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Dawei Shen

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par  
**Dawei Shen**

**Problèmes de stabilité en relativité générale**

après avis des rapporteurs

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devant le jury composé de

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# Chapter 1

## Introduction

## 1 The evolution problem in general relativity

In this section, we introduce Einstein vacuum equations and review the corresponding Cauchy problem.

### 1.1 Einstein vacuum equations

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \quad \text{in } \mathcal{M}, \quad (1.1)$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ .

The prime example of a vacuum spacetime is Minkowski spacetime  $(\mathcal{M}, \eta)$ :

$$\mathcal{M} = \mathbb{R}^4, \quad \eta := -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad (1.2)$$

where  $\eta$  is expressed here in Cartesian coordinates. Each component of the Ricci curvature tensor  $\mathbf{Ric}(g)_{\mu\nu}$  is a second-order nonlinear partial differential operator on the metric components  $\mathbf{g}_{\alpha\beta}$  for  $\alpha, \beta = 0, 1, 2, 3$ . Therefore, Einstein vacuum equations (1.1) form a system of 10 nonlinear coupled partial differential equations on the 10 unknowns  $\mathbf{g}_{\alpha\beta}$ .

Note that we have, for a diffeomorphism  $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ ,

$$\mathbf{Ric}(\Phi^*\mathbf{g}) = \Phi^*(\mathbf{Ric}(\mathbf{g})).$$

Hence, if  $(\mathcal{M}, \mathbf{g})$  is a solution to (1.1), and if  $(\mathcal{M}', \mathbf{g}')$  is isometric to  $(\mathcal{M}, \mathbf{g})$ , i.e. if there exists a diffeomorphism:

$$\Phi : \mathcal{M} \rightarrow \mathcal{M}', \quad \Phi^*\mathbf{g}' = \mathbf{g},$$

then  $(\mathcal{M}', \mathbf{g}')$  also solves (1.1). Thus, when we talk about a solution to (1.1), we consider the equivalence class of  $(\mathcal{M}, \mathbf{g})$ , i.e. all the spacetimes  $(\mathcal{M}', \mathbf{g}')$  isometric to  $(\mathcal{M}, \mathbf{g})$ . In particular, we have the freedom of choice of the represent in this class, which corresponds to the choice of a coordinates system. This is called the *general covariance* of Einstein vacuum equations (1.1).

### 1.2 The Cauchy problem

In this section, we recall that (1.1) corresponds to a Cauchy problem. To illustrate this fact we start by relying on general covariance to rewrite (1.1) in so-called *wave coordinates*.

#### 1.2.1 Wave coordinates

In this section, we consider the wave coordinates gauge, which is the choice of four coordinates satisfying

$$\square_{\mathbf{g}}(x^\mu) := \mathbf{D}^\alpha \mathbf{D}_\alpha(x^\mu) = 0, \quad \mu = 0, 1, 2, 3,$$

where  $\square_{\mathbf{g}}$  denotes the d'Alembertian operator associated to the metric  $\mathbf{g}$  and  $\mathbf{D}$  denotes the Levi-Civita connection of  $\mathbf{g}$ . Under this gauge choice, it can be shown that (1.1) rewrites as the following system of coupled nonlinear wave equations:

$$\square_{\mathbf{g}}(\mathbf{g}_{\mu\nu}) = \mathcal{N}_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}), \quad \mu, \nu = 0, 1, 2, 3, \quad (1.3)$$

where  $\mathcal{N}_{\mu\nu}$  are nonlinearities, quadratic in  $\partial\mathbf{g}$ . Thus, Einstein vacuum equations in the wave coordinates gauge can be cast as a system of coupled quasilinear wave equations, which illustrates the fact that (1.1) corresponds to a Cauchy problem with Cauchy data  $(\mathbf{g}_{\alpha\beta}, \partial_t \mathbf{g}_{\alpha\beta})|_{t=0}$ .

### 1.2.2 Cauchy data

In the previous section we have exhibited Cauchy data in a particular coordinates system. In this section, we recall the coordinate invariant definition of Cauchy data for (1.1), see for example Chapter 10 in [59].

The initial data for the Einstein vacuum equations is given by specifying a triplet  $(\Sigma, g, k)$  where  $(\Sigma, g)$  is a Riemannian 3-manifold and  $k$  is the traceless symmetric 2-tensor on  $\Sigma$  satisfying the constraint equations:

$$\begin{aligned} R &= |k|^2 - (\operatorname{tr} k)^2, \\ \nabla^j k_{ij} &= \nabla_i \operatorname{tr} k, \end{aligned} \tag{1.4}$$

where  $R$  denotes the scalar curvature of  $g$ ,  $\nabla$  denotes the Levi-Civita connection of  $g$  and

$$|k|^2 := g^{im} g^{jl} k_{ij} k_{lm}, \quad \operatorname{tr} k := g^{ij} k_{ij}.$$

A vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  satisfies the Cauchy problem for Einstein equations (1.1) with Cauchy data  $(\Sigma, g, k)$  if  $\Sigma \subset \mathcal{M}$  is a spacelike hypersurface with first fundamental form  $g$  and second fundamental form  $k$ . That is,  $g$  is the metric induced by  $\mathbf{g}$  on  $\Sigma$  and

$$k_{ij} = -\mathbf{D}_i T_j,$$

where  $T$  is the future directed unit normal to  $\Sigma$  in  $\mathcal{M}$ .

**Remark 1.1.** *The standard Cauchy data for Minkowski spacetime  $(\mathbb{R}^4, \eta)$  are given by*

$$\Sigma = \mathbb{R}^3, \quad g = \delta = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad k = 0.$$

### 1.2.3 Local existence

The seminal well-posedness results for the Cauchy problem obtained in [8, 9] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  solution of Einstein equations (1.1) such that  $\Sigma \subset \mathcal{M}$  and  $g, k$  are respectively the first and second fundamental forms of  $\Sigma$  in  $\mathcal{M}$ . More precisely, we have the following theorem.

**Theorem 1.2** (Choquet-Bruhat-Geroch [9]). *Let  $(\Sigma, g, k)$  be smooth Cauchy data for Einstein vacuum equations. Then, there exists a unique smooth vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  which satisfies the following properties:*

- $(\mathcal{M}, \mathbf{g})$  admits  $(\Sigma, g, k)$  as Cauchy data.
- $(\mathcal{M}, \mathbf{g})$  is globally hyperbolic and admits  $\Sigma$  as a Cauchy hypersurface, i.e.  $\Sigma$  is achronal in  $\mathcal{M}$  and  $\mathcal{M}$  coincides with the domain of dependence of  $\Sigma$  in  $\mathcal{M}$ .<sup>1</sup>
- $(\mathcal{M}, \mathbf{g})$  is maximal for the inclusion among all the spacetimes satisfying the above conditions.

*The spacetime  $(\mathcal{M}, \mathbf{g})$  is called the maximal globally hyperbolic development of  $(\Sigma, g, k)$ .*

**Remark 1.3.** *The maximal development of arbitrary initial data sets  $(\Sigma, g, k)$  in Theorem 1.2 is in general out of reach. Thus, we restrict ourselves in this thesis to the stability of particular stationary solutions of Einstein vacuum equations (1.1): the Minkowski spacetime introduced in (1.2), and Schwarzschild and Kerr black holes introduced below in Section 5.1.*

In the next section, we introduce the relevant foliations used in this thesis.

<sup>1</sup>See Chapter 8 in [59] for the precise definition of achronal sets and of global hyperbolicity.



## 2 Spacetime foliations

Einstein equations (1.1) form a well-determined system of partial differential equations, provided that additional gauge conditions are prescribed. In Section 1.2.1, we discussed the wave coordinates gauge. In this section, we introduce the particular gauge choices used in this thesis. The material of this section is taken from in Chapters 1 and 7 in [10] and Chapter 3 in [29].

### 2.1 Foliation by spacelike hypersurfaces

In this section, we introduce the decompositions associated to foliations of the spacetime by spacelike hypersurfaces, which are level sets of a time function  $t$ . This formalism is called 1 + 3. In this context, we formulate the maximal gauge condition and present the Einstein vacuum equations in that gauge. This gauge has first been used in [10], see also Chapter 3 in this thesis.

#### 2.1.1 Spacelike hypersurfaces

In the sequel, we always assume that  $(\mathcal{M}, \mathbf{g})$  is a time-oriented vacuum spacetime.

We consider  $t$  a time function on  $\mathcal{M}$ , i.e. such that

$$\mathbf{g}(\mathbf{D}t, \mathbf{D}t) < 0.$$

We denote  $\Sigma_t$  its level sets, which are spacelike hypersurfaces of  $(\mathcal{M}, \mathbf{g})$ . We define the first fundamental form  $g$  of  $\Sigma_t$  to be the induced Riemannian metric on  $\Sigma_t$ . We define the second fundamental form  $k$  of  $\Sigma_t$  to be the  $\Sigma_t$ -tangent tensor given by

$$k(Y, Z) := -\mathbf{g}(\mathbf{D}_Y T, Z), \quad (2.1)$$

where  $T$  is the future directed unit normal of  $\Sigma_t$  and  $Y, Z \in T\Sigma_t$ .

For simplicity, we shall from now on consider functions  $x^i$  which are transported without shift along the flow of  $t$ . Such a prescription can be written as

$$T(x^i) = 0, \quad i = 1, 2, 3. \quad (2.2)$$

We shall also assume that  $(t, x^i)$  forms a coordinate system on  $\mathcal{M}$ , and we note that the assumptions (2.2) consist in the 3 gauge conditions

$$\mathbf{g}_{0i} = 0.$$

We define the *time lapse function* of the foliation  $\Sigma_t$  by

$$\phi^{-2} := -\mathbf{g}(\mathbf{D}t, \mathbf{D}t).$$

With these definitions, the spacetime metric  $\mathbf{g}$  in the coordinates  $(t, x^i)$  decomposes as

$$\mathbf{g} = -\phi^2 dt^2 + g_{ij} dx^i dx^j.$$

Using the above definitions, the Einstein vacuum equations (1.1) rewrite in these coordinates as

$$\begin{aligned} \partial_t g_{ij} &= -2\phi k_{ij}, \\ \partial_t k_{ij} &= -\nabla_i \nabla_j \phi + \phi(\text{Ric}(g)_{ij} + (\text{tr}_g k)k_{ij} - 2k_{il}k^l{}_j), \end{aligned} \quad (2.3)$$

see (1.0.6) in [10]. The system of equations (2.3) is a well-determined system of evolution equations, up to a choice of time function  $t$ , which can be turned into a choice for  $\phi$  or  $\text{tr}_g k$ .

### 2.1.2 Maximal hypersurfaces

Additionally to the gauge choice (2.2), we make the following maximal gauge choice

$$\operatorname{tr}_g k = 0. \quad (2.4)$$

Recall the constraint equations:

$$\begin{aligned} R(g) &= |k|^2 - (\operatorname{tr}_g k)^2, \\ (\operatorname{div} k)_i &= \nabla_i (\operatorname{tr}_g k). \end{aligned} \quad (2.5)$$

We remark that taking the trace in (2.3) and applying (2.5), we obtain the following lapse equation:

$$\Delta_g \phi = |k|^2 \phi, \quad (2.6)$$

see (1.0.13) in [10].

**Remark 2.1.** *One can show that for a vacuum spacetime, any compact perturbation of each separate hypersurface  $\Sigma_t$  yield hypersurfaces with smaller volume. This justifies that the hypersurfaces  $\Sigma_t$  are called maximal hypersurfaces.*

To highlight the main features of equations (2.3) and (2.5) together with the maximal gauge choice (2.4), we introduce the electric-magnetic tensors  $E$  and  $H$ , which are the  $\Sigma_t$ -tangent tensors defined by

$$E(X, Y) := \mathbf{R}(T, X, T, Y), \quad H(X, Y) := *\mathbf{R}(T, X, T, Y), \quad (2.7)$$

for all  $X, Y \in T\Sigma_t$ , where  $*\mathbf{R}$  is the Hodge dual of the spacetime curvature tensor  $\mathbf{R}$ .

Einstein vacuum equations (1.1) together with the additional gauge choices (2.2) and (2.4) corresponds to Einstein vacuum equations in maxime gauge. It is equivalent to the system of coupled quasilinear transport-elliptic-Maxwell equations, displayed below in Propositions 2.2–2.4.

**Proposition 2.2.** *We impose a maximal foliation on  $\Sigma_t$  in the sense of (2.4). Then, we have:*

- *Maximal constraint equations:*

$$\begin{aligned} \operatorname{tr} k &= 0, \\ \operatorname{div}_g k &= 0, \\ R &= |k|^2. \end{aligned} \quad (2.8)$$

- *Evolution equations:*

$$\begin{aligned} \partial_t g_{ij} &= -2\phi k_{ij}, \\ \partial_t k_{ij} &= -\nabla_i \nabla_j \phi + \phi (\operatorname{Ric}_{ij} - 2k_{il} k^l_j). \end{aligned} \quad (2.9)$$

- *Lapse equation:*

$$\Delta \phi = \phi |k|^2. \quad (2.10)$$

*Proof.* See (1.0.11)–(1.0.13) in [10]. □

We also have the following proposition.

**Proposition 2.3.** *We have the following equations:*

$$\begin{aligned} \nabla_i k_{jm} - \nabla_j k_{im} &= \epsilon_{ij}{}^l H_{lm}, \\ \operatorname{Ric}_{ij} - k_{il} k^l_j &= E_{ij}. \end{aligned}$$

*Proof.* See (1.0.14a)–(1.0.14b) in [10].  $\square$

Finally, we have the following Maxwell-type equations for  $E$  and  $H$ .

**Proposition 2.4.** *Let  $E, H$  be the electric-magnetic decomposition defined in (2.7). Then, the following identities hold:*

$$\begin{aligned} \operatorname{div}_g E &= k \wedge H, \\ \operatorname{div}_g H &= -k \wedge E, \\ -\hat{\mathcal{L}}_T H + \operatorname{curl}_g E &= -\nabla \log \phi \wedge E - \frac{1}{2} k \times H, \\ \hat{\mathcal{L}}_T E + \operatorname{curl}_g H &= -\nabla \log \phi \wedge H + \frac{1}{2} k \times E, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} (\hat{\mathcal{L}}_T E)_{ij} &:= (\mathcal{L}_T E)_{ij} - \frac{2}{3} (k \cdot E) g_{ij}, \\ (\hat{\mathcal{L}}_T H)_{ij} &:= (\mathcal{L}_T H)_{ij} - \frac{2}{3} (k \cdot H) g_{ij}, \end{aligned} \tag{2.12}$$

and  $\mathcal{L}_T$  denotes the Lie derivative in the direction  $T$ .

*Proof.* See Proposition 7.2.1 in [10].  $\square$

The 1 + 3 decomposition and the maximal gauge are used in a large number of work in general relativity, see for example [10, 5] and Chapter 3 in this thesis.

The first motivation for that choice is that the system of equations in Propositions 2.2–2.4 displays a clear hyperbolic structure due to the Maxwell-type equations (2.11) for the electric-magnetic tensors  $E$  and  $H$ . In particular, energy estimates can be obtained for  $E$  and  $H$ . The metric and connection coefficients  $g, k$  and  $\phi$  are then determined by solving the transport and elliptic equations in Propositions 2.2 and 2.3, for which the electric-magnetic tensors  $E$  and  $H$  are source terms.

**Remark 2.5.** *The time lapse function  $\phi$  is only defined through the elliptic equation (2.10) on each maximal hypersurface  $\Sigma_t$ , and is therefore well-determined up to a choice of a boundary condition at the boundary of  $\Sigma_t$ . In [10, 5] and Chapter 3 of this thesis where  $\Sigma_t \simeq \mathbb{R}^3$ , the chosen condition is  $\phi \rightarrow 1$  at spatial infinity, which physically corresponds to considering a centre-of-mass frame for the system, see Chapter 1 in [10].*

## 2.2 $S$ -foliations

### 2.2.1 Definition of $S$ -foliations

Many important works in general relativity rely on a geometric formalism based on  $S$ -foliations, i.e. foliations by topological 2-spheres, and adapted null pairs  $(e_3, e_4)$

$$\mathbf{g}(e_4, e_3) = -2, \quad \mathbf{g}(e_4, e_4) = \mathbf{g}(e_3, e_3) = 0, \quad TS = \langle e_3, e_4 \rangle^\perp,$$

i.e.  $\langle e_3, e_4 \rangle^\perp$  spans the tangent space of  $S$ .

**Remark 2.6.** *In Minkowski spacetime, the prime example of  $S$ -foliation is given in  $(t, r, \theta, \varphi)$  coordinates by  $S = S(t, r)$ ,  $e_3 = \partial_t - \partial_r$  and  $e_4 = \partial_t + \partial_r$ .*

**Remark 2.7.** *The terminology  $S$ -foliation is introduced in [33] to distinguish from the non integrable formalism introduced in [17] where  $\langle e_3, e_4 \rangle^\perp \neq TS$  in the context of the Kerr stability problem.*

### 2.2.2 Ricci coefficients and curvature components

Let a foliation of  $\mathcal{M}$  by a 2-parameter family of spacelike 2-spheres which are intersections of the level sets of two scalar functions and let  $(e_3, e_4)$  a null pair such that  $\langle e_3, e_4 \rangle^\perp = TS$ . We denote  $\mathcal{g}$  the Riemannian metric induced by  $\mathbf{g}$  on  $S$ , and we denote  $r$  its area radius, i.e.

$$r := \frac{\sqrt{|S|}}{4\pi}. \quad (2.13)$$

**Remark 2.8.** A null pair  $(e_3, e_4)$  decomposes the spacetime metric  $\mathbf{g}$  as follows:

$$\mathbf{g} = -\frac{1}{2}(e_3)^\flat \otimes (e_4)^\flat - \frac{1}{2}(e_4)^\flat \otimes (e_3)^\flat + \mathcal{g},$$

where  $(e_3)^\flat, (e_4)^\flat \in T^*\mathcal{M}$  are the 1-forms canonically associated to  $e_3$  and  $e_4$ .

We define the null connection coefficients relative to a null pair  $(e_3, e_4)$  to be the  $S$ -tangent tensors defined by:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), & \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), \\ \underline{\xi}_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_A), & \xi_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_A), \\ \underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_A), & \eta_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_A), \\ \underline{\zeta}_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_A} e_4, e_3), \end{aligned} \quad (2.14)$$

where  $e_A$ ,  $A = 1, 2$  denotes an orthonormal basis of  $TS$ . The null second fundamental forms  $\chi, \underline{\chi}$  are further decomposed in their traces  $\text{tr } \chi$  and  $\text{tr } \underline{\chi}$ , and traceless parts  $\widehat{\chi}$  and  $\widehat{\underline{\chi}}$ :

$$\begin{aligned} \text{tr } \chi &:= \mathcal{g}^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2}\mathcal{g}_{AB} \text{tr } \chi, \\ \text{tr } \underline{\chi} &:= \mathcal{g}^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2}\mathcal{g}_{AB} \text{tr } \underline{\chi}. \end{aligned}$$

The coefficients  $\text{tr } \chi$ ,  $\text{tr } \underline{\chi}$  are called the null expansions and  $\widehat{\chi}$ ,  $\widehat{\underline{\chi}}$  are called the shears.

We also define the null curvature components relative to a null pair  $(e_3, e_4)$  by:

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= \mathbf{R}(e_A, e_3, e_B, e_3), \\ \beta_A &= \frac{1}{2}\mathbf{R}(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A, e_3, e_3, e_4), \\ \rho &= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), & \sigma &= \frac{1}{4}{}^*\mathbf{R}(e_3, e_4, e_3, e_4), \end{aligned} \quad (2.15)$$

where  ${}^*\mathbf{R}$  denotes the Hodge dual of the spacetime curvature tensor  $\mathbf{R}$  of  $\mathbf{g}$ .

We define the horizontal covariant operator  $\nabla$  as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2}\underline{\chi}(X, Y)e_4 - \frac{1}{2}\chi(X, Y)e_3,$$

where  $X, Y \in TS$ . We also define  $\nabla_4 X$  and  $\nabla_3 X$  to be the horizontal projections of  $\mathbf{D}_4 X$  and  $\mathbf{D}_3 X$ :

$$\begin{aligned} \nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_4 e_3)e_4 - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_4 e_4)e_3, \\ \nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_3 e_3)e_4 - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_3 e_4)e_3. \end{aligned}$$

A tensor field  $\psi$  defined on  $\mathcal{M}$  is called tangent to  $S(t, u)$  if it is a priori defined on the spacetime  $\mathcal{M}$  and all the possible contractions of  $\psi$  with either  $e_3$  or  $e_4$  are zero. We use  $\nabla_3\psi$  and  $\nabla_4\psi$  to denote the projection to  $S(t, u)$  of usual derivatives  $\mathbf{D}_3\psi$  and  $\mathbf{D}_4\psi$ .

As a direct consequence of (2.14), we have the Ricci formulae:

$$\begin{aligned}
\mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2}\chi_{AB}e_3 + \frac{1}{2}\underline{\chi}_{AB}e_4, \\
\mathbf{D}_A e_3 &= \underline{\chi}_{AB}e_B + \zeta_A e_3, \\
\mathbf{D}_A e_4 &= \chi_{AB}e_B - \zeta_A e_4, \\
\mathbf{D}_3 e_A &= \nabla_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, \\
\mathbf{D}_4 e_A &= \nabla_4 e_A + \underline{\eta}_A e_4 + \xi_A e_3, \\
\mathbf{D}_3 e_3 &= -2\underline{\omega}e_3 + 2\underline{\xi}_B e_B, \\
\mathbf{D}_3 e_4 &= 2\underline{\omega}e_4 + 2\underline{\eta}_B e_B, \\
\mathbf{D}_4 e_4 &= -2\underline{\omega}e_4 + 2\xi_B e_B, \\
\mathbf{D}_4 e_3 &= 2\underline{\omega}e_3 + 2\underline{\eta}_B e_B.
\end{aligned} \tag{2.16}$$

### 2.2.3 Main equations

We recall the null structure equations, see Proposition 7.4.1 in [10].

**Proposition 2.9.** *We have the null structure equations:*

$$\begin{aligned}
\nabla_4 \widehat{\chi} + \text{tr } \chi \widehat{\chi} &= \nabla \widehat{\otimes} \xi - 2\underline{\omega} \widehat{\chi} + (\eta + \underline{\eta} + 2\underline{\zeta}) \widehat{\otimes} \xi - \alpha, \\
\nabla_4 \text{tr } \chi + \frac{1}{2}(\text{tr } \chi)^2 &= 2\text{d}\text{iv } \xi - 2\underline{\omega} \text{tr } \chi + 2\underline{\xi} \cdot (\eta + \underline{\eta} + 2\underline{\zeta}) - |\widehat{\chi}|^2, \\
\nabla_3 \widehat{\chi} + \text{tr } \underline{\chi} \widehat{\chi} &= \nabla \widehat{\otimes} \underline{\xi} - 2\underline{\omega} \widehat{\chi} + (\eta + \underline{\eta} - 2\underline{\zeta}) \widehat{\otimes} \underline{\xi} - \underline{\alpha}, \\
\nabla_3 \text{tr } \underline{\chi} + \frac{1}{2}(\text{tr } \underline{\chi})^2 &= 2\text{d}\text{iv } \underline{\xi} - 2\underline{\omega} \text{tr } \underline{\chi} + 2\underline{\xi} \cdot (\eta + \underline{\eta} - 2\underline{\zeta}) - |\widehat{\chi}|^2, \\
\nabla_4 \widehat{\chi} + \frac{1}{2} \text{tr } \chi \widehat{\chi} &= \nabla \widehat{\otimes} \underline{\eta} + 2\underline{\omega} \widehat{\chi} - \frac{1}{2} \text{tr } \underline{\chi} \widehat{\chi} + \underline{\xi} \widehat{\otimes} \underline{\xi} + \underline{\eta} \widehat{\otimes} \underline{\eta}, \\
\nabla_4 \text{tr } \underline{\chi} + \frac{1}{2} \text{tr } \chi \text{tr } \underline{\chi} &= 2\underline{\omega} \text{tr } \underline{\chi} - \widehat{\chi} \cdot \widehat{\chi} + 2\text{d}\text{iv } \underline{\eta} + 2(\underline{\xi} \cdot \xi + |\underline{\eta}|^2) + 2\rho, \\
\nabla_3 \widehat{\chi} + \frac{1}{2} \text{tr } \underline{\chi} \widehat{\chi} &= \nabla \widehat{\otimes} \eta + 2\underline{\omega} \widehat{\chi} - \frac{1}{2} \text{tr } \chi \widehat{\chi} + \xi \widehat{\otimes} \xi + \eta \widehat{\otimes} \eta, \\
\nabla_3 \text{tr } \chi + \frac{1}{2} \text{tr } \underline{\chi} \text{tr } \chi &= 2\underline{\omega} \text{tr } \chi - \widehat{\chi} \cdot \widehat{\chi} + 2\text{d}\text{iv } \eta + 2(\xi \cdot \underline{\xi} + |\eta|^2) + 2\rho, \\
\nabla_4 \eta - \nabla_3 \xi &= -4\underline{\omega} \xi - \chi \cdot (\eta - \underline{\eta}) - \beta, \\
\nabla_3 \underline{\eta} - \nabla_4 \underline{\xi} &= -4\underline{\omega} \underline{\xi} - \underline{\chi} \cdot (\underline{\eta} - \eta) + \underline{\beta}, \\
\nabla_3 \zeta &= -2\underline{\nabla} \underline{\omega} - \underline{\chi} \cdot (\zeta + \eta) + 2\underline{\omega}(\zeta - \eta) + \chi \cdot \underline{\xi} + 2\underline{\omega} \underline{\xi} - \underline{\beta}, \\
\nabla_4 \zeta &= 2\underline{\nabla} \omega + \chi \cdot (-\zeta + \underline{\eta}) + 2\underline{\omega}(\zeta + \underline{\eta}) - \underline{\chi} \cdot \xi - 2\underline{\omega} \xi - \beta, \\
\nabla_3 \omega + \nabla_4 \underline{\omega} &= \xi \cdot \underline{\xi} + \zeta \cdot (\eta - \underline{\eta}) - \eta \cdot \underline{\eta} + 4\underline{\omega} \omega + \rho.
\end{aligned}$$

Also, we have the Codazzi equations:

$$\begin{aligned}
\text{d}\text{iv } \widehat{\chi} &= \frac{1}{2} \nabla \text{tr } \chi - \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \text{tr } \chi \right) - \beta, \\
\text{d}\text{iv } \widehat{\chi} &= \frac{1}{2} \nabla \text{tr } \underline{\chi} + \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \text{tr } \underline{\chi} \right) + \underline{\beta},
\end{aligned} \tag{2.17}$$

the torsion equation:

$$\begin{aligned}
\text{curl } \eta &= \sigma + \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi} - \xi \wedge \underline{\xi}, \\
\text{curl } \underline{\eta} &= -\sigma - \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi} + \xi \wedge \underline{\xi}, \\
\text{curl } \xi &= \xi \wedge (\eta + \underline{\eta} + 2\zeta), \\
\text{curl } \underline{\xi} &= \underline{\xi} \wedge (\eta + \underline{\eta} - 2\zeta),
\end{aligned} \tag{2.18}$$

and the Gauss equation:

$$\mathbf{K} = -\frac{1}{4} \text{tr } \chi \text{tr } \underline{\chi} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho. \tag{2.19}$$

We recall the Bianchi equations, see Proposition 7.3.2 in [10].

**Proposition 2.10.** *The Bianchi equations take the following form*

$$\begin{aligned}
\nabla_3 \alpha + \frac{1}{2} \text{tr } \underline{\chi} \alpha &= \nabla \widehat{\otimes} \beta + 4\underline{\omega} \alpha - 3(\widehat{\chi} \rho + {}^* \widehat{\chi} \sigma) + (\zeta + 4\underline{\eta}) \widehat{\otimes} \beta, \\
\nabla_4 \beta + 2 \text{tr } \chi \beta &= \text{div } \alpha - 2\underline{\omega} \beta + (2\zeta + \underline{\eta}) \cdot \alpha + 3(\xi \rho + {}^* \xi \sigma), \\
\nabla_3 \beta + \text{tr } \underline{\chi} \beta &= \nabla \rho + {}^* \nabla \sigma + 2\underline{\omega} \beta + \underline{\xi} \cdot \alpha + 2\widehat{\chi} \cdot \underline{\beta} + 3(\eta \rho + {}^* \eta \sigma), \\
\nabla_4 \rho + \frac{3}{2} \text{tr } \chi \rho &= \text{div } \beta - \frac{1}{2} \widehat{\chi} \cdot \alpha + \zeta \cdot \beta + 2(\underline{\eta} \cdot \beta - \xi \cdot \underline{\beta}), \\
\nabla_3 \rho + \frac{3}{2} \text{tr } \underline{\chi} \rho &= -\text{div } \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} + 2(\underline{\xi} \cdot \beta - \eta \cdot \underline{\beta}), \\
\nabla_4 \sigma + \frac{3}{2} \text{tr } \chi \sigma &= -\text{curl } \beta + \frac{1}{2} \widehat{\chi} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2(\underline{\eta} \cdot {}^* \beta + 2\underline{\xi} \cdot {}^* \beta), \\
\nabla_3 \sigma + \frac{3}{2} \text{tr } \underline{\chi} \sigma &= -\text{curl } \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot {}^* \underline{\alpha} + \zeta \cdot {}^* \underline{\beta} - 2(\eta \cdot {}^* \underline{\beta} + \underline{\xi} \cdot {}^* \beta), \\
\nabla_4 \underline{\beta} + \text{tr } \chi \underline{\beta} &= -\nabla \rho + {}^* \nabla \sigma + 2\underline{\omega} \underline{\beta} + 2\widehat{\chi} \cdot \beta - \xi \cdot \underline{\alpha} - 3(\eta \rho - {}^* \eta \sigma), \\
\nabla_3 \underline{\beta} + 2 \text{tr } \underline{\chi} \underline{\beta} &= -\text{div } \alpha - 2\underline{\omega} \underline{\beta} + (2\zeta - \underline{\eta}) \cdot \underline{\alpha} + 3(-\underline{\xi} \rho + {}^* \underline{\xi} \sigma), \\
\nabla_4 \underline{\alpha} + \frac{1}{2} \text{tr } \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + 4\underline{\omega} \underline{\alpha} - 3(\widehat{\chi} \rho - {}^* \widehat{\chi} \sigma) + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta}.
\end{aligned}$$

## 2.3 Main examples of $S$ -foliations

### 2.3.1 Maximal-null foliation

In this section, we present the maximal-null foliation. This gauge choice is used in [10, 5], see also Chapter 3 in this thesis.

We define an *optical function*  $u$ , i.e a scalar function satisfying

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = 0. \tag{2.20}$$

The level sets of  $u$ , denoted by  $C_u$  are null hypersurfaces, which foliate the spacetime  $\mathcal{M}$ . We define the null lapse function  $a$  by

$$a := |\nabla u|^{-1},$$

where  $\nabla$  is the connection on  $\Sigma_t$  induced by  $\mathbf{D}$ .

Combining with the time function  $t$  and maximal hypersurfaces  $\Sigma_t$  defined in Section 2.1.2, we obtain the foliation by a 2-parameter family of spacelike 2-spheres

$$S(t, u) := C_u \cap \Sigma_t.$$

For any  $S = S(t, u)$ , we denote  $N$  the outward unit normal vectorfield of  $S$  on  $\Sigma_t$ . We also denote  $T$  the future directed unit normal vectorfield on  $\Sigma_t$ . We define

$$e_4 := T + N, \quad e_3 := T - N,$$

which form an adapted null pair on  $S$ .

Next, we decompose  $k^2$  as follows:

$$\delta := k_{NN}, \quad \epsilon_A := k_{AN}, \quad \kappa_{AB} := k_{AB}. \quad (2.21)$$

The following identities hold for a maximal-null foliation:

$$\begin{aligned} \chi_{AB} &= \theta_{AB} - \kappa_{AB}, & \underline{\chi}_{AB} &= -\theta_{AB} - \kappa_{AB}, \\ \xi_A &= 0, & \underline{\xi}_A &= \nabla_A \log \phi - \nabla_A \log a, \\ \eta_A &= \nabla_A \log a + \epsilon_A, & \underline{\eta}_A &= \nabla_A \log \phi - \epsilon_A, \\ 2\omega &= -\nabla_N \log \phi + \delta, & 2\underline{\omega} &= \nabla_N \log \phi + \delta, \\ \zeta_A &= \epsilon_A, & & \end{aligned} \quad (2.22)$$

see (7.5.2b) in [10].

### 2.3.2 Double null foliation

In this section, we present the double null foliation. This gauge choice is used in a large number of works in general relativity, see for example [29, 12, 11]. See also Chapters 2 and 4 in this thesis for more applications of double null foliations.

We first introduce two optical functions  $u$  and  $\underline{u}$ , i.e.

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = 0, \quad \mathbf{g}(\mathbf{D}\underline{u}, \mathbf{D}\underline{u}) = 0, \quad (2.23)$$

in which case their level sets are null hypersurfaces  $C_u$  and  $\underline{C}_{\underline{u}}$  which foliate  $\mathcal{M}$  by a family of 2-parameter spacelike 2-spheres:

$$S(u, \underline{u}) := C_u \cap \underline{C}_{\underline{u}}.$$

We define the null lapse function  $\Omega$  by

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}\underline{u}) = -2\Omega^{-2}.$$

The adapted null pair  $(e_3, e_4)$  on  $S$  is defined by:

$$e_4 = -2\Omega \mathbf{D}u, \quad e_3 = -2\Omega \mathbf{D}\underline{u}.$$

The following identities hold for a double null foliation:

$$\begin{aligned} \nabla \log \Omega &= \frac{1}{2}(\eta + \underline{\eta}), & \omega &= -\frac{1}{2}\mathbf{D}_4(\log \Omega), & \underline{\omega} &= -\frac{1}{2}\mathbf{D}_3(\log \Omega), \\ \eta &= \zeta + \nabla \log \Omega, & \underline{\eta} &= -\zeta + \nabla \log \Omega, & \xi &= \underline{\xi} = 0, \end{aligned} \quad (2.24)$$

see for example (6) in [32].

---

<sup>2</sup>Recall that  $k$  is defined in (2.1).

### 2.3.3 Geodesic foliation

In this section, we present the geodesic foliation by 2-spheres of a null hypersurface. This gauge choice is used in Chapter 5, as well as in Chapters 2 and 4 on the last null incoming slice.

Let  $\mathcal{H}$  be a null hypersurface of  $\mathcal{M}$ , which we assume is emanating from a 2-sphere  $S_0$ . Let  $e_4$  be a null geodesic generator of  $\mathcal{H}$ , i.e. such that  $e_4$  is everywhere null and tangent to  $\mathcal{H}$  and satisfies the following geodesic equation

$$\mathbf{D}_{e_4} e_4 = 0.$$

To  $e_4$ , we associate an affine geodesic parameter  $s$  on  $\mathcal{H}$  by

$$e_4(s) = 1, \quad s|_{S_0} = 1.$$

We call  $S(s)$  the geodesic foliation on  $\mathcal{H}$  where  $S(s)$  are the level sets of  $s$ . We then define  $e_3$  by the following conditions:

$$\mathbf{g}(e_3, e_3) = 0, \quad \mathbf{g}(e_3, e_4) = -2, \quad e_3 \perp TS.$$

To describe the geometry of the foliation  $S(s)$ , we use the null connection coefficients defined in (2.14), which make sense in the context of a null pair only defined on  $\mathcal{H}$  (but not on  $\mathcal{M}$ ), that is we consider

$$\chi, \quad \underline{\chi}, \quad \zeta, \quad \underline{\eta}, \quad \xi, \quad \omega.$$

We have the following identities for an outgoing geodesic foliation:

$$\xi = 0, \quad \omega = 0, \quad \zeta + \underline{\eta} = 0.$$

**Remark 2.11.** *We can also define an incoming geodesic foliation by exchanging the role of  $e_4$  and  $e_3$ . In this case, the following quantities make sense on  $\mathcal{H}$ :*

$$\chi, \quad \underline{\chi}, \quad \zeta, \quad \eta, \quad \underline{\xi}, \quad \underline{\omega}.$$

*More precisely, we have the following identities for an incoming geodesic foliation:*

$$\mathbf{D}_{e_3} e_3 = 0, \quad e_3(s) = -1, \quad \underline{\xi} = 0, \quad \underline{\omega} = 0, \quad \eta = \zeta.$$

*Incoming geodesic foliations are used in Chapters 2 and 4 while outgoing geodesic foliations are used in Chapter 5.*

**Remark 2.12.** *Another particular choice of foliation of a null hypersurface, not used in this thesis, is given by the canonical foliation, see for example [29] and [19].*

## 3 The $r^p$ -weighted estimate method

In Chapters 2, 3 and 4 of this thesis, we apply  $r^p$ -weighted estimates to Bianchi equations to show decay estimates for the curvature terms. In this section, we recall the  $r^p$ -weighted estimate method. We start by reviewing shortly  $r^p$ -weighted estimates for the scalar wave equation. Then, we illustrate the method for Bianchi equations on a toy model.



### 3.1 $r^p$ -weighted estimates for the scalar wave equation

The  $r^p$ -weighted estimate method has been originally introduced in order to derive decay for the scalar wave equation

$$\square_{\mathbf{g}}\psi = 0 \tag{3.1}$$

by Dafermos and Rodnianski in [13] as an alternative to the vectorfield method. The vectorfield method has been initiated by Klainerman [27, 28] in the context of the decay of solutions to quasilinear wave equations in Minkowski spacetime. The method relies in particular on:

- good communication properties of the scalar wave equation with Killing and conformally Killing vectorfields,
- the fact that Minkowski has a maximal number of such vectorfields: Lorentz boosts, space rotations, translations and scaling,
- and the fact that some of these vectorfields<sup>3</sup> have weights in  $t$ .

The vectorfield method is very robust to nonlinear perturbations of Minkowski spacetime and hence has been successfully used in proving the stability of Minkowski, see for example [10, 29, 5, 41] and see also Section 4 for more references. However, the vectorfield method is difficult to extend to more general spacetimes, including black hole spacetimes, as such spacetimes contain much fewer Killing and conformal Killing vectorfields.

The  $r^p$ -weighted estimate method is introduced by Dafermos and Rodnianski in [13], in conjunction with energy-Morawetz estimates, to circumvent these difficulties. We introduce below the main idea of  $r^p$ -weighted estimates in Minkowski spacetime for the scalar wave equation, see Section 3 of [13] for more details.

We assume that the Minkowski spacetime is foliated by a family of outgoing null hypersurfaces  $\Sigma(\tau)$ . In the sequel, we only consider the region where  $\tau \geq 1$ . Let  $\tau_1 \leq \tau_2$ . We denote  $\mathcal{M}(\tau_1, \tau_2)$  the domain between  $\Sigma(\tau_1)$  and  $\Sigma(\tau_2)$ . Let  $\psi$  be a solution of (3.1). We define

$$\begin{aligned} B_p[\psi](\tau_1, \tau_2) &:= \int_{\mathcal{M}(\tau_1, \tau_2)} r^{p-3} |e_4(r\psi)|^2 + r^{p-1} |\nabla\psi|^2, \\ E_p[\psi](\tau) &:= \int_{\Sigma(\tau)} r^{p-2} |e_4(r\psi)|^2. \end{aligned}$$

We rewrite the wave equation (3.1) in Minkowski in the form:

$$-e_3 e_4(r\psi) + \Delta(r\psi) = 0.$$

Multiplying this equation by  $r^{p-2} e_4(r\psi)$  and integrating by parts in  $\mathcal{M}(\tau_1, \tau_2)$ , we deduce

$$B_p[\psi](\tau_1, \tau_2) + E_p[\psi](\tau_2) \lesssim E_p[\psi](\tau_1), \quad 0 < p < 2, \tag{3.2}$$

see Section 3 in [13] for more details.

Notice that (3.2) only implies decay in  $r$ . The  $\tau$ -decay of  $\psi$  can then be obtained by the mean value method as follows. Assume that we have the following initial assumption

$$E_p[\psi](1) \lesssim \varepsilon_0^2. \tag{3.3}$$

---

<sup>3</sup>I.e. Lorentz boosts and scaling.

We have by definition

$$B_p[\psi](\tau_1, \tau_2) \gtrsim \int_{\tau_1}^{\tau_2} E_{p-1}[\psi](\tau) d\tau.$$

Hence, (3.2) and (3.3) imply that for all  $j \in \mathbb{N}$  and  $1 < p < 2$

$$\int_{2^j}^{2^{j+1}} E_{p-1}[\psi](\tau) d\tau \lesssim \varepsilon_0^2.$$

Thus, there exists  $\tau_0 \in [2^j, 2^{j+1}]$  such that

$$E_{p-1}[\psi](\tau_0) \lesssim \frac{\varepsilon_0^2}{2^j}.$$

Taking  $\tau_1 = \tau_0$  and  $\tau_2 = \tau \in [2^{j+1}, 2^{j+2}]$  in (3.2), we deduce

$$B_{p-1}[\psi](\tau_0, \tau) + E_{p-1}[\psi](\tau) \lesssim E_{p-1}[\psi](\tau_0) \lesssim \frac{\varepsilon_0^2}{2^j} \lesssim \frac{\varepsilon_0^2}{\tau}.$$

Applying once again (3.2), we infer

$$B_{p-1}[\psi](\tau_1, \tau_2) + E_{p-1}[\psi](\tau_2) \lesssim \frac{\varepsilon_0^2}{\tau_1}.$$

This implies that we can trade  $r$ -decay for  $\tau$ -decay thanks to the above mean value argument.

The  $r^p$ -weighted estimate method has the following advantages:

- It captures the radiative properties of waves in a natural way.
- It can be applied to a vast class of asymptotically flat spacetimes, as once Morawetz estimate is proved, the  $r^p$ -weighted estimate method is performed in the asymptotically flat region only.
- Unlike the vectorfield method, it does not make use of multipliers or commutators with weights growing in  $t$  which could be problematic in the region  $r \lesssim t$ .

Dafermos-Rodnianski [13] considered  $r^p$ -weighted estimates for (3.1) on Minkowski spacetime and Schwarzschild black holes. Later on, Schlue [50] extended it to higher-dimensional Schwarzschild black holes. It also has been generalized to a broad class of asymptotically flat spacetime  $(\mathcal{M}, \mathbf{g})$ , including Kerr spacetimes in the full subextremal range  $|a| < M$  by Moschidis [48]. Moreover, a new vectorfield approach based on the  $r^p$ -weighted estimate method has been introduced by Angelopoulos-Aretakis-Gajic [2], which permits to improve the decay of  $\psi$  obtained by the method in [13]. More recently, Ma-Zhang [45, 46] obtained the sharp decay for Teukolsky equation in Schwarzschild and Kerr spacetimes relying in particular on the methods in [13, 2]. See also [3] by Angelopoulos-Aretakis-Gajic in the case of subextremal Reissner-Nordström black holes.

### 3.2 $r^p$ -weighted estimates for Bianchi equations

The  $r^p$ -weighted estimate method was applied to Bianchi equations first by Holzegel [22], see also [33, 18]. We illustrate the method in the case of the toy model of the linearization of the Bianchi equations in Minkowski for the Bianchi pair  $(\alpha, \beta)$ .

**Lemma 3.1.** *The linearized Bianchi equations for  $(\alpha, \beta)$  in Minkowski take the following form:*

$$\begin{aligned}\nabla_3 \alpha - \frac{\alpha}{r} &= \nabla \widehat{\otimes} \beta, \\ \nabla_4 \beta + \frac{4\beta}{r} &= \mathfrak{d}\not\!v \alpha.\end{aligned}\tag{3.4}$$

*Proof.* We define the following null pair in Minkowski spacetime:

$$e_4 = \partial_t + \partial_r, \quad e_3 = \partial_t - \partial_r.$$

The Ricci coefficients then take the following values in Minkowski:

$$\begin{aligned}\mathrm{tr} \chi &= \frac{2}{r}, & \mathrm{tr} \underline{\chi} &= -\frac{2}{r}, & \widehat{\chi} &= \widehat{\underline{\chi}} = 0, & \eta &= \underline{\eta} = \zeta = 0, & \omega &= \underline{\omega} = 0, \\ \alpha &= \underline{\alpha} = 0, & \beta &= \underline{\beta} = 0, & \rho &= \sigma = 0.\end{aligned}$$

Thus, (3.4) follows directly from Proposition 2.10.  $\square$

**Lemma 3.2.** *Assume that  $(\alpha, \beta)$  satisfies the linearized Bianchi equations in Lemma 3.1. Then, we have for any  $p \in \mathbb{R}$*

$$\mathrm{Div}(r^p |\alpha|^2 e_3) + 2\mathrm{Div}(r^p |\beta|^2 e_4) + pr^{p-1} |\alpha|^2 + 2(6-p)r^{p-1} |\beta|^2 = 4r^p \mathfrak{d}\not\!v (\alpha \cdot \beta).\tag{3.5}$$

*Proof.* Recall that we have in Minkowski spacetime

$$\mathrm{Div}(e_4) = \frac{2}{r}, \quad \mathrm{Div}(e_3) = -\frac{2}{r}, \quad e_4(r) = 1, \quad e_3(r) = -1.$$

We then compute

$$\begin{aligned}& \mathrm{Div}(r^p |\alpha|^2 e_3) + 2\mathrm{Div}(r^p |\beta|^2 e_4) \\ &= \nabla_3(r^p |\alpha|^2) + \mathrm{Div}(e_3) r^p |\alpha|^2 + 2\nabla_4(r^p |\beta|^2) + 2\mathrm{Div}(e_4) r^p |\beta|^2 \\ &= pr^{p-1} e_3(r) |\alpha|^2 + 2r^p \alpha \cdot \nabla_3 \alpha - 2r^{p-1} |\alpha|^2 + 2pr^{p-1} |\beta|^2 + 4r^p \beta \cdot \nabla_4 \beta + 4r^{p-1} |\beta|^2 \\ &= -pr^{p-1} |\alpha|^2 + 2r^p \alpha \cdot \left( \frac{\alpha}{r} + \nabla \widehat{\otimes} \beta \right) - 2r^{p-1} |\alpha|^2 + 2pr^{p-1} |\beta|^2 + 4r^p \beta \cdot \left( -\frac{4\beta}{r} + \mathfrak{d}\not\!v \alpha \right) + 4r^{p-1} |\beta|^2 \\ &= 2r^p \alpha \cdot \nabla \widehat{\otimes} \beta - pr^{p-1} |\alpha|^2 + 4r^p \beta \cdot \mathfrak{d}\not\!v \alpha + 2(-6+p)r^{p-1} |\beta|^2 \\ &= 4r^p \mathfrak{d}\not\!v (\alpha \cdot \beta) - pr^{p-1} |\alpha|^2 + 2(p-6)r^{p-1} |\beta|^2,\end{aligned}$$

where we used

$$\begin{aligned}\mathfrak{d}\not\!v (\alpha \cdot \beta) &= \nabla^A (\alpha_{AB} \beta_B) = (\nabla^A \alpha_{AB}) \beta^B + \alpha_{AB} \nabla^A \beta^B \\ &= (\mathfrak{d}\not\!v \alpha) \cdot \beta + \frac{1}{2} \alpha_{AB} (\nabla^A \beta^B + \nabla^B \beta^A - \not\!g^{AB} \mathfrak{d}\not\!v \beta) \\ &= (\mathfrak{d}\not\!v \alpha) \cdot \beta + \frac{1}{2} \alpha \cdot \nabla \widehat{\otimes} \beta.\end{aligned}$$

Then, we obtain

$$\mathrm{Div}(r^p |\alpha|^2 e_3) + 2\mathrm{Div}(r^p |\beta|^2 e_4) + pr^{p-1} |\alpha|^2 + 2(6-p)r^{p-1} |\beta|^2 = 4r^p \mathfrak{d}\not\!v (\alpha \cdot \beta)$$

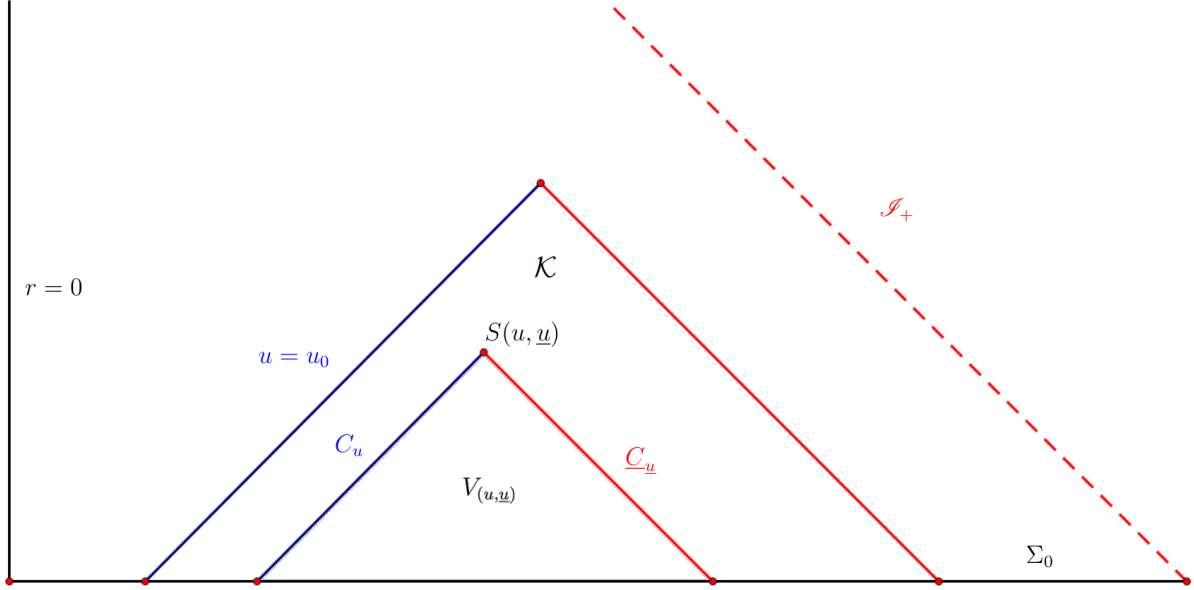
which implies (3.5). This concludes the proof of Lemma 3.2.  $\square$

We consider the following choice of optical functions  $u$  and  $\underline{u}$  in Minkowski spacetime:

$$u = t - r, \quad \underline{u} = t + r.$$

The following proposition provides the flux estimates of  $\alpha$  and  $\beta$  on respectively outgoing null cones  $C_u$  and incoming null cones  $\underline{C}_{\underline{u}}$ .<sup>4</sup>

<sup>4</sup>Here,  $C_u$  and  $\underline{C}_{\underline{u}}$  are respectively level sets of  $u$  and  $\underline{u}$ .

Figure 1.1: Domain of integration  $V(u, \underline{u})$ 

**Proposition 3.3.** *We denote<sup>5</sup>*

$$V := V(u, \underline{u}), \quad C_u^V := C_u \cap V, \quad \underline{C}_u^V := \underline{C}_u \cap V. \quad (3.6)$$

*Then, we have for  $(\alpha, \beta)$  satisfying the linearized Bianchi equations in Lemma 3.1, in the case  $0 \leq p \leq 6$ :*

$$\int_{C_u^V} r^p |\alpha|^2 + \int_{\underline{C}_u^V} r^p |\beta|^2 \lesssim \int_{\Sigma_0 \cap V} r^p |\alpha|^2 + r^p |\beta|^2. \quad (3.7)$$

*Proof.* Applying Stokes formula, we obtain<sup>6</sup>

$$\begin{aligned} & \int_V \mathbf{Div}(r^p |\alpha|^2 e_3) + \int_V \mathbf{Div}(r^p |\beta|^2 e_4) \\ &= \int_{\Sigma_0 \cap V} r^p |\alpha|^2 e_3 \cdot \mathcal{N}_{\Sigma_0} + \int_{\underline{C}_u^V} r^p |\alpha|^2 e_3 \cdot (-e_3) + \int_{C_u^V} r^p |\alpha|^2 e_3 \cdot (-e_4) \\ & \quad + \int_{\Sigma_0 \cap V} r^p |\beta|^2 e_4 \cdot \mathcal{N}_{\Sigma_0} + \int_{\underline{C}_u^V} r^p |\beta|^2 e_4 \cdot (-e_3) + \int_{C_u^V} r^p |\beta|^2 e_4 \cdot (-e_4) \\ &= - \int_{\Sigma_0 \cap V} (r^p |\alpha|^2 + r^p |\beta|^2) + 2 \int_{C_u^V} r^p |\alpha|^2 + 2 \int_{\underline{C}_u^V} r^p |\beta|^2. \end{aligned}$$

Combining with (3.5), we deduce

$$\int_V p r^{p-1} |\alpha|^2 + 2(6-p) r^{p-1} |\beta|^2 - \int_{\Sigma_0 \cap V} (r^p |\alpha|^2 + r^p |\beta|^2) + 2 \int_{C_u^V} r^p |\alpha|^2 + 2 \int_{\underline{C}_u^V} r^p |\beta|^2 = 0,$$

which implies (3.7) in the case  $0 \leq p \leq 6$ . This concludes the proof of Proposition 3.3.  $\square$

This type of  $r^p$ -weighted estimates, for the full Bianchi equations, are used repeatedly in Chapters 2, 3 and 4 to prove decay for curvature components. These estimates are sufficient to prove decay in the exterior region  $r \geq u$ , while in the interior region  $r \leq u$ , the mean value argument outlined at the end of Section 3.1 is also needed.

<sup>5</sup>See Figure 1.1 for a picture.

<sup>6</sup>Here,  $\mathcal{N}_{\Sigma_0}$  denotes the unit future normal vectorfield on  $\Sigma_0$ .

## 4 Minkowski stability

In this section, we review the state of the art of Minkowski stability.

### 4.1 Previous works on the stability of Minkowski spacetime

In 1993, Christodoulou and Klainerman [10] proved the global stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2007, Bieri [5] gave a proof of global stability of Minkowski requiring less derivative and less vectorfield compared to [10]. In 2003, Klainerman and Nicolò [29] proved the Minkowski stability in the exterior of an outgoing cone. Moreover, Klainerman and Nicolò [30] showed that under stronger asymptotic decay and regularity properties than those used in [10, 29], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. Given that the goal of this thesis is to extend the result of [10, 5, 29], we will state their results in Section 4.2.

The proofs in [10, 5, 29] are based respectively on the maximal foliation and the double null foliation. We now mention proofs of stability of Minkowski using other gauges. Lindblad and Rodnianski [40, 41] gave a new proof of the stability of the Minkowski spacetime using *wave-coordinates* and showing that the Einstein equations verify the so called *weak null structure* in that gauge. Huneau [23] proved the nonlinear stability of Minkowski spacetime with a translation Killing field using generalised wave-coordinates. Using the framework of Melrose's b-analysis, Hintz and Vasy [21] reproved the stability of Minkowski space. Graf [19] proved the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations, which together with [29] allows to reobtain [10]. More recently, Hintz [20] reproved the Minkowski stability in exterior regions by using the framework of [21].

There are also stability results concerning Einstein's equations coupled with non trivial matter fields:

- Einstein-Maxwell system: Zipser [62] extended the framework of [10] to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system. In [43], Loizelet used the framework of [41] to demonstrate the stability of the Minkowski spacetime solution of the Einstein-scalar field-Maxwell system in  $(1 + n)$ -dimensions ( $n \geq 3$ ). Speck [56] gave a proof of the global nonlinear stability of the  $(1 + 3)$ -dimensional Minkowski spacetime solution to the coupled system for a family of electromagnetic fields, which includes the standard Maxwell fields.
- Einstein-Klein-Gordon system: Lefloch and Ma [38] and Wang [60] proved the global stability of Minkowski for the Einstein-Klein-Gordon system with initial data coinciding with the Schwarzschild solution with small mass outside a compact set. Ionescu and Pausader [24] proved the global stability of Minkowski for the Einstein-Klein-Gordon system for general initial data. See also [39] by Lefloch and Ma using a foliation that merges hyperboloids with constant time slices.
- Einstein-Vlasov system: Taylor [57] considered the massless case where the initial data for the Vlasov part is compactly supported on the mass shell. Fajman, Joudioux and Smulevici [14] considered the massive case where the initial data coincides with Schwarzschild in the exterior region and with compact support assumption only in space on the Vlasov part. Lindblad and Taylor [42] considered the massive case where the initial data has compact support for the Vlasov part. Bigorgne, Fajman, Joudioux, Smulevici and Thaller [6] considered the massless case for general initial data. Wang [61] considered the massive case for general initial data.

## 4.2 Minkowski stability in [10, 5, 29]

We recall in this section the results in [10, 5, 29], which are most relevant to the results in this thesis. We first introduce the notion of  $(s, q)$ -asymptotically flat initial data.

**Definition 4.1.** *Given  $s > 1$  and  $q \in \mathbb{N}$ , we say that a data set  $(\Sigma_0, g, k)$  is  $(s, q)$ -asymptotically flat if there exists a coordinate system  $(x^1, x^2, x^3)$  defined outside a sufficiently large compact set such that:*

- In the case of  $s \geq 3$ <sup>7</sup>

$$\begin{aligned} g_{ij} &= \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_q(r^{-\frac{s+1}{2}}). \end{aligned} \quad (4.1)$$

- In the case of  $1 < s < 3$

$$\begin{aligned} g_{ij} &= \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_q(r^{-\frac{s+1}{2}}). \end{aligned} \quad (4.2)$$

**Definition 4.2.** *We denote  $d_0$  the geodesic distance from a fixed point  $O \in \Sigma_0$ ,  $B_{ij} := (\text{curl } \widehat{R})_{ij}$  the Bach tensor and  $\widehat{R}$  is the traceless part of Ric. Then, we define for  $q \geq 2$ :*

- In the case of  $s \geq 3$ :

$$\begin{aligned} J_0^{(q)}(\Sigma_0, g, k) &:= \sup_{\Sigma_0} \left( (d_0^2 + 1)^3 |\text{Ric}|^2 \right) + \sum_{l=0}^q \int_{\Sigma_0} (d_0^2 + 1)^{l+\frac{s-2}{2}} |\nabla^l k|^2 \\ &+ \sum_{l=0}^{q-2} \int_{\Sigma_0} (d_0^2 + 1)^{l+\frac{s+2}{2}} |\nabla^l B|^2. \end{aligned} \quad (4.3)$$

- In the case of  $1 < s < 3$ :

$$J_0^{(q)}(\Sigma_0, g, k) := \sum_{l=0}^q \int_{\Sigma_0} (d_0^2 + 1)^{l+\frac{s-2}{2}} |\nabla^l k|^2 + \sum_{l=0}^{q-1} \int_{\Sigma_0} (d_0^2 + 1)^{l+\frac{s}{2}} |\nabla^l \text{Ric}|^2. \quad (4.4)$$

We can now state the main theorems of [10] and [5].

**Theorem 4.3** (Global stability of Minkowski spacetime [10]). *There exists an  $\varepsilon_0 > 0$  sufficiently small such that if  $J_0^{(3)}(\Sigma_0, g, k) \leq \varepsilon_0^2$ , then the initial data set  $(\Sigma_0, g, k)$ ,  $(4, 3)$ -asymptotically flat (in the sense of Definition 4.1) and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e. the Riemann curvature tensor tends to zero along any causal or space-like geodesic. Moreover, there exists a global maximal time function  $t$  and an optical function  $u$  defined everywhere in an external region.*

**Theorem 4.4** (Global stability of Minkowski spacetime [5]). *There exists an  $\varepsilon_0 > 0$  sufficiently small such that if  $J_0^{(2)}(\Sigma_0, g, k) \leq \varepsilon_0^2$ , then the initial data set  $(\Sigma_0, g, k)$ ,  $(2, 2)$ -asymptotically flat and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e. the Riemann curvature tensor tends to zero along any causal or space-like geodesic. Moreover, there exists a global maximal time function  $t$  and an optical function  $u$  defined everywhere in an external region.*

<sup>7</sup>The notation  $f = o_q(r^{-m})$  means  $\partial^\alpha f = o(r^{-m-|\alpha|})$ ,  $|\alpha| \leq q$ .

The goal of Chapter 3 in this thesis is to extend the results of [5] to more general asymptotic behaviors, see Section 6.2.

We also state the main theorem of [29].

**Theorem 4.5** (Minkowski Stability in the exterior region [29]). *Consider an initial data set  $(\Sigma_0, g, k)$ , 4-asymptotically flat and maximal, and assume  $J_0(\Sigma_0, g, k)$  is bounded. Then, given a sufficiently large compact set  $K \subset \Sigma_0$  such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \bar{B}_1$ , and under additional smallness assumptions, there exists a unique development  $(\mathcal{M}, \mathbf{g})$  with the following properties:*

1.  $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $\{C_u\}$  and  $\{\underline{C}_u\}$  whose outgoing leaves  $C_u$  are complete.
2. We have detailed control of all the quantities associated with the double null foliations of the spacetime, see Theorem 3.7.1 of [29].

**Remark 4.6.** [30] extends Theorem 4.5 to the case  $s > 7$  by assuming sufficient regular initial data  $(\Sigma_0, g, k)$ .

The goal of Chapter 2 in this thesis is to reprove [29] by a different method and to treat more general asymptotic behaviors, see Section 6.1.

## 5 Kerr stability

In this section, we recall the definition of Schwarzschild and Kerr solutions, we state the Kerr stability conjecture and review the state of the art on Kerr stability in external regions as well as the notion of GCM spheres introduced in [33, 34, 35].

### 5.1 Schwarzschild and Kerr solutions

Minkowski spacetime is the simplest solution of (1.1). We recall two other relevant solutions of (1.1):

- *Schwarzschild spacetime.* (1.1) admits a remarkable family of explicit, stationary, solutions given by the one parameter family of solutions of mass  $M > 0$  discovered by Schwarzschild in [51]

$$\mathbf{g}_M = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \quad (5.1)$$

Though the metric seems singular at  $r = 2M$ , it turns out that one can glue together two regions  $r > 2M$  and  $r < 2M$  of the Schwarzschild metric to obtain a metric which is smooth along  $\mathcal{H} = \{r = 2M\}$ , called the Schwarzschild *event horizon*. The portion of  $r < 2M$  to the future of the hypersurface  $t = 0$  is a *black hole* whose future boundary  $r = 0$  is singular. The similar region to the past of  $t = 0$  is called a *white hole*. The region  $r > 2M$ , called the *domain of outer communication* (DOC), is free of singularities. The Penrose diagram of Schwarzschild space is recalled in Figure 1.2.

- *Kerr spacetime.* The Schwarzschild family is included in a larger two parameter family of solutions  $K(a, M)$  discovered by Kerr in [26]. In Boyer-Lindquist coordinates, the Kerr metric has the form:

$$\mathbf{g}_{a,M} = - \frac{(\Delta - a^2 \sin^2 \theta)}{q^2} dt^2 - \frac{4aMr}{q^2} \sin^2 \theta dt d\varphi + \frac{q^2}{\Delta} dr^2 + q^2 d\theta^2 + \frac{\Sigma^2}{q^2} \sin^2 \theta d\varphi^2, \quad (5.2)$$

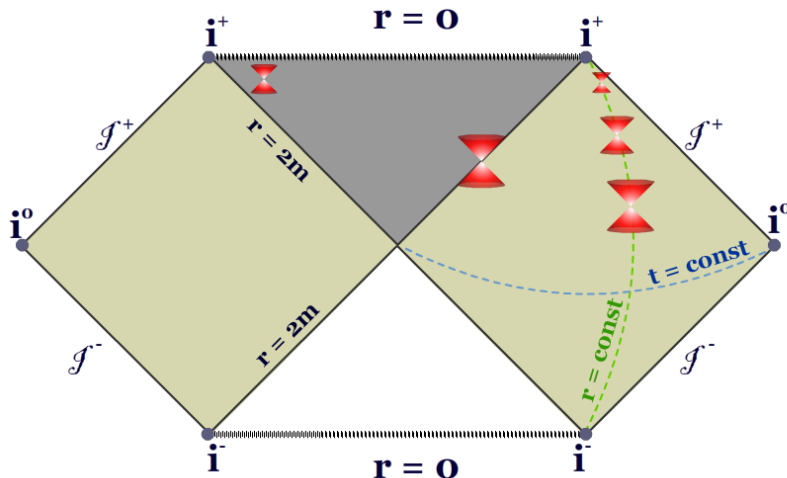


Figure 1.2: Complete Penrose diagram of Schwarzschild spacetime

where

$$q^2 = r^2 + a^2 \cos 2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2 \Delta.$$

Kerr spacetimes  $g_{a,M}$ , with  $0 \leq |a| \leq M$  possess well defined event horizons  $r = r_+ := M + (M^2 - a^2)^{\frac{1}{2}}$ , domain of outer communication  $r > r_+$  and smooth future null infinity  $\mathcal{I}^+$  where  $r = +\infty$ . The metric can be extended smoothly inside the black hole region  $r < r_+$ , see Figure 1.3 for the Penrose diagram of Kerr spacetime. The boundary  $r = r_- := M - (M^2 - a^2)^{\frac{1}{2}}$  inside black hole region is a Cauchy horizon across which predictability fails<sup>8</sup>.

## 5.2 Stability of Kerr conjecture

**Conjecture 5.1** (Stability of Kerr conjecture). *Vacuum initial data sets, sufficiently close to Kerr initial data, have a maximal development with complete future null infinity<sup>9</sup> and with domain of outer communication which approaches (globally) a nearby Kerr solution.*

The stability of Kerr has been proved in the case of small angular momentum,  $|a| \ll M$ , in the sequence of works by Klainerman-Szeftel [34], [35], [36], Giorgi-Klainerman-Szeftel [18] and myself [52] (see Chapter 5 in this thesis), but the conjecture is open in general. For a complete introduction to this problem, we refer the reader to the review [37] and to the introduction in [36] and [18].

In order to attack the conjecture, it is convenient to divide the domain of outer communication to the future of an initial spacelike hypersurface  $\Sigma_0$  into three causal regions as in Figure 1.4:

- Region I lies to the future of  $\Sigma_0$  and to the past of an incoming null cone  $\underline{C}_0$ . The stability of Kerr in spacetime region I is a direct consequence of the local existence results in [8, 9].
- Region II lies to the future of  $\Sigma_0$  and to the past of an outgoing null cone  $C_0$  sufficiently far away from the event horizon. See Section 5.3 for a discussion of Kerr stability in this

<sup>8</sup>Infinitely many smooth extensions are possible beyond the boundary.

<sup>9</sup>This means, roughly, that observers which are far away from the black hole may live forever.



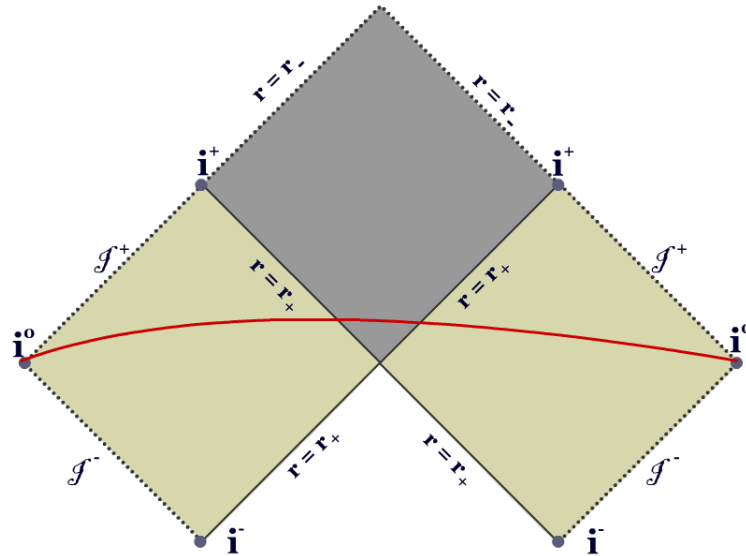


Figure 1.3: Penrose diagram of Kerr for  $0 < |a| < M$ . The surface  $r = r_+$ , the larger root of  $\Delta = 0$ , is the event horizon of the black hole,  $r > r_+$  the domain of outer communication,  $\mathcal{I}^+$  is the future null infinity, corresponding to  $r = +\infty$ .

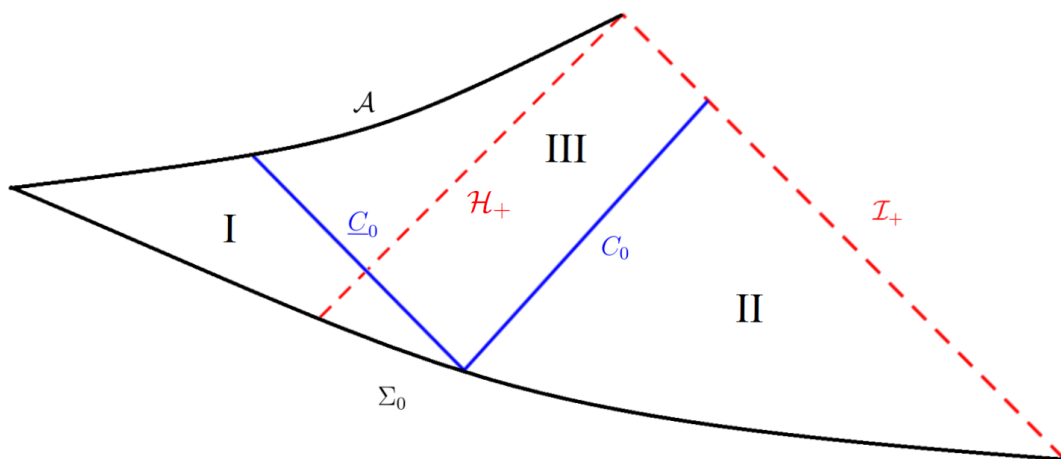


Figure 1.4: The Penrose diagram of a perturbation of Kerr

region. The goal of Chapter 4 in this thesis is to provide a new proof of the stability of Kerr in Region II.

- Region III lies to the future of the incoming null cone  $\underline{C}_0$  and of the outgoing null cone  $C_0$ . The stability of Kerr in region III for small angular momentum  $|a| \ll M$  was proved recently in the sequence of works [34], [35], [36], [18] and [52].

As mentioned above, the goal of Chapter 4 in this thesis is to provide a new proof of the stability of Kerr in Region II. In the next section, we review the state of the art on this problem.

### 5.3 Previous works on Kerr stability in external regions

In 2009, Nicolò [49] proved that the asymptotic behaviour of the Riemann curvature components is in agreement with the *Peeling theorem* if the corrections to the Kerr initial data decay sufficiently fast. In 2010, Caciotta and Nicolò [7] proved the nonlinear stability of the Kerr spacetime for any  $|a| \leq M$  and appropriate class of initial data close to Kerr, in an external region where  $r \geq R_0$  and  $\frac{M}{R_0} \ll 1$ . Their main result can be stated as follows.

**Theorem 5.2** (Caciotta-Nicolò [7]). *Assume that the initial data are given on  $\Sigma_0$  outside of a ball centered on the origin of radius  $R_0$ , denoted  $K := B_{R_0}$ . They are perturbations of the initial data of a Kerr spacetime with mass  $M$  and angular momentum  $a$  satisfying*

$$\frac{M}{R_0} \ll 1, \quad |a| \leq M.$$

Let  $s > 7$ , and assume that the initial data satisfies for  $q \geq 4$ :

$$\begin{aligned} g_{ij} &= (g_{Kerr})_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= (k_{Kerr})_{ij} + o_q(r^{-\frac{s+1}{2}}). \end{aligned} \tag{5.3}$$

Then, the initial data set  $(\Sigma_0, g, k)$  has a unique development  $(\mathcal{M}, \mathbf{g})$  defined outside  $I^+(K)$ , the domain of influence of  $K$ . Moreover,  $\mathcal{M}$  can be foliated by a canonical double null foliation  $(u, \underline{u})$  whose outgoing leaves  $C_u$  are complete and suitable norms of Ricci coefficients and curvature components are sufficiently small.

The goal of Chapter 4 is to provide a new proof of Theorem 5.2, see Section 6.3.

### 5.4 General Covariant Modulated (GCM) procedure

In this section, we review the *General Covariant Modulated* (GCM) procedure introduced firstly in [33] for polarized perturbations of Schwarzschild, and extended to the case of general perturbation of Kerr in [34, 35, 52], where [52] corresponds to Chapter 5 in this thesis.

#### 5.4.1 GCM admissible spacetimes in [36]

In the sequence of works [34], [35], [18], [36] and [52] establishing Kerr stability for small angular momentum, the proof is centered around a limiting argument for a family of carefully constructed finite *GCM admissible spacetimes*  $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ . As can be seen in Figure 1.5, the future boundary of the spacetime is given by  $\mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*$  where  $\Sigma_*$  is a spacelike, GCM hypersurface, that is a hypersurface verifying a set of crucial, well-specified, geometric conditions, essential to the proof of convergence to a final state. See Section 6.4 and Chapter 5 in this thesis for more details on GCM hypersurfaces.

Once  $\Sigma_*$  is specified, the whole GCM admissible spacetime  $\mathcal{M}$  is determined by a more conventional construction, based on geometric transport type equations. More precisely  ${}^{(ext)}\mathcal{M}$  can

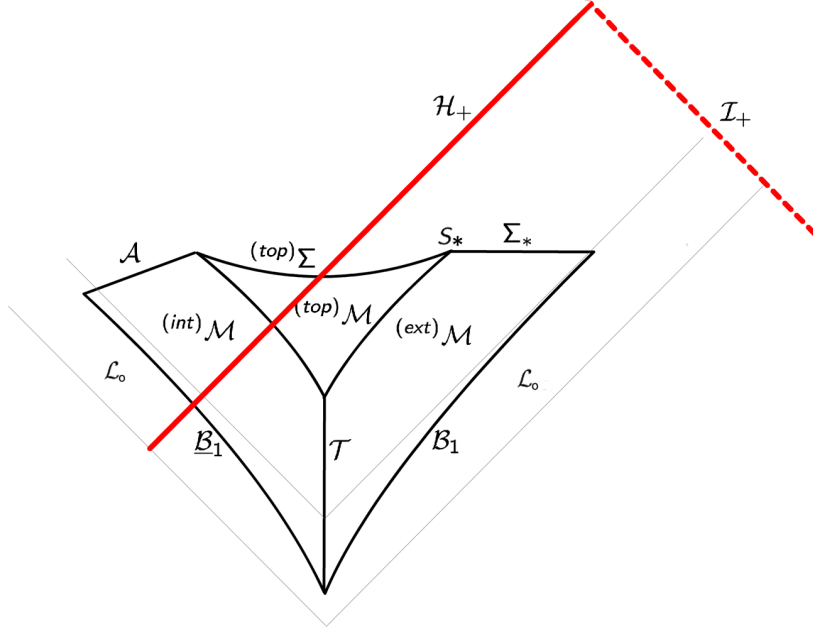


Figure 1.5: The Penrose diagram of a finite GCM admissible spacetime  $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ .

be determined from  $\Sigma_*$  by a specified outgoing foliation terminating in the timelike boundary  $\mathcal{T}$ .  ${}^{(int)}\mathcal{M}$  is determined from  $\mathcal{T}$  by a specified incoming one, and  ${}^{(top)}\mathcal{M}$  is a complement of  ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$  which makes  $\mathcal{M}$  a causal domain. The past boundary  $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$  of  $\mathcal{M}$ , which is itself to be constructed, is included in the initial layer  $\mathcal{L}_0$  in which the spacetime is assumed to be known, i.e. a small perturbation of a Kerr solution. The passage from the initial data specified on  $\Sigma_0$  to  $\mathcal{L}_0$  is justified in Chapter 4 in this thesis, see also the discussion on regions I, II and III in Section 5.2.

The construction of the GCM hypersurfaces is performed in Chapter 5 of this thesis. It consists of the concatenation of a suitable choice of a 1-parameter family of GCM spheres. In the next section, we review the notion of GCM spheres.

#### 5.4.2 GCM spheres in [34]

As mentioned above, the crucial concept in the proof of Kerr stability for small angular momentum is that of a GCM admissible spacetime, whose construction is anchored by the GCM sphere  $S_*$  in Figure 1.5. GCM spheres, defined in [34] and [35], are codimension 2 compact surfaces, unrelated to the initial conditions, on which specific geometric quantities take Schwarzschildian values.

Recall that in an outgoing geodesic foliation of Schwarzschild spacetime, we have:

$$\kappa = \frac{2}{r}, \quad \underline{\kappa} = -\frac{2\Upsilon}{r}, \quad \mu = \frac{2m}{r^3}, \quad (5.4)$$

where  $\kappa := \text{tr } \chi$ ,  $\underline{\kappa} := \text{tr } \underline{\chi}$ ,  $\Upsilon := 1 - \frac{2m}{r}$  with  $r$ ,  $m$  denoting the area radius and Hawking mass of  $S$  and  $\mu$  denoting the mass aspect function, i.e.

$$\mu := -\text{div } \zeta - \rho + \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}}, \quad r := \sqrt{\frac{|S|}{4\pi}}, \quad \frac{2m}{r} := 1 + \frac{1}{16\pi} \int_S \kappa \underline{\kappa}. \quad (5.5)$$

The idea to construct GCM spheres is to mimic the conditions (5.4) in a spacetime which is a perturbation of Kerr.

More precisely, the GCM spheres are topological spheres  $\mathbf{S}$  embedded in a spacetime region  $\mathcal{R}$  which is a perturbation of Kerr with  $r \gg m$ , endowed with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  adapted to  $\mathbf{S}$  (i.e.  $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$  are tangent to  $\mathbf{S}$ ), relative to which the null expansions  $\kappa^{\mathbf{S}} = \text{tr} \chi^{\mathbf{S}}$ ,  $\underline{\kappa}^{\mathbf{S}} = \text{tr} \underline{\chi}^{\mathbf{S}}$  and mass aspect function  $\mu^{\mathbf{S}}$  satisfy:<sup>10</sup>

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (5.6)$$

where  $r^{\mathbf{S}}$  and  $m^{\mathbf{S}}$  denote the area radius and Hawking mass of  $\mathbf{S}$ .

Here is a short version of the main result in [34].

**Theorem 5.3** (Existence of GCM spheres in [34]). *Let  $\mathcal{R}$  be fixed spacetime region, endowed with an outgoing geodesic foliation  $S(u, s)$ , verifying specific asymptotic assumptions<sup>11</sup> expressed in terms of two parameters  $0 < \delta \leq \overset{\circ}{\varepsilon}$ . In particular we assume that the GCM quantities of the background spheres in  $\mathcal{R}$ , i.e.*

$$\kappa - \frac{2}{r}, \quad \left( \underline{\kappa} + \frac{2\Upsilon}{r} \right)_{\ell \geq 2}, \quad \left( \mu - \frac{2m}{r^3} \right)_{\ell \geq 2}, \quad (5.7)$$

are small with respect to the parameter  $\delta$ . Let  $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$  be a fixed sphere of the foliation with  $\beta^{(1)}$  and  $\overset{\circ}{m}$  denoting respectively its area radius and Hawking mass, with  $\beta^{(1)} \gg \overset{\circ}{m}$ . Then, for any fixed triplets  $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$  verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \delta,$$

there exists a unique sphere  $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ , together with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ , which is GCM, i.e.  $\mathbf{S}$  is a deformation of  $\overset{\circ}{S}$ , such that

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0. \quad (5.8)$$

**Remark 5.4.** Notice that the spheres of Kerr spacetime  $S(t, r)$  are close to that of Minkowski for  $r \gg m$ . The triplets  $\Lambda, \underline{\Lambda}$  of parameters in Theorem 5.3 take into account the fact that Minkowski contains an 8 parameter family of spheres. Indeed, in addition to the 2 parameters  $(t, r)$ , 3 parameters correspond to space translations and 3 parameters correspond to Lorentzian boosts.

See Chapter 5 in this thesis for more details on GCM spheres.

## 6 Overview of the main results of the thesis

In this section, we present the main results obtained in this thesis.

<sup>10</sup>The  $\ell = 1$  modes are defined by  $J^{(0)} := \cos \theta$ ,  $J^{(-)} := \sin \theta \sin \varphi$  and  $J^{(+)} := \sin \theta \cos \varphi$ , which is a natural generalization of  $\ell = 1$  spherical harmonics.

<sup>11</sup>Compatible with small perturbations of Kerr.

## 6.1 Stability of Minkowski spacetime in the exterior regions

In Chapter 2, we obtain the following result, which provides a new proof of [29].

**Theorem 6.1** (Minkowski stability in the exterior regions [53]). *Let  $s > 3$ , and let an initial data set  $(\Sigma_0, g, k)$  which is  $s$ -asymptotically flat in the sense of Definition 4.1. Let a sufficiently large compact set  $K \subset \Sigma_0$  such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \overline{B}_1$ . Assume that we have a smallness conditions in an initial layer region  $\mathcal{K}_{(0)}$  near  $\Sigma_0 \setminus K$ . Then, there exists a unique future development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:*

- $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $(C_u, \underline{C}_u)$  whose outgoing leaves  $C_u$  are complete for all  $u \leq u_0$ ;
- We have detailed control of all the quantities associated with the double null foliations of the spacetime.

**Remark 6.2.** *In the particular case  $s = 4$ , we reobtain the results of [29]. Also, in the case  $s > 7$ , we reobtain the strong peeling properties of [30].*

The proof of Theorem 6.1 has the same structure as in [29]. Below, we compare the proof of Theorem 6.1 and that of [29]:

1. In [29], the functional (4.3) is introduced to fix the initial conditions on the initial hypersurface. Here we fix the initial conditions in an initial layer region  $\mathcal{K}_{(0)}$  near the initial hypersurface  $\Sigma_0$ .
2. To estimate the norms of curvature components, [29] uses the vectorfield method introduced in [10]. Here, we estimate the curvature norms by  $r^p$ -weighted estimate, a method introduced by Dafermos and Rodnianski in [13]. This allows for a simpler treatment of the curvature estimates.
3. [29] uses second order derivatives of curvature and third order derivatives of Ricci coefficients. Thanks to the use of  $r^p$ -weighted estimates, we only need first order derivatives of both curvature and Ricci coefficients.
4. In order to control one more derivative for Ricci coefficients compared to curvature, [29] relies on the canonical foliation on the last slice  $\underline{C}_*$ . Since we control the same number of derivatives of curvature and Ricci coefficients, instead of introducing the canonical foliation, we use the geodesic foliation to simplify the estimates on the last slice  $\underline{C}_*$ .

## 6.2 Global stability of Minkowski spacetime with minimal decay

In this section, we state a simple version of the main theorem in Chapter 3.

**Theorem 6.3** (Global stability of Minkowski spacetime [55]). *Let  $s \in (1, 2]$  and let an initial data set  $(\Sigma_0, g, k)$  which is  $(s, 2)$ -asymptotically flat in the sense of Definition 4.1. Assume that we have smallness conditions in an initial layer region  $\mathcal{K}_{(0)}$  near  $\Sigma_0$ . Then, there exists a unique future development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:*

- $(\mathcal{M}, \mathbf{g})$  can be foliated by a maximal-null foliation  $(C_u, \Sigma_t)$  whose outgoing leaves  $C_u$  are complete for all  $u$ ;
- We have detailed control of all the quantities associated with the maximal-null foliation of the spacetime.

**Remark 6.4.** *In the particular case  $s = 2$ , we reobtain the results of [5].*

The structure of the proof of Theorem 6.3 is similar to [10, 5]. Below, we compare the proof of Theorem 6.3 and that of [5]:

1. In [5], the functional (4.4) is introduced to fix the initial conditions on the initial hypersurface. Here we fix the initial conditions in an initial layer region  $\mathcal{K}_{(0)}$  near the initial hypersurface  $\Sigma_0$ .
2. To estimate the norms of curvature components, [5] uses the vectorfield method introduced in [10]. Here, we estimate the curvature norms by  $r^p$ -weighted estimate, a method introduced by Dafermos and Rodnianski in [13]. This allows for a simpler treatment of the curvature estimates.
3. In order to obtain optimal decay for all the connection coefficients, [5] uses two foliations constructed respectively forward and backward. In this paper, we derive decay for the connection coefficients based on only one foliation constructed forward from the initial hypersurface  $\Sigma_0$  and the symmetric axis  $\Phi$ . This avoids having to compare forward and backward gauges.

### 6.3 Kerr stability in external regions

In this section, we state a simple version of the main theorem in Chapter 4.

**Theorem 6.5** (Kerr stability in external regions [54]). *Let  $K = B_{R_0} \subset \Sigma_0$  a ball centered at the origin of radius  $R_0$  and let  $\mathbf{g}_{a,M}$  a Kerr metric with mass  $M > 0$  and angular momentum  $a$  satisfying*

$$\frac{M}{R_0} \ll 1, \quad |a| \leq M.$$

*Let  $s > 3$  and let an initial data set  $(\Sigma_0, g, k)$  satisfying (5.3) with  $q = 2$ . Let  $\mathcal{K}_{(0)}$  an initial layer region near  $\Sigma_0 \setminus K$  and assume that the initial data in  $\mathcal{K}_{(0)}$  are perturbations from the initial data of  $\mathbf{g}_{a,M}$  of size  $\varepsilon_0 \ll \frac{M}{R_0}$ .*

*Then, the initial data set  $(\Sigma_0, g, k)$  has a unique development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:*

- $\mathcal{M}$  can be foliated by a double null foliation  $(u, \underline{u})$  whose outgoing leaves  $C_u$  are complete for all  $u \leq u_0$ ;
- we have detailed control of all the quantities associated with the double null foliation of the spacetime.

**Remark 6.6.** *In the particular case of  $s$ -asymptotically flat initial data with  $s > 7$ , we reobtain the results of [7].*

**Remark 6.7.** *Let  $C_{u_0}$  be the outgoing null cone located near  $\partial K$  and choose  $s = 4 + 2\delta_B$  with  $\delta_B > 0$ . Then, the estimates of Theorem 6.5 on  $C_{u_0}$  are consistent with the initial data considered in [33] and [36].*

**Remark 6.8.** *Theorem 6.5 is the analog for the Kerr spacetime of Theorem 6.1 for Minkowski spacetime.*

The proof of Theorem 6.5 has the same structure as in [29, 7] and is closely related to Theorem 6.1. Below, we compare the proof of Theorem 6.5 and that of [7]:

1. To estimate the norms of curvature components, [7] uses the vectorfield method introduced in [10, 29]. Here, we estimate the curvature norms by the  $r^p$ -weighted estimate method introduced in [13]. This allows for a simple treatment of the curvature estimates.

2. [7] uses fourth order derivatives of Ricci coefficients and third order derivatives of curvature components. Thanks to the use of  $r^p$ -weighted estimates, we only need first order derivatives of both Ricci coefficients and curvature components.
3. In order to control one more derivative for Ricci coefficients compared to curvature, [7] relies on the canonical foliation on the last slice  $\underline{C}_*$ . Since we control the same number of derivatives for curvature and Ricci coefficients, instead of introducing the canonical foliation, we use a geodesic type foliation to simplify the estimates on the last slice  $\underline{C}_*$ .

## 6.4 Construction of GCM hypersurfaces in perturbations of Kerr

In this section, we state a simple version of the main theorem of Chapter 5 in this thesis.

**Theorem 6.9** (Existence of GCM hypersurfaces [52]). *Let  $\mathcal{R}$  be fixed spacetime region, endowed with an outgoing geodesic foliation  $S(u, s)$ , verifying same assumptions as Theorem 5.3.*

Let  $\mathbf{S}_0$  be a fixed sphere included in the region  $\mathcal{R}$ , let a pair of triplets  $\Lambda_0, \underline{\Lambda}_0 \in \mathbb{R}^3$  such that

$$|\Lambda_0|, |\underline{\Lambda}_0| \lesssim \delta,$$

and let  $J^{(\mathbf{S}_0, p)}$  a basis of  $\ell = 1$  modes on  $\mathbf{S}_0$ , such that we have on  $\mathbf{S}_0$

$$\kappa^{\mathbf{S}_0} - \frac{2}{r^{\mathbf{S}_0}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}_0} + \frac{2\Upsilon^{\mathbf{S}_0}}{r^{\mathbf{S}_0}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}_0} - \frac{2m^{\mathbf{S}_0}}{(r^{\mathbf{S}_0})^3} \right)_{\ell \geq 2} = 0.$$

Then, there exists a unique, local, smooth, spacelike hypersurface  $\Sigma_0$  passing through  $\mathbf{S}_0$ , a scalar function  $u^{\mathbf{S}}$  defined on  $\Sigma_0$ , whose level surfaces are topological spheres denoted by  $\mathbf{S}$ , a smooth collection of triplets of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  and a triplet of functions  $J^{(\mathbf{S}, p)}$  defined on  $\Sigma_0$  verifying,

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \quad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0, \quad J^{(\mathbf{S}, p)}|_{\mathbf{S}_0} = J^{(\mathbf{S}_0, p)},$$

such that the following conditions are verified:

1. The surfaces  $\mathbf{S}$  of constant  $u^{\mathbf{S}}$ , together with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ , verify

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (6.1)$$

for the triplets of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  and  $\ell = 1$  modes  $J^{(\mathbf{S}, p)}$ .

2. The following transversality conditions

$$\xi^{\mathbf{S}} = 0, \quad \omega^{\mathbf{S}} = 0, \quad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \quad (6.2)$$

and

$$e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0, \quad e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1 \quad (6.3)$$

are assumed on  $\Sigma_0$ .

3. We have, for some constant  $c_0$ ,

$$u^{\mathbf{S}} + r^{\mathbf{S}} = c_0, \quad \text{along } \Sigma_0. \quad (6.4)$$

4. Let  $\nu^{\mathbf{S}}$  be the unique vectorfield tangent to the hypersurface  $\Sigma_0$ , normal to  $\mathbf{S}$ , and normalized by  $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$ . Let  $b^{\mathbf{S}}$  be the unique scalar function on  $\Sigma_0$  such that  $\nu^{\mathbf{S}}$  is given by

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}. \quad (6.5)$$

The following normalization condition holds true

$$\overline{b^{\mathbf{S}}} = -1 - \frac{2m_{(0)}}{r^{\mathbf{S}}}, \quad (6.6)$$

where  $\overline{b^{\mathbf{S}}}$  is the average value of  $b^{\mathbf{S}}$  over  $\mathbf{S}$  and  $m_{(0)}$  is a constant.

5. We have the following identities on  $\Sigma_0$ :

$$(\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} = 0, \quad (\operatorname{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} = 0. \quad (6.7)$$

**Remark 6.10.** Theorem 6.9 is the generalization of Theorem 9.52 in [33] in the absence of symmetry. It plays a central role in the proof of Theorems M6 and M7 in [36], see Sections 8.4 and 8.5 in [36].

**Remark 6.11.** We provide below more explanations for the statements 1-5 in Theorem 6.9:

1. Since we concatenate a family of GCM spheres  $\mathbf{S}(\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}})$  emanating from  $\mathbf{S}_0$  to construct the GCM hypersurfaces  $\Sigma_0$ , by Theorem 5.3, we have automatically (6.1) on every  $\mathbf{S}$ .
2. The transversality conditions (6.2) and (6.3) are consistent with a local extension by an outgoing geodesic foliation initialized on  $\Sigma_0$ . The role of these transversality conditions is to make sense of  $\eta^{\mathbf{S}}$  and  $\underline{\xi}^{\mathbf{S}}$  on  $\Sigma_0$ , see item 5 below.
3.  $u^{\mathbf{S}}$  should be chosen to be constant on the GCM spheres foliating  $\Sigma_0$ . The choice (6.4) is simple and fulfills this condition but other choices making  $\Sigma_0$  spacelike are possible.
4. The value  $\overline{b^{\mathbf{S}}}$  is free and should be prescribed. Note that the choice (6.6) coincides with the value for the hypersurface  $\{u + r = c_0\}$  in Schwarzschild spacetime.
5. In (6.7),  $\eta^{\mathbf{S}}$  and  $\underline{\xi}^{\mathbf{S}}$  are defined intrinsically on  $\Sigma_0$  by

$$\eta_a^{\mathbf{S}} = \frac{1}{2} \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_4^{\mathbf{S}}, e_a^{\mathbf{S}}), \quad \underline{\xi}_a^{\mathbf{S}} = \frac{1}{2} \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_3^{\mathbf{S}}, e_a^{\mathbf{S}}) + b^{\mathbf{S}} \zeta_a^{\mathbf{S}}.$$

These definitions are consistent with the standard ones provided  $\Sigma_0$  satisfies (6.2) which is equivalent to extending  $\Sigma_0$  locally by an outgoing geodesic foliation.

**Remark 6.12.** As in Section 9.8 of [33],  $\Sigma_0$  is chosen to be spacelike. One may wonder whether  $\Sigma_0$  could be chosen to be null<sup>12</sup>. The reason for choosing it to be spacelike is that it allows more flexibility: all spheres foliating  $\Sigma_0$  in Theorem 6.9 are GCM spheres, while only one could be a GCM sphere on a null hypersurface.

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<sup>12</sup>In the context of the stability of Minkowski, the last slice in the original proof by Christodoulou and Klainerman in [10] is spacelike, while it is null in the proof by Klainerman and Nicolò in [29] in the case of the exterior of an outgoing null cone.





## Chapter 2

# Stability of Minkowski spacetime in exterior regions

## 1 Introduction

### 1.1 Einstein vacuum equations and the Cauchy problem

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \quad \text{in } \mathcal{M}, \quad (1.1)$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for  $\mathbf{g}$ . In suitable coordinates, for example so-called wave coordinates, it can be shown that (1.1) is hyperbolic and hence admits an initial value formulation.

The corresponding initial data for the Einstein vacuum equations is given by specifying a triplet  $(\Sigma, g, k)$  where  $(\Sigma, g)$  is a Riemannian 3-manifold and  $k$  is the traceless symmetric 2-tensor on  $\Sigma$  satisfying the constraint equations:

$$\begin{aligned} R &= |k|^2 - (\text{tr } k)^2, \\ D^j k_{ij} &= D_i(\text{tr } k), \end{aligned} \quad (1.2)$$

where  $R$  denotes the scalar curvature of  $g$ ,  $D$  denotes the Levi-Civita connection of  $g$  and

$$|k|^2 := g^{ad} g^{bc} k_{ab} k_{cd}, \quad \text{tr } k := g^{ij} k_{ij}.$$

In the future development  $(\mathcal{M}, \mathbf{g})$  of such initial data  $(\Sigma, g, k)$ ,  $\Sigma \subset \mathcal{M}$  is a spacelike hypersurface with induced metric  $g$  and second fundamental form  $k$ .

The seminal well-posedness results for the Cauchy problem obtained in [8, 9] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  solution of Einstein equations (1.1) such that  $\Sigma \subset \mathcal{M}$  and  $g, k$  are respectively the first and second fundamental forms of  $\Sigma$  in  $\mathcal{M}$ .

The prime example of a vacuum spacetime is Minkowski space:

$$\mathcal{M} = \mathbb{R}^4, \quad \mathbf{g} = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \quad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad k = 0.$$

In the present work, we consider the problem of the stability of Minkowski spacetime and start with reviewing the literature on this problem.

### 1.2 Previous works of the stability of Minkowski spacetime

In 1993, Christodoulou and Klainerman [10] proved the stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2003, Klainerman and Nicolò [29] gave a second proof of this result in the exterior of an outgoing cone. Moreover, Klainerman and Nicolò [30] showed that under stronger asymptotic decay and regularity properties than those used in [10, 29], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. Given that the goal of this paper is to provide a new proof of the stability of Minkowski in the exterior region, we will state the results of [10] and [29] in section 1.3.

The proofs in [10] and [29] are based respectively on the maximal foliation and the double null foliation. Lindblad and Rodnianski [40, 41] gave a new proof of the stability of the Minkowski spacetime using *wave-coordinates* and showing that the Einstein equations verify the so called *weak null structure* in that gauge. Bieri [5] gave a proof requiring less derivative and less vector-field compared to [10]. Huneau [23] proved the nonlinear stability of Minkowski spacetime with a translation Killing field using generalised wave-coordinates. Using the framework of Melrose's b-analysis, Hintz and Vasy [21] reproved the stability of Minkowski space. Graf [19] proved the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations, which together with [29] allows to reobtain [10].

There are also stability results concerning Einstein's equations coupled with non trivial matter fields:

- Einstein-Maxwell system: Zipser [62] extended the framework of [10] to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system. In [43], Loizelet used the framework of [40, 41] to demonstrate the stability of the Minkowski spacetime solution of the Einstein-scalar field-Maxwell system in  $(1+n)$ -dimensions ( $n \geq 3$ ). Speck [56] gave a proof of the global nonlinear stability of the  $(1+3)$ -dimensional Minkowski spacetime solution to the coupled system for a family of electromagnetic fields, which includes the standard Maxwell fields.
- Einstein-Klein-Gordon system: Lefloch and Ma [38] and Wang [60] proved the global stability of Minkowski for the Einstein-Klein-Gordon system with initial data coinciding with the Schwarzschild solution with small mass outside a compact set. Ionescu and Pausader [24] proved the global stability of Minkowski for the Einstein-Klein-Gordon system for general initial data.
- Einstein-Vlasov system: Taylor [57] considered the massless case where the initial data for the Vlasov part is compactly supported on the mass shell. Fajman, Joudioux and Smulevici [14] considered the massive case where the initial data coincides with Schwarzschild in the exterior region and with compact support assumption only in space on the Vlasov part. Lindblad and Taylor [42] considered the massive case where the initial data has compact support for the Vlasov part. Bigorgne, Fajman, Joudioux, Smulevici and Thaller [6] considered the massless case for general initial data. Wang [61] considered the massive case for general initial data.

### 1.3 Minkowski Stability in [10] and [29]

We recall in this section the results in [10, 29]. First, we recall the definition of a *maximal hypersurface*, which plays an important role in the statements of the main theorems in [10, 29].

**Definition 1.1.** *An initial data  $(\Sigma, g, k)$  is posed on a maximal hypersurface if it satisfies*

$$\operatorname{tr} k = 0. \quad (1.3)$$

*In this case, we say that  $(\Sigma, g, k)$  is a maximal initial data set, and the constraint equations (1.2) reduce to*

$$R = |k|^2, \quad \operatorname{div} k = 0, \quad \operatorname{tr} k = 0. \quad (1.4)$$

We introduce the notion of *s-asymptotically flat initial data*.

**Definition 1.2.** *Let  $s$  be a real number with  $s > 3$ . We say that a data set  $(\Sigma_0, g, k)$  is *s-asymptotically flat* if there exists a coordinate system  $(x^1, x^2, x^3)$  defined outside a sufficiently*

large compact set such that<sup>1</sup>

$$\begin{aligned} g_{ij} &= \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_4(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_3(r^{-\frac{s+1}{2}}). \end{aligned} \quad (1.5)$$

We also introduce the following functional associated to any asymptotically flat initial data set:

$$J_0(\Sigma_0, g, k) := \sup_{\Sigma_0} \left( (d_0^2 + 1)^3 |\text{Ric}|^2 \right) + \int_{\Sigma_0} \sum_{l=0}^3 (d_0^2 + 1)^{l+1} |D^l k|^2 + \int_{\Sigma_0} \sum_{l=0}^1 (d_0^2 + 1)^{l+3} |D^l B|^2, \quad (1.6)$$

where  $d_0$  is the geodesic distance from a fixed point  $O \in \Sigma_0$ , and  $B_{ij} := (\text{curl } \widehat{R})_{ij}$  is the *Bach tensor*,  $\widehat{R}$  is the traceless part of  $\text{Ric}$ . Now, we can state the main theorems of [10] and [29].

**Theorem 1.3** (Global stability of Minkowski space [10]). *There exists an  $\epsilon > 0$  sufficiently small such that if  $J_0(\Sigma_0, g, k) \leq \epsilon^2$ , then the initial data set  $(\Sigma_0, g, k)$ , 4-asymptotically flat (in the sense of Definition 1.2) and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e. the Riemann curvature tensor tends to zero along any causal or space-like geodesic. Moreover, there exists a global maximal time function  $t$  and an optical function  $u^2$  defined everywhere in an external region.*

**Theorem 1.4** (Minkowski Stability in the exterior region [29]). *Consider an initial data set  $(\Sigma_0, g, k)$ , 4-asymptotically flat and maximal, and assume  $J_0(\Sigma_0, g, k)$  is bounded. Then, given a sufficiently large compact set  $K \subset \Sigma_0$  such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \overline{B}_1$ , and under additional smallness assumptions, there exists a unique development  $(\mathcal{M}, \mathbf{g})$  with the following properties:*

- (1)  $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $\{C_u\}$  and  $\{\underline{C}_u\}$  whose outgoing leaves  $C_u$  are complete.
- (2) We have detailed control of all the quantities associated with the double null foliations of the spacetime, see Theorem 3.7.1 of [29].

**Remark 1.5.** [30] extends Theorem 1.4 to the case  $s > 7$  by assuming sufficient regular initial data  $(\Sigma_0, g, k)$ .

The goal of this paper is to reprove [29] by a different method and to treat more general asymptotic behavior than  $s = 4$  in (1.5).

## 1.4 Rough version of the main theorem

In this section, we state a simple version of our main theorem. For the explicit statement, see Theorem 3.1.

**Theorem 1.6** (Main Theorem (first version)). *Let  $s > 3$ , and let an initial data set  $(\Sigma_0, g, k)$  which is  $s$ -asymptotically flat in the sense of Definition 1.2. Let a sufficiently large compact set  $K \subset \Sigma_0$  such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \overline{B}_1$ . Assume that we have a smallness conditions in an initial layer region  $\mathcal{K}_{(0)}$ <sup>3</sup> near  $\Sigma_0 \setminus K$ . Then, there exists a unique future development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:*

- $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $(C_u, \underline{C}_u)$  whose outgoing leaves  $C_u$  are complete for all  $u \leq u_0$ ;

<sup>1</sup>The notation  $f = o_l(r^{-m})$  means  $\partial^\alpha f = o(r^{-m-|\alpha|})$ ,  $|\alpha| \leq l$ .

<sup>2</sup>An optical function  $u$  is a scalar function satisfying  $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ .

<sup>3</sup>The initial data layer  $\mathcal{K}_{(0)}$  is defined in section 2.1.2.

- We have detailed control of all the quantities associated with the double null foliations of the spacetime, see Theorem 3.1.

**Remark 1.7.** *In the particular case  $s = 4$ , we reobtain the results of [29]. Also, in the case  $s > 7$ , we reobtain the strong peeling properties of [30].*

The proof of Theorem 1.6 has the same structure as in [29], see section 3.4. Below, we compare the proof of this paper and that of [29]:

1. In [29], the functional (1.6) is introduced to fix the initial conditions on the initial hypersurface. Here we fix the initial conditions in an initial layer region  $\mathcal{K}_{(0)}$  near the initial hypersurface  $\Sigma_0$ .
2. To estimate the norms of curvature components, [29] uses the vectorfield method introduced in [10]. Here, we estimate the curvature norms by  $r^p$ -weighted estimate, a method introduced by Dafermos and Rodnianski in [13]. This allows for a simpler treatment of the curvature estimates.
3. [29] uses second order derivatives of curvature and third order derivatives of Ricci coefficients. Thanks to the use of  $r^p$ -weighted estimates, we only need first order derivatives of both curvature and Ricci coefficients.
4. In order to control one more derivative for Ricci coefficients compared to curvature, [29] relies on the canonical foliation on the last slice  $\underline{C}_*$ . Since we control the same number of derivatives of curvature and Ricci coefficients, instead of introducing the canonical foliation, we use the geodesic foliation to simplify the estimates on the last slice  $\underline{C}_*$ .

## 1.5 Structure of the paper

- In section 2, we recall the fundamental notions and the basic equations.
- In section 3, we present the main theorem. We then state intermediate results, and prove the main theorem. The rest of the paper focuses on the proof of these intermediary results. In sections 4–6, we consider the case  $s \in [4, 6]$  and deal with the other cases in Appendices B and C.
- In section 4, we apply  $r^p$ -weighted estimates to Bianchi equations to control curvature.
- In section 5, we estimate the Ricci coefficients using the null structure equations.
- In section 6, we estimate the norms of curvature components and Ricci coefficients in the initial layer region  $\mathcal{K}_{(0)}$  and on the last slice  $\underline{C}_*$ . We also show how to extend the spacetime in the context of a bootstrap argument.
- In Appendix A, we prove the equivalence between optical functions and area radius associated to various frames used in the proof.
- In Appendix B, we prove Theorem 1.6 in the case  $s \in (3, 4)$ .
- In Appendix C, we prove Theorem 1.6 in the case  $s > 6$  by applying the  $r^p$ -weighted method to the Teukolsky equation of  $\alpha$ .

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## 2 Preliminaries

### 2.1 Geometric set-up

#### 2.1.1 Double null foliation

We first introduce the geometric setup. We denote  $(\mathcal{M}, \mathbf{g})$  a spacetime  $\mathcal{M}$  with the Lorentzian metric  $\mathbf{g}$  and  $\mathbf{D}$  its Levi-Civita connection. Let  $u$  and  $\underline{u}$  be two optical functions on  $\mathcal{M}$ , that is

$$\mathbf{g}(\text{grad } u, \text{grad } u) = \mathbf{g}(\text{grad } \underline{u}, \text{grad } \underline{u}) = 0.$$

The spacetime  $\mathcal{M}$  is foliated by the level sets of  $u$  and  $\underline{u}$  respectively, and the functions  $u, \underline{u}$  are required to increase towards the future. We use  $C_u$  to denote the outgoing null hypersurfaces which are the level sets of  $u$  and use  $\underline{C}_{\underline{u}}$  to denote the incoming null hypersurfaces which are the level sets of  $\underline{u}$ . We denote

$$S(u, \underline{u}) := C_u \cap \underline{C}_{\underline{u}}, \quad (2.1)$$

which are space-like 2-spheres. We introduce the vectorfields  $L$  and  $\underline{L}$  by

$$L := -\text{grad } u, \quad \underline{L} := -\text{grad } \underline{u}.$$

We define a positive function  $\Omega$  by the formula

$$\mathbf{g}(L, \underline{L}) = -\frac{1}{2\Omega^2},$$

where  $\Omega$  is called the lapse function. We then define the normalized null pair  $(e_3, e_4)$  by

$$e_3 = 2\Omega\underline{L}, \quad e_4 = 2\Omega L,$$

and define another null pair by

$$\underline{N} = \Omega e_3, \quad N = \Omega e_4.$$

On a given two sphere  $S(u, \underline{u})$ , we choose a local frame  $(e_1, e_2)$ , we call  $(e_1, e_2, e_3, e_4)$  a null frame. As a convention, throughout the paper, we use capital Latin letters  $A, B, C, \dots$  to denote an index from 1 to 2 and Greek letters  $\alpha, \beta, \gamma, \dots$  to denote an index from 1 to 4, e.g.  $e_A$  denotes either  $e_1$  or  $e_2$ .

The spacetime metric  $\mathbf{g}$  induces a Riemannian metric  $\gamma$  on  $S(u, \underline{u})$ . We use  $\nabla$  to denote the Levi-Civita connection of  $\gamma$  on  $S(u, \underline{u})$ . Using  $(u, \underline{u})$ , we introduce a local coordinate system  $(u, \underline{u}, \phi^A)$  on  $\mathcal{M}$  with  $e_4(\phi^A) = 0$ . In that coordinates system, the metric  $\mathbf{g}$  takes the form:

$$\mathbf{g} = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \gamma_{AB}(d\phi^A - \underline{b}^A du) \otimes (d\phi^B - \underline{b}^B du), \quad (2.2)$$

and we have

$$\underline{N} = \partial_u + \underline{b}, \quad N = \partial_{\underline{u}}, \quad \underline{b} := \underline{b}^A \partial_{\phi^A}.$$

We recall the null decomposition of the Ricci coefficients and curvature components of the null frame  $(e_1, e_2, e_3, e_4)$  as follows:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), & \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), \\ \underline{\xi}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_A), & \xi_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_A), \\ \underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_A), & \eta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_A), \\ \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_A} e_4, e_3), \end{aligned} \quad (2.3)$$

and

$$\begin{aligned}
\alpha_{AB} &= \mathbf{R}(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= \mathbf{R}(e_A, e_3, e_B, e_3), \\
\beta_A &= \frac{1}{2}\mathbf{R}(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A, e_3, e_3, e_4), \\
\rho &= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), & \sigma &= \frac{1}{4}{}^*\mathbf{R}(e_3, e_4, e_3, e_4),
\end{aligned} \tag{2.4}$$

where  ${}^*\mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ . The null second fundamental forms  $\chi, \underline{\chi}$  are further decomposed in their traces  $\text{tr } \chi$  and  $\text{tr } \underline{\chi}$ , and traceless parts  $\widehat{\chi}$  and  $\widehat{\underline{\chi}}$ :

$$\begin{aligned}
\text{tr } \chi &:= \delta^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \chi, \\
\text{tr } \underline{\chi} &:= \delta^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \underline{\chi}.
\end{aligned}$$

We define the horizontal covariant operator  $\nabla$  as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3.$$

We also define  $\nabla_4 X$  and  $\nabla_3 X$  to be the horizontal projections:

$$\begin{aligned}
\nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_4) e_3, \\
\nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_3) e_3 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_4) e_4.
\end{aligned}$$

A tensor field  $\psi$  defined on  $\mathcal{M}$  is called tangent to  $S$  if it is a priori defined on the spacetime  $\mathcal{M}$  and all the possible contractions of  $\psi$  with either  $e_3$  or  $e_4$  are zero. We use  $\nabla_3 \psi$  and  $\nabla_4 \psi$  to denote the projection to  $S(u, \underline{u})$  of usual derivatives  $\mathbf{D}_3 \psi$  and  $\mathbf{D}_4 \psi$ . As a direct consequence of (2.3), we have the Ricci formulas:

$$\begin{aligned}
\mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4, \\
\mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + \zeta_A e_3, \\
\mathbf{D}_A e_4 &= \chi_{AB} e_B - \zeta_A e_4, \\
\mathbf{D}_3 e_A &= \nabla_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, \\
\mathbf{D}_4 e_A &= \nabla_4 e_A + \underline{\eta}_A e_4 + \xi_A e_3, \\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_B e_B, \\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\underline{\eta}_B e_B, \\
\mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_B e_B, \\
\mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_B e_B.
\end{aligned} \tag{2.5}$$

In addition to

$$\xi = \underline{\xi} = 0, \tag{2.6}$$

the following identities hold for a double null foliation:

$$\begin{aligned}
\nabla \log \Omega &= \frac{1}{2}(\eta + \underline{\eta}), & \omega &= -\frac{1}{2} \mathbf{D}_4(\log \Omega), & \underline{\omega} &= -\frac{1}{2} \mathbf{D}_3(\log \Omega), \\
\eta &= \zeta + \nabla \log \Omega, & \underline{\eta} &= -\zeta + \nabla \log \Omega,
\end{aligned} \tag{2.7}$$

see for example (6) in [32].



### 2.1.2 Initial layer region

Let  $\Sigma_0$  be a spacelike hypersurface. Let  $K \subset \Sigma_0$  be a compact subset such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \overline{B_1}$  where  $\overline{B_1}$  is the unit closed ball in  $\mathbb{R}^3$ . We fix a foliation on the initial hypersurface  $\Sigma_0 \setminus K$  by the level sets of a scalar function  $w$ . The leaves are denoted by

$$S_{(0)}(w_1) = \{p \in \Sigma_0 / w(p) = w_1\}, \quad (2.8)$$

where  $w_1 \in \mathbb{R}$ . We assume that

$$\partial K = \{p \in \Sigma_0 / w(p) = w_0\}, \quad K = \{p \in \Sigma_0 / w(p) \leq w_0\}, \quad (2.9)$$

where  $w_0$  is the area radius of  $\partial K$  defined by

$$w_0 := \sqrt{\frac{|\partial K|}{4\pi}}. \quad (2.10)$$

We can construct a double null foliation in a neighborhood of  $\Sigma_0 \setminus K$ . For this, we denote  $T$  the normal vectorfield of  $\Sigma_0$  oriented towards the future and  $N$  the unit vectorfield tangent to  $\Sigma_0$ , oriented towards infinity and orthogonal to the leaves  $S_0(w)$ . We define two null vectors on  $\Sigma_0 \setminus K$  by

$$L_{(0)} := N(w)(T + N), \quad \underline{L} := N(w)(T - N). \quad (2.11)$$

We extend the definition of  $L_{(0)}$  and  $\underline{L}$  to a neighborhood of  $\Sigma_0 \setminus K$  by the geodesic equations:

$$\mathbf{D}_{L_{(0)}} L_{(0)} = 0, \quad \mathbf{D}_{\underline{L}} \underline{L} = 0. \quad (2.12)$$

The lapse function  $\Omega_{(0)}$  is defined by

$$\mathbf{g}(L_{(0)}, \underline{L}) = -\frac{1}{2\Omega_{(0)}^2}. \quad (2.13)$$

Then, we define two optical functions  $u_{(0)}$  and  $\underline{u}$  satisfying the initial conditions

$$u_{(0)}|_{\Sigma_0 \setminus K} = -w, \quad \underline{u}|_{\Sigma_0 \setminus K} = w,$$

and the equations

$$L_{(0)} = -\text{grad } u_{(0)}, \quad \underline{L} = -\text{grad } \underline{u}.$$

Recall that  $u_{(0)} + \underline{u} = 0$  on  $\Sigma_0 \setminus K$ , we define the region

$$\mathcal{K}_{(0)} := \{p / 0 \leq u_{(0)}(p) + \underline{u}(p) \leq 2\delta_0\}, \quad (2.14)$$

where  $0 < \delta_0 < 1$  is a constant. We call  $\mathcal{K}_{(0)}$  the *initial layer region* of height  $\delta_0$ , the double null foliation  $(u_{(0)}, \underline{u})$  is called the *initial layer foliation*. Its leaves are denoted by

$$S_{(0)}(u_{(0)}, \underline{u}) := (C_{(0)})_{u_{(0)}} \cap \underline{C}_{\underline{u}}.$$

We define

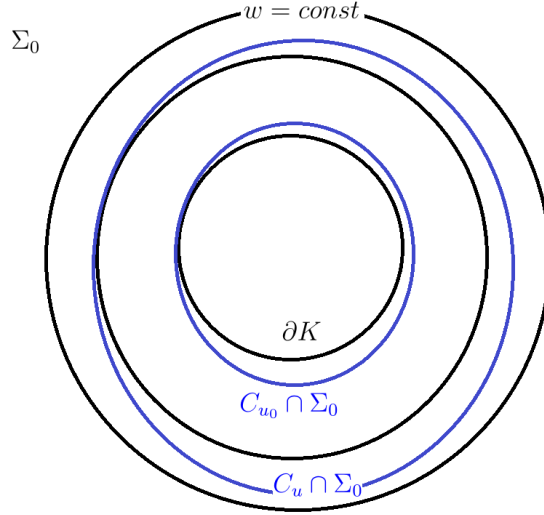
$$w := \frac{\underline{u} - u_{(0)}}{2} \quad \text{in } \mathcal{K}_{(0)}, \quad (2.15)$$

which extend the definition of  $w$  from  $\Sigma_0 \setminus K$  to  $\mathcal{K}_{(0)}$ . Moreover, we transport the coordinates  $\phi_{(0)}^A$  from  $\Sigma_0 \setminus K$  to  $\mathcal{K}_{(0)}$  by  $\underline{L}(\phi_{(0)}^A) = 0$ . Hence, we deduce that the metric  $\mathbf{g}$  in  $\mathcal{K}_{(0)}$  takes the following form in the  $(\underline{u}, u_{(0)}, \phi_{(0)}^A)$  coordinates system:

$$\mathbf{g} = -2\Omega_{(0)}^2 (d\underline{u} \otimes du_{(0)} + du_{(0)} \otimes d\underline{u}) + (\gamma_{(0)})_{AB} (d\phi_{(0)}^A - b^A d\underline{u})(d\phi_{(0)}^B - b^B d\underline{u}), \quad (2.16)$$

where

$$b^A = \Omega_{(0)}(e_4)_{(0)}(\phi_{(0)}^A).$$

Figure 2.1: The leaves of  $\Sigma_0$ 

### 2.1.3 Bootstrap region

Now, we define the bootstrap region and its double null foliation. We choose a value  $\underline{u}_* > w_0$ , then  $w = \underline{u}_*$  defines a leaf of  $\Sigma_0 \setminus K$ , denoted by  $S_0(\underline{u}_*)$ . We construct an incoming cone  $\underline{C}_*$  called the *last slice* from  $S_0(\underline{u}_*)$ . We denote

$$\underline{L}_* := \underline{L}|_{\underline{C}_*}, \quad (2.17)$$

which is fixed by (2.11) and (2.12). Notice that  $\underline{L}_*$  is a generator of  $\underline{C}_*$ . We denote  $u^*$  an affine parameter, that is a function defined on  $\underline{C}_*$  satisfying

$$\underline{L}_*(u^*) = 2. \quad (2.18)$$

Thus,  $\underline{C}_*$  is endowed with a geodesic foliation of affine parameter  $u^*$ . For every  $u$ ,  $\{u^* = u\}$  is a sphere on  $\underline{C}_*$ . We can therefore construct an outgoing cone from  $\{u^* = u\}$  to the past, denoted by  $C_u$ . Then,  $C_u \cap \Sigma_0$  is a sphere on  $\Sigma_0$ , which does a priori not coincide with the leaves of  $\Sigma_0 \setminus K$ , see Figure 2.1. There exists a unique cone  $C_{u_0}$  located outside of the  $J^+(K)$ , the causal future of  $K$ , but touch the boundary  $\partial K$ , see Figure 2.1. By (2.18),  $u^*$  is unique up to a constant. We choose  $u_0 = -w_0$  where  $w_0$  is defined in (2.10). Combining with (2.18), this fixes  $u^*$ .

**Remark 2.1.** *Remark that we fix  $u^*$  by choosing the value near  $\partial K$ . We do not have*

$$u^*|_{S_0(\underline{u}_*)} = -\underline{u}_*.$$

*The reason is that we need to integrate the quantities in the initial layer forward instead of backward. This will lead to a logarithm difference which is sufficient to prove the equivalences, see Appendix A.*

Now we define the null vector  $L$  on  $\underline{C}_*$  by the relation

$$\mathbf{g}(L, \underline{L}) = -2, \quad (2.19)$$

Hence, we have  $\Omega = \frac{1}{2}$  on  $\underline{C}_*$ . By the equations

$$-\text{grad } u = L, \quad u|_{\underline{C}_*} = u^*, \quad (2.20)$$

we can extend  $u$  to the causal past of  $\underline{C}_*$ . So we have constructed a double null foliation  $(u, \underline{u})$  which is geodesic on  $\underline{C}_*$ . For every  $(u, \underline{u})$ , we denote

$$V(u, \underline{u}) := J^-(S(u, \underline{u})) \cap J^+(\Sigma_0), \quad (2.21)$$

where for a domain  $\mathcal{M}$ ,  $J^+(\mathcal{M})$  and resp.  $J^-(\mathcal{M})$  denotes the causal future and resp. causal past of  $\mathcal{M}$ . We also denote the bootstrap region by

$$\mathcal{K} := V(u_0, \underline{u}_*). \quad (2.22)$$

Remark that  $\mathcal{K}$  is covered by the double null foliation  $(u, \underline{u})$ , and that the boundaries of  $\mathcal{K}$  are:

1. a finite region of  $\Sigma_0 \setminus K$ ;
2. a portion of an outgoing cone  $C_0$ , where  $C_0 := \{p \in \mathcal{K} | u(p) = u_0\}$ ;
3. a portion of the last slice  $\underline{C}_*$ .

Note that we have the following types of manifolds in this paper: spacetime regions  $\mathcal{K}$  and  $\mathcal{K}_{(0)}$ , initial hypersurface  $\Sigma_0$ , and the spheres  $S(u, \underline{u})$  and  $S_{(0)}(u_{(0)}, \underline{u})$ . Every manifold has its metric, Levi-Civita connection and curvature tensor:

$$(\mathcal{K}, \mathbf{g}, \mathbf{D}, \mathbf{R}), \quad (\mathcal{K}_{(0)}, \mathbf{g}_{(0)}, \mathbf{D}_{(0)}, \mathbf{R}_{(0)}), \quad (\Sigma_0, g, D, R), \quad (S, \gamma, \nabla, \mathbf{K}), \quad (S_{(0)}, \gamma_{(0)}, \nabla_{(0)}, \mathbf{K}_{(0)}),$$

where  $\mathbf{K}$  (resp.  $\mathbf{K}_{(0)}$ ) is the Gauss curvature of  $S$  (resp.  $S_{(0)}$ ).

## 2.2 Integral formulas

We define the  $S$ -average of scalar functions.

**Definition 2.2.** *Given any scalar function  $f$ , we denote its average and its average free part by*

$$\bar{f} := \frac{1}{|S|} \int_S f d\gamma, \quad \check{f} := f - \bar{f}.$$

The following lemma follows immediately from the definition.

**Lemma 2.3.** *For any two scalar functions  $u$  and  $v$ , we have*

$$\overline{uv} = \bar{u}\bar{v} + \overline{\check{u}\check{v}},$$

and

$$uv - \overline{uv} = \check{u}\bar{v} + \bar{u}\check{v} + (\check{u}\check{v} - \overline{\check{u}\check{v}}).$$

We recall the following integral formulas, which will be used repeatedly in this paper.

