H_\beta$  on  $\mathbb{R}$ , which has a surprisingly simple form. To do so, we follow a method which is very similar with what is done in [71] for the continuous Anderson model.
- In the discrete-space case, many identities in law involving the  $\beta$  potential are known. Taking the scaling-limit in these formulas, we prove new identities in law involving exponential functionals of the Brownian motion. These identities generalize the famous Dufresne identity (originally proved in [57]) which states that

$$\int_0^\infty e^{2\alpha_s - s} ds \stackrel{\text{law}}{=} \frac{1}{2\gamma}$$

where  $\alpha$  is a Brownian motion and  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ .

- Considering the scaling limit of the  $\beta$ -potential on  $\mathbb{N}^*$ , we give a new proof of the Matsumoto-Yor properties (see [121] and [122]), which concern exponential functionals of the Brownian motion. More precisely, we give a discrete time version of the Matsumoto-Yor properties which involves natural functions of the  $\beta$ -potential. Specified to our context, the Matsumoto-Yor properties state that the process  $Z_t = \frac{T_t}{e_t}$ , where  $e$  is the geometric

Brownian motion and  $T_t = \int_0^t e_s^2 ds$ , is a Markov process in its own filtration. Moreover the filtration of  $Z$  is strictly smaller than the filtration of  $e$ , and there is an explicit intertwining between  $e$  and  $Z$  involving Inverse Gaussian distributions. While somehow mysterious at first sight, Matsumoto-Yor properties seem to be rather fundamental and have been generalized in different directions, in particular in relation with properties of Lie groups and solvable polymer models (see [134, 23, 22]). Note that some Matsumoto-Yor properties on graphs also appear in a different way in Chapter 6 of this thesis.

Finally, let us mention some related works concerning a different operator, the continuous Anderson model on  $\mathbb{R}$  (where the random potential is given by a white noise). In [58], Dumaz and Labbé gave a very accurate description of the spectrum and of the eigenstates for this operator. It is probably possible to apply their ideas in order to give the precise behaviour of the spectrum of the continuous-space version of  $H_\beta$ . However we do not do it in this article. Furthermore, an interesting question would be to find continuous versions of  $H_\beta$  on topological spaces which are not one-dimensional. In the case of the Anderson model, in [105], Labbé managed to do it on  $(-L, L)^d$  with  $d \in \{1, 2, 3\}$  and  $L > 0$ . It would be interesting to know if it is possible to do the same thing for  $H_\beta$  but this question remains open for now.

## 5.2 Context and statement of the results

### 5.2.1 General notation

For every  $n \in \mathbb{N}^*$ ,  $\mathcal{C}_n$  denotes the circle graph with  $2n + 1$  points. More precisely, the vertex set of  $\mathcal{C}_n$  is  $\{-n, \dots, 0, \dots, n\}$  and for every  $i \in \{-n, \dots, 0, \dots, n-1\}$  there is an edge between  $i$  and  $i+1$  and there is an edge between  $n$  and  $-n$ . In any graph  $(V, E)$  the fact that two vertices  $i$  and  $j$  are related by an edge is denoted by  $i \sim j$ .

If  $V_1$  and  $V_2$  are two finite sets and  $H$  is a matrix indexed by  $V_1 \times V_2$ , we denote the coefficient of  $H$  at  $(i, j) \in V_1 \times V_2$  by  $H(i, j)$ . If  $H$  is a squared symmetric matrix on a set  $V$ , then we write  $H > 0$  (resp.  $H \geq 0$ ) when  $H$  is positive definite (resp. when  $H$  is non-negative). If  $H$  is a squared matrix, its determinant is denoted by  $|H|$ . If  $V$  is a finite set and  $v_1$  and  $v_2$  are two vectors of  $\mathbb{R}^V$ , the standard scalar product between  $v_1$  and  $v_2$  is denoted by  $\langle v_1, v_2 \rangle$ . If  $H$  is a matrix on a finite set  $V \times V$  and if  $V_1$  and  $V_2$  are two subsets of  $V$ , the restriction of  $H$  to  $V_1 \times V_2$  is denoted by  $H_{V_1, V_2}$ . Moreover, if  $v$  is a vector of  $\mathbb{R}^V$  and  $V_1$  is a subset of  $V$ , then the restriction of  $v$  to  $V_1$  is denoted by  $v_{V_1}$ .

We denote by  $\mathcal{C}^{(\lambda)}$  the continuous circle  $\mathbb{R}/2\lambda\mathbb{R}$ . For sake of convenience, we will often write  $\mathcal{C}^{(\lambda)}$  as  $[-\lambda, \lambda]$  (for example when we write integrals) where it is implicit that  $-\lambda$  and  $\lambda$  are topologically identified. If  $t, t' \in \mathcal{C}^{(\lambda)}$  we write sometimes  $t \leq t'$  or  $t < t'$  referring to the natural order on  $[-\lambda, \lambda]$ .

Recall that an Inverse Gaussian distribution with parameters  $(\mu, \lambda)$  has density

$$\mathbf{1}\{x > 0\} \frac{\sqrt{\lambda}}{\sqrt{2\pi x^3}} e^{-\lambda \frac{(x-\mu)^2}{2x\mu^2}} dx.$$

The Inverse Gaussian distribution with parameters  $(\mu, \lambda)$  will be denoted by  $IG(\mu, \lambda)$ . Recall that  $\mathbb{E}[IG(\mu, \lambda)] = \mu$  and  $\mathbb{E}[IG(\mu, \lambda)^2] = \mu^2 + \frac{\mu^3}{\lambda}$ .

### 5.2.2 The random potential $\beta$

Let  $(V, E)$  be a finite graph with  $n$  vertices. Let  $W$  be a matrix of symmetric non-negative weights  $(W_{i,j})_{i,j \in V \times V}$  such that  $W_{i,j} > 0$  if and only if  $\{i, j\} \in E$ . For every  $\beta \in \mathbb{R}_+^V$ , let us define the matrix  $H_\beta$  on  $V \times V$  such that for every  $i, j \in V$ ,

$$H_\beta(i, j) = \mathbf{1}\{i = j\} 2\beta_i - W_{i,j}. \quad (5.1)$$

For every  $\beta \in \mathbb{R}_+^V$  such that  $H_\beta$  is positive definite, one can define its inverse  $G_\beta$  which has only positive entries. In [153], in order to study the VRJP, Sabot, Tarrès and Zeng introduced a probability measure  $\nu_V^W$  on  $\mathbb{R}_+^V$  which will be crucial in this paper. It is defined by means of the following proposition :

**Proposition 5.1** (Proposition 1 and Theorem 3 in [153]).

(i) *The function*

$$\beta \mapsto \left(\frac{2}{\pi}\right)^{n/2} \mathbf{1}\{H_\beta > 0\} e^{-\frac{1}{2}\langle 1, H_\beta 1 \rangle} \frac{1}{\sqrt{|H_\beta|}} \quad (5.2)$$

is a density. In the formula (5.2),  $1$  stands for the vector of  $\mathbb{R}^V$  whose entries are all equal to 1. There exists a probability measure  $\nu_V^W$  on  $\mathbb{R}_+^V$  which is associated with the density of equation (5.2).

(ii) *One can compute explicitly the Laplace transform of  $\nu_V^W$ . For every  $t \in \mathbb{R}_+^V$ ,*

$$\int e^{-\langle t, \beta \rangle} \nu_V^W(d\beta) = \exp\left(-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} \left(\sqrt{(t_i+1)(t_j+1)} - 1\right)\right) \prod_{i \in V} \frac{1}{\sqrt{1+t_i}}. \quad (5.3)$$

In this article, we will often make a small abuse of notation by using the notation  $\beta$  to designate a random vector and a variable inside the density of this random vector. Now, let  $\beta$  be a random vector with distribution  $\nu_V^W$ .

(iii) *For every  $i \in V$ ,  $1/(2\beta_i - W_{i,i})$  is an Inverse Gaussian ditribution with parameters*

$$\left(\frac{1}{\sum_{i \neq j} W_{i,j}}, 1\right).$$

(iv) *The random potential  $\beta$  is 1-dependent, that is, if  $V_1$  and  $V_2$  are two subsets of  $V$  which are not related by an edge, then  $(\beta_i)_{i \in V_1}$  and  $(\beta_i)_{i \in V_2}$  are independent.*

(v) *For every  $i \in V$ ,*

$$G_\beta(i, i) \stackrel{\text{law}}{=} \frac{1}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ .

In [153], Sabot, Tarrès and Zeng used the  $\beta$ -potential in order to study the VRJP on any finite graph  $V$ . They showed that the VRJP (actually a time-changed version of the VRJP) on  $V$  starting from  $i_0 \in V$  with weight matrix  $W$  is a mixture of random processes which jumps from  $i$  to  $j$  at rate

$$\frac{W_{i,j} G_\beta(i_0, j)}{2 G_\beta(i_0, i)}.$$

It is natural to wonder whether there exists a similar representation of the VRJP on infinite graphs. In [154], Sabot and Zeng proved that this is possible to extend the measure  $\nu_V^W$  on an infinite graph which enables us to look at the interesting case of  $\mathbb{Z}^d$ .

**Proposition 5.2** (Section 4.2 in [154]). *Let  $(V, E, W)$  be an infinite locally finite graph with conductances where  $W$  is a symmetric conductance operator  $(W_{i,j})_{i,j \in V}$  such that  $W_{i,j} > 0$  if and only if  $\{i, j\} \in E$ . Then, there exists an infinite-volume measure  $\nu_V^W$  on  $\mathbb{R}_+^V$  such that for every finite subset  $V_1$  which is included in  $V$ , for any  $t \in \mathbb{R}_+^{V_1}$ ,*

$$\int e^{-\langle t, \beta \rangle} \nu_V^W(d\beta) = e^{-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} (\sqrt{(t_i+1)(t_j+1)} - 1) - \sum_{\{i,j\} \in E, j \notin V_1} W_{i,j} (\sqrt{t_i+1} - 1)} \prod_{i \in V_1} \frac{1}{\sqrt{1+t_i}}. \quad (5.4)$$

Now, let  $(V_n)_{n \in \mathbb{N}}$  be an increasing sequence of boxes such that  $\bigcup_{n \in \mathbb{N}} V_n = V$ . For every  $n \in \mathbb{N}$ , for every  $i, j \in V$  let us define

$$\hat{G}_\beta^{(n)}(i, j) = (H_\beta)_{V_n, V_n}^{-1}(i, j)$$

if  $i, j \in V_n$  and  $\hat{G}_\beta^{(n)}(i, j) = 0$  otherwise. Moreover, for every  $n \in \mathbb{N}$ , let us define  $\psi_\beta^{(n)}$  as the unique solution of the equation

$$\begin{aligned} (H_\beta \psi_\beta^{(n)})(i) &= 0, & \text{for every } i \in V_n, \\ \psi_\beta^{(n)}(i) &= 1, & \text{for every } i \in V_n^c. \end{aligned}$$

Note that for every  $n \in \mathbb{N}$ , and for every  $i \in V_n$ , there is another useful expression of  $\psi_\beta^{(n)}(i)$  which is

$$\psi_\beta^{(n)}(i) = \sum_{j \in V_n} \hat{G}_\beta^{(n)}(i, j) \eta_j^{(n)}$$

where for every  $j \in V_n$ ,  $\eta_j^{(n)} = \sum_{k \sim j, j \notin V_n} W_{j,k}$ . These objects were introduced by Sabot and Zeng in [154] and were crucial in order to study the VRJP on infinite graphs because of the following proposition :

**Proposition 5.3** (Proposition 9 and Theorem 1 in [154]). *Let  $\beta \sim \nu_V^W$ . It holds that,*

- (i) *For every  $i \in V$ ,  $(\psi_\beta^{(n)}(i))_{n \in \mathbb{N}}$  is a non-negative martingale. In particular, it has almost surely a limit  $\psi_\beta(i)$ .*
- (ii) *The bracket of  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  is  $(\hat{G}_\beta^{(n)})_{n \in \mathbb{N}}$  in the sense that for every  $i, j \in V$ ,  $(\psi_\beta^{(n)}(i)\psi_\beta^{(n)}(j) - \hat{G}_\beta^{(n)}(i, j))_{n \in \mathbb{N}}$  is a martingale.*
- (iii) *The VRJP is almost surely recurrent if and only if almost surely  $\psi_\beta(i) = 0$  for every  $i \in V$ .*
- (iv) *The VRJP is almost surely transient if and only if almost surely  $\psi_\beta(i) > 0$  for every  $i \in V$ .*
- (v) *Let  $\gamma$  be a Gamma random variable with parameters  $(1/2, 1)$  which is independent of  $\beta$ . For every  $i, j \in V$ , let us define*

$$G_{\beta, \gamma}(i, j) = \hat{G}_\beta(i, j) + \frac{1}{2\gamma} \psi_\beta(i) \psi_\beta(j).$$

*Then, the VRJP on  $V$  starting from  $i_0 \in V$  with weight matrix  $W$  is a mixture of random processes which jumps from  $i$  to  $j$  at rate*

$$\frac{W_{i,j} G_{\beta, \gamma}(i_0, j)}{2 G_{\beta, \gamma}(i_0, i)}.$$

In this article, we want to understand what is going on when we consider a scaling limit for the  $\beta$ -field on a one-dimensional graph. We will show that it is possible to define a continuous version of  $H_\beta$  on  $\mathbb{R}$  or continuous circles. But before that, we prove a discrete-time version of the Matsumoto-Yor properties by means of the  $\beta$ -field on  $\mathbb{N}^*$ ,  $(\psi_\beta^{(n)})_{n \in \mathbb{N}}$  and  $(\hat{G}_\beta^{(n)})_{n \in \mathbb{N}}$ . We will take a scaling limit in this discrete-time version in order to recover the continuous-time version of the Matsumoto-Yor properties. This strong connection between the  $\beta$ -field and the Matsumoto-Yor properties explains the origin of the surprising identities in law in the forthcoming subsection 5.2.4.4.

### 5.2.3 A new approach of the Matsumoto-Yor properties in relation with the mixing measure of the VRJP

First let us recall the Matsumoto-Yor properties. Let  $\alpha$  be a standard Brownian motion on  $\mathbb{R}_+$ . Then we can define the associated geometric Brownian motion  $e$  as  $(e_t)_{t \geq 0} = (\exp(\alpha_t - t/2))_{t \geq 0}$ . Moreover, let us define the related exponential functionals  $T$  and  $Z$  such that for every  $t > 0$ ,

$$T_t = \int_0^t e_s^2 ds \text{ and } Z_t = \frac{T_t}{e_t}. \quad (5.5)$$

For every  $t \geq 0$ , we define two sigma-fields  $\mathcal{A}_t = \sigma(\alpha_s, s \leq t)$  and  $\mathcal{Z}_t = \sigma(Z_s, s \leq t)$ . Then, Matsumoto and Yor proved the following results :

**Theorem 5.4** (Theorem 1.6 and Proposition 1.7 in [122].).

(i) For every  $t > 0$ ,  $\mathcal{Z}_t \subsetneq \mathcal{A}_t$ .

(ii)  $Z$  is a diffusion process whose infinitesimal generator is

$$\frac{1}{2} z^2 \frac{d^2}{dz^2} + (1+z) \frac{d}{dz}.$$

(iii) For every  $t > 0$ , the conditional distribution of  $e_t$  knowing  $\mathcal{Z}_t$  is an Inverse Gaussian distribution with parameters  $(1, 1/Z_t)$ . More precisely, for every  $t > 0$ , the conditional distribution of  $e_t$  knowing  $\mathcal{Z}_t$  is an Inverse Gaussian distribution with parameter  $(1, 1/Z_t)$ , i.e. it has density

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi Z_t x^3}} e^{-\frac{1}{Z_t} \frac{(x-1)^2}{2x}} dx.$$

Theorem 5.4 will be called "Matsumoto-Yor properties" in the sequel of this paper. Now, let us give a discrete-time counterpart of those Matsumoto-Yor properties. Now, let  $m > 0$  and let  $K_m$  be a weight operator on the line graph  $\mathbb{N}^*$  such that for every  $i \in \mathbb{N}^*$ ,  $K_m(i, i+1) = K_m(i+1, i) = m$ . All other entries of  $K_m$  are zero. Then we can define the random operator  $\mathbf{H}_\beta^{(m)}$  on the discrete half-line  $\mathbb{N}^*$  associated with the random field  $\beta \sim \nu_{\mathbb{N}^*}^{K_m}$ . We write  $\mathbf{H}_\beta^{(m)}$  in bold letters in order to avoid the confusion with  $H_\beta^{(\lambda, n)}$  on the discrete circle which shall be introduced later. Now, for every  $n \in \mathbb{N}^*$ , let us define  $\hat{\mathbf{G}}_\beta^{(n, m)} = \left( (\mathbf{H}_\beta^{(m)})_{[[1, n], [[1, n]]} \right)^{-1}$ . For every  $n \in \mathbb{N}^*$ , we define also

$$\psi_\beta^{(n, m)} = \hat{\mathbf{G}}_\beta^{(n, m)}(1, n)m \text{ and } Z_\beta^{(n, m)} = \frac{\hat{\mathbf{G}}_\beta^{(n, m)}(1, 1)}{\psi_\beta^{(n, m)}}.$$

For every  $n \in \mathbb{N}^*$ , we define

$$\mathcal{A}_{n, m} = \sigma(\psi_\beta^{(k, m)}, 1 \leq k \leq n) \text{ and } \mathcal{Z}_{n, m} = \sigma(Z_\beta^{(k, m)}, 1 \leq k \leq n).$$

By Proposition 5.3,  $(\psi_\beta^{(n, m)})_{n \in \mathbb{N}^*}$  is a martingale whose bracket is  $(\hat{\mathbf{G}}_\beta^{(n, m)}(1, 1))_{n \in \mathbb{N}^*}$ . Remark that it is analogous to the case of the geometric Brownian motion in equation (5.5) because  $(e_t)_{t \geq 0}$  is a martingale whose bracket is  $(T_t)_{t \geq 0}$ . The interest of these discrete objects is that they give a discrete version of the results of Matsumoto and Yor :

**Proposition 5.5** (Discrete version of the Matsumoto-Yor properties). *Let  $m \in \mathbb{N}^*$  be fixed.*

(i) For every  $n \in \mathbb{N}^*$ ,  $\mathcal{Z}_{n, m} \subsetneq \mathcal{A}_{n, m}$ .

(ii)  $\left( Z_\beta^{(n, m)} \right)_{n \in \mathbb{N}^*}$  is a Markov process. More precisely, for every  $n \in \mathbb{N}^*$ , the law of  $Z_\beta^{(n+1, m)}$  conditionally on  $\mathcal{Z}_{n, m}$  is

$$\frac{Z_\beta^{(n, m)}}{m} \times \frac{1}{IG \left( \frac{1}{m + \frac{1}{Z_\beta^{(n, m)}}}, 1 \right)}.$$

(iii) For every  $n \in \mathbb{N}^*$ , the conditional distribution of  $\psi_\beta^{(n,m)}$  knowing  $\mathcal{Z}_{n,m}$  is an Inverse Gaussian distribution with parameters  $(1, 1/Z_\beta^{(n,m)})$ . More precisely, for every  $n \in \mathbb{N}^*$  the conditional density of  $\psi_\beta^{(n,m)}$  knowing  $\mathcal{Z}_{n,m}$  is

$$\mathbf{1}\{x > 0\} \frac{1}{\sqrt{2\pi Z_\beta^{(n,m)} x^3}} e^{-\frac{1}{Z_\beta^{(n,m)}} \frac{(x-1)^2}{2x}} dx.$$

Remark that Theorem 5.4 and Proposition 5.5 are very similar. Actually, we can recover Theorem 5.4 by taking a scaling limit in Proposition 5.5 thanks to the following proposition :

**Proposition 5.6.** *Let  $(\tilde{\psi}^{(m)}(t))_{t \geq 0}$  and  $(\tilde{Z}^{(m)}(t))_{t \geq 0}$  be the continuous linear interpolations of  $(\psi_\beta^{(\lfloor mt \rfloor, m)})_{t \geq 0}$  and  $(Z_\beta^{(\lfloor mt \rfloor, m)})_{t \geq 0}$ . Then, the following convergence does hold for the topology of uniform convergence on compact sets :*

$$\left( \tilde{\psi}^{(m)}(t), \tilde{Z}^{(m)}(t) \right)_{t \geq 0} \xrightarrow[m \rightarrow +\infty]{law} (e_t, Z_t)_{t \geq 0}.$$

In section 5.5, we will prove Propositions 5.5 and 5.6. Moreover, we will use these results in order to give a new proof of Theorem 5.4.

**Remark 5.1.** *Thanks to the discrete processes  $(\psi_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  and  $(Z_\beta^{(n,m)})_{n \in \mathbb{N}^*}$ , we were able to define a discrete one-dimensional analogue of Matsumoto-Yor exponential functionals of the Brownian motion and we recovered the results of Matsumoto and Yor by taking a scaling limit. However, the processes  $(\psi_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  and  $(Z_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  can be constructed on any graph and one could prove (i) and (iii) of Proposition 5.5 on any graph. (However, we do not know whether (ii) is true on a general graph.) Consequently, we can define  $(\psi_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  and  $(Z_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  for example on  $\mathbb{Z}^d$  with any  $d \geq 2$ . If one could take a scaling limit on  $\mathbb{Z}^d$ , it would give new continuous-time processes which should exhibit properties like (i) and (iii) in Theorem 5.4.*

## 5.2.4 Scaling limit and continuous version of $H_\beta$ on the circle

### 5.2.4.1 Definition of the discrete operator on the circle

The main goal of this paper is to define a version of  $H_\beta$  and  $G_\beta$  on continuous unidimensional spaces. In order to do this, we define a model on a discretized version of the circle and we will make the size of the mesh go to 0. Let  $n \in \mathbb{N}^*$ . Let  $\lambda > 0$ . Let  $W_n^{(\lambda)}$  be a matrix on the discretized circle  $\mathcal{C}_{[\lambda n]}$  such that  $(W_n^{(\lambda)})_{i,j}$  is 0 if  $i$  and  $j$  are not connected and is  $n$  otherwise. Let us denote  $H_\beta^{(\lambda,n)}$  the matrix associated with the random potential  $\beta$  with distribution  $\nu_{\mathcal{C}_{[\lambda n]}}^{W_n^{(\lambda)}}$ . Moreover, we denote by  $G_\beta^{(\lambda,n)}$  the inverse of  $H_\beta^{(\lambda,n)}$ . We define also a rescaled continuous bilinear interpolation  $(\tilde{G}_\beta^{(\lambda,n)})_{t,t' \in \mathcal{C}(\lambda)}$  of  $(G_\beta^{(\lambda,n)}(\lfloor nt \rfloor, \lfloor nt' \rfloor))_{t,t' \in \mathcal{C}(\lambda)}$ . More precisely, if  $i/n \leq t < (i+1)/n$  and  $j/n \leq t' < (j+1)/n$ ,

$$\begin{aligned} \tilde{G}_\beta^{(\lambda,n)}(t, t') &= G_\beta^{(\lambda,n)}(i, j) + n(t - i/n) \left( G_\beta^{(\lambda,n)}(i+1, j) - G_\beta^{(\lambda,n)}(i, j) \right) \\ &\quad + n(t' - j/n) \left( G_\beta^{(\lambda,n)}(i, j+1) - G_\beta^{(\lambda,n)}(i, j) \right) \\ &\quad + n^2(t' - j/n)(t - i/n) \left( G_\beta^{(\lambda,n)}(i, j) + G_\beta^{(\lambda,n)}(i+1, j+1) - G_\beta^{(\lambda,n)}(i, j+1) - G_\beta^{(\lambda,n)}(i+1, j) \right). \end{aligned} \tag{5.6}$$

### 5.2.4.2 Definition of the continuous limit

Let  $B$  be a Brownian motion on  $\mathbb{R}$  such that  $B(0) = 0$  almost surely. We define the geometric Brownian motion  $M$  by

$$(M_t)_{t \in \mathbb{R}} = (e^{B_t - t/2})_{t \in \mathbb{R}}.$$

Let  $\lambda > 0$ . Then, we introduce the symmetric random kernel  $\mathcal{G}^{(\lambda)}$  on  $\mathcal{C}^{(\lambda)}$  by

$$\mathcal{G}^{(\lambda)}(t, t') = \frac{M_{t'} M_t}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_{t'}^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^{t'} \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right)$$

for every  $t \leq t' \in \mathcal{C}^{(\lambda)}$ .

### 5.2.4.3 Results of convergence

First, the rescaled continuous interpolation  $\tilde{G}_\beta^{(\lambda, n)}$  of the matrix  $G_\beta^{(\lambda, n)}$  has a limit in law when  $n$  goes to infinity.

**Theorem 5.7.** *Let  $\lambda > 0$ . Then*

$$\tilde{G}_\beta^{(\lambda, n)} \xrightarrow[n \rightarrow +\infty]{law} \mathcal{G}^{(\lambda)}$$

for the topology of uniform convergence on  $(\mathcal{C}^{(\lambda)})^2$ .

Moreover,  $\mathcal{G}^{(\lambda)}$  can be seen as a bilinear form with the following expression.

**Proposition 5.8.** *Let  $\lambda > 0$ . Let  $f \in L^2([-\lambda, \lambda])$ . Then*

$$\begin{aligned} & \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda f(t) \mathcal{G}^{(\lambda)}(t, t') \bar{f}(t') dt dt' \\ &= \frac{1}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{1}{M_u^2} \left| M_{-\lambda} \int_u^\lambda f(t) M_t dt + M_\lambda \int_{-\lambda}^u f(t) M_t dt \right|^2 du. \end{aligned}$$

In particular,  $\mathcal{G}^{(\lambda)}$  is positive definite almost surely.

### 5.2.4.4 Dufresne's type identities in law

In [57] (see also [167] for an alternative proof), Dufresne proved the following famous identity in law :

$$\int_0^{+\infty} e^{2\alpha s - s} ds \stackrel{law}{=} \frac{1}{2\gamma}$$

where  $\gamma$  has Gamma distribution with parameters  $(1/2, 1)$ . Recall that, by Proposition (5.1), in the discrete-time setting,  $G_\beta^{(\lambda, n)}(i, i)$  is also distributed as the inverse of a Gamma distribution for every  $i \in \mathcal{C}_{[\lambda n]}$ . Actually, this is not a coincidence and one can recover Dufresne's identity by making  $n$  go to infinity in  $G_\beta^{(\lambda, n)}(i, i)$ . Moreover, by means of the limiting random kernel  $\mathcal{G}^{(\lambda)}$ , we can prove some new identities in law which generalize Dufresne's identity.

**Proposition 5.9.** *Let  $t \in [-\lambda, \lambda]$ . Then, the following identity in law does hold :*

$$\mathcal{G}^{(\lambda)}(t, t) = \frac{M_t^2}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_t^\lambda \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) \stackrel{law}{=} \frac{1}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ .

A particular case of Proposition 5.9 implies the following corollary :

**Corollary 5.10.** *If  $\alpha$  is a standard Brownian motion on  $\mathbb{R}_+$ , then for every  $\lambda > 0$ ,*

$$\frac{\int_0^\lambda e^{2\alpha_s - s} ds}{(e^{\alpha_\lambda - \lambda/2} - 1)^2} \stackrel{\text{law}}{=} \frac{1}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ .

**Remark 5.2.** *The identity of Corollary 5.10 is not really new. Actually, it can be deduced also from the Matsumoto-Yor properties. This second approach is also explained in the proof of Corollary 5.10. Remark also that making  $\lambda$  go to infinity in Corollary 5.10 gives the Dufresne identity. The fact that the second proof of Corollary 5.10 involves the Matsumoto-Yor properties is not very surprising because we explained in section 5.2.3 that the Matsumoto-Yor properties can be deduced from the  $\beta$ -field.*

More generally, we can compute the distribution of  $\langle \eta, G_\beta^{(\lambda, n)} \eta \rangle$  for every  $\eta \in (\mathbb{R}_+^*)^{\mathcal{C}[\lambda, n]}$ . Combining this with the limit obtained in Theorem 5.7 gives new identities for the geometric Brownian motion.

**Proposition 5.11.** *Let  $f$  be a deterministic continuous non-negative function on  $\mathcal{C}(\lambda)$ . Then, the following identity in law does hold :*

$$\frac{1}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{1}{M_u^2} \left( M_{-\lambda} \int_u^\lambda f(t) M_t dt + M_\lambda \int_{-\lambda}^u f(t) M_t dt \right)^2 du \stackrel{\text{law}}{=} \frac{\left( \int_{-\lambda}^\lambda f(t) dt \right)^2}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ .

**Remark 5.3.** *Actually, in Proposition 5.11, the continuity of  $f$  is a very strong assumption. Indeed, we can allow discontinuity points for  $f$  but we only focus on the continuous case for sake of convenience.*

### 5.2.4.5 The continuous random operator $\mathcal{H}^{(\lambda)}$

Let us define the domain

$$\mathcal{D} \left( \mathcal{H}^{(\lambda)} \right) = \left\{ \begin{array}{l} g \in L^2([-\lambda, \lambda]), \quad \left( \frac{g}{M} \right)' \in L^2([-\lambda, \lambda]), \quad \left( M^2 \left( \frac{g}{M} \right)' \right)' \in L^2([-\lambda, \lambda]), \\ g(-\lambda) = g(\lambda), \quad M_{-\lambda} \left( \frac{g}{M} \right)'(-\lambda) = M_\lambda \left( \frac{g}{M} \right)'(\lambda) \end{array} \right\}.$$

In the definition above,  $\frac{g}{M}$  means the random function  $x \mapsto \frac{g(x)}{M_x}$ . The derivative  $'$  is defined in the sense of distributions. Moreover, if  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ ,  $g$  and  $M \left( \frac{g}{M} \right)'$  are well-defined at  $-\lambda$  and  $\lambda$  because they are actually continuous functions. This stems from Sobolev injections in dimension 1. (Indeed the Sobolev space  $H^1([-\lambda, \lambda])$  can be injected in the set of continuous functions.) For every  $f \in L^2([-\lambda, \lambda])$ , we define the function  $\mathcal{G}^{(\lambda)} f$  such that for every  $x \in \mathcal{C}(\lambda)$ ,

$$\mathcal{G}^{(\lambda)} f(x) = \int_{-\lambda}^\lambda \mathcal{G}^{(\lambda)}(x, t) f(t) dt.$$

Consequently,  $\mathcal{G}^{(\lambda)}$  can be viewed as an operator on  $L^2([-\lambda, \lambda])$ . Now, we can state our next result :

**Theorem 5.12.** *Let  $\lambda > 0$ . The image of  $\mathcal{G}^{(\lambda)}$  is exactly  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Therefore,  $\mathcal{G}^{(\lambda)}$  has a bijective inverse  $\mathcal{H}^{(\lambda)}$  from  $\mathcal{D}(\mathcal{H}^{(\lambda)})$  onto  $L^2([-\lambda, \lambda])$ . For every  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ ,*

$$\mathcal{H}^{(\lambda)} g = -\frac{1}{M} \left( M^2 \left( \frac{g}{M} \right)' \right)'.$$

Furthermore,  $\mathcal{H}^{(\lambda)}$  is a positive self-adjoint operator (for the classical inner-product on  $L^2([-\lambda, \lambda])$ ) with domain  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ .

As  $\mathcal{C}^{(\lambda)}$  is compact, the operator  $\mathcal{H}^{(\lambda)}$  is localized. More precisely, its spectrum  $\sigma(\mathcal{H}^{(\lambda)})$ , that is, the set of real numbers  $E$  such that  $\mathcal{H}^{(\lambda)} - E$  is not invertible, consists only in a sequence of isolated eigenvalues.

**Proposition 5.13.** *Almost surely, there exists an increasing positive random sequence  $(E_k(\lambda))_{k \geq 0} \in (\mathbb{R}_+^*)^{\mathbb{N}}$  which diverges toward infinity such that*

$$\sigma(\mathcal{H}^{(\lambda)}) = (E_k(\lambda))_{k \geq 0}.$$

Moreover, for every  $k \in \mathbb{N}$ ,  $E_k(\lambda)$  is an eigenvalue of  $\mathcal{H}^{(\lambda)}$  with finite multiplicity. The eigenvalues of  $\mathcal{H}^{(\lambda)}$  are counted with multiplicity.

### 5.2.5 Continuous version of $H_\beta$ on the real line

One can wonder what is the limit of  $\mathcal{G}^{(\lambda)}$  and  $\mathcal{H}^{(\lambda)}$  when  $\lambda$  goes to infinity. It would define some operators  $\mathcal{G}^{(\infty)}$  and  $\mathcal{H}^{(\infty)}$  on  $\mathbb{R}$  which are associated with the VRJP on  $\mathbb{R}$ . The following proposition gives a partial answer to this question.

**Proposition 5.14.** *For the topology of uniform convergence on compact sets in  $\mathbb{R}^2$ , it holds that*

$$\mathcal{G}^{(\lambda)} \xrightarrow[\lambda \rightarrow +\infty]{law} \mathcal{G}^{(\infty)}$$

where  $\mathcal{G}^{(\infty)}$  is a symmetric random kernel on  $\mathbb{R}^2$  such that for every  $t, t' \in \mathbb{R}$  such that  $t \leq t'$ ,

$$\mathcal{G}^{(\infty)}(t, t') = M_{t'} M_t \int_{-\infty}^t \frac{ds}{M_s^2}.$$

**Remark 5.4.** *It is important to notice that  $\mathcal{G}^{(\infty)}$  is not well-defined on the whole Hilbert space  $L^2(\mathbb{R})$ . (Contrary to the case of the circle.) We could define for every "nice function"  $f$ ,*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) \bar{f}(t') \mathcal{G}^{(\infty)}(t, t') dt dt' = \int_{-\infty}^{\infty} \frac{1}{M_u^2} \left| \int_u^{+\infty} f(t) M_t dt \right|^2 du.$$

However, this quantity is not almost surely finite for every  $f$  and the problem is to give sense to these "nice functions". We strongly suspect that the set of "nice functions" is actually  $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$  (because of Proposition 5.11). However, we are not able to prove it for now. It would also be possible to define  $\mathcal{H}^{(\infty)}$  as the inverse of  $\mathcal{G}^{(\infty)}$ , that is, by

$$\mathcal{H}^{(\infty)} g = -\frac{1}{M} \left( M^2 \left( \frac{g}{M} \right)' \right)'$$

Nevertheless, this is not clear what should be the domain of  $\mathcal{H}^{(\infty)}$ . We will try to solve this problem as soon as possible.

**Remark 5.5.** *At first sight, the expression of  $\mathcal{G}^{(\infty)}$  is not symmetrical in the sense that when  $t'$  increases, then only  $M_{t'}$  changes and when  $t$  decreases, only  $M_t \int_{-\infty}^t \frac{ds}{M_s^2}$  changes. The dissymmetry between  $M_{t'}$  and  $M_t \int_{-\infty}^t \frac{ds}{M_s^2}$  is surprising because the law of  $G_\beta^{(\lambda, n)}$  is totally symmetric in the two directions of the circle. However, recall that we chose an orientation of the circle when we described the potential  $\beta$  by means of  $(A_i)_{i \in \mathcal{C}_{[\lambda n]}}$  which explains this apparent dissymmetry. Corollary 5.15 explains why this dissymmetry does not exist "in law". It exists only "almost surely".*

Thanks to Proposition 5.14, one can also prove a new functional identity in law :

**Corollary 5.15.** *Let  $\alpha$  be a standard Brownian motion on  $\mathbb{R}_+$ . Then the process*

$$\left( e^{-\alpha_t + t/2} \frac{\int_t^{+\infty} e^{2\alpha_s - s} ds}{\int_0^{+\infty} e^{2\alpha_s - s} ds} \right)_{t \geq 0}$$

is a geometric Brownian motion starting from 1.

### 5.2.6 The asymptotic density of states

Moreover, one can look for the asymptotic density of states of  $\mathcal{H}^{(\lambda)}$  when  $\lambda$  goes to infinity. For every  $E \in \mathbb{R}_+^*$ , let us define the random variable  $N_\lambda(E)$  which is the number of eigenvalues of  $\mathcal{H}^{(\lambda)}$  which are lower than  $E$ . Then, we have the following result :

**Theorem 5.16.** *For every  $E > 0$ ,*

$$\frac{N_\lambda(E)}{2\lambda} \xrightarrow[\lambda \rightarrow +\infty]{\mathbb{P}} \frac{\sqrt{E}}{\pi} := N_\infty(E).$$

In some way,  $N_\infty$  is the integrated density of states of  $\mathcal{H}^{(\infty)}$ . (However  $\mathcal{H}^{(\infty)}$  can not be defined rigorously as an operator for now.)

**Remark 5.6.** *The behaviour of the density of states around 0 is reminiscent of what is going on for the operator  $H_\beta$  on  $\mathbb{Z}^d$ . Indeed, by Theorem 1 and 3 in [55] (see also Chapter 4), if the weights  $W$  are small enough, then the density of states of the discrete Schrödinger operator  $H_\beta$  behaves like  $\sqrt{E}$  near 0.*

**Remark 5.7.** *The density of states  $E \mapsto \frac{\sqrt{E}}{\pi}$  is exactly the density of states of  $-\Delta$  on  $\mathbb{R}$ . However it is possible to see a difference between the eigenvalues of  $\mathcal{H}^{(\infty)}$  and  $-\Delta$  if we look at the microscopic scale. Moreover, in the case of the Anderson model, the integrated density of states can be computed explicitly. (see [68], [85] and [71]) Then the behaviour at infinity of the integrated density of states is also  $\frac{\sqrt{E}}{\pi}$ , exactly as for our operator  $\mathcal{H}^{(\infty)}$  and  $-\Delta$ .*

### 5.2.7 Organisation of the paper

- In section 5.3, we remind several important facts regarding the  $\beta$  potential.
- Section 5.4 is devoted to the proof of a few lemmas which will be useful in the sequel of this paper.
- In section 5.5, we give a new proof of Matsumoto-Yor properties by means of the  $\beta$ -potential. This section is independent from the next ones.
- In other sections, we prove the results concerning the continuous versions of the operator  $H_\beta$ .