**Lemma 2.4.** *For any scalar function  $f$ , the following equations hold:*

$$\begin{aligned} \Omega e_4 \left( \int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (\Omega e_4(f) + \Omega \operatorname{tr} \chi f) d\gamma, \\ \Omega e_3 \left( \int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (\Omega e_3(f) + \Omega \operatorname{tr} \underline{\chi} f) d\gamma. \end{aligned}$$

Taking  $f = 1$ , we obtain

$$e_4(r) = \frac{\overline{\Omega \operatorname{tr} \chi}}{2\Omega} r, \quad e_3(r) = \frac{\overline{\Omega \operatorname{tr} \underline{\chi}}}{2\Omega} r,$$

where  $r$  is the area radius defined by

$$r(u, \underline{u}) := \sqrt{\frac{|S(u, \underline{u})|}{4\pi}}.$$

*Proof.* See Lemma 2.26 of [31] or Lemma 3.1.3 of [29].  $\square$

**Corollary 2.5.** *For any scalar function  $f$ , the following equations hold:*

$$\begin{aligned}\Omega e_4(\bar{f}) &= \overline{\Omega e_4(f)} + \overline{\Omega \operatorname{tr} \chi \check{f}}, \\ \Omega e_3(\bar{f}) &= \overline{\Omega e_3(f)} + \overline{\Omega \operatorname{tr} \underline{\chi} \check{f}}.\end{aligned}$$

*Proof.* Applying Lemma 2.4, we infer

$$\begin{aligned}\Omega e_4(\bar{f}) &= \Omega e_4\left(\frac{1}{|S|} \int_S f d\gamma\right) = -\frac{\Omega e_4(|S|)}{|S|^2} \int_S f d\gamma + \frac{1}{|S|} \int_S (\Omega e_4(f) + \Omega \operatorname{tr} \chi f) d\gamma \\ &= -\frac{2\Omega e_4(r)}{r} \bar{f} + \overline{\Omega e_4(f)} + \overline{\Omega \operatorname{tr} \chi f} = -\overline{\Omega \operatorname{tr} \chi \check{f}} + \overline{\Omega e_4(f)} + \overline{\Omega \operatorname{tr} \chi f} \\ &= \overline{\Omega e_4(f)} + \overline{\Omega \operatorname{tr} \chi \check{f}},\end{aligned}$$

which implies the first identity. The second identity can be obtained similarly. This concludes the proof of Corollary 2.5.  $\square$

### 2.3 Hodge systems

**Definition 2.6.** *For tensor fields defined on a 2-sphere  $S$ , we denote by  $\mathfrak{s}_0 := \mathfrak{s}_0(S)$  the set of pairs of scalar functions,  $\mathfrak{s}_1 := \mathfrak{s}_1(S)$  the set of 1-forms and  $\mathfrak{s}_2 := \mathfrak{s}_2(S)$  the set of symmetric traceless 2-tensors.*

**Definition 2.7.** *Given  $\xi \in \mathfrak{s}_1$ , we define its Hodge dual*

$$*\xi_A := \epsilon_{AB} \xi^B.$$

Clearly  $*\xi \in \mathfrak{s}_1$  and

$$*(*\xi) = -\xi.$$

Given  $U \in \mathfrak{s}_2$ , we define its Hodge dual

$$*U_{AB} := \epsilon_{AC} U^C{}_B.$$

Observe that  $*U \in \mathfrak{s}_2$  and

$$*(*U) = -U.$$

**Definition 2.8.** *Given  $\xi, \eta \in \mathfrak{s}_1$ , we denote*

$$\begin{aligned}\xi \cdot \eta &:= \delta^{AB} \xi_A \eta_B, \\ \xi \wedge \eta &:= \epsilon^{AB} \xi_A \eta_B, \\ (\widehat{\xi \otimes \eta})_{AB} &:= \xi_A \eta_B + \xi_B \eta_A - \delta_{AB} \xi \cdot \eta.\end{aligned}$$

Given  $\xi \in \mathfrak{s}_1$ ,  $U \in \mathfrak{s}_2$ , we denote

$$(\xi \cdot U)_A := \delta^{BC} \xi_B U_{AC}.$$

Given  $U, V \in \mathfrak{s}_2$ , we denote

$$(U \wedge V)_{AB} := \epsilon^{AB} U_{AC} V_{CB}.$$

**Definition 2.9.** *For a given  $\xi \in \mathfrak{s}_1$ , we define the following differential operators:*

$$\begin{aligned}\operatorname{div} \xi &:= \delta^{AB} \nabla_A \xi_B, \\ \operatorname{curl} \xi &:= \epsilon^{AB} \nabla_A \xi_B, \\ (\widehat{\nabla \otimes \xi})_{AB} &:= \nabla_A \xi_B + \nabla_B \xi_A - \delta_{AB} (\operatorname{div} \xi).\end{aligned}$$

**Definition 2.10.** We define the following Hodge type operators, as introduced in section 2.2 in [10]:

- $d_1$  takes  $\mathfrak{s}_1$  into  $\mathfrak{s}_0$  and is given by:

$$d_1\xi := (\operatorname{div} \xi, \operatorname{curl} \xi),$$

- $d_2$  takes  $\mathfrak{s}_2$  into  $\mathfrak{s}_1$  and is given by:

$$(d_2U)_A := \nabla^B U_{AB},$$

- $d_1^*$  takes  $\mathfrak{s}_0$  into  $\mathfrak{s}_1$  and is given by:

$$d_1^*(f, f_*)_A := -\nabla_A f + \epsilon_{AB} \nabla_B f_*,$$

- $d_2^*$  takes  $\mathfrak{s}_1$  into  $\mathfrak{s}_2$  and is given by:

$$d_2^*\xi := -\frac{1}{2}\nabla\widehat{\otimes}\xi.$$

We have the following identities:

$$\begin{aligned} d_1^*d_1 &= -\Delta_1 + \mathbf{K}, & d_1d_1^* &= -\Delta_0, \\ d_2^*d_2 &= -\frac{1}{2}\Delta_2 + \mathbf{K}, & d_2d_2^* &= -\frac{1}{2}(\Delta_1 + \mathbf{K}). \end{aligned} \quad (2.23)$$

where  $\mathbf{K}$  denotes the Gauss curvature on  $S$ . See for example (2.2.2) in [10] for a proof of (2.23).

**Definition 2.11.** We define the weighted angular derivatives  $\wp$  as follows:

$$\begin{aligned} \wp U &:= rd_2U, & \forall U \in \mathfrak{s}_2, \\ \wp \xi &:= rd_1\xi, & \forall \xi \in \mathfrak{s}_1, \\ \wp f &:= rd_1^*f, & \forall f \in \mathfrak{s}_0. \end{aligned}$$

We denote for any tensor  $h \in \mathfrak{s}_k$ ,  $k = 0, 1, 2$ ,

$$h^{(0)} := h, \quad h^{(1)} := (h, \wp h).$$

## 2.4 Elliptic estimates for Hodge systems

**Definition 2.12.** For a tensor field  $f$  on a 2-sphere  $S$ , we denote its  $L^p$ -norm:

$$|f|_{p,S} := \left( \int_S |f|^p \right)^{\frac{1}{p}}. \quad (2.24)$$

**Proposition 2.13** ( $L^p$  estimates for Hodge systems). Assume that  $S$  is an arbitrary compact 2-surface with positive bounded Gauss curvature. Then the following statements hold for all  $p \in (1, +\infty)$ :

1. Let  $\phi \in \mathfrak{s}_0$  be a solution of  $\Delta\phi = f$ . Then we have

$$|\nabla^2\phi|_{p,S} + r^{-1}|\nabla\phi|_{p,S} + r^{-2}|\phi - \bar{\phi}|_{p,S} \lesssim |f|_{p,S}.$$

2. Let  $\xi \in \mathfrak{s}_1$  be a solution of  $d_1\xi = (f, f_*)$ . Then we have

$$|\nabla\xi|_{p,S} + r^{-1}|\xi|_{p,S} \lesssim |(f, f_*)|_{p,S}.$$

3. Let  $U \in \mathfrak{s}_2$  be a solution of  $d_2U = f$ . Then we have

$$|\nabla U|_{p,S} + r^{-1}|U|_{p,S} \lesssim |f|_{p,S}.$$

*Proof.* See Corollary 2.3.1.1 of [10]. □

## 2.5 Schematic notation $\Gamma_g$ , $\Gamma_b$ and $\Gamma_a$

We introduce the following schematic notations for the Ricci coefficients.

**Definition 2.14.** *We divide the Ricci coefficients into three parts:*

$$\begin{aligned}\Gamma_g &:= \{\eta, \underline{\eta}, \zeta, \widehat{\chi}, \overline{\Omega \operatorname{tr} \chi}, \overline{\Omega \operatorname{tr} \underline{\chi}}, \check{\omega}, \nabla \log \Omega\}, \\ \Gamma_b &:= \{\widehat{\underline{\chi}}, \underline{\check{\omega}}\}, \\ \Gamma_a &:= \left\{ \omega, \underline{\omega}, \operatorname{tr} \chi - \frac{2}{r}, \operatorname{tr} \underline{\chi} + \frac{2}{r} \right\}.\end{aligned}$$

We also denote:

$$\Gamma_g^{(1)} := (r\nabla)^{\leq 1} \Gamma_g \cup \{r\beta, r\check{\rho}, r\sigma\}, \quad \Gamma_b^{(1)} := (r\nabla)^{\leq 1} \Gamma_b \cup \{r\underline{\beta}\}, \quad \Gamma_a^{(1)} := (r\nabla)^{\leq 1} \Gamma_a \cup \{r\bar{\rho}\}.$$

**Remark 2.15.** *The justification of Definition 2.14 has to do with the expected decay properties of the Ricci coefficients, see Lemma 4.1.*

**Remark 2.16.** *In the sequel, we choose the following conventions:*

- For a quantity  $h$  satisfying the same or even better decay as  $\Gamma_i$ , for  $i = g, b, a$ , we write

$$h \in \Gamma_i, \quad i = g, b, a.$$

- For a sum of schematic notations, we ignore the terms which have same or even better decay. For example, we write

$$\Gamma_g + \Gamma_b = \Gamma_b, \quad \Gamma_g + \Gamma_a = \Gamma_a,$$

since  $\Gamma_g$  has better decay than  $\Gamma_b$  and  $\Gamma_a$  throughout this paper.

## 2.6 Null structure equations

We recall the null structure equations for a double null foliation, see for example (3.1)-(3.5) of [32].

**Proposition 2.17.** *We have the null structure equations:*

$$\begin{aligned}\nabla_4 \eta &= -\chi \cdot (\eta - \underline{\eta}) - \beta, \\ \nabla_3 \underline{\eta} &= -\underline{\chi} \cdot (\underline{\eta} - \eta) + \underline{\beta}, \\ \nabla_4 \widehat{\chi} + (\operatorname{tr} \chi) \widehat{\chi} &= -2\omega \widehat{\chi} - \alpha, \\ \nabla_4 \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \chi)^2 &= -|\widehat{\chi}|^2 - 2\omega \operatorname{tr} \chi, \\ \nabla_3 \widehat{\underline{\chi}} + (\operatorname{tr} \underline{\chi}) \widehat{\underline{\chi}} &= -2\underline{\omega} \widehat{\underline{\chi}} - \underline{\alpha}, \\ \nabla_3 \operatorname{tr} \underline{\chi} + \frac{1}{2} (\operatorname{tr} \underline{\chi})^2 &= -|\widehat{\underline{\chi}}|^2 - 2\underline{\omega} \operatorname{tr} \underline{\chi}, \\ \nabla_4 \widehat{\underline{\chi}} + \frac{1}{2} (\operatorname{tr} \chi) \widehat{\underline{\chi}} &= \nabla \widehat{\otimes} \underline{\eta} + 2\omega \widehat{\underline{\chi}} - \frac{1}{2} (\operatorname{tr} \underline{\chi}) \widehat{\underline{\chi}} + \underline{\eta} \widehat{\otimes} \eta, \\ \nabla_3 \widehat{\chi} + \frac{1}{2} (\operatorname{tr} \underline{\chi}) \widehat{\chi} &= \nabla \widehat{\otimes} \eta + 2\underline{\omega} \widehat{\chi} - \frac{1}{2} (\operatorname{tr} \chi) \widehat{\chi} + \eta \widehat{\otimes} \underline{\eta}, \\ \nabla_4 \operatorname{tr} \underline{\chi} + \frac{1}{2} (\operatorname{tr} \chi) \operatorname{tr} \underline{\chi} &= 2\omega \operatorname{tr} \underline{\chi} + 2\rho - \widehat{\chi} \cdot \widehat{\underline{\chi}} + 2 \operatorname{div} \underline{\eta} + 2|\underline{\eta}|^2, \\ \nabla_3 \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \underline{\chi}) \operatorname{tr} \chi &= 2\underline{\omega} \operatorname{tr} \chi + 2\rho - \widehat{\underline{\chi}} \cdot \widehat{\chi} + 2 \operatorname{div} \eta + 2|\eta|^2.\end{aligned}\tag{2.25}$$

We have the Codazzi equations:

$$\begin{aligned}\operatorname{div} \widehat{\chi} &= \frac{1}{2} \nabla \operatorname{tr} \chi - \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \operatorname{tr} \chi \right) - \underline{\beta}, \\ \operatorname{div} \widehat{\underline{\chi}} &= \frac{1}{2} \nabla \operatorname{tr} \underline{\chi} + \zeta \cdot \left( \widehat{\underline{\chi}} - \frac{1}{2} \operatorname{tr} \underline{\chi} \right) + \underline{\beta},\end{aligned}\tag{2.26}$$

the torsion equation:

$$\operatorname{curl} \eta = -\operatorname{curl} \underline{\eta} = \sigma + \frac{1}{2} \widehat{\underline{\chi}} \wedge \widehat{\chi},\tag{2.27}$$

and the Gauss equation:

$$\mathbf{K} = -\frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} + \frac{1}{2} \widehat{\underline{\chi}} \cdot \widehat{\chi} - \rho.\tag{2.28}$$

Moreover,

$$\begin{aligned}\nabla_4 \underline{\omega} &= 2\omega \underline{\omega} + \frac{3}{4} |\eta - \underline{\eta}|^2 - \frac{1}{4} (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8} |\eta + \underline{\eta}|^2 + \frac{1}{2} \rho, \\ \nabla_3 \omega &= 2\omega \underline{\omega} + \frac{3}{4} |\eta - \underline{\eta}|^2 + \frac{1}{4} (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8} |\eta + \underline{\eta}|^2 + \frac{1}{2} \rho.\end{aligned}\tag{2.29}$$

*Proof.* See Section 7.4 of [10]. □

## 2.7 Bianchi equations

We recall the Bianchi equations in double null foliation, see Proposition 3.2.4 of [29].

**Proposition 2.18.** *In a double null frame, the Bianchi equations take the form*

$$\begin{aligned}\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + h[\underline{\alpha}_4], \\ h[\underline{\alpha}_4] &= 4\omega \underline{\alpha} - 3(\widehat{\underline{\chi}} \rho - {}^* \widehat{\underline{\chi}} \sigma) + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta}, \\ \nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -\operatorname{div} \underline{\alpha} + h[\underline{\beta}_3], \\ h[\underline{\beta}_3] &= -2\omega \underline{\beta} + (2\zeta - \eta) \cdot \underline{\alpha}, \\ \nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\nabla \rho + {}^* \nabla \sigma + h[\underline{\beta}_4], \\ h[\underline{\beta}_4] &= 2\omega \underline{\beta} + 2\widehat{\underline{\chi}} \cdot \underline{\beta} - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma), \\ \nabla_3 \rho + \frac{3}{2} \operatorname{tr} \underline{\chi} \rho &= -\operatorname{div} \underline{\beta} + h[\rho_3], \\ h[\rho_3] &= -\frac{1}{2} \widehat{\underline{\chi}} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} - 2\underline{\eta} \cdot \underline{\beta}, \\ \nabla_4 \rho + \frac{3}{2} \operatorname{tr} \chi \rho &= \operatorname{div} \beta + h[\rho_4], \\ h[\rho_4] &= -\frac{1}{2} \widehat{\underline{\chi}} \cdot \alpha + \zeta \cdot \beta + 2\underline{\eta} \cdot \beta, \\ \nabla_3 \sigma + \frac{3}{2} \operatorname{tr} \underline{\chi} \sigma &= -\operatorname{div} {}^* \underline{\beta} + h[\sigma_3], \\ h[\sigma_3] &= \frac{1}{2} \widehat{\underline{\chi}} \cdot {}^* \underline{\alpha} - \zeta \cdot {}^* \underline{\beta} + 2\underline{\eta} \cdot {}^* \underline{\beta}, \\ \nabla_4 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{div} {}^* \beta + h[\sigma_4], \\ h[\sigma_4] &= \frac{1}{2} \widehat{\underline{\chi}} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2\underline{\eta} \cdot {}^* \beta, \\ \nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= \nabla \rho + {}^* \nabla \sigma + h[\beta_3], \\ h[\beta_3] &= 2\omega \beta + 2\widehat{\underline{\chi}} \cdot \underline{\beta} + 3(\underline{\eta} \rho + {}^* \underline{\eta} \sigma),\end{aligned}$$

$$\begin{aligned}
\nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= \operatorname{div} \alpha + h[\beta_4], \\
h[\beta_4] &= -2\omega\beta + (2\zeta + \underline{\eta})\alpha, \\
\nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \alpha &= \nabla \widehat{\otimes} \beta + h[\alpha_3], \\
h[\alpha_3] &= 4\underline{\omega}\alpha - 3(\widehat{\chi}\rho + {}^* \widehat{\chi}\sigma) + (\zeta + 4\underline{\eta})\widehat{\otimes} \beta.
\end{aligned}$$

We also derive equations for  $\check{\rho}$  and  $\bar{\rho}$ .

**Lemma 2.19.** *We have the following equations:*

$$\begin{aligned}
\nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\nabla \check{\rho} + {}^* \nabla \sigma + h[\underline{\beta}_4], \\
\nabla_3 \underline{\beta} + \operatorname{tr} \underline{\chi} \underline{\beta} &= \nabla \check{\rho} + {}^* \nabla \sigma + h[\underline{\beta}_3], \\
\nabla_3 \check{\rho} + \frac{3}{2} \operatorname{tr} \underline{\chi} \check{\rho} &= -\operatorname{div} \underline{\beta} + h[\check{\rho}_3], \\
h[\check{\rho}_3] &:= \Gamma_g \cdot \underline{\beta} + h[\rho_3] - \overline{h[\rho_3]}, \\
\nabla_3 \bar{\rho} + \frac{3}{2} \operatorname{tr} \underline{\chi} \bar{\rho} &= h[\bar{\rho}_3], \\
h[\bar{\rho}_3] &:= \Gamma_g \cdot \underline{\beta} + \overline{h[\rho_3]}, \\
\nabla_4 \check{\rho} + \frac{3}{2} \operatorname{tr} \chi \check{\rho} &= \operatorname{div} \beta + h[\check{\rho}_4], \\
h[\check{\rho}_4] &:= \Gamma_g \cdot \check{\rho} + h[\rho_4] - \overline{h[\rho_4]}, \\
\nabla_4 \bar{\rho} + \frac{3}{2} \operatorname{tr} \chi \bar{\rho} &= h[\bar{\rho}_4], \\
h[\bar{\rho}_4] &:= \Gamma_g \cdot \check{\rho} + \overline{h[\rho_4]},
\end{aligned}$$

where  $h[\underline{\beta}_4]$ ,  $h[\underline{\beta}_3]$ ,  $h[\rho_3]$  and  $h[\rho_4]$  are defined in Proposition 2.18.

*Proof.* Note that  $\nabla \check{\rho} = \nabla \rho$ , hence the first two equations follow directly from Proposition 2.18. On the other hand, applying Corollary 2.5, we infer

$$\begin{aligned}
\Omega \nabla_3 \bar{\rho} &= \overline{\Omega \operatorname{tr} \underline{\chi} \check{\rho}} + \overline{\Omega \nabla_3 \rho} \\
&= -\frac{3}{2} \Omega \operatorname{tr} \underline{\chi} \rho - \Omega \operatorname{div} \underline{\beta} + h[\rho_3] + \Gamma_g \cdot \check{\rho} \\
&= -\frac{3}{2} \Omega \operatorname{tr} \underline{\chi} \bar{\rho} + \overline{h[\rho_3]} + \underline{\beta} \cdot \nabla \Omega + \Gamma_g \cdot \check{\rho}, \\
&= -\frac{3}{2} \Omega \operatorname{tr} \underline{\chi} \bar{\rho} + \overline{h[\rho_3]} + \Gamma_g \cdot \underline{\beta}.
\end{aligned}$$

So, we have

$$\nabla_3 \bar{\rho} + \frac{3}{2} \operatorname{tr} \underline{\chi} \bar{\rho} = \Gamma_g \cdot \underline{\beta} + \overline{h[\rho_3]},$$

which implies the fourth equation. Combining with the equation of  $\nabla_3 \rho$  in Proposition 2.18, we infer

$$\nabla_3 \check{\rho} + \frac{3}{2} \operatorname{tr} \underline{\chi} \check{\rho} = -\operatorname{div} \underline{\beta} + \Gamma_g \cdot \underline{\beta} + h[\rho_3] - \overline{h[\rho_3]},$$

which implies the third equation. The proof of the fifth and sixth equations are similar. This concludes the proof of Lemma 2.19.  $\square$

We rewrite the Bianchi equations of Proposition 2.18 and Lemma 2.19 in the following Corollary.



**Corollary 2.20.**

$$\begin{aligned}
\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + h[\underline{\alpha}_4], \\
\nabla_3 \underline{\beta} + 2 \operatorname{tr} \chi \underline{\beta} &= -\operatorname{div} \underline{\alpha} + h[\underline{\beta}_3], \\
\nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\nabla \check{\rho} + {}^* \nabla \sigma + h[\underline{\beta}_4], \\
\nabla_3 \check{\rho} + \frac{3}{2} \operatorname{tr} \chi \check{\rho} &= -\operatorname{div} \underline{\beta} + h[\check{\rho}_3], \\
\nabla_4 \check{\rho} + \frac{3}{2} \operatorname{tr} \chi \check{\rho} &= \operatorname{div} \beta + h[\check{\rho}_4], \\
\nabla_3 \bar{\rho} + \frac{3}{2} \operatorname{tr} \chi \bar{\rho} &= h[\bar{\rho}_3], \\
\nabla_4 \bar{\rho} + \frac{3}{2} \operatorname{tr} \chi \bar{\rho} &= h[\bar{\rho}_4], \\
\nabla_3 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{div} {}^* \underline{\beta} + h[\sigma_3], \\
\nabla_4 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{div} {}^* \beta + h[\sigma_4], \\
\nabla_3 \beta + \operatorname{tr} \chi \beta &= \nabla \check{\rho} + {}^* \nabla \sigma + h[\beta_3], \\
\nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= \operatorname{div} \alpha + h[\beta_4], \\
\nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha &= \nabla \widehat{\otimes} \beta + h[\alpha_3],
\end{aligned}$$

where  $h[\underline{\alpha}_4], h[\underline{\beta}_3], \dots, h[\alpha_3]$  are defined in Proposition 2.18 and Lemma 2.19.

## 2.8 Commutation identities

We recall the following commutation identities for scalar functions, see Proposition 4.8.1 in [29].

**Proposition 2.21.** *For any scalar function  $f$ , we have:*

$$\begin{aligned}
[\nabla_4, \nabla]f &= -\chi \cdot \nabla f + (\nabla \log \Omega) \nabla_4 f, \\
[\nabla_3, \nabla]f &= -\underline{\chi} \cdot \nabla f + (\nabla \log \Omega) \nabla_3 f, \\
[\nabla_4, \nabla_3]f &= 2\omega \nabla_3 f - 2\underline{\omega} \nabla_4 f - 4\zeta \cdot \nabla f.
\end{aligned}$$

The following corollary is a direct consequence of Proposition 2.21.

**Corollary 2.22.** *For any scalar function  $f$ , we have:*

$$\begin{aligned}
[\Omega \nabla_4, \nabla]f &= -\Omega \chi \cdot \nabla f, \\
[\Omega \nabla_3, \nabla]f &= -\Omega \underline{\chi} \cdot \nabla f, \\
[\Omega \nabla_4, r \nabla]f &= \Gamma_g \cdot r \nabla f, \\
[\Omega \nabla_3, r \nabla]f &= \Gamma_b \cdot r \nabla f.
\end{aligned}$$

We also need the commutation identities for more general tensor fields. For this, we record the following commutation lemma.

**Lemma 2.23.** *Let  $U_{A_1 \dots A_k}$  be an  $S$ -tangent  $k$ -covariant tensor on  $(\mathcal{M}, \mathbf{g})$ . Then*

$$[\nabla_4, \nabla_B]U_{A_1 \dots A_k} = -\chi_{BC} \nabla_C U_{A_1 \dots A_k} + F_{4BA_1 \dots A_k},$$

$$F_{4BA_1 \dots A_k} := (\zeta_B + \underline{\eta}_B) \nabla_4 U_{A_1 \dots A_k} + \sum_{i=1}^k (\chi_{A_i B} \underline{\eta}_C - \chi_{BC} \underline{\eta}_{A_i} + \epsilon_{A_i C} \beta_B) U_{A_1 \dots C \dots A_k},$$

$$[\nabla_3, \nabla_B]U_{A_1 \dots A_k} = -\underline{\chi}_{BC} \nabla_C U_{A_1 \dots A_k} + F_{3BA_1 \dots A_k},$$

$$F_{3BA_1 \dots A_k} := (\eta_B - \zeta_B) \nabla_3 U_{A_1 \dots A_k} + \sum_{i=1}^k (\underline{\chi}_{A_i B} \eta_C - \underline{\chi}_{BC} \eta_{A_i} + \epsilon_{A_i C} \beta_B) U_{A_1 \dots C \dots A_k},$$

$$[\nabla_3, \nabla_4]U_{A_1 \dots A_k} = F_{34A_1 \dots A_k},$$

$$F_{34A_1 \dots A_k} := -2\omega \nabla_3 U + 2\underline{\omega} \nabla_4 U + 4\zeta_B \nabla_B U_{A_1 \dots A_k} + 2 \sum_{i=1}^k (\underline{\eta}_{A_i} \eta_C - \underline{\eta}_{A_i} \eta_C + \epsilon_{A_i C} \sigma) U_{A_1 \dots C \dots A_k}.$$

*Proof.* It is a direct consequence of Lemma 7.3.3 in [10].  $\square$

Notice that Lemma 2.23 implies the following proposition.

**Proposition 2.24.** *We have the following simple schematic consequences of the commutator identities*

$$\begin{aligned} [r\nabla, \Omega\nabla_4] &= \Gamma_g \cdot r\nabla + \Gamma_g^{(1)}, \\ [r\nabla, \Omega\nabla_3] &= \Gamma_b \cdot r\nabla + \Gamma_b^{(1)}, \\ [\Omega\nabla_4, \Omega\nabla_3] &= \Gamma_g \cdot \nabla + r^{-1}\Gamma_g^{(1)}. \end{aligned}$$

## 2.9 Teukolsky equations

We state the following *Teukolsky equations* first derived by Teukolsky in the linearized setting in [58].

**Proposition 2.25.** *We have the following Teukolsky equations for  $\alpha$  and  $\underline{\alpha}$ :*

$$\begin{aligned} \Omega\nabla_3(\Omega\nabla_4(r\Omega^2\alpha)) + 2r\Omega^2 d_2^* d_2(\Omega^2\alpha) &= -\frac{2\Omega}{r} \nabla_3(r\Omega^2\alpha) + (\Gamma_g \cdot (\beta, \alpha))^{(1)}, \\ \Omega\nabla_4(\Omega\nabla_3(r\Omega^2\underline{\alpha})) + 2r\Omega^2 d_2^* d_2(\Omega^2\underline{\alpha}) &= \frac{2\Omega}{r} \nabla_4(r\Omega^2\underline{\alpha}) + (\Gamma_b \cdot (\underline{\alpha}, \underline{\beta}))^{(1)}. \end{aligned} \tag{2.30}$$

*Proof.* See for example Propositions 3.4.6 and 3.4.7 in [11].<sup>4</sup>  $\square$

**Lemma 2.26.** *We define the following quantities:*

$$\begin{aligned} \dot{\alpha} &:= \frac{1}{r^4} \nabla_4(r^5\alpha) \in \mathfrak{s}_2, & \phi &:= rd_2\alpha \in \mathfrak{s}_1, \\ \dot{\underline{\alpha}} &:= \frac{1}{r^4} \nabla_3(r^5\underline{\alpha}) \in \mathfrak{s}_2, & \underline{\phi} &:= rd_2\underline{\alpha} \in \mathfrak{s}_1. \end{aligned} \tag{2.31}$$

*Then, we have the following equations:*

$$\begin{aligned} \nabla_3 \dot{\alpha} &= -2d_2^* \phi + \frac{4\alpha}{r} + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha), \\ \nabla_4 \phi + \frac{5}{2} \operatorname{tr} \chi \phi &= d_2 \dot{\alpha} + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g \cdot \dot{\alpha} + \Gamma_g^{(1)} \cdot (\beta, \alpha), \end{aligned} \tag{2.32}$$

<sup>4</sup>Remark that  $\Omega = 1 + O(\varepsilon)$  in [11] while  $\Omega = \frac{1}{2} + O(\varepsilon)$  in this paper.

and

$$\begin{aligned}\nabla_4 \dot{\underline{\alpha}} &= -2d_2^* \underline{\phi} + \frac{4\underline{\alpha}}{r} + \Gamma_a \cdot (\underline{\beta}, \underline{\alpha})^{(1)} + (\Gamma_b \cdot (\underline{\beta}, \underline{\alpha}))^{(1)}, \\ \nabla_3 \underline{\phi} + \frac{5}{2} \operatorname{tr} \chi \underline{\phi} &= d_2 \dot{\underline{\alpha}} + \Gamma_a \cdot (\underline{\beta}, \underline{\alpha})^{(1)} + \Gamma_b \cdot \dot{\underline{\alpha}} + (\Gamma_b \cdot (\underline{\beta}, \underline{\alpha}))^{(1)}.\end{aligned}\tag{2.33}$$

*Proof.* We have from (2.30)

$$\nabla_3(\nabla_4(r\alpha)) + 2rd_2^*d_2\alpha = -\frac{4}{r}\nabla_3(r\alpha) + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha).$$

Hence, we have

$$\begin{aligned}\nabla_3 \dot{\underline{\alpha}} &= \nabla_3(r^{-4}\nabla_4(r^5\alpha)) \\ &= \nabla_3(r^{-4}(4r^3e_4(r)r\alpha + r^4\nabla_4(r\alpha))) \\ &= \nabla_3(4\alpha + \nabla_4(r\alpha)) + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha) \\ &= \frac{4}{r}\nabla_3\alpha + \nabla_3\nabla_4(r\alpha) + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha) \\ &= \frac{4}{r}\nabla_3(r\alpha) - 4e_3(r)\frac{\alpha}{r} - 2rd_2^*d_2\alpha - \frac{4}{r}\nabla_3(r\alpha) + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha) \\ &= \frac{4\alpha}{r} - 2d_2^*\underline{\phi} + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha),\end{aligned}$$

which implies the first equation in (2.32). Applying Proposition 2.24, we have

$$\begin{aligned}\Omega\nabla_4\underline{\phi} &= \Omega\nabla_4(r^{-5}(rd_2(r^5\alpha))) \\ &= -5r^{-6}\Omega e_4(r)r^5(rd_2)\alpha + r^{-5}\Omega\nabla_4rd_2(r^5\alpha) \\ &= -\frac{5\Omega}{r}rd_2\alpha + r^{-1}rd_2r^{-4}(\Omega\nabla_4(r^5\alpha)) + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g^{(1)} \cdot (\beta, \alpha) \\ &= -\frac{5}{2}\Omega \operatorname{tr} \chi \underline{\phi} + \Omega d_2 \dot{\underline{\alpha}} + \Gamma_a \cdot (\beta, \alpha)^{(1)} + \Gamma_g \cdot \dot{\underline{\alpha}} + \Gamma_g^{(1)} \cdot (\beta, \alpha),\end{aligned}$$

which implies the second equation in (2.32). The proof of (2.33) is similar. This concludes the proof of Lemma 2.26.  $\square$

## 2.10 Sobolev inequalities

**Definition 2.27.** For a tensor field  $h$  on a null cone  $C$ , we define its  $L^2$ -flux:

$$\|h\|_{2,C} := \left( \int_C |h|^2 \right)^{\frac{1}{2}}.\tag{2.34}$$

We recall the following Sobolev inequalities.

**Proposition 2.28.** Let  $F$  be a tensor field, tangent to  $S := S(u, \underline{u})$  at every point. We have the following estimates:

$$\begin{aligned}|rF|_{4,S} &\lesssim \|F\|_{2,C_u \cap V(u, \underline{u})} + \|r\nabla F\|_{2,C_u \cap V(u, \underline{u})} + \|r\nabla_4 F\|_{2,C_u \cap V(u, \underline{u})}, \\ |r^{\frac{1}{2}}|u|^{\frac{1}{2}}F|_{4,S} &\lesssim \|F\|_{2,\underline{C}_u \cap V(u, \underline{u})} + \|r\nabla F\|_{2,\underline{C}_u \cap V(u, \underline{u})} + |u|\|\nabla_3 F\|_{2,\underline{C}_u \cap V(u, \underline{u})}.\end{aligned}$$

*Proof.* See Corollary 3.2.1.1 in [10] and Corollary 4.1.1 in [29].  $\square$

We also recall the following standard Sobolev inequalities.

**Proposition 2.29.** Let  $F$  be a tensor field on a 2-sphere  $S$ . Then, we have

$$\sup_S r^{\frac{1}{2}}|F| \lesssim |F|_{4,S} + |r\nabla F|_{4,S}.$$

*Proof.* See Lemma 4.1.3 of [29].  $\square$

### 2.11 Evolution lemma

We recall the following evolution lemma, which will be used repeatedly in Sections 5 and 6.3.

**Lemma 2.30** (Evolution lemma). *Consider the spacetime  $\mathcal{K}$  foliated by a double null foliation  $(u, \underline{u})$ . Assume that, for  $\varepsilon > 0$  small enough, we have*

$$|\Gamma_g| \leq \frac{\varepsilon}{r^2}. \quad (2.35)$$

Then, the following holds:

1. Let  $U, F$  be  $k$ -covariant  $S$ -tangent tensor fields satisfying the outgoing evolution equation

$$\nabla_4 U + \lambda_0 \operatorname{tr} \chi U = F, \quad (2.36)$$

where  $\lambda_0 \geq 0$ . Denoting  $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$ , we have along  $C_u$

$$|r^{\lambda_1} U|_{p,S}(u, \underline{u}) \lesssim |r^{\lambda_1} U|_{p,S}(u, \underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1} F|_{p,S}(u, \underline{u}') d\underline{u}'. \quad (2.37)$$

Here  $\underline{u}_*$  is the value that the function  $\underline{u}(p)$  assumes on  $\underline{C}_*$ .

2. Let  $V, \underline{F}$  be  $k$ -covariant and  $S$ -tangent tensor fields satisfying the incoming evolution equation

$$\nabla_3 V + \lambda_0 \operatorname{tr} \underline{\chi} V = \underline{F}. \quad (2.38)$$

Denoting  $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$ , we have along  $\underline{C}_u$

$$|r^{\lambda_1} V|_{p,S}(u, \underline{u}) \lesssim |r^{\lambda_1} V|_{p,S}(u_0(\underline{u}), \underline{u}) + \int_{u_0(\underline{u})}^u |r^{\lambda_1} \underline{F}|_{p,S}(u', \underline{u}) du', \quad (2.39)$$

where  $S(u_0(\underline{u}), \underline{u}) = C_{u_0(\underline{u})} \cap \underline{C}_u$  and  $u_0(\underline{u})$  is the unique value of  $u$  such that  $S(u, \underline{u})$  is in the future of  $\Sigma_0$  and touches  $\Sigma_0$ .

*Proof.* See Lemma 4.1.5 in [29]. □

## 3 Main theorem

### 3.1 Fundamental norms

Our result holds for  $s$ -asymptotically flat initial data sets in the sense of Definition 1.2 with  $s > 3$ . We will focus on the case  $s \in [4, 6]$  in sections 3 to 6, and we postpone the necessary adaptations to the case  $s \in (3, 4)$  and  $s > 6$  to Appendices B and C. The norms in sections 3.1.1-3.1.4 are defined with respect to the foliation  $(u, \underline{u})$  in  $\mathcal{K}$  while the norms in section 3.1.5 are defined with respect to the foliation  $(u_{(0)}, \underline{u})$  in  $\mathcal{K}_{(0)}$ .

#### 3.1.1 $\mathcal{R}$ norms (curvature)

We define the norms of Ricci coefficients in the bootstrap region  $\mathcal{K}$ . We denote

$$\mathcal{R}_0^S := \mathcal{R}_0^S[\alpha] + \mathcal{R}_0^S[\beta], \quad \underline{\mathcal{R}}_0^S := \underline{\mathcal{R}}_0^S[\beta] + \underline{\mathcal{R}}_0^S[\check{\rho}, \sigma] + \underline{\mathcal{R}}_0^S[\check{\beta}] + \underline{\mathcal{R}}_0^S[\underline{\alpha}] + \underline{\mathcal{R}}_0^S[\check{\rho}],$$

where

$$\begin{aligned}
\mathcal{R}_0^S[\alpha] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{\frac{s+3}{2} - \frac{2}{p}} \alpha|_{p, S(u, \underline{u})}, \\
\mathcal{R}_0^S[\beta] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{\frac{7}{2} - \frac{2}{p}} |u|^{\frac{s-4}{2}} \beta|_{p, S(u, \underline{u})}, \\
\underline{\mathcal{R}}_0^S[\beta] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{\frac{s+2}{2} - \frac{2}{p}} |u|^{\frac{1}{2}} \beta|_{p, S(u, \underline{u})}, \\
\underline{\mathcal{R}}_0^S[\check{\rho}, \sigma] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{3 - \frac{2}{p}} |u|^{\frac{s-3}{2}} (\check{\rho}, \sigma)|_{p, S(u, \underline{u})}, \\
\underline{\mathcal{R}}_0^S[\underline{\beta}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{2 - \frac{2}{p}} |u|^{\frac{s-1}{2}} \underline{\beta}|_{p, S(u, \underline{u})}, \\
\underline{\mathcal{R}}_0^S[\underline{\alpha}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{1 - \frac{2}{p}} |u|^{\frac{s+1}{2}} \underline{\alpha}|_{p, S(u, \underline{u})}, \\
\underline{\mathcal{R}}_0^S[\underline{\rho}] &:= \sup_{\mathcal{K}} |r^3 \underline{\rho}|,
\end{aligned}$$

where the  $L^p$ -norms are defined in Definition 2.12. Then we define the norms of flux of curvature components. We denote

$$\mathcal{R} := \mathcal{R}_{[1]} + \underline{\mathcal{R}}_{[1]},$$

where

$$\mathcal{R}_{[1]} := \mathcal{R}_0 + \mathcal{R}_0^S + \mathcal{R}_1, \quad \underline{\mathcal{R}}_{[1]} := \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_0^S + \underline{\mathcal{R}}_1,$$

with

$$\begin{aligned}
\mathcal{R}_0 &:= (\mathcal{R}_0[\alpha]^2 + \mathcal{R}_0[\beta]^2 + \mathcal{R}_0[(\check{\rho}, \sigma)]^2 + \mathcal{R}_0[\underline{\beta}]^2)^{\frac{1}{2}}, \\
\underline{\mathcal{R}}_0 &:= (\underline{\mathcal{R}}_0[\beta]^2 + \underline{\mathcal{R}}_0[(\check{\rho}, \sigma)]^2 + \underline{\mathcal{R}}_0[\underline{\beta}]^2 + \underline{\mathcal{R}}_0[\underline{\alpha}]^2)^{\frac{1}{2}}, \\
\mathcal{R}_1 &:= (\mathcal{R}_1[\alpha]^2 + \mathcal{R}_1[\beta]^2 + \mathcal{R}_1[(\check{\rho}, \sigma)]^2 + \mathcal{R}_1[\underline{\beta}]^2 + \mathcal{R}_1[\alpha_4]^2)^{\frac{1}{2}}, \\
\underline{\mathcal{R}}_1 &:= (\underline{\mathcal{R}}_1[\beta]^2 + \underline{\mathcal{R}}_1[(\check{\rho}, \sigma)]^2 + \underline{\mathcal{R}}_1[\underline{\beta}]^2 + \underline{\mathcal{R}}_1[\alpha]^2 + \underline{\mathcal{R}}_1[\alpha_3]^2)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{0,1}[w] &:= \sup_{\mathcal{K}} \mathcal{R}_{0,1}[w](u, \underline{u}), \\
\underline{\mathcal{R}}_{0,1}[w] &:= \sup_{\mathcal{K}} \underline{\mathcal{R}}_{0,1}[w](u, \underline{u}).
\end{aligned}$$

It remains to define  $\mathcal{R}_q[w](u, \underline{u})$  and  $\underline{\mathcal{R}}_q[w](u, \underline{u})$  for  $q = 0, 1$ :

$$\begin{aligned}
\mathcal{R}_q[\alpha](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \alpha\|_{2, C_u \cap V(u, \underline{u})}, \\
\mathcal{R}_q[\beta](u, \underline{u}) &:= \|r^2 |u|^{\frac{s-4}{2}} (r\nabla)^q \beta\|_{2, C_u \cap V(u, \underline{u})}, \\
\mathcal{R}_q[(\check{\rho}, \sigma)](u, \underline{u}) &:= \|r |u|^{\frac{s-2}{2}} (r\nabla)^q (\check{\rho}, \sigma)\|_{2, C_u \cap V(u, \underline{u})}, \\
\mathcal{R}_q[\underline{\beta}](u, \underline{u}) &:= \| |u|^{\frac{s}{2}} (r\nabla)^q \underline{\beta}\|_{2, C_u \cap V(u, \underline{u})}, \\
\underline{\mathcal{R}}_q[\beta](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \beta\|_{2, \underline{C}_{\underline{u}} \cap V(u, \underline{u})}, \\
\underline{\mathcal{R}}_q[(\check{\rho}, \sigma)](u, \underline{u}) &:= |u|^{\frac{s-4}{2}} \|r^2 (r\nabla)^q (\check{\rho}, \sigma)\|_{2, \underline{C}_{\underline{u}} \cap V(u, \underline{u})}, \\
\underline{\mathcal{R}}_q[\underline{\beta}](u, \underline{u}) &:= |u|^{\frac{s-2}{2}} \|r (r\nabla)^q \underline{\beta}\|_{2, \underline{C}_{\underline{u}} \cap V(u, \underline{u})}, \\
\underline{\mathcal{R}}_q[\underline{\alpha}](u, \underline{u}) &:= |u|^{\frac{s}{2}} \|(r\nabla)^q \underline{\alpha}\|_{2, \underline{C}_{\underline{u}} \cap V(u, \underline{u})}, \\
\mathcal{R}_1[\alpha_4](u, \underline{u}) &:= \|r^{\frac{s+2}{2}} \nabla_4 \alpha\|_{2, C_u \cap V(u, \underline{u})}, \\
\underline{\mathcal{R}}_1[\alpha_3](u, \underline{u}) &:= |u|^{\frac{s+2}{2}} \|\nabla_3 \alpha\|_{2, \underline{C}_{\underline{u}} \cap V(u, \underline{u})}.
\end{aligned}$$

### 3.1.2 $\mathcal{O}$ norms (Ricci coefficients)

We define the norms of Ricci coefficients in the bootstrap region  $\mathcal{K}$ . Denoting

$$\mathcal{O} := \mathcal{O}_{[1]} + \underline{\mathcal{O}}_{[1]}, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{O}_{[1]} &:= \mathcal{O}_1 + \mathcal{O}_0 + \mathcal{O}_0(\Omega\underline{\omega}) + \sup_{\mathcal{K}} \left| r^2 \left( \overline{\text{tr } \chi} - \frac{2}{r} \right) \right| + \sup_{\mathcal{K}} \left| r \left( \Omega - \frac{1}{2} \right) \right|, \\ \underline{\mathcal{O}}_{[1]} &:= \underline{\mathcal{O}}_1 + \underline{\mathcal{O}}_0 + \mathcal{O}_0(\Omega\omega) + \sup_{\mathcal{K}} \left| r|u| \left( \overline{\text{tr } \underline{\chi}} + \frac{2}{r} \right) \right|. \end{aligned}$$

We define  $\mathcal{O}_q$  and  $\underline{\mathcal{O}}_q$  as follows:

$$\begin{aligned} \mathcal{O}_q &:= \mathcal{O}_q(\widetilde{\Omega \text{tr } \chi}) + \mathcal{O}_q(\widehat{\chi}) + \mathcal{O}_q(\eta) + \mathcal{O}_q(\widetilde{\Omega\omega}) + \mathcal{O}_q(\check{\Omega}), & q = 0, 1, \\ \underline{\mathcal{O}}_q &:= \mathcal{O}_q(\widetilde{\Omega \text{tr } \underline{\chi}}) + \mathcal{O}_q(\widehat{\underline{\chi}}) + \mathcal{O}_q(\underline{\eta}) + \mathcal{O}_q(\widetilde{\Omega\omega}), & q = 0, 1, \end{aligned}$$

where

$$\mathcal{O}_q(\Gamma) := \sup_{p \in [2,4]} \sup_{\mathcal{K}} \mathcal{O}_q^{p,S}(\Gamma)(u, \underline{u}).$$

It remains to define  $\mathcal{O}_q^{p,S}(\Gamma)(u, \underline{u})$  for  $q = 0, 1$ :

$$\begin{aligned} \mathcal{O}_q^{p,S}(\widetilde{\Omega \text{tr } \chi})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \widetilde{\Omega \text{tr } \chi}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega \text{tr } \underline{\chi}})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \widetilde{\Omega \text{tr } \underline{\chi}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widehat{\chi})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \widehat{\chi}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widehat{\underline{\chi}})(u, \underline{u}) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-1}{2}} \nabla^q \widehat{\underline{\chi}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\eta)(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \eta|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\underline{\eta})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \underline{\eta}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_0^{p,S}(\Omega\omega)(u, \underline{u}) &:= |r^{2-\frac{2}{p}}\Omega\omega|_{p,S(u, \underline{u})}, \\ \mathcal{O}_0^{p,S}(\Omega\underline{\omega})(u, \underline{u}) &:= |r^{2-\frac{2}{p}}\Omega\underline{\omega}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega\omega})(u, \underline{u}) &:= |r^{2+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}} (r\nabla)^q \widetilde{\Omega\omega}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega\underline{\omega}})(u, \underline{u}) &:= |r^{1-\frac{2}{p}}|u|^{\frac{s-1}{2}} (r\nabla)^q \widetilde{\Omega\underline{\omega}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\Omega})(u, \underline{u}) &:= |r^{1-\frac{2}{p}}|u|^{\frac{s-3}{2}} (r\nabla)^q \check{\Omega}|_{p,S(u, \underline{u})}. \end{aligned}$$

We also define

$$\check{\mathcal{O}} := \sum_{q=0}^1 \left( \mathcal{O}_q(\widetilde{\Omega \text{tr } \chi}) + \mathcal{O}_q(\eta) + \mathcal{O}_q(\check{\Omega}) \right) + \sup_{\mathcal{K}} r \left| \Omega - \frac{1}{2} \right| + \sup_{\mathcal{K}} r^2 \left| \overline{\Omega \text{tr } \chi} - \frac{1}{r} \right|,$$

which appears in sections 5 and 6.2.

### 3.1.3 $\underline{\mathcal{O}}(\Sigma_0 \setminus K)$ -norms and $\mathfrak{R}_0$ -norms (Initial data on the foliation of $\mathcal{K}$ )

Notice that for every  $\underline{u}$ , there exists a unique leaf  $S(u_0(\underline{u}), \underline{u})$  of  $\underline{C}_{\underline{u}}$ , which located in the future of  $\Sigma_0 \setminus K$  and touches  $\Sigma_0 \setminus K$ . Moreover, we have

$$S(u_0(\underline{u}), \underline{u}) \subset \mathcal{K}_{(0)}.$$

The following norms are defined on the union of spheres  $S(u_0(\underline{u}), \underline{u})$ . We define

$$\underline{\mathcal{O}}(\Sigma_0 \setminus K) := \underline{\mathcal{O}}_1(\Sigma_0 \setminus K) + \underline{\mathcal{O}}_0(\Sigma_0 \setminus K) + \underline{\mathcal{O}}_0(\Sigma_0 \setminus K)(\Omega\omega) + \sup_{\underline{u}} \left| r^2 \left( \Omega \operatorname{tr} \underline{\chi} + \frac{1}{r} \right) \right|,$$

where

$$\underline{\mathcal{O}}_q(\Sigma_0 \setminus K) := \sum_{\Gamma \in \{\widetilde{\Omega^{-1} \operatorname{tr} \underline{\chi}}, \widetilde{\eta}, \widetilde{\Omega\omega}\}} \underline{\mathcal{O}}_q(\Sigma_0 \setminus K)(\Gamma), \quad q = 0, 1,$$

with

$$\underline{\mathcal{O}}_q(\Sigma_0 \setminus K)(\Gamma) := \sup_{p \in [2, 4]} \sup_{\underline{u}} \mathcal{O}_q^{p, S}(\Sigma_0 \setminus K)(\Gamma)(\underline{u}).$$

It remains to define  $\underline{\mathcal{O}}_q^{p, S}(\Sigma_0 \setminus K)(\Gamma)(\underline{u})$ :

$$\begin{aligned} \mathcal{O}_q^{p, S}(\Sigma_0 \setminus K)(\widetilde{\Omega^{-1} \operatorname{tr} \underline{\chi}})(\underline{u}) &:= |r^{\frac{s+1}{2} - \frac{2}{p}} (r\nabla)^q \widetilde{\Omega^{-1} \operatorname{tr} \underline{\chi}}|_{p, S(u_0(\underline{u}), \underline{u})}, \\ \mathcal{O}_q^{p, S}(\Sigma_0 \setminus K)(\widetilde{\eta})(\underline{u}) &:= |r^{\frac{s+1}{2} - \frac{2}{p}} (r\nabla)^q \widetilde{\eta}|_{p, S(u_0(\underline{u}), \underline{u})}, \\ \mathcal{O}_0^{p, S}(\Sigma_0 \setminus K)(\Omega\omega)(\underline{u}) &:= |r^{2 - \frac{2}{p}} \Omega\omega|_{p, S(u_0(\underline{u}), \underline{u})}, \\ \mathcal{O}_q^{p, S}(\Sigma_0 \setminus K)(\widetilde{\Omega\omega})(\underline{u}) &:= |r^{\frac{s+1}{2} - \frac{2}{p}} (r\nabla)^q \widetilde{\Omega\omega}|_{p, S(u_0(\underline{u}), \underline{u})}. \end{aligned}$$

We also define

$$\check{\mathcal{O}}(\Sigma_0 \setminus K) := \sum_{q=0}^1 \left( \underline{\mathcal{O}}_q(\Sigma_0 \setminus K)(\widetilde{\Omega^{-1} \operatorname{tr} \underline{\chi}}) + \underline{\mathcal{O}}_q(\Sigma_0 \setminus K)(\widetilde{\Omega\omega}) \right) + \underline{\mathcal{O}}_0(\Sigma_0 \setminus K)(\Omega\omega),$$

which appears in sections 5 and 6.2.

We can extend the foliation  $(u, \underline{u})$  to a neighborhood of  $\Sigma_0 \setminus K$  in  $J^-(\Sigma_0 \setminus K)$ <sup>5</sup> such that it is well defined on  $\Sigma_0 \setminus K$ . The curvature flux on  $\Sigma_0 \setminus K$  is defined by:

$$\mathfrak{R}_0^2 := \int_{\Sigma_0 \setminus K} \sum_{l=0}^1 r^s \left( |\mathfrak{d}^l \alpha|^2 + |\mathfrak{d}^l \beta|^2 + |\mathfrak{d}^l(\check{\rho}, \sigma)|^2 + |\mathfrak{d}^l \underline{\beta}|^2 + |\mathfrak{d}^l \underline{\alpha}|^2 \right) + \sup_{\Sigma_0 \setminus K} |r^3 \check{\rho}|^2, \quad (3.2)$$

with

$$\mathfrak{d} := (r\nabla, r\nabla_3, r\nabla_4).$$

### 3.1.4 $\mathcal{O}^*(\underline{\mathcal{C}}_*)$ norms (Ricci coefficients on the last slice)

We define the following norms on the last slice  $\underline{\mathcal{C}}_*$  in the foliation  $(u, \underline{u})$  of  $\mathcal{K}$ . We denote

$$\mathcal{O}^*(\underline{\mathcal{C}}_*) := \mathcal{O}_1^* + \mathcal{O}_0^* + \sup_{\underline{\mathcal{C}}_*} \left| r^2 \left( \operatorname{tr} \underline{\chi} - \frac{2}{r} \right) \right| + \mathcal{O}_2^*(\widetilde{\operatorname{tr} \underline{\chi}}),$$

where

$$\mathcal{O}_q^* := \mathcal{O}_q^*(\widetilde{\operatorname{tr} \underline{\chi}}) + \mathcal{O}_q^*(\widehat{\chi}) + \mathcal{O}_q^*(\widetilde{\underline{\chi}}) + \mathcal{O}_q^*(\widehat{\underline{\chi}}) + \mathcal{O}_q^*(\zeta),$$

with

$$\mathcal{O}_q^*(\Gamma) := \sup_{p \in [2, 4]} \sup_{\underline{\mathcal{C}}_*} \mathcal{O}_q^{*p, S}(\Gamma).$$

<sup>5</sup>Recall that  $J^-(\Sigma_0 \setminus K)$  denotes the causal past of  $\Sigma_0 \setminus K$ .

It remains to define  $\mathcal{O}_q^{*p,S}(\Gamma)$ :

$$\begin{aligned}\mathcal{O}_q^{*p,S}(\widehat{\chi}) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\widehat{\chi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\widetilde{\chi}) &:= |r^{1-\frac{2}{p}}|u|^{\frac{s-1}{2}}(r\nabla)^q\widetilde{\chi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\widetilde{\text{tr}}\chi) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\widetilde{\text{tr}}\chi|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\widetilde{\text{tr}}\underline{\chi}) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\widetilde{\text{tr}}\underline{\chi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\zeta) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\zeta|_{p,S(u,\underline{u}_*)}.\end{aligned}$$

### 3.1.5 $\mathcal{O}_{(0)}$ and $\mathfrak{R}_{(0)}$ norms (Initial data on the foliation of $\mathcal{K}_{(0)}$ )

In this section, all the norms are defined by the initial layer foliation  $(u_{(0)}, \underline{u})$  in  $\mathcal{K}_{(0)}$ . We define

$$\begin{aligned}\mathcal{O}_{(0)} &:= \sum_{\Gamma_{(0)}} \mathcal{O}_{(0)}(\Gamma_{(0)}) + \sup_{\mathcal{K}_{(0)}} \left| r_{(0)}^2 \left( \text{tr} \chi_{(0)} - \frac{2}{r_{(0)}} \right) \right| + \sup_{\mathcal{K}_{(0)}} \left| r_{(0)}^2 \left( \text{tr} \underline{\chi}_{(0)} + \frac{2}{r_{(0)}} \right) \right| \\ &\quad + \sup_{\mathcal{K}_{(0)}} \left| r_{(0)}^2 \omega_{(0)} \right| + \sup_{\mathcal{K}_{(0)}} \left| r_{(0)}^2 \underline{\omega}_{(0)} \right| + \sup_{\mathcal{K}_{(0)}} \left| r_{(0)} \left( \Omega_{(0)} - \frac{1}{2} \right) \right|,\end{aligned}$$

where

$$\mathcal{O}_{(0)}(\Gamma_{(0)}) := \sum_{q=0}^2 \sup_{\mathcal{K}_{(0)}} \sup_{p \in [2,4]} \left| r_{(0)}^{\frac{s+1}{2}-\frac{2}{p}} (r_{(0)}\nabla)^q \Gamma_{(0)} \right|_{p,S_{(0)}(u_{(0)},\underline{u})},$$

with

$$\Gamma_{(0)} \in \{\widetilde{\text{tr}}\chi_{(0)}, \widetilde{\text{tr}}\underline{\chi}_{(0)}, \widehat{\chi}_{(0)}, \widehat{\underline{\chi}}_{(0)}, \eta_{(0)}, \underline{\eta}_{(0)}, \check{\omega}_{(0)}, \check{\underline{\omega}}_{(0)}\}.$$

We define the curvature flux on the initial hypersurface:

$$\mathfrak{R}_{(0)}^2 := \sum_{l=0}^2 \int_{\Sigma_0 \setminus K} r_{(0)}^s \left| \mathfrak{d}_{(0)}^l \left( \alpha_{(0)}, \beta_{(0)}, \check{\rho}_{(0)}, \sigma_{(0)}, \underline{\beta}_{(0)}, \underline{\alpha}_{(0)} \right) \right|^2 + \sup_{\Sigma_0 \setminus K} |r_{(0)}^3 \rho_{(0)}|,$$

where

$$\mathfrak{d}_{(0)} := \{r_{(0)}\nabla_3, r_{(0)}\nabla_4, r_{(0)}\nabla\}. \quad (3.3)$$

### 3.1.6 Osc-norms (Oscillation control)

We have two foliations  $(u, \underline{u})$  and  $(u_{(0)}, \underline{u})$  in the initial layer  $\mathcal{K}_{(0)}$  which will be compared using Osc-norms. We denote  $(f, \lambda)$  the change of frame from  $(u, \underline{u})$  to  $(u_{(0)}, \underline{u})$ , defined in Lemma 6.1 in section 6.1.1. We define

$$\text{Osc} := \text{Osc}(f) + \text{Osc}(\lambda) + \text{Osc}(r),$$

where

$$\text{Osc}(f) = \sup_{\mathcal{K}_{(0)}} \left| r^{\frac{s-1}{2}} \mathfrak{d}^{\leq 1} f \right|, \quad \text{Osc}(\lambda) = \sup_{\mathcal{K}_{(0)}} r |\overset{\circ}{\lambda}|, \quad \text{Osc}(r) := \sup_{\mathcal{K}_{(0)}} |r_{(0)} - r|,$$

with

$$\overset{\circ}{\lambda} := \lambda - 1, \quad \mathfrak{d} := \{r\nabla_3, r\nabla_4, r\nabla\}.$$

See Definition 6.9 in section 6.4 for a generalized definition of Osc-norms.



### 3.2 Main theorem

The goal of this paper is to prove the following theorem, which provides a new proof of the seminal result obtained by Klainerman and Nicolò in [29].

**Theorem 3.1** (Main Theorem). *Consider an initial data set  $(\Sigma_0, g, k)$   $s$ -asymptotically flat in the sense of Definition 1.2 with  $s > 3$ . Assume that we have the following control of the initial layer region  $\mathcal{K}_{(0)}$  defined in section 2.1.2*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0, \quad (3.4)$$

where  $\mathcal{O}_{(0)}, \mathfrak{R}_{(0)}$  are defined in section 3.1.5 and  $\varepsilon_0 > 0$  is small enough.

Then, the initial layer  $\mathcal{K}_{(0)}$  has a unique development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:

1.  $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $(u, \underline{u})$ . Moreover, the outgoing cones  $C_u$  are complete for all  $u \leq u_0$ .
2. The norms  $\mathcal{O}$  and  $\mathcal{R}$  defined in section 3.1 for  $s \in [4, 6]$ , and in sections B.1 and C.1 respectively for  $s \in (3, 4)$  and  $s > 6$  satisfy

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.5)$$

**Remark 3.2.** *Theorem 3.1 contains also a number of important conclusions following from (3.5): peeling properties, complete future infinity, Bondi mass formula and so on, see Chapter 8 in [29].*

**Remark 3.3.** *Theorem 3.1 is proved in section 3.4 for  $s \in [4, 6]$ , and extended to  $s \in (3, 4)$  and  $s > 6$  in Appendices B and C.*

The proof of Theorem 3.1 is given in section 3.4. It hinges on a sequence of basic theorems stated in section 3.3, concerning estimates for  $\mathcal{O}$  and  $\mathcal{R}$  norms.

We choose  $\varepsilon_0$  and  $\varepsilon$  small enough such that

$$\varepsilon \ll \delta_0 < 1, \quad \varepsilon := \varepsilon_0^{\frac{2}{3}},$$

where we recall that  $\delta_0$  is the height of the initial layer, see (2.14). Here,  $A \ll B$  means that  $CA < B$  where  $C$  is the largest universal constant among all the constants involved in the proof via  $\lesssim$ .

### 3.3 Main intermediate results

The following lemma provides comparisons between  $r_{(0)}, r, w, u_{(0)}, \underline{u}, u$ .

**Lemma 3.4.** *Under the assumptions*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon, \quad (3.6)$$

we have in the initial layer region  $\mathcal{K}_{(0)}$

$$|r_{(0)} - w| \lesssim \varepsilon_0 \log r_{(0)}, \quad |u_{(0)} - u| \lesssim \varepsilon \log r_{(0)}, \quad (3.7)$$

and in the bootstrap region  $\mathcal{K}$

$$\left| r - \frac{u - \underline{u}}{2} \right| \lesssim \varepsilon \log r. \quad (3.8)$$

*Proof.* See Appendix A.  $\square$

**Remark 3.5.** Recall that we have  $|u| \leq \underline{u}$ , and  $u < 0$  in  $\mathcal{K}$ . Together with (3.8), this yields

$$|u| \leq \underline{u} \simeq r \quad \text{in } \mathcal{K}. \quad (3.9)$$

In the sequel, we will use (3.7), (3.8) and (3.9) frequently without explicitly mentioning them.

The following theorems are important intermediate steps in the proof of Theorem 3.1.

**Theorem M0.** Under the assumptions

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon,$$

we have

$$\mathfrak{R}_0 \lesssim \varepsilon_0, \quad \check{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (3.10)$$

If in addition we assume that

$$\check{\mathcal{O}} \lesssim \varepsilon_0,$$

then, we have

$$\text{Osc} \lesssim \varepsilon_0, \quad \underline{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (3.11)$$

Theorem M0 is proved in section 6.2. The proof is based on null frame transformation formulae introduced by Klainerman and Szeftel in [34], see Proposition 6.2 in section 6.1.1.

**Remark 3.6.** The second part of Theorem M0 implies that we need an estimate for  $\check{\mathcal{O}}$  to control Osc. To this end, we first estimate the quantities in  $\check{\mathcal{O}}(\Sigma_0 \setminus K)$ , which are nonlinearly dependent on Osc. Next, we apply them as initial data to estimate part of the Ricci coefficients, i.e.  $\check{\mathcal{O}}$ , see (3.17) in Theorem M3. Then, we obtain the oscillation control, which allows us to estimate all the initial data, i.e. the control (3.11) for  $\underline{\mathcal{O}}(\Sigma_0 \setminus K)$ .

**Theorem M1.** Assume that

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \underline{\mathcal{R}}_0^S \leq \varepsilon. \quad (3.12)$$

Then, we have

$$\mathcal{R} \lesssim \mathfrak{R}_0. \quad (3.13)$$

Theorem M1 is proved in section 4. The proof is based on the  $r^p$ -weighted method introduced by Dafermos and Rodnianski in [13].

**Theorem M2.** Assume that

$$\mathcal{O}^*(\underline{C}_*) \leq \varepsilon, \quad \underline{\mathcal{R}}_0^S \leq \Delta_0, \quad \mathcal{O}_{(0)}(\Sigma_0 \cap \underline{C}_*) \leq \mathcal{I}_0. \quad (3.14)$$

Then, we have

$$\mathcal{O}^*(\underline{C}_*) \lesssim \Delta_0 + \mathcal{I}_0 + \varepsilon^2. \quad (3.15)$$

Theorem M2 is proved in section 6.3. The proof is done by integrating the transport equations along the null generator  $\underline{L}$  of the last slice  $\underline{C}_*$  and applying elliptic estimates on 2-spheres of the geodesic foliation of  $\underline{C}_*$ .

**Theorem M3.** Assume that

$$\mathcal{O} \leq \varepsilon, \quad \underline{\mathcal{R}}_0^S + \underline{\mathcal{R}}_0^S \leq \Delta_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \mathcal{I}_*, \quad \check{\mathcal{O}}(\Sigma_0 \setminus K) \leq \check{\mathcal{I}}_0, \quad (3.16)$$

then, we have

$$\check{\mathcal{O}} \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (3.17)$$

If we assume in addition that

$$\underline{\mathcal{O}}(\Sigma_0 \setminus K) \leq \mathcal{I}_0, \quad (3.18)$$

then, we have

$$\mathcal{O} \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (3.19)$$

Theorem M3 is proved in section 5. The proof is done by integrating the transport equations along the outgoing and incoming null cones and applying elliptic estimate on 2–spheres of the double null foliation of the spacetime  $\mathcal{K}$ .

**Theorem M4.** *We consider the spacetime  $\mathcal{K}$  and its double null foliation  $(u, \underline{u})$ , which satisfy the hypotheses*

$$\mathcal{O} \lesssim \epsilon_0, \quad \mathcal{R} \lesssim \epsilon_0, \quad \text{Osc} \lesssim \epsilon_0. \quad (3.20)$$

*Then, we can extend the spacetime  $\mathcal{K} = V(u_0, \underline{u}_*)$  and the double null foliation  $(u, \underline{u})$  to a new spacetime  $\widehat{\mathcal{K}} = \widehat{V}(u_0, \underline{u}_* + \nu)$ , where  $\nu$  is sufficiently small, and an associated double null foliation  $(\widehat{u}, \underline{u})$ . Moreover, the new foliation  $(\widehat{u}, \underline{u})$  is geodesic on the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$  and the new norms satisfy*

$$\widehat{\mathcal{O}} \lesssim \epsilon_0, \quad \widehat{\mathcal{R}} \lesssim \epsilon_0, \quad \widehat{\text{Osc}} \lesssim \epsilon_0. \quad (3.21)$$

Theorem M4 is proved in section 6.4. The proof is based on local existence, and the null frame transformation formulas of Proposition 6.2. We also need to reapply Theorems M1, M2 and M3 in the extended spacetime region  $\widehat{\mathcal{K}}$ .

**Remark 3.7.** *When compared to [29], we have the following similarities and differences:*

1. *We treat the general decay  $s > 3$  in Theorem 3.1 while the main result of [29]<sup>6</sup> correspond to the particular case  $s = 4$  in Theorem 3.1.*
2. *In [29], third order derivatives of Ricci coefficients and second order derivatives of curvature are estimated. In this paper, we only estimate first order derivative of both Ricci coefficients and curvature respectively in Theorems M0–M2–M3 and M1.*
3. *To estimate the curvature components in Theorem M1, instead of using vectorfield method as in [29], we use the  $r^p$ –weighted estimates introduced by Dafermos and Rodnianski in [13].*
4. *On the last slice  $\underline{C}_*$ , a canonical foliation is used in [29] while we use a geodesic foliation in Theorem M2.*
5. *The estimates of Ricci coefficients and the extension argument, i.e. the proof of Theorems M3 and M4, are similar to [29].*

### 3.4 Proof of the main theorem

We now use Theorems M0–M4 to prove Theorem 3.1.

**Definition 3.8.** *Let  $\mathfrak{N}(\underline{u}_*)$  the set of spacetimes  $\mathcal{K}$  associated with a double null foliation  $(u, \underline{u})$  which satisfy the following properties:*

1.  $\mathcal{K} = V(u_0, \underline{u}_*)$ .
2. *The foliation on  $\underline{C}_*$  is geodesic.*
3. *We have the following bootstrap bounds:*

$$\mathcal{O} \leq \epsilon, \quad \mathcal{R} \leq \epsilon, \quad \text{Osc} \leq \epsilon. \quad (3.22)$$

**Definition 3.9.** *We denote  $\mathcal{U}$  the set of values  $\underline{u}_*$  such that  $\mathfrak{N}(\underline{u}_*) \neq \emptyset$ .*

<sup>6</sup>We also reobtain the strong peeling properties of [30] that correspond to  $s > 7$ .

Applying the local existence result of Theorem 10.2.1 in [10] and the assumption  $\mathcal{O}_{(0)} \leq \varepsilon_0$ , we deduce that (3.22) holds if  $\underline{u}_*$  is sufficiently small. So, we have  $\mathcal{U} \neq \emptyset$ .

Define  $\underline{u}_*$  to be the supremum of the set  $\mathcal{U}$ . We want to prove  $\underline{u}_* = +\infty$ . We assume by contradiction that  $\underline{u}_*$  is finite. In particular we may assume  $\underline{u}_* \in \mathcal{U}$ . We consider the region  $\mathcal{K} = V(u_0, \underline{u}_*)$ . Recall that we have

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0, \quad (3.23)$$

according to the assumptions of Theorem 3.1. Applying Theorems M0 and M1, we obtain

$$\mathcal{R} \lesssim \mathfrak{R}_0 \lesssim \varepsilon_0, \quad \check{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (3.24)$$

Then, applying Theorem M2 and the hypothesis  $\mathcal{O} \leq \varepsilon$ , we obtain<sup>7</sup>

$$\mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0.$$

In view of the above, the quantities in Theorem M3 satisfy the following conditions:

$$\mathcal{I}_* \lesssim \varepsilon_0, \quad \Delta_0 \lesssim \varepsilon_0, \quad \check{\mathcal{I}}_0 \lesssim \varepsilon_0.$$

Applying the first part of Theorem M3, we obtain

$$\check{\mathcal{O}} \lesssim \varepsilon_0. \quad (3.25)$$

Then, we can apply the second part of Theorem M0 to deduce

$$\text{Osc} \lesssim \varepsilon_0, \quad \underline{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (3.26)$$

Note that (3.26) implies  $\mathcal{I}_0 \leq \varepsilon_0$  where  $\mathcal{I}_0$  is defined in Theorem M3. Thus, we may apply the second part of Theorem M3 to obtain

$$\mathcal{O} \lesssim \varepsilon_0. \quad (3.27)$$

Notice that (3.24), (3.26) and (3.27) implies that we can apply Theorem M4 to extend  $\mathcal{K}$  to  $\widehat{\mathcal{K}} := \widehat{V}(u_0, \underline{u}_* + \nu)$  for a  $\nu$  sufficiently small. We denote  $\widehat{\mathcal{O}}$  and  $\widehat{\mathcal{R}}$  the norms associated to the new foliation  $(\widehat{u}, \underline{u})$ . We have

$$\widehat{\mathcal{O}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}} \lesssim \varepsilon_0, \quad \widehat{\text{Osc}} \lesssim \varepsilon_0,$$

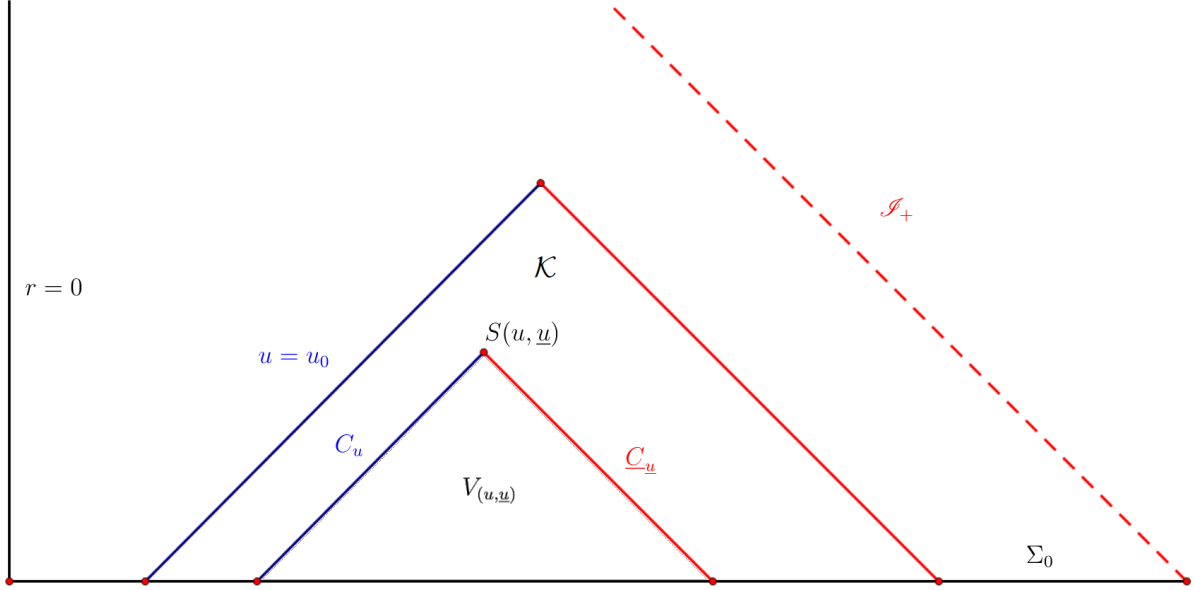
as a consequence of Theorem M4. We deduce that  $\widehat{V}(u_0, \underline{u}_* + \nu)$  with the double null foliation  $(\widehat{u}, \underline{u})$  satisfies all the properties in Definition 3.8, and so  $\mathfrak{N}(\underline{u}_* + \nu) \neq \emptyset$ , which is a contradiction. Thus, we have  $\underline{u}_* = +\infty$ , which implies property 1 of Theorem 3.1. Moreover, we have

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad (3.28)$$

in the whole exterior region, which implies property 2 of Theorem 3.1. This concludes the proof of Theorem 3.1.

## 4 Curvature estimates (Theorem M1)

In this section, we prove Theorem M1 by the  $r^p$ -weighted estimate method introduced in [13] and applied to Bianchi equations in [22] and [33]. For convenience, we recall the statement below.


 Figure 2.2: Domain of integration  $V(u, \underline{u})$ 

**Theorem M1.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \underline{\mathcal{R}}_0^S \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \quad (4.1)$$

*Then, we have*

$$\mathcal{R} \lesssim \mathfrak{R}_0. \quad (4.2)$$

**Lemma 4.1.** *We have the following estimates:*

$$|\Gamma_g| \lesssim \frac{\varepsilon}{r^2 |u|^{\frac{s-3}{2}}}, \quad |\Gamma_b| \lesssim \frac{\varepsilon}{r |u|^{\frac{s-1}{2}}}, \quad |\Gamma_a| \lesssim \frac{\varepsilon}{r^2}, \quad (4.3)$$

*and*

$$|r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} \Gamma_g^{(1)}|_{p,S} \lesssim \varepsilon, \quad |r^{1-\frac{2}{p}} |u|^{\frac{s-1}{2}} \Gamma_b^{(1)}|_{p,S} \lesssim \varepsilon, \quad p \in [2, 4]. \quad (4.4)$$

*Proof.* It follows directly from the assumption  $\mathcal{O} \leq \varepsilon$ , Definition 2.14 and the Sobolev inequality of Proposition 2.29.  $\square$

#### 4.1 Estimates for general Bianchi pairs

The following lemma provides the general structure of Bianchi pairs. It will be used repeatedly in this section. See Lemma 8.24 in [33] for an analog version under the assumption of axial polarization.

**Lemma 4.2.** *Let  $k = 1, 2$  and  $a_{(1)}, a_{(2)}$  real numbers. Then, we have the following properties.*

1. *If  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_k$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_{k-1}$  satisfying*

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \text{tr} \underline{\chi} \psi_{(1)} &= -kd_k^*(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \text{tr} \chi \psi_{(2)} &= d_k(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (4.5)$$

<sup>7</sup>By definition,  $\mathcal{O} \leq \varepsilon$  implies  $\mathcal{O}^*(\underline{C}_*) \leq \varepsilon$ .

Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$

$$\begin{aligned}
& \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\
& + \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + k \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\
& = 2kr^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} - 2r^p \underline{\omega} |\psi_{(1)}|^2 \\
& - 2kr^p \omega |\psi_{(2)}|^2 + pr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + kpr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \chi\right) |\psi_{(2)}|^2.
\end{aligned} \tag{4.6}$$

2. If  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_{k-1}$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_k$  satisfying

$$\begin{aligned}
& \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} = d_k(\psi_{(2)}) + h_{(1)}, \\
& \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \psi_{(2)} = -kd_k^*(\psi_{(1)}) + h_{(2)}.
\end{aligned} \tag{4.7}$$

Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$

$$\begin{aligned}
& k \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\
& + k \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\
& = 2r^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2kr^p \psi_{(1)} \cdot h_{(1)} + 2r^p \psi_{(2)} \cdot h_{(2)} - 2kr^p \underline{\omega} |\psi_{(1)}|^2 \\
& - 2r^p \omega |\psi_{(2)}|^2 + kpr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + pr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \chi\right) |\psi_{(2)}|^2.
\end{aligned} \tag{4.8}$$

**Remark 4.3.** Note that the Bianchi equations can be written as systems of equations of the type (4.5) and (4.7). In particular

- the Bianchi pair  $(\alpha, \beta)$  satisfies (4.5) with  $k = 2$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,
- the Bianchi pair  $(\beta, (\check{\rho}, -\sigma))$  satisfies (4.5) with  $k = 1$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,
- the Bianchi pair  $((\check{\rho}, \sigma), \underline{\beta})$  satisfies (4.7) with  $k = 1$ ,  $a_{(1)} = \frac{3}{2}$ ,  $a_{(2)} = 1$ ,
- the Bianchi pair  $(\underline{\beta}, \underline{\alpha})$  satisfies (4.7) with  $k = 2$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ .

*Proof.* By a direct computation, we have

$$\mathbf{Dive}_4 = \mathbf{D}_\gamma e_4^\gamma = -\frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_3) - \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_4) + \mathbf{g}^{AB} \mathbf{g}(\mathbf{D}_A e_4, e_B) = -2\omega + \operatorname{tr} \chi,$$

and similarly

$$\mathbf{Dive}_3 = -2\underline{\omega} + \operatorname{tr} \underline{\chi}.$$

For  $\psi_{(1)}$  and  $\psi_{(2)}$  satisfying (4.5), we compute

$$\begin{aligned}
& \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\
& = \nabla_3(r^p |\psi_{(1)}|^2) + \mathbf{Div}(e_3) r^p |\psi_{(1)}|^2 + k \nabla_4(r^p |\psi_{(2)}|^2) + k \mathbf{Div}(e_4) r^p |\psi_{(2)}|^2 \\
& = pr^{p-1} e_3(r) |\psi_{(1)}|^2 + 2r^p \psi_{(1)} \cdot \nabla_3 \psi_{(1)} + r^p (\operatorname{tr} \underline{\chi} - 2\underline{\omega}) |\psi_{(1)}|^2 \\
& + kpr^{p-1} e_4(r) |\psi_{(2)}|^2 + 2kr^p \psi_{(2)} \cdot \nabla_4 \psi_{(2)} + kr^p (\operatorname{tr} \chi - 2\omega) |\psi_{(2)}|^2 \\
& = pr^{p-1} e_3(r) |\psi_{(1)}|^2 + 2r^p \psi_{(1)} \cdot (-a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} - kd_k^* \psi_{(2)} + h_{(1)}) + r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 - 2r^p \underline{\omega} |\psi_{(1)}|^2 \\
& + kpr^{p-1} e_4(r) |\psi_{(2)}|^2 + 2kr^p \psi_{(2)} \cdot (-a_{(2)} \operatorname{tr} \chi \psi_{(2)} + d_k \psi_{(1)} + h_{(2)}) + kr^p \operatorname{tr} \chi |\psi_{(2)}|^2 - 2kr^p \omega |\psi_{(2)}|^2 \\
& = pr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + 2r^p \psi_{(1)} \cdot (-kd_k^* \psi_{(2)} + h_{(1)}) + \left(1 - 2a_{(1)} + \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2
\end{aligned}$$

$$\begin{aligned}
& +kpr^{p-1} \left( e_4(r) - \frac{r}{2} \operatorname{tr} \chi \right) |\psi_{(2)}|^2 + 2kr^p \psi_{(2)} \cdot (d_k \psi_{(1)} + h_{(2)}) + \left( 1 - 2a_{(2)} + \frac{p}{2} \right) kr^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\
& - 2r^p \underline{\omega} |\psi_{(1)}|^2 - 2kr^p \omega |\psi_{(2)}|^2 \\
& = 2k \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + pr^{p-1} \left( e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} \right) |\psi_{(1)}|^2 + kpr^{p-1} \left( e_4(r) - \frac{r}{2} \operatorname{tr} \chi \right) |\psi_{(2)}|^2 \\
& + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} + \left( 1 - 2a_{(1)} + \frac{p}{2} \right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left( 1 - 2a_{(2)} + \frac{p}{2} \right) kr^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\
& - 2kr^p \omega |\psi_{(2)}|^2 - 2r^p \underline{\omega} |\psi_{(1)}|^2.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
& \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) + \left( 2a_{(1)} - 1 - \frac{p}{2} \right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left( 2a_{(2)} - 1 - \frac{p}{2} \right) kr^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\
& = 2kr^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} - 2r^p \underline{\omega} |\psi_{(1)}|^2 - 2kr^p \omega |\psi_{(2)}|^2 \\
& + pr^{p-1} \left( e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} \right) |\psi_{(1)}|^2 + kpr^{p-1} \left( e_4(r) - \frac{r}{2} \operatorname{tr} \chi \right) |\psi_{(2)}|^2,
\end{aligned}$$

which implies (4.6). The proof of (4.8) is similar and left to the reader. This concludes the proof of Lemma 4.2.  $\square$

**Proposition 4.4.** *We denote*

$$V := V(u, \underline{u}), \quad C_u^V := C_u \cap V, \quad \underline{C}_{\underline{u}}^V := \underline{C}_{\underline{u}} \cap V. \quad (4.9)$$

For  $\psi_{(1)}$ ,  $\psi_{(2)}$  and  $h_{(1)}$ ,  $h_{(2)}$  satisfying (4.5) or (4.7), we have the following properties for  $q = 0, 1$  and all  $(u, \underline{u}) \in \mathcal{K}$ .

- In the case of  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p > 0$ , we have

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}^{(q)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^{p-1} |\psi_{(1)}^{(q)}|^2 + r^{p-1} |\psi_{(2)}^{(q)}|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}^{(q)}|^2 + r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^p |\psi_{(1)}^{(q)}| |h_{(1)}^{(q)}| + r^p |\psi_{(2)}^{(q)}| |h_{(2)}^{(q)}|.
\end{aligned} \quad (4.10)$$

- In the case of  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p = 0$ , we have

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}^{(q)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^{p-1} |\psi_{(1)}^{(q)}|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}^{(q)}|^2 + r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^p |\psi_{(1)}^{(q)}| |h_{(1)}^{(q)}| + r^p |\psi_{(2)}^{(q)}| |h_{(2)}^{(q)}|.
\end{aligned} \quad (4.11)$$

- In the case of  $2 + p - 4a_{(1)} \leq 0$  and  $4a_{(2)} - 2 - p \geq 0$ , we have

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}^{(q)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}^{(q)}|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}^{(q)}|^2 + r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^{p-1} |\psi_{(1)}^{(q)}|^2 + r^p |\psi_{(1)}^{(q)}| |h_{(1)}^{(q)}| + r^p |\psi_{(2)}^{(q)}| |h_{(2)}^{(q)}|.
\end{aligned} \quad (4.12)$$

- In the case of  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p \leq 0$ , we have

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}^{(q)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^{p-1} |\psi_{(1)}^{(q)}|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}^{(q)}|^2 + r^p |\psi_{(2)}^{(q)}|^2 + \int_V r^{p-1} |\psi_{(2)}^{(q)}|^2 + r^p |\psi_{(1)}^{(q)}| |h_{(1)}^{(q)}| + r^p |\psi_{(2)}^{(q)}| |h_{(2)}^{(q)}|.
\end{aligned} \quad (4.13)$$

**Remark 4.5.** In the sequel, For a sum of terms of the form  $\Gamma \cdot R^{(1)}$ , we ignore the terms have same or even better decay. For example, we write

$$\Gamma_a \cdot \beta^{(1)} + \Gamma_a \cdot \alpha^{(1)} = \Gamma_a \cdot \beta^{(1)},$$

since  $\alpha^{(1)}$  decays better than  $\beta^{(1)}$ .

*Proof.* Recall that  $\omega, \underline{\omega} \in \Gamma_a$ . Applying Lemma 2.4, we obtain

$$e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} = \frac{\overline{\Omega \operatorname{tr} \underline{\chi}}}{2\Omega} r - \frac{r}{2} \operatorname{tr} \underline{\chi} = \frac{r}{2\Omega} (\overline{\Omega \operatorname{tr} \underline{\chi}} - \Omega \operatorname{tr} \underline{\chi}) \in r\Gamma_g,$$

and similarly

$$e_4(r) - \frac{r}{2} \operatorname{tr} \chi \in r\Gamma_g.$$

Applying (4.3), we infer

$$\begin{aligned} & \int_V r^p |\Gamma_a| (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \\ & \lesssim \varepsilon \int_V r^{p-2} (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \\ & \lesssim \varepsilon \int_{u_0(\underline{u})}^u \left( |u|^{-2} \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) du + \varepsilon \int_{|u_0|}^u \left( |\underline{u}|^{-2} \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 \right) d\underline{u} \\ & \lesssim \varepsilon \sup_u \left( \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) \int_{u_0(\underline{u})}^u |u|^{-2} du + \varepsilon \sup_{\underline{u}} \left( \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 \right) \int_{|u_0|}^u |\underline{u}|^{-2} d\underline{u} \\ & \lesssim \varepsilon \sup_u \left( \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) + \varepsilon \sup_{\underline{u}} \left( \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 \right). \end{aligned} \quad (4.14)$$

Integrating (4.6) or (4.8), reminding that  $\operatorname{tr} \chi - \frac{2}{r} \in \Gamma_a$ ,  $\operatorname{tr} \underline{\chi} + \frac{2}{r} \in \Gamma_a$ , and that  $\Gamma_g$  decays better than  $\Gamma_a$ , we obtain

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 + \int_V (2 + p - 4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)} - 2 - p) r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}| + r^p |\Gamma_a| |\psi_{(1)}|^2 + r^p |\Gamma_a| |\psi_{(2)}|^2. \end{aligned}$$

Taking the supremum of  $u$  and  $\underline{u}$  and applying (4.14), we obtain for  $\varepsilon$  small enough

$$\begin{aligned} & \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 + \sup_{\underline{u}} \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 \\ & + \int_V (2 + p - 4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)} - 2 - p) r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|, \end{aligned} \quad (4.15)$$

which implies (4.10)–(4.13) hold in correspond cases when  $q = 0$ .

Next, we consider the case  $q = 1$ . Assume that  $(\psi_{(1)}, \psi_{(2)})$  satisfies (4.5).<sup>8</sup> We multiply (4.5) by

<sup>8</sup>The case of  $(\psi_{(1)}, \psi_{(2)})$  satisfies (4.7) is similar.



$\Omega$  and differentiate it by  $\not\partial$  to obtain

$$\begin{aligned} & [\not\partial, \Omega \nabla_3] \psi_{(1)} + \Omega \nabla_3(\not\partial \psi_{(1)}) + a_{(1)} r \nabla(\Omega \operatorname{tr} \chi) \cdot \psi_{(1)} + a_{(1)} \Omega \operatorname{tr} \chi \not\partial \psi_{(1)} \\ &= -k(r \nabla \Omega) d_k^*(\psi_{(2)}) - k \Omega d_k^*(\not\partial \psi_{(2)}) + (r \nabla \Omega) h_{(1)} + \Omega \not\partial h_{(1)}, \\ & [\not\partial, \Omega \nabla_4] \psi_{(2)} + \Omega \nabla_4(\not\partial \psi_{(2)}) + a_{(2)} r \nabla(\Omega \operatorname{tr} \chi) \cdot \psi_{(2)} + a_{(2)} \Omega \operatorname{tr} \chi \not\partial \psi_{(2)} \\ &= (r \nabla \Omega) d_k(\psi_{(1)}) + \Omega d_k(\not\partial \psi_{(1)}) + (r \nabla \Omega) \cdot h_{(2)} + \Omega \not\partial h_{(2)}. \end{aligned}$$

Applying Proposition 2.24, we infer

$$\begin{aligned} \nabla_3(\not\partial \psi_{(1)}) + a_{(1)} \operatorname{tr} \chi \not\partial \psi_{(1)} &= -k d_k^*(\not\partial \psi_{(2)}) + h_{(1)}^{(1)} + (\Gamma_b \cdot \psi_{(1)})^{(1)} + (\Gamma_g \cdot \psi_{(2)})^{(1)}, \\ \nabla_4(\not\partial \psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \not\partial \psi_{(2)} &= d_k^*(\not\partial \psi_{(1)}) + h_{(2)}^{(1)} + (\Gamma_g \cdot (\psi_{(1)}, \psi_{(2)}))^{(1)}. \end{aligned} \quad (4.16)$$

Integrating (4.16), and proceeding similarly as in (4.14), we deduce

$$\begin{aligned} & \sup_u \int_{C_u^V} r^p |\not\partial \psi_{(1)}|^2 + \sup_{\underline{u}} \int_{\underline{C}_{\underline{u}}^V} r^p |\not\partial \psi_{(2)}|^2 \\ &+ \int_V (2+p-4a_{(1)}) r^{p-1} |\not\partial \psi_{(1)}|^2 + (4a_{(2)}-2-p) r^{p-1} |\not\partial \psi_{(2)}|^2 \\ &\lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}^{(1)}|^2 + r^p |\psi_{(2)}^{(1)}|^2 + \int_V r^p |\psi_{(1)}^{(1)}| |h_{(1)}^{(1)}| + r^p |\psi_{(2)}^{(1)}| |h_{(2)}^{(1)}|. \end{aligned} \quad (4.17)$$

Combining (4.15) and (4.17), this concludes the proof of Proposition 4.4.  $\square$

The following lemma allows us to obtain  $|u|$ -decay of curvature along  $\Sigma_0 \cap V(u, \underline{u}_*)$ .

**Lemma 4.6.** *We have the following estimate for  $p \leq s$ :*

$$\int_{\Sigma_0 \cap V(u, \underline{u})} r^p (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \lesssim \frac{\mathfrak{R}_0^2}{|u|^{s-p}}.$$

*Proof.* For fixed  $u$  and  $\underline{u}$ , we have from Lemma 3.4

$$\begin{aligned} & \int_{\Sigma_0 \cap V(u, \underline{u})} r^p (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \\ &\lesssim \int_{\Sigma_0 \cap \{2|u'| \geq |u|\}} r'^p (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \\ &\lesssim \int_{\Sigma_0 \cap \{2|u'| \geq |u|\}} |u'|^{p-s} r'^s (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \\ &\lesssim |u|^{p-s} \int_{\Sigma_0 \cap \{2|u'| \geq |u|\}} r'^s (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \\ &\lesssim \frac{\mathfrak{R}_0^2}{|u|^{s-p}}. \end{aligned}$$

This concludes the proof of Lemma 4.6.  $\square$

## 4.2 Estimates for the Bianchi pair $(\alpha, \beta)$

**Proposition 4.7.** *We have the following estimate:*

$$\mathcal{R}_0[\alpha]^2 + \underline{\mathcal{R}}_0[\beta]^2 + \mathcal{R}_1[\alpha]^2 + \underline{\mathcal{R}}_1[\beta]^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \quad (4.18)$$

*Proof.* We recall the following equations of Corollary 2.20:

$$\begin{aligned}\nabla_4\beta + 2\operatorname{tr}\chi\beta &= \not{d}_2\alpha + h[\beta_4], \\ \nabla_3\alpha + \frac{1}{2}\operatorname{tr}\underline{\chi}\alpha &= -2\not{d}_2^*\beta + h[\alpha_3],\end{aligned}\tag{4.19}$$

which correspond to  $\psi_{(1)} = \alpha$ ,  $\psi_{(2)} = \beta$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,  $h_{(1)} = h[\alpha_3]$ ,  $h_{(2)} = h[\beta_4]$  and  $k = 2$  in (4.5). Taking  $p = s$  and recalling that  $s \in [4, 6]$ , we have

$$2 + p - 4a_{(1)} = s > 0, \quad 4a_{(2)} - 2 - p = 6 - s \geq 0.\tag{4.20}$$

We apply (4.10) to obtain

$$\begin{aligned}& \int_{C_u^V} r^s |\alpha^{(1)}|^2 + \int_{\underline{C}_u^V} r^s |\beta^{(1)}|^2 + \int_V r^{s-1} |\alpha^{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^s |\alpha^{(1)}|^2 + r^s |\beta^{(1)}|^2 + \int_V r^s |\alpha^{(1)}| |h[\alpha_3]^{(1)}| + r^s |\beta^{(1)}| |h[\beta_4]^{(1)}|.\end{aligned}$$

It remains to estimate  $|\alpha^{(1)}| |h[\alpha_3]^{(1)}|$  and  $|\beta^{(1)}| |h[\beta_4]^{(1)}|$ . Recalling from Corollary 2.20 that<sup>9</sup>

$$h[\alpha_3] = 4\underline{\omega}\alpha + \Gamma_g \cdot \rho, \quad h[\beta_4] = (\Gamma_a, \Gamma_g) \cdot \beta,\tag{4.21}$$

we have<sup>10</sup>

$$|\alpha^{(1)}| |h[\alpha_3]^{(1)}| + |\beta^{(1)}| |h[\beta_4]^{(1)}| \lesssim |\Gamma_b^{(1)}| |\alpha^{(1)}| |\alpha| + |\Gamma_a| |\beta^{(1)}|^2 + |\Gamma_g| |\alpha^{(1)}| |\rho^{(1)}| + |\Gamma_b^{(1)}| |\alpha| |\rho|.$$

Applying (4.3) to estimate  $\Gamma_b^{(1)}$ , we have

$$\begin{aligned}\int_V r^s |\Gamma_b^{(1)}| |\alpha^{(1)}| |\alpha| & \lesssim \int_{u_0(\underline{u})}^u \left( \int_{C_u^V} r^s |\alpha^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^s |\Gamma_b^{(1)}|^2 |\alpha|^2 \right)^{\frac{1}{2}} du \\ & \lesssim \mathcal{R} \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u \left( \int_{S(u, \underline{u})} r^s |\Gamma_b^{(1)}|^2 |\alpha|^2 \right) d\underline{u} \right)^{\frac{1}{2}} du \\ & \lesssim \mathcal{R} \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u r^{-3} |u|^{1-s} |r^{\frac{1}{2}} |u|^{\frac{s-1}{2}} \Gamma_b^{(1)}|_{4,S}^2 |r^{\frac{s+2}{2}} \alpha|_{4,S}^2 d\underline{u} \right)^{\frac{1}{2}} du \tag{4.22} \\ & \lesssim \varepsilon \mathcal{R}^2 \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u r^{-3} |u|^{1-s} d\underline{u} \right)^{\frac{1}{2}} du \\ & \lesssim \varepsilon \mathcal{R}^2 \int_{u_0(\underline{u})}^u |u|^{-1} |u|^{\frac{1-s}{2}} du \lesssim \varepsilon \mathcal{R}^2.\end{aligned}$$

Applying (4.3) to estimate  $\Gamma_a$ , we infer

$$\int_V r^s |\Gamma_a| |\beta^{(1)}|^2 \lesssim \varepsilon \int_V r^{s-2} |\beta^{(1)}|^2 \lesssim \varepsilon \int_{|u_0|}^u |u|^{-2} d\underline{u} \left( \int_{\underline{C}_u^V} r^s |\beta^{(1)}|^2 \right) \lesssim \varepsilon \mathcal{R}^2.\tag{4.23}$$

<sup>9</sup>We used the fact that  $\sigma$ ,  $\beta$  decay better than  $\rho$ .

<sup>10</sup>We ignore the terms which decay better.

Applying (4.3) to estimate  $\Gamma_g$ , we infer

$$\begin{aligned}
\int_V r^s |\Gamma_g| |\rho^{(1)}| |\alpha^{(1)}| &\lesssim \int_{u_0(\underline{u})}^u \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} \left( \int_{C_u^V} r^s |\alpha^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{s-4} |\rho^{(1)}|^2 \right)^{\frac{1}{2}} du \\
&\lesssim \int_{u_0(\underline{u})}^u \frac{\varepsilon \mathcal{R}}{|u|^{\frac{s-3}{2}}} \left( \int_{|u|}^u r^{s-8} d\underline{u} \int_S |r^2 \rho^{(1)}|^2 \right)^{\frac{1}{2}} du \\
&\lesssim \int_{u_0(\underline{u})}^u \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{s-3}{2}}} \left( \int_{|u|}^u r^{s-8} d\underline{u} \right)^{\frac{1}{2}} du \\
&\lesssim \int_{u_0(\underline{u})}^u \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{s-3}{2}} |u|^{\frac{7-s}{2}}} du \lesssim \varepsilon \mathcal{R}^2.
\end{aligned} \tag{4.24}$$

We also have

$$\begin{aligned}
\int_V r^s |\Gamma_g^{(1)}| |\rho| |\alpha| &\lesssim \int_{u_0(\underline{u})}^u \left( \int_{C_u^V} r^s |\alpha|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^s |\Gamma_g^{(1)}|^2 |\rho|^2 \right)^{\frac{1}{2}} du \\
&\lesssim \mathcal{R} \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u \left( \int_{S(u, \underline{u})} r^s |\Gamma_g^{(1)}|^2 |\rho|^2 \right) d\underline{u} \right)^{\frac{1}{2}} du \\
&\lesssim \mathcal{R} \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u r^{s-8} |u|^{3-s} |r^{\frac{3}{2}} |u|^{\frac{s-3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{5}{2}} \rho|_{4,S}^2 d\underline{u} \right)^{\frac{1}{2}} du \\
&\lesssim \varepsilon \mathcal{R}^2 \int_{u_0(\underline{u})}^u \left( \int_{|u|}^u r^{s-8} |u|^{3-s} d\underline{u} \right)^{\frac{1}{2}} du \\
&\lesssim \varepsilon \mathcal{R}^2 \int_{u_0(\underline{u})}^u |u|^{\frac{s-7}{2}} |u|^{\frac{3-s}{2}} du \lesssim \varepsilon \mathcal{R}^2.
\end{aligned} \tag{4.25}$$

Thus, we obtain

$$\int_{C_u^V} r^s |\alpha^{(1)}|^2 + \int_{\underline{C}_u^V} r^s |\beta^{(1)}|^2 + \int_V r^{s-1} |\alpha^{(1)}|^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \tag{4.26}$$

This concludes the proof of Proposition 4.7.  $\square$

### 4.3 Estimates for the Bianchi pair $(\beta, (\check{\rho}, -\sigma))$

**Proposition 4.8.** *We have the following estimate:*

$$\mathcal{R}_0[\beta]^2 + \underline{\mathcal{R}}_0[(\check{\rho}, \sigma)]^2 + \mathcal{R}_1[\beta]^2 + \underline{\mathcal{R}}_1[(\check{\rho}, \sigma)]^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \tag{4.27}$$

*Proof.* We recall the following Bianchi equations

$$\begin{aligned}
\nabla_4 \check{\rho} + \frac{3}{2} \operatorname{tr} \chi \check{\rho} &= \operatorname{div} \beta + h[\check{\rho}_4], \\
\nabla_4 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{div}^* \beta + h[\sigma_4], \\
\nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= \nabla \check{\rho} + {}^* \nabla \sigma + h[\beta_3],
\end{aligned}$$

which can be written in the form

$$\begin{aligned}
\nabla_4(\check{\rho}, -\sigma) + \frac{3}{2} \operatorname{tr} \chi(\check{\rho}, -\sigma) &= \not{d}_1 \beta + h[\check{\rho}_4, \sigma_4], \\
\nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= -\not{d}_1^*(\check{\rho}, -\sigma) + h[\beta_3],
\end{aligned} \tag{4.28}$$

where  $h[\check{\rho}_4, \sigma_4] = (h[\check{\rho}_4], h[\sigma_4])$ . Applying (4.11) with  $\psi_{(1)} = \beta$ ,  $\psi_{(2)} = (\check{\rho}, \sigma)$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = h[\beta_3]$ ,  $h_{(2)} = h[\check{\rho}_4, \sigma_4]$  and  $p = 4$ , we obtain from Lemma 4.6

$$\begin{aligned}
& \int_{C_u^V} r^4 |\beta^{(1)}|^2 + \int_{\underline{C}_u^V} r^4 |(\check{\rho}^{(1)}, \sigma^{(1)})|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} (r^4 |\beta^{(1)}|^2 + r^4 |(\check{\rho}^{(1)}, \sigma^{(1)})|^2) \\
& \quad + \int_V r^4 |\beta^{(1)} \cdot h[\beta_3]^{(1)}| + \int_V r^4 |(\check{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\check{\rho}_4, \sigma_4]^{(1)}| \\
& \lesssim |u|^{4-s} \mathfrak{R}_0^2 + \int_V r^4 |\beta^{(1)} \cdot h[\beta_3]^{(1)}| + \int_V r^4 |(\check{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\check{\rho}_4, \sigma_4]^{(1)}|.
\end{aligned} \tag{4.29}$$

Recall from Corollary 2.20 that

$$h[\beta_3] = \Gamma_a \cdot \beta + \Gamma_g \cdot \underline{\beta}, \quad h[\check{\rho}_4, \sigma_4] = \Gamma_b \cdot \alpha + \Gamma_g \cdot \check{\rho}.$$

Hence, we have

$$h[\beta_3]^{(1)} = \Gamma_a \cdot \beta^{(1)} + \Gamma_g \cdot \underline{\beta}^{(1)} + \Gamma_g^{(1)} \cdot \underline{\beta}, \quad h[\check{\rho}_4, \sigma_4]^{(1)} = \Gamma_b \cdot \alpha^{(1)} + \Gamma_g \cdot \check{\rho}^{(1)} + \Gamma_b^{(1)} \cdot \alpha + \Gamma_g^{(1)} \cdot \check{\rho}.$$

Hence, we infer

$$\begin{aligned}
& \int_V r^4 |\beta^{(1)}| |h[\beta_3]^{(1)}| + r^4 |(\check{\rho}, \sigma)^{(1)}| |h[\check{\rho}_4, \sigma_4]^{(1)}| \\
& \lesssim \int_V r^4 |\Gamma_a| |\beta^{(1)}|^2 + r^4 |\Gamma_g| |\beta^{(1)}| |\underline{\beta}^{(1)}| + r^4 |\Gamma_g^{(1)}| |\beta| |\underline{\beta}|.
\end{aligned} \tag{4.30}$$

For the first term in (4.30) we have

$$\int_V r^4 |\Gamma_a| |\beta^{(1)}|^2 \lesssim \varepsilon \int_{u_0(\underline{u})}^u |u|^{-2} du \left( \int_{C_u^V} r^4 |\beta^{(1)}|^2 \right) \lesssim \varepsilon \mathcal{R}^2 |u|^{4-s}.$$

For the second term in (4.30), we infer

$$\begin{aligned}
\int_V r^4 |\Gamma_g| |\beta^{(1)}| |\underline{\beta}^{(1)}| & \lesssim \varepsilon \int_{u_0(\underline{u})}^u \frac{1}{|u|^{\frac{s-3}{2}}} du \left( \int_{C_u^V} r^4 |\beta^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\underline{\beta}^{(1)}|^2 \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon \int_{u_0(\underline{u})}^u \frac{du}{|u|^{\frac{s-3}{2}}} \frac{\mathcal{R}}{|u|^{\frac{s-4}{2}}} \frac{\mathcal{R}}{|u|^{\frac{s}{2}}} \lesssim \frac{\varepsilon \mathcal{R}^2}{|u|^{s-4}}.
\end{aligned}$$

For the third term in (4.30), we have

$$\begin{aligned}
\int_V r^4 |\Gamma_g^{(1)}| |\beta| |\underline{\beta}| & \lesssim \int_V r^4 \frac{\varepsilon}{|u|^{\frac{s-1}{2}}} |\beta|^2 + \int_V r^4 |u|^{\frac{s-1}{2}} \varepsilon^{-1} |\Gamma_g^{(1)}|^2 |\beta|^2 \\
& \lesssim \varepsilon \int_{u_0(\underline{u})}^u \frac{du}{|u|^{\frac{s-1}{2}}} \int_{C_u^V} r^4 |\beta|^2 + \int_{|u|}^u \frac{d\underline{u}}{|u|} \int_{\underline{C}_{\underline{u}}^V} r^4 |u|^{\frac{s-1}{2}} \varepsilon^{-1} |\Gamma_g^{(1)}|^2 |\beta|^2 \\
& \lesssim \varepsilon \int_{u_0(\underline{u})}^u \frac{du}{|u|^{\frac{s-1}{2}}} \frac{\mathcal{R}^2}{|u|^{s-4}} + \int_{|u|}^u \frac{d\underline{u}}{|u|} \int_{u_0(\underline{u})}^u r^{-2} |u|^{\frac{s-1}{2}} \varepsilon^{-1} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{3}{2}} \underline{\beta}|_{4,S}^2 du \\
& \lesssim \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{s-3}{2}} |u|^{s-4}} + \int_{|u|}^u \frac{d\underline{u}}{|u|} \int_{u_0(\underline{u})}^u r^{-2} |u|^{\frac{s-1}{2}} \frac{\varepsilon \mathcal{R}^2}{|u|^{2s-4}} du \\
& \lesssim \frac{\varepsilon \mathcal{R}^2}{|u|^{s-4}} + \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{3}{2}s - \frac{5}{2}}} \int_{|u|}^u \frac{d\underline{u}}{r^2} \\
& \lesssim \frac{\varepsilon \mathcal{R}^2}{|u|^{s-4}}.
\end{aligned}$$

Injecting all the estimates above into (4.30), we deduce

$$\int_V r^4 |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + r^4 |(\underline{\rho}^{(1)}, \sigma^{(1)})| |h[\underline{\rho}_4, \sigma_4]^{(1)}| \lesssim \varepsilon \mathcal{R}^2 |u|^{4-s}.$$

Combining with (4.29), we obtain (4.27). This concludes the proof of Proposition 4.8.  $\square$

#### 4.4 Estimates for the Bianchi pair $((\underline{\rho}, \sigma), \underline{\beta})$

**Proposition 4.9.** *We have the following estimates:*

$$\mathcal{R}_0[(\underline{\rho}, \sigma)]^2 + \underline{\mathcal{R}}_0[\underline{\beta}]^2 + \mathcal{R}_1[(\underline{\rho}, \sigma)]^2 + \underline{\mathcal{R}}_1[\underline{\beta}]^2 \lesssim \mathfrak{A}_0^2 + \varepsilon \mathcal{R}^2. \quad (4.31)$$

*Proof.* We recall the following Bianchi equations:

$$\begin{aligned} \nabla_4 \underline{\beta} + \text{tr } \chi \underline{\beta} &= \mathfrak{d}_1^*(\underline{\rho}, \sigma) + h[\underline{\beta}_4], \\ \nabla_3(\underline{\rho}, \sigma) + \frac{3}{2} \text{tr } \underline{\chi}(\underline{\rho}, \sigma) &= -\mathfrak{d}_1 \underline{\beta} + h[\underline{\rho}_3, \sigma_3], \end{aligned}$$

where

$$h[\underline{\rho}_3, \sigma_3] := (h[\underline{\rho}_3], h[\sigma_3]) = \Gamma_g \cdot \underline{\alpha}, \quad h[\underline{\beta}_4] = \Gamma_a \cdot \underline{\beta}. \quad (4.32)$$

We apply (4.12) with  $\psi_{(1)} = (\underline{\rho}, \sigma)$ ,  $\psi_{(2)} = \underline{\beta}$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = h[\underline{\rho}_3, \sigma_3]$ ,  $h_{(2)} = h[\underline{\beta}_4]$  and  $p = 2$ , we obtain from Lemma 4.6 and Proposition 4.8 that

$$\begin{aligned} & \int_{C_u^V} r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)})|^2 + \int_{\underline{C}_u^V} r^2 |\underline{\beta}^{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} (r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)})|^2 + r^2 |\underline{\beta}^{(1)}|^2) + \int_V r |(\underline{\rho}^{(1)}, \sigma^{(1)})|^2 \\ & \quad + \int_V r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\underline{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}| \\ & \lesssim |u|^{2-s} \mathfrak{A}_0^2 + \int_{|u|}^u |\underline{u}|^{-3} d\underline{u} \int_{\underline{C}_u^V} r^4 |(\underline{\rho}^{(1)}, \sigma^{(1)})|^2 \\ & \quad + \int_V r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\underline{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}| \\ & \lesssim |u|^{2-s} \mathfrak{A}_0^2 + (\mathfrak{A}_0^2 + \varepsilon \mathcal{R}^2) \int_{|u|}^u |u|^{4-s} |\underline{u}|^{-3} d\underline{u} \\ & \quad + \int_V r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\underline{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}| \\ & \lesssim |u|^{2-s} (\mathfrak{A}_0^2 + \varepsilon \mathcal{R}^2) + \int_V r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\underline{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}|. \end{aligned} \quad (4.33)$$

Applying (4.32) we obtain

$$\begin{aligned} & \int_V r^2 |(\underline{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\underline{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}| \\ & \lesssim \int_V r^2 |\Gamma_g| |\underline{\alpha}^{(1)}| |\underline{\rho}^{(1)}| + r^2 |\Gamma_a| |\underline{\beta}^{(1)}|^2 + |\Gamma_g^{(1)}| |\underline{\beta}|^2. \end{aligned} \quad (4.34)$$

For the first term in (4.34), we infer

$$\begin{aligned}
\int_V r^2 |\Gamma_g| |\underline{\alpha}^{(1)}| |\check{\rho}^{(1)}| &\lesssim \varepsilon \int_{|u|}^u du \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}| |\check{\rho}^{(1)}| \\
&\lesssim \varepsilon \int_{|u|}^u \frac{du}{|u|^2} \left( \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} r^4 |\check{\rho}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \int_{|u|}^u \frac{du}{|u|^2} \frac{\mathcal{R}}{|u|^{\frac{s}{2}}} \frac{\mathcal{R}}{|u|^{\frac{s-4}{2}}} \lesssim \varepsilon \mathcal{R}^2 |u|^{2-s}.
\end{aligned}$$

For the second term in (4.34), we have

$$\int_V r^2 |\Gamma_a| |\underline{\beta}^{(1)}|^2 \lesssim \varepsilon \int_{|u|}^u |u|^{-2} du \int_{\underline{C}_u^V} r^2 |\underline{\beta}^{(1)}|^2 \lesssim \varepsilon \mathcal{R}^2 |u|^{2-s} \int_{|u|}^u |u|^{-2} du \lesssim \varepsilon \mathcal{R}^2 |u|^{2-s}.$$

For the third term in (4.34), we infer

$$\begin{aligned}
\int_V r^2 |r \nabla \Gamma_g| |\underline{\beta}|^2 &\lesssim \int_{|u|}^u du \left( \int_{\underline{C}_u^V} r^2 |\underline{\beta}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} r^2 |\Gamma_g^{(1)}|^2 |\underline{\beta}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{|u|}^u du \frac{\mathcal{R}}{|u|^{\frac{s-2}{2}}} \left( \int_{u_0(u)}^u \frac{du}{|u|^4} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{3}{2}} \underline{\beta}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \mathcal{R} \int_{|u|}^u du \frac{1}{|u|^{\frac{s-2}{2}}} \left( \int_{u_0(u)}^u \frac{du}{|u|^4} \frac{\varepsilon^2}{|u|^{s-3}} \frac{\mathcal{R}^2}{|u|^{s-1}} \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \mathcal{R}^2 \int_{|u|}^u du \frac{1}{|u|^{\frac{s-2}{2}}} \frac{1}{|u|^2} \frac{1}{|u|^{s-\frac{5}{2}}} \lesssim \varepsilon \mathcal{R}^2 |u|^{2-s}.
\end{aligned}$$

Thus, we obtain

$$\int_V r^2 |(\check{\rho}^{(1)}, \sigma^{(1)}) \cdot h[\check{\rho}_3, \sigma_3]^{(1)}| + r^2 |\underline{\beta}^{(1)} \cdot h[\underline{\beta}_4]^{(1)}| \lesssim \varepsilon \mathcal{R}^2 |u|^{2-s}.$$

Combining with (4.33), we obtain (4.31). This concludes the proof of Proposition 4.9.  $\square$

#### 4.5 Estimates for the Bianchi pair $(\underline{\beta}, \underline{\alpha})$

**Proposition 4.10.** *We have the following estimates:*

$$\mathcal{R}_0[\underline{\beta}]^2 + \mathcal{R}_0[\underline{\alpha}]^2 + \mathcal{R}_1[\underline{\beta}]^2 + \mathcal{R}_1[\underline{\alpha}]^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \quad (4.35)$$

*Proof.* We recall the following Bianchi equations:

$$\begin{aligned}
\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + h[\underline{\alpha}_4], \\
\nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -\operatorname{div} \underline{\alpha} + h[\underline{\beta}_3],
\end{aligned} \quad (4.36)$$

where

$$h[\underline{\alpha}_4] = \Gamma_a \cdot \underline{\alpha}, \quad h[\underline{\beta}_3] = \Gamma_a \cdot \underline{\beta} + \Gamma_g \cdot \underline{\alpha}. \quad (4.37)$$

Applying (4.12) with  $\psi_{(1)} = \underline{\beta}$ ,  $\psi_{(2)} = \underline{\alpha}$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ ,  $h_{(1)} = h[\underline{\beta}_3]$ ,  $h_{(2)} = h[\underline{\alpha}_4]$  and  $p = 0$ , we obtain from Lemma 4.6

$$\begin{aligned}
& \int_{C_u^V} |\underline{\beta}^{(1)}|^2 + \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}|^2 \\
& \lesssim \int_{\Sigma_0 \cap V} |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2 + \int_V r^{-1} |\underline{\beta}^{(1)}|^2 + \int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}| \\
& \lesssim |u|^{-s} \mathfrak{R}_0^2 + \int_{|u_0|}^u r^{-3} d\underline{u} \int_{\underline{C}_u^V} r^2 |\underline{\beta}^{(1)}|^2 + \int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}| \quad (4.38) \\
& \lesssim |u|^{-s} \mathfrak{R}_0^2 + (\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2) \int_{|u_0|}^u r^{-3} d\underline{u} |u|^{2-s} + \int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}| \\
& \lesssim |u|^{-s} (\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2) + \int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}|.
\end{aligned}$$

Applying (4.37), we infer<sup>11</sup>

$$\int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}| \lesssim \int_V |\Gamma_a| |\underline{\alpha}^{(1)}|^2 + |\Gamma_g^{(1)}| |\underline{\alpha}|^2. \quad (4.39)$$

For the first term in (4.39), we have

$$\int_V |\Gamma_a| |\underline{\alpha}^{(1)}|^2 \lesssim \varepsilon \int_{|u|}^u |\underline{u}|^{-2} d\underline{u} \int_{\underline{C}_u^V} |\underline{\alpha}|^2 \lesssim \varepsilon \mathcal{R}^2 |u|^{-s}.$$

For the second term in (4.39), we infer

$$\begin{aligned}
\int_V |\Gamma_g^{(1)}| |\underline{\alpha}|^2 & \lesssim \int_{|u|}^u d\underline{u} \left( \int_{\underline{C}_u^V} |\underline{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |\Gamma_g^{(1)}|^2 |\underline{\alpha}|^2 \right)^{\frac{1}{2}} \\
& \lesssim \mathcal{R} \int_{|u|}^u d\underline{u} |\underline{u}|^{-\frac{s}{2}} \left( \int_{u_0(\underline{u})}^u \frac{d\underline{u}}{|\underline{u}|^4} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{1}{2}} \underline{\alpha}|_{4,S}^2 \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon \mathcal{R}^2 \int_{|u|}^u d\underline{u} |\underline{u}|^{-2} |\underline{u}|^{-\frac{s}{2}} \left( \int_{u_0(\underline{u})}^u \frac{d\underline{u}}{|\underline{u}|^{2s-2}} \right)^{\frac{1}{2}} \\
& \lesssim \varepsilon \mathcal{R}^2 \int_{|u|}^u d\underline{u} |\underline{u}|^{-\frac{s}{2}} |\underline{u}|^{-2} |\underline{u}|^{\frac{3-2s}{2}} \lesssim \varepsilon \mathcal{R}^2 |u|^{-s}.
\end{aligned}$$

Thus, we obtain

$$\int_V |\underline{\beta}^{(1)}| |h[\underline{\beta}_3]^{(1)}| + |\underline{\alpha}^{(1)}| |h[\underline{\alpha}_4]^{(1)}| \lesssim \varepsilon \mathcal{R}^2 |u|^{-s}.$$

Combining with (4.38), we obtain (4.35). This concludes the proof of Proposition 4.10.  $\square$

## 4.6 Estimate for $\nabla_4 \alpha$

**Proposition 4.11.** *We have the following estimates:*

$$\mathcal{R}_1[\alpha_4]^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \quad (4.40)$$

*Proof.* We recall from (2.32)

$$\begin{aligned}
\nabla_3 \dot{\alpha} & = -2d_2^* \phi + \frac{4\alpha}{r} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta, \\
\nabla_4 \phi + \frac{5}{2} \text{tr} \chi \phi & = d_2 \dot{\alpha} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta.
\end{aligned}$$

<sup>11</sup>Recall that the  $\Gamma_a$  before  $\underline{\alpha}$  only contains  $\omega$ .

Applying (4.10) with  $\psi_{(1)} = \dot{\alpha}$ ,  $\psi_{(2)} = \dot{\phi}$ ,  $a_{(1)} = 0$ ,  $a_{(2)} = \frac{5}{2}$ ,  $h_{(1)} = \frac{4\alpha}{r} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta$ ,  $h_{(2)} = \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta$ ,  $k = 2$  and  $p = s$ , we obtain

$$\begin{aligned} & \int_{C_u} r^s |\dot{\alpha}|^2 + \int_{\underline{C}_u} r^s |\dot{\phi}|^2 + \int_V r^{s-1} (|\dot{\alpha}|^2 + |\dot{\phi}|^2) \\ & \lesssim \mathfrak{R}_0^2 + \int_V r^{s-1} |\dot{\alpha}| |\alpha| + r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_a| |\beta^{(1)}| + r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_g^{(1)}| |\beta|. \end{aligned} \quad (4.41)$$

First, we have for all  $\delta > 0$

$$\int_V r^{s-1} |\dot{\alpha}| |\alpha| \leq \delta \int_V r^{s-1} |\dot{\alpha}|^2 + \frac{1}{4\delta} \int_V r^{s-1} |\alpha|^2.$$

Combining with (4.26), we deduce from (4.41), for  $\delta$  small enough

$$\begin{aligned} & \int_{C_u} r^s |\dot{\alpha}|^2 + \int_{\underline{C}_u} r^s |\dot{\phi}|^2 + \int_V r^{s-1} (|\dot{\alpha}|^2 + |\dot{\phi}|^2) \\ & \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2 + \int_V r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_a| |\beta^{(1)}| + r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_g^{(1)}| |\beta|. \end{aligned}$$

We have

$$\begin{aligned} \int_V r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_a| |\beta^{(1)}| & \lesssim \varepsilon \int_V r^{s-2} |\dot{\alpha}, \dot{\phi}| |\beta^{(1)}| \\ & \lesssim \varepsilon \int_{-\underline{u}}^u du \frac{1}{|u|^{\frac{8-s}{2}}} \left( \int_{C_u^V} r^s |\dot{\alpha}, \dot{\phi}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^4 |\beta^{(1)}|^2 \right)^{\frac{1}{2}} \\ & \lesssim \int_{-\underline{u}}^u du \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{8-s}{2}} |u|^{\frac{s-4}{2}}} \\ & \lesssim \varepsilon \mathcal{R}^2. \end{aligned}$$

We also have

$$\begin{aligned} \int_V r^s |(\dot{\alpha}, \dot{\phi})| |\Gamma_g^{(1)}| |\beta| & \lesssim \int_V r^s |\dot{\alpha}| |\Gamma_g^{(1)}| |\beta| + \int_V r^s |\dot{\phi}| |\Gamma_g^{(1)}| |\beta| \\ & \lesssim \int_{-\underline{u}}^u du \left( \int_{C_u^V} r^s |\dot{\alpha}, \dot{\phi}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^s |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\ & \lesssim \mathcal{R} \int_{-\underline{u}}^u du \left( \int_{|u|}^u du r^s \int_S |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\ & \lesssim \mathcal{R} \int_{-\underline{u}}^u du \left( \int_{|u|}^u du r^{s-10} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{7}{2}} \beta|_{4,S}^2 \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon \mathcal{R}^2 \int_{-\underline{u}}^u du \left( \int_{|u|}^u r^{s-10} |u|^{8-2s} du \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon \mathcal{R}^2 \int_{-\underline{u}}^u du |u|^{\frac{s-9}{2}} |u|^{4-s} \\ & \lesssim \varepsilon \mathcal{R}^2. \end{aligned}$$

Combining the above estimates, we obtain

$$\int_{C_u^V} r^s |\dot{\alpha}|^2 + \int_{\underline{C}_u^V} r^s |\dot{\phi}|^2 + \int_V r^{s-1} (|\dot{\alpha}|^2 + |\dot{\phi}|^2) \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2.$$



Recalling that

$$\dot{\alpha} = r^{-4} \nabla_4 (r^5 \alpha) = r \nabla_4 \alpha + 5e_4(r) \alpha,$$

we deduce from Proposition 4.7

$$\int_{C_u^V} r^{s+2} |\nabla_4 \alpha|^2 \lesssim \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2.$$

This concludes the proof of Proposition 4.11.  $\square$

#### 4.7 Estimate for $\nabla_3 \alpha$

**Proposition 4.12.** *We have the following estimates:*

$$\mathcal{R}_1[\alpha_3]^2 \lesssim \mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}} + \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2. \quad (4.42)$$

*Proof.* We recall from (2.33)

$$\begin{aligned} \nabla_4 \dot{\alpha} &= -2d_2^* \underline{\phi} + \frac{4\underline{\alpha}}{r} + (\Gamma_a, \Gamma_b) \cdot \underline{\alpha}^{(1)} + \Gamma_b^{(1)} \cdot \underline{\alpha}, \\ \nabla_3 \underline{\phi} + \frac{5}{2} \text{tr} \chi \underline{\phi} &= d_2 \dot{\alpha} + (\Gamma_a, \Gamma_b) \cdot \underline{\alpha}^{(1)} + \Gamma_b \cdot \dot{\alpha} + \Gamma_b^{(1)} \cdot \underline{\alpha}. \end{aligned}$$

Applying (4.12) with  $\psi_{(1)} = \underline{\phi}$ ,  $\psi_{(2)} = \dot{\alpha}$ ,  $a_{(1)} = \frac{5}{2}$ ,  $a_{(2)} = 0$ ,  $h_{(1)} = \Gamma_b^{(1)} \cdot \underline{\alpha} + \Gamma_b \cdot \dot{\alpha} + (\Gamma_a, \Gamma_b) \cdot \underline{\alpha}^{(1)}$ ,  $h_{(2)} = \frac{4\underline{\alpha}}{r} + \Gamma_b^{(1)} \cdot \underline{\alpha} + (\Gamma_a, \Gamma_b) \cdot \underline{\alpha}^{(1)}$ ,  $k = 2$  and  $p = -2$ , we obtain

$$\begin{aligned} &\int_{C_u^V} r^{-2} |\underline{\phi}|^2 + \int_{\underline{C}_u^V} r^{-2} |\dot{\alpha}|^2 \\ &\lesssim \int_{\Sigma_0 \cap V} r^{-2} (|\underline{\phi}|^2 + |\dot{\alpha}|^2) + \int_V r^{-3} |\underline{\phi}|^2 + r^{-3} |\dot{\alpha}| |\underline{\alpha}^{(1)}| + r^{-2} |\Gamma_b^{(1)}| |\underline{\alpha}| (|\dot{\alpha}| + |\underline{\phi}|). \end{aligned}$$

First, we have from Lemma 4.6

$$\int_{\Sigma_0 \cap V} r^{-2} (|\underline{\phi}|^2 + |\dot{\alpha}|^2) \lesssim \frac{\mathfrak{R}_0^2}{|u|^{s+2}}.$$

Notice that we have from Proposition 4.10

$$\int_V r^{-3} |\underline{\phi}|^2 \lesssim \int_{|u|}^u r^{-3} du \int_{\underline{C}_u^V} |\underline{\phi}|^2 \lesssim \frac{\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2}{|u|^{s+2}},$$

and

$$\begin{aligned} \int_V r^{-3} |\dot{\alpha}| |\underline{\alpha}^{(1)}| &\lesssim \int_{|u|}^u r^{-2} \left( \int_{\underline{C}_u^V} r^{-2} |\dot{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} du \\ &\lesssim \int_{|u|}^u r^{-2} \left( \frac{\mathcal{R}^2}{|u|^{s+2}} \right)^{\frac{1}{2}} \left( \frac{\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2}{|u|^s} \right)^{\frac{1}{2}} du \\ &\lesssim \int_{|u|}^u \frac{\mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}}}{r^2 |u|^{s+1}} du \\ &\lesssim \frac{\mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}}}{|u|^{s+2}}. \end{aligned}$$

We also have

$$\begin{aligned}
\int_V r^{-2} |\Gamma_b^{(1)}| |\underline{\alpha}| (|\underline{\dot{\alpha}}| + |\underline{\dot{\alpha}}|) &\lesssim \int_{|u|}^u r^{-2} d\underline{u} \int_{\underline{C}_{\underline{u}}^V} (|\underline{\dot{\alpha}}| + |\underline{\alpha}^{(1)}|) |\underline{\alpha}| |\Gamma_b^{(1)}| \\
&\lesssim \int_{|u|}^u d\underline{u} \left( \int_{\underline{C}_{\underline{u}}^V} |\underline{\dot{\alpha}}|^2 + |\underline{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_{\underline{u}}^V} |\underline{\alpha}|^2 |\Gamma_b^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\mathcal{R}}{|u|^{\frac{s}{2}}} \int_{|u|}^u r^{-3} d\underline{u} \left( \int_{|u|}^u du |r^{\frac{1}{2}} \underline{\alpha}|_{4,S}^2 |r^{\frac{1}{2}} \Gamma_b^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\mathcal{R}}{|u|^{\frac{s}{2}}} \int_{|u|}^u r^{-3} d\underline{u} \left( \int_{|u|}^u du \frac{\varepsilon^2 \mathcal{R}^2}{|u|^{2s}} \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon \mathcal{R}^2}{|u|^{\frac{3s+3}{2}}}.
\end{aligned}$$

Combining the above estimates, we deduce

$$\int_{\underline{C}_{\underline{u}}^V} r^{-2} |\underline{\dot{\alpha}}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^{-2} |\underline{\dot{\alpha}}|^2 \lesssim \frac{\mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}} + \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2}{|u|^{s+2}}.$$

Combining with Proposition 4.10, we infer

$$\int_{\underline{C}_{\underline{u}}^V} |\nabla_3 \underline{\alpha}|^2 \lesssim \int_{\underline{C}_{\underline{u}}^V} r^{-10} |\nabla_3 (r^5 \underline{\alpha})|^2 + r^{-2} |\underline{\alpha}|^2 \lesssim \int_{\underline{C}_{\underline{u}}^V} r^{-2} |\underline{\dot{\alpha}}|^2 + r^{-2} |\underline{\alpha}|^2 \lesssim \frac{\mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}} + \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2}{|u|^{s+2}}.$$

This concludes the proof of Proposition 4.12.  $\square$

#### 4.8 Estimate for $\bar{\rho}$

**Proposition 4.13.** *We have the following estimate for  $p \in [2, 4]$ :*

$$|r^3 \bar{\rho}|_{\infty, S}(u, \underline{u}) \lesssim \mathfrak{R}_0 + \varepsilon \mathcal{R}. \tag{4.43}$$

*Proof.* We recall the Bianchi equation

$$\nabla_3 \bar{\rho} + \frac{3}{2} \text{tr} \chi \bar{\rho} = h[\bar{\rho}_3].$$

Applying Lemma 2.30, we infer for  $p \in [2, 4]$

$$\begin{aligned}
|r^{3-\frac{2}{p}} \bar{\rho}|_{p, S}(u, \underline{u}) &\lesssim \mathfrak{R}_0 + \int_{u_0(\underline{u})}^u |r^{3-\frac{2}{p}} h[\bar{\rho}_3]|_{p, S} du \\
&\lesssim \mathfrak{R}_0 + \int_{u_0(\underline{u})}^u |r^{3-\frac{2}{p}} \Gamma_g \cdot (\underline{\alpha}, \bar{\rho})|_{p, S} du \\
&\lesssim \mathfrak{R}_0 + \int_{u_0(\underline{u})}^u \left( \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} \frac{\mathcal{R}}{|u|^{\frac{s+1}{2}}} + \frac{\varepsilon \mathcal{R}}{r^2 |u|^{\frac{s-3}{2}}} \right) du \\
&\lesssim \mathfrak{R}_0 + \varepsilon \mathcal{R}.
\end{aligned}$$

Since  $\bar{\rho}$  is constant on  $S(u, \underline{u})$ , this concludes the proof of Proposition 4.13.  $\square$

#### 4.9 End of the proof of Theorem M1

**Proposition 4.14.** *We have the following estimate:*

$$\mathcal{R}_0^S + \underline{\mathcal{R}}_0^S \lesssim \mathcal{R}_0 + \mathcal{R}_1 + \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1. \quad (4.44)$$

*Proof.* It is sufficient to prove that

$$\mathcal{R}_0^S \lesssim \mathcal{R}_0 + \mathcal{R}_1, \quad \underline{\mathcal{R}}_0^S \lesssim \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1,$$

which is a direct consequence of Propositions 2.20 and 2.28.  $\square$

Finally, we deduce from Proposition 4.7-4.14 that

$$\mathcal{R}^2 \lesssim \mathcal{R}(\mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2)^{\frac{1}{2}} + \mathfrak{R}_0^2 + \varepsilon \mathcal{R}^2.$$

Applying Cauchy-Schwarz inequality, we obtain for  $\varepsilon$  small enough,

$$\mathcal{R} \lesssim \mathfrak{R}_0,$$

this concludes the proof of Theorem M1.

### 5 Ricci coefficients estimates (Theorem M3)

The goal of this section is to prove Theorem M3, which we recall below for convenience.

**Theorem M3.** *Assume that*

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{R}_0^S + \underline{\mathcal{R}}_0^S \leq \Delta_0, \quad \mathcal{O}^*(\underline{\mathcal{C}}_*) \leq \mathcal{I}_*, \quad \check{\mathcal{O}}(\Sigma_0 \setminus K) \leq \check{\mathcal{I}}_0, \quad (5.1)$$

then, we have

$$\check{\mathcal{O}} \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (5.2)$$

If we assume in addition that

$$\underline{\mathcal{O}}(\Sigma_0 \setminus K) \leq \mathcal{I}_0, \quad (5.3)$$

then, we have

$$\mathcal{O} \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (5.4)$$

In this section, we always assume  $p \in [2, 4]$ .

#### 5.1 Estimates for $\mathcal{O}_0(\Omega\omega)$ and $\mathcal{O}_0(\Omega\underline{\omega})$

**Proposition 5.1.** *We have the following estimates:*

$$\begin{aligned} |r^{2-\frac{2}{p}}\Omega\omega|_{p,S}(u, \underline{u}) &\lesssim \check{\mathcal{I}}_0 + \Delta_0 + \varepsilon^2 \\ |r^{2-\frac{2}{p}}\Omega\underline{\omega}|_{p,S}(u, \underline{u}) &\lesssim \mathcal{I}_* + \Delta_0 + \varepsilon^2. \end{aligned} \quad (5.5)$$

*Proof.* By the null structure equations (2.29), we have

$$\mathbf{D}_3(\Omega\omega) = \widehat{F} + \frac{1}{2}\Omega\rho, \quad \mathbf{D}_4(\Omega\underline{\omega}) = \widehat{\underline{F}} + \frac{1}{2}\Omega\underline{\rho}, \quad (5.6)$$

where

$$\begin{aligned} \widehat{F} &:= -\frac{1}{2}\Omega\zeta \cdot (\eta + \underline{\eta}) - \frac{1}{2}\Omega(\underline{\eta} \cdot \eta - 2\zeta^2) = \Gamma_g \cdot \Gamma_g, \\ \widehat{\underline{F}} &:= \frac{1}{2}\Omega\zeta \cdot (\eta + \underline{\eta}) - \frac{1}{2}\Omega(\underline{\eta} \cdot \eta - 2\zeta^2) = \Gamma_g \cdot \Gamma_g. \end{aligned}$$

Apply Lemma 2.30, we obtain

$$\begin{aligned} |r^{-\frac{2}{p}}\Omega\underline{\omega}|_{p,S}(u,\underline{u}) &\lesssim |r^{-\frac{2}{p}}\Omega\underline{\omega}|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{u_*} |r^{-\frac{2}{p}}\widehat{F}|_{p,S} + \int_{\underline{u}}^{u_*} |r^{-\frac{2}{p}}\Omega\rho|_{p,S}, \\ |r^{-\frac{2}{p}}\Omega\omega|_{p,S}(u,\underline{u}) &\lesssim |r^{-\frac{2}{p}}\Omega\omega|_{p,S}(u_0(\underline{u}),\underline{u}) + \int_{u_0(\underline{u})}^u |r^{-\frac{2}{p}}\widehat{F}|_{p,S} + \int_{u_0(\underline{u})}^u |r^{-\frac{2}{p}}\Omega\rho|_{p,S}. \end{aligned}$$

Notice that we have

$$|r^{-\frac{2}{p}}(\widehat{F},\widehat{F})|_{p,S} \lesssim \frac{\epsilon^2}{r^4|u|^{s-3}}, \quad |r^{-\frac{2}{p}}\Omega\rho|_{p,S} \lesssim r^{-3}|r^{3-\frac{2}{p}}\rho|_{p,S} \lesssim \frac{\Delta_0}{r^3}.$$

Hence, we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}}\Omega\underline{\omega}|_{p,S}(u,\underline{u}) &\lesssim |r^{2-\frac{2}{p}}\Omega\underline{\omega}|_{p,S}(u,\underline{u}_*) + \epsilon^2 + \Delta_0 \lesssim \mathcal{I}_* + \Delta_0 + \epsilon^2, \\ |r^{2-\frac{2}{p}}\Omega\omega|_{p,S}(u,\underline{u}) &\lesssim |r^{2-\frac{2}{p}}\Omega\omega|_{p,S}(u_0(\underline{u}),\underline{u}) + \epsilon^2 + \Delta_0 \lesssim \check{\mathcal{I}}_0 + \Delta_0 + \epsilon^2. \end{aligned}$$

which concludes the proof.  $\square$

## 5.2 Estimates for $\mathcal{O}_{0,1}(\widetilde{\Omega}\omega)$ and $\mathcal{O}_{0,1}(\widetilde{\Omega}\underline{\omega})$

**Proposition 5.2.** *We have the following estimates:*

$$\begin{aligned} |r^{2+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}}(r\nabla)^q\widetilde{\Omega}\omega|_{p,S}(u,\underline{u}) &\lesssim \check{\mathcal{I}}_0 + \Delta_0 + \epsilon^2, \quad q = 0, 1, \\ |r^{1-\frac{2}{p}}|u|^{\frac{s-1}{2}}(r\nabla)^q\widetilde{\Omega}\underline{\omega}|_{p,S}(u,\underline{u}) &\lesssim \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1. \end{aligned} \quad (5.7)$$

*Proof.* First, we derive an evolution equation for  $\nabla(\Omega\omega)$ . Applying (2.29) and Corollary 2.22, we obtain

$$\begin{aligned} \Omega\nabla_3\nabla(\Omega\omega) &= [\Omega\nabla_3, \nabla](\Omega\omega) + \nabla(\Omega\nabla_3(\Omega\omega)) \\ &= -\Omega\underline{\chi} \cdot \nabla(\Omega\omega) + \nabla(\Omega\nabla_3(\Omega\omega)) \\ &= -\Omega\underline{\chi} \cdot \nabla(\Omega\omega) + \nabla\left(\frac{1}{2}\Omega^2\rho + \Gamma_g \cdot \Gamma_g\right) \\ &= -\frac{1}{2}\Omega \operatorname{tr} \underline{\chi} \nabla(\Omega\omega) + \frac{1}{2}\Omega^2\nabla\rho + (\Gamma_a, \Gamma_b) \cdot \nabla\Gamma_g, \end{aligned}$$

which implies

$$\nabla_3\nabla(\Omega\omega) + \frac{1}{2}\operatorname{tr} \underline{\chi} \nabla(\Omega\omega) = \frac{1}{2}\nabla(\Omega\rho) + r^{-1}(\Gamma_a, \Gamma_b) \cdot \Gamma_g^{(1)}. \quad (5.8)$$

Since we cannot control  $\nabla\rho$  in  $L^p(S)$ , we use a renormalization method. We define  $\omega^\dagger$  by

$$\nabla_3\omega^\dagger = \frac{1}{2}\Omega\sigma \quad \text{in } \mathcal{K}, \quad \omega^\dagger = 0 \quad \text{on } S(u_0(\underline{u}), \underline{u}). \quad (5.9)$$

Applying Corollary 2.22, we have

$$\Omega\nabla_3^*\nabla\omega^\dagger = *\nabla(\Omega\nabla_3\omega^\dagger) + [\Omega\nabla_3, *\nabla]\omega^\dagger = \frac{1}{2}*\nabla(\Omega^2\sigma) - \Omega\underline{\chi} \cdot *\nabla\omega^\dagger,$$

which implies

$$\nabla_3^*\nabla\omega^\dagger + \frac{1}{2}\operatorname{tr} \underline{\chi}(*\nabla\omega^\dagger) = \frac{1}{2}*\nabla(\Omega\sigma) + \Gamma_b \cdot (\sigma, *\nabla\omega^\dagger). \quad (5.10)$$

We recall the following Bianchi equation from Proposition 2.18:

$$\begin{aligned} \nabla_3\beta + \operatorname{tr} \underline{\chi}\beta &= \nabla\rho + *\nabla\sigma + 2\underline{\omega}\beta + 2\widehat{\chi} \cdot \underline{\beta} + 3(\eta\rho + *\eta\sigma) \\ &= \nabla\rho + *\nabla\sigma + \Gamma_a \cdot \beta + \Gamma_g \cdot \underline{\beta}, \end{aligned}$$

which implies<sup>12</sup>

$$\nabla_3(\Omega\beta) + \text{tr}\chi(\Omega\beta) = \nabla(\Omega\rho) + *\nabla(\Omega\sigma) + r^{-1}\Gamma_a \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_g \cdot \Gamma_b^{(1)}. \quad (5.11)$$

We introduce a new quantity:

$$\varkappa := \nabla(\Omega\omega) + *\nabla\omega^\dagger - \frac{1}{2}\Omega\beta. \quad (5.12)$$

Thus, we have the following equation:

$$\begin{aligned} \nabla_3\varkappa &= \nabla_3\nabla(\Omega\omega) + \nabla_3*\nabla\omega^\dagger - \frac{1}{2}\nabla_3(\Omega\beta) \\ &= -\frac{1}{2}\text{tr}\chi\nabla(\Omega\omega) - \frac{1}{2}\text{tr}\chi(*\nabla\omega^\dagger) + \frac{1}{2}\Omega\text{tr}\chi\beta + r^{-1}(\Gamma_a, \Gamma_b) \cdot \Gamma_g^{(1)} + \Gamma_b \cdot *\nabla\omega^\dagger \\ &= -\frac{1}{2}\text{tr}\chi\varkappa + \frac{1}{4}\Omega\text{tr}\chi\beta + r^{-1}(\Gamma_a, \Gamma_b) \cdot \Gamma_g^{(1)} + \Gamma_b \cdot *\nabla\omega^\dagger. \end{aligned} \quad (5.13)$$

Denoting  $\langle\omega\rangle := (-\Omega\omega, \omega^\dagger)$ , then<sup>13</sup>

$$d_1^*\langle\omega\rangle = \varkappa + \frac{1}{2}\Omega\beta. \quad (5.14)$$

Applying Proposition 2.13 to (5.14), we obtain

$$|r^{1-\frac{2}{p}}\nabla(\Omega\omega)|_{p,S} + |r^{1-\frac{2}{p}}\nabla\omega^\dagger|_{p,S} \lesssim |r^{1-\frac{2}{p}}\varkappa|_{p,S} + |r^{1-\frac{2}{p}}\beta|_{p,S}. \quad (5.15)$$

Also, applying Lemma 2.30 to (5.13), we have

$$\begin{aligned} &|r^{1-\frac{2}{p}}\varkappa|_{p,S}(u, \underline{u}) \\ &\lesssim |r^{1-\frac{2}{p}}\varkappa|_{p,S}(u_0(\underline{u}), \underline{u}) + \int_{u_0(\underline{u})}^u |r^{-\frac{2}{p}}\beta|_{p,S} + |r^{-\frac{2}{p}}(\Gamma_a, \Gamma_b) \cdot \Gamma_g^{(1)}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_b \cdot *\nabla\omega^\dagger|_{p,S} \\ &\lesssim \check{\mathcal{I}}_0 + \Delta_0 + \int_{u_0(\underline{u})}^u \frac{\Delta_0}{r^{\frac{7}{2}}|u|^{\frac{s-4}{2}}} + \frac{\varepsilon^2}{r^4|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^3|u|^{s-2}} + \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}(*\nabla\omega^\dagger)|_{p,S} \\ &\lesssim \check{\mathcal{I}}_0 + \Delta_0 + \varepsilon^2 + \int_{u_0(\underline{u})}^u \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}\varkappa|_{p,S}, \end{aligned}$$

where at the last step we used for all  $s > 3$

$$2 + \frac{s-3}{6} \leq \min\left\{3, \frac{s+1}{2}\right\}. \quad (5.16)$$

Applying Gronwall inequality, we infer

$$|r^{3+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}}\varkappa|_{p,S}(u, \underline{u}) \lesssim \check{\mathcal{I}}_0 + \Delta_0 + \varepsilon^2.$$

Injecting it into (5.15) and recall that

$$|r^{3+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}}\beta|_{p,S} \lesssim |r^{\frac{7}{2}-\frac{2}{p}}|u|^{\frac{s-4}{2}}\beta|_{p,S} \lesssim \Delta_0,$$

we infer

$$|r^{3+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}}\nabla(\Omega\omega)|_{p,S}(u, \underline{u}) \lesssim \check{\mathcal{I}}_0 + \Delta_0 + \varepsilon^2. \quad (5.17)$$

Applying Poincaré inequality, we conclude the first estimate of (5.7). The second estimate is similar and left to the reader. This concludes the proof of Proposition 5.2.  $\square$

<sup>12</sup>Notice that  $\beta\nabla_3\Omega = -2\Omega\omega\beta = \Gamma_a \cdot \beta$  and  $(\rho, \sigma)\nabla\Omega = \Gamma_a \cdot \nabla\Gamma_g$  decay better than the R.H.S. of (5.11).

<sup>13</sup>Recall that  $d_1^*(f, g) = -\nabla f + *\nabla g$ .

### 5.3 Estimates for $\mathcal{O}_{0,1}(\check{\Omega})$ and $r \left| \Omega - \frac{1}{2} \right|$

**Proposition 5.3.** *We have the following estimates:*

$$\left| r^{1-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \check{\Omega} \right|_{p,S} (u, \underline{u}) \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1, \quad (5.18)$$

$$\sup r \left| \Omega - \frac{1}{2} \right| \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2. \quad (5.19)$$

*Proof.* We recall that

$$\Omega \nabla_4 \Omega = \Omega^2 \nabla_4 \log \Omega = -2\Omega^2 \omega.$$

Differentiating by  $r\nabla$  and applying Proposition 2.24, we deduce

$$\begin{aligned} \Omega \nabla_4 (r\nabla \Omega) &= [\Omega \nabla_4, r\nabla] \Omega - 2\Omega r\nabla(\Omega\omega) - 2r\nabla\Omega(\Omega\omega) \\ &= O(r\nabla(\Omega\omega)) + r\Gamma_a \cdot \Gamma_g. \end{aligned}$$

Applying Lemma 2.30 and Proposition 5.2, we infer

$$\begin{aligned} |r^{1-\frac{2}{p}} \nabla \Omega|_{p,S}(u, \underline{u}) &\lesssim |r^{1-\frac{2}{p}} \nabla \Omega|_{p,S}(u, \underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} \left( |r^{1-\frac{2}{p}} \nabla(\Omega\omega)|_{p,S} + \frac{\epsilon^2}{r^3 |u|^{\frac{s-3}{2}}} \right) \\ &\lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2}{r |u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used the fact that  $\Omega = \frac{1}{2}$  on  $\underline{C}_*$ . Combining with Poincaré inequality, we deduce (5.18). On the other hand, we have

$$\Omega \nabla_4 \left( \Omega - \frac{1}{2} \right) = -2\Omega^2 \omega.$$

Recall that Propositions 5.1, 5.2 and 2.29 implies

$$|\Omega\omega|_{\infty,S} \lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2}{r^2}.$$

Applying Lemma 2.30, we obtain

$$\left| \Omega - \frac{1}{2} \right| \lesssim \int_{\underline{u}}^{\underline{u}_*} |\Omega\omega|_{\infty,S} \lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2}{r},$$

which implies (5.19). This concludes the proof of Proposition 5.3.  $\square$

### 5.4 Estimate for $\mathcal{O}_{0,1}(\widetilde{\Omega \operatorname{tr} \chi})$

**Proposition 5.4.** *We have the following estimate:*

$$\left| r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \widetilde{\Omega \operatorname{tr} \chi} \right|_{p,S} (u, \underline{u}) \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1. \quad (5.20)$$

*Proof.* We recall the following equation from Proposition 2.17:

$$\Omega \nabla_4 (\Omega \operatorname{tr} \chi) + \frac{1}{2} \Omega \operatorname{tr} \chi (\Omega \operatorname{tr} \chi) = -4\Omega \operatorname{tr} \chi (\Omega\omega) - \Omega^2 |\widehat{\chi}|^2. \quad (5.21)$$

We derive an evolution equation for  $r\nabla(\Omega \operatorname{tr} \chi)$ . Differentiating (5.21) by  $r\nabla$  and applying Proposition 2.24 to obtain

$$\begin{aligned} &\Omega \nabla_4 (r\nabla(\Omega \operatorname{tr} \chi)) + \Omega \operatorname{tr} \chi (r\nabla \Omega \operatorname{tr} \chi) \\ &= [\Omega \nabla_4, r\nabla] (\Omega \operatorname{tr} \chi) - 4r\nabla(\Omega \operatorname{tr} \chi) \Omega\omega - 4\Omega \operatorname{tr} \chi r\nabla(\Omega\omega) + \Gamma_g \cdot \Gamma_g^{(1)} \\ &= \Gamma_a \cdot \Gamma_g^{(1)} + O(\nabla(\Omega\omega)). \end{aligned} \quad (5.22)$$

Applying Lemma 2.30 and Proposition 5.2, we obtain

$$\begin{aligned}
|r^{3-\frac{2}{p}}\nabla(\Omega \operatorname{tr} \chi)|_{p,S}(u, \underline{u}) &\lesssim |r^{3-\frac{2}{p}}\nabla(\Omega \operatorname{tr} \chi)|_{p,S}(u, \underline{u}_*) + \int_{\underline{u}}^{u_*} |r^{2-\frac{2}{p}}\nabla(\Omega\omega)|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_g^{(1)} \cdot \Gamma_a|_{p,S} \\
&\lesssim \frac{\mathcal{I}_*}{|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \frac{\Delta_0 + \check{\mathcal{I}}_0 + \varepsilon^2}{r^{1+\frac{s-3}{6}}|u|^{\frac{s-3}{3}}} \\
&\lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}.
\end{aligned}$$

Applying Poincaré inequality, we obtain (5.20). This concludes the proof of Proposition 5.4.  $\square$

## 5.5 Estimate for $|r^2(\overline{\Omega \operatorname{tr} \chi} - \frac{1}{r})|$

**Proposition 5.5.** *We have the following estimate:*

$$\sup \left| r^2 \left( \overline{\Omega \operatorname{tr} \chi} - \frac{1}{r} \right) \right| \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (5.23)$$

*Proof.* Applying Lemma 2.4, we obtain

$$\frac{d}{d\underline{u}} \frac{1}{r} = -\frac{1}{r^2} \frac{\partial r}{\partial \underline{u}} = -\frac{1}{2r} \overline{\Omega \operatorname{tr} \chi} = -\frac{1}{2r} \Omega \operatorname{tr} \chi + \frac{\widetilde{\Omega \operatorname{tr} \chi}}{2r}. \quad (5.24)$$

Recalling (5.21) and denoting

$$W := \overline{\Omega \operatorname{tr} \chi} - \frac{1}{r}, \quad (5.25)$$

we deduce

$$\begin{aligned}
\frac{d}{d\underline{u}} W + \frac{1}{2} \Omega \operatorname{tr} \chi W &= \frac{1}{2} W(\widetilde{\Omega \operatorname{tr} \chi}) + \frac{1}{2} \overline{(\Omega \operatorname{tr} \chi)^2} - \overline{4\Omega \operatorname{tr} \chi(\Omega\omega) + \Omega^2|\widehat{\chi}|^2} \\
&= -4\overline{\Omega \operatorname{tr} \chi \Omega\omega} + W \cdot \Gamma_g + \Gamma_g \cdot W.
\end{aligned} \quad (5.26)$$

Applying Proposition 5.1 and  $W \in \Gamma_a$ , we infer

$$\begin{aligned}
|r^{1-\frac{2}{p}}\Omega \operatorname{tr} \chi \Omega\omega|_{p,S}(u, \underline{u}) &\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r^2}, \\
|r^{1-\frac{2}{p}}W \cdot \Gamma_g|_{p,S}(u, \underline{u}) + |r^{1-\frac{2}{p}}\Gamma_g \cdot W|_{p,S}(u, \underline{u}) &\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r^3|u|^{\frac{s-3}{2}}}.
\end{aligned}$$

Using Lemma 2.30, we have

$$|r^{1-\frac{2}{p}}W|_{p,S}(u, \underline{u}) \lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r}.$$

Recalling that  $\nabla W = 0$ , applying Proposition 2.29, we deduce

$$|r^2W|_{\infty,S} \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2.$$

In view of (5.25), this concludes the proof of Proposition 5.5.  $\square$

### 5.6 Estimate for $\mathcal{O}_{0,1}(\widetilde{\Omega \operatorname{tr} \underline{\chi}})$

**Proposition 5.6.** *We have the following estimate:*

$$\left| r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \widetilde{\Omega \operatorname{tr} \underline{\chi}} \right|_{p,S}(u, \underline{u}) \lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1. \quad (5.27)$$

*Proof.* We recall the following equation from Proposition 2.17:

$$\Omega \nabla_3 (\Omega \operatorname{tr} \underline{\chi}) + \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} (\Omega \operatorname{tr} \underline{\chi}) = -4\Omega \operatorname{tr} \underline{\chi} (\Omega \underline{\omega}) - \Omega^2 |\widehat{\underline{\chi}}|^2. \quad (5.28)$$

As in (5.22), we derive an evolution equation for  $r\nabla(\Omega \operatorname{tr} \underline{\chi})$ . Differentiating (5.28) by  $r\nabla$  and applying Proposition 2.24 to obtain

$$\begin{aligned} & \Omega \nabla_3 (r\nabla(\Omega \operatorname{tr} \underline{\chi})) + \Omega \operatorname{tr} \underline{\chi} (r\nabla \Omega \operatorname{tr} \underline{\chi}) \\ &= [\Omega \nabla_3, r\nabla](\Omega \operatorname{tr} \underline{\chi}) - 4r\nabla(\Omega \operatorname{tr} \underline{\chi}) \Omega \underline{\omega} - 4\Omega \operatorname{tr} \underline{\chi} r\nabla(\Omega \underline{\omega}) + \Gamma_b \cdot \Gamma_b^{(1)} \\ &= \Gamma_a \cdot \Gamma_g^{(1)} + \Gamma_b \cdot \Gamma_b^{(1)} + O(\nabla(\Omega \underline{\omega})). \end{aligned} \quad (5.29)$$

Applying Lemma 2.30, we obtain

$$\begin{aligned} |r^{3-\frac{2}{p}} \nabla(\Omega \operatorname{tr} \underline{\chi})|_{p,S}(u, \underline{u}) &\lesssim |r^{3-\frac{2}{p}} \nabla(\Omega \operatorname{tr} \underline{\chi})|_{p,S}(u_0(\underline{u}), \underline{u}) \\ &\quad + \int_{u_0(\underline{u})}^u \left( |r^{2-\frac{2}{p}} \nabla(\Omega \underline{\omega})|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_b^{(1)}| + |r^{2-\frac{2}{p}} \Gamma_a \cdot \Gamma_g^{(1)}|_{p,S} \right) \\ &\lesssim |r^{3-\frac{2}{p}} \nabla(\Omega^{-1} \operatorname{tr} \underline{\chi})|_{p,S}(u_0(\underline{u}), \underline{u}) + |r^{2-\frac{2}{p}} \nabla \Omega|_{p,S} \\ &\quad + \int_{u_0(\underline{u})}^u \left( |r^{2-\frac{2}{p}} \nabla(\Omega \underline{\omega})|_{p,S} + \frac{\epsilon^2}{r^2 |u|^{\frac{s-3}{2}}} + \frac{\epsilon^2}{|u|^{s-1}} \right) \\ &\lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used Propositions 5.2, 5.3 and the fact that  $\check{\mathcal{O}} \leq \check{\mathcal{I}}_0$  at the last step. Combining with Poincaré inequality, this concludes the proof of Proposition 5.6.  $\square$

### 5.7 Estimates for $\mathcal{O}_{0,1}(\eta)$ and $\mathcal{O}_{0,1}(\underline{\eta})$

The following proposition plays an essential role in section 5.

**Proposition 5.7.** *We have the following estimates:*

$$\begin{aligned} |r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \eta|_{p,S}(u, \underline{u}) &\lesssim \check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1, \\ |r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \underline{\eta}|_{p,S}(u, \underline{u}) &\lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \epsilon^2, \quad q = 0, 1. \end{aligned} \quad (5.30)$$

*Proof.* We recall the following null structure equations from Proposition 2.17:

$$\begin{aligned} \nabla_4 \eta &= -\underline{\chi} \cdot \eta + \underline{\chi} \cdot \underline{\eta} - \underline{\beta}, \\ \nabla_3 \underline{\eta} &= -\underline{\chi} \cdot \underline{\eta} + \underline{\chi} \cdot \eta + \underline{\beta}. \end{aligned}$$

We introduce the mass aspect functions

$$\begin{aligned} \mu &:= -\operatorname{div} \eta + \frac{1}{2} \widehat{\underline{\chi}} \cdot \widehat{\underline{\chi}} - \rho, \\ \underline{\mu} &:= -\operatorname{div} \underline{\eta} + \frac{1}{2} \widehat{\underline{\chi}} \cdot \widehat{\underline{\chi}} - \rho. \end{aligned} \quad (5.31)$$



By a direct computation, we obtain<sup>14</sup>

$$\begin{aligned}\nabla_4 \mu + \operatorname{tr} \chi \mu &= G + \frac{1}{2} \operatorname{tr} \chi \underline{\mu}, \\ \nabla_3 \underline{\mu} + \operatorname{tr} \underline{\chi} \underline{\mu} &= \underline{G} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\mu},\end{aligned}\tag{5.32}$$

where

$$G = \operatorname{tr} \chi \rho + \Gamma_b \cdot \nabla \Gamma_g, \quad \underline{G} = \operatorname{tr} \underline{\chi} \rho + \Gamma_b \cdot \nabla \Gamma_b.$$

We introduce the following modifications of  $\mu$  and  $\underline{\mu}$ :

$$[\mu] := \mu + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi}, \quad [\underline{\mu}] := \underline{\mu} + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi}.\tag{5.33}$$

We also define

$$\begin{aligned}\widetilde{[\mu]} &:= [\mu] - \overline{[\mu]} = -\operatorname{div} \eta + \frac{1}{2} (\widehat{\chi} \cdot \widehat{\chi} - \overline{\widehat{\chi}} \cdot \overline{\widehat{\chi}}) + \frac{1}{4} \operatorname{tr} \overline{\chi} \operatorname{tr} \underline{\chi} - \check{\rho}, \\ \widetilde{[\underline{\mu}]} &:= [\underline{\mu}] - \overline{[\underline{\mu}]} = -\operatorname{div} \underline{\eta} + \frac{1}{2} (\widehat{\chi} \cdot \widehat{\chi} - \overline{\widehat{\chi}} \cdot \overline{\widehat{\chi}}) + \frac{1}{4} \operatorname{tr} \overline{\chi} \operatorname{tr} \underline{\chi} - \check{\rho},\end{aligned}\tag{5.34}$$

and remark that  $\widetilde{[\mu]}, \widetilde{[\underline{\mu}]} \in r^{-1} \Gamma_g^{(1)}$ . To simplify the notations, we denote

$$I := \frac{1}{2} (\widehat{\chi} \cdot \widehat{\chi} - \overline{\widehat{\chi}} \cdot \overline{\widehat{\chi}}) + \frac{1}{4} \operatorname{tr} \overline{\chi} \operatorname{tr} \underline{\chi}.$$

By the torsion equation (2.27) and (5.34), we obtain

$$\begin{aligned}\operatorname{div} \eta &= -\widetilde{[\mu]} + I - \check{\rho}, \\ \operatorname{curl} \eta &= \sigma + \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi},\end{aligned}\tag{5.35}$$

and

$$\begin{aligned}\operatorname{div} \underline{\eta} &= -\widetilde{[\underline{\mu}]} + I - \check{\rho}, \\ \operatorname{curl} \underline{\eta} &= -\sigma - \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi}.\end{aligned}\tag{5.36}$$

According to (5.31) and applying Proposition 2.17, we infer

$$\begin{aligned}\nabla_4 [\mu] &= \nabla_4 \left( \mu + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} \right) \\ &= \nabla_4 \mu + \frac{1}{4} \Omega^{-1} \operatorname{tr} \chi \nabla_4 (\Omega \operatorname{tr} \underline{\chi}) + \frac{1}{4} \nabla_4 (\Omega^{-1} \operatorname{tr} \chi) \Omega \operatorname{tr} \underline{\chi} \\ &= -\operatorname{tr} \chi \mu + G + \frac{1}{2} \operatorname{tr} \chi \left( \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho - \operatorname{div} \eta \right) \\ &\quad + \frac{1}{4} \operatorname{tr} \chi \left[ -\frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} - \widehat{\chi} \cdot \widehat{\chi} + 2 \operatorname{div} \underline{\eta} + 2 |\underline{\eta}|^2 + 2 \rho \right] + \frac{1}{4} \operatorname{tr} \underline{\chi} \left[ -\frac{1}{2} (\operatorname{tr} \chi)^2 - |\widehat{\chi}|^2 \right] \\ &= -\operatorname{tr} \chi [\mu] + G + \frac{1}{2} \operatorname{tr} \chi |\underline{\eta}|^2 - \frac{1}{4} \operatorname{tr} \underline{\chi} |\widehat{\chi}|^2.\end{aligned}$$

Thus, we obtain

$$\nabla_4 [\mu] + \operatorname{tr} \chi [\mu] = \operatorname{tr} \chi \rho + \Gamma_b \cdot \nabla \Gamma_g.\tag{5.37}$$

<sup>14</sup>See Lemma 4.3.1 of [29].

By Corollary 2.5, we infer

$$\Omega \nabla_4 \overline{[\mu]} = \overline{\Omega \operatorname{tr} \chi [\mu]} + \overline{\Omega \nabla_4 [\mu]} = -\overline{\Omega \operatorname{tr} \chi [\mu]} + \overline{\Omega \operatorname{tr} \chi \rho} + \Gamma_b \cdot \nabla \Gamma_g.$$

Then, we obtain

$$\nabla_4 \widetilde{[\mu]} + \operatorname{tr} \chi \widetilde{[\mu]} = r^{-1} O(\check{\rho}) + \Gamma_b \cdot \nabla \Gamma_g. \quad (5.38)$$

Applying Lemma 2.30, we deduce

$$\begin{aligned} |r^{2-\frac{2}{p}} \widetilde{[\mu]}|_{p,S}(u, \underline{u}) &\lesssim |r^{2-\frac{2}{p}} \widetilde{[\mu]}|_{p,S}(u, \underline{u}_*) + \int_{\underline{u}}^{u_*} |r^{1-\frac{2}{p}} \check{\rho}|_{p,S} + \int_{\underline{u}}^{u_*} |r^{2-\frac{2}{p}} \Gamma_b \cdot \nabla \Gamma_g|_{p,S} \\ &\lesssim \frac{\mathcal{I}_* + \Delta_0}{r|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \frac{\Delta_0}{r^2|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \frac{\varepsilon^2}{r^2|u|^{s-2}} \\ &\lesssim \frac{\mathcal{I}_* + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-3}{2}}}. \end{aligned} \quad (5.39)$$

Applying Proposition 2.13 to (5.35), recalling that

$$I = \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} + \Gamma_g \cdot \Gamma_b = r^{-1} O(\operatorname{tr} \chi) + r^{-1} O(\operatorname{tr} \underline{\chi}) + \Gamma_g \cdot \Gamma_b,$$

we have for  $q = 0, 1$

$$\begin{aligned} |r^{2-\frac{2}{p}} (r \nabla)^q \eta|_{p,S}(u, \underline{u}) &\lesssim |r^{3-\frac{2}{p}} \widetilde{[\mu]}|_{p,S}(u, \underline{u}) + |r^{2-\frac{2}{p}} (\operatorname{tr} \chi, \operatorname{tr} \underline{\chi})|_{p,S}(u, \underline{u}) \\ &\quad + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_b|_{p,S}(u, \underline{u}) + |r^{3-\frac{2}{p}} (\check{\rho}, \sigma)|_{p,S}(u, \underline{u}) \\ &\lesssim \frac{\check{\mathcal{I}}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used Propositions 5.3, 5.4 and 5.6. This concludes the first estimate of (5.30). The second estimate is similar and left to the reader. This concludes the proof of Proposition 5.7.  $\square$

Remark that Propositions 5.4 and 5.7 implies (5.2), which concludes the first part of Theorem M3. In the sequel, we focus on the second part and hence we assume  $\mathcal{Q}(\Sigma_0 \setminus K) \leq \mathcal{I}_0$ .

### 5.8 Estimates for $\mathcal{O}_{0,1}(\widehat{\chi})$ and $\mathcal{O}_{0,1}(\widehat{\underline{\chi}})$

**Proposition 5.8.** *We have the following estimate:*

$$\begin{aligned} |r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r \nabla)^q \widehat{\chi}|_{p,S}(u, \underline{u}) &\lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2, & q = 0, 1, \\ |r^{1-\frac{2}{p}} |u|^{\frac{s-1}{2}} (r \nabla)^q \widehat{\underline{\chi}}|_{p,S}(u, \underline{u}) &\lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2, & q = 0, 1. \end{aligned} \quad (5.40)$$

*Proof.* We recall the following Codazzi equations from Proposition 2.17

$$\begin{aligned} \operatorname{div} \widehat{\chi} &= \frac{1}{2} \nabla \operatorname{tr} \chi - \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \operatorname{tr} \chi \right) - \underline{\beta}, \\ \operatorname{div} \widehat{\underline{\chi}} &= \frac{1}{2} \nabla \operatorname{tr} \underline{\chi} + \zeta \cdot \left( \widehat{\underline{\chi}} - \frac{1}{2} \operatorname{tr} \underline{\chi} \right) + \underline{\beta}. \end{aligned}$$

Applying Propositions 5.3, 5.4, 5.7 and the fact that  $\zeta = \frac{\eta - \underline{\eta}}{2}$ , we obtain

$$\begin{aligned} |r^{3-\frac{2}{p}} \operatorname{div} \widehat{\chi}|_{p,S} &\lesssim |r^{3-\frac{2}{p}} \nabla \operatorname{tr} \chi|_{p,S} + |r^{2-\frac{2}{p}} \zeta|_{p,S} + |r^{3-\frac{2}{p}} \underline{\beta}|_{p,S} + |r^{3-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p,S} \\ &\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{s-3}} \\ &\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

and respectively

$$\begin{aligned}
|r^{2-\frac{2}{p}} \operatorname{div} \underline{\chi}|_{p,S} &\lesssim |r^{2-\frac{2}{p}} \nabla \operatorname{tr} \chi|_{p,S} + |r^{1-\frac{2}{p}} \zeta|_{p,S} + |r^{2-\frac{2}{p}} \underline{\beta}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_b|_{p,S} \\
&\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r|u|^{s-2}} \\
&\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-1}{2}}}.
\end{aligned}$$

Combining with Proposition 2.13, these conclude the proof of Proposition 5.8.  $\square$

### 5.9 Estimate for $\mathcal{O}_2(\check{\Omega})$

**Proposition 5.9.** *We have the following estimate for  $q = 0, 1, 2$ :*

$$|r^{1+q-\frac{2}{p}} |u|^{\frac{s-3}{2}} \nabla^q \check{\Omega}|_{p,S}(u, \underline{u}) \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (5.41)$$

*Proof.* Recall that  $\eta + \underline{\eta} = 2\nabla \log \Omega$ , which implies

$$\Delta \log \Omega = \frac{1}{2} \operatorname{div}(\eta + \underline{\eta}).$$

Applying Propositions 2.13 and 5.7, we obtain

$$|r^{1-\frac{2}{p}} (r\nabla)^q (\log \Omega)|_{p,S} \lesssim |r^{3-\frac{2}{p}} \operatorname{div}(\eta + \underline{\eta})|_{p,S} \lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \quad q = 0, 1, 2. \quad (5.42)$$

Combining with (5.18), this concludes the proof of Proposition 5.9.  $\square$

### 5.10 Estimate for $\left| r^2 \left( \overline{\Omega \operatorname{tr} \underline{\chi}} + \frac{1}{r} \right) \right|$

**Proposition 5.10.** *We have the following estimate:*

$$\sup \left| r^2 \left( \overline{\Omega \operatorname{tr} \underline{\chi}} + \frac{1}{r} \right) \right| \lesssim \mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2. \quad (5.43)$$

*Proof.* We introduce the following notation:

$$\underline{W} := \overline{\Omega \operatorname{tr} \underline{\chi}} + \frac{1}{r}.$$

As in (5.26), we have

$$\Omega \nabla_3 \underline{W} + \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} \underline{W} = -4 \overline{\Omega \operatorname{tr} \underline{\chi} (\Omega \underline{\omega})} + \Gamma_g \cdot \underline{W} + \Gamma_b \cdot \Gamma_b, \quad (5.44)$$

Using Lemma 2.30 and  $\underline{W} \in \Gamma_a$ , we have

$$\begin{aligned}
\left| r^{1-\frac{2}{p}} \underline{W} \right|_{p,S}(u, \underline{u}) &\lesssim \left| r^{1-\frac{2}{p}} \underline{W} \right|_{p,S}(u_0(\underline{u}), \underline{u}) + \int_{u_0(\underline{u})}^u \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r^2} + \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r|u|^{s-1}} \\
&\lesssim \frac{\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \varepsilon^2}{r}.
\end{aligned}$$

Noticing that  $\nabla \underline{W} = 0$ , taking  $p = 4$  and applying Proposition 2.29, we conclude the proof of Proposition 5.10.  $\square$

In view of Propositions 5.1-5.10, we obtain (5.4). This concludes the proof of Theorem M3.

## 6 Initialization and extension (Theorems M0, M2 and M4)

### 6.1 Preliminaries

#### 6.1.1 Null frame transformations

**Lemma 6.1.** *Let two null frames  $(e_3, e_4, e_1, e_2)$  and  $(e'_3, e'_4, e'_1, e'_2)$  associated to double null foliations  $(u, \underline{u})$  and  $(u', \underline{u}')$ , and assume that they have the same generator  $\underline{L}$  for the incoming direction. Then, a null frame transformation from the null frame  $(e_3, e_4, e_1, e_2)$  to  $(e'_3, e'_4, e'_1, e'_2)$  can be written in the form:*

$$\begin{aligned} e'_4 &= \lambda \left( e_4 + f_B e_B + \frac{1}{4} |f|^2 e_3 \right), \\ e'_3 &= \lambda^{-1} e_3, \\ e'_A &= e_A + \frac{1}{2} f_A e_3, \end{aligned} \tag{6.1}$$

where

$$\lambda = \frac{\Omega}{\Omega'} \tag{6.2}$$

is a scalar function and  $f$  is a 1-form. Moreover, the inverse transform of (6.1) is given by

$$\begin{aligned} e_4 &= \lambda^{-1} e'_4 - f^A e'_A + \frac{\lambda}{4} |f|^2 e'_3, \\ e_3 &= \lambda e'_3, \\ e_A &= e'_A - \frac{\lambda}{2} f_A e'_3. \end{aligned} \tag{6.3}$$

*Proof.* Applying Lemma 3.1 in [34] with  $\underline{f} = 0$ , we obtain (6.1). Notice that (6.3) is a direct consequence of (6.1). Recalling

$$e'_3 = 2\Omega' \underline{L}, \quad e_3 = 2\Omega \underline{L},$$

we obtain (6.2) immediately. □

**Proposition 6.2.** *Under the null frame transform (6.1), some of the Ricci coefficients transform as follows:*

$$\begin{aligned} \lambda^{-1} \operatorname{tr} \chi' &= \operatorname{tr} \chi + \operatorname{div}' f + l.o.t., \\ 0 &= \operatorname{curl}' f + l.o.t., \\ \lambda \underline{\chi}' &= \underline{\chi}, \\ \eta' &= \eta + \frac{1}{2} \lambda \nabla'_{e'_3} f - \underline{\omega} f + l.o.t., \\ \underline{\eta}' &= \underline{\eta} + \frac{1}{4} \operatorname{tr} \underline{\chi} f + l.o.t., \\ \lambda^{-2} \xi' &= \xi + \frac{1}{2} \nabla'_{\lambda^{-1} e'_4} f + \frac{1}{4} \operatorname{tr} \chi f + \omega f + l.o.t., \\ \lambda^{-1} \omega' &= \omega + l.o.t., \end{aligned}$$

where the terms denoted by *l.o.t.* have the following schematic structure

$$l.o.t. := r^{-1} O(f^2) + \Gamma_b \cdot f.$$

The curvature components transform as follows:

$$\begin{aligned}\lambda^{-2}\alpha' &= \alpha + O(f)\beta + O(f^2)\rho + O(f^3)(\sigma, \underline{\beta}) + O(f^4)\underline{\alpha}, \\ \lambda^{-1}\beta' &= \beta + O(f)(\rho, \sigma) + O(f^2)\underline{\beta} + O(f^3)\underline{\alpha}, \\ \rho' &= \rho + O(f)\underline{\beta} + O(f^2)\underline{\alpha}, \\ \sigma' &= \sigma + O(f)\underline{\beta} + O(f^2)\underline{\alpha}, \\ \lambda\underline{\beta}' &= \underline{\beta} + O(f)\underline{\alpha}, \\ \lambda^2\underline{\alpha}' &= \underline{\alpha}.\end{aligned}$$

*Proof.* We only prove the transform formula of  $\omega$  since the others are direct consequences of Proposition 3.3 in [34] for  $\underline{f} = 0$ . Applying (3.13) in [34], we have

$$\lambda^{-1}\omega' = \omega - \frac{1}{2}\lambda^{-1}e'_4(\log \lambda) + \text{Err},$$

where

$$\text{Err} = r^{-1}O(f^2) + \Gamma_b \cdot f.$$

Applying (6.2), we infer

$$\begin{aligned}\lambda^{-1}\omega' &= \omega - \frac{1}{2\lambda}e'_4(\log \Omega - \log \Omega') + \text{Err} \\ &= \omega - \frac{1}{2}\left(e_4 + f^B e_B + \frac{1}{4}|f|^2 e_3\right) \log \Omega - \frac{1}{2\lambda}(2\omega') \\ &= 2\omega - \lambda^{-1}\omega' + \text{Err},\end{aligned}$$

which implies the transformation formula of  $\omega$ .  $\square$

### 6.1.2 Deformations of spheres

**Lemma 6.3.** *Given two double null foliations  $(u, \underline{u})$  and  $(u', \underline{u})$  and we denote their leaves*

$$S := S(u, \underline{u}), \quad S' := S'(u', \underline{u}).$$

*Assume that  $S \cap S' \neq \emptyset$  and the oscillation of  $u$  on  $S'$  satisfies*

$$\sup_{S'} |u - \bar{u}'| \leq \delta_1, \quad \bar{u}' := \frac{1}{|S'|} \int_{S'} u. \quad (6.4)$$

*Let  $V$  be a tensor field satisfying*

$$\nabla_{\partial_u} V = F, \quad (6.5)$$

*where for all  $S$*

$$|r^{-\frac{2}{p}} F|_{L^p(S)} \leq \delta, \quad p \in [1, \infty]. \quad (6.6)$$

*Then, we have*

$$|r^{-\frac{2}{p}} V|_{L^p(S')} \lesssim |r^{-\frac{2}{p}} V|_{L^p(S)} + \delta_1 \delta. \quad (6.7)$$

*Proof.* The proof is largely analogous to Lemma 4.1.7 in [29].

Notice from (6.5) that

$$|\partial_u |V|^2| = |V \cdot (\nabla_{\partial_u} F)| \leq |V| |F|,$$

which implies

$$|\partial_u |V|| \leq |F|.$$

Taking  $p \in S \cap S'$ , we have from (6.4) that

$$\sup_{S'} |u - u(p)| \leq \sup_{S'} |u - \bar{u}'| + |u(p) - \bar{u}'| \leq 2\delta_1. \quad (6.8)$$

In a sphere coordinates  $\phi = (\phi^1, \phi^2)$ , we have

$$|V|(u', \underline{u}, \phi) - |V|(u, \underline{u}, \phi) = \int_0^1 \partial_u(|V|)(u + t(u' - u), \underline{u}, \phi)(u' - u) dt \quad (6.9)$$

Hence, we infer

$$\begin{aligned} |r^{-\frac{2}{p}} V|_{L^p(S')}^p &= \int_{\mathbb{S}^2} r^{-2} |V|^p(u', \underline{u}, \phi) \sqrt{\det(g)|_{S'}} d\phi^1 d\phi^2 \\ &\lesssim \int_{\mathbb{S}^2} |V|^p(u', \underline{u}, \phi) d\phi^1 d\phi^2 \\ &\lesssim \int_{\mathbb{S}^2} |V|^p(u, \underline{u}, \phi) d\phi^1 d\phi^2 + \int_{\mathbb{S}^2} \left( \int_0^1 |F|(u + t(u' - u), \underline{u}, \phi) |u' - u| dt \right)^p d\phi^1 d\phi^2 \\ &\lesssim \int_{\mathbb{S}^2} |r^{-\frac{2}{p}} V|^p \sqrt{\det(g)|_S} d\phi^1 d\phi^2 + \delta_1^p \int_{\mathbb{S}^2} \left( \int_0^1 |F|(u + t(u' - u), \underline{u}, \phi) dt \right)^p d\phi^1 d\phi^2 \\ &\lesssim |r^{-\frac{2}{p}} V|_{L^p(S)}^p + \delta_1^p |F(u + t(u' - u), \underline{u}, \phi)|_{L^p(\mathbb{S}^2) L_t^1(0,1)}^p \\ &\lesssim |r^{-\frac{2}{p}} V|_{L^p(S)}^p + \delta_1^p |r^{-\frac{2}{p}} F|_{L_t^1(0,1) L^p(S(u+t(u'-u), \underline{u}))}^p \\ &\lesssim |r^{-\frac{2}{p}} V|_{L^p(S)}^p + \delta_1^p \delta^p, \end{aligned}$$

where we used (6.6), (6.8), (6.9) and General Minkowski inequality. This concludes the proof of Lemma 6.3.  $\square$

**Remark 6.4.** *In the remainder of this paper, for any quantity  $X$ , we denote the quantity  $X_{(0)}$  associated to the initial data layer region  $\mathcal{K}_{(0)}$  by*

$$X' := X_{(0)},$$

to simplify the notations. For example:

$$u' := u_{(0)}, \quad \eta' := \eta_{(0)}, \quad \mathcal{O}' := \mathcal{O}_{(0)}, \quad \mathfrak{R}' := \mathfrak{R}_{(0)}, \quad S' := S_{(0)}, \quad \mathcal{K}' := \mathcal{K}_{(0)}.$$

## 6.2 The initial hypersurface (Theorem M0)

The goal of this section is to prove Theorem M0 which we recall below for convenience.

**Theorem M0.** *Under the assumptions*

$$\mathcal{O}' \leq \varepsilon_0, \quad \mathfrak{R}' \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon, \quad (6.10)$$

we have

$$\mathfrak{R}_0 \lesssim \varepsilon_0, \quad \check{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (6.11)$$

If in addition we assume that

$$\check{\mathcal{O}} \lesssim \varepsilon_0, \quad (6.12)$$

then, we have

$$\text{Osc} \lesssim \varepsilon_0, \quad \underline{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (6.13)$$

Recall that we denoted by  $(f, \lambda)$  the frame transformation from  $(u, \underline{u})$  to  $(u', \underline{u}')$ , and let  $(f', \lambda')$  its inverse transformation. Notice from (6.1) and (6.3) that we have

$$\lambda' = \lambda^{-1}, \quad f' = -\lambda f. \quad (6.14)$$

**Remark 6.5.** In  $\mathcal{K}'$ , we have  $|u| \simeq r$ . Hence,  $\Gamma_b$  has the same decay as  $\Gamma_g$  in  $\mathcal{K}'$ . The assumption  $\text{Osc} \leq \varepsilon$  implies that  $r \simeq r'$  and  $f \in r\Gamma_g$ .

We first prove (6.11) under the assumptions in (6.10).

**Lemma 6.6.** Under the assumptions

$$\mathcal{O}' \leq \varepsilon_0, \quad \mathfrak{R}' \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon,$$

we have

$$\mathfrak{R}_0 \lesssim \varepsilon_0.$$

*Proof.* Recall that

$$\mathfrak{R}_0^2 = \int_{\Sigma_0 \setminus K} \sum_{l=0}^1 r^s \left( |\mathfrak{d}^l \alpha|^2 + |\mathfrak{d}^l \beta|^2 + |\mathfrak{d}^l(\check{\rho}, \sigma)|^2 + |\mathfrak{d}^l \underline{\beta}|^2 + |\mathfrak{d}^l \underline{\alpha}|^2 \right) + \sup_{\Sigma_0 \setminus K} |r^3 \bar{\rho}|^2,$$

Applying Proposition 6.2 with (6.14), we infer<sup>15</sup>

$$\begin{aligned} \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1} \alpha|^2 &\lesssim \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1}(\lambda^{-2} \alpha')|^2 + r^s \left| \mathfrak{d}^{\leq 1}(\lambda^{-1} f \cdot \beta') \right|^2 + r^s \left| \mathfrak{d}^{\leq 1}(f^2 \cdot \rho') \right|^2 + \text{l.o.t.} \\ &\lesssim \mathfrak{R}'^2 + \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1}(f^2) \rho'|^2 + r^s |f^2 \mathfrak{d}^{\leq 1} \rho'|^2 \\ &\lesssim \mathfrak{R}'^2 + \int_{\Sigma_0 \setminus K} r^s \left| \frac{\varepsilon^3}{r^{s+2}} \right|^2 \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1} \beta|^2 &\lesssim \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1}(\lambda^{-1} \beta')|^2 + r^s |\mathfrak{d}^{\leq 1} f \cdot (\rho', \sigma')|^2 \\ &\lesssim \mathfrak{R}'^2 + \int_{\Sigma_0 \setminus K} r^s \left| \frac{\varepsilon^2}{r^{\frac{s+5}{2}}} \right|^2 \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

By the same method, we obtain:

$$\int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1} \check{\rho}|^2 + r^s |\mathfrak{d}^{\leq 1} \sigma|^2 + r^s |\mathfrak{d}^{\leq 1} \underline{\beta}|^2 + r^s |\mathfrak{d}^{\leq 1} \underline{\alpha}|^2 \lesssim \varepsilon_0^2.$$

Finally, we estimate

$$|r^3 \bar{\rho}| \lesssim |r^3 \rho|_\infty \lesssim |r^3 \rho'|_\infty + |r^4 \Gamma_g \cdot \underline{\beta}'|_\infty \lesssim \varepsilon_0.$$

This concludes the proof of Lemma 6.6. □

<sup>15</sup>Recall that  $\mathfrak{d}^{\leq 1} f = O(\varepsilon)r^{-\frac{s-1}{2}}$  and  $\rho' = O(\varepsilon)r^{-3}$ .

**Lemma 6.7.** *Under the assumptions*

$$\mathcal{O}' \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon,$$

we have

$$\check{\mathcal{O}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0.$$

*Proof.* We recall from Proposition 6.2 that<sup>16</sup>

$$\begin{aligned} \Omega'^{-1} \text{tr } \underline{\chi}' &= \Omega^{-1} \text{tr } \underline{\chi} + r\Gamma_g \cdot \Gamma_g, \\ \Omega'\omega' &= \Omega\omega + r\Gamma_g \cdot \Gamma_g. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}}\Omega\omega|_{p,S(u_0(\underline{u}),\underline{u})} &\lesssim |r^{2-\frac{2}{p}}\Omega'\omega'|_{p,S(u_0(\underline{u}),\underline{u})} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S(u_0(\underline{u}),\underline{u})} \\ &\lesssim |r^2\Omega'\omega'|_{\infty,S(u_0(\underline{u}),\underline{u})} + \frac{\varepsilon^2}{r^{s-2}} \\ &\lesssim \varepsilon_0. \end{aligned}$$

Notice from (6.1) and Proposition 2.17 that

$$\begin{aligned} re_A(\Omega\omega) &= re'_A(\Omega'\omega') - \frac{\lambda}{2}rf_Ae'_3(\Omega'\omega') + r\Gamma_g \cdot \Gamma_g^{(1)} \\ &= re'_A(\Omega'\omega') + O(r\rho')f + r\Gamma_g \cdot \Gamma_g^{(1)}. \end{aligned}$$

We deduce

$$\begin{aligned} |r^{1-\frac{2}{p}}\nabla(\Omega\omega)|_{p,S(u_0(\underline{u}),\underline{u})} &\lesssim |r^{1-\frac{2}{p}}\nabla'\Omega'\omega'|_{p,S(u_0(\underline{u}),\underline{u})} + |r^{2-\frac{2}{p}}\Gamma_g^{(1)} \cdot \Gamma_g|_{p,S(u_0(\underline{u}),\underline{u})} \\ &\lesssim |r\nabla'\Omega'\omega'|_{\infty,S(u_0(\underline{u}),\underline{u})} + \frac{\varepsilon^2}{r^{\frac{s+1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned}$$

Similarly, we have

$$|r^{1-\frac{2}{p}}\nabla(\Omega^{-1} \text{tr } \underline{\chi})|_{p,S(u_0(\underline{u}),\underline{u})} \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Applying Poincaré inequality and recalling the definition of  $\check{\mathcal{O}}(\Sigma_0 \setminus K)$ , we obtain

$$\check{\mathcal{O}}(\Sigma_0 \setminus K) = \sum_{q=0}^1 \left( \mathcal{O}_q(\Sigma_0 \setminus K)(\widetilde{\Omega^{-1} \text{tr } \underline{\chi}}) + \mathcal{O}_q(\Sigma_0 \setminus K)(\widetilde{\Omega\omega}) \right) + \mathcal{O}_0(\Sigma_0 \setminus K)(\Omega\omega) \lesssim \varepsilon_0.$$

This concludes the proof of Lemma 6.7.  $\square$

Remark that Lemmas 6.6 and 6.7 imply (6.11), which concludes the first part of Theorem M0. We now focus on the second part of Theorem M0 under the additional assumption (6.12).

**Lemma 6.8** (Oscillation lemma). *Under the assumptions:*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon, \quad \check{\mathcal{O}} \lesssim \varepsilon_0, \quad (6.15)$$

we have

$$\text{Osc} \lesssim \varepsilon_0. \quad (6.16)$$

<sup>16</sup>Recall that  $\Gamma_b = \Gamma_g$  in  $\mathcal{K}_{(0)}$  and  $f \in r\Gamma_g$ .



*Proof. Step 1.* Estimate for  $\text{Osc}(f)$ .

We have from Proposition 6.2 and Remark 6.5

$$\begin{aligned}\lambda^{-1} \text{tr} \chi' &= \text{tr} \chi + \text{div}' f + f \cdot (\Gamma_g, r^{-1} f), \\ 0 &= \text{curl}' f + f \cdot (\Gamma_g, r^{-1} f),\end{aligned}$$

which implies from (6.2) and the fact that  $f \in r\Gamma_g$ <sup>17</sup>

$$\begin{aligned}\Omega' \text{tr} \chi' &= \Omega \text{tr} \chi + \Omega \text{div} f + f \cdot \Gamma_g, \\ 0 &= \text{curl} f + f \cdot \Gamma_g.\end{aligned}\tag{6.17}$$

Differentiating it by  $e_A$  and applying (6.1), we obtain

$$e'_A(\Omega' \text{tr} \chi') - \frac{\lambda}{2} f_A e'_3(\Omega' \text{tr} \chi') = e_A(\Omega \text{tr} \chi) + \Omega e_A(\text{div} f) + \Gamma_g \cdot \Gamma_g^{(1)}.$$

Recall that we have from Proposition 2.17

$$\begin{aligned}e'_3(\Omega' \text{tr} \chi') &= -\frac{1}{2} \Omega' \text{tr} \chi' \text{tr} \underline{\chi}' + r^{-1} \Gamma_a + r^{-1} \Gamma_g^{(1)} + \Gamma_g \cdot \Gamma_g \\ &= \frac{1}{r^2} + r^{-1} \Gamma_a + \Gamma_g \cdot \Gamma_g.\end{aligned}$$

Hence, we have

$$\Omega e_A(\text{div} f) + \frac{1}{2r^2} f_A = e'_A(\Omega' \text{tr} \chi') - e_A(\Omega \text{tr} \chi) + \Gamma_a \cdot \Gamma_g^{(1)},$$

which implies

$$e_A(\text{div} f) + \frac{1}{r^2} f_A = \Omega^{-1}(e'_A(\Omega' \text{tr} \chi') - e_A(\Omega \text{tr} \chi)) + \Gamma_a \cdot \Gamma_g^{(1)}.$$

We also have from (6.17)

$$e_A(\text{curl} f) = \Gamma_g \cdot \Gamma_g^{(1)}.$$

Hence, we infer

$$\begin{aligned}\left(d_1^* d_1 - \frac{1}{r^2}\right) f &= -\nabla \text{div} f + {}^* \nabla \text{curl} f - \frac{1}{r^2} f \\ &= -\Omega^{-1} \nabla'(\Omega' \text{tr} \chi') + \Omega^{-1} \nabla(\Omega \text{tr} \chi) + \Gamma_a \cdot \Gamma_g^{(1)}.\end{aligned}\tag{6.18}$$

Recall from (2.23) and Corollary 2.20 that

$$\begin{aligned}d_1^* d_1 - \frac{1}{r^2} &= -\Delta_1 + \mathbf{K} - \frac{1}{r^2} \\ &= -\Delta_1 - \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} + \Gamma_g \cdot \Gamma_g - \rho - \frac{1}{r^2} \\ &= -\Delta_1 + r^{-1} \Gamma_a + r^{-1} \Gamma_g^{(1)}.\end{aligned}$$

Injecting it into (6.18), we deduce

$$-\Delta_1 f = -\Omega^{-1} \nabla'(\Omega' \text{tr} \chi') + \Omega^{-1} \nabla(\Omega \text{tr} \chi) + \Gamma_a \cdot \Gamma_g^{(1)}.$$

---

<sup>17</sup>We have from (6.1) and Proposition 6.2 that  $\nabla' f - \nabla f = O(f) \cdot \nabla_3 f = f \cdot \Gamma_g$ .

We have from Proposition 2.13

$$|r^{\frac{s-1}{2}-\frac{2}{p}}(r\nabla)^{\leq 2}f|_{p,S} \lesssim |r^{\frac{s+3}{2}-\frac{2}{p}}(\nabla'\Omega' \operatorname{tr} \chi', \nabla\Omega \operatorname{tr} \chi)|_{p,S} + |r^{\frac{s+3}{2}-\frac{2}{p}}\Gamma_a \cdot \Gamma_g^{(1)}|_{p,S} \lesssim \varepsilon_0.$$

Applying Proposition 2.29, we deduce

$$|r^{\frac{s-1}{2}}(r\nabla)^{\leq 1}f|_{\infty,S} \lesssim \varepsilon_0. \quad (6.19)$$

We have from Proposition 6.2 that

$$\begin{aligned} \sup_{\mathcal{K}'} |\nabla_3 f| &\lesssim \sup_{\mathcal{K}'} |\eta' - \eta| + \sup_{\mathcal{K}'} |f \cdot \Gamma_a| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{\frac{s+3}{2}}} \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}, \\ \sup_{\mathcal{K}'} |\nabla_4 f| &\lesssim \sup_{\mathcal{K}'} |r^{-1}f| + \sup_{\mathcal{K}'} |f \cdot \Gamma_a| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{\frac{s+3}{2}}} \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned} \quad (6.20)$$

Combining (6.19) and (6.20), we deduce

$$\sup_{\mathcal{K}'} (|f| + |\mathfrak{d}f|) \lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}},$$

which implies

$$\operatorname{Osc}(f) \lesssim \varepsilon_0. \quad (6.21)$$

**Step 2.** Estimate for  $\operatorname{Osc}(\lambda)$ .

We recall from (6.2) that

$$|r\dot{\lambda}| = |r(\lambda - 1)| \lesssim r|\Omega' - \Omega| \lesssim r \left| \Omega' - \frac{1}{2} + \frac{1}{2} - \Omega \right| \lesssim \varepsilon_0,$$

where we used (6.15) in the last step. Hence, we obtain

$$\operatorname{Osc}(\lambda) \lesssim \varepsilon_0. \quad (6.22)$$

**Step 3.** Estimate for  $\operatorname{Osc}(r)$ .

We have

$$\begin{aligned} \frac{1}{r} - \frac{1}{r'} &= \Omega' \operatorname{tr} \chi' - \frac{1}{r'} - \Omega \operatorname{tr} \chi + \frac{1}{r} + (\Omega \operatorname{tr} \chi - \Omega' \operatorname{tr} \chi') \\ &= \widetilde{\Omega' \operatorname{tr} \chi'} - \widetilde{\Omega \operatorname{tr} \chi} + \overline{\Omega' \operatorname{tr} \chi'} - \frac{1}{r'} + \overline{\Omega \operatorname{tr} \chi} - \frac{1}{r} + (\Omega \operatorname{tr} \chi - \Omega' \operatorname{tr} \chi'). \end{aligned} \quad (6.23)$$

Applying (6.17) and (6.19), we deduce

$$|\Omega \operatorname{tr} \chi - \Omega' \operatorname{tr} \chi'| \lesssim |\operatorname{div} f| + |r\Gamma_g \cdot \Gamma_g| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \quad (6.24)$$

Moreover, we have from (6.15) that

$$r^{\frac{s+1}{2}} \left( |\widetilde{\Omega' \operatorname{tr} \chi'}| + |\widetilde{\Omega \operatorname{tr} \chi}| \right) + r^2 \left( \left| \overline{\Omega' \operatorname{tr} \chi'} - \frac{1}{r'} \right| + \left| \overline{\Omega \operatorname{tr} \chi} - \frac{1}{r} \right| \right) \lesssim \varepsilon_0. \quad (6.25)$$

Hence, we obtain from (6.23), (6.24) and (6.25) that

$$\left| \frac{1}{r} - \frac{1}{r'} \right| \lesssim \frac{\varepsilon_0}{r^2},$$

which implies

$$\operatorname{Osc}(r) \lesssim \varepsilon_0. \quad (6.26)$$

Combining (6.21), (6.22) and (6.26), we infer

$$\operatorname{Osc} \lesssim \varepsilon_0. \quad (6.27)$$

This concludes the proof of Lemma 6.8.  $\square$

*Proof of Theorem M0.* We have from Lemma 6.8 that

$$\text{Osc} \lesssim \varepsilon_0. \quad (6.28)$$

We recall from Proposition 6.2 that

$$\underline{\eta}' = \underline{\eta} + \frac{1}{4} \text{tr} \underline{\chi} f + r \Gamma_g \cdot \Gamma_g.$$

Applying (6.1), we have

$$\nabla' \underline{\eta}' = \nabla \underline{\eta} + r^{-1} O(\nabla f) + r \Gamma_g \cdot \Gamma_g^{(1)}.$$

Thus, we obtain from (6.10) and (6.28) that

$$\sup_{\mathcal{K}'} |\underline{\eta}| \lesssim \sup_{\mathcal{K}'} |\underline{\eta}'| + \sup_{\mathcal{K}'} |r^{-1} f| + \sup_{\mathcal{K}'} |r \Gamma_g \cdot \Gamma_g| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}, \quad (6.29)$$

and for  $p \in [2, 4]$

$$\begin{aligned} |r^{1-\frac{2}{p}} \nabla \underline{\eta}|_{p, S(u_0(\underline{u}), \underline{u})} &\lesssim |r^{1-\frac{2}{p}} \nabla' \underline{\eta}'|_{p, S(u_0(\underline{u}), \underline{u})} + |r^{-\frac{2}{p}} \nabla f|_{p, S(u_0(\underline{u}), \underline{u})} + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_g^{(1)}|_{p, S(u_0(\underline{u}), \underline{u})} \\ &\lesssim |r \nabla' \underline{\eta}'|_{\infty, S(u_0(\underline{u}), \underline{u})} + |\nabla f|_{\infty, S(u_0(\underline{u}), \underline{u})} + \frac{\varepsilon^2}{r^{s-1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned} \quad (6.30)$$

Finally, we recall from Proposition 6.2 that

$$\Omega \text{tr} \underline{\chi} = \frac{\Omega^2}{\Omega'} \text{tr} \underline{\chi}'.$$

Hence, we infer

$$\begin{aligned} \Omega \text{tr} \underline{\chi} + \frac{1}{r} &= \frac{\Omega^2}{\Omega'} \text{tr} \underline{\chi}' + \frac{1}{r} \\ &= \left(1 - \frac{\Omega^2}{\Omega'^2}\right) \text{tr} \underline{\chi}' + \Omega' \text{tr} \underline{\chi}' + \frac{1}{r'} + \frac{1}{r} - \frac{1}{r'}. \end{aligned}$$

Thus, we have from (6.10) and (6.28)

$$r^2 \left| \Omega \text{tr} \underline{\chi} + \frac{1}{r} \right| \lesssim r^2 \left| \Omega' \text{tr} \underline{\chi}' + \frac{1}{r'} \right| + r |\overset{\circ}{\lambda}| + |r' - r| \lesssim \varepsilon_0. \quad (6.31)$$

Combining (6.29), (6.30), (6.31) and Lemma 6.7 and recalling the definition of  $\underline{\mathcal{Q}}(\Sigma_0 \setminus K)$ , we obtain

$$\underline{\mathcal{Q}}(\Sigma_0 \setminus K) \lesssim \varepsilon_0. \quad (6.32)$$

This concludes the proof of Theorem M0.  $\square$

### 6.3 The last slice (Theorem M2)

We now assume that the last slice  $\underline{\mathcal{C}}_*$  is endowed with a geodesic foliation. Notice that  $\Omega = \frac{1}{2}$  and  $\underline{\omega} = 0$  on  $\underline{\mathcal{C}}_*$ . We write the null structure equations along the last slice in the following form:

$$\begin{aligned} \nabla_3 \zeta + \text{tr} \underline{\chi} \zeta + 2 \widehat{\underline{\chi}} \cdot \zeta &= -\beta, \\ \nabla_3 \text{tr} \underline{\chi} + \frac{1}{2} (\text{tr} \underline{\chi})^2 + |\widehat{\underline{\chi}}|^2 &= 0, \\ \nabla_3 \widehat{\underline{\chi}} + \frac{1}{2} \text{tr} \underline{\chi} \widehat{\underline{\chi}} + \frac{1}{2} \text{tr} \chi \widehat{\underline{\chi}} - \nabla \widehat{\otimes} \zeta - \zeta \widehat{\otimes} \zeta &= 0, \\ \nabla_3 \text{tr} \chi + \frac{1}{2} \text{tr} \underline{\chi} \text{tr} \chi + \widehat{\underline{\chi}} \cdot \widehat{\chi} - 2 \text{div} \zeta - 2|\zeta|^2 &= 2\rho. \end{aligned} \quad (6.33)$$

We also have the Codazzi equations:

$$\begin{aligned}\operatorname{div} \widehat{\chi} &= \frac{1}{2} \nabla \operatorname{tr} \chi - \zeta \left( \widehat{\chi} - \frac{1}{2} \operatorname{tr} \chi \right) - \underline{\beta}, \\ \operatorname{div} \widehat{\underline{\chi}} &= \frac{1}{2} \nabla \operatorname{tr} \underline{\chi} + \zeta \left( \widehat{\underline{\chi}} - \frac{1}{2} \operatorname{tr} \underline{\chi} \right) + \underline{\beta}.\end{aligned}\tag{6.34}$$

For convenience, we recall the statement of Theorem M2.

**Theorem M2.** *Let  $\underline{C}_*$  endowed with a geodesic foliation. We assume that*

$$\mathcal{O}^*(\underline{C}_*) \leq \varepsilon, \quad \mathcal{R}_0^S \leq \Delta_0, \quad \mathcal{O}' \leq \mathcal{I}_0.\tag{6.35}$$

Then, we have

$$\mathcal{O}^*(\underline{C}_*) \lesssim \Delta_0 + \mathcal{I}_0 + \varepsilon^2.\tag{6.36}$$

In the remainder of this section, we always assume  $p \in [2, 4]$  and we denote

$$S := S(u, \underline{u}_*), \quad S_* := \underline{C}_* \cap \Sigma_0.\tag{6.37}$$

*Proof of Theorem M2.* Notice that the assumption  $\mathcal{O}' \leq \mathcal{I}_0$  controls the initial data on the last sphere  $S_* := \underline{C}_* \cap \Sigma_0$ . The idea of the proof is to transport the estimates in the direction of  $\nabla_3$  along  $\underline{C}_*$  using (6.33). The proof is similar to section 5, and in fact easier since  $\Omega = \frac{1}{2}$  on  $\underline{C}_*$ . We only provide a sketch.

**Step 1.** Estimate for  $\mathcal{O}_{0,1,2}^*(\widetilde{\operatorname{tr} \underline{\chi}})$ .

The estimates for  $\mathcal{O}_0^*(\operatorname{tr} \underline{\chi})$  and  $\mathcal{O}_1^*(\operatorname{tr} \underline{\chi})$  follow directly from the results of section 5.6 in the particular case  $\Omega = \frac{1}{2}$  and  $\underline{\omega} = 0$ .

We then estimate  $\nabla^2 \operatorname{tr} \underline{\chi}$ . According to the assumption  $\mathcal{O}^*(\underline{C}_*) \leq \varepsilon$ , we have on  $\underline{C}_*$ :

$$r \nabla \operatorname{tr} \underline{\chi} \in \Gamma_g, \quad (r \nabla)^2 \operatorname{tr} \underline{\chi} \in \Gamma_g^{(1)}.\tag{6.38}$$

We recall from (6.33) that

$$\nabla_3 \operatorname{tr} \underline{\chi} + \frac{1}{2} (\operatorname{tr} \underline{\chi})^2 = -|\widehat{\underline{\chi}}|^2.$$

Differentiating it by  $r^2 \Delta$  and applying Proposition 2.24, we deduce

$$\nabla_3 (r^2 \Delta \operatorname{tr} \underline{\chi}) + \operatorname{tr} \underline{\chi} (r^2 \Delta \operatorname{tr} \underline{\chi}) = -r^2 \Delta |\widehat{\underline{\chi}}|^2 + \Gamma_b \cdot \Gamma_b^{(1)}.\tag{6.39}$$

We recall from (6.34) that

$$\operatorname{div} \widehat{\underline{\chi}} = \frac{1}{2} \nabla \operatorname{tr} \underline{\chi} - \frac{1}{2} \operatorname{tr} \underline{\chi} \zeta + \underline{\beta} + \Gamma_g \cdot \Gamma_b = \underline{\beta} + r^{-1} \Gamma_g.\tag{6.40}$$

Applying Proposition 2.13, we obtain<sup>18</sup>

$$\nabla \widehat{\underline{\chi}} = \underline{\beta}^{(0)} + r^{-1} \Gamma_g.$$

Hence, we have

$$|\nabla \widehat{\underline{\chi}}|^2 = \underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)} + r^{-2} \Gamma_g \cdot \Gamma_b^{(1)}.$$

<sup>18</sup>Here,  $\underline{\beta}^{(0)}$  denotes a quantity which has same or even better decay and regularity than  $\underline{\beta}$ .

Moreover, we have from (2.23) and (6.40)

$$2\Delta_2 \underline{\hat{\chi}} \cdot \underline{\hat{\chi}} = 4((\mathbf{K} - d_2^* d_2) \underline{\hat{\chi}}) \cdot \underline{\hat{\chi}} = -4d_2^* d_2 \underline{\hat{\chi}} \cdot \underline{\hat{\chi}} + r^{-2} \Gamma_b \cdot \Gamma_b^{(1)} = -4(d_2^* \underline{\beta}) \cdot \underline{\hat{\chi}} + r^{-2} \Gamma_b \cdot \Gamma_b^{(1)}.$$

We then compute

$$\Delta |\underline{\hat{\chi}}|^2 = 2\nabla^C \left( (\nabla_C \underline{\hat{\chi}}_{AB}) \underline{\hat{\chi}}^{AB} \right) = 2(\Delta_2 \underline{\hat{\chi}}) \cdot \underline{\hat{\chi}} + 2|\nabla \underline{\hat{\chi}}|^2 = -4(d_2^* \underline{\beta}) \cdot \underline{\hat{\chi}} + \underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)} + r^{-2} \Gamma_b \cdot \Gamma_b^{(1)}.$$

Injecting it into (6.39), we deduce

$$\nabla_3(r^2 \Delta \operatorname{tr} \underline{\chi}) + \operatorname{tr} \underline{\chi}(r^2 \Delta \operatorname{tr} \underline{\chi}) = 4r^2(d_2^* \underline{\beta}) \cdot \underline{\hat{\chi}} + r^2 \underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)} + \Gamma_b \cdot \Gamma_b^{(1)}. \quad (6.41)$$

Applying Proposition 2.13, we have

$$|d_2^* \underline{\beta}|_{p,S} \lesssim |d_1 \underline{\beta}|_{p,S},$$

which implies that there exists a bounded elliptic operator  $A : L^p(S) \rightarrow L^p(S)$  of order 0 such that

$$d_2^* \underline{\beta} = A(d_1 \underline{\beta}).$$

We have from Proposition 2.23

$$\begin{aligned} [\Omega \nabla_3, A] &= [\Omega \nabla_3, rd_2^* \circ (rd_1)^{-1}] = [\Omega \nabla_3, rd_2^*] \circ (rd_1)^{-1} + rd_2^* \circ [\Omega \nabla_3, (rd_1)^{-1}] \\ &= \Gamma_b \cdot rd_2^* \circ (rd_1)^{-1} + r \underline{\beta}^{(0)} \cdot (rd_1)^{-1} - rd_2^* \circ (rd_1)^{-1} \circ [\Omega \nabla_3, rd_1] \circ (rd_1)^{-1} \\ &= \Gamma_b \cdot A + r \underline{\beta}^{(0)} \cdot (rd_1)^{-1} - A \circ (\Gamma_b \cdot rd_1 + r \underline{\beta}^{(0)}) \circ (rd_1)^{-1} \\ &= \Gamma_b + r \underline{\beta}^{(0)} \cdot (rd_1)^{-1}. \end{aligned} \quad (6.42)$$

We recall from Corollary 2.20 and Lemma 2.5 that

$$\nabla_3(\underline{\rho}, \underline{\sigma}) + \frac{3}{2} \operatorname{tr} \underline{\chi}(\underline{\rho}, \underline{\sigma}) = -d_1 \underline{\beta} + \Gamma_g \cdot \underline{\alpha}.$$

Combining with (6.42), we deduce

$$\nabla_3(r^2 A(\underline{\rho}, \underline{\sigma})) = r(\underline{\rho}, \underline{\sigma})^{(0)} - A(r^2 d_1 \underline{\beta}) + r^3 \underline{\beta}^{(0)} \cdot (\underline{\rho}, \underline{\sigma})^{(0)} + r^2 \Gamma_g \cdot \underline{\alpha}.$$

Hence, we have

$$4\underline{\hat{\chi}} \cdot (r^2 d_2^* \underline{\beta}) = -4\nabla_3(r^2 A(\underline{\rho}, \underline{\sigma}) \underline{\hat{\chi}}) + r^2 \underline{\alpha}^{(0)} \cdot (\underline{\rho}^{(0)}, \underline{\sigma}^{(0)}) + \Gamma_b \cdot \Gamma_b^{(1)}.$$

Injecting it into (6.41), we infer<sup>19</sup>

$$\nabla_3(r^2 \Delta \operatorname{tr} \underline{\chi} + 4r^2 A(\underline{\rho}, \underline{\sigma}) \underline{\hat{\chi}}) + \operatorname{tr} \underline{\chi}(r^2 \Delta \operatorname{tr} \underline{\chi} + 4r^2 A(\underline{\rho}, \underline{\sigma}) \underline{\hat{\chi}}) = r^2 \underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)} + \Gamma_b \cdot \Gamma_b^{(1)}. \quad (6.43)$$

Applying Lemma 2.30, we have

$$\begin{aligned} & \left| r^{4-\frac{2}{p}} (\Delta \operatorname{tr} \underline{\chi} + 4A(\underline{\rho}, \underline{\sigma}) \underline{\hat{\chi}}) \right|_{p,S} \\ & \lesssim \left| r^{4-\frac{2}{p}} (\Delta \operatorname{tr} \underline{\chi} + 4A(\underline{\rho}, \underline{\sigma}) \underline{\hat{\chi}}) \right|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{4-\frac{2}{p}} \underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_b^{(1)}|_{p,S} \\ & \lesssim \frac{\mathcal{I}_0 + \Delta_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{8}} \underline{\beta}^{(0)}|_{8,S}^2 + \frac{\varepsilon^2}{|u|^{s-1}} \\ & \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{2}} \underline{\beta}^{(1)}|_{2,S}^2 \\ & \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

<sup>19</sup>Note that  $\underline{\alpha}^{(0)} \cdot (\underline{\rho}, \underline{\sigma})^{(0)}$  has the same decay as  $\underline{\beta}^{(0)} \cdot \underline{\beta}^{(0)}$ .

where we used the Sobolev inequality

$$|r^{-\frac{2}{8}}\underline{\beta}^{(0)}|_{8,S} \lesssim |r^{-\frac{2}{2}}\underline{\beta}^{(1)}|_{2,S}.$$

Hence, we obtain from Proposition 2.13

$$\left| r^{4-\frac{2}{p}} \nabla^2 \operatorname{tr} \underline{\chi} \right|_{p,S} \lesssim \left| r^{4-\frac{2}{p}} \Delta \operatorname{tr} \underline{\chi} \right|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}} + \left| r^{4-\frac{2}{p}} A(\underline{\rho}, \underline{\sigma}) \widehat{\underline{\chi}} \right|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}. \quad (6.44)$$

Hence, we obtain

$$\mathcal{O}_0^*(\widetilde{\operatorname{tr} \underline{\chi}}) + \mathcal{O}_1^*(\widetilde{\operatorname{tr} \underline{\chi}}) + \mathcal{O}_2^*(\widetilde{\operatorname{tr} \underline{\chi}}) \lesssim \mathcal{I}_0 + \Delta_0 + \varepsilon^2. \quad (6.45)$$

**Step 2.** Estimate for  $[\underline{\mu}]$ .

We introduce the mass aspect function  $\underline{\mu}$  as in (5.31):

$$\underline{\mu} := \operatorname{div} \zeta + \frac{1}{2} \widehat{\underline{\chi}} \cdot \widehat{\underline{\chi}} - \rho. \quad (6.46)$$

We recall from Lemma 4.3.1 in [29] that

$$\nabla_3 \underline{\mu} + \operatorname{tr} \underline{\chi} \underline{\mu} = \underline{G} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\mu},$$

where

$$\underline{G} = \operatorname{tr} \underline{\chi} \rho - \frac{1}{2} \widehat{\underline{\chi}} \cdot (\nabla \widehat{\underline{\chi}}) - \zeta \cdot \underline{\beta} + \Gamma_g \cdot \nabla \operatorname{tr} \underline{\chi} + r^{-1} \Gamma_b \cdot \Gamma_b.$$

Applying (6.38), we have

$$\underline{G} = \operatorname{tr} \underline{\chi} \rho - \frac{1}{2} \widehat{\underline{\chi}} \cdot (\nabla \widehat{\underline{\chi}}) - \zeta \cdot \underline{\beta} + r^{-1} \Gamma_b \cdot \Gamma_b.$$

As in (5.33), we define

$$[\underline{\mu}] := \underline{\mu} + \frac{1}{4} \operatorname{tr} \underline{\chi} \operatorname{tr} \underline{\chi}, \quad (6.47)$$

and deduce the following analog of (5.37):

$$\nabla_3 [\underline{\mu}] + \operatorname{tr} \underline{\chi} [\underline{\mu}] = \operatorname{tr} \underline{\chi} \rho - \frac{1}{2} \widehat{\underline{\chi}} \cdot (\nabla \widehat{\underline{\chi}}) - \zeta \cdot \underline{\beta} + r^{-1} \Gamma_b \cdot \Gamma_b. \quad (6.48)$$

Differentiating (6.48) by  $r\nabla$  and applying Corollary 2.22, we obtain

$$\begin{aligned} \nabla_3 (r\nabla [\underline{\mu}]) + \operatorname{tr} \underline{\chi} (r\nabla [\underline{\mu}]) &= \operatorname{tr} \underline{\chi} (r\nabla \rho) - \frac{1}{2} \widehat{\underline{\chi}} \cdot (r\nabla \nabla \widehat{\underline{\chi}}) - \zeta \cdot (r\nabla \underline{\beta}) \\ &\quad + r^{-1} \Gamma_a \cdot \Gamma_g^{(1)} + r^{-1} \Gamma_b \cdot \Gamma_b^{(1)}. \end{aligned} \quad (6.49)$$

Recall from (6.33) and Corollary 2.20 that, we have

$$\begin{aligned} \nabla_3 \widehat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \widehat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \widehat{\underline{\chi}} &= \nabla \widehat{\underline{\chi}} + \Gamma_g \cdot \Gamma_g, \\ \nabla_3 (\underline{\rho}, \underline{\sigma}) + \frac{3}{2} \operatorname{tr} \underline{\chi} (\underline{\rho}, \underline{\sigma}) &= -d_1 \underline{\beta} + \Gamma_g \cdot \underline{\alpha}. \end{aligned}$$

We apply a renormalization method similar to the one in (6.43), and leave the details to the reader. We obtain

$$\nabla_3 \Xi + \operatorname{tr} \underline{\chi} \Xi = O(\nabla \rho) + r^{-1} \Gamma_a \cdot \Gamma_g^{(1)} + r^{-1} \Gamma_b \cdot \Gamma_b^{(1)}, \quad (6.50)$$

where

$$\Xi := r\nabla[\underline{\mu}] + \frac{1}{2}\widehat{\underline{\chi}} \cdot (r\nabla\widehat{\underline{\chi}}) + r\zeta \cdot \check{\rho}. \quad (6.51)$$

Next, we use a renormalization method similar to the one in section 5.2. We define  $\underline{\mu}^\dagger$  by:

$$\nabla_3 \underline{\mu}^\dagger + \text{tr} \underline{\chi} \underline{\mu}^\dagger = \text{tr} \underline{\chi} \sigma \quad \text{on } \underline{C}_*, \quad \underline{\mu}^\dagger = 0 \quad \text{on } S_*. \quad (6.52)$$

We assume that

$$\underline{\mu}^\dagger \in r^{-1}\Gamma_g, \quad (6.53)$$

which will be improved in (6.58) and (6.59). Differentiating (6.52) by  $r^*\nabla$  and applying Corollary 2.22, we infer

$$\nabla_3(r^*\nabla \underline{\mu}^\dagger) + \text{tr} \underline{\chi}(r^*\nabla \underline{\mu}^\dagger) = r \text{tr} \underline{\chi}^* \nabla \sigma + r^{-1}\Gamma_g \cdot \Gamma_g^{(1)}. \quad (6.54)$$

Recall from Corollary 2.20 that

$$\nabla_3 \beta + \text{tr} \underline{\chi} \beta = \nabla \check{\rho} + {}^*\nabla \sigma + \Gamma_a \cdot \beta + \Gamma_g \cdot \underline{\beta},$$

which implies

$$\nabla_3(r \text{tr} \underline{\chi} \beta) + \text{tr} \underline{\chi}(r \text{tr} \underline{\chi} \beta) = \nabla_3(r \text{tr} \underline{\chi}) \beta + \text{tr} \underline{\chi}(r \nabla \check{\rho} + r^*\nabla \sigma) + r^{-1}\Gamma_a \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_g \cdot \Gamma_b^{(1)}. \quad (6.55)$$

Combining (6.50), (6.54) and (6.55), we obtain

$$\begin{aligned} & \nabla_3(\Xi + r^*\nabla \underline{\mu}^\dagger - r \text{tr} \underline{\chi} \beta) + \text{tr} \underline{\chi}(\Xi + r^*\nabla \underline{\mu}^\dagger - r \text{tr} \underline{\chi} \beta) \\ &= r^{-1}O(\beta) + r^{-1}\Gamma_a \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_b^{(1)}. \end{aligned} \quad (6.56)$$

Applying Lemma 2.30, we deduce

$$|r^{2-\frac{2}{p}}(\Xi + r^*\nabla \underline{\mu}^\dagger - r \text{tr} \underline{\chi} \beta)|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-3}{2}}}. \quad (6.57)$$

Recall from (6.51) that

$$\Xi - r\nabla[\widetilde{\underline{\mu}}] = \Gamma_b \cdot \Gamma_g^{(1)},$$

we obtain

$$|r^{2-\frac{2}{p}}(r\nabla[\widetilde{\underline{\mu}}] + r^*\nabla \underline{\mu}^\dagger)|_{p,S} \lesssim |r^{2-\frac{2}{p}}\beta|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} + \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-3}{2}}} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-3}{2}}}.$$

Then, we have from Proposition 2.13 that

$$\left| r^{4-\frac{2}{p}} \nabla([\underline{\mu}], \underline{\mu}^\dagger) \right|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}. \quad (6.58)$$

Finally, applying Lemma 2.30 to (6.52), we obtain

$$|r^{2-\frac{2}{p}}\underline{\mu}^\dagger|_{p,S} \lesssim |r^{2-\frac{2}{p}}\underline{\mu}^\dagger|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{1-\frac{2}{p}}\sigma|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0}{r|u|^{\frac{s-3}{2}}}. \quad (6.59)$$

Remark that (6.58) and (6.59) improves (6.53).

**Step 3.** Estimates for  $\mathcal{O}_{0,1}(\widetilde{\text{tr}} \underline{\chi})$  and  $\mathcal{O}_{0,1}(\zeta)$ .

We recall from (6.33) that

$$\nabla_3 \operatorname{tr} \chi + \frac{1}{2} \operatorname{tr} \underline{\chi} \operatorname{tr} \chi - 2 \operatorname{div} \zeta = 2\rho + \Gamma_b \cdot \Gamma_g.$$

Applying (6.46) and (6.47), we obtain

$$\begin{aligned} \nabla_3 \operatorname{tr} \chi + \operatorname{tr} \underline{\chi} \operatorname{tr} \chi &= 2 \operatorname{div} \zeta + 2\rho + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} + \Gamma_b \cdot \Gamma_g \\ &= 2\underline{\mu} + 4\rho + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} + \Gamma_b \cdot \Gamma_g \\ &= 2[\underline{\mu}] + 4\rho + \Gamma_b \cdot \Gamma_g. \end{aligned}$$

Differentiating it by  $r\nabla$ , we deduce

$$\nabla_3(r\nabla \operatorname{tr} \chi) + \operatorname{tr} \underline{\chi}(r\nabla \operatorname{tr} \chi) = -\operatorname{tr} \chi(r\nabla \operatorname{tr} \underline{\chi}) + 2r\nabla[\underline{\mu}] + 4r\nabla\rho + \Gamma_b \cdot \Gamma_g^{(1)}.$$

We use a renormalization argument as in Step 2. For this, we define  $\chi^\dagger$  by:

$$\nabla_3 \chi^\dagger + \operatorname{tr} \underline{\chi} \chi^\dagger = 4\sigma, \quad \text{on } \underline{C}_*, \quad \chi^\dagger = 0, \quad \text{on } S_*. \quad (6.60)$$

We add the bootstrap assumption

$$\chi^\dagger \in \Gamma_g, \quad (6.61)$$

which will be improved by (6.62) and (6.63). Differentiating it by  $r^*\nabla$  and applying Corollary 2.22, we infer

$$\nabla_3(r^*\nabla \chi^\dagger) = 4r^*\nabla\sigma + \Gamma_g \cdot \Gamma_g^{(1)}.$$

We recall from Corollary 2.20 that

$$\nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta = \nabla \check{\rho} + {}^*\nabla\sigma + \Gamma_a \cdot \beta + \Gamma_g \cdot \underline{\beta},$$

which implies

$$\nabla_3(r\beta) + \operatorname{tr} \underline{\chi}(r\beta) = -e_3(r)\beta + r\nabla\check{\rho} + r^*\nabla\sigma + \Gamma_a \cdot \Gamma_g^{(1)} + \Gamma_g \cdot \Gamma_b^{(1)}.$$

Hence, we obtain the following analog of (6.56):

$$\begin{aligned} &\nabla_3(r\nabla \operatorname{tr} \chi + r^*\nabla \chi^\dagger - 4r\beta) + \operatorname{tr} \underline{\chi}(r\nabla \operatorname{tr} \chi + r^*\nabla \chi^\dagger - 4r\beta) \\ &= -e_3(r)\beta - \operatorname{tr} \chi(r\nabla \operatorname{tr} \underline{\chi}) + 2r\nabla[\underline{\mu}] + \Gamma_b \cdot \Gamma_g^{(1)} + \Gamma_a \cdot \Gamma_g^{(1)}. \end{aligned}$$

Applying Lemma 2.30, (6.45) and (6.58), we deduce

$$\begin{aligned} |r^{2-\frac{2}{p}}(r\nabla \operatorname{tr} \chi + r^*\nabla \chi^\dagger - 4r\beta)|_{p,S} &\lesssim |r^{2-\frac{2}{p}}(r\nabla \operatorname{tr} \chi + r^*\nabla \chi^\dagger - 4r\beta)|_{p,S_*} \\ &\quad + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}}\beta|_{p,S} + |r^{2-\frac{2}{p}}\nabla \operatorname{tr} \underline{\chi}|_{p,S} + |r^{3-\frac{2}{p}}\nabla[\underline{\mu}]|_{p,S} \\ &\quad + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}}\Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_a \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \frac{\mathcal{I}_0 + \Delta_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{s-2}} + \frac{\varepsilon^2}{r^2|u|^{\frac{s-3}{2}}} \\ &\lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$



Hence, we obtain from Proposition 2.13 that

$$|r^{3-\frac{2}{p}}\nabla(\operatorname{tr}\chi, \chi^\dagger)|_{p,S} \lesssim |r^{3-\frac{2}{p}}\beta|_{p,S} + \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}. \quad (6.62)$$

Applying Lemma 2.30 to (6.60), we easily obtain

$$|r^{2-\frac{2}{p}}\chi^\dagger|_{p,S} \lesssim |r^{2-\frac{2}{p}}\chi^\dagger|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}}\sigma|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0}{|u|^{\frac{s-3}{2}}}, \quad (6.63)$$

which improved (6.61). By Poincaré inequality, we infer

$$\mathcal{O}_0(\widetilde{\operatorname{tr}\chi}) + \mathcal{O}_1(\widetilde{\operatorname{tr}\chi}) \lesssim \mathcal{I}_0 + \Delta_0 + \varepsilon^2. \quad (6.64)$$

Next, we recall from (5.36) that

$$\begin{aligned} \operatorname{div}\zeta &= [\underline{\mu}] + \check{\rho} + \frac{1}{4}\widetilde{\operatorname{tr}\chi} \operatorname{tr}\underline{\chi} + \Gamma_b \cdot \Gamma_g, \\ \operatorname{curl}\zeta &= \sigma + \Gamma_b \cdot \Gamma_g. \end{aligned}$$

Hence, we have from Proposition 2.13 that

$$\begin{aligned} |r^{2-\frac{2}{p}}(r\nabla)^{\leq 1}\zeta|_{p,S} &\lesssim |r^{3-\frac{2}{p}}[\underline{\mu}]|_{p,S} + |r^{3-\frac{2}{p}}(\check{\rho}, \sigma)|_{p,S} + |r^{1-\frac{2}{p}}(\widetilde{\operatorname{tr}\chi}, \operatorname{tr}\underline{\chi})|_{p,S} \\ &\lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used (6.45), (6.58) and (6.64) in the last step. Hence, we obtain

$$\mathcal{O}_0(\zeta) + \mathcal{O}_1(\zeta) \lesssim \mathcal{I}_0 + \Delta_0 + \varepsilon^2. \quad (6.65)$$

**Step 4.** Estimates for  $\mathcal{O}_{0,1}(\widehat{\chi})$  and  $\mathcal{O}_{0,1}(\underline{\widehat{\chi}})$ .

Applying Proposition 2.13 to (6.34) and recalling (6.45), (6.64) and (6.65), we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}}(r\nabla)^{\leq 1}\widehat{\chi}|_{p,S} &\lesssim |r^{3-\frac{2}{p}}\nabla\widetilde{\operatorname{tr}\chi}|_{p,S} + |r^{2-\frac{2}{p}}\zeta|_{p,S} + |r^{3-\frac{2}{p}}\beta| + |r^{2-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{|u|^{\frac{s-3}{2}}}, \\ |r^{1-\frac{2}{p}}(r\nabla)^{\leq 1}\underline{\widehat{\chi}}|_{p,S} &\lesssim |r^{2-\frac{2}{p}}\nabla\widetilde{\operatorname{tr}\underline{\chi}}|_{p,S} + |r^{2-\frac{2}{p}}\zeta|_{p,S} + |r^{2-\frac{2}{p}}\underline{\beta}| + |r^{1-\frac{2}{p}}\Gamma_b \cdot \Gamma_g|_{p,S} \lesssim \frac{\mathcal{I}_0 + \Delta_0 + \varepsilon^2}{r|u|^{\frac{s-1}{2}}}. \end{aligned}$$

Hence, we have

$$\mathcal{O}_0(\widehat{\chi}) + \mathcal{O}_1(\widehat{\chi}) + \mathcal{O}_0(\underline{\widehat{\chi}}) + \mathcal{O}_1(\underline{\widehat{\chi}}) \lesssim \mathcal{I}_0 + \Delta_0 + \varepsilon^2. \quad (6.66)$$

**Step 5.** Estimate for  $r^2|\overline{\operatorname{tr}\chi} - \frac{2}{r}|$ .

We have from Corollary 2.5 and (6.33) that<sup>20</sup>

$$\nabla_3(\overline{\operatorname{tr}\chi}) = \overline{\nabla_3(\operatorname{tr}\chi)} + \overline{\operatorname{tr}\underline{\chi} \operatorname{tr}\chi} = -\frac{1}{2}\overline{\operatorname{tr}\chi \operatorname{tr}\underline{\chi}} + 2\overline{\rho} + \Gamma_g \cdot \Gamma_b.$$

Notice that

$$\nabla_3\left(\frac{2}{r}\right) = -\frac{1}{r}\overline{\operatorname{tr}\underline{\chi}}.$$

<sup>20</sup>Recall that  $\Omega = \frac{1}{2}$  on  $\underline{C}_*$ .

Hence, we infer from Lemma 2.3 that

$$\begin{aligned} \nabla_3 \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) + \frac{1}{2} \text{tr} \underline{\chi} \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) &= \frac{1}{2} \text{tr} \underline{\chi} \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) - \frac{1}{2} \overline{\text{tr} \chi} \text{tr} \underline{\chi} + \frac{1}{r} \overline{\text{tr} \chi} + 2\bar{\rho} + \Gamma_g \cdot \Gamma_b \\ &= \frac{1}{2} \overline{\text{tr} \chi} \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) - \frac{1}{2} \overline{\text{tr} \chi} \text{tr} \underline{\chi} + \frac{1}{r} \overline{\text{tr} \chi} + 2\bar{\rho} + \Gamma_g \cdot (\Gamma_a, \Gamma_b) \\ &= 2\bar{\rho} + \Gamma_g \cdot (\Gamma_a, \Gamma_b). \end{aligned}$$

Applying Lemma 2.30, we obtain

$$\begin{aligned} \left| r \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) \right|_{\infty, S} &\lesssim \left| r \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) \right|_{\infty, S_*} + \int_{u_0(\underline{u}_*)}^u |r\bar{\rho}|_{\infty, S} + |r\Gamma_g \cdot (\Gamma_a, \Gamma_b)|_{\infty, S} \\ &\lesssim \frac{\mathcal{I}_0}{r} + \int_{u_0}^u \frac{\Delta_0 + \mathcal{I}_0 + \varepsilon^2}{r^2} \\ &\lesssim \frac{\Delta_0 + \mathcal{I}_0 + \varepsilon^2}{r}. \end{aligned}$$

Hence, we deduce

$$\sup_{\underline{C}_*} \left| r^2 \left( \overline{\text{tr} \chi} - \frac{2}{r} \right) \right| \lesssim \Delta_0 + \mathcal{I}_0 + \varepsilon^2. \quad (6.67)$$

In view of (6.45), (6.64)–(6.67), we obtain

$$\mathcal{O}^*(\underline{C}_*) \lesssim \mathcal{I}_0 + \Delta_0 + \varepsilon^2.$$

This concludes the proof of Theorem M2.  $\square$

#### 6.4 Extension argument (Theorem M4)

In this section, we prove Theorem M4, which we recall below for convenience.

**Theorem M4.** *We consider the spacetime  $\mathcal{K}$  and its double null foliation  $(u, \underline{u})$ , which satisfy the assumptions:*

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad \text{Osc} \lesssim \varepsilon_0. \quad (6.68)$$

*Then, we can extend the spacetime  $\mathcal{K} = V(u_0, \underline{u}_*)$  and the double null foliation  $(u, \underline{u})$  to a new spacetime  $\widehat{\mathcal{K}} = \widehat{V}(u_0, \underline{u}_* + \nu)$ , where  $\nu$  is sufficiently small, and an associated double null foliation  $(\widehat{u}, \underline{u})$ . Moreover, the new foliation  $(\widehat{u}, \underline{u})$  is geodesic on the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$  and the new norms satisfy*

$$\widehat{\mathcal{O}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}} \lesssim \varepsilon_0, \quad \widehat{\text{Osc}} \lesssim \varepsilon_0. \quad (6.69)$$

Recall that we have introduced three different double null foliations in different regions:

- $(u, \underline{u})$  in the bootstrap region  $\mathcal{K}$ ;
- $(u', \underline{u})$  in the initial layer region  $\mathcal{K}'$ ;
- $(\widehat{u}, \underline{u})$  in the extended bootstrap region  $\widehat{\mathcal{K}}$ .

We introduce the following definition.

**Definition 6.9.** *For the change of frame  $(f, \lambda)$  from a double null foliation  $(u^{(1)}, \underline{u})$  to another double null foliation  $(u^{(2)}, \underline{u})$  in their common domain  $\mathcal{M}$ , we define the following norms:*

$$\text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M}) := \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M})(f) + \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M})(\lambda) + \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M})(r),$$

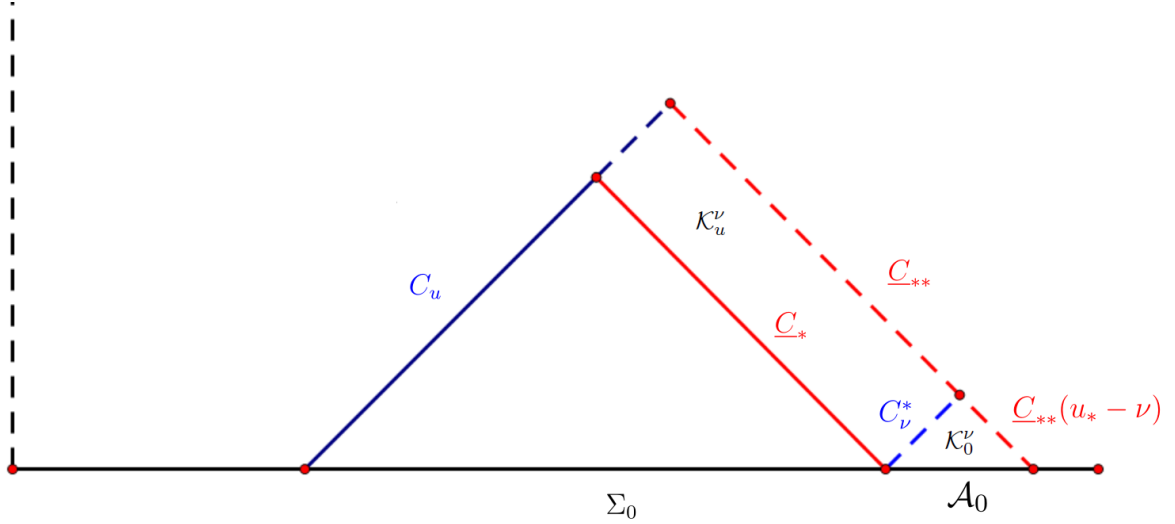


Figure 2.3: Extension argument

where

$$\begin{aligned} \text{Osc}[u^{(1)}, u^{(2)}](f) &= \sup_{\mathcal{M}} \left| r^{(1)} |u^{(1)}|^{\frac{s-3}{2}} \mathfrak{d} \leq 1 f \right|, \\ \text{Osc}[u^{(1)}, u^{(2)}](\lambda) &= \sup_{\mathcal{M}} \left| r^{(1)} \lambda^\circ \right|, \\ \text{Osc}[u^{(1)}, u^{(2)}](r) &= \sup_{\mathcal{M}} \left| r^{(1)} - r^{(2)} \right|. \end{aligned}$$

**Remark 6.10.** *Definition 6.9 immediately implies*

$$\begin{aligned} \text{Osc}[u^{(1)}, u^{(3)}](\mathcal{M}) &\lesssim \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M}) + \text{Osc}[u^{(2)}, u^{(3)}](\mathcal{M}), \\ \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M}_1 \cup \mathcal{M}_2) &\lesssim \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M}_1) + \text{Osc}[u^{(1)}, u^{(2)}](\mathcal{M}_2). \end{aligned} \quad (6.70)$$

To simplify the notations, we ignore the domain  $\mathcal{M}$  and the optical functions  $[u^{(1)}, u^{(2)}]$  when they are clear in the context. For example, we denoted

$$\text{Osc} = \text{Osc}[u, u'](\mathcal{K} \cap \mathcal{K}'), \quad \widehat{\text{Osc}} = \text{Osc}[\widehat{u}, u'](\widehat{\mathcal{K}} \cap \mathcal{K}'),$$

in the statements of Theorems  $M0$  and  $M4$ .

*Proof of Theorem  $M4$ .* **Step 1.** Extension of the domain.

We define the region

$$\mathcal{A}_0 := \{p \in \Sigma_0 / w(p) \in [\underline{u}_*, \underline{u}_* + \nu]\}.$$

For  $\nu$  small enough, we consider the future dependence domain  $\mathcal{K}_0^\nu$  of  $\mathcal{A}_0$ , with

$$\mathcal{K}_0^\nu := V'(u_*, \underline{u}_* + \nu) \subset \mathcal{K}'.$$

The boundary of  $\mathcal{K}_0^\nu$  contain

- (1) the part  $\mathcal{A}_0$ ;
- (2) a part of an incoming cone denoted by

$$\underline{C}_{**}(u_* - \nu) := \underline{C}_{\underline{u}_* + \nu} \cap \{u_* - \nu \leq u' \leq u_*\},$$

which is a part of the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$ , where

$$u_* - \nu := u' \big|_{\underline{C}_{\underline{u}_* + \nu} \cap \Sigma_0}, \quad u_* = u' \big|_{\underline{C}_{\underline{u}_*} \cap \Sigma_0};$$

(3) a part of the outgoing cone emanating from  $S_0(\underline{u}_*) = \underline{C}_{\underline{u}_*} \cap \Sigma_0$  and in  $V'(u_*, \underline{u}_* + \nu)$ , denoted by  $C_\nu^*$ , see Figure 2.3.

For the double null foliation  $(u', \underline{u})$  in  $\mathcal{K}_0^\nu$ , we have:

$$\mathcal{O}'(\mathcal{K}_0^\nu) \lesssim \varepsilon_0, \quad \mathcal{R}'(\mathcal{K}_0^\nu) \lesssim \varepsilon_0, \quad (6.71)$$

for  $\nu$  small enough by local existence. By the local existence theorem of the characteristic Cauchy problem introduced in [44], we can extend the spacetime  $\mathcal{K} \cup \mathcal{K}_0^\nu$  to the future domain of dependence of  $C_\nu^* \cup \underline{C}_*$ , denoted by  $\mathcal{K}_u^\nu$ . Moreover, we extend the outgoing cones of  $\mathcal{K}$  to this new domain, construct the incoming cones from  $C_\nu^*$  and denote  $(\tilde{u}, \underline{u})$  the global foliation in  $\tilde{\mathcal{K}} := \mathcal{K} \cup \mathcal{K}_u^\nu \cup \mathcal{K}_0^\nu$ . As a consequence of Theorem 1 of [44], we have

$$\tilde{\mathcal{O}}(\mathcal{K}_u^\nu) \lesssim \varepsilon_0, \quad \tilde{\mathcal{R}}(\mathcal{K}_u^\nu) \lesssim \varepsilon_0, \quad \text{Osc}[\tilde{u}, u'](\mathcal{K}_u^\nu \cap \mathcal{K}') \lesssim \varepsilon_0. \quad (6.72)$$

Combining (6.68), (6.71) and (6.72), we obtain

$$\tilde{\mathcal{O}}(\tilde{\mathcal{K}}) \lesssim \varepsilon_0, \quad \tilde{\mathcal{R}}(\tilde{\mathcal{K}}) \lesssim \varepsilon_0, \quad \text{Osc}[\tilde{u}, u'](\tilde{\mathcal{K}} \cap \mathcal{K}') \lesssim \varepsilon_0. \quad (6.73)$$

**Step 2.** Estimates on  $\underline{C}_{**}$ .

Notice that the double null foliation  $(\tilde{u}, \underline{u})$  is in general not geodesic on  $\underline{C}_{**}$  and only continuous across  $C_\nu^*$ . Next, we construct from  $S'(u_* - \nu, \underline{u}_* + \nu)$  as in section 2.1.3 another new double null foliation  $(\hat{u}, \underline{u})$  which is a geodesic foliation on  $\underline{C}_{**}$ . We add the following bootstrap assumptions on  $\underline{C}_*$ , which will be improved at the end of this step:

$$\text{Osc}[\tilde{u}, \hat{u}](\underline{C}_{**}) \leq \varepsilon, \quad \widehat{\mathcal{O}}^*(\underline{C}_{**}) \leq \varepsilon. \quad (6.74)$$

Applying (6.74) and Proposition 6.2 and recalling that two foliations have the same incoming cones, we deduce by proceeding similarly to the proof of Lemma 6.6 that<sup>21</sup>

$$\widehat{\mathcal{R}}_{[1]}(\underline{C}_{**}) \lesssim \tilde{\mathcal{R}}_{[1]}(\underline{C}_{**}) \lesssim \varepsilon_0. \quad (6.75)$$

As a direct consequence of Theorem M2, we obtain

$$\widehat{\mathcal{O}}^*(\underline{C}_{**}) \lesssim \varepsilon_0. \quad (6.76)$$

Then, by proceeding similarly to the proof of Lemma 6.8, we obtain

$$\text{Osc}[\tilde{u}, \hat{u}](\underline{C}_{**}) \lesssim \varepsilon_0, \quad (6.77)$$

which improves (6.74).

**Step 3.** Estimates in  $\tilde{\mathcal{K}}$ .

Let  $U_1$  the set of  $|u_0| \leq \underline{u}_1 \leq \underline{u}_{**} + \nu$  such that the following holds:

$$\widehat{\mathcal{O}}(\widehat{V}_1) \leq \varepsilon, \quad \text{Osc}[\tilde{u}, \hat{u}](\widehat{V}_1) \leq \frac{\varepsilon}{2}, \quad (6.78)$$

<sup>21</sup>The estimate for  $\bar{\rho}$  can be done similarly as in Proposition 4.13.

where

$$\widehat{V}_1 := \widehat{V}(u_0, \underline{u}_* + \nu) \cap \{u_1 \leq u \leq \underline{u}_{**} + \nu\}. \quad (6.79)$$

Notice that (6.76) and (6.77) imply that  $U_1 \neq \emptyset$ . We apply Proposition 6.2 and proceed similarly to the proof of Lemma 6.6 to deduce<sup>22</sup>

$$\widehat{\mathcal{R}}_0^S \lesssim \widetilde{\mathcal{R}}_0^S \lesssim \varepsilon_0. \quad (6.80)$$

Next, we estimate  $\widehat{\mathcal{R}}_0^S[\beta]$ . For this, recall that

$$|r^{-\frac{2}{p}} \nabla_3 \beta|_{p, \widehat{S}} = |r^{-2-\frac{2}{p}} \Gamma_g^{(1)}|_{p, \widehat{S}} \lesssim \frac{\varepsilon}{r^4 |u|^{\frac{s-3}{2}}}.$$

For any sphere  $\widehat{S} = \widehat{S}(\widehat{u}, \underline{u})$ , we take a sphere  $\widetilde{S} = \widetilde{S}(\widetilde{u}, \underline{u})$  satisfying  $\widehat{S} \cap \widetilde{S} \neq \emptyset$ . The assumption (6.78) implies

$$\sup_{\widehat{S}} \left| \widetilde{u} - \frac{1}{|\widehat{S}|} \int_{\widehat{S}} \widetilde{u} \right| \lesssim \frac{\varepsilon}{r |u|^{\frac{s-3}{2}}}.$$

Applying Proposition 6.2 and Lemma 6.3, we deduce

$$|r^{-\frac{2}{p}} \widehat{\beta}|_{p, \widehat{S}} \lesssim |r^{-\frac{2}{p}} \widetilde{\beta}|_{p, \widehat{S}} + \text{l.o.t.} \lesssim \frac{\widetilde{\mathcal{R}}_0^S[\beta]}{r^{\frac{7}{2}} |u|^{\frac{s-4}{2}}} + \frac{\varepsilon^2}{r^5 |u|^{s-3}} \lesssim \frac{\varepsilon_0}{r^{\frac{7}{2}} |u|^{\frac{s-4}{2}}}.$$

Hence, we deduce

$$\widehat{\mathcal{R}}_0^S + \widetilde{\mathcal{R}}_0^S \lesssim \varepsilon_0. \quad (6.81)$$

Combining (6.78) and (6.73) and applying (6.70), we infer

$$\text{Osc}[\widehat{u}, u'](\widehat{V}_1 \cap \mathcal{K}') \leq \frac{\varepsilon}{2} + C\varepsilon_0 \leq \varepsilon.$$

We now apply Theorems M0 and M3 to conclude<sup>23</sup>

$$\text{Osc}[\widehat{u}, u'](\widehat{V}_1 \cap \mathcal{K}') \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}(\widehat{V}_1) \lesssim \varepsilon_0. \quad (6.82)$$

Proceeding similarly to the proof of Lemma 6.8, we obtain

$$\text{Osc}[\widetilde{u}, \widehat{u}](\widehat{V}_1) \lesssim \varepsilon_0,$$

which improves (6.78). Thus, we have  $\inf U_1 = |u_0|$  and hence,

$$\text{Osc}[\widehat{u}, \widetilde{u}](\widehat{\mathcal{K}}) \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}(\widehat{\mathcal{K}}) \lesssim \varepsilon_0. \quad (6.83)$$

Notice that (6.73) and (6.83) implies

$$\text{Osc}[\widehat{u}, u'](\widehat{\mathcal{K}} \cap \mathcal{K}') \lesssim \varepsilon_0. \quad (6.84)$$

Finally, we apply Theorem M1 with (6.81), (6.83) and (6.84) to conclude

$$\widehat{\mathcal{R}} \lesssim \varepsilon_0. \quad (6.85)$$

Combining (6.83), (6.84) and (6.85), this concludes the proof of Theorem M4.  $\square$

<sup>22</sup>Recall that the incoming cones are invariant and that  $\bar{p}$  can be estimated as in Proposition 4.13.

<sup>23</sup>Remark that Theorem M3 also applied in the region  $\widehat{V}_1$ .

## A Proof of Lemma 3.4

In this appendix, we use the notations introduced in Remark 6.4.

First, we compare  $r'$  and  $w$ . By construction, we have

$$r' = w = w_0 \quad \text{on } \partial K.$$

We add the bootstrap assumption:

$$|r' - w| \leq \varepsilon \log r', \quad \text{on } \Sigma_0 \setminus K, \quad (\text{A.1})$$

to ensure the equivalence  $r' \simeq w$ . Applying (6.1) and Lemma 2.4, we deduce

$$(\Omega' e'_4 - \Omega' e'_3)(r' - w) = \frac{\overline{\Omega' \operatorname{tr} \chi'}}{2} r' - \frac{\overline{\Omega' \operatorname{tr} \underline{\chi}'}}{2} r' - 1 + \frac{|f|^2}{4} = r' \Gamma'_a + O(f^2).$$

Integrating from  $w_0$  along  $(\Omega' e'_4 - \Omega' e'_3)$  and using  $\operatorname{Osc}(f) \leq \varepsilon$ , we infer

$$|r' - w| \lesssim \varepsilon + \int_{w_0}^{r'} \left( \frac{\varepsilon^2}{r'^{s-1}} + \frac{\varepsilon_0}{r'} \right) dr' \lesssim \varepsilon_0 \log r', \quad (\text{A.2})$$

which improves (A.1).