## 5.3 Background on the $\beta$ -potential

Let  $(V, E)$  a finite graph. The  $\beta$ -potential with distribution  $\nu_V^W$  which is defined in (5.3) is a special case of a more general family of random potentials which appears naturally when taking restrictions. Let us consider a symmetric matrix  $W$  on  $V \times V$  with non-negative entries  $(W_{i,j})_{i,j \in V \times V}$  and let  $\eta := (\eta_i)_{i \in V}$  be a vector on  $V$  with non-negative entries. Recall that for every  $\beta \in \mathbb{R}_+^V$ ,  $H_\beta$  is a matrix such that for every  $i, j \in V$ ,

$$H_\beta(i, j) = \mathbf{1}\{i = j\}2\beta_i - W_{i,j}.$$

We can generalize the measure  $\nu_V^W$  thanks to the following proposition.

**Proposition 5.17** (Theorem 2.2 in [110] or Lemma 4 in [154]). *Let us define the measure  $\nu_V^{W,\eta}$  on  $\mathbb{R}_+^V$  by*

$$\nu_V^{W,\eta}(d\beta) := \mathbf{1}\{H_\beta > 0\} \left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac{1}{2}\langle \mathbf{1}, H_\beta \mathbf{1} \rangle - \frac{1}{2}\langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \mathbf{1} \rangle} \frac{d\beta_V}{\sqrt{\det(H_\beta)}}$$

where  $\mathbf{1}$  stands for a vector whose entries are all equal to 1. Then  $\nu_V^{W,\eta}$  is a probability measure. Moreover, its Laplace transform is, for any  $t \in \mathbb{R}_+^V$ ,

$$\int e^{-\langle t, \beta \rangle} \nu_V^{W,\eta}(d\beta) = e^{-\sum_{i \in V} \eta_i (\sqrt{t_i+1}-1) - \frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} (\sqrt{(1+t_i)(1+t_j)}-1)} \prod_{i \in V} \frac{1}{\sqrt{1+t_i}}.$$

Remark that the Laplace transform in (5.3) is the same as the Laplace transform of  $\nu_V^{W,0}$  given in Proposition 5.17. Further, the measures of type  $\nu_V^{W,\eta}$  are convenient when we want to manipulate them because they form a family which is stable under restriction and conditioning.

**Proposition 5.18** (Lemma 5 in [154] or Proposition 4.3 in [110]). *Let  $U$  be a subset of  $V$ . If  $\beta \sim \nu_V^{W,\eta}$ , it holds that*

(i)  $\beta_U$  follows the distribution  $\nu_U^{W_{U,U},\hat{\eta}}$ , where

$$\hat{\eta} = \eta_U + W_{U,U^c}(\mathbf{1}_{U^c}),$$

(ii) Conditionally on  $\beta_U$ ,  $\beta_{U^c}$  follows the distribution  $\nu_{U^c}^{\check{W},\check{\eta}}$ , where

$$\check{W} = W_{U^c,U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}W_{U,U^c}, \quad \check{\eta} = \eta_{U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}\eta_U.$$

Let  $\beta \in \mathbb{R}_+^V$  be such that  $H_\beta$  is positive definite. Let  $G_\beta$  be the inverse of  $H_\beta$ . Let  $i, j \in V$ . A path in the graph  $(V, E)$  from  $i$  to  $j$  consists in a finite sequence  $\sigma = (\sigma_0, \dots, \sigma_m)$  in  $V$  such that  $\sigma_0 = i$ ,  $\sigma_m = j$  and for every  $k \in \{0, \dots, m-1\}$ ,  $\{\sigma_k, \sigma_{k+1}\} \in E$ . The length  $m$  of  $\sigma$  will be denoted by  $|\sigma|$ . Let  $\mathcal{P}_{i,j}^V$  be the set of paths from  $i$  to  $j$ . We define also the set  $\overline{\mathcal{P}}_{i,j}^V$  which is the collection of paths  $\sigma$  from  $i$  to  $j$  such that  $\sigma_k \neq j$  for every  $k \in \{0, \dots, m-1\}$ . Moreover, for any path  $\sigma$ , we define

$$W_\sigma = \prod_{k=0}^{|\sigma|-1} W_{\sigma_k, \sigma_{k+1}}, \quad (2\beta)_\sigma = \prod_{k=0}^{|\sigma|} (2\beta_{\sigma_k}), \quad (2\beta)_\sigma^- = \prod_{k=0}^{|\sigma|-1} (2\beta_{\sigma_k}).$$

Then, we have the following useful description of  $G_\beta$  :

**Proposition 5.19.** *Let  $(\beta_i)_{i \in V}$  be a random field on  $V$  with distribution  $\nu_V^{W,\eta}$ . Then for every  $i, j \in V$ , almost surely,*

$$G_\beta(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}, \quad \frac{G_\beta(i, j)}{G_\beta(i, i)} = \sum_{\sigma \in \overline{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma^-}.$$

## 5.4 Preliminary lemmas

In this section we will prove a few lemmas about Inverse Gaussian random variables which will be crucial in the sequel of this paper.

**Lemma 5.20.** *Let  $A^{(n)}$  be an Inverse Gaussian random variable with parameters  $(1, n)$ . Then we know that*

$$\mathbb{E} \left[ \ln(A^{(n)}) \right] = -\frac{1}{2n} + o\left(\frac{1}{n}\right) \quad \text{and} \quad \text{Var} \left[ \ln(A^{(n)}) \right] = \frac{1}{n} + o\left(\frac{1}{n}\right).$$

Moreover, for any  $n \in \mathbb{N} \setminus \{0, 1\}$  and for any  $v > 0$ , it holds that

$$\mathbb{P} \left[ \left| \ln(A^{(n)}) \right| > v \right] \leq \frac{2e}{v\sqrt{\pi(n-1)}} e^{-(n-1)v^2/2}.$$

*Proof.*

This proof is very similar with the proof of Lemma 3.4 in [113] but we do it again here for the paper to be self-contained. Let  $n \in \mathbb{N}^*$ . The density of  $\ln(A^{(n)})$  is

$$\frac{\sqrt{n}}{\sqrt{2\pi}} e^{-u/2 - 2n \sinh(u/2)^2} du.$$

Therefore,

$$\begin{aligned}\mathbb{E} \left[ \ln(A^{(n)}) \right] &= \int_{-\infty}^{+\infty} \frac{\sqrt{n}}{\sqrt{2\pi}} u e^{-u/2 - 2n \sinh(\frac{u}{2})^2} du \\ &= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_0^{+\infty} \left( u e^{-u/2} - u e^{u/2} \right) e^{-2n \sinh(\frac{u}{2})^2} du \\ &= -\frac{8\sqrt{n}}{\sqrt{2\pi}} \int_0^{+\infty} u \sinh(u) e^{-2n \sinh(u)^2} du.\end{aligned}\tag{5.7}$$

$$(5.8)$$

Now, let us do the change of variable  $t = \sinh(u)$  in (5.7). It yields

$$\begin{aligned}\mathbb{E} \left[ \ln(A^{(n)}) \right] &= -\frac{8\sqrt{n}}{\sqrt{2\pi}} \int_0^{+\infty} \frac{t \times \operatorname{argsinh}(t)}{\sqrt{1+t^2}} e^{-2nt^2} dt \\ &= -\frac{1}{n} \times \frac{8}{\sqrt{2\pi}} \int_0^{+\infty} \frac{\sqrt{nt} \times \operatorname{argsinh}(t/\sqrt{n})}{\sqrt{1+t^2/n}} e^{-2t^2} dt.\end{aligned}\tag{5.9}$$

Besides, for every  $t > 0$  and for every  $n \in \mathbb{N}^*$ ,  $\frac{\sqrt{nt} \times \operatorname{argsinh}(t/\sqrt{n})}{\sqrt{1+t^2/n}} e^{-2t^2} \leq t^2 e^{-2t^2}$ . Therefore, we can apply the dominated convergence theorem in (5.9) which implies

$$\mathbb{E} \left[ \ln(A^{(n)}) \right] = -\frac{1}{n} \times \frac{8}{\sqrt{2\pi}} \int_0^{+\infty} t^2 e^{-2t^2} dt + o\left(\frac{1}{n}\right) = -\frac{1}{2n} + o\left(\frac{1}{2n}\right).\tag{5.10}$$

Now, let us define  $B^{(n)} = \sqrt{n} (\ln(A^{(n)}) + \frac{1}{2n})$ . Observe that the density of  $B^{(n)}$  is

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{v}{\sqrt{n}} - \frac{1}{2n} \right) - 2n \sinh \left( \frac{1}{2} \left( \frac{v}{\sqrt{n}} - \frac{1}{2n} \right) \right)^2} dv.$$

Therefore, as  $\sinh(x)^2 \geq x^2$  for every  $x \in \mathbb{R}$ , for any positive function  $F$  of  $\mathbb{R}$  into itself,

$$\begin{aligned}\mathbb{E} \left[ F(B^{(n)}) \right] &= \int_{-\infty}^{+\infty} F(v) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{v}{\sqrt{n}} - \frac{1}{2n} \right) - 2n \sinh \left( \frac{1}{2} \left( \frac{v}{\sqrt{n}} - \frac{1}{2n} \right) \right)^2} dv \\ &\leq \frac{1}{\sqrt{2\pi}} e^{\frac{1}{8n}} \int_{-\infty}^{+\infty} F(v) e^{-v^2/2} dv.\end{aligned}\tag{5.11}$$

Consequently, by the dominated convergence theorem,

$$\mathbb{E} \left[ B^{(n)2} \right] \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v^2 e^{-v^2/2} dv = 1.\tag{5.12}$$

Combining (5.12) with (5.10), we get

$$\operatorname{Var} \left[ \ln(A^{(n)}) \right] = \frac{1}{n} + o\left(\frac{1}{n}\right).$$

Now, let us look at the tail of  $\ln(A^{(n)})$ . Let  $v > 0$ .

$$\begin{aligned}\mathbb{P} \left( |\ln(A^{(n)})| > v \right) &= \int_{\mathbb{R} \setminus [-v, v]} \frac{\sqrt{n}}{\sqrt{2\pi}} \exp \left( -2n \sinh(x/2)^2 - x/2 \right) dx \\ &\leq \int_{\mathbb{R} \setminus [-v, v]} \frac{\sqrt{n}}{\sqrt{2\pi}} \exp \left( -nx^2/2 - x/2 \right) dx \\ &\leq 2e \int_v^{+\infty} \frac{\sqrt{n}}{\sqrt{2\pi}} \exp \left( -(n-1)x^2/2 \right) dx\end{aligned}\tag{5.13}$$

where in the first inequality we used the fact that  $\sinh(x) \geq x$  for every  $x > 0$  and in the second inequality we used the fact that for every  $x \in \mathbb{R}$ ,  $e^{-x/2} \leq e \times e^{x^2/2}$ . Therefore, by (5.13), for every  $v > 0$ ,

$$\begin{aligned} \mathbb{P}\left(|\ln(A^{(n)})| > v\right) &\leq 2e \int_{v\sqrt{n-1}}^{+\infty} \frac{\sqrt{n}}{\sqrt{2\pi(n-1)}} \exp(-x^2/2) dx \\ &\leq \frac{2e}{\sqrt{\pi}} \int_{v\sqrt{n-1}}^{+\infty} \exp(-x^2/2) dx \\ &\leq \frac{2e}{v\sqrt{\pi(n-1)}} e^{-(n-1)v^2/2} \end{aligned} \quad (5.14)$$

where in the second inequality we used the fact that  $n \leq 2(n-1)$  for every  $n \geq 2$  and in the last inequality we used the fact that for  $x \geq v\sqrt{n-1}$ , it holds that  $1 \leq x/(v\sqrt{n-1})$ . It concludes the proof of Lemma 5.20.  $\square$

**Lemma 5.21.** *Let  $c > 0$ . Let  $n \in \mathbb{N}^*$ . Let  $(A_i^{(n)})_{i \in \mathbb{N}^*}$  be a sequence of independent Inverse Gaussian random variables with parameters  $(1, n)$ . Then, for every  $\varepsilon > 0$ , it holds that*

$$\lim_{n \rightarrow +\infty} \mathbb{P}\left(\sup_{i \in [1, [cn]]} |\ln(A_i^{(n)})| > \varepsilon\right) = 0.$$

*Proof.*

Let  $\varepsilon > 0$ . By Lemma 5.21, we know that

$$\begin{aligned} \mathbb{P}\left(\sup_{i \in [1, [cn]]} |\ln(A_i^{(n)})| > \varepsilon\right) &= 1 - \left(1 - \mathbb{P}\left(|\ln(A_1^{(n)})| > \varepsilon\right)\right)^{[cn]} \\ &\leq 1 - \left(1 - \frac{2e}{\varepsilon\sqrt{\pi(n-1)}} e^{-(n-1)\varepsilon^2/2}\right)^{[cn]} \end{aligned} \quad (5.15)$$

which goes to 0 as  $n$  goes to infinity.  $\square$

**Lemma 5.22.** *Let  $n \in \mathbb{N}^*$ . Let  $(A_i^{(n)})_{i \in \mathbb{N}^*}$  be a sequence of independent random variables which are distributed as an Inverse Gaussian random variable with parameters  $(1, n)$ . Let us define the process  $t \mapsto Y_t^{(n)}$  which is a random continuous function such that if  $j/n \leq t < (j+1)/n$ ,*

$$Y_t^{(n)} = \prod_{i=1}^j A_i^{(n)} + n(t - j/n) \left( \prod_{i=1}^{j+1} A_i^{(n)} - \prod_{i=1}^j A_i^{(n)} \right).$$

*Then, the following convergence holds for the topology of uniform convergence on compact sets :*

$$(Y_t^{(n)})_{t \geq 0} \xrightarrow[n \rightarrow +\infty]{law} (e^{\alpha t - t/2})_{t \geq 0}$$

*where  $\alpha$  is a Brownian motion.*

*Proof.*

By Lemma 5.21, for every  $T > 0$ ,

$$\left(\ln(Y_t^{(n)})\right)_{t \in [0, T]} = \left(\sum_{i=1}^{[tn]} \ln(A_i^{(n)}) - \mathbb{E}\left[\ln(A_i^{(n)})\right] + \sum_{i=1}^{[tn]} \mathbb{E}\left[\ln(A_i^{(n)})\right]\right)_{t \in [0, T]} + o_{n, T}^{\mathbb{P}}(1)$$

where  $o_{n, T}^{\mathbb{P}}(1)$  is a random function whose supremum goes toward 0 in probability when  $n$  goes to infinity. By Lemma 5.20, we know that  $\sum_{i=1}^{[tn]} \mathbb{E}\left[\ln(A_i^{(n)})\right]$  converges toward  $-t/2$ . Moreover,

$t \mapsto \sum_{i=1}^{\lfloor tn \rfloor} \ln(A_i^{(n)}) - \mathbb{E} \left[ \ln(A_i^{(n)}) \right]$  is a martingale. Furthermore, as in the proof of Lemma 3.4 in [113], one can combine the estimates of Lemma 5.20 and the martingale functional central limit theorem (see Theorem 1.4, Section 7.1 in [62]) in order to prove that

$$\left( \sum_{i=1}^{\lfloor tn \rfloor} \ln(A_i^{(n)}) - \mathbb{E} \left[ \ln(A_i^{(n)}) \right] \right)_{t \geq 0}$$

converges toward a Brownian motion. □

**Lemma 5.23.** *Let  $K \geq 1$ . Let  $c > 0$ . Let  $A_1$  be an Inverse Gaussian random variable with parameters  $(1, K)$ . Then, it is possible to find a coupling with a random variable  $A_2$  which has an Inverse Gaussian distribution with parameters  $(1, K + c)$  such that*

$$|\ln(A_1) - \ln(A_2)| \leq \left( \frac{1}{A_1} + \frac{1}{A_2} \right) \times \left( \frac{cR^{(1)}}{K^{3/2}} + Ber \times \frac{cR^{(2)} + \sqrt{c}R^{(3)} + R^{(4)}}{\sqrt{K}} \right)$$

where for every  $i \in \{1, 2, 3, 4\}$ ,  $R^{(i)}$  is a positive random variable and conditionally on  $\{R^{(i)}, i \in \{1, 2, 3, 4\}\}$ ,  $Ber$  is a Bernoulli random variable whose parameter is smaller than  $\frac{cR^{(1)}}{K^{3/2}}$ . Moreover, there exists a positive constant  $\kappa$  which does not depend on  $K$  and  $c$  such that for every  $i \in \{1, 2, 3, 4\}$ ,  $\mathbb{E} \left[ R^{(i)4} \right] \leq \kappa$ .

*Proof.*

Following [132], we can construct  $A_1$  and  $A_2$  in the following way : Let  $\gamma$  be a Gamma random variable with parameters  $(1/2, 1)$ . Let  $U$  be a uniform random variable which is independent of  $\gamma$ . Now, let us consider

$$X_1 = 1 + \frac{\gamma}{K} - \frac{\sqrt{\gamma}}{K} \sqrt{2K + \gamma}$$

and

$$X_2 = 1 + \frac{\gamma}{K + c} - \frac{\sqrt{\gamma}}{K + c} \sqrt{2(K + c) + \gamma}.$$

For every  $i \in \{1, 2\}$ , if  $U \leq \frac{1}{1+X_i}$ , then we define  $A_i = X_i$  and if  $U > \frac{1}{1+X_i}$ , then we define  $A_i = 1/X_i$ . According to [132],  $A_1 \sim IG(1, K)$  and  $A_2 \sim IG(1, K + c)$ . Now, let us show that this coupling satisfies the required estimate.

First remark that

$$\begin{aligned} |\ln(A_1) - \ln(A_2)| &= \left| \ln \left( \frac{A_1}{A_2} \right) \right| \\ &\leq \ln \left( 1 + \frac{|A_1 - A_2|}{A_1} \right) + \ln \left( 1 + \frac{|A_1 - A_2|}{A_2} \right) \\ &\leq \left( \frac{1}{A_1} + \frac{1}{A_2} \right) |A_1 - A_2|. \end{aligned} \tag{5.16}$$

Therefore, it is enough to find an upper bound for  $|A_1 - A_2|$  in order to prove Lemma 5.23. To do this, we have to consider three situations.

**Situation 1 :** Let us assume that  $U \leq \frac{1}{1+X_1}$  and  $U \leq \frac{1}{1+X_2}$ . Then, it holds that

$$\begin{aligned}
|A_1 - A_2| &= |X_1 - X_2| \\
&\leq \frac{\gamma c}{K^2} + \sqrt{\gamma} \left| \frac{1}{K} \sqrt{2K + \gamma} - \frac{1}{K + c} \sqrt{2(K + c) + \gamma} \right| \\
&\leq \frac{\gamma c}{K^2} + \sqrt{\gamma} \sqrt{2K + \gamma} \left( \frac{1}{K} - \frac{1}{K + c} \right) + \sqrt{\gamma} \frac{1}{K + c} \left( \sqrt{2(K + c) + \gamma} - \sqrt{2K + \gamma} \right) \\
&\leq \frac{\gamma c}{K^2} + c\sqrt{\gamma} \frac{\sqrt{2K} + \sqrt{\gamma}}{K^2} + \sqrt{\gamma} \frac{1}{K} \frac{2c}{\sqrt{2(K + c) + \gamma} + \sqrt{2K + \gamma}} \\
&\leq \frac{\gamma c}{K^2} + c\sqrt{\gamma} \frac{\sqrt{2K} + \sqrt{\gamma}}{K^2} + \frac{\sqrt{\gamma} c}{\sqrt{2} K^{3/2}}.
\end{aligned} \tag{5.17}$$

Therefore, by (5.17), in the situation 1, there exists a positive random variable  $R^{(1)}$  whose fourth moment is bounded by some constant  $\kappa^{(1)}$  which does not depend on  $K$  and  $c$  such that

$$|A_1 - A_2| \leq \frac{cR^{(1)}}{K^{3/2}}. \tag{5.18}$$

Remark that  $U$  and  $R^{(1)}$  are independent.

**Situation 2 :** Now, let us assume that  $U > \frac{1}{1+X_1}$  and  $U > \frac{1}{1+X_2}$ . Remark that

$$\frac{1}{X_1} = 1 + \frac{\gamma}{K} + \frac{\sqrt{\gamma}}{K} \sqrt{2K + \gamma}$$

and

$$\frac{1}{X_2} = 1 + \frac{\gamma}{K + c} + \frac{\sqrt{\gamma}}{K + c} \sqrt{2(K + c) + \gamma}.$$

Therefore, exactly as in the first situation, one can show that

$$|A_1 - A_2| \leq \frac{cR^{(1)}}{K^{3/2}}. \tag{5.19}$$

**Situation 3 :** Now, let us consider the case where  $U \leq \frac{1}{1+X_1}$  and  $U > \frac{1}{1+X_2}$  or the case where  $U > \frac{1}{1+X_1}$  and  $U \leq \frac{1}{1+X_2}$ . These two subcases are similar. Thus we will only treat the first one. If we assume that  $U \leq \frac{1}{1+X_1}$  and  $U > \frac{1}{1+X_2}$ , then

$$\begin{aligned}
|A_1 - A_2| &= \left| X_1 - \frac{1}{X_2} \right| \\
&\leq \frac{\gamma c}{K^2} + \frac{\sqrt{\gamma}}{K} \sqrt{2K + \gamma} + \frac{\sqrt{\gamma}}{K + c} \sqrt{2(K + c) + \gamma} \\
&\leq \frac{cR^{(2)} + \sqrt{c}R^{(3)} + R^{(4)}}{\sqrt{K}}
\end{aligned} \tag{5.20}$$

where  $R^{(2)}$ ,  $R^{(3)}$  and  $R^{(4)}$  are positive random variables whose fourth moments are bounded by some constant  $\kappa^{(2)}$  which does not depend on  $c$  and  $K$ . Remark that  $U$  is independent from  $R^{(1)}$ ,  $R^{(2)}$ ,  $R^{(3)}$  and  $R^{(4)}$ . Moreover, in this situation 3, we know that  $U \in \left[ \frac{1}{1+X_1}, \frac{1}{1+X_2} \right]$  or  $U \in \left[ \frac{1}{1+X_2}, \frac{1}{1+X_1} \right]$ . Therefore,  $U$  belongs to some interval  $\mathcal{I}$  whose size is

$$\begin{aligned}
\left| \frac{1}{1+X_1} - \frac{1}{1+X_2} \right| &= \frac{|X_1 - X_2|}{(1+X_1)(1+X_2)} \\
&\leq |X_1 - X_2| \\
&\leq \frac{cR^{(1)}}{K^{3/2}}
\end{aligned} \tag{5.21}$$

where in the last inequality, we used (5.18). Together with (5.20), it implies that, in situation 3,

$$|A_1 - A_2| \leq \left( \frac{cR^{(2)} + \sqrt{c}R^{(3)} + R^{(4)}}{\sqrt{K}} \right) \mathbf{1}\{U \in \mathcal{I}\} \quad (5.22)$$

where the size of  $\mathcal{I}$  is lower than  $\frac{cR^{(1)}}{K^{3/2}}$  with  $R^{(1)}$  independent of  $U$ . Finally, choosing  $\kappa = \max(\kappa^{(1)}, \kappa^{(2)})$  and combining (5.18), (5.19) and (5.22) concludes the proof.  $\square$

## 5.5 Proof of the results of section 5.2.3

First, let us prove Proposition 5.5.

*Proof of Proposition 5.5.*

**Step 1 : Proof of (i) and (iii).** One remarks that (iii) is just a particular case of Lemma 3.12 in Chapter 3. Actually, in Lemma 3.12 in Chapter 3, we condition with respect to the sigma-field  $\sigma(\beta_i, i \in \llbracket 2, n \rrbracket)$  and not with respect to  $\mathcal{Z}_{n,m}$ . Nevertheless, we will see in the proof of (ii) that for every  $n \in \mathbb{N}^*$  and for every  $m \in \mathbb{N}^*$ ,

$$\mathcal{Z}_{n,m} \subset \sigma(\beta_i, i \in \llbracket 2, n \rrbracket).$$

Moreover (i) stems directly from (iii).

**Step 2 : Proof of (ii).** Let  $m \geq 1$ . For sake of convenience, for every  $n \in \mathbb{N}^*$ , we denote  $\mathbf{H}_n = (\mathbf{H}_\beta^{(m)})_{\llbracket 1, n \rrbracket, \llbracket 1, n \rrbracket}$  and  $\hat{\mathbf{G}}_n = \mathbf{H}_n^{-1}$ . The strategy of the proof is the following one : first, we will establish an algebraic relation between  $(\hat{\mathbf{G}}_n, \psi_\beta^{(n,m)})$  and  $(\hat{\mathbf{G}}_{n-1}, \psi_\beta^{(n-1,m)})$  by means of the Schur complements. Then we will condition this algebraic relation with respect to the  $\sigma$ -field  $\sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)$  thanks to the conditioning properties of  $\beta$  given by Proposition 5.18. We will divide this proof into two main lemmas. Here is the first one.

**Lemma 5.24.** *Let  $n \in \mathbb{N}^* \setminus \{1\}$ . It holds that,*

$$Z_\beta^{(n,m)} = \frac{Z_\beta^{(n-1,m)}}{m} \left( 2\beta_n - m^2 \hat{\mathbf{G}}_{2,n-1}(n-1, n-1) \right) \quad (5.23)$$

where  $\hat{\mathbf{G}}_{2,n-1}$  is the inverse of  $(\mathbf{H}_\beta^{(m)})_{\llbracket 2, n-1 \rrbracket, \llbracket 2, n-1 \rrbracket}$ .

*Proof of Lemma 5.24.*

For every  $n \in \mathbb{N}^*$ , let  $C_n$  be a vector of size  $n$  such that

$$C_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -m \end{pmatrix}.$$

With this notation, remark that for every integer  $n \geq 2$ ,

$$\mathbf{H}_n = \begin{pmatrix} \mathbf{H}_{n-1} & C_{n-1} \\ C_{n-1}^T & 2\beta_n \end{pmatrix}.$$

For every  $n \geq 2$ , let us define  $D_n = 2\beta_n - C_{n-1}^T \hat{\mathbf{G}}_{n-1} C_{n-1}$ . Using the Schur complement, we get that for every integer  $n \geq 2$ ,

$$\hat{\mathbf{G}}_n = \left( \begin{array}{c|c} \hat{\mathbf{G}}_{n-1} + \hat{\mathbf{G}}_{n-1} C_{n-1} D_n^{-1} C_{n-1}^T \hat{\mathbf{G}}_{n-1} & -\hat{\mathbf{G}}_{n-1} C_{n-1} D_n^{-1} \\ \hline * & * \end{array} \right). \quad (5.24)$$

Now, let us fix an integer  $n \geq 2$ . If we apply (5.24) at points  $(1, 1)$  and  $(1, n)$ , we obtain

$$\hat{\mathbf{G}}_n(1, 1) = \hat{\mathbf{G}}_{n-1}(1, 1) + m^2 D_n^{-1} \hat{\mathbf{G}}_{n-1}(1, n-1)^2, \quad (5.25)$$

$$\psi_\beta^{(n,m)} = m \hat{\mathbf{G}}_n(1, n) = m^2 D_n^{-1} \hat{\mathbf{G}}_{n-1}(1, n-1) = m D_n^{-1} \psi_\beta^{(n-1,m)}. \quad (5.26)$$

Therefore, combining (5.25) and (5.26), we get

$$\begin{aligned} Z_\beta^{(n,m)} &= \frac{\hat{\mathbf{G}}_n(1, 1)}{\psi_\beta^{(n,m)}} = \frac{\hat{\mathbf{G}}_{n-1}(1, 1)}{m D_n^{-1} \psi_\beta^{(n-1,m)}} + \hat{\mathbf{G}}_{n-1}(1, n-1) \\ &= Z_\beta^{(n-1,m)} \times \frac{D_n}{m} + \hat{\mathbf{G}}_{n-1}(1, n-1). \end{aligned} \quad (5.27)$$

Moreover,  $D_n = 2\beta_n - m^2 \hat{\mathbf{G}}_{n-1}(n-1, n-1)$ . Together with (5.27), it yields

$$\begin{aligned} Z_\beta^{(n,m)} &= \frac{Z_\beta^{(n-1,m)}}{m} \times 2\beta_n - m Z_\beta^{(n-1,m)} \hat{\mathbf{G}}_{n-1}(n-1, n-1) + \hat{\mathbf{G}}_{n-1}(1, n-1) \\ &= \frac{Z_\beta^{(n-1,m)}}{m} \times 2\beta_n - \hat{\mathbf{G}}_{n-1}(n-1, n-1) \frac{\hat{\mathbf{G}}_{n-1}(1, 1)}{\hat{\mathbf{G}}_{n-1}(1, n-1)} + \hat{\mathbf{G}}_{n-1}(1, n-1) \\ &= \frac{Z_\beta^{(n-1,m)}}{m} \times 2\beta_n - \frac{\hat{\mathbf{G}}_{n-1}(1, 1)}{\hat{\mathbf{G}}_{n-1}(1, n-1)} \left( \hat{\mathbf{G}}_{n-1}(n-1, n-1) - \frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)} \hat{\mathbf{G}}_{n-1}(1, n-1) \right) \end{aligned} \quad (5.28)$$

where the second equality comes from the definition of  $Z_\beta^{(n-1,m)}$ . Besides, according to Proposition 5.19,  $\hat{\mathbf{G}}_{n-1}(n-1, n-1)$  can be interpreted as a sum over the set of paths from  $n-1$  to  $n-1$  in  $\llbracket 1, n-1 \rrbracket$ . Moreover, thanks to Proposition 5.19 again,  $\frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)} \hat{\mathbf{G}}_{n-1}(1, n-1)$  can be interpreted as a sum over the set of paths from  $n-1$  to  $n-1$  in  $\llbracket 1, n-1 \rrbracket$  which go through 1. Therefore the difference,

$$\hat{\mathbf{G}}_{n-1}(n-1, n-1) - \frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)} \hat{\mathbf{G}}_{n-1}(1, n-1)$$

can be interpreted as a sum over the set of paths from  $n-1$  to  $n-1$  in  $\llbracket 2, n-1 \rrbracket$ . Consequently, by Proposition 5.19 again,

$$\hat{\mathbf{G}}_{n-1}(n-1, n-1) - \frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)} \hat{\mathbf{G}}_{n-1}(1, n-1) = \hat{\mathbf{G}}_{2,n-1}(n-1, n-1)$$

where  $\hat{\mathbf{G}}_{2,n-1}$  is the inverse of  $(\mathbf{H}_\beta^{(m)})_{\llbracket 2, n-1 \rrbracket, \llbracket 2, n-1 \rrbracket}$ . Together with (5.28), it implies that

$$\begin{aligned} Z_\beta^{(n,m)} &= Z_\beta^{(n-1,m)} \times \frac{2\beta_n}{m} - m Z_\beta^{(n-1,m)} \hat{\mathbf{G}}_{2,n-1}(n-1, n-1) \\ &= \frac{Z_\beta^{(n-1,m)}}{m} \left( 2\beta_n - m^2 \hat{\mathbf{G}}_{2,n-1}(n-1, n-1) \right). \end{aligned} \quad (5.29)$$

□

Now, let us enounce the second fundamental lemma of this proof.

**Lemma 5.25.** *For every  $n \in \mathbb{N}^* \setminus \{1\}$ , it holds that*

$$\mathcal{L}(2\beta_n | \sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)) = m^2 \hat{\mathbf{G}}_{2,n-1}(n-1, n-1) + \frac{1}{IG \left( \frac{1}{m + \frac{1}{Z_\beta^{(n-1,m)}}}, 1 \right)}.$$

*Proof of Lemma 5.25.*

Now, let us condition  $\beta_n$  in Lemma 5.24 with respect to  $\sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)$ . Recall that we assumed  $\beta \sim \nu_{\mathbb{N}^*}^{K_m}$ . Thanks to Proposition 5.18, conditionally on  $\sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)$ ,  $\beta_n$  is distributed as  $\nu_{\{n\}}^{\eta_{n,m}, W_{n,m}}$  where

$$W_{n,m} = m^2 \hat{\mathbf{G}}_{2,n-1}(n-1, n-1)$$

and

$$\eta_{n,m} = m + m^2 \hat{\mathbf{G}}_{2,n-1}(2, n-1).$$

However, thanks to Proposition 5.19,

$$m \hat{\mathbf{G}}_{2,n-1}(2, n-1) = \frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)}.$$

Consequently,

$$\eta_{n,m} = m + m \frac{\hat{\mathbf{G}}_{n-1}(1, n-1)}{\hat{\mathbf{G}}_{n-1}(1, 1)} = m + \frac{1}{Z_\beta^{(n-1,m)}}.$$

Therefore,

$$\begin{aligned} \mathcal{L}(2\beta_n | \sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)) &= W_{n,m} + \frac{1}{IG\left(\frac{1}{\eta_{n,m}}, 1\right)} \\ &= m^2 \hat{\mathbf{G}}_{2,n-1}(n-1, n-1) + \frac{1}{IG\left(\frac{1}{m + \frac{1}{Z_\beta^{(n-1,m)}}}, 1\right)}. \end{aligned}$$

□

Combining Lemmas 5.24 and 5.25, it holds that

$$\mathcal{L}\left(Z_\beta^{(n,m)} | \sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)\right) = \frac{Z_\beta^{(n-1,m)}}{m} \times \frac{1}{IG\left(\frac{1}{m + \frac{1}{Z_\beta^{(n-1,m)}}}, 1\right)}. \quad (5.30)$$

Moreover, by Proposition 5.19, for every  $k \in \llbracket 1, n-1 \rrbracket$ ,

$$\frac{1}{Z_\beta^{(k,m)}} = \frac{m \hat{\mathbf{G}}_k(1, k)}{\hat{\mathbf{G}}_k(1, 1)} = m \sum_{\sigma \in \overline{\mathcal{P}}_{k,1}^{\llbracket 1, k \rrbracket}} \frac{W_\sigma}{(2\beta)_\sigma}. \quad (5.31)$$

Remark that the sum in the right-hand side of (5.31) never contains  $\beta_1$ . Thus, for every  $k \in \llbracket 1, n-1 \rrbracket$ ,  $Z_\beta^{(k,m)}$  is measurable with respect to  $\sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket)$ . This implies that

$$\mathcal{Z}_{n-1,m} \subset \sigma(\beta_i, i \in \llbracket 2, n-1 \rrbracket).$$

Together with (5.30), it yields

$$\mathcal{L}\left(Z_\beta^{(n,m)} | \mathcal{Z}_{n-1,m}\right) = \frac{Z_\beta^{(n-1,m)}}{m} \times \frac{1}{IG\left(\frac{1}{m + \frac{1}{Z_\beta^{(n-1,m)}}}, 1\right)}.$$

It concludes the proof of (ii). □

Now, we still have to prove that  $(\psi_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  and  $(Z_\beta^{(n,m)})_{n \in \mathbb{N}^*}$  converge toward the exponential functionals of the Brownian motion introduced by Matsumoto and Yor when we take the scaling limit as  $m$  goes to infinity. To do so, we need first to prove a lemma which gives a useful representation of the  $\beta$ -field with distribution  $\nu_{\mathbb{N}^*}^{K_m}$ . Note that this construction is very specific to the one-dimensional structure of the graph.

**Lemma 5.26.** *Let  $m \in \mathbb{N}^*$ . Let  $(A_i^{(m)})_{i \in \mathbb{N}^*}$  be a sequence of independent Inverse Gaussian random variables with parameters  $(1, m)$ . We define  $\beta_1 = \frac{m}{2A_1^{(m)}}$  and for every  $i \in \mathbb{N}^* \setminus \{1\}$ ,*

$$\beta_i = \frac{m}{2} A_{i-1}^{(m)} + \frac{m}{2A_i^{(m)}}.$$

Then,  $\beta \sim \nu_{\mathbb{N}^*}^{K_m}$ .