Next, we compare  $u$  and  $u'$ . By construction, we have

$$u' = u_0 \quad \text{on } \partial K,$$

and  $u = u_0$  at a point  $p \in \partial K$ . Hence, we have from (6.1) and  $\operatorname{Osc}(f) \leq \varepsilon$  that

$$\sup_{\partial K} |u' - u| \lesssim |r \nabla'(u' - u)|_{\infty, \partial K} + |u' - u|(p) \lesssim \left| r \nabla(u) + \frac{1}{2} e_3(u) f \right|_{\infty, \partial K} \lesssim |f|_{\infty, \partial K} \lesssim \varepsilon_0.$$

Applying (6.1), we obtain

$$\Omega' e'_4(u) - \Omega' e'_3(u) = \frac{\lambda \Omega'}{4} |f|^2 e_3(u) - \frac{\Omega'}{\lambda \Omega} = \frac{\Omega}{4} |f|^2 - \frac{\Omega'^2}{\Omega^2}.$$

On the other hand, we have

$$(\Omega' e'_4 - \Omega' e'_3)(u') = -1.$$

Then, we obtain

$$(\Omega' e'_4 - \Omega' e'_3)(u - u') = \frac{\Omega}{4} |f|^2 + 1 - \frac{\Omega'^2}{\Omega^2}.$$

Integrating it along  $(\Omega' e'_4 - \Omega' e'_3)$ , we infer from  $\operatorname{Osc} \leq \varepsilon$  that

$$|u' - u| \lesssim \varepsilon + \int_{w_0}^{r'} \left( \frac{\varepsilon^2}{r'^{s-1}} + \frac{\varepsilon}{r'} \right) dr' \lesssim \varepsilon \log r', \quad \text{on } \Sigma_0 \setminus K. \quad (\text{A.3})$$

Notice that

$$e'_4(u - u') = \frac{|f|^2}{4\Omega'} = O\left(\frac{\varepsilon^2}{r'^{s-1}}\right). \quad (\text{A.4})$$

Integrating (A.4) from  $\Sigma_0 \setminus K$  to  $\mathcal{K}'$  and recalling that the height of  $\mathcal{K}'$  is finite<sup>24</sup> we obtain immediately from (A.3) that

$$|u' - u| \lesssim \varepsilon \log r', \quad \text{in } \mathcal{K}'_{(0)}. \quad (\text{A.5})$$

<sup>24</sup>Recall that we defined  $\mathcal{K}' := \{p/0 \leq u'(p) + \underline{u}(p) \leq 2\delta_0\}$  and  $\delta_0 < 1$ .

Finally, we compare  $r$  and  $\frac{u-u'}{2}$ . Recalling that  $w = \frac{u-u'}{2}$  and applying (A.5), we obtain

$$\left| w - \frac{u-u'}{2} \right| \lesssim |u' - u| \lesssim \varepsilon \log r', \quad \text{in } \mathcal{K}'.$$

Then, we have, using also (A.2) and  $\text{Osc}(r) \leq \varepsilon$ ,

$$\left| r - \frac{u-u'}{2} \right| \lesssim |r' - r| + |r' - w| + \left| w - \frac{u-u'}{2} \right| \lesssim \varepsilon \log r' \quad \text{in } \mathcal{K}'.$$

Next, we have from Lemma 2.4

$$\frac{d}{d\underline{u}} \left( r - \frac{u-u'}{2} \right) = \frac{r}{2} \left( \overline{\Omega \text{tr} \chi} - \frac{1}{r} \right) = r\Gamma_a.$$

We integrate it along  $C_u$  from a sphere  $(u, v_0) \in \mathcal{K}_{(0)}$  to obtain on  $\mathcal{K}$

$$\left| r(u, \underline{u}) - \frac{u-u'}{2} \right| \lesssim \varepsilon \log r'(u, v_0) + \varepsilon \log r(u, \underline{u}) \lesssim \varepsilon \log r(u, \underline{u}).$$

This concludes the proof of Lemma 3.4.

## B Proof of the case $s \in (3, 4)$

In sections 3-6, we have provided the proof of Theorem 3.1 in the case of  $s \in [4, 6]$ . In this appendix, we outline the proof in the case  $s \in (3, 4)$ , which is similar to the case of  $s \in [4, 6]$ .

### B.1 Fundamental norms

The definitions of some of the norms are different from section 3.1. In the sequel, we only mention the norms that differ from the ones in section 3.1.

We define  $\mathcal{R}_0^S[\beta]$  and  $\underline{\mathcal{R}}_0^S[\check{\rho}, \sigma]$  as follows:

$$\begin{aligned} \mathcal{R}_0^S[\beta] &:= \sup_{\mathcal{K}} \sup_{p \in [2, 4]} |r^{\frac{s+3}{2} - \frac{2}{p}} \beta|_{p, S}, \\ \underline{\mathcal{R}}_0^{p, S}[\check{\rho}, \sigma] &:= \sup_{\mathcal{K}} \sup_{p \in [2, 4]} |r^{\frac{s+2}{2} - \frac{2}{p}} |u|^{\frac{1}{2}} (\check{\rho}, \sigma)|_{p, S}. \end{aligned}$$

The following flux of curvature components are different from the case  $s \in [4, 6]$  for  $q = 0, 1$ :

$$\begin{aligned} \mathcal{R}_q[\beta](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \beta\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[(\check{\rho}, \sigma)](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q (\check{\rho}, \sigma)\|_{2, \underline{C}_u^V}. \end{aligned}$$

The definition of  $\mathcal{O}_q^{p, S}(\Omega \underline{\omega})(u, \underline{u})$  is different from the case  $s \in [4, 6]$ :

$$\mathcal{O}_0^{p, S}(\Omega \underline{\omega})(u, \underline{u}) := |r^{\frac{s}{2} - \frac{2}{p}} |u|^{\frac{4-s}{2}} \underline{\omega}|_{p, S(u, \underline{u})}.$$

The other  $\mathcal{O}$  norms are the same as in the case  $s \in [4, 6]$ .

## B.2 Estimates for Ricci coefficients and curvature components

We discuss the curvature estimates of section 4 in this case. Recall that we have four Bianchi pairs:  $(\alpha, \beta)$ ,  $(\beta, (\check{\rho}, -\sigma))$ ,  $((\check{\rho}, \sigma), \underline{\beta})$  and  $(\underline{\beta}, \underline{\alpha})$ . In section 4, we took respectively  $p = s, 4, 2, 0$  to estimate the Bianchi pairs. In the case  $s \in (3, 4)$ , we take respectively  $p = s, s, 2, 0$  to estimate the Bianchi pairs. The method is then exactly the same as in section 4.

Concerning the control of Ricci coefficients, we can proceed by the same method as in section 5. Recall that Proposition 5.2 used the fact that there exists a constant  $\delta > 0$  such that

$$\sup_{p \in [2, 4]} |r^{3+\delta-\frac{2}{p}} \beta|_{p, S} \lesssim \Delta_0.$$

This still holds true since we have  $\frac{s+3}{2} > 3$  for  $s > 3$ . The other propositions still hold true since we only used  $s > 3$  in their proofs.

## B.3 Conclusion

The proof of Theorems M0, M2, M4 and Lemma 3.4 remain exactly the same since all the arguments also applied to the case  $s > 3$ . Consequently, we have Theorem 3.1 in this case. Hence, we deduce that Theorem 3.1 holds true for  $s \in (3, 6]$ .

# C Proof of the case $s > 6$

In this appendix, we prove Theorem 3.1 in the case  $s > 6$ . Then, we compare the result to the peeling decay for curvature components obtained in [30].

## C.1 Fundamental norms

The definitions of  $\mathcal{R}$ -norms for  $\alpha$  and  $\beta$  are different from section 3.1. We denote

$$s_0 := \min \left\{ s, \frac{29}{4} \right\}. \quad (\text{C.1})$$

We define for  $q = 0, 1$ :

$$\begin{aligned} \underline{\mathcal{R}}_q[\beta] &:= \|r^3 |u|^{\frac{s-6}{2}} (r\nabla)^q \beta\|_{2, \underline{C}_u^V}, \\ \mathcal{R}_q[\alpha] &:= \|r^3 |u|^{\frac{s-6}{2}} (r\nabla)^q \alpha\|_{2, C_u^V}, \\ \mathcal{R}_1[\alpha_4] &:= |u|^{\frac{s-s_0}{2}} \|r^{\frac{s_0}{2}} \hat{\alpha}\|_{2, C_u^V}. \end{aligned} \quad (\text{C.2})$$

We also define:

$$\underline{\mathcal{R}}_0^S[\beta] := \sup_{\mathcal{K}} \sup_{p \in [2, 4]} |r^{4-\frac{2}{p}} |u|^{\frac{s-5}{2}} \beta|_{p, S}. \quad (\text{C.3})$$

The norms  $\mathcal{R}_0^S[\alpha]$  are defined as follows:

$$\begin{aligned} \mathcal{R}_0^S[\alpha] &:= \sup_{\mathcal{K}} |r^{\frac{s+1}{2}} \alpha|_{2, S}, & s \in (6, 7), \\ \mathcal{R}_0^S[\alpha] &:= \sup_{\mathcal{K}} |r^4 (\log r)^{-\frac{1}{2}} \alpha|_{2, S}, & s = 7, \\ \mathcal{R}_0^S[\alpha] &:= \sup_{\mathcal{K}} |r^4 |u|^{\frac{s-7}{2}} \alpha|_{2, S}, & s > 7. \end{aligned} \quad (\text{C.4})$$

All the other norms are defined as in section 3.1.



## C.2 Optimal constant for Poincaré inequality

**Proposition C.1.** *For any  $\alpha \in \mathfrak{s}_2$ , we have the following Poincaré inequality:*

$$|rd_2\alpha|_{2,S}^2 \geq c_2|\alpha|_{2,S}^2,$$

for sphere  $S = S(u, \underline{u})$ , where  $c_2$  is a constant satisfying:

$$c_2 = 2 - O(\varepsilon). \quad (\text{C.5})$$

*Proof.* See for example Proposition 9.3.2 in [11].  $\square$

## C.3 Curvature estimates

In section 4, we took respectively  $p = s, 4, 2, 0$  to estimate the Bianchi pairs  $(\alpha, \beta)$ ,  $(\beta, (\check{\rho}, \sigma))$ ,  $((\check{\rho}, \sigma), \beta)$  and  $(\beta, \underline{\alpha})$ . In the case  $s > 6$ , we take respectively  $p = 6, 4, 2, 0$  for the Bianchi pairs. Proceeding as in section 4, we deduce from Propositions 4.7–4.13 and 2.28

$$\begin{aligned} & \sum_{q=0}^1 (\mathcal{R}_q[\alpha] + \mathcal{R}_q[\beta] + \underline{\mathcal{R}}_q[\beta] + \mathcal{R}_q[\check{\rho}, \sigma] + \underline{\mathcal{R}}_q[\check{\rho}, \sigma] + \mathcal{R}_q[\underline{\beta}] + \underline{\mathcal{R}}_q[\underline{\beta}] + \underline{\mathcal{R}}_q[\underline{\alpha}]) \\ & + \underline{\mathcal{R}}_1[\underline{\alpha}_3] + \mathcal{R}_0^S[\beta] + \underline{\mathcal{R}}_0^S[\beta] + \underline{\mathcal{R}}_0^S[\check{\rho}, \sigma] + \underline{\mathcal{R}}_0^S[\underline{\beta}] + \underline{\mathcal{R}}_0^S[\underline{\alpha}] + \underline{\mathcal{R}}_0^S[\underline{\rho}] \lesssim \varepsilon_0. \end{aligned} \quad (\text{C.6})$$

It remains to estimate  $\alpha$  and  $\check{\alpha}$ . To this end, we prove the following divergence identity.

**Lemma C.2.** *We have the following identity for any real number  $p$ :*

$$\begin{aligned} & \mathbf{Div}(r^p|\check{\alpha}|^2 e_3) + 2\mathbf{Div}(r^p|\check{\phi}|^2 e_4) + (p+2)r^{p-1}|\check{\alpha}|^2 + 2(8-p)r^{p-1}|\check{\phi}|^2 - 8r^{p-1}\check{\alpha} \cdot \alpha \\ & = 4r^p d_1(\check{\alpha} \cdot \check{\phi}) + r^p(\check{\alpha}, \check{\phi}) \cdot (\Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta) + r^p \Gamma_a \cdot (|\check{\alpha}|^2, |\check{\phi}|^2). \end{aligned} \quad (\text{C.7})$$

*Proof.* We recall from (2.32)

$$\begin{aligned} \nabla_3 \check{\alpha} &= -2d_2^* \check{\phi} + \frac{4\alpha}{r} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta, \\ \nabla_4 \check{\phi} + \frac{5}{2} \text{tr} \chi \check{\phi} &= d_2 \check{\alpha} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta. \end{aligned}$$

Applying Lemma 4.2 with  $\psi_{(1)} = \check{\alpha}$ ,  $\psi_{(2)} = \check{\phi}$ ,  $a_{(1)} = 0$ ,  $a_{(2)} = \frac{5}{2}$ ,  $h_{(1)} = \frac{4\alpha}{r} + \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta$ ,  $h_{(2)} = \Gamma_a \cdot \beta^{(1)} + \Gamma_g^{(1)} \cdot \beta$  and  $k = 2$ , we obtain (C.7) as stated.  $\square$

Next, we prove the following analog of Proposition 4.11.

**Proposition C.3.** *We have the following estimate:*

$$\mathcal{R}_1[\alpha_4] \lesssim \varepsilon_0.$$

*Proof.* Integrating (C.7) with  $p = s_0$  and proceeding as in Proposition 4.7, we obtain

$$\begin{aligned} & \int_{C_u} r^{s_0} |\check{\alpha}|^2 + \int_{\underline{C}_{\underline{u}}} r^{s_0} |\check{\phi}|^2 + \int_V r^{s_0-1} ((s_0+2)|\check{\alpha}|^2 + 2(8-s_0)|\check{\phi}|^2 - 8\check{\alpha} \cdot \alpha) \\ & \lesssim \int_{\Sigma_0 \cap V} r^{s_0} (|\check{\alpha}|^2 + |\check{\phi}|^2) + \int_V r^{s_0} |(\check{\alpha}, \check{\phi})| |\Gamma_a| |\beta^{(1)}| + r^{s_0} |(\check{\alpha}, \check{\phi})| |\Gamma_g^{(1)}| |\beta| + r^{s_0} |\Gamma_a| (|\check{\alpha}|^2 + |\check{\phi}|^2). \end{aligned}$$

First, we proceed as in Lemma 4.6 to obtain

$$\int_{\Sigma_0 \cap V} r^{s_0} (|\check{\alpha}|^2 + |\check{\phi}|^2) \lesssim |u|^{s_0-s} \int_{\Sigma_0 \cap V} r^s (|\check{\alpha}|^2 + |\check{\phi}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

Next, we have

$$\begin{aligned}
\int_V r^{s_0} |(\dot{\alpha}, \phi)| |\Gamma_a| |\beta^{(1)}| &\lesssim \varepsilon \int_V r^{s_0-2} |\dot{\alpha}| |\beta^{(1)}| + \varepsilon \int_V r^{s_0-2} |\phi| |\beta^{(1)}| \\
&\lesssim \varepsilon \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{8-s_0}{2}}} \left( \int_{C_u^V} r^{s_0} |\dot{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^4 |\beta^{(1)}|^2 \right)^{\frac{1}{2}} \\
&+ \varepsilon \int_{|u|}^u \frac{d\underline{u}}{r^{\frac{10-s_0}{2}}} \left( \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\phi|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_{\underline{u}}^V} r^6 |\beta^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{8-s_0}{2}} |u|^{\frac{s-s_0}{2}} |u|^{\frac{s-4}{2}}} \varepsilon^3 + \int_{-\underline{u}}^u \frac{d\underline{u}}{r^{\frac{10-s_0}{2}} |u|^{\frac{s-s_0}{2}} |u|^{\frac{s-6}{2}}} \varepsilon^3 \\
&\lesssim \int_{-\underline{u}}^u \frac{\varepsilon^3}{|u|^{2+s-s_0}} du + \int_{-\underline{u}}^u \frac{\varepsilon^3}{r^{\frac{10-s_0}{2}} |u|^{\frac{2s-s_0-6}{2}}} du \\
&\lesssim \frac{\varepsilon_0^2}{|u|^{1+s-s_0}},
\end{aligned}$$

where we used  $s_0 < 8$ . Then, we compute

$$\begin{aligned}
\int_V r^{s_0} |(\dot{\alpha}, \phi)| |\Gamma_g^{(1)}| |\beta| &\lesssim \int_V r^{s_0} |\dot{\alpha}| |\Gamma_g^{(1)}| |\beta| + \int_V r^{s_0} |\phi| |\Gamma_g^{(1)}| |\beta| \\
&\lesssim \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{8-s_0}{2}}} \left( \int_{C_u^V} r^{s_0} |\dot{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{s_0} |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\
&+ \int_{|u|}^u \frac{d\underline{u}}{|u|^{\frac{8-s_0}{2}}} \left( \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\phi|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{s-s_0}{2}}} \left( \int_{|u|}^u \frac{d\underline{u}}{r^{s_0}} \int_S |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\
&+ \varepsilon \int_{|u|}^u \frac{d\underline{u}}{|u|^{\frac{s-s_0}{2}}} \left( \int_{-\underline{u}}^u \frac{d\underline{u}}{r^{s_0}} \int_S |\Gamma_g^{(1)}|^2 |\beta|^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{s-s_0}{2}}} \left( \int_{|u|}^u \frac{d\underline{u}}{r^{s_0-10}} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{7}{2}} \beta|_{4,S}^2 \right)^{\frac{1}{2}} \\
&+ \varepsilon \int_{|u|}^u \frac{d\underline{u}}{|u|^{\frac{s-s_0}{2}}} \left( \int_{-\underline{u}}^u \frac{d\underline{u}}{r^{s_0-10}} |r^{\frac{3}{2}} \Gamma_g^{(1)}|_{4,S}^2 |r^{\frac{7}{2}} \beta|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon^3 \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{s-s_0}{2}}} \left( \int_{|u|}^u r^{s_0-10} |u|^{8-2s} d\underline{u} \right)^{\frac{1}{2}} \\
&+ \varepsilon^3 \int_{|u|}^u \frac{d\underline{u}}{|u|^{\frac{s-s_0}{2}}} \left( \int_{-\underline{u}}^u r^{s_0-10} |u|^{8-2s} d\underline{u} \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon^3 \int_{-\underline{u}}^u \frac{du}{|u|^{\frac{s-s_0}{2}}} r^{\frac{s_0-9}{2}} |u|^{4-s} + \varepsilon^3 \int_{|u|}^u \frac{d\underline{u}}{|u|^{\frac{s-s_0}{2}}} r^{\frac{s_0}{2}-5} |u|^{\frac{9}{2}-s} \\
&\lesssim \frac{\varepsilon^3}{|u|^{\frac{s-s_0}{2}} |u|^{s-5} |u|^{\frac{9-s_0}{2}}} + \frac{\varepsilon^3}{r^{4-\frac{s_0}{2}} |u|^{\frac{s-s_0}{2}} |u|^{\frac{2s-9}{2}}} \\
&\lesssim \frac{\varepsilon_0^2}{|u|^{\frac{s-1}{2}+s-s_0}}.
\end{aligned}$$

We also have

$$\int_V r^{s_0} |\Gamma_a| (|\dot{\alpha}|^2 + |\not\alpha|^2) \lesssim \varepsilon \int_V r^{s_0-2} (|\dot{\alpha}|^2 + |\not\alpha|^2).$$

Moreover, applying Proposition C.1, we infer

$$\begin{aligned} & \int_V r^{s_0-1} ((s_0+2)|\dot{\alpha}|^2 + 2(8-s_0)|\not\alpha|^2 - 8\dot{\alpha} \cdot \alpha) \\ & \geq \int_V r^{s_0-1} \left( (s_0-6)|\dot{\alpha}|^2 + \frac{8-s_0}{4}|\not\alpha|^2 + \left(\frac{7}{4}(8-s_0)c_2 - 2\right)|\alpha|^2 + 8|\dot{\alpha}|^2 - 8\dot{\alpha} \cdot \alpha + 2|\alpha|^2 \right) \\ & \geq \int_V r^{s_0-1} \left( (s_0-6)|\dot{\alpha}|^2 + \frac{3}{16}|\not\alpha|^2 + \frac{19}{64}|\alpha|^2 + 2|\alpha - 2\dot{\alpha}|^2 \right) \\ & \gtrsim \int_V r^{s_0-1} (|\dot{\alpha}|^2 + |\not\alpha|^2 + |\alpha|^2), \end{aligned}$$

where we used  $6 < s_0 \leq \frac{29}{4}$  and  $c_2 \geq \frac{7}{4}$  for  $\varepsilon$  small enough. Combining the above estimates, we obtain

$$\int_{C_u^V} r^{s_0} |\dot{\alpha}|^2 + \int_{\underline{C}_u^V} r^{s_0} |\not\alpha|^2 + \int_V r^{s_0-1} (|\dot{\alpha}|^2 + |\not\alpha|^2 + |\alpha|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}} + \varepsilon \int_V r^{s_0-2} (|\dot{\alpha}|^2 + |\not\alpha|^2).$$

For  $\varepsilon$  small enough, we deduce

$$\int_{C_u^V} r^{s_0} |\dot{\alpha}|^2 + \int_{\underline{C}_u^V} r^{s_0} |\not\alpha|^2 + \int_V r^{s_0-1} (|\dot{\alpha}|^2 + |\not\alpha|^2 + |\alpha|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

This concludes the proof of Proposition C.3.  $\square$

**Proposition C.4.** *We have the following estimate:*

$$\mathcal{R}_0^S[\alpha] \lesssim \varepsilon_0.$$

*Proof.* We have from Lemma 2.4 that

$$\begin{aligned} \Omega e_4 \left( \int_S |r^4 \alpha|^2 \right) &= \int_S 2\Omega \nabla_4(r^4 \alpha) \cdot (r^4 \alpha) + \Omega \operatorname{tr} \chi |r^4 \alpha|^2 \\ &= \int_S 2\Omega r^4 \alpha \cdot (\nabla_4(r^4 \alpha) + r^3 \alpha) + \Omega \left( \operatorname{tr} \chi - \frac{2}{r} \right) |r^4 \alpha|^2 \\ &= \int_S 2\Omega r^4 \alpha \cdot (\nabla_4(r^5 \alpha) r^{-1} + e_4(r^{-1}) r^5 \alpha + r^3 \alpha) + \Gamma_a |r^4 \alpha|^2 \\ &= \int_S 2\Omega r^4 \alpha \cdot (\nabla_4(r^5 \alpha) r^{-1}) + \Gamma_a |r^4 \alpha|^2 \\ &= \int_S 2\Omega r^4 \alpha \cdot r^3 \dot{\alpha} + \Gamma_a |r^4 \alpha|^2. \end{aligned}$$

Hence, we obtain

$$\left| e_4 \left( \int_S |r^4 \alpha|^2 \right) \right| \lesssim |r^4 \alpha|_{2,S} |r^3 \dot{\alpha}|_{2,S} + \varepsilon |r^4 \alpha|_{2,S} |r^2 \alpha|_{2,S},$$

which implies

$$e_4(|r^4 \alpha|_{2,S}) \lesssim |r^3 \dot{\alpha}|_{2,S} + \varepsilon |r^2 \alpha|_{2,S}.$$

Integrating along  $C_u$ , applying initial assumption and Proposition C.3, we infer

$$\begin{aligned}
|r^4 \alpha|_{2,S} &\lesssim \varepsilon_0 + \int_{|u|}^u |r^3 \dot{\alpha}|_{2,S} d\underline{u} + \varepsilon |r^2 \alpha|_{2,S} \\
&\lesssim \varepsilon_0 + \int_{|u|}^u r^{3-\frac{s_0}{2}} |r^{\frac{s_0}{2}} \dot{\alpha}|_{2,S} d\underline{u} + \int_{|u|}^u \frac{\varepsilon}{r^{\frac{3}{2}}} |r^{\frac{7}{2}} \alpha|_{2,S} d\underline{u} \\
&\lesssim \varepsilon_0 + \left( \int_{|u|}^u r^{6-s_0} d\underline{u} \right)^{\frac{1}{2}} \left( \int_{|u|}^u |r^{\frac{s_0}{2}} \dot{\alpha}|_{2,S}^2 d\underline{u} \right)^{\frac{1}{2}} + \int_{|u|}^u \frac{\varepsilon^2}{r^{\frac{3}{2}}} d\underline{u} \\
&\lesssim \varepsilon_0 + \frac{\varepsilon_0}{|u|^{\frac{s-s_0}{2}}} \left( \int_{|u|}^u r^{6-s_0} d\underline{u} \right)^{\frac{1}{2}}.
\end{aligned} \tag{C.8}$$

Notice that we have

$$\int_{|u|}^u r^{6-s_0} d\underline{u} = \begin{cases} \frac{1}{7-s_0} (\underline{u}^{7-s_0} - |u|^{7-s_0}), & s_0 \in (6, 7), \\ \log \left( \frac{\underline{u}}{|u|} \right), & s_0 = 7, \\ \frac{1}{s_0-7} (|u|^{7-s_0} - \underline{u}^{7-s_0}), & s_0 > 7. \end{cases}$$

Hence, we obtain from (C.8)

$$|r^4 \alpha|_{2,S} \lesssim \begin{cases} \varepsilon_0 r^{\frac{7-s}{2}}, & s \in (6, 7), \\ \varepsilon_0 (\log r)^{\frac{1}{2}}, & s = 7, \\ \varepsilon_0 |u|^{\frac{7-s}{2}}, & s > 7, \end{cases} \tag{C.9}$$

which implies from (C.4) that

$$\mathcal{R}_0^S[\alpha] \lesssim \varepsilon_0. \tag{C.10}$$

This concludes the proof of Proposition C.4.  $\square$

Combining Propositions C.4 with (C.6), this concludes the proof of Theorem M1 in the case of  $s > 6$ . Notice that Theorems M0, M2, M3, M4 also hold true in the case  $s > 6$  since we only used  $s > 3$  in their proofs<sup>25</sup>. Combining with section B.3, this concludes the proof of Theorem 3.1 for  $s > 3$  as stated.

#### C.4 Peeling decay for curvature components

As a consequence of Theorem M1 in the case  $s > 7$ , the curvature components satisfy the following decay:

$$\begin{aligned}
|r^{5-\frac{2}{p}} |u|^{\frac{s-7}{2}} \alpha|_{p=2,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{4-\frac{2}{p}} |u|^{\frac{s-5}{2}} \beta|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{3-\frac{2}{p}} |u|^{\frac{s-3}{2}} (\check{\rho}, \sigma)|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{2-\frac{2}{p}} |u|^{\frac{s-1}{2}} \underline{\beta}|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{1-\frac{2}{p}} |u|^{\frac{s+1}{2}} \underline{\alpha}|_{p,S} &\lesssim \varepsilon_0.
\end{aligned} \tag{C.11}$$

<sup>25</sup>See the discussion in section B.2.

**Remark C.5.** Assume  $s > 7$  and that the initial data in  $\mathcal{K}_{(0)}$  has sufficient regularity properties. Then, commuting  $r\nabla$  with the Bianchi equations in Corollary 2.20 and the Teukolsky equation (2.30), proceeding as in section C.3, we deduce the following strong peeling properties:

$$\begin{aligned} |\alpha|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^5 |u|^{\frac{s-7}{2}}}, & |\beta|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^4 |u|^{\frac{s-5}{2}}}, \\ |\check{\rho}, \sigma|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^3 |u|^{\frac{s-3}{2}}}, & |\underline{\beta}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-1}{2}}}, & |\underline{\alpha}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r |u|^{\frac{s+1}{2}}}, \end{aligned} \quad (\text{C.12})$$

which recovers the results obtained in [30] by the vectorfield method.

## Chapter 3

# Global stability of Minkowski spacetime with minimal decay

## 1 Introduction

### 1.1 Einstein vacuum equations and the Cauchy problem

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \quad \text{in } \mathcal{M}, \quad (1.1)$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for  $\mathbf{g}$ . In suitable coordinates, for example so-called wave coordinates, it can be shown that (1.1) is hyperbolic and hence admits an initial value formulation.

The corresponding initial data for the Einstein vacuum equations is given by specifying a triplet  $(\Sigma, g, k)$  where  $(\Sigma, g)$  is a Riemannian 3-manifold and  $k$  is the traceless symmetric 2-tensor on  $\Sigma$  satisfying the constraint equations:

$$\begin{aligned} R &= |k|^2 - (\operatorname{tr} k)^2, \\ \nabla^j k_{ij} &= \nabla_i \operatorname{tr} k, \end{aligned} \quad (1.2)$$

where  $R$  denotes the scalar curvature of  $g$ ,  $\nabla$  denotes the Levi-Civita connection of  $g$  and

$$|k|^2 := g^{im} g^{jl} k_{ij} k_{lm}, \quad \operatorname{tr} k := g^{ij} k_{ij}.$$

In the future development  $(\mathcal{M}, \mathbf{g})$  of such initial data  $(\Sigma, g, k)$ ,  $\Sigma \subset \mathcal{M}$  is a spacelike hypersurface with induced metric  $g$  and second fundamental form  $k$ .

The seminal well-posedness results for the Cauchy problem obtained in [8, 9] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  solution of Einstein equations (1.1) such that  $\Sigma \subset \mathcal{M}$  and  $g, k$  are respectively the first and second fundamental forms of  $\Sigma$  in  $\mathcal{M}$ .

The prime example of a vacuum spacetime is Minkowski space  $(\mathcal{M}, \eta)$ :

$$\mathcal{M} = \mathbb{R}^4, \quad \eta = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \quad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad k = 0.$$

In the present work, we consider the problem of the stability of Minkowski spacetime and start by reviewing the state of the art on this problem.

### 1.2 Previous works on the stability of Minkowski spacetime

In 1993, Christodoulou and Klainerman [10] proved the global stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2007, Bieri [5] gave a new proof of global stability of Minkowski requiring one less derivative and less vectorfields compared to [10]. Both [10] and [5] rely on the maximal foliation. Given that the goal of this paper is to extend the result of [5], we will state the results of [10, 5] in Section 1.3.

We now mention proofs of Minkowski stability using other gauges. In 2003, Klainerman and Nicolò [29] proved the Minkowski stability in the exterior of an outgoing cone using the double null foliation. Moreover, Klainerman and Nicolò [30] showed that under stronger asymptotic

decay and regularity properties than those used in [10, 29], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. Lindblad and Rodnianski [40, 41] gave a new proof of the stability of the Minkowski spacetime using *wave-coordinates* and showing that the Einstein equations verify the so called *weak null structure* in that gauge. Huneau [23] proved the nonlinear stability of Minkowski spacetime with a translation Killing field using generalised wave-coordinates. Using the framework of Melrose's b-analysis, Hintz and Vasy [21] reproved the stability of Minkowski space. Graf [19] proved the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations, which together with [29] allows to reobtain [10]. Under the framework of [29] and using  $r^p$ -weighted estimates of Dafermos and Rodnianski [13], the author [53] reproved the Minkowski stability in exterior regions. More recently, Hintz [20] reproved the Minkowski stability in exterior regions by using the framework of [21].

There are also stability results concerning Einstein's equations coupled with non trivial matter fields:

- Einstein-Maxwell system: Zipser [62] extended the framework of [10] to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system. In [43], Loizelet used the framework of [41] to demonstrate the stability of the Minkowski spacetime solution of the Einstein-scalar field-Maxwell system in  $(1 + n)$ -dimensions ( $n \geq 3$ ). Speck [56] gave a proof of the global nonlinear stability of the  $(1 + 3)$ -dimensional Minkowski spacetime solution to the coupled system for a family of electromagnetic fields, which includes the standard Maxwell fields.
- Einstein-Klein-Gordon system: Lefloch and Ma [38] and Wang [60] proved the global stability of Minkowski for the Einstein-Klein-Gordon system with initial data coinciding with the Schwarzschild solution with small mass outside a compact set. Ionescu and Pausader [24] proved the global stability of Minkowski for the Einstein-Klein-Gordon system for general initial data. See also [39] by Lefloch and Ma using a foliation that merges hyperboloids with constant time slices.
- Einstein-Vlasov system: Taylor [57] considered the massless case where the initial data for the Vlasov part is compactly supported on the mass shell. Fajman, Joudioux and Smulevici [14] considered the massive case where the initial data coincides with Schwarzschild in the exterior region and with compact support assumption only in space on the Vlasov part. Lindblad and Taylor [42] considered the massive case where the initial data has compact support for the Vlasov part. Bigorgne, Fajman, Joudioux, Smulevici and Thaller [6] considered the massless case for general initial data. Wang [61] considered the massive case for general initial data.

### 1.3 Minkowski stability in [10, 5]

We recall in this section the results in [10, 5]. First, we recall the definition of a *maximal hypersurface*, which plays an important role in the statements of the main theorems in [10, 5].

**Definition 1.1.** *An initial data  $(\Sigma, g, k)$  is posed on a maximal hypersurface if it satisfies*

$$\operatorname{tr} k = 0. \tag{1.3}$$

*In this case, we say that  $(\Sigma, g, k)$  is a maximal initial data set, and the constraint equations (1.2) reduce to*

$$R = |k|^2, \quad \operatorname{div} k = 0, \quad \operatorname{tr} k = 0. \tag{1.4}$$

We introduce the notion of  $(s, q)$ -*asymptotically flat initial data*.



**Definition 1.2.** Given  $s > 1$  and  $q \in \mathbb{N}$ , we say that a data set  $(\Sigma_0, g, k)$  is  $(s, q)$ -asymptotically flat if there exists a coordinate system  $(x^1, x^2, x^3)$  defined outside a sufficiently large compact set such that:

- In the case  $s \geq 3$ <sup>1</sup>

$$\begin{aligned} g_{ij} &= \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_q(r^{-\frac{s+1}{2}}). \end{aligned} \quad (1.5)$$

- In the case  $1 < s < 3$

$$\begin{aligned} g_{ij} &= \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_q(r^{-\frac{s+1}{2}}). \end{aligned} \quad (1.6)$$

**Remark 1.3.** According to (1.6), the assumption  $s > 1$  is necessary to define an asymptotic perturbation of the flat metric on  $\Sigma_0$ . See also Remark 3.7 for more explanations.

**Definition 1.4.** We denote  $d_0$  the geodesic distance from a fixed point  $O \in \Sigma_0$ ,  $B_{ij} := (\text{curl } \widehat{R})_{ij}$  the Bach tensor and  $\widehat{R}$  is the traceless part of Ric. Then, we define for  $q \geq 2$ :

- In the case  $s \geq 3$ :

$$\begin{aligned} J_0^{(q)}(\Sigma_0, g, k) &:= \sup_{\Sigma_0} \left( (d_0^2 + 1)^3 |\text{Ric}|^2 \right) + \sum_{l=0}^q \int_{\Sigma_0} (d_0^2 + 1)^{l + \frac{s-2}{2}} |\nabla^l k|^2 \\ &\quad + \sum_{l=0}^{q-2} \int_{\Sigma_0} (d_0^2 + 1)^{l + \frac{s+2}{2}} |\nabla^l B|^2. \end{aligned} \quad (1.7)$$

- In the case  $1 < s < 3$ :

$$J_0^{(q)}(\Sigma_0, g, k) := \sum_{l=0}^q \int_{\Sigma_0} (d_0^2 + 1)^{l + \frac{s-2}{2}} |\nabla^l k|^2 + \sum_{l=0}^{q-1} \int_{\Sigma_0} (d_0^2 + 1)^{l + \frac{s}{2}} |\nabla^l \text{Ric}|^2. \quad (1.8)$$

We can now state the main theorems of [10] and [5].

**Theorem 1.5** (Global stability of Minkowski spacetime [10]). *There exists an  $\varepsilon_0 > 0$  sufficiently small such that if  $J_0^{(3)}(\Sigma_0, g, k) \leq \varepsilon_0^2$ , then the initial data set  $(\Sigma_0, g, k)$ ,  $(4, 3)$ -asymptotically flat (in the sense of Definition 1.2) and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e. the Riemann curvature tensor tends to zero along any causal or space-like geodesic. Moreover, there exists a global maximal time function  $t$  and an optical function<sup>2</sup>  $u$  defined everywhere in an external region.*

**Theorem 1.6** (Global stability of Minkowski spacetime [5]). *There exists an  $\varepsilon_0 > 0$  sufficiently small such that if  $J_0^{(2)}(\Sigma_0, g, k) \leq \varepsilon_0^2$ , then the initial data set  $(\Sigma_0, g, k)$ ,  $(2, 2)$ -asymptotically flat and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat. Moreover, there exists a global maximal time function  $t$  and an optical function  $u$  defined everywhere in an external region.*

<sup>1</sup>The notation  $f = o_q(r^{-m})$  means  $\partial^\alpha f = o(r^{-m-|\alpha|})$ ,  $|\alpha| \leq q$ .

<sup>2</sup>An optical function  $u$  is a scalar function satisfying  $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ .

**Remark 1.7.** *Various choices of the parameter  $s$  in Definition 1.2 have been made in the literature. In particular.<sup>3</sup>*

- $s = 4$  is used for example in [10].
- $s = 3 + \delta^4$  is used for example in [41].
- $s = 3 - \delta$  is used in [24] for the Einstein-Klein-Gordon system.
- $s \in (2, 3)$  is used in [39] for the Einstein-Klein-Gordon system.
- $s = 2$  is used in [5].

The goal of this paper is to extend the results of [5] to lower values of  $s$ . More precisely, we prove the global stability of Minkowski spacetime for  $(s, 2)$ -asymptotically flat initial data for all  $s \in (1, 2]$ .

#### 1.4 Rough version of the main theorem

In this section, we state a simple version of our main theorem. For the precise statement, see Theorem 3.6.

**Theorem 1.8** (Main Theorem, first version). *Let  $s \in (1, 2]$  and let an initial data set  $(\Sigma_0, g, k)$  which is  $(s, 2)$ -asymptotically flat in the sense of Definition 1.2. Assume that we have a smallness conditions in an initial layer region  $\mathcal{K}_{(0)}$ <sup>5</sup> near  $\Sigma_0$ . Then, there exists a unique future development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:*

- $(\mathcal{M}, \mathbf{g})$  can be foliated by a maximal-null foliation  $(C_u, \Sigma_t)$  whose outgoing leaves  $C_u$  are complete for all  $u$ ;
- We have detailed control of all the quantities associated with the maximal-null foliation of the spacetime, see Theorem 3.6.

**Remark 1.9.** *In the particular case  $s = 2$ , we reobtain the results of [5] restated above in Theorem 1.6.*

The structure of the proof of Theorem 1.8 is similar to [10, 5], see Section 3.5. Below, we compare the proof of this paper and that of [5]:

1. In [5], the functional (1.8) is introduced to fix the initial conditions on the initial hypersurface  $\Sigma_0$ . Here we fix the initial conditions in an initial layer region  $\mathcal{K}_{(0)}$  near the initial hypersurface  $\Sigma_0$ .
2. To estimate the norms of curvature components, [5] uses the vectorfield method introduced in [10]. Here, we estimate the curvature norms by relying on  $r^p$ -weighted estimates, a method introduced by Dafermos and Rodnianski in [13]. This allows for a simpler treatment of the curvature estimates.
3. In order to obtain optimal decay for all the connection coefficients, [5] uses two foliations constructed respectively forward and backward. In this paper, we derive decay for the connection coefficients based on only one foliation constructed forward from the initial hypersurface  $\Sigma_0$  and the symmetry axis  $\Phi$ . This avoids having to compare forward and backward gauges.

<sup>3</sup>This list focuses on low values of  $s$ . For large values of  $s$ , note for example [30] valid for  $s > 7$  and [53] valid for  $s > 3$ .

<sup>4</sup>Here and in the line below,  $\delta$  denotes a constant satisfying  $0 < \delta \ll 1$ .

<sup>5</sup>See (2.9) for the definition of  $\mathcal{K}_{(0)}$ .

## 1.5 Structure of the paper

- In Section 2, we introduce the geometric set-up and the basic equations.
- In Section 3, we state a precise version of the main theorem. We then state intermediate results, and use them to prove the main theorem. The rest of the paper then focuses on the proof of these intermediary results.
- In Section 4, we make bootstrap assumptions and prove first consequences. These consequences will be used frequently in the rest of the paper.
- In Section 5, we apply  $r^p$ -weighted estimates to Bianchi equations to control curvature.
- In Section 6, we estimate maximal connection coefficients and the lapse function using the elliptic systems satisfied on maximal hypersurfaces.
- In Section 7, we estimate the null connection coefficients using the null structure equations.
- In Appendix A, we prove Theorem 5.8, which provides a unified treatment of nonlinear error terms appearing in the curvature estimates of Section 5.

## 1.6 Acknowledgements

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# 2 Preliminaries

## 2.1 Geometric set-up

### 2.1.1 Maximal foliation

We foliate the spacetime  $(\mathcal{M}, \mathbf{g})$  by spacelike hypersurfaces  $\Sigma_t$  as level hypersurfaces of a time function  $t$ . We denote by  $T$  the unit, future oriented, normal to  $\Sigma_t$ . We then introduce a coordinates system  $(x^1, x^2, x^3)$  on  $\Sigma_0$ , and extend it to  $\mathcal{M}$  by

$$T(x^i) = 0, \quad i = 1, 2, 3.$$

Relative to this foliation, the spacetime metric takes the form:

$$\mathbf{g} = -\phi^2 dt^2 + \sum_{i,j=1}^3 g_{ij} dx^i dx^j. \quad (2.1)$$

We define the *lapse function* of the foliation by

$$\phi(t, x) := (\mathbf{g}(\mathbf{D}t, \mathbf{D}t))^{-1}.$$

Then, we have

$$T = \phi \mathbf{D}t.$$

We define the second fundamental form:

$$k_{ij} := -\mathbf{g}(\mathbf{D}_i T, \partial_j). \quad (2.2)$$

We denote by  $\nabla$  and  $\text{Ric}$  respectively the induced covariant derivative and the corresponding Ricci curvature tensor on the leaves  $\Sigma_t$ . Relative to an orthonormal frame  $e_1, e_2, e_3$  tangent to the leaves of the foliation, we have the formulae:

$$\begin{aligned}\mathbf{D}_i e_j &= \nabla_i e_j - k_{ij} T, \\ \mathbf{D}_i T &= -k_{ij} e_j, \\ \mathbf{D}_T e_i &= (\phi^{-1} \nabla_i \phi) T, \\ \mathbf{D}_T T &= (\phi^{-1} \nabla_i \phi) e_i,\end{aligned}$$

where  $\nabla_T e_i$  denotes the projection of  $\mathbf{D}_T e_i$  to the tangent space of  $\Sigma_t$ .

**Lemma 2.1.** *We have the following structure equations of the foliation:*

$$\begin{aligned}\partial_t k_{ij} &= -\nabla_i \nabla_j \phi + \phi(\mathbf{R}_{i T j T} - k_{il} k^l{}_j), \\ \nabla_i k_{jl} - \nabla_j k_{il} &= \mathbf{R}_{l T i j}, \\ \text{Ric}_{ij} - k_{il} k^l{}_j + k_{ij} \text{tr } k &= \mathbf{R}_{i T j T} + \mathbf{R}_{ij}.\end{aligned}$$

*Proof.* See (1.0.3a)–(1.0.3c) in [10]. □

Taking the trace of the formulae in Lemma 2.1, we obtain the following corollary.

**Corollary 2.2.** *If  $(\mathcal{M}, \mathbf{g})$  is a solution of Einstein equations, we have the following equations:*

$$\begin{aligned}R - |k|^2 + (\text{tr } k)^2 &= 0, \\ \nabla^j k_{ji} - \nabla^i \text{tr } k &= 0, \\ \partial_t \text{tr } k &= -\Delta \phi + \phi |k|^2.\end{aligned}$$

Combining Lemma 2.1 and Corollary 2.2, we obtain the following proposition.

**Proposition 2.3.** *We impose a maximal foliation on  $\Sigma_t$  in the sense of Definition 1.1. Then, we have:*

- *Maximal constraint equations:*

$$\begin{aligned}\text{tr } k &= 0, \\ \nabla^j k_{ji} &= 0, \\ R &= |k|^2.\end{aligned}\tag{2.3}$$

- *Evolution equations:*

$$\begin{aligned}\partial_t g_{ij} &= -2\phi k_{ij}, \\ \partial_t k_{ij} &= -\nabla_i \nabla_j \phi + \phi(\text{Ric}_{ij} - 2k_{il} k^l{}_j).\end{aligned}\tag{2.4}$$

- *Lapse equation:*

$$\Delta \phi = \phi |k|^2.\tag{2.5}$$

*Proof.* See (1.0.11)–(1.0.13) in [10]. □

### 2.1.2 Maximal-null foliation

Let  $O \in \Sigma_0$ . We define  $\Phi$  as the integral curve of  $O$  along the vectorfield  $T$ , called the *symmetry axis*. Starting from  $O$ , we construct an outgoing null cone  $C_0$  in the future of  $\Sigma_0$ . The spacetime  $\mathcal{M}$  is then divided into

$$\mathcal{M} = I^+(O) \cup C_0 \cup D^+(\Sigma_0 \setminus O),$$

where  $I^+(O)$  denotes the *future domain of influence* of  $O$  and  $D^+(\Sigma_0 \setminus O)$  denotes the *future domain of dependence* of  $\Sigma_0 \setminus O$ .

We now construct an outgoing optical function  $u$  in  $\mathcal{M}$  as follows. First, to allow ourselves some room, we assume that a spacetime slab  $\bigcup_{t \in [-1, 0]} \Sigma_t$  in the past of  $\Sigma_0$  has been constructed. We denote

$$O_{-1} := \Phi \cap \Sigma_{-1}.$$

We define  $u$  on  $\Phi$  by

$$T(u) = 1 \quad \text{on } \Phi, \quad u(O) = 0.$$

Then, for any  $p \in \Phi \cap I^+(O_{-1})$ , we construct, emanating from  $p$ , an outgoing null cone  $C_{u(p)}$ . Thus, the spacetime  $I^+(O_{-1})$  is foliated by the outgoing null cones  $C_u$  for  $u > u(O_{-1})$ . We then extend  $u$  to  $I^+(O_{-1})$  by assuming that  $C_u$  are level sets of  $u$  and we denote

$$L := -\text{grad } u \quad \text{in } I^+(O_{-1}).$$

Note that

$$I^+(O) \cup C_0 \subseteq I^+(O_{-1}).$$

It thus remains to define  $u$  in  $D^+(\Sigma_0 \setminus O)$ .

For  $u(O_{-1}) < u \leq 0$ , the 2-spheres  $C_u \cap \Sigma_0$  define a radial foliation on  $\Sigma_0 \cap I^+(O_{-1})$  centered at  $O$ . We define a scalar function  $w$  on  $\Sigma_0$  by assuming

$$w := -u, \quad \text{on } \Sigma_0 \cap I^+(O_{-1}),$$

and extending it smoothly to  $\Sigma_0$ . We thus fixed a radial foliation on the initial hypersurface  $\Sigma_0$  centered at  $O$  by the level sets of  $w$ . The corresponding leaves are denoted by

$$S_{(0)}(w_1) = \{p \in \Sigma_0 / w(p) = w_1\}, \quad (2.6)$$

where  $w_1 \geq 0$  and  $w(O) = 0$ . Then, we define the outgoing optical function  $u$  in  $D^+(\Sigma_0 \setminus O)$  by the following *Eikonal equation*:

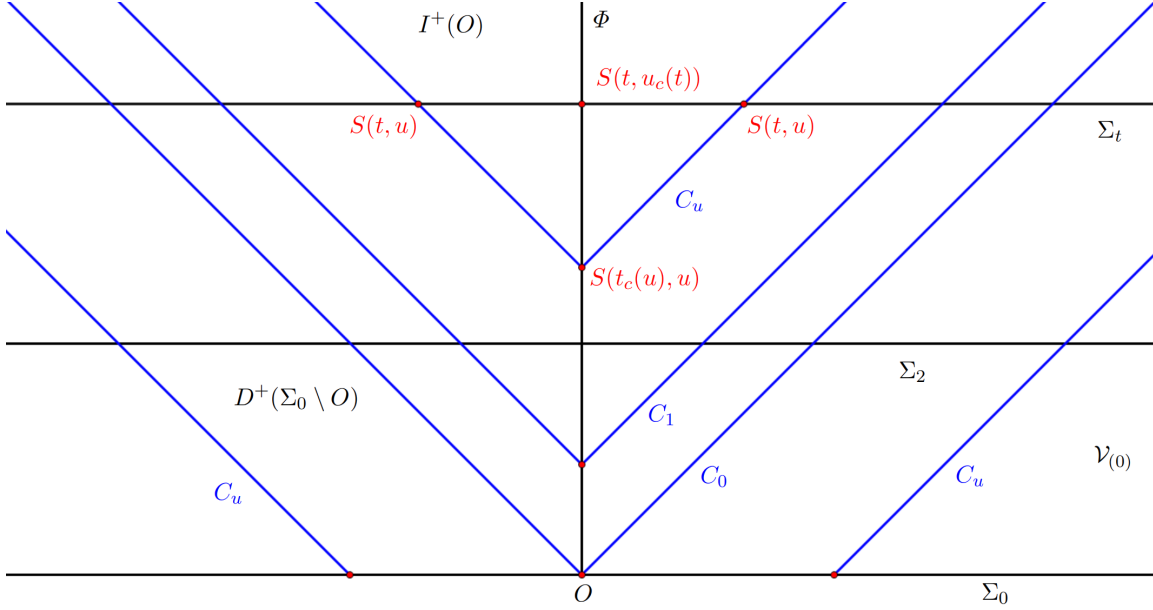
$$\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad u|_{\Sigma_0 \setminus O} = -w. \quad (2.7)$$

Denoting

$$L := -\text{grad } u \quad \text{in } D^+(\Sigma_0 \setminus K),$$

we have the following geodesic equation:

$$\mathbf{D}_L L = 0.$$

Figure 3.1: Maximal-null foliation of the spacetime  $\mathcal{M}$ 

Thus,  $u$  and  $L$  are well-defined everywhere in  $\mathcal{M}$ . Moreover,  $u$  is smooth in  $\mathcal{M}$  by construction.

We consider the points on  $\Phi$ :

$$S(t, u_c(t)) := \Sigma_t \cap \Phi, \quad S(t_c(u), u) := C_u \cap \Phi, \quad (2.8)$$

as spheres of radius 0. Thus, the spacetime  $\mathcal{M}$  is foliated by 2-spheres:

$$S(t, u) := \Sigma_t \cap C_u, \quad u \in \mathbb{R}, \quad t \geq 0.$$

We define the *initial layer region*:

$$\mathcal{K}_{(0)} := \mathcal{M} \cap \{0 \leq t \leq 2\} = J^-(\Sigma_2) \cap J^+(\Sigma_0), \quad (2.9)$$

where  $J^\pm(\Sigma)$  denotes the *causal future* (resp. *past*) of  $\Sigma$ . Note that we have

$$u > 0 \text{ in } I^+(O), \quad u = 0 \text{ on } C_0, \quad u < 0 \text{ in } D^+(\Sigma_0 \setminus O),$$

see Figure 3.1 for a picture. We also define for  $t \geq 2$

$$V_t := J^-(\Sigma_t) \cap J^+(\Sigma_2).$$

For any  $t \geq 2$ , we define the *exterior region*:

$${}^e V_t := \{p \in V_t / u \leq 1\},$$

and the *interior region*:

$${}^i V_t := \{p \in V_t / u \geq 1\}.$$

We then define the null vectorfield  $e_4$  and the *null lapse function*  $a$  by

$$a := |\nabla u|^{-1}, \quad e_4 := aL.$$

By definition, we have

$$0 = \mathbf{g}(\mathbf{D}u, \mathbf{D}u) = |\nabla u|^2 - |T(u)|^2 = a^{-2} - |T(u)|^2,$$

which implies

$$a = T(u)^{-1}.$$

Thus, we have in particular  $a = 1$  on  $\Phi$ . Next, we define

$$N := e_4 - T,$$

which is a vectorfield on  $\mathcal{M} \setminus \Phi$ . By a direct computation, we have

$$\begin{aligned} \mathbf{g}(T, N) &= \mathbf{g}(T, e_4 - T) = a\mathbf{g}(T, L) - \mathbf{g}(T, T) = -a\mathbf{g}(T, \mathbf{D}u) + 1 = -aT(u) + 1 = 0, \\ \mathbf{g}(N, N) &= \mathbf{g}(e_4 - T, e_4 - T) = -2\mathbf{g}(e_4, T) + \mathbf{g}(T, T) = -2a\mathbf{g}(L, T) - 1 = 2aT(u) - 1 = 1. \end{aligned}$$

Thus,  $N$  denotes the unit vectorfield tangent to  $\Sigma_t$ , oriented towards infinity and orthogonal to the leaves  $S(t, u)$ .

We denote by  $\theta$  the second fundamental form of the surfaces  $S(t, u)$  relative to  $\Sigma_t$ :

$$\theta_{AB} := \mathbf{g}(\mathbf{D}_A N, e_B).$$

We then define the null vectorfield  $e_3$  by

$$e_3 := T - N.$$

By a direct computation, we have

$$\mathbf{g}(e_3, e_3) = \mathbf{g}(T - N, T - N) = 0, \quad \mathbf{g}(e_3, e_4) = \mathbf{g}(T - N, T + N) = 0.$$

On a given 2-sphere  $S(t, u)$ , we choose a local orthonormal frame  $(e_1, e_2)$  and we call  $(e_1, e_2, e_3, e_4)$  a null frame.<sup>6</sup>

**Remark 2.4.** *As a convention, throughout the paper, we use capital Latin letters  $A, B, C, \dots$  to denote an index from 1 to 2, lower case Latin letters  $i, j, k, \dots$  to denote an index from 1 to 3 and Greek letters  $\alpha, \beta, \gamma, \dots$  to denote an index from 1 to 4, e.g.  $e_A$  denotes either  $e_1$  or  $e_2$ .*

The spacetime metric  $\mathbf{g}$  induces a Riemannian metric  $\mathcal{g}$  on  $S(t, u)$ . We use  $\nabla$  to denote the Levi-Civita connection of  $\mathcal{g}$  on  $S(t, u)$ . We recall the null decomposition of the Ricci coefficients and curvature components of the null frame  $(e_1, e_2, e_3, e_4)$  as follows:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), & \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), \\ \underline{\xi}_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_A), & \xi_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_A), \\ \underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_A), & \eta_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_A), \\ \zeta_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_A} e_4, e_3), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= \mathbf{R}(e_A, e_3, e_B, e_3), \\ \beta_A &= \frac{1}{2}\mathbf{R}(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A, e_3, e_3, e_4), \\ \rho &= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), & \sigma &= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), \end{aligned} \tag{2.11}$$

<sup>6</sup>Note that the null frame  $(e_1, e_2, e_3, e_4)$  is well-defined on  $\mathcal{M} \setminus \Phi$ .

where  $^*\mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ .

The null second fundamental forms  $\chi, \underline{\chi}$  are further decomposed in their traces  $\text{tr } \chi$  and  $\text{tr } \underline{\chi}$ , and traceless parts  $\widehat{\chi}$  and  $\widehat{\underline{\chi}}$ :

$$\begin{aligned} \text{tr } \chi &:= \not{g}^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \not{g}_{AB} \text{tr } \chi, \\ \text{tr } \underline{\chi} &:= \not{g}^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \not{g}_{AB} \text{tr } \underline{\chi}. \end{aligned}$$

We define the horizontal covariant operator  $\nabla$  as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3.$$

We also define  $\nabla_4 X$  and  $\nabla_3 X$  to be the horizontal projections of  $\mathbf{D}_4 X$  and  $\mathbf{D}_3 X$ :

$$\begin{aligned} \nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_4) e_3, \\ \nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_4) e_3. \end{aligned}$$

A tensor field  $\psi$  defined on  $\mathcal{M}$  is called tangent to  $S(t, u)$  if it is a priori defined on the spacetime  $\mathcal{M}$  and all the possible contractions of  $\psi$  with either  $e_3$  or  $e_4$  are zero. We use  $\nabla_3 \psi$  and  $\nabla_4 \psi$  to denote the projection to  $S(t, u)$  of usual derivatives  $\mathbf{D}_3 \psi$  and  $\mathbf{D}_4 \psi$ .

As a direct consequence of (2.10), we have the Ricci formulae:

$$\begin{aligned} \mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4, \\ \mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + \zeta_A e_3, \\ \mathbf{D}_A e_4 &= \chi_{AB} e_B - \zeta_A e_4, \\ \mathbf{D}_3 e_A &= \nabla_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, \\ \mathbf{D}_4 e_A &= \nabla_4 e_A + \underline{\eta}_A e_4 + \xi_A e_3, \\ \mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_B e_B, \\ \mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\underline{\eta}_B e_B, \\ \mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_B e_B, \\ \mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_B e_B. \end{aligned} \tag{2.12}$$

Next, we decompose  $k$  as follows:

$$\delta := k_{NN}, \quad \epsilon_A := k_{AN}, \quad \kappa_{AB} := k_{AB}. \tag{2.13}$$

The following identities hold for a maximal-null foliation:

$$\begin{aligned} \chi_{AB} &= \theta_{AB} - \kappa_{AB}, & \underline{\chi}_{AB} &= -\theta_{AB} - \kappa_{AB}, \\ \xi_A &= 0, & \underline{\xi}_A &= \nabla_A \log \phi - \nabla_A \log a, \\ \eta_A &= \nabla_A \log a + \epsilon_A, & \underline{\eta}_A &= \nabla_A \log \phi - \epsilon_A, \\ 2\omega &= -\nabla_N \log \phi + \delta, & 2\underline{\omega} &= \nabla_N \log \phi + \delta, \\ \zeta_A &= \epsilon_A, \end{aligned} \tag{2.14}$$

see (7.5.2b) in [10].



Note that we have the following types of manifolds in this paper: the spacetime region  $\mathcal{M}$ , the spacelike hypersurfaces  $\Sigma_t$ , and the spheres  $S(t, u)$ . Every type of manifold has its metric, Levi-Civita connection and curvature tensor:

$$(\mathcal{M}, \mathbf{g}, \mathbf{D}, \mathbf{R}), \quad (\Sigma_t, g, \nabla, R), \quad (S(t, u), \mathcal{g}, \nabla, \mathcal{R}).$$

## 2.2 Integral formulae

We define the  $S$ -average of scalar functions.

**Definition 2.5.** *Given any scalar function  $f$ , we denote its average and its average free part by*

$$\bar{f} := \frac{1}{|S|} \int_S f d\mathcal{g}, \quad \check{f} := f - \bar{f},$$

where  $|S|$  denotes the area of  $S$ .

The following lemma follows immediately from the definition.

**Lemma 2.6.** *For any two scalar functions  $u$  and  $v$ , we have*

$$\overline{uv} = \bar{u}\bar{v} + \overline{\check{u}\check{v}},$$

and

$$uv - \overline{uv} = \check{u}\bar{v} + \bar{u}\check{v} + (\check{u}\check{v} - \overline{\check{u}\check{v}}).$$

We recall the following integral formulae.

**Lemma 2.7.** *For any scalar function  $f$ , the following equations hold:*

$$\begin{aligned} \phi e_4 \left( \int_{S(t,u)} f d\mathcal{g} \right) &= \int_{S(t,u)} (\phi e_4(f) + \phi \operatorname{tr} \chi f) d\mathcal{g}, \\ aN \left( \int_{S(t,u)} f d\mathcal{g} \right) &= \int_{S(t,u)} (aN(f) + a \operatorname{tr} \theta f) d\mathcal{g}. \end{aligned}$$

Taking  $f = 1$ , we obtain

$$e_4(r) = \frac{\overline{\phi \operatorname{tr} \chi}}{2\phi} r, \quad N(r) = \frac{\overline{a \operatorname{tr} \theta}}{2a} r, \quad (2.15)$$

where  $r$  is the area radius defined by

$$r(t, u) := \sqrt{\frac{|S(t, u)|}{4\pi}}.$$

*Proof.* See Lemma 2.26 in [31]. □

## 2.3 Hodge systems in $2D$ -geometry

**Definition 2.8.** *For tensor fields defined on a 2-sphere  $S$ , we denote by  $\mathfrak{s}_0 := \mathfrak{s}_0(S)$  the set of pairs of scalar functions,  $\mathfrak{s}_1 := \mathfrak{s}_1(S)$  the set of 1-forms and  $\mathfrak{s}_2 := \mathfrak{s}_2(S)$  the set of symmetric traceless 2-tensors.*

**Definition 2.9.** *Given  $\xi \in \mathfrak{s}_1$ , we define its Hodge dual  ${}^*\xi$  by*

$${}^*\xi_A := \epsilon_{AB} \xi^B,$$

where  $\epsilon_{AB}$  denotes the volume element on  $S$ . Clearly  ${}^*\xi \in \mathfrak{s}_1$  and

$${}^*({}^*\xi) = -\xi.$$

Given  $U \in \mathfrak{s}_2$ , we define its Hodge dual  $*U$  by

$$*U_{AB} := \epsilon_{AC} U^C{}_B.$$

Observe that  $*U \in \mathfrak{s}_2$  and

$$*(U) = -U.$$

**Definition 2.10.** Given  $\xi, \eta \in \mathfrak{s}_1$ , we denote

$$\begin{aligned} \xi \cdot \eta &:= \delta^{AB} \xi_A \eta_B, \\ \xi \wedge \eta &:= \epsilon^{AB} \xi_A \eta_B, \\ (\widehat{\xi \otimes \eta})_{AB} &:= \xi_A \eta_B + \xi_B \eta_A - \delta_{AB} \xi \cdot \eta. \end{aligned}$$

Given  $\xi \in \mathfrak{s}_1$ ,  $U \in \mathfrak{s}_2$ , we denote

$$(\xi \cdot U)_A := \delta^{BC} \xi_B U_{AC}.$$

Given  $U, V \in \mathfrak{s}_2$ , we denote

$$(U \wedge V)_{AB} := \epsilon^{AB} U_{AC} V_{CB}.$$

**Definition 2.11.** For a given  $\xi \in \mathfrak{s}_1$ , we define the following differential operators:

$$\begin{aligned} \text{div} \xi &:= \delta^{AB} \nabla_A \xi_B, \\ \text{curl} \xi &:= \epsilon^{AB} \nabla_A \xi_B, \\ (\widehat{\nabla \otimes \xi})_{AB} &:= \nabla_A \xi_B + \nabla_B \xi_A - g_{AB} (\text{div} \xi). \end{aligned}$$

**Definition 2.12.** We define the following Hodge type operators, as introduced in Section 2.2 in [10]:

- $\mathfrak{d}_1$  takes  $\mathfrak{s}_1$  into pairs of  $\mathfrak{s}_0$  and is given by:

$$\mathfrak{d}_1 \xi := (\text{div} \xi, \text{curl} \xi),$$

- $\mathfrak{d}_2$  takes  $\mathfrak{s}_2$  into  $\mathfrak{s}_1$  and is given by:

$$(\mathfrak{d}_2 U)_A := \nabla^B U_{AB},$$

- $\mathfrak{d}_1^*$  takes pairs of  $\mathfrak{s}_0$  into  $\mathfrak{s}_1$  and is given by:

$$\mathfrak{d}_1^*(f, f_*)_A := -\nabla_A f + \epsilon_A{}^B \nabla_B f_*,$$

- $\mathfrak{d}_2^*$  takes  $\mathfrak{s}_1$  into  $\mathfrak{s}_2$  and is given by:

$$\mathfrak{d}_2^* \xi := -\frac{1}{2} \widehat{\nabla \otimes \xi}.$$

**Proposition 2.13.** We have the following identities:

$$\begin{aligned} \mathfrak{d}_1^* \mathfrak{d}_1 &= -\Delta_1 + \mathbf{K}, & \mathfrak{d}_1 \mathfrak{d}_1^* &= -\Delta_0, \\ \mathfrak{d}_2^* \mathfrak{d}_2 &= -\frac{1}{2} \Delta_2 + \mathbf{K}, & \mathfrak{d}_2 \mathfrak{d}_2^* &= -\frac{1}{2} (\Delta_1 + \mathbf{K}), \end{aligned} \tag{2.16}$$

where  $\mathbf{K}$  denotes the Gauss curvature on  $S$  and  $\Delta_k$  denotes the Laplacian on  $\mathfrak{s}_k$ .

*Proof.* See (2.2.2) in [10]. □

**Definition 2.14.** We define the weighted angular derivatives  $\mathfrak{D}$  as follows:

$$\begin{aligned}\mathfrak{D}U &:= r\mathfrak{d}_2U, & \forall U \in \mathfrak{s}_2, \\ \mathfrak{D}\xi &:= r\mathfrak{d}_1\xi, & \forall \xi \in \mathfrak{s}_1, \\ \mathfrak{D}(f, f_*) &:= r\mathfrak{d}_1^*(f, f_*), & \forall (f, f_*) \in \mathfrak{s}_0 \times \mathfrak{s}_0.\end{aligned}$$

We denote for any tensor  $h \in \mathfrak{s}_k$ ,  $k = 0, 1, 2$ ,

$$h^{(0)} := h, \quad h^{(1)} := (h, \mathfrak{D}h), \quad h^{(2)} := (h, \mathfrak{D}h, \mathfrak{D}^2h).$$

**Proposition 2.15.** We have the following commutator identities:

- For any  $U \in \mathfrak{s}_2$

$$\mathfrak{D}\mathfrak{d}_2U = \mathfrak{d}_1\mathfrak{D}U.$$

- For any  $\xi \in \mathfrak{s}_1$

$$\begin{aligned}\mathfrak{D}\mathfrak{d}_1\xi &= \mathfrak{d}_1^*\mathfrak{D}\xi, \\ \mathfrak{D}(2\mathfrak{d}_2^*\xi) &= \mathfrak{d}_1^*\mathfrak{D}\xi - 2r\mathbf{K}\xi.\end{aligned}$$

- For any  $(f, f_*) \in \mathfrak{s}_0$

$$\mathfrak{D}\mathfrak{d}_1^*(f, f_*) = \mathfrak{d}_1\mathfrak{D}(f, f_*).$$

*Proof.* We recall from (2.16) that for any  $\xi \in \mathfrak{s}_1$

$$2\mathfrak{D}\mathfrak{d}_2^*\xi = 2r\mathfrak{d}_2\mathfrak{d}_2^*\xi = -r(\mathfrak{A}_1 + \mathbf{K})\xi = r(\mathfrak{d}_1^*\mathfrak{d}_1 - 2\mathbf{K})\xi = \mathfrak{d}_1^*\mathfrak{D}\xi - 2r\mathbf{K}\xi,$$

which implies the third identity. The other identities follow directly from Definition 2.14. This concludes the proof of Proposition 2.15.  $\square$

**Definition 2.16.** For a tensor field  $f$  on a 2-sphere  $S$ , we denote its  $L^p$ -norm as follows:

$$|f|_{p,S} := \left( \int_S |f|^p \right)^{\frac{1}{p}}. \quad (2.17)$$

## 2.4 Null structure equations

We recall the null structure equations, see Proposition 7.4.1 in [10].

**Proposition 2.17.** *We have the null structure equations:*

$$\begin{aligned}
\mathring{\nabla}_4 \widehat{\chi} + \operatorname{tr} \chi \widehat{\chi} &= \mathring{\nabla} \widehat{\otimes} \xi - 2\omega \widehat{\chi} + (\eta + \underline{\eta} + 2\zeta) \widehat{\otimes} \xi - \alpha, \\
\mathring{\nabla}_4 \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \chi)^2 &= 2\operatorname{div} \xi - 2\omega \operatorname{tr} \chi + 2\xi \cdot (\eta + \underline{\eta} + 2\zeta) - |\widehat{\chi}|^2, \\
\mathring{\nabla}_3 \widehat{\underline{\chi}} + \operatorname{tr} \underline{\chi} \widehat{\underline{\chi}} &= \mathring{\nabla} \widehat{\otimes} \underline{\xi} - 2\omega \widehat{\underline{\chi}} + (\eta + \underline{\eta} - 2\zeta) \widehat{\otimes} \underline{\xi} - \underline{\alpha}, \\
\mathring{\nabla}_3 \operatorname{tr} \underline{\chi} + \frac{1}{2} (\operatorname{tr} \underline{\chi})^2 &= 2\operatorname{div} \underline{\xi} - 2\omega \operatorname{tr} \underline{\chi} + 2\underline{\xi} \cdot (\eta + \underline{\eta} - 2\zeta) - |\widehat{\underline{\chi}}|^2, \\
\mathring{\nabla}_4 \widehat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \chi \widehat{\underline{\chi}} &= \mathring{\nabla} \widehat{\otimes} \underline{\eta} + 2\omega \widehat{\underline{\chi}} - \frac{1}{2} \operatorname{tr} \chi \widehat{\underline{\chi}} + \underline{\xi} \widehat{\otimes} \underline{\xi} + \underline{\eta} \widehat{\otimes} \underline{\eta}, \\
\mathring{\nabla}_4 \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} &= 2\omega \operatorname{tr} \underline{\chi} - \widehat{\chi} \cdot \widehat{\underline{\chi}} + 2\operatorname{div} \underline{\eta} + 2(\underline{\xi} \cdot \xi + |\underline{\eta}|^2) + 2\rho, \\
\mathring{\nabla}_3 \widehat{\chi} + \frac{1}{2} \operatorname{tr} \underline{\chi} \widehat{\chi} &= \mathring{\nabla} \widehat{\otimes} \eta + 2\omega \widehat{\chi} - \frac{1}{2} \operatorname{tr} \chi \widehat{\chi} + \xi \widehat{\otimes} \xi + \eta \widehat{\otimes} \eta, \\
\mathring{\nabla}_3 \operatorname{tr} \chi + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \chi &= 2\omega \operatorname{tr} \chi - \widehat{\chi} \cdot \widehat{\underline{\chi}} + 2\operatorname{div} \eta + 2(\xi \cdot \underline{\xi} + |\eta|^2) + 2\rho, \\
\mathring{\nabla}_4 \eta - \mathring{\nabla}_3 \xi &= -4\omega \xi - \chi \cdot (\eta - \underline{\eta}) - \beta, \\
\mathring{\nabla}_3 \underline{\eta} - \mathring{\nabla}_4 \underline{\xi} &= -4\omega \underline{\xi} - \underline{\chi} \cdot (\underline{\eta} - \eta) + \beta, \\
\mathring{\nabla}_3 \zeta &= -2\mathring{\nabla} \omega - \underline{\chi} \cdot (\zeta + \eta) + 2\omega (\zeta - \eta) + \chi \cdot \underline{\xi} + 2\omega \underline{\xi} - \beta, \\
\mathring{\nabla}_4 \zeta &= 2\mathring{\nabla} \omega + \chi \cdot (-\zeta + \underline{\eta}) + 2\omega (\zeta + \underline{\eta}) - \underline{\chi} \cdot \xi - 2\omega \xi - \beta, \\
\mathring{\nabla}_3 \omega + \mathring{\nabla}_4 \omega &= \xi \cdot \underline{\xi} + \zeta \cdot (\eta - \underline{\eta}) - \eta \cdot \underline{\eta} + 4\omega \omega + \rho.
\end{aligned}$$

Also, we have the Codazzi equations:

$$\begin{aligned}
\operatorname{div} \widehat{\chi} &= \frac{1}{2} \mathring{\nabla} \operatorname{tr} \chi - \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \operatorname{tr} \chi \right) - \beta, \\
\operatorname{div} \widehat{\underline{\chi}} &= \frac{1}{2} \mathring{\nabla} \operatorname{tr} \underline{\chi} + \zeta \cdot \left( \widehat{\underline{\chi}} - \frac{1}{2} \operatorname{tr} \underline{\chi} \right) + \underline{\beta},
\end{aligned} \tag{2.18}$$

the torsion equation:

$$\begin{aligned}
\operatorname{curl} \eta &= \sigma + \frac{1}{2} \widehat{\underline{\chi}} \wedge \widehat{\chi} - \xi \wedge \underline{\xi}, \\
\operatorname{curl} \underline{\eta} &= -\sigma - \frac{1}{2} \widehat{\underline{\chi}} \wedge \widehat{\chi} + \xi \wedge \underline{\xi}, \\
\operatorname{curl} \xi &= \xi \wedge (\eta + \underline{\eta} + 2\zeta), \\
\operatorname{curl} \underline{\xi} &= \underline{\xi} \wedge (\eta + \underline{\eta} - 2\zeta),
\end{aligned} \tag{2.19}$$

and the Gauss equation:

$$\mathbf{K} = -\frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} + \frac{1}{2} \widehat{\underline{\chi}} \cdot \widehat{\chi} - \rho. \tag{2.20}$$

## 2.5 Bianchi equations

We recall the Bianchi equations, see Proposition 7.3.2 in [10].

**Proposition 2.18.** *The Bianchi equations take the following form*

$$\begin{aligned}
\mathring{\nabla}_3 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha &= \mathring{\nabla} \widehat{\otimes} \beta + 4\underline{\omega} \alpha - 3(\widehat{\chi} \rho + {}^* \widehat{\chi} \sigma) + (\zeta + 4\underline{\eta}) \widehat{\otimes} \beta, \\
\mathring{\nabla}_4 \beta + 2 \operatorname{tr} \chi \beta &= \operatorname{div} \alpha - 2\underline{\omega} \beta + (2\zeta + \underline{\eta}) \cdot \alpha + 3(\xi \rho + {}^* \xi \sigma), \\
\mathring{\nabla}_3 \beta + \operatorname{tr} \chi \beta &= \mathring{\nabla} \rho + {}^* \mathring{\nabla} \sigma + 2\underline{\omega} \beta + \underline{\xi} \cdot \alpha + 2\widehat{\chi} \cdot \beta + 3(\eta \rho + {}^* \eta \sigma), \\
\mathring{\nabla}_4 \rho + \frac{3}{2} \operatorname{tr} \chi \rho &= \operatorname{div} \beta - \frac{1}{2} \widehat{\chi} \cdot \alpha + \zeta \cdot \beta + 2(\underline{\eta} \cdot \beta - \underline{\xi} \cdot \beta), \\
\mathring{\nabla}_3 \rho + \frac{3}{2} \operatorname{tr} \chi \rho &= -\operatorname{div} \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} + 2(\underline{\xi} \cdot \beta - \eta \cdot \underline{\beta}), \\
\mathring{\nabla}_4 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{curl} \beta + \frac{1}{2} \widehat{\chi} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2(\underline{\eta} \cdot {}^* \beta + 2\underline{\xi} \cdot {}^* \beta), \\
\mathring{\nabla}_3 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{curl} \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot {}^* \underline{\alpha} + \zeta \cdot {}^* \underline{\beta} - 2(\eta \cdot {}^* \underline{\beta} + \underline{\xi} \cdot {}^* \beta), \\
\mathring{\nabla}_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\mathring{\nabla} \rho + {}^* \mathring{\nabla} \sigma + 2\underline{\omega} \underline{\beta} + 2\widehat{\chi} \cdot \beta - \underline{\xi} \cdot \alpha - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma), \\
\mathring{\nabla}_3 \underline{\beta} + 2 \operatorname{tr} \chi \underline{\beta} &= -\operatorname{div} \alpha - 2\underline{\omega} \underline{\beta} + (2\zeta - \underline{\eta}) \cdot \underline{\alpha} + 3(-\underline{\xi} \rho + {}^* \underline{\xi} \sigma), \\
\mathring{\nabla}_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\mathring{\nabla} \widehat{\otimes} \underline{\beta} + 4\underline{\omega} \underline{\alpha} - 3(\widehat{\chi} \rho - {}^* \widehat{\chi} \sigma) + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta}.
\end{aligned}$$

## 2.6 Commutation identities

We recall the following commutation lemma.

**Lemma 2.19.** *Let  $U_{A_1 \dots A_k}$  be an  $S$ -tangent  $k$ -covariant tensor on  $(\mathcal{M}, \mathbf{g})$ . Then*

$$\begin{aligned}
[\mathring{\nabla}_4, \mathring{\nabla}_B] U_{A_1 \dots A_k} &= -\chi_{BC} \mathring{\nabla}_C U_{A_1 \dots A_k} + F_{4BA_1 \dots A_k}, \\
F_{4BA_1 \dots A_k} &:= \xi_B \mathring{\nabla}_3 U_{A_1, \dots, A_k} + (\zeta_B + \underline{\eta}_B) \mathring{\nabla}_4 U_{A_1 \dots A_k} \\
&\quad + \sum_{i=1}^k (\chi_{A_i B} \xi_C - \chi_{BC} \xi_{A_i} + \chi_{A_i B} \underline{\eta}_C - \chi_{BC} \underline{\eta}_{A_i} + \epsilon_{A_i C} {}^* \beta_B) U_{A_1 \dots C \dots A_k}, \\
[\mathring{\nabla}_3, \mathring{\nabla}_B] U_{A_1 \dots A_k} &= -\chi_{BC} \mathring{\nabla}_C U_{A_1 \dots A_k} + F_{3BA_1 \dots A_k}, \\
F_{3BA_1 \dots A_k} &:= \underline{\xi}_B \mathring{\nabla}_4 U_{A_1, \dots, A_k} + (\eta_B - \zeta_B) \mathring{\nabla}_3 U_{A_1 \dots A_k} \\
&\quad + \sum_{i=1}^k (\chi_{A_i B} \underline{\xi}_C - \chi_{BC} \underline{\xi}_{A_i} + \chi_{A_i B} \eta_C - \chi_{BC} \eta_{A_i} + \epsilon_{A_i C} {}^* \underline{\beta}_B) U_{A_1 \dots C \dots A_k}, \\
[\mathring{\nabla}_3, \mathring{\nabla}_4] U_{A_1 \dots A_k} &= F_{34A_1 \dots A_k}, \\
F_{34A_1 \dots A_k} &:= -2\underline{\omega} \mathring{\nabla}_3 U + 2\underline{\omega} \mathring{\nabla}_4 U + (\eta_B - \underline{\eta}_B) \mathring{\nabla}_B U_{A_1 \dots A_k} \\
&\quad + 2 \sum_{i=1}^k (\xi_{A_i} \underline{\xi}_C - \xi_C \underline{\xi}_{A_i} + \underline{\eta}_{A_i} \eta_C - \underline{\eta}_{A_i} \eta_C + \epsilon_{A_i C} \sigma) U_{A_1 \dots C \dots A_k}.
\end{aligned}$$

*Proof.* See Lemma 7.3.3 in [10]. □

## 2.7 The electric-magnetic decomposition of curvature

**Proposition 2.20.** *We define the electric-magnetic decomposition of  $\mathbf{R}$ :*

$$E_{ij} := \mathbf{R}(T, e_i, T, e_j), \quad H_{ij} := {}^* \mathbf{R}(T, e_i, T, e_j), \quad i, j = 1, 2, 3. \quad (2.21)$$

Then, the following identities hold for a maximal foliation:

$$\begin{aligned} E_{AB} &= \frac{1}{4}\alpha_{AB} + \frac{1}{4}\underline{\alpha}_{AB} - \frac{1}{2}\rho\delta_{AB}, & H_{AB} &= -\frac{1}{4}^*\alpha_{AB} + \frac{1}{4}^*\underline{\alpha}_{AB} - \frac{1}{2}\sigma\delta_{AB}, \\ E_{AN} &= \frac{1}{2}\underline{\beta}_A + \frac{1}{2}\beta_A, & H_{AN} &= \frac{1}{2}^*\beta_A - \frac{1}{2}^*\beta_A, \\ E_{NN} &= \rho, & H_{NN} &= \sigma. \end{aligned} \tag{2.22}$$

*Proof.* See (7.3.3e) in [10]. □

**Proposition 2.21.** *The following equations hold for a maximal foliation:*

$$\begin{aligned} \nabla_i k_{jm} - \nabla_j k_{im} &= \epsilon_{ij}{}^l H_{lm}, \\ \text{Ric}_{ij} - k_{il} k^l{}_j &= E_{ij}. \end{aligned}$$

*Proof.* See (1.0.8a)–(1.0.8b) in [10]. □

**Definition 2.22.** *Let  $\Sigma$  be a 3-dimensional manifold diffeomorphic to  $\mathbb{R}^3$ . We define the following differential operators on  $\Sigma$ :*

- For any scalar function  $\phi$  on  $\Sigma$ , we define

$$\Delta\phi = \nabla^i \nabla_i \phi.$$

- For any 1-form  $\xi$  on  $\Sigma$ , we define

$$\begin{aligned} \text{div } \xi &:= \nabla^i \xi_i, \\ (\text{curl } \xi)_i &:= \epsilon_i{}^{lm} \nabla_l \xi_m. \end{aligned}$$

- For any traceless symmetric 2-tensor  $U$  on  $\Sigma$ , we define

$$\begin{aligned} (\text{div } U)_i &:= \nabla^j U_{ij}, \\ (\text{curl } U)_{ij} &:= \frac{1}{2} \left( \epsilon_i{}^{lm} \nabla_l U_{mj} + \epsilon_j{}^{lm} \nabla_l U_{mi} \right). \end{aligned}$$

**Proposition 2.23.** *Let  $E, H$  be the electric-magnetic decomposition defined in (2.21). Then, the following identities hold:*

$$\begin{aligned} \text{div } E &= k \wedge H, \\ \text{div } H &= -k \wedge E, \\ -\hat{\mathcal{L}}_T H + \text{curl } E &= -\nabla \log \phi \wedge E - \frac{1}{2} k \times H, \\ \hat{\mathcal{L}}_T E + \text{curl } H &= -\nabla \log \phi \wedge H + \frac{1}{2} k \times E, \end{aligned}$$

where

$$\begin{aligned} (\hat{\mathcal{L}}_T E)_{ij} &:= (\mathcal{L}_T E)_{ij} - \frac{2}{3} (k \cdot E) g_{ij}, \\ (\hat{\mathcal{L}}_T H)_{ij} &:= (\mathcal{L}_T H)_{ij} - \frac{2}{3} (k \cdot H) g_{ij}, \end{aligned} \tag{2.23}$$

and where  $\mathcal{L}_X U$  denotes the Lie derivative of a tensor  $U$  relative to a vectorfield  $X$ .

*Proof.* See Proposition 7.2.1 in [10]. □

### 3 Main theorem

#### 3.1 Preliminary definitions

**Definition 3.1.** For any  $t \geq 2$  and  $-\infty \leq u_1 < u_2 \leq +\infty$ , we denote

$$V_t(u_1, u_2) := J^+(\Sigma_2) \cap J^-(\Sigma_t) \cap \{u_1 \leq u \leq u_2\},$$

and for  $V = V_t(u_1, u_2)$  and  $u_1 \leq u \leq u_2$

$$C_u^V := C_u \cap V, \quad \Sigma_t^V := \Sigma_t \cap V.$$

We then define for any tensorfield  $\Psi$ :

$$\begin{aligned} B_p^q[\Psi](u_1, u_2) &:= \int_V r^{p-1} |\Psi^{(q)}|^2, \\ E_p^q[\Psi](u) &:= \int_{C_u^V} r^p |\Psi^{(q)}|^2, \\ F_p^q[\Psi](u_1, u_2) &:= \int_{\Sigma_t^V} r^p |\Psi^{(q)}|^2, \\ D_\ell^q[\Psi](u) &:= \sup_{S \subseteq C_u^V} \sup_{p \in [2, 4]} |r^{\frac{\ell+3}{2} - \frac{2}{p}} \Psi^{(q)}|_{p, S}^2, \\ D_\ell^{(\infty)}[\Psi](u) &:= \sup_{C_u^V} |r^{\frac{\ell+3}{2}} \Psi|^2, \end{aligned}$$

where  $q = 0, 1, 2, 3$  and where the notation  $\Psi^{(q)}$  has been introduced in Definition 2.14. We also use the following shorthand notations:

$$V_t(u) := V_t(-\infty, u), \quad X_p^q[\Psi](u) := X_p^q[\Psi](-\infty, u), \quad \text{where } X = B, F.$$

We also denote

$$X_p[\Psi](u_1, u_2) := X_p^0[\Psi](u_1, u_2), \quad \text{where } X = B, E, F, D.$$

Moreover, we denote

$$XY_p^q[\Psi](u_1, u_2) := X_p^q[\Psi](u_1, u_2) + Y_p^q[\Psi](u_1, u_2) \quad \text{where } X, Y \in \{B, E, F\},$$

and

$$E_p^q[\Psi](u_1, u_2) := \sup_{u \in [u_1, u_2]} E_p^q[\Psi](u).$$

**Definition 3.2.** Let  $2 < t_* < +\infty$  and  $q = 0, 1, 2, 3$ . We define:

$$\begin{aligned} {}^e\mathcal{B}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{u \leq 1} \langle u \rangle^d B_p^q[\Psi](u), \\ {}^e\mathcal{E}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{u \leq 1} \langle u \rangle^d E_p^q[\Psi](u), \\ {}^e\mathcal{F}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{u \leq 1} \langle u \rangle^d F_p^q[\Psi](u), \\ {}^e\mathcal{D}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{u \leq 1} \langle u \rangle^d D_p^q[\Psi](u), \\ {}^e\mathcal{D}_{p,d}^{(\infty)}[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{u \leq 1} \langle u \rangle^d D_p^{(\infty)}[\Psi](u), \end{aligned}$$

where

$$\langle u \rangle := \sqrt{1 + |u|^2}. \quad (3.1)$$

We also define:

$$\begin{aligned}
{}^i\mathcal{B}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{1 \leq u_1 \leq u_2 \leq u_c(t)} u_1^d B_p^q[\Psi](u_1, u_2), \\
{}^i\mathcal{E}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{1 \leq u \leq u_c(t)} u^d E_p^q[\Psi](u), \\
{}^i\mathcal{F}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{1 \leq u_1 \leq u_2 \leq u_c(t)} u_1^d F_p^q[\Psi](u_1, u_2), \\
{}^i\mathcal{D}_{p,d}^q[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{1 \leq u \leq u_c(t)} u^d D_p^q[\Psi](u), \\
{}^i\mathcal{D}_{p,d}^{(\infty)}[\Psi] &:= \sup_{2 \leq t \leq t_*} \sup_{1 \leq u \leq u_c(t)} u^d D_{p,d}^{(\infty)}[\Psi](u).
\end{aligned}$$

Moreover, we denote

$$\mathcal{X}_{p,d}^q[\Psi] := {}^e\mathcal{X}_{p,d}^q[\Psi] + {}^i\mathcal{X}_{p,d}^q[\Psi], \quad \text{where } \mathcal{X} \in \{\mathcal{B}, \mathcal{E}, \mathcal{F}, \mathcal{D}\}.$$

We also use the following shorthand notations:

$$\mathcal{X}_{p,d}[\Psi] := \mathcal{X}_{p,d}^0[\Psi], \quad \mathcal{X}_{p,0}^q[\Psi] := \mathcal{X}_{p,0}^q[\Psi], \quad \mathcal{X}_p[\Psi] := \mathcal{X}_{p,0}^0[\Psi], \quad \text{where } \mathcal{X} \in \{\mathcal{B}, \mathcal{E}, \mathcal{F}, \mathcal{D}\}.$$

**Definition 3.3.** We define the renormalized quantity  $\Psi$  and mass aspect function  $\mu$ :<sup>7</sup>

$$\Psi := \nabla \text{tr } \chi + \text{tr } \chi \zeta, \quad \mu := -\text{div } \eta - \rho + \frac{1}{2} \widehat{\chi} \cdot \underline{\widehat{\chi}}. \quad (3.2)$$

We also define the following renormalized curvature components:<sup>8</sup>

$$\check{\rho} := \rho - \frac{1}{2} \widehat{\chi} \cdot \underline{\widehat{\chi}}, \quad \check{\sigma} := \sigma - \frac{1}{2} \widehat{\chi} \wedge \underline{\widehat{\chi}}. \quad (3.3)$$

**Remark 3.4.** The purpose of defining the renormalized quantities  $\check{\rho}$  and  $\check{\sigma}$  is to get rid of the terms  $\widehat{\chi} \cdot \underline{\alpha}$  and  $\widehat{\chi} \wedge \underline{\alpha}$  in the right hand side of the Bianchi equations for  $\nabla_3 \rho$  and  $\nabla_3 \sigma$ , as they are the most dangerous among all the nonlinear terms appearing in the Bianchi equations. See Lemma 4.9 for the Bianchi equations satisfied by  $(\check{\rho}, \check{\sigma})$ . Note that the terms  $\widehat{\chi} \cdot \underline{\alpha}$  and  $\widehat{\chi} \wedge \underline{\alpha}$  are emphasized as being borderline for the case  $s = 2$  in [5].

## 3.2 Main norms

The norms in Sections 3.2.1–3.2.3 are defined in the bootstrap region  $V_{t_*}$  while the norms in Section 3.2.4 are defined in the initial layer region  $\mathcal{K}_{(0)}$ . Also, throughout Sections 3.2.1–3.2.3, the norms  ${}^E\mathcal{R}$ ,  ${}^E\mathcal{K}$  and  ${}^E\mathcal{O}$  are defined in the whole region  $V_{t_*} = {}^eV \cup {}^iV$  while the norms  ${}^I\mathcal{R}$ ,  ${}^I\mathcal{K}$ ,  ${}^I\mathcal{O}$  and  ${}^T\mathcal{O}$  are defined in the interior region  ${}^iV$  only.

### 3.2.1 $\mathcal{R}$ norms (curvature components)

We define

$${}^E\mathcal{R} := {}^E\mathcal{R}[\alpha] + {}^E\mathcal{R}[\beta] + {}^E\mathcal{R}[\check{\rho}, \check{\sigma}] + {}^E\mathcal{R}[\beta] + {}^E\mathcal{R}[\alpha] + {}^E\mathcal{R}[\alpha_4] + {}^E\mathcal{R}[\underline{\alpha}_3],$$

<sup>7</sup>The quantities  $\Psi$  and  $\mu$  appeared first in [10].

<sup>8</sup>This renormalization appeared first in [31].



where

$$\begin{aligned}
({}^E\mathcal{R}[\alpha])^2 &:= \mathcal{B}_s^1[\alpha] + \mathcal{E}_s^1[\alpha] + \mathcal{F}_s^1[\alpha] + \mathcal{D}_s[\alpha], \\
({}^E\mathcal{R}[\beta])^2 &:= \mathcal{B}_s^1[\beta] + \mathcal{E}_s^1[\beta] + \mathcal{F}_s^1[\beta] + \mathcal{D}_s[\beta], \\
({}^E\mathcal{R}[\check{\rho}, \check{\sigma}])^2 &:= \mathcal{B}_s^1[\check{\rho}, \check{\sigma}] + \mathcal{E}_s^1[\check{\rho}, \check{\sigma}] + \mathcal{F}_s^1[\check{\rho}, \check{\sigma}] + \mathcal{D}_s[\check{\rho}, \check{\sigma}], \\
({}^E\mathcal{R}[\underline{\beta}])^2 &:= \mathcal{E}_{0,s}^1[\underline{\beta}] + \mathcal{F}_s^1[\underline{\beta}] + \mathcal{D}_{s-1,1}[\underline{\beta}], \\
({}^E\mathcal{R}[\underline{\alpha}])^2 &:= \mathcal{F}_{0,s}^1[\underline{\alpha}] + \mathcal{D}_{-1,s+1}[\underline{\alpha}], \\
({}^E\mathcal{R}[\alpha_4])^2 &:= \mathcal{B}_s[\alpha_4] + \mathcal{E}_s[\alpha_4] + \mathcal{F}_s[\alpha_4], \\
({}^E\mathcal{R}[\underline{\alpha}_3])^2 &:= \mathcal{F}_{-2,s+2}[\underline{\alpha}_3],
\end{aligned}$$

with

$$\alpha_4 := r^{-4}\check{\nabla}_4(r^5\alpha), \quad \underline{\alpha}_3 := r^{-4}\check{\nabla}_3(r^5\underline{\alpha}).$$

Next, we define

$${}^I\mathcal{R} := {}^I\mathcal{R}[\alpha] + {}^I\mathcal{R}[\beta] + {}^I\mathcal{R}[\check{\rho}, \check{\sigma}] + {}^I\mathcal{R}[\underline{\beta}] + {}^I\mathcal{R}[\underline{\alpha}] + {}^I\mathcal{R}[\alpha_4],$$

where

$$\begin{aligned}
({}^I\mathcal{R}[R])^2 &:= {}^i\mathcal{E}_{0,s}^1[R] + {}^i\mathcal{F}_{-2,s+2}^1[R] + {}^i\mathcal{D}_{-2,s+2}[R], \quad \forall R \in \{\alpha, \beta, \check{\rho}, \check{\sigma}, \underline{\beta}, \underline{\alpha}\}, \\
({}^I\mathcal{R}[\alpha_4])^2 &:= {}^i\mathcal{E}_{0,s}[\alpha_4] + {}^i\mathcal{F}_{-2,s+2}[\alpha_4] + {}^i\mathcal{D}_{-2,s+2}[\alpha_4].
\end{aligned}$$

We also define the following time derivative norms:

$${}^T\mathcal{R} := {}^T\mathcal{R}[\alpha] + {}^T\mathcal{R}[\beta] + {}^T\mathcal{R}[\rho, \sigma] + {}^T\mathcal{R}[\underline{\beta}] + {}^T\mathcal{R}[\underline{\alpha}],$$

where

$$\begin{aligned}
({}^T\mathcal{R}[R])^2 &:= {}^i\mathcal{E}_{0,s+2}[\check{\nabla}_T R] + {}^i\mathcal{F}_{0,s+2}[\check{\nabla}_T R], \quad \forall R \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}\}, \\
({}^T\mathcal{R}[\underline{\alpha}])^2 &:= {}^i\mathcal{F}_{0,s+2}[\check{\nabla}_T \underline{\alpha}].
\end{aligned}$$

Finally, we denote

$$\mathcal{R} := {}^E\mathcal{R} + {}^I\mathcal{R} + {}^T\mathcal{R}.$$

### 3.2.2 $\mathcal{K}$ norms (maximal connection coefficients)

We define

$${}^E\mathcal{K} := {}^E\mathcal{K}[\delta] + {}^E\mathcal{K}[\epsilon] + {}^E\mathcal{K}[\widehat{\kappa}] + {}^E\mathcal{K}[\phi],$$

where

$$\begin{aligned}
({}^E\mathcal{K}[\delta])^2 &:= \mathcal{F}_{s-2}^2[\delta] + \mathcal{F}_s^1[\check{\nabla}_N \delta] + \mathcal{D}_{s-2}^1[\delta] + \mathcal{D}_{s-1,1}[\check{\nabla}_N \delta], \\
({}^E\mathcal{K}[\epsilon])^2 &:= \mathcal{F}_{s-2}^2[\epsilon] + \mathcal{F}_s^1[\check{\nabla}_N \epsilon] + \mathcal{D}_{s-2}^1[\epsilon] + \mathcal{D}_{s-1,1}[\check{\nabla}_N \epsilon], \\
({}^E\mathcal{K}[\widehat{\kappa}])^2 &:= \mathcal{F}_{s-2}^2[\widehat{\kappa}] + \mathcal{F}_s^1[\check{\nabla}_N \widehat{\kappa}] + \mathcal{D}_{s-3,1}^1[\widehat{\kappa}] + \mathcal{D}_{-1,s-1}[\widehat{\kappa}] + \mathcal{D}_{-1,s+1}[\check{\nabla}_N \widehat{\kappa}], \\
({}^E\mathcal{K}[\phi])^2 &:= \mathcal{F}_{s-4}^3[\log \phi] + \mathcal{F}_{s-2}^2[\nabla_N \phi] + \mathcal{F}_s^1[\nabla_N \nabla_N \phi] \\
&\quad + \mathcal{D}_{s-4}^2[\log \phi] + \mathcal{D}_{s-2}^1[\nabla_N \phi] + \mathcal{D}_{s-1,1}[\nabla_N \nabla_N \phi].
\end{aligned}$$

Next, we define

$${}^I\mathcal{K} := {}^I\mathcal{K}[k] + {}^I\mathcal{K}[\phi],$$

where

$$\begin{aligned} ({}^I\mathcal{K}[k])^2 &:= {}^i\mathcal{F}_{-2,s}[k] + {}^i\mathcal{F}_{0,s}[\nabla k] + {}^i\mathcal{F}_{0,s+2}[\nabla^2 k] + {}^iD_{-3,s+1}^{(\infty)}[k] + {}^i\mathcal{D}_{-2,s+2}[\nabla k], \\ ({}^I\mathcal{K}[\phi])^2 &:= {}^i\mathcal{F}_{-3,s-1}[\log \phi] + {}^i\mathcal{F}_{-2,s}[\nabla \phi] + {}^i\mathcal{F}_{0,s}[\nabla^2 \phi] + {}^i\mathcal{F}_{0,s+2}[\nabla^3 \phi] \\ &\quad + {}^iD_{-3,s-1}^{(\infty)}[\log \phi] + {}^iD_{-3,s+1}^{(\infty)}[\nabla \phi] + {}^i\mathcal{D}_{-2,s+2}[\nabla^2 \phi]. \end{aligned}$$

We also define the following time derivative norms:

$${}^T\mathcal{K} := {}^T\mathcal{K}[\delta] + {}^T\mathcal{K}[\epsilon] + {}^T\mathcal{K}[\widehat{\kappa}] + {}^T\mathcal{K}[\phi],$$

where

$$\begin{aligned} ({}^T\mathcal{K}[\delta])^2 &:= \sup_{p \in [-2, s-1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \delta], \\ ({}^T\mathcal{K}[\epsilon])^2 &:= \sup_{p \in [-2, s-1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \epsilon], \\ ({}^T\mathcal{K}[\widehat{\kappa}])^2 &:= \sup_{p \in [-2, -1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \widehat{\kappa}], \\ ({}^T\mathcal{K}[\phi])^2 &:= \sup_{p \in [-2, s-1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \nabla \phi]. \end{aligned}$$

Finally, we denote

$$\mathcal{K} := {}^E\mathcal{K} + {}^I\mathcal{K} + {}^T\mathcal{K}.$$

### 3.2.3 $\mathcal{O}$ norms (null connection coefficients)

In the rest of this paper, we always denote by  $\delta_B$  a fixed constant satisfying

$$0 < \delta_B \ll s - 1. \quad (3.4)$$

We define

$${}^E\mathcal{O} := {}^E\mathcal{O}[\Psi] + {}^E\mathcal{O}[\mu] + {}^E\mathcal{O}[\text{tr } \chi] + {}^E\mathcal{O}[\widehat{\chi}] + {}^E\mathcal{O}[\eta],$$

where

$$\begin{aligned} ({}^E\mathcal{O}[\Psi])^2 &:= \mathcal{D}_s[\Psi] + \sup_{S \subseteq V} |r^{\frac{s+3}{2}} \nabla \Psi|_{2,S} + \mathcal{E}_{s-\delta_B, \delta_B}^1[\Psi] + \mathcal{F}_{s-\delta_B, \delta_B}^1[\Psi], \\ ({}^E\mathcal{O}[\mu])^2 &:= \mathcal{D}_{1, s-1}[\mu] + \mathcal{F}_{s-\delta_B, \delta_B}^1[\mu], \\ ({}^E\mathcal{O}[\text{tr } \chi])^2 &:= \mathcal{D}_{s-2}^1[\widetilde{\text{tr } \chi}] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\widetilde{\text{tr } \chi}] + \mathcal{D}_{-1, s-1}^{(\infty)} \left[ \text{tr } \chi - \frac{2}{r} \right], \\ ({}^E\mathcal{O}[\widehat{\chi}])^2 &:= \mathcal{D}_{s-2}^1[\widehat{\chi}] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\widehat{\chi}], \\ ({}^E\mathcal{O}[\eta])^2 &:= \mathcal{D}_{-1, s-1}^1[\eta] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\eta]. \end{aligned}$$

Next, we define

$${}^I\mathcal{O} := {}^I\mathcal{O}[\Psi] + {}^I\mathcal{O}[\mu] + {}^I\mathcal{O}[\text{tr } \chi] + {}^I\mathcal{O}[\widehat{\chi}] + {}^I\mathcal{O}[\eta],$$

where

$$\begin{aligned} ({}^I\mathcal{O}[\Psi])^2 &:= {}^i\mathcal{D}_{-2, s+2}[\Psi] + \sup_{S \subseteq {}^iV} |ru^{\frac{s+1}{2}} \nabla \Psi|_{2,S} + {}^i\mathcal{E}_{0,s}^1[\Psi] + {}^i\mathcal{F}_{0,s}^1[\Psi], \\ ({}^I\mathcal{O}[\mu])^2 &:= {}^i\mathcal{D}_{-2, s+2}[\mu] + {}^i\mathcal{F}_{0,s}^1[\mu], \\ ({}^I\mathcal{O}[\text{tr } \chi])^2 &:= {}^i\mathcal{D}_{-3, s+1}^1[\widetilde{\text{tr } \chi}] + {}^i\mathcal{F}_{-2, s}^2[\widetilde{\text{tr } \chi}] + {}^i\mathcal{D}_{-3, s+1}^{(\infty)} \left[ \text{tr } \chi - \frac{2}{r} \right], \\ ({}^I\mathcal{O}[\widehat{\chi}])^2 &:= {}^i\mathcal{D}_{-3, s+1}^1[\widehat{\chi}] + {}^i\mathcal{F}_{-2, s}^2[\widehat{\chi}], \\ ({}^I\mathcal{O}[\eta])^2 &:= {}^i\mathcal{D}_{-3, s+1}^1[\eta] + {}^i\mathcal{F}_{-2, s}^2[\eta]. \end{aligned}$$

We also define

$$({}^T\mathcal{O})^2 := ({}^T\mathcal{O}[\eta])^2 := \sup_{p \in [-2, -1]} {}^i\mathcal{D}_{p, s-p} [{}^{\nabla}T\eta].$$

Finally, we denote

$$\mathcal{O} := {}^E\mathcal{O} + {}^I\mathcal{O} + {}^T\mathcal{O}.$$

**Remark 3.5.** *By definition, the norms  $\mathcal{O}$  and  $\mathcal{K}$  control  $\delta$ ,  $\epsilon$ ,  $\kappa$ ,  $\nabla \log \phi$ ,  $\eta$  and  $\chi$ . Moreover, we have from (2.14)*

$$\begin{aligned} \underline{\chi}_{AB} &= -\chi_{AB} - 2k_{AB}, & \underline{\xi}_A &= \nabla_A \log \phi + \epsilon_A - \eta_A, \\ 2\underline{\omega} &= -\nabla_N \log \phi + \delta, & 2\underline{\omega} &= \nabla_N \log \phi + \delta, \\ \nabla_A \log a &= \eta_A - \epsilon_A, & \zeta_A &= \epsilon_A, \\ \xi &= 0, & \underline{\eta}_A &= \nabla_A \log \phi - \epsilon_A. \end{aligned}$$

Thus, the norms  $\mathcal{O}$  and  $\mathcal{K}$  control all the connection coefficients.

### 3.2.4 $\mathcal{O}_{(0)}$ , $\mathcal{K}_{(0)}$ and $\mathfrak{R}_{(0)}$ norms (initial data)

We define the curvature flux in the initial layer region  $\mathcal{K}_{(0)}$  introduced in (2.9):

$$(\mathfrak{R}_{(0)})^2 := \sup_{t \in [0, 2]} \int_{\Sigma_t} r^s |\mathfrak{d}^{\leq 1}(\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha})|^2, \quad \mathfrak{d} := \{r\nabla_3, r\nabla_4, r\nabla\}.$$

Next, we define the norms of null connection coefficients in the initial layer  $\mathcal{K}_{(0)}$ :

$$\mathcal{O}_{(0)} := \sum_{\Gamma \in \{\eta, \widehat{\text{tr}}\chi, \widehat{\chi}\}} \mathcal{O}_{(0)}[\Gamma] + \sup_{S \subseteq \mathcal{K}_{(0)}} \left| r^{\frac{s+1}{2}} \left( \text{tr} \chi - \frac{2}{r} \right) \right|_{\infty, S} + \sup_{\Sigma_0} |r + u|,$$

where

$$(\mathcal{O}_{(0)}[\Gamma])^2 := \sup_{t \in [0, 2]} \int_{\Sigma_t} r^{s-2} |\mathfrak{d}^{\leq 2} \Gamma|^2 + \sup_{S \subseteq \mathcal{K}_{(0)}} |r^{\frac{s+1}{2}} \mathfrak{d}^{\leq 2} \Gamma|_{\infty, S}^2.$$

We also define the norms of maximal connection coefficients in  $\mathcal{K}_{(0)}$ :

$$\mathcal{K}_{(0)} := \sum_{k \in \{\epsilon, \delta, \widehat{\kappa}, \nabla \phi\}} \mathcal{K}_{(0)}[k] + \sup_{S \subseteq \mathcal{K}_{(0)}} |r^{\frac{s-1}{2}} (\phi - 1)|_{\infty, S},$$

where

$$(\mathcal{K}_{(0)}[k])^2 := \sup_{t \in [0, 2]} \int_{\Sigma_t} r^{s-2} |\mathfrak{d}^{\leq 2} k|^2.$$

## 3.3 Main theorem

We are ready to state a precise version of our main theorem.

**Theorem 3.6** (Main Theorem, version 2). *Consider an initial data set  $(\Sigma_0, g, k)$ ,  $(s, 2)$ -asymptotically flat in the sense of Definition 1.2 with  $1 < s \leq 2$ . Assume that we have the following control in the initial layer region  $\mathcal{K}_{(0)}$ :*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{K}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0, \quad (3.5)$$

where  $\mathcal{O}_{(0)}$ ,  $\mathcal{K}_{(0)}$  and  $\mathfrak{R}_{(0)}$  are defined in Section 3.2.4 and  $\varepsilon_0 > 0$  is a small enough constant.

Then, the initial layer  $\mathcal{K}_{(0)}$  has a unique development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:

1.  $(\mathcal{M}, \mathbf{g})$  can be foliated by a maximal-null foliation  $(t, u)$ . Moreover, the outgoing cones  $C_u$  are complete for all  $u \in \mathbb{R}$ .
2. The norms  $\mathcal{O}$ ,  $\mathcal{K}$  and  $\mathcal{R}$  defined in Sections 3.2.1–3.2.3 satisfy

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{K} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.6)$$

**Remark 3.7.** The lower bound  $s > 1$  is almost sharp in the following sense. Denoting  $\Gamma$  the connection coefficients,  $R$  the curvature components and  $\mu = 3, 4$ , we have from Propositions 2.17 and 2.18, schematically,

$$\begin{aligned} \nabla_\mu \Gamma &= \nabla \Gamma + R + \Gamma \cdot \Gamma, \\ \nabla_\mu R &= \nabla R + \Gamma \cdot R. \end{aligned} \quad (3.7)$$

Also, we have from (1.6) that the order of decay of  $\Gamma$  is  $-\frac{s+1}{2}$  while that of  $R$  is  $-\frac{s+3}{2}$ . Thus, since the nonlinear terms should have better order of decay than the linear terms, we should have in view of (3.7)

$$\frac{s+1}{2} + \frac{s+1}{2} > \frac{s+3}{2}, \quad \frac{s+1}{2} + \frac{s+3}{2} > \frac{s+5}{2},$$

where both restrictions are equivalent to  $s > 1$ .

The proof of Theorem 3.6 is given in Section 3.5. It hinges on three theorems stated in Section 3.4, concerning estimates for  $\mathcal{O}$ ,  $\mathcal{K}$  and  $\mathcal{R}$  norms.

We choose  $\varepsilon_0$  small enough such that

$$\varepsilon := \varepsilon_0^{\frac{2}{3}}, \quad 0 < \varepsilon \ll 1,$$

where  $\varepsilon$  will be the smallness constant involved in bootstrap assumptions. Here,  $A \ll B$  means that  $CA < B$  where  $C$  is the largest universal constant among all the constants involved in the proof via  $\lesssim$ .

### 3.4 Main intermediate results

The following three theorems are the main intermediate results in the proof of Theorem 3.6.

**Theorem 3.8.** Assume that

$$\mathfrak{R}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{K} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (3.8)$$

Then, we have

$$\mathcal{R} \lesssim \varepsilon_0. \quad (3.9)$$

Theorem 3.8 is proved in Section 5. The proof is based on the  $r^p$ -weighted estimate method introduced by Dafermos and Rodnianski in [13] and applied to the Bianchi equations.

**Theorem 3.9.** Assume that

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{K} \leq \varepsilon, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.10)$$

Then, we have

$$\mathcal{K} \lesssim \varepsilon_0. \quad (3.11)$$

Theorem 3.9 is proved in Section 6. The proof is done by applying elliptic estimates on maximal hypersurfaces  $\Sigma_t$ .

**Theorem 3.10.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{K} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.12)$$

Then, we have

$$\mathcal{O} \lesssim \varepsilon_0. \quad (3.13)$$

Theorem 3.10 is proved in Section 7. The proof is done by integrating transport equations along the outgoing null cones and applying elliptic estimates on 2-spheres of the maximal-null foliation of the spacetime  $\mathcal{M}$ .

### 3.5 Proof of the main theorem

We now use Theorems 3.8–3.10 to prove Theorem 3.6.

**Definition 3.11.** *We denote  $\mathcal{U}$  the set of values  $t_*$  such that the spacetime*

$$\mathcal{K}_{t_*} := V_{t_*} \cup \mathcal{K}_{(0)} = J^-(\Sigma_{t_*}) \cap J^+(\Sigma_0)$$

*associated with the maximal-null foliation  $(t, u)$  defined in Section 2.1 satisfies the following bounds:*

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{K} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (3.14)$$

The assumption (3.5) implies that (3.14) holds if  $t_* \in [0, 2]$ . So, we have  $\mathcal{U} \neq \emptyset$ . Next, we define  $t_*$  to be the supremum of the set  $\mathcal{U}$ . We want to prove  $t_* = +\infty$ . We assume by contradiction that  $t_* < \infty$ . In particular we may assume  $t_* \in \mathcal{U}$ . We have from (3.5)

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{K}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0. \quad (3.15)$$

Applying Theorems 3.8, 3.9 and 3.10 one by one in that order, we obtain

$$\mathcal{R} \lesssim \varepsilon_0, \quad \mathcal{K} \lesssim \varepsilon_0, \quad \mathcal{O} \lesssim \varepsilon_0. \quad (3.16)$$

By local existence, we can extend  $V_{t_*}$  to  $\widehat{V} := V_{t_* + \nu}$  for a  $\nu > 0$  sufficiently small. We denote  $\widehat{\mathcal{O}}$ ,  $\widehat{\mathcal{K}}$  and  $\widehat{\mathcal{R}}$  the norms in the extended region  $\widehat{V}$ . We have

$$\widehat{\mathcal{O}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{K}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}} \lesssim \varepsilon_0,$$

as a consequence of (3.16) and local existence in  $t_* \leq t \leq t_* + \nu$ . We deduce that  $\widehat{V}$  satisfies (3.14), which is a contradiction. Thus, we have  $t_* = +\infty$ , which implies property 1 of Theorem 3.6. Moreover, we have

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{K} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad \text{in } \mathcal{M} = \mathcal{K}_\infty, \quad (3.17)$$

which implies property 2 of Theorem 3.6. This concludes the proof of Theorem 3.6.

## 4 Bootstrap assumptions and first consequences

In the rest of the paper, we always assume the following bootstrap assumptions:

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{K} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (4.1)$$

In this section, we derive first consequences of (4.1) in the region  $V_{t_*}$ . In the sequel, the results of this section will be used frequently without explicitly mentioning them.

### 4.1 Schematic notation $\Gamma_g$ , $\Gamma_b$ and $\Gamma_w$

We introduce the following schematic notations.

**Definition 4.1.** *We divide the Ricci coefficients into three parts:*

$$\begin{aligned}\Gamma_g &:= \{\widetilde{\text{tr}} \chi, \widetilde{\text{tr}} \underline{\chi}, \widehat{\chi}, \omega, \underline{\omega}, \epsilon, \delta, \nabla \log \phi, \zeta, \underline{\eta}\}, \\ \Gamma_b &:= \left\{ \eta, \nabla \log a, \underline{\xi}, \text{tr} \chi - \frac{2}{r} \right\}, \\ \Gamma_w &:= \{\widehat{\chi}, \widehat{\kappa}, \widehat{\theta}\}.\end{aligned}$$

We also denote:

$$\begin{aligned}\Gamma_g^{(1)} &:= (r\nabla)^{\leq 1} \Gamma_g \cup \{r\alpha, r\beta, r\rho, r\sigma, r\psi\}, \\ \Gamma_b^{(1)} &:= (r\nabla)^{\leq 1} \Gamma_b \cup \{r\mu\}, \\ \Gamma_w^{(1)} &:= (r\nabla)^{\leq 1} \Gamma_w \cup \{r\underline{\beta}\},\end{aligned}$$

and

$$\Gamma_i^{(2)} := (r\nabla)^{\leq 1} \Gamma_i^{(1)}, \quad \text{where } i = g, b, w.$$

**Lemma 4.2.** *We have the following estimates:*

$$\mathcal{D}_{s-2}^1[\Gamma_g] \lesssim \varepsilon^2, \quad \mathcal{D}_{-1, s-1}^1[\Gamma_b] \lesssim \varepsilon^2, \quad \mathcal{D}_{-1, s-1}[\Gamma_w] \lesssim \varepsilon^2, \quad \mathcal{D}_{s-3, 1}^1[\Gamma_w] \lesssim \varepsilon^2,$$

and

$${}^i \mathcal{D}_{-3, s+1}^{(\infty)}[\Gamma_g, \Gamma_b, \Gamma_w] \lesssim \varepsilon^2, \quad {}^i \mathcal{D}_{-2, s}^1[\Gamma_g, \Gamma_b, \Gamma_w] \lesssim \varepsilon^2.$$

Moreover, we have

$$\mathcal{F}_{s-2-\delta_B, \delta_B}^2[\Gamma_g, \Gamma_b, \Gamma_w] \lesssim \varepsilon^2, \tag{4.2}$$

where  $\delta_B$  is introduced in (3.4).

*Proof.* It follows directly from (2.14), (4.1) and Definitions 3.2 and 4.1.  $\square$

**Lemma 4.3.** *For two quantities  $X_{(1)}$  and  $X_{(2)}$ , we denote*

$$X_{(1)} \preceq X_{(2)}$$

*if  $X_{(1)}^{(q)}$  decays better than  $X_{(2)}^{(q)}$  for any  $q = 0, 1, 2$ . Then, throughout this paper, we have*

$$\alpha \preceq \beta \preceq (\rho, \sigma) \preceq \underline{\beta} \preceq \underline{\alpha}, \quad \Gamma_g \preceq \Gamma_b \preceq \Gamma_w.$$

*Proof.* It is a direct consequence of (4.1).  $\square$

**Remark 4.4.** *In the sequel, we choose the following conventions:*

- For a quantity  $h$  satisfying the same or even better decay and regularity as  $\Gamma_i$ , for  $i = g, b, w$ , we write

$$h \in \Gamma_i, \quad i = g, b, w.$$

- For a sum of schematic notations, we ignore the terms which have same or even better decay and regularity. For example, we write

$$\Gamma_g + \Gamma_b = \Gamma_b, \quad \Gamma_b + \Gamma_w = \Gamma_w, \quad \left( \frac{\langle u \rangle}{\langle u \rangle + r} \right)^{\frac{s-1}{2}} \Gamma_b = \Gamma_g.$$

- Since  $\Gamma_w$  has better decay and regularity than  $\Gamma_b^{(1)}$ , we write

$$\Gamma_w = \Gamma_b^{(1)}.$$

- For a quantity  $h$  satisfying the same or even better decay and regularity than the curvature component  $R^{(q)}$ , we write

$$h = R^{(q)}.$$

For example, we have

$$r^{-1}\Gamma_w^{(1)} = \underline{\alpha}^{(0)}, \quad \Gamma_b \cdot \rho = \Gamma_b \cdot \underline{\beta}, \quad \Gamma_g \cdot \beta + \Gamma_g \cdot \rho = \Gamma_g \cdot \rho.$$

## 4.2 Commutation identities in schematic form

**Proposition 4.5.** *We have the following simple schematic consequences of the commutator identities*

$$\begin{aligned} [r\mathring{\nabla}, \mathring{\nabla}_4] &= \Gamma_g \cdot r\mathring{\nabla} + \Gamma_g \cdot r\mathring{\nabla}_4 + \Gamma_g^{(1)}, \\ [r\mathring{\nabla}, \mathring{\nabla}_3] &= \Gamma_w \cdot r\mathring{\nabla} + \Gamma_b \cdot r\mathring{\nabla}_4 + \Gamma_b \cdot r\mathring{\nabla}_3 + \Gamma_w^{(1)}, \\ [\mathring{\nabla}_3, \mathring{\nabla}_4] &= \Gamma_b \cdot \mathring{\nabla} + \Gamma_g \cdot \mathring{\nabla}_3 + \Gamma_g \cdot \mathring{\nabla}_4 + r^{-1}\Gamma_g^{(1)}. \end{aligned}$$

*Proof.* It follows directly from Lemma 2.19 and (2.14).  $\square$

**Corollary 4.6.** *We have the following commutator identities:*

$$\begin{aligned} [\mathring{\nabla}_T, \mathring{\nabla}_4] &= \Gamma_b \cdot \mathring{\nabla} + \Gamma_g \cdot \mathring{\nabla}_T + \Gamma_g \cdot \mathring{\nabla}_4 + r^{-1}\Gamma_g^{(1)}, \\ [\mathring{\nabla}_T, \mathring{\nabla}_3] &= \Gamma_b \cdot \mathring{\nabla} + \Gamma_g \cdot \mathring{\nabla}_T + \Gamma_g \cdot \mathring{\nabla}_4 + r^{-1}\Gamma_g^{(1)}, \\ [\mathring{\nabla}_T, \mathring{\nabla}] &= \Gamma_w \cdot \mathring{\nabla} + \Gamma_b \cdot \mathring{\nabla}_T + \Gamma_b \cdot \mathring{\nabla}_4 + r^{-1}\Gamma_w^{(1)}. \end{aligned}$$

*Proof.* It follows directly from Proposition 4.5 and the fact that  $2T = e_4 + e_3$ .  $\square$

## 4.3 Main equations in schematic form

**Proposition 4.7.** *We have the null structure equations:*

$$\begin{aligned} \mathring{\nabla}_4 \eta &= r^{-1}\Gamma_b^{(1)}, & \mathring{\nabla}_3 \widehat{\chi} &= r^{-1}\Gamma_b^{(1)}, \\ \mathring{\nabla}_4 \widehat{\chi} &= r^{-1}\Gamma_g^{(1)}, & \mathring{\nabla}_3 \text{tr } \chi + \frac{1}{2} \text{tr } \underline{\chi} \text{tr } \chi &= r^{-1}\Gamma_b^{(1)}, \\ \mathring{\nabla}_4 \text{tr } \chi + \frac{1}{2} (\text{tr } \chi)^2 &= r^{-1}\Gamma_g, & \mathring{\nabla}_3 \widehat{\underline{\chi}} &= -\underline{\alpha} + r^{-1}\Gamma_b^{(1)}, \\ \mathring{\nabla}_3 \widehat{\underline{\chi}} &= -\underline{\alpha} + r^{-1}\Gamma_b^{(1)}, & \mathring{\nabla}_4 \widehat{\underline{\chi}} &= r^{-1}\Gamma_w^{(1)}, \\ \mathring{\nabla}_3 \text{tr } \underline{\chi} + \frac{1}{2} (\text{tr } \underline{\chi})^2 &= r^{-1}\Gamma_w^{(1)}, & \mathring{\nabla}_4 \text{tr } \underline{\chi} + \frac{1}{2} \text{tr } \chi \text{tr } \underline{\chi} &= r^{-1}\Gamma_g^{(1)}, \\ \mathring{\nabla}_3 \zeta &= r^{-1}\Gamma_w^{(1)}, & \mathring{\nabla}_4 \zeta &= r^{-1}\Gamma_g^{(1)}, \\ \text{div } \widehat{\chi} &= \frac{1}{2} \mathring{\mathcal{U}} - \beta + \Gamma_g \cdot \Gamma_g, & \text{div } \widehat{\underline{\chi}} &= r^{-1}\Gamma_w^{(1)}. \end{aligned}$$

*Proof.* It follows directly from Proposition 2.17.  $\square$

**Corollary 4.8.** *We have the following identities:*

$$\begin{aligned} \mathring{\nabla}_T \widehat{\chi} &= r^{-1}\Gamma_b^{(1)}, & \mathring{\nabla}_T \zeta &= r^{-1}\Gamma_w^{(1)}, & \mathring{\nabla}_T \widehat{\underline{\chi}} &= \underline{\alpha}^{(0)}, \\ \mathring{\nabla}_T \text{tr } \chi &= r^{-1}\Gamma_b^{(1)}, & \mathring{\nabla}_T \text{tr } \underline{\chi} &= r^{-1}\Gamma_w^{(1)}. \end{aligned} \tag{4.3}$$

*Proof.* It follows directly from Proposition 4.7 and the fact that  $2T = e_4 + e_3$ .  $\square$

**Lemma 4.9.** *We have the following Bianchi equations for  $((\check{\rho}, \check{\sigma}), \underline{\beta})$ :*

$$\begin{aligned}\nabla_4 \underline{\beta} + \text{tr } \underline{\chi} \underline{\beta} &= \mathcal{d}_1^*(\check{\rho}, \check{\sigma}) + \Gamma_w \cdot (\underline{\psi}, \underline{\beta}) + r^{-1} \Gamma_g \cdot \Gamma_w^{(1)}, \\ \nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr } \underline{\chi}(\check{\rho}, \check{\sigma}) &= -\mathcal{d}_1 \underline{\beta} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}.\end{aligned}$$

*Proof.* We have from Proposition 4.7

$$\nabla_3 \widehat{\chi} = -\underline{\alpha} + r^{-1} \Gamma_b^{(1)}, \quad \nabla_3 \widehat{\chi} = r^{-1} \Gamma_b^{(1)}.$$

Hence, we have from Proposition 2.18 and Remark 4.4

$$\begin{aligned}\nabla_3 \check{\rho} + \frac{3}{2} \text{tr } \underline{\chi} \check{\rho} &= \nabla_3 \rho + \frac{3}{2} \text{tr } \underline{\chi} \rho - \frac{1}{2} \nabla_3(\widehat{\chi} \cdot \widehat{\chi}) + r^{-1} \Gamma_g \cdot \Gamma_w \\ &= -\mathcal{d}_1 \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot \underline{\alpha} + \Gamma_b \cdot (\underline{\beta}, \underline{\beta}) - \frac{1}{2} \widehat{\chi} \cdot \nabla_3 \widehat{\chi} - \frac{1}{2} \nabla_3 \widehat{\chi} \cdot \widehat{\chi} + r^{-1} \Gamma_g \cdot \Gamma_w \\ &= -\mathcal{d}_1 \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot \underline{\alpha} + r^{-1} \Gamma_b \cdot \Gamma_w^{(1)} - \frac{1}{2} \widehat{\chi} \cdot (-\underline{\alpha} + r^{-1} \Gamma_b^{(1)}) + r^{-1} \Gamma_b^{(1)} \cdot \Gamma_w \\ &= -\mathcal{d}_1 \underline{\beta} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}\end{aligned}$$

as stated. The derivation of the equation of  $\check{\sigma}$  is similar and left to the reader.

Next, we have from (2.18), (3.2), (3.3) and Proposition 2.18

$$\begin{aligned}\nabla_4 \underline{\beta} + \text{tr } \underline{\chi} \underline{\beta} &= \mathcal{d}_1^*(\rho, \sigma) + \Gamma_w \cdot \underline{\beta} + \Gamma_g \cdot (\rho, \sigma, \underline{\beta}) \\ &= \mathcal{d}_1^*(\check{\rho}, \check{\sigma}) + \frac{1}{2} \mathcal{d}_1^*(\widehat{\chi} \cdot \widehat{\chi}, \widehat{\chi} \wedge \widehat{\chi}) + \Gamma_w \cdot \underline{\beta} + \Gamma_g \cdot (\rho, \sigma, \underline{\beta}) \\ &= \mathcal{d}_1^*(\check{\rho}, \check{\sigma}) + \Gamma_w \cdot (\underline{\psi}, \underline{\beta}) + r^{-1} \Gamma_g \cdot \Gamma_w^{(1)}\end{aligned}$$

as stated. This concludes the proof of Proposition 4.9.  $\square$

**Proposition 4.10.** *The Bianchi equations take the following form:*

$$\begin{aligned}\nabla_3 \underline{\alpha} + \frac{1}{2} \text{tr } \underline{\chi} \underline{\alpha} &= -2\mathcal{d}_2^* \underline{\beta} + \Gamma_g \cdot \underline{\rho} + \Gamma_b \cdot \underline{\beta}, \\ \nabla_4 \underline{\beta} + 2 \text{tr } \underline{\chi} \underline{\beta} &= \mathcal{d}_2 \underline{\alpha} + \Gamma_g \cdot \underline{\rho}, \\ \nabla_3 \underline{\beta} + \text{tr } \underline{\chi} \underline{\beta} &= -\mathcal{d}_1^*(\rho, -\sigma) + \Gamma_g \cdot \underline{\beta} + \Gamma_b \cdot \underline{\rho}, \\ \nabla_4(\rho, -\sigma) + \frac{3}{2} \text{tr } \underline{\chi}(\rho, -\sigma) &= \mathcal{d}_1 \underline{\beta} + \Gamma_w \cdot \underline{\beta}, \\ \nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr } \underline{\chi}(\check{\rho}, \check{\sigma}) &= -\mathcal{d}_1 \underline{\beta} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}, \\ \nabla_4 \underline{\beta} + \text{tr } \underline{\chi} \underline{\beta} &= \mathcal{d}_1^*(\check{\rho}, \check{\sigma}) + \Gamma_w \cdot (\underline{\psi}, \underline{\beta}) + r^{-1} \Gamma_g \cdot \Gamma_w^{(1)}, \\ \nabla_3 \underline{\beta} + 2 \text{tr } \underline{\chi} \underline{\beta} &= -\mathcal{d}_2 \underline{\alpha} + \Gamma_b \cdot \underline{\alpha}, \\ \nabla_4 \underline{\alpha} + \frac{1}{2} \text{tr } \underline{\chi} \underline{\alpha} &= 2\mathcal{d}_2^* \underline{\beta} + \Gamma_g \cdot \underline{\alpha} + \Gamma_w \cdot \underline{\rho}.\end{aligned}$$

*Proof.* It follows directly from Proposition 2.18 and Lemma 4.9.  $\square$

**Corollary 4.11.** *We have the following identities:*

$$\begin{aligned}\nabla_T \underline{\alpha} &= r^{-1} \underline{\beta}^{(1)}, & \nabla_T \underline{\beta} &= r^{-1} \underline{\rho}^{(1)} + \Gamma_g \cdot \underline{\beta}, \\ \nabla_T(\rho, \sigma) &= r^{-1} \underline{\beta}^{(1)} + \Gamma_g \cdot \underline{\alpha}, & \nabla_T \underline{\beta} &= r^{-1} \underline{\alpha}^{(1)}.\end{aligned}$$

*Proof.* It follows directly from Proposition 4.10 and the fact that  $2T = e_4 + e_3$ .  $\square$



#### 4.4 Elliptic estimates in 2D–geometry

Throughout Section 4.4, we assume that  $t \geq 0$  and  $u \in \mathbb{R}$  are fixed constants and we denote  $S := S(t, u)$  the leaf of the maximal-null foliation constructed in Section 2.1.2.

**Proposition 4.12** ( $L^p$  estimates for Hodge systems). *The following statements hold for all  $p \in (1, +\infty)$ :*

1. Let  $\phi \in \mathfrak{s}_0$  be a solution to  $\mathring{\Delta}\phi = f$ . Then we have

$$|\mathring{\nabla}^2 \phi|_{p,S} + r^{-1} |\mathring{\nabla} \phi|_{p,S} + r^{-2} |\check{\phi}|_{p,S} \lesssim |f|_{p,S}.$$

2. Let  $\xi \in \mathfrak{s}_1$  be a solution to  $\mathring{\mathcal{L}}_1 \xi = (f, f_*)$ . Then we have

$$|\mathring{\nabla} \xi|_{p,S} + r^{-1} |\xi|_{p,S} \lesssim |(f, f_*)|_{p,S}.$$

3. Let  $U \in \mathfrak{s}_2$  be a solution to  $\mathring{\mathcal{L}}_2 U = f$ . Then we have

$$|\mathring{\nabla} U|_{p,S} + r^{-1} |U|_{p,S} \lesssim |f|_{p,S}.$$

*Proof.* See Corollary 2.3.1.1 in [10]. □

**Proposition 4.13.** *Let  $F$  be a tensor field on  $S$ . Then, we have*

$$|r^{\frac{1}{2}} F|_{\infty,S} \lesssim |F|_{4,S} + |r \mathring{\nabla} F|_{4,S}.$$

*Proof.* See Lemma 4.1.3 in [29]. □

#### 4.5 Elliptic estimates in 3D–geometry

Throughout Section 4.5, we assume that  $t \geq 0$  is a fixed constant and we denote  $\Sigma := \Sigma_t$  the maximal hypersurface constructed in Section 2.1.1.

**Definition 4.14.** *For a tensor field  $h$  on  $\Sigma$ , we define its  $L^p$ –norm as follows:*

$$\|h\|_{p,\Sigma} := \left( \int_{\Sigma} |h|^p \right)^{\frac{1}{p}}. \quad (4.4)$$

**Proposition 4.15** (Hardy). *We have for any tensor field  $F$  on  $\Sigma$*

$$\|r^{-1} F\|_{2,\Sigma} \leq c_H \|\nabla F\|_{2,\Sigma},$$

where

$$c_H = 2 + O(\varepsilon).$$

*Proof.* We have

$$\operatorname{div} N = g^{NN} g(D_N N, N) + g^{AB} g(D_A N, e_B) = \mathring{g}^{AB} \theta_{AB} = \operatorname{tr} \theta = \frac{2}{r} + \Gamma_b, \quad (4.5)$$

which implies from  $r\Gamma_b = O(\varepsilon)$

$$\operatorname{div} \left( \frac{N}{r} \right) = \frac{\operatorname{div} N}{r} - \frac{N(r)}{r^2} = \frac{1}{r^2} + r^{-1} \Gamma_b = \frac{1 + O(\varepsilon)}{r^2}.$$

Thus, we have

$$(1 + O(\varepsilon)) \int_{\Sigma} \frac{|F|^2}{r^2} = \int_{\Sigma} \operatorname{div} \left( \frac{N}{r} \right) |F|^2 = - \int_{\Sigma} \frac{2}{r} F \cdot \nabla_N F \leq 2 \left( \int_{\Sigma} \frac{|F|^2}{r^2} \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla_N F|^2 \right)^{\frac{1}{2}},$$

which implies

$$\|r^{-1} F\|_{2,\Sigma} \leq (2 + O(\varepsilon)) \|\nabla F\|_{2,\Sigma}.$$

This concludes the proof of Proposition 4.15. □

**Proposition 4.16** (Sobolev). *Let  $F$  be a tensor field, tangent to  $S$  at every point. We have the following inequalities:*

$$\begin{aligned} \|F\|_{6,\Sigma} &\lesssim \|\nabla F\|_{2,\Sigma}, \\ \sup_{S \subseteq \Sigma} |F|_{4,S} &\lesssim \|\nabla F\|_{2,\Sigma}, \\ \sup_{S \subseteq \Sigma} |r^{\frac{1}{2}} \langle u \rangle^{\frac{1}{2}} F|_{4,S} &\lesssim \|F\|_{2,\Sigma} + \|r \nabla F\|_{2,\Sigma} + \|\langle u \rangle \nabla_N F\|_{2,\Sigma}. \end{aligned}$$

*Proof.* See Corollaries 3.2.1.1 and 3.2.1.2 in [10]. □

**Proposition 4.17** (Gagliardo-Nirenberg). *Given  $F$  a tensor field on  $\Sigma$ , we have the following inequality:*

$$\|F\|_{\infty,\Sigma} \lesssim \|F\|_{6,\Sigma}^{\frac{1}{2}} \|\nabla F\|_{6,\Sigma}^{\frac{1}{2}}.$$

*Proof.* See Page 308 in [10]. □

**Proposition 4.18.** *Let  $\xi$  be a 1-form on  $\Sigma$  satisfying*

$$\begin{aligned} \operatorname{div} \xi &= D(\xi), \\ \operatorname{curl} \xi &= A(\xi). \end{aligned}$$

*Then, the following integral identity holds:*

$$\int_{\Sigma} |\nabla \xi|^2 = \int_{\Sigma} |A(\xi)|^2 + |D(\xi)|^2 - \operatorname{Ric}^{ij} \xi_i \xi_j.$$

*Proof.* See Lemma 4.4.1 in [10]. □

**Proposition 4.19.** *Let  $U$  be a 2-traceless symmetric tensor on  $\Sigma$  satisfying*

$$\begin{aligned} \operatorname{div} U &= D(U), \\ \operatorname{curl} U &= A(U). \end{aligned}$$

*Then, the following estimates hold:*

$$\begin{aligned} \int_{\Sigma} |\nabla U|^2 &\lesssim \int_{\Sigma} |A(U)|^2 + |D(U)|^2 + |\operatorname{Ric}| |U|^2, \\ \int_{\Sigma} |\nabla^2 U|^2 &\lesssim \int_{\Sigma} |\nabla A(U)|^2 + |\nabla D(U)|^2 + |\operatorname{Ric}| |\nabla U|^2 + |\operatorname{Ric}|^2 |U|^2. \end{aligned}$$

*Proof.* See Propositions 4.4.1 and 4.4.2 in [10]. □

**Proposition 4.20.** *Let  $\phi$  be a scalar function on  $\Sigma$  satisfying*

$$\Delta \phi = f.$$

*Then, the following estimates hold:*

$$\begin{aligned} \int_{\Sigma} |\nabla \phi|^2 + r^2 |\nabla^2 \phi|^2 &\lesssim \int_{\Sigma} r^2 |f|^2, \\ \int_{\Sigma} r^4 |\nabla^3 \phi|^2 + r^4 |\nabla^2 \nabla_N \phi|^2 + r^4 |\nabla \nabla_N^2 \phi|^2 &\lesssim \int_{\Sigma} r^2 |f|^2 + r^4 |\nabla f|^2. \end{aligned}$$

*Proof.* See Propositions 4.2.2 and 4.2.3 in [10]. □

#### 4.6 Comparison of $t$ , $u$ and $r$

**Lemma 4.21.** *We have the following estimates on the symmetry axis:*

$$|t(p) - u(p)| \lesssim \varepsilon t(p), \quad \forall p \in \Phi. \quad (4.6)$$

*In particular, we have*

$$|u_c(t) - t| \lesssim \varepsilon t, \quad |t_c(u) - u| \lesssim \varepsilon t.$$

*Proof.* Recalling that  $\partial_t = \phi T$  and

$$T(u) = 1, \quad \text{on } \Phi,$$

we infer from (4.1)

$$|\partial_t(t - u)| = |1 - \phi T(u)| = |1 - \phi| \lesssim \varepsilon, \quad \text{on } \Phi.$$

Integrating it along  $\Phi$ , we deduce

$$|t(p) - u(p)| \leq |t(O) - u(O)| + \int_0^t |\partial_t(t - u)| dt \lesssim 0 + \int_0^t \varepsilon dt \lesssim \varepsilon t, \quad \forall p \in \Phi.$$

This concludes the proof of Lemma 4.21. □

**Lemma 4.22.** *We have the following estimates:*

$$|r - (t - u)| \lesssim \varepsilon t.$$

*Proof.* We have from Lemma 2.7

$$e_4(r) = \frac{\overline{\phi \operatorname{tr} \chi}}{2\phi} r.$$

Recalling that

$$\phi e_4(t) = \phi T(t) = 1, \quad \phi e_4(u) = 0,$$

we infer from (4.1)

$$|\phi e_4(r - (t - u))| = \left| \frac{\overline{\phi \operatorname{tr} \chi}}{2} r - 1 \right| \lesssim \varepsilon.$$

Hence, we obtain in  ${}^eV$

$$|r(t, u) - (t - u)| \lesssim |r(0, u) + u| + \int_0^t |\phi e_4(r - (t - u))| dt \lesssim \varepsilon t,$$

where we used the fact that  $|r + u| \leq \mathcal{O}_{(0)} \leq \varepsilon_0$  on  $\Sigma_0$ .

Applying Lemma 4.21, we have in  ${}^iV$

$$|r(t, u) - (t - u)| \lesssim |t_c(u) - u| + \int_{t_c(u)}^t |\phi e_4(r - (t - u))| dt \lesssim \varepsilon t.$$

This concludes the proof of Lemma 4.22. □

We define

$${}^iV_{t_*} := \left\{ p \in {}^iV_{t_*} / r \geq \frac{t}{2} \right\}, \quad {}^iV_{t_*} := \left\{ p \in {}^iV_{t_*} / r \leq \frac{t}{2} \right\}. \quad (4.7)$$

**Corollary 4.23.** *We have the following equivalences on the symmetry axis:*

$$u_c(t) \simeq t, \quad t_c(u) \simeq u, \quad \text{on } \Phi.$$

We also have the following comparison results:

$$t + \langle u \rangle \lesssim r \quad \text{in } {}^eV_{t_*}, \quad 1 \leq u, \langle r \rangle \lesssim t \quad \text{in } {}^iV_{t_*}.$$

Moreover, we have

$$1 \leq u \lesssim t \simeq r, \quad \text{in } {}^iV_{t_*}, \quad r \lesssim t \simeq u \quad \text{in } {}^iV_{t_*}.$$

*Proof.* It follows directly from Lemmas 4.21 and 4.22, and from (4.7).  $\square$

#### 4.7 Evolution lemma

We recall the following evolution lemma, which will be used in Sections 6 and 7.

**Lemma 4.24.** *Under the assumption (4.1), the following holds:*

1. Let  $U, F$  be  $k$ -covariant  $S$ -tangent tensor fields satisfying the outgoing evolution equation

$$\nabla_4 U + \lambda_0 \operatorname{tr} \chi U = F.$$

Denoting  $\lambda_1 := 2(\lambda_0 - \frac{1}{p})$ , we have for all  $p \in (1, \infty)$ :

$$\begin{aligned} |r^{\lambda_1} U|_{p,S}(t, u) &\lesssim |r^{\lambda_1} U|_{p,S}(t_c(u), u) + \int_{t_c(u)}^t |r^{\lambda_1} F|_{p,S}(t', u) dt' & \forall u \geq 1, \\ |r^{\lambda_1} U|_{p,S}(t, u) &\lesssim |r^{\lambda_1} U|_{p,S}(2, u) + \int_{\langle u \rangle}^r |r^{\lambda_1} F|_{p,S}(t', u) dr' & \forall u \leq 1. \end{aligned}$$

2. Let  $V, \underline{F}$  be  $k$ -covariant and  $S$ -tangent tensor fields satisfying the normal evolution equation

$$\nabla_N V + \lambda_0 \operatorname{tr} \theta V = \underline{F}.$$

Denoting  $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$ , we have for all  $p \in (1, \infty)$ :

$$\begin{aligned} |r^{\lambda_1} V|_{p,S}(t, u) &\lesssim \lim_{u \rightarrow -\infty} |r^{\lambda_1} V|_{p,S}(t, u) + \int_{-\infty}^u |r^{\lambda_1} \underline{F}|_{p,S}(t, u') du', \\ |r^{\lambda_1} V|_{p,S}(t, u) &\lesssim \lim_{u \rightarrow u_c(t)} |r^{\lambda_1} V|_{p,S}(t, u) + \int_u^{u_c(t)} |r^{\lambda_1} \underline{F}|_{p,S}(t, u') du'. \end{aligned}$$

*Proof.* See Lemma 13.1.1 in [10] or Lemma 7 in [5].  $\square$

## 5 Curvature estimates (Theorem 3.8)

In this section, we prove Theorem 3.8 by the  $r^p$ -weighted estimate method introduced in [13] and applied to Bianchi equations in [22, 33], see also [53].

Throughout Section 5, we denote

$$\begin{aligned} {}^eV &:= V_t(-\infty, u), & {}^iV &:= V_t(u_1, u_2), \\ {}^eV &:= {}^iV \cap \left\{ r \geq \frac{t}{2} \right\}, & {}^iV &:= {}^iV \cap \left\{ r \leq \frac{t}{2} \right\}. \end{aligned}$$

## 5.1 Preliminaries

### 5.1.1 Schematic notation $\mathbb{E}_q$ , $\mathbb{F}_q$ and $\mathbb{O}_\ell$

**Definition 5.1.** For a quantity  $X$ , we denote

$$X \in \mathbb{E}_q(s),$$

if  $X$  satisfies the following estimates:

$$\mathcal{E}_{q,s-q}^1[X] + \mathcal{D}_{q,s-q}[X] + {}^i\mathcal{E}_{0,s}^1[X] + {}^i\mathcal{D}_{-2,s+2}[X] \lesssim \varepsilon^2.$$

Similarly, we denote

$$X \in \mathbb{F}_q(s),$$

if  $X$  satisfies the following estimates:

$$\mathcal{F}_{q,s-q}^1[X] + \mathcal{D}_{q-1,s-q+1}[X] + {}^i\mathcal{F}_{0,s}^1[X] + {}^i\mathcal{D}_{-2,s+2}[X] \lesssim \varepsilon^2.$$

Moreover, we denote

$$X \in \mathbb{O}_\ell(s),$$

if  $X$  satisfies the following estimates:

$$\mathcal{D}_{\ell-2,s-\ell}^1[X] + {}^i\mathcal{D}_{-3,s+1}^1[X] \lesssim \varepsilon^2.$$

We also denote

$$\begin{aligned} X \in \mathbb{E}_q^{(1)}(s) & \quad \text{if} & \quad \mathcal{E}_{q,s-q}[X] + {}^i\mathcal{E}_{0,s}[X] \lesssim \varepsilon^2, \\ X \in \mathbb{F}_q^{(1)}(s) & \quad \text{if} & \quad \mathcal{F}_{q,s-q}[X] + {}^i\mathcal{F}_{0,s}[X] \lesssim \varepsilon^2, \\ X \in \mathbb{O}_\ell^{(1)}(s) & \quad \text{if} & \quad \mathcal{D}_{\ell-2,s-\ell}[X] + {}^i\mathcal{D}_{-3,s+1}[X] \lesssim \varepsilon^2. \end{aligned}$$

We also use the following shorthand notation:

$$\mathbb{X}_q := \mathbb{X}_q(s), \quad \forall \mathbb{X} \in \{\mathbb{E}, \mathbb{F}, \mathbb{O}\}.$$

**Remark 5.2.** Under the assumptions (4.1), we have

$$\begin{aligned} (\alpha, \beta, \check{\rho}, \check{\sigma}) & \in \mathbb{E}_s \cap \mathbb{F}_s, & \quad \underline{\beta} & \in \mathbb{E}_0 \cap \mathbb{F}_s, & \quad \underline{\alpha} & \in \mathbb{F}_0, \\ \alpha_4 & \in \mathbb{E}_s^{(1)}, & \quad \underline{\alpha}_3 & \in \mathbb{F}_{-2}^{(1)}. \end{aligned}$$

We also have<sup>9</sup>

$$(\rho, \sigma, \Psi) \in \mathbb{E}_{s-\delta_B}, \quad r^{-1}\Gamma_i^{(1)} \in \mathbb{F}_{s-\delta_B}, \quad i = g, b, w.$$

Moreover, we have from Lemma 4.2

$$\Gamma_g \in \mathbb{O}_s, \quad \Gamma_b \in \mathbb{O}_1, \quad \Gamma_w \in \mathbb{O}_{s-1}.$$

We also have from Corollary 4.11

$$\begin{aligned} \nabla_T \alpha & \in \mathbb{E}_{s+2}^{(1)}(s+2), & \quad \nabla_T \beta & \in \mathbb{E}_{2s-\delta_B}^{(1)}(s+2), & \quad \nabla_T(\rho, \sigma) & \in \mathbb{F}_{s+1}^{(1)}(s+2), \\ \nabla_T \underline{\beta} & \in \mathbb{F}_2^{(1)}(s+2), & \quad \nabla_T \underline{\alpha} & \in \mathbb{F}_0^{(1)}(s+2). \end{aligned}$$

<sup>9</sup>Recall that  $\delta_B$  is defined in (3.4).

## 5.1.2 Bianchi equations in schematic form

**Proposition 5.3.** *We have the following Bianchi equations:*

$$\begin{aligned}
\mathring{\nabla}_3 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha &= -2\mathring{d}_2^* \beta + \mathbb{O}_1 \cdot \mathbb{E}_s, \\
\mathring{\nabla}_4 \beta + 2 \operatorname{tr} \chi \beta &= \mathring{d}_2 \alpha + \mathbb{O}_s \cdot \mathbb{E}_s, \\
\mathring{\nabla}_3 \beta + \operatorname{tr} \chi \beta &= -\mathring{d}_1^*(\rho, -\sigma) + \mathbb{O}_s \cdot \mathbb{F}_s + \mathbb{O}_1 \cdot \mathbb{E}_{s-\delta_B}, \\
\mathring{\nabla}_4(\rho, -\sigma) + \frac{3}{2} \operatorname{tr} \chi(\rho, -\sigma) &= \mathring{d}_1 \beta + \mathbb{O}_{s-1} \cdot \mathbb{E}_s, \\
\mathring{\nabla}_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \operatorname{tr} \chi(\check{\rho}, \check{\sigma}) &= -\mathring{d}_1 \beta + \mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B}, \\
\mathring{\nabla}_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= \mathring{d}_1^*(\check{\rho}, \check{\sigma}) + \mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B} + \mathbb{O}_s \cdot \mathbb{F}_{s-\delta_B}, \\
\mathring{\nabla}_3 \underline{\beta} + 2 \operatorname{tr} \chi \underline{\beta} &= -\mathring{d}_2 \alpha + \mathbb{O}_1 \cdot \mathbb{F}_0, \\
\mathring{\nabla}_4 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha &= 2\mathring{d}_2^* \underline{\beta} + \mathbb{O}_s \cdot \mathbb{F}_0.
\end{aligned} \tag{5.1}$$

*Proof.* It follows directly from Remark 5.2, Lemma 4.9 and Proposition 4.10.  $\square$

Commuting (5.1) with  $\mathring{\phi}$ , applying Propositions 2.15 and 4.5 and ignoring the terms that decay better, we obtain the following corollary.<sup>10</sup>

**Corollary 5.4.** *We have the following equations:*

$$\begin{aligned}
\mathring{\nabla}_3(\mathring{\phi}\alpha) + \frac{1}{2} \operatorname{tr} \chi(\mathring{\phi}\alpha) &= -\mathring{d}_1^*(\mathring{\phi}\beta) + r^{-1}O(\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}, \\
\mathring{\nabla}_4(\mathring{\phi}\beta) + 2 \operatorname{tr} \chi(\mathring{\phi}\beta) &= \mathring{d}_1(\mathring{\phi}\alpha) + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}, \\
\mathring{\nabla}_3(\mathring{\phi}\beta) + \operatorname{tr} \chi(\mathring{\phi}\beta) &= -\mathring{d}_1(\mathring{\phi}\rho, -\mathring{\phi}\sigma) + (\mathbb{O}_s \cdot \mathbb{F}_s)^{(1)} + (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B})^{(1)}, \\
\mathring{\nabla}_4(\mathring{\phi}\rho, -\mathring{\phi}\sigma) + \frac{3}{2} \operatorname{tr} \chi(\mathring{\phi}\rho, -\mathring{\phi}\sigma) &= \mathring{d}_1^*(\mathring{\phi}\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}, \\
\mathring{\nabla}_3(\mathring{\phi}\check{\rho}, \mathring{\phi}\check{\sigma}) + \frac{3}{2} \operatorname{tr} \chi(\mathring{\phi}\check{\rho}, \mathring{\phi}\check{\sigma}) &= -\mathring{d}_1^*(\mathring{\phi}\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B})^{(1)}, \\
\mathring{\nabla}_4(\mathring{\phi}\beta) + \operatorname{tr} \chi(\mathring{\phi}\beta) &= \mathring{d}_1(\mathring{\phi}\check{\rho}, \mathring{\phi}\check{\sigma}) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B})^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_{s-\delta_B})^{(1)}, \\
\mathring{\nabla}_3(\mathring{\phi}\beta) + 2 \operatorname{tr} \chi(\mathring{\phi}\beta) &= -\mathring{d}_1(\mathring{\phi}\alpha) + (\mathbb{O}_1 \cdot \mathbb{F}_0)^{(1)}, \\
\mathring{\nabla}_4(\mathring{\phi}\alpha) + \frac{1}{2} \operatorname{tr} \chi(\mathring{\phi}\alpha) &= \mathring{d}_1^*(\mathring{\phi}\beta) + r^{-1}O(\beta) + (\mathbb{O}_s \cdot \mathbb{F}_0)^{(1)}.
\end{aligned}$$

**Lemma 5.5.** *We define the following quantities:*

$$\begin{aligned}
\alpha_4 &:= \frac{1}{r^4} \mathring{\nabla}_4(r^5 \alpha) \in \mathfrak{s}_2, & \underline{\alpha} &:= r \mathring{d}_2 \alpha \in \mathfrak{s}_1, \\
\underline{\alpha}_3 &:= \frac{1}{r^4} \mathring{\nabla}_3(r^5 \underline{\alpha}) \in \mathfrak{s}_2, & \underline{\alpha} &:= r \mathring{d}_2 \underline{\alpha} \in \mathfrak{s}_1.
\end{aligned} \tag{5.2}$$

Then, we have the following equations:

$$\begin{aligned}
\mathring{\nabla}_3 \alpha_4 &= -2\mathring{d}_2^* \underline{\alpha} + \frac{4\underline{\alpha}}{r} + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}, \\
\mathring{\nabla}_4 \underline{\alpha} + \frac{5}{2} \operatorname{tr} \chi \underline{\alpha} &= \mathring{d}_2 \alpha_4 + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)},
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\mathring{\nabla}_4 \dot{\underline{\alpha}} &= -2\mathring{d}_2^* \underline{\alpha} + \frac{4\underline{\alpha}}{r} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}, \\
\mathring{\nabla}_3 \dot{\underline{\alpha}} + \frac{5}{2} \operatorname{tr} \chi \dot{\underline{\alpha}} &= \mathring{d}_2 \dot{\underline{\alpha}} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}.
\end{aligned} \tag{5.4}$$

<sup>10</sup>The linear terms  $r^{-1}O(\beta)$  and  $r^{-1}O(\underline{\beta})$  come from the fact that  $[\mathring{\phi}, 2\mathring{d}_2^*] = O(r^{-1})$ , see Proposition 2.15.

*Proof.* See Lemma 2.26 in [53]. □

### 5.1.3 General Bianchi pairs

The following lemma provides the general structure of Bianchi pairs. It will be used repeatedly in Sections 5.2–5.4.

**Lemma 5.6.** *Let  $k = 1, 2$  and  $a_{(1)}, a_{(2)}$  real numbers. Then, we have the following properties.*

1. *If  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_k$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_{k-1}$  satisfying*

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= -k \underline{d}_k^*(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \underline{\chi} \psi_{(2)} &= \underline{d}_k(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (5.5)$$

*Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$*

$$\begin{aligned} & \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + k \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(2)}|^2 \\ & = 2kr^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} - 2r^p \underline{\omega} |\psi_{(1)}|^2 \\ & - 2kr^p \underline{\omega} |\psi_{(2)}|^2 + pr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + kpr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(2)}|^2. \end{aligned} \quad (5.6)$$

2. *If  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_{k-1}$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_k$  satisfying*

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= \underline{d}_k(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \underline{\chi} \psi_{(2)} &= -k \underline{d}_k^*(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (5.7)$$

*Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$*

$$\begin{aligned} & k \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + k \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(2)}|^2 \\ & = 2r^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2kr^p \psi_{(1)} \cdot h_{(1)} + 2r^p \psi_{(2)} \cdot h_{(2)} - 2kr^p \underline{\omega} |\psi_{(1)}|^2 \\ & - 2r^p \underline{\omega} |\psi_{(2)}|^2 + kpr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + pr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(2)}|^2. \end{aligned} \quad (5.8)$$

*Proof.* See Lemma 4.2 in [53]. □

**Remark 5.7.** *Note that the Bianchi equations in (5.1), (5.3) and (5.4) can be written as systems of equations of the type (5.5) and (5.7). In particular*

- *the Bianchi pair  $(\alpha_4, \phi)$  satisfies (5.5) with  $k = 2$ ,  $a_{(1)} = 0$ ,  $a_{(2)} = \frac{5}{2}$ ,*
- *the Bianchi pair  $(\alpha, \beta)$  satisfies (5.5) with  $k = 2$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,*
- *the Bianchi pair  $(\beta, (\rho, -\sigma))$  satisfies (5.5) with  $k = 1$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,*
- *the Bianchi pair  $((\check{\rho}, \check{\sigma}), \beta)$  satisfies (5.7) with  $k = 1$ ,  $a_{(1)} = \frac{3}{2}$ ,  $a_{(2)} = 1$ ,*
- *the Bianchi pair  $(\underline{\beta}, \underline{\alpha})$  satisfies (5.7) with  $k = 2$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ ,*
- *the Bianchi pair  $(\underline{\phi}, \underline{\alpha}_3)$  satisfies (5.7) with  $k = 2$ ,  $a_{(1)} = \frac{5}{2}$ ,  $a_{(2)} = 0$ .*

### 5.1.4 Estimates for nonlinear error terms

The following theorem provides a unified treatment of all the nonlinear error terms in curvature estimates.

**Theorem 5.8.** *We define*<sup>11</sup>

$$\mathcal{N}_p^q [\psi_{(1)}, \psi_{(2)}] (V) := \int_V r^p \left| \psi_{(1)}^{(q)} \cdot \psi_{(2)}^{(q)} \right|. \quad (5.9)$$

Let  $p_1, p_2, p, \ell \leq s$  and let  $s > 1$ . Then, we have the following properties:

1. In the case  $2p < p_1 + p_2 + \ell + 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}] ({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.$$

2. In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}] ({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}},$$

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}] ({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.$$

3. In the case  $2p < p_1 + p_2 + \ell - 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}] ({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.$$

If in addition we assume that  $p_1, p_2, p \geq 0$ . Then, we have the following properties:

1. In the case  $2p < p_1 + p_2 + \ell + 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}] ({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}.$$

2. In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}] ({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}},$$

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}] ({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}.$$

3. In the case  $2p < p_1 + p_2 + \ell - 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}] ({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}.$$

*Proof.* See Appendix A. □

<sup>11</sup>In order to simplify the notations,  $V$  can be ignored if it is clear in the context.



### 5.1.5 Strategy of the proof of Theorem 3.8

The proof of Theorem 3.8 proceeds in 4 steps which we summarize below for convenience:

- In Section 5.2, we apply  $r^p$ -weighted estimates with  $-2 \leq p \leq s$  to the Bianchi equations to control the curvature components in the exterior region  ${}^eV$ .
- In Section 5.3, we apply  $r^p$ -weighted estimates with  $0 \leq p \leq s^{12}$  to the Bianchi equations to control the curvature components in the interior region  ${}^iV = {}^eV \cup {}^iV$ . In  ${}^eV$ , the desired decay of curvature components follows directly from the fact that  $\langle u \rangle \lesssim r$  holds in  ${}^eV$ . On the other hand, in  ${}^iV$ , by the mean value method introduced in [13], we can only deduce  $r^{-\frac{3}{2}}u^{-\frac{s}{2}}$ -decay for curvature components. The goal of the last two steps is then to recover the expected  $r^{-\frac{1}{2}}u^{-\frac{s+2}{2}}$ -decay for curvature components in  ${}^iV$ .
- In Section 5.4, we first commute the Bianchi equations with  $\nabla_T$ . We then apply  $r^p$ -weighted estimates with  $0 \leq p \leq s+2$  to the commuted Bianchi equations to control  $\nabla_T \mathbf{R}$ . By the mean value method, we deduce  $r^{-\frac{3}{2}}u^{-\frac{s+2}{2}}$ -decay for  $\nabla_T \mathbf{R}$  in  ${}^iV$ .
- Finally, in Section 5.5, we apply the elliptic estimates of Proposition 4.19 to the Maxwell type systems in Proposition 2.23 to deduce the desired  $r^{-\frac{1}{2}}u^{-\frac{s+2}{2}}$ -decay for  $\mathbf{R}$  from the estimate of  $\nabla_T \mathbf{R}$  in  ${}^iV$ .

## 5.2 Exterior region estimates

Throughout Section 5.2, we always assume that  $u \leq 1$  and we denote

$$V := V_t(u) = V_t(-\infty, u).$$

**Proposition 5.9.** *For  $\psi_{(1)}$ ,  $\psi_{(2)}$  and  $h_{(1)}$ ,  $h_{(2)}$  satisfying (5.5) or (5.7), we have the following properties:*

- In the case  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p > 0$ , we have

$$\begin{aligned} & BEF_p[\psi_{(1)}](u) + BF_p[\psi_{(2)}](u) \\ & \lesssim F_p[\psi_{(1)}, \psi_{(2)}](\Sigma_2 \cap V) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}]. \end{aligned} \quad (5.10)$$

- In the case  $2 + p - 4a_{(1)} \leq 0$  and  $4a_{(2)} - 2 - p > 0$ , we have

$$\begin{aligned} & EF_p[\psi_{(1)}](u) + BF_p[\psi_{(2)}](u) \\ & \lesssim F_p[\psi_{(1)}, \psi_{(2)}](\Sigma_2 \cap V) + B_p[\psi_{(1)}](u) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}]. \end{aligned} \quad (5.11)$$

- In the case  $2 + p - 4a_{(1)} \leq 0$  and  $4a_{(2)} - 2 - p \geq 0$ , we have

$$\begin{aligned} & EF_p[\psi_{(1)}](u) + F_p[\psi_{(2)}](u) \\ & \lesssim F_p[\psi_{(1)}, \psi_{(2)}](\Sigma_2 \cap V) + B_p[\psi_{(1)}](u) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}]. \end{aligned} \quad (5.12)$$

*Proof.* We have from Lemma 2.7

$$e_4(r) - \frac{r}{2} \operatorname{tr} \chi = \frac{\overline{\phi \operatorname{tr} \chi}}{2\phi} r - \frac{r}{2} \operatorname{tr} \chi = \frac{r}{2\phi} (\overline{\phi \operatorname{tr} \chi} - \phi \operatorname{tr} \chi) \in r\Gamma_g,$$

and similarly

$$e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} = e_4(r) - 2N(r) - \frac{r}{2} (\operatorname{tr} \chi - 2 \operatorname{tr} \theta) = r\Gamma_g - 2 \left( N(r) - \frac{r}{2} \operatorname{tr} \theta \right) \in r\Gamma_b.$$

<sup>12</sup>Notice that the restriction  $p \geq 0$  is necessary according to the fact that  $r^p$  is unbounded near  $\mathcal{H}$  for  $p < 0$ .

Applying Lemma 4.2, we infer

$$\begin{aligned}
& \int_V r^p (|\Gamma_b|\psi_{(1)}|^2 + |\Gamma_g|\psi_{(2)}|^2) \\
& \lesssim \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} r^p |\psi_{(1)}|^2 + \int_V \frac{\varepsilon}{t^{\frac{s+1}{2}}} r^p |\psi_{(2)}|^2 \\
& \lesssim \varepsilon \int_{-\infty}^u \left( \langle u \rangle^{-\frac{s+1}{2}} \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) du + \varepsilon \int_2^t \left( t^{-\frac{s+1}{2}} \int_{\Sigma_t^V} r^p |\psi_{(2)}|^2 \right) dt \\
& \lesssim \varepsilon \sup_u \left( \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) \int_{-\infty}^u \langle u \rangle^{-\frac{s+1}{2}} du + \varepsilon \sup_t \left( \int_{\Sigma_t^V} r^p |\psi_{(2)}|^2 \right) \int_2^t t^{-\frac{s+1}{2}} dt \\
& \lesssim \varepsilon \sup_u \left( \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) + \varepsilon \sup_t \left( \int_{\Sigma_t^V} r^p |\psi_{(2)}|^2 \right).
\end{aligned} \tag{5.13}$$

Integrating (5.6) or (5.8) in  $V^{13}$ , applying Stokes formula and reminding that  $\text{tr } \chi \simeq \frac{2}{r}$ ,  $\text{tr } \underline{\chi} \simeq -\frac{2}{r}$ ,  $\omega, \underline{\omega} \in \Gamma_g$ , we obtain

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\Sigma_t^V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) + \int_{\mathcal{I}_* \cap V} r^p |\psi_{(2)}|^2 \\
& + \int_V (2+p-4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)}-2-p) r^{p-1} |\psi_{(2)}|^2 \\
& \lesssim \int_{\Sigma_2 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}| + r^p |\Gamma_b| |\psi_{(1)}|^2 + r^p |\Gamma_g| |\psi_{(2)}|^2,
\end{aligned}$$

where  $\mathcal{I}_*$  denotes the future null infinity. Taking the supremum of  $u$  and  $t$  and applying (5.13), we obtain for  $\varepsilon$  small enough

$$\begin{aligned}
& \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 + \sup_t \int_{\Sigma_t^V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \\
& + \int_V (2+p-4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)}-2-p) r^{p-1} |\psi_{(2)}|^2 \\
& \lesssim \int_{\Sigma_2 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|,
\end{aligned} \tag{5.14}$$

which implies

$$\begin{aligned}
& EF_p[\psi_{(1)}](u) + F_p[\psi_{(2)}](u) + (2+p-4a_{(1)})B_p[\psi_{(1)}](u) + (4a_{(2)}-2-p)B_p[\psi_{(2)}](u) \\
& \lesssim F_p[\psi_{(1)}, \psi_{(2)}](\Sigma_2 \cap V) + B_p[\psi_{(2)}](u) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}].
\end{aligned}$$

Hence, (5.10)–(5.12) hold in the corresponding range of parameters. This concludes the proof of Proposition 5.9.  $\square$

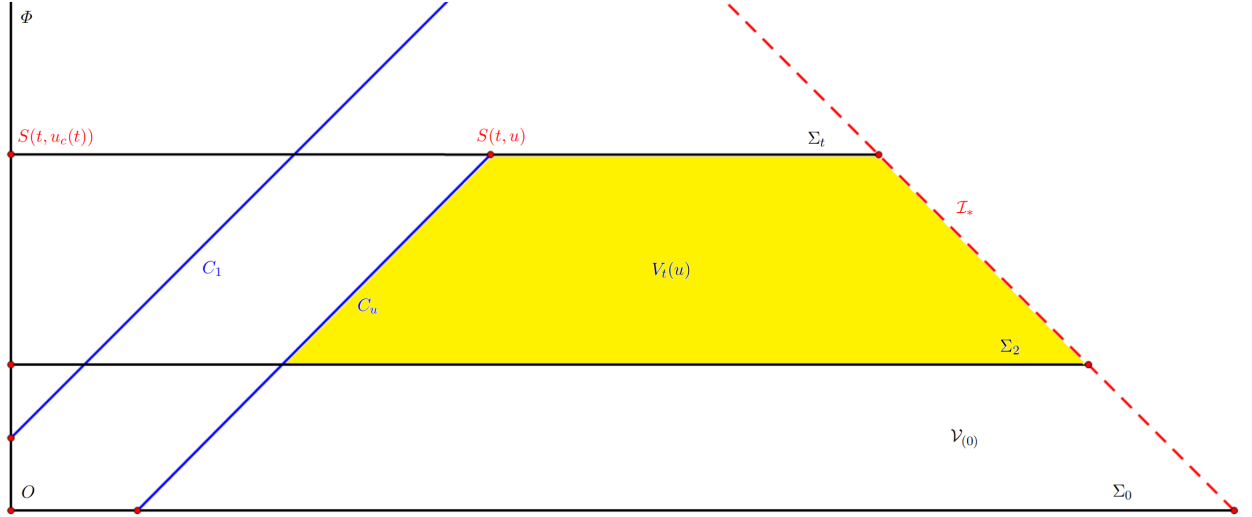
The following lemma allows us to obtain  $\langle u \rangle$ -decay of curvature along  $\Sigma_2 \cap V$ .

**Lemma 5.10.** *We have the following estimate for  $p \leq s$  and  $u \leq 1$ :*

$$\int_{\Sigma_2 \cap V} r^p (|\alpha^{(1)}|^2 + |\beta^{(1)}|^2 + |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 + |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2) \lesssim \frac{\mathfrak{R}_0^2}{\langle u \rangle^{s-p}}.$$

*Proof.* It follows directly from the fact that  $\langle u \rangle \lesssim r$  on  $\Sigma_2 \cap V$ .  $\square$

<sup>13</sup>See Figure 3.2 for a geometric description of  $V$ .

Figure 3.2: Domain of integration  $V_t(u)$ 

### 5.2.1 Estimates for the Bianchi pair $(\alpha, \beta)$

**Proposition 5.11.** *We have the following estimate:*

$$BEF_s^1[\alpha](u) + BF_s^1[\beta](u) \lesssim \varepsilon_0^2. \quad (5.15)$$

*Proof.* We recall from Proposition 5.3

$$\begin{aligned} \nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha &= -2 \not{d}_2^* \beta + \mathbb{O}_1 \cdot \mathbb{E}_s, \\ \nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= \not{d}_2 \alpha + \mathbb{O}_s \cdot \mathbb{E}_s. \end{aligned}$$

Applying (5.10) with  $\psi_{(1)} = \alpha$ ,  $\psi_{(2)} = \beta$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,  $h_{(1)} = \mathbb{O}_1 \cdot \mathbb{E}_s$ ,  $h_{(2)} = \mathbb{O}_s \cdot \mathbb{E}_s$ ,  $p = s$  and noticing that

$$2 + p - 4a_{(1)} = s > 0, \quad 4a_{(2)} - 2 - p = 6 - s > 0,$$

we obtain from (5.9) and Lemma 5.10

$$BEF_s[\alpha](u) + BF_s[\beta](u) \lesssim \varepsilon_0^2 + \mathcal{N}_s^0[\mathbb{E}_s, \mathbb{O}_1 \cdot \mathbb{E}_s] + \mathcal{N}_s^0[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s].$$

Applying Theorem 5.8, we obtain

$$\mathcal{N}_s^0[\mathbb{E}_s, \mathbb{O}_1 \cdot \mathbb{E}_s] + \mathcal{N}_s^0[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s] \lesssim \varepsilon_0^2.$$

Hence, we have

$$BEF_s[\alpha](u) + BF_s[\beta](u) \lesssim \varepsilon_0^2. \quad (5.16)$$

Next, we recall from Corollary 5.4

$$\begin{aligned} \nabla_3(\not{\vartheta}\alpha) + \frac{1}{2} \operatorname{tr} \chi(\not{\vartheta}\alpha) &= -\not{d}_1^*(\not{\vartheta}\beta) + r^{-1}O(\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}, \\ \nabla_4(\not{\vartheta}\beta) + 2 \operatorname{tr} \chi(\not{\vartheta}\beta) &= \not{d}_1(\not{\vartheta}\alpha) + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}. \end{aligned}$$

Applying (5.10) with  $\psi_{(1)} = \not{\vartheta}\alpha$ ,  $\psi_{(2)} = \not{\vartheta}\beta$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,  $h_{(1)} = r^{-1}O(\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}$  and  $p = s$ , we infer

$$BEF_s^1[\alpha](u) + BF_s^1[\beta](u) \lesssim \varepsilon_0^2 + \mathcal{N}_{s-1}^0[\alpha^{(1)}, \beta] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s].$$

We have from Cauchy-Schwarz inequality that for all  $\delta_0 > 0$

$$\mathcal{N}_{s-1}^0[\alpha^{(1)}, \beta] = \int_V r^{s-1} |\alpha^{(1)}| |\beta| \leq \delta_0 B_s^1[\alpha](u) + (4\delta_0)^{-1} B_s[\beta](u).$$

Moreover, applying Theorem 5.8, we have

$$\mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s] \lesssim \varepsilon_0^2.$$

Hence, combining with (5.16), we have for  $\delta_0$  small enough

$$BEF_s^1[\alpha](u) + BF_s^1[\beta](u) \lesssim \varepsilon_0^2. \quad (5.17)$$

This concludes the proof of Proposition 5.11.  $\square$

### 5.2.2 Estimates for the Bianchi pair $(\beta, (\rho, -\sigma))$

**Proposition 5.12.** *We have the following estimate:*

$$BEF_s^1[\beta](u) + BF_s^1[\check{\rho}, \check{\sigma}](u) \lesssim \varepsilon_0^2. \quad (5.18)$$

*Proof.* We recall from Corollary 5.4

$$\begin{aligned} \nabla_3(\check{\rho}\beta) + \text{tr} \underline{\chi}(\check{\rho}\beta) &= -\mathfrak{d}_1(\check{\rho}\rho, -\check{\rho}\sigma) + (\mathbb{O}_s \cdot \mathbb{F}_s)^{(1)} + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}, \\ \nabla_4(\check{\rho}\rho, -\check{\rho}\sigma) + \frac{3}{2} \text{tr} \chi(\check{\rho}\rho, -\check{\rho}\sigma) &= \mathfrak{d}_1^*(\check{\rho}\beta) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}. \end{aligned}$$

Applying (5.11) with  $\psi_{(1)} = \check{\rho}\beta$ ,  $\psi_{(2)} = (\check{\rho}\rho, -\check{\rho}\sigma)$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = (\mathbb{O}_s \cdot \mathbb{F}_s)^{(1)} + (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}$  and  $p = s$ , we obtain

$$\begin{aligned} &BEF_s^1[\beta](u) + BF_s^1[\rho, \sigma](u) \\ &\lesssim F_s^1[\beta, \rho, \sigma](\Sigma_2 \cap V) + B_s^1[\beta](u) + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{F}_s] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] + \mathcal{N}_s^1[\mathbb{E}_{s-\delta_B}, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] \\ &\lesssim \varepsilon_0^2 + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{F}_s] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] + \mathcal{N}_s^1[\mathbb{E}_{s-\delta_B}, \mathbb{O}_{s-1} \cdot \mathbb{E}_s], \end{aligned}$$

where we used (5.17) and Lemma 5.10 in the last step. Next, applying Theorem 5.8 in corresponding cases, we infer

$$\mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{F}_s] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] + \mathcal{N}_s^1[\mathbb{E}_{s-\delta_B}, \mathbb{O}_{s-1} \cdot \mathbb{E}_s] \lesssim \varepsilon_0^2.$$

Hence, we deduce

$$BEF_s^1[\beta](u) + BF_s^1[\rho, \sigma](u) \lesssim \varepsilon_0^2. \quad (5.19)$$

Recalling (3.3), we infer

$$\begin{aligned} F_s^1[\check{\rho}, \check{\sigma}](u) &\lesssim F_s^1[\rho, \sigma](u) + \int_{-\infty}^u r^{s+2} |r^{-\frac{2}{4}} \Gamma_g|_{4,S}^2 |r^{-\frac{2}{4}} \Gamma_w|_{4,S}^2 \\ &\lesssim \varepsilon_0^2 + \int_{-\infty}^u r^{s+2} \frac{\varepsilon^2}{r^{s+1}} \frac{\varepsilon^2}{r^2 \langle u \rangle^{s-1}} du \\ &\lesssim \varepsilon_0^2, \end{aligned}$$

and

$$\begin{aligned} B_s^1[\check{\rho}, \check{\sigma}](u) &\lesssim B_s^1[\rho, \sigma](u) + \int_2^t dt \int_{-\infty}^u r^{s+1} |r^{-\frac{2}{4}} \Gamma_g|_{4,S}^2 |r^{-\frac{2}{4}} \Gamma_w|_{4,S}^2 du \\ &\lesssim \varepsilon_0^2 + \int_2^t dt \int_{-\infty}^u r^{s+1} \frac{\varepsilon^2}{r^{s+1}} \frac{\varepsilon^2}{r^2 \langle u \rangle^{s-1}} du \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

This concludes the proof of Proposition 5.12.  $\square$

### 5.2.3 Estimates for the Bianchi pair $((\check{\rho}, \check{\sigma}), \underline{\beta})$

**Proposition 5.13.** *We have the following estimate:*

$$BEF_s^1[\check{\rho}, \check{\sigma}](u) + F_s^1[\underline{\beta}](u) \lesssim \varepsilon_0^2. \quad (5.20)$$

*Proof.* We recall from Corollary 5.4

$$\begin{aligned} \nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \operatorname{tr} \underline{\chi}(\check{\rho}, \check{\sigma}) &= -\mathcal{d}_1^*(\check{\rho}, \check{\sigma}) + (\mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B})^{(1)}, \\ \nabla_4(\check{\rho}, \check{\sigma}) + \operatorname{tr} \chi(\check{\rho}, \check{\sigma}) &= \mathcal{d}_1(\check{\rho}, \check{\sigma}) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B})^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_{s-\delta_B})^{(1)}. \end{aligned}$$

Applying (5.12) with  $\psi_{(1)} = (\check{\rho}, \check{\sigma})$ ,  $\psi_{(2)} = \check{\rho}, \underline{\beta}$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_{s-1} \cdot \mathbb{E}_s)^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_s)^{(1)}$  and  $p = s$ , we obtain from (5.18) and Lemma 5.10

$$\begin{aligned} BEF_s^1[\check{\rho}, \check{\sigma}](u) + F_s^1[\underline{\beta}](u) \\ \lesssim \varepsilon_0^2 + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B}] + \mathcal{N}_s^1[\mathbb{F}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B}] + \mathcal{N}_s^1[\mathbb{F}_s, \mathbb{O}_s \cdot \mathbb{F}_{s-\delta_B}]. \end{aligned}$$

Applying Theorem 5.8 in corresponding cases, we deduce

$$\mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B}] + \mathcal{N}_s^1[\mathbb{F}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B}] + \mathcal{N}_s^1[\mathbb{F}_s, \mathbb{O}_s \cdot \mathbb{F}_{s-\delta_B}] \lesssim \varepsilon_0^2. \quad (5.21)$$

Thus, we obtain

$$BEF_s^1[\check{\rho}, \check{\sigma}](u) + F_s^1[\underline{\beta}](u) \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 5.13.  $\square$

### 5.2.4 Estimates for the Bianchi pair $(\underline{\beta}, \underline{\alpha})$

**Proposition 5.14.** *We have the following estimate:*

$$BEF_0^1[\underline{\beta}](u) + F_0^1[\underline{\alpha}](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s}. \quad (5.22)$$

*Proof.* We have from Corollary 5.4

$$\begin{aligned} \nabla_3(\underline{\beta}) + 2 \operatorname{tr} \underline{\chi}(\underline{\beta}) &= -\mathcal{d}_1(\underline{\beta}, \underline{\alpha}) + (\mathbb{O}_1 \cdot \mathbb{F}_0)^{(1)}, \\ \nabla_4(\underline{\beta}, \underline{\alpha}) + \frac{1}{2} \operatorname{tr} \chi(\underline{\beta}, \underline{\alpha}) &= \mathcal{d}_1^*(\underline{\beta}, \underline{\alpha}) + r^{-1}O(\underline{\beta}) + (\mathbb{O}_s \cdot \mathbb{F}_0)^{(1)}. \end{aligned}$$

Applying (5.12) with  $\psi_{(1)} = \underline{\beta}, \underline{\alpha}$ ,  $\psi_{(2)} = \underline{\beta}, \underline{\alpha}$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ ,  $h_{(1)} = (\mathbb{O}_1 \cdot \mathbb{F}_0)^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_s \cdot \mathbb{F}_0)^{(1)}$  and  $p = 0$ , we obtain from Lemma 5.10

$$\begin{aligned} EF_0^1[\underline{\beta}](u) + F_0^1[\underline{\alpha}](u) \\ \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s} + B_0^1[\underline{\beta}](u) + \mathcal{N}_{-1}^0[\underline{\alpha}^{(1)}, \underline{\beta}] + \mathcal{N}_0^1[\mathbb{F}_s, \mathbb{O}_1 \cdot \mathbb{F}_0] + \mathcal{N}_0^1[\mathbb{F}_0, \mathbb{O}_s \cdot \mathbb{F}_0]. \end{aligned} \quad (5.23)$$

We first have

$$\begin{aligned} \mathcal{N}_{-1}^0[\underline{\alpha}^{(1)}, \underline{\beta}] &= \int_2^t \frac{dt}{(t+|u|)^{\frac{s+2}{2}}} \left( \int_{\Sigma_t^Y} |\underline{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\Sigma_t^Y} r^s |\underline{\beta}|^2 \right)^{\frac{1}{2}} \\ &\lesssim \int_2^t \frac{dt}{(t+|u|)^{\frac{s+2}{2}}} F_0^1[\underline{\alpha}](u)^{\frac{1}{2}} F_s[\underline{\beta}](u)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s}{2}}} \sup_{2 \leq t \leq t_*} F_0^1[\underline{\alpha}](u)^{\frac{1}{2}}, \end{aligned}$$

which implies that for all  $\delta_0 > 0$

$$\mathcal{N}_{-1}^0[\underline{\alpha}^{(1)}, \underline{\beta}] \leq \delta_0 \sup_{2 \leq t \leq t_*} F_0^1[\underline{\alpha}](u) + C\delta_0^{-1} \frac{\varepsilon_0^2}{\langle u \rangle^s}, \quad (5.24)$$

where  $C$  is a positive constant independent of  $\delta_0$ . Next, applying (5.12) to the Bianchi pair  $((\not\partial\rho, \not\partial\check{\sigma}), \not\partial\underline{\beta})$  with  $p = 0$  and proceeding as in Proposition 5.13, we obtain from (5.18), Lemma 5.10 and Theorem 5.8

$$\begin{aligned} B_0^1[\underline{\beta}](u) &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s} + \mathcal{N}_0^1[\mathbb{E}_s, \mathbb{O}_{s-1} \cdot \mathbb{F}_{s-\delta_B}] + \mathcal{N}_0^1[\mathbb{F}_s, \mathbb{O}_{s-1} \cdot \mathbb{E}_{s-\delta_B}] + \mathcal{N}_0^1[\mathbb{F}_s, \mathbb{O}_s \cdot \mathbb{E}_{s-\delta_B}] \\ &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s}. \end{aligned} \quad (5.25)$$

Finally, applying Theorem 5.8 once again, we deduce

$$\mathcal{N}_0^1[\mathbb{F}_s, \mathbb{O}_1 \cdot \mathbb{F}_0] + \mathcal{N}_0^1[\mathbb{F}_0, \mathbb{O}_s \cdot \mathbb{F}_0] \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s}. \quad (5.26)$$

Injecting (5.24), (5.25) and (5.26) into (5.23), we have for  $\delta_0$  small enough

$$BEF_0^1[\underline{\beta}](u) + F_0^1[\underline{\alpha}](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^s}.$$

This concludes the proof of Proposition 5.14.  $\square$

### 5.2.5 Estimate for $\alpha_4$

**Proposition 5.15.** *We have the following estimate:*

$$BEF_s[\alpha_4](u) \lesssim \varepsilon_0^2. \quad (5.27)$$

*Proof.* We have from (5.3)

$$\begin{aligned} \not\nabla_3 \alpha_4 &= -2\not\mathcal{L}_2^* \not\phi + \frac{4\alpha}{r} + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}, \\ \not\nabla_4 \not\phi + \frac{5}{2} \operatorname{tr} \chi \not\phi &= \not\mathcal{L}_2 \alpha_4 + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}. \end{aligned}$$

Applying (5.10) with  $\psi_{(1)} = \alpha_4$ ,  $\psi_{(2)} = \not\phi$ ,  $a_{(1)} = 0$ ,  $a_{(2)} = \frac{5}{2}$ ,  $h_{(1)} = \frac{4\alpha}{r} + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}$  and  $p = s$ , we deduce

$$BEF_s[\alpha_4](u) + BF_s^1[\alpha](u) \lesssim \varepsilon_0^2 + \mathcal{N}_{s-1}^0[\alpha_4, \alpha] + \mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s].$$

Applying Cauchy-Schwarz inequality and (5.17), we obtain for any  $\delta_0 > 0$

$$\mathcal{N}_{s-1}^0[\alpha_4, \alpha] \leq \delta_0 \int_V r^{s-1} |\alpha_4|^2 + \frac{1}{4\delta_0} \int_V r^{s-1} |\alpha|^2 \leq \delta_0 B_s[\alpha_4](u) + (4\delta_0)^{-1} B_s[\alpha].$$

Moreover, applying Theorem 5.8, we have

$$\mathcal{N}_s^1[\mathbb{E}_s, \mathbb{O}_s \cdot \mathbb{E}_s] \lesssim \varepsilon_0^2.$$

Thus, we obtain for  $\delta_0$  small enough

$$BEF_s[\alpha_4](u) + BF_s^1[\alpha](u) \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 5.15.  $\square$

### 5.2.6 Estimate for $\underline{\alpha}_3$

**Proposition 5.16.** *We have the following estimate:*

$$EF_{-2}^1[\underline{\alpha}](u) + F_{-2}[\underline{\alpha}_3](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s+2}}. \quad (5.28)$$

*Proof.* We recall from (5.4)

$$\begin{aligned} \nabla_4 \dot{\underline{\alpha}} &= -2\dot{\underline{d}}_2^* \underline{\phi} + \frac{4\underline{\alpha}}{r} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}, \\ \nabla_3 \underline{\phi} + \frac{5}{2} \operatorname{tr} \underline{\chi} \underline{\phi} &= \dot{\underline{d}}_2 \dot{\underline{\alpha}} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}. \end{aligned}$$

Applying (5.12) with  $\psi_{(1)} = \underline{\phi}$ ,  $\psi_{(2)} = \dot{\underline{\alpha}}$ ,  $a_{(1)} = \frac{5}{2}$ ,  $a_{(2)} = 0$ ,  $h_{(1)} = (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}$ ,  $h_{(2)} = \frac{4\underline{\alpha}}{r} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}$  and  $p < -2$ , we obtain from Lemma 5.10

$$EF_p^1[\underline{\alpha}](u) + BF_p[\underline{\alpha}_3](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-p}} + B_p^1[\underline{\alpha}](u) + \mathcal{N}_{p-1}^0[\underline{\alpha}_3, \underline{\alpha}] + \mathcal{N}_p^0[\mathbb{F}_{-2}^{(1)}, (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}].$$

Applying (5.12) to the Bianchi pair  $(\beta, \underline{\alpha})$  with  $p < 0$  and proceeding as in Proposition 5.14, we obtain from Lemma 5.10 and Theorem 5.8

$$B_p^1[\underline{\alpha}](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-p}}, \quad \forall p < 0. \quad (5.29)$$

We have from Cauchy-Schwarz inequality that for any  $\delta_0 > 0$ :

$$\mathcal{N}_{p-1}^0[\underline{\alpha}_3, \underline{\alpha}] \leq \int_V r^{p-1} |\underline{\alpha}_3| |\underline{\alpha}| \leq \delta_0 B_p[\underline{\alpha}_3](u) + (4\delta_0)^{-1} B_p[\underline{\alpha}](u).$$

We also have from Theorem 5.8

$$\mathcal{N}_p^0[\mathbb{F}_{-2}^{(1)}, (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}] \lesssim \mathcal{N}_p^1[\mathbb{F}_{-2}, \mathbb{O}_{s-1} \cdot \mathbb{F}_0] \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-p}}.$$

Combining the above estimates, we deduce for  $\delta_0$  small enough

$$B_p[\underline{\alpha}_3](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-p}}, \quad \forall p < -2. \quad (5.30)$$

Next, applying (5.12) with  $\psi_{(1)} = \underline{\phi}$ ,  $\psi_{(2)} = \dot{\underline{\alpha}}$ ,  $a_{(1)} = \frac{5}{2}$ ,  $a_{(2)} = 0$ ,  $h_{(1)} = (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}$ ,  $h_{(2)} = \frac{4\underline{\alpha}}{r} + (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}$  and  $p = -2$ , we obtain from Lemma 5.10 and (5.29)

$$EF_{-2}^1[\underline{\alpha}](u) + F_{-2}[\underline{\alpha}_3](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s+2}} + \mathcal{N}_{-3}^0[\underline{\alpha}_3, \underline{\alpha}] + \mathcal{N}_{-2}^0[\mathbb{F}_{-2}^{(1)}, (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}].$$

We then have from (5.29) and (5.30)

$$\mathcal{N}_{-3}^0[\underline{\alpha}_3, \underline{\alpha}] \lesssim B_{-3}[\underline{\alpha}_3](u)^{\frac{1}{2}} B_{-1}[\underline{\alpha}](u)^{\frac{1}{2}} \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s+2}}.$$

We also have from Theorem 5.8

$$\mathcal{N}_{-2}^0[\mathbb{F}_{-2}^{(1)}, (\mathbb{O}_{s-1} \cdot \mathbb{F}_0)^{(1)}] \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s+2}}.$$

Combining the above estimates, we deduce

$$EF_{-2}^1[\underline{\alpha}](u) + F_{-2}[\underline{\alpha}_3](u) \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s+2}}.$$

This concludes the proof of Proposition 5.16.  $\square$

### 5.3 Interior region estimates

Throughout Section 5.3, we always assume that  $1 \leq u_1 \leq u_2 \leq u_c(t)$  and we denote

$$V := V_t(u_1, u_2).$$

Also, we recall that

$$V = {}^i V \cup {}^e V,$$

where

$${}^i V := V \cap \left\{ r \leq \frac{t}{2} \right\}, \quad {}^e V := V \cap \left\{ r \geq \frac{t}{2} \right\}.$$

#### 5.3.1 Domain of integration

**Lemma 5.17.** *Let  $\mathbf{C}_\delta$  be the cylinder of radius  $\delta > 0$  and centered on  $\Phi$ , i.e.*

$$\mathbf{C}_\delta := \{q \in \mathcal{M} / r(q) \leq \delta\}.$$

*Then, for any  $p \geq -1$ , there exists a sequence  $\delta_n \rightarrow 0$  satisfying*

$$\lim_{n \rightarrow \infty} \int_{\partial \mathbf{C}_{\delta_n} \cap V} r^p |\Psi^{(1)}|^2 = 0, \quad \Psi \in \{\alpha, \beta, \rho, \sigma, \check{\rho}, \check{\sigma}, \underline{\beta}, \underline{\alpha}\},$$

where

$$\partial \mathbf{C}_\delta = \{q \in \mathcal{M} / r(q) = \delta\},$$

denotes the boundary of  $\mathbf{C}_\delta$ .

*Proof.* We have from (4.1)

$${}^i \mathcal{F}_{-2, s+2}^1[\Psi] \lesssim \varepsilon^2,$$

which implies in particular

$$\int_0^1 d\delta \int_{\partial \mathbf{C}_\delta \cap V} r^{-2} |\Psi^{(1)}|^2 \lesssim \varepsilon^2 t.$$

Noticing that  $r = \delta$  on  $\partial \mathbf{C}_\delta$ , we infer

$$\int_0^1 \frac{d\delta}{\delta^{p+2}} \int_{\partial \mathbf{C}_\delta \cap V} r^p |\Psi^{(1)}|^2 \lesssim \varepsilon^2 t < +\infty.$$

Hence, for  $p \geq -1$ , we deduce that there exists a sequence  $\delta_n \rightarrow 0$  satisfying

$$\lim_{n \rightarrow \infty} \int_{\partial \mathbf{C}_{\delta_n} \cap V} r^p |\Psi^{(1)}|^2 = 0.$$

This concludes the proof of Lemma 5.17. □

We have the following analog of Proposition 5.9.

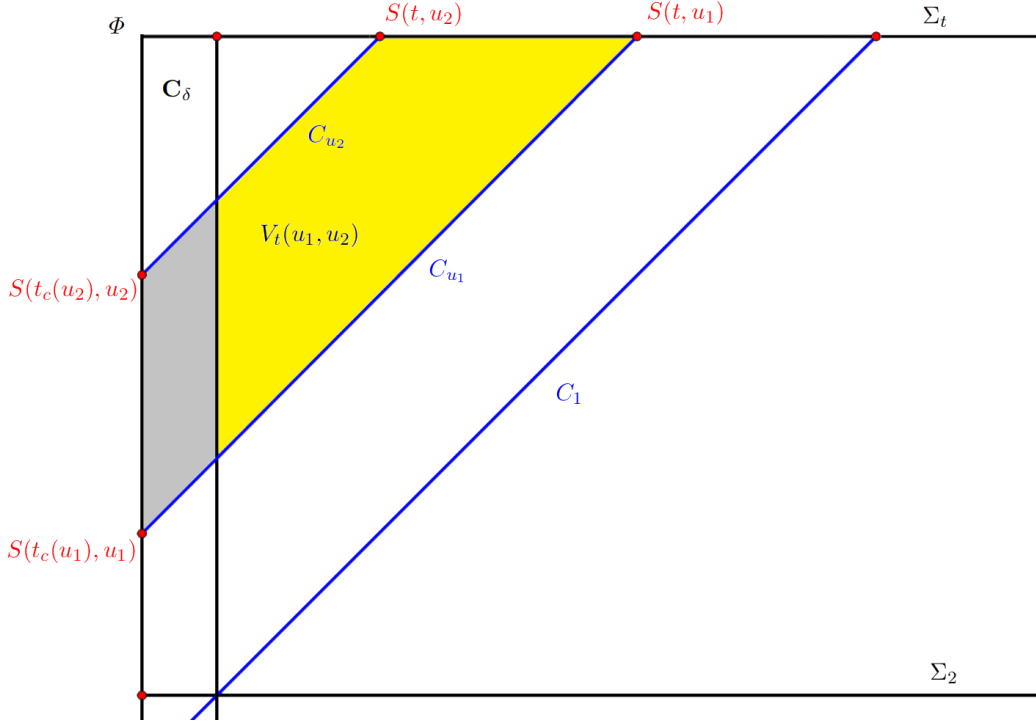
**Proposition 5.18.** *Let  $p \geq -1$  and let  $\psi_{(1)}$  and  $\psi_{(2)}$  be curvature components, i.e.*

$$\psi_{(1)}, \psi_{(2)} = R^{(q)}, \quad \text{where } R \in \{\alpha, \beta, (\rho, -\sigma), (\check{\rho}, \check{\sigma}), \underline{\beta}, \underline{\alpha}\}.$$

*Assume that  $\psi_{(1)}$  and  $\psi_{(2)}$  satisfy (5.5) or (5.7). Then, we have*

$$\begin{aligned} & EF_p[\psi_{(1)}](u_2) + F_p[\psi_{(2)}](u_1, u_2) \\ & + (2 + p - 4a_{(1)})B_p[\psi_{(1)}](u_1, u_2) + (4a_{(2)} - 2 - p)B_p[\psi_{(2)}](u_1, u_2) \\ & \lesssim E_p[\psi_{(1)}](u_1) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}]. \end{aligned}$$



Figure 3.3: Domain of integration  $V_t(u_1, u_2) \setminus C_\delta$ 

*Proof.* Integrating (5.6) or (5.8) in  $V \setminus C_\delta$  described in Figure 3.3 and proceeding as in Proposition 5.9, we obtain for all  $\delta > 0$

$$\begin{aligned}
& \int_{C_{u_2}^V \setminus C_\delta} r^p |\psi_{(1)}|^2 + \int_{\Sigma_t^V \setminus C_\delta} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \\
& + \int_V (2 + p - 4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)} - 2 - p) r^{p-1} |\psi_{(2)}|^2 \\
& \lesssim \int_{C_{u_1}^V} r^p |\psi_{(1)}|^2 + \int_{V \cap \partial C_\delta(u_1, u_2)} |\psi_{(1)}|^2 + |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|.
\end{aligned}$$

Taking  $\delta = \delta_n$  defined in Lemma 5.17 and letting  $n \rightarrow \infty$ , we obtain

$$\begin{aligned}
& \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\Sigma_t^V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \\
& + \int_V (2 + p - 4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)} - 2 - p) r^{p-1} |\psi_{(2)}|^2 \\
& \lesssim \int_{C_{u_1}^V} r^p |\psi_{(1)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|.
\end{aligned}$$

This concludes the proof of Proposition 5.18.  $\square$

### 5.3.2 Estimates for Bianchi pairs

The following lemmas allows us to obtain  $u$ -decay of curvature in the interior region.

**Lemma 5.19.** *Let  $s > 1$ ,  $q = 0, 1$  and let  $1 \leq u_1 \leq u_2 \leq u_c(t)$ . We assume that*

$$pB_p^q[\Psi](u_1, u_2) + EF_p^q[\Psi](u_1, u_2) \lesssim E_p^q[\Psi](u_1) + \frac{\varepsilon_0^2}{u^{s-p}}, \quad \forall p \in [0, s], \quad (5.31)$$

and

$$E_s^q[\Psi](1) \lesssim \varepsilon_0^2.$$

Then, we have

$$EF_p^q[\Psi](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s].$$

*Proof.* We first have for all  $u \geq 1$

$$BEF_s^q[\Psi](1, u) \lesssim E_s^q[\Psi](1) \lesssim \varepsilon_0^2,$$

which implies

$$E_s^q[\Psi](u) \lesssim \varepsilon_0^2, \quad \forall 1 \leq u \leq u_c(t). \quad (5.32)$$

Next, taking  $u_1 = 2^j$  and  $u_2 = 2^{j+1}$  for all  $j \in \mathbb{N}$  in (5.31), we infer

$$B_s^q[\Psi](2^j, 2^{j+1}) \lesssim E_s^q[\Psi](2^j) \lesssim \varepsilon_0^2.$$

By definition, we have

$$\int_{2^j}^{2^{j+1}} E_{s-1}^q[\Psi](u) du \lesssim \varepsilon_0^2,$$

which implies that there exists a sequence  $u^{(j)} \in [2^j, 2^{j+1}]$  such that

$$E_{s-1}^q[\Psi](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{u^{(j)}}.$$

Interpolating with (5.32), we obtain

$$E_1^q[\Psi](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{(u^{(j)})^{s-1}}.$$

Thus, applying (5.31) with  $p = 1$ ,  $u_1 = u^{(j-1)}$  and  $u_2 = u \in [2^j, 2^{j+1}]$ , we have

$$E_1^q[\Psi](u) \lesssim \frac{\varepsilon_0^2}{(u^{(j-1)})^{s-1}} \lesssim \frac{\varepsilon_0^2}{(2^j)^{s-1}} \lesssim \frac{\varepsilon_0^2}{u^{s-1}}.$$

Next, we apply once again (5.31) with  $p = 1$  to deduce

$$BEF_1^q[\Psi](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-1}}.$$

Then, we obtain for  $j \in \mathbb{N}$

$$B_1^q[\Psi](2^j, 2^{j+1}) \lesssim \frac{\varepsilon_0^2}{(2^j)^{s-1}},$$

which implies

$$\int_{2^j}^{2^{j+1}} E_0^q[\Psi](u) du \lesssim \frac{\varepsilon_0^2}{(2^j)^{s-1}}.$$

Then, there exists  $v^{(j)} \in [2^j, 2^{j+1}]$  such that

$$E_0^q[\Psi](v^{(j)}) \lesssim \frac{\varepsilon_0^2}{(2^j)^s} \lesssim \frac{\varepsilon_0^2}{(v^{(j)})^s}.$$

Thus, applying (5.31) with  $p = 0$ ,  $u_1 = v^{(j-1)}$  and  $u_2 = u \in [2^j, 2^{j+1}]$ , we have

$$E_0^q[\Psi](u) \lesssim \frac{\varepsilon_0^2}{(v^{(j-1)})^s} \lesssim \frac{\varepsilon_0^2}{(2^j)^s} \lesssim \frac{\varepsilon_0^2}{u^s}.$$

Next, we apply once again (5.31) with  $p = 0$  to deduce

$$EF_0^q[\Psi](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

Interpolating with (5.32), this concludes the proof of Lemma 5.19.  $\square$

**Lemma 5.20.** *Let  $s > 1$ ,  $q = 0, 1$  and let  $1 \leq u_1 \leq u_2 \leq u_c(t)$ . We assume that*

$$BF_p^q[\Psi](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad p \in [0, 1], \quad (5.33)$$

and

$$E_0^q[\Psi](u_2) \lesssim E_0^q[\Psi](u_1) + \frac{\varepsilon_0^2}{u_1^s}. \quad (5.34)$$

Then, we have

$$BEF_0^q[\Psi](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}. \quad (5.35)$$

*Proof.* Applying (5.33) with  $p = 1$ ,  $u_1 = 2^j$  and  $u_2 = 2^{j+1}$ , we obtain

$$\int_{2^j}^{2^{j+1}} E_0[\Psi](u) du \lesssim \frac{\varepsilon_0^2}{(2^j)^{s-1}},$$

which implies that there exists a sequence  $u^{(j)} \in [2^j, 2^{j+1}]$  such that

$$E_0[\Psi](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{(2^j)^s}.$$

Next, applying (5.34) with  $u_1 = u^{(j-1)}$  and  $u_2 = u \in [2^j, 2^{j+1}]$ , we infer

$$E_0^q[\Psi](u) \lesssim \frac{\varepsilon_0^2}{(2^j)^s} \lesssim \frac{\varepsilon_0^2}{u^s}. \quad (5.36)$$

Finally, applying once again (5.33) with  $p = 0$  and (5.34) and combining with (5.36), we obtain (5.35). This concludes the proof of Lemma 5.20.  $\square$

**Remark 5.21.** *The nonlinear error terms  $\mathcal{N}_p^q[\psi_{(1)}, h_{(1)}]$  and  $\mathcal{N}_p^q[\psi_{(2)}, h_{(2)}]$  in the proof of Propositions 5.22 and 5.24 below are the same as in Propositions 5.11–5.15. Thus, we have from the second part of Theorem 5.8<sup>14</sup>*

$$\mathcal{N}_p^q[\psi_{(1)}, h_{(1)}] \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \mathcal{N}_p^q[\psi_{(2)}, h_{(2)}] \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}.$$

*In the proof of Propositions 5.22 and 5.24 below, we thus focus on the linear terms.*

<sup>14</sup>We always have  $p_1, p_2, p \geq 0$  in the proof of Propositions 5.22 and 5.24, so that the additional assumption of the second part of Theorem 5.8 is satisfied.

**Proposition 5.22.** *We have the following estimates for all  $0 \leq p \leq s$ :*

$$\begin{aligned} EF_p^1[\alpha](u_1, u_2) + pB_p^1[\alpha](u_1, u_2) + BF_p^1[\beta](u_1, u_2) &\lesssim E_p^1[\alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \\ BEF_p^1[\beta](u_1, u_2) + BF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) &\lesssim E_p^1[\alpha, \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \\ BEF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) + (2-p)B_p^1[\underline{\beta}](u_1, u_2) + F_p^1[\underline{\beta}](u_1, u_2) &\lesssim E_p^1[\alpha, \beta, \check{\rho}, \check{\sigma}](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \\ BEF_0^1[\underline{\beta}](u_1, u_2) + F_0^1[\underline{\alpha}](u_1, u_2) &\lesssim E_0^1[\alpha, \beta, \check{\rho}, \check{\sigma}, \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^s}. \end{aligned}$$

*Proof.* The proof is largely analogous to Propositions 5.11–5.15. So we only provide a sketch. Throughout the proof, we always assume that  $0 \leq p \leq s$ .

Applying Proposition 5.18 with  $\psi_{(1)} = \check{\rho}^{\leq 1}\alpha$ ,  $\psi_{(2)} = \check{\rho}^{\leq 1}\beta$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ , we obtain

$$EF_p^1[\alpha](u_1, u_2) + pB_p^1[\alpha](u_1, u_2) + BF_p^1[\beta](u_1, u_2) \lesssim E_p^1[\alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \quad (5.37)$$

Next, applying Proposition 5.18 with  $\psi_{(1)} = \check{\rho}\beta$ ,  $\psi_{(2)} = (\check{\rho}\rho, -\check{\rho}\sigma)$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ , we deduce

$$EF_p^1[\beta](u_1, u_2) + BF_p^1[\rho, \sigma](u_1, u_2) \lesssim E_p^1[\beta](u_1) + B_p^1[\beta](u_1, u_2) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \quad (5.38)$$

Combining with (5.37) and (3.3), we obtain

$$BEF_p^1[\beta](u_1, u_2) + BF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim E_p^1[\alpha, \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \quad (5.39)$$

Next, applying Proposition 5.18 with  $\psi_{(1)} = (\check{\rho}\check{\rho}, \check{\rho}\check{\sigma})$ ,  $\psi_{(2)} = \check{\rho}\underline{\beta}$ ,  $a_{(1)} = \frac{3}{2}$ ,  $a_{(2)} = 1$ , we infer

$$\begin{aligned} &EF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) + (2-p)B_p^1[\underline{\beta}](u_1, u_2) + F_p^1[\underline{\beta}](u_1, u_2) \\ &\lesssim E_p^1[\check{\rho}, \check{\sigma}](u_1) + B_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \end{aligned} \quad (5.40)$$

Combining with (5.39), we have

$$\begin{aligned} &BEF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) + (2-p)B_p^1[\underline{\beta}](u_1, u_2) + F_p^1[\underline{\beta}](u_1, u_2) \\ &\lesssim E_p^1[\alpha, \beta, \check{\rho}, \check{\sigma}](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \end{aligned} \quad (5.41)$$

Finally, applying Proposition 5.18 with  $\psi_{(1)} = \check{\rho}^{\leq 1}\underline{\beta}$ ,  $\psi_{(2)} = \check{\rho}^{\leq 1}\underline{\alpha}$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$  and  $p = 0$ , we deduce

$$EF_0^1[\underline{\beta}](u_1, u_2) + F_0^1[\underline{\alpha}](u_1, u_2) \lesssim E_0^1[\underline{\beta}](u_1) + B_0^1[\underline{\beta}](u_1, u_2) + \frac{\varepsilon_0^2}{u_1^s}.$$

Combining with (5.41) in the case  $p = 0$ , we obtain

$$BEF_0^1[\underline{\beta}](u_1, u_2) + F_0^1[\underline{\alpha}](u_1, u_2) \lesssim E_0^1[\alpha, \beta, \check{\rho}, \check{\sigma}, \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^s}.$$

This concludes the proof of Proposition 5.22.  $\square$

**Proposition 5.23.** *We have the following estimates for all  $0 \leq p \leq s$ :*

$$\begin{aligned} pB_p^1[\alpha](u_1, u_2) + B_p^1[\beta, \check{\rho}, \check{\sigma}, \underline{\beta}](u_1, u_2) + EF_p^1[\alpha, \beta, \check{\rho}, \check{\sigma}](u_1, u_2) + F_p^1[\underline{\beta}](u_1, u_2) &\lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \\ E_0^1[\underline{\beta}](u_1, u_2) + F_0^1[\alpha](u_1, u_2) &\lesssim \frac{\varepsilon_0^2}{u_1^s}. \end{aligned} \quad (5.42)$$

*Proof.* Recall that we have from Propositions 5.11–5.15

$$E_s^1[\alpha, \beta, \check{\rho}, \check{\sigma}](1) + E_0^1[\underline{\beta}](1) \lesssim \varepsilon_0^2. \quad (5.43)$$

Applying Proposition 5.22 in the case  $p = s$ , we obtain immediately the first estimate of (5.42) in the case  $p = s$ . We then focus on the case  $p = 0$ .

We have from Proposition 5.22

$$pB_p^1[\alpha](u_1, u_2) + EF_p^1[\alpha](u_1, u_2) \lesssim E_p^1[\alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \quad (5.44)$$

Applying Lemma 5.19 with  $\Psi = \alpha$ , we obtain

$$EF_p^1[\alpha](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s]. \quad (5.45)$$

Next, we have from Proposition 5.22 and (5.45)

$$BF_p^1[\beta](u_1, u_2) \lesssim E_p^1[\alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s],$$

and

$$E_p^1[\beta](u_1, u_2) \lesssim E_p^1[\alpha, \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} \lesssim E_p^1[\beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s].$$

Applying Lemma 5.20 with  $\Psi = \beta$ , we obtain

$$BEF_0^1[\beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

We also have from Proposition 5.22 and (5.43)

$$BEF_s^1[\beta](u_1, u_2) \lesssim E_s^1[\alpha, \beta](u_1) + \varepsilon_0^2 \lesssim E_s^1[\alpha, \beta](1) + \varepsilon_0^2 \lesssim \varepsilon_0^2.$$

Combining the above estimates, we infer

$$BEF_p^1[\beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s]. \quad (5.46)$$

Then, we have from Proposition 5.22, (5.45) and (5.46)

$$BF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim E_p^1[\alpha, \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s],$$

and

$$E_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim E_p^1[\alpha, \beta, \check{\rho}, \check{\sigma}](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} \lesssim E_p^1[\check{\rho}, \check{\sigma}](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s].$$

Applying Lemma 5.20, we obtain

$$BEF_0^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

We also have from Proposition 5.22 and (5.43)

$$BEF_s^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim \varepsilon_0^2.$$

Combining the above estimates, we deduce

$$BEF_p^1[\check{\rho}, \check{\sigma}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s]. \quad (5.47)$$

Next, we have from Proposition 5.22 and (5.45)–(5.47)

$$BF_p^1[\underline{\beta}](u_1, u_2) \lesssim E_p^1[\alpha, \beta, \check{\rho}, \check{\sigma}](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad p \in [0, s],$$

and

$$E_0^1[\underline{\beta}](u_1, u_2) \lesssim E_0^1[\alpha, \beta, \check{\rho}, \check{\sigma}, \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^s} \lesssim E_0^1[\underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^s}.$$

Applying Lemma 5.20, we deduce

$$BEF_0^1[\underline{\beta}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}. \quad (5.48)$$

Finally, we have from Proposition 5.22 and (5.45)–(5.48)

$$F_0^1[\underline{\alpha}](u_1, u_2) \lesssim E_0^1[\alpha, \beta, \check{\rho}, \check{\sigma}, \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^s} \lesssim \frac{\varepsilon_0^2}{u_1^s}. \quad (5.49)$$

Combining the above estimates, this concludes the proof of Proposition 5.23.  $\square$

**Proposition 5.24.** *We have the following estimate:*

$$EF_p[\alpha_4](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s].$$

*Proof.* We have from (5.3)

$$\begin{aligned} \nabla_3 \alpha_4 &= -2\mathfrak{d}_2^* \mathfrak{d} + \frac{4\alpha}{r} + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}, \\ \nabla_4 \mathfrak{d} + \frac{5}{2} \operatorname{tr} \chi \mathfrak{d} &= \mathfrak{d}_2 \alpha_4 + (\mathbb{O}_s \cdot \mathbb{E}_s)^{(1)}. \end{aligned}$$

Applying Proposition 5.18 with  $\psi_{(1)} = \alpha_4$ ,  $\psi_{(2)} = \mathfrak{d}$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$  and  $0 \leq p \leq s$ , we deduce from Remark 5.21

$$BEF_p[\alpha_4](u_1, u_2) + BF_p^1[\alpha](u_1, u_2) \lesssim E_p[\alpha_4](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}} + \mathcal{N}_{p-1}^0[\alpha_4, \alpha].$$

Applying Cauchy-Schwarz inequality, we obtain for any  $\delta_0 > 0$

$$\mathcal{N}_{p-1}^0[\alpha_4, \alpha] \leq \delta_0 \int_V r^{p-1} |\alpha_4|^2 + \frac{1}{4\delta_0} \int_V r^{p-1} |\alpha|^2 \leq \delta_0 B_p[\alpha_4](u_1, u_2) + (4\delta_0)^{-1} B_p[\alpha](u_1, u_2).$$

Combining with Proposition 5.22, we obtain for  $\delta_0$  small enough,

$$BEF_p[\alpha_4](u_1, u_2) + BF_p^1[\alpha](u_1, u_2) \lesssim E_p[\alpha_4](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall p \in [0, s].$$

Applying Lemma 5.19, we deduce

$$EF_p[\alpha_4](u) \lesssim \frac{\varepsilon_0^2}{u^{s-p}}, \quad \forall p \in [0, s].$$

This concludes the proof of Proposition 5.24.  $\square$

## 5.4 Time derivative estimates in the interior region

### 5.4.1 Bianchi equations commuted with $\nabla_T$

**Lemma 5.25.** *We have*

$$\begin{aligned} \nabla_T \widehat{\chi} &= r^{-1} \mathbb{O}_1^{(1)}, & \nabla_T \zeta &= r^{-1} \mathbb{O}_{s-1}^{(1)}, & \nabla_T \widehat{\underline{\chi}} &= r^{-1} \mathbb{O}_{-1}^{(1)}, \\ \nabla_T \operatorname{tr} \chi &= r^{-1} \mathbb{O}_1^{(1)}, & \nabla_T \operatorname{tr} \underline{\chi} &= r^{-1} \mathbb{O}_{s-1}^{(1)}, \end{aligned} \quad (5.50)$$

and

$$\begin{aligned} \nabla_T \eta &\in r^{-1} \mathbb{O}_{-1}^{(1)}, & \nabla_T \underline{\omega} &\in r^{-1} \mathbb{O}_{s-1}^{(1)}, & \nabla_T \underline{\xi} &\in r^{-1} \mathbb{O}_{-1}^{(1)}, \\ \nabla_T \underline{\eta} &\in r^{-1} \mathbb{O}_{s-1}^{(1)}, & \nabla_T \omega &\in r^{-1} \mathbb{O}_{s-1}^{(1)}. \end{aligned} \quad (5.51)$$

Moreover, we have

$$\begin{aligned} \nabla_T \alpha &\in \mathbb{E}_{s+2}^{(1)}(s+2), & \nabla_T \beta &\in \mathbb{E}_{2s-\delta_B}^{(1)}(s+2), \\ \nabla_T(\rho, \sigma) &\in \mathbb{F}_{s+1}^{(1)}(s+2), & \nabla_T \underline{\beta} &\in \mathbb{F}_2^{(1)}(s+2), & \nabla_T \underline{\alpha} &\in \mathbb{F}_0^{(1)}(s+2), \end{aligned} \quad (5.52)$$

and

$$\begin{aligned} r^{-1} \alpha &\in \mathbb{E}_{s+2}(s+2), & r^{-1} \beta &\in \mathbb{E}_{s+2}(s+2), & r^{-1}(\rho, \sigma) &\in \mathbb{E}_{s+2}(s+2), \\ r^{-1} \underline{\beta} &\in \mathbb{F}_{s+2}(s+2), & r^{-1} \underline{\alpha} &\in \mathbb{F}_2(s+2). \end{aligned} \quad (5.53)$$

*Proof.* It follows directly from (2.14), (4.1) and Corollaries 4.8 and 4.11.  $\square$

**Proposition 5.26.** *We have the following equations:<sup>15</sup>*

$$\begin{aligned} \nabla_3(\nabla_T \alpha) + \frac{1}{2} \operatorname{tr} \underline{\chi}(\nabla_T \alpha) &= -2\mathcal{d}_2^*(\nabla_T \beta) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{2s-\delta_B}(s+2))^{(1)} \\ &\quad + (\mathbb{O}_s \cdot \mathbb{F}_{s+1}(s+2))^{(1)}, \\ \nabla_4(\nabla_T \beta) + 2 \operatorname{tr} \chi(\nabla_T \beta) &= \mathcal{d}_2(\nabla_T \alpha) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}, \\ \nabla_3(\nabla_T \beta) + \operatorname{tr} \underline{\chi}(\nabla_T \beta) &= -\mathcal{d}_1^*(\nabla_T(\rho, -\sigma)) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2))^{(1)} \\ &\quad + (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)}, \\ \nabla_4(\nabla_T(\rho, -\sigma)) + \frac{3}{2} \operatorname{tr} \chi(\nabla_T(\rho, -\sigma)) &= \mathcal{d}_2(\nabla_T \beta) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}, \\ \nabla_3(\nabla_T(\rho, \sigma)) + \frac{3}{2} \operatorname{tr} \underline{\chi}(\nabla_T(\rho, \sigma)) &= -\mathcal{d}_1(\nabla_T \underline{\beta}) + (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}, \\ \nabla_4(\nabla_T \underline{\beta}) + \operatorname{tr} \chi(\nabla_T \underline{\beta}) &= \mathcal{d}_1^*(\nabla_T \rho) + (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)} + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}, \\ \nabla_3(\nabla_T \underline{\beta}) + 2 \operatorname{tr} \underline{\chi}(\nabla_T \underline{\beta}) &= -\mathcal{d}_2(\nabla_T \underline{\alpha}) + (\mathbb{O}_1 \cdot \mathbb{F}_0(s+2))^{(1)}, \\ \nabla_4(\nabla_T \underline{\alpha}) + \frac{1}{2} \operatorname{tr} \chi(\nabla_T \underline{\alpha}) &= 2\mathcal{d}_2^*(\nabla_T \underline{\beta}) + (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}. \end{aligned}$$

*Proof.* It follows directly from Proposition 2.18, Corollary 4.6 and Lemma 5.25.  $\square$

**Theorem 5.27.** *Let  $0 \leq p_1, p_2, p \leq s+2$ ,  $\ell \leq s$  and let  $s > 1$ . Then, we have the following properties.*

1. *In the case  $2p < p_1 + p_2 + \ell + 1$ , we have*

$$\mathcal{N}_p^1[\mathbb{E}_{p_1}(s+2), \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}(s+2)] \lesssim \frac{\varepsilon^3}{u_1^{s+2-p}}.$$

<sup>15</sup>Here, we use the equations for  $\nabla_T(\rho, \sigma)$  instead of  $\nabla_T(\check{\rho}, \check{\sigma})$  since the decay of nonlinear terms are sufficient.

2. In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have

$$\begin{aligned}\mathcal{N}_p^1 [\mathbb{E}_{p_1}(s+2), \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}(s+2)] &\lesssim \frac{\varepsilon^3}{u_1^{s+2-p}}, \\ \mathcal{N}_p^1 [\mathbb{F}_{p_1}(s+2), \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}(s+2)] &\lesssim \frac{\varepsilon^3}{u_1^{s+2-p}}.\end{aligned}$$

3. In the case  $2p < p_1 + p_2 + \ell - 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}(s+2), \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}(s+2)] \lesssim \frac{\varepsilon^3}{u_1^{s+2-p}}.$$

*Proof.* The proof is largely analogous to Theorem 5.8 and left to the reader.  $\square$

**Proposition 5.28.** *Let  $\psi_{(1)}$  and  $\psi_{(2)}$  be time derivative of curvature components, i.e.*

$$\psi_{(1)}, \psi_{(2)} \in \{\nabla_T \alpha, \nabla_T \beta, \nabla_T \rho, \nabla_T \sigma, \nabla_T \underline{\beta}, \nabla_T \underline{\alpha}\}.$$

We also assume that  $\psi_{(1)}$  and  $\psi_{(2)}$  satisfy (5.5) or (5.7). Then, we have for  $p \geq 0$

$$\begin{aligned}&EF_p[\psi_{(1)}](u_2) + F_p[\psi_{(2)}](u_1, u_2) \\ &+ (2 + p - 4a_{(1)})B_p[\psi_{(1)}](u_1, u_2) + (4a_{(2)} - 2 - p)B_p[\psi_{(2)}](u_1, u_2) \\ &\lesssim E_p[\psi_{(1)}](u_1) + \mathcal{N}_p^0[\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0[\psi_{(2)}, h_{(2)}].\end{aligned}$$

*Proof.* Applying Theorem 7.1 in [47], we construct a sequence of initial data  $(\Sigma_0, g_n, k_n)$  which approximate  $(\Sigma_0, g, k)$  in  $H^6(\Sigma_0)$ . By local existence and Sobolev inequality, we obtain a sequence of  $C^4$ -solutions  $(\mathcal{M}_n, \mathbf{g}_n)$  of Einstein vacuum equations with initial data  $(\Sigma_0, g_n, k_n)$  which approximate  $(\mathcal{M}, \mathbf{g})$ . Denoting  $(\mathbf{D}_n, \mathbf{R}_n)$  the Levi-Civita connection and curvature components of  $\mathbf{g}_n$ , we have near the symmetry axis  $\Phi$

$$E_n, H_n = O(1), \quad (\mathbf{D}_n)_{T_n}(E_n, H_n) = O(1).$$

Recalling that  $(e_4)_n = T_n + N_n$ ,  $(e_3)_n = T_n - N_n$  and  $\|T_n\|, \|N_n\| = 1$ , we deduce on  $\partial\mathbf{C}_\delta$

$$(\nabla_n)_{T_n} R_n = O(1), \quad \text{where } R \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}.$$

Integrating (5.6) or (5.8) in  $V \setminus \mathbf{C}_\delta$ <sup>16</sup> and proceeding as in Proposition 5.18, we obtain<sup>17</sup>

$$\begin{aligned}&\int_{C_{u_2}^V \setminus \mathbf{C}_\delta} r^p |(\psi_n)_{(1)}|^2 + \int_{\Sigma_t^V \setminus \mathbf{C}_\delta} r^p (|(\psi_n)_{(1)}|^2 + |(\psi_n)_{(2)}|^2) \\ &+ \int_V (2 + p - 4a_{(1)})r^{p-1} |(\psi_n)_{(1)}|^2 + (4a_{(2)} - 2 - p)r^{p-1} |(\psi_n)_{(2)}|^2 \\ &\lesssim \int_{C_{u_1}^V} r^p |(\psi_n)_{(1)}|^2 + \int_{V \cap \partial\mathbf{C}_\delta(u_1, u_2)} |(\psi_n)_{(1)}|^2 + |(\psi_n)_{(2)}|^2 \\ &+ \int_V r^p |(\psi_n)_{(1)}| |(h_n)_{(1)}| + r^p |(\psi_n)_{(2)}| |(h_n)_{(2)}|,\end{aligned}\tag{5.54}$$

where we used  $r^p \leq 1$  on  $\partial\mathbf{C}_\delta$  since  $p \geq 0$ . Note that we have

$$\lim_{\delta \rightarrow 0} \int_{V \cap \partial\mathbf{C}_\delta(u_1, u_2)} |(\psi_n)_{(1)}|^2 + |(\psi_n)_{(2)}|^2 \lesssim C(n)t \lim_{\delta \rightarrow 0} \delta^2 = 0,\tag{5.55}$$

<sup>16</sup>Recall that  $\mathbf{C}_\delta$  is defined in Lemma 5.17.

<sup>17</sup>Note that in (5.54)–(5.56), the constants involved in  $\lesssim$  are independent of  $n$ .



where  $C(n)$  is a constant which depends on  $n$ . Taking  $\delta \rightarrow 0$  in (5.54), we obtain

$$\begin{aligned} & \int_{C_{u_2}^V} r^p |(\psi_n)_{(1)}|^2 + \int_{\Sigma_t^V} r^p (|(\psi_n)_{(1)}|^2 + |(\psi_n)_{(2)}|^2) \\ & + \int_V (2 + p - 4a_{(1)}) r^{p-1} |(\psi_n)_{(1)}|^2 + (4a_{(2)} - 2 - p) r^{p-1} |(\psi_n)_{(2)}|^2 \\ & \lesssim \int_{C_{u_1}^V} r^p |(\psi_n)_{(1)}|^2 + \int_V r^p |(\psi_n)_{(1)}| |(h_n)_{(1)}| + r^p |(\psi_n)_{(2)}| |(h_n)_{(2)}|. \end{aligned} \quad (5.56)$$

Letting  $n \rightarrow +\infty$  in (5.56), this concludes the proof of Proposition 5.28.  $\square$

#### 5.4.2 Estimates for the Bianchi pair $(\nabla_T \alpha, \nabla_T \beta)$

**Proposition 5.29.** *We have the following estimate for all  $0 \leq p \leq s+1$ :*

$$pB_p[\nabla_T \alpha](u_1, u_2) + EF_p[\nabla_T \alpha](u_1, u_2) + BF_p[\nabla_T \beta](u_1, u_2) \lesssim E_p[\nabla_T \alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \quad (5.57)$$

*Proof.* We recall from Proposition 5.26 that

$$\begin{aligned} \nabla_3(\nabla_T \alpha) + \frac{1}{2} \operatorname{tr} \chi(\nabla_T \alpha) &= -2\mathfrak{d}_2^*(\nabla_T \beta) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{2s-\delta_B}(s+2))^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_{s+1}(s+2))^{(1)}, \\ \nabla_4(\nabla_T \beta) + 2 \operatorname{tr} \chi(\nabla_T \beta) &= \mathfrak{d}_2(\nabla_T \alpha) + (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}. \end{aligned}$$

Applying Proposition 5.18 with  $\psi_{(1)} = \nabla_T \alpha$ ,  $\psi_{(2)} = \nabla_T \beta$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,  $h_{(1)} = (\mathbb{O}_{-1} \cdot \mathbb{E}_{2s-\delta_B}(s+2))^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_{s+1}(s+2))^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_{s-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}$  and  $0 \leq p \leq s+1$ , we obtain from (5.9) and Remark 5.2

$$\begin{aligned} & pB_p[\nabla_T \alpha](u_1, u_2) + EF_p[\nabla_T \alpha](u_1, u_2) + BF_p[\nabla_T \beta](u_1, u_2) \\ & \lesssim E_p[\nabla_T \alpha](u_1) + \mathcal{N}_p^1[\mathbb{E}_{s+2}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{2s-\delta_B}(s+2)] \\ & + \mathcal{N}_p^1[\mathbb{E}_{s+2}(s+2), \mathbb{O}_s \cdot \mathbb{F}_{s+1}(s+2)] + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_{s-1} \cdot \mathbb{E}_{s+2}(s+2)]. \end{aligned}$$

Applying Theorem 5.27, we obtain for  $0 \leq p \leq s+1$

$$\begin{aligned} & \mathcal{N}_p^1[\mathbb{E}_{s+2}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{2s-\delta_B}(s+2)] + \mathcal{N}_p^1[\mathbb{E}_{s+2}(s+2), \mathbb{O}_s \cdot \mathbb{F}_{s+1}(s+2)] \\ & + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_{s-1} \cdot \mathbb{E}_{s+2}(s+2)] \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned}$$

Hence, we have for  $0 \leq p \leq s+1$

$$pB_p[\nabla_T \alpha](u_1, u_2) + EF_p[\nabla_T \alpha](u_1, u_2) + BF_p[\nabla_T \beta](u_1, u_2) \lesssim E_p[\nabla_T \alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}.$$

This concludes the proof of Proposition 5.29.  $\square$

#### 5.4.3 Estimates for the Bianchi pair $(\nabla_T \beta, \nabla_T(\rho, -\sigma))$

**Proposition 5.30.** *We have the following estimate for  $0 \leq p \leq 2$ :*

$$BEF_p[\nabla_T \beta](u_1, u_2) + BF_p[\nabla_T \rho, \nabla_T \sigma](u_1, u_2) \lesssim E_p[\nabla_T \alpha, \nabla_T \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \quad (5.58)$$

*Proof.* We recall from Proposition 5.26

$$\begin{aligned} \nabla_3(\nabla_T\beta) + \text{tr } \underline{\chi}(\nabla_T\beta) &= -\mathfrak{d}_1^*(\nabla_T(\rho, -\sigma)) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2))^{(1)} \\ &\quad + (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)}, \\ \nabla_4(\nabla_T(\rho, -\sigma)) + \frac{3}{2} \text{tr } \chi(\nabla_T(\rho, -\sigma)) &= \mathfrak{d}_2(\nabla_T\beta) + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}. \end{aligned}$$

Applying Proposition 5.18 with  $\psi_{(1)} = \nabla_T\beta$ ,  $\psi_{(2)} = \nabla_T(\rho, -\sigma)$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2))^{(1)} + (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}$  and  $0 \leq p \leq 2$  and combining with Proposition 5.29, we obtain

$$\begin{aligned} &BEF_p[\nabla_T\beta](u_1, u_2) + BF_p[\nabla_T\rho, \nabla_T\sigma](u_1, u_2) \\ &\lesssim E_p[\nabla_T\beta](u_1) + B_p[\nabla_T\beta](u_1, u_2) + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2)] \\ &\quad + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_s \cdot \mathbb{F}_2(s+2)] + \mathcal{N}_p^1[\mathbb{F}_{s+1}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2)] \\ &\lesssim E_p[\nabla_T\alpha, \nabla_T\beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}} + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2)] \\ &\quad + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_s \cdot \mathbb{F}_2(s+2)] + \mathcal{N}_p^1[\mathbb{F}_{s+1}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2)]. \end{aligned}$$

Applying Theorem 5.27 in corresponding cases, we infer for  $0 \leq p \leq 2$

$$\begin{aligned} &\mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2-\delta_B}(s+2)] + \mathcal{N}_p^1[\mathbb{E}_{2s-\delta_B}(s+2), \mathbb{O}_s \cdot \mathbb{F}_2(s+2)] \\ &+ \mathcal{N}_p^1[\mathbb{F}_{s+1}(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2)] \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned}$$

Hence, we deduce for  $0 \leq p \leq 2$

$$BEF_p[\nabla_T\beta](u_1, u_2) + BF_p[\nabla_T\rho, \nabla_T\sigma](u_1, u_2) \lesssim E_p[\nabla_T\alpha, \nabla_T\beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}.$$

This concludes the proof of Proposition 5.30.  $\square$

#### 5.4.4 Estimates for the Bianchi pair $(\nabla_T(\rho, \sigma), \nabla_T\underline{\beta})$

**Proposition 5.31.** *We have the following estimate for  $0 \leq p \leq 1$ :*

$$\begin{aligned} &BEF_p[\nabla_T\rho, \nabla_T\sigma](u_1, u_2) + BF_p[\nabla_T\underline{\beta}](u_1, u_2) \\ &\lesssim E_p[\nabla_T\alpha, \nabla_T\beta, \nabla_T\rho, \nabla_T\sigma](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned} \tag{5.59}$$

*Proof.* We recall from Proposition 5.26

$$\begin{aligned} \nabla_3(\nabla_T(\rho, \sigma)) + \frac{3}{2} \text{tr } \underline{\chi}(\nabla_T(\rho, \sigma)) &= -\mathfrak{d}_1(\nabla_T\underline{\beta}) + (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}, \\ \nabla_4(\nabla_T\underline{\beta}) + \text{tr } \chi(\nabla_T\underline{\beta}) &= \mathfrak{d}_1^*(\nabla_T\rho) + (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)} + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}. \end{aligned}$$

Applying Proposition 5.18 with  $\psi_{(1)} = \nabla_T(\rho, \sigma)$ ,  $\psi_{(2)} = \nabla_T\underline{\beta}$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_s \cdot \mathbb{F}_2(s+2))^{(1)} + (\mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2))^{(1)}$  and  $0 \leq p \leq 1$  and combining with Proposition 5.30, we obtain

$$\begin{aligned} &BEF_p[\nabla_T\rho, \nabla_T\sigma](u_1, u_2) + BF_p[\nabla_T\underline{\beta}](u_1, u_2) \\ &\lesssim E_p[\nabla_T\alpha, \nabla_T\beta, \nabla_T\rho, \nabla_T\sigma](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}} + \mathcal{N}_p^1[\mathbb{F}_{s+1}(s+2), \mathbb{O}_s \cdot \mathbb{F}_0(s+2)] \\ &\quad + \mathcal{N}_p^1[\mathbb{F}_2(s+2), \mathbb{O}_s \cdot \mathbb{F}_2(s+2)] + \mathcal{N}_p^1[\mathbb{F}_2(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2)]. \end{aligned}$$

Applying Theorem 5.27 in corresponding cases, we deduce for  $0 \leq p \leq 1$

$$\begin{aligned} & \mathcal{N}_p^1[\mathbb{F}_{s+1}(s+2), \mathbb{O}_s \cdot \mathbb{F}_0(s+2)] + \mathcal{N}_p^1[\mathbb{F}_2(s+2), \mathbb{O}_s \cdot \mathbb{F}_2(s+2)] \\ & + \mathcal{N}_p^1[\mathbb{F}_2(s+2), \mathbb{O}_{-1} \cdot \mathbb{E}_{s+2}(s+2)] \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned}$$

Thus, we obtain for  $0 \leq p \leq 1$

$$BEF_p[\nabla_T \rho, \nabla_T \sigma](u_1, u_2) + BF_p[\nabla_T \underline{\beta}](u_1, u_2) \lesssim E_p[\nabla_T \alpha, \nabla_T \beta, \nabla_T \rho, \nabla_T \sigma](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}.$$

This concludes the proof of Proposition 5.31.  $\square$

#### 5.4.5 Estimates for the Bianchi pair $(\nabla_T \underline{\beta}, \nabla_T \underline{\alpha})$

**Proposition 5.32.** *We have the following estimate:*

$$BEF_0[\nabla_T \underline{\beta}](u_1, u_2) + F_0[\nabla_T \underline{\alpha}](u_1, u_2) \lesssim E_0[\nabla_T \alpha, \nabla_T \beta, \nabla_T \rho, \nabla_T \sigma, \nabla_T \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2}}. \quad (5.60)$$

*Proof.* We have from Proposition 5.26

$$\begin{aligned} \nabla_3(\nabla_T \underline{\beta}) + 2 \operatorname{tr} \chi(\nabla_T \underline{\beta}) &= -\not{d}_2(\nabla_T \underline{\alpha}) + (\mathbb{O}_1 \cdot \mathbb{F}_0(s+2))^{(1)}, \\ \nabla_4(\nabla_T \underline{\alpha}) + \frac{1}{2} \operatorname{tr} \chi(\nabla_T \underline{\alpha}) &= 2\not{d}_2^*(\nabla_T \underline{\beta}) + (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}. \end{aligned}$$

Applying Proposition 5.18 with  $\psi_{(1)} = \nabla_T \underline{\beta}$ ,  $\psi_{(2)} = \nabla_T \underline{\alpha}$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ ,  $h_{(1)} = (\mathbb{O}_1 \cdot \mathbb{F}_0(s+2))^{(1)}$ ,  $h_{(2)} = (\mathbb{O}_s \cdot \mathbb{F}_0(s+2))^{(1)}$  and  $p = 0$  and combining with Proposition 5.31

$$\begin{aligned} & BEF_0[\nabla_T \underline{\beta}](u_1, u_2) + F_0[\nabla_T \underline{\alpha}](u_1, u_2) \\ & \lesssim E_0[\nabla_T \alpha, \nabla_T \beta, \nabla_T \rho, \nabla_T \sigma, \nabla_T \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2}} \\ & + \mathcal{N}_0^1[\mathbb{F}_2(s+2), \mathbb{O}_1 \cdot \mathbb{F}_0(s+2)] + \mathcal{N}_0^1[\mathbb{F}_0(s+2), \mathbb{O}_s \cdot \mathbb{F}_0(s+2)]. \end{aligned}$$

Applying Theorem 5.27 in corresponding cases, we obtain

$$\mathcal{N}_0^1[\mathbb{F}_2(s+2), \mathbb{O}_1 \cdot \mathbb{F}_0(s+2)] + \mathcal{N}_0^1[\mathbb{F}_0(s+2), \mathbb{O}_s \cdot \mathbb{F}_0(s+2)] \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

Hence, we infer

$$BEF_0[\nabla_T \underline{\beta}](u_1, u_2) + F_0[\nabla_T \underline{\alpha}](u_1, u_2) \lesssim E_0[\nabla_T \alpha, \nabla_T \beta, \nabla_T \rho, \nabla_T \sigma, \nabla_T \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

This concludes the proof of Proposition 5.32.  $\square$

#### 5.4.6 $u$ -decay estimates

Combining Propositions 5.29–5.32, we have the following proposition.

**Proposition 5.33.** *We have the following estimates:*

- We have for  $0 \leq p \leq s+1$

$$\begin{aligned} & pB_p[\nabla_T \alpha](u_1, u_2) + EF_p[\nabla_T \alpha](u_1, u_2) + BF_p[\nabla_T \beta](u_1, u_2) \\ & \lesssim E_p[\nabla_T \alpha](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned} \quad (5.61)$$

- We have for  $0 \leq p \leq 2$

$$\begin{aligned} & BEF_p[\nabla_T\beta](u_1, u_2) + BF_p[\nabla_T\check{\rho}, \nabla_T\check{\sigma}](u_1, u_2) \\ & \lesssim E_p[\nabla_T\alpha, \nabla_T\beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned} \quad (5.62)$$

- We have for  $0 \leq p \leq 1$

$$\begin{aligned} & BEF_p[\nabla_T\rho, \nabla_T\sigma](u_1, u_2) + BF_p[\nabla_T\underline{\beta}](u_1, u_2) \\ & \lesssim E_p[\nabla_T\alpha, \nabla_T\beta, \nabla_T\rho, \nabla_T\sigma](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}. \end{aligned} \quad (5.63)$$

- We have

$$\begin{aligned} & BEF_0[\nabla_T\underline{\beta}](u_1, u_2) + F_0[\nabla_T\underline{\alpha}](u_1, u_2) \\ & \lesssim E_0[\nabla_T\alpha, \nabla_T\beta, \nabla_T\check{\rho}, \nabla_T\check{\sigma}, \nabla_T\underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2}}. \end{aligned} \quad (5.64)$$

**Proposition 5.34.** *We have the following estimates:*

$$EF_0[\nabla_T\alpha, \nabla_T\beta, \nabla_T\rho, \nabla_T\sigma, \nabla_T\underline{\beta}](u_1, u_2) + F_0[\nabla_T\underline{\alpha}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

*Proof.* The proof is largely analogous to Proposition 5.23. So, we only provide a sketch.

Recalling that

$$\nabla_T\alpha = r^{-1}O(\alpha_4, \alpha^{(1)}, \beta^{(1)}),$$

we have from Proposition 5.22

$$BEF_{s+2}[\nabla_T\alpha](u_1, u_2) \lesssim BEF_s[\alpha_4](u_1, u_2) + BEF_s^1[\alpha, \beta](u_1, u_2) \lesssim \varepsilon_0^2. \quad (5.65)$$

Taking  $u_1 = 2^j$  and  $u_2 = 2^{j+1}$ , we obtain

$$\int_{2^j}^{2^{j+1}} E_{s+1}[\nabla_T\alpha](u) du \lesssim \varepsilon_0^2,$$

which implies that there exists a sequence  $u^{(j)} \in [2^j, 2^{j+1}]$  such that

$$E_{s+1}[\nabla_T\alpha](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{u^{(j)}}.$$

Applying (5.61) with  $p = s + 1$ ,  $u_1 = u^{(j-1)}$  and  $u_2 = u \in [2^j, 2^{j+1}]$ , we deduce

$$BEF_{s+1}[\nabla_T\alpha](u^{(j-1)}, u) \lesssim E_{s+1}[\nabla_T\alpha](u^{(j-1)}) + \frac{\varepsilon_0^2}{u^{(j-1)}} \lesssim \frac{\varepsilon_0^2}{u},$$

which implies

$$E_{s+1}[\nabla_T\alpha](u) \lesssim \frac{\varepsilon_0^2}{u}.$$

Applying once again (5.61) with  $p = s + 1$ , we deduce

$$BEF_{s+1}[\nabla_T\alpha](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1}.$$

Next, proceeding as in Lemma 5.19, we obtain

$$EF_0[\nabla_T \alpha](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

Combining with (5.65), we deduce

$$EF_p[\nabla_T \alpha](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, \quad \forall p \in [0, s+2]. \quad (5.66)$$

Next, we have from (5.61), (5.62) and (5.66)

$$\begin{aligned} BF_p[\nabla_T \beta](u_1, u_2) &\lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, & p \in [0, 2], \\ E_p[\nabla_T \beta](u_2) &\lesssim E_p[\nabla_T \beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}, & p \in [0, 2]. \end{aligned} \quad (5.67)$$

Applying Lemma 5.20<sup>18</sup>, we infer

$$BEF_0[\nabla_T \beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

Moreover, we have from Corollary 4.11 and Proposition 5.22

$$BEF_2[\nabla_T \beta](u_1, u_2) \lesssim BEF_0^1[\alpha, \beta, \rho, \sigma, \underline{\beta}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

Hence, we obtain

$$BEF_p[\nabla_T \beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, \quad \forall p \in [0, 2]. \quad (5.68)$$

Then, we have from (5.63), (5.66) and (5.68)

$$\begin{aligned} BF_p[\nabla_T \rho, \nabla_T \sigma](u_1, u_2) &\lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, & \forall p \in [0, 2], \\ E_p[\nabla_T \rho, \nabla_T \sigma](u_2) &\lesssim E_p[\nabla_T \rho, \nabla_T \sigma](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2-p}}, & \forall p \in [0, 1]. \end{aligned} \quad (5.69)$$

Applying Lemma 5.20, we obtain

$$BEF_0[\nabla_T \rho, \nabla_T \sigma](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}.$$

Moreover, taking  $p = 2$  in the first estimate in (5.69), we deduce

$$\int_{u_1}^{u_2} E_1[\nabla_T \rho, \nabla_T \sigma](u) du \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

Proceeding as in Lemma 5.19, we obtain a sequence  $u^{(j)} \sim 2^j$  satisfying

$$E_1[\nabla_T \rho, \nabla_T \sigma](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{(u^{(j)})^{s+1}}.$$

---

<sup>18</sup>In fact, we apply Lemma 5.20 by replacing  $s$  with  $s+2$ .

Combining with the second estimate in (5.69), we deduce

$$E_1[\nabla_T \rho, \nabla_T \sigma](u) \lesssim \frac{\varepsilon_0^2}{u^{s+1}}.$$

Hence, we have

$$BEF_p[\nabla_T \rho, \nabla_T \sigma](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, \quad \forall p \in [0, 1]. \quad (5.70)$$

Finally, we have from (5.64), (5.66), (5.68) and (5.70)

$$\begin{aligned} BF_p[\nabla_T \underline{\beta}](u_1, u_2) &\lesssim \frac{\varepsilon_0^2}{u_1^{s+2-p}}, \quad \forall p \in [0, 1], \\ E_0[\nabla_T \underline{\beta}](u_2) &\lesssim E_0[\nabla_T \underline{\beta}](u_1) + \frac{\varepsilon_0^2}{u_1^{s+2}}. \end{aligned} \quad (5.71)$$

Applying Lemma 5.20, we obtain

$$BEF_0[\nabla_T \underline{\beta}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}. \quad (5.72)$$

Injecting it into (5.64), we infer

$$F_0[\nabla_T \underline{\alpha}](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s+2}}. \quad (5.73)$$

Combining the above estimates, this concludes the proof of Proposition 5.34.  $\square$

## 5.5 End of the proof of Theorem 3.8

**Proposition 5.35.** *We have the following estimates:*

$$\mathcal{D}_s[\alpha, \beta, \rho, \sigma] + \mathcal{D}_{s-1,1}[\underline{\beta}] + \mathcal{D}_{-1,s+1}[\underline{\alpha}] \lesssim \varepsilon_0^2. \quad (5.74)$$

*Proof.* We have from Propositions 5.11–5.16

$$\begin{aligned} {}^e\mathcal{E}_s^1[\alpha, \beta, \check{\rho}, \check{\sigma}] &\lesssim \varepsilon_0^2, & {}^e\mathcal{F}_s^1[\underline{\beta}] &\lesssim \varepsilon_0^2, & {}^e\mathcal{F}_{0,s}^1[\underline{\alpha}] &\lesssim \varepsilon_0^2, \\ {}^e\mathcal{E}_{0,s}^1[\underline{\beta}] &\lesssim \varepsilon_0^2, & {}^e\mathcal{E}_s[\alpha_4] &\lesssim \varepsilon_0^2, & {}^e\mathcal{F}_{-2,s+2}[\underline{\alpha}_3] &\lesssim \varepsilon_0^2. \end{aligned}$$

Applying Propositions 4.16 and 5.3, we deduce

$${}^e\mathcal{D}_s[\alpha, \beta, \check{\rho}, \check{\sigma}] \lesssim \varepsilon_0^2, \quad {}^e\mathcal{D}_{s-1,1}[\underline{\beta}] \lesssim \varepsilon_0^2, \quad {}^e\mathcal{D}_{-1,s+1}[\underline{\alpha}] \lesssim \varepsilon_0^2.$$

Recalling (3.3), we obtain

$$\begin{aligned} \sup_{p \in [2,4]} |r^{-\frac{2}{p}}(\rho, \sigma)|_{p,S} &\lesssim \sup_{p \in [2,4]} |r^{-\frac{2}{p}}(\check{\rho}, \check{\sigma})|_{p,S} + |\Gamma_g|_{\infty,S} \sup_{p \in [2,4]} |r^{-\frac{2}{p}}\Gamma_w|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} + \frac{\varepsilon^2}{r^{\frac{s+1}{2}} r \langle u \rangle^{\frac{s-1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}, \end{aligned} \quad (5.75)$$

which implies

$${}^e\mathcal{D}_s[\rho, \sigma] \lesssim \varepsilon_0^2.$$

Similarly, we have from Propositions 5.32 and 5.33 that in  ${}^iV$

$$\begin{aligned} {}^i\mathcal{E}_s^1[\alpha, \beta, \check{\rho}, \check{\sigma}] &\lesssim \varepsilon_0^2, & {}^i\mathcal{F}_s^1[\underline{\beta}] &\lesssim \varepsilon_0^2, & {}^i\mathcal{F}_{0,s}^1[\underline{\alpha}] &\lesssim \varepsilon_0^2, \\ {}^i\mathcal{E}_{0,s}^1[\underline{\beta}] &\lesssim \varepsilon_0^2, & {}^i\mathcal{E}_s[\alpha_4] &\lesssim \varepsilon_0^2, & {}^i\mathcal{F}_{0,s+2}[\nabla_T \underline{\alpha}] &\lesssim \varepsilon_0^2. \end{aligned}$$

Recalling that  $2T = e_4 + e_3$  and applying Propositions 4.16 and 5.3, we obtain

$${}^i\mathcal{D}_s[\alpha, \beta, \rho, \sigma] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{s-1,1}[\underline{\beta}] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{-1,s+1}[\underline{\alpha}] \lesssim \varepsilon_0^2,$$

where we used (5.75). This concludes the proof of Proposition 5.35.  $\square$

We then focus on the  $u$ -decay estimates in the interior region.

**Definition 5.36.** *We define the truncation function on  $\Sigma_t$*

$$f(x) := h\left(\frac{r(x)}{t}\right),$$

where  $h$  is a smooth cut-off function such that  $h = 1$  for  $s \leq \frac{1}{2}$  and  $h = 0$  for  $s \geq \frac{3}{4}$ . For any tensor field  $X$  on  $\Sigma_t$ , we denote:

$$\overset{\circ}{X} := fX.$$

**Proposition 5.37.** *We have the following estimates:*

$$\begin{aligned} {}^i\mathcal{F}_{-2,s+2}^1[\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}] &\lesssim \varepsilon_0^2, \\ {}^i\mathcal{D}_{-2,s+2}[\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}] &\lesssim \varepsilon_0^2. \end{aligned}$$

*Proof.* We have from Propositions 5.23 and 5.34

$${}^i\mathcal{F}_{0,s+2}[\nabla_T(\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha})] + {}^i\mathcal{F}_{0,s}^1[\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}] \lesssim \varepsilon_0^2,$$

which implies from (2.22) and (2.23)

$${}^i\mathcal{F}_{0,s+2}[\hat{\mathcal{L}}_T E] + {}^i\mathcal{F}_{0,s+2}[\hat{\mathcal{L}}_T H] \lesssim \varepsilon_0^2.$$

We recall from Proposition 2.23

$$\begin{aligned} \operatorname{div} E &= k \wedge H, \\ \operatorname{div} H &= -k \wedge E, \\ \operatorname{curl} E &= \hat{\mathcal{L}}_T H - \nabla \log \phi \wedge E - \frac{1}{2}k \times H, \\ \operatorname{curl} H &= -\hat{\mathcal{L}}_T E - \nabla \log \phi \wedge H + \frac{1}{2}k \times E. \end{aligned}$$

Commuting with  $f$ , we easily deduce<sup>19</sup>

$$\begin{aligned} \operatorname{div} \overset{\circ}{E} &= \Gamma_w \cdot \overset{\circ}{H} + \nabla f \cdot \overset{\circ}{E}, \\ \operatorname{div} \overset{\circ}{H} &= \Gamma_w \cdot \overset{\circ}{E} + \nabla f \cdot \overset{\circ}{H}, \\ \operatorname{curl} \overset{\circ}{E} &= f \hat{\mathcal{L}}_T H + \Gamma_w \cdot (\overset{\circ}{E}, \overset{\circ}{H}) + \nabla f \cdot \overset{\circ}{E}, \\ \operatorname{curl} \overset{\circ}{H} &= -f \hat{\mathcal{L}}_T E + \Gamma_w \cdot (\overset{\circ}{E}, \overset{\circ}{H}) + \nabla f \cdot \overset{\circ}{H}. \end{aligned}$$

<sup>19</sup>See (7.7.6g)–(7.7.6j) in [10] for the explicit formulae.