*Proof.*

For every  $i \in \mathbb{N}^*$ , let us write  $A_i$  for  $A_i^{(m)}$  for sake of convenience. Let  $k \in \mathbb{N}^*$ . Let  $(t_i)_{i \in \llbracket 1, k \rrbracket} \in \mathbb{R}_+^k$ . Then, it holds that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( -\frac{t_1 m}{2A_1} - \sum_{i=2}^k \frac{t_i m}{2} \left( A_{i-1} + \frac{1}{A_i} \right) \right) \right] \\ &= \mathbb{E} \left[ \exp \left( -\sum_{i=1}^{k-1} \frac{m}{2} \left( t_{i+1} A_i + \frac{t_i}{A_i} \right) \right) \times \exp \left( -\frac{m}{2} \frac{t_k}{A_k} \right) \right] \\ &= \prod_{i=1}^{k-1} \mathbb{E} \left[ \exp \left( -\frac{m}{2} \left( t_{i+1} A_i + \frac{t_i}{A_i} \right) \right) \right] \times \mathbb{E} \left[ \exp \left( -\frac{m}{2} \frac{t_k}{A_k} \right) \right] \\ &= \prod_{i=1}^{k-1} \int_0^{+\infty} \frac{\sqrt{m}}{\sqrt{2\pi x^3}} e^{-\frac{m(x-1)^2}{2x}} e^{-m \frac{t_{i+1} x + t_i/x}{2}} dx \times \int_0^{+\infty} \frac{\sqrt{m}}{\sqrt{2\pi x^3}} e^{-\frac{m(x-1)^2}{2x}} e^{-m \frac{t_k}{2x}} dx. \end{aligned}$$

Moreover, for every  $i \in \llbracket 1, k-1 \rrbracket$ , remark that

$$\begin{aligned} \int_0^{+\infty} \frac{\sqrt{m}}{\sqrt{2\pi x^3}} e^{-\frac{m(x-1)^2}{2x}} e^{-m \frac{t_{i+1} x + t_i/x}{2}} dx &= \int_0^{+\infty} \frac{\sqrt{m(1+t_i)}}{\sqrt{2\pi x^3}} \exp \left( -m(1+t_i) \frac{\left( x - \sqrt{\frac{1+t_i}{1+t_{i+1}}} \right)^2}{2x \frac{1+t_i}{1+t_{i+1}}} \right) dx \\ &\quad \times \frac{1}{\sqrt{1+t_i}} \times \exp \left( -m \left( \sqrt{1+t_i} \sqrt{1+t_{i+1}} - 1 \right) \right) \\ &= \frac{1}{\sqrt{1+t_i}} \exp \left( -m \left( \sqrt{1+t_i} \sqrt{1+t_{i+1}} - 1 \right) \right) \end{aligned}$$

because in the first equality we recognised the density of an Inverse Gaussian random variable with parameters  $\left( \sqrt{\frac{1+t_i}{1+t_{i+1}}}, m(1+t_i) \right)$  for every  $i \in \llbracket 1, k-1 \rrbracket$ . Besides, one can prove in the same way that

$$\int_0^{+\infty} \frac{\sqrt{m}}{\sqrt{2\pi x^3}} e^{-\frac{m(x-1)^2}{2x}} e^{-m \frac{t_k}{2x}} dx = \frac{1}{\sqrt{1+t_k}} \exp \left( -m \left( \sqrt{1+t_k} - 1 \right) \right).$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( -\frac{t_1 m}{2A_1} - \sum_{i=2}^k \frac{t_i m}{2} \left( A_{i-1} + \frac{1}{A_i} \right) \right) \right] \\ &= \exp \left( -\sum_{i=1}^{k-1} m \left( \sqrt{1+t_i} \sqrt{1+t_{i-1}} - 1 \right) \right) \times \exp \left( -m \left( \sqrt{1+t_k} - 1 \right) \right) \times \prod_{i=1}^k \frac{1}{\sqrt{1+t_i}}. \end{aligned}$$

This is exactly the Laplace Transform in (5.4).  $\square$

*Proof of Proposition 5.6.*

Let  $m \in \mathbb{N}^*$ . First, let us use the construction of the  $\beta$ -field given in Lemma 5.26. For every  $i \in \mathbb{N}^*$ , we write  $A_i^{(m)} = A_i$  for sake of convenience. Let  $n \in \mathbb{N}^*$ . For every  $i \in \llbracket 1, n \rrbracket$ , let us define

$$V_i = \frac{1}{m} \prod_{k=1}^{i-1} A_k \times \left( A_i + \sum_{k=i}^{n-1} \prod_{r=i}^k A_r^2 \times A_{k+1} \right)$$

where the product on an empty set is 1. Let  $\mathbf{V}_n$  be the vector of  $\mathbb{R}^n$  whose coordinates are  $(V_i)_{i \in \llbracket 1, n \rrbracket}$ . Then, it is rather tedious but not difficult to check that

$$(\mathbf{H}_\beta^{(m)})_{\llbracket 1, n \rrbracket, \llbracket 1, n \rrbracket} \mathbf{V}_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5.32)$$

We did not write the computation of (5.32) here because it is long and very similar to the computation which is done in the Proposition 5.28 below. The equality (5.32) implies for every  $i \in \llbracket 1, n \rrbracket$ ,

$$\hat{\mathbf{G}}_n(1, i) = V_i = \frac{1}{m} \prod_{k=1}^{i-1} A_k \times \left( A_i + \sum_{k=i}^{n-1} \prod_{r=i}^k A_r^2 \times A_{k+1} \right).$$

In particular,

$$\psi_\beta^{(n, m)} = \prod_{i=1}^n A_i \text{ and } \hat{\mathbf{G}}_n(1, 1) = \frac{1}{m} \left( A_1 + \sum_{k=1}^{n-1} \prod_{r=1}^k A_r^2 \times A_{k+1} \right). \quad (5.33)$$

Therefore,

$$Z_\beta^{(n, m)} = \frac{\frac{1}{m} \left( A_1 + \sum_{k=1}^{n-1} \prod_{r=1}^k A_r^2 \times A_{k+1} \right)}{\prod_{i=1}^n A_i}. \quad (5.34)$$

Recall that  $(\tilde{\psi}^{(m)}(t))_{t \geq 0}$  and  $(\tilde{Z}^{(m)}(t))_{t \geq 0}$  are respectively the continuous linear interpolations of  $(\psi_\beta^{(\lfloor mt \rfloor, m)})_{t \geq 0}$  and  $(Z_\beta^{(\lfloor mt \rfloor, m)})_{t \geq 0}$ . Therefore, combining (5.33), (5.34) and Lemma 5.21, for every  $T > 0$ , for every  $m \in \mathbb{N}^*$ , it holds that

$$\left( \tilde{\psi}^{(m)}(t), \tilde{Z}^{(m)}(t) \right)_{t \in [0, T]} = \left( Y_t^{(m)}, \frac{\int_0^t (Y_s^{(m)})^2 ds}{Y_t^{(m)}} \right)_{t \in [0, T]} + o_{m, T}^{\mathbb{P}}(1) \quad (5.35)$$

where  $o_{m, T}^{\mathbb{P}}(1)$  is a random function whose supremum on  $[0, T]$  goes to 0 in probability and  $Y^{(m)}$  is defined in Lemma 5.22. Thus, we can use Lemma 5.22 in (5.35) in order to conclude the proof of Proposition 5.6.  $\square$

Now, we are ready to give a new proof of Theorem 5.4 thanks to the discrete version of the Matsumoto Yor properties given by Proposition 5.5.

*New proof of Theorem 5.4.*

First, let us prove point (iii) of Theorem 5.4. Let  $t > 0$ . Let  $k \in \mathbb{N}^*$  and let  $t_1 < t_2 < \dots < t_k \leq t$

be  $k$  positive real numbers which are smaller than  $t$ . Let  $m \in \mathbb{N}^*$  and let  $F$  be a bounded continuous function from  $\mathbb{R}^{k+1}$  into  $\mathbb{R}$ . By (iii) in Proposition 5.5, it holds that,

$$\begin{aligned} & \mathbb{E} \left[ F \left( \psi_\beta^{(\lfloor mt \rfloor, m)}, Z_{\lfloor mt_1 \rfloor}^{(m)}, \dots, Z_{\lfloor mt_k \rfloor}^{(m)} \right) \right] \\ &= \mathbb{E} \left[ \int_0^{+\infty} F \left( x, Z_\beta^{(\lfloor mt_1 \rfloor, m)}, \dots, Z_\beta^{(\lfloor mt_k \rfloor, m)} \right) \frac{1}{\sqrt{2\pi Z_\beta^{(\lfloor mt_1 \rfloor, m)} x^3}} \exp \left( -\frac{1}{Z_\beta^{(\lfloor mt_1 \rfloor, m)} 2x} (x-1)^2 \right) dx \right]. \end{aligned} \quad (5.36)$$

Thanks to Proposition 5.6, one can make  $m$  go to infinity in (5.36) which yields

$$\mathbb{E} [F(e_t, Z_{t_1}, \dots, Z_{t_k})] = \mathbb{E} \left[ \int_0^{+\infty} F(x, Z_{t_1}, \dots, Z_{t_k}) \frac{1}{\sqrt{2\pi Z_t x^3}} \exp \left( -\frac{1}{Z_t} \frac{(x-1)^2}{2x} \right) dx \right].$$

It proves (iii) in Theorem 5.4. Moreover (i) in Theorem 5.4 is a direct consequence of (iii). Now, let us prove (ii). Let  $\varepsilon > 0$ . Let  $T > \varepsilon$ . Let  $t \in [\varepsilon, T]$ . Let  $m \in \mathbb{N}^*$ . Let us define

$$\mathcal{M}_t^{(\varepsilon, m)} = \ln(Z_\beta^{(\lfloor mt \rfloor, m)}) - \ln(Z_\beta^{(\lfloor m\varepsilon \rfloor, m)}) - \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \mathbb{E} \left[ \ln(Z_\beta^{(k+1, m)}) - \ln(Z_\beta^{(k, m)}) \mid \mathcal{Z}_{k, m} \right].$$

Remark that  $(\mathcal{M}_t^{(\varepsilon, m)}, t \geq \varepsilon)$  is a martingale. We would like to show that it converges toward a Brownian motion. First, by (ii) in Proposition 5.5, one can get the following useful representation of  $\mathcal{M}_t^{(\varepsilon, m)}$  :

$$\begin{aligned} \mathcal{M}_t^{(\varepsilon, m)} &= \ln(Z_\beta^{(\lfloor mt \rfloor, m)}) - \ln(Z_\beta^{(\lfloor m\varepsilon \rfloor, m)}) + \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \ln \left( \frac{m}{m + 1/Z_\beta^{(k, m)}} \right) \\ &+ \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \mathbb{E} \left[ \ln(IG(1, m + 1/Z_\beta^{(k, m)})) \mid \mathcal{Z}_{k, m} \right]. \end{aligned} \quad (5.37)$$

Besides, another useful representation of  $\mathcal{M}_t^{(\varepsilon, m)}$  is the following one :

$$\mathcal{M}_t^{(\varepsilon, m)} = \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \Delta_k^{(m)} \quad (5.38)$$

where for every  $k \in \mathbb{N}^*$ ,

$$\Delta_k^{(m)} = \ln(Z_\beta^{(k+1, m)}) - \ln(Z_\beta^{(k, m)}) + \ln \left( \frac{m}{m + 1/Z_\beta^{(k, m)}} \right) + \mathbb{E} \left[ \ln(IG(1, m + 1/Z_\beta^{(k, m)})) \mid \mathcal{Z}_{k, m} \right].$$

By (ii) in Proposition 5.5, for every  $k \in \mathbb{N}^*$ ,

$$\Delta_k^{(m)} = -\ln(IG_k(1, m + 1/Z_\beta^{(k, m)})) + \mathbb{E} \left[ \ln(IG_k(1, m + 1/Z_\beta^{(k, m)})) \mid \mathcal{Z}_{k, m} \right] \quad (5.39)$$

where conditionally on  $(Z_\beta^{(k, m)})_{k \in \mathbb{N}^*} = (z_k)_{k \in \mathbb{N}^*}$ ,  $(IG_k(1, m + 1/z_k))_{k \in \mathbb{N}^*}$  is a sequence of independent random variables such that for every  $k \in \mathbb{N}^*$ ,  $IG_k(1, m + 1/z_k)$  is an Inverse Gaussian random variable with parameters  $(1, m + 1/z_k)$ . Let  $\delta > 0$ . By Proposition 5.6,

$$\sup_{k \in \llbracket \lfloor m\varepsilon \rfloor, \lfloor mT \rfloor \rrbracket} \frac{1}{Z_\beta^{(k, m)}} \xrightarrow[m \rightarrow +\infty]{law} \sup_{s \in [\varepsilon, T]} \frac{1}{Z_s}. \quad (5.40)$$

Remark that in the limit above, the restriction on  $[\varepsilon, T]$  is crucial. Indeed,  $\varepsilon = 0$  would give an infinite limit. Furthermore, by (5.40), there exists a positive constant  $C_\delta$  such that for every  $m \in \mathbb{N}^*$ ,

$$\mathbb{P} \left( \sup_{k \in \llbracket m\varepsilon, \llbracket mT \rrbracket \rrbracket} \frac{1}{Z_\beta^{(k,m)}} > C_\delta \right) \leq \delta. \quad (5.41)$$

In the sequel, we use the notation  $A(\delta)$  to designate the event

$$\left\{ \sup_{k \in \llbracket m\varepsilon, \llbracket mT \rrbracket \rrbracket} \frac{1}{Z_\beta^{(k,m)}} \leq C_\delta \right\}.$$

Conditionally on  $A(\delta)$ , one can use the estimate of the expectation of an Inverse Gaussian random variable given by Lemma 5.20 to obtain that for every  $k \in \llbracket m\varepsilon, \llbracket mT \rrbracket \rrbracket$ ,

$$\Delta_k^{(m)} = -\ln(IG_k(1, m + 1/Z_\beta^{(k,m)})) + \mathbb{E}[\ln(IG_k(1, m))] + \frac{1}{m} o_{m,\delta}(1) \quad (5.42)$$

where  $o_{m,\delta}(1)$  goes to zero when  $m$  goes to infinity and does not depend on  $k$ . Now, let us use the coupling of Lemma 5.23 in (5.42). Therefore, for every  $k \in \llbracket m\varepsilon, \llbracket mT \rrbracket \rrbracket$ , conditionally on  $A(\delta)$ , there exist random variables  $(R_k^{(i)}, i \in \{1, 2, 3, 4\})$  and  $Ber_k$  exactly as in Lemma 5.23 such that

$$\Delta_k^{(m)} = -\ln(IG_k(1, m)) + \mathbb{E}[\ln(IG_k(1, m))] + \frac{1}{m} o_{m,\delta}(1) + J_k^{(m)} \quad (5.43)$$

where  $(IG_k(1, m))_{k \in \mathbb{N}^*}$  is a sequence of independent Inverse Gaussian random variables with parameters  $(1, m)$  and for every  $k \in \mathbb{N}^*$

$$|J_k^{(m)}| \leq \left( \frac{1}{IG_k(1, m)} + \frac{1}{IG_k(1, m + 1/Z_\beta^{(k,m)})} \right) \times \left( \frac{C_\delta R_k^{(1)}}{m^{3/2}} + Ber_k \times \frac{C_\delta R_k^{(2)} + \sqrt{C_\delta} R_k^{(3)} + R_k^{(4)}}{\sqrt{m}} \right).$$

Moreover, for every  $i \in \{1, 2, 3, 4\}$ ,  $R_k^{(i)}$  is a positive random variable and conditionally on  $\{R_k^{(i)}, i \in \{1, 2, 3, 4\}\}$ ,  $Ber_k$  is a Bernoulli random variable whose parameter is smaller than  $\frac{C_\delta R_k^{(1)}}{m^{3/2}}$ . Moreover, there exists a positive constant  $\kappa$  which does not depend on  $m, k$  and  $C_\delta$  such that for every  $i \in \{1, 2, 3, 4\}$ ,  $\mathbb{E} \left[ R_k^{(i)4} \right] \leq \kappa$ . Consequently, conditionally on  $A(\delta)$ , using (5.43) in (5.38) yields for every  $t \in [\varepsilon, T]$ ,

$$\mathcal{M}_t^{(\varepsilon, m)} = \sum_{k=\llbracket m\varepsilon \rrbracket}^{\llbracket mt \rrbracket} (-\ln(IG_k(1, m)) + \mathbb{E}[\ln(IG_k(1, m))] + \sum_{k=\llbracket m\varepsilon \rrbracket}^{\llbracket mt \rrbracket} J_k^{(m)} + (T - \varepsilon) o_{m,\delta}(1). \quad (5.44)$$

Now, let us show that the term  $\sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} J_k^{(m)}$  is negligible under  $A(\delta)$ . Remark that

$$\begin{aligned} \mathbb{E} \left[ \mathbf{1}\{A(\delta)\} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} J_k^{(m)} \right] &\leq \sqrt{2} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} \mathbb{E} \left[ \frac{1}{IG_k(1, m)^2} + \frac{1}{IG_k(1, m + 1/Z_\beta^{(k, m)})^2} \right]^{1/2} \\ &\quad \times \mathbb{E} \left[ \left( \frac{C_\delta R_k^{(1)}}{m^{3/2}} + Ber_k \times \frac{C_\delta R_k^{(2)} + \sqrt{C_\delta} R_k^{(3)} + R_k^{(4)}}{\sqrt{m}} \right)^2 \right]^{1/2} \\ &\leq 2\sqrt{7} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} \mathbb{E} \left[ \left( \frac{C_\delta R_k^{(1)}}{m^{3/2}} + Ber_k \times \frac{C_\delta R_k^{(2)} + \sqrt{C_\delta} R_k^{(3)} + R_k^{(4)}}{\sqrt{m}} \right)^2 \right]^{1/2} \end{aligned} \quad (5.45)$$

where we used the fact that for every  $K \geq 1$ ,  $\mathbb{E} [IG(1, K)^{-2}] = (1 + 1/K)^2 + 1/K + 2/K^2 \leq 7$ . Now, in (5.45), we can apply the estimates concerning the random variables  $Ber_k$  and  $(R_k^{(i)}, i \in \{1, 2, 3, 4\})$  given by Lemma 5.23. Therefore, it holds that

$$\begin{aligned} \mathbb{E} \left[ \mathbf{1}\{A(\delta)\} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} J_k^{(m)} \right] &\leq 4\sqrt{7} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} \mathbb{E} \left[ \frac{C_\delta^2 (R_k^{(1)})^2}{m^3} + Ber_k^2 \left( \frac{C_\delta^2 (R_k^{(2)})^2 + C_\delta (R_k^{(3)})^2 + (R_k^{(4)})^2}{m} \right) \right]^{1/2} \\ &\leq 4\sqrt{7} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} \mathbb{E} \left[ \frac{C_\delta^2 (R_k^{(1)})^2}{m^3} + \frac{C_\delta R_k^{(1)}}{m^{3/2}} \left( \frac{C_\delta^2 (R_k^{(2)})^2 + C_\delta (R_k^{(3)})^2 + (R_k^{(4)})^2}{m} \right) \right]^{1/2} \\ &\leq 4\sqrt{7}(T - \varepsilon) \left( \frac{C_\delta \kappa^{1/4}}{\sqrt{m}} + \frac{\kappa^{3/8}}{m^{1/4}} (C_\delta^{3/2} + C_\delta + C_\delta^{1/2}) \right). \end{aligned} \quad (5.46)$$

Therefore, we can use the estimate (5.46) in (5.44) which implies that, on the event  $A(\delta)$ , for every  $t \in [\varepsilon, T]$ ,

$$\mathcal{M}_t^{(\varepsilon, m)} = \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor} (-\ln(IG_k(1, m)) + \mathbb{E} [\ln(IG_k(1, m))]) + (T - \varepsilon) o_{m, \delta}(1) + o_{m, \delta, \varepsilon, T}^{\mathbb{P}}(1) \quad (5.47)$$

where  $o_{m, \delta, \varepsilon, T}^{\mathbb{P}}(1)$  is a random function whose supremum on  $[\varepsilon, T]$  goes to 0 in probability when  $m$  goes to infinity. Moreover, exactly as in the proof of Lemma 5.22, we know that the continuous linear interpolation of

$$\left( \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor m(t+\varepsilon) \rfloor} (-\ln(IG_k(1, m)) + \mathbb{E} [\ln(IG_k(1, m))]) \right)_{t \in [0, T-\varepsilon]}$$

converges in law toward some standard Brownian motion  $\alpha^{(\varepsilon)}$  for the topology of uniform convergence. Therefore, on the event  $A(\delta)$ , the continuous linear interpolation of  $(\mathcal{M}_{\varepsilon+t}^{(\varepsilon, m)})_{t \in [0, T-\varepsilon]}$  converges in law toward  $\alpha^{(\varepsilon)}$ . Moreover, the probability of the event  $A(\delta)$  can be made as close as we want from 1. Therefore, the linear interpolation of  $(\mathcal{M}_{\varepsilon+t}^{(\varepsilon, m)})_{t \in [0, T-\varepsilon]}$  converges in law toward

$\alpha^{(\varepsilon)}$ . Moreover, recall from (5.37) that for every  $t \in [\varepsilon, T]$ ,

$$\begin{aligned} \mathcal{M}_t^{(\varepsilon, m)} &= \ln(Z_\beta^{(\lfloor mt \rfloor, m)}) - \ln(Z_\beta^{(\lfloor m\varepsilon \rfloor, m)}) + \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \ln \left( \frac{m}{m + 1/Z_\beta^{(k, m)}} \right) \\ &+ \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \mathbb{E} \left[ \ln(IG(1, m + 1/Z_\beta^{(k, m)})) | \mathcal{Z}_{k, m} \right]. \end{aligned} \quad (5.48)$$

Moreover, as before, one can show that  $\sup_{k \in \llbracket \lfloor m\varepsilon \rfloor, \lfloor mT \rfloor \rrbracket} \frac{1}{Z_\beta^{(k, m)}}$  is tight and by using a Taylor expansion of the logarithm, we get that

$$\begin{aligned} \left( \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \ln \left( \frac{m}{m + 1/Z_\beta^{(k, m)}} \right) \right)_{t \in [\varepsilon, T]} &= \left( -\frac{1}{m} \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \frac{1}{Z_\beta^{(k, m)}} \right)_{t \in [\varepsilon, T]} + o_{m, \varepsilon, T}^{\mathbb{P}}(1) \\ &= \left( -\int_\varepsilon^t \frac{1}{\tilde{Z}^{(m)}(s)} ds \right)_{t \in [\varepsilon, T]} + o_{m, \varepsilon, T}^{\mathbb{P}}(1) \end{aligned} \quad (5.49)$$

where  $o_{m, \varepsilon, T}^{\mathbb{P}}(1)$  is a random function whose supremum on  $[\varepsilon, T]$  goes to 0 in probability as  $m$  goes to infinity. In the last equality, we used the definition of  $\tilde{Z}^{(m)}$  given in Proposition 5.6. Besides, using Lemma 5.20 and the tightness of  $\sup_{k \in \llbracket \lfloor m\varepsilon \rfloor, \lfloor mT \rfloor \rrbracket} \frac{1}{Z_\beta^{(k, m)}}$  again, we obtain that

$$\left( \sum_{k=\lfloor m\varepsilon \rfloor}^{\lfloor mt \rfloor - 1} \mathbb{E} \left[ IG \left( 1, m + 1/Z_\beta^{(k, m)} \right) | \mathcal{Z}_{k, m} \right] \right)_{t \in [\varepsilon, T]} = (-t - \varepsilon/2)_{t \in [\varepsilon, T]} + o_{m, \varepsilon, T}^{\mathbb{P}}(1). \quad (5.50)$$

Therefore, one can combine (5.50), (5.49), Proposition 5.6 and the fact that the linear interpolation of  $(\mathcal{M}_{\varepsilon+t}^{(\varepsilon, m)})_{t \in [0, T-\varepsilon]}$  converges in law toward  $\alpha^{(\varepsilon)}$  to make  $m$  go to infinity in equation (5.48). The convergence in law in equation (5.48) holds on the compact set  $[\varepsilon, T]$  for the topology of uniform convergence. Actually, we consider this convergence for the continuous linear interpolations of the random functions in (5.48). However, this does not change anything since the error which is associated with this approximation goes to 0 in probability uniformly on  $[\varepsilon, T]$ . It implies that, almost surely, for every  $t \in [0, T - \varepsilon]$ ,

$$\ln(Z_{t+\varepsilon}) = \ln(Z_\varepsilon) + \alpha_t^{(\varepsilon)} + \int_0^t \frac{1}{Z_{s+\varepsilon}} ds + \frac{t}{2}. \quad (5.51)$$

Finally, using Ito formula in (5.51), almost surely, for every  $t \in [0, T - \varepsilon]$ ,

$$Z_{t+\varepsilon} = Z_\varepsilon + \int_0^t Z_{s+\varepsilon} d\alpha_s^{(\varepsilon)} + \int_0^t Z_s ds + t. \quad (5.52)$$

Moreover, the SDE (5.52) is satisfied for every  $\varepsilon > 0$  and for every  $T > \varepsilon$ . This gives exactly (ii) in Theorem 5.4.  $\square$

## 5.6 Study of the discrete operator on the circle

In this section, we will give a simple description of  $H_\beta^{(\lambda, n)}$  and we will use it to compute  $G_\beta^{(\lambda, n)}$  explicitly. It is the first step in order to construct the continuous-space operator  $\mathcal{H}^{(\lambda)}$ .

### 5.6.1 A simple description of the random potential $\beta$

Recall that an Inverse Gaussian distribution with parameters  $(\mu, \lambda)$  has density

$$\mathbf{1}\{x > 0\} \frac{\sqrt{\lambda}}{\sqrt{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}.$$

Let  $\lambda > 0$ . Let  $n \in \mathbb{N}$ . Let us consider a family  $(A_i^{(n)})_{i \in \mathcal{C}_{[\lambda n]}}$  of i.i.d Inverse Gaussian random variables with parameters  $(1, n)$ . Most of the time we will write  $A_i^{(n)} = A_i$  for sake of convenience. For every  $i \in \mathcal{C}_{[\lambda n]}$ , let us define

$$\beta_i = \frac{n}{2} \left( A_{i+1} + \frac{1}{A_i} \right). \quad (5.53)$$

In the definition above, if  $i = [\lambda n]$ , then  $i + 1$  is  $-\lceil \lambda n \rceil$  in order to respect the circular structure of  $\mathcal{C}_{[\lambda n]}$ . Recall that  $W_n^{(\lambda)}$  is a matrix on the discretized circle  $\mathcal{C}_{[\lambda n]}$  such that  $(W_n^{(\lambda)})_{i,j}$  is 0 if  $i$  and  $j$  are not connected and is  $n$  otherwise. Then, we have the following result

**Lemma 5.27.** *The distribution of the random potential  $(\beta_i)_{i \in \mathcal{C}_{[\lambda n]}}$  which is defined in (5.53) is  $\nu_{\mathcal{C}_{[\lambda n]}}^{W_n^{(\lambda)}}$ .*

*Proof of Lemma 5.27.*

The proof follows the same lines as the proof of Lemma 5.26. □

### 5.6.2 Computation of $G_\beta^{(\lambda, n)}$

In the sequel of this article, we will assume that the random potential  $\beta$  with distribution  $\nu_{\mathcal{C}_{[\lambda n]}}^{W_n^{(\lambda)}}$  is constructed with the random field  $A$  introduced in the previous section. A remarkable fact is that this is possible to compute explicitly the matrix  $G_\beta^{(\lambda, n)}$  as a function of the field  $A$ . In order to make the computation simpler, let us make a small change of variables. For every  $i \in \mathcal{C}_{[\lambda n]}$ , let us introduce  $u_i = \sqrt{A_i A_{i+1}}$  and  $D$  the matrix on  $\mathcal{C}_{[\lambda n]}^2$  whose diagonal coefficients are  $\sqrt{A_{-\lceil \lambda n \rceil}}, \sqrt{A_{-\lceil \lambda n \rceil + 1}}, \dots$  and so on. Moreover, we introduce the matrix  $R_u^{(\lambda, n)}$  which is defined by

- $R_u^{(\lambda, n)}(i, i) = u_i^2 + 1$  for every  $i \in \mathcal{C}_{[\lambda n]}$ .
- $R_u^{(\lambda, n)}(i, i + 1) = -u_i$  for every  $i \in \mathcal{C}_{[\lambda n]}$ .
- $R_u^{(\lambda, n)}(i, i - 1) = -u_{i-1}$  for every  $i \in \mathcal{C}_{[\lambda n]}$ .
- $R_u^{(\lambda, n)}(i, j) = 0$  elsewhere.

Remark that

$$H_\beta^{(\lambda, n)} = nD^{-1}R_u^{(\lambda, n)}D^{-1}. \quad (5.54)$$

Therefore, this is enough to compute the inverse of  $R_u^{(\lambda, n)}$ . Now, we are going to describe the inverse of  $R_u^{(\lambda, n)}$ . However, some new notation is required. Recall that  $\mathcal{C}_{[\lambda n]} = \{-\lceil \lambda n \rceil, \dots, \lceil \lambda n \rceil\}$ . This means that the discrete circle  $\mathcal{C}_{[\lambda n]}$  is oriented by the  $+1$  increment. For  $i, j \in \mathcal{C}_{[\lambda n]}$ , when we write  $\overrightarrow{\prod}_{k=i}^j$  or  $\overleftarrow{\sum}_{k=i}^j$ , we mean that  $k$  is in the set  $\{i, i + 1, \dots, j - 1, j\}$ . Then, the inverse of  $R_u^{(\lambda, n)}$  is given by the following proposition.

**Proposition 5.28.** *Let  $n \in \mathbb{N}^*$ . Let  $\lambda > 0$ . Let  $i, j \in \mathcal{C}_{[\lambda n]}$ . If  $j \notin \{i-1, i, i+1\}$ , then*

$$(R_u^{(\lambda, n)})^{-1}(i, j) = \frac{\prod_{k=i}^{j-1} u_k \times \left(1 + \sum_{k=j+1}^{i-1} \prod_{l=k}^{i-1} u_l^2\right) + \prod_{k=j}^{i-1} u_k \times \left(1 + \sum_{k=i+1}^{j-1} \prod_{l=k}^{j-1} u_l^2\right)}{\left(\prod_{k \in \mathcal{C}_{[\lambda n]}} u_k - 1\right)^2}. \quad (5.55)$$

If  $j = i+1$ , then

$$(R_u^{(\lambda, n)})^{-1}(i, i+1) = \frac{u_i \times \left(1 + \sum_{k=i+2}^{i-1} \prod_{l=k}^{i-1} u_l^2\right) + \prod_{k=i+1}^{i-1} u_k}{\left(\prod_{k \in \mathcal{C}_{[\lambda n]}} u_k - 1\right)^2} \quad (5.56)$$

if  $j = i-1$ , then

$$(R_u^{(\lambda, n)})^{-1}(i, i-1) = \frac{u_{i-1} \times \left(1 + \sum_{k=i+1}^{i-2} \prod_{l=k}^{i-2} u_l^2\right) + \prod_{k=i}^{i-2} u_k}{\left(\prod_{k \in \mathcal{C}_{[\lambda n]}} u_k - 1\right)^2} \quad (5.57)$$

Moreover,

$$(R_u^{(\lambda, n)})^{-1}(i, i) = \frac{1 + \sum_{k=i+1}^{i-1} \prod_{l=k}^{i-1} u_l^2}{\left(\prod_{k \in \mathcal{C}_{[\lambda n]}} u_k - 1\right)^2}. \quad (5.58)$$

*Proof of Proposition 5.28.*

Let  $L_u^{(\lambda, n)}$  be a matrix with the coefficients above. We have to check that  $R_u^{(\lambda, n)} L_u^{(\lambda, n)}$  is the identity matrix. The computation is quite awful. That is why we will only show that for every  $i \in \mathcal{C}_{[\lambda n]}$ ,  $(R_u^{(\lambda, n)} L_u^{(\lambda, n)})(i, i)$  equals 1 and  $(R_u^{(\lambda, n)} L_u^{(\lambda, n)})(i, i+1)$  equals 0. The other computations are not more difficult and we omit it in this paper for sake of convenience. Let  $i \in \mathcal{C}_{[\lambda n]}$ . Then, by definition of  $R_u^{(\lambda, n)}$ ,

$$(R_u^{(\lambda, n)} L_u^{(\lambda, n)})(i, i) = (1 + u_i^2) L_u^{(\lambda, n)}(i, i) - u_{i-1} L_u^{(\lambda, n)}(i, i-1) - u_i L_u^{(\lambda, n)}(i, i+1). \quad (5.59)$$

For sake of convenience, we multiply identity (5.59) by

$$\left(\prod_{k \in \mathcal{C}_{[\lambda n]}} u_k - 1\right)^2.$$

Then, using the definition of the coefficients of  $L_u^{(\lambda,n)}$ , we get

$$\begin{aligned}
& (R_u^{(\lambda,n)} L_u^{(\lambda,n)})(i, i) \times \left( \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k - 1 \right)^2 \\
&= (1 + u_i^2) \times \left( 1 + \sum_{k=i+1}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 \right) - u_{i-1} \times \left( u_{i-1} \times \left( 1 + \sum_{k=i+1}^{i-2} \vec{\prod}_{l=k}^{i-2} u_l^2 \right) + \vec{\prod}_{k=i}^{i-2} u_k \right) \\
&\quad - u_i \times \left( u_i \times \left( 1 + \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 \right) + \vec{\prod}_{k=i+1}^{i-1} u_k \right) \\
&= 1 + u_i^2 + \sum_{k=i+1}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 + \sum_{k=i+1}^{i-1} \vec{\prod}_{l=k}^i u_l^2 - u_{i-1}^2 - \sum_{k=i+1}^{i-2} \vec{\prod}_{l=k}^{i-1} u_l^2 - \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k \\
&\quad - u_i^2 - \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^i u_l^2 - \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k \\
&= 1 - 2 \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k - u_{i-1}^2 + \left( \sum_{k=i+1}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 - \sum_{k=i+1}^{i-2} \vec{\prod}_{l=k}^{i-1} u_l^2 \right) \\
&\quad + \left( \sum_{k=i+1}^{i-1} \vec{\prod}_{l=k}^i u_l^2 - \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^i u_l^2 \right) \\
&= 1 - 2 \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k - u_{i-1}^2 + u_{i-1}^2 + \vec{\prod}_{l=i+1}^i u_l^2 \\
&= \left( 1 - \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k \right)^2.
\end{aligned}$$

Thus, we get that

$$(R_u^{(\lambda,n)} L_u^{(\lambda,n)})(i, i) = 1.$$

Now, let us look at  $(R_u^{(\lambda,n)} L_u^{(\lambda,n)})(i, i+1)$ . By definition of  $R_{(\lambda,n)}$ ,

$$\begin{aligned}
& (R_u^{(\lambda,n)} L_u^{(\lambda,n)})(i, i+1) \\
&= (1 + u_i^2) L_u^{(\lambda,n)}(i, i+1) - u_{i-1} L_u^{(\lambda,n)}(i-1, i+1) - u_i L_u^{(\lambda,n)}(i+1, i+1). \quad (5.60)
\end{aligned}$$

As previously, we multiply identity (5.60) by  $\left( \prod_{k \in \mathcal{C}_{[\lambda_n]}} u_k - 1 \right)^2$ . Then, using the definition of the coefficients of  $L_u^{(\lambda,n)}$ , we get

$$\begin{aligned}
& (1 + u_i^2) \times \left( u_i \times \left( 1 + \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 \right) + \vec{\prod}_{k=i+1}^{i-1} u_k \right) \\
&\quad - u_{i-1} \times \left( u_{i-1} u_i \left( 1 + \sum_{k=i+2}^{i-2} \vec{\prod}_{l=k}^{i-2} u_l^2 \right) + \vec{\prod}_{k=i+1}^{i-2} u_k \times (1 + u_i^2) \right) \\
&\quad - u_i \times \left( 1 + \sum_{k=i+2}^i \vec{\prod}_{l=k}^i u_l^2 \right) \\
&= u_i + u_i^3 + u_i \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^i u_l^2 + u_i \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 + (1 + u_i^2) \vec{\prod}_{k=i+1}^{i-1} u_k \\
&\quad - u_i u_{i-1}^2 - u_i \sum_{k=i+2}^{i-2} \vec{\prod}_{l=k}^{i-1} u_l^2 - (1 + u_i^2) \vec{\prod}_{k=i+1}^{i-1} u_k - u_i - u_i \sum_{k=i+2}^i \vec{\prod}_{l=k}^i u_l^2 \\
&= u_i^3 - u_i u_{i-1}^2 + u_i \left( \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^{i-1} u_l^2 - \sum_{k=i+2}^{i-2} \vec{\prod}_{l=k}^{i-1} u_l^2 \right) \\
&\quad + u_i \left( \sum_{k=i+2}^{i-1} \vec{\prod}_{l=k}^i u_l^2 - \sum_{k=i+2}^i \vec{\prod}_{l=k}^i u_l^2 \right) \\
&= u_i^3 - u_i u_{i-1}^2 + u_i u_{i-1}^2 - u_i^3 \\
&= 0.
\end{aligned}$$

This concludes the proof.  $\square$

### 5.7 Convergence of the discretized Green function

Before proving Theorem 5.7, we need to prove a first lemma.

For every  $t \in [-\lambda, \lambda]$  and for every  $n \in \mathbb{N}^*$ , let us define a continuous random fonction  $t \mapsto X_t^{(n)}$  such that if  $j/n \leq t < (j + 1)/n$ ,

$$X_t^{(n)} = \prod_{i=-[\lambda n]}^j u_i + n(t - j/n) \left( \prod_{i=-[\lambda n]}^{j+1} u_i - \prod_{i=-[\lambda n]}^j u_i \right)$$

where for every  $i \in \mathcal{C}_{[\lambda n]}$ ,  $u_i = \sqrt{A_i A_{i+1}}$  where  $(A_i)_{i \in \mathcal{C}_{[\lambda n]}}$  is a family of independent Inverse Gaussian random variables with parameters  $(1, n)$ . Then we have the following lemma :

**Lemma 5.29.** *For the topology of uniform convergence on  $[-\lambda, \lambda]$ ,*

$$(X_t^{(n)})_{t \in [-\lambda, \lambda]} \xrightarrow[n \rightarrow +\infty]{law} (X_t)_{t \in [-\lambda, \lambda]} := \left( e^{B_t + \lambda - (t+\lambda)/2} \right)_{t \in [-\lambda, \lambda]}.$$

*Proof of lemma 5.29.*

First, remark that for every  $t \in [-\lambda, \lambda]$  and for every  $n \in \mathbb{N}^*$ ,

$$X_t^{(n)} = Y_t^{(n)} + E_t^{(n)} \tag{5.61}$$

where  $t \mapsto Y_t^{(n)}$  is a continuous function such that if  $j/n \leq t < (j + 1)/n$ ,

$$Y_t^{(n)} = \prod_{i=-[\lambda n]}^j A_i + n(t - j/n) \left( \prod_{i=-[\lambda n]}^{j+1} A_i - \prod_{i=-[\lambda n]}^j A_i \right)$$

and  $t \mapsto E_t^{(n)}$  is a random error function. By Lemma 5.21,  $\sup_{t \in [-\lambda, \lambda]} E_t^{(n)}$  goes to zero in probability

as  $n$  goes to infinity. Consequently, we only have to focus on  $Y^{(n)}$ . It converges toward a Brownian motion according to Lemma 5.22. □

Now, we are ready to prove Theorem 5.7.

*Proof of Theorem 5.7.*

Let us define a rescaled bilinear continuous interpolation  $(\tilde{I}_u^{(\lambda, n)})$  of  $(R_u^{(\lambda, n)})^{-1}$  exactly as in (5.6). Besides, recall that by Lemma 5.29

$$(X_t^{(n)})_{t \in [-\lambda, \lambda]} \xrightarrow[n \rightarrow +\infty]{law} (X_t)_{t \in [-\lambda, \lambda]} \tag{5.62}$$

where the convergence holds for the topology of uniform convergence. Now, the idea of the proof is to write  $\frac{1}{n} \tilde{I}_u^{(\lambda, n)}$  as a function of  $X^{(n)}$ . Remark that

— If  $j > i$  (for the usual order in  $\{-[\lambda n], \dots, [\lambda n]\}$ ) then

$$\prod_{l=j}^{\rightarrow i} u_l = X_{i/n}^{(n)} \frac{X_{[\lambda n]/n}^{(n)}}{X_{(j-1)/n}^{(n)}}.$$

— If  $j < i$ , then

$$\prod_{l=j}^{\rightarrow i} u_l = \frac{X_{i/n}^{(n)}}{X_{(j-1)/n}^{(n)}}.$$

Therefore, by Proposition 5.28, it holds that :

— From (5.58), if  $t \in [-\lambda, \lambda]$ ,

$$\frac{1}{n} \tilde{I}_u^{(\lambda, n)}(t, t) = \frac{\int_t^\lambda \left( \frac{X_\lambda^{(n)} X_t^{(n)}}{X_s^{(n)}} \right)^2 ds + \int_{-\lambda}^t \left( \frac{X_t^{(n)}}{X_s^{(n)}} \right)^2 ds}{(X_\lambda^{(n)} - 1)^2} + o_n^{\mathbb{P}}(1) \quad (5.63)$$

where  $o_n^{\mathbb{P}}(1)$  is a random variable which goes to zero in probability uniformly in  $t$ . (The uniformity in  $t$  comes from Lemma 5.21.)