Applying Proposition 4.19, we obtain<sup>20</sup>

$$\begin{aligned}
\int_{\Sigma_t} |\nabla \mathring{E}|^2 + |\nabla \mathring{H}|^2 &\lesssim \int_{\Sigma_t} f^2 |\hat{\mathcal{L}}_T E, \hat{\mathcal{L}}_T H|^2 + |\Gamma_w \cdot (\mathring{E}, \mathring{H})|^2 + |\nabla f|^2 |(\mathring{E}, \mathring{H})|^2 + |(\mathring{E}, \mathring{H})|^3 \\
&\lesssim \int_{\Sigma_t(r \leq \frac{3t}{4})} |\hat{\mathcal{L}}_T E, \hat{\mathcal{L}}_T H|^2 + \int_{\frac{t}{4}}^t r^2 |r^{-\frac{2}{4}} \Gamma_w|_{4,S}^2 |r^{-\frac{2}{4}} (\mathring{E}, \mathring{H})|_{4,S}^2 du \\
&\quad + \int_{\frac{t}{2}}^{\frac{3t}{4}} \frac{r^2}{t^2} |r^{-\frac{2}{2}} (\mathring{E}, \mathring{H})|_{2,S}^2 du + \int_{\frac{t}{4}}^t r^2 |r^{-\frac{2}{3}} (\mathring{E}, \mathring{H})|_{3,S}^3 du \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}} + \int_{\frac{t}{4}}^t r^2 \frac{\varepsilon^4}{t^{s+1} t^{s+3}} du + \int_{\frac{t}{2}}^{\frac{3t}{4}} \frac{\varepsilon_0^2}{r^{s+3}} du + \int_{\frac{t}{4}}^t r^2 \frac{\varepsilon^3}{t^{\frac{3(s+3)}{2}}} du \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}} + \frac{\varepsilon^4}{t^{2s+1}} + \frac{\varepsilon_0^2}{t^{s+2}} + \frac{\varepsilon^3}{t^{\frac{3s+5}{2}}} \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}}.
\end{aligned}$$

Combining with Proposition 4.15, we obtain

$$F_{-2}^1[\mathring{E}, \mathring{H}](^iV) \lesssim \frac{\varepsilon_0^2}{t^{s+2}}.$$

Next, applying Proposition 4.16, we infer

$$|(\mathring{E}, \mathring{H})|_{4,S} \lesssim \frac{\varepsilon_0}{t^{\frac{s+2}{2}}}.$$

Recalling Definition 5.36, we deduce

$${}^i\mathcal{F}_{-2,s+2}^1[E, H](^iV) \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{-2,s+2}[E, H](^iV) \lesssim \varepsilon_0^2.$$

The estimates in  ${}^iV$  follows directly from Propositions 5.23 and 5.34. Combining with (2.22), this concludes the proof of Proposition 5.37.  $\square$

Combining Propositions 5.11–5.16, 5.22–5.23, 5.29–5.35 and 5.37, this concludes the proof of Theorem 3.8.

## 5.6 Improved estimate for $\beta$

In this section, we prove an improved estimate for  $\beta$  which will be used in Section 7.2.

**Lemma 5.38.** *We have the following estimate:*

$$\sup_{p \in [-1, s]} {}^i\mathcal{B}_{p, s-p}^1[\beta] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Theorem 3.8

$$\int_{\Sigma_t \cap {}^iV} r^p |\beta^{(1)}|^2 \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}, \quad \forall -2 \leq p \leq s.$$

Thus, we have

$$\begin{aligned}
\int_{{}^iV} r^{-2} |\beta^{(1)}|^2 &\lesssim \int_{t_c(u_1)}^t dt \int_{\Sigma_t \cap {}^iV} r^{-2} |\beta^{(1)}|^2 + \int_{t_c(u_1)}^t dt \int_{\Sigma_t \cap {}^i_eV} r^{-2} |\beta^{(1)}|^2 \\
&\lesssim \varepsilon_0^2 \int_{t_c(u_1)}^t \frac{dt}{t^{s+2}} + \int_{t_c(u_1)}^t \frac{dt}{t^2} \int_{\Sigma_t \cap {}^i_eV} |\beta^{(1)}|^2 \\
&\lesssim \frac{\varepsilon_0^2}{u_1^{s+1}}.
\end{aligned}$$

<sup>20</sup>Notice from Proposition 2.21 that  $\text{Ric} = O(E) + \Gamma_w \cdot \Gamma_w$ .



Interpolating with

$$\int_{iV} r^{s-1} |\beta^{(1)}|^2 \lesssim \varepsilon_0^2,$$

which follows directly from Theorem 3.8, this concludes the proof of Lemma 5.38.  $\square$

**Proposition 5.39.** *We have the following estimate:*

$${}^i\mathcal{E}_{-1,s+1}^1[\beta] \lesssim \varepsilon_0^2.$$

*Proof.* We recall from Proposition 4.10

$$\begin{aligned} \nabla_3 \beta + \text{tr} \chi \beta &= -\not{d}_1^*(\rho, -\sigma) + \Gamma_g \cdot \underline{\beta} + \Gamma_b \cdot \rho, \\ \nabla_4(\rho, -\sigma) + \frac{3}{2} \text{tr} \chi(\rho, -\sigma) &= \not{d}_1 \beta + \Gamma_w \cdot \beta. \end{aligned}$$

Applying Propositions 4.5 and 5.18, we deduce for  $-1 \leq p \leq 0$

$$EF_p^1[\beta](u_1, u_2) + BF_p^1[\rho, \sigma](u_1, u_2) \lesssim B_p^1[\beta](u_1, u_2) + E_p^1[\beta](u_1) + \mathcal{N}_p^1[R, \Gamma_w \cdot R]({}^iV), \quad (5.76)$$

where

$$R \in \{\beta, \rho, \sigma, \underline{\beta}\}.$$

We first have from Lemma 5.38 that for  $-1 \leq p \leq 0$

$$B_p^1[\beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}.$$

Moreover, we have from the proof of Proposition 5.12 that for  $-1 \leq p \leq 0$

$$\mathcal{N}_p^1[R, \Gamma_w \cdot R]({}^iV) \lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}.$$

We also have

$$\begin{aligned} \mathcal{N}_p^1[R, \Gamma_w \cdot R]({}^iV) &\lesssim \int_{iV} r^p |R^{(1)}| |\Gamma_w| |R^{(1)}| + \int_{iV} r^p |R^{(1)}| |\Gamma_w^{(1)}| |R| \\ &\lesssim \varepsilon \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}}} \int_{\Sigma_t \cap iV} r^p |R^{(1)}|^2 + \left( \int_{iV} r^{2p} |R^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{iV} |\Gamma_w^{(1)}|^2 |R|^2 \right)^{\frac{1}{2}} \\ &\lesssim \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}}} \frac{\varepsilon_0^2}{u_1^{s-p}} + \left( \int_{t_c(u_1)}^t dt \int_{\Sigma_t \cap iV} r^{2p} |R^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{u_1}^{u_2} du \int_{t_c(u)}^t \frac{\varepsilon^4}{t^{2(s+1)}} dt \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon_0^2}{u_1^{s-p}} + \left( \int_{t_c(u_1)}^t \frac{\varepsilon^2}{t^{s-2p}} dt \right)^{\frac{1}{2}} \frac{\varepsilon^2}{u_1^s} \\ &\lesssim \frac{\varepsilon_0^2}{u_1^{s-p}}. \end{aligned}$$

Injecting the above estimates into (5.76), we infer

$$EF_p^1[\beta](u_1, u_2) + BF_p^1[\rho, \sigma](u_1, u_2) \lesssim E_p^1[\beta](u_1) + \frac{\varepsilon_0^2}{u_1^{s-p}}. \quad (5.77)$$

We recall from Lemma 5.38

$$B_0^1[\beta](u_1, u_2) \lesssim \frac{\varepsilon_0^2}{u_1^s},$$

which implies that there exists a sequence  $u^{(j)} \sim 2^j$  satisfying

$$E_{-1}^1[\beta](u^{(j)}) \lesssim \frac{\varepsilon_0^2}{(u^{(j)})^{s+1}}.$$

Applying (5.77) with  $u_1 = u^{(j-1)}$ ,  $u_2 = u \in [2^j, 2^{j+1}]$  and  $p = -1$ , we obtain

$$E_{-1}^1[\beta](u) \lesssim \frac{\varepsilon_0^2}{u^{s+1}}.$$

This concludes the proof of Proposition 5.39.  $\square$

## 6 Maximal connection estimates (Theorem 3.9)

In this section, we apply the elliptic estimates in Section 4.5 to prove Theorem 3.9.

### 6.1 Preliminaries

**Proposition 6.1.** *We have the following elliptic system on  $\Sigma_t$ :*

$$\begin{aligned} \operatorname{div} k &= 0, \\ \operatorname{curl} k &= H, \\ \operatorname{tr} k &= 0. \end{aligned}$$

*Proof.* See (11.1.1a) in [10].  $\square$

**Proposition 6.2.** *We have the following system on  $\Sigma_t$ :*

$$\begin{aligned} \operatorname{div} \epsilon &= -\nabla_N \delta - \frac{3}{2} \operatorname{tr} \theta \delta + \widehat{\kappa} \cdot \widehat{\theta} - 2(a^{-1} \nabla a) \cdot \epsilon, \\ \operatorname{curl} \epsilon &= \sigma + \widehat{\theta} \wedge \widehat{\kappa}, \\ \nabla_N \epsilon + \operatorname{tr} \theta \epsilon &= \frac{1}{2}(\beta - \underline{\beta}) + \nabla \delta - 2\widehat{\theta} \cdot \epsilon + \frac{3}{2}(a^{-1} \nabla a) \cdot \delta - \widehat{\kappa} \cdot (a^{-1} \nabla a), \\ \operatorname{div} \widehat{\kappa} &= \frac{1}{2}(-\underline{\beta} + \beta) - \frac{1}{2} \nabla \delta + \widehat{\theta} \cdot \epsilon - \frac{1}{2} \operatorname{tr} \theta \epsilon, \\ \nabla_N \widehat{\kappa} + \frac{1}{2} \operatorname{tr} \theta \widehat{\kappa} &= \frac{1}{4}(-\underline{\alpha} + \alpha) + \frac{1}{2} \nabla \widehat{\otimes} \epsilon + \frac{3}{2} \delta \widehat{\theta} + (a^{-1} \nabla a) \widehat{\otimes} \epsilon, \end{aligned} \tag{6.1}$$

where we recall

$$\kappa_{AB} = k_{AB}, \quad \epsilon_A = k_{AN}, \quad \delta = k_{NN},$$

and  $\widehat{\kappa}$  denotes the traceless part of  $\kappa$ .

*Proof.* See (11.1.2) in [10].  $\square$

**Proposition 6.3.** *We have the following identities:*

$$\begin{aligned} \nabla_T \delta &= -\phi^{-1} \nabla_N \nabla_N \phi + \rho + \delta^2 - \eta \cdot \underline{\eta} + \eta \cdot \epsilon - \underline{\eta} \cdot \epsilon, \\ \nabla_T \epsilon &= -\phi^{-1} \nabla \nabla_N \phi + \frac{1}{2}(\beta + \underline{\beta}) + (a^{-1} \nabla a) \phi^{-1} \nabla_N \phi \\ &\quad - \frac{3}{2}(\eta - \phi^{-1} \nabla \phi) \delta + (\eta - \phi^{-1} \nabla \phi + \epsilon) \cdot \widehat{\kappa} + \frac{1}{2} \delta \epsilon, \\ \nabla_T \widehat{\kappa} &= -\phi^{-1} \nabla \widehat{\otimes} \nabla \phi + \frac{1}{4}(\alpha + \underline{\alpha}) - \delta \widehat{\kappa} + \epsilon \widehat{\otimes} \epsilon - (\eta - \phi^{-1} \nabla \phi) \widehat{\otimes} \epsilon. \end{aligned}$$

*Proof.* See (11.4.2) in [10]. □

**Proposition 6.4.** *We define the position vectorfield:*

$$Z := rN. \quad (6.2)$$

We denote  $\pi$  the deformation tensor of  $Z$ :

$$\pi_{ij} := \frac{1}{2} (\mathbf{g}(\mathbf{D}_{e_i} Z, e_j) + \mathbf{g}(\mathbf{D}_{e_j} Z, e_i)), \quad (6.3)$$

and  $\widehat{\pi}$  its traceless part. Then, we have

$$\begin{aligned} \widehat{\pi}_{ij} &= 2r\widehat{\theta}_{ij} + \frac{1}{3}ra^{-1}\widetilde{a\operatorname{tr}\theta}(g_{ij} - N_iN_j) - ra^{-1}(N_i\mathring{\nabla}_j a + N_j\mathring{\nabla}_i a), \\ \operatorname{tr} \pi &= ra^{-1}(2a\operatorname{tr} \theta + \overline{a\operatorname{tr} \theta}), \\ \pi_{ij} &= \widehat{\pi}_{ij} + \frac{1}{3}\operatorname{tr} \pi g_{ij}. \end{aligned}$$

*Proof.* See (4.1.3) in [10]. □

**Proposition 6.5.** *We define the following 1-form:*

$$(i_Z k)_i := k_{ij}Z^j. \quad (6.4)$$

Then, we have

$$\begin{aligned} \operatorname{div}(i_Z k) &= \frac{1}{2}\widehat{\pi}^{ij}k_{ij}, \\ \operatorname{curl}(i_Z k)_i &= H_{ij}Z^j + \frac{1}{2}\in_i{}^{mn}(\widehat{\pi}_m^j - ra^{-1}(N_m\mathring{\nabla}^j a - N^j\mathring{\nabla}_m a))k_{nj}. \end{aligned}$$

*Proof.* See (11.2.1) in [10]. □

**Proposition 6.6.** *Let  $\xi$  be a 1-form on  $\Sigma$  satisfying*

$$\begin{aligned} \operatorname{div} \xi &= D(\xi), \\ \operatorname{curl} \xi &= A(\xi). \end{aligned}$$

Then, the following estimate holds for all  $|p| < 1$ :

$$\int_{\Sigma_t} r^p |\nabla \xi|^2 + r^{p-2} |\xi|^2 \lesssim \int_{\Sigma_t} r^p (|A(\xi)|^2 + |D(\xi)|^2 + |\operatorname{Ric}||\xi|^2).$$

*Proof.* Note that we have

$$\operatorname{div}(r^{\frac{p}{2}}\xi) = \nabla^i(r^{\frac{p}{2}}\xi_i) = r^{\frac{p}{2}}D(\xi) + \frac{p}{2}r^{\frac{p-2}{2}}N(r)\xi_N.$$

We also have

$$\begin{aligned} \operatorname{curl}(r^{\frac{p}{2}}\xi)_i &= r^{\frac{p}{2}}A(\xi)_i + \in_i{}^{lm}\nabla_l(r^{\frac{p}{2}}\xi_m) \\ &= r^{\frac{p}{2}}A(\xi)_i + \frac{p}{2}\in_i{}^{Nm}r^{\frac{p-2}{2}}N(r)\xi_m, \end{aligned}$$

which implies

$$\operatorname{curl}(r^{\frac{p}{2}}\xi) = r^{\frac{p}{2}}A(\xi) + \frac{p}{2}r^{\frac{p-2}{2}}N(r)(-\xi_2, \xi_1, 0).$$

Applying Proposition 4.18, we deduce that for any constant  $\delta_0 > 0$

$$\begin{aligned} \int_{\Sigma_t} |\nabla(r^{\frac{p}{2}}\xi)|^2 &= \frac{p^2(1+\delta_0)}{4} \int_{\Sigma_t} r^{p-2} N(r)^2 |\xi|^2 + \left(1 + \frac{1}{\delta_0}\right) \int_{\Sigma_t} r^p (|A(\xi)|^2 + |D(\xi)|^2) \\ &\quad - \int_{\Sigma_t} r^p \text{Ric}^{ij} \xi_i \xi_j. \end{aligned}$$

Recalling that  $N(r) = 1 + O(\varepsilon)$ , we obtain

$$\int_{\Sigma_t} |\nabla(r^{\frac{p}{2}}\xi)|^2 - \frac{p^2(1+\delta_0)(1+O(\varepsilon))}{4} \frac{|r^{\frac{p}{2}}\xi|^2}{r^2} \lesssim \int_{\Sigma_t} r^p (|A(\xi)|^2 + |D(\xi)|^2 + |\text{Ric}||\xi|^2).$$

Note that we have for  $\delta_0$  and  $\varepsilon$  small enough

$$\frac{p^2(1+\delta_0)(1+O(\varepsilon))}{4} < c_H^{-2} - \delta_0,$$

where  $c_H = 2 + O(\varepsilon)$  is defined in Proposition 4.15. Thus, we obtain

$$\int_{\Sigma_t} |\nabla(r^{\frac{p}{2}}\xi)|^2 - (c_H^{-2} - \delta_0) \int_{\Sigma_t} r^{p-2} |\xi|^2 \lesssim \int_{\Sigma_t} r^p (|A(\xi)|^2 + |D(\xi)|^2 + |\text{Ric}||\xi|^2).$$

Applying Proposition 4.15, we deduce

$$\int_{\Sigma_t} |\nabla(r^{\frac{p}{2}}\xi)|^2 + r^{p-2} |\xi|^2 \lesssim \int_{\Sigma_t} r^p (|A(\xi)|^2 + |D(\xi)|^2 + |\text{Ric}||\xi|^2).$$

Recalling that  $|\nabla r| = 1 + O(\varepsilon)$ , this concludes the proof of Proposition 6.6.  $\square$

## 6.2 Estimates for $\epsilon$ and $\delta$

**Proposition 6.7.** *We have the following estimate:*

$$\mathcal{F}_s[\nabla\epsilon, \nabla\delta] \lesssim \epsilon_0^2, \quad \mathcal{F}_{s-2}[\epsilon, \delta] \lesssim \epsilon_0^2, \quad \mathcal{D}_{s-2}[\epsilon, \delta] \lesssim \epsilon_0^2.$$

*Proof.* We have from Propositions 6.4 and 6.5

$$\begin{aligned} \text{div}(i_Z k) &= r\Gamma_w \cdot \Gamma_w, \\ \text{curl}(i_Z k) &= i_Z H + r\Gamma_w \cdot \Gamma_w. \end{aligned}$$

Applying Proposition 6.6 with  $p = s - 2$ , we obtain

$$\begin{aligned} &\int_{\Sigma_t} r^{s-4} |i_Z k|^2 + r^{s-2} |\nabla i_Z k|^2 \\ &\lesssim \int_{\Sigma_t} r^{s-2} |i_Z H|^2 + r^s |\Gamma_w \cdot \Gamma_w|^2 + r^{s-2} |\text{Ric}||i_Z k|^2 \\ &\lesssim \int_{\Sigma_t} r^s |(\beta, \sigma, \underline{\beta})|^2 + r^s |\Gamma_w \cdot \Gamma_w|^2 + r^s |E||\Gamma_g \cdot \Gamma_g| \\ &\lesssim \epsilon_0^2 + \int_{-\infty}^{u_c(t)} \left( r^{s-2} |r^{1-\frac{2}{4}} \Gamma_w|_{4,S}^2 |r^{1-\frac{2}{4}} \Gamma_w|_{4,S}^2 + |E|_{2,S} |r^{\frac{s}{2}} \Gamma_g|_{4,S} |r^{\frac{s}{2}} \Gamma_g|_{4,S} \right) du \\ &\lesssim \epsilon_0^2 + \int_{-\infty}^1 \left( r^{s-2} \frac{\epsilon^4}{\langle u \rangle^{2(s-1)}} + \frac{\epsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} \right) du + \int_1^{u_c(t)} \left( \frac{\epsilon^4}{u^s} + \frac{\epsilon^3}{u^{\frac{s+1}{2}}} \right) du \\ &\lesssim \epsilon_0^2. \end{aligned}$$

Combining with (6.4), we obtain

$$\mathcal{F}_{s-2}[\epsilon, \delta] \lesssim \epsilon_0^2, \quad \mathcal{F}_s[\nabla\epsilon, \nabla\delta] \lesssim \epsilon_0^2.$$

Applying Proposition 4.16, we deduce

$$\mathcal{D}_{s-2}[\epsilon, \delta] \lesssim \epsilon_0^2.$$

This concludes the proof of Proposition 6.7.  $\square$

**Proposition 6.8.** *We have the following estimates:*

$$\mathcal{F}_s^1[(\nabla, \nabla_N)\epsilon, (\nabla, \nabla_N)\delta] \lesssim \epsilon_0^2, \quad \mathcal{D}_{s-2}^1[\epsilon, \delta] \lesssim \epsilon_0^2, \quad \mathcal{D}_{s-1,1}[\nabla_N\epsilon, \nabla_N\delta] \lesssim \epsilon_0^2.$$

*Proof.* We have from Proposition 6.5

$$\begin{aligned} \operatorname{div}(i_Z k) &= r\Gamma_w \cdot \Gamma_w, \\ \operatorname{curl}(i_Z k) &= i_Z H + r\Gamma_w \cdot \Gamma_w. \end{aligned}$$

Commuting with  $r\nabla_A$ , we obtain<sup>21</sup>

$$\begin{aligned} \operatorname{div}(r\nabla_A(i_Z k)) &= r\Gamma_w \cdot \Gamma_w^{(1)}, \\ \operatorname{curl}(r\nabla_A(i_Z k)) &= r\nabla_A(i_Z H) + r\Gamma_w \cdot \Gamma_w^{(1)}. \end{aligned}$$

Applying Proposition 6.6 with  $p = s - 2$ , we obtain

$$\begin{aligned} & \int_{\Sigma_t} r^s |\nabla \nabla_A i_Z k|^2 + r^{s-2} |\nabla_A i_Z k|^2 \\ & \lesssim \int_{\Sigma_t} r^s |\nabla i_Z H|^2 + r^s |\Gamma_w \cdot \Gamma_w^{(1)}|^2 + r^s |\operatorname{Ric}| |\nabla i_Z k|^2 \\ & \lesssim \int_{\Sigma_t} r^s |(\beta, \sigma, \underline{\beta})^{(1)}|^2 + r^s |\Gamma_w \cdot \Gamma_w|^2 + r^s |E| |\Gamma_g^{(1)} \cdot \Gamma_g^{(1)}|, \\ & \lesssim \epsilon_0^2 + \int_{-\infty}^{u_c(t)} \left( |r^{1-\frac{2}{4}} \Gamma_w|_{4,S}^2 |r^{\frac{s}{2}-\frac{2}{4}} \Gamma_w^{(1)}|_{4,S}^2 + |E|_{2,S} |r^{\frac{s}{2}} \Gamma_g^{(1)}|_{4,S} |r^{\frac{s}{2}} \Gamma_g^{(1)}|_{4,S} \right) du \\ & \lesssim \epsilon_0^2 + \int_{-\infty}^1 \left( \frac{\epsilon^4}{\langle u \rangle^s} + \frac{\epsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} \right) du + \int_1^{u_c(t)} \left( \frac{\epsilon^4}{u^s} + \frac{\epsilon^3}{u^{\frac{s+1}{2}}} \right) du \\ & \lesssim \epsilon_0^2. \end{aligned}$$

Combining with Proposition 6.7, we infer

$$\mathcal{F}_s^1[(\nabla, \nabla_N)\epsilon, (\nabla, \nabla_N)\delta] \lesssim \epsilon_0^2.$$

Applying Proposition 4.16, we deduce

$$\mathcal{D}_{s-2}^1[\epsilon, \delta] \lesssim \epsilon_0^2.$$

Finally, we have from Proposition 6.2

$$\begin{aligned} \operatorname{div} \epsilon &= -\nabla_N \delta - \frac{3}{2} \operatorname{tr} \theta \delta + \Gamma_w \cdot \Gamma_w, \\ \nabla_N \epsilon + \operatorname{tr} \theta \epsilon &= \frac{1}{2} (\beta - \underline{\beta}) + \nabla \delta + \Gamma_w \cdot \Gamma_b, \end{aligned}$$

which implies

$$|r^{-\frac{2}{p}} \nabla_N(\epsilon, \delta)|_{p,S} \lesssim |r^{-\frac{2}{p}}(\beta, \underline{\beta})| + |r^{-\frac{2}{p}}(\nabla \epsilon, \nabla \delta)|_{p,S} + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w|_{p,S} \lesssim \frac{\epsilon_0}{r^{\frac{s+2}{2}} u^{\frac{1}{2}}}.$$

Hence, we obtain

$$\mathcal{D}_{s-1,1}[\nabla_N \epsilon, \nabla_N \delta] \lesssim \epsilon_0^2.$$

This concludes the proof of Proposition 6.8.  $\square$

<sup>21</sup>See (6.119) in [5] for the precise equations and its derivation.

### 6.3 Estimate for $\widehat{\kappa}$

**Proposition 6.9.** *We have the following estimates:*

$$\mathcal{F}_{s-2}^2[\widehat{\kappa}] \lesssim \varepsilon_0^2, \quad \mathcal{F}_s^1[\nabla_N \widehat{\kappa}] \lesssim \varepsilon_0^2, \quad \mathcal{D}_{s-3,1}^1[\widehat{\kappa}] \lesssim \varepsilon_0^2, \quad \mathcal{D}_{-1,s+1}[\nabla_N \widehat{\kappa}] \lesssim \varepsilon_0^2. \quad (6.5)$$

Moreover, we have

$$\mathcal{D}_{-1,s-1}[\widehat{\kappa}] \lesssim \varepsilon_0^2. \quad (6.6)$$

**Remark 6.10.** *Note that (6.6) implies that  $\widehat{\kappa}$  decays better than  $\phi \widehat{\kappa}$ .*

*Proof.* We have from Proposition 6.2

$$\begin{aligned} d\widehat{\kappa} &= \frac{1}{2}(-\underline{\beta} + \beta) - \frac{1}{2}\nabla\delta - \frac{1}{2}\text{tr}\theta\epsilon + \Gamma_g \cdot \Gamma_w, \\ \nabla_N \widehat{\kappa} + \frac{1}{2}\text{tr}\theta\widehat{\kappa} &= \frac{1}{4}(-\underline{\alpha} + \alpha) + \frac{1}{2}\nabla\widehat{\otimes}\epsilon + \Gamma_g \cdot \Gamma_w. \end{aligned}$$

Applying Propositions 4.12 and 6.8, we immediately obtain (6.5). Next, applying Lemma 4.24, we have for  $p \in [2, 4]$  and  $u \leq 1$

$$\begin{aligned} |r^{1-\frac{2}{p}}\widehat{\kappa}|_{p,S(t,u)} &\lesssim \lim_{u \rightarrow \infty} |r^{1-\frac{2}{p}}\widehat{\kappa}|_{p,S(t,u)} + \int_{-\infty}^u \left( |r^{1-\frac{2}{p}}(\alpha, \underline{\alpha}, \nabla\epsilon)|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_w|_{p,S} \right) du \\ &\lesssim \int_{-\infty}^u \left( \frac{\varepsilon_0}{\langle u \rangle^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{\langle u \rangle^s} \right) du \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}}, \end{aligned}$$

and similarly for  $1 \leq u \leq u_c(t)$

$$\begin{aligned} |r^{1-\frac{2}{p}}\widehat{\kappa}|_{p,S(t,u)} &\lesssim \lim_{u \rightarrow u_c(t)} |r^{1-\frac{2}{p}}\widehat{\kappa}|_{p,S(t,u)} + \int_u^{u_c(t)} \left( |r^{1-\frac{2}{p}}(\alpha, \underline{\alpha}, \nabla\epsilon)|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_w|_{p,S} \right) du \\ &\lesssim \int_u^{u_c(t)} \left( \frac{\varepsilon_0}{u^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{u^s} \right) du \\ &\lesssim \frac{\varepsilon_0}{u^{\frac{s-1}{2}}}. \end{aligned}$$

Hence, we obtain (6.6). This concludes the proof of Proposition 6.9.  $\square$

### 6.4 Estimate for the lapse function $\phi$

We have from (2.5) and (4.1) that

$$\Delta\phi = \phi|k|^2, \quad \lim_{u \rightarrow -\infty} \phi = 1.$$

In view of the maximum principle and Harnarck inequality<sup>22</sup>, we infer

$$0 < \phi \leq 1.$$

Thus, we can define the following negative scalar function:

$$\varphi := \log \phi. \quad (6.7)$$

<sup>22</sup>See for example Theorem 3.1 in [15] for the maximum principle and Theorem 8.20 in [15] for Harnarck inequality.

**Lemma 6.11.** *We have the following equation for  $\varphi$ :*

$$\Delta\varphi = |k|^2 - |\nabla\varphi|^2, \quad \lim_{u \rightarrow -\infty} \varphi = 0.$$

*Proof.* We have from (6.7)

$$\Delta\varphi = \nabla^i \nabla_i (\log \phi) = \nabla^i (\phi^{-1} \nabla_i \phi) = \phi^{-1} \Delta\phi - \phi^{-2} \nabla^i \phi \nabla_i \phi = |k|^2 - |\nabla\varphi|^2,$$

as stated. We also have from (6.7)

$$\lim_{u \rightarrow -\infty} \varphi = \log \left( \lim_{u \rightarrow -\infty} \phi \right) = 0.$$

This concludes the proof of Lemma 6.11. □

**Proposition 6.12.** *We have the following estimate:*

$$\mathcal{F}_{s-2}[\nabla\varphi] \lesssim \varepsilon_0^2, \quad \mathcal{D}_{s-4}[\varphi] \lesssim \varepsilon_0^2.$$

*Proof.* Recalling that  $N(r) = 1 + r\Gamma_b$ , we have from Lemma 6.11

$$\begin{aligned} r^{s-2} |\nabla\varphi|^2 &= \nabla^i (r^{s-2} \varphi \nabla_i \varphi) - \nabla^i (r^{s-2}) \varphi \nabla_i \varphi - r^{s-2} \varphi \Delta\varphi \\ &= \operatorname{div}(r^{s-2} \varphi \nabla\varphi) - (s-2)r^{s-3} N(r) \varphi N(\varphi) - r^{s-2} \varphi (|k|^2 - |\nabla\varphi|^2) \\ &= \operatorname{div}(r^{s-2} \varphi \nabla\varphi) - (s-2)r^{s-3} \varphi N(\varphi) + r^{s-1} \Gamma_g \cdot \Gamma_w \cdot \Gamma_w. \end{aligned}$$

We also have from (4.5)

$$\begin{aligned} \int_{\Sigma_t} 2r^{s-3} \varphi N(\varphi) &= - \int_{\Sigma_t} \operatorname{div}(r^{s-3} N) |\varphi|^2 \\ &= - \int_{\Sigma_t} r^{s-3} \operatorname{div}(N) |\varphi|^2 - (s-3) \int_{\Sigma_t} r^{s-4} N(r) |\varphi|^2 \\ &= -(s-1) \int_{\Sigma_t} r^{s-4} |\varphi|^2 + \int_{\Sigma_t} r^{s-1} \Gamma_g \cdot \Gamma_g \cdot \Gamma_b. \end{aligned}$$

Combining the above identities, we infer

$$\begin{aligned} \int_{\Sigma_t} r^{s-2} |\nabla\varphi|^2 &= - \int_{\Sigma_t} (s-2)r^{s-3} \varphi N(\varphi) + \int_{\Sigma_t} r^{s-1} \Gamma_g \cdot \Gamma_w \cdot \Gamma_w \\ &= \frac{(s-2)(s-1)}{2} \int_{\Sigma_t} r^{s-4} |\varphi|^2 + \int_{\Sigma_t} r^{s-1} \Gamma_g \cdot \Gamma_w \cdot \Gamma_w \\ &\lesssim \int_{-\infty}^u r^{s+1} |r^{-\frac{2}{2}} \Gamma_g|_{2,S} |r^{-\frac{2}{4}} \Gamma_w|_{4,S} |r^{-\frac{2}{4}} \Gamma_w|_{4,S} \\ &\lesssim \int_{-\infty}^1 r^{s+1} \frac{\varepsilon}{r^{\frac{s+1}{2}}} \frac{\varepsilon}{r\langle u \rangle^{\frac{s-1}{2}}} \frac{\varepsilon}{r\langle u \rangle^{\frac{s-1}{2}}} + \int_1^{u_c(t)} r^{s+1} \frac{\varepsilon}{r^{\frac{s+1}{2}}} \frac{\varepsilon}{ru^{\frac{s-1}{2}}} \frac{\varepsilon}{r^{\frac{s-1}{2}} u} \\ &\lesssim \int_{-\infty}^1 \frac{\varepsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} du + \int_1^{u_c(t)} \frac{\varepsilon^3}{u^{\frac{s+1}{2}}} du \\ &\lesssim \varepsilon_0^2, \end{aligned}$$

where we used the fact that

$$(s-2)(s-1) \leq 0.$$

Hence, we deduce

$$\mathcal{F}_{s-2}[\nabla\varphi] \lesssim \varepsilon_0^2.$$

Applying Proposition 4.16, we obtain

$$\mathcal{D}_{s-4}[\varphi] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 6.12.  $\square$

**Proposition 6.13.** *We have the following estimates:*

$$\mathcal{F}_s[\nabla^2\varphi] \lesssim \varepsilon_0^2, \quad \mathcal{D}_{s-2}[\nabla\varphi] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Lemma 6.11

$$\operatorname{div}(\nabla\varphi) = \Gamma_w \cdot \Gamma_w, \quad \operatorname{curl}(\nabla\varphi) = 0.$$

Thus, we obtain

$$\operatorname{div}(r^{\frac{s}{2}}\nabla\varphi) = \nabla^i(r^{\frac{s}{2}}\nabla_i\varphi) = \frac{s}{2}r^{\frac{s-2}{2}}N(r)\nabla_N\varphi + r^{\frac{s}{2}}\operatorname{div}(\nabla\varphi) = r^{\frac{s-2}{2}}O(\nabla\varphi) + r^{\frac{s}{2}}\Gamma_w \cdot \Gamma_w,$$

and similarly

$$\operatorname{curl}(r^{\frac{s}{2}}\nabla\varphi) = r^{\frac{s-2}{2}}O(\nabla\varphi).$$

Applying Propositions 4.16 and 6.12, we infer

$$\begin{aligned} \int_{\Sigma_t} |\nabla(r^{\frac{s}{2}}\nabla\varphi)|^2 &\lesssim \int_{\Sigma_t} r^{s-2}|\nabla\varphi|^2 + r^s|\Gamma_w \cdot \Gamma_w|^2 + |\operatorname{Ric}||r^{\frac{s}{2}}\nabla\varphi|^2 \\ &\lesssim \varepsilon_0^2 + \int_{-\infty}^{u_c(t)} |r^{1-\frac{2}{4}}\Gamma_w|_{4,S}^2 |r^{\frac{s}{2}-\frac{2}{4}}\Gamma_w|_{4,S}^2 + |r^{1-\frac{2}{2}}E|_{2,S} |r^{\frac{s+1}{2}-\frac{2}{4}}\Gamma_g|_{4,S} |r^{\frac{s+1}{2}-\frac{2}{4}}\Gamma_g|_{4,S} \\ &\lesssim \varepsilon_0^2 + \int_{-\infty}^1 \left( \frac{\varepsilon^4}{\langle u \rangle^s} + \frac{\varepsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} \right) du + \int_1^{u_c(t)} \left( \frac{\varepsilon^4}{u^s} + \frac{\varepsilon^3}{u^{\frac{s+1}{2}}} \right) du \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

Combining with Proposition 6.12, we obtain

$$\mathcal{F}_s[\nabla^2\varphi] \lesssim \varepsilon_0^2.$$

Finally, applying Proposition 4.16, we deduce

$$\mathcal{D}_{s-2}[\nabla\varphi] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 6.13.  $\square$

**Proposition 6.14.** *We have the following estimates:*

$$\begin{aligned} \mathcal{F}_s^1[\nabla_N\nabla_N\varphi] + \mathcal{F}_{s-2}^2[\nabla_N\varphi] + \mathcal{F}_{s-4}^3[\varphi] &\lesssim \varepsilon_0^2, \\ \mathcal{D}_{s-1,1}[\nabla_N\nabla_N\varphi] + \mathcal{D}_{s-2}^1[\nabla_N\varphi] + \mathcal{D}_{s-4}^2[\varphi] &\lesssim \varepsilon_0^2. \end{aligned}$$

*Proof.* We have from Lemma 6.11

$$\Delta\varphi = \Gamma_w \cdot \Gamma_w,$$

which implies

$$\begin{aligned} \Delta(r^{\frac{s-2}{2}}\varphi) &= \Delta(r^{\frac{s-2}{2}})\varphi + 2\nabla^i(r^{\frac{s-2}{2}})\nabla_i\varphi + r^{\frac{s-2}{2}}\Delta\varphi \\ &= r^{\frac{s-6}{2}}O(\varphi) + r^{\frac{s-4}{2}}O(\nabla\varphi) + r^{\frac{s-2}{2}}\Gamma_w \cdot \Gamma_w. \end{aligned}$$



Applying Proposition 4.20, we deduce

$$\begin{aligned}
& \int_{\Sigma_t} r^4 |\mathring{\nabla}^3 (r^{\frac{s-2}{2}} \varphi)|^2 + r^4 |\mathring{\nabla}^2 \nabla_N (r^{\frac{s-2}{2}} \varphi)|^2 + r^4 |\mathring{\nabla} \nabla_N^2 (r^{\frac{s-2}{2}} \varphi)|^2 \\
& \lesssim \int_{\Sigma_t} r^{s-4} |\varphi|^2 + r^{s-2} |\nabla \varphi|^2 + r^s |\mathring{\nabla} \nabla \varphi|^2 + r^s |\Gamma_w \cdot \Gamma_w^{(1)}|^2 \\
& \lesssim \varepsilon_0^2 + \int_{-\infty}^{u_c(t)} |r^{1-\frac{2}{4}} \Gamma_w|_{4,S}^2 |r^{\frac{s}{2}-\frac{2}{4}} \Gamma_w^{(1)}|_{4,S}^2 \\
& \lesssim \varepsilon_0^2 + \int_{-\infty}^1 \frac{\varepsilon^2}{\langle u \rangle^{s-1}} \frac{\varepsilon^2}{\langle u \rangle} du + \int_1^{u_c(t)} \frac{\varepsilon^2}{u^{s-1}} \frac{\varepsilon^2}{u} du \\
& \lesssim \varepsilon_0^2.
\end{aligned}$$

Combining with Proposition 6.13, we obtain

$$\mathcal{F}_s^1[\nabla_N \nabla_N \varphi] + \mathcal{F}_{s-2}^2[\nabla_N \varphi] + \mathcal{F}_{s-4}^3[\varphi] \lesssim \varepsilon_0^2.$$

Applying Proposition 4.16, we infer

$$\mathcal{D}_{s-2}^1[\nabla_N \varphi] + \mathcal{D}_{s-4}^2[\varphi] \lesssim \varepsilon_0^2.$$

Finally, we have from Lemma 6.11

$$\nabla_N \nabla_N \varphi = -g^{AB} \nabla_A \nabla_B \varphi + \Gamma_w \cdot \Gamma_w,$$

which implies

$$\mathcal{D}_{s-1,1}[\nabla_N \nabla_N \varphi] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 6.14.  $\square$

## 6.5 $u$ -decay estimates

In this section, we always assume that  $u \geq 1$ .

**Proposition 6.15.** *We have the following estimates:*

$$\begin{aligned}
{}^i \mathcal{F}_{-2,s}[k] &\lesssim \varepsilon_0^2, & {}^i \mathcal{F}_{0,s}[\nabla k] &\lesssim \varepsilon_0^2, & {}^i \mathcal{F}_{0,s+2}[\nabla^2 k] &\lesssim \varepsilon_0^2, \\
{}^i \mathcal{D}_{-3,s+1}^{(\infty)}[k] &\lesssim \varepsilon_0^2, & {}^i \mathcal{D}_{-2,s+2}[\nabla k] &\lesssim \varepsilon_0^2.
\end{aligned}$$

*Proof.* We recall from Proposition 6.1

$$\begin{aligned}
\operatorname{div} k &= 0, \\
\operatorname{curl} k &= H, \\
\operatorname{tr} k &= 0.
\end{aligned}$$

Commuting with the truncation function  $f$  defined in Definition 5.36, we deduce

$$\begin{aligned}
\operatorname{div} \overset{\circ}{k} &= O(k) \nabla f, \\
\operatorname{curl} \overset{\circ}{k} &= \overset{\circ}{H} + O(k) \nabla f, \\
\operatorname{tr} \overset{\circ}{k} &= 0.
\end{aligned}$$

Applying Proposition 4.19, we infer

$$\begin{aligned}
\int_{\Sigma_t} |\nabla \overset{\circ}{k}|^2 &\lesssim \int_{\Sigma_t} |\overset{\circ}{H}|^2 + |\nabla f|^2 |k|^2 + |\text{Ric}| |\overset{\circ}{k}|^2 \\
&\lesssim \frac{\varepsilon_0^2}{t^s} + \int_{\frac{t}{2} \leq r \leq \frac{3t}{4}} \frac{\varepsilon_0^2}{t^{s+1}} + \int_{0 \leq r \leq \frac{3t}{4}} |r^{1-\frac{2}{2}} \text{Ric}|_{2,S} |r^{\frac{1}{2}-\frac{2}{4}} \overset{\circ}{k}|_{4,S}^2 \\
&\lesssim \frac{\varepsilon_0^2}{t^s} + \int_{0 \leq r \leq \frac{3t}{4}} \frac{\varepsilon^3}{t^{\frac{s+1}{2}} t^s} \\
&\lesssim \frac{\varepsilon_0^2}{t^s},
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{\Sigma_t} |\nabla^2 \overset{\circ}{k}|^2 &\lesssim \int_{\Sigma_t} |\nabla \overset{\circ}{H}|^2 + |\nabla^2 f|^2 |k|^2 + |\nabla f|^2 |\nabla k|^2 + |\text{Ric}| |\nabla k|^2 + |\text{Ric}|^2 |k|^2 \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}} + \int_{\frac{t}{2} \leq r \leq \frac{3t}{4}} \frac{\varepsilon_0^2}{t^{s+3}} + \frac{\varepsilon_0^2}{t^{s+3}} + \int_{0 \leq r \leq \frac{3t}{4}} \frac{\varepsilon^3}{t^{\frac{3s+7}{2}}} + \frac{\varepsilon^4}{t^{2s+2}} \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}},
\end{aligned}$$

which implies from Proposition 4.16 that

$${}^i \mathcal{D}_{-2,s+2}[\nabla \overset{\circ}{k}] \lesssim \varepsilon_0^2.$$

Moreover, applying Proposition 4.15, we infer

$$\int_{\Sigma_t} r^{-2} |\nabla \overset{\circ}{k}|^2 \lesssim \frac{\varepsilon_0^2}{t^{s+2}}.$$

Finally, applying Propositions 4.16 and 4.17, we obtain

$$|\overset{\circ}{k}|_{\infty, S(t,u)} \lesssim \|\overset{\circ}{k}\|_{6, \Sigma_t}^{\frac{1}{2}} \|\nabla \overset{\circ}{k}\|_{6, \Sigma_t}^{\frac{1}{2}} \lesssim \|\nabla \overset{\circ}{k}\|_{2, \Sigma_t}^{\frac{1}{2}} \|\nabla^2 \overset{\circ}{k}\|_{2, \Sigma_t}^{\frac{1}{2}} \lesssim \frac{\varepsilon_0}{t^{\frac{s+1}{2}}}.$$

Combining with Propositions 6.7–6.9, this concludes the proof of Proposition 6.15.  $\square$

**Proposition 6.16.** *We have the following estimates:*

$$\begin{aligned}
{}^i \mathcal{F}_{-4,s}[\varphi] &\lesssim \varepsilon_0^2, & {}^i \mathcal{F}_{-2,s}[\nabla \varphi] &\lesssim \varepsilon_0^2, & {}^i \mathcal{F}_{0,s}[\nabla^2 \varphi] &\lesssim \varepsilon_0^2, & {}^i \mathcal{F}_{0,s+2}[\nabla^3 \varphi] &\lesssim \varepsilon_0^2, \\
{}^i \mathcal{D}_{-3,s-1}^{(\infty)}[\varphi] &\lesssim \varepsilon_0^2, & {}^i \mathcal{D}_{-3,s+1}^{(\infty)}[\nabla \varphi] &\lesssim \varepsilon_0^2, & {}^i \mathcal{D}_{-2,s+2}[\nabla^2 \varphi] &\lesssim \varepsilon_0^2.
\end{aligned}$$

*Proof.* We recall from Lemma 6.11

$$\text{div}(\nabla \varphi) = \Gamma_w \cdot \Gamma_w, \quad \text{curl}(\nabla \varphi) = 0,$$

which implies

$$\begin{aligned}
\text{div}(f \nabla \varphi) &= f \Gamma_w \cdot \Gamma_w + \nabla f \cdot \nabla \varphi, \\
\text{curl}(f \nabla \varphi) &= \nabla f \cdot \nabla \varphi.
\end{aligned}$$

Applying Proposition 4.19 and proceeding as in Proposition 6.15, we deduce

$$\begin{aligned}
\int_{\Sigma_t} f^2 |\nabla^2 \varphi|^2 &\lesssim \int_{\Sigma_t} f^2 |\Gamma_w \cdot \Gamma_w|^2 + |\nabla f|^2 |\nabla \varphi|^2 + f^2 |\text{Ric}| |\nabla \varphi|^2 \\
&\lesssim \int_{0 \leq r \leq \frac{3t}{4}} |\Gamma_w|_{4,S}^2 |\Gamma_w|_{4,S}^2 + |\text{Ric}|_{2,S} |\nabla \varphi|_{4,S}^2 + \int_{\frac{t}{2} \leq r \leq \frac{3t}{4}} |\nabla \varphi|^2 \\
&\lesssim \frac{\varepsilon_0^2}{t^s}.
\end{aligned}$$

We also have from Proposition 4.19

$$\begin{aligned}
\int_{\Sigma_t} f^2 |\nabla^3 \varphi|^2 &\lesssim \int_{\Sigma_t} |\nabla(f\Gamma_w \cdot \Gamma_w)|^2 + |\nabla(\nabla f \cdot \nabla \varphi)|^2 + |\text{Ric}||\nabla(f\nabla \varphi)|^2 + |\text{Ric}|^2 |f\nabla \varphi|^2 \\
&\lesssim \int_{\Sigma_t} f^2 |\Gamma_w \cdot \nabla \Gamma_w|^2 + |\nabla f|^2 |\Gamma_w \cdot \Gamma_w|^2 + |\nabla^2 f|^2 |\nabla \varphi|^2 + |\nabla f|^2 |\nabla^2 \varphi|^2 \\
&\quad + \int_{\Sigma_t} |\text{Ric}||\nabla f|^2 |\nabla \varphi|^2 + f^2 |\text{Ric}||\nabla^2 \varphi|^2 + f^2 |\text{Ric}|^2 |\nabla \varphi|^2 \\
&\lesssim \int_{0 \leq r \leq \frac{3t}{4}} |\Gamma_w|_{4,S}^2 |\nabla \Gamma_w|_{4,S}^2 + |\text{Ric}|_{2,S} |\nabla^2 \varphi|_{4,S}^2 + |\text{Ric}|_{4,S}^2 |\nabla \varphi|_{4,S}^2 \\
&\quad + \int_{\frac{t}{2} \leq r \leq \frac{3t}{4}} r^{-2} |\Gamma_w|_{4,S}^2 |\Gamma_w|_{4,S}^2 + r^{-4} |\nabla \varphi|_{2,S}^2 + r^{-2} |\nabla^2 \varphi|_{2,S}^2 + r^{-2} |\text{Ric}|_{2,S} |\nabla \varphi|_{4,S}^2 \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}} + \int_{0 \leq r \leq \frac{3t}{4}} \frac{\varepsilon^4}{t^{2s+2}} \\
&\lesssim \frac{\varepsilon_0^2}{t^{s+2}}.
\end{aligned}$$

Applying Proposition 4.15, we deduce

$${}^i\mathcal{F}_{-4,s}[\overset{\circ}{\varphi}] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{F}_{-2,s}[\nabla \overset{\circ}{\varphi}] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{F}_{0,s}[\nabla^2 \overset{\circ}{\varphi}] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{F}_{0,s+2}[\nabla^3 \overset{\circ}{\varphi}] \lesssim \varepsilon_0^2.$$

Thus, we have from Proposition 4.16

$${}^i\mathcal{D}_{-4,s}[\overset{\circ}{\varphi}] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{-2,s+2}[\nabla^2 \overset{\circ}{\varphi}] \lesssim \varepsilon_0^2.$$

Finally, applying Propositions 4.16 and 4.17, we obtain

$$|\nabla \overset{\circ}{\varphi}|_{\infty, S(t,u)} \lesssim \|\nabla \overset{\circ}{\varphi}\|_{6, \Sigma_t}^{\frac{1}{2}} \|\nabla^2 \overset{\circ}{\varphi}\|_{6, \Sigma_t}^{\frac{1}{2}} \lesssim \|\nabla^2 \overset{\circ}{\varphi}\|_{2, \Sigma_t}^{\frac{1}{2}} \|\nabla^3 \overset{\circ}{\varphi}\|_{2, \Sigma_t}^{\frac{1}{2}} \lesssim \frac{\varepsilon_0}{t^{\frac{s+1}{2}}},$$

which implies

$${}^i\mathcal{D}_{-3,s}^{(\infty)}[\nabla \overset{\circ}{\varphi}] \lesssim \varepsilon_0^2.$$

Combining with Propositions 6.12–6.14, this concludes the proof of Proposition 6.16.  $\square$

## 6.6 Time derivative estimates in the interior region

**Proposition 6.17.** *We have the following estimates:*

$$\sup_{p \in [-2, s-1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \delta, \nabla_T \epsilon] \lesssim \varepsilon_0^2, \quad \sup_{p \in [-2, -1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \widehat{\kappa}] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Proposition 6.3

$$\begin{aligned}
\nabla_T \delta &= -\phi^{-1} \nabla_N \nabla_N \phi + \rho + \Gamma_g \cdot \Gamma_b, \\
\nabla_T \epsilon &= -\phi^{-1} \nabla \nabla_N \phi + \frac{1}{2}(\beta + \underline{\beta}) + \Gamma_b \cdot \Gamma_w, \\
\nabla_T \widehat{\kappa} &= -\phi^{-1} \nabla \widehat{\otimes} \nabla \phi + \frac{1}{4}(\alpha + \underline{\alpha}) + \Gamma_g \cdot \Gamma_w.
\end{aligned}$$

Combining with Proposition 6.14, we infer

$${}^i\mathcal{D}_{s-1,1}[\nabla_T \delta] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{s-1,1}[\nabla_T \epsilon] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{D}_{-1, s+1}[\nabla_T \widehat{\kappa}] \lesssim \varepsilon_0^2.$$

Moreover, we have from Proposition 6.16

$${}^i\mathcal{D}_{-2, s+2}[\nabla_T \delta, \nabla_T \epsilon, \nabla_T \widehat{\kappa}] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 6.17.  $\square$

**Proposition 6.18.** *We have the following estimate:*

$$\sup_{p \in [-2, s-2]} {}^i\mathcal{D}_{p, s-2-p}[T\phi] \lesssim \varepsilon_0^2, \quad \sup_{p \in [-2, s-1]} {}^i\mathcal{D}_{p, s-p}[\nabla_T \nabla \phi] \lesssim \varepsilon_0^2.$$

*Proof.* We first introduce the spacetime scaling operator

$$S := \bar{t}T + Z, \quad \bar{t} := r + u, \quad (6.8)$$

where  $Z = rN$  is defined in (6.2). Commuting  $S$  with (2.5), we obtain<sup>23</sup>

$$\Delta(S\phi) = 2\phi\widehat{\kappa}^{AB}(\phi\bar{t}\text{Ric}_{AB} + r\nabla_N\widehat{\kappa}_{AB}) + \Gamma_w \cdot \Gamma_w. \quad (6.9)$$

Moreover, we have from Proposition 6.2

$$\nabla_N\widehat{\kappa} = -\frac{1}{4}\underline{\alpha} + r^{-1}\Gamma_b^{(1)}.$$

Combining with Propositions 2.20 and 2.21, we have

$$\begin{aligned} 2\phi\widehat{\kappa}^{AB}(\phi\bar{t}\text{Ric}_{AB} + r\nabla_N\widehat{\kappa}_{AB}) &= \Gamma_w \cdot \left( \bar{t}E - \frac{1}{4}r\underline{\alpha} + \Gamma_b^{(1)} \right) \\ &= \Gamma_w \cdot (u\underline{\alpha}) + \Gamma_w \cdot \Gamma_b^{(1)} \\ &= \Gamma_w \cdot \Gamma_b^{(1)}, \end{aligned}$$

where we used  $u\underline{\alpha} \in \Gamma_b^{(1)}$ . Injecting it into (6.9), we infer

$$\Delta(S\phi) = \Gamma_w \cdot \Gamma_b^{(1)}.$$

Thus, we deduce that  $S\phi$  satisfies a similar equation as  $\varphi$ . Proceeding as in Propositions 6.12, 6.13 and 6.16, we obtain

$$\sup_{p \in [-3, s-4]} {}^i\mathcal{D}_{p, s-4-p}[S\phi] \lesssim \varepsilon_0^2, \quad \sup_{p \in [-2, s-2]} {}^i\mathcal{D}_{p, s-2-p}[\nabla(S\phi)] \lesssim \varepsilon_0^2.$$

Next, we recall from Propositions 6.14 and 6.16

$$\sup_{p \in [-2, s-2]} {}^i\mathcal{D}_{p, s-2-p}[\nabla\varphi] \lesssim \varepsilon_0^2, \quad \sup_{p \in [-2, s-3]} {}^i\mathcal{D}_{p, s-2-p}[\nabla\nabla_Z\varphi] \lesssim \varepsilon_0^2.$$

Combining the above estimates, we infer

$$\sup_{p \in [-2, s-4]} {}^i\mathcal{D}_{p, s-4-p}[\bar{t}T\phi] \lesssim \varepsilon_0^2, \quad \sup_{p \in [-2, s-3]} {}^i\mathcal{D}_{p, s-2-p}[\nabla(\bar{t}T\phi)] \lesssim \varepsilon_0^2.$$

Recalling that  $\bar{t} = r + u$ , this concludes the proof of Proposition 6.18.  $\square$

Combining Propositions 6.12–6.18, this concludes the proof of Theorem 3.9.

## 7 Null connection estimates (Theorem 3.10)

In this section, we apply Lemma 4.24 to prove Theorem 3.10. Throughout this section, we always assume that  $p \in [2, 4]$  and we denote

$$S := S(t, u), \quad S_2 := S(2, u), \quad S_c := \lim_{t \rightarrow t_c(u)} S(t, u).$$

<sup>23</sup>See (12.0.5f) in [10] for the precise equation and its derivation.

## 7.1 Null transport equations

The following equations will be used repeatedly in Section 7.

**Proposition 7.1.** *We have the following transport equations:*

$$\begin{aligned}\nabla_4 \Psi + \frac{3}{2} \operatorname{tr} \chi \Psi &= -\operatorname{tr} \chi \beta - \widehat{\chi} \cdot \Psi - \nabla |\widehat{\chi}|^2 + \operatorname{tr} \chi \widehat{\chi} \cdot \underline{\eta} - 2|\widehat{\chi}|^2 \eta, \\ \nabla_4 \mu + \operatorname{tr} \chi \mu &= \frac{1}{2} \operatorname{tr} \chi (-\operatorname{div} \underline{\eta} - \check{\rho} - |\underline{\eta}|^2) + \widehat{\chi} \cdot \nabla \widehat{\otimes} \eta \\ &\quad + (\eta - \underline{\eta}) \cdot \Psi - \frac{1}{4} \operatorname{tr} \chi |\widehat{\chi}|^2 + 2\eta \cdot \widehat{\chi} \cdot \underline{\eta} - 2\eta \cdot \beta.\end{aligned}$$

*Proof.* See (13.1.6c) and (13.1.11) in [10]. □

**Proposition 7.2.** *We have the following equations:*

$$\begin{aligned}\nabla_4 \eta + \frac{1}{2} \operatorname{tr} \chi \eta &= \frac{1}{2} \operatorname{tr} \chi \underline{\eta} - \widehat{\chi} \cdot (\eta - \underline{\eta}) - \beta, \\ \operatorname{div} \eta &= -\mu - \check{\rho}, \\ \operatorname{curl} \eta &= \check{\sigma}, \\ \nabla_4 \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \chi)^2 &= -2\omega \operatorname{tr} \chi - |\widehat{\chi}|^2, \\ \operatorname{div} \widehat{\chi} &= \frac{1}{2} \Psi - \beta - \zeta \cdot \widehat{\chi}.\end{aligned}$$

*Proof.* It follows directly from Proposition 2.17, (3.2) and (3.3). □

## 7.2 Estimate for $\Psi$

**Proposition 7.3.** *We have the following estimates:*

$$\mathcal{D}_s[\Psi] + {}^i\mathcal{D}_{-2,s+2}[\Psi] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Proposition 7.1

$$\nabla_4 \Psi + \frac{3}{2} \operatorname{tr} \chi \Psi = -\operatorname{tr} \chi \beta + r^{-1} \Gamma_g \cdot \Gamma_g^{(1)}.$$

Applying Lemma 4.24, we obtain for  $u \leq 1$

$$\begin{aligned}|r^{3-\frac{2}{p}} \Psi|_{p,S} &\lesssim |r^{3-\frac{2}{p}} \Psi|_{p,S_2} + \int_{\langle u \rangle}^r |r^{2-\frac{2}{p}} \beta|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-3}{2}}} + \int_{|u|}^r \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r^{s-1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}},\end{aligned}$$

which implies

$${}^e\mathcal{D}_s[\Psi] \lesssim \varepsilon_0. \tag{7.1}$$

Note that we have from (4.1)

$$\lim_{t \rightarrow t_c(u)} |r^{3-\frac{2}{p}} \Psi|_{p,S} = \lim_{t \rightarrow t_c(u)} r^{\frac{5}{2}} |r^{\frac{1}{2}-\frac{2}{p}} \Psi|_{p,S} \lesssim \lim_{r \rightarrow 0} \frac{\varepsilon r^{\frac{5}{2}}}{u^{\frac{s+2}{2}}} = 0.$$

Thus, for any  $S \subseteq {}^i_e V$ , we have from Lemma 4.24

$$\begin{aligned} |r^{3-\frac{2}{p}}\Psi|_{p,S} &\lesssim |r^{3-\frac{2}{p}}\Psi|_{p,S_c} + \int_{t_c(u)}^t |r^{2-\frac{2}{p}}\beta|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{t^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{t^{s-1}} \\ &\lesssim \frac{\varepsilon_0}{t^{\frac{s-3}{2}}}, \end{aligned}$$

which implies

$$|r^{\frac{s+3}{2}-\frac{2}{p}}\Psi|_{p,S} \lesssim \varepsilon_0^2, \quad \forall S \subseteq {}^i_e V. \quad (7.2)$$

Moreover, for any  $S \subseteq {}^i_i V$ , we have from Lemma 4.24

$$\begin{aligned} |r^{3-\frac{2}{p}}\Psi|_{p,S} &\lesssim |r^{3-\frac{2}{p}}\Psi|_{p,S_c} + \int_{t_c(u)}^t |r^{2-\frac{2}{p}}\beta|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0 r^{\frac{3}{2}}}{u^{\frac{s+2}{2}}} + \frac{\varepsilon^2 r^{\frac{3}{2}}}{u^{\frac{2s+1}{2}}} \\ &\lesssim \frac{\varepsilon_0 r^{\frac{5}{2}}}{u^{\frac{s+2}{2}}}, \end{aligned}$$

where we used  $|t_c(u) - t| \lesssim r$  for any  $S(t, u) \subseteq {}^i_i V$ . Hence, we obtain

$$|r^{\frac{1}{2}-\frac{2}{p}}\Psi|_{p,S} \lesssim \frac{\varepsilon_0}{u^{\frac{s+2}{2}}}, \quad \forall S \subseteq {}^i_i V. \quad (7.3)$$

Combining (7.2) and (7.3), we infer<sup>24</sup>

$${}^i\mathcal{D}_{2,s-2}[\Psi] + {}^i\mathcal{D}_s[\Psi] \lesssim \varepsilon_0^2.$$

Combining with (7.1), this concludes the proof of Proposition 7.3.  $\square$

**Proposition 7.4.** *We have the following estimates:*

$$|r\nabla\Psi|_{2,S} \lesssim \frac{\varepsilon_0}{t^{\frac{s+1}{2}}}, \quad \forall S \subseteq {}^i_e V, \quad |r\nabla\Psi|_{2,S} \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}, \quad \forall S \subseteq {}^e_e V.$$

We also have

$$\mathcal{E}_{s-\delta_B, \delta_B}^1[\Psi] + {}^i\mathcal{E}_{0,s}^1[\Psi] \lesssim \varepsilon_0^2, \quad \mathcal{F}_{s-\delta_B, \delta_B}^1[\Psi] + {}^i\mathcal{F}_{0,s}^1[\Psi] \lesssim \varepsilon_0^2,$$

where  $\delta_B$  is defined in (3.4).

*Proof.* We have from Propositions 4.5 and 7.1

$$\nabla_4(r\nabla\Psi) + \frac{3}{2} \operatorname{tr} \chi(r\nabla\Psi) = O(\nabla\beta) + \Gamma_g \cdot (\Psi^{(1)}, \beta^{(1)}) + r^{-1}\Gamma_g^{(1)} \cdot \Gamma_g^{(1)}.$$

Applying Lemma 4.24, we obtain for  $u \leq 1$

$$\begin{aligned} |r^3\nabla\Psi|_{2,S} &\lesssim |r^3\nabla\Psi|_{2,S_2} + \int_{\langle u \rangle}^r |r\beta^{(1)}|_{2,S} + |\Gamma_g|_{\infty,S} |r^2\Psi^{(1)}|_{2,S} + |r\Gamma_g^{(1)} \cdot \Gamma_g^{(1)}|_{2,S} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-3}{2}}} + \left( \int_{\langle u \rangle}^r r^s |\beta^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \left( \int_{\langle u \rangle}^r r^{2-s} dr \right)^{\frac{1}{2}} + \int_{\langle u \rangle}^r \frac{\varepsilon^2}{r^{\frac{s+1}{2}} r^{\frac{s-3}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}}, \end{aligned}$$

<sup>24</sup>Recall that  $u \lesssim r$  in  ${}^i_e V$  and  $r \lesssim u$  in  ${}^i_i V$ .

which implies

$$|r^{\frac{s+3}{2}} \nabla \Psi|_{2,S} \lesssim \varepsilon_0, \quad \forall S \subseteq {}^e V.$$

Similarly, we have

$$|r^{\frac{s+3}{2}} \nabla \Psi|_{2,S} \lesssim \varepsilon_0, \quad \forall S \subseteq {}^i V.$$

Next, we have from Lemma 4.24 that for any  $S \subseteq {}^i V$

$$\begin{aligned} |r^3 \nabla \Psi|_{2,S} &\lesssim |r^3 \nabla \Psi|_{2,S_c} + \int_{t_c(u)}^t |r\beta^{(1)}|_{2,S} + |\Gamma_g|_{\infty,S} |r^2 \Psi^{(1)}|_{2,S} + |r\Gamma_g^{(1)} \cdot \Gamma_g^{(1)}|_{2,S} \\ &\lesssim \left( \int_{t_c(u)}^t r^{-1} |\beta^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u)}^t r^3 dt \right)^{\frac{1}{2}} + \int_{t_c(u)}^t |\Gamma_g|_{\infty,S} |r^2 \Psi^{(1)}|_{2,S} + |r\Gamma_g^{(1)}|_{4,S} |\Gamma_g^{(1)}|_{4,S}, \\ &\lesssim r^2 \frac{\varepsilon_0}{u^{\frac{s+1}{2}}} + \int_{t_c(u)}^t \frac{\varepsilon}{u^{\frac{s+1}{2}}} \frac{\varepsilon r}{u^{\frac{s+1}{2}}} + \frac{\varepsilon^2 r^2}{u^{s+2}} \\ &\lesssim \frac{\varepsilon_0 r^2}{u^{\frac{s+1}{2}}}, \end{aligned}$$

where we used Proposition 5.39 at the third step. Hence, we obtain

$$|r \nabla \Psi|_{2,S} \lesssim \frac{\varepsilon_0}{u^{\frac{s+1}{2}}}, \quad \forall S \subseteq {}^i V.$$

Combining the above estimates, we deduce

$$|r \nabla \Psi|_{2,S} \lesssim \frac{\varepsilon_0}{t^{\frac{s+1}{2}}}, \quad \forall S \subseteq {}^i V, \quad |r \nabla \Psi|_{2,S} \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}, \quad \forall S \subseteq {}^e V. \quad (7.4)$$

Thus, we obtain for  $u \geq 1$

$$\int_{C_u^V} |r \nabla \Psi|^2 \lesssim \int_{t_c(u)}^t |r \nabla \Psi|_{2,S}^2 dt \lesssim \int_{t_c(u)}^t \frac{\varepsilon_0^2}{t^{s+1}} dt \lesssim \frac{\varepsilon_0^2}{u^s},$$

and for  $u \leq 1$

$$\int_{C_u^V} r^{s-\delta_B} |r \nabla \Psi|^2 \lesssim \int_{\langle u \rangle}^r r^{s-\delta_B} |r \nabla \Psi|_{2,S}^2 \lesssim \int_{\langle u \rangle}^r \frac{\varepsilon_0^2}{r^{1+\delta_B}} \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{\delta_B}}.$$

Hence, we deduce

$$\mathcal{E}_{s-\delta_B, \delta_B}^1[\Psi] \lesssim \varepsilon_0^2, \quad {}^i \mathcal{E}_{0,s}^1[\Psi] \lesssim \varepsilon_0^2. \quad (7.5)$$

Next, we have from (7.4) that for  $u \leq 1$

$$\int_{\Sigma_t(-\infty, u)} r^{s-\delta_B} |r \nabla \Psi|^2 \lesssim \int_{-\infty}^u \frac{\varepsilon_0^2}{r^{1+\delta_B}} du \lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{\delta_B}},$$

and similarly for  $1 \leq u_1 \leq u_2 \leq u_c(t)$

$$\int_{\Sigma_t(u_1, u_2)} r^{s-\delta_B} |r \nabla \Psi|^2 \lesssim \int_{u_1}^{u_2} \frac{\varepsilon_0^2}{u^{1+\delta_B}} du \lesssim \frac{\varepsilon_0^2}{u_1^{\delta_B}}.$$

Finally, we have from Lemma 4.24 that for  $u \geq 1$

$$\begin{aligned} |r^3 \nabla \Psi|_{2,S} &\lesssim \int_{t_c(u)}^t r |\beta^{(1)}|_{2,S} + |\Gamma_g|_{\infty,S} |r^2 \Psi^{(1)}|_{2,S} + |r\Gamma_g^{(1)} \cdot \Gamma_g^{(1)}|_{2,S} \\ &\lesssim \left( \int_{t_c(u)}^t r^{-1} |\beta^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u)}^t r^3 dt \right)^{\frac{1}{2}} + \int_{t_c(u)}^t \frac{\varepsilon^2 r^2}{t^{s+1}} \\ &\lesssim r^2 E_{-1}^1[\beta](u)^{\frac{1}{2}} + \frac{\varepsilon_0 r^2}{u^s}, \end{aligned}$$

which implies

$$|r\nabla\psi|_{2,S} \lesssim E_{-1}^1[\beta](u)^{\frac{1}{2}} + \frac{\varepsilon_0}{u^s}.$$

Hence, we obtain

$$\int_{u_1}^{u_2} |r\nabla\psi|_{2,S}^2 \lesssim B_0^1[\beta](u_1, u_2) + \frac{\varepsilon_0^2}{u_1^{2s-1}} \lesssim \frac{\varepsilon_0^2}{u_1^s}.$$

Combining the above estimates, we deduce

$$\mathcal{F}_{s-\delta_B, \delta_B}^1[\psi] + {}^i\mathcal{F}_{0,s}^1[\psi] \lesssim \varepsilon_0^2. \quad (7.6)$$

This concludes the proof of Proposition 7.4.  $\square$

### 7.3 Estimate for $\mu$

**Proposition 7.5.** *We have the following estimate:*

$$\mathcal{D}_{1,s-1}[\mu] + {}^i\mathcal{D}_{-2,s+2}[\mu] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Proposition 7.1

$$\nabla_4\mu + \text{tr}\chi\mu = r^{-1}O(\rho, \nabla\underline{\eta}) + r^{-1}\Gamma_g \cdot \Gamma_b^{(1)}.$$

Applying Lemma 4.24, we deduce for  $u \leq 1$

$$\begin{aligned} |r^{2-\frac{2}{p}}\mu|_{p,S} &\lesssim |r^{2-\frac{2}{p}}\mu|_{p,S_2} + \int_{\langle u \rangle}^r |r^{1-\frac{2}{p}}(\rho, \nabla\underline{\eta})|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}} + \int_{\langle u \rangle}^r \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{\frac{s+1}{2}}\langle u \rangle^{\frac{s-1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}}. \end{aligned}$$

Note that we have from (4.1)

$$|r^{2-\frac{2}{p}}\mu|_{p,S_c} = \lim_{t \rightarrow t_c(u)} |r^{2-\frac{2}{p}}\mu|_{p,S} \lesssim \lim_{r \rightarrow 0} \frac{\varepsilon r^{\frac{3}{2}}}{u^{\frac{s+2}{2}}} = 0.$$

Applying Lemma 4.24, we have for  $u \geq 1$

$$\begin{aligned} |r^{2-\frac{2}{p}}\mu|_{p,S} &\lesssim |r^{2-\frac{2}{p}}\mu|_{p,S_c} + \int_{t_c(u)}^t |r^{1-\frac{2}{p}}(\rho, \nabla\underline{\eta})|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{t^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{t^s} \\ &\lesssim \frac{\varepsilon_0}{u^{\frac{s-1}{2}}}. \end{aligned}$$

We also have for  $u \geq 1$

$$\begin{aligned} |r^{2-\frac{2}{p}}\mu|_{p,S} &\lesssim |r^{2-\frac{2}{p}}\mu|_{p,S_c} + \int_{t_c(u)}^t |r^{1-\frac{2}{p}}(\rho, \nabla\underline{\eta})|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0 r^{\frac{1}{2}}}{t^{\frac{s+2}{2}}} + \frac{\varepsilon^2 r^{\frac{1}{2}}}{t^{\frac{s+2}{2}}} \\ &\lesssim \frac{\varepsilon_0 r^{\frac{3}{2}}}{u^{\frac{s+2}{2}}}, \end{aligned}$$



which implies

$$|r^{\frac{1}{2}-\frac{2}{p}}\mu|_{p,S} \lesssim \frac{\varepsilon_0}{u^{\frac{s+2}{2}}}.$$

Combining the above estimates, we infer

$$\mathcal{D}_{1,s-1}[\mu] + {}^i\mathcal{D}_{-2,s+2}[\mu] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 7.5.  $\square$

**Proposition 7.6.** *We have the following estimate:*

$$\mathcal{F}_{s-\delta_B,\delta_B}^1[\mu] + {}^i\mathcal{F}_{0,s}^1[\mu] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Propositions 4.5 and 7.1

$$\nabla_4(r\nabla\mu) + \text{tr}\chi(r\nabla\mu) = O(\nabla\rho) + O(\nabla^2\eta) + r^{-1}(\Gamma_g \cdot \Gamma_b)^{(2)}.$$

Applying Lemma 4.24, we infer for  $u \leq 1$

$$|r^2\nabla\mu|_{2,S} \lesssim |r^2\nabla\mu|_{2,S_2} + \int_{\langle u \rangle}^r |X|_{2,S} \lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}} + \int_{\langle u \rangle}^r |X|_{2,S},$$

where we denoted

$$X := \left( r\nabla\rho, r\nabla^2\eta, (\Gamma_b \cdot \Gamma_g)^{(2)} \right),$$

which satisfies

$$\mathcal{F}_s[X] \lesssim \varepsilon_0^2, \quad {}^i\mathcal{F}_{-2,s+2}[X] \lesssim \varepsilon_0^2. \quad (7.7)$$

Hence, we obtain for  $u \leq 1$

$$\begin{aligned} |r^2\nabla\mu|_{2,S}^2 &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-1}} + \left( \int_{\langle u \rangle}^r |X|_{2,S} \right)^2 \\ &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-1}} + \int_{\langle u \rangle}^r r^{s-1-\delta_B} |X|_{2,S}^2 \int_{\langle u \rangle}^r r^{1-s+\delta_B} \\ &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{s-1}} + r^{2-s+\delta_B} E_{s-1-\delta_B}[X](u), \end{aligned}$$

which implies

$$\begin{aligned} \int_{-\infty}^u |r^{\frac{s+2-\delta_B}{2}}\nabla\mu|_{2,S}^2 &\lesssim \int_{-\infty}^u \frac{\varepsilon_0^2}{r^{s-2+\delta_B}\langle u \rangle^{s-1}} + B_{s-\delta_B}[X](u) \\ &\lesssim \int_{-\infty}^u \frac{\varepsilon_0^2}{\langle u \rangle^{1+\delta_B}} + \int_2^t \frac{dt}{(t+|u|)^{1+\delta_B}} \int_{\Sigma_t^Y} r^s |X|^2 \\ &\lesssim \frac{\varepsilon_0^2}{\langle u \rangle^{\delta_B}}. \end{aligned}$$

Moreover, we have from Lemma 4.24 that for  $u \geq 1$

$$|r^2\nabla\mu|_{2,S}^2 \lesssim \left( \int_{t_c(u)}^t |X|_{2,S} \right)^2 \lesssim \int_{t_c(u)}^t r^{s-1-\delta_B} |X|_{2,S}^2 \int_0^r r^{1-s+\delta_B} \lesssim r^{2-s+\delta_B} E_{s-1-\delta_B}[X],$$

which implies for  $1 \leq u_1 \leq u_2 \leq u_c(t)$ <sup>25</sup>

$$\begin{aligned}
\int_{u_1}^{u_2} |r^{\frac{s+2-\delta_B}{2}} \nabla \mu|_{2,S}^2 &\lesssim \int_{eV} r^{s-1-\delta_B} |X|^2 + \int_{iV} r^{s-1-\delta_B} |X|^2 \\
&\lesssim \int_{t_c(u_1)}^t t^{-1-\delta_B} \int_{\Sigma_t^Y \cap eV} r^s |X|^2 + \int_{t_c(u_1)}^t \int_{\Sigma_t^Y \cap iV} r^{s-1-\delta_B} |X|^2 \\
&\lesssim \int_{t_c(u_1)}^t \frac{\varepsilon_0^2}{t^{1+\delta_B}} + \int_{t_c(u_1)}^t \frac{\varepsilon_0^2}{t^{1+\delta_B}} \\
&\lesssim \frac{\varepsilon_0^2}{u_1^{\delta_B}}.
\end{aligned}$$

Thus, we obtain

$$\mathcal{F}_{s-\delta_B, \delta_B}^1[\mu] \lesssim \varepsilon_0^2.$$

Similarly, we have for  $u \geq 1$

$$|r \nabla \mu|_{2,S}^2 \lesssim \int_{t_c(u)}^t r^{-1} |X|_{2,S}^2 = E_{-1}[X](u),$$

which implies for  $1 \leq u_1 \leq u_2 \leq u_c(t)$

$$\begin{aligned}
\int_{u_1}^{u_2} |r \nabla \mu|_{2,S}^2 &\lesssim \int_{eV} r^{-1} |X|^2 + \int_{iV} r^{-1} |X|^2 \\
&\lesssim \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta_B}} \int_{\Sigma_t^Y \cap eV} r^{\delta_B} |X|^2 + \int_{t_c(u_1)}^t dt \int_{\Sigma_t^Y \cap iV} r^{-1} |X|^2 \\
&\lesssim \frac{\varepsilon_0^2}{u_1^{s-\delta_B}} \frac{1}{u_1^{\delta_B}} + \int_{t_c(u_1)}^t \frac{\varepsilon_0^2}{t^{s+1}} dt \\
&\lesssim \frac{\varepsilon_0^2}{u_1^s}.
\end{aligned}$$

Combining the above estimates, we deduce

$$\mathcal{F}_{s-\delta_B, \delta_B}^1[\mu] + {}^i\mathcal{F}_{0,s}^1[\mu] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 7.6. □

#### 7.4 Estimates for $\text{tr } \chi$ , $\widehat{\chi}$ and $\eta$

**Proposition 7.7.** *We have the following estimates:*

$$\begin{aligned}
\mathcal{D}_{-1,s-1}^1[\eta] + {}^i\mathcal{D}_{-3,s+1}^1[\eta] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\eta] + {}^i\mathcal{F}_{-2,s}^2[\eta] &\lesssim \varepsilon_0^2, \\
\mathcal{D}_{s-2}^1[\widehat{\chi}] + {}^i\mathcal{D}_{-3,s+1}^1[\widehat{\chi}] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\widehat{\chi}] + {}^i\mathcal{F}_{-2,s}^2[\widehat{\chi}] &\lesssim \varepsilon_0^2, \\
\mathcal{D}_{s-2}^1[\widetilde{\text{tr } \chi}] + {}^i\mathcal{D}_{-3,s+1}^1[\widetilde{\text{tr } \chi}] + \mathcal{F}_{s-2-\delta_B, \delta_B}^2[\widetilde{\text{tr } \chi}] + {}^i\mathcal{F}_{-2,s}^2[\widetilde{\text{tr } \chi}] &\lesssim \varepsilon_0^2, \\
\mathcal{D}_{-1,s-1}^{(\infty)} \left[ \text{tr } \chi - \frac{2}{r} \right] + {}^i\mathcal{D}_{-3,s+1}^{(\infty)} \left[ \text{tr } \chi - \frac{2}{r} \right] &\lesssim \varepsilon_0^2.
\end{aligned}$$

*Proof.* We have from Proposition 7.2

$$\text{div } \eta = -\mu - \check{\rho}, \quad \text{curl } \eta = \check{\sigma}.$$

<sup>25</sup>Recall that  $t \lesssim r$  in  $eV$  while  $t \lesssim u$  in  $iV$ .

Applying Proposition 4.12, we have

$$|r^{-\frac{2}{p}}(r\mathbb{V})^{\leq 1}\eta|_{p,S} \lesssim |r^{1-\frac{2}{p}}(\mu, \check{\rho}, \check{\sigma})|_{p,S}.$$

Combining with Proposition 7.5, we infer

$$\mathcal{D}_{-1,s-1}^1[\eta] + {}^i\mathcal{D}_{-3,s+1}^1[\eta] \lesssim \varepsilon_0^2.$$

Moreover, combining with Proposition 7.6, we obtain

$$\mathcal{F}_{s-2-\delta_B,\delta_B}^1[\eta] + {}^i\mathcal{F}_{-2,s}^1[\eta] \lesssim \varepsilon_0^2.$$

Similarly, we have from Proposition 7.2

$$\mathring{d}\mathbb{V}\widehat{\chi} = \frac{1}{2}\mathbb{V}\psi - \beta + \Gamma_g \cdot \Gamma_g.$$

Applying Proposition 4.12, we deduce

$$|r^{-\frac{2}{p}}(r\mathbb{V})^{\leq 1}\widehat{\chi}|_{p,S} \lesssim |r^{1-\frac{2}{p}}(\mathbb{V}\psi, \beta, \Gamma_g \cdot \Gamma_g)|_{p,S}.$$

Combining with Propositions 7.3 and 7.4, we infer

$$\mathcal{D}_{s-2}^1[\widehat{\chi}] + {}^i\mathcal{D}_{-3,s+1}^1[\widehat{\chi}] \lesssim \varepsilon_0^2,$$

and

$$\mathcal{F}_{s-2-\delta_B,\delta_B}^2[\widehat{\chi}] + {}^i\mathcal{F}_{-2,s}^2[\widehat{\chi}] \lesssim \varepsilon_0^2.$$

Similarly, we have from (2.14), (3.2) and Proposition 7.4

$$\begin{aligned} \mathcal{D}_{s-2}^1[\widetilde{\text{tr}}\chi] + {}^i\mathcal{D}_{-3,s+1}^1[\widetilde{\text{tr}}\chi] &\lesssim \varepsilon_0^2, \\ \mathcal{F}_{s-2-\delta_B,\delta_B}^2[\widetilde{\text{tr}}\chi] + {}^i\mathcal{F}_{-2,s}^2[\widetilde{\text{tr}}\chi] &\lesssim \varepsilon_0^2. \end{aligned} \tag{7.8}$$

Next, we recall from Proposition 7.2

$$\mathbb{V}_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = -2\omega \text{tr}\chi + \Gamma_g \cdot \Gamma_g.$$

On the other hand, we have

$$\mathbb{V}_4 \left(\frac{2}{r}\right) + \frac{1}{2} \text{tr}\chi \left(\frac{2}{r}\right) = -\frac{2e_4(r)}{r^2} + \frac{\text{tr}\chi}{r} = \frac{\widetilde{\phi \text{tr}\chi}}{\phi r}.$$

Taking the difference, we obtain

$$\mathbb{V}_4 \left(\text{tr}\chi - \frac{2}{r}\right) + \frac{1}{2} \text{tr}\chi \left(\text{tr}\chi - \frac{2}{r}\right) = r^{-1}O(\omega, \widetilde{\phi \text{tr}\chi}) + \Gamma_g \cdot \Gamma_g.$$

Applying Lemma 4.24, we infer for  $u \leq 1$

$$\begin{aligned} \left| r^{1-\frac{2}{p}} \left( \text{tr}\chi - \frac{2}{r} \right) \right|_{p,S} &\lesssim \left| r^{1-\frac{2}{p}} \left( \text{tr}\chi - \frac{2}{r} \right) \right|_{p,S_2} + \int_{\langle u \rangle}^r |r^{-\frac{2}{p}}(\omega, \widetilde{\phi \text{tr}\chi})|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}} + \int_{\langle u \rangle}^r \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^s} \\ &\lesssim \frac{\varepsilon_0}{\langle u \rangle^{\frac{s-1}{2}}}. \end{aligned}$$

Noticing that

$$\left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S_c} = \lim_{t \rightarrow t_c(u)} \left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S} \lesssim \lim_{r \rightarrow 0} \frac{\varepsilon r}{u^{\frac{s+1}{2}}} = 0,$$

we have for  $u \geq 1$

$$\begin{aligned} \left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S} &\lesssim \left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S_c} + \int_{t_c(u)}^t |r^{-\frac{2}{p}}(\omega, \overline{\phi \operatorname{tr} \chi})|_{p, S} + |r^{1-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p, S} \\ &\lesssim \frac{\varepsilon_0}{u^{\frac{s-1}{2}}}. \end{aligned}$$

We also have for  $u \geq 1$

$$\begin{aligned} \left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S} &\lesssim \left| r^{1-\frac{2}{p}} \left( \operatorname{tr} \chi - \frac{2}{r} \right) \right|_{p, S_c} + \int_{t_c(u)}^t |r^{-\frac{2}{p}}(\omega, \overline{\phi \operatorname{tr} \chi})|_{p, S} + |r^{1-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p, S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{t^{\frac{s+1}{2}}} + \frac{r\varepsilon^2}{t^{s+1}} \\ &\lesssim \frac{\varepsilon_0 r}{t^{\frac{s+1}{2}}}. \end{aligned}$$

Combining the above estimates and (7.8), we infer

$$\mathcal{D}_{-1, s-1}^{(\infty)} \left[ \operatorname{tr} \chi - \frac{2}{r} \right] + {}^i \mathcal{D}_{-3, s+1}^{(\infty)} \left[ \operatorname{tr} \chi - \frac{2}{r} \right] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 7.7.  $\square$

## 7.5 Estimate for $\nabla_T \eta$

**Proposition 7.8.** *We have the following estimate:*

$${}^i \mathcal{D}_{-1, s+1}[\nabla_T \eta] + {}^i \mathcal{D}_{-2, s+2}[\nabla_T \eta] \lesssim \varepsilon_0^2.$$

*Proof.* We have from Propositions 7.2 and 4.6

$$\nabla_4(\nabla_T \eta) + \frac{1}{2} \operatorname{tr} \chi(\nabla_T \eta) = r^{-1} O(\nabla_T \eta) - \nabla_T \beta + r^{-1} \Gamma_g \cdot \Gamma_b^{(1)}.$$

We have from (2.14) that in  ${}^i V$

$$|r^{-\frac{2}{p}} \nabla_T \eta|_{p, S} \lesssim |r^{-\frac{2}{p}} \nabla_T \epsilon|_{p, S} + |r^{-\frac{2}{p}} \nabla_T \nabla \phi|_{p, S} \lesssim \frac{\varepsilon_0}{r^{\frac{1}{2}} u^{\frac{1}{2}} t^{\frac{s+1}{2}}}.$$

Applying Lemma 4.24, we infer for  $u \geq 1$

$$\begin{aligned} |r^{1-\frac{2}{p}} \nabla_T \eta|_{p, S} &\lesssim \int_{t_c(u)}^t |r^{-\frac{2}{p}} \nabla_T \eta|_{p, S} + |r^{1-\frac{2}{p}} \nabla_T \beta|_{p, S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_b^{(1)}|_{p, S} \\ &\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{r^{\frac{1}{2}} t^{\frac{s+1}{2}} u^{\frac{1}{2}}} + \int_{t_c(u)}^t \frac{\varepsilon^2}{t^{\frac{s+3}{2}} u^{\frac{s-1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{u^{\frac{s+1}{2}}}, \end{aligned}$$

which implies

$${}^i \mathcal{D}_{-1, s+1}[\nabla_T \eta] \lesssim \varepsilon_0^2.$$

Similarly, applying Lemma 4.24, we obtain for  $u \geq 1$

$$\begin{aligned}
|r^{1-\frac{2}{p}} \nabla_T \eta|_{p,S} &\lesssim \int_{t_c(u)}^t |r^{-\frac{2}{p}} \nabla_T \eta|_{p,S} + |r^{1-\frac{2}{p}} \nabla_T \beta|_{p,S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} \\
&\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{r^{\frac{1}{2}} t^{\frac{s+1}{2}} u^{\frac{1}{2}}} + \int_{t_c(u)}^t \frac{\varepsilon^2}{t^{\frac{s+3}{2}} u^{\frac{s-1}{2}}} \\
&\lesssim \int_{t_c(u)}^t \frac{\varepsilon_0}{r^{\frac{1}{2}} u^{\frac{s+2}{2}}} \\
&\lesssim \frac{\varepsilon_0 r^{\frac{1}{2}}}{u^{\frac{s+2}{2}}},
\end{aligned}$$

which implies

$${}^i \mathcal{D}_{-2, s+2}[\nabla_T \eta] \lesssim \varepsilon_0^2.$$

This concludes the proof of Proposition 7.8.  $\square$

Combining Propositions 7.3–7.8, this concludes the proof of Theorem 3.10.

## A Proof of Theorem 5.8

We prove the following lemmas, which directly imply Theorem 5.8.

**Lemma A.1.** *Let  $p_1, p_2, p, \ell \leq s$  and let  $s > 1$ . Then, we have the following properties:*

1. *In the case  $2p < p_1 + p_2 + \ell + 1$ , we have*

$$\mathcal{N}_p^1[\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}. \quad (\text{A.1})$$

2. *In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have*

$$\mathcal{N}_p^1[\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}, \quad (\text{A.2})$$

$$\mathcal{N}_p^1[\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}. \quad (\text{A.3})$$

3. *In the case  $2p < p_1 + p_2 + \ell - 1$ , we have*

$$\mathcal{N}_p^1[\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^e V) \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}. \quad (\text{A.4})$$

*Proof.* Throughout the proof, we always denote  $V := {}^e V$  and  $0 < \delta \ll s - 1$  a constant sufficiently small.

We first assume that  $2p < p_1 + p_2 + \ell + 1$ . Then, we have

$$\begin{aligned}
\int_V r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}^{(1)}| &\lesssim \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \frac{1}{r^{\frac{p_1+p_2+\ell+1}{2}-p}} r^{\frac{p_1}{2}} |\mathbb{E}_{p_1}^{(1)}| r^{\frac{p_2}{2}} |\mathbb{E}_{p_2}^{(1)}| \\
&\lesssim \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{\frac{p_1-p}{2}}} r^{\frac{p_1}{2}} |\mathbb{E}_{p_1}^{(1)}| \frac{1}{\langle u \rangle^{\frac{p_2-p}{2}}} r^{\frac{p_2}{2}} |\mathbb{E}_{p_2}^{(1)}| \\
&\lesssim \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_1-p}} r^{p_1} |\mathbb{E}_{p_1}^{(1)}|^2 + \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_2-p}} r^{p_2} |\mathbb{E}_{p_2}^{(1)}|^2 \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_1-p}} \int_{C_u^V} r^{p_1} |\mathbb{E}_{p_1}^{(1)}|^2 + \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_2-p}} \int_{C_u^V} r^{p_2} |\mathbb{E}_{p_2}^{(1)}|^2 \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_1-p}} \frac{\varepsilon^2}{\langle u \rangle^{s-p_1}} + \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{p_2-p}} \frac{\varepsilon^2}{\langle u \rangle^{s-p_2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s+1}{2}} \langle u \rangle^{s-p}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

We also have

$$\begin{aligned}
\int_V r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}^{(1)}| &\lesssim \int_{-\infty}^u du \left( \int_{C_u^V} r^{p_1} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{2p-p_1} |\mathbb{E}_{p_2}^{(1)}|^2 |\mathbb{O}_\ell^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1} |\mathbb{E}_{p_2}|_{4,S}^2 |\mathbb{O}_\ell^{(1)}|_{4,S}^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-2} |r^{\frac{p_2+3}{2}-\frac{2}{4}} \mathbb{E}_{p_2}|_{4,S}^2 |r^{\frac{\ell+1}{2}-\frac{2}{4}} \mathbb{O}_\ell^{(1)}|_{4,S}^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-2} \frac{\varepsilon^2}{\langle u \rangle^{s-p_2}} \frac{\varepsilon^2}{\langle u \rangle^{s-\ell}} dt \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s-p_1}{2}}} \frac{1}{\langle u \rangle^{s-\frac{p_2+\ell}{2}}} \left( \langle u \rangle^{2p-p_1-p_2-\ell-1+\delta} \int_2^t r^{-1-\delta} dt \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s-p_1}{2}}} \frac{1}{\langle u \rangle^{s-\frac{p_2+\ell}{2}}} \langle u \rangle^{p-\frac{p_1+p_2+\ell+1-\delta}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{s-p-\delta}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

Hence, we obtain (A.1).

Next, we assume that  $2p < p_1 + p_2 + \ell$ . Then, we have

$$\begin{aligned}
\int_V r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{F}_{p_2}| &\lesssim \int_{-\infty}^u du \left( \int_{C_u^V} r^p |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{2p-p_1} |\mathbb{F}_{p_2}|^2 |\mathbb{O}_\ell^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1} |\mathbb{F}_{p_2}|_{4,S}^2 |\mathbb{O}_\ell^{(1)}|_{4,S}^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-1} |r^{\frac{p_2+2}{2}-\frac{2}{4}} \mathbb{F}_{p_2}|_{4,S}^2 |r^{\frac{\ell+1}{2}-\frac{2}{4}} \mathbb{O}_\ell^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-1} \frac{\varepsilon^2}{\langle u \rangle^{s-p_2+1}} \frac{\varepsilon^2}{\langle u \rangle^{s-\ell}} \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s-p_1}{2}}} \frac{1}{\langle u \rangle^{s-\frac{p_2}{2}+1-\frac{\ell+1}{2}}} \left( \langle u \rangle^{2p-p_1-p_2-\ell-1+\delta} \int_2^t r^{-1-\delta} dt \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon^3}{\langle u \rangle^{\frac{s+1}{2}}} \frac{1}{\langle u \rangle^{s-p-\delta}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

We also have

$$\begin{aligned}
\int_V r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell \cdot \mathbb{F}_{p_2}^{(1)}| &\lesssim \int_V \frac{\varepsilon}{r^{\frac{\ell+1}{2}-p} \langle u \rangle^{\frac{s-\ell}{2}}} |\mathbb{E}_{p_1}^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \int_V \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell-1-\delta}{2}}} \frac{1}{r^{\frac{p_1+p_2+\ell-\delta}{2}-p}} r^{\frac{p_1}{2}} \langle u \rangle^{-\frac{1+\delta}{2}} |\mathbb{E}_{p_1}^{(1)}| r^{\frac{p_2}{2}} r^{-\frac{1+\delta}{2}} |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{p_1+p_2+s-1}{2}-p-\delta}} \int_V r^{\frac{p_1}{2}} \langle u \rangle^{-\frac{1+\delta}{2}} |\mathbb{E}_{p_1}^{(1)}| r^{\frac{p_2}{2}} r^{-\frac{1+\delta}{2}} |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{p_1+p_2+s-1}{2}-p-\delta}} \left( \int_V r^{p_1} \langle u \rangle^{-1-\delta} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_V r^{p_2} r^{-1-\delta} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{p_1+p_2+s-1}{2}-p-\delta}} \left( \int_{-\infty}^u \langle u \rangle^{-1-\delta} \int_{C_u^V} r^{p_1} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_2^t t^{-1-\delta} \int_{\Sigma_t^V} r^{p_2} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{p_1+p_2+s-1}{2}-p-\delta}} \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_2}{2}}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}},
\end{aligned}$$

where we used  $p_1 + p_2 + s - 1 > 2p$ . Hence, we obtain (A.2).

Proceeding as above, we deduce for  $p_1 + p_2 + \ell > 2p$  and  $p_1 + p_2 + s - 1 > 2p$

$$\int_V r^p |\mathbb{F}_{p_1}^{(1)} \cdot \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}^{(1)}| \lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.$$

We also have

$$\begin{aligned}
\int_V r^p |\mathbb{E}_{p_2}^{(1)}| |\mathbb{O}_\ell^{(1)} \cdot \mathbb{F}_{p_1}| &\lesssim \int_{-\infty}^u du \left( \int_{C_u^V} r^{p_2} |\mathbb{E}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{2p-p_2} |\mathbb{F}_{p_1}|^2 |\mathbb{O}_\ell^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_2}{2}}} \left( \int_2^t r^{2p-p_2} |\mathbb{F}_{p_1}|_{4,S}^2 |\mathbb{O}_\ell^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_2}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-1} |r^{\frac{p_1+2}{2}-\frac{2}{4}} \mathbb{F}_{p_1}|_{4,S}^2 |r^{\frac{\ell+1}{2}-\frac{2}{4}} \mathbb{O}_\ell^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_2}{2}}} \left( \int_2^t r^{2p-p_1-p_2-\ell-1} \frac{\varepsilon^2}{\langle u \rangle^{s-p_1+1}} \frac{\varepsilon^2}{\langle u \rangle^{s-\ell}} \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{-\infty}^u du \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_2}{2}}} \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1+1}{2}}} \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \frac{1}{\langle u \rangle^{\frac{p_1+p_2+\ell}{2}-p}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}} \int_{-\infty}^u \frac{1}{\langle u \rangle^{\frac{s+1}{2}}} du \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

Hence, we obtain (A.3).

Finally, we assume that  $2p+1 < p_1+p_2+\ell$ . Then, we have

$$\begin{aligned}
\int_V r^k |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| \cdot \mathbb{F}_{p_2} &\lesssim \int_2^t dt \left( \int_{\Sigma_t^V} r^{p_1} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\Sigma_t^V} r^{2p-p_1} |\mathbb{F}_{p_2}|^2 |\mathbb{O}_\ell^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_2^t dt \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_{-\infty}^u r^{2p-p_1} |\mathbb{F}_{p_2}|_{4,S}^2 |\mathbb{O}_\ell^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_2^t dt \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} \left( \int_{-\infty}^u r^{2p-p_1-p_2-\ell-1} |r^{\frac{p_2+2}{2}-\frac{2}{4}} \mathbb{F}_{p_2}|_{4,S}^2 |r^{\frac{\ell+1}{2}-\frac{2}{4}} \mathbb{O}_\ell^{(1)}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_2^t dt \frac{\varepsilon}{\langle u \rangle^{\frac{s-p_1}{2}}} r^{p-\frac{p_1+p_2+\ell+1}{2}} \left( \int_{-\infty}^u \frac{\varepsilon^2}{\langle u \rangle^{s-p_2+1}} \frac{\varepsilon^2}{\langle u \rangle^{s-\ell}} \right)^{\frac{1}{2}} \\
&\lesssim \int_2^t dt \frac{\varepsilon^3}{\langle u \rangle^{\frac{s-p_1}{2}}} r^{p-\frac{p_1+p_2+\ell+1-\delta}{2}} \left( \frac{1}{\langle u \rangle^{2s-p_2-\ell+\delta}} \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{\frac{s-p_1}{2}}} \langle u \rangle^{p-\frac{p_1+p_2-\delta}{2}} \frac{1}{\langle u \rangle^{s-\frac{p_2+1}{2}}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

We also have

$$\begin{aligned}
\int_V r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{F}_{p_2}^{(1)}| &\lesssim \int_2^t dt \int_{\Sigma_t^V} r^{p-\frac{p_1+p_2}{2}} \frac{\varepsilon}{r^{\frac{\ell+1}{2}} \langle u \rangle^{\frac{s-\ell}{2}}} r^{\frac{p_1}{2}} |\mathbb{F}_{p_1}^{(1)}| r^{\frac{p_2}{2}} |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \int_2^t r^{p-\frac{p_1+p_2+\ell+1}{2}} dt \int_{\Sigma_t^V} r^{\frac{p_1}{2}} |\mathbb{F}_{p_1}^{(1)}| r^{\frac{p_2}{2}} |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \int_2^t r^{p-\frac{p_1+p_2+\ell+1}{2}} dt \left( \int_{\Sigma_t^V} r^{p_1} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\Sigma_t^V} r^{p_2} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \int_2^t r^{p-\frac{p_1+p_2+\ell+1}{2}} dt \frac{\varepsilon^2}{\langle u \rangle^{\frac{s-p_1}{2}} \langle u \rangle^{\frac{s-p_2}{2}}} \\
&\lesssim \frac{\varepsilon}{\langle u \rangle^{\frac{s-\ell}{2}}} \langle u \rangle^{p-\frac{p_1+p_2+\ell-1-\delta}{2}} \frac{\varepsilon^2}{\langle u \rangle^{\frac{s-p_1}{2}} \langle u \rangle^{\frac{s-p_2}{2}}} \\
&\lesssim \frac{\varepsilon^3}{\langle u \rangle^{s-p}}.
\end{aligned}$$

Hence, we obtain (A.4). This concludes the proof of Lemma A.1.  $\square$

**Lemma A.2.** *Let  $p_1, p_2, p, \ell \leq s$  and  $s > 1$ . Then, we have the following properties.*



1. In the case  $2p < p_1 + p_2 + \ell + 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.5})$$

2. In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.6})$$

3. In the case  $2p < p_1 + p_2 + \ell$  and  $2p + 1 < p_1 + p_2 + s$ , we have

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.7})$$

4. In the case  $2p < p_1 + p_2 + \ell - 1$ , we have

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.8})$$

*Proof.* The proof follows directly by replacing  $\int_{-\infty}^u du$  with  $\int_{u_1}^{u_2} du$  and  $\int_2^t dt$  with  $\int_{t_c(u)}^t dt$  in the proof of Lemma A.1.  $\square$

**Lemma A.3.** *Let  $0 \leq p_1, p_2, p \leq s$ ,  $\ell \leq s$  and let  $s > 1$ . Then, we have the following properties.*

1. We have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.9})$$

2. In the case  $2p + 1 < p_1 + p_2 + s$ , we have

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}, \quad (\text{A.10})$$

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{E}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}, \quad (\text{A.11})$$

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}, \mathbb{O}_\ell \cdot \mathbb{F}_{p_2}]({}^i V) \lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \quad (\text{A.12})$$

*Proof.* Throughout the proof, we always denote  $V := {}^i V$  and  $0 < \delta \ll s - 1$  a constant small enough. Note that we have

$$\mathbb{E}_{p_1} \subseteq \mathbb{E}_{p_2}, \quad \mathbb{F}_{p_1} \subseteq \mathbb{F}_{p_2} \quad \forall p_1 \geq p_2 \geq 0.$$

We also recall that

$$r \lesssim u, \quad t \quad \text{in } {}^i V.$$

We first have

$$\begin{aligned} \int_{{}^i V} r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{E}_{p_2}^{(1)}| &\lesssim \int_{u_1}^{u_2} \frac{\varepsilon}{u^{\frac{s-2p+1}{2}}} du \left( \int_{C_u^V} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\mathbb{E}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\ &\lesssim \int_{u_1}^{u_2} \frac{\varepsilon^3}{u^{\frac{3s-2p+1}{2}}} du \\ &\lesssim \frac{\varepsilon^3}{u_1^{s-p}}. \end{aligned}$$

We also have

$$\begin{aligned}
\int_{i_V} r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| &\lesssim \int_{u_1}^{u_2} u^p du \int_{C_u^V} |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| \\
&\lesssim \int_{u_1}^{u_2} u^p du \left( \int_{C_u^V} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\mathbb{O}_\ell^{(1)}|^2 |\mathbb{E}_{p_2}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{u_1}^{u_2} u^p du \frac{\varepsilon}{u^{\frac{s}{2}}} \left( \int_{t_c(u)}^t dt |\mathbb{O}_\ell^{(1)}|_{4,S}^2 |\mathbb{E}_{p_2}|_{4,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim \int_{u_1}^{u_2} du \frac{\varepsilon}{u^{\frac{s-2p}{2}}} \left( \int_{t_c(u)}^t dt \frac{\varepsilon^2}{u^s} \frac{\varepsilon^2}{u^{s+2}} \right)^{\frac{1}{2}} \\
&\lesssim \int_{u_1}^{u_2} du \frac{\varepsilon}{u^{\frac{s-2p}{2}}} \frac{\varepsilon^2}{u^{\frac{2s+1}{2}}} \\
&\lesssim \frac{\varepsilon^3}{u_1^{s-p}}.
\end{aligned} \tag{A.13}$$

Thus, we obtain (A.9).

In the rest of the proof, we always assume that

$$2p + 1 < p_1 + p_2 + s. \tag{A.14}$$

We have for  $p < \frac{s-1}{2}$

$$\begin{aligned}
\int_{i_V} r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{F}_{p_2}^{(1)}| &\lesssim \varepsilon \int_{i_V} \frac{1}{u^{\frac{s+1}{2}-p}} |\mathbb{E}_{p_1}^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \varepsilon \left( \int_{u_1}^{u_2} \frac{du}{u^{\frac{s+1}{2}-p}} \int_{C_u^V} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \int_{\Sigma_t^V} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon \left( \int_{u_1}^{u_2} \frac{du}{u^{\frac{s+1}{2}-p}} \frac{\varepsilon^2}{u^s} \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \frac{\varepsilon^2}{u_1^s} \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon^3}{u_1^s} \frac{1}{u_1^{\frac{s-1}{2}-p}} \\
&\lesssim \frac{\varepsilon^3}{u_1^{s-p}},
\end{aligned}$$

and for  $p \geq \frac{s-1}{2}$  and  $0 < \delta \ll s - 1$  sufficiently small

$$\begin{aligned}
&\int_{i_V} r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \varepsilon \int_{i_V} \frac{1}{u^{1+\delta}} r^{p+\delta-\frac{s-1}{2}} |\mathbb{E}_{p_1}^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \varepsilon \int_{i_V} \frac{1}{u^{1+\delta}} r^{(p+\delta-\frac{s-1}{2})\frac{p_1}{p_1+p_2}} |\mathbb{E}_{p_1}^{(1)}| r^{(p+\delta-\frac{s-1}{2})\frac{p_2}{p_1+p_2}} |\mathbb{F}_{p_2}^{(1)}| \\
&\lesssim \varepsilon \left( \int_{u_1}^{u_2} \frac{du}{u^{1+\delta}} \int_{C_u^V} r^{(p+\delta-\frac{s-1}{2})\frac{2p_1}{p_1+p_2}} |\mathbb{E}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta}} \int_{\Sigma_t^V} r^{(p+\delta-\frac{s-1}{2})\frac{2p_2}{p_1+p_2}} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{\varepsilon^3}{u_1^{s-(p+\delta-\frac{s-1}{2})}} \left( \int_{u_1}^{u_2} \frac{du}{u^{1+\delta}} \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta}} \right)^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \frac{\varepsilon^3}{u_1^{s-p}},$$

where we used

$$\left(p + \delta - \frac{s-1}{2}\right) \frac{2p_1}{p_1 + p_2} < p_1, \quad \left(p + \delta - \frac{s-1}{2}\right) \frac{2p_2}{p_1 + p_2} < p_2,$$

which follows from (A.14) and the smallness of  $\delta$ . Proceeding as in (A.13), we have

$$\int_{iV} r^p |\mathbb{E}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| \lesssim \frac{\varepsilon^3}{u_1^{s-p}}.$$

Thus, we obtain (A.10).

Next, we have for  $p \leq \frac{p_1}{2}$  and  $0 < \delta \ll s-1$  sufficiently small

$$\begin{aligned} \int_{iV} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| &\lesssim \int_{iV} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| \\ &\lesssim \left( \int_{iV} \frac{r^{2p}}{u^{1+\delta}} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{iV} u^{1+\delta} |\mathbb{O}_\ell^{(1)}|^2 |\mathbb{E}_{p_2}|^2 \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta}} \int_{\Sigma_t^V} r^{2p} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{u_1}^{u_2} du \int_{t_c(u)}^t dt u^{1+\delta} |\mathbb{O}_\ell^{(1)}|_{4,S}^2 |\mathbb{E}_{p_2}|_{4,S}^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon}{u_1^{\frac{s-2p}{2}}} \left( \int_{u_1}^{u_2} du \int_{t_c(u)}^t dt \frac{\varepsilon^4}{u^{2s+1-\delta}} \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon^3}{u_1^{s-p}}, \end{aligned}$$

and also for  $p \geq \frac{p_1}{2}$

$$\begin{aligned} \int_{iV} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| &\lesssim \int_{iV} r^{\frac{p_1}{2}} |\mathbb{F}_{p_1}^{(1)}| r^{p-\frac{p_1}{2}} |\mathbb{O}_\ell^{(1)}| |\mathbb{E}_{p_2}| \\ &\lesssim \left( \int_{iV} u^{-1-\delta} r^{p_1} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{iV} u^{2p-p_1+1+\delta} |\mathbb{O}_\ell^{(1)}|^2 |\mathbb{E}_{p_2}|^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon}{u_1^{\frac{s-p_1}{2}}} \left( \int_{u_1}^{u_2} du u^{2p-p_1+1+\delta} \int_{t_c(u)}^t dt |\mathbb{O}_\ell^{(1)}|_{4,S}^2 |\mathbb{E}_{p_2}|_{4,S}^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon}{u_1^{\frac{s-p_1}{2}}} \left( \int_{u_1}^{u_2} du u^{2p-p_1+1+\delta} \int_{t_c(u)}^t dt \frac{\varepsilon^4}{t^{2s+2}} \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon^3}{u_1^{\frac{s-p_1}{2}}} \left( \int_{u_1}^{u_2} du \frac{1}{u^{2s-2p+p_1-\delta}} \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon^3}{u_1^{s-p}}, \end{aligned} \tag{A.15}$$

where we used  $2p+1 < p_1+2s$ .<sup>26</sup> Thus, we obtain (A.11).

<sup>26</sup>Recalling that  $p_2 \leq s$  and  $p_1 + p_2 + s > 2p+1$ , we deduce that  $2p+1 < p_1+2s$ .

Finally, we have for  $p < \frac{s-1}{2}$

$$\begin{aligned}
 \int_{iV} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{F}_{p_2}^{(1)}| &\lesssim \varepsilon \int_{iV} \frac{1}{u^{\frac{s+1}{2}-p}} |\mathbb{F}_{p_1}^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \\
 &\lesssim \varepsilon \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \int_{\Sigma_t^V} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \int_{\Sigma_t^V} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
 &\lesssim \varepsilon \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \frac{\varepsilon^2}{u_1^s} \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{\frac{s+1}{2}-p}} \frac{\varepsilon^2}{u_1^s} \right)^{\frac{1}{2}} \\
 &\lesssim \frac{\varepsilon^3}{u_1^{s-p}},
 \end{aligned}$$

and also for  $p \geq \frac{s-1}{2}$  and  $0 < \delta \ll s-1$  sufficiently small

$$\begin{aligned}
 &\int_{iV} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell| |\mathbb{F}_{p_2}^{(1)}| \\
 &\lesssim \int_{iV} \frac{\varepsilon}{u^{1+\delta}} r^{p+\delta-\frac{s-1}{2}} |\mathbb{F}_{p_1}^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \\
 &\lesssim \varepsilon \left( \int_{iV} u^{-1-\delta} r^{(p+\delta-\frac{s-1}{2})\frac{2p_1}{p_1+p_2}} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{iV} u^{-1-\delta} r^{(p+\delta-\frac{s-1}{2})\frac{2p_2}{p_1+p_2}} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
 &\lesssim \varepsilon \left( \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta}} \int_{\Sigma_t^V} r^{(p+\delta-\frac{s-1}{2})\frac{2p_1}{p_1+p_2}} |\mathbb{F}_{p_1}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{t_c(u_1)}^t \frac{dt}{t^{1+\delta}} \int_{\Sigma_t^V} r^{(p+\delta-\frac{s-1}{2})\frac{2p_2}{p_1+p_2}} |\mathbb{F}_{p_2}^{(1)}|^2 \right)^{\frac{1}{2}} \\
 &\lesssim \frac{\varepsilon^3}{u_1^{s-p}},
 \end{aligned}$$

where we used  $2p+1 < p_1+p_2+s$ . Proceeding as in (A.15), we have

$$\int_{V_i} r^p |\mathbb{F}_{p_1}^{(1)}| |\mathbb{O}_\ell^{(1)}| |\mathbb{F}_{p_2}^{(1)}| \lesssim \frac{\varepsilon^3}{u_1^{s-p}}.$$

Thus, we obtain (A.12). This concludes the proof of Lemma A.3. □

Combining Lemmas A.1–A.3, this concludes the proof of Theorem 5.8.<sup>27</sup>

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<sup>27</sup>Note that the assumptions in Lemma A.3 are weaker than those of Lemma A.2.



## Chapter 4

# Kerr stability in external regions

## 1 Introduction

### 1.1 Einstein vacuum equations and the Cauchy problem

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \quad \text{in } \mathcal{M}, \quad (1.1)$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a nonlinear geometric coupled system of second order partial differential equations for  $\mathbf{g}$ . In suitable coordinates, for example so-called wave coordinates, it can be shown that (1.1) is hyperbolic and hence admits an initial value formulation.

The corresponding initial data for the Einstein vacuum equations is given by specifying a triplet  $(\Sigma, g, k)$  where  $(\Sigma, g)$  is a Riemannian 3-dimension manifold and  $k$  is a symmetric 2-tensor on  $\Sigma$  satisfying the constraint equations:

$$\begin{aligned} R &= |k|^2 - (\operatorname{tr} k)^2, \\ D^j k_{ij} &= D_i(\operatorname{tr} k), \end{aligned} \quad (1.2)$$

where  $R$  denotes the scalar curvature of  $g$ ,  $D$  denotes the Levi-Civita connection of  $g$  and

$$|k|^2 := g^{ad}g^{bc}k_{ab}k_{cd}, \quad \operatorname{tr} k := g^{ij}k_{ij}.$$

In the future development  $(\mathcal{M}, \mathbf{g})$  of such initial data  $(\Sigma, g, k)$ ,  $\Sigma \subset \mathcal{M}$  is a spacelike hypersurface with induced metric  $g$  and second fundamental form  $k$ .

#### 1.1.1 Local existence theorem

The seminal well-posedness results for the Cauchy problem obtained in [8, 9] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  solution of Einstein equations (1.1) such that  $\Sigma \subset \mathcal{M}$  and  $g, k$  are respectively the first and second fundamental forms of  $\Sigma$  in  $\mathcal{M}$ .

The simplest example of a vacuum spacetime is Minkowski spacetime:

$$\mathcal{M} = \mathbb{R}^4, \quad \mathbf{g} = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \quad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad k = 0.$$

#### 1.1.2 Special solutions

Minkowski spacetime is the simplest solution of (1.1). We recall two other relevant solutions of (1.1):

- Schwarzschild spacetime  
(1.1) admits a remarkable family of explicit, stationary, solutions given by the one parameter family of Schwarzschild solutions of mass  $M > 0$

$$\mathbf{g}_M = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \quad (1.3)$$

Though the metric seems singular at  $r = 2M$ , it turns out that one can glue together two regions  $r > 2M$  and  $r < 2M$  of the Schwarzschild metric to obtain a metric which

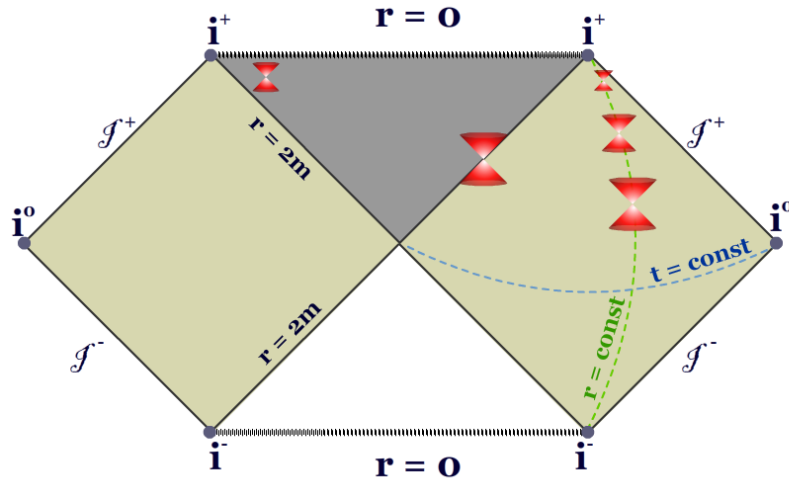


Figure 4.1: Complete Penrose diagram of Schwarzschild spacetime

is smooth along  $\mathcal{H} = \{r = 2M\}$ , called the Schwarzschild *event horizon*. The portion of  $r < 2M$  to the future of the hypersurface  $t = 0$  is a *black hole* whose future boundary  $r = 0$  is singular. The similar region to the past of  $t = 0$  is called a *white hole*. The region  $r > 2M$ , called the *domain of outer communication* (DOC), is free of singularities. The Penrose diagram of Schwarzschild space is recalled in Figure 4.1.

- Kerr spacetime

The Schwarzschild family is included in a larger two parameter family of solutions  $K(a, M)$  discovered by Kerr in [26]. A given Kerr spacetime, with  $0 \leq |a| \leq M$  has a well defined domain of outer communication  $r > r_+ := M + (M^2 - a^2)^{\frac{1}{2}}$ . In Boyer-Lindquist coordinates, well adapted to  $r > r_+$ , the Kerr metric has the form:

$$\mathbf{g}_{a,M} = -\frac{(\Delta - a^2 \sin^2 \theta)}{q^2} dt^2 - \frac{4aMr}{q^2} \sin^2 \theta dt d\varphi + \frac{q^2}{\Delta} dr^2 + q^2 d\theta^2 + \frac{\Sigma^2}{q^2} \sin^2 \theta d\varphi^2, \quad (1.4)$$

with

$$q^2 = r^2 + a^2 \cos 2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2 \Delta.$$

See Figure 4.2 for the Penrose diagram of Kerr spacetime.

## 1.2 Stability of Kerr conjecture

**Conjecture 1.1** (Stability of Kerr conjecture). *Vacuum initial data sets, sufficiently close to Kerr initial data, have a maximal development with complete future null infinity<sup>1</sup> and with domain of outer communication which approaches (globally) a nearby Kerr solution.*

The stability of Kerr has been proved in the case of small angular momentum,  $|a| \ll M$ , in the sequence of works by Klainerman-Szeftel [34], [35], [36], Giorgi-Klainerman-Szeftel [18] and the author [52], but the conjecture is open in general. For a complete introduction to this problem, we refer the reader to [37] and to the introduction in [36] and [18].

<sup>1</sup>This means, roughly, that observers which are far away from the black hole may live forever.



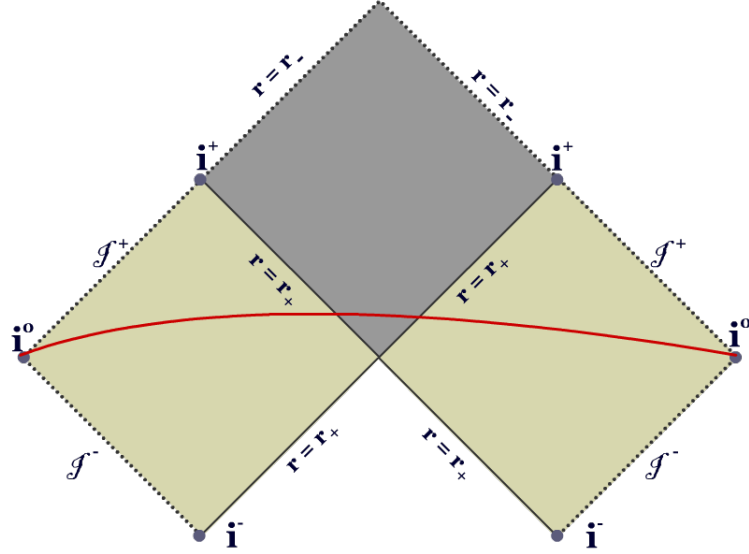


Figure 4.2: Penrose diagram of Kerr for  $0 < |a| < M$ . The surface  $r = r_+$ , the larger root of  $\Delta = 0$ , is the event horizon of the black hole,  $r > r_+$  the domain of outer communication,  $\mathcal{I}^+$  is the future null infinity, corresponding to  $r = +\infty$ .

In order to attack the conjecture, it is convenient to divide the domain of outer communication to the future of an initial spacelike hypersurface  $\Sigma_0$  into three causal regions as in Figure 4.3:

- Region I lies to the future of  $\Sigma_0$  and to the past of an ingoing null cone  $\underline{C}_0$ . The stability of Kerr in spacetime region I is a direct consequence of the local existence results in [8, 9].
- Region II lies to the future of  $\Sigma_0$  and to the past of an outgoing null cone  $C_0$  sufficiently far away from the event horizon. See Section 1.4 for a discussion of Kerr stability in this region. The goal of this paper is to provide a new proof of the stability of Kerr in Region II.
- Region III lies to the future of the ingoing null cone  $\underline{C}_0$  and of the outgoing null cone  $C_0$ . The stability of Kerr in region III for small angular momentum  $|a| \ll M$  was proved recently in the sequence of works [34], [35], [36], [18] and [52].

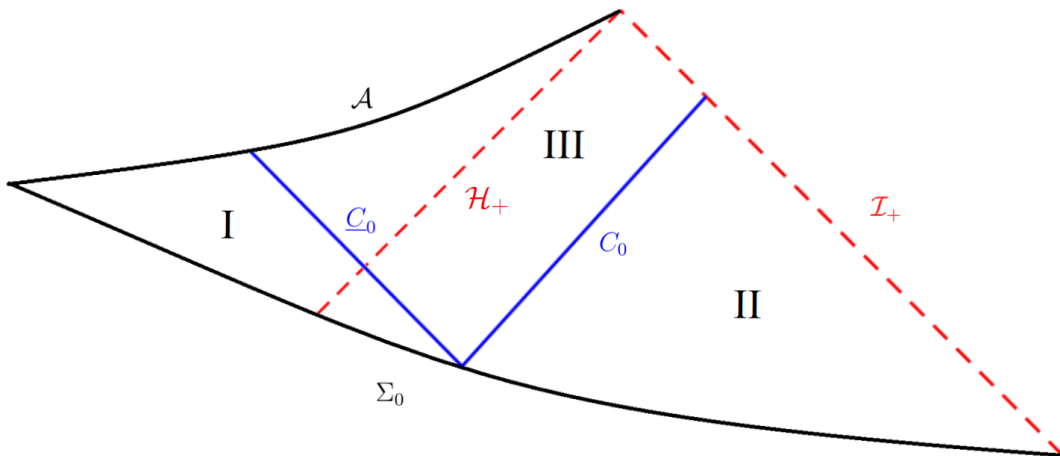


Figure 4.3: The Penrose diagram of a perturbation of Kerr

As mentioned above, the goal of this paper is to provide a new proof of the stability of Kerr in Region II. In the next section, we review the corresponding problem for Minkowski.