— From (5.55), if  $t, t' \in [-\lambda, \lambda]$  with  $t < t'$ ,

$$\begin{aligned} \frac{1}{n} \tilde{I}_u^{(\lambda, n)}(t, t') &= \\ \frac{X_{t'}^{(n)}}{X_t^{(n)}} &\left( \frac{\int_t^\lambda \left( \frac{X_\lambda^{(n)} X_t^{(n)}}{X_s^{(n)}} \right)^2 ds + \int_{-\lambda}^t \left( \frac{X_t^{(n)}}{X_s^{(n)}} \right)^2 ds}{(X_\lambda^{(n)} - 1)^2} + \frac{X_\lambda^{(n)} X_t^{(n)}}{X_{t'}^{(n)}} \int_t^{t'} \left( \frac{X_{t'}^{(n)}}{X_s^{(n)}} \right)^2 ds \right) + o_n^{\mathbb{P}}(1) \end{aligned} \quad (5.64)$$

where  $o_n^{\mathbb{P}}(1)$  goes to 0 in probability uniformly in  $t$  and  $t'$  thanks to Lemma 5.21.

Remark that (5.63) is just a special case of (5.64). Therefore, in the rest of the proof, we will only focus on (5.64). From (5.64) and (5.62) and the fact that the Lebesgue integral is a continuous functional for the topology of uniform convergence, we obtain that  $\frac{1}{n} \tilde{I}_u^{(\lambda, n)}(\cdot, \cdot)$  converges in law for the topology of uniform convergence toward some symmetric random kernel  $(\mathcal{R}^{(\lambda)}(t, t'))_{(t, t') \in (\mathcal{C}(\lambda))^2}$  which is defined for  $t, t' \in [-\lambda, \lambda]$  such that  $t \leq t'$  by the formula

$$\begin{aligned} \mathcal{R}^{(\lambda)}(t, t') &= \frac{\frac{X_{t'}}{X_t} \left( \int_t^\lambda \left( \frac{X_\lambda X_t}{X_s} \right)^2 ds + \int_{-\lambda}^t \left( \frac{X_t}{X_s} \right)^2 ds \right) + \frac{X_\lambda X_t}{X_{t'}} \int_t^{t'} \left( \frac{X_{t'}}{X_s} \right)^2 ds}{(X_\lambda - 1)^2} \\ &= \frac{X_{t'} X_t}{(X_\lambda - 1)^2} \left( X_\lambda^2 \int_{t'}^\lambda \frac{ds}{X_s^2} + X_\lambda \int_t^{t'} \frac{ds}{X_s^2} + \int_{-\lambda}^t \frac{ds}{X_s^2} \right). \end{aligned} \quad (5.65)$$

Now, let us remark that  $(X_t/X_0)_{t \in [-\lambda, \lambda]}$  is distributed as  $(M_t)_{t \in [-\lambda, \lambda]}$  where  $M$  was defined in subsection 5.2.4.2. Moreover, by (5.65), for every  $t, t' \in [-\lambda, \lambda]$  such that  $t \leq t'$

$$\begin{aligned} \mathcal{R}^{(\lambda)}(t, t') &= \\ &= \frac{(X_{t'}/X_0)(X_t/X_0)}{(X_\lambda/X_0 - 1/X_0)^2} \left( (X_\lambda/X_0)^2 \int_{t'}^\lambda \frac{X_0^2}{X_s^2} ds + (X_\lambda/X_0) X_0^{-1} \int_t^{t'} \frac{X_0^2}{X_s^2} ds + X_0^{-2} \int_{-\lambda}^t \frac{X_0^2}{X_s^2} ds \right). \end{aligned}$$

Consequently, it holds that

$$\begin{aligned} &\left( \mathcal{R}^{(\lambda)}(t, t') \right)_{-\lambda \leq t \leq t' \leq \lambda} \\ &\stackrel{\text{law}}{=} \left( \frac{M_{t'} M_t}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_{t'}^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^{t'} \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) \right)_{-\lambda \leq t \leq t' \leq \lambda}. \end{aligned}$$

Therefore,  $\mathcal{R}^{(\lambda)}$  has the same distribution as  $\mathcal{G}^{(\lambda)}$  which is introduced in subsection 5.2.4.2. In order to conclude the proof, we only have to justify that  $\frac{1}{n} \tilde{I}_u^{(\lambda, n)}(\cdot, \cdot)$  has the same limit as  $\tilde{G}_\beta^{(\lambda, n)}(\cdot, \cdot)$ . Recall that, by (5.54)

$$G_\beta^{(\lambda, n)} = \frac{1}{n} D(R_u^{(\lambda, n)})^{-1} D$$

where  $D$  is a diagonal matrix whose entries are  $(\sqrt{A_i})_{i \in \mathcal{C}_{[\lambda n]}}$  and  $(A_i)_{i \in \mathcal{C}_{[\lambda n]}}$  are independent Inverse Gaussian random variables with parameters  $(1, n)$ . However by Lemma 5.21,  $\sup \{|\ln(A_i)|, i \in \mathcal{C}_{[\lambda n]}\}$  goes to 0 in probability as  $n$  goes to infinity. Therefore,  $D$  goes to the identity matrix in probability uniformly in its coefficients as  $n$  goes to infinity. Consequently, the limits in law of  $n^{-1}\tilde{I}_u^{(\lambda, n)}$  and  $\tilde{G}_\beta^{(\lambda, n)}$  are the same, that is,  $\mathcal{R}^{(\lambda)} \stackrel{\text{law}}{=} \mathcal{G}^{(\lambda)}$ . This concludes the proof.  $\square$

## 5.8 Study of $\mathcal{G}^{(\lambda)}$

*Proof of Proposition 5.8.*

Remark that for every  $t < t'$ ,  $\mathcal{G}^{(\lambda)}(t, t')$  can be divided into three parts as follows

$$(M_\lambda - M_{-\lambda})^2 \frac{\mathcal{G}^{(\lambda)}(t, t')}{M_t M_{t'}} = M_\lambda^2 A(t, t') + M_\lambda M_{-\lambda} B(t, t') + M_{-\lambda}^2 C(t, t') \quad (5.66)$$

with

$$A(t, t') = \int_{t'}^\lambda \frac{ds}{M_s^2}, \quad B(t, t') = \int_t^{t'} \frac{ds}{M_s^2} \quad \text{and} \quad C(t, t') = \int_{-\lambda}^t \frac{ds}{M_s^2}.$$

Therefore, we have three symmetric kernels  $A$ ,  $B$  and  $C$ . Let  $f \in L^2([-\lambda, \lambda])$ . We can observe that

$$\int_{-\lambda}^\lambda \int_{-\lambda}^\lambda A(t, t') f(t) \bar{f}(t') dt dt' = \int_{-\lambda}^\lambda \int_{-\lambda}^{t'} \int_t^\lambda \frac{ds}{M_s^2} f(t) \bar{f}(t') dt dt' ds + \int_{-\lambda}^\lambda \int_{t'}^\lambda \int_t^\lambda \frac{ds}{M_s^2} f(t) \bar{f}(t') dt dt' ds.$$

By Fubini's theorem, this implies that

$$\begin{aligned} \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda A(t, t') f(t) \bar{f}(t') dt dt' &= \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s \int_{-\lambda}^{t'} f(t) \bar{f}(t') dt dt' ds + \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s \int_{t'}^s f(t) \bar{f}(t') dt dt' ds \\ &= \int_{-\lambda}^\lambda \frac{1}{M_s^2} \left| \int_{-\lambda}^s f(t) dt \right|^2 ds. \end{aligned} \quad (5.67)$$

In the same way, one can show that,

$$\int_{-\lambda}^\lambda \int_{-\lambda}^\lambda C(t, t') f(t) \bar{f}(t') dt dt' = \int_{-\lambda}^\lambda \frac{1}{M_s^2} \left| \int_s^\lambda f(t) dt \right|^2 ds. \quad (5.68)$$

By Fubini's theorem again, we get

$$\begin{aligned} \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda B(t, t') f(t) \bar{f}(t') dt dt' &= \int_{-\lambda}^\lambda \int_{-\lambda}^{t'} \int_t^{t'} \frac{ds}{M_s^2} f(t) \bar{f}(t') dt dt' ds + \int_{-\lambda}^\lambda \int_{t'}^\lambda \int_t^{t'} \frac{ds}{M_s^2} f(t) \bar{f}(t') dt dt' ds \\ &= \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s f(t) dt \int_s^\lambda \bar{f}(t) dt ds + \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s \bar{f}(t) dt \int_s^\lambda f(t) dt ds. \end{aligned} \quad (5.69)$$

These identities hold for every  $f \in L^2([-\lambda, \lambda])$ . Let  $f \in L^2([-\lambda, \lambda])$ . Combining identities (5.67), (5.68), (5.69) and (5.66), we get

$$\begin{aligned} &(M_\lambda - M_{-\lambda})^2 \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda \mathcal{G}^{(\lambda)}(t, t') f(t) \bar{f}(t') dt dt' \\ &= M_\lambda^2 \int_{-\lambda}^\lambda \frac{1}{M_s^2} \left| \int_{-\lambda}^s f(t) M_t dt \right|^2 ds + M_\lambda M_{-\lambda} \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s f(t) M_t dt \int_s^\lambda \bar{f}(t) M_t dt ds \\ &\quad + M_\lambda M_{-\lambda} \int_{-\lambda}^\lambda \frac{1}{M_s^2} \int_{-\lambda}^s \bar{f}(t) M_t dt \int_s^\lambda f(t) M_t dt ds + M_{-\lambda}^2 \int_{-\lambda}^\lambda \frac{1}{M_s^2} \left| \int_s^\lambda f(t) M_t dt \right|^2 ds \\ &= \int_{-\lambda}^\lambda \frac{1}{M_s^2} \left| M_\lambda \int_{-\lambda}^s f(t) M_t dt + M_{-\lambda} \int_s^\lambda f(t) M_t dt \right|^2 ds. \end{aligned}$$

This proves that  $\mathcal{G}^\lambda$  is non-negative. Now, let us check that it is positive. Let  $f \in L^2([-\lambda, \lambda])$  such that

$$\int_{-\lambda}^{\lambda} \frac{1}{M_s^2} \left| M_\lambda \int_{-\lambda}^s f(t) M_t dt + M_{-\lambda} \int_s^{\lambda} f(t) M_t dt \right|^2 ds = 0.$$

Then, for almost every  $s \in [-\lambda, \lambda]$ ,

$$M_\lambda \int_{-\lambda}^s f(t) M_t dt + M_{-\lambda} \int_s^{\lambda} f(t) M_t dt = 0.$$

Thus, by Lebesgue differentiation theorem, we get that for almost every  $s \in [-\lambda, \lambda]$ ,

$$M_\lambda f(s) M_s - M_{-\lambda} f(s) M_s = 0.$$

Moreover  $M_\lambda - M_{-\lambda} \neq 0$  almost surely. Therefore, almost surely,  $f$  is zero almost everywhere. This concludes the proof.  $\square$

## 5.9 Proof of identities in law

*Proof of Proposition 5.9.*

Let  $t \in \mathcal{C}^{(\lambda)}$ . By Theorem 5.7,

$$G_\beta^{(\lambda, n)}([tn], [tn]) \xrightarrow[n \rightarrow +\infty]{law} \mathcal{G}^{(\lambda)}(t, t).$$

However, by Theorem 3 in [153], for every  $n \in \mathbb{N}^*$ ,  $G_\beta^{(\lambda, n)}([tn], [tn])$  is distributed as  $1/(2\gamma)$  where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ . Therefore, we get that

$$\frac{M_t^2}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_t^\lambda \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) = \mathcal{G}^{(\lambda)}(t, t) \stackrel{law}{=} \frac{1}{2\gamma}.$$

$\square$

*Proof of corollary 5.10.*

**First proof :** Let us use Proposition 5.9 with  $t = -\lambda$ . This gives that

$$\frac{M_{-\lambda}^2 M_\lambda^2}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{ds}{M_s^2} \stackrel{law}{=} \frac{1}{2\gamma}$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ . However, we can rewrite the left-hand side as

$$\begin{aligned} \frac{M_{-\lambda}^2 M_\lambda^2}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{ds}{M_s^2} &= \frac{1}{\left(\frac{M_\lambda}{M_{-\lambda}} - 1\right)^2} \int_0^{2\lambda} \frac{M_\lambda^2}{M_{s-\lambda}^2} ds \\ &= \frac{1}{\left(\frac{M_\lambda}{M_{-\lambda}} - 1\right)^2} \int_0^{2\lambda} \frac{M_\lambda^2}{M_{\lambda-s}^2} ds. \end{aligned}$$

However, recall that for every  $s \in [-\lambda, \lambda]$ ,  $M_s = e^{B_s - s/2}$  where  $B$  is a Brownian motion such that  $B(0) = 0$ . Consequently, we get that

$$\frac{M_{-\lambda}^2 M_\lambda^2}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^\lambda \frac{ds}{M_s^2} = \frac{1}{(e^{B_\lambda - B_{-\lambda} - \lambda} - 1)^2} \int_0^{2\lambda} e^{2(B_\lambda - B_{\lambda-s}) - s} ds.$$

Remark that  $\tilde{B} := (B_\lambda - B_{\lambda-s})_{s \geq 0}$  is a standard Brownian motion such that  $\tilde{B}(0) = 0$ . This gives exactly the formula in Corollary 5.10 with  $\tilde{B}$  and  $2\lambda$ . This first proof of Corollary 5.10 uses directly the new tools developed in this paper. However, this is also possible to prove it thanks to the Matsumoto-Yor properties whose a new proof is given in this paper.

**Second proof :** Let  $\lambda > 0$ . Let us consider a Brownian motion  $\alpha$  on  $\mathbb{R}_+$  such that  $\alpha_0 = 0$ . For every  $t \geq 0$ , we define  $e_t = e^{\alpha_t - t/2}$ ,  $T_t = \int_0^t e_s^2 ds$  and  $Z_t = T_t/e_t$ . By (iii) in Theorem 5.4, the law of  $e_t$  conditionally on  $Z_t$  is an Inverse Gaussian distribution with parameters  $(1, 1/Z_t)$ . Therefore, conditionally on  $Z_t = z$ ,

$$\frac{(e_t - 1)^2}{T_t} = \frac{(e_t - 1)^2}{ze_t}$$

is distributed as

$$\frac{(X - 1)^2}{zX}$$

where  $X$  is an Inverse Gaussian distribution with parameters  $(1, 1/z)$ . However, it is true generally that if  $Y$  is an Inverse Gaussian distribution with parameters  $(\mu, r)$ , then

$$r \frac{(Y - \mu)^2}{\mu^2 Y} \stackrel{\text{law}}{=} 2\gamma$$

where  $\gamma$  is a Gamma distribution with parameters  $(1/2, 1)$ . (This stems from a direct computation involving the density of an Inverse Gaussian distribution.) Consequently,  $\frac{(e_t - 1)^2}{T_t}$  is distributed like  $2\gamma$ . This is exactly what we wanted to prove. □

Now, let us prove functional identities in law.

*Proof of Proposition 5.11.*

Let  $f$  be a deterministic continuous non-negative function on  $\mathcal{C}^\lambda$ . By Lemma 8.1 in [77], for every  $n \in \mathbb{N}^*$ , for every  $\eta \in \mathbb{R}_+^{\mathcal{C}_{[\lambda n]}}$ ,

$$\sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_\beta^{(\lambda, n)}(i, j) \eta_i \eta_j \stackrel{\text{law}}{=} \frac{\left( \sum_{i \in \mathcal{C}_{[\lambda n]}} \eta_i \right)^2}{2\gamma}$$

where  $\gamma$  is distributed like a gamma distribution with parameters  $(1/2, 1)$ . Now, let us apply this fact with  $\eta_i = n^{-1} f(i/n)$ . Then we obtain that

$$\left( \frac{1}{n} \sum_{i \in \mathcal{C}_{[\lambda n]}} f(i/n) \right)^2 \frac{1}{2\gamma} \stackrel{\text{law}}{=} \frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_\beta^{(\lambda, n)}(i, j) f(i/n) f(j/n).$$

The left-hand side converges in law toward  $\left( \int_{-\lambda}^\lambda f(x) dx \right)^2 \frac{1}{2\gamma}$  because  $f$  is assumed to be continuous. Now, let us focus on the right-hand side. By Lemma 5.21, it holds that,

$$\begin{aligned} & \frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_\beta^{(\lambda, n)}(i, j) f(i/n) f(j/n) \\ &= \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} \int_{i/n}^{(i+1)/n} \int_{j/n}^{(j+1)/n} \tilde{G}_\beta^{(\lambda, n)}(t, t') dt dt' f(i/n) f(j/n) + o_n^{\mathbb{P}}(1) \end{aligned} \tag{5.70}$$

where  $o_n^{\mathbb{P}}(1)$  goes to 0 in probability. Moreover, by (5.70) and the continuity of  $f$ , we get that

$$\frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_\beta^{(\lambda, n)}(i, j) f(i/n) f(j/n) = \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \tilde{G}_\beta^{(\lambda, n)}(t, t') f(t) f(t') dt dt' + o_n^{\mathbb{P}}(1). \quad (5.71)$$

We know that  $\tilde{G}_\beta^{(\lambda, n)}$  converges in law toward  $\mathcal{G}^{(\lambda)}$ . Besides, if  $W(\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)})$  is the normed vector space of continuous functions on  $\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)}$  with the uniform norm, then

$$H \mapsto \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} H(t, t') f(t) f(t') dt dt'$$

is a continuous function on  $W(\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)})$ . Together with (5.71), this implies that

$$\frac{1}{n^2} \sum_{i \in \mathcal{C}_{[\lambda n]}} \sum_{j \in \mathcal{C}_{[\lambda n]}} G_\beta^{(\lambda, n)}(i, j) f(i/n) f(j/n) \xrightarrow[n \rightarrow +\infty]{law} \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \mathcal{G}^{(\lambda)}(t, t') f(t) f(t') dt dt'.$$

Moreover, by Proposition 5.8,

$$\begin{aligned} & \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \mathcal{G}^{(\lambda)}(t, t') f(t) f(t') dx dy \\ &= \frac{1}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^{\lambda} \frac{1}{M_u^2} \left( M_{-\lambda} \int_u^{\lambda} f(t) M_t dt + M_\lambda \int_{-\lambda}^u M_t f(t) dt \right)^2 du. \end{aligned}$$

It concludes the proof.  $\square$

## 5.10 Study of $\mathcal{H}^{(\lambda)}$

*Proof of Theorem 5.12.*

**Step 1 :** First let us show that the range of  $\mathcal{G}^{(\lambda)}$  is included in  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Let  $f \in L^2([-\lambda, \lambda])$ . For every  $x \in \mathcal{C}^{(\lambda)}$ , it holds that

$$\begin{aligned} (M_\lambda - M_{-\lambda})^2 \mathcal{G}^{(\lambda)} f(x) &= (M_\lambda - M_{-\lambda})^2 \left( \int_{-\lambda}^x f(t) \mathcal{G}^{(\lambda)}(x, t) dt + \int_x^{\lambda} f(t) \mathcal{G}^{(\lambda)}(x, t) dt \right) \\ &= \int_{-\lambda}^x f(t) M_x M_t \left( M_\lambda^2 \int_x^{\lambda} \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^x \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) dt \\ &\quad + \int_x^{\lambda} f(t) M_x M_t \left( M_\lambda^2 \int_t^{\lambda} \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_x^t \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^x \frac{ds}{M_s^2} \right) dt. \end{aligned} \quad (5.72)$$

Therefore,  $\mathcal{G}^{(\lambda)} f$  is continuous. In particular it is in  $L^2([-\lambda, \lambda])$ . Besides, by looking at (5.72), one can remark that  $\mathcal{G}^{(\lambda)} f(-\lambda)$  and  $\mathcal{G}^{(\lambda)} f(\lambda)$  are both equal to

$$\frac{1}{(M_\lambda - M_{-\lambda})^2} \int_{-\lambda}^{\lambda} f(t) M_t \left( M_\lambda^2 M_{-\lambda} \int_t^{\lambda} \frac{ds}{M_s^2} + M_\lambda M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) dt.$$

Now, we have to look at the derivative of  $\left(\frac{\mathcal{G}^{(\lambda)}}{M}\right)$ . By differentiating (5.72), we get that for every  $x \in \mathcal{C}^{(\lambda)}$

$$\begin{aligned} & (M_\lambda - M_{-\lambda})^2 \left( \frac{\mathcal{G}^{(\lambda)} f}{M} \right)'(x) \\ &= \frac{1}{M_x^2} \left( M_\lambda (M_{-\lambda} - M_\lambda) \int_{-\lambda}^x f(t) M_t dt + M_{-\lambda} (M_{-\lambda} - M_\lambda) \int_x^{\lambda} f(t) M_t dt \right). \end{aligned} \quad (5.73)$$

By (5.73), It is clear that  $\left(\frac{\mathcal{G}^{(\lambda)}f}{M}\right)'$  is continuous. In particular, this is in  $L^2([-\lambda, \lambda])$ . Moreover, by looking at (5.73), one can remark that  $M_{-\lambda}\left(\frac{\mathcal{G}^{(\lambda)}f}{M}\right)'(-\lambda)$  and  $M_\lambda\left(\frac{\mathcal{G}^{(\lambda)}f}{M}\right)'(\lambda)$  are both equal to

$$\frac{1}{M_{-\lambda} - M_\lambda} \int_{-\lambda}^{\lambda} f(t)M_t dt.$$

Finally, one can differentiate also (5.73). This gives that for almost every  $x \in \mathcal{C}^{(\lambda)}$ ,

$$\left(M^2 \left(\frac{\mathcal{G}^{(\lambda)}f}{M}\right)'\right)'(x) = -f(x)M_x. \quad (5.74)$$

Recall that  $f \in L^2([-\lambda, \lambda])$ . Together with (5.74), this implies that  $\left(M^2 \left(\frac{\mathcal{G}^{(\lambda)}f}{M}\right)'\right)' \in L^2([-\lambda, \lambda])$ .

Therefore,  $\mathcal{G}^{(\lambda)}f \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ .

**Step 2 :** Now, we have to show that  $\mathcal{D}(\mathcal{H}^{(\lambda)})$  is included in the range of  $\mathcal{G}^{(\lambda)}$ . Let  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ . Let us define

$$f = -\frac{1}{M} \left(M^2 \left(\frac{g}{M}\right)'\right)'.$$

Let us show that  $g = \mathcal{G}^{(\lambda)}f$ . Exactly as in (5.72), it holds that for every  $x \in \mathcal{C}^{(\lambda)}$ ,

$$\begin{aligned} (M_\lambda - M_{-\lambda})^2 \mathcal{G}^{(\lambda)}f(x) &= (M_\lambda - M_{-\lambda})^2 \left( \int_{-\lambda}^x f(t)\mathcal{G}^{(\lambda)}(x,t)dt + \int_x^\lambda f(t)\mathcal{G}^{(\lambda)}(x,t)dt \right) \\ &= \int_{-\lambda}^x -\frac{1}{M_t} \left(M^2 \left(\frac{g}{M}\right)'\right)'(t)M_x M_t \left( M_\lambda^2 \int_x^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^x \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) dt \\ &\quad + \int_x^\lambda -\frac{1}{M_t} \left(M^2 \left(\frac{g}{M}\right)'\right)'(t)M_x M_t \left( M_\lambda^2 \int_t^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_x^t \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^x \frac{ds}{M_s^2} \right) dt \\ &= - \int_{-\lambda}^x \left(M^2 \left(\frac{g}{M}\right)'\right)'(t)M_x \left( M_\lambda^2 \int_x^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^x \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) dt \\ &\quad - \int_x^\lambda \left(M^2 \left(\frac{g}{M}\right)'\right)'(t)M_x \left( M_\lambda^2 \int_t^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_x^t \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^x \frac{ds}{M_s^2} \right) dt. \end{aligned} \quad (5.75)$$

Thanks to integration by parts, one can check that for every  $x \in \mathcal{C}^{(\lambda)}$ ,

$$\begin{aligned} \frac{(M_\lambda - M_{-\lambda})^2}{M_x} \mathcal{G}^{(\lambda)}f(x) &= -M_\lambda^2 \int_x^\lambda \frac{ds}{M_s^2} \left( M_x^2 \left(\frac{g}{M}\right)'(x) - M_{-\lambda}^2 \left(\frac{g}{M}\right)'(-\lambda) \right) \\ &\quad - M_\lambda M_{-\lambda} \int_{-\lambda}^x \left(\frac{g}{M}\right)'(t)dt + M_\lambda M_{-\lambda}^3 \left(\frac{g}{M}\right)'(-\lambda) \int_{-\lambda}^x \frac{ds}{M_s^2} \\ &\quad + M_{-\lambda}^2 \int_{-\lambda}^x \left(\frac{g}{M}\right)'(t)dt - M_{-\lambda}^2 M_x^2 \left(\frac{g}{M}\right)'(x) \int_{-\lambda}^x \frac{ds}{M_s^2} \\ &\quad - M_\lambda^2 \int_x^\lambda \left(\frac{g}{M}\right)'(t)dt + M_\lambda^2 M_x^2 \left(\frac{g}{M}\right)'(x) \int_x^\lambda \frac{ds}{M_s^2} \\ &\quad + M_\lambda M_{-\lambda} \int_x^\lambda \left(\frac{g}{M}\right)'(t)dt - M_\lambda^3 M_{-\lambda} \left(\frac{g}{M}\right)'(\lambda) \int_x^\lambda \frac{ds}{M_s^2} \\ &\quad - M_{-\lambda}^2 \int_{-\lambda}^x \frac{ds}{M_s^2} \left( M_\lambda^2 \left(\frac{g}{M}\right)'(\lambda) - M_x^2 \left(\frac{g}{M}\right)'(x) \right). \end{aligned} \quad (5.76)$$

This expression seems to be quite awful. However, recall from the definition of  $\mathcal{D}(\mathcal{H}^{(\lambda)})$  that  $M_{-\lambda}\left(\frac{g}{M}\right)'(-\lambda) = M_\lambda\left(\frac{g}{M}\right)'(\lambda)$ . Therefore, we can simplify many terms in (5.76) and we obtain

that for every  $x \in \mathcal{C}^{(\lambda)}$ ,

$$\begin{aligned}
\frac{(M_\lambda - M_{-\lambda})^2}{M_x} \mathcal{G}^{(\lambda)} f(x) &= -M_\lambda M_{-\lambda} \int_{-\lambda}^x \left(\frac{g}{M}\right)'(t) dt + M_{-\lambda}^2 \int_{-\lambda}^x \left(\frac{g}{M}\right)'(t) dt \\
&\quad - M_\lambda^2 \int_x^\lambda \left(\frac{g}{M}\right)'(t) dt + M_\lambda M_{-\lambda} \int_x^\lambda \left(\frac{g}{M}\right)'(t) dt \\
&= M_\lambda (g(\lambda) - g(-\lambda)) + M_{-\lambda} (g(\lambda) - g(-\lambda)) \\
&\quad + \frac{g(x)}{M_x} (-M_\lambda M_{-\lambda} + M_{-\lambda}^2 - M_\lambda M_{-\lambda} + M_\lambda^2) \\
&= \frac{g(x)}{M_x} (M_\lambda - M_{-\lambda})^2
\end{aligned} \tag{5.77}$$

where in the last equality, we used the fact that  $g(\lambda) = g(-\lambda)$ . Finally, we proved that for every  $x \in \mathcal{C}^{(\lambda)}$ ,

$$\mathcal{G}^{(\lambda)} f(x) = g(x).$$

Therefore,  $g$  is in the range of  $\mathcal{G}^{(\lambda)}$ . This concludes the proof of the fact that  $\mathcal{D}(\mathcal{H}^{(\lambda)})$  is exactly the image of  $\mathcal{G}^{(\lambda)}$ .

**Step 3 :** We know that  $\mathcal{G}^{(\lambda)}$  is a surjection from  $L^2([-\lambda, \lambda])$  onto  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Moreover, by Proposition 5.8,  $\mathcal{G}^{(\lambda)}$  is positive definite. In particular it is injective. Therefore,  $\mathcal{G}^{(\lambda)}$  is a one-to-one mapping from  $L^2([-\lambda, \lambda])$  onto  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Let us denote its inverse by  $\mathcal{H}^{(\lambda)}$ . The computation in the two first steps implies directly that for every  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ ,

$$H^{(\lambda)} g = -\frac{1}{M} \left( M^2 \left( \frac{g}{M} \right)' \right)'.$$

Now, let us show that  $\mathcal{H}^{(\lambda)}$  is self-adjoint. On the first hand,  $\mathcal{H}^{(\lambda)}$  is clearly symmetric. In particular,  $\mathcal{D}(\mathcal{H}^{(\lambda)}) \subset \mathcal{D}(\mathcal{H}^{(\lambda)*})$ . On the second hand, let  $\phi \in \mathcal{D}(\mathcal{H}^{(\lambda)*})$ . By definition, there exists  $\eta \in L^2([-\lambda, \lambda])$  such that for every  $g \in \mathcal{D}(\mathcal{H}^{(\lambda)})$ ,

$$\int_{-\lambda}^\lambda \mathcal{H}^{(\lambda)} g(x) \bar{\phi}(x) dx = \int_{-\lambda}^\lambda g(x) \bar{\eta}(x) dx.$$

As the image of  $\mathcal{G}^{(\lambda)}$  is included in  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ , it holds that for every  $f \in L^2([-\lambda, \lambda])$ ,

$$\begin{aligned}
\int_{-\lambda}^\lambda \mathcal{H}^{(\lambda)} \mathcal{G}^{(\lambda)} f(x) \bar{\phi}(x) dx &= \int_{-\lambda}^\lambda \mathcal{G}^{(\lambda)} f(x) \bar{\eta}(x) dx \\
&= \int_{-\lambda}^\lambda f(x) \mathcal{G}^{(\lambda)} \bar{\eta}(x) dx
\end{aligned}$$

where in the second equality we used the fact that  $\mathcal{G}^{(\lambda)}$  is self-adjoint. Therefore, for every  $f \in L^2([-\lambda, \lambda])$ ,

$$\int_{-\lambda}^\lambda f(x) \bar{\phi}(x) dx = \int_{-\lambda}^\lambda f(x) \mathcal{G}^{(\lambda)} \bar{\eta}(x) dx.$$

Consequently,  $\phi = \mathcal{G}^{(\lambda)} \eta$ . Thus,  $\phi$  is in the image of  $\mathcal{G}^{(\lambda)}$ , that is,  $\mathcal{D}(\mathcal{H}^{(\lambda)})$ . Finally, we proved that  $\mathcal{D}(\mathcal{H}^{(\lambda)}) = \mathcal{D}(\mathcal{H}^{(\lambda)*})$  which means that  $\mathcal{H}^{(\lambda)}$  is self-adjoint. Moreover,  $\mathcal{H}^{(\lambda)}$  is positive definite because it is the inverse of  $\mathcal{G}^{(\lambda)}$  which is itself positive definite by Proposition 5.8.  $\square$

Now, let us show that the spectrum of  $\mathcal{H}^{(\lambda)}$  consists in a sequence of increasing positive eigenvalues.

*Proof of Proposition 5.13.*

Recall that the operator  $\mathcal{G}^{(\lambda)}$  on  $L^2([-\lambda, \lambda])$  comes from a continuous kernel on the compact set  $\mathcal{C}^{(\lambda)} \times \mathcal{C}^{(\lambda)}$ . Then, a classical proof using Arzelà-Ascoli Theorem shows that  $\mathcal{G}^{(\lambda)}$  is a compact operator on  $L^2([-\lambda, \lambda])$ . (See the beginning of section VI.5 in [143].) Moreover, we know that  $\mathcal{G}^{(\lambda)}$  is self-adjoint on  $L^2([-\lambda, \lambda])$ . Therefore, by the theorem of diagonalization of self-adjoint compact operators (see Theorem VI.16 in [143]), there exists a random sequence  $(A_k(\lambda))_{k \geq 0}$  which represents the eigenvalues of  $\mathcal{G}^{(\lambda)}$  which are counted with multiplicity. The sequence  $(A_k(\lambda))_{k \geq 0}$  is bounded, decreasing and converges to 0 when  $k$  goes to infinity. Moreover,

$$\sigma(\mathcal{G}^{(\lambda)}) = \{0\} \cup \{(A_k(\lambda))_{k \geq 0}\}.$$

Indeed, 0 is also a spectral value of  $\mathcal{G}^{(\lambda)}$  but not an eigenvalue because we know by Theorem 5.12 that  $\mathcal{G}^{(\lambda)}$  is injective but not surjective because its image is  $\mathcal{D}(\mathcal{H}^{(\lambda)}) \subsetneq L^2([-\lambda, \lambda])$ . Now, remark that  $E$  is a spectral value of  $\mathcal{H}^{(\lambda)}$  if and only if  $Id - E\mathcal{G}^{(\lambda)}$  is invertible. Therefore,

$$\sigma(\mathcal{H}^{(\lambda)}) \cap \mathbb{R}^* = \{(1/A_k(\lambda))_{k \geq 0}\} := \{(E_k(\lambda))_{k \geq 0}\}.$$

Moreover, 0 is not a spectral value of  $\mathcal{H}^{(\lambda)}$  because it is a bijection from its domain onto  $L^2([-\lambda, \lambda])$  by Theorem 5.12.  $\square$

## 5.11 Proof of asymptotic results on $\mathcal{H}^{(\lambda)}$

Now, let us prove Proposition 5.14.

*Proof of Proposition 5.14.*

By subsection 5.2.4.2 for every  $\lambda > 0$  and for every  $t, t' \in [-\lambda, \lambda]$  such that  $t \geq t'$ ,

$$\begin{aligned} \mathcal{G}^{(\lambda)}(t, t') &= \frac{M_{t'}M_t}{(M_\lambda - M_{-\lambda})^2} \left( M_\lambda^2 \int_{t'}^\lambda \frac{ds}{M_s^2} + M_\lambda M_{-\lambda} \int_t^{t'} \frac{ds}{M_s^2} + M_{-\lambda}^2 \int_{-\lambda}^t \frac{ds}{M_s^2} \right) \\ &= \frac{M_{t'}M_t}{(M_\lambda/M_{-\lambda} - 1)^2} \left( \frac{M_\lambda^2}{M_{-\lambda}^2} \int_{t'}^\lambda \frac{ds}{M_s^2} + \frac{M_\lambda}{M_{-\lambda}} \int_t^{t'} \frac{ds}{M_s^2} + \int_{-\lambda}^t \frac{ds}{M_s^2} \right). \end{aligned} \quad (5.78)$$

We know that  $M_\lambda/M_{-\lambda}$  goes to 0 when  $\lambda$  goes to infinity. Therefore, we can replace the term  $\frac{1}{(M_\lambda/M_{-\lambda} - 1)^2}$  by 1 when  $\lambda$  goes to infinity. Moreover, for every  $\lambda > 0$ , for every  $t' \in [-\lambda, \lambda]$ , it holds almost surely that

$$\frac{M_\lambda^2}{M_{-\lambda}^2} \int_{t'}^\lambda \frac{ds}{M_s^2} = e^{2B_\lambda - 2B_{-\lambda} - 2\lambda} \int_{t'}^\lambda e^{-2B_s + s} ds = O(e^{-\lambda})$$

because  $|B_t|_{t \rightarrow +\infty} = o(t^{3/4})$ . In the same way, one can show that

$$\frac{M_\lambda}{M_{-\lambda}} \int_t^{t'} \frac{ds}{M_s^2} \xrightarrow[\lambda \rightarrow +\infty]{a.s.} 0.$$

Using this in (5.78) implies that for every  $T > 0$ ,

$$\sup_{-T \leq t \leq t' \leq T} \left| \mathcal{G}^{(\lambda)}(t, t') - M_{t'}M_t \int_{-\infty}^t \frac{ds}{M_s^2} \right| \xrightarrow[\lambda \rightarrow +\infty]{a.s.} 0. \quad (5.79)$$

It concludes the proof.  $\square$

Thanks to Proposition 5.14, we can prove a surprising identity in law.

*Proof of Corollary 5.15.*

By Proposition 5.14,

$$\left( \frac{\mathcal{G}^{(\infty)}(0, t)}{\mathcal{G}^{(\infty)}(0, 0)} \right)_{t \geq 0} = (M_t)_{t \geq 0} \quad (5.80)$$

and

$$\left( \frac{\mathcal{G}^{(\infty)}(0, -t)}{\mathcal{G}^{(\infty)}(0, 0)} \right)_{t \geq 0} = \left( M_{-t} \frac{\int_{-\infty}^{-t} \frac{ds}{M_s^2}}{\int_{-\infty}^0 \frac{ds}{M_s^2}} \right)_{t \geq 0}. \quad (5.81)$$

Besides,

$$\left( \frac{\mathcal{G}^{(\infty)}(0, t)}{\mathcal{G}^{(\infty)}(0, 0)} \right)_{t \geq 0} \stackrel{\text{law}}{=} \left( \frac{\mathcal{G}^{(\infty)}(0, -t)}{\mathcal{G}^{(\infty)}(0, 0)} \right)_{t \geq 0}. \quad (5.82)$$

Indeed, the law of the discrete process  $\mathcal{G}^{(\lambda, n)}$  is clearly symmetric and this property remains true when we take the limit. Recall that  $M_t = e^{B_t - t/2}$  for every  $t \in \mathbb{R}$ . Therefore, combining (5.80), (5.81) and (5.82), we get that

$$\left( e^{B_{-t} + t/2} \frac{\int_t^{+\infty} e^{-2B_{-s} - s} ds}{\int_0^{+\infty} e^{-2B_{-s} - s} ds} \right)_{t \geq 0}$$

is a geometric Brownian motion starting from 1. Moreover,  $(-B_{-t})_{t \geq 0}$  is a standard Brownian motion starting from 0. Therefore, we proved Corollary 5.15.  $\square$

Now, let us prove the asymptotic behaviour of the density of states of  $\mathcal{H}^{(\lambda)}$ . Our proof is inspired from a paper of Fukushima and Nakao (see [71]) and it is based on the study of a Sturm-Liouville equation.

**Strategy of the proof :** The equation  $\mathcal{H}^{(\lambda)}\varphi_E = E\varphi_E$  is equivalent to the Sturm-Liouville equation

$$(M^2\varphi'_E)' + EM^2\varphi_E = 0.$$

Some classical results on Sturm-Liouville equations imply that  $\varphi_E$  can be written as  $\varphi_E = R_E \sin(\theta_E)$  where  $R_E$  never vanishes. Therefore,  $\varphi_E$  vanishes  $k$  times in  $[-\lambda, \lambda]$  if and only if  $\theta_E(\lambda) \in [k\pi, (k+1)\pi]$ . Moreover, Sturm-Liouville oscillation theorem states that  $\varphi_{E_k(\lambda)}$  vanishes approximately  $k$  times. Consequently, if  $N_\lambda(E) = k$ , then  $E_k(\lambda) \leq E < E_{k+1}(\lambda)$  which implies that

$$N_\lambda(E) = k \simeq \frac{\theta_{E_k(\lambda)}(\lambda)}{\pi} \simeq \frac{\theta_E(\lambda)}{\pi}.$$

Therefore, for every fixed  $E > 0$ , we only have to study the asymptotic behaviour of  $\theta_E(\lambda)$  when  $\lambda$  goes to infinity. However, according to the theory of Sturm-Liouville equations,  $\theta_E$  is solution of the following ODE with random coefficients :

$$\theta'_E = M_t^{-2} \cos(\theta_E)^2 + EM_t^2 \sin(\theta_E)^2.$$

Moreover a very surprising fact is that  $\zeta_E := -\frac{\cot(\theta_E)}{M^2}$  is a Markov process with explosions. Actually explosions of  $\zeta_E$  occur precisely when  $\theta_E$  is a multiple of  $\pi$ . Therefore, we only have to count the explosions of  $\zeta_E$ . However, by the Markov property, the explosion times of  $\zeta_E$  are i.i.d random variables. As a consequence, the number of explosions of  $\zeta_E$  before time  $t$ , that is the

number of times where  $\theta_E$  is a multiple of  $\pi$  before  $t$  is a renewal process which can be studied thanks to classical tools. However, one problem of the Sturm-Liouville equation is that  $M$  is not differentiable. Consequently, we can not apply directly the Sturm-Liouville theory. However, we can apply it with a regularized version of  $M$ . Let us replace  $M$  by  $M^{(n)}$  which is a  $C^2$  random function such that  $M^{(n)}$  converges uniformly to  $M$  almost surely. (For example,  $M^{(n)}$  can be taken as a polynomial interpolation of  $M$ .) Let us prove the following lemma about the effects of this approximation on the spectrum.

**Lemma 5.30.** *Let  $\lambda > 0$ . Let  $(M^{(n)})_{n \in \mathbb{N}^*}$  be a sequence of  $C^2$  functions which converges almost surely uniformly to  $M$  on  $[-\lambda, \lambda]$ . Let  $\mathcal{H}_n^{(\lambda)}$  be a random self-adjoint positive operator defined by*

$$\mathcal{H}_n^{(\lambda)} g = -\frac{1}{M^{(n)}} \left( (M^{(n)})^2 \left( \frac{g}{M^{(n)}} \right)' \right)'$$

where  $g$  is in the domain  $\mathcal{D}(\mathcal{H}_n^{(\lambda)})$  which is defined by

$$\left\{ \begin{array}{l} g \in L^2([-\lambda, \lambda]), \quad \left( \frac{g}{M^{(n)}} \right)' \in L^2([-\lambda, \lambda]), \quad \left( (M^{(n)})^2 \left( \frac{g}{M^{(n)}} \right)' \right)' \in L^2([-\lambda, \lambda]), \\ g(-\lambda) = g(\lambda), \quad M_{-\lambda}^{(n)} \left( \frac{g}{M^{(n)}} \right)'(-\lambda) = M_{\lambda}^{(n)} \left( \frac{g}{M^{(n)}} \right)'(\lambda) \end{array} \right\}.$$

Then, the spectrum of  $\mathcal{H}_n^{(\lambda)}$  is a random increasing positive sequence  $(E_{n,k}(\lambda))_{k \geq 0}$  which diverges to infinity. Moreover, these spectral values of  $\mathcal{H}_n^{(\lambda)}$  are eigenvalues which are counted with multiplicity. Furthermore, for every  $k \in \mathbb{N}$ ,

$$E_{n,k}(\lambda) \xrightarrow[n \rightarrow +\infty]{a.s.} E_k(\lambda).$$

In particular, if  $N_{n,\lambda}(E)$  is the number of eigenvalues of  $\mathcal{H}_n^{(\lambda)}$  which are lower than  $E$ , then for every  $E > 0$

$$N_{n,\lambda}(E) \xrightarrow[n \rightarrow +\infty]{a.s.} N_{\lambda}(E).$$

*Proof of Lemma 5.30.*

The fact that  $\mathcal{H}_n^{(\lambda)}$  is positive and self-adjoint can be proved exactly as for  $\mathcal{H}^{(\lambda)}$ . Moreover, exactly as for  $\mathcal{H}^{(\lambda)}$  in the proof of Proposition 5.13, the eigenvalues of  $\mathcal{H}_n^{(\lambda)}$  are the inverse of the eigenvalues of  $\mathcal{G}_n^{(\lambda)}$  where  $\mathcal{G}_n^{(\lambda)}(t, t')$  is defined by

$$\frac{M_{t'}^{(n)} M_t^{(n)}}{(M_{\lambda}^{(n)} - M_{-\lambda}^{(n)})^2} \left( (M_{\lambda}^{(n)})^2 \int_{t'}^{\lambda} \frac{ds}{(M_s^{(n)})^2} + M_{\lambda}^{(n)} M_{-\lambda}^{(n)} \int_t^{t'} \frac{ds}{(M_s^{(n)})^2} + (M_{-\lambda}^{(n)})^2 \int_{-\lambda}^t \frac{ds}{(M_s^{(n)})^2} \right)$$

for every  $t \leq t' \in \mathcal{C}^{(\lambda)}$ . By the expression above, this is clear that  $(t, t') \mapsto \mathcal{G}_n^{(\lambda)}(t, t')$  converges uniformly toward  $(t, t') \mapsto \mathcal{G}^{(\lambda)}(t, t')$ . This implies that  $\mathcal{G}_n^{(\lambda)} f$  converges almost surely in  $L^2([-\lambda, \lambda])$  toward  $\mathcal{G}^{(\lambda)} f$  uniformly in  $f \in L^2([-\lambda, \lambda])$  such that  $\|f\|_2 = 1$ . Furthermore, by the min-max principle,

$$\frac{1}{E_{n,k}(\lambda)} = \max_{V_k} \min_{f \in V_k, \|f\|_2=1} \int_{-\lambda}^{\lambda} \left| \mathcal{G}_n^{(\lambda)} f(t) \right|^2 dt$$

where  $V_k$  is in the set of vector spaces of dimension  $k$  in  $L^2([-\lambda, \lambda])$ . This implies directly that  $E_{n,k}(\lambda)$  converges toward  $E_k(\lambda)$  when  $n$  goes to infinity. □

*Proof of Theorem 5.16.*

**Step 1 : Link with the Sturm-Liouville equation.** Let  $\lambda > 0$ . Let  $M^{(n)}$  be a  $C^2$  sequence of random functions which converges almost surely uniformly toward  $M$  and let  $\mathcal{H}_n^{(\lambda)}$  be the

operator which is associated with  $M^{(n)}$ . Let  $E_{n,k}(\lambda)$  be some eigenvalue of  $\mathcal{H}_n^{(\lambda)}$ . There exists an eigenvector  $\psi_{E_{n,k}(\lambda)} \in \mathcal{D}\left(\mathcal{H}_n^{(\lambda)}\right)$  such that

$$\left( (M^{(n)})^2 \left( \frac{\psi_{E_{n,k}(\lambda)}}{M^{(n)}} \right)' \right)' + E_{n,k}(\lambda) M^{(n)} \psi_{E_{n,k}(\lambda)} = 0. \quad (5.83)$$

Then, let us define  $\varphi_{n,k} = \psi_{E_{n,k}(\lambda)}/M^{(n)}$ . We omit the dependence in  $\lambda$  for sake of convenience. Then,  $\varphi_{n,k}$  belongs to the set

$$\left\{ \begin{array}{l} g \in L^2([- \lambda, \lambda]), \quad g' \in L^2([- \lambda, \lambda]), \quad ((M^{(n)})^2 g')' \in L^2([- \lambda, \lambda]), \\ M_{-\lambda}^{(n)} g(-\lambda) = M_\lambda^{(n)} g(\lambda), \quad M_{-\lambda}^{(n)} g'(-\lambda) = M_\lambda^{(n)} g'(\lambda) \end{array} \right\}.$$

Moreover,  $\varphi_{n,k}$  is solution of

$$\left( (M^{(n)})^2 \varphi'_{n,k} \right)' + E_{n,k}(\lambda) (M^{(n)})^2 \varphi_{n,k} = 0. \quad (5.84)$$

Remark that the ODE in (5.84) is a Sturm-Liouville equation. By Theorem 2.1 in Chapter 8 of [42], there exists an increasing sequence of real numbers  $(\mu_{n,k}(\lambda))_{k \geq 0}$  which goes to infinity such that for every  $k \in \mathbb{N}$ , there exists a non trivial  $C^2$  function  $\xi_{n,k}$  such that

$$\left( (M^{(n)})^2 \xi'_{n,k} \right)' + \mu_{n,k}(\lambda) (M^{(n)})^2 \xi_{n,k} = 0$$

and  $\xi_{n,k}(-\lambda) = \xi_{n,k}(\lambda) = 0$ . These numbers  $(\mu_{n,k}(\lambda))_{k \geq 0}$  are called the Dirichlet eigenvalues of our Sturm Liouville equation.  $(\xi_{n,k})_{k \in \mathbb{N}}$  are the Dirichlet eigenstates which are associated with  $(\mu_{n,k}(\lambda))_{k \geq 0}$ . For every  $E > 0$ , let us define  $\tilde{N}_{n,\lambda}(E)$  be the number of Dirichlet eigenvalues which are lower than  $E$ . Moreover, by the Liouville transformation (see 4.3 in [63]) the eigenvalues of the Sturm-Liouville problem

$$(I) \left\{ \begin{array}{l} ((M^{(n)})^2 \varphi')' + E(M^{(n)})^2 \varphi = 0 \\ M_{-\lambda}^{(n)} \varphi(-\lambda) = M_\lambda^{(n)} \varphi(\lambda) \\ M_{-\lambda}^{(n)} \varphi'(-\lambda) = M_\lambda^{(n)} \varphi'(\lambda) \end{array} \right.$$

are the same as the eigenvalues of the Sturm-Liouville problem

$$(II) \left\{ \begin{array}{l} \Phi'' - Q\Phi + E\Phi = 0 \\ \Phi(-\lambda) = \Phi(\lambda) \\ \Phi'(-\lambda) = \Phi'(\lambda) + \left( M_\lambda^{(n)} \left( \frac{1}{M^{(n)}} \right)'_\lambda - M_{-\lambda}^{(n)} \left( \frac{1}{M^{(n)}} \right)'_{-\lambda} \right) \Phi(\lambda) \end{array} \right.$$

where  $Q = 1 - \frac{1}{M^{(n)}} \left( (M^{(n)})^2 \left( \frac{1}{M^{(n)}} \right)' \right)'$ . Besides, by Theorem 1.3 in [138], the eigenvalues of (II) are interlaced with the eigenvalues of the following Sturm-Liouville problem :

$$(III) \left\{ \begin{array}{l} \Phi'' - Q\Phi + E\Phi = 0 \\ \Phi(-\lambda) = \Phi(\lambda) = 0 \end{array} \right. .$$

Furthermore, by the Liouville transformation again, the eigenvalues of (III) are the same as the eigenvalues of the following Sturm-Liouville problem :

$$(IV) \left\{ \begin{array}{l} ((M^{(n)})^2 \varphi')' + E(M^{(n)})^2 \varphi = 0 \\ \varphi(-\lambda) = \varphi(\lambda) = 0 \end{array} \right. .$$

Therefore, the eigenvalues of (I) and (IV) are interlaced. In particular, almost surely, for every  $E > 0$

$$|N_{n,\lambda}(E) - \tilde{N}_{n,\lambda}(E)| \leq 1. \quad (5.85)$$

Now, let  $\mu_{n,k}(\lambda)$  be an eigenvalue of the Sturm Liouville equation with Dirichlet condition. Let  $\xi_{n,k}$  be the associated eigenstate. According to chapter 8 in [42], there exists two functions  $R_{n,\mu_{n,k}(\lambda)}$  and  $\theta_{n,\mu_{n,k}(\lambda)}$  such that

$$\xi_{n,k} = R_{n,\mu_{n,k}(\lambda)} \sin\left(\theta_{n,\mu_{n,k}(\lambda)}\right) \quad (5.86)$$

$$R_{n,\mu_{n,k}(\lambda)}^2 = ((M^{(n)})^2 \xi'_{n,k})^2 + \xi_{n,k}^2 \quad (5.87)$$

$$\theta'_{n,\mu_{n,k}(\lambda)} = \frac{1}{(M^{(n)})^2} \cos(\theta_{n,\mu_{n,k}(\lambda)})^2 + \mu_{n,k}(\lambda) (M^{(n)})^2 \sin(\theta_{n,\mu_{n,k}(\lambda)})^2 \quad (5.88)$$

where  $\theta_{n,\mu_{n,k}(\lambda)}(-\lambda) = 0$ . Remark that for any  $\mu > 0$ , the function  $\theta_{n,\mu}$  depends also on  $\lambda$  but we omit this dependence in the notation for sake of convenience. By (5.87),  $R_{n,\mu_{n,k}(\lambda)}$  can never be zero at any point. Otherwise, this would imply that there exists some point  $t_0$  such that  $\xi'_{n,k}(t_0) = \xi_{n,k}(t_0) = 0$ . By Cauchy-Lipschitz theorem, this would imply that  $\xi_{n,k}$  is zero everywhere which is false because this is an eigenvector. Therefore, as  $R_{n,\mu_{n,k}(\lambda)}$  never vanishes, (5.86) implies that  $\xi_{n,k}$  vanishes when  $\theta_{n,\mu_{n,k}(\lambda)}$  is a multiple of  $\pi$ . Moreover, by Theorem 2.1 in chapter 8 of [42], the number of zeros of  $\xi_{n,k}$  in  $[-\lambda, \lambda]$  is always  $k + 2$ . Now, let  $E \in \mathbb{R}_+^*$ . There exists some  $k \in \mathbb{N}$  such that  $\tilde{N}_{n,\lambda}(E) = k$ , that is,  $\mu_{n,k-1}(\lambda) \leq E < \mu_{n,k}(\lambda)$  with the convention stating that  $\mu_{n,-1}(\lambda) = 0$ . We said that  $\xi_{n,k-1}$  vanishes exactly  $k + 1$  times in  $[-\lambda, \lambda]$ . Therefore,

$$\theta_{n,\mu_{n,k-1}(\lambda)}(\lambda) = k\pi.$$

Consequently,

$$\tilde{N}_{n,\lambda}(E) = \frac{\theta_{n,\mu_{n,k-1}(\lambda)}(\lambda)}{\pi} \leq \frac{\theta_{n,E}(\lambda)}{\pi}$$

where the inequality stems from the increasingness of  $E \mapsto \theta_{n,E}(\lambda)$  where  $\theta_{n,E}$  is solution of

$$\theta'_{n,E} = \frac{1}{(M^{(n)})^2} \cos(\theta_{n,E})^2 + E \times (M^{(n)})^2 \sin(\theta_{n,E})^2.$$

with  $\theta_{n,E}(-\lambda) = 0$ . We can obtain a lower bound in the same way which yields

$$-1 + \frac{\theta_{n,E}(\lambda)}{\pi} \leq \tilde{N}_{n,\lambda}(E) \leq \frac{\theta_{n,E}(\lambda)}{\pi}. \quad (5.89)$$

Together with (5.85), this yields

$$-2 + \frac{\theta_{n,E}(\lambda)}{\pi} \leq N_{n,\lambda}(E) \leq 1 + \frac{\theta_{n,E}(\lambda)}{\pi}. \quad (5.90)$$

As  $M^{(n)}$  converges uniformly to  $M$  on  $[-\lambda, \lambda]$ ,  $\theta_{n,E}$  converges uniformly on  $[-\lambda, \lambda]$  toward  $\theta_E$  which is the solution of the random ODE

$$\theta'_E = \frac{1}{M^2} \cos(\theta_E)^2 + EM^2 \sin(\theta_E)^2$$

with  $\theta_E(-\lambda) = 0$ . This solution is well-defined on  $[-\lambda, \lambda]$  for every initial condition because  $(t, \theta) \mapsto \frac{1}{M_t^2} \cos(\theta)^2 + EM_t^2 \sin(\theta)^2$  is continuous and globally Lipschitz in the second variable

on any set of the type  $[-r, r] \times \mathbb{R}$ . Now, let us assume that  $E$  and  $\lambda$  are fixed. By Lemma 5.30,  $N_{n,\lambda}(E)$  converges toward  $N_\lambda(E)$  almost surely. Moreover, we proved that  $\theta_{n,E}$  converges uniformly toward  $\theta_E$  almost surely. As a consequence, we can take the limit in (5.90) which implies that for every  $E > 0$  and every  $\lambda > 0$ , almost surely,

$$-2 + \frac{\theta_E(\lambda)}{\pi} \leq N_\lambda(E) \leq 1 + \frac{\theta_E(\lambda)}{\pi}. \quad (5.91)$$

**Step 2 : Study of  $\theta_E$ .** Let us shift  $\theta_E$  in order to start from 0. For every  $t \geq 0$ , let us define  $\tilde{\theta}_E$  such that for every  $t \geq 0$ ,  $\tilde{\theta}_E(t) = \theta_E(t - \lambda)$ . This implies that  $\tilde{\theta}_E$  is a solution of the random ODE

$$\tilde{\theta}'_E(t) = \frac{1}{M_{-\lambda}^2 \tilde{M}_t^2} \cos(\tilde{\theta}_E)^2 + EM_{-\lambda}^2 \tilde{M}_t^2 \sin(\tilde{\theta}_E)^2 \quad (5.92)$$

where  $M_{-\lambda} = e^{B_{-\lambda} + \lambda/2}$  and  $\tilde{M}_t = e^{B_{t-\lambda} - B_{-\lambda} - t/2}$  which is a geometric Brownian motion which is equal to 1 at 0. Moreover,  $\tilde{M}$  is independent of  $M_{-\lambda}$ . Of course,  $\tilde{\theta}_E$  depends on  $\lambda$  (as well as  $\theta_E$  actually) but we omit this in the notation for sake of convenience. For every  $t \geq 0$ , we define  $\mathcal{F}_t$  as the right-continuous completion of  $\sigma(M_{-\lambda}, \tilde{M}_s, s \leq t)$ . This is clear that  $\tilde{\theta}_E$  is adapted with the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . By (5.92), remark that  $\tilde{\theta}_E$  is strictly increasing. Therefore, it goes to infinity or it has a finite random limit  $\Theta^*$ . As  $(1/\tilde{M}_t)_{t \geq 0}$  goes to infinity at exponential speed almost surely, (5.92) implies that  $\Theta^*$  must be of the form  $\pi/2 + K^*\pi$  with  $K^*$  which is random and possibly infinite. For every  $k \in \mathbb{N}$ , let us define the stopping times

$$T_k = \inf\{t \geq 0, \tilde{\theta}_E(t) = k\pi\} \text{ and } \tau_k = \inf\{t \geq 0, \tilde{\theta}_E(t) = \pi/2 + k\pi\}.$$

By definition we have a sequence of inequalities

$$0 = T_0 < \tau_0 < T_1 < \tau_1 < \dots < T_{K^*} < \tau_{K^*} = +\infty.$$

In order to prove that  $\tilde{\theta}_E$  goes to infinity almost surely, we will prove that for every  $k \in \mathbb{N}^*$ ,  $\mathbb{E}(\tau_k) < +\infty$ . Further, for technical purposes, we need to introduce other stopping times : for every  $k \in \mathbb{N}$  and for every  $n \in \mathbb{N}^*$ , let us define

$$T_{k,n}^+ = \inf\{t \geq 0, \tilde{\theta}_E(t) = k\pi + 1/n\} \text{ and } T_{k,n}^- = \inf\{t \geq 0, \tilde{\theta}_E(t) = k\pi - 1/n\}.$$

Now, we need a lemma whose proof is postponed to the end of this section.

**Lemma 5.31.** *For every  $n \in \mathbb{N}^*$ ,  $\mathbb{E}[T_{0,n}^+] < +\infty$ .*

Thanks to this lemma, let us show that  $\mathbb{E}[\tau_0] < +\infty$  and  $\mathbb{E}[T_1] < +\infty$ . Let us introduce  $j_E := -\cot(\tilde{\theta}_E)$ . Observe that  $j_E(t)$  explodes when  $t$  is some stopping time  $T_k$  for any  $k$  and vanishes when  $t$  is some stopping time  $\tau_k$  for any  $k$ . Moreover, on each interval of type  $]T_k, T_{k+1}[$ ,  $j_E$  is solution of the following Riccati equation :

$$j'_E = \frac{1}{M_{-\lambda}^2 \tilde{M}^2} j_E^2 + EM_{-\lambda}^2 \tilde{M}^2. \quad (5.93)$$

Now, consider  $n \in \mathbb{N}^*$  and let us work on the interval  $]T_{0,n}^+, \tau_0[$ . (For now, we do not know whether  $\tau_0$  is finite or not.) Then, we define the stochastic process  $\zeta_E := \frac{j_E}{M_{-\lambda}^2 \tilde{M}^2}$ . This stochastic process explodes at some times. However, it is a continuous locally bounded process on  $]T_{0,n}^+, \tau_0[$  almost surely. Therefore, we can use stochastic calculus on this interval. Recall that  $\tilde{M}$  is a geometric Brownian motion on  $[0, 2\lambda]$  which is 1 at time 0. In particular,  $\tilde{M}$  satisfies the SDE

$$d\tilde{M}_t = \tilde{M}_t dB_t. \quad (5.94)$$

Therefore, on  $]T_{0,n}^+, \tau_0[$ , it holds that

$$d\zeta_E(t) = \frac{1}{M_{-\lambda}^2} \left( j_E \cdot d(\tilde{M}_t^{-2}) + \frac{j'_E(t)}{\tilde{M}_t^2} dt \right).$$

Then, using Ito's formula together with (5.93) and (5.94), we get

$$\begin{aligned} d\zeta_E(t) &= \frac{1}{M_{-\lambda}^2} \left( -\frac{2j_E(t)}{\tilde{M}_t^3} d\tilde{M}_t + \frac{3j_E(t)}{\tilde{M}_t^4} d\langle \tilde{M} \rangle_t + \frac{1}{\tilde{M}_t^2} \left( \frac{j_E(t)^2}{M_{-\lambda}^2 \tilde{M}_t^2} + EM_{-\lambda}^2 \tilde{M}_t^2 \right) dt \right) \\ &= -2 \frac{j_E(t)}{M_{-\lambda}^2 \tilde{M}_t^2} dB_t + 3 \frac{j_E(t)}{M_{-\lambda}^2 \tilde{M}_t^2} dt + \frac{j_E(t)^2}{M_{-\lambda}^4 \tilde{M}_t^4} dt + E dt \\ &= -2\zeta_E(t) dB_t + (\zeta_E(t)^2 + 3\zeta_E(t) + E) dt. \end{aligned}$$

Finally, on  $]T_{0,n}^+, \tau_0[$ ,  $\zeta_E$  is solution of the SDE :

$$d\zeta_E(t) = -2\zeta_E(t) dB_t + (\zeta_E(t)^2 + 3\zeta_E(t) + E) dt. \quad (5.95)$$

Therefore, until its explosion, this is a Markov process with generator,

$$\mathcal{L}f = 2z^2 f''(z) + (z^2 + 3z + E) f'(z)$$

for any regular function  $f$ . Now, let us define a function  $f_-$  on  $\mathbb{R}_-$  such that for every  $z \in \mathbb{R}_-$ ,

$$f_-(z) = - \int_z^0 \frac{e^{-u/2+E/(2u)}}{2|u|^{3/2}} \int_{-\infty}^u \frac{1}{|t|^{1/2}} e^{t/2-E/(2t)} dt du. \quad (5.96)$$

One can check that  $f_-$  is well-defined (the singularity at 0 in the integral is not a real one) and smooth on  $\mathbb{R}_^*$ , that  $f_-$  is bounded and has a finite limit at  $-\infty$ . Besides  $f_-$  satisfies  $\mathcal{L}f_- = 1$  on  $\mathbb{R}_^*$ . Then, by Propositions 2.6 and 2.2 in Chapter VII of [148], conditionally on  $\mathcal{F}_{T_{0,n}^+}$ , it holds that

$$\left( f_- \left( \zeta_E((T_{0,n}^+ + t) \wedge \tau_0) \right) - (T_{0,n}^+ + t) \wedge \tau_0 \right)_{t \geq 0}$$

is a martingale with respect to the filtration  $(\mathcal{F}_{T_{0,n}^+ + t})_{t \geq 0}$ . (The integrability of this martingale comes from the boundedness of  $f_-$  and Lemma 5.31.) This implies that

$$\mathbb{E} \left[ (T_{0,n}^+ + t) \wedge \tau_0 \right] = \mathbb{E} \left[ T_{0,n}^+ \right] + \mathbb{E} \left[ f_- \left( \zeta_E((T_{0,n}^+ + t) \wedge \tau_0) \right) \right] - \mathbb{E} \left[ f_- \left( \zeta_E(T_{0,n}^+) \right) \right]. \quad (5.97)$$

By monotone convergence and dominated convergence (recall that  $f_-$  is bounded), we can make  $t$  go toward infinity in (5.97) which implies that

$$\mathbb{E} [\tau_0] = \mathbb{E} \left[ T_{0,n}^+ \right] + \mathbb{E} \left[ f_- \left( \zeta_E(\tau_0) \right) \right] - \mathbb{E} \left[ f_- \left( \zeta_E(T_{0,n}^+) \right) \right].$$

Thanks to Lemma 5.31 and the boundedness of  $f_-$ , this implies that  $\mathbb{E} [\tau_0] < +\infty$ . Actually, this is not difficult to iterate this method in order to prove that  $\mathbb{E} [\tau_k] < +\infty$  for every  $k \in \mathbb{N}^*$ . In particular, this yields

$$\lim_{t \rightarrow +\infty} \tilde{\theta}_E(t) = +\infty \quad a.s. \quad (5.98)$$

**Step 3 : Link with a renewal process.** On any interval of the form  $[T_{k,n}^+, T_{k+1,n}^-]$ ,  $\zeta_E$  is a continuous stochastic process which is the solution of the SDE (5.95). This is very tempting to say that  $\zeta_E$  is a Markov diffusion process on  $\mathbb{R}$  whose values are in  $\mathbb{R} \cup \{\infty\}$  which would imply that

the lengths of time  $T_{k+1} - T_k$  are i.i.d as the successive hitting times of  $\infty$  by a Markov process. However, to our knowledge, the case of a process whose values can be  $\infty$  is not contained in the theory of Markov diffusions. In order to avoid this theoretical problem, let us make a change of variable. We define  $\Gamma_E := \frac{\zeta_E - i}{\zeta_E + i}$ . The idea under this transformation is to map  $\mathbb{R} \cup \{\infty\}$  on the unit circle. Remark that  $\Gamma_E = 1$  when  $\zeta_E = \infty$ . Moreover,  $\zeta_E$  is continuous on intervals of the form  $]T_k, T_{k+1}[$ . Furthermore, for every  $k \in \mathbb{N}$ ,  $\zeta_E(t)$  goes to  $+\infty$  when  $t$  goes to  $T_k$  on the left and  $\zeta_E(t)$  goes to  $-\infty$  when  $t$  goes to  $T_k$  on the right. Therefore,  $\Gamma_E$  is continuous on  $\mathbb{R}_+$ . Further, remark that  $\zeta_E = -i \frac{\Gamma_E + 1}{\Gamma_E - 1}$ . Then, on each interval of the form  $[T_{k,n}^+, T_{k+1,n}^-]$ , by (5.95) and Ito's formula,

$$\begin{aligned} d\Gamma_E(t) &= \frac{2i}{(\zeta_E(t) + i)^2} d\zeta_E(t) - \frac{2i}{(\zeta_E(t) + i)^3} d\langle \zeta_E \rangle_t \\ &= 4i \frac{\zeta_E(t)}{(\zeta_E(t) + i)^2} dB_t + 2i \left( \frac{\zeta_E(t)^2 + 3\zeta_E(t) + E}{(\zeta_E(t) + i)^2} - 4 \frac{\zeta_E(t)^2}{(\zeta_E(t) + i)^3} \right) dt. \end{aligned} \quad (5.99)$$

Moreover, by a straightforward computation :

$$\frac{1}{(\zeta_E + i)^2} = -\frac{(1 - \Gamma_E)^2}{4}, \quad \frac{\zeta_E}{(\zeta_E + i)^2} = -\frac{i}{4}(1 - \Gamma_E^2), \quad \frac{\zeta_E^2}{(\zeta_E + i)^2} = \frac{(1 + \Gamma_E)^2}{4}$$

and

$$\frac{\zeta_E^2}{(\zeta_E + i)^3} = -\frac{i}{8}(1 + \Gamma_E)^2(1 - \Gamma_E).$$

Together with (5.99), this implies that on each interval of the form  $]T_{k,n}^+, T_{k+1,n}^-]$ ,

$$\begin{aligned} d\Gamma_E(t) &= (1 - \Gamma_E^2) dB_t + 2i \left( \frac{1}{4}(1 + \Gamma_E)^2 - \frac{3i}{4}(1 - \Gamma_E^2) - \frac{E}{4}(1 - \Gamma_E)^2 + \frac{i}{2}(1 + \Gamma_E)^2(1 - \Gamma_E) \right) dt. \end{aligned} \quad (5.100)$$

Besides,  $\Gamma_E$  is a continuous and uniformly bounded stochastic process. Therefore, we can prove that  $\Gamma_E$  satisfies the SDE (5.100) on the whole set  $\mathbb{R}_+$  and not only on intervals of the form  $[T_{k,n}^+, T_{k+1,n}^-]$  by using the continuity of  $\Gamma_E$  and the dominated convergence theorem for the stochastic integral. (See Theorem 2.12 in [148].) Now, we want to apply Definition 7.1.1 of [135] in order to say that  $\Gamma_E$  is an Ito diffusion. However, Definition 7.1.1 in [135] requires that the functions inside the SDE are globally lipschitz. Here, this is not true at first sight. Nevertheless,  $|\Gamma_E|$  is bounded by 1. Consequently, we can consider compactly supported functions which coincide with  $\gamma \mapsto 1 - \gamma^2, \gamma \mapsto (1 + \gamma)^2, \gamma \mapsto (1 - \gamma)^2$  and  $\gamma \mapsto (1 + \gamma)^2(1 - \gamma)$  on the unit circle. Then, these compactly supported functions are lipschitz and we can consider a modified version of the SDE (5.100) with lipschitz coefficients which is satisfied by  $\Gamma_E$ . Therefore, according to Definition 7.1.1 in [135],  $\Gamma_E$  is a complex Itô diffusion. Now, recall that the stopping times  $(T_k)_{k \geq 0}$  are the successive hitting times of  $k\pi$  by  $\tilde{\theta}_E$ . This implies that the stopping times  $(T_k)_{k \geq 0}$  are the successive hitting times of  $\infty$  by  $\zeta_E$ , that is, the successive hitting times of 1 by  $\Gamma_E$ . As  $\Gamma_E$  is an Itô diffusion, by Theorem 7.2.4 in [135] it satisfies the strong Markov property. Then, by classical arguments, this implies that  $(T_{k+1} - T_k)_{k \in \mathbb{N}}$  is an i.i.d sequence of random variables. Therefore

$$\left( \left[ \frac{\tilde{\theta}_E(t)}{\pi} \right] \right)_{t \geq 0} = (\#\{k \in \mathbb{N}^*, T_k \leq t\})_{t \geq 0} =: (R_t)_{t \geq 0} \quad (5.101)$$

where  $(R_t)_{t \geq 0}$  is a renewal process whose  $k$ -th interarrival time is  $T_k - T_{k-1}$  for every  $k \in \mathbb{N}^*$ . One can refer to chapter 10 in [83] for more information on renewal processes. Combining (5.101) with (5.91) implies that for every  $E > 0$ ,

$$\frac{N_\lambda(E)}{2\lambda} = \frac{R_{2\lambda}}{2\lambda} + o_\lambda^\mathbb{P}(1) \quad (5.102)$$

where  $o_\lambda^\mathbb{P}(1)$  is a random variable which goes to 0 in probability when  $\lambda$  goes to infinity. Therefore, by Theorem 10.2(1) and (5.102), we know that

$$\frac{N_\lambda(E)}{2\lambda} \xrightarrow[\lambda \rightarrow +\infty]{\mathbb{P}} \frac{1}{\mathbb{E}[T_1]}.$$

Finally, we only have to compute explicitly  $\mathbb{E}[T_1]$ .

**Step 4 : Computation of  $\mathbb{E}[T_1]$ .** Let  $n \in \mathbb{N}^*$ . By (5.95), on the interval  $[T_{0,n}^+, T_{1,n}^-]$ , recall that  $\zeta_E$  is a solution of the SDE

$$d\zeta_E(t) = -2\zeta_E(t)dB_t + (\zeta_E(t)^2 + 3\zeta_E(t) + E)dt.$$

As before, the generator associated with this SDE is given by

$$\mathcal{L}f = 2z^2f''(z) + (z^2 + 3z + E)f'(z)$$

for any function  $f$  which is enough regular. Now, for every  $z \in \mathbb{R}_-$ , we define as before

$$f^*(z) = f_-(z) = - \int_z^0 \frac{e^{-u/2+E/(2u)}}{2|u|^{3/2}} \int_{-\infty}^u \frac{1}{|t|^{1/2}} e^{t/2-E/(2t)} dt du.$$

Moreover, for any  $z \in \mathbb{R}_+$ , let us define

$$f^*(z) := \int_0^z \frac{e^{-u/2+E/(2u)}}{2u^{3/2}} \int_0^u \frac{1}{\sqrt{t}} e^{t/2-E/(2t)} dt du.$$

This is not difficult to check that  $f^*$  is well-defined, bounded, smooth everywhere, excepted at 0 where it may be only  $C^1$ . Moreover,  $\mathcal{L}f^* = 1$  on  $\mathbb{R}^*$ . Consequently, conditionally on  $\mathcal{F}_{T_{0,n}^+}$ ,

$$\left( f^* \left( \zeta_E((T_{0,n}^+ + t) \wedge T_{1,n}^-) \right) - (T_{0,n}^+ + t) \wedge T_{1,n}^- \right)_{t \geq 0}$$

is a martingale with respect to the filtration  $(\mathcal{F}_{T_{0,n}^+ + t})_{t \geq 0}$ . In particular, for every  $n \in \mathbb{N}^*$  and for every  $t \geq 0$ ,

$$\mathbb{E} \left[ f^* \left( \zeta_E((T_{0,n}^+ + t) \wedge T_{1,n}^-) \right) \right] - \mathbb{E} \left[ f^* \left( \zeta_E(T_{0,n}^+) \right) \right] = \mathbb{E} \left[ (T_{0,n}^+ + t) \wedge T_{1,n}^- \right] - \mathbb{E} \left[ T_{0,n}^+ \right]. \quad (5.103)$$

By monotone convergence and dominated convergence theorem (recall that  $f^*$  is bounded), we can make  $t$  go to infinity in (5.103). This yields for every  $n \in \mathbb{N}^*$

$$\mathbb{E} \left[ f^* \left( \zeta_E(T_{1,n}^-) \right) \right] - \mathbb{E} \left[ f^* \left( \zeta_E(T_{0,n}^+) \right) \right] = \mathbb{E} \left[ T_{1,n}^- \right] - \mathbb{E} \left[ T_{0,n}^+ \right]. \quad (5.104)$$

Then, by monotone convergence and by dominated convergence again, we can make  $n$  go to infinity in (5.104) which yields

$$\begin{aligned} & \int_{-\infty}^0 \frac{e^{-u/2+E/(2u)}}{2|u|^{3/2}} \int_{-\infty}^u \frac{1}{|t|^{1/2}} e^{t/2-E/(2t)} dt du + \int_0^{\infty} \frac{e^{-u/2+E/(2u)}}{2u^{3/2}} \int_0^u \frac{1}{\sqrt{t}} e^{t/2-E/(2t)} dt du \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[ f^* \left( \zeta_E((T_{1,n}^-)) \right) \right] - \mathbb{E} \left[ f^* \left( \zeta_E(T_{0,n}^+) \right) \right] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[ T_{1,n}^- \right] - \mathbb{E} \left[ T_{0,n}^+ \right] \\ &= \mathbb{E} [T_1]. \end{aligned}$$

Moreover, by using the change of variable  $(u', t') = (\pm 1/u, \pm 1/t)$ , then by using Fubini's theorem and finally by making a simple translation change of variables, one can check that for every  $E > 0$ ,

$$\begin{aligned} & \int_{-\infty}^0 \frac{e^{-u/2+E/(2u)}}{2|u|^{3/2}} \int_{-\infty}^u \frac{1}{|t|^{1/2}} e^{t/2-E/(2t)} dt du + \int_0^{\infty} \frac{e^{-u/2+E/(2u)}}{2u^{3/2}} \int_0^u \frac{1}{\sqrt{t}} e^{t/2-E/(2t)} dt du \\ &= \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \frac{e^{-\frac{t}{2u(t+u)} - Et/2}}{\sqrt{u}\sqrt{t+u}} \times \frac{2u+t}{u(t+u)} dudt. \end{aligned} \quad (5.105)$$

Now, we are going to compute explicitly the integral of (5.105). Let us make the change of variables

$$t = \frac{s}{E}, \quad \frac{t}{u(t+u)} = \frac{s}{v^2}$$

which maps  $(\mathbb{R}_+^*)^2$  onto itself. This is equivalent to

$$(t, u) := \phi(v, s) = \left( \frac{s}{E}, \frac{1}{2E} \left( -s + \sqrt{s^2 + 4Ev^2} \right) \right).$$

We can compute the Jacobian of  $\phi$  which is

$$Jac(\phi) = \frac{2v}{E\sqrt{s^2 + 4Ev^2}}.$$

Moreover, we can remark that

$$u(u+t) = \frac{v^2}{E} \text{ and } 2u+t = \frac{1}{E}\sqrt{s^2 + 4Ev^2}.$$

Therefore,

$$\begin{aligned} & \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \frac{e^{-\frac{t}{2u(t+u)} - Et/2}}{\sqrt{u}\sqrt{t+u}} \times \frac{2u+t}{u(t+u)} dudt \\ &= \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} e^{-s/(2v^2) - s/2} \frac{1}{(v^2/E)^{3/2}} \times \frac{1}{E} \sqrt{s^2 + 4Ev^2} \times \frac{1}{E} \frac{2v}{\sqrt{s^2 + 4Ev^2}} ds dv \\ &= \frac{1}{\sqrt{E}} \int_0^{+\infty} \int_0^{+\infty} \frac{e^{-s/(2v^2) - s/2}}{v^2} ds dv \\ &= \frac{1}{\sqrt{E}} \int_0^{+\infty} \frac{2}{1+v^2} dv \\ &= \frac{\pi}{\sqrt{E}}. \end{aligned}$$

This completes the proof of Theorem 5.16. □

Now, let us prove Lemma 5.31.