### 1.3 Minkowski stability in [29]

In 1993, Christodoulou and Klainerman [10] proved the stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2003, Klainerman and Nicolò [29] gave a second proof of this result in the exterior of an outgoing cone. Moreover, Klainerman and Nicolò [30] showed that under stronger asymptotic decay and regularity properties than those used in [10, 29], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. For more references on the stability of Minkowski for Einstein vacuum equations or Einstein equations coupled to matter, see for example the introduction of [53].

In this section, we present the main result of [29], which is the motivation for the analog results in Kerr recalled in Section 1.4 and for our main result stated in Section 1.5. First, we recall the definition of a *maximal hypersurface*, which plays an important role in the statements of the main theorems in [29].

**Definition 1.2.** *An initial data  $(\Sigma, g, k)$  is posed on a maximal hypersurface if it satisfies*

$$\operatorname{tr} k = 0. \tag{1.5}$$

*In this case, we say that  $(\Sigma, g, k)$  is a maximal initial data set, and the constraint equations (1.2) reduce to*

$$R = |k|^2, \quad \operatorname{div} k = 0, \quad \operatorname{tr} k = 0. \tag{1.6}$$

We introduce the notion of *s-asymptotically flat initial data*.

**Definition 1.3.** *Let  $s$  be a real number with  $s > 3$ . We say that a data set  $(\Sigma_0, g, k)$  is s-asymptotically flat if there exists a coordinates system  $(x^1, x^2, x^3)$  defined outside a sufficiently large compact set such that<sup>2</sup>*

$$\begin{aligned} g_{ij} &= \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_4(r^{-\frac{s-1}{2}}), \\ k_{ij} &= o_3(r^{-\frac{s+1}{2}}). \end{aligned} \tag{1.7}$$

We also introduce the following functional associated to any asymptotically flat initial data set:

$$J_0(\Sigma_0, g, k) := \sup_{\Sigma_0} \left( (d_0^2 + 1)^3 |\operatorname{Ric}|^2 \right) + \int_{\Sigma_0} \sum_{l=0}^3 (d_0^2 + 1)^{l+1} |D^l k|^2 + \int_{\Sigma_0} \sum_{l=0}^1 (d_0^2 + 1)^{l+3} |D^l B|^2, \tag{1.8}$$

where  $d_0$  is the geodesic distance from a fixed point  $O \in \Sigma_0$ , and  $B_{ij} := (\operatorname{curl} \widehat{R})_{ij}$  is the *Bach tensor*,  $\widehat{R}$  is the traceless part of  $\operatorname{Ric}$ . Now, we can state the main theorem of [29].

**Theorem 1.4** (Klainerman-Nicolò [29]). *Consider an initial data set  $(\Sigma_0, g, k)$ , 4-asymptotically flat in the sense of Definition 1.3 and maximal, and assume that  $J_0(\Sigma_0, g, k)$  is bounded. Then, given a sufficiently large compact set  $K \subset \Sigma_0$  such that  $\Sigma_0 \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus \overline{B}_1$ , and assuming that  $J_0(\Sigma_0, g, k)$  is small enough, there exists a unique development  $(\mathcal{M}, \mathbf{g})$  with the following properties:*

1.  $(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $\{C_u\}$  and  $\{\underline{C}_u\}$  whose outgoing leaves  $C_u$  are complete.

---

<sup>2</sup>The notation  $f = o_l(r^{-m})$  means  $\partial^\alpha f = o(r^{-m-|\alpha|})$ ,  $|\alpha| \leq l$ .

2. We have detailed control of all the quantities associated with the double null foliation of the spacetime, see Theorem 3.7.1 of [29].

**Remark 1.5.** [30] and [53] extend Theorem 1.4 to the case of  $s$ -asymptotically flat initial data with  $s > 7$  and  $s > 3$  respectively. See also [20] for a proof in the case  $3 < s < 5$  using wave coordinates, constraint damping, spacetime compactification,  $b$ -Sobolev spaces and Nash-Moser iteration.

#### 1.4 Previous works of Kerr stability in external regions

In 2009, Nicolò [49] proved that the asymptotic behaviour of the Riemann components is in agreement with the *Peeling theorem* if the corrections to the Kerr initial data decay sufficiently fast. In 2010, Caciotta and Nicolò [7] proved the nonlinear stability of the Kerr spacetime for any  $|a| \leq M$  and appropriate class of initial data close to Kerr, in an external region where  $r \geq R_0$  and  $\frac{M}{R_0} \ll 1$ . Their main results can be stated as follows.

**Theorem 1.6** (Caciotta-Nicolò [7]). *Assume that the initial data are given on  $\Sigma_0$  outside of a ball centered on the origin of radius  $R_0$ , denoted  $K := B_{R_0}$ . They are perturbations of the initial data of a Kerr spacetime with mass  $M$  and angular momentum  $a$  satisfying*

$$\frac{M}{R_0} \ll 1, \quad |a| \leq M.$$

Let  $s > 7$ , and assume that the initial data satisfies for  $q \geq 4$ :

$$\begin{aligned} g_{ij} &= (g_{Kerr})_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \\ k_{ij} &= (k_{Kerr})_{ij} + o_q(r^{-\frac{s+1}{2}}). \end{aligned} \tag{1.9}$$

Then, the initial data set  $(\Sigma_0, g, k)$  has a unique development  $(\mathcal{M}, \mathbf{g})$  defined outside  $I^+(K)$ , the domain of influence of  $K$ . Moreover,  $\mathcal{M}$  can be foliated by a canonical double null foliation  $(u, \underline{u})$  whose outgoing leaves  $C_u$  are complete and suitable norms of Ricci coefficients and curvature components are sufficiently small.

The goal of this paper is to provide a new proof of Theorem 1.6.

#### 1.5 Rough version of the main theorem

In this section, we state a simple version of our main theorem. For the precise statement, see Theorem 3.2.

**Theorem 1.7** (Main Theorem, first version). *Let  $K = B_{R_0} \subset \Sigma_0$  a ball centered at the origin of radius  $R_0$  and let  $\mathbf{g}_{a,M}$  a Kerr metric with mass  $M > 0$  and angular momentum  $a$  satisfying*

$$\frac{M}{R_0} \ll 1, \quad |a| \leq M.$$

Let  $s > 3$  and let an initial data set  $(\Sigma_0, g, k)$  satisfying (1.9) with  $q = 2$ . Let  $\mathcal{K}_{(0)}$ <sup>3</sup> an initial layer region near  $\Sigma_0 \setminus K$  and assume that the initial data in  $\mathcal{K}_{(0)}$  are perturbations from the initial data of  $\mathbf{g}_{a,M}$  of size  $\varepsilon_0 \ll \frac{M}{R_0}$ .<sup>4</sup>

Then, the initial data set  $(\Sigma_0, g, k)$  has a unique development  $(\mathcal{M}, \mathbf{g})$  in its future domain of dependence with the following properties:

<sup>3</sup> $\mathcal{K}_{(0)}$  is defined in Section 2.1.4.

<sup>4</sup>See (3.6) for the explicit initial data assumptions and see Section 3.1.5 for the definitions of the norms on the initial data.

- $\mathcal{M}$  can be foliated by a double null foliation  $(u, \underline{u})$  whose outgoing leaves  $C_u$  are complete for all  $u \leq u_0$ ;
- we have detailed control of all the quantities associated with the double null foliation of the spacetime, see Theorem 3.2.

**Remark 1.8.** *In the particular case of  $s$ -asymptotically flat initial data with  $s > 7$ , we reobtain the results of [7].*

**Remark 1.9.** *Let  $C_{u_0}$  be the outgoing null cone located near  $\partial K$  and choose  $s = 4 + 2\delta_B$  with  $\delta_B > 0$ . Then, the estimates of Theorem 1.7 on  $C_{u_0}$  are consistent with the initial data considered in [33] and [36].*

**Remark 1.10.** *Theorem 1.7 is the analog for the Kerr spacetime of the result for Minkowski in [53].*

The proof of Theorem 1.7 has the same structure as in [29, 7] and is closely related to the one in [53]. Below, we compare the proof of this paper and that of [7]:

1. To estimate the norms of curvature components, [7] uses the vectorfield method introduced in [10, 29]. Here, we estimate the curvature norms by the  $r^p$ -weighted estimate method introduced in [13]. This allows for a simple treatment of the curvature estimates.
2. [7] uses fourth order derivatives of Ricci coefficients and third order derivatives of curvature components. Thanks to the use of  $r^p$ -weighted estimates, we only need first order derivatives of both Ricci coefficients and curvature components.
3. In order to control one more derivative for Ricci coefficients compared to curvature, [7] relies on the canonical foliation on the last slice  $\underline{C}_*$ . Since we control the same number of derivatives for curvature and Ricci coefficients, instead of introducing the canonical foliation, we use a geodesic type foliation to simplify the estimates on the last slice  $\underline{C}_*$ .

## 1.6 Structure of the paper

- In Section 2, we recall the fundamental notions and basic equations used in this paper. We also introduce the linearization procedure and the linearized equations.
- In Section 3, we state the precise version of the main theorem. We then state intermediate results, and rely on them to prove the main theorem. The rest of the paper focuses on the proof of these intermediary results. In Sections 4–6, we consider the case  $s \in [4, 6]$  and deal with the other cases in Appendices B and C.
- In Section 4, we apply  $r^p$ -weighted estimates to the linearized Bianchi equations to control curvature.
- In Section 5, we estimate the Ricci coefficients using the linearized null structure equations.
- In Section 6, we estimate the curvature in the initial layer region  $\mathcal{K}_{(0)}$ . Then, we estimate Ricci coefficients on the last slice  $\underline{C}_*$ . We also show how to extend the spacetime in the context of a bootstrap argument.
- In Appendix A, we provide the derivation of the linearized equations stated in Section 2.8.
- In Appendix B, we prove Theorem 1.7 in the case  $s \in (3, 4)$ .
- In Appendix C, we prove Theorem 1.7 in the case  $s > 6$  by applying  $r^p$ -weighted estimates to the linearized Teukolsky equation.

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## 2 Preliminaries

### 2.1 Geometric set-up

#### 2.1.1 Double null foliation

We first introduce the geometric setup. The material in this section is classical, see for example Chapter 3 in [29].

We denote  $(\mathcal{M}, \mathbf{g})$  a spacetime  $\mathcal{M}$  with the Lorentzian metric  $\mathbf{g}$  and  $\mathbf{D}$  its Levi-Civita connection. Let  $u$  and  $\underline{u}$  be two optical functions on  $\mathcal{M}$ , that is

$$\mathbf{g}(\text{grad } u, \text{grad } u) = \mathbf{g}(\text{grad } \underline{u}, \text{grad } \underline{u}) = 0.$$

The spacetime  $\mathcal{M}$  is foliated by the level sets of  $u$  and  $\underline{u}$  respectively, and the functions  $u, \underline{u}$  are required to increase towards the future. We use  $C_u$  to denote the outgoing null cones which are the level sets of  $u$  and use  $\underline{C}_{\underline{u}}$  to denote the ingoing null cones which are the level sets of  $\underline{u}$ . We denote

$$S(u, \underline{u}) := C_u \cap \underline{C}_{\underline{u}}, \quad (2.1)$$

which is a space-like 2-sphere. We define the area radius of  $S(u, \underline{u})$ :

$$r(u, \underline{u}) := \sqrt{\frac{|S(u, \underline{u})|}{4\pi}}. \quad (2.2)$$

We introduce the vector fields  $L$  and  $\underline{L}$  by

$$L := -\text{grad } u, \quad \underline{L} := -\text{grad } \underline{u}.$$

We define the a positive scalar function  $\Omega$  by:

$$\mathbf{g}(L, \underline{L}) = -\frac{1}{2\Omega^2},$$

called the lapse function. We then define the normalized null pair  $(e_3, e_4)$  by

$$e_3 = 2\Omega\underline{L}, \quad e_4 = 2\Omega L,$$

and another null pair

$$\underline{N} = \Omega e_3, \quad N = \Omega e_4.$$

The flows generated by  $\underline{N}$  and  $N$  preserve the double null foliation. On a given two sphere  $S(u, \underline{u})$ , we choose a local frame  $(e_1, e_2)$ , and we call  $(e_1, e_2, e_3, e_4)$  a null frame. As a convention, throughout the paper, we use capital Latin letters  $A, B, C, \dots$  to denote an index from 1 to 2 and Greek letters  $\alpha, \beta, \gamma, \dots$  to denote an index from 1 to 4, e.g.  $e_A$  denotes either  $e_1$  or  $e_2$ .

The spacetime metric  $\mathbf{g}$  induces a Riemannian metric  $\gamma$  on  $S(u, \underline{u})$ . We use  $\nabla$  to denote the Levi-Civita connection of  $\gamma$  on  $S(u, \underline{u})$ . Using  $(u, \underline{u})$ , we can introduce a local coordinates system  $(u, \underline{u}, x^A)$  on  $\mathcal{M}$  with  $e_4(x^A) = 0$ . The metric  $\mathbf{g}$  takes the form:

$$\mathbf{g} = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \gamma_{AB}(dx^A - \underline{b}^A du) \otimes (dx^B - \underline{b}^B du), \quad (2.3)$$

and we have  $\underline{N} = \partial_u + \underline{b}$  and  $N = \partial_{\underline{u}}$  where

$$\underline{b} := \underline{b}^A \partial_{x^A}.$$

We recall the null decomposition of the Ricci coefficients and curvature components of the null frame  $(e_1, e_2, e_3, e_4)$ . We have:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), & \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), \\ \underline{\xi}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_A), & \xi_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_A), \\ \underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_A), & \eta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_A), \\ \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_A} e_4, e_3), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= \mathbf{R}(e_A, e_3, e_B, e_3), \\ \beta_A &= \frac{1}{2} \mathbf{R}(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= \frac{1}{2} \mathbf{R}(e_A, e_3, e_3, e_4), \\ \rho &= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4), & \sigma &= \frac{1}{4} \mathbf{R}^*(e_3, e_4, e_3, e_4), \end{aligned} \tag{2.5}$$

where  $\mathbf{R}$  denotes the curvature tensor of  $\mathbf{g}$  and  $\mathbf{R}^*$  denotes the Hodge dual of  $\mathbf{R}$ . The null second fundamental forms  $\chi, \underline{\chi}$  are further decomposed in their traces  $\text{tr } \chi$  and  $\text{tr } \underline{\chi}$ , and traceless parts  $\widehat{\chi}$  and  $\widehat{\underline{\chi}}$ :

$$\begin{aligned} \text{tr } \chi &:= \delta^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \chi, \\ \text{tr } \underline{\chi} &:= \delta^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \underline{\chi}. \end{aligned}$$

We define the horizontal covariant operator  $\nabla$  as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3.$$

We also define  $\nabla_4 X$  and  $\nabla_3 X$  to be the horizontal projections:

$$\begin{aligned} \nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_4) e_3, \\ \nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_3) e_3 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_4) e_4. \end{aligned}$$

A tensor field  $\psi$  defined on  $\mathcal{M}$  is called tangent to  $S$  if it is a priori defined on the spacetime  $\mathcal{M}$  and all the possible contractions of  $\psi$  with either  $e_3$  or  $e_4$  are zero. We use  $\nabla_3 \psi$  and  $\nabla_4 \psi$  to denote the projection to  $S(u, \underline{u})$  of usual derivatives  $\mathbf{D}_3 \psi$  and  $\mathbf{D}_4 \psi$ . As a direct consequence of

(2.4), we have the Ricci formulae:

$$\begin{aligned}
\mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4, \\
\mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + \zeta_A e_3, \\
\mathbf{D}_A e_4 &= \chi_{AB} e_B - \zeta_A e_4, \\
\mathbf{D}_3 e_A &= \nabla_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, \\
\mathbf{D}_4 e_A &= \nabla_4 e_A + \underline{\eta}_A e_4 + \xi_A e_3, \\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_B e_B, \\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\underline{\eta}_B e_B, \\
\mathbf{D}_4 e_3 &= -2\omega e_3 + 2\xi_B e_B, \\
\mathbf{D}_4 e_4 &= 2\omega e_4 + 2\underline{\eta}_B e_B.
\end{aligned} \tag{2.6}$$

In addition to

$$\xi = \underline{\xi} = 0, \tag{2.7}$$

the following identities hold for a double null foliation:

$$\begin{aligned}
\nabla \log \Omega &= \frac{1}{2}(\eta + \underline{\eta}), & \omega &= -\frac{1}{2} \mathbf{D}_4(\log \Omega), & \underline{\omega} &= -\frac{1}{2} \mathbf{D}_3(\log \Omega), \\
\eta &= \zeta + \nabla \log \Omega, & \underline{\eta} &= -\zeta + \nabla \log \Omega,
\end{aligned} \tag{2.8}$$

see for example (6) in [32].

### 2.1.2 Geometry of Kerr spacetime

In this section, we recall the basic notions of the geometry of Kerr spacetime used in the sequel. See Section 4 in [25] and Appendix A in [12] for more details.

Let  $\mathcal{M}_{Kerr}$  be a 4-dimensional manifold diffeomorphic to  $\mathbb{R}^2 \times \mathbb{S}^2$  which describe the Kerr black hole. The Kerr metric  $\mathbf{g}_{a,M}$  with mass  $M$  and angular momentum  $a$  is a smooth metric on  $\mathcal{M}_{Kerr}$ , which takes the following form in the Boyer-Lindquist local coordinates  $(t, r, \theta, \phi)$ :

$$\begin{aligned}
\mathbf{g}_{a,M} &= - \left( 1 - \frac{2Mr}{\Sigma} \right) dt \otimes dt + \frac{\Sigma}{\Delta} dr \otimes dr + \Sigma d\theta \otimes d\theta, \\
&\quad + R^2 \sin^2 \theta d\phi \otimes d\phi - \frac{2Mar \sin^2 \theta}{\Sigma} (d\phi \otimes dt + dt \otimes d\phi),
\end{aligned} \tag{2.9}$$

where

$$\Sigma := r^2 + a^2 \cos^2 \theta, \quad R^2 := r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma}, \quad \Delta := r^2 + a^2 - 2Mr.$$

We denote

$$r_+ := M + (M^2 - a^2)^{\frac{1}{2}},$$

and recall that  $r = r_+$  defines the event horizons of Kerr black hole. Note that (2.9) is valid in  $r > r_+$ .

**Remark 2.1.** *Note that  $(\theta, \phi)$  are spherical coordinates on  $S(t, r)$  and have a coordinate singularity at  $\theta = 0, \pi$ . We thus also introduce*

$$x^1 := \sin \theta \cos \phi, \quad x^2 := \sin \theta \sin \phi.$$

*Then,  $(x^1, x^2)$  are regular coordinates when  $0 \leq \theta \leq \frac{\pi}{3}$  and  $\frac{2\pi}{3} \leq \theta \leq \pi$ . Hence,  $S(t, r)$  can be covered by three regular coordinates charts. In the sequel, we focus on the spherical coordinates chart  $(\theta, \phi)$ .*

**Proposition 2.2.** *We can define two new functions  $r_* = r(r, \theta)$  and  $\theta_* = \theta_*(r, \theta)$  such that the Kerr metric takes the following form in  $r > r_+$ :*

$$\begin{aligned} \mathbf{g}_{a,M} = & \frac{\Delta}{R^2} (-dt \otimes dt + dr_* \otimes dr_*) + \frac{\ell^2}{R^2} d\theta_* \otimes d\theta_* \\ & + R^2 \sin^2 \theta \left( d\phi - \frac{2Mar}{\Sigma R^2} dt \right) \otimes \left( d\phi - \frac{2Mar}{\Sigma R^2} dt \right), \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} \ell & := -Ma \sqrt{\sin^2 \theta_* - \sin^2 \theta} \sqrt{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta}, \\ \Delta & := r^2 + a^2 - 2Mr, \\ R^2 & := r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma}, \\ \Sigma & := r^2 + a^2 \cos^2 \theta. \end{aligned}$$

*Proof.* See Appendices A.1–A.3 in [12]. □

We can also write the Kerr metric in double null foliation. We define

$$\underline{u}_{Kerr} := t + r_*, \quad u_{Kerr} := t - r_*, \quad \phi_* := \phi - h(r_*, \theta_*),$$

where

$$\frac{dh}{dr_*}(r_*, \theta_*) = \frac{2Mar}{\Sigma R^2}.$$

We easily obtain in  $r > r_+$ <sup>5</sup>

$$\begin{aligned} \mathbf{g}_{a,M} = & -2\Omega_{Kerr}^2 (d\underline{u}_{Kerr} \otimes du_{Kerr} + du_{Kerr} \otimes d\underline{u}_{Kerr}) \\ & + (\gamma_{Kerr})_{AB} (dx_{Kerr}^A - \underline{b}_{Kerr}^A du_{Kerr}) \otimes (dx_{Kerr}^B - \underline{b}_{Kerr}^B du_{Kerr}), \end{aligned}$$

where  $(x_{Kerr}^1, x_{Kerr}^2) = (\theta_*, \phi_*)$  or  $(x_{Kerr}^1, x_{Kerr}^2) = (\sin \theta_* \cos \phi_*, \sin \theta_* \sin \phi_*)$ .

We also denote

$$L_{Kerr} := -\text{grad } u_{Kerr}, \quad \underline{L}_{Kerr} := -\text{grad } \underline{u}_{Kerr}, \quad \mathbf{g}(L_{Kerr}, \underline{L}_{Kerr}) = -\frac{1}{2\Omega_{Kerr}^2},$$

and

$$(e_4)_{Kerr} := 2\Omega_{Kerr} L_{Kerr}, \quad (e_3)_{Kerr} := 2\Omega_{Kerr} \underline{L}_{Kerr}.$$

The leaves of  $\mathcal{M}_{Kerr}$  are defined by

$$S_{Kerr}(u_{Kerr}, \underline{u}_{Kerr}) := (C_{Kerr})_{u_{Kerr}} \cap (\underline{C}_{Kerr})_{\underline{u}_{Kerr}},$$

where  $(C_{Kerr})_{u_{Kerr}}$  and  $(\underline{C}_{Kerr})_{\underline{u}_{Kerr}}$  respectively denote the outgoing and ingoing null cones of the  $(u_{Kerr}, \underline{u}_{Kerr})$ -foliation in  $\mathcal{M}_{Kerr}$ . Let also  $r_{Kerr}$  denote the area radius of the  $S(u_{Kerr}, \underline{u}_{Kerr})$ , i.e.

$$r_{Kerr} := \sqrt{\frac{|S(u_{Kerr}, \underline{u}_{Kerr})|}{4\pi}}.$$

<sup>5</sup>See (A.42) in [12] and notice the exchange of role between  $\underline{u}$  and  $u$ .



Note that  $r_{Kerr}$  is a function of  $r_*(r, \theta)$ .

We define

$$\Sigma_{Kerr} := \{p \in \mathcal{M}_{Kerr} / u_{Kerr} + \underline{u}_{Kerr} = 0\},$$

and

$$w_{Kerr} := \frac{\underline{u}_{Kerr} - u_{Kerr}}{2}.$$

Let  $\delta_0 > 0$  be a fixed small constant. The initial layer region of  $\mathcal{M}_{Kerr}$  is defined by

$$(\mathcal{K}_{(0)})_{Kerr} := \{p \in \mathcal{M}_{Kerr} / 0 \leq u_{Kerr} + \underline{u}_{Kerr} \leq 2\delta_0\} \setminus I^+(K_{Kerr}), \quad (2.11)$$

where

$$K_{Kerr} := \{p \in \Sigma_{Kerr} / w_{Kerr}(p) \leq R_0\}, \quad (2.12)$$

$I^+(K_{Kerr})$  denotes the future domain of influence of  $K_{Kerr}$  and  $R_0 > 0$  is a constant.

### 2.1.3 Decay of quantities in Kerr spacetime

In this section, we provide the decay of quantities associated to the double null foliation in  $\mathcal{M}_{Kerr}$  induced by  $(u_{Kerr}, \underline{u}_{Kerr})$  introduced in Section 2.1.2.

**Definition 2.3.** For any  $S_{Kerr}$ -tangent tensor field  $h_{Kerr}$  defined in  $\mathcal{M}_{Kerr}$ , we denote

$$h_{Kerr} = O_q^p, \quad (2.13)$$

if for any integer  $l \geq 0$

$$|\nabla^{\leq l} h_{Kerr}| \lesssim \frac{M^p}{r_{Kerr}^{q+l}},$$

where

$$\nabla := \left\{ \nabla_{\partial_{\underline{u}_{Kerr}}}, \nabla_{\partial_{u_{Kerr}}}, r_{Kerr}^{-1} \nabla_{\partial_{x_{Kerr}^A}} \right\},$$

where the coordinates  $(u_{Kerr}, \underline{u}_{Kerr}, x_{Kerr}^A)$  and the area radius  $r_{Kerr}$  of  $S(u_{Kerr}, \underline{u}_{Kerr})$  have been introduced in Section 2.1.2.

**Proposition 2.4.** The Ricci coefficients of the  $(u_{Kerr}, \underline{u}_{Kerr})$  double null foliation satisfy in  $\mathcal{M}_{Kerr}$ :

$$\begin{aligned} (\operatorname{tr} \chi)_{Kerr} - \frac{2}{r_{Kerr}} &= O_2^1, & (\operatorname{tr} \underline{\chi})_{Kerr} + \frac{2}{r_{Kerr}} &= O_2^1, \\ \omega_{Kerr} &= O_2^1, & \underline{\omega}_{Kerr} &= O_2^1, \\ \widehat{\chi}_{Kerr} &= O_3^2, & \widehat{\underline{\chi}}_{Kerr} &= O_3^2, \\ (\nabla \operatorname{tr} \chi)_{Kerr} &= O_4^2, & (\nabla \operatorname{tr} \underline{\chi})_{Kerr} &= O_4^2, \\ \eta_{Kerr} &= O_3^2, & \underline{\eta}_{Kerr} &= O_3^2, \\ (\nabla \omega)_{Kerr} &= O_4^2, & (\nabla \underline{\omega})_{Kerr} &= O_4^2, \\ \underline{b}_{Kerr} &= O_2^2, & \Omega_{Kerr} - \frac{1}{2} &= O_1^1. \end{aligned}$$

*Proof.* See for example Section 2.2.3 in [7] and Appendix A.4 in [12].  $\square$

**Proposition 2.5.** The curvature components of the  $(u_{Kerr}, \underline{u}_{Kerr})$  double null foliation satisfy in  $\mathcal{M}_{Kerr}$ :

$$\begin{aligned} \alpha_{Kerr} &= O_5^3, & \underline{\alpha}_{Kerr} &= O_5^3, \\ \beta_{Kerr} &= O_4^2, & \underline{\beta}_{Kerr} &= O_4^2, \\ \rho_{Kerr} &= O_3^1, & \sigma_{Kerr} &= O_4^2. \end{aligned}$$

*Proof.* See Section 3.3.5 of [7].  $\square$

### 2.1.4 Initial layer region

Given  $\Sigma_0$  a spacelike hypersurface, called the *initial hypersurface*, we construct an *initial layer region*  $\mathcal{K}_{(0)}$  near  $\Sigma_0$  endowed with a double null foliation  $(u_{(0)}, \underline{u})$  as in Section 2.1.1 such that

$$\Sigma_0 = \{p \in \mathcal{K}_{(0)} / u_{(0)}(p) + \underline{u}(p) = 0\}, \quad (2.14)$$

and

$$0 \leq u_{(0)}(p) + \underline{u}(p) \leq 2\delta_0, \quad \forall p \in \mathcal{K}_{(0)}, \quad (2.15)$$

where  $\delta_0 > 0$  is the small constant appearing in (2.11). The double null foliation  $(u_{(0)}, \underline{u})$  is called the *initial layer foliation*. Its leaves are denoted by

$$S_{(0)}(u_{(0)}, \underline{u}) := (C_{(0)})_{u_{(0)}} \cap \underline{C}_{\underline{u}}, \quad (2.16)$$

where  $(C_{(0)})_{u_{(0)}}$  is the outgoing null cone emanating from  $S_{(0)}(u_{(0)}, -u_{(0)})$  and  $\underline{C}_{\underline{u}}$  is the ingoing null cone emanating from  $S_{(0)}(-\underline{u}, \underline{u})$ . Also, as in Section 2.1.1,  $\mathcal{K}_{(0)}$  is covered by coordinates systems  $(u_{(0)}, \underline{u}, x_{(0)}^A)$  with  $L_{(0)}(x_{(0)}^A) = 0$  and the metric  $\mathbf{g}$  in  $\mathcal{K}_{(0)}$  has the form:

$$\mathbf{g} = -2\Omega_{(0)}^2 (d\underline{u} \otimes du_{(0)} + du_{(0)} \otimes d\underline{u}) + (\gamma_{(0)})_{AB} (dx_{(0)}^A - \underline{b}_{(0)}^A du_{(0)}) \otimes (dx_{(0)}^B - \underline{b}_{(0)}^B du_{(0)}). \quad (2.17)$$

We also define

$$K := \{p \in \Sigma_0 / w(p) \leq R_0\}, \quad (2.18)$$

where  $R_0 > 0$  is the constant appearing in (2.12) and

$$w := \frac{\underline{u} - u_{(0)}}{2}. \quad (2.19)$$

Let  $\Phi_{(0)}$  be the map defined on  $\Sigma_0$  such that

$$\begin{aligned} \Phi_{(0)} : \Sigma_0 &\longrightarrow \Sigma_{Kerr}, \\ p &\longrightarrow \Phi_{(0)}(p) / u_{Kerr}(\Phi_{(0)}(p)) = u(p), \quad \underline{u}_{Kerr}(\Phi_{(0)}(p)) = \underline{u}(p), \quad x_{Kerr}^A(\Phi_{(0)}(p)) = x_{(0)}^A(p). \end{aligned}$$

The map  $\Phi_{(0)}$  being defined on  $\Sigma_0$ , we now extend it to  $\mathcal{K}_{(0)}$ . We denote  $L_{(0)}$  and  $L_{Kerr}$  respectively the generator of  $(C_{(0)})_{u_{(0)}}$  and  $C_{Kerr}(u_{Kerr})$ . For any  $p \in \mathcal{K}_{(0)}$ , we denote  $\phi_t^{L_{(0)}}(p)$  the flow of  $L_{(0)}$  emanating from  $p$  to the past. Then, we denote

$$p_{(0)} := \phi_t^{L_{(0)}}(p) \cap \Sigma_0, \quad q_{(0)} := \Phi_{(0)}(p_{(0)}).$$

Next, we denote  $\phi_t^{L_{Kerr}}(q_{(0)})$  the flow of  $L_{Kerr}$  emanating from  $q_{(0)}$  to the future. We define

$$q := \phi_t^{L_{Kerr}}(q_{(0)}) \cap S_{Kerr}(u_{Kerr} = u_{(0)}(p), \underline{u}_{Kerr} = \underline{u}(p)).$$

Finally, we define

$$\Phi_{(0)}(p) = q. \quad (2.20)$$

By construction, we have for any  $p \in \mathcal{K}_{(0)}$ <sup>6</sup>

$$\Phi_{(0)}(p) \in \mathcal{K}_{Kerr}, \quad u_{Kerr}(\Phi_{(0)}(p)) = u(p), \quad \underline{u}_{Kerr}(\Phi_{(0)}(p)) = \underline{u}(p), \quad x_{Kerr}^A(\Phi_{(0)}(p)) = x_{(0)}^A(p).$$

which implies that the map  $\Phi_{(0)}$  is equivalent to an identification of  $\mathcal{K}_{(0)}$  and  $\mathcal{K}_{Kerr}$  via the values of the coordinates functions  $(u_{(0)}, \underline{u}, x_{(0)}^1, x_{(0)}^2)$  and  $(u_{Kerr}, \underline{u}_{Kerr}, x_{Kerr}^1, x_{Kerr}^2)$ .<sup>7</sup>

<sup>6</sup>Note that the identity  $x_{Kerr}^A(\Phi_{(0)}(p)) = x_{(0)}^A(p)$  follows from the fact that it holds on  $\Sigma_0$  and from the identities  $L_{(0)}(x_{(0)}^A) = 0$  and  $L_{Kerr}(x_{Kerr}^A) = 0$ .

<sup>7</sup>See Section 4.3 in [12] for more explanations concerning this identification.

### 2.1.5 Bootstrap region

For every value  $\underline{u}_* > R_0$ ,

$$S_* := S_{(0)}(-\underline{u}_*, \underline{u}_*)$$

denotes a leaf of the  $(u_{(0)}, \underline{u})$ -foliation on  $\Sigma_0 \setminus K$ . Emanating from this sphere  $S_*$ , we construct an ingoing cone  $\underline{C}_*$  called *the last slice*. For an optical function  $u$  defined on  $\underline{C}_*$  such that  $u = -\underline{u}_*$  on  $S_*$ , the level sets of  $u$  induce a foliation on  $\underline{C}_*$ , and its leaves are denoted by  $S(u, \underline{u}_*)$ . We will take a particular choice of the coordinates systems  $(u, x^A)$  on  $\underline{C}_*$  in the context of a geodesic type foliation, see Definition 6.1, with  $x^A = x_{(0)}^A$  on  $S_*$ .

We define on  $\underline{C}_*$  a map  $\Phi^8$  by

$$\begin{aligned} \Phi : \underline{C}_* &\longrightarrow (\underline{C}_*)_{Kerr}, \\ p &\longrightarrow \Phi(p) / u_{Kerr}(\Phi(p)) = u(p), \quad \underline{u}_{Kerr}(\Phi(p)) = \underline{u}(p), \quad x_{Kerr}^A(\Phi(p)) = x^A(p), \end{aligned}$$

where

$$(\underline{C}_*)_{Kerr} = \{p \in \mathcal{M}_{Kerr} / \underline{u}_{Kerr} = \underline{u}_*\}.$$

For every  $u \geq -\underline{u}_*$ , we construct an outgoing null cone emanating from  $S(u, \underline{u}_*)$  on  $\underline{C}_*$  to the past, denoted  $C_u$ . Then,  $C_u \cap \Sigma_0$  is a sphere on  $\Sigma_0$ , which in general does not coincide with the leaves of the  $(u_{(0)}, \underline{u})$ -foliation on  $\Sigma_0$ . We also define  $\underline{C}_{\underline{u}}$  the ingoing null cone emanating from  $S_{(0)}(-\underline{u}, \underline{u})$ . Hence, we have constructed a double null foliation  $(u, \underline{u})$  in  $\mathcal{K}$  and its leaves are given by:

$$S(u, \underline{u}) := C_u \cap \underline{C}_{\underline{u}}. \quad (2.21)$$

For every  $(u, \underline{u})$ , we define<sup>9</sup>

$$V(u, \underline{u}) := J^-(S(u, \underline{u})) \cap J^+(\Sigma_0). \quad (2.22)$$

Next, we denote

$$\mathcal{K} := V(u_0, \underline{u}_*), \quad u_0 = -R_0, \quad (2.23)$$

called the *bootstrap region*, see Figure 4.4. Remark that  $\mathcal{K}$  is covered by the double null foliation  $(u, \underline{u})$ , and that the boundary of  $\mathcal{K}$  consists of:

- a bounded subset of  $\Sigma_0 \setminus K$ ;
- a part of the outgoing null cone  $C_{u_0}$ ;
- a part of the last slice  $\underline{C}_*$ .

Recalling that the map  $\Phi$  is defined on  $\underline{C}_*$ , we now extend it to  $\mathcal{K}$ . As in Section 2.1.1, we denote  $L$  the generator of  $C_u$  and extend  $x^A$  to  $\mathcal{K}$  by

$$L(x^A) = 0, \quad A = 1, 2.$$

For any  $p \in \mathcal{K}$ , we denote  $\phi_t^L(p)$  the flow of  $L$  emanating from  $p$  to the future. Then, we denote

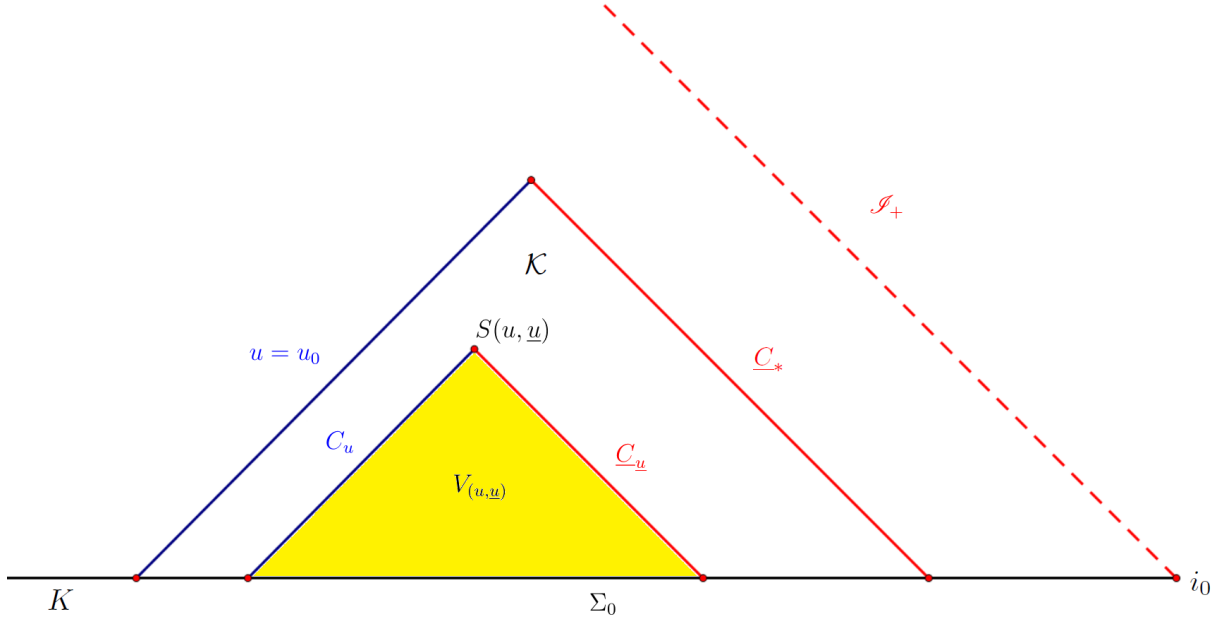
$$p_* := \phi_t^L(p) \cap \underline{C}_*, \quad q_* := \Phi(p_*).$$

Next, we denote  $\phi_t^{LKerr}(q_*)$  the flow of  $L_{Kerr}$  emanating from  $q_*$  to the past. We denote

$$q := \phi_t^{LKerr}(q_*) \cap S_{Kerr}(u(p), \underline{u}(p)).$$

<sup>8</sup>Notice that  $\Phi = \Phi_{(0)}$  on  $S_*$  since  $u = u_{(0)}$  and  $x^A = x_{(0)}^A$  on  $S_*$ .

<sup>9</sup>For any set  $A$ ,  $J^\pm(A)$  denotes the causal future (and resp. causal past) of  $A$ .

Figure 4.4: Bootstrap region  $\mathcal{K}$ 

Finally, we define

$$\Phi(p) = q. \quad (2.24)$$

Thus, we have extended the domain of definition of  $\Phi$  to  $\mathcal{K}$  and we have for any  $p \in \mathcal{K}^{10}$

$$\Phi(p) \in \mathcal{M}_{Kerr}, \quad u_{Kerr}(\Phi(p)) = u(p), \quad \underline{u}_{Kerr}(\Phi(p)) = \underline{u}(p), \quad x_{Kerr}^A(\Phi(p)) = x^A(p). \quad (2.25)$$

In other words, the expression of the application  $\Phi$  in the local coordinates charts  $(u, \underline{u}, x^1, x^2)$  and  $(u_{Kerr}, \underline{u}_{Kerr}, x_{Kerr}^1, x_{Kerr}^2)$  is the identity map, or equivalently

$$u = u_{Kerr} \circ \Phi, \quad \underline{u} = \underline{u}_{Kerr} \circ \Phi, \quad x^A = x_{Kerr}^A \circ \Phi, \quad A = 1, 2.$$

## 2.2 Integral formulae

We define the  $S$ -average of scalar functions.

**Definition 2.6.** *Given any scalar function  $f$ , we denote its average by*

$$\bar{f} := \frac{1}{|S|} \int_S f d\gamma.$$

We recall the following integral formulae, which will be used repeatedly in this paper.

**Lemma 2.7.** *Let  $S(u, \underline{u})$  a leaf of a double null foliation  $(u, \underline{u})$ . For any scalar function  $f$ , the following equations hold:*

$$\begin{aligned} \Omega e_4 \left( \int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (\Omega e_4(f) + \Omega \operatorname{tr} \chi f) d\gamma, \\ \Omega e_3 \left( \int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (\Omega e_3(f) + \Omega \operatorname{tr} \underline{\chi} f) d\gamma. \end{aligned}$$

<sup>10</sup>Note that the identity  $x_{Kerr}^A(\Phi(p)) = x^A(p)$  follows from the fact that it holds on  $\underline{C}_*$  and from the identities  $L(x^A) = 0$  and  $L_{Kerr}(x_{Kerr}^A) = 0$ .

Taking  $f = 1$ , we obtain

$$e_4(r) = \frac{\overline{\Omega \operatorname{tr} \chi}}{2\Omega} r, \quad e_3(r) = \frac{\overline{\Omega \operatorname{tr} \underline{\chi}}}{2\Omega} r,$$

where we recall that  $r$  is the area radius defined by

$$r(u, \underline{u}) = \sqrt{\frac{|S(u, \underline{u})|}{4\pi}}.$$

*Proof.* See Lemma 3.1.3 in [29]. □

### 2.3 Hodge systems

**Definition 2.8.** For tensor fields defined on a 2-sphere  $S$ , we denote by  $\mathfrak{s}_0 := \mathfrak{s}_0(S)$  the set of pairs of scalar functions,  $\mathfrak{s}_1 := \mathfrak{s}_1(S)$  the set of 1-forms and  $\mathfrak{s}_2 := \mathfrak{s}_2(S)$  the set of symmetric traceless 2-tensors.

**Definition 2.9.** Given  $\xi \in \mathfrak{s}_1$ , we define its Hodge dual

$$*\xi_A := \epsilon_{AB} \xi^B.$$

Clearly  $*\xi \in \mathfrak{s}_1$  and

$$*(*\xi) = -\xi.$$

Given  $U \in \mathfrak{s}_2$ , we define its Hodge dual

$$*U_{AB} := \epsilon_{AC} U^C{}_B.$$

Observe that  $*U \in \mathfrak{s}_2$  and

$$*(*U) = -U.$$

**Definition 2.10.** Given  $\xi, \eta \in \mathfrak{s}_1$ , we denote

$$\begin{aligned} \xi \cdot \eta &:= \delta^{AB} \xi_A \eta_B, \\ \xi \wedge \eta &:= \epsilon^{AB} \xi_A \eta_B, \\ (\widehat{\xi \otimes \eta})_{AB} &:= \xi_A \eta_B + \xi_B \eta_A - \delta_{AB} \xi \cdot \eta, \end{aligned}$$

where  $\widehat{\xi \otimes \eta} \in \mathfrak{s}_2$ . Given  $\xi \in \mathfrak{s}_1$ ,  $U \in \mathfrak{s}_2$ , we denote

$$(\xi \cdot U)_A := \delta^{BC} \xi_B U_{AC}.$$

Given  $U, V \in \mathfrak{s}_2$ , we denote

$$U \wedge V := \epsilon^{AB} U_{AC} V_{CB}.$$

**Definition 2.11.** For a given  $\xi \in \mathfrak{s}_1$ , we define the following differential operators:

$$\begin{aligned} \operatorname{div} \xi &:= \delta^{AB} \nabla_A \xi_B, \\ \operatorname{curl} \xi &:= \epsilon^{AB} \nabla_A \xi_B, \\ (\widehat{\nabla \otimes \xi})_{AB} &:= \nabla_A \xi_B + \nabla_B \xi_A - \delta_{AB} (\operatorname{div} \xi). \end{aligned}$$

**Definition 2.12.** We define the Hodge type operators, as introduced in Section 2.2 in [10]:

- $d_1$  takes  $\mathfrak{s}_1$  into  $\mathfrak{s}_0$  and is given by:

$$d_1 \xi := (\operatorname{div} \xi, \operatorname{curl} \xi),$$

- $d_2$  takes  $\mathfrak{s}_2$  into  $\mathfrak{s}_1$  and is given by:

$$(d_2U)_A := \nabla^A U_{AB},$$

- $d_1^*$  takes  $\mathfrak{s}_0$  into  $\mathfrak{s}_1$  and is given by:

$$d_1^*(f, f_*) := -\nabla_A f + \epsilon_{AB} \nabla_B f_*,$$

- $d_2^*$  takes  $\mathfrak{s}_1$  into  $\mathfrak{s}_2$  and is given by:

$$d_2^*\xi := -\frac{1}{2}\nabla\widehat{\otimes}\xi.$$

Then, we have the following identities:

$$\begin{aligned} d_1^*d_1 &= -\Delta_1 + \mathbf{K}, & d_1d_1^* &= -\Delta_0, \\ d_2^*d_2 &= -\frac{1}{2}\Delta_2 + \mathbf{K}, & d_2d_2^* &= -\frac{1}{2}(\Delta_1 + \mathbf{K}), \end{aligned} \quad (2.26)$$

where  $\mathbf{K}$  denotes the Gauss curvature of  $S$ . See for example (2.2.2) in [10] for a proof of (2.26).

**Definition 2.13.** We define weighted angular derivatives  $\wp$  as follows:

$$\begin{aligned} \wp U &:= rd_2U, & \forall U \in \mathfrak{s}_2, \\ \wp \xi &:= rd_1\xi, & \forall \xi \in \mathfrak{s}_1, \\ \wp f &:= rd_1^*f, & \forall f \in \mathfrak{s}_0. \end{aligned}$$

Also, we denote for any tensor  $h \in \mathfrak{s}_k$ ,  $k = 0, 1, 2$ ,

$$h^{(0)} = h, \quad h^{(1)} = (h, r\nabla h).$$

## 2.4 Elliptic estimates for Hodge systems

**Definition 2.14.** For a tensor field  $f$  on a 2-sphere  $S$ , we denote its  $L^p$ -norm by

$$|f|_{L^p} := \left( \int_S |f|^p \right)^{\frac{1}{p}}, \quad (2.27)$$

where  $|f|^2 := f^{A_1 \dots A_k} f_{A_1 \dots A_k}$  for a  $k$ -tensor  $f$  on  $S$ .

**Proposition 2.15** ( $L^p$  estimates for Hodge systems). Assume that  $S$  is an arbitrary compact 2-surface satisfying  $c_1 \leq r^2K \leq c_2$  where  $K$  denotes its Gauss curvature and  $0 < c_1 \leq c_2 < +\infty$  are two positive constants. Then the following statements hold for all<sup>11</sup>  $p \in (1, +\infty)$ :

1. Let  $\phi \in \mathfrak{s}_0$  be a solution of  $\Delta\phi = f$ . Then we have

$$|\nabla^2\phi|_{L^p} + r^{-1}|\nabla\phi|_{L^p} + r^{-2}|\phi - \bar{\phi}|_{L^p} \lesssim |f|_{L^p}.$$

2. Let  $\xi \in \mathfrak{s}_1$  be a solution of  $d_1\xi = (f, f_*)$ . Then we have

$$\int_S |\nabla\xi|^p + r^{-p}|\xi|^p \lesssim \int_S |f|^p + |f_*|^p.$$

3. Let  $U \in \mathfrak{s}_2$  be a solution of  $d_2U = f$ . Then we have

$$\int_S |\nabla U|^p + r^{-p}|U|^p \lesssim \int_S |f|^p.$$

*Proof.* See Corollary 2.3.1.1 of [10]. □

<sup>11</sup>In practice, we only use Proposition 2.15 with  $p = 2$  and  $p = 4$ .

## 2.5 Null structure equations

We recall the standard null structure equations in a double null foliation.

**Proposition 2.16.** *We have the following null structure equations in a double null foliation:*

$$\begin{aligned}
\nabla_4 \underline{\eta} &= -\underline{\chi} \cdot (\underline{\eta} - \underline{\eta}) - \underline{\beta}, \\
\nabla_3 \underline{\eta} &= -\underline{\chi} \cdot (\underline{\eta} - \underline{\eta}) + \underline{\beta}, \\
\nabla_4 \widehat{\chi} + (\text{tr } \underline{\chi}) \widehat{\chi} &= -2\omega \widehat{\chi} - \alpha, \\
\nabla_4 \text{tr } \underline{\chi} + \frac{1}{2} (\text{tr } \underline{\chi})^2 &= -|\widehat{\chi}|^2 - 2\omega \text{tr } \underline{\chi}, \\
\nabla_3 \widehat{\chi} + (\text{tr } \underline{\chi}) \widehat{\chi} &= -2\omega \widehat{\chi} - \alpha, \\
\nabla_3 \text{tr } \underline{\chi} + \frac{1}{2} (\text{tr } \underline{\chi})^2 &= -|\widehat{\chi}|^2 - 2\omega \text{tr } \underline{\chi}, \\
\nabla_4 \widehat{\chi} + \frac{1}{2} (\text{tr } \underline{\chi}) \widehat{\chi} &= \nabla \widehat{\otimes} \underline{\eta} + 2\omega \widehat{\chi} - \frac{1}{2} (\text{tr } \underline{\chi}) \widehat{\chi} + \underline{\eta} \widehat{\otimes} \underline{\eta}, \\
\nabla_3 \widehat{\chi} + \frac{1}{2} (\text{tr } \underline{\chi}) \widehat{\chi} &= \nabla \widehat{\otimes} \underline{\eta} + 2\omega \widehat{\chi} - \frac{1}{2} (\text{tr } \underline{\chi}) \widehat{\chi} + \underline{\eta} \widehat{\otimes} \underline{\eta}, \\
\nabla_4 \text{tr } \underline{\chi} + \frac{1}{2} (\text{tr } \underline{\chi}) \text{tr } \underline{\chi} &= 2\omega \text{tr } \underline{\chi} + 2\rho - \widehat{\chi} \cdot \widehat{\chi} + 2 \text{div } \underline{\eta} + 2|\underline{\eta}|^2, \\
\nabla_3 \text{tr } \underline{\chi} + \frac{1}{2} (\text{tr } \underline{\chi}) \text{tr } \underline{\chi} &= 2\omega \text{tr } \underline{\chi} + 2\rho - \widehat{\chi} \cdot \widehat{\chi} + 2 \text{div } \underline{\eta} + 2|\underline{\eta}|^2.
\end{aligned}$$

We also have the Codazzi equations

$$\begin{aligned}
\text{div } \widehat{\chi} &= \frac{1}{2} \nabla \text{tr } \underline{\chi} - \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \text{tr } \underline{\chi} \right) - \underline{\beta}, \\
\text{div } \widehat{\chi} &= \frac{1}{2} \nabla \text{tr } \underline{\chi} + \zeta \cdot \left( \widehat{\chi} - \frac{1}{2} \text{tr } \underline{\chi} \right) + \underline{\beta},
\end{aligned}$$

the torsion equation

$$\text{curl } \underline{\eta} = -\text{curl } \underline{\eta} = \sigma + \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi},$$

and the Gauss equation

$$\mathbf{K} = -\frac{1}{4} \text{tr } \underline{\chi} \text{tr } \underline{\chi} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho.$$

Moreover, we have

$$\begin{aligned}
\nabla_4 \omega &= 2\omega \omega + \frac{3}{4} |\underline{\eta} - \underline{\eta}|^2 - \frac{1}{4} (\underline{\eta} - \underline{\eta}) \cdot (\underline{\eta} + \underline{\eta}) - \frac{1}{8} |\underline{\eta} + \underline{\eta}|^2 + \frac{1}{2} \rho, \\
\nabla_3 \omega &= 2\omega \omega + \frac{3}{4} |\underline{\eta} - \underline{\eta}|^2 + \frac{1}{4} (\underline{\eta} - \underline{\eta}) \cdot (\underline{\eta} + \underline{\eta}) - \frac{1}{8} |\underline{\eta} + \underline{\eta}|^2 + \frac{1}{2} \rho.
\end{aligned}$$

*Proof.* See (3.1)–(3.5) in [32]. □

**Definition 2.17.** *We define the mass aspect functions as follows:*

$$\mu := -\text{div } \underline{\eta} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho, \quad \underline{\mu} := -\text{div } \underline{\eta} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho. \quad (2.28)$$

We also define the following renormalized quantities which will be used in Sections 5 and 6:

$$[\mu] := \mu + \frac{1}{4} \text{tr } \underline{\chi} \text{tr } \underline{\chi}, \quad [\underline{\mu}] := \underline{\mu} + \frac{1}{4} \text{tr } \underline{\chi} \text{tr } \underline{\chi}. \quad (2.29)$$

**Proposition 2.18.** *We have the following null structure equations:*

$$\begin{aligned}
\Omega \nabla_4(\Omega \operatorname{tr} \chi) + \frac{1}{2}(\Omega \operatorname{tr} \chi)^2 &= -4\Omega \operatorname{tr} \chi(\Omega \omega) - \Omega^2 |\widehat{\chi}|^2, \\
\Omega \nabla_3(\Omega \operatorname{tr} \underline{\chi}) + \frac{1}{2}(\Omega \operatorname{tr} \underline{\chi})^2 &= -4\Omega \operatorname{tr} \underline{\chi}(\Omega \omega) - \Omega^2 |\widehat{\underline{\chi}}|^2, \\
\Omega \nabla_4(\Omega \operatorname{tr} \underline{\chi}) &= -2\Omega^2 [\underline{\mu}] + 2\Omega^2 |\underline{\eta}|^2, \\
\Omega \nabla_3(\Omega \operatorname{tr} \chi) &= -2\Omega^2 [\mu] + 2\Omega^2 |\eta|^2.
\end{aligned} \tag{2.30}$$

We have the following equations for the renormalized mass aspect functions:

$$\begin{aligned}
\Omega \nabla_4[\mu] + \Omega \operatorname{tr} \chi[\mu] &= \Omega[F] + \Omega(\operatorname{tr} \chi \rho - 2\underline{\eta} \cdot \beta), \\
[F] &:= \widehat{\chi} \cdot (\nabla \widehat{\otimes} \eta) + (\eta - \underline{\eta}) \cdot (\nabla \operatorname{tr} \chi + \operatorname{tr} \chi \zeta) + \frac{1}{2} \operatorname{tr} \chi (|\eta|^2 - |\underline{\eta}|^2) \\
&\quad - \frac{1}{2} (\operatorname{tr} \underline{\chi} |\widehat{\chi}|^2 + \operatorname{tr} \chi (\widehat{\chi} \cdot \widehat{\underline{\chi}})) + 2\underline{\eta} \cdot \widehat{\chi} \cdot \eta,
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
\Omega \nabla_3[\underline{\mu}] + \Omega \operatorname{tr} \underline{\chi}[\underline{\mu}] &= \Omega[\underline{F}] + \Omega(\operatorname{tr} \underline{\chi} \rho + 2\underline{\eta} \cdot \underline{\beta}), \\
[\underline{F}] &:= \widehat{\underline{\chi}} \cdot (\nabla \widehat{\otimes} \eta) + (\underline{\eta} - \eta) \cdot (\nabla \operatorname{tr} \underline{\chi} - \operatorname{tr} \underline{\chi} \zeta) + \frac{1}{2} \operatorname{tr} \underline{\chi} (|\underline{\eta}|^2 - |\eta|^2) \\
&\quad - \frac{1}{2} (\operatorname{tr} \chi |\widehat{\underline{\chi}}|^2 + \operatorname{tr} \underline{\chi} (\widehat{\chi} \cdot \widehat{\underline{\chi}})) + 2\underline{\eta} \cdot \widehat{\underline{\chi}} \cdot \eta.
\end{aligned}$$

*Proof.* Notice that (2.30) follows directly from Proposition 2.16 and (2.8). See Lemma 4.3.2 in [29] for the proof of (2.31).  $\square$

## 2.6 Bianchi equations

We recall the Bianchi equations in a double null foliation.

**Proposition 2.19.** *The Bianchi equations take the following form in a double null foliation:*

$$\begin{aligned}
\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + 4\underline{\omega} \underline{\alpha} - 3(\widehat{\underline{\chi}} \rho - {}^* \widehat{\underline{\chi}} \sigma) + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta}, \\
\nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -\operatorname{div} \underline{\alpha} - 2\underline{\omega} \underline{\beta} + (2\underline{\zeta} - \underline{\eta}) \cdot \underline{\alpha}, \\
\nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\nabla \rho + {}^* \nabla \sigma + 2\underline{\omega} \underline{\beta} + 2\underline{\widehat{\chi}} \cdot \beta - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma), \\
\nabla_3 \rho + \frac{3}{2} \operatorname{tr} \underline{\chi} \rho &= -\operatorname{div} \underline{\beta} - \frac{1}{2} \widehat{\underline{\chi}} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} - 2\underline{\eta} \cdot \underline{\beta}, \\
\nabla_4 \rho + \frac{3}{2} \operatorname{tr} \chi \rho &= \operatorname{div} \beta - \frac{1}{2} \widehat{\underline{\chi}} \cdot \alpha + \zeta \cdot \beta + 2\underline{\eta} \cdot \beta, \\
\nabla_3 \sigma + \frac{3}{2} \operatorname{tr} \underline{\chi} \sigma &= -\operatorname{curl} \underline{\beta} + \frac{1}{2} \widehat{\underline{\chi}} \cdot {}^* \underline{\alpha} - \zeta \cdot {}^* \underline{\beta} + 2\underline{\eta} \cdot {}^* \underline{\beta}, \\
\nabla_4 \sigma + \frac{3}{2} \operatorname{tr} \chi \sigma &= -\operatorname{curl} \beta + \frac{1}{2} \widehat{\underline{\chi}} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2\underline{\eta} \cdot {}^* \beta, \\
\nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= \nabla \rho + {}^* \nabla \sigma + 2\underline{\omega} \beta + 2\underline{\widehat{\chi}} \cdot \underline{\beta} + 3(\underline{\eta} \rho + {}^* \underline{\eta} \sigma), \\
\nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= \operatorname{div} \alpha - 2\underline{\omega} \beta + (2\underline{\zeta} + \underline{\eta}) \alpha, \\
\nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \alpha &= \nabla \widehat{\otimes} \beta + 4\underline{\omega} \alpha - 3(\widehat{\underline{\chi}} \rho + {}^* \widehat{\underline{\chi}} \sigma) + (\zeta + 4\underline{\eta}) \widehat{\otimes} \beta.
\end{aligned}$$

*Proof.* See Proposition 3.2.4 in [29].  $\square$



## 2.7 Transport equations for metric components

We introduce the following transport equations for metric components.

**Proposition 2.20.** *The metric components satisfy the following equations:*

$$\begin{aligned}\nabla_4(\gamma_{AB}) &= 2\chi_{AB}, \\ \nabla_4(\underline{b}^A) &= -4\Omega\zeta^A, \\ \nabla_4(\in_{AB}) &= \text{tr } \chi \in_{AB}, \\ \nabla_3(\gamma_{AB}) &= 2\underline{\chi}_{AB} - \Omega^{-1}\partial_A(\underline{b}^C)\gamma_{BC} - \Omega^{-1}\partial_B(\underline{b}^C)\gamma_{AC}, \\ \nabla_3(\in_{AB}) &= \text{tr } \underline{\chi} \in_{AB} - \Omega^{-1}\partial_A(\underline{b}^C) \in_{BC} - \Omega^{-1}\partial_B(\underline{b}^C) \in_{BC}.\end{aligned}$$

*Proof.* We have from  $[\partial_{\underline{u}}, \partial_A] = 0$  that

$$\begin{aligned}0 &= \nabla_{\partial_{\underline{u}}}(\gamma) \left( \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B} \right) = \frac{\partial}{\partial \underline{u}}\gamma_{AB} - \gamma(\mathbf{D}_{\partial_{\underline{u}}}\partial_A, \partial_B) - \gamma(\partial_A, \mathbf{D}_{\partial_{\underline{u}}}\partial_B) \\ &= \Omega\nabla_4(\gamma_{AB}) - \Omega\mathbf{g}(\mathbf{D}_{e_4}\partial_A, \partial_B) - \Omega\mathbf{g}(\partial_A, \mathbf{D}_{e_4}\partial_B) \\ &= \Omega\nabla_4(\gamma_{AB}) - 2\Omega\chi_{AB},\end{aligned}$$

which implies

$$\nabla_4(\gamma_{AB}) = 2\chi_{AB}$$

as stated in the first equation. For the proof of the second equation, see (3.1.62) in [29].

We have from  $[\partial_{\underline{u}}, \partial_A] = 0$  that

$$\begin{aligned}0 &= \nabla_{\partial_{\underline{u}}}(\in) \left( \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B} \right) = \frac{\partial}{\partial \underline{u}}\in_{AB} - \in(\mathbf{D}_{\partial_{\underline{u}}}\partial_A, \partial_B) - \in(\partial_A, \mathbf{D}_{\partial_{\underline{u}}}\partial_B) \\ &= \frac{\partial}{\partial \underline{u}}\in_{AB} - \in(\mathbf{D}_{\partial_A}\partial_{\underline{u}}, \partial_B) - \in(\partial_A, \mathbf{D}_{\partial_B}\partial_{\underline{u}}) \\ &= \frac{\partial}{\partial \underline{u}}\in_{AB} - \Omega\in(\chi_A^C\partial_C, \partial_B) - \Omega\in(\partial_A, \chi_B^C\partial_C) \\ &= \Omega\nabla_4(\in_{AB}) - \Omega\chi_A^C \in_{CB} - \Omega\chi_B^C \in_{AC},\end{aligned}$$

which implies

$$\nabla_4(\in_{AB}) = \chi_A^C \in_{CB} + \chi_B^C \in_{AC} = \text{tr } \chi \in_{AB}.$$

Next, we compute

$$\begin{aligned}0 &= (\nabla_{\Omega e_3}\gamma) \left( \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B} \right) \\ &= \Omega\nabla_3(\gamma_{AB}) - \gamma(\nabla_{\Omega e_3}\partial_A, \partial_B) - \gamma(\partial_A, \nabla_{\Omega e_3}\partial_B) \\ &= \Omega\nabla_3(\gamma_{AB}) - \Omega\mathbf{g}(\mathbf{D}_{\partial_A}e_3, \partial_B) - \Omega\mathbf{g}(\partial_A, \mathbf{D}_{\partial_B}e_3) - \mathbf{g}([\Omega e_3, \partial_A], \partial_B) - \mathbf{g}(\partial_A, [\Omega e_3, \partial_B]) \\ &= \Omega\nabla_3(\gamma_{AB}) - 2\Omega\underline{\chi}_{AB} - \mathbf{g}([\underline{b}^B\partial_C, \partial_A], \partial_B) - \mathbf{g}(\partial_A, [\underline{b}^C\partial_C, \partial_B]) \\ &= \Omega\nabla_3(\gamma_{AB}) - 2\Omega\underline{\chi}_{AB} + \partial_A(\underline{b}^C)\gamma_{BC} + \partial_B(\underline{b}^C)\gamma_{AC},\end{aligned}$$

which implies

$$\nabla_3(\gamma_{AB}) = 2\underline{\chi}_{AB} - \Omega^{-1}\partial_A(\underline{b}^C)\gamma_{BC} - \Omega^{-1}\partial_B(\underline{b}^C)\gamma_{AC}. \quad (2.32)$$

Similarly, we have

$$\nabla_3(\in_{AB}) = \text{tr } \underline{\chi} \in_{AB} - \Omega^{-1}\partial_A(\underline{b}^C) \in_{BC} - \Omega^{-1}\partial_B(\underline{b}^C) \in_{AC}.$$

This concludes the proof of Proposition 2.20.  $\square$

## 2.8 The linearized equations

### 2.8.1 Linearization procedure

Let  $\mathcal{K}$  the bootstrap region defined in Section 2.1.5. By construction, we have

$$L(x^A) = 0.$$

Hence, the metric in  $\mathcal{K}$  takes the following form, see (2.3):

$$\mathbf{g} = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \gamma_{AB}(dx^A - \underline{b}^A du) \otimes (dx^B - \underline{b}^B du). \quad (2.33)$$

The inverse metric components are given by:

$$\begin{aligned} \mathbf{g}^{AB} &= \gamma^{AB}, & \mathbf{g}^{uu} &= -\frac{1}{2\Omega^2}, & \mathbf{g}^{uA} &= -\frac{\underline{b}^A}{2\Omega^2}, \\ \mathbf{g}^{uu} &= \mathbf{g}^{\underline{u}\underline{u}} = 0, & \mathbf{g}^{uA} &= 0, & & A, B = 1, 2. \end{aligned} \quad (2.34)$$

Let  $X$  be a tensor defined on  $S := S(u, \underline{u})$ . We denote  $X_{Kerr}$  the corresponding  $S_{Kerr}$ -tangent tensor in  $\mathcal{M}_{Kerr}$ . Then, its pullback  $\Phi^* X_{Kerr}$ <sup>12</sup> is a  $S$ -tangent tensor in  $\mathcal{K}$  denoted by

$$X_K := \Phi^* X_{Kerr}. \quad (2.35)$$

We now define the linearization of  $X$  as follows

$$\check{X} := X - X_K. \quad (2.36)$$

We also denote  $\nabla_K$ , (resp.  $\Gamma_K$ ,  $\text{div}_K$  and  $\text{curl}_K$ ) the Levi-Civita connection, (resp. Christoffel symbols, divergence operator and curl operator) of the metric  $\mathbf{g}_K = \Phi^*(\mathbf{g}_{Kerr})$ .

The following lemma plays an important role in the linearization of the equations.

**Lemma 2.21.** *We have the following formulae:*

$$\begin{aligned} \widetilde{\Omega e_4} &:= \Omega e_4 - \Omega_K(e_4)_K = 0, \\ \widetilde{\Omega e_3} &:= \Omega e_3 - \Omega_K(e_3)_K = \check{\underline{b}}^A \partial_{x^A}. \end{aligned}$$

*Proof.* Notice that we have

$$\begin{aligned} e_4(u) &= 0, & e_4(\underline{u}) &= \mathbf{g}(e_4, \text{grad } \underline{u}) = \mathbf{g}(2\Omega L, -\underline{L}) = \Omega^{-1}, \\ e_3(\underline{u}) &= 0, & e_3(u) &= \mathbf{g}(e_3, \text{grad } u) = \mathbf{g}(2\Omega \underline{L}, -L) = \Omega^{-1}, \\ e_4(x^A) &= 0, & e_3(x^A) &= \Omega^{-1} \underline{b}^A. \end{aligned} \quad (2.37)$$

Hence, we obtain

$$\Omega e_4 = \Omega e_4(u) \partial_u + \Omega e_4(\underline{u}) \partial_{\underline{u}} + \Omega e_4(x^A) \partial_{x^A} = \partial_{\underline{u}}. \quad (2.38)$$

Thus, we infer

$$\Phi_*(\Omega e_4) = \Phi_*(\partial_{\underline{u}}) = \partial_{\underline{u}_{Kerr}} = \Omega_{Kerr}(e_4)_{Kerr},$$

which implies

$$\Omega_K(e_4)_K = \Phi^*(\Omega_{Kerr}(e_4)_{Kerr}) = \Omega e_4.$$

Hence, we deduce

$$\widetilde{\Omega e_4} = \Omega e_4 - \Omega_K(e_4)_K = \partial_{\underline{u}} - \partial_{\underline{u}} = 0. \quad (2.39)$$

<sup>12</sup>Recall that the map  $\Phi : \mathcal{K} \rightarrow \mathcal{M}_{Kerr}$  is defined in (2.24) and satisfies (2.25).

Similarly, we have

$$\Omega e_3 = \partial_u + \underline{b}^A \partial_{x^A},$$

which implies

$$\Phi_*(\Omega e_3) = \partial_{u_{Kerr}} + (\Phi^{-1})^*(\underline{b}^A) \partial_{x_{Kerr}^A}.$$

Hence, we obtain

$$\begin{aligned} \widetilde{\Omega e_3} &= \Omega e_3 - \Omega_K(e_3)_K = \Phi_*^{-1}(\Phi_*(\Omega e_3) - \Omega_{Kerr}(e_3)_{Kerr}) \\ &= \Phi_*^{-1}\left(\left((\Phi^{-1})^*\underline{b}^A - \underline{b}_{Kerr}^A\right)\partial_{x_{Kerr}^A}\right) \\ &= (\underline{b}^A - \underline{b}_K^A)\partial_{x^A} = \check{\underline{b}}^A \partial_{x^A}. \end{aligned} \quad (2.40)$$

This concludes the proof of Lemma 2.21.  $\square$

**Proposition 2.22.** Denoting  $\Gamma$  and  $\Gamma_K$  respectively the Christoffel symbols of  $\nabla$  and  $\nabla_K$ . For any  $U \in \mathfrak{s}_2(S)$  and  $\phi \in \mathfrak{s}_1(S)$  where  $S = S(u, \underline{u})$ , we have the following formulae:

$$\begin{aligned} (\nabla - \nabla_K)_A U_{BC} &= -(\Gamma - \Gamma_K)_{AB}^D U_{CD} - (\Gamma - \Gamma_K)_{AC}^D U_{BD}, \\ (\nabla - \nabla_K)_A \phi_B &= -(\Gamma - \Gamma_K)_{AB}^C \phi_C, \\ (\nabla - \nabla_K)_{\underline{u}} U_{AB} &= -(\Gamma - \Gamma_K)_{\underline{u}A}^C U_{CB} - (\Gamma - \Gamma_K)_{\underline{u}B}^C U_{CA}, \\ (\nabla - \nabla_K)_{\underline{u}} \phi_A &= -(\Gamma - \Gamma_K)_{\underline{u}A}^B \phi_B, \\ (\nabla - \nabla_K)_{\partial_u} U_{AB} &= -(\Gamma - \Gamma_K)_{uA}^C U_{CB} - (\Gamma - \Gamma_K)_{uB}^C U_{CA}, \\ (\nabla - \nabla_K)_{\partial_u} \phi_A &= -(\Gamma - \Gamma_K)_{uA}^B \phi_B. \end{aligned} \quad (2.41)$$

We also have:

$$\begin{aligned} (\operatorname{div} - \operatorname{div}_K) U &= (\gamma^{AB} - \gamma_K^{AB}) \nabla_A U_{B\bullet} - \gamma_K^{AB} (\Gamma - \Gamma_K)_{AB}^D U_{\bullet D} \\ &\quad - \gamma_K^{AB} (\Gamma - \Gamma_K)_{A\bullet}^D U_{BD}, \\ (\operatorname{div} - \operatorname{div}_K) \phi &= (\gamma^{AB} - (\gamma_K)^{AB}) \nabla_A \phi_B - (\gamma_K)^{AB} (\Gamma - \Gamma_K)_{AB}^C \phi_C, \\ (\operatorname{curl} - \operatorname{curl}_K) \phi &= (\epsilon^{AB} - (\epsilon_K)^{AB}) \nabla_A \phi_B - (\epsilon_K)^{AB} (\Gamma - \Gamma_K)_{AB}^C \phi_C, \\ (\nabla \widehat{\otimes} \phi - \nabla_K \widehat{\otimes} \phi_K)_{AB} &= (\nabla \widehat{\otimes} \check{\phi})_{AB} - 2(\Gamma - \Gamma_K)_{AB}^C (\phi_K)_C \\ &\quad + \delta_{AB} [(\gamma^{CD} - (\gamma_K)^{CD}) \nabla_C (\phi_K)_D - (\gamma_K)^{CD} (\Gamma - \Gamma_K)_{CD}^E (\phi_K)_E]. \end{aligned} \quad (2.42)$$

*Proof.* The proof follows from a direct computation. For example, we have

$$\begin{aligned} (\nabla - \nabla_K)_A U_{BC} &= \left( \frac{\partial}{\partial x^A} (U_{BC}) - \Gamma_{AB}^D U_{CD} - \Gamma_{AC}^D U_{BD} \right) \\ &\quad - \left( \frac{\partial}{\partial x^A} (U_{BC}) - (\Gamma_K)_{AB}^D U_{CD} - (\Gamma_K)_{AC}^D U_{BD} \right) \\ &= -(\Gamma - \Gamma_K)_{AB}^D U_{CD} - (\Gamma - \Gamma_K)_{AC}^D U_{BD}, \end{aligned}$$

which implies the first equation in (2.41). The proof of the other equations in (2.41) are similar. We also have

$$\begin{aligned} (\operatorname{div} - \operatorname{div}_K) U_C &= \gamma^{AB} \nabla_A U_{BC} - \gamma_K^{AB} (\nabla_K)_A U_{BC} \\ &= (\gamma^{AB} - \gamma_K^{AB}) \nabla_A U_{BC} + \gamma_K^{AB} (\nabla_A U_{BC} - (\nabla_K)_A U_{BC}) \\ &= (\gamma^{AB} - \gamma_K^{AB}) \nabla_A U_{BC} - \gamma_K^{AB} (\Gamma - \Gamma_K)_{AB}^D U_{CD} - \gamma_K^{AB} (\Gamma - \Gamma_K)_{AC}^D U_{BD}, \end{aligned}$$

which implies the first equation in (2.42). The proof of the other equations in (2.42) are similar and left to the reader. This concludes the proof of Proposition 2.22.  $\square$

**Lemma 2.23.** *We have the following identities:*

$$\begin{aligned}\Gamma_{\underline{u}A}^B &= \frac{1}{2}\gamma^{BC}\partial_{\underline{u}}(\gamma_{AC}), \\ \Gamma_{\underline{u}A}^B &= \frac{1}{2}\gamma^{BC}\left(\partial_{\underline{u}}(\gamma_{AC}) - \frac{\partial}{\partial x^A}(\gamma_{CD}\underline{b}^D) + \frac{\partial}{\partial x^C}(\gamma_{AD}\underline{b}^D)\right) + \frac{\underline{b}^B}{4\Omega^2}\left(2\frac{\partial}{\partial x^A}(\Omega^2) - \frac{\partial}{\partial \underline{u}}(\gamma_{AD}\underline{b}^D)\right), \\ \Gamma_{BC}^A &= \frac{1}{2}\gamma^{AD}\left(\frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} - \frac{\partial\gamma_{BC}}{\partial x^D}\right) + \frac{\underline{b}^A}{4\Omega^2}\frac{\partial\gamma_{BC}}{\partial \underline{u}}.\end{aligned}$$

*Proof.* We have from (2.33) and (2.34)

$$\Gamma_{\underline{u}A}^B = \frac{1}{2}\mathbf{g}^{BC}\left(\frac{\partial\mathbf{g}_{uC}}{\partial x^A} + \frac{\partial\mathbf{g}_{AC}}{\partial \underline{u}} - \frac{\partial\mathbf{g}_{uA}}{\partial x^C}\right) + \frac{1}{2}\mathbf{g}^{Bu}\left(\frac{\partial\mathbf{g}_{uu}}{\partial x^A} + \frac{\partial\mathbf{g}_{Au}}{\partial \underline{u}} - \frac{\partial\mathbf{g}_{uA}}{\partial \underline{u}}\right) = \frac{1}{2}\gamma^{BC}\partial_{\underline{u}}(\gamma_{AC}),$$

and similarly

$$\begin{aligned}\Gamma_{\underline{u}A}^B &= \frac{1}{2}\mathbf{g}^{BC}\left(\frac{\partial\mathbf{g}_{uC}}{\partial x^A} + \frac{\partial\mathbf{g}_{AC}}{\partial \underline{u}} - \frac{\partial\mathbf{g}_{uA}}{\partial x^C}\right) + \frac{1}{2}\mathbf{g}^{Bu}\left(\frac{\partial\mathbf{g}_{uu}}{\partial x^A} + \frac{\partial\mathbf{g}_{Au}}{\partial \underline{u}} - \frac{\partial\mathbf{g}_{uA}}{\partial \underline{u}}\right) \\ &= \frac{1}{2}\gamma^{BC}\left(-\frac{\partial}{\partial x^A}(\gamma_{CD}\underline{b}^D) + \frac{\partial\gamma_{AC}}{\partial \underline{u}} + \frac{\partial}{\partial x^C}(\gamma_{AD}\underline{b}^D)\right) + \frac{\underline{b}^B}{4\Omega^2}\left(2\frac{\partial}{\partial x^A}(\Omega^2) - \frac{\partial}{\partial \underline{u}}(\gamma_{AD}\underline{b}^D)\right).\end{aligned}$$

Finally, we compute

$$\begin{aligned}\Gamma_{BC}^A &= \frac{1}{2}\gamma^{AD}\left(\frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} - \frac{\partial\gamma_{BC}}{\partial x^D}\right) + \frac{1}{2}\mathbf{g}^{Au}\left(\frac{\partial\mathbf{g}_{Bu}}{\partial x^C} + \frac{\partial\mathbf{g}_{Cu}}{\partial x^B} - \frac{\partial\mathbf{g}_{BC}}{\partial \underline{u}}\right) \\ &= \frac{1}{2}\gamma^{AD}\left(\frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} - \frac{\partial\gamma_{BC}}{\partial x^D}\right) + \frac{\underline{b}^A}{4\Omega^2}\frac{\partial\gamma_{BC}}{\partial \underline{u}}.\end{aligned}$$

This concludes the proof of Lemma 2.23.  $\square$

The following tensors play an important role in the linearization of the null structure equations and Bianchi equations.

**Definition 2.24.** *We define the following tensors<sup>13</sup>*

$$\check{J}_A^B := (\Gamma - \Gamma_K)_{uA}^B, \quad \check{J}_A^B := (\Gamma - \Gamma_K)_{\underline{u}A}^B, \quad \check{L}_{AB}^C := (\Gamma - \Gamma_K)_{AB}^C. \quad (2.43)$$

### 2.8.2 Schematic notation $\Gamma_g$ and $\Gamma_b$

We introduce the following schematic notations for metric components, Ricci coefficients and curvature components.

**Definition 2.25.** *We divide the linearized quantities into two parts:*

$$\begin{aligned}\Gamma_g &:= \Gamma_g^{(0)} := \{\check{\eta}, \check{\eta}, \check{\zeta}, \check{\chi}, \widetilde{\text{tr}}\check{\chi}, \widetilde{\text{tr}}\check{\chi}, \check{\omega}, r^{-1}\check{b}, r^{-1}\check{\Omega}, r^{-1}\check{\gamma}, r^{-1}\check{\epsilon}, r^{-2}\check{r}\}, \\ \Gamma_b &:= \Gamma_b^{(0)} := \{\check{\chi}, \check{\omega}\}.\end{aligned}$$

We also denote:

$$\begin{aligned}\Gamma_g^{(1)} &:= (r\nabla)^{\leq 1}\Gamma_g^{(0)} \cup \{\check{J}, \check{L}, r\check{b}, r\check{\rho}, r\check{\sigma}\}, \\ \Gamma_b^{(1)} &:= (r\nabla)^{\leq 1}\Gamma_b^{(0)} \cup \{\check{J}, r\check{\beta}\}.\end{aligned}$$

**Remark 2.26.** *Throughout this paper,  $\Gamma_g$  decays better than  $\Gamma_b$ , see Lemma 4.1.*

<sup>13</sup>Remark that the difference of two Christoffel symbols is a tensor.

### 2.8.3 Schematic notation $\mathbf{O}_q^p$

**Definition 2.27.** For any  $S$ -tangent tensor field  $X_K = \Phi^*(X_{Kerr})$ , we denote

$$X_K = \mathbf{O}_q^p,$$

if for any integer  $l \geq 0$  we have

$$|(\partial_K)^l X_K| \lesssim \frac{M^p}{r_K^{q+l}},$$

where  $r_K = \Phi^*(r_{Kerr})$  and

$$\partial_K := \{(\nabla_K)_{(e_3)_K}, \nabla_K, (\nabla_K)_{(e_4)_K}\}.$$

**Proposition 2.28.** We have the following formulae:

$$\begin{aligned} \operatorname{tr}_K \chi_K - \frac{2}{r_K} &= \mathbf{O}_2^1, & \operatorname{tr}_K \underline{\chi}_K + \frac{2}{r_K} &= \mathbf{O}_2^1, \\ \omega_K &= \mathbf{O}_2^1, & \underline{\omega}_K &= \mathbf{O}_2^1, \\ \widehat{\chi}_K &= \mathbf{O}_3^2, & \widehat{\underline{\chi}}_K &= \mathbf{O}_3^2, \\ \nabla_K \operatorname{tr}_K \chi_K &= \mathbf{O}_4^2, & \nabla_K \operatorname{tr}_K \underline{\chi}_K &= \mathbf{O}_4^2, \\ \eta_K &= \mathbf{O}_3^2, & \underline{\eta}_K &= \mathbf{O}_3^2, \\ \underline{b}_K &= \mathbf{O}_2^2, & \Omega_K - \frac{1}{2} &= \mathbf{O}_1^1. \end{aligned}$$

We also have:

$$\begin{aligned} \alpha_K &= \mathbf{O}_5^3, & \underline{\alpha}_K &= \mathbf{O}_5^3, \\ \beta_K &= \mathbf{O}_4^2, & \underline{\beta}_K &= \mathbf{O}_4^2, \\ \rho_K &= \mathbf{O}_3^1, & \sigma_K &= \mathbf{O}_4^2. \end{aligned}$$

*Proof.* It follows directly from Propositions 2.4, 2.5 and Definition 2.27.  $\square$

**Remark 2.29.** In the sequel, we consider the following conventions:

- For a quantity  $h$  satisfying the same or even better decay as  $\Gamma_b$ , (resp.  $\Gamma_g$ ), we write

$$h \in \Gamma_b, \quad (\text{resp. } h \in \Gamma_g).$$

- For a sum of schematic notations, we ignore the terms that decay better. For example, we write

$$\Gamma_g + \Gamma_b = \Gamma_b, \quad \Gamma_g \cdot \mathbf{O}_q^p + \Gamma_b \cdot \mathbf{O}_q^p = \Gamma_b \cdot \mathbf{O}_q^p,$$

since  $\Gamma_g$  decays better than  $\Gamma_b$  throughout this paper.

**Lemma 2.30.** We have the following formulae:

$$\begin{aligned} \operatorname{tr} \chi - \frac{2}{r} &= \mathbf{O}_2^1 + \Gamma_g, & \operatorname{tr} \underline{\chi} + \frac{2}{r} &= \mathbf{O}_2^1 + \Gamma_b, \\ \widehat{\chi} &= \mathbf{O}_3^2 + \Gamma_g, & \widehat{\underline{\chi}} &= \mathbf{O}_3^2 + \Gamma_b, \\ \eta &= \mathbf{O}_3^2 + \Gamma_g, & \underline{\eta} &= \mathbf{O}_3^2 + \Gamma_g, \\ \omega &= \mathbf{O}_2^1 + \Gamma_g, & \underline{\omega} &= \mathbf{O}_2^1 + \Gamma_b, \\ \underline{b} &= \mathbf{O}_2^2 + r\Gamma_g, & \Omega - \frac{1}{2} &= \mathbf{O}_2^1 + r\Gamma_g, \\ \gamma &= \mathbf{O}_0^0 + r\Gamma_g, & \epsilon &= \mathbf{O}_0^0 + r\Gamma_g. \end{aligned}$$

We also have:

$$\begin{aligned}\alpha &= \mathbf{O}_5^3 + \check{\alpha}, & \underline{\alpha} &= \mathbf{O}_5^3 + \check{\alpha}, \\ \beta &= \mathbf{O}_4^2 + \check{\beta}, & \underline{\beta} &= \mathbf{O}_4^2 + \check{\beta}, \\ \rho &= \mathbf{O}_3^1 + \check{\rho}, & \sigma &= \mathbf{O}_4^2 + \check{\sigma}.\end{aligned}$$

*Proof.* It follows directly from Definitions 2.25, 2.27 and Proposition 2.28.  $\square$

The following lemma allows us to compute the derivatives of  $\mathbf{O}_q^p$ .

**Lemma 2.31.** *Let  $X_K$  be a  $S$ -tangent tensor field satisfying*

$$X_K = \mathbf{O}_q^p.$$

*Then, we have*

$$\begin{aligned}\Omega \nabla_3 X_K &= \mathbf{O}_{q+1}^p + \mathbf{O}_q^p \cdot \Gamma_g + \mathbf{O}_q^p \cdot \check{J} + \mathbf{O}_q^p \cdot \check{L}, \\ \Omega \nabla_4 X_K &= \mathbf{O}_{q+1}^p + \mathbf{O}_q^p \cdot \check{J}, \\ \nabla X_K &= \mathbf{O}_{q+1}^p + \mathbf{O}_q^p \cdot \check{L}.\end{aligned}$$

*Proof.* We have from Lemma 2.21 and Proposition 2.22

$$\begin{aligned}\Omega \nabla_3 X_K - \Omega_K \nabla_{(e_3)_K} X_K &= (\nabla_{\Omega e_3} - \nabla_{\Omega_K(e_3)_K}) X_K + (\nabla_{\Omega_K(e_3)_K} - (\nabla_K)_{\Omega_K(e_3)_K}) X_K \\ &= \check{b}^A \nabla_{\partial_{x^A}} X_K + (\nabla - \nabla_K)_{\partial_u} X_K + \check{b}_K^A (\nabla - \nabla_K)_{\partial_{x^A}} X_K \\ &= \check{b}^A (\nabla - \nabla_K)_{\partial_{x^A}} X_K + \check{b}^A (\nabla_K)_{\partial_{x^A}} X_K + \check{J} \cdot \mathbf{O}_q^p + \check{L} \cdot \mathbf{O}_q^p \\ &= \Gamma_g \cdot \mathbf{O}_q^p + \check{J} \cdot \mathbf{O}_q^p + \check{L} \cdot \mathbf{O}_q^p,\end{aligned}$$

where we used Remark 2.29 to ignore the terms having better decay. Combining with Definition 2.27, we obtain

$$\Omega \nabla_3 X_K = \mathbf{O}_{q+1}^p + \Gamma_g \cdot \mathbf{O}_q^p + \check{J} \cdot \mathbf{O}_q^p + \check{L} \cdot \mathbf{O}_q^p.$$

Similarly, we have from Lemma 2.21 and Proposition 2.22

$$\begin{aligned}\Omega \nabla_4 X_K - \Omega_K \nabla_{(e_4)_K} X_K &= (\nabla_{\Omega e_4} - \nabla_{\Omega_K(e_4)_K}) X_K + (\nabla_{\Omega_K(e_4)_K} - (\nabla_K)_{\Omega_K(e_4)_K}) X_K \\ &= (\nabla - \nabla_K)_{\partial_u} X_K \\ &= \check{J} \cdot \mathbf{O}_q^p.\end{aligned}$$

Combining with Definition 2.27, we deduce

$$\Omega \nabla_4 X_K = \mathbf{O}_{q+1}^p + \check{J} \cdot \mathbf{O}_q^p.$$

Finally, we have from Proposition 2.22 and Definition 2.27

$$\nabla X_K = \nabla_K X_K + (\nabla - \nabla_K) X_K = \mathbf{O}_{q+1}^p + \check{L} \cdot \mathbf{O}_q^p.$$

This concludes the proof of Lemma 2.31.  $\square$

**Corollary 2.32.** *We have the following schematic computation:*

$$\nabla(\mathbf{O}_q^p \cdot \Gamma_g) = \mathbf{O}_{q+1}^p \cdot \Gamma_g^{(1)}, \quad \nabla(\mathbf{O}_q^p \cdot \Gamma_b) = \mathbf{O}_{q+1}^p \cdot \Gamma_b^{(1)}.$$

*Proof.* We have from Lemma 2.31

$$\begin{aligned}\nabla(\mathbf{O}_q^p \cdot \Gamma_i) &= \mathbf{O}_q^p \cdot \nabla \Gamma_i + \nabla \mathbf{O}_q^p \cdot \Gamma_i \\ &= \mathbf{O}_{q+1}^p \cdot \Gamma_i^{(1)} + (\mathbf{O}_{q+1}^p + \mathbf{O}_q^p \cdot \check{L}) \cdot \Gamma_i \\ &= \mathbf{O}_{q+1}^p \cdot \Gamma_i^{(1)},\end{aligned}$$

for  $i = g, b$ . This concludes the proof of Corollary 2.32.  $\square$

**Remark 2.33.** *In the remainder of this paper, Lemma 2.31 and Corollary 2.32 will be used frequently without explicit mention.*

### 2.8.4 The linearized null structure equations

In this section, we introduce the following linearized null structure equations, which play a fundamental role in Ricci coefficients estimates.

**Proposition 2.34.** *We have the following linearized null structure equations:*

$$\begin{aligned}\nabla_4 \widetilde{\Omega \operatorname{tr} \chi} + \operatorname{tr} \chi \widetilde{\Omega \operatorname{tr} \chi} &= \mathbf{O}_1^0 \cdot \widetilde{\Omega \omega} + \mathfrak{L}[\nabla_4 \widetilde{\operatorname{tr} \chi}] + \operatorname{Err}[\nabla_4 \widetilde{\operatorname{tr} \chi}], \\ \mathfrak{L}[\nabla_4 \widetilde{\operatorname{tr} \chi}] &:= \Gamma_g \cdot \mathbf{O}_2^1, \\ \operatorname{Err}[\nabla_4 \widetilde{\operatorname{tr} \chi}] &:= \Gamma_g \cdot \Gamma_g,\end{aligned}$$

$$\begin{aligned}\nabla_3 \widetilde{\Omega \operatorname{tr} \underline{\chi}} + \operatorname{tr} \underline{\chi} \widetilde{\Omega \operatorname{tr} \underline{\chi}} &= \mathbf{O}_1^0 \cdot \widetilde{\Omega \omega} + \mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \underline{\chi}}] + \operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \underline{\chi}}], \\ \mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \underline{\chi}}] &:= \Gamma_g \cdot \mathbf{O}_2^1, \\ \operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \underline{\chi}}] &:= \Gamma_b \cdot \Gamma_b,\end{aligned}$$

$$\begin{aligned}\nabla_4(\widetilde{\Omega \operatorname{tr} \underline{\chi}}) &= -2\Omega[\widetilde{\mu}] + \mathfrak{L}[\nabla_4 \widetilde{\operatorname{tr} \underline{\chi}}] + \operatorname{Err}[\nabla_4 \widetilde{\operatorname{tr} \underline{\chi}}], \\ \mathfrak{L}[\nabla_4 \widetilde{\operatorname{tr} \underline{\chi}}] &:= \Gamma_g \cdot \mathbf{O}_3^2, \\ \operatorname{Err}[\nabla_4 \widetilde{\operatorname{tr} \underline{\chi}}] &:= \Gamma_g \cdot \Gamma_g,\end{aligned}$$

$$\begin{aligned}\nabla_3(\widetilde{\Omega \operatorname{tr} \chi}) &= -2\Omega[\widetilde{\mu}] + \mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \chi}] + \operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \chi}], \\ \mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \chi}] &:= \Gamma_g \cdot \mathbf{O}_3^2, \\ \operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \chi}] &:= \Gamma_g \cdot \Gamma_g.\end{aligned}$$

We also have the linearized Codazzi equations:

$$\begin{aligned}\operatorname{div}(\check{\chi}) &= -\check{b} + \mathbf{O}_0^0 \cdot \nabla \widetilde{\operatorname{tr} \chi} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathfrak{L}[\operatorname{div}(\check{\chi})] + \operatorname{Err}[\operatorname{div}(\check{\chi})], \\ \mathfrak{L}[\operatorname{div}(\check{\chi})] &:= \Gamma_g \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{div}(\check{\chi})] &:= \Gamma_g \cdot \Gamma_g,\end{aligned}$$

$$\begin{aligned}\operatorname{div}(\check{\underline{\chi}}) &= \check{\beta} + \mathbf{O}_0^0 \cdot \nabla \widetilde{\operatorname{tr} \underline{\chi}} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathfrak{L}[\operatorname{div}(\check{\underline{\chi}})] + \operatorname{Err}[\operatorname{div}(\check{\underline{\chi}})], \\ \mathfrak{L}[\operatorname{div}(\check{\underline{\chi}})] &:= \Gamma_b \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{div}(\check{\underline{\chi}})] &:= \Gamma_b \cdot \Gamma_g.\end{aligned}$$

The linearized mass aspect functions are given by:

$$\begin{aligned}\operatorname{div} \check{\eta} &= -\check{\mu} - \check{\rho} + \mathfrak{L}[\operatorname{div}(\check{\eta})] + \operatorname{Err}[\operatorname{div}(\check{\eta})], \\ \mathfrak{L}[\operatorname{div}(\check{\eta})] &= \Gamma_b \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{div}(\check{\eta})] &= \Gamma_g \cdot \Gamma_b,\end{aligned}$$

$$\begin{aligned}\operatorname{div} \check{\eta} &= -\check{\mu} - \check{\rho} + \mathfrak{L}[\operatorname{div}(\check{\eta})] + \operatorname{Err}[\operatorname{div}(\check{\eta})], \\ \mathfrak{L}[\operatorname{div}(\check{\eta})] &= \Gamma_b \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{div}(\check{\eta})] &= \Gamma_g \cdot \Gamma_b.\end{aligned}$$

The linearized torsion equations are given by:

$$\begin{aligned}\operatorname{curl} \check{\eta} &= \check{\sigma} + \mathfrak{L}[\operatorname{curl} \check{\eta}] + \operatorname{Err}[\operatorname{curl} \check{\eta}], \\ \mathfrak{L}[\operatorname{curl} \check{\eta}] &:= \Gamma_b \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{curl} \check{\eta}] &:= \Gamma_b \cdot \Gamma_g,\end{aligned}$$

$$\begin{aligned}\operatorname{curl} \check{\eta} &= -\check{\sigma} + \mathfrak{L}[\operatorname{curl} \check{\eta}] + \operatorname{Err}[\operatorname{curl} \check{\eta}], \\ \mathfrak{L}[\operatorname{curl} \check{\eta}] &:= \Gamma_b \cdot \mathbf{O}_3^2 + \mathbf{O}_3^2 \cdot \check{L}, \\ \operatorname{Err}[\operatorname{curl} \check{\eta}] &:= \Gamma_b \cdot \Gamma_g.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\Omega \nabla_4(\check{\omega}) &= \frac{1}{2} \check{\omega}^2 \rho + \mathfrak{L}[\nabla_4 \check{\omega}] + \operatorname{Err}[\nabla_4 \check{\omega}], \\ \mathfrak{L}[\nabla_4 \check{\omega}] &:= \Gamma_g \cdot \mathbf{O}_2^1, \\ \operatorname{Err}[\nabla_4 \check{\omega}] &:= \Gamma_g \cdot \Gamma_g,\end{aligned}$$

$$\begin{aligned}\Omega \nabla_3(\check{\omega}) &= \frac{1}{2} \check{\omega}^2 \rho + \mathfrak{L}[\nabla_3 \check{\omega}] + \operatorname{Err}[\nabla_3 \check{\omega}], \\ \mathfrak{L}[\nabla_3 \check{\omega}] &:= \Gamma_g \cdot \mathbf{O}_2^1, \\ \operatorname{Err}[\nabla_3 \check{\omega}] &:= \Gamma_g \cdot \Gamma_g.\end{aligned}$$

*Proof.* See Appendix A.1. □

**Remark 2.35.** In the linearized equations of Proposition 2.34, as well as in the other linearized equations below,  $\mathfrak{L}[\dots]$  contains linear terms which display more decay in  $\frac{M}{r}$  and  $\operatorname{Err}[\dots]$  contains the nonlinear terms.

**Proposition 2.36.** We have the following linearized equations:

$$\begin{aligned}\Omega \nabla_4[\check{\mu}] + \Omega \operatorname{tr} \chi[\check{\mu}] &:= \Omega \operatorname{tr} \chi \check{\rho} + \mathfrak{L}[\nabla_4[\check{\mu}]] + \operatorname{Err}[\nabla_4[\check{\mu}]], \\ \mathfrak{L}[\nabla_4[\check{\mu}]] &:= \mathbf{O}_3^2 \cdot (d_2^* \check{\eta}, \nabla \operatorname{tr} \check{\chi}, \check{b}) + \mathbf{O}_3^1 \cdot (\Gamma_g, \check{L}), \\ \operatorname{Err}[\nabla_4[\check{\mu}]] &:= -2\check{\chi} \cdot (d_2^* \check{\eta}) + (\check{\eta} - \check{\eta}) \cdot \nabla \operatorname{tr} \check{\chi} - 2\check{\Omega} \check{\eta} \cdot \check{b} + r^{-1} \Gamma_g \cdot \check{L} + r^{-1} \Gamma_g \cdot \Gamma_b,\end{aligned}$$

$$\begin{aligned}\Omega \nabla_3[\check{\mu}] + \Omega \operatorname{tr} \chi[\check{\mu}] &:= \Omega \operatorname{tr} \chi \check{\rho} + \mathfrak{L}[\nabla_3[\check{\mu}]] + \operatorname{Err}[\nabla_3[\check{\mu}]], \\ \mathfrak{L}[\nabla_3[\check{\mu}]] &:= \mathbf{O}_3^2 \cdot (d_2^* \check{\eta}, \nabla \operatorname{tr} \check{\chi}, \check{\beta}) + \mathbf{O}_3^1 \cdot (\Gamma_g, \check{L}), \\ \operatorname{Err}[\nabla_3[\check{\mu}]] &:= -2\check{\chi} \cdot (d_2^* \check{\eta}) + (\check{\eta} - \check{\eta}) \cdot \nabla \operatorname{tr} \check{\chi} + 2\check{\Omega} \check{\eta} \cdot \check{\beta} + r^{-1} \Gamma_b \cdot \check{L} + r^{-1} \Gamma_b \cdot \Gamma_b.\end{aligned}$$

*Proof.* See Appendix A.2. □



### 2.8.5 The linearized Bianchi equations

In this section, we introduce the linearized Bianchi equations, which play a fundamental role in curvature estimates.

**Proposition 2.37.** *We have the following linearized Bianchi equations:*

$$\begin{aligned}\nabla_4 \underline{\dot{\alpha}} + \frac{1}{2} \operatorname{tr} \chi \underline{\dot{\alpha}} &= \nabla \widehat{\otimes} \underline{\check{\beta}} + \mathfrak{L}[\nabla_4 \underline{\dot{\alpha}}] + \operatorname{Err}[\nabla_4 \underline{\dot{\alpha}}], \\ \mathfrak{L}[\nabla_4 \underline{\dot{\alpha}}] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\dot{\alpha}} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}), \\ \operatorname{Err}[\nabla_4 \underline{\dot{\alpha}}] &:= \Gamma_g \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}) + \Gamma_b \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \underline{\check{\beta}} + 2 \operatorname{tr} \underline{\chi} \underline{\check{\beta}} &= -\operatorname{div} \underline{\dot{\alpha}} + \mathfrak{L}[\nabla_3 \underline{\check{\beta}}] + \operatorname{Err}[\nabla_3 \underline{\check{\beta}}], \\ \mathfrak{L}[\nabla_3 \underline{\check{\beta}}] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}} + \mathbf{O}_3^2 \cdot \underline{\dot{\alpha}}, \\ \operatorname{Err}[\nabla_3 \underline{\check{\beta}}] &:= \Gamma_b \cdot \underline{\check{\beta}} + \Gamma_g \cdot \underline{\dot{\alpha}},\end{aligned}$$

$$\begin{aligned}\nabla_4 \underline{\check{\beta}} + \operatorname{tr} \chi \underline{\check{\beta}} &= -\nabla \underline{\check{\rho}} + * \nabla \underline{\check{\sigma}} + \mathfrak{L}[\nabla_4 \underline{\check{\beta}}] + \operatorname{Err}[\nabla_4 \underline{\check{\beta}}], \\ \mathfrak{L}[\nabla_4 \underline{\check{\beta}}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}} + \mathbf{O}_3^2 \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}, \underline{\check{b}}), \\ \operatorname{Err}[\nabla_4 \underline{\check{\beta}}] &:= \Gamma_g \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_b \cdot \underline{\check{b}},\end{aligned}$$

$$\begin{aligned}\nabla_3 (\underline{\check{\rho}}, \underline{\check{\sigma}}) + \frac{3}{2} \operatorname{tr} \underline{\chi} (\underline{\check{\rho}}, \underline{\check{\sigma}}) &= -d_1 \underline{\check{\beta}} + \mathfrak{L}[\nabla_3 (\underline{\check{\rho}}, \underline{\check{\sigma}})] + \operatorname{Err}[\nabla_3 (\underline{\check{\rho}}, \underline{\check{\sigma}})], \\ \mathfrak{L}[\nabla_3 (\underline{\check{\rho}}, \underline{\check{\sigma}})] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}), \\ \operatorname{Err}[\nabla_3 (\underline{\check{\rho}}, \underline{\check{\sigma}})] &:= \Gamma_g \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}),\end{aligned}$$

$$\begin{aligned}\nabla_4 (\underline{\check{\rho}}, -\underline{\check{\sigma}}) + \frac{3}{2} \operatorname{tr} \chi (\underline{\check{\rho}}, -\underline{\check{\sigma}}) &= d_1 \underline{\check{b}} + \mathfrak{L}[\nabla_4 (\underline{\check{\rho}}, -\underline{\check{\sigma}})] + \operatorname{Err}[\nabla_4 (\underline{\check{\rho}}, -\underline{\check{\sigma}})], \\ \mathfrak{L}[\nabla_4 (\underline{\check{\rho}}, -\underline{\check{\sigma}})] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\check{\alpha}}, \underline{\check{b}}), \\ \operatorname{Err}[\nabla_4 (\underline{\check{\rho}}, -\underline{\check{\sigma}})] &:= \Gamma_b \cdot \underline{\check{\alpha}} + \Gamma_g \cdot \underline{\check{b}},\end{aligned}$$

$$\begin{aligned}\nabla_3 \underline{\check{b}} + \operatorname{tr} \underline{\chi} \underline{\check{b}} &= \nabla \underline{\check{\rho}} + * \nabla \underline{\check{\sigma}} + \mathfrak{L}[\nabla_3 \underline{\check{b}}] + \operatorname{Err}[\nabla_3 \underline{\check{b}}], \\ \mathfrak{L}[\nabla_3 \underline{\check{b}}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{b}} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}), \\ \operatorname{Err}[\nabla_3 \underline{\check{b}}] &:= \Gamma_b \cdot \underline{\check{b}} + \Gamma_g \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}),\end{aligned}$$

$$\begin{aligned}\nabla_4 \underline{\check{b}} + 2 \operatorname{tr} \chi \underline{\check{b}} &= \operatorname{div} \underline{\check{\alpha}} + \mathfrak{L}[\nabla_4 \underline{\check{b}}] + \operatorname{Err}[\nabla_4 \underline{\check{b}}], \\ \mathfrak{L}[\nabla_4 \underline{\check{b}}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\underline{\check{b}}, \underline{\check{\alpha}}), \\ \operatorname{Err}[\nabla_4 \underline{\check{b}}] &:= \Gamma_g \cdot (\underline{\check{b}}, \underline{\check{\alpha}}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \underline{\check{\alpha}} + \frac{1}{2} \operatorname{tr} \chi \underline{\check{\alpha}} &= \nabla \widehat{\otimes} \underline{\check{b}} + \mathfrak{L}[\nabla_3 \underline{\check{\alpha}}] + \operatorname{Err}[\nabla_3 \underline{\check{\alpha}}], \\ \mathfrak{L}[\nabla_3 \underline{\check{\alpha}}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{\alpha}} + \mathbf{O}_3^2 \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}, \underline{\check{b}}), \\ \operatorname{Err}[\nabla_3 \underline{\check{\alpha}}] &:= \Gamma_b \cdot \underline{\check{\alpha}} + \Gamma_g \cdot \underline{\check{b}}.\end{aligned}$$

*Proof.* See Appendix A.3. □

**Proposition 2.38.** *We have the following linearized equations:<sup>14</sup>*

$$\begin{aligned}\nabla_4 \widetilde{\vartheta}_\alpha + \frac{1}{2} \operatorname{tr} \chi \widetilde{\vartheta}_\alpha &= -\nabla \widehat{\otimes} \widetilde{\vartheta}_\beta + \mathcal{L}[\nabla_4 \widetilde{\vartheta}_\alpha] + \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_\alpha], \\ \mathcal{L}[\nabla_4 \widetilde{\vartheta}_\alpha] &= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\dot{\alpha}}^{(1)} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}), \\ \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_\alpha] &= \Gamma_g^{(1)} \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}) + \Gamma_b^{(1)} \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\dot{\alpha}}^{(1)}, \underline{\check{\beta}}^{(1)}) + \Gamma_b \cdot (\underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \widetilde{\vartheta}_\beta + 2 \operatorname{tr} \chi \widetilde{\vartheta}_\beta &= -\operatorname{div} \widetilde{\vartheta}_\alpha + \mathcal{L}[\nabla_3 \widetilde{\vartheta}_\beta] + \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_\beta], \\ \mathcal{L}[\nabla_3 \widetilde{\vartheta}_\beta] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}}^{(1)} + \mathbf{O}_3^2 \cdot \underline{\dot{\alpha}}^{(1)}, \\ \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_\beta] &:= \Gamma_b^{(1)} \cdot \underline{\check{\beta}} + \Gamma_b \cdot \underline{\check{\beta}}^{(1)} + \Gamma_g^{(1)} \cdot \underline{\dot{\alpha}} + \Gamma_g \cdot \underline{\dot{\alpha}}^{(1)},\end{aligned}$$

$$\begin{aligned}\nabla_4 \widetilde{\vartheta}_\beta + \operatorname{tr} \chi \widetilde{\vartheta}_\beta &:= -\nabla \widetilde{\vartheta}_\rho + * \nabla \widetilde{\vartheta}_\sigma + \mathcal{L}[\nabla_4 \widetilde{\vartheta}_\beta] + \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_\beta], \\ \mathcal{L}[\nabla_4 \widetilde{\vartheta}_\beta] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}}^{(1)} + \mathbf{O}_3^2 \cdot (\underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}, \underline{\check{b}}^{(1)}), \\ \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_\beta] &:= \Gamma_g^{(1)} \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}) + \Gamma_b^{(1)} \cdot \underline{\check{b}} + \Gamma_b \cdot \underline{\check{b}}^{(1)},\end{aligned}$$

$$\begin{aligned}\nabla_3(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma) + \frac{3}{2} \operatorname{tr} \chi(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma) &= -d_1 \widetilde{\vartheta}_\beta + \mathcal{L}[\nabla_3(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma)] + \operatorname{Err}[\nabla_3(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma)], \\ \mathcal{L}[\nabla_3(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma)] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\dot{\alpha}}^{(1)}, \underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}), \\ \operatorname{Err}[\nabla_3(\widetilde{\vartheta}_\rho, \widetilde{\vartheta}_\sigma)] &:= \Gamma_g^{(1)} \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}) + \Gamma_g \cdot (\underline{\dot{\alpha}}^{(1)}, \underline{\check{\beta}}^{(1)}) + \Gamma_b \cdot (\underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}) + \Gamma_b^{(1)} \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}),\end{aligned}$$

$$\begin{aligned}\nabla_4(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma) + \frac{3}{2} \operatorname{tr} \chi(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma) &= d_1 \widetilde{\vartheta}_\beta + \mathcal{L}[\nabla_4(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma)] + \operatorname{Err}[\nabla_4(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma)], \\ \mathcal{L}[\nabla_4(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma)] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\check{\alpha}}^{(1)}, \underline{\check{b}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}), \\ \operatorname{Err}[\nabla_4(\widetilde{\vartheta}_\rho, -\widetilde{\vartheta}_\sigma)] &:= \Gamma_b^{(1)} \cdot \underline{\check{\alpha}} + \Gamma_b \cdot \underline{\check{\alpha}}^{(1)} + \Gamma_g^{(1)} \cdot (\underline{\check{b}}, \underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\check{b}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \widetilde{\vartheta}_b + \operatorname{tr} \chi \widetilde{\vartheta}_b &= \nabla \widetilde{\vartheta}_\rho + * \nabla \widetilde{\vartheta}_\sigma + \mathcal{L}[\nabla_3 \widetilde{\vartheta}_b] + \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_b], \\ \mathcal{L}[\nabla_3 \widetilde{\vartheta}_b] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{b}}^{(1)} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}), \\ \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_b] &:= \Gamma_b^{(1)} \cdot \underline{\check{b}} + \Gamma_b \cdot \underline{\check{b}}^{(1)} + \Gamma_g^{(1)} \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_4 \widetilde{\vartheta}_b + 2 \operatorname{tr} \chi \widetilde{\vartheta}_b &= \operatorname{div} \widetilde{\vartheta}_\alpha + \mathcal{L}[\nabla_4 \widetilde{\vartheta}_b] + \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_b], \\ \mathcal{L}[\nabla_4 \widetilde{\vartheta}_b] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\underline{\check{b}}^{(1)}, \underline{\check{\alpha}}^{(1)}), \\ \operatorname{Err}[\nabla_4 \widetilde{\vartheta}_b] &:= \Gamma_g^{(1)} \cdot (\underline{\check{b}}, \underline{\check{\alpha}}) + \Gamma_g \cdot (\underline{\check{b}}^{(1)}, \underline{\check{\alpha}}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \widetilde{\vartheta}_\alpha + \frac{1}{2} \operatorname{tr} \chi \widetilde{\vartheta}_\alpha &= \nabla \widehat{\otimes} \widetilde{\vartheta}_b + \mathcal{L}[\nabla_3 \widetilde{\vartheta}_\alpha] + \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_\alpha], \\ \mathcal{L}[\nabla_3 \widetilde{\vartheta}_\alpha] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\check{\alpha}}^{(1)} + \mathbf{O}_3^2 \cdot (\underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}, \underline{\check{b}}^{(1)}), \\ \operatorname{Err}[\nabla_3 \widetilde{\vartheta}_\alpha] &:= \Gamma_b^{(1)} \cdot \underline{\check{\alpha}} + \Gamma_b \cdot \underline{\check{\alpha}}^{(1)} + \Gamma_g^{(1)} \cdot \underline{\check{b}} + \Gamma_g \cdot \underline{\check{b}}^{(1)}.\end{aligned}$$

**Remark 2.39.** *Notice that Proposition 2.38 involves the quantities  $\widetilde{\vartheta}_\alpha$ ,  $\widetilde{\vartheta}_\beta$ ,  $\widetilde{\vartheta}_\rho$ ,  $\widetilde{\vartheta}_\sigma$ ,  $\widetilde{\vartheta}_b$  and*

<sup>14</sup>Recall that  $\vartheta$  is introduced in Definition 2.13.

$\check{\varphi}\underline{\alpha}$  instead of  $\check{\varphi}\check{\alpha}$ ,  $\check{\varphi}\check{b}$ ,  $\check{\varphi}\check{\rho}$ ,  $\check{\varphi}\check{\sigma}$ ,  $\check{\varphi}\check{\beta}$  and  $\check{\varphi}\check{\hat{\alpha}}$ . This allows to avoid a loss of a derivative of  $\check{J}$ ,  $\check{J}$  and  $\check{L}$  in the linearization procedure. Indeed, we have for example

$$\check{\varphi}\check{\beta} = \check{\varphi}\check{b} + (\check{J}, \check{J}, \check{L}) \cdot \beta_K,$$

and the norm  $\mathcal{O}_\Gamma$  in Section 3.1.2 only controls  $(\check{J}, \check{J}, \check{L})$  but not its first order derivatives.

*Proof.* See Appendix A.3. □

### 2.8.6 Linearized equations for the metric components

**Proposition 2.40.** *We have the following transport equations:*

$$\begin{aligned} \nabla_4(\check{\gamma}_{AB}) - \text{tr } \chi(\check{\gamma}_{AB}) &= \mathbf{O}_{-2}^0 \cdot (\check{\chi}, \widetilde{\text{tr } \chi}) + O(r)\check{\Omega} + r^3\Gamma_g \cdot \Gamma_g, \\ \nabla_4(\check{\Omega}) &= \mathbf{O}_0^0 \cdot \check{\Omega}\omega + \mathbf{O}_1^1 \cdot \Gamma_g + r\Gamma_g \cdot \Gamma_g, \\ \nabla_4(\check{b}^A) &= \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_g + \Gamma_g \cdot \Gamma_g, \\ \nabla_4(\check{\epsilon}_{AB}) - \text{tr } \chi(\check{\epsilon}_{AB}) &= \mathbf{O}_{-2}^0 \cdot \widetilde{\text{tr } \chi} + r^3\Gamma_g \cdot \Gamma_g. \end{aligned}$$

Moreover, we have the following transport equations in the case  $\check{\Omega} = 0$  and  $\check{b} = 0$ .<sup>15</sup>

$$\begin{aligned} \nabla_3(\check{\gamma}_{AB}) - \text{tr } \underline{\chi}(\check{\gamma}_{AB}) &= \mathbf{O}_{-2}^0 \cdot (\widetilde{\text{tr } \underline{\chi}}, \check{\chi}) + \mathbf{O}_3^2 \cdot \check{\gamma}_{AB}, \\ \nabla_3(\check{\epsilon}_{AB}) - \text{tr } \underline{\chi}(\check{\epsilon}_{AB}) &= \mathbf{O}_{-2}^0 \cdot \widetilde{\text{tr } \underline{\chi}} + \mathbf{O}_3^2 \cdot \check{\epsilon}_{AB}. \end{aligned}$$

*Proof.* See Appendix A.4. □

## 2.9 Commutator identities

We recall the following commutator identities for scalar functions, see Proposition 4.8.1 of [29].

**Proposition 2.41.** *For a scalar function  $f$ , we have*

$$\begin{aligned} [\Omega\nabla_4, \nabla]f &= -\Omega\underline{\chi} \cdot \nabla f, \\ [\Omega\nabla_3, \nabla]f &= -\Omega\underline{\chi} \cdot \nabla f, \\ [\nabla_4, \nabla_3]f &= 2\omega\nabla_3 f - 2\underline{\omega}\nabla_4 f - 4\underline{\zeta} \cdot \nabla f. \end{aligned}$$

We also need the commutation identities for more general tensor fields. For this, we record the following commutation lemma.

**Lemma 2.42.** *Let  $U_{a_1 \dots a_k}$  be an  $S$ -tangent  $k$ -covariant tensor on  $(\mathcal{M}, \mathbf{g})$ . Then*

$$\begin{aligned} [\nabla_4, \nabla_b]U_{a_1 \dots a_k} &= -\chi_{BC}\nabla_c U_{a_1 \dots a_k} + F_{4ba_1 \dots a_k}, \\ F_{4ba_1 \dots a_k} &:= (\zeta_b + \underline{\eta}_b)\nabla_4 U_{a_1 \dots a_k} + \sum_{i=1}^k (\chi_{a_i b} \underline{\eta}_c - \chi_{BC} \underline{\eta}_{a_i} + \epsilon_{a_i c} \beta_b) U_{a_1 \dots c \dots a_k}, \\ [\nabla_3, \nabla_b]U_{a_1 \dots a_k} &= -\underline{\chi}_{BC}\nabla_c U_{a_1 \dots a_k} + F_{3ba_1 \dots a_k}, \\ F_{3ba_1 \dots a_k} &:= (\eta_b - \zeta_b)\nabla_3 U_{a_1 \dots a_k} + \sum_{i=1}^k (\underline{\chi}_{a_i b} \eta_c - \underline{\chi}_{BC} \eta_{a_i} + \epsilon_{a_i c} \beta_b) U_{a_1 \dots c \dots a_k}, \\ [\nabla_3, \nabla_4]U_{a_1 \dots a_k} &= F_{34a_1 \dots a_k}, \\ F_{34a_1 \dots a_k} &:= -2\omega\nabla_3 U + 2\underline{\omega}\nabla_4 U + 4\underline{\zeta}_b \nabla_b U_{a_1 \dots a_k} + 2 \sum_{i=1}^k (\underline{\eta}_{a_i} \eta_c - \underline{\eta}_{a_i} \eta_c + \epsilon_{a_i c} \sigma) U_{a_1 \dots c \dots a_k}. \end{aligned}$$

<sup>15</sup>The conditions  $\check{\Omega} = 0$  and  $\check{b} = 0$  hold on  $\underline{C}_*$ , see Definition 6.1.

*Proof.* It is a direct consequence of Lemma 7.3.3 in [10]. □

**Corollary 2.43.** *We have the following schematic commutation identities:*

$$\begin{aligned}
 [r\nabla, \Omega\nabla_4] &= \Gamma_g \cdot r\nabla + \mathbf{O}_3^2 \cdot r\nabla + \Gamma_g^{(1)} + \mathbf{O}_3^2, \\
 [r\nabla, \Omega\nabla_3] &= \Gamma_b \cdot r\nabla + \mathbf{O}_3^2 \cdot r\nabla + \Gamma_b^{(1)} + \mathbf{O}_3^2, \\
 [\nabla, \Omega\nabla_4] &= -\frac{1}{2}\Omega \operatorname{tr} \chi \nabla + \Gamma_g \cdot \nabla + \mathbf{O}_3^2 \cdot \nabla + r^{-1}\Gamma_g^{(1)} + \mathbf{O}_4^2, \\
 [\nabla, \Omega\nabla_3] &= -\frac{1}{2}\Omega \operatorname{tr} \underline{\chi} \nabla + \Gamma_b \cdot \nabla + \mathbf{O}_3^2 \cdot \nabla + r^{-1}\Gamma_b^{(1)} + \mathbf{O}_4^2, \\
 [\nabla_4, \nabla_3] &= \Gamma_g \cdot \nabla_3 + \Gamma_b \cdot (\nabla_4, \nabla) + \mathbf{O}_2^1 \cdot (\nabla_3, \nabla_4, \nabla) + r^{-1}\Gamma_g^{(1)} + \mathbf{O}_4^2, \\
 [\Omega\nabla_4, \Omega\nabla_3] &= \Gamma_b \cdot \nabla + \mathbf{O}_3^2 \cdot \nabla + r^{-1}\Gamma_g^{(1)} + \mathbf{O}_4^2.
 \end{aligned}$$

*Proof.* It follows directly from Lemmas 2.42 and 2.30 together with the identities (2.8). □

### 2.10 Sobolev inequalities

**Definition 2.44.** *For a tensor field  $h$  on a null cone  $C$ , we define its  $L^2$ -flux:*

$$\|h\|_{2,C} := \left( \int_C |h|^2 \right)^{\frac{1}{2}}. \tag{2.44}$$

We recall the following Sobolev inequalities.

**Proposition 2.45.** *Let  $F$  be a tensor field, tangent to  $S := S(u, \underline{u})$  at every point. We have the following estimates:*

$$\begin{aligned}
 |rF|_{4,S} &\lesssim \|F\|_{2,C_u \cap V(u, \underline{u})} + \|r\nabla F\|_{2,C_u \cap V(u, \underline{u})} + \|r\nabla_4 F\|_{2,C_u \cap V(u, \underline{u})}, \\
 |r^{\frac{1}{2}}|u|^{\frac{1}{2}}F|_{4,S} &\lesssim \|F\|_{2,\underline{C}_u \cap V(u, \underline{u})} + \|r\nabla F\|_{2,\underline{C}_u \cap V(u, \underline{u})} + |u| \|\nabla_3 F\|_{2,\underline{C}_u \cap V(u, \underline{u})}.
 \end{aligned}$$

*Proof.* See Corollary 3.2.1.1 in [10] and Corollary 4.1.1 in [29]. □

We also need the following standard Sobolev inequalities, see Lemma 4.1.3 in [29].

**Proposition 2.46.** *Let  $F$  be a tensor field tangent to the sphere  $S := S(u, \underline{u})$ . Then, we have*

$$\sup_S r^{\frac{1}{2}}|F| \lesssim \left( \int_S |F|^4 + |r\nabla F|^4 \right)^{\frac{1}{4}}.$$

## 3 Main theorem

### 3.1 Fundamental norms

Our result holds for initial data sets satisfying (1.9) with  $s > 3$ . We will focus on the case  $s \in [4, 6]$  in Sections 3–6, and we postpone the necessary adaptations to the case  $s \in (3, 4)$  and  $s > 6$  to Appendices B and C. The norms in Sections 3.1.1–3.1.4 are defined with respect to the foliation  $(u, \underline{u})$  in the bootstrap region  $\mathcal{K}$  while the norms in Section 3.1.5 are defined with respect to the foliation  $(u_{(0)}, \underline{u})$  in the initial data layer  $\mathcal{K}_{(0)}$ .