*Proof of lemma 5.31.*

Let  $n \in \mathbb{N}^*$ . By (5.92),

$$\tilde{\theta}_E(T_{0,n}^+) - \tilde{\theta}_E(0) \geq \frac{1}{M_{-\lambda}^2} \int_0^{T_{0,n}^+} \frac{\cos(\tilde{\theta}_E(s))^2}{\tilde{M}_s^2} ds.$$

Therefore,

$$\begin{aligned} \frac{M_{-\lambda}^2}{n} &\geq \cos(1/n)^2 \int_0^{T_{0,n}^+} e^{-2\tilde{B}_s + s} ds \\ &\geq \cos(1/n)^2 \inf_{s \in [0, +\infty]} e^{-2\tilde{B}_s + s/2} \times 2 \left( e^{T_{0,n}^+/2} - 1 \right). \end{aligned} \quad (5.106)$$

Then, (5.106) and the independence between  $M_{-\lambda}$  and  $\tilde{B}$  yield

$$\begin{aligned} \mathbb{E} \left[ \left( e^{T_{0,n}^+} - 1 \right)^{1/8} \right] &\leq \frac{\mathbb{E} \left[ M_{-\lambda}^{1/4} \right]}{(2n)^{1/8} \cos(1/n)^{1/4}} \mathbb{E} \left[ \sup_{s \geq 0} e^{\tilde{B}_s/4 - s/16} \right] \\ &= \frac{\mathbb{E} \left[ M_{-\lambda}^{1/4} \right]}{(2n)^{1/8} \cos(1/n)^{1/4}} \mathbb{E} \left[ \sup_{s \geq 0} e^{\tilde{B}_s - s} \right] \end{aligned} \quad (5.107)$$

where in the equality, we used a change of time for the Brownian motion. Then, by Girsanov's theorem (see Theorem VIII.1.7 in [148]), it holds that for every  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [0, t]} e^{\tilde{B}_s - s} \right] &= e^{-t/2} \mathbb{E} \left[ \sup_{s \in [0, t]} e^{\tilde{B}_s - \tilde{B}_t} \right] \\ &= e^{-t/2} \mathbb{E} \left[ e^{|\tilde{B}_t|} \right] \\ &\leq e^{-t/2} \mathbb{E} \left[ e^{B_t} + e^{-B_t} \right] \\ &\leq 2 \end{aligned} \quad (5.108)$$

where in the second equality we used Theorem 2.23 in [133]. The combination of (5.107) and (5.108) concludes the proof of Lemma 5.31.  $\square$



## Chapitre 6

# A multi-dimensional version of the Matsumoto-Yor properties

### Abstract

This paper is based on a joint work with Thomas Gerard, Christophe Sabot and Xiaolin Zeng. Actually, at the beginning, this paper was written only by Gerard, Sabot and Zeng and was posted on arxiv as a preprint. (See [78].) It consisted in applying a Lamperti time-scale to a system of SDEs which was found by Sabot and Zeng in [155] and which is related to the  $\beta$  potential. Thanks to this Lamperti time-scale, they proved a multidimensional generalization of the Matsumoto-Yor opposite drift theorem which was originally proved in dimension 1 in [119]. However, they were not totally satisfied of their preprint because they did not manage to relate their results to the Matsumoto-Yor properties given by Theorem 1.64 even if such a link was strongly suspected. My contribution consists in having made this link rigorous. More precisely, under the supervision of Christophe Sabot, I proved Theorems 6.4, 6.5 and 6.6 in this chapter. These new results will be published soon in a new version of [78].

### 6.1 Introduction

We start with a discussion on several classical properties of one-dimensional Brownian motion, for which we provide their multi-dimensional counterparts. Let  $\theta > 0$  and  $\eta \geq 0$  be fixed and let  $(B(t))_{t \geq 0}$  be a standard 1-dimensional Brownian motion. Further, let us consider the drifted Brownian motion

$$(X(t))_{t \geq 0} = (\theta + B(t) - \eta t)_{t \geq 0}.$$

Then, by [165, 162], the distribution of the first hitting time  $\tau$  of 0 by  $X$  has a density which is known explicitly and conditionally on  $\tau$ ,  $(X(t))_{0 \leq t \leq \tau}$  is a 3-dimensional Bessel bridge. We will give more details in Proposition A. Another important result is the Matsumoto-Yor opposite drift theorem [120]. This theorem concerns a Brownian motion with negative drift  $-\mu$  which, when conditioned on some exponential functional of its sample path, can be represented as a Brownian motion with opposite drift  $\mu$ , with an additional explicit corrective term. A version of this result is stated in Theorem B, in the special case where  $\mu = \frac{1}{2}$ . One proof of the theorem relies on applying Lamperti's relation to the classic result on hitting times of Brownian motion. Lamperti's relation, presented in Proposition C, provides a way to write any Bessel process with index  $\mu$  as the exponential of a time-changed Brownian motion with drift  $\mu$ . Moreover, in [121] and [122], Matsumoto and Yor proved another interesting results concerning exponential functionals of the Brownian motion. More precisely, if  $B$  is a standard Brownian motion and  $\mu \in \mathbb{R}$ , let us define for every  $t > 0$ ,

$$e_t^{(\mu)} = \exp(B_t + \mu t), \quad A_t^{(\mu)} = \int_0^t (e_s^{(\mu)})^2 ds \quad \text{and} \quad Z_t^{(\mu)} = A_t^{(\mu)} / e_t^{(\mu)}.$$

Then, the conditional distribution of  $e_t^{(\mu)}$  knowing  $Z_t^{(\mu)} = z$  is known explicitly and is a generalized Inverse Gaussian distribution. This result implies a property of intertwining between the processes  $e$  and  $Z$ . Moreover,  $Z$  is a diffusion with respect to a filtration which is strictly smaller than the filtration of  $B$ .

In [155], Sabot and Zeng gave a multivariate version of the process  $X$  which is presented above. This multivariate version concerns a family of Brownian motions indexed by a finite graph with interacting drifts, defined as the solution of a system of stochastic differential equations (SDEs). The hitting times of 0 for this family are related to a random potential, which we denote  $\beta$ , introduced by Sabot, Tarrès and Zeng in [153] and generalized in [110]. The distribution of this random potential can be interpreted as a multi-dimensional version of the Inverse Gaussian distribution. It is closely related to the supersymmetric hyperbolic sigma model studied by Disertori, Spencer and Zirnbauer in [54] and [53], and was central in the analysis of the Vertex Reinforced Jump Process in [152], [153] and [154]. See also [113, 17, 16, 43, 36, 130] for related models in statistical mechanics and random operators.

The goal of this paper is two-fold :

1. Our first goal is to obtain a multi-dimensional version of the opposite drift theorem, by applying an analogue of Lamperti's time change to the family of interacting Brownian motions given in Theorem G. Difficulties arise in applying a time change to the interaction term, since the time change is different on every coordinate of the process. We can overcome this problem in two different ways : either by using the representation given in Theorem G, and applying the time change to each independent 3-dimensional Bessel bridge ; or by using a form of strong Markov property verified by these interacting Brownian motions (c.f. Theorem H or Theorem 2 in [155]).
2. Another goal of this paper is to prove a multidimensional version of the conditional Matsumoto Yor property for  $\mu = -1/2$ . The proof of this property will use the multidimensional version of the opposite drift theorem. Furthermore, thanks to the multidimensional version of the conditional Matsumoto Yor property, we can prove some intertwining and identities in law which generalize previous results of Matsumoto and Yor in [121], [122] and [123].

### Organisation of the paper :

- In section 6.2.1, we recall several results for one-dimensional Brownian motion.
- In section 6.2.2, we recall previous results of Sabot, Zeng and Tarrès about the random potential  $\beta$  and its realisation as the limit of interacting Brownian motions.
- In section 6.2.3, we give multi-dimensional counterparts of the results of section 6.2.1.
- In section 6.2.4, we focus on two open questions.
- The remaining sections are devoted to the proofs.

## 6.2 Context and statement of the results

### 6.2.1 Results in dimension one

**Proposition A.** *Let  $\theta > 0$  and  $\eta \geq 0$  be fixed, and let  $B = (B(t))_{t \geq 0}$  be a standard one-dimensional Brownian motion. We define the Brownian motion  $X = (X(t))_{t \geq 0}$  with drift  $\eta$  by*

$$X(t) = \theta + B(t) - \eta t, \quad t \geq 0.$$

*If  $\tau$  is the first hitting time of 0 by  $X$ , i.e.*

$$\tau = \inf\{t \geq 0, B(t) + \theta - \eta t = 0\}, \quad (6.1)$$

then the distribution of  $\tau$  is given by an inverse Gaussian distribution  $IG(\frac{\theta}{\eta}, \theta^2)$ , its density is

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbf{1}\{t \geq 0\} dt. \quad (6.2)$$

Moreover, conditionally on  $\tau$ ,  $(X(t))_{0 \leq t \leq \tau}$  has the same distribution as a 3-dimensional Bessel bridge from  $\theta$  to 0 on the time interval  $[0, \tau]$ .

In particular, when  $\eta = 0$ , i.e.  $X$  is a Brownian motion without drift, then (6.2) is the density of inverse Gamma, i.e. it has the distribution of  $\frac{1}{2\gamma}$ , where  $\gamma$  is a Gamma random variable with parameter  $(\frac{1}{2}, \theta^2)$ .

Now, let us present a version of the Matsumoto-Yor opposite drift theorem from [120], in the specific case where the drift  $\mu$  is  $\frac{1}{2}$ , and with an added term depending on  $\eta$ .

**Theorem B.** [Theorem 2.2 and Proposition 3.1 in [120]] Let  $\theta > 0$  and  $\eta \geq 0$  be fixed, and let  $B$  be a standard one-dimensional Brownian motion. We define the process  $\varrho$  as the solution of the following SDE :

$$\varrho(u) = \log(\theta) + B(u) - \frac{1}{2}u - \int_0^u \eta e^{\varrho(v)} dv \quad (6.3)$$

for  $u \geq 0$ .

Let us define  $T(u) = \int_0^u e^{2\varrho(v)} dv$ . Then :

(i) We have

$$T(u) \xrightarrow[u \rightarrow \infty]{a.s.} T^\infty,$$

where  $T^\infty$  has the same distribution as  $\tau$  in Proposition A : it is distributed according to

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbf{1}\{t \geq 0\} dt.$$

(ii) Conditionally on  $T^\infty$ , there exists a standard one-dimensional Brownian motion  $B^*$  such that for  $u \geq 0$ ,

$$\varrho(u) = \log(\theta) + B^*(u) + \frac{1}{2}u + \log\left(\frac{T^\infty - T(u)}{T^\infty}\right). \quad (6.4)$$

Note that Theorem B holds for any  $\mu > 0$ , except that the law of  $T^\infty$  needs to be adjusted accordingly. One proof of Theorem B relies on applying a time change to the result on hitting times of the Brownian motion introduced in Proposition A. The relevant time change is the one that appears in Lamperti's relation, presented below (see e.g. [148] p.452) :

**Proposition C.** [Lamperti's relation] Let  $(\varrho(u))_{u \geq 0}$  be a drifted Brownian motion with drift  $\mu \in \mathbb{R}$ . For  $u \geq 0$ , define

$$T(u) = \int_0^u \exp(2\varrho(v)) dv.$$

Then there exists a Bessel process  $(X(t))_{t \geq 0}$  with index  $\mu$ , starting from 1, such that for  $u \geq 0$ ,

$$e^{\varrho(u)} = X(T(u)).$$

Let us sketch the proof of Theorem B using this time change. We use the same notations as in the introduction : fix  $\theta > 0$ ,  $\eta > 0$ , and let  $X(t) = \theta + B(t) - \eta t$  where  $B$  is a standard Brownian motion. Let  $U(t) = \int_0^t \frac{1}{X(s)^2} ds$ , and denote  $T = U^{-1}$ . Note that  $T$  is the analogue of the time change featured in Lamperti's relation, where  $X$  plays the role of a drifted Bessel process with index  $-\frac{1}{2}$ .

If  $\varrho(u) = \log(X(T(u)))$ , then  $e^{\varrho(u)} = X(T(u))$ , and  $\varrho$  has the same distribution as the solution of Equation (6.3). Moreover, when  $u \rightarrow \infty$ , we have  $T(u) \rightarrow \tau$ , where  $\tau$  is the first hitting time of 0 by  $X$ . By Proposition A, conditionally on  $\tau$ ,  $X$  has the distribution of a 3-dimensional Bessel bridge, i.e. a Bessel bridge with index  $\frac{1}{2}$ , consequently the conditional law of  $\varrho$  equals the law of the solution to Equation (6.4).

**Remark 6.1.** In [120], the opposite drift theorem is stated in a different form, where  $\eta = 0$  and the drift  $\mu$  can be different from  $\frac{1}{2}$ . Its proof still relies on applying Lamperti's relation, but this time to a result concerning hitting times of Bessel processes with any index  $-\mu$  (see [107] and [137]).

Now, let us recall the conditional Matsumoto-Yor property [121, 122]. For every  $\alpha \in \mathbb{R}$ ,  $K_\alpha$  is the modified Bessel function of the second kind with index  $\alpha$ .

**Theorem D.** Let  $\mu \in \mathbb{R}$ . Let  $B$  be a Brownian motion. For every  $t > 0$ , define

$$e_t^{(\mu)} = \exp(B_t + \mu t), A_t^{(\mu)} = \int_0^t (e_s^{(\mu)})^2 ds \text{ and } Z_t^{(\mu)} = A_t^{(\mu)} / e_t^{(\mu)}.$$

Let  $(\mathcal{Z}_t^{(\mu)})_{t \geq 0}$  be the natural filtration associated with the process  $(Z_t^{(\mu)})_{t \geq 0}$ .

(i) For every  $t > 0$ ,

$$\mathcal{Z}_t^{(\mu)} \not\subseteq \sigma(B_s, s \leq t).$$

(ii) There exists a Brownian motion  $\tilde{B}$  such that  $(Z_t^{(\mu)})_{t \geq 0}$  is solution of the SDE

$$dZ_t^{(\mu)} = Z_t^{(\mu)} d\tilde{B}_t + \left(\frac{1}{2} - \mu\right) Z_t^{(\mu)} dt + \frac{K_{1+\mu}}{K_\mu} \left(\frac{1}{Z_t^{(\mu)}}\right) dt.$$

(iii) For any  $t > 0$ ,

$$\mathbb{P}(e_t^{(\mu)} \in dx | \mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)} = z) = \frac{x^{\mu-1}}{2K_\mu(1/z)} \exp\left(-\frac{1}{2z} \left(\frac{1}{x} + x\right)\right) dx.$$

Moreover, in [122], Matsumoto and Yor showed that Theorem D implies the following property of intertwinings.

**Theorem E.** Let  $\mu \in \mathbb{R}$ . Let  $(\tilde{P}_t^{(\mu)})_{t \geq 0}$  be the semigroup of  $e^{(\mu)}$  and let  $(\tilde{Q}_t^{(\mu)})_{t \geq 0}$  be the semigroup of the diffusion  $Z^{(\mu)}$ . Then, for every  $t \geq 0$ ,

$$\tilde{Q}_t \tilde{K}^{(\mu)} = \tilde{K}^{(\mu)} \tilde{P}_t$$

where for every measurable function  $g$  from  $\mathbb{R}$  into  $\mathbb{R}_+$ ,

$$\tilde{K}^{(\mu)}(g)(z) = \int_0^{+\infty} g(x) \frac{x^{\mu-1}}{2K_\mu(1/z)} \exp\left(-\frac{1}{2z} \left(\frac{1}{x} + x\right)\right) dx.$$

The aim of this article is now to obtain a multi-dimensional version of Theorem B, D and E, by applying the time change from Lamperti's relation to Theorem G, which gives a multi-dimensional counterparts to these three theorems.

### 6.2.2 Brownian motions with interacting drifts and the random $\beta$ potential

Let  $\mathcal{G} = (V, E)$  be a finite, connected, and non-oriented graph, endowed with conductances  $(W_e)_{e \in E} \in (\mathbb{R}_+^*)^E$ . For  $i, j \in V$ , we denote by  $W_{i,j} = W_{\{i,j\}}$  if  $\{i, j\} \in E$ , and  $W_{i,j} = 0$  otherwise. Note that it is possible to have  $W_{i,i} > 0$ . For  $\beta \in \mathbb{R}^V$ , we define  $H_\beta = 2\beta - W$ , where  $W$  is the graph adjacency matrix  $W = (W_{i,j})_{i,j \in V}$ , and  $\beta$  denotes here abusively the diagonal matrix with diagonal coefficients  $(\beta_i)_{i \in V}$ ; in particular,  $H_\beta$  is a  $V \times V$  matrix. **Notation :** For a vector  $x = (x_i)_{i \in V} \in \mathbb{R}^V$ , we sometimes simply write  $x$  for the diagonal matrix with diagonal coefficients  $x_i$ , there will be no ambiguity thanks to the context by considering dimension.

**Proposition F** (Theorem 4 in [153], Theorem 2.2 in [110]). *For all  $\theta \in (\mathbb{R}_+^*)^V$  and  $\eta \in (\mathbb{R}_+)^V$ , the measure  $\nu_V^{W,\theta,\eta}$  defined by*

$$\nu_V^{W,\theta,\eta}(d\beta) = \mathbf{1}\{H_\beta > 0\} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle\theta, H_\beta\theta\rangle - \frac{1}{2}\langle\eta, (H_\beta)^{-1}\eta\rangle + \langle\eta, \theta\rangle\right) \frac{\prod_{i \in V} \theta_i}{\sqrt{|H_\beta|}} d\beta$$

*is a probability distribution. Moreover, for all  $i \in V$ , the random variable  $\frac{1}{2\beta_i - W_{i,i}}$  has Inverse Gaussian distribution with parameter  $(\frac{\theta_i}{\eta_i + \sum_{j \neq i} W_{i,j}\theta_j}, \theta_i^2)$ . Furthermore, for every  $(\lambda_i) \in \mathbb{R}_+^V$ ,*

$$\int \exp\left(-\sum_{i \in V} \lambda_i \beta_i\right) d\nu^{W,\theta,\eta}(\beta) = e^{-\frac{1}{2}\langle\sqrt{\theta^2 + \lambda}, W\sqrt{\theta^2 + \lambda}\rangle + \frac{1}{2}\langle\theta, W\theta\rangle + \langle\eta, \theta - \sqrt{\theta^2 + \lambda}\rangle} \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}}.$$

For  $t \in (\mathbb{R}_+)^V$ , we also denote by  $K_t$  the matrix  $K_t = \text{Id} - tW$ , where  $t$  still denotes the diagonal matrix with coefficients  $(t_i)_{i \in V}$ . Note that if  $t \in (\mathbb{R}_+^*)^V$ , we have  $K_t = tH_{\frac{1}{2t}}$ , where  $\frac{1}{2t} = \left(\frac{1}{2t_i}\right)_{i \in V}$ . Finally, for  $t \in (\mathbb{R}_+)^V$  and  $T \in (\overline{\mathbb{R}_+})^V$ , we define the vector  $t \wedge T = (t_i \wedge T_i)_{i \in V}$ .

**Theorem G.** [Lemma 1 and Theorem 1 in [155]] *Let  $\theta \in (\mathbb{R}_+^*)^V$  and  $\eta \in (\mathbb{R}_+)^V$  be fixed, and let  $(B_i(t))_{i \in V, t \geq 0}$  be a standard  $|V|$ -dimensional Brownian motion.*

(i) *The following stochastic differential equation (SDE) has a unique pathwise solution :*

$$X_i(t) = \theta_i + \int_0^t \mathbf{1}\{s < \tau_i\} dB_i(s) - \int_0^t \mathbf{1}\{s < \tau_i\} ((W\psi)(s) + \eta)_i ds \quad (E_V^{W,\theta,\eta}(X))$$

*for  $i \in V$  and  $t \geq 0$ , where for  $i \in V$ ,  $\tau_i$  is the first hitting time of 0 by  $X_i$ , and for  $t \geq 0$ ,*

$$\psi(t) = K_{t \wedge \tau}^{-1}(X(t) + (t \wedge \tau)\eta).$$

(ii) *If  $(X_i)_{i \in V}$  is solution of  $(E_V^{W,\theta,\eta}(X))$ , the vector  $\left(\frac{1}{2\tau_i}\right)_{i \in V}$  has distribution  $\nu_V^{W,\theta,\eta}$ , and conditionally on  $(\tau_i)_{i \in V}$ , the paths  $(X_i(t))_{0 \leq t \leq \tau_i}$  are independent 3-dimensional Bessel bridges.*

To obtain an analogue of the opposite drift theorem as in Section 6.2.1, we want to apply the time change from Lamperti's relation to solutions  $(X_i)_{i \in V}$  of  $(E_V^{W,\theta,\eta}(X))$ . A problem will arise in the interaction term, since the time change will be different on every coordinate of  $X$ . To solve this, we will use a form of strong Markov property verified by solutions of  $(E_V^{W,\theta,\eta}(X))$ , which is a consequence of Theorem G(ii). This Markov property will be true with respect to multi-stopping times, defined as follows.

**Definition 6.1.** *Let  $X$  be a multi-dimensional càdlàg process indexed by  $V$ . A random vector  $T = (T_i)_{i \in V} \in \overline{\mathbb{R}_+}^V$  is called a multi-stopping time with respect to  $X$  if for all  $t \in \mathbb{R}_+^V$ , the event  $\cap_{i \in V} \{T_i \leq t_i\}$  is  $\mathcal{F}_t^X$ -measurable, where*

$$\mathcal{F}_t^X = \sigma((X_i(s))_{0 \leq s \leq t_i}, i \in V).$$

*In this case, we denote by  $\mathcal{F}_T^X$  the  $\sigma$ -algebra of events anterior to  $T$ , i.e.*

$$\mathcal{F}_T^X = \{A \in \mathcal{F}_\infty^X, \forall t \in \mathbb{R}_+^V, A \cap \{T_i \leq t_i\} \in \mathcal{F}_t^X\}$$

Let us now formulate the strong Markov property for solutions of  $(E_V^{W,\theta,\eta}(X))$ .

**Theorem H.** [Theorem 2 (iv) in [155]] Let  $X$  be a solution of  $(E_V^{W,\theta,\eta}(X))$ , and  $T = (T_i)_{i \in V}$  be a multi-stopping time with respect to  $X$ .

Define the shifted process  $Y$  by

$$Y_i(t) = X_i(T_i + t)$$

for  $i \in V$  and  $t \geq 0$ . Moreover, we denote

$$\widetilde{W}^{(T)} = W(K_{T \wedge \tau})^{-1}, \widetilde{\eta}^{(T)} = \eta + \widetilde{W}^{(T)}((T \wedge \tau)\eta), \text{ and } X(T) = (X_i(T_i))_{i \in V}.$$

Then on the event  $\cap_{i \in V} \{T_i < \infty\}$ , conditionally on  $T$  and  $\mathcal{F}_T^X$ , the process  $Y$  has the same distribution as the solution of  $(E_V^{\widetilde{W}^{(T)}, X(T), \widetilde{\eta}^{(T)}}(X))$ .

### 6.2.3 Main results : A multi-dimensional version of the opposite drift theorem, the conditional Matsumoto-Yor property and consequences

Let  $(X_i)_{i \in V}$  be a solution of  $(E_V^{W,\theta,\eta}(X))$ . As in the usual case of Lamperti's relation, let us introduce the functional that will define the time change. For  $i \in V$  and  $t \geq 0$ , we set

$$U_i(t) = \int_0^t \frac{\mathbf{1}\{s < \tau_i\}}{X_i(s)^2} ds.$$

It turns out that for any  $i \in V$ ,  $\lim_{t \rightarrow \tau_i} U_i(t) = +\infty$  a.s. It will be proved in Lemma 6.7. Therefore, for all  $i \in V$ , we can define  $T_i = (U_i|_{[0, \tau_i]})^{-1}$ . In particular, for all  $u \geq 0$ ,  $T_i(u) < \tau_i$ . Moreover  $\lim_{u \rightarrow \infty} T_i(u) = \tau_i$ . Thus, in this time scale, it is natural to prefer the notation  $T_i^\infty := \tau_i$  for every  $i \in V$ .

We will show that the time-changed solution  $(X_i \circ T_i)_{i \in V}$  can be written as

$$X_i(T_i(u)) = e^{\varrho_i(u)}$$

for  $u \geq 0$ , where  $(\varrho_i)_{i \in V}$  is solution of a new system of stochastic differential equations :

**Theorem 6.2.** (i) For  $i \in V$  and  $u \geq 0$ , let us define  $\varrho_i(u) = \log(X_i(T_i(u)))$ . Then  $(\varrho, T)$  is solution of the following system of SDEs :

$$\begin{cases} \varrho_i(v) = \log(\theta_i) + \widetilde{B}_i(v) + \int_0^v \left( -\frac{1}{2} - e^{\varrho_i(u)} \left( \widetilde{W}^{(u)}(e^{\varrho(u)} + T(u)\eta) \right)_i \right) du, \\ T_i(v) = \int_0^v e^{2\varrho_i(u)} du, \end{cases} \quad (E_V^{W,\theta,\eta}(\varrho))$$

for  $i \in V$  and  $v \geq 0$ , where  $(\widetilde{B}_i)_{i \in V}$  is a  $|V|$ -dimensional standard Brownian motion,  $e^{\varrho(u)}$  denotes the vector  $(e^{\varrho_i(u)})_{i \in V}$ , and

$$\widetilde{W}^{(u)} = WK_{T(u)}^{-1} = W(\text{Id} - T(u)W)^{-1}.$$

(ii) The equation  $(E_V^{W,\theta,\eta}(\varrho))$  admits a unique pathwise solution  $u \mapsto (\varrho(u), T(u))$ , which is a.s. well defined on all of  $\mathbb{R}_+$ .

As a consequence of Theorems G(ii) and 6.2, we can relate the solutions of  $(E_V^{W,\theta,\eta}(\varrho))$  to time-changed Bessel bridges and the distribution  $\nu_V^{W,\theta,\eta}$ . This is stated in Theorem 6.3 below, which is the multi-dimensional version of Theorem B.

**Theorem 6.3.** Let  $(\varrho, T)$  be solution of  $(E_V^{W,\theta,\eta}(\varrho))$ .

(i) For all  $i \in V$ , we have

$$T_i(u) = \int_0^u e^{2\rho_i(v)} dv \xrightarrow[u \rightarrow \infty]{a.s.} T_i^\infty,$$

where  $\left(\frac{1}{2T_i^\infty}\right)_{i \in V}$  is distributed according to  $\nu_V^{W,\theta,\eta}$ .

(ii) There exists a standard  $|V|$ -dimensional Brownian motion  $B^*$  which is independent of  $T^\infty$  such that for  $i \in V$  and  $u \geq 0$ ,

$$\rho_i(u) = \log(\theta_i) + B_i^*(u) + \frac{1}{2}u + \log\left(\frac{T_i^\infty - T_i(u)}{T_i^\infty}\right).$$

In particular, the processes  $(\rho_i, T_i)_{i \in V}$  are independent conditionally on  $T^\infty$ .

The multidimensional counterpart of Theorem D is as follows.

**Theorem 6.4.** Let  $(\rho, T)$  be a solution of  $E_V^{W,\theta,\eta}(\rho)$ . For every  $u \geq 0$  and for every  $i \in V$ , we define  $Z_i(u) = T_i(u) \exp(-\rho_i(u))$ . Let us denote by  $(\mathcal{Z}_u)_{u \geq 0}$  the natural filtration associated with  $(Z(u))_{u \geq 0}$ . Then, it holds that

(i) For every  $u > 0$ ,

$$\mathcal{Z}_u \not\subseteq \sigma(\rho_v, v \leq u).$$

(ii) The process  $(Z(u))_{u \geq 0}$  has independent components and for every  $i \in V$ , there exists a Brownian motion  $\hat{B}_i$  such that  $(Z_i(u))_{u \geq 0}$  is solution of the SDE

$$dZ_i(u) = Z_i(u)d\hat{B}_i(u) + (\theta_i + Z_i(u))du.$$

(iii)  $(Z(u))_{u \geq 0}$  is independent of  $T^\infty$ .

(iv) For every  $u > 0$ , let us define  $\beta(u) = (\beta_i(u))_{i \in V} = (1/(2T_i(u)))_{i \in V}$ . Then, for every  $u > 0$ , the conditional law of  $\beta(u)$  given  $\mathcal{Z}_u$  is

$$\mathcal{L}\left(\beta(u) \middle| \mathcal{Z}_u, Z(u) = z\right) = \nu_V^{W,\theta,\eta+1/z}$$

where  $\eta + 1/z$  is the vector  $(\eta_i + 1/z_i)_{i \in V}$ .

If  $\rho$  be a solution of  $E_V^{W,\theta,\eta}(\rho)$ , for every  $u \geq 0$  and for every  $i \in V$ , we define  $Z_i(u) = T_i(u) \exp(-\rho_i(u))$ . An important consequence of the previous result is that  $(\rho, T)$  and  $Z$  are related via Markov intertwining which generalize Theorem E. In order to make this more precise, let us introduce some notations. By Theorem 6.4,  $(Z(u))_{u \geq 0}$  is a Markov process in its own sigma-field  $(\mathcal{Z}_u)_{u \geq 0}$ . Let us denote its semigroup by  $(Q_u)_{u \geq 0}$ . Moreover, by Theorem 6.2,  $(\rho, T)$  is also a Markov process. Let us denote its semigroup by  $(P_u)_{u \geq 0}$ .

**Theorem 6.5.** Let  $\theta \in (\mathbb{R}_+^*)^V$  and  $\eta \in (\mathbb{R}_+)^V$ . Let  $(\rho, T)$  be a solution of  $E_V^{W,\theta,\eta}(\rho)$  and let  $Z = Te^{-\rho}$ . Then,  $(\rho, T)$  and  $Z$  are intertwined in the following sense : for every  $u \in \mathbb{R}_+$ ,

$$Q_u \circ K = K \circ P_u$$

where for every measurable function  $g$  from  $(\mathbb{R}^V)^2$  into  $\mathbb{R}_+$ ,

$$K(g)(z) = \int g\left(\left(-\ln(2\beta_i z_i)\right)_{i \in V}, \left(\frac{1}{2\beta_i}\right)_{i \in V}\right) d\nu_V^{W,\theta,\eta+1/z}(\beta).$$

The kernel  $K$  in Theorem 6.5 comes from (iv) in Theorem 6.4. Moreover, thanks to Theorem 6.4, we are able to give some new identities involving the measure  $\nu_V^{W,\theta,\eta}$ . These identities look like other identities in law which are known in the 1-dimensional case. We give more details in the next section.

**Theorem 6.6.** *Let  $\theta \in (\mathbb{R}_+^*)^V$ ,  $\eta \in \mathbb{R}_+^V$  and  $z \in (\mathbb{R}_+^*)^V$ . On the one hand, let  $(\beta_i)_{i \in V}$  be a random potential with distribution  $\nu_V^{W, \theta, \eta + 1/z}$ . Conditionally on  $\beta$ , let  $(\alpha_i)_{i \in V}$  be a random potential with distribution  $\nu_V^{\widetilde{W}, \tilde{\theta}, \tilde{\eta}}$  where  $\widetilde{W} = WK_{1/(2\beta)}^{-1}$ ,  $\tilde{\eta} = \eta + WH_\beta^{-1}\eta$  and for every  $i \in V$ ,  $\tilde{\theta}_i = 1/(2\beta_i z_i)$ . On the other hand, let  $(\delta_i)_{i \in V}$  be a random potential with distribution  $\nu_V^{W, \theta, \eta}$ . Let  $(A_i)_{i \in V}$  be a family of independent random variables such that for every  $i \in V$ ,  $A_i$  is distributed like  $IG(1/(\theta_i z_i), 1/z_i^2)$ . Moreover we assume that  $(A_i)_{i \in V}$  is independent of  $\delta$ . Then it holds that,*

$$\left( (2\beta_i)_{i \in V}, \left( \frac{(2\beta_i)^2}{2\alpha_i} \right)_{i \in V} \right) \stackrel{law}{=} \left( (2\delta_i + A_i)_{i \in V}, \left( A_i + \frac{A_i^2}{2\delta_i} \right)_{i \in V} \right).$$

### 6.2.4 Two open questions

#### The Matsumoto-Yor property

The Gamma and Inverse Gaussian distributions, as well as the inverse Gamma and reciprocal Inverse Gaussian distributions, all fall into the family of the so-called generalized Inverse Gaussian distributions.

A random variable is said to have generalized Inverse Gaussian distribution with parameter  $(q, a, b)$  where  $q \in \mathbb{R}$  and  $a, b > 0$ , and denoted  $GIG(q, a, b)$  if it has the following density :

$$\left(\frac{a}{b}\right)^{q/2} \frac{1}{2K_q(\sqrt{ab})} t^{q-1} e^{-\frac{1}{2}(at+b/t)} \mathbb{1}_{t>0}. \tag{6.5}$$

In particular, we have the following special cases (where zero parameter is understood as in [91]) :

$$IG\left(\frac{\theta}{\eta}, \theta^2\right) = GIG\left(-\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2}\right), \quad \text{Gamma}\left(\frac{1}{2}, \theta^2\right) = GIG\left(\frac{1}{2}, 0, \frac{\theta^2}{2}\right)$$

and

$$X \sim GIG\left(-\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2}\right) \Leftrightarrow 1/X \sim GIG\left(\frac{1}{2}, \frac{\theta^2}{2}, \frac{\eta^2}{2}\right).$$

Define the last visit of 0 of our drifted Brownian motion to be  $\tilde{\tau} = \sup\{t \geq 0 : B_t + \theta - \eta t = 0\}$ . By a time inversion argument, i.e. setting

$$\tilde{B}_t = \begin{cases} -tB_{1/t} & t > 0 \\ 0 & t = 0 \end{cases},$$

the Gaussian process  $\tilde{B}$  is also a Brownian motion and we deduce that  $\tilde{\tau}^{-1}$  is the first visit time to 0 of  $\tilde{B}_t + \eta - \theta t$ , hence  $\tilde{\tau}$  is  $GIG(\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2})$  distributed, moreover,  $\tilde{\tau} - \tau$  is  $\text{Gamma}(\frac{1}{2}, \theta^2)$  distributed and by Strong Markov property of Brownian motion, it is independent of  $\tau$ . There exists also an interpretation of the generalized Inverse Gaussian distribution with any index  $q$  as stopping time of some diffusion process [14, 162] but we focus on the case  $q = \pm 1/2$  in this paper.

More generally, we have the following identity in distribution, which is known as the Matsumoto-Yor property [164, 123] :

**Proposition I.** *Let  $(\tau, \tilde{\tau})$  be a random vector, then there is equivalence between the following statements :*

- (i)  $\left(\frac{1}{\tau}, \tilde{\tau} - \tau\right) \sim GIG(\frac{1}{2}, \frac{\theta^2}{2}, \frac{\eta^2}{2}) \otimes \text{Gamma}(\frac{1}{2}, \theta^2)$
- (ii)  $\left(\frac{1}{\tau} - \frac{1}{\tilde{\tau}}, \tilde{\tau}\right) \sim \text{Gamma}(\frac{1}{2}, \eta^2) \otimes GIG(\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2})$ .

This is not clear how we can get a multi-dimensional generalization of Proposition I. However, the identity of Theorem 6.6 can be rewritten as

$$\left( \frac{1}{A_i + 2\delta_i}, \frac{1}{A_i} - \frac{1}{A_i + 2\delta_i} \right)_{i \in V} \stackrel{\text{law}}{=} \left( \frac{1}{2\beta_i}, \frac{2\alpha_i}{(2\beta_i)^2} \right)_{i \in V}.$$

This identity can be viewed as a weaker form of (i) in Proposition I where we lost the independence property. Recall that for every  $i \in V$ ,  $1/(2\beta_i)$  is the first hitting time of zero of  $X_i$  (that is, a drifted Brownian motion with interaction). Moreover,  $1/A_i$  is distributed like  $GIG\left(1/2, \frac{1}{2z_i^2}, \frac{\theta_i^2}{2}\right)$ , that is, like the last hitting time of zero by a Brownian motion with drift  $1/z_i$  without interaction. Perhaps the identity of Theorem 6.6 can be interpreted in term of a coupling between interacting and non-interacting Brownian motions.

Properties in Proposition I has been called the Matsumoto–Yor (MY) property by Stirzaker [160] p. 43. Letac and Wesolowski [109] provided a characterization theorem related to MY property, namely, if  $\tau, \tilde{\tau}$  are random variables s.t.  $\frac{1}{\tau}, \tilde{\tau} - \tau$  are independent and  $\frac{1}{\tau} - \frac{1}{\tilde{\tau}}, \tilde{\tau}$  are independent, then they necessarily follow the law prescribed in Proposition I. It is tempting to say that a multi-dimensional counterpart of such characterization law also holds.

### An opposite-drift theorem for other values of the drift

The multi-dimensional opposite-drift theorem proved in this paper is limited to the case of the drift  $-\frac{1}{2}$ , since it results from Theorem G, which concerns Bessel processes with index  $-\frac{1}{2}$  and  $\frac{1}{2}$  (i.e. Brownian motion and 3-dimensional Bessel bridges). We could try to obtain a similar result for other values of the drift  $\mu$ . This necessitates the use of a random potential analogous to  $\beta$ , whose marginals would relate to the hitting times of Bessel processes with other indices, that is, generalized Inverse Gaussian distributions. A natural candidate for the distribution of the potential associated with the drift  $-\mu$  with  $\mu > 0$  is the measure  $\nu_{V,\mu}^{W,\eta}$  with density :

$$\nu_{V,\mu}^{W,\theta,\eta}(d\beta) = C(\mu, W, \eta) \mathbf{1}\{H_\beta > 0\} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle\theta, H_\beta\theta\rangle - \frac{1}{2}\langle\eta, (H_\beta)^{-1}\eta\rangle\right) \frac{\prod_{i \in V} \theta_i}{|H_\beta|^{\mu-1}} d\beta$$

where  $C(\mu, W, \eta)$  is a normalizing constant. Nonetheless, the explicit density of  $\beta$  plays an important role in our proof. When  $\mu \neq \frac{1}{2}$ , its normalizing constant  $C(\mu, W, \eta)$  is no longer a constant, but depends on the underlying graph and our proof does not apply directly to such cases.

The case of index  $\mu = \frac{3}{2}$  might be solvable thanks to recent developments by Bauerschmidt, Crawford, Helmuth and Swan in [18], and by Crawford in [46]. These articles concern other sigma models, in particular  $\mathbb{H}^{2|4}$ , which is related to random spanning forests, The normalizing constant  $C(3/2, W, \eta)$  of  $\beta$  associated to this model is the partition function of random forests. It might be a candidate if one looks for a generalization of the  $\beta$  potential corresponding to index  $\frac{3}{2}$ . Moreover, the SDE given by (i) in Theorem 6.2 should be much more complicated.

## 6.3 Multi-dimensional time change : Proof of Theorem 6.2 and Theorem 6.3

### 6.3.1 Justification of the Lamperti time change

Recall that

$$U_i(t) = \int_0^t \frac{\mathbf{1}\{s < \tau_i\}}{X_i(s)^2} ds$$

where  $(X_i)_{i \in V}$  is a solution of  $(E_V^{W,\theta,\eta}(X))$ .

**Lemma 6.7.** For any  $i \in V$ ,

$$\lim_{t \rightarrow \tau_i} U_i(t) = +\infty,$$

consequently  $U_i : [0, \tau_i[ \rightarrow [0, +\infty[$  is a.s. a bijection.

*Proof.*

Let  $X = (X_i)_{i \in V}$  be a solution of  $(E_V^{W, \theta, \eta}(X))$ . According to Theorem G, conditionally on  $(\tau_i)_{i \in V}$ , the trajectories  $(X_i(t))_{0 \leq t \leq \tau_i}$  are independent three-dimensional Bessel bridges. As a consequence, to prove Lemma 6.7, it suffices to show the same result for a three-dimensional Bessel bridge.

Fix  $\theta > 0$  and  $\tau > 0$ , let  $X$  be a three-dimensional Bessel bridge from  $\theta$  to 0 in the time interval  $[0, \tau]$ . Since  $X$  is a three-dimensional Bessel bridge, there exists a standard Brownian motion  $B$  such that

$$dX(t) = dB(t) + \frac{1}{X(t)}dt - \frac{X(t)}{\tau - t}dt,$$

therefore by Ito's lemma, for  $t < \tau$ ,

$$\begin{aligned} d \log(X(t)) &= \frac{dB(t)}{X(t)} + \frac{dt}{X(t)^2} - \frac{dt}{\tau - t} - \frac{1}{2} \frac{dt}{X(t)^2} \\ &= dM(t) + \frac{1}{2}dU(t) + d \log(\tau - t), \end{aligned}$$

where  $M(t) = \int_0^t \frac{dB(s)}{X(s)}$  is a martingale, and  $\langle M \rangle_t = U(t)$ . Therefore, there exists a standard Brownian motion  $\hat{B}$  such that  $M(t) = \hat{B}(U(t))$ . Thus, for  $t \geq 0$ , we have

$$\log(X(t)) = \log(\theta) + \hat{B}(U(t)) + \frac{1}{2}U(t) + \log\left(\frac{\tau - t}{\tau}\right), \quad (6.6)$$

*i.e.*

$$\frac{X(t)}{\tau - t} = \frac{\theta}{\tau} e^{\hat{B}(U(t)) + \frac{1}{2}U(t)}.$$

On the other hand, since  $X$  is a three-dimensional Bessel bridge, there exists (see [148] p.467) a three-dimensional Bessel process  $Y$  such that for  $t \geq 0$ ,

$$X(t) = (\tau - t)Y\left(\frac{t}{\tau(\tau - t)}\right).$$

Therefore, when  $t \rightarrow \tau$ , we have a.s.  $\frac{X(t)}{\tau - t} \rightarrow +\infty$ . Since  $u \mapsto \hat{B}(u) + \frac{1}{2}u$  cannot explode in finite time, we have necessarily

$$U(t) \xrightarrow[t \rightarrow \tau]{a.s.} +\infty.$$

□

### 6.3.2 Proof of Theorem 6.2

*Proof of Theorem 6.2 (ii).*

Assume that Theorem 6.2 (i) is proven, *i.e.* that  $(E_V^{W, \theta, \eta}(\varrho))$  has almost surely a unique pathwise solution defined on all of  $\mathbb{R}_+$ . Let  $\tilde{B}$  be a  $|V|$ -dimensional Brownian motion. Thanks to Theorem 6.2 (i), we know that  $(E_V^{W, \theta, \eta}(\varrho))$  admits a solution that is well defined on  $\mathbb{R}_+$ . Let us now show that this solution is necessarily unique.

Let  $(\varrho^*, T^*)$  be another solution of  $(E_V^{W, \theta, \eta}(\varrho))$  with the Brownian motion  $\tilde{B}$ . Let also  $\mathcal{K}$  be a compact subset of  $\mathbb{R}^V \times \{t \in \mathbb{R}_+, K_t > 0\}$  containing  $(\log(\theta_i), 0)_{i \in V}$ . Then the function

$$\left\{ \begin{array}{l} \mathcal{K} \rightarrow \mathbb{R}^V \times \mathbb{R}^V \\ (\varrho, t) \mapsto \left( -\frac{1}{2} - e^{\varrho_i} \left( W K_t^{-1} (e^{\varrho} + t\eta) + \eta \right)_i, e^{2\varrho_i} \right)_{i \in V} \end{array} \right.$$

is bounded and Lipschitz. Therefore, up to the stopping time  $U_{\mathcal{K}} = \inf\{u \geq 0, (\varrho(u), T(u)) \notin \mathcal{K}\}$ , we have  $(\varrho(u), T(u)) = (\varrho^*(u), T^*(u))$  from Theorem 2.1, p.375 of [148]. Since this is true for all compact subset  $\mathcal{K}$  of  $\mathbb{R}^V \times \{t \in \mathbb{R}_+^V, K_t > 0\}$ , we have a.s.  $(\varrho, T) = (\varrho^*, T^*)$ . This concludes the proof of Theorem 6.2 (ii).  $\square$

Let's prove Theorem 6.2 (i). Let  $B$  be a standard  $|V|$ -dimensional Brownian motion, and let  $(X_i)_{i \in V}$  be a solution of  $(E_V^{W, \theta, \eta}(X))$ . For  $i \in V$ , recall that  $T_i$  is the inverse function of

$$U_i : \begin{cases} [0, T_i^\infty[ \rightarrow [0, +\infty[ \\ t \mapsto \int_0^t \frac{ds}{X_i(s)^2} \end{cases}$$

and  $\varrho_i(u) = \log(X_i(T_i(u)))$  for  $u \geq 0$ . In order to show that  $(\varrho, T)$  is solution of  $(E_V^{W, \theta, \eta}(\varrho))$ , we want to apply the same time change as in Lamperti's relation. However, in the equation  $(E_V^{W, \theta, \eta}(X))$ , the interactive drifts provided by  $\psi(t)$  to each coordinates  $X_i(t)$  are calculated at the same time  $t \geq 0$ , while in Lamperti's time scale, i.e. for  $X_i(t) = e^{\varrho_i(U_i(t))}$ , the times  $U_i(t)$  are different at each coordinates  $i \in V$ .

We present here two different ways to overcome this problem. The first proof relies on identifying the infinitesimal generator of the process  $(\varrho_i, T_i)_{i \in V}$ , using the strong Markov property presented in Theorem H. The second one uses Theorem G (ii), i.e.  $X$  is a mixture of independent Bessel bridges, to which we can apply the time change separately, and then identify the law of the annealed process using Girsanov's theorem.

### 6.3.3 First proof of (i) : using the strong Markov property of Theorem H

*Proof of Theorem 6.2 (i).*

Firstly, let  $u \geq 0$  be fixed, and  $f : \mathbb{R}^V \times \mathbb{R}^V \rightarrow \mathbb{R}$  be a compactly supported  $\mathcal{C}^2$  function. To identify the infinitesimal generator of  $(\varrho, T)$ , let us compute

$$\lim_{v \rightarrow u^+} \frac{\mathbb{E}[f(\varrho(v), T(v)) | \mathcal{F}_u^{(\varrho, T)}] - f(\varrho(u), T(u))}{v - u}.$$

Note that  $(T_i(u))_{i \in V}$  is a multi-stopping time in the sense of Definition 6.1 and that  $\mathcal{F}_u^{(\varrho, T)} = \mathcal{F}_{T(u)}^X$ . Define

$$\widetilde{W}^{(u)} = W(K_{T(u)})^{-1}, \quad \widetilde{K}_t^{(u)} = \text{Id} - t\widetilde{W}^{(u)}, \quad \text{and} \quad \widetilde{\eta}^{(u)} = \eta + \widetilde{W}^{(u)}(T(u)\eta).$$

Thanks to Theorem H, conditionally on  $\mathcal{F}_{T(u)}^X$ , the shifted process

$$Y = Y^{(u)} : t \mapsto (X_i(T_i(u) + t))_{i \in V}$$

is the solution of

$$\begin{cases} dY_i(t) = \mathbf{1}\{t \leq \widehat{T}_i^0\} d\widehat{B}_i(t) - \mathbf{1}\{t \leq \widehat{T}_i^0\} \left( \widetilde{W}^{(u)}(\widetilde{K}_t^{(u)})^{-1} \left( Y(t) + (t \wedge \widehat{T}_i^0) \widetilde{\eta}^{(u)} \right) + \widetilde{\eta}^{(u)} \right)_i dt, & i \in V, t \geq 0 \\ Y_i(0) = X_i(T_i(u)), & i \in V \end{cases}$$

where  $\widehat{B}$  is a  $|V|$ -dimensional standard Brownian motion, independent of  $\mathcal{F}_{T(u)}^X$ , and  $\widehat{T}_i^0$  is the first hitting time of 0 by  $Y_i$ .

Fix  $v > u$ , define the interrupted process

$$Z = Z^{(u,v)} : t \mapsto \left( X_i((T_i(u) + t) \wedge T_i(v)) \right)_{i \in V}.$$

For all  $i \in V$  and  $t \geq 0$ , we have  $Z_i(t) = Y_i(t \wedge \widehat{T}_i(v))$ , where  $\widehat{T}_i(v) = T_i(v) - T_i(u)$ . Therefore,  $Z$  is the solution of

$$\begin{cases} dZ_i(t) = \mathbf{1}\{t \leq \widehat{T}_i(v)\} d\widehat{B}_i(t) - \mathbf{1}\{t \leq \widehat{T}_i(v)\} \left( \widetilde{W}^{(u)}(\widetilde{K}_t^{(u)})^{-1} \left( Y(t) + (t \wedge \widehat{T}_i^0) \widetilde{\eta}^{(u)} \right) + \widetilde{\eta}^{(u)} \right)_i dt, & i \in V, t \geq 0 \\ Z_i(0) = X_i(T_i(u)), & i \in V. \end{cases}$$

Moreover, since  $\widehat{T}_i(v) < \widehat{T}_i^0 < \infty$  a.s. for all  $i \in V$ , there exists a.s.  $\widehat{T}^\infty$  large enough so that  $Z_i(t) = Y_i(\widehat{T}_i(v)) = X_i(T_i(v))$  for all  $i \in V$  and  $t \geq \widehat{T}^\infty$ .

By Ito's lemma, for all  $t \geq 0$  we have

$$\begin{aligned} d \log(Z_i(t)) &= \mathbf{1}\{t \leq \widehat{T}_i(v)\} \frac{d\widehat{B}_i(t)}{Z_i(t)} - \mathbf{1}\{t \leq \widehat{T}_i(v)\} \frac{dt}{2Z_i(t)^2} \\ &\quad - \mathbf{1}\{t \leq \widehat{T}_i(v)\} \left( \widetilde{W}^{(u)}(\widetilde{K}_t^{(u)})^{-1} \left( Y(t) + (t \wedge \widehat{T}^0) \widetilde{\eta}^{(u)} \right) + \widetilde{\eta}^{(u)} \right)_i \frac{dt}{Z_i(t)}, \end{aligned}$$

where we can replace  $t \wedge \widehat{T}^0$  with  $t$ , as  $\widehat{T}_i(v) < \widehat{T}^0$ . For  $i \in V$ , let  $\widehat{M}_i$  be the martingale  $\widehat{M}_i(t) = \int_0^t \frac{d\widehat{B}_i(s)}{Z_i(s)}$  for  $t \geq 0$ .

If we denote

$$\Phi(t) = (\log(Z_i(t)), (T_i(u) + t) \wedge T_i(v))_{i \in V} \in \mathbb{R}^V \times \mathbb{R}^V$$

for  $t \geq 0$ , then, applying Ito's lemma to  $t \mapsto f(\Phi(t))$ , we get

$$\begin{aligned} f(\Phi(t)) - f(\Phi(0)) &= \sum_{i \in V} \int_0^{t \wedge \widehat{T}_i(v)} \frac{\partial f}{\partial \varrho_i}(\Phi(s)) d\widehat{M}_i(s) + \sum_{i \in V} \int_0^{t \wedge \widehat{T}_i(v)} \frac{1}{2} \frac{\partial^2 f}{\partial \varrho_i^2}(\Phi(s)) \frac{ds}{Z_i(s)^2} \\ &\quad + \sum_{i \in V} \int_0^{t \wedge \widehat{T}_i(v)} \frac{\partial f}{\partial \varrho_i}(\Phi(s)) \left( -\frac{1}{2} - Z_i(s) \left( \widetilde{W}^{(u)}(\widetilde{K}_s^{(u)})^{-1} (Y(s) + s \widetilde{\eta}^{(u)}) + \widetilde{\eta}^{(u)} \right)_i \right) \frac{ds}{Z_i(s)^2} \\ &\quad + \sum_{i \in V} \int_0^{t \wedge \widehat{T}_i(v)} \frac{\partial f}{\partial t_i}(\Phi(s)) ds. \end{aligned}$$

Taking  $t \geq \widehat{T}^\infty$ , we get  $t \wedge \widehat{T}_i(v) = \widehat{T}_i(v)$  for all  $i \in V$ , and

$$f(\Phi(t)) - f(\Phi(0)) = f(\varrho(v), T(v)) - f(\varrho(u), T(u)),$$

since  $\varrho_i(w) = \log(X_i(T_i(w)))$  for  $w \in \mathbb{R}_+$  and  $i \in V$ . For all  $i \in V$ , we can now use the following time change in the corresponding integrals above :  $s = T_i(w) - T_i(u) = \widehat{T}_i(w)$ , i.e.  $w = U_i(T_i(u) + s)$ . Note that for  $0 \leq s \leq \widehat{T}_i(v)$ ,

$$\frac{d}{ds} U_i(T_i(u) + s) = \frac{1}{X_i(T_i(u) + s)^2} = \frac{1}{Z_i(s)^2},$$

and for  $u \leq w \leq v$ ,

$$\frac{d}{dw} \widehat{T}_i(w) = X_i(T_i(w))^2 = e^{2\varrho_i(w)}.$$

Thus,

$$\begin{aligned} &f(\varrho(v), T(v)) - f(\varrho(u), T(u)) \\ &= \sum_{i \in V} \left( \int_u^v \frac{\partial f}{\partial \varrho_i}(\varrho(w), T(w)) d\widehat{M}_i(\widehat{T}_i(w)) + \int_u^v \frac{1}{2} \frac{\partial^2 f}{\partial \varrho_i^2}(\varrho(w), T(w)) dw \right. \\ &\quad \left. + \int_u^v \frac{\partial f}{\partial \varrho_i}(\varrho(w), T(w)) \left( -\frac{1}{2} - e^{\varrho_i(w)} \left( \widetilde{W}^{(u)}(\widetilde{K}_{\widehat{T}_i(w)}^{(u)})^{-1} (X(T_i(w)) + \widehat{T}_i(w) \widetilde{\eta}^{(u)}) + \widetilde{\eta}^{(u)} \right)_i \right) dw \right. \\ &\quad \left. + \int_u^v \frac{\partial f}{\partial t_i}(\varrho(w), T(w)) e^{2\varrho_i(w)} dw \right). \end{aligned}$$

Note that, the vector  $X(T_i(w)) = (X_j(T_i(w)))_{j \in V}$  is different from  $e^{\varrho(w)} = (X_j(T_j(w)))_{j \in V}$ . This is why we need to take  $v \rightarrow u$  and identify the generator.

Since  $\widehat{B}$  is independent from  $\mathcal{F}_{T(u)}^X$ , we have

$$\mathbb{E} \left[ \int_u^v \frac{\partial f}{\partial \varrho_i}(\varrho(w), T(w)) d\widehat{M}_i(\widehat{T}_i(w)) \middle| \mathcal{F}_u^{(\varrho, T)} \right] = \mathbb{E} \left[ \int_0^{\widehat{T}_i(v)} \frac{\partial f}{\partial \varrho_i}(\Phi(t)) \frac{d\widehat{B}_i(s)}{Z_i(s)} \middle| \mathcal{F}_{T(u)}^X \right] = 0$$

for all  $i \in V$ , and therefore

$$\begin{aligned} & \mathbb{E} \left[ f(\varrho(v), T(v)) - f(\varrho(u), T(u)) \middle| \mathcal{F}_u^{(\varrho, T)} \right] \\ &= \mathbb{E} \left[ \sum_{i \in V} \left( \int_u^v \frac{1}{2} \frac{\partial^2 f}{\partial \varrho_i^2}(\varrho(w), T(w)) dw \right. \right. \\ & \quad \left. \left. + \int_u^v \frac{\partial f}{\partial \varrho_i}(\varrho(w), T(w)) \left( -\frac{1}{2} - e^{\varrho_i(w)} \left( \widetilde{W}^{(u)}(\tilde{K}_{\widehat{T}_i(w)}^{(u)})^{-1} (X(T_i(w)) + \widehat{T}_i(w) \tilde{\eta}^{(u)}) + \tilde{\eta}^{(u)} \right)_i \right) dw \right. \right. \\ & \quad \left. \left. + \int_u^v \frac{\partial f}{\partial t_i}(\varrho(w), T(w)) e^{2\varrho_i(w)} dw \right) \middle| \varrho(u), T(u) \right]. \end{aligned}$$

By continuity and dominated convergence, we conclude that

$$\begin{aligned} & \lim_{v \rightarrow u^+} \frac{1}{v - u} \mathbb{E} \left[ f(\varrho(v), T(v)) \middle| \mathcal{F}_u^{(\varrho, T)} \right] - f(\varrho(u), T(u)) \\ &= \sum_{i \in V} \left( \frac{1}{2} \frac{\partial^2 f}{\partial \varrho_i^2}(\varrho(u), T(u)) \right. \\ & \quad \left. + \frac{\partial f}{\partial \varrho_i}(\varrho(u), T(u)) \left( -\frac{1}{2} - e^{\varrho_i(u)} \left( \widetilde{W}^{(u)} e^{\varrho(u)} + \tilde{\eta}^{(u)} \right)_i \right) + \frac{\partial f}{\partial t_i}(\varrho(u), T(u)) e^{2\varrho_i(u)} \right), \end{aligned}$$

which is  $\mathcal{L}f(u)$ , where  $\mathcal{L}$  is the infinitesimal generator associated with the system of SDEs  $(E_V^{W, \theta, \eta}(\varrho))$ .  $\square$

### 6.3.4 Second proof of (i) : using the mixing measure and Girsanov's theorem

This proof follows the same structure as that of Theorem G. We start from the distribution of the process as a mixture of simpler quenched processes, and we compute the integral in order to identify the annealed distribution, using Girsanov's theorem.

*Alternative proof of Theorem 6.2 (i).*

Let  $X = (X_i(t))_{i \in V, t \geq 0}$  be the canonical process in  $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^V)$ , and  $\mathbb{P}$  be the distribution on  $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^V)$  under which  $X$  is solution of  $(E_V^{W, \theta, \eta}(X))$ . According to Theorem G (ii), the vector  $(\beta_i)_{i \in V} = \left( \frac{1}{2\tau_i} \right)_{i \in V} = \left( \frac{1}{2T_i^\infty} \right)_{i \in V}$  has distribution  $\nu_V^{W, \theta, \eta}$ . Moreover, conditionally on  $(T_i^\infty)_{i \in V}$ , the marginal processes  $X_i$  for  $i \in V$  are independent 3-dimensional Bessel bridges from  $\theta_i$  to 0 on  $[0, T_i^\infty]$ . In other words, we can write

$$\mathbb{P}[\cdot] = \int \left( \bigotimes_{i \in V} \mathbb{P}_i^{\beta_i}[\cdot] \right) \nu_V^{W, \theta, \eta}(d\beta),$$

where for  $i \in V$ ,  $\mathbb{P}_i^{\beta_i}$  is the distribution on  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  under which the canonical process  $X_i$  is a 3-dimensional Bessel bridge from  $\theta_i$  to 0 on  $[0, T_i^\infty]$ .

Conditionally on  $(\beta_i)_{i \in V}$ , we can apply the time change independently to each marginal  $X_i$ . According to the computations done in the proof of Lemma 6.7, In particular, Equation (6.6), we know that under  $\mathbb{P}_i^{\beta_i}$ , there exists a standard Brownian motion  $\widehat{B}_i$  such that

$$\varrho_i(u) = \log(\theta_i) + \widehat{B}_i(u) + \frac{1}{2}u + \log \left( \frac{T_i^\infty - T_i(u)}{T_i^\infty} \right), u \geq 0$$

where  $T_i^\infty = \frac{1}{2\beta}$  and  $T_i(u) = \int_0^u e^{2\varrho_i(v)} dv$ , i.e.

$$\varrho_i(u) = \log(\theta_i) + \widehat{B}_i(u) + \frac{1}{2}u + \log \left( 1 - 2\beta_i \int_0^u e^{2\varrho_i(v)} dv \right).$$

For each  $i \in V$ , define a martingale  $L_i$  by

$$L_i(u) = \int_0^u \left( -\frac{1}{2} + \frac{2\beta_i e^{2\varrho_i(v)}}{1 - 2\beta_i \int_0^v e^{2\varrho_i(s)} ds} \right) d\widehat{B}_i(v), \quad u \geq 0.$$

Clearly  $\varrho_i(u) = \widehat{B}_i(u) - \langle \widehat{B}_i, L_i \rangle_u$ . We can then introduce a probability distribution  $\widehat{\mathbb{P}}_i$  such that for all  $u \geq 0$ ,

$$\mathbb{E} \left[ \frac{d\widehat{\mathbb{P}}_i}{d\mathbb{P}_i^{\beta_i}} \middle| \mathcal{F}_u^i \right] = \mathcal{E}(L_i)(u),$$

where  $\mathcal{F}_u^i = \sigma \left( \widehat{B}_i(v), 0 \leq v \leq u \right)$ , and  $\mathcal{E}(L_i)(u) = e^{L_i(u) - \frac{1}{2} \langle L_i, L_i \rangle_u}$  is the exponential martingale associated with  $L_i$ . By Girsanov's theorem,  $\varrho_i$  is a standard Brownian motion under  $\widehat{\mathbb{P}}_i$ . Note that  $\widehat{\mathbb{P}}_i$  does not depend on  $\beta_i$ .

From now on, let us write  $\phi_i(u) = 1 - 2\beta_i \int_0^u e^{2\varrho_i(v)} dv = \frac{T_i^\infty - T_i(u)}{T_i^\infty}$  for  $u \geq 0$  and  $i \in V$ . The following lemma gives an expression of  $\mathcal{E}(L_i)$ .

**Lemma 6.8.** *For  $i \in V$  and  $u \geq 0$ , define*

$$E_i(u) = \exp \left( -\theta_i^2 \beta_i + \frac{\beta_i e^{2\varrho_i(u)}}{\phi_i(u)} - \frac{1}{2} \varrho_i(u) + \frac{1}{8} u \right) \phi_i(u)^{3/2} \sqrt{\theta_i}.$$

We have  $\mathcal{E}(L_i) = E_i$ .

*Proof of Lemma 6.8.*

Since  $E_i(0) = 1$  almost surely, it suffices to show that  $\frac{dE_i(u)}{E_i(u)} = dL_i(u)$  for all  $u \geq 0$ . Note that  $\varrho_i(u) = \widehat{B}_i(u) + \frac{1}{2}u + \log(\phi_i(u))$ , therefore

$$E_i(u) = \exp \left( -\theta_i^2 \beta_i + \beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} - \frac{1}{2} \widehat{B}_i(u) - \frac{1}{8} u \right) \phi_i(u) \sqrt{\theta_i}.$$

By Ito's lemma, for  $u \geq 0$  we have

$$\begin{aligned} dE_i(u) &= \left( 2\beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} - \frac{1}{2} \right) E_i(u) d\widehat{B}_i(u) \\ &\quad + \frac{1}{2} \left( \left( 2\beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} - \frac{1}{2} \right)^2 + 4\beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} \right) E_i(u) du \\ &\quad + \left( \beta_i (\phi_i(u) + \phi_i'(u)) e^{2\widehat{B}_i(u)+u} - \frac{1}{8} + \frac{\phi_i'(u)}{\phi_i(u)} \right) E_i(u) du. \end{aligned}$$

Since  $\phi_i'(u) = -2\beta_i e^{2\varrho_i(u)} = -2\beta_i \phi_i(u)^2 e^{2\widehat{B}_i(u)+u}$ , we get

$$\begin{aligned} \frac{dE_i(u)}{E_i(u)} &= \left( -\frac{1}{2} + \frac{2\beta_i e^{2\varrho_i(u)}}{\phi_i(u)} \right) d\widehat{B}_i(u) + \left( 2\beta_i^2 \phi_i(u)^2 e^{4\widehat{B}_i(u)+2u} + \frac{1}{8} + \beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} \right. \\ &\quad \left. + \beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} - 2\beta_i^2 \phi_i(u)^2 e^{4\widehat{B}_i(u)+2u} - \frac{1}{8} - 2\beta_i \phi_i(u) e^{2\widehat{B}_i(u)+u} \right) du \\ &= dL_i(u). \end{aligned}$$

□

Fix  $u \geq 0$ , for any event  $A_u \in \mathcal{F}_u^\varrho = \sigma(\varrho(v), 0 \leq v \leq u)$ , we have

$$\begin{aligned} \mathbb{P}[A_u] &= \int \left( \bigotimes_{i \in V} \mathbb{P}_i^{\beta_i} [A_u] \right) \nu_V^{W, \theta, \eta} = \int \int_{A_u} \prod_{i \in V} \left( E_i(u)^{-1} d\hat{\mathbb{P}}_i \right) \nu_V^{W, \theta, \eta}(d\beta) \\ &= \int_{A_u} D(u) d\hat{\mathbb{P}} \end{aligned}$$

where  $\hat{\mathbb{P}} = \bigotimes_{i \in V} \hat{\mathbb{P}}_i$  and for  $u \geq 0$ ,

$$D(u) = \int \left( \prod_{i \in V} E_i(u)^{-1} \right) \nu_V^{W, \theta, \eta}(d\beta).$$

Let's compute  $D(u)$  now, and express it as an exponential martingale, so as we can apply Girsanov's theorem once again, and identify the distribution of  $\varrho$  under  $\mathbb{P}$ .

For  $u \geq 0$ , we have

$$\begin{aligned} D(u) &= \int \exp \left( \sum_{i \in V} \left( \theta_i^2 \beta_i - \frac{\beta_i e^{2\varrho_i(u)}}{\phi_i(u)} + \frac{1}{2} \varrho_i(u) - \frac{1}{8} u \right) \right) \frac{1}{\prod_{i \in V} \phi_i(u)^{3/2} \sqrt{\theta_i}} \\ &\quad \mathbf{1}\{H_\beta > 0\} \left( \frac{2}{\pi} \right)^{|V|/2} \exp \left( -\frac{1}{2} \langle \theta, H_\beta \theta \rangle - \frac{1}{2} \langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \theta \rangle \right) \frac{\prod_{i \in V} \theta_i d\beta_i}{\sqrt{|H_\beta|}}. \end{aligned}$$

In order to compute this integral, we will introduce a change of variables, and obtain an integral against the distribution  $\nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}}$ , where  $\tilde{W}^{(u)}$ ,  $\tilde{\theta}^{(u)}$  and  $\tilde{\eta}^{(u)}$  are new parameters depending on the trajectory of  $\varrho$  up to time  $u$  defined below.

Let us introduce the following notations : for  $u \geq 0$ ,

$$\begin{cases} \beta_i^{(u)} = \frac{1}{2T_i(u)} \text{ for } i \in V, \\ H^{(u)} = 2\beta^{(u)} - W, \\ K^{(u)} = T(u)H^{(u)} = \text{Id} - T(u)W. \end{cases}$$

Now, we define the new following parameters

$$\begin{cases} \tilde{W}^{(u)} = W(K^{(u)})^{-1} = W + W(H^{(u)})^{-1}W, \\ \tilde{\eta}^{(u)} = \tilde{W}^{(u)}T(u)\eta + \eta, \\ \tilde{\theta}_i^{(u)} = e^{\varrho_i(u)} \text{ for } i \in V \end{cases}$$

as well as the following associated quantities

$$\begin{cases} \tilde{T}_i(u) = \frac{1}{2\tilde{\beta}_i} - T_i(u) = \frac{\phi_i(u)}{2\beta_i} \text{ for } i \in V, \\ \tilde{\beta}_i^{(u)} = \frac{1}{2\tilde{T}_i(u)} = \frac{\beta_i}{\phi_i(u)} \text{ for } i \in V, \\ \tilde{H}^{(u)} = 2\tilde{\beta}^{(u)} - \tilde{W}^{(u)}, \\ \tilde{K}^{(u)} = \tilde{T}(u)\tilde{H}^{(u)} = \text{Id} - \tilde{T}(u)\tilde{W}^{(u)}. \end{cases}$$

Using these new notations, we can write

$$\sum_{i \in V} \frac{\beta_i e^{2\varrho_i(u)}}{\phi_i(u)} = \sum_{i \in V} (\tilde{\theta}_i^{(u)})^2 \tilde{\beta}_i^{(u)} = \frac{1}{2} \left\langle \tilde{\theta}^{(u)}, \left( \tilde{H}^{(u)} + \tilde{W}^{(u)} \right) \tilde{\theta}^{(u)} \right\rangle \quad (6.7)$$

for  $u \geq 0$ . The following technical lemma will allow us to express  $D(u)$  as an integral against  $\nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}}$ .

**Lemma J** (Lemma 2 in [155]). *For  $u \geq 0$ , with the quantities defined just above, we have*

- (i)  $K_{1/2\beta} = \tilde{K}^{(u)} K^{(u)}$
- (ii)  $\tilde{\eta}^{(u)} = T(u)^{-1} (H^{(u)})^{-1} \eta$
- (iii)  $\langle \tilde{\eta}^{(u)}, (\tilde{H}^{(u)})^{-1} \tilde{\eta}^{(u)} \rangle = \langle \eta, H_\beta^{-1} \eta \rangle - \langle \eta, (H^{(u)})^{-1} \eta \rangle$ .

Using Lemma J (i), we get that for  $u \geq 0$ ,

$$H_\beta = 2\beta K_{1/2\beta} = 2\beta \tilde{K}^{(u)} K^{(u)} = 2\beta \tilde{T}(u) \tilde{H}^{(u)} K^{(u)},$$

where  $2\beta_i \tilde{T}_i(u) = 1 - \frac{\beta_i}{\beta_i^{(u)}} = \phi_i(u)$  for  $i \in V$ . Therefore, we have

$$\prod_{i \in V} \phi_i(u)^{3/2} \sqrt{|H_\beta|} = \prod_{i \in V} \phi_i(u)^2 \sqrt{|\tilde{H}^{(u)}|} \sqrt{|K^{(u)}|}, \quad (6.8)$$

where

$$\frac{d\tilde{\beta}_i^{(u)}}{d\beta_i} = \frac{1}{\left(1 - \frac{\beta_i}{\beta_i^{(u)}}\right)^2} = \frac{1}{\phi_i(u)^2}.$$

Moreover, for all  $u \geq 0$  we have

$$\mathbf{1}\{H_\beta > 0\} = \mathbf{1}\{H^{(u)} > 0\} \mathbf{1}\{\tilde{H}^{(u)} > 0\}. \quad (6.9)$$

Combining equations (6.7), (6.8) and (6.9), as well as Lemma J (iii), we finally obtain :

$$\begin{aligned} D(u) &= \left[ \int \mathbf{1}\{\tilde{H}^{(u)} > 0\} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle \tilde{\theta}^{(u)}, \tilde{H}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2}\langle \tilde{\eta}^{(u)}, (\tilde{H}^{(u)})^{-1} \tilde{\eta}^{(u)} \rangle + \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle\right) \right. \\ &\quad \left. \frac{\prod_{i \in V} \tilde{\theta}_i^{(u)}}{\sqrt{|\tilde{H}^{(u)}|}} \prod_{i \in V} \frac{d\tilde{\beta}_i^{(u)}}{d\beta_i} d\beta_i \right] \mathbf{1}\{H^{(u)} > 0\} \exp\left(\frac{1}{2}\langle \theta, W\theta \rangle + \langle \eta, \theta \rangle\right) \prod_{i \in V} \sqrt{\theta_i} \\ &\quad \exp\left(-\frac{1}{2}\langle \tilde{\theta}^{(u)}, \tilde{W}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2}\langle \eta, (H^{(u)})^{-1} \eta \rangle - \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle\right) \frac{\prod_{i \in V} \exp\left(\frac{1}{2}\varrho_i(u) - \frac{1}{8}u\right)}{\sqrt{|K^{(u)}|} \prod_{i \in V} \tilde{\theta}_i^{(u)}} \\ &= \mathbf{1}\{H^{(u)} > 0\} \exp\left(-\frac{1}{2}\langle \tilde{\theta}^{(u)}, \tilde{W}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2}\langle \eta, (H^{(u)})^{-1} \eta \rangle - \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle\right) \\ &\quad \frac{\prod_{i \in V} \exp\left(\frac{1}{2}\varrho_i(u) - \frac{1}{8}u\right)}{\sqrt{|K^{(u)}|}} \exp\left(\frac{1}{2}\langle \theta, W\theta \rangle + \langle \eta, \theta \rangle\right) \prod_{i \in V} \sqrt{\theta_i}, \end{aligned}$$

since the integral between brackets becomes

$$\int \nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}}(d\tilde{\beta}^{(u)}) = 1.$$

We are ready to show that  $D$  is the exponential martingale associated with a certain  $\mathcal{F}_u^o$ -

martingale. By Ito's lemma, for  $u \geq 0$  we have

$$\begin{aligned} dD(u) &= \sum_{i \in V} \left( -(\widetilde{W}^{(u)} e^{\varrho(u)})_i e^{\varrho_i(u)} - \widetilde{\eta}_i^{(u)} e^{\varrho_i(u)} - \frac{1}{2} \right) D(u) d\varrho_i(u) \\ &\quad + \frac{1}{2} \sum_{i \in V} \left( \left( -(\widetilde{W}^{(u)} e^{\varrho(u)})_i e^{\varrho_i(u)} - \widetilde{\eta}_i^{(u)} e^{\varrho_i(u)} - \frac{1}{2} \right)^2 \right. \\ &\quad \left. + \left( -(\widetilde{W}^{(u)} e^{\varrho(u)})_i e^{\varrho_i(u)} - \widetilde{W}_{i,i}^{(u)} e^{2\varrho_i(u)} - \widetilde{\eta}_i^{(u)} e^{\varrho_i(u)} \right) \right) D(u) du \\ &\quad + \left( -\frac{1}{2} \langle e^{\varrho(u)}, \partial_u (\widetilde{W}^{(u)} e^{\varrho(u)}) \rangle - \frac{1}{2} \langle \eta, \partial_u (H^{(u)})^{-1} \eta \rangle \right. \\ &\quad \left. - \langle \partial_u \widetilde{\eta}^{(u)}, e^{\varrho(u)} \rangle - \frac{|V|}{8} - \frac{1}{2} \frac{\partial_u |K^{(u)}|}{|K^{(u)}|} \right) D(u) du. \end{aligned}$$

Since  $H^{(u)} = 2\beta^{(u)} - W = 1/T(u) - W$ , we have

$$\partial_u (H^{(u)})^{-1} = (H^{(u)})^{-1} T(u)^{-1} \partial_u (T(u)) T(u)^{-1} (H^{(u)})^{-1},$$

using Lemma J (ii), we get

$$\begin{aligned} \langle \eta, \partial_u (H^{(u)})^{-1} \eta \rangle &= \langle T(u)^{-1} (H^{(u)})^{-1} \eta, e^{2\varrho(u)} T(u)^{-1} (H^{(u)})^{-1} \eta \rangle \\ &= \langle \widetilde{\eta}^{(u)}, e^{2\varrho(u)} \widetilde{\eta}^{(u)} \rangle = \sum_{i \in V} (\widetilde{\eta}_i^{(u)})^2 e^{2\varrho_i(u)}. \end{aligned}$$

Moreover,  $\widetilde{W}^{(u)} = W + W(H^{(u)})^{-1}W$ , therefore

$$\begin{aligned} \langle e^{\varrho(u)}, \partial_u (\widetilde{W}^{(u)} e^{\varrho(u)}) \rangle &= \langle e^{\varrho(u)}, W(H^{(u)})^{-1} T(u)^{-1} e^{2\varrho(u)} T(u)^{-1} (H^{(u)})^{-1} W e^{\varrho(u)} \rangle \\ &= \langle e^{\varrho(u)}, \widetilde{W}^{(u)} e^{2\varrho(u)} \widetilde{W}^{(u)} e^{\varrho(u)} \rangle = \sum_{i \in V} (\widetilde{W}^{(u)} e^{\varrho(u)})_i^2 e^{2\varrho_i(u)}, \end{aligned}$$

and  $\widetilde{\eta}^{(u)} = \widetilde{W}^{(u)} T(u) \eta + \eta$ , thus

$$\begin{aligned} \langle \partial_u \widetilde{\eta}^{(u)}, e^{\varrho(u)} \rangle &= \langle \partial_u (\widetilde{W}^{(u)} T(u) \eta + \widetilde{W}^{(u)} \partial_u (T(u)) \eta), e^{\varrho(u)} \rangle \\ &= \langle \widetilde{W}^{(u)} e^{2\varrho(u)} \widetilde{W}^{(u)} T(u) \eta + \widetilde{W}^{(u)} e^{2\varrho(u)} \eta, e^{\varrho(u)} \rangle \\ &= \langle \widetilde{W}^{(u)} e^{2\varrho(u)} \widetilde{\eta}^{(u)}, e^{\varrho(u)} \rangle = \sum_{i \in V} (\widetilde{W}^{(u)} e^{\varrho(u)})_i \widetilde{\eta}_i^{(u)} e^{2\varrho_i(u)}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \partial_u |K^{(u)}| &= \text{Tr}(|K^{(u)}| (K^{(u)})^{-1} \partial_u K^{(u)}) = -|K^{(u)}| \text{Tr}(W (K^{(u)})^{-1} e^{2\varrho(u)}) \\ &= -|K^{(u)}| \sum_{i \in V} \widetilde{W}_{i,i}^{(u)} e^{2\varrho_i(u)}. \end{aligned}$$

Plug in the above computations in  $dD(u)$ , we get that

$$\begin{aligned} \frac{dD(u)}{D(u)} &= \sum_{i \in V} \left( -(\widetilde{W}^{(u)} (e^{\varrho(u)} + T(u) \eta))_i e^{\varrho_i(u)} - \frac{1}{2} \right) d\varrho_i(u) \\ &\quad + \frac{1}{2} \sum_{i \in V} \left( (\widetilde{W}^{(u)} e^{\varrho(u)})_i^2 e^{2\varrho_i(u)} + (\widetilde{\eta}_i^{(u)})^2 e^{2\varrho_i(u)} + \frac{1}{4} + 2 (\widetilde{W}^{(u)} e^{\varrho(u)})_i \widetilde{\eta}_i^{(u)} e^{2\varrho_i(u)} \right. \\ &\quad \left. + (\widetilde{W}^{(u)} e^{\varrho(u)})_i e^{\varrho_i(u)} + \widetilde{\eta}_i^{(u)} e^{\varrho_i(u)} - (\widetilde{W}^{(u)} e^{\varrho(u)})_i e^{\varrho_i(u)} - \widetilde{W}_{i,i}^{(u)} e^{2\varrho_i(u)} - \widetilde{\eta}_i^{(u)} e^{\varrho_i(u)} \right. \\ &\quad \left. - (\widetilde{W}^{(u)} e^{\varrho(u)})_i^2 e^{2\varrho_i(u)} - (\widetilde{\eta}_i^{(u)})^2 e^{2\varrho_i(u)} - 2 (\widetilde{W}^{(u)} e^{\varrho(u)})_i \widetilde{\eta}_i^{(u)} e^{2\varrho_i(u)} - \frac{1}{4} + \widetilde{W}_{i,i}^{(u)} e^{2\varrho_i(u)} \right) du \\ &= \sum_{i \in V} \left( -(\widetilde{W}^{(u)} (e^{\varrho(u)} + T(u) \eta))_i e^{\varrho_i(u)} - \frac{1}{2} \right) d\varrho_i(u) = d\widetilde{L}(u), \end{aligned}$$

where for  $i \in V$  and  $u \geq 0$ ,

$$\tilde{L}_i(u) = \int_0^u \left( -\frac{1}{2} - \left( \tilde{W}(u) \left( e^{\varrho(u)} + T(u)\eta \right) + \eta \right)_i e^{\varrho_i(u)} \right) d\varrho_i(u).$$

Therefore,  $D$  is the exponential martingale associated with  $\tilde{L}$ .

Recall that for  $u \geq 0$  and any event  $A_u \in \mathcal{F}_u^\varrho = \sigma(\varrho(v), 0 \leq v \leq u)$ , we have

$$\mathbb{P}[A_u] = \int_{A_u} D(u) d\hat{\mathbb{P}},$$

i.e.  $\mathbb{P}$  is such that

$$\mathbb{E} \left[ \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \middle| \mathcal{F}_u^\varrho \right] = \mathcal{E}(\tilde{L})(u)$$

for all  $u \geq 0$ . Moreover,  $\hat{\mathbb{P}} = \otimes_{i \in V} \hat{\mathbb{P}}_i$ , therefore  $\varrho$  is a  $|V|$ -dimensional standard Brownian motion under  $\hat{\mathbb{P}}$ . By Girsanov's theorem, the process  $\tilde{B}(u) = \varrho(u) - \langle \varrho, \tilde{L} \rangle_u$  is a standard Brownian motion under  $\mathbb{P}$ . In other words, under  $\mathbb{P}$ , the process  $\varrho$  verifies the following SDE : for all  $i \in V$  and  $u \geq 0$ ,

$$d\varrho_i(u) = d\tilde{B}_i(u) - \frac{1}{2} du - \left( \tilde{W}(u) (e^{\varrho(u)} + T(u)\eta) + \eta \right)_i e^{\varrho_i(u)} du.$$

□

### 6.3.5 Time change on the conditional process

*Proof of Theorem 6.3.*

Let  $(\tilde{B}_i)_{i \in V}$  be a  $|V|$ -dimensional standard Brownian motion. According to Theorem 6.2, there exists a  $|V|$ -dimensional standard Brownian motion  $(B_i)_{i \in V}$  such that, if  $(X_i)_{i \in V}$  is the solution of  $(E_V^{W,\theta,\eta}(X))$  with the Brownian motion  $B$ , and  $T_i$  is the inverse function of  $U_i : t \mapsto \int_0^t \frac{ds}{X_i(s)^2}$  for all  $i \in V$ , then  $(\varrho, T)$  is the solution of  $(E_V^{W,\theta,\eta}(\varrho))$  with the Brownian motion  $\tilde{B}$ , where  $\varrho_i(u) = \log(X_i(T_i(u)))$  for  $u \geq 0$ .

Therefore, according to Lemma 6.7, we have a.s. for all  $i \in V$  :

$$\lim_{u \rightarrow +\infty} T_i(u) = \tau_i,$$

where  $\tau_i$  is the hitting time of 0 by  $X_i$ . In this coupling between  $(\varrho, T)$  and  $X$ , it is natural to prefer the notation  $T^\infty = \tau$ . Moreover, we can apply Theorem G (ii) to  $X$  : the vector  $\left( \frac{1}{2T_i^\infty} \right)_{i \in V}$  is distributed according to  $\nu_V^{W,\theta,\eta}$ , and conditionally on  $(T_i^\infty)_{i \in V}$ , the trajectories  $(X_i(t))_{0 \leq t \leq T_i^\infty}$  are independent three-dimensional Bessel bridges from  $\theta_i$  to 0 respectively.

Since  $\varrho_i(u) = \log(X_i(T_i(u)))$  for  $u \geq 0$ , conditionally on  $(T_i^\infty)_{i \in V}$ , the processes  $(\varrho_i, T_i)_{i \in V}$  are independent, and their distribution is given by applying the time change from Lamperti's relation to a three-dimensional Bessel bridge. This time-change was already realized in the proof of Lemma 6.7, see Equation (6.6), and the result is as follows : conditionally on  $(T_i^\infty)_{i \in V}$ , for all  $i \in V$ , there exists a standard Brownian motion  $B_i^*$  such that for  $u \geq 0$ .

$$\begin{cases} \varrho_i(u) = \log(\theta_i) + B_i^*(u) + \frac{1}{2}u + \log\left(\frac{T_i^\infty - T_i(u)}{T_i^\infty}\right) \\ T_i(u) = \int_0^u e^{2\varrho_i(w)} dw. \end{cases}$$

□

## 6.4 Conditioning in Lamperti time scale : Proof of Theorem 6.4, 6.5 and 6.6

### 6.4.1 Proof of Theorem 6.4

In this section, we prove Theorem 6.4 by using the multidimensional opposite drift Theorem obtained in Theorem 6.3. Besides, our proof follows the same computations as in the one dimensional case which is treated by Matsumoto and Yor in [122].

*Proof of Theorem 6.4.*

By (ii) in Theorem 6.3, there exists a  $|V|$ -dimensional Brownian motion  $B^*$  which is independent of  $T^\infty$  such that for every  $u \geq 0$ ,

$$\varrho(u) = \log(\theta) + B^*(u) + \frac{1}{2}u + \log\left(\frac{T^\infty - T(u)}{T^\infty}\right).$$

For every  $i \in V$  and for every  $u \geq 0$ , let us define

$$e_i^*(u) := e^{B_i^*(u) + u/2}.$$

Therefore, for every  $i \in V$ , for every  $u \geq 0$ ,

$$\frac{T_i^\infty}{T_i^\infty - T_i(u)} \frac{e^{\varrho_i(u)}}{\theta_i} = e_i^*(u). \quad (6.10)$$

Furthermore, for every  $i \in V$  and for every  $u \geq 0$ , let us define

$$T_i^*(u) = \int_0^u e_i^*(v)^2 dv.$$

Integrating the square of identity (6.10), we have, for every  $i \in V$  and for every  $u \geq 0$ ,

$$\begin{aligned} \theta_i^2 T_i^*(u) &= (T_i^\infty)^2 \int_0^u \frac{e^{2\varrho_i(v)}}{(T_i^\infty - T_i(v))^2} dv \\ &= (T_i^\infty)^2 \left[ \frac{1}{T_i^\infty - T_i(v)} \right]_0^u \\ &= (T_i^\infty)^2 \left( \frac{1}{T_i^\infty - T_i(u)} - \frac{1}{T_i^\infty} \right). \end{aligned}$$

Therefore, almost surely, for every  $i \in V$  and for every  $u \geq 0$ ,

$$\theta_i^2 T_i^*(u) = \frac{T_i^\infty T_i(u)}{T_i^\infty - T_i(u)}. \quad (6.11)$$

For every  $i \in V$  and for every  $u \geq 0$ , identity (6.11) yields

$$\frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(u)} = \frac{1}{T_i(u)}. \quad (6.12)$$

Let us differentiate (6.12). This gives that for every  $i \in V$  and for every  $u \geq 0$ ,

$$\frac{1}{\theta_i^2} \frac{e_i^*(u)^2}{T_i^*(u)^2} = \frac{e^{2\varrho_i(u)}}{T_i(u)^2}. \quad (6.13)$$

For every  $i \in V$  and for every  $u \geq 0$ , we denote  $Z_i^*(u) = \frac{T_i^*(u)}{e_i^*(u)}$ . Thus, by (6.13), for every  $i \in V$ , almost surely

$$(\theta_i Z_i^*(u))_{u \geq 0} = (Z_i(u))_{u \geq 0}. \quad (6.14)$$

The components of  $Z^*$  are independent. Moreover, by Theorem D with  $\mu = 1/2$ , for every  $i \in V$ ,  $(Z_i^*(u))_{u \geq 0}$  is solution of

$$dZ_i^*(u) = Z_i^*(u)d\widehat{B}_i(u) + \frac{K_{3/2}}{K_{1/2}} \left( \frac{1}{Z_i^*(u)} \right) du$$

for some Brownian motion  $\widehat{B}$  which is different from  $B^*$ . However, it is not difficult to see that for every  $x \in \mathbb{R}$ ,  $\frac{K_{3/2}(x)}{K_{1/2}(x)} = 1 + \frac{1}{x}$ . Therefore, for every  $i \in V$ ,  $(Z_i^*(u))_{u \geq 0}$  is solution of the SDE :

$$dZ_i^*(u) = Z_i^*(u)d\widehat{B}_i(u) + (1 + Z_i^*(u))du.$$

Together with (6.14), this yields (ii) of Theorem 6.4. By (6.12) and (6.14), we know that almost surely,

$$\left( Z_i(u), \frac{1}{T_i(v)} \right)_{u \geq 0, v \geq 0, i \in V} = \left( \theta_i Z_i^*(u), \frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(v)} \right)_{u \geq 0, v \geq 0, i \in V}. \quad (6.15)$$

Remark that  $T^\infty$  is independent of  $(Z^*(u))_{u \geq 0}$  because  $(Z^*(u))_{u \geq 0}$  depends only on  $B^*$ . Therefore, making  $v$  go to infinity in (6.15), we get that  $(Z(u))_{u \geq 0}$  is independent of  $T^\infty$  which is (iii) in Theorem 6.4.

Now, let  $(\lambda_i)_{i \in V} \in \mathbb{R}_+^V$ . Let  $z \in (\mathbb{R}_+^*)^V$ . Recall that for every  $i \in V$  and for every  $u \geq 0$ ,  $\beta_i(u) = \frac{1}{2T_i(u)}$ . Let us look at the Laplace transform of  $\beta(u)$ , conditionally on  $Z(u) = z$ . By (6.15), we get

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \lambda_i \beta_i(u) \right) \middle| \mathcal{Z}_u, Z_u = z \right] = \\ \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \frac{\lambda_i}{2T_i^\infty} \right) \times \exp \left( - \sum_{i \in V} \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z(u) = z \right]. \end{aligned}$$

By (i) in Theorem 6.3, we know that the random vector  $1/(2T^\infty)$  is distributed according to  $\nu_V^{W, \theta, \eta}$ . Moreover, we know that  $T^\infty$  is independent of  $B^*$ , that is, of  $(Z^*, T^*)$ . By (6.14), this implies that  $T^\infty$  is independent of  $(Z, T^*)$ . Therefore, by Proposition F,

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \lambda_i \beta_i(u) \right) \middle| \mathcal{Z}_u, Z_u = z \right] = e^{-\frac{1}{2} \langle \sqrt{\theta^2 + \lambda}, W \sqrt{\theta^2 + \lambda} \rangle + \frac{1}{2} \langle \theta, W \theta \rangle + \langle \eta, \theta - \sqrt{\theta^2 + \lambda} \rangle} \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}} \\ \times \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z(u) = z \right]. \end{aligned}$$

Besides, by (6.14),  $\theta Z^* = Z$ . Therefore,

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \lambda_i \beta_i(u) \right) \middle| \mathcal{Z}_u, Z_u = z \right] = e^{-\frac{1}{2} \langle \sqrt{\theta^2 + \lambda}, W \sqrt{\theta^2 + \lambda} \rangle + \frac{1}{2} \langle \theta, W \theta \rangle + \langle \eta, \theta - \sqrt{\theta^2 + \lambda} \rangle} \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}} \\ \times \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z_i^*(u) = z_i / \theta_i \right]. \end{aligned} \quad (6.16)$$

Furthermore, we know that  $(Z_i^*, T_i^*)_{i \in V}$  is site by site independent because for every  $i \in V$ ,

$(Z_i^*, T_i^*)$  is a functional of  $B_i^*$ . Consequently, we get that

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \lambda_i \beta_i(u) \right) \middle| \mathcal{Z}_u, Z_u = z \right] &= e^{-\frac{1}{2} \langle \sqrt{\theta^2 + \lambda}, W \sqrt{\theta^2 + \lambda} \rangle + \frac{1}{2} \langle \theta, W \theta \rangle + \langle \eta, \theta - \sqrt{\theta^2 + \lambda} \rangle} \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}} \\ &\quad \times \prod_{i \in V} \mathbb{E} \left[ \exp \left( - \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z_i^*(u) = z_i/\theta_i \right]. \end{aligned} \quad (6.17)$$

Now, for every  $i \in V$ , let us compute

$$\mathbb{E} \left[ \exp \left( - \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z(u) = z \right] = \mathbb{E} \left[ \exp \left( - \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z_i^*(u) = z_i/\theta_i \right].$$

Recall that  $IG(\mu, r)$  designates an Inverse Gaussian distribution with parameter  $(\mu, r)$ . By (iii) in Theorem D, the distribution of  $e_i^*(u)$  conditionally on  $Z_i^*(u) = z_i/\theta_i$  is

$$\mathcal{L}(e_i^*(u) | \mathcal{Z}_u, Z_i^*(u) = z_i/\theta_i) = \frac{1}{IG\left(1, \frac{\theta_i}{z_i}\right)}.$$

Moreover, for every  $t, \mu, r \in (0, +\infty)$ ,

$$t \times IG(\mu, r) \stackrel{law}{=} IG(t\mu, tr).$$

Therefore, as  $T^* = e^* Z^*$ , it holds that

$$\mathcal{L} \left( \frac{1}{2\theta_i^2 T_i^*(u)} \middle| \mathcal{Z}_u, Z_i^*(u) = z_i/\theta_i \right) = IG \left( \frac{1}{2\theta_i z_i}, \frac{1}{2z_i^2} \right). \quad (6.18)$$

It is well known that the Laplace transform of an Inverse Gaussian random variable  $X$  with parameters  $(\mu, r)$  is given by

$$\mathbb{E}(e^{-tX}) = \exp \left( \frac{r}{\mu} \left( 1 - \sqrt{1 + \frac{2\mu^2 t}{r}} \right) \right).$$

Consequently,

$$\mathbb{E} \left[ \exp \left( - \frac{\lambda_i}{2\theta_i^2 T_i^*(u)} \right) \middle| \mathcal{Z}_u, Z(u) = z \right] = \exp \left( \frac{1}{z_i} \left( \theta_i - \sqrt{\theta_i^2 + \lambda_i} \right) \right).$$

Combining this with (6.17) yields

$$\begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in V} \lambda_i \beta_i(u) \right) \middle| \mathcal{Z}_u, Z_u = z \right] \\ = e^{-\frac{1}{2} \langle \sqrt{\theta^2 + \lambda}, W \sqrt{\theta^2 + \lambda} \rangle + \frac{1}{2} \langle \theta, W \theta \rangle + \langle \eta + \frac{1}{z}, \theta - \sqrt{\theta^2 + \lambda} \rangle} \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}}. \end{aligned}$$

This is exactly the Laplace Transform of  $\nu_V^{W, \theta, \eta + 1/z}$ . This proves (iv) in Theorem 6.4. Remark that (iv) implies directly (i).  $\square$

### 6.4.2 Proof of Theorem 6.5

Let us prove the link between the solution  $(\varrho, T)$  of  $E_V^{W, \theta, \eta}(\varrho)$  and  $Z$  via intertwinings.

*Proof of Theorem 6.5.*

Let  $v, u \in \mathbb{R}_+$  such that  $v < u$ . Let  $f$  be a measurable function from  $(\mathbb{R}^V)^2$  into  $\mathbb{R}_+$ . On the one hand, it holds that,

$$\begin{aligned} \mathbb{E}[f(\varrho(u), T(u)) | \mathcal{Z}_v] &= \mathbb{E}[\mathbb{E}[f(\varrho(u), T(u)) | \sigma(\varrho(w), T(w), w \leq v)] | \mathcal{Z}_v] \\ &= \mathbb{E}[P_{u-v} f(\varrho(v), T(v)) | \mathcal{Z}_v]. \end{aligned}$$

Consequently, by (iv) in Theorem 6.4,

$$\begin{aligned} \mathbb{E}[f(\varrho(u), T(u)) | \mathcal{Z}_v] &= \mathbb{E}\left[ P_{u-v} f\left(\left(-\ln(2\beta_i(v)Z_i(v))\right)_{i \in V}, \left(\frac{1}{2\beta_i(v)}\right)_{i \in V}\right) \middle| \mathcal{Z}_v \right] \\ &= KP_{u-v} f(Z(v)). \end{aligned}$$

On the other hand, by (iv) in Theorem 6.4 again remark that,

$$\begin{aligned} \mathbb{E}[f(\varrho(u), T(u)) | \mathcal{Z}_v] &= \mathbb{E}[\mathbb{E}[f(\varrho(u), T(u)) | \mathcal{Z}_u] | \mathcal{Z}_v] \\ &= \mathbb{E}[Kf(Z(u)) | \mathcal{Z}_v] \\ &= Q_{u-v} Kf(Z(v)). \end{aligned}$$

Therefore, almost surely,

$$Q_{u-v} Kf(Z(v)) = KP_{u-v} f(Z(v)).$$

□

### 6.4.3 Proof of Theorem 6.6

In the proof of Theorem 6.6, we use the same notation as in the beginning of the proof of Theorem 6.4. For example,  $B^*$ ,  $e^*$ ,  $T^*$  and  $Z^*$  are defined in the same way as before.

*Proof of Theorem 6.6.*

Recall that, by (6.15),

$$\left(Z_i(u), \frac{1}{T_i(v)}\right)_{u \geq 0, v \geq 0, i \in V} = \left(\theta_i Z_i^*(u), \frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(v)}\right)_{u \geq 0, v \geq 0, i \in V}.$$

Let  $u \geq 0$ . By (6.15) and the fact that for every  $i \in V$ ,  $T_i(u) = Z_i(u)e^{\varrho_i(u)}$  and  $T_i^*(u) = Z_i^*(u)e_i^*(u)$ , we get

$$\frac{1}{e_i^*(u)} = \frac{\theta_i}{e^{\varrho_i(u)}} - \frac{\theta_i Z_i(u)}{T_i^\infty}. \quad (6.19)$$

Following [123], remark that by (6.19) it holds that

$$\begin{aligned} \theta_i^2 \frac{Z_i^*(u)}{e_i^*(u)} + \frac{T_i^\infty}{e_i^*(u)^2} &= \theta_i^2 \frac{Z_i(u)}{\theta_i} \left(\frac{\theta_i}{e^{\varrho_i(u)}} - \frac{\theta_i Z_i(u)}{T_i^\infty}\right) + T_i^\infty \left(\frac{\theta_i}{e^{\varrho_i(u)}} - \frac{\theta_i Z_i(u)}{T_i^\infty}\right)^2 \\ &= \theta_i^2 \left(\frac{T_i^\infty}{e^{2\varrho_i(u)}} - \frac{Z_i(u)}{e^{\varrho_i(u)}}\right). \end{aligned}$$

As  $T_i(u)e^{-\varrho_i(u)} = Z_i(u)$ , this entails

$$\begin{aligned} \theta_i^2 \frac{Z_i^*(u)}{e_i^*(u)} + \frac{T_i^\infty}{e_i^*(u)^2} &= \theta_i^2 \frac{T_i^\infty - T_i(u)}{e^{2\varrho_i(u)}} \\ &= \theta_i^2 \int_0^{+\infty} e^{2\varrho_i(u+v) - 2\varrho_i(u)} dv. \end{aligned} \quad (6.20)$$

Therefore, combining (6.15) and (6.20) yields

$$\left( \frac{1}{T_i(u)}, \int_0^{+\infty} e^{2\varrho_i(u+v)-2\varrho_i(u)} dv \right)_{i \in V} = \left( \frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(u)}, \frac{Z_i^*(u)}{e_i^*(u)} + \frac{T_i^\infty}{\theta_i^2 e_i^*(u)^2} \right)_{i \in V}. \quad (6.21)$$

Now, the idea is to condition both sides of (6.21) on  $Z(u) = z$  and to use (iv) in Theorem 6.4. Let us begin with the left-hand side. This term can be rewritten as

$$\left( \frac{1}{T_i(u)}, \frac{Z_i(u)^2}{T_i(u)^2} \int_0^{+\infty} e^{2\varrho_i(u+v)} dv \right)_{i \in V}. \quad (6.22)$$

First, let us condition on  $\sigma(\varrho(v), v \leq u)$ . By Theorem H in the exponential scale, conditionally on  $\sigma(\varrho(v), v \leq u)$ ,  $(\varrho(u+v))_{v \geq 0}$  is distributed as a solution of  $E_V^{\widetilde{W}(u), \widetilde{\theta}(u), \widetilde{\eta}(u)}$  where  $\widetilde{W}(u) = WK_{T(u)}^{-1}$ ,  $\widetilde{\eta}(u) = \eta + WK_{T(u)}^{-1} T(u)\eta$  and for every  $i \in V$ ,  $\widetilde{\theta}_i(u) = e^{\varrho_i(u)} = 1/(2\beta_i(u)Z_i(u))$ . Thus, by (i) in Theorem 6.3, conditionally on  $\sigma(\varrho(v), v \leq u)$ ,

$$\left( \left( 2 \int_0^{+\infty} e^{2\varrho_i(u+v)} dv \right)^{-1} \right)_{i \in V}$$

is distributed as  $\alpha \sim \nu_V^{\widetilde{W}(u), \widetilde{\theta}(u), \widetilde{\eta}(u)}$ . Recall that, by (iv) in Theorem 6.4, conditionally on  $\mathcal{Z}_u$ ,  $1/(2T(u))$  is distributed as  $\beta \sim \nu_V^{W, \theta, \eta}$ . Therefore, if we condition (6.22), first on  $\sigma(\varrho(v), v \leq u)$  and then on  $Z(u) = z$ , we obtain

$$\left( (2\beta_i)_{i \in V}, \left( \frac{z_i^2 (2\beta_i)^2}{2\alpha_i} \right)_{i \in V} \right)$$

where  $\beta \sim \nu_V^{W, \theta, \eta+1/z}$  and conditionally on  $\beta$ ,  $\alpha \sim \nu_V^{\widetilde{W}, \widetilde{\theta}, \widetilde{\eta}}$ .

Now, let us look at the right-hand side in (6.21). This right-hand side can be rewritten as

$$\left( \frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(u)}, \frac{Z_i^*(u)^2}{T_i^*(u)} + \frac{T_i^\infty Z_i^*(u)^2}{\theta_i^2 T_i^*(u)^2} \right)_{i \in V}.$$

We know that  $T^\infty$  is independent of  $B^*$ , thus of  $(Z^*, T^*)$ . By (6.14), this implies that  $T^\infty$  is independent of  $(Z, T^*)$ . Moreover, by (i) in Theorem 6.3,  $\delta = 1/(2T^\infty)$  is distributed as  $\nu_V^{W, \theta, \eta}$ . Thus, conditionally on  $(Z_i(u) = z_i)_{i \in V}$ , that is,  $(Z_i^*(u) = z_i/\theta_i)_{i \in V}$ , we have

$$\left( \frac{1}{T_i^\infty} + \frac{1}{\theta_i^2 T_i^*(u)}, \frac{Z_i^*(u)^2}{T_i^*(u)} + \frac{T_i^\infty Z_i^*(u)^2}{\theta_i^2 T_i^*(u)^2} \right)_{i \in V} \stackrel{law}{=} \left( 2\delta_i + \frac{1}{\theta_i^2 T_i^*(u)}, \frac{z_i^2}{\theta_i^2 T_i^*(u)} + \frac{z_i^2}{2\delta_i (\theta_i^2 T_i^*(u))^2} \right)_{i \in V} \quad (6.23)$$

where  $(T_i^*(u))_{i \in V}$  is independent of  $(\delta_i)_{i \in V}$ . Moreover, by (6.18), conditionally on  $(Z_i^*(u) = z_i/\theta_i)_{i \in V}$ ,  $(T_i^*(u))_{i \in V}$  are independent site by site and for every  $i \in V$ ,  $1/(\theta_i^2 T_i^*(u))$  is distributed as

$$IG(1/(\theta_i z_i), 1/z_i^2).$$

□

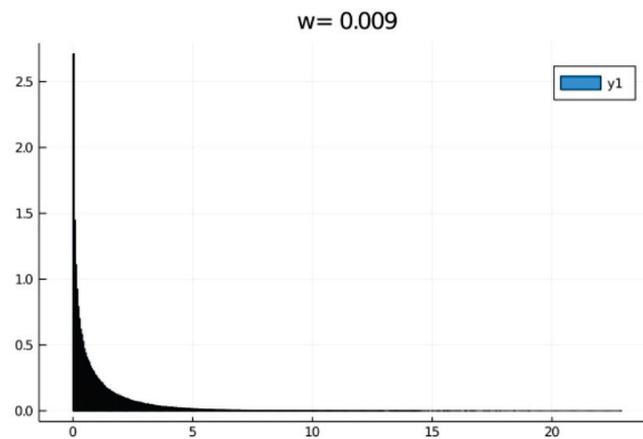


# Annexe A

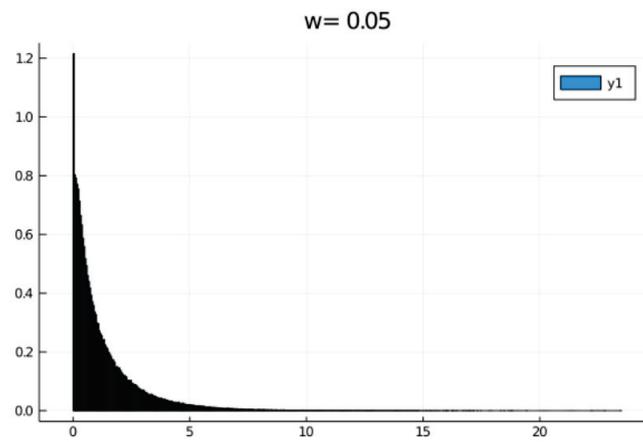
In this appendix, we show numerical simulations of the non-integrated density of states which has been studied in Chapter 4.

## A.1 Simulations

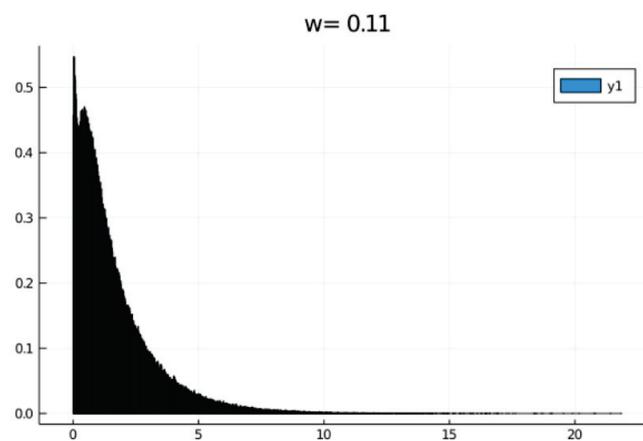
Let us simulate the non-integrated density of states of the restriction of the operator  $H_\beta$  on a box of size  $15 \times 15 \times 15$ . We change the value of  $W$  in the different figures.



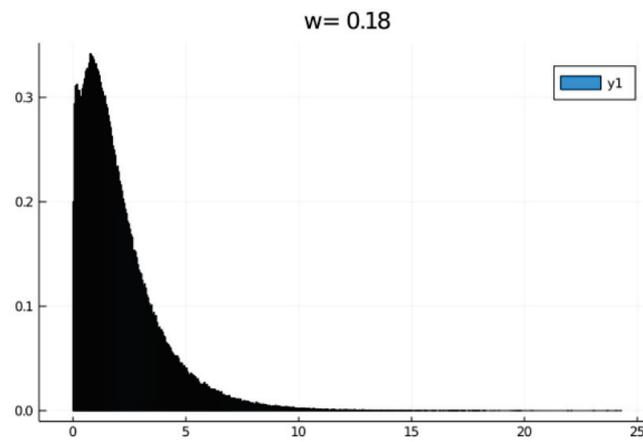
**FIGURE A.1** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0,009$



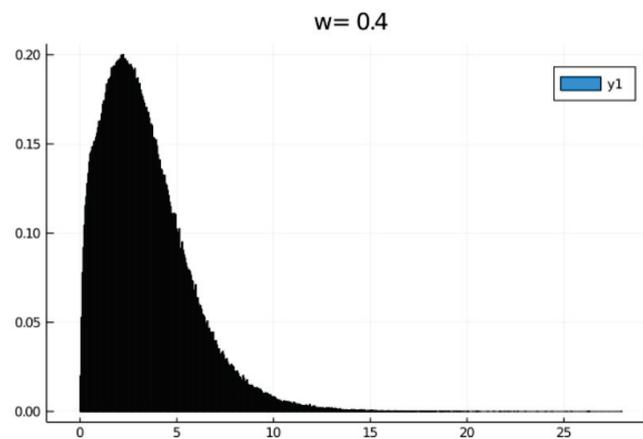
**FIGURE A.2** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0,05$



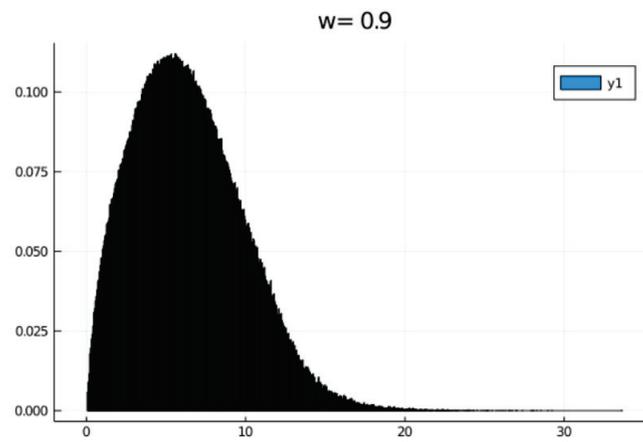
**FIGURE A.3** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0,11$



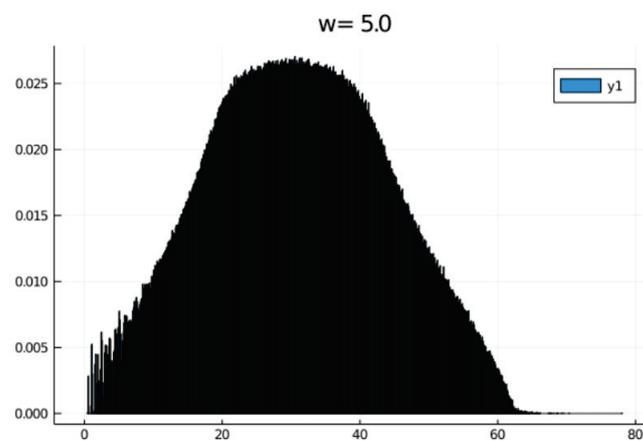
**FIGURE A.4** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0, 18$



**FIGURE A.5** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0, 4$



**FIGURE A.6** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 0,9$



**FIGURE A.7** – Non-integrated density of states of the restriction of  $H_\beta$  on a grid  $15 \times 15 \times 15$  with  $W = 5$

## A.2 Interpretation

When  $W$  is small, we observe a singularity at 0 of the non-integrated density of states. It is consistent with Theorem 4.4 in Chapter 4. In this regime, we proved that the integrated density of states  $E \mapsto n(E)$  behaves roughly as  $n(E) \simeq \sqrt{E}$  when  $E$  goes to 0. It implies that the non-integrated density of states  $E \mapsto \alpha(E)$  should behave as  $\alpha(E) \simeq 1/\sqrt{E}$  which explains the singularity at 0 in the pictures above when  $W$  is small. On the contrary, when  $W$  is large, we observe that the singularity of the non-integrated density of states disappears. It is consistent with Theorem 4.5 in Chapter 4. Indeed, in this regime, we proved that  $n(E) = O(E)$  when  $E$  goes to 0 which suggests that  $\alpha(E) = O(1)$  when  $E$  goes to 0. We conjecture that when  $d \geq 3$  and  $W$  is large,  $H_\beta$  could exhibit Van Hove asymptotics, that is,  $n(E) \simeq E^{d/2}$  and  $\alpha(E) \simeq E^{d/2-1}$ . Indeed, when  $W$  is large,  $H_\beta/W$  should be very similar with the opposite of the discrete Laplacian on  $\mathbb{Z}^d$  which is known to exhibit such asymptotics at the bottom of its spectrum. Moreover, we conjecture that the phase transition with respect to the density of states of  $H_\beta$  on  $\mathbb{Z}^d$  with  $d \geq 3$  should coincide with the phase transition between recurrence and transience for the VRJP.



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# Étude de quelques problèmes liés aux marches aléatoires branchantes et aux marches aléatoires renforcées.

**Résumé :** Dans cette thèse nous étudions des marches aléatoires branchantes spatiales critiques partant de processus ponctuels et un processus renforcé nommé VRJP. Nous donnons une nouvelle preuve de la caractérisation des mesures invariantes pour les marches branchantes spatiales en grande dimension. Par ailleurs, nous nous intéressons au comportement asymptotique d'une martingale associée au VRJP. Nous étudions également la densité d'état d'un opérateur de Schrödinger aléatoire lié au VRJP. De plus, en considérant des limites d'échelle du potentiel aléatoire associé au VRJP, nous fournissons une nouvelle preuve des propriétés de Matsumoto-Yor concernant des fonctionnelles exponentielles du mouvement Brownien. A l'aide des mêmes limites d'échelle, nous construisons une version continue de l'opérateur de Schrödinger associé au VRJP. Enfin, nous prouvons une version multidimensionnelle des propriétés de Matsumoto-Yor.

**Mots clés :** Marches aléatoires branchantes ; Marches aléatoires renforcées ; Opérateurs aléatoires ; Propriétés de Matsumoto-Yor.

## Study of several problems linked to branching random walks and reinforced random walks.

**Abstract :** In this thesis we study spatial critical branching random walks starting from point processes and a reinforced process called the VRJP. We give a new proof of the characterization of invariant measures for spatial branching random walks in high dimension. Moreover, we look at the asymptotic behaviour of some martingale which is related with the VRJP. We also study the density of states of a random Schrödinger operator which is associated with the VRJP. Furthermore, considering the scaling limits of a random potential which is linked to the VRJP, we give a new proof of Matsumoto-Yor properties regarding some exponential functionals of the Brownian motion. By means of these scaling limits, we also construct a continuous version of the Schrödinger operator associated with the VRJP. Finally, we prove a multidimensional version of the Matsumoto-Yor properties.

**Keywords :** Branching random walks ; Reinforced random walks ; Random operators ; Matsumoto-Yor properties.

Image en couverture : [L'arbre de vie](#), Gustav Klimt, 1909.