### 3.1.1 $\mathcal{R}$ norms (Curvature components)

We define

$$\mathcal{R}_0^S := \mathcal{R}_0^S[\check{b}], \quad \underline{\mathcal{R}}_0^S := \underline{\mathcal{R}}_0^S[\check{b}] + \underline{\mathcal{R}}_0^S[\check{\rho}, \check{\sigma}] + \underline{\mathcal{R}}_0^S[\check{\beta}],$$

where

$$\begin{aligned} \mathcal{R}_0^S[\check{b}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{\frac{7}{2} - \frac{2}{p}} |u|^{\frac{s-4}{2}} \check{b}|_{p, S(u, \underline{u})}, \\ \underline{\mathcal{R}}_0^S[\check{b}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{\frac{s+2}{2} - \frac{2}{p}} |u|^{\frac{1}{2}} \check{b}|_{p, S(u, \underline{u})}, \\ \underline{\mathcal{R}}_0^S[\check{\rho}, \check{\sigma}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{3 - \frac{2}{p}} |u|^{\frac{s-3}{2}} (\check{\rho}, \check{\sigma})|_{p, S(u, \underline{u})}, \\ \underline{\mathcal{R}}_0^S[\check{\beta}] &:= \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{2 - \frac{2}{p}} |u|^{\frac{s-1}{2}} \check{\beta}|_{p, S(u, \underline{u})}. \end{aligned}$$

Then we define the norms of flux of curvature components. We denote

$$\mathcal{R} := \sum_{q=0}^1 (\mathcal{R}_q + \underline{\mathcal{R}}_q) + \mathcal{R}_0^S + \underline{\mathcal{R}}_0^S,$$

where

$$\begin{aligned} \mathcal{R}_q &:= \left( \mathcal{R}_q[\check{\alpha}]^2 + \mathcal{R}_q[\check{b}]^2 + \mathcal{R}_q[(\check{\rho}, \check{\sigma})]^2 + \mathcal{R}_q[\check{\beta}]^2 \right)^{\frac{1}{2}}, \\ \underline{\mathcal{R}}_q &:= \left( \underline{\mathcal{R}}_q[\check{b}]^2 + \underline{\mathcal{R}}_q[(\check{\rho}, \check{\sigma})]^2 + \underline{\mathcal{R}}_q[\check{\beta}]^2 + \underline{\mathcal{R}}_q[\check{\alpha}]^2 \right)^{\frac{1}{2}}, \end{aligned}$$

with

$$\begin{aligned} \mathcal{R}_q[w] &:= \sup_{\mathcal{K}} \mathcal{R}_q[w](u, \underline{u}), \\ \underline{\mathcal{R}}_q[w] &:= \sup_{\mathcal{K}} \underline{\mathcal{R}}_q[w](u, \underline{u}). \end{aligned}$$

We denote

$$V := V(u, \underline{u}), \quad C_u^V := C_u \cap V, \quad \underline{C}_u^V := \underline{C}_u \cap V. \quad (3.1)$$

It remains to define  $\mathcal{R}_q[w](u, \underline{u})$  and  $\underline{\mathcal{R}}_q[w](u, \underline{u})$ :

$$\begin{aligned} \mathcal{R}_q[\check{\alpha}](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \check{\alpha}\|_{2, C_u^V}, \\ \mathcal{R}_q[\check{b}](u, \underline{u}) &:= \|r^2 |u|^{\frac{s-4}{2}} (r\nabla)^q \check{b}\|_{2, C_u^V}, \\ \mathcal{R}_q[(\check{\rho}, \check{\sigma})](u, \underline{u}) &:= \|r |u|^{\frac{s-2}{2}} (r\nabla)^q (\check{\rho}, \check{\sigma})\|_{2, C_u^V}, \\ \mathcal{R}_q[\check{\beta}](u, \underline{u}) &:= \| |u|^{\frac{s}{2}} (r\nabla)^q \check{\beta}\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[\check{b}](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \check{b}\|_{2, \underline{C}_u^V}, \\ \underline{\mathcal{R}}_q[(\check{\rho}, \check{\sigma})](u, \underline{u}) &:= |u|^{\frac{s-4}{2}} r^2 \| (r\nabla)^q (\check{\rho}, \check{\sigma})\|_{2, \underline{C}_u^V}, \\ \underline{\mathcal{R}}_q[\check{\beta}](u, \underline{u}) &:= |u|^{\frac{s-2}{2}} r \| (r\nabla)^q \check{\beta}\|_{2, \underline{C}_u^V}, \\ \underline{\mathcal{R}}_q[\check{\alpha}](u, \underline{u}) &:= |u|^{\frac{s}{2}} \| (r\nabla)^q \check{\alpha}\|_{2, \underline{C}_u^V}. \end{aligned}$$

### 3.1.2 $\mathcal{O}$ norms (Ricci coefficients and metric components)

We denote for any linearized quantity:

$$\mathcal{O}_q(\check{X}) := \sup_{\mathcal{K}} \sup_{p \in [2,4]} \mathcal{O}_q^{p,S}(\check{X})(u, \underline{u}). \quad (3.2)$$

We first define

$$\mathcal{O}_{[1]} := \mathcal{O}_1 + \mathcal{O}_0 + \underline{\mathcal{O}}_1 + \underline{\mathcal{O}}_0 + \sum_{q=0}^2 \mathcal{O}_q(\check{\Omega}),$$

where

$$\begin{aligned} \mathcal{O}_q &:= \mathcal{O}_q(\widetilde{\Omega \operatorname{tr} \chi}) + \mathcal{O}_q(\check{\chi}) + \mathcal{O}_q(\check{\eta}) + \mathcal{O}_q(\widetilde{\Omega \omega}), & q = 0, 1, \\ \underline{\mathcal{O}}_q &:= \mathcal{O}_q(\widetilde{\Omega \operatorname{tr} \underline{\chi}}) + \mathcal{O}_q(\check{\underline{\chi}}) + \mathcal{O}_q(\check{\underline{\eta}}) + \mathcal{O}_q(\widetilde{\Omega \underline{\omega}}), & q = 0, 1. \end{aligned}$$

It remains to define  $\mathcal{O}_q^{p,S}(\check{X})(u, \underline{u})$ :

$$\begin{aligned} \mathcal{O}_q^{p,S}(\widetilde{\Omega \operatorname{tr} \chi})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q(\widetilde{\Omega \operatorname{tr} \chi})|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega \operatorname{tr} \underline{\chi}})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q(\widetilde{\Omega \operatorname{tr} \underline{\chi}})|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\chi})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\chi}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\underline{\chi}})(u, \underline{u}) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-1}{2}} \nabla^q \check{\underline{\chi}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\eta})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\eta}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\underline{\eta}})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\underline{\eta}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega \omega})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \widetilde{\Omega \omega}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\widetilde{\Omega \underline{\omega}})(u, \underline{u}) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-1}{2}} \nabla^q \widetilde{\Omega \underline{\omega}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\Omega})(u, \underline{u}) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\Omega}|_{p,S(u, \underline{u})}. \end{aligned}$$

Next, we define the norms of metric components:

$$\mathcal{O}_\gamma := \sum_{q=0}^1 \left( \mathcal{O}_q(\check{\gamma}) + \mathcal{O}_q(\check{\epsilon}) + \mathcal{O}_q(\check{\underline{b}}) \right),$$

where

$$\begin{aligned} \mathcal{O}_q^{p,S}(\check{\gamma})(u, \underline{u}) &:= |r^{-1+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\gamma}_{AB}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\epsilon})(u, \underline{u}) &:= |r^{-1+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\epsilon}_{AB}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_q^{p,S}(\check{\underline{b}})(u, \underline{u}) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\underline{b}}^A|_{p,S(u, \underline{u})}. \end{aligned}$$

We also define

$$\mathcal{O}_\Gamma := \mathcal{O}_0(\check{J}) + \mathcal{O}_0(\check{\underline{J}}) + \mathcal{O}_0(\check{\underline{L}}),$$

where

$$\begin{aligned} \mathcal{O}_0^{p,S}(\check{\underline{L}})(u, \underline{u}) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}} \check{\underline{L}}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_0^{p,S}(\check{J})(u, \underline{u}) &:= |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}} \check{J}|_{p,S(u, \underline{u})}, \\ \mathcal{O}_0^{p,S}(\check{\underline{J}})(u, \underline{u}) &:= |r^{1-\frac{2}{p}}|u|^{\frac{s-1}{2}} \check{\underline{J}}|_{p,S(u, \underline{u})}. \end{aligned}$$

Finally, we denote

$$\mathcal{O} := \mathcal{O}_{[1]} + \mathcal{O}_\gamma + \mathcal{O}_\Gamma. \quad (3.3)$$

### 3.1.3 $\mathring{O}(\Sigma_0)$ norms and $\mathfrak{R}_0$ norms (Initial data for the foliation of $\mathcal{K}$ )

Notice that for every  $\underline{u}$ , there exists a unique leaf  $S(u_0(\underline{u}), \underline{u})$  of  $\underline{C}_{\underline{u}}$ , which located in the future of  $\Sigma_0$  and touches  $\Sigma_0 \setminus K$ . Moreover, we have

$$S(u_0(\underline{u}), \underline{u}) \subset \mathcal{K}_{(0)}.$$

We define the following norms on the union of spheres  $S(u_0(\underline{u}), \underline{u})$ :

$$\mathring{O}_q(\check{\Gamma}) := \sup_{p \in [2,4]} \sup_{\Sigma_0 \setminus K} \mathring{O}_q^{p,S}(\check{\Gamma})(\underline{u}), \quad q = 0, 1.$$

We denote

$$\mathring{O}(\Sigma_0) := \mathring{O}_1 + \mathring{O}_0,$$

where

$$\mathring{O}_q := \mathring{O}_q(\widetilde{\Omega \operatorname{tr} \chi}) + \mathring{O}_q(\widetilde{\Omega \omega}) + \mathring{O}_q(\check{\eta}), \quad q = 0, 1.$$

It remains to define  $\mathring{O}_q^{p,S}(\check{\Gamma})(\underline{u})$ :

$$\begin{aligned} \mathring{O}_q^{p,S}(\widetilde{\Omega \operatorname{tr} \chi})(\underline{u}) &:= |r^{\frac{s+1}{2}+q-\frac{2}{p}} \nabla^q(\widetilde{\Omega \operatorname{tr} \chi})|_{p,S(u_0(\underline{u}), \underline{u})}, \\ \mathring{O}_q^{p,S}(\check{\eta})(\underline{u}) &:= |r^{\frac{s+1}{2}+q-\frac{2}{p}} \nabla^q \check{\eta}|_{p,S(u_0(\underline{u}), \underline{u})}, \\ \mathring{O}_q^{p,S}(\widetilde{\Omega \omega})(\underline{u}) &:= |r^{\frac{s+1}{2}+q-\frac{2}{p}} \nabla^q(\widetilde{\Omega \omega})|_{p,S(u_0(\underline{u}), \underline{u})}. \end{aligned}$$

We can extend the foliation  $(u, \underline{u})$  to a neighborhood of  $\Sigma_0 \setminus K$  in  $J^-(\Sigma_0 \setminus K)$  such that it is well defined on  $\Sigma_0 \setminus K$ . The curvature flux on  $\Sigma_0 \setminus K$  is defined by:

$$\mathfrak{R}_0^2 := \int_{\Sigma_0 \setminus K} \sum_{l=0}^1 r^s \left( |\mathfrak{d}^l \check{\alpha}|^2 + |\mathfrak{d}^l \check{b}|^2 + |\mathfrak{d}^l(\check{\rho}, \check{\sigma})|^2 + |\mathfrak{d}^l \check{\beta}|^2 + |\mathfrak{d}^l \check{\alpha}|^2 \right),$$

where

$$\mathfrak{d} := (r\nabla, r\nabla_4, r\nabla_3).$$

### 3.1.4 $\mathcal{O}^*(\underline{C}_*)$ norms (Ricci coefficients and metric components on $\underline{C}_*$ )

For any linearized quantity  $\check{\Gamma}$  on  $\underline{C}_*$ , we define

$$\mathcal{O}_q^*(\check{\Gamma}) := \sup_{p \in [2,4]} \sup_{\underline{C}_*} \mathcal{O}_q^{*p,S}(\check{\Gamma})(u).$$

We denote

$$\mathcal{O}^*(\underline{C}_*) := \mathcal{O}_1^* + \mathcal{O}_0^* + \mathcal{O}_\gamma^* + \mathcal{O}_\Gamma^* + \mathcal{O}_2^*(\widetilde{\operatorname{tr} \chi}) + \mathcal{O}_1^*(\check{[\mu]}),$$

where

$$\begin{aligned} \mathcal{O}_\gamma^* &:= \sum_{q=0}^2 (\mathcal{O}_q^*(\check{\gamma}) + \mathcal{O}_q^*(\check{\epsilon})), \\ \mathcal{O}_\Gamma^* &:= \sum_{q=0}^1 (\mathcal{O}_q^*(\check{L}) + \mathcal{O}_q^*(\check{J})), \\ \mathcal{O}_q^* &:= \mathcal{O}_q^*(\widetilde{\operatorname{tr} \chi}) + \mathcal{O}_q^*(\check{\chi}) + \mathcal{O}_q^*(\check{\zeta}) + \mathcal{O}_q^*(\widetilde{\operatorname{tr} \chi}) + \mathcal{O}_q^*(\check{\chi}), \quad q = 0, 1. \end{aligned}$$

It remains to define  $\mathcal{O}_q^{*p,S}(\check{\Gamma})(u)$  for  $q = 0, 1, 2$ :

$$\begin{aligned} \mathcal{O}_q^{*p,S}(\check{\chi})(u) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\chi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\underline{\chi}})(u) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-1}{2}} \nabla^q \check{\underline{\chi}}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\text{tr}} \chi)(u) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\text{tr}} \chi|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\text{tr}} \underline{\chi})(u) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\text{tr}} \underline{\chi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\zeta})(u) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\eta}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\gamma})(u) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\gamma}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{\xi})(u) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{\xi}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{L})(u) &:= |r^{2+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{L}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{J})(u) &:= |r^{1+q-\frac{2}{p}}|u|^{\frac{s-1}{2}} \nabla^q \check{J}|_{p,S(u,\underline{u}_*)}, \\ \mathcal{O}_q^{*p,S}(\check{[\underline{\mu}]})(u) &:= |r^{3+q-\frac{2}{p}}|u|^{\frac{s-3}{2}} \nabla^q \check{[\underline{\mu}]}|_{p,S(u,\underline{u}_*)}. \end{aligned}$$

We also define the flux of curvature components on  $\underline{C}_*$  as follows:

$$\underline{\mathcal{R}}(\underline{C}_*) := \underline{\mathcal{R}}_1|_{\underline{C}_*} + \underline{\mathcal{R}}_0|_{\underline{C}_*},$$

where  $\underline{\mathcal{R}}_q$ ,  $q = 0, 1$  are defined in Section 3.1.1.

### 3.1.5 $\mathcal{O}_{(0)}$ norms and $\mathfrak{R}_{(0)}$ norms (Initial data for the foliation of $\mathcal{K}_{(0)}$ )

In this section, all the norms are defined with respect to the initial layer foliation  $(u_{(0)}, \underline{u})$  of  $\mathcal{K}_{(0)}$ . We define

$$\mathcal{O}_{(0)} := \sum_{q=0}^2 \sum_{\check{\Gamma}_{(0)}} (\mathcal{O}_{(0)})_q(\check{\Gamma}_{(0)}),$$

where

$$(\mathcal{O}_{(0)})_q(\check{\Gamma}_{(0)}) := \sup_{p \in [2,4]} \sup_{\mathcal{K}_{(0)}} (\mathcal{O}_{(0)})_q^{p,S}(\check{\Gamma}_{(0)}),$$

with

$$(\mathcal{O}_{(0)})_q^{p,S}(\check{\Gamma}_{(0)}) := \left| r_{(0)}^{\frac{s+1}{2}-\frac{2}{p}} (r_{(0)} \nabla_{(0)})^q \check{\Gamma}_{(0)} \right|_{p,S_{(0)}(u_{(0)}, \underline{u})},$$

and

$$\check{\Gamma}_{(0)} \in \left\{ \check{\text{tr}} \chi_{(0)}, \check{\text{tr}} \underline{\chi}_{(0)}, \check{\chi}_{(0)}, \check{\underline{\chi}}_{(0)}, \check{\eta}_{(0)}, \check{\underline{\eta}}_{(0)}, \check{\omega}_{(0)}, \check{\underline{\omega}}_{(0)}, r_{(0)}^{-1} \check{\gamma}_{(0)}, r_{(0)}^{-1} \check{\xi}_{(0)}, r_{(0)}^{-1} \check{b}_{(0)}, r_{(0)}^{-1} \check{\Omega}_{(0)} \right\}.$$

The initial curvature flux on  $\Sigma_0 \setminus K$  is defined by:

$$\mathfrak{R}_{(0)}^2 := \int_{\Sigma_0 \setminus K} \sum_{l=0}^2 r_{(0)}^s \left( |\mathfrak{d}_{(0)}^l \check{\alpha}_{(0)}|^2 + |\mathfrak{d}_{(0)}^l \check{b}_{(0)}|^2 + |\mathfrak{d}_{(0)}^l (\check{\rho}_{(0)}, \check{\sigma}_{(0)})|^2 + |\mathfrak{d}_{(0)}^l \check{\beta}_{(0)}|^2 + |\mathfrak{d}_{(0)}^l \check{\alpha}_{(0)}|^2 \right),$$

where

$$\mathfrak{d}_{(0)} := (r_{(0)} \nabla_{(0)}, r_{(0)} (\nabla_{(0)})_4, r_{(0)} (\nabla_{(0)})_3).$$



### 3.1.6 Osc norms (Oscillation control)

We have two foliations  $(u, \underline{u})$  and  $(u_{(0)}, \underline{u})$  in the initial layer region  $\mathcal{K}_{(0)}$  which will be compared using Osc norms. We denote  $(f, \lambda)$  the change of frame from  $(u, \underline{u})$  to  $(u_{(0)}, \underline{u})$  introduced in Lemma 5.2. We define

$$\text{Osc} := \text{Osc}(f) + \text{Osc}(u) + \text{Osc}(x) + \text{Osc}(r),$$

where

$$\begin{aligned} \text{Osc}(f) &:= \sup_{\mathcal{K}_{(0)}} |(r_{(0)})^{\frac{s-1}{2}} \mathfrak{d}_{(0)}^{\leq 1} f|, \\ \text{Osc}(u) &:= \sup_{\mathcal{K}_{(0)}} |(r_{(0)})^{\frac{s-3}{2}} (u' - u)|, \\ \text{Osc}(x) &:= \sup_{\mathcal{K}_{(0)}} \sum_{A=1}^2 |(r_{(0)})^{\frac{s-1}{2}} (x_{(0)}^A - x^A)|, \\ \text{Osc}(r) &:= \sup_{\mathcal{K}_{(0)}} |(r_{(0)})^{\frac{s-3}{2}} (r_{(0)} - r)|, \end{aligned}$$

with

$$\mathfrak{d}_{(0)} := \{r_{(0)}(\nabla_{(0)})_3, r_{(0)}\nabla_{(0)}, r_{(0)}(\nabla_{(0)})_4\}.$$

## 3.2 Smallness constants

Before starting our main theorem, we first introduce the following constants that will be used throughout this paper.

- The constant  $M > 0$  and the constant  $a$  are the mass and angular momentum of the initial Kerr spacetime relative to which our initial perturbation is measured.
- $R_0 > 0$  measures the radius of the compact set  $K \subset \Sigma_0$  relative to which the external region is defined by  $\mathcal{M} := D^+(\Sigma_0 \setminus K)$ , the future domain of dependence of  $\Sigma_0 \setminus K$ .
- The size of the initial data layer norm is measured by  $\varepsilon_0 > 0$ .
- The size of the bootstrap assumption norms are measured by  $\varepsilon > 0$ .
- $\delta_0 > 0$  measures the height of the initial layer region  $\mathcal{K}_{(0)}$ .

In what follows  $M$  and  $a$  are fixed constants such that  $0 \leq |a| \leq M$ . We define

$$\mathcal{E}_0 := \frac{M}{R_0}, \quad \varepsilon := \mathcal{E}_0^{-\frac{1}{3}} \varepsilon_0. \quad (3.4)$$

We take  $\varepsilon_0$  small enough and  $R_0$  large enough such that

$$\varepsilon^2 \ll \varepsilon \mathcal{E}_0 \ll \varepsilon \mathcal{E}_0^{\frac{1}{2}} \ll \varepsilon_0 \ll \varepsilon \ll \mathcal{E}_0 \ll \delta_0 < 1. \quad (3.5)$$

**Remark 3.1.** In (3.5),  $A \ll B$  means that  $CA < B$  where  $C$  is the largest universal constant among all the constants involved in the proof via  $\lesssim$ .

### 3.3 Main theorem

The goal of this paper is to prove the following theorem, which is similar to Theorem 4.1 in [7].

**Theorem 3.2** (Main theorem, version 2). *Let  $\mathbf{g}_{a,M}$  a Kerr metric in  $\mathcal{M}_{Kerr}$  with  $|a| \leq M$ . Let  $\mathcal{K}_{(0)}$  the initial layer region and  $\Phi_{(0)}$  the map defined in Section 2.1.4. We also assume that<sup>16</sup>*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0. \tag{3.6}$$

*Then,  $\mathcal{K}_{(0)}$  has a unique future development  $(\mathcal{M}, \mathbf{g})$ , defined outside its future domain of influence  $I^+(\mathcal{K}_{(0)})$  with the following properties:*

1.  *$(\mathcal{M}, \mathbf{g})$  can be foliated by a double null foliation  $(u, \underline{u})$ . Moreover, the outgoing cones  $C_u$  are complete for all  $u \leq u_0$ .*
2. *The norms  $\mathcal{O}, \mathcal{R}$  satisfy*

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0. \tag{3.7}$$

The proof of Theorem 3.2 is given in Section 3.5. It hinges on a sequence of basic theorems stated in Section 3.4, concerning estimates for  $\mathcal{O}$  and  $\mathcal{R}$  norms.

**Remark 3.3.** *Theorem 3.2 is proved in Section 3.5 for  $s \in [4, 6]$ , and extended to  $s \in (3, 4)$  and  $s > 6$  in Appendices B and C.*

### 3.4 Intermediate results

**Theorem M0.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \tag{3.8}$$

*Then, we have*

$$\mathfrak{R}_0 \lesssim \varepsilon_0. \tag{3.9}$$

Theorem M0 is proved in Section 6.1. The proof is based on null frame transformation formulae introduced by Klainerman and Szeftel in [34], see Proposition 5.3 in Section 5.1.1.

**Theorem M1.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathfrak{R}_0 \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \tag{3.10}$$

*Then, we have*

$$\mathcal{R} \lesssim \varepsilon_0. \tag{3.11}$$

Theorem M1 is proved in Section 4. The proof is based on the  $r^p$ -weighted estimate method introduced by Dafermos and Rodnianski in [13].

**Theorem M2.** *Assume that  $\underline{C}_*$  is foliated by the geodesic type foliation of Definition 6.1 and that we have the following assumptions:*

$$\mathcal{O}_{(0)}(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0, \quad \underline{\mathcal{R}}(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon.$$

*Then, we have*

$$\mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0.$$

---

<sup>16</sup>The norms  $\mathcal{O}_{(0)}$  and  $\mathfrak{R}_{(0)}$  are defined in Section 3.1.5.

Theorem M2 is proved in Section 6.2. The proof is done by integrating the transport equations along the null generator  $\underline{L}$  of the last slice  $\underline{C}_*$  and applying elliptic estimates on the 2-spheres of the geodesic type foliation of  $\underline{C}_*$ .

**Theorem M3.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{R} \leq \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \quad (3.12)$$

Then, we have

$$\mathcal{O} \lesssim \varepsilon_0, \quad \text{Osc} \lesssim \varepsilon_0.$$

Theorem M3 is proved in Section 5. The proof is done by integrating the transport equations along the outgoing and ingoing null cones and applying elliptic estimate on the 2-spheres of the double null foliation of the spacetime  $\mathcal{K}$ .

**Theorem M4.** *We consider the spacetime  $\mathcal{K}$  and its double null foliation  $(u, \underline{u})$  which satisfies the assumptions:*

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad \text{Osc} \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0. \quad (3.13)$$

Then, we can extend the spacetime  $\mathcal{K} = V(u_0, \underline{u}_*)$  and the double null foliation to a new spacetime  $\widehat{\mathcal{K}} = \widehat{V}(u_0, \underline{u}_* + \nu)$ , where  $\nu > 0$  is sufficiently small, and an associated double null foliation  $(\widehat{u}, \underline{u})$ . Moreover, the new foliation  $(\widehat{u}, \underline{u})$  is geodesic on the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$  and the new corresponding norms satisfy

$$\widehat{\mathcal{O}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}} \lesssim \varepsilon_0, \quad \widehat{\text{Osc}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}^*(\underline{C}_{**}) \lesssim \varepsilon_0. \quad (3.14)$$

Theorem M4 is proved in Section 6.3. The proof is based on local existence type arguments.

**Remark 3.4.** *When compared to [7], we have the following similarities and differences:*

1. *We treat the general decay  $s > 3$  in Theorem 3.2 while the main result of [7] corresponds to the particular case  $s > 7$  in Theorem 3.2.*
2. *In [7], fourth order derivatives of Ricci coefficients and third order derivatives of curvature are estimated. In this paper, we only estimate first order derivatives of both Ricci coefficients and curvature respectively in Theorems M2–M3 and M1.*
3. *To estimate the curvature components in Theorem M1, instead of using the classical vectorfield method as in [7], we use  $r^p$ -weighted estimates of [13].*
4. *On the last slice  $\underline{C}_*$ , a canonical foliation is used in [7] while we use a geodesic type foliation in Theorem M2.*
5. *The estimates of Ricci coefficients and the extension argument, i.e. the proof of Theorems M3 and M4, are similar to [7].*

### 3.5 Proof of the main theorem

Now, we use Theorems M0–M4 to prove Theorem 3.2.

**Definition 3.5.** *Let  $\mathfrak{N}(\underline{u}_*)$  the set of spacetimes  $\mathcal{K}$  associated with a double null foliation  $(u, \underline{u})$  which satisfy the following properties:*

1.  $\mathcal{K} = V(u_0, \underline{u}_*)$ .
2. *The foliation on  $\underline{C}_*$  is the geodesic type foliation of Definition 6.1.*

3. We have the following bootstrap assumptions:

$$\mathcal{O} \leq \epsilon, \quad \mathcal{R} \leq \epsilon, \quad \text{Osc} \leq \epsilon, \quad \mathcal{O}^*(\underline{C}_*) \leq \epsilon. \tag{3.15}$$

**Definition 3.6.** We denote  $\mathcal{U}$  the set of values  $\underline{u}_*$  such that  $\aleph(\underline{u}_*) \neq \emptyset$ .

Applying the local existence result of Theorem 10.2.1 in [10] and the assumption  $\mathcal{O}_{(0)} \leq \epsilon_0$ , we deduce that (3.15) holds if  $\underline{u}_*$  is sufficiently close to  $R_0$ . So, we have  $\mathcal{U} \neq \emptyset$ .

Define  $\underline{u}_*$  to be the supremum of the set  $\mathcal{U}$ . We want to prove  $\underline{u}_* = \infty$ . We assume by contradiction that  $\underline{u}_*$  is finite. In particular, we may assume that  $\underline{u}_* \in \mathcal{U}$ . We consider the region  $\mathcal{K} = V(u_0, \underline{u}_*)$ . Recall that we have

$$\mathcal{O}_{(0)} \leq \epsilon_0, \quad \mathfrak{R}_{(0)} \leq \epsilon_0.$$

Combining with (3.15) and applying Theorem M0, we obtain

$$\mathfrak{R}_0 \lesssim \epsilon_0. \tag{3.16}$$

Then, we have from Theorem M1

$$\mathcal{R} \lesssim \epsilon_0. \tag{3.17}$$

Applying Theorem M2, we obtain

$$\mathcal{O}^*(\underline{C}_*) \lesssim \epsilon_0. \tag{3.18}$$

Combining (3.15), (3.17), (3.18) and Theorem M3, we deduce

$$\mathcal{O} \lesssim \epsilon_0, \quad \text{Osc} \lesssim \epsilon_0. \tag{3.19}$$

Next, applying Theorem M4, we can extend  $\mathcal{K}$  to  $\widehat{\mathcal{K}} := \widehat{V}(u_0, \underline{u}_* + \nu)$  endowed with a new double null foliation  $(\widehat{u}, \underline{u})$  and

$$\widehat{\mathcal{O}}(\widehat{\mathcal{K}}) \lesssim \epsilon_0, \quad \widetilde{\mathcal{R}}(\widehat{\mathcal{K}}) \lesssim \epsilon_0, \quad \widehat{\text{Osc}}(\widehat{\mathcal{K}}) \lesssim \epsilon_0, \quad \widehat{\mathcal{O}}^*(\underline{C}_{**} \cap \widehat{\mathcal{K}}) \lesssim \epsilon_0,$$

where  $\underline{C}_{**} = \underline{C}_{\underline{u}_* + \nu}$  is the new last slice. Thus, the region  $\widehat{V}(u_0, \underline{u}_* + \nu)$  with the new double null foliation  $(\widehat{u}, \underline{u})$  satisfies all the properties in Definition 3.5, and so  $\aleph(\underline{u}_* + \nu) \neq \emptyset$ , a contradiction. So, we have  $\underline{u}_* = \infty$ , which implies property 1 of Theorem 3.2. Moreover, we also have

$$\mathcal{O} \lesssim \epsilon_0, \quad \mathcal{R} \lesssim \epsilon_0, \tag{3.20}$$

in the whole external region, which implies property 2 of Theorem 3.2. This concludes the proof of Theorem 3.2.

### 3.6 Conclusions

The following proposition is a consequence of Theorem 3.2.

**Proposition 3.7.** *Under the assumptions of Theorem 3.2, we have the following estimate in the external region<sup>17</sup>:*

$$|\mathbf{g} - \mathbf{g}_K|_{\infty, S(u, \underline{u})} \lesssim \frac{\epsilon_0}{r|u|^{\frac{s-3}{2}}}. \tag{3.21}$$

---

<sup>17</sup>See Figure 4.5 for a description of the external region.

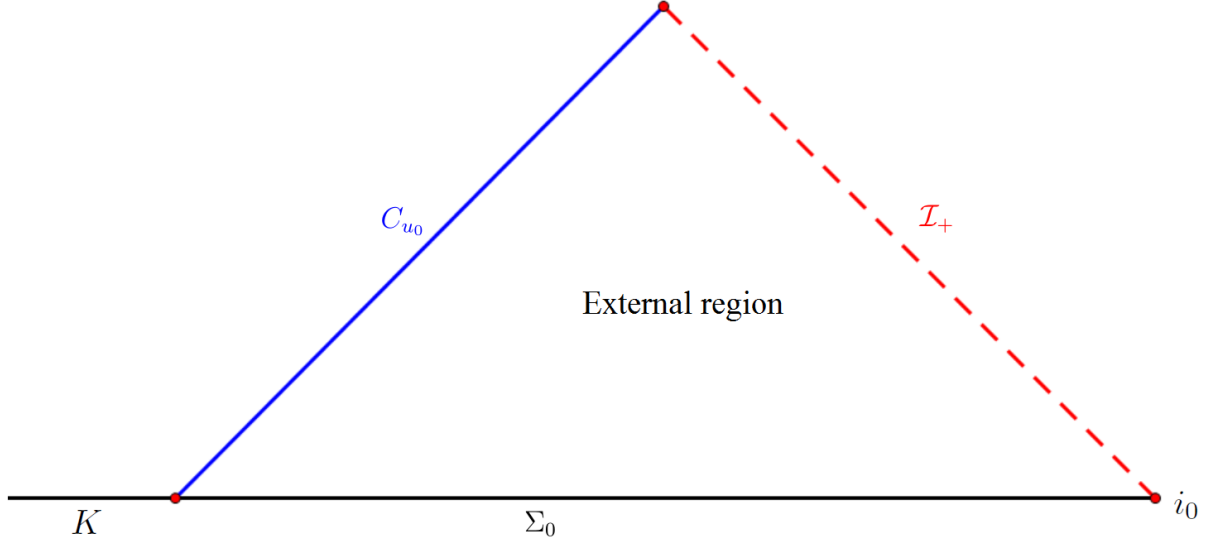


Figure 4.5: External region

*Proof.* We recall that we have

$$\mathbf{g} = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \gamma_{AB}(dx^A - \underline{b}^A du) \otimes (dx^B - \underline{b}^B du),$$

and

$$\mathbf{g}_K = -2\Omega_K^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + (\gamma_K)_{AB}(dx^A - \underline{b}_K^A du) \otimes (dx^B - \underline{b}_K^B du).$$

We compare these two metrics as follows:

$$\begin{aligned} \mathbf{g}_{\underline{u}\underline{u}} - (\mathbf{g}_K)_{\underline{u}\underline{u}} &= 0, \\ \mathbf{g}_{\underline{u}u} - (\mathbf{g}_K)_{\underline{u}u} &= -4\Omega^2 + 4\Omega_K^2 = -4\check{\Omega}(\Omega + \Omega_K) = O\left(\frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}\right), \\ \mathbf{g}_{\underline{u}A} - (\mathbf{g}_K)_{\underline{u}A} &= 0, \\ \mathbf{g}_{uu} - (\mathbf{g}_K)_{uu} &= \check{\gamma}_{AB}\underline{b}^A\underline{b}^B + (\gamma_K)_{AB}(\underline{b}^A\underline{b}^B - \underline{b}_K^A\underline{b}_K^B) = O\left(\frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}\right), \\ \mathbf{g}_{uA} - (\mathbf{g}_K)_{uA} &= -\check{\gamma}_{AB}\underline{b}^B - (\gamma_K)_{AB}\check{\underline{b}}^B = O\left(\frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}\right), \\ \mathbf{g}_{AB} - (\mathbf{g}_K)_{AB} &= \check{\gamma}_{AB} = O\left(\frac{\varepsilon_0 r}{|u|^{\frac{s-3}{2}}}\right). \end{aligned}$$

This concludes the proof of Proposition 3.7.  $\square$

**Remark 3.8.** *Theorem 3.2 contains also a number of important conclusions following from (3.7): peeling properties, complete future infinity, Bondi mass formula and so on, see Chapter 8 in [29]. See also Section 3.8 in [36] for various conclusions of the Kerr stability in the entire domain of outer communication.*

## 4 Curvature estimates (Theorem M1)

In this section, we prove Theorem M1 by the  $r^p$ -weighted estimate method introduced in [13] and applied to Bianchi equations in [22], [33] and [18]. For convenience, we recall the statement below.

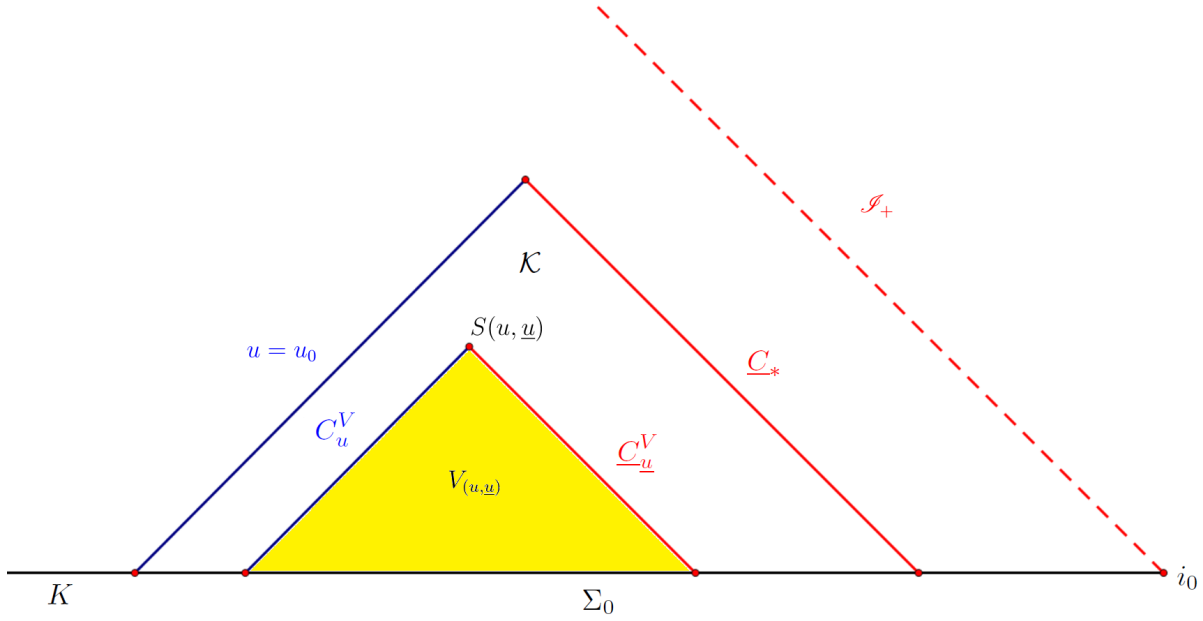


Figure 4.6: Domain of integration

**Theorem M1.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon, \quad \mathfrak{R}_0 \lesssim \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \quad (4.1)$$

*Then, we have*

$$\mathcal{R} \lesssim \varepsilon_0. \quad (4.2)$$

In the sequel, we denote<sup>18</sup>

$$V := V(u, \underline{u}), \quad C_u^V := C_u \cap V, \quad \underline{C}_u^V := \underline{C}_u \cap V. \quad (4.3)$$

The following lemma allows us to estimate the linear terms and error terms in Propositions 2.37 and 2.38.

**Lemma 4.1.** *Under the assumptions  $\mathcal{O} \leq \varepsilon$  and  $\mathcal{R} \leq \varepsilon$ , we have*

$$\begin{aligned} r^2 |u|^{\frac{s-3}{2}} |\Gamma_g|_{\infty, S} &\lesssim \varepsilon, & r |u|^{\frac{s-1}{2}} |\Gamma_b|_{\infty, S} &\lesssim \varepsilon \\ |r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} \Gamma_g^{(1)}|_{p, S} &\lesssim \varepsilon, & |r^{1-\frac{2}{p}} |u|^{\frac{s-1}{2}} \Gamma_b^{(1)}|_{p, S} &\lesssim \varepsilon, \end{aligned} \quad p \in [2, 4]. \quad (4.4)$$

*Proof.* It follows directly from the assumption (4.1), Definition 2.25 and the Sobolev inequalities in Propositions 2.45 and 2.46.  $\square$

### 4.1 General Bianchi pairs

The following lemma provides the general structure of Bianchi pairs. It will be used repeatedly in Section 4. See Lemma 8.24 in [33] for an analog under the assumption of axial polarization and Lemma 15.3.8 in [18] for an analog in the context of non integrable foliations.

**Lemma 4.2.** *Let  $k = 1, 2$  and  $a_{(1)}, a_{(2)}$  real numbers. Then, we have the following properties.*

<sup>18</sup>See Figure 4.6 for a description of  $V(u, \underline{u})$ ,  $C_u^V$  and  $\underline{C}_u^V$ .

1. Let  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_k$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_{k-1}$  satisfying

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= -kd_k^*(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \psi_{(2)} &= d_k(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (4.5)$$

Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$

$$\begin{aligned} & \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + k \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\ & = 2kr^p d_1(\psi_{(1)} \cdot \psi_{(2)}) + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} - 2r^p \underline{\omega} |\psi_{(1)}|^2 \\ & - 2kr^p \omega |\psi_{(2)}|^2 + pr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + kpr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \chi\right) |\psi_{(2)}|^2. \end{aligned} \quad (4.6)$$

2. Let  $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_{k-1}$  and  $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_k$  satisfying

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= d_k(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \psi_{(2)} &= -kd_k^*(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (4.7)$$

Then, the pair  $(\psi_{(1)}, \psi_{(2)})$  satisfies for any real number  $p$

$$\begin{aligned} & k \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + k \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\ & = 2r^p d_1(\psi_{(1)} \cdot \psi_{(2)}) + 2kr^p \psi_{(1)} \cdot h_{(1)} + 2r^p \psi_{(2)} \cdot h_{(2)} - 2kr^p \underline{\omega} |\psi_{(1)}|^2 \\ & - 2r^p \omega |\psi_{(2)}|^2 + kpr^{p-1} \left(e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi}\right) |\psi_{(1)}|^2 + pr^{p-1} \left(e_4(r) - \frac{r}{2} \operatorname{tr} \chi\right) |\psi_{(2)}|^2. \end{aligned} \quad (4.8)$$

*Proof.* See Lemma 4.2 in [53]. □

**Remark 4.3.** Note that the linearized Bianchi equations can be written as systems of equations of the type (4.5) and (4.7). In particular

- the Bianchi pair  $(\check{\alpha}, \check{b})$  satisfies (4.5) with  $k = 2$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,
- the Bianchi pair  $(\check{b}, (\check{\rho}, -\check{\sigma}))$  satisfies (4.5) with  $k = 1$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,
- the Bianchi pair  $((\check{\rho}, \check{\sigma}), \check{\beta})$  satisfies (4.7) with  $k = 1$ ,  $a_{(1)} = \frac{3}{2}$ ,  $a_{(2)} = 1$ ,
- the Bianchi pair  $(\check{\beta}, \check{\alpha})$  satisfies (4.7) with  $k = 2$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ .

**Proposition 4.4.** For  $\psi_{(1)}, \psi_{(2)}$  and  $h_{(1)}, h_{(2)}$  satisfying (4.5) or (4.7), we have the following properties for all  $(u, \underline{u}) \in \mathcal{K}$ :

(a) In the case  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p > 0$ , we have

$$\begin{aligned} & \int_{C_{\underline{u}}^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(1)}|^2 + r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|. \end{aligned} \quad (4.9)$$

(b) In the case  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p = 0$ , we have

$$\begin{aligned} & \int_{C_{\underline{u}}^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|. \end{aligned} \quad (4.10)$$

(c) In the case  $2 + p - 4a_{(1)} \leq 0$  and  $4a_{(2)} - 2 - p \geq 0$ , we have

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(1)}|^2 + r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|. \end{aligned} \quad (4.11)$$

(d) In the case  $2 + p - 4a_{(1)} > 0$  and  $4a_{(2)} - 2 - p \leq 0$ , we have

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(2)}|^2 + r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|. \end{aligned} \quad (4.12)$$

**Remark 4.5.** In the sequel, For a sum of terms of the form  $\Gamma \cdot R$ , we ignore the terms having the same or even better decay. For example, we write

$$\Gamma_g \cdot \beta + \Gamma_g \cdot \alpha = \Gamma_g \cdot \beta, \quad \Gamma_g \cdot (\check{\rho}, \check{\sigma}) + \Gamma_g \cdot \underline{\beta} = \Gamma_g \cdot \underline{\beta},$$

since  $\alpha$  decays better than  $\beta$  and  $\check{\rho}$  and  $\check{\sigma}$  decay better than  $\underline{\beta}$ .

*Proof.* We obtain from Proposition 2.30 that

$$\omega = \Gamma_g + \mathbf{O}_2^1, \quad \underline{\omega} = \Gamma_b + \mathbf{O}_2^1.$$

Applying Lemma 2.7, we obtain

$$e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} = \frac{\overline{\Omega \operatorname{tr} \underline{\chi}}}{2\Omega} r - \frac{r}{2} \operatorname{tr} \underline{\chi} = \frac{r}{2\Omega} (\overline{\Omega \operatorname{tr} \underline{\chi}} - \Omega \operatorname{tr} \underline{\chi}) = r\Gamma_g + \mathbf{O}_2^1, \quad (4.13)$$

and similarly

$$e_4(r) - \frac{r}{2} \operatorname{tr} \chi = r\Gamma_g + \mathbf{O}_2^1. \quad (4.14)$$

Applying Lemma 4.1, we infer

$$\begin{aligned} & \int_V r^p (|\underline{\omega}| |\psi_{(1)}|^2 + |\omega| |\psi_{(2)}|^2) \\ & + \int_V r^{p-1} \left( e_3(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} \right) |\psi_{(1)}|^2 + r^{p-1} \left( e_4(r) - \frac{r}{2} \operatorname{tr} \underline{\chi} \right) |\psi_{(2)}|^2 \\ & \lesssim \int_V r^p (\Gamma_b + \mathbf{O}_2^1) |\psi_{(1)}|^2 + \int_V r^p (\Gamma_g + \mathbf{O}_2^1) |\psi_{(2)}|^2 \\ & \lesssim M \int_V r^{p-2} |\psi_{(1)}|^2 + \varepsilon \int_V r^{p-1} |u|^{-1} |\psi_{(1)}|^2 + M \int_V r^{p-2} |\psi_{(2)}|^2 \\ & \lesssim M \int_{u_0(\underline{u})}^u \left( |u|^{-2} \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) du + M \int_{|u|}^{\underline{u}} \left( |\underline{u}|^{-2} \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 \right) d\underline{u} \\ & \lesssim M \int_{u_0(\underline{u})}^u |u|^{-2} du \left( \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) + M \int_{|u|}^{\underline{u}} |\underline{u}|^{-2} d\underline{u} \left( \sup_{\underline{u}} \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 \right) \\ & \lesssim \mathcal{E}_0 \left( \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 \right) + \mathcal{E}_0 \left( \sup_{\underline{u}} \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 \right). \end{aligned} \quad (4.15)$$



Integrating (4.6) or (4.8) and recalling that

$$\operatorname{tr} \chi - \frac{2}{r} = \Gamma_g + \mathbf{O}_2^1, \quad \operatorname{tr} \underline{\chi} + \frac{2}{r} = \Gamma_g + \mathbf{O}_2^1,$$

we obtain

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 + \int_V (2+p-4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)}-2-p) r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}| \\ & \quad + \mathcal{E}_0 \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 + \mathcal{E}_0 \sup_{\underline{u}} \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2. \end{aligned}$$

Taking the supremum in  $u$  and  $\underline{u}$  and applying (4.15), we obtain for  $\varepsilon$  and  $\mathcal{E}_0$  small enough

$$\begin{aligned} & \sup_u \int_{C_u^V} r^p |\psi_{(1)}|^2 + \sup_{\underline{u}} \int_{\underline{C}_{\underline{u}}^V} r^p |\psi_{(2)}|^2 \\ & + \int_V (2+p-4a_{(1)}) r^{p-1} |\psi_{(1)}|^2 + (4a_{(2)}-2-p) r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 + r^p |\psi_{(2)}|^2 + \int_V r^p |\psi_{(1)}| |h_{(1)}| + r^p |\psi_{(2)}| |h_{(2)}|. \end{aligned} \tag{4.16}$$

This concludes the proof of Proposition 4.4.  $\square$

The following lemma allows us to compare  $\check{\phi}\check{\psi}$  and  $\check{\phi}\psi$ .

**Lemma 4.6.** *We denote for  $\psi \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$*

$$\widetilde{\psi^{(1)}} := (\check{\psi}, \check{\phi}\psi). \tag{4.17}$$

Then, we have for  $p < 7$

$$\begin{aligned} \int_{C_u^V} r^p |\check{\psi}^{(1)}|^2 & \lesssim \int_{C_u^V} r^p |\widetilde{\psi^{(1)}}|^2 + \frac{\varepsilon_0^2}{|u|^{s-p}}, \\ \int_{\underline{C}_{\underline{u}}^V} r^p |\check{\psi}^{(1)}|^2 & \lesssim \int_{\underline{C}_{\underline{u}}^V} r^p |\widetilde{\psi^{(1)}}|^2 + \frac{\varepsilon_0^2}{|u|^{s-p}}. \end{aligned} \tag{4.18}$$

**Remark 4.7.** *Note that Proposition 2.38 involves the quantities  $\check{\phi}\underline{\alpha}$ ,  $\check{\phi}\underline{\beta}$ ,  $\check{\phi}\rho$ ,  $\check{\phi}\sigma$ ,  $\check{\phi}\beta$  and  $\check{\phi}\alpha$ , while the curvature norms  $\mathcal{R}_1$  and  $\underline{\mathcal{R}}_1$  involve  $\check{\phi}\underline{\alpha}$ ,  $\check{\phi}\underline{\beta}$ ,  $\check{\phi}\check{\rho}$ ,  $\check{\phi}\check{\sigma}$ ,  $\check{\phi}\check{\beta}$  and  $\check{\phi}\check{\alpha}$ . We thus need to compare  $\check{\phi}\check{\psi}$  and  $\check{\phi}\psi$  which is the purpose of Lemma 4.6.*

*Proof.* For  $\psi \in \{\rho, \sigma\}$  which is a scalar function, we have

$$\check{\phi}\check{\psi} = \check{\phi}\psi.$$

The estimate (4.18) is trivial in this case. Next, we focus on the case  $\psi \in \{\alpha, \beta, \underline{\beta}, \underline{\alpha}\}$ . We have from Proposition 2.22

$$\check{\phi}\check{\psi} = \check{\phi}\psi - \check{\phi}_K \psi_K = \check{\phi}\psi + (\check{\phi} - \check{\phi}_K) \psi_K = \check{\phi}\psi + r\check{L} \cdot \psi_K.$$

Recalling that  $\psi_K = \mathbf{O}_4^2$ , we deduce

$$\begin{aligned} \int_{C_u^V} r^p |\check{\psi}|^2 &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + \int_{C_u^V} r^p |\Gamma_g^{(1)} \cdot \mathbf{O}_3^2|^2 \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + M^4 \int_{C_u^V} r^{p-6} |\Gamma_g^{(1)}|^2 \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + M^4 \int_{|u|}^u r^{p-8} du \int_S |r\Gamma_g^{(1)}|^2 \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + M^4 \varepsilon^2 \int_{|u|}^u r^{p-8} \frac{du}{|u|^{s-3}} \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + M^4 \varepsilon^2 |u|^{p-s-4} \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + \frac{\mathcal{E}_0^4 \varepsilon^2}{|u|^{s-p}} \\ &\lesssim \int_{C_u^V} r^p |\check{\psi}|^2 + \frac{\varepsilon_0^2}{|u|^{s-p}}, \end{aligned}$$

which implies the first inequality in (4.18). The second inequality in (4.18) is similar. This concludes the proof of Lemma 4.6.  $\square$

**Remark 4.8.** *The proof of Theorem M1 is analogous to Theorem M1 in [53]. In particular, the estimates for the error terms  $\text{Err}[\dots]$  in Propositions 2.37 and 2.38 have already been done in [53]. In the proof of Theorem M1 below, we thus focus on the linear terms  $\mathfrak{L}[\dots]$ .*

### 4.2 Estimates for the Bianchi pair $(\check{\alpha}, \check{b})$

**Proposition 4.9.** *We have the following estimate:*

$$\mathcal{R}_0[\check{\alpha}]^2 + \mathcal{R}_0[\check{b}]^2 + \mathcal{R}_1[\check{\alpha}]^2 + \mathcal{R}_1[\check{b}]^2 \lesssim \varepsilon_0^2. \tag{4.19}$$

*Proof.* We recall the following equations from Propositions 2.37 and 2.38:

$$\begin{aligned} \nabla_3 \widetilde{\alpha}^{(1)} + \frac{1}{2} \text{tr} \chi \widetilde{\alpha}^{(1)} &= -2d_2^*(\widetilde{\beta}^{(1)}) + \mathfrak{L}[\nabla_3 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\alpha}^{(1)}], \\ \nabla_4 \widetilde{\beta}^{(1)} + 2 \text{tr} \chi \widetilde{\beta}^{(1)} &= d_2(\widetilde{\alpha}^{(1)}) + \mathfrak{L}[\nabla_4 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\beta}^{(1)}]. \end{aligned} \tag{4.20}$$

Applying (4.9) with  $\psi_{(1)} = \widetilde{\alpha}^{(1)}$ ,  $\psi_{(2)} = \widetilde{\beta}^{(1)}$ ,  $a_{(1)} = \frac{1}{2}$ ,  $a_{(2)} = 2$ ,  $h_{(1)} = \mathfrak{L}[\nabla_3 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\alpha}^{(1)}]$ ,  $h_{(2)} = \mathfrak{L}[\nabla_4 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\beta}^{(1)}]$  and  $p = s$  we obtain

$$\begin{aligned} &\int_{C_u^V} r^s |\widetilde{\alpha}^{(1)}|^2 + \int_{\underline{C}_u^V} r^s |\widetilde{\beta}^{(1)}|^2 + \int_V r^{s-1} |\widetilde{\alpha}^{(1)}|^2 \\ &\lesssim \int_{\Sigma_0 \cap V} r^s |\widetilde{\alpha}^{(1)}|^2 + r^s |\widetilde{\beta}^{(1)}|^2 + \int_V r^s |\widetilde{\alpha}^{(1)}| \left| \mathfrak{L}[\nabla_3 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\alpha}^{(1)}] \right| \\ &\quad + \int_V r^s |\widetilde{\beta}^{(1)}| \left| \text{Err}[\nabla_4 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\beta}^{(1)}] \right|. \end{aligned} \tag{4.21}$$

We recall from Proposition 2.37<sup>19</sup>

$$\begin{aligned} &\int_V r^s |\widetilde{\alpha}^{(1)}| |\mathfrak{L}[\nabla_3 \widetilde{\alpha}^{(1)}]| + r^s |\widetilde{\beta}^{(1)}| |\mathfrak{L}[\nabla_4 \widetilde{\beta}^{(1)}]| \\ &\lesssim \int_V r^s \left( \frac{M}{r^3} |\Gamma_g^{(1)}| |\widetilde{\alpha}^{(1)}| + \frac{M^2}{r^3} |\widetilde{\alpha}^{(1)}| |\check{\rho}^{\leq 1}| + \frac{M}{r^2} |\check{\rho}^{\leq 1} \check{b}|^2 \right). \end{aligned} \tag{4.22}$$

<sup>19</sup>We ignore the terms which decay better.

We estimate (4.22) term by term. For the first term:

$$\begin{aligned}
\int_V M r^{s-3} |\widetilde{\alpha}^{(1)}| |\Gamma_g^{(1)}| &\lesssim M \int_{u_0(\underline{u})}^u du \left( \int_{C_u^V} r^s |\widetilde{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^{s-6} |\Gamma_g^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim M \varepsilon \int_{u_0(\underline{u})}^u du \left( \int_{|u|}^u r^{s-8} |r \Gamma_g^{(1)}|_{2,S}^2 du \right)^{\frac{1}{2}} \\
&\lesssim M \varepsilon^2 \int_{u_0(\underline{u})}^u du \left( \int_{|u|}^u r^{s-8} |u|^{3-s} du \right)^{\frac{1}{2}} \\
&\lesssim M \varepsilon^2 \int_{u_0(\underline{u})}^u |u|^{-2} du \\
&\lesssim \varepsilon_0^2.
\end{aligned}$$

For the second term:

$$\begin{aligned}
\int_V M^2 r^{s-3} |\widetilde{\alpha}^{(1)}| |\check{\rho}^{(1)}| &\lesssim M^2 \int_{u_0(\underline{u})}^u du \left( \int_{C_u^V} r^s |\widetilde{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} r^2 |\check{\rho}^{(1)}|^2 \right)^{\frac{1}{2}} r^{\frac{s}{2}-4} \\
&\lesssim M^2 \int_{u_0(\underline{u})}^u du \frac{\varepsilon^2}{|u|^{\frac{s-2}{2}}} |u|^{\frac{s}{2}-4} \\
&\lesssim M^2 \varepsilon^2 |u|^{-2} \\
&\lesssim \varepsilon_0^2.
\end{aligned}$$

For the last term:

$$\int_V M r^{s-2} |\widetilde{\beta}^{(1)}|^2 \lesssim M \int_{|u|}^u \frac{du}{r^2} \int_{C_u^V} r^s |\widetilde{\beta}^{(1)}|^2 \lesssim M \varepsilon^2 \int_{|u|}^u \frac{du}{r^2} \lesssim \varepsilon_0^2.$$

Injecting all the estimates into (4.22), we deduce

$$\int_V r^s |\widetilde{\alpha}^{(1)}| |\mathfrak{L}[\nabla_3 \widetilde{\alpha}^{(1)}]| + r^s |\widetilde{\beta}^{(1)}| |\mathfrak{L}[\nabla_4 \widetilde{\beta}^{(1)}]| \lesssim \varepsilon_0^2. \quad (4.23)$$

Next, we proceed as in Proposition 4.7 in [53] to obtain

$$\int_V r^s \left( |\widetilde{\alpha}^{(1)}| |\text{Err}[\nabla_3 \widetilde{\alpha}^{(1)}]| + |\widetilde{\beta}^{(1)}| |\text{Err}[\nabla_4 \widetilde{\beta}^{(1)}]| \right) \lesssim \varepsilon_0^2. \quad (4.24)$$

Combining (4.23) and (4.24), we obtain

$$\int_{C_u^V} r^s |\widetilde{\alpha}^{(1)}|^2 + \int_{C_u^V} r^s |\widetilde{\beta}^{(1)}|^2 + \int_V r^{s-1} |\widetilde{\alpha}^{(1)}|^2 \lesssim \varepsilon_0^2. \quad (4.25)$$

Applying Lemma 4.6, we deduce (4.19). This concludes the proof of Proposition 4.9.  $\square$

### 4.3 Estimates for the Bianchi pair $(\check{b}, (\check{\rho}, \check{\sigma}))$

**Proposition 4.10.** *We have the following estimate:*

$$\mathcal{R}_0[\check{b}]^2 + \underline{\mathcal{R}}_0[(\check{\rho}, \check{\sigma})]^2 + \mathcal{R}_1[\check{b}]^2 + \underline{\mathcal{R}}_1[(\check{\rho}, \check{\sigma})]^2 \lesssim \varepsilon_0^2. \quad (4.26)$$

*Proof.* We recall from Propositions 2.37 and 2.38:

$$\begin{aligned}
\nabla_4(\widetilde{\rho}^{(1)}, -\widetilde{\sigma}^{(1)}) + \frac{3}{2} \text{tr} \chi(\widetilde{\rho}^{(1)}, -\widetilde{\sigma}^{(1)}) &= d_1 \check{b} + \mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}], \\
\nabla_3 \widetilde{\beta}^{(1)} + \text{tr} \underline{\chi} \widetilde{\beta}^{(1)} &= -d_1^*(\widetilde{\rho}^{(1)}, -\widetilde{\sigma}^{(1)}) + \mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\beta}^{(1)}],
\end{aligned}$$

where

$$\mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}] = \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot \check{\rho}^{(1)}, \quad \mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}] = \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b}^{(1)} + \mathbf{O}_3^2 \cdot \check{\underline{\beta}}^{(1)}.$$

Applying (4.10) with  $\psi_{(1)} = \widetilde{\beta}^{(1)}$ ,  $\psi_{(2)} = (\widetilde{\rho}^{(1)}, -\widetilde{\sigma}^{(1)})$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = \mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\beta}^{(1)}]$ ,  $h_{(2)} = \mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}]$  and  $p = 4$ , we infer

$$\begin{aligned} & \int_{C_u^V} r^4 |\widetilde{\beta}^{(1)}|^2 + \int_{\underline{C}_u^V} r^4 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} \left( r^4 |\widetilde{\beta}^{(1)}|^2 + r^4 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})|^2 \right) + \int_V r^4 \left| \widetilde{\beta}^{(1)} \cdot \left( \mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\beta}^{(1)}] \right) \right| \\ & \quad + \int_V r^4 \left| (\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)}) \cdot \left( \mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, \nabla_4 \widetilde{\sigma}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\rho}^{(1)}, \nabla_4 \widetilde{\sigma}^{(1)}] \right) \right| \tag{4.27} \\ & \lesssim |u|^{4-s} \varepsilon_0^2 + \int_V r^4 \left| \widetilde{\beta}^{(1)} \cdot \left( \mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}] + \text{Err}[\nabla_3 \widetilde{\beta}^{(1)}] \right) \right| \\ & \quad + \int_V r^4 \left| (\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)}) \cdot \left( \mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, \nabla_4 \widetilde{\sigma}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\rho}^{(1)}, \nabla_4 \widetilde{\sigma}^{(1)}] \right) \right|. \end{aligned}$$

Notice that we have

$$\begin{aligned} & \int_V r^4 |\widetilde{\beta}^{(1)}| |\mathfrak{L}[\nabla_3 \widetilde{\beta}^{(1)}]| + r^4 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})| |\mathfrak{L}[\nabla_4 \widetilde{\rho}^{(1)}, -\nabla_4 \widetilde{\sigma}^{(1)}]| \\ & \lesssim \int_V M r^2 |\check{b}^{(1)}|^2 + M^2 r |\check{b}^{(1)}| |\check{\underline{\beta}}^{(1)}| + M r |\Gamma_g^{(1)}| |\check{\rho}^{(1)}| \\ & \lesssim M \int_{|u|}^u \frac{du}{r^2} \int_{\underline{C}_u^V} r^4 |\check{b}^{(1)}|^2 + M^2 \int_{|u|}^u \frac{du}{r^{\frac{s}{2}}} \int_{\underline{C}_u^V} |r^{\frac{s}{2}} \check{b}^{(1)}| |r \check{\underline{\beta}}^{(1)}| \\ & \quad + M \int_{|u|}^u \frac{du}{r^2} \int_{\underline{C}_u^V} |r \Gamma_g^{(1)}| |r^2 \check{\rho}^{(1)}| \\ & \lesssim M \int_{|u|}^u \frac{du}{r^2} \frac{\varepsilon^2}{|u|^{s-4}} + M^2 \int_{|u|}^u \frac{du}{r^{\frac{s}{2}}} \left( \int_{\underline{C}_u^V} |r^{\frac{s}{2}} \check{b}^{(1)}| \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |r \check{\underline{\beta}}^{(1)}|^2 \right)^{\frac{1}{2}} \tag{4.28} \\ & \quad + M \int_{|u|}^u \frac{du}{r^2} \left( \int_{u_0(u)}^u du |r \Gamma_g^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |r^2 \check{\rho}^{(1)}|^2 \right)^{\frac{1}{2}} \\ & \lesssim \frac{M \varepsilon^2}{r |u|^{s-4}} + M^2 \varepsilon \int_{|u|}^u \frac{du}{r^{\frac{s}{2}}} \frac{\varepsilon}{|u|^{\frac{s-2}{2}}} + M \int_{|u|}^u \frac{du}{r^2} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} \\ & \lesssim \frac{M \varepsilon^2}{r |u|^{s-4}} + \frac{M^2 \varepsilon^2}{|u|^{s-2}} + \frac{M \varepsilon^2}{r |u|^{s-4}} \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-4}}. \end{aligned}$$

Next, we proceed as in Proposition 4.8 in [53] to obtain

$$\int_V r^4 |\widetilde{\beta}^{(1)}| |\text{Err}[\nabla_3 \widetilde{\beta}^{(1)}]| + r^4 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})| |\text{Err}[\nabla_4 \widetilde{\rho}^{(1)}, \nabla_4 \widetilde{\sigma}^{(1)}]| \lesssim \frac{\varepsilon_0^2}{|u|^{s-4}}. \tag{4.29}$$

Combining (4.27), (4.28) and (4.29), we deduce

$$\int_{C_u^V} r^4 |\widetilde{\beta}^{(1)}|^2 + \int_{\underline{C}_u^V} r^4 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{s-4}}. \tag{4.30}$$

Combining with Lemma 4.6, this concludes the proof of Proposition 4.10.  $\square$

#### 4.4 Estimates for the Bianchi pair $((\check{\rho}, \check{\sigma}), \check{\underline{\beta}})$

**Proposition 4.11.** *We have the following estimates:*

$$\mathcal{R}_0[(\check{\rho}, \check{\sigma})]^2 + \mathcal{R}_0[\check{\underline{\beta}}]^2 + \mathcal{R}_1[(\check{\rho}, \check{\sigma})]^2 + \mathcal{R}_1[\check{\underline{\beta}}]^2 \lesssim \varepsilon_0^2. \quad (4.31)$$

*Proof.* We recall the following equations from Propositions 2.37 and 2.38:

$$\begin{aligned} \nabla_4 \check{\underline{\beta}}^{(1)} + \text{tr} \chi \check{\underline{\beta}}^{(1)} &= d_1^*(\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \mathfrak{L}[\nabla_4 \check{\underline{\beta}}^{(1)}] + \text{Err}[\nabla_4 \check{\underline{\beta}}^{(1)}], \\ \nabla_3(\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \frac{3}{2} \text{tr} \underline{\chi}(\check{\rho}^{(1)}, \check{\sigma}^{(1)}) &= -d_1 \check{\underline{\beta}}^{(1)} + \mathfrak{L}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}] + \text{Err}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}], \end{aligned}$$

where

$$\mathfrak{L}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}] = \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}}^{(1)}, \quad \mathfrak{L}[\nabla_4 \check{\underline{\beta}}^{(1)}] = \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}}^{(1)}.$$

Applying (4.11) with  $\psi_{(1)} = (\check{\rho}^{(1)}, \check{\sigma}^{(1)})$ ,  $\psi_{(2)} = \check{\underline{\beta}}^{(1)}$ ,  $a_{(1)} = 1$ ,  $a_{(2)} = \frac{3}{2}$ ,  $h_{(1)} = \mathfrak{L}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}] + \text{Err}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}]$ ,  $h_{(2)} = \mathfrak{L}[\nabla_4 \check{\underline{\beta}}^{(1)}] + \text{Err}[\nabla_4 \check{\underline{\beta}}^{(1)}]$  and  $p = 2$ , we obtain

$$\begin{aligned} & \int_{C_u^V} r^2 |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 + \int_{\underline{C}_u^V} r^2 |\check{\underline{\beta}}^{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^2 \left( |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 + |\check{\underline{\beta}}^{(1)}|^2 \right) + \int_V r |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 \\ & \quad + \int_V r^2 |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})| |\mathfrak{L}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}] + \text{Err}[\nabla_3 \check{\rho}^{(1)}, \nabla_3 \check{\sigma}^{(1)}]| \\ & \quad + \int_V r^2 |\check{\underline{\beta}}^{(1)}| |\mathfrak{L}[\nabla_4 \check{\underline{\beta}}^{(1)}] + \text{Err}[\nabla_4 \check{\underline{\beta}}^{(1)}]|. \end{aligned}$$

First, we have

$$\int_{\Sigma_0 \cap V} r^2 \left( |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 + |\check{\underline{\beta}}^{(1)}|^2 \right) \lesssim |u|^{2-s} \int_{\Sigma_0 \cap V} r^s \left( |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 + |\check{\underline{\beta}}^{(1)}|^2 \right) \lesssim \frac{\varepsilon_0^2}{|u|^{s-2}}. \quad (4.32)$$

Notice that we have from Proposition 4.10 that

$$\int_V r |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 \lesssim \int_{|u|}^u r^{-3} d\underline{u} \int_{\underline{C}_u^V} r^4 |(\check{\rho}^{(1)}, \check{\sigma}^{(1)})|^2 \lesssim \int_{|u|}^u r^{-3} d\underline{u} \frac{\varepsilon_0^2}{|u|^{s-4}} \lesssim \frac{\varepsilon_0^2}{|u|^{s-2}}. \quad (4.33)$$

Moreover, we have

$$\begin{aligned}
 & \int_V r^2 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})| |\mathfrak{L}[\nabla_3 \widetilde{\rho}^{(1)}, \nabla_3 \widetilde{\sigma}^{(1)}]| + r^2 |\widetilde{\beta}^{(1)}| |\mathfrak{L}[\nabla_4 \widetilde{\beta}^{(1)}]| \\
 & \lesssim \int_V M r^{-1} |\check{\rho}^{(1)}| |\Gamma_b^{(1)}| + M^2 r^{-1} |\check{\rho}^{(1)}| |\check{\alpha}^{(1)}| \\
 & \lesssim M \int_{|u|}^u \frac{du}{r^3} \int_{\underline{C}_u^V} |r^2 \check{\rho}^{(1)}| |\Gamma_b^{(1)}| + M^2 \int_{|u|}^u \frac{du}{r^3} \int_{\underline{C}_u^V} |r^2 \check{\rho}^{(1)}| |\check{\alpha}^{(1)}| \\
 & \lesssim M \int_{|u|}^u \frac{du}{r^3} \left( \int_{\underline{C}_u^V} |r^2 \check{\rho}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{u_0(u)}^u du |\Gamma_b^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \\
 & \quad + M^2 \int_{|u|}^u \frac{du}{r^3} \left( \int_{\underline{C}_u^V} |r^2 \check{\rho}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |\check{\alpha}^{(1)}|^2 \right)^{\frac{1}{2}} \\
 & \lesssim M \int_{|u|}^u \frac{du}{r^3} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} \frac{\varepsilon}{|u|^{\frac{s-2}{2}}} + M^2 \int_{|u|}^u \frac{du}{r^3} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \\
 & \lesssim \frac{M\varepsilon^2}{|u|^{s-1}} + \frac{M^2\varepsilon^2}{|u|^s} \\
 & \lesssim \frac{\varepsilon_0^2}{|u|^{s-2}}.
 \end{aligned} \tag{4.34}$$

Next, we have from Proposition 4.12 in [53] that

$$\int_V r^2 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})| |\text{Err}[\nabla_3 \widetilde{\rho}^{(1)}, \nabla_3 \widetilde{\sigma}^{(1)}]| + r^2 |\widetilde{\beta}^{(1)}| |\text{Err}[\nabla_4 \widetilde{\beta}^{(1)}]| \lesssim \frac{\varepsilon_0^2}{|u|^{s-2}}. \tag{4.35}$$

Combining (4.32), (4.33), (4.34) and (4.35), we obtain

$$\int_{C_u^V} r^2 |(\widetilde{\rho}^{(1)}, \widetilde{\sigma}^{(1)})|^2 + \int_{\underline{C}_u^V} r^2 |\widetilde{\beta}^{(1)}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{s-2}}.$$

Combining with Lemma 4.6, this concludes the proof of Proposition 4.11.  $\square$

### 4.5 Estimates for the Bianchi pair $(\check{\beta}, \check{\alpha})$

**Proposition 4.12.** *We have the following estimate:*

$$\mathcal{R}_0[\check{\beta}]^2 + \mathcal{R}_0[\check{\alpha}]^2 + \mathcal{R}_1[\check{\beta}]^2 + \mathcal{R}_1[\check{\alpha}]^2 \lesssim \varepsilon_0^2. \tag{4.36}$$

*Proof.* We recall the following Bianchi equations from Propositions 2.37 and 2.38:

$$\begin{aligned}
 \nabla_4 \widetilde{\alpha}^{(1)} + \frac{1}{2} \text{tr} \chi \widetilde{\alpha}^{(1)} &= -\nabla \widehat{\otimes} \widetilde{\beta}^{(1)} + \mathfrak{L}[\nabla_4 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\alpha}^{(1)}], \\
 \nabla_3 \check{\beta} + 2 \text{tr} \chi \check{\beta} &= -\text{div} \widetilde{\alpha}^{(1)} + \mathfrak{L}[\nabla_3 \check{\beta}^{(1)}] + \text{Err}[\nabla_3 \check{\beta}^{(1)}],
 \end{aligned} \tag{4.37}$$

where

$$\mathfrak{L}[\nabla_4 \widetilde{\alpha}^{(1)}] = \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha}^{(1)}, \quad \mathfrak{L}[\nabla_3 \check{\beta}^{(1)}] = \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\beta}^{(1)}.$$

Applying (4.11) with  $\psi_{(1)} = \check{\beta}^{(1)}$ ,  $\psi_{(2)} = \check{\alpha}^{(1)}$ ,  $a_{(1)} = 2$ ,  $a_{(2)} = \frac{1}{2}$ ,  $h_{(1)} = \mathfrak{L}[\nabla_3 \check{\beta}^{(1)}] + \text{Err}[\nabla_3 \check{\beta}^{(1)}]$ ,

$h_{(2)} = \mathfrak{L}[\nabla_4 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\alpha}^{(1)}]$  and  $p = 0$ , we obtain

$$\begin{aligned} & \int_{C_u^V} |\underline{\beta}^{(1)}|^2 + \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2 + \int_V r^{-1} |\underline{\beta}^{(1)}|^2 \\ & \quad + \int_V |\underline{\beta}^{(1)}| |\mathfrak{L}[\nabla_3 \underline{\beta}^{(1)}] + \text{Err}[\nabla_3 \underline{\beta}^{(1)}]| + |\underline{\alpha}^{(1)}| |\mathfrak{L}[\nabla_4 \widetilde{\alpha}^{(1)}] + \text{Err}[\nabla_4 \widetilde{\alpha}^{(1)}]|. \end{aligned} \quad (4.38)$$

First, we have

$$\int_{\Sigma_0 \cap V} |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2 \lesssim |u|^{-s} \int_{\Sigma_0 \cap V} r^s \left( |\underline{\beta}^{(1)}|^2 + |\underline{\alpha}^{(1)}|^2 \right) \lesssim \frac{\varepsilon_0^2}{|u|^s}.$$

We recall from Proposition 4.11 that

$$\int_V r^{-1} |\underline{\beta}^{(1)}|^2 \lesssim \int_{|u|}^u \frac{du}{r^3} \int_{\underline{C}_u^V} |r \check{\beta}^{(1)}|^2 \lesssim \int_{|u|}^u \frac{du}{r^3} \frac{\varepsilon_0^2}{|u|^{s-2}} \lesssim \frac{\varepsilon_0^2}{|u|^s}.$$

Next, we estimate

$$\begin{aligned} & \int_V |\underline{\beta}^{(1)}| |\mathfrak{L}[\nabla_3 \underline{\beta}^{(1)}]| + |\underline{\alpha}^{(1)}| |\mathfrak{L}[\nabla_4 \widetilde{\alpha}^{(1)}]| \\ & \lesssim \int_V M r^{-3} |\dot{\underline{\alpha}}^{(1)}| |\Gamma_b^{(1)}| + M r^{-2} |\dot{\underline{\alpha}}^{(1)}|^2 \\ & \lesssim M \int_{|u|}^u \frac{du}{r^3} \left( \int_{\underline{C}_u^V} |\dot{\underline{\alpha}}^{(1)}|^2 \right)^{\frac{1}{2}} \left( \int_{u_0(u)}^u |\Gamma_b^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} + M \int_{|u|}^u \frac{du}{r^2} \int_{\underline{C}_u^V} |\dot{\underline{\alpha}}^{(1)}|^2 \\ & \lesssim M \int_{|u|}^u \frac{du}{r^3} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \frac{\varepsilon}{|u|^{\frac{s-2}{2}}} + M \int_{|u|}^u \frac{du}{r^2} \frac{\varepsilon^2}{|u|^s} \\ & \lesssim \frac{\varepsilon_0^2}{|u|^s}. \end{aligned}$$

We recall from Proposition 4.13 in [53]

$$\int_V |\underline{\beta}^{(1)}| |\text{Err}[\nabla_3 \underline{\beta}^{(1)}]| + |\underline{\alpha}^{(1)}| |\text{Err}[\nabla_4 \widetilde{\alpha}^{(1)}]| \lesssim \frac{\varepsilon_0^2}{|u|^s}.$$

Hence, we obtain

$$\int_{C_u^V} |\underline{\beta}^{(1)}|^2 + \int_{\underline{C}_u^V} |\underline{\alpha}^{(1)}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^s}.$$

This concludes the proof of Proposition 4.12.  $\square$

## 4.6 End of the proof of Theorem M1

**Proposition 4.13.** *We have the following estimate:*

$$\mathcal{R}_0^S + \underline{\mathcal{R}}_0^S \lesssim \mathcal{R}_0 + \mathcal{R}_1 + \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1. \quad (4.39)$$

*Proof.* It is sufficient to prove that

$$\mathcal{R}_0^S \lesssim \mathcal{R}_0 + \mathcal{R}_1, \quad \underline{\mathcal{R}}_0^S \lesssim \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1, \quad (4.40)$$

which is a direct consequence of Propositions 2.37, 2.38 and 2.45.  $\square$

Remark that Propositions 4.9–4.13 imply

$$\mathcal{R}^2 \lesssim \varepsilon_0^2.$$

This concludes the proof of Theorem M1.

## 5 Ricci coefficients estimates (Theorem M3)

The goal of this section is to prove Theorem M3, which we recall below for convenience.

**Theorem M3.** *Assume that*

$$\mathcal{O}_{(0)} \leq \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \quad (5.1)$$

Then, we have

$$\mathcal{O} \lesssim \varepsilon_0, \quad \text{Osc} \lesssim \varepsilon_0. \quad (5.2)$$

**Remark 5.1.** *Throughout Section 5, we denote*

$$S_0(\underline{u}) := S(u_0(\underline{u}), \underline{u}), \quad S_*(u) := S(u, \underline{u}_*), \quad S := S(u, \underline{u}).$$

### 5.1 Preliminaries

#### 5.1.1 Null frame transformation

In order to control Osc, we will need to compare the double null frame of  $\mathcal{K}$  and the one of  $\mathcal{K}_{(0)}$  in  $\mathcal{K}_{(0)}$ . To this end, we will rely on the null frame transformations introduced in [33] and [34].

**Lemma 5.2.** *Given two null frames  $(e_3, e_4, e_1, e_2)$  and  $(e'_3, e'_4, e'_1, e'_2)$  associated to double-null foliations, assume that they have the same generator  $\underline{L}$  for the ingoing direction. Then, a null frame transformation from the null frame  $(e_3, e_4, e_1, e_2)$  to  $(e'_3, e'_4, e'_1, e'_2)$  can be written in the form:*

$$\begin{aligned} e'_4 &= \lambda \left( e_4 + f^B e_B + \frac{1}{4} |f|^2 e_3 \right), \\ e'_3 &= \lambda^{-1} e_3, \\ e'_A &= e_A + \frac{1}{2} f_A e_3, \end{aligned} \quad (5.3)$$

where

$$\lambda = \frac{\Omega}{\Omega'} \quad (5.4)$$

is a scalar function and  $f$  is a 1-form. Moreover, the inverse transform of (5.3) is given by

$$\begin{aligned} e_4 &= \lambda^{-1} e'_4 - f^A e'_A + \frac{\lambda}{4} |f|^2 e'_3, \\ e_3 &= \lambda e'_3, \\ e_A &= e'_A - \frac{\lambda}{2} f_A e'_3. \end{aligned} \quad (5.5)$$

*Proof.* Applying Lemma 3.1 in [34] with  $\underline{f} = 0$ , we obtain (5.3). Notice that (5.5) is a direct consequence of (5.3). Recalling

$$e'_3 = 2\Omega' \underline{L}, \quad e_3 = 2\Omega \underline{L},$$

we deduce immediately (5.4). This concludes the proof of Lemma 5.2. □



**Proposition 5.3.** *Under the null frame transformation (5.3), some of the Ricci coefficients transform as follows:*

$$\begin{aligned}
\lambda^{-1} \operatorname{tr} \chi' &= \operatorname{tr} \chi + \operatorname{div}' f + l.o.t., \\
\operatorname{curl}' f &= l.o.t., \\
\lambda \underline{\chi}' &= \underline{\chi}, \\
\eta' &= \eta + \frac{1}{2} \lambda \nabla'_{e'_3} f + l.o.t., \\
\underline{\eta}' &= \underline{\eta} + \frac{1}{4} \operatorname{tr} \chi f + l.o.t., \\
\lambda^{-2} \xi' &= \xi + \frac{1}{2} \nabla'_{\lambda^{-1} e'_4} f + \frac{1}{4} \operatorname{tr} \chi f + l.o.t., \\
\lambda^{-1} \omega' &= \omega + l.o.t.,
\end{aligned}$$

where the terms denoted by *l.o.t.* have the following schematic form:

$$l.o.t. = \Gamma_b \cdot f + \mathbf{O}_2^1 \cdot f.$$

The curvature components transform as follows:

$$\begin{aligned}
\lambda^{-2} \alpha' &= \alpha + (f \widehat{\otimes} \beta - * f \widehat{\otimes} * \beta) + \left( f \widehat{\otimes} f - \frac{1}{2} * f \widehat{\otimes} * f \right) \rho + \frac{3}{2} (f \widehat{\otimes} * f)^* \rho + l.o.t., \\
\lambda^2 \underline{\alpha}' &= \underline{\alpha}, \\
\lambda^{-1} \beta' &= \beta + \frac{3}{2} (f \rho + * f \sigma) + l.o.t., \\
\lambda \underline{\beta}' &= \underline{\beta} - \frac{1}{2} \underline{\alpha} \cdot f, \\
\rho' &= \rho - f \cdot \underline{\beta} + l.o.t., \\
\sigma' &= \sigma - f \cdot * \underline{\beta} + l.o.t.,
\end{aligned}$$

where the terms denoted by *l.o.t.* have the following schematic form:

$$l.o.t. = O(f^3)(\rho, \sigma) + O(f^2)(\alpha, \beta, \underline{\beta}, \underline{\alpha}). \quad (5.6)$$

*Proof.* All the transformation formulae, with the exception of the one for  $\omega$ , follow immediately from Proposition 3.3 in [34], in view of the fact that  $\underline{f} = 0$ . See Proposition 6.2 in [53] for the proof of the transformation formula for  $\omega$  which uses in addition that  $\lambda = \frac{\Omega}{\Omega'}$ .  $\square$

### 5.1.2 Difference of the pull backs of Kerr values

The goal of this section is to compare the difference of the pull backs of Kerr values by  $\Phi$  and  $\Phi_{(0)}$  in  $\mathcal{K}_{(0)}$  which will appear in the control of Osc. In the remainder of this paper, for any quantity  $X$ , we denote the quantity  $X_{(0)}$  associated to the initial data layer region  $\mathcal{K}_{(0)}$  by

$$X' := X_{(0)}, \quad (5.7)$$

to simplify the notations. For example:

$$u' := u_{(0)}, \quad \eta' := \eta_{(0)}, \quad \mathcal{O}' := \mathcal{O}_{(0)}, \quad \mathfrak{R}' := \mathfrak{R}_{(0)}, \quad S' := S_{(0)}, \quad \mathcal{K}' := \mathcal{K}_{(0)}. \quad (5.8)$$

**Lemma 5.4.** *Let  $X$  be a scalar defined in  $\mathcal{K}'$ . We denote*

$$X'_K := \Phi'^*(X_{Kerr}), \quad X_K := \Phi^*(X_{Kerr}),$$

and we assume that

$$X_K = \mathbf{O}_q^k.$$

Then, we have:

$$\sup_S |X'_K - X_K| \lesssim \frac{M^k \varepsilon}{r^{q+\frac{s-1}{2}}} \lesssim \frac{\varepsilon_0}{r^{q-k+\frac{s-1}{2}}}.$$

*Proof.* Let  $p \in \mathcal{K}$  which has the coordinates  $(u, \underline{u}, x^1, x^2)$  and resp.  $(u', \underline{u}, x'^1, x'^2)$  in two different foliations. Then, we have

$$\begin{aligned} |X'_K(p) - X_K(p)| &\lesssim |\Phi'^*(X_{Kerr})(p) - \Phi^*(X_{Kerr})(p)| \\ &\lesssim |X_{Kerr}(\Phi'(p)) - X_{Kerr}(\Phi(p))| \\ &\lesssim \left| \frac{\partial X_{Kerr}}{\partial u_{Kerr}}(u' - u) \right| + \left| \frac{\partial X_{Kerr}}{\partial x^A_{Kerr}}(x'^A - x^A) \right| \\ &\lesssim \frac{M^k}{r^{q+1}} |u' - u| + \frac{M^k}{r^q} |x'^A - x^A| \\ &\lesssim \frac{M^k}{r^{q+1}} \frac{\text{Osc}(u)}{r^{\frac{s-3}{2}}} + \frac{M^k}{r^q} \frac{\text{Osc}(x)}{r^{\frac{s-1}{2}}} \\ &\lesssim \frac{M^k}{r^{q+1}} \frac{\varepsilon}{r^{\frac{s-3}{2}}} + \frac{M^k}{r^q} \frac{\varepsilon}{r^{\frac{s-1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{q-k+\frac{s-1}{2}}}, \end{aligned} \tag{5.9}$$

where we used (5.1) to bound  $\text{Osc}(u)$  and  $\text{Osc}(x)$ . This concludes the proof of Lemma 5.4.  $\square$

### 5.1.3 Evolution lemma

**Lemma 5.5** (Evolution lemma). *Assume that the spacetime  $\mathcal{K}$  is foliated by  $(u, \underline{u})$ .*

(1) *Let  $U, F$  be  $k$ -covariant  $S$ -tangent tensor fields satisfying the outgoing evolution equation*

$$\nabla_4 U + \lambda_0 \text{tr } \chi U = F, \tag{5.10}$$

where  $\lambda_0 \geq 0$ . Denoting  $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$ , we have along  $C_u$

$$|r^{\lambda_1} U|_{p,S} \lesssim |r^{\lambda_1} U|_{p,S_*(u)} + \int_{\underline{u}(p)}^{\underline{u}_*(p)} |r^{\lambda_1} F|_{p,S}(u, \underline{u}') du', \tag{5.11}$$

where  $\underline{u}_*(p)$  is such that  $C_{u(p)} \cap \underline{C}_* = S(u(p), \underline{u}_*(p))$  and where  $S_*(u) = S(u(p), \underline{u}_*(p))$ .

(2) *Let  $V, \underline{F}$  be  $k$ -covariant  $S$ -tangent tensor fields satisfying the ingoing evolution equation*

$$\nabla_3 V + \lambda_0 \text{tr } \underline{\chi} V = \underline{F}. \tag{5.12}$$

Denoting  $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$ , we have along  $\underline{C}_u$

$$|r^{\lambda_1} V|_{p,S} \lesssim |r^{\lambda_1} V|_{p,S_0(\underline{u})} + \int_{u_0(\underline{u})}^u |r^{\lambda_1} \underline{F}|_{p,S}(u', \underline{u}) du', \tag{5.13}$$

where  $S(u_0(\underline{u}), \underline{u}) := C_{u_0(\underline{u})} \cap \underline{C}_u \subset \mathcal{K}$  is the unique sphere of  $\underline{C}_u$  which is located in the future of  $\Sigma_0$  and touches  $\Sigma_0$  and where  $S_0(\underline{u}) = S(u_0(\underline{u}), \underline{u})$ .

*Proof.* See Lemma 4.1.5 in [29].  $\square$

### 5.1.4 Bootstrap control in the initial data layer

**Lemma 5.6.** *We denote*

$$\Gamma_w := \Gamma_g \cup \Gamma_b \cup \Gamma'_g \cup \Gamma'_b. \quad (5.14)$$

*Then, we have the following estimates in  $\mathcal{K}'$ :*

$$\begin{aligned} \sup_{\mathcal{K}'} r^{\frac{s+1}{2}} |\Gamma_w| &\lesssim \varepsilon, \\ \sup_{S \subset \mathcal{K}'} r^{\frac{s+1}{2}} |r^{-\frac{2}{p}} \Gamma_w^{(1)}|_{p,S} &\lesssim \varepsilon, \quad p \in [2, 4], \end{aligned} \quad (5.15)$$

*Moreover, we have*

$$f \in r\Gamma_w, \quad \lambda \in r\Gamma_w, \quad u' - u \in r^2\Gamma_w, \quad r' - r \in r^2\Gamma_w.$$

*Proof.* It follows directly from Lemma 4.1 and the assumption (5.1).  $\square$

## 5.2 Estimates for $\mathcal{O}_{[1]}$ , $\mathcal{O}_\gamma$ and Osc

In the remainder of Section 5, we always assume  $p \in [2, 4]$ .

### 5.2.1 Estimate for $\mathring{\mathcal{O}}_{0,1}(\widetilde{\Omega}\omega)$

**Proposition 5.7.** *We have the following estimate for  $q = 0, 1$ :*

$$|r^{\frac{s+1}{2} - \frac{2}{p}} (r\nabla)^q \widetilde{\Omega}\omega|_{p, S_0(\underline{u})} \lesssim \varepsilon_0, \quad (5.16)$$

where we recall that  $S_0(\underline{u})$  is the unique leaf of  $\underline{C}_{\underline{u}}$  which is located in the future of  $\Sigma_0$  and touches  $\Sigma_0$ .

*Proof.* We recall from Proposition 5.3 and  $\lambda = \frac{\Omega}{\Omega'}$  that

$$\Omega'\omega' = \Omega\omega + f \cdot \Gamma_w + \mathbf{O}_2^1 \cdot f. \quad (5.17)$$

Notice from Proposition 2.28 and Lemma 5.4 that

$$|\Omega_K\omega_K - \Omega'_K\omega'_K| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Hence, we have

$$\begin{aligned} |\widetilde{\Omega'\omega'} - \widetilde{\Omega}\omega| &\lesssim |\Omega'\omega' - \Omega\omega| + |\Omega'_K\omega'_K - \Omega_K\omega_K| \\ &\lesssim |\Gamma_w \cdot f| + |\mathbf{O}_2^1 \cdot f| + \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned}$$

Thus, we infer

$$|r^{\frac{s+1}{2} - \frac{2}{p}} \widetilde{\Omega}\omega|_{p, S_0(\underline{u})} \lesssim |r^{\frac{s+1}{2} - \frac{2}{p}} \widetilde{\Omega'\omega'}|_{p, S_0(\underline{u})} + \varepsilon_0 \lesssim \varepsilon_0.$$

We have from (5.17) and (5.5)

$$\begin{aligned} e_A(\Omega\omega) &= e'_A(\Omega'\omega') - \frac{\lambda}{2} f_A e'_3(\Omega'\omega') + \Gamma_w^{(1)} \cdot f + \Gamma_w \cdot f^{(1)} + \mathbf{O}_2^1 \cdot f^{(1)} \\ &= e'_A(\Omega'\omega') + \Gamma_w \cdot \Gamma_w^{(1)} + \mathbf{O}_2^1 \cdot \Gamma_w^{(1)}. \end{aligned}$$

Notice that we have from Lemma 5.4

$$|e_A(\Omega_K \omega_K) - e'_A(\Omega'_K \omega'_K)| \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}.$$

Hence, we have

$$|r^{-\frac{2}{p}} e_A(\widetilde{\Omega \omega})|_{p, S_0(\underline{u})} \lesssim |e'_A(\widetilde{\Omega' \omega'})|_{\infty, S_0(\underline{u})} + \frac{M}{r^2} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{s+1}} + \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}},$$

which implies

$$|r^{\frac{s+3}{2} - \frac{2}{p}} e_A(\widetilde{\Omega \omega})|_{p, S_0(\underline{u})} \lesssim \varepsilon_0.$$

This concludes the proof of Proposition 5.7. □

### 5.2.2 Estimates for $\mathcal{O}_0(\widetilde{\Omega \omega})$ and $\mathcal{O}_0(\widetilde{\Omega \underline{\omega}})$

**Proposition 5.8.** *We have the following estimates:*

$$|r^{2 + \frac{s-3}{6} - \frac{2}{p}} |u|^{\frac{s-3}{3}} \widetilde{\Omega \omega}|_{p, S} \lesssim \varepsilon_0, \tag{5.18}$$

$$|r^{2 - \frac{2}{p}} |u|^{\frac{s-3}{2}} \widetilde{\Omega \underline{\omega}}|_{p, S} \lesssim \varepsilon_0. \tag{5.19}$$

*Proof.* We recall from Proposition 2.34

$$\nabla_4(\widetilde{\Omega \underline{\omega}}) = \mathbf{O}_0^0 \cdot \check{\rho} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g + r \Gamma_g \cdot \check{\rho}.$$

Applying Lemma 5.5, we infer

$$\begin{aligned} |r^{-\frac{2}{p}} \widetilde{\Omega \underline{\omega}}|_{p, S} &\lesssim |r^{-\frac{2}{p}} \widetilde{\Omega \underline{\omega}}|_{p, S_*(\underline{u})} + \int_{\underline{u}}^{\underline{u}_*} \left( |r^{-\frac{2}{p}} \check{\rho}|_{p, S} + \frac{M}{r^2} |r^{-\frac{2}{p}} \Gamma_g|_{p, S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p, S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{\underline{u}_*} \left( \frac{\varepsilon_0}{r^3 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon M}{r^4 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^4 |u|^{s-3}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}}, \end{aligned}$$

which implies (5.19). Next, we recall from Proposition 2.34

$$\nabla_3(\widetilde{\Omega \omega}) = \mathbf{O}_0^0 \cdot \check{\rho} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g + r \Gamma_g \cdot \check{\rho}.$$

Applying Lemma 5.5 and Proposition 5.7, we deduce

$$\begin{aligned} |r^{-\frac{2}{p}} \widetilde{\Omega \omega}|_{p, S} &\lesssim |r^{-\frac{2}{p}} \widetilde{\Omega \omega}|_{p, S_0(\underline{u})} + \int_{u_0(\underline{u})}^u \left( |r^{-\frac{2}{p}} \check{\rho}|_{p, S} + \frac{M}{r^2} |r^{-\frac{2}{p}} \Gamma_g|_{p, S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p, S} \right) du \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \int_{u_0(\underline{u})}^u \left( \frac{\varepsilon_0}{r^3 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon M}{r^3 |u|^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r^4 |u|^{s-3}} \right) du \\ &\lesssim \frac{\varepsilon_0}{r^{2 + \frac{s-3}{6}} |u|^{\frac{s-3}{3}}}, \end{aligned}$$

which implies (5.18). This concludes the proof of Proposition 5.8. □

### 5.2.3 Estimates for $\mathcal{O}_1(\widetilde{\Omega\omega})$ and $\mathcal{O}_1(\widetilde{\Omega\underline{\omega}})$

**Proposition 5.9.** *We have the following estimates:*

$$|r^{3+\frac{s-3}{6}-\frac{2}{p}}|u|^{\frac{s-3}{3}}\nabla\widetilde{\Omega\omega}|_{p,S}(u,\underline{u}) \lesssim \varepsilon_0, \quad (5.20)$$

$$|r^{2-\frac{2}{p}}|u|^{\frac{s-1}{2}}\nabla\widetilde{\Omega\underline{\omega}}|_{p,S}(u,\underline{u}) \lesssim \varepsilon_0. \quad (5.21)$$

*Proof.* We recall from Propositions 2.34

$$\Omega\nabla_4(\widetilde{\Omega\underline{\omega}}) = \frac{1}{2}\widetilde{\Omega^2\rho} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g.$$

Differentiating it by  $r\nabla$  and applying Corollary 2.43, we infer

$$\nabla_4(r\nabla\widetilde{\Omega\underline{\omega}}) = \frac{\Omega}{2}r\nabla\check{\rho} + \Gamma_g^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g^{(1)}. \quad (5.22)$$

Since we cannot control  $\nabla\check{\rho}$  in  $L^p$  directly, we use a renormalization argument. We recall from Proposition 2.37

$$\nabla_4\check{\underline{\beta}} + \text{tr}\chi\check{\underline{\beta}} = -\nabla\check{\rho} + {}^*\nabla\check{\sigma} + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + r^{-1}\Gamma_g \cdot \Gamma_b^{(1)},$$

which implies from (4.14)

$$\nabla_4(\Omega r\check{\underline{\beta}}) + \frac{1}{2}\text{tr}\chi(\Omega r\check{\underline{\beta}}) = -\Omega r\nabla\check{\rho} + \Omega r{}^*\nabla\check{\sigma} + \Gamma_b^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_b^{(1)}. \quad (5.23)$$

We define  $\check{\underline{\omega}}^\dagger$  by

$$\nabla_4\check{\underline{\omega}}^\dagger = \frac{1}{2}\Omega\check{\sigma}, \quad \check{\underline{\omega}}^\dagger = 0 \quad \text{on } \underline{C}_*.$$

Applying Proposition 2.41, we deduce

$$\Omega\nabla_4(r{}^*\nabla\check{\underline{\omega}}^\dagger) = r{}^*\nabla(\Omega\nabla_4\check{\underline{\omega}}^\dagger) + [\Omega\nabla_4, r{}^*\nabla]\check{\underline{\omega}}^\dagger = \frac{1}{2}\Omega r{}^*\nabla\check{\sigma} + \Gamma_g \cdot r\nabla\check{\underline{\omega}}^\dagger + \Gamma_g \cdot \Gamma_g^{(1)}. \quad (5.24)$$

We introduce a new quantity

$$\underline{\varkappa} := -r\nabla\widetilde{\Omega\underline{\omega}} + r{}^*\nabla\check{\underline{\omega}}^\dagger - \frac{1}{2}r\check{\underline{\beta}}.$$

Then, we obtain from (5.22), (5.23) and (5.24)

$$\begin{aligned} \nabla_4\underline{\varkappa} &= -\nabla_4(r\nabla\widetilde{\Omega\underline{\omega}}) + \nabla_4(r{}^*\nabla\check{\underline{\omega}}^\dagger) - \frac{1}{2}\nabla_4(r\check{\underline{\beta}}) \\ &= \Gamma_b^{(1)} \cdot O\left(\frac{M}{r^2}\right) + \Gamma_g \cdot \Gamma_b^{(1)} + \Gamma_g \cdot r\nabla\check{\underline{\omega}}^\dagger. \end{aligned} \quad (5.25)$$

Applying Lemma 5.5, we obtain

$$\begin{aligned} |r^{-\frac{2}{p}}\underline{\varkappa}|_{p,S} &\lesssim |r^{-\frac{2}{p}}\underline{\varkappa}|_{p,S_*(u)} + \int_{\underline{u}}^{\underline{u}_*} \left( |r^{-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} + \frac{M}{r^2}|r^{-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S} + |r^{-\frac{2}{p}}\Gamma_g \cdot r\nabla\check{\underline{\omega}}^\dagger|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-1}{2}}} + \int_{\underline{u}}^{\underline{u}_*} \left( \frac{\varepsilon^2}{r^3|u|^{s-2}} + \frac{M}{r^3} \frac{\varepsilon}{|u|^{\frac{s-1}{2}}} + \frac{\varepsilon}{r^2|u|^{\frac{s-3}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\underline{\omega}}^\dagger|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-1}{2}}} + \int_{\underline{u}}^{\underline{u}_*} \left( \frac{\varepsilon}{r^2|u|^{\frac{s-3}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\underline{\omega}}^\dagger|_{p,S} \right) d\underline{u}. \end{aligned}$$

Denoting  $\langle \underline{\omega} \rangle := (\widetilde{\Omega}\underline{\omega}, \widetilde{\omega}^\dagger)$ , then<sup>20</sup>

$$d_1^* \langle \underline{\omega} \rangle = r^{-1} \underline{\varkappa} + \frac{1}{2} \underline{\beta}. \quad (5.26)$$

Applying Proposition 2.15 to (5.26), we obtain

$$\begin{aligned} |r^{1-\frac{2}{p}} \nabla \widetilde{\Omega}\underline{\omega}|_{p,S} + |r^{1-\frac{2}{p}} \nabla \widetilde{\omega}^\dagger|_{p,S} &\lesssim |r^{-\frac{2}{p}} \underline{\varkappa}|_{p,S} + |r^{1-\frac{2}{p}} \underline{\beta}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-1}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon}{r^2|u|^{\frac{s-3}{2}}} |r^{1-\frac{2}{p}} \nabla \widetilde{\omega}^\dagger|_{p,S} \right) d\underline{u}. \end{aligned}$$

By Gronwall inequality, we deduce

$$|r^{1-\frac{2}{p}} \nabla \widetilde{\Omega}\underline{\omega}|_{p,S} + |r^{1-\frac{2}{p}} \nabla \widetilde{\omega}^\dagger|_{p,S} \lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-1}{2}}},$$

which implies (5.21).

Next, we recall from Proposition 2.34

$$\Omega \nabla_3(\widetilde{\Omega}\omega) = \frac{1}{2} \widetilde{\Omega}^2 \rho + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g.$$

Differentiating it by  $r\nabla$  and applying Corollary 2.43, we infer

$$\nabla_3(r\nabla \widetilde{\Omega}\omega) = \frac{\Omega}{2} r\nabla \rho + \Gamma_g^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g^{(1)}. \quad (5.27)$$

We recall from Proposition 2.37

$$\nabla_3 \check{b} + \text{tr} \underline{\chi} \check{b} = \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + r^{-1} \Gamma_g \cdot \Gamma_b^{(1)},$$

which implies from (4.13)

$$\nabla_3(\Omega r \check{b}) + \frac{1}{2} \text{tr} \underline{\chi}(\Omega r \check{b}) = \Omega r \nabla \check{\rho} + \Omega r {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_b^{(1)}. \quad (5.28)$$

We define  $\check{\omega}^\dagger$  by

$$\nabla_3 \check{\omega}^\dagger = \frac{1}{2} \Omega \check{\sigma}, \quad \check{\omega}^\dagger = 0 \quad \text{on } S_0(\underline{u}) \quad \forall \underline{u}.$$

Applying Proposition 2.41, we deduce

$$\nabla_3(r {}^* \nabla \check{\omega}^\dagger) = \frac{1}{2} r {}^* \nabla \check{\sigma} + \Gamma_b \cdot r \nabla \check{\omega}^\dagger + \Gamma_g \cdot \Gamma_b^{(1)}. \quad (5.29)$$

Combining (5.27), (5.28) and (5.29), we obtain

$$\nabla_3 \varkappa = \Gamma_b^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_b^{(1)} + \Gamma_b \cdot r \nabla \check{\omega}^\dagger, \quad (5.30)$$

where

$$\varkappa := r \nabla \widetilde{\Omega}\omega + r {}^* \nabla \check{\omega}^\dagger - \frac{1}{2} r \check{b}. \quad (5.31)$$

---

<sup>20</sup>Recall that  $d_1^*(f, g) = -\nabla f + {}^* \nabla g$ .

Applying Lemma 5.5, we infer

$$\begin{aligned}
|r^{1-\frac{2}{p}}\mathcal{X}|_{p,S} &\lesssim |r^{1-\frac{2}{p}}\mathcal{X}|_{p,S_0(\underline{u})} + \int_{-\underline{u}}^u \left( |r^{-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} + \frac{M}{r^2} |r^{-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S} + |r^{-\frac{2}{p}}\Gamma_b \cdot r\nabla\check{\omega}^\dagger|_{p,S} \right) d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \int_{-\underline{u}}^u \left( \frac{\varepsilon^2}{r^3|u|^{s-2}} + \frac{M\varepsilon}{r^3|u|^{\frac{s-1}{2}}} + \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} \right) d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r^{2+\frac{s-3}{6}}|u|^{\frac{s-3}{3}}} + \int_{-\underline{u}}^u \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} d\underline{u}.
\end{aligned}$$

Applying Proposition 2.15, we obtain

$$\begin{aligned}
|r^{2-\frac{2}{p}}\nabla\check{\Omega}\check{\omega}|_{p,S} + |r^{2-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} &\lesssim |r^{1-\frac{2}{p}}\mathcal{X}|_{p,S} + |r^{2-\frac{2}{p}}\check{b}|_{p,S} + \int_{-\underline{u}}^u \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r^{2+\frac{s-3}{6}}|u|^{\frac{s-3}{3}}} + \int_{-\underline{u}}^u \frac{\varepsilon}{r|u|^{\frac{s-1}{2}}} |r^{1-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} d\underline{u}.
\end{aligned}$$

By Gronwall inequality, we deduce

$$|r^{2-\frac{2}{p}}\nabla\check{\Omega}\check{\omega}|_{p,S} + |r^{2-\frac{2}{p}}\nabla\check{\omega}^\dagger|_{p,S} \lesssim \frac{\varepsilon_0}{r^{2+\frac{s-3}{6}}|u|^{\frac{s-3}{3}}},$$

which implies (5.20). This concludes the proof of Proposition 5.9.  $\square$

#### 5.2.4 Estimate for $\mathcal{O}_{0,1}(\check{\Omega})$

**Proposition 5.10.** *We have the following estimate for  $q = 0, 1$ :*

$$\left| r^{1-\frac{2}{p}}(r\nabla)^q |u|^{\frac{s-3}{2}} \check{\Omega} \right|_{p,S} \lesssim \varepsilon_0. \quad (5.32)$$

*Proof.* We recall from Proposition 2.40

$$\nabla_4 \check{\Omega} = \mathbf{O}_0^0 \cdot \check{\Omega}\check{\omega} + \mathbf{O}_1^1 \cdot \Gamma_g + r\Gamma_g \cdot \Gamma_g,$$

which implies from Corollary 2.43

$$\nabla_4(r\nabla\check{\Omega}) = \mathbf{O}_0^0 \cdot r\nabla(\check{\Omega}\check{\omega}) + \mathbf{O}_1^1 \cdot \Gamma_g^{(1)} + r\Gamma_g \cdot \Gamma_g^{(1)}.$$

Applying Lemma 5.5, we have

$$\begin{aligned}
|r^{-\frac{2}{p}}\check{\Omega}|_{p,S} &\lesssim |r^{-\frac{2}{p}}\check{\Omega}|_{p,S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{-\frac{2}{p}}\check{\omega}|_{p,S} + \frac{M}{r} |r^{-\frac{2}{p}}\Gamma_g|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_g| \right) d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon_0}{r^2|u|^{\frac{s-3}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{r|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^3|u|^{s-3}} \right) d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}},
\end{aligned}$$

and similarly

$$\begin{aligned}
|r^{1-\frac{2}{p}}\nabla\check{\Omega}|_{p,S} &\lesssim |r^{1-\frac{2}{p}}\nabla\check{\Omega}|_{p,S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{1-\frac{2}{p}}\nabla\check{\Omega}\check{\omega}|_{p,S} + \frac{M}{r} |r^{-\frac{2}{p}}\Gamma_g^{(1)}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}| \right) d\underline{u} \\
&\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}.
\end{aligned}$$

This concludes the proof of Proposition 5.10.  $\square$

**5.2.5 Estimate for  $\text{Osc}(u)$** 

**Proposition 5.11.** *We have the following estimate:*

$$\sup_{\mathcal{K}'} r^{\frac{s-3}{2}} |u' - u| \lesssim \varepsilon_0. \quad (5.33)$$

*Proof.* We have from (5.3)

$$\begin{aligned} (\Omega' e'_4 - \Omega' e'_3)(u' - u) &= -1 - (\Omega' e'_4 - \Omega' e'_3) u \\ &= -1 - \Omega \left( e_4 + f^B e_B + \frac{1}{4} |f|^2 e_3 \right) u + \frac{\Omega'^2}{\Omega^2} \\ &= -\frac{1}{4\Omega} |f|^2 + \frac{\Omega'^2 - \Omega^2}{\Omega^2} \\ &= \mathbf{O}_0^0 \cdot (\Omega' - \Omega) + r^2 \Gamma_w \cdot \Gamma_w. \end{aligned}$$

Applying Propositions 5.10, 2.46 and Lemma 5.4, we obtain

$$|\Omega' - \Omega| \lesssim |\check{\Omega}' - \check{\Omega}| + \left| \left( \Omega'_K - \frac{1}{2} \right) - \left( \Omega_K - \frac{1}{2} \right) \right| \lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}}.$$

Integrating it along  $e'_4 - e'_3$ , we deduce

$$|u' - u|_{\infty, S'} \lesssim \int_w^{u_*} \left( \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r^{s-1}} \right) dw \lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}}.$$

This concludes the proof of Proposition 5.11.  $\square$

**5.2.6 Estimate for  $\mathcal{O}_{0,1}(\widetilde{\Omega \text{tr} \chi})$** 

**Proposition 5.12.** *We have the following estimate for  $q = 0, 1$ :*

$$\left| r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \widetilde{\Omega \text{tr} \chi} \right|_{p,S} \lesssim \varepsilon_0. \quad (5.34)$$

*Proof.* We recall from Proposition 2.34

$$\nabla_4(\widetilde{\Omega \text{tr} \chi}) + \text{tr} \chi \widetilde{\Omega \text{tr} \chi} = \mathbf{O}_1^0 \cdot \widetilde{\Omega \omega} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g. \quad (5.35)$$

Applying Propositions 5.8 and Lemma 5.5, we infer

$$\begin{aligned} |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr} \chi}|_{p,S} &\lesssim |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr} \chi}|_{p, S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{1-\frac{2}{p}} \widetilde{\Omega \omega}|_{p,S} + \frac{M}{r^2} |r^{2-\frac{2}{p}} \Gamma_g|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon_0}{r^{1+\frac{s-3}{6}} |u|^{\frac{s-3}{3}}} + \frac{M}{r^2} \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^2 |u|^{s-3}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

Next, differentiating (5.35) by  $r\nabla$  and applying Corollary 2.43, we obtain

$$\nabla_4(r\nabla \widetilde{\Omega \text{tr} \chi}) + \text{tr} \chi (r\nabla \widetilde{\Omega \text{tr} \chi}) = \mathbf{O}_0^0 \cdot \nabla \widetilde{\Omega \omega} + \Gamma_g^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_g^{(1)}.$$

Applying Lemma 5.5, we deduce

$$\begin{aligned} &|r^{3-\frac{2}{p}} \nabla \widetilde{\Omega \text{tr} \chi}|_{p,S} \\ &\lesssim |r^{3-\frac{2}{p}} \nabla \widetilde{\Omega \text{tr} \chi}|_{p, S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{2-\frac{2}{p}} \nabla \widetilde{\Omega \omega}|_{p,S} + M |r^{-\frac{2}{p}} \Gamma_g^{(1)}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon_0}{r^{1+\frac{s-3}{6}} |u|^{\frac{s-3}{3}}} + \frac{M}{r^2 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^2 |u|^{s-3}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$



This concludes the proof of Proposition 5.12.  $\square$

### 5.2.7 Estimate for $\widetilde{\mathcal{O}}_0(\Omega^{-1} \text{tr } \underline{\chi})$

**Proposition 5.13.** *We have the following estimate:*

$$|r^{\frac{s+1}{2}-\frac{2}{p}} \widetilde{\Omega^{-1} \text{tr } \underline{\chi}}|_{p, S_0(\underline{u})} \lesssim \varepsilon_0. \quad (5.36)$$

*Proof.* We have from Proposition 5.3

$$\Omega'^{-1} \text{tr } \underline{\chi}' = \Omega^{-1} \text{tr } \underline{\chi},$$

which implies

$$\widetilde{\Omega'^{-1} \text{tr } \underline{\chi}'} = \widetilde{\Omega^{-1} \text{tr } \underline{\chi}} + \Omega_K^{-1} \text{tr}_K \underline{\chi}_K - \Omega_K'^{-1} \text{tr}_K \underline{\chi}'_K. \quad (5.37)$$

Proceeding as in (5.9) and applying Propositions 2.16 and 5.11

$$\begin{aligned} \left| \Omega_K^{-1} \text{tr}_K \underline{\chi}_K - \Omega_K'^{-1} \text{tr}_K \underline{\chi}'_K \right| &\lesssim \left| \partial_u(\Omega^{-1} \text{tr } \underline{\chi})_{Kerr} \right| |u' - u| + \left| \partial_{x^A}(\Omega^{-1} \text{tr } \underline{\chi})_{Kerr} \right| |x'^A - x^A| \\ &\lesssim \frac{1}{r^2} |u' - u| + \frac{M^2}{r^3} |r \Gamma_w| \\ &\lesssim \frac{1}{r^2} \frac{\varepsilon_0}{r^{\frac{s-3}{2}}} + \frac{M^2}{r^3} \frac{\varepsilon}{r^{\frac{s-1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned}$$

Injecting it into (5.37), we deduce

$$\sup_{\mathcal{K}'} r^{\frac{s+1}{2}} |\widetilde{\Omega^{-1} \text{tr } \underline{\chi}}| \lesssim \sup_{\mathcal{K}'} r^{\frac{s+1}{2}} |\widetilde{\Omega'^{-1} \text{tr } \underline{\chi}'}| + \sup_{\mathcal{K}'} r^{\frac{s+1}{2}} |\Omega_K^{-1} \text{tr}_K \underline{\chi}_K - \Omega_K'^{-1} \text{tr}_K \underline{\chi}'_K| \lesssim \varepsilon_0.$$

This concludes the proof of Proposition 5.13.  $\square$

### 5.2.8 Estimate for $\mathcal{O}_0(\widetilde{\Omega \text{tr } \underline{\chi}})$

**Proposition 5.14.** *We have the following estimate:*

$$|r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} \widetilde{\Omega \text{tr } \underline{\chi}}|_{p, S} \lesssim \varepsilon_0. \quad (5.38)$$

*Proof.* Next, we recall from Proposition 2.34

$$\nabla_3(\widetilde{\Omega \text{tr } \underline{\chi}}) + \text{tr } \underline{\chi} \widetilde{\Omega \text{tr } \underline{\chi}} = \mathbf{O}_1^0 \cdot \widetilde{\Omega \omega} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_g \cdot \Gamma_b.$$

Notice that we have from Propositions 5.10 and 5.13

$$|r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \underline{\chi}}|_{p, S_0(\underline{u})} \lesssim |r^{2-\frac{2}{p}} \widetilde{\Omega^{-1} \text{tr } \underline{\chi}}|_{p, S_0(\underline{u})} + |r^{2-\frac{2}{p}} \widetilde{\Omega^2}|_{p, S_0(\underline{u})} + |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \underline{\chi}} \widetilde{\Omega^2}|_{p, S_0(\underline{u})} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Applying Lemma 5.5, we infer

$$\begin{aligned} |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \underline{\chi}}|_{p, S} &\lesssim |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \underline{\chi}}|_{p, S_0(\underline{u})} + \int_{u_0(\underline{u})}^u \left( |r^{1-\frac{2}{p}} \widetilde{\Omega \omega}|_{p, S} + \frac{M}{r^2} |r^{2-\frac{2}{p}} \Gamma_g|_{p, S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_b|_{p, S} \right) du \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u})}^u \left( \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{|u|^{s-1}} \right) du \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

This concludes the proof of Proposition 5.14.  $\square$

**5.2.9 Estimate for  $\mathcal{O}_{0,1}(\tilde{\eta})$**

**Proposition 5.15.** *We have the following estimate for  $q = 0, 1$ :*

$$|r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\tilde{\eta}|_{p,S} \lesssim \varepsilon_0. \tag{5.39}$$

*Proof.* We recall from Proposition 2.36

$$\nabla_4[\tilde{\mu}] + \text{tr } \chi[\tilde{\mu}] = \mathbf{O}_1^0 \cdot \check{\rho} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_g^{(1)}. \tag{5.40}$$

Applying Lemma 5.5, we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}}[\tilde{\mu}]|_{p,S} &\lesssim |r^{2-\frac{2}{p}}[\tilde{\mu}]|_{p,S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{1-\frac{2}{p}}\check{\rho}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} + \frac{M}{r^3}|r^{2-\frac{2}{p}}\Gamma_g^{(1)}|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon_0}{r^2|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^2|u|^{s-1}} + \frac{M}{r^3} \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}. \end{aligned}$$

We have from Proposition 2.34

$$\text{div } \check{\eta} = -\check{\mu} - \check{\rho} + \mathbf{O}_3^2 \cdot \Gamma_b^{(1)} + \Gamma_g \cdot \Gamma_b.$$

We recall from (2.29) that

$$[\mu] = \mu + \frac{1}{4} \text{tr } \chi \text{ tr } \underline{\chi},$$

which implies

$$[\tilde{\mu}] = \check{\mu} + \mathbf{O}_1^0 \cdot (\widetilde{\text{tr } \chi}, \widetilde{\text{tr } \underline{\chi}}, r^{-1}\check{\Omega}) + \Gamma_g \cdot \Gamma_g. \tag{5.41}$$

Thus, we obtain

$$\text{div } \check{\eta} = -[\tilde{\mu}] - \check{\rho} + \mathbf{O}_3^2 + \mathbf{O}_1^0 \cdot (\widetilde{\Omega \text{tr } \chi}, \widetilde{\Omega \text{tr } \underline{\chi}}, r^{-1}\check{\Omega}) + \Gamma_g \cdot \Gamma_b. \tag{5.42}$$

Recall the linearized torsion equation in Proposition 2.34:

$$\text{curl } \check{\eta} = \check{\sigma} + \mathbf{O}_3^2 \cdot \Gamma_b^{(1)} + \Gamma_b \cdot \Gamma_g. \tag{5.43}$$

Combining (5.42) and (5.43), applying Propositions 2.15, 5.10 and 5.12, we obtain for  $q = 0, 1$ :

$$\begin{aligned} &|r^{2+q-\frac{2}{p}}\nabla^q\check{\eta}|_{p,S} \\ &\lesssim |r^{3-\frac{2}{p}}([\tilde{\mu}], \check{\rho}, \check{\sigma})|_{p,S} + |r^{2-\frac{2}{p}}(\widetilde{\Omega \text{tr } \chi}, \widetilde{\Omega \text{tr } \underline{\chi}}, r^{-1}\check{\Omega})|_{p,S} + \frac{M^2}{r^3}|r^{3-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_b|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

This concludes the proof of Proposition 5.15. □

**5.2.10 Estimate for  $\text{Osc}(f)$**

**Proposition 5.16.** *We have the following estimate:*

$$\sup_{\mathcal{K}'} |r^{\frac{s-1}{2}} \mathfrak{d}^{\leq 1} f| \lesssim \varepsilon_0. \tag{5.44}$$

*Proof.* We have from Proposition 5.3

$$\Omega' \operatorname{tr} \chi' = \Omega \operatorname{tr} \chi + \operatorname{div} f + f \cdot \Gamma_w + \mathbf{O}_2^1 \cdot f.$$

Applying (5.5) and Proposition 2.34, we have

$$\begin{aligned} e_A(\Omega \operatorname{tr} \chi) &= e_A(\Omega' \operatorname{tr} \chi') - e_A(\operatorname{div} f) + r^{-1} f \cdot \Gamma_w^{(1)} + r^{-1} \Gamma_w \cdot f^{(1)} + \mathbf{O}_3^1 \cdot f^{(1)} \\ &= e'_A(\Omega' \operatorname{tr} \chi') - \frac{\lambda}{2} f_A e'_3(\Omega' \operatorname{tr} \chi') - e_A(\operatorname{div} f) + \Gamma_w \cdot \Gamma_w^{(1)} + \mathbf{O}_2^1 \cdot \Gamma_w^{(1)} \\ &= e'_A(\Omega' \operatorname{tr} \chi') + \frac{\Omega'}{4} f_A \operatorname{tr} \chi' \operatorname{tr} \underline{\chi}' - e_A(\operatorname{div} f) + \Gamma_w \cdot \Gamma_w^{(1)} + \mathbf{O}_2^1 \cdot \Gamma_w^{(1)}. \end{aligned}$$

Recall from Lemma 5.4 that

$$|e'_A(\Omega'_K \operatorname{tr}_K \chi'_K) - e_A(\Omega_K \operatorname{tr}_K \chi_K)| \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}.$$

Thus, we infer

$$\begin{aligned} &\left| r^{-\frac{2}{p}} \left( e_A(\operatorname{div} f) - \frac{\Omega'}{4} f_A \operatorname{tr} \chi' \operatorname{tr} \underline{\chi}' + e_A(\widetilde{\Omega \operatorname{tr} \chi}) - e'_A(\widetilde{\Omega' \operatorname{tr} \chi'}) \right) \right|_{p,S} \\ &\lesssim |e'_A(\Omega'_K \operatorname{tr}_K \chi'_K) - e_A(\Omega_K \operatorname{tr}_K \chi_K)|_{\infty,S} + \frac{M^2}{r^3} |r^{-\frac{2}{p}} \cdot \Gamma_w^{(1)}| + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} + \frac{M^2}{r^3} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{s+1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

Recall that

$$-\frac{\Omega'}{4} f \operatorname{tr} \chi' \operatorname{tr} \underline{\chi}' = \frac{f}{2r^2} + \mathbf{O}_3^1 \cdot f + \Gamma_w \cdot \Gamma_w.$$

Hence, we deduce from Proposition 5.12

$$\begin{aligned} &\left| r^{-\frac{2}{p}} \left( e_A(\operatorname{div} f) + \frac{f}{2r^2} \right) \right|_{p,S} \\ &\lesssim |r^{-\frac{2}{p}} e_A(\widetilde{\Omega \operatorname{tr} \chi})|_{p,S} + |e'_A(\widetilde{\Omega' \operatorname{tr} \chi'})|_{\infty,S} + \frac{M}{r^2} |r^{-\frac{2}{p}} \cdot \Gamma_w^{(1)}|_{p,S} + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w^{(1)}|_{p,S} + \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} \quad (5.45) \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{s+1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

Next, we have from Proposition 5.3

$$\operatorname{curl} f = \mathbf{O}_2^1 \cdot \Gamma_w + r \Gamma_w \cdot \Gamma_w,$$

which implies

$$\begin{aligned} |r^{-\frac{2}{p}} e_A(\operatorname{curl} f)|_{p,S} &\lesssim \frac{M^2}{r^3} |r^{-\frac{2}{p}} \cdot \Gamma_w^{(1)}|_{p,S} + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w^{(1)}|_{p,S} \\ &\lesssim \frac{M^2}{r^3} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{s+1}} \quad (5.46) \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

Combining (5.45) and (5.46), we deduce<sup>21</sup>

$$\left| r^{-\frac{2}{p}} \left( d_1^* d_1 f - \frac{f}{2r^2} \right) \right|_{p,S} \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}.$$

We have from Proposition 2.16

$$\begin{aligned} d_1^* d_1 f - \frac{f}{2r^2} &= (-\Delta_1 + \mathbf{K})f - \frac{f}{2r^2} \\ &= -\Delta_1 f - \frac{f}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} - \frac{f}{2r^2} + r^{-1} \Gamma_w^{(1)} \cdot f + \mathbf{O}_3^1 \cdot f \\ &= -\Delta_1 f + \frac{f}{2r^2} + \mathbf{O}_2^1 \cdot \Gamma_w^{(1)} + \Gamma_w \cdot \Gamma_w^{(1)}. \end{aligned}$$

Thus, we infer

$$\begin{aligned} \left| r^{-\frac{2}{p}} \left( -\Delta_1 + \frac{1}{2r^2} \right) f \right|_{p,S} &\lesssim \left| r^{-\frac{2}{p}} \left( d_1^* d_1 - \frac{1}{2r^2} \right) f \right|_{p,S} + \frac{M}{r^2} |r^{-\frac{2}{p}} \Gamma_w^{(1)}|_{p,S} + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^{s+1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

Applying Proposition 2.15, we deduce for  $q = 0, 1, 2$ ,

$$|r^{-\frac{2}{p}} (r\nabla)^q f|_{p,S} \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}.$$

Then, applying Proposition 2.46, we obtain

$$\sup_{S \subset \mathcal{K}'} |r^{\frac{s+3}{2}} (r\nabla)^{\leq 1} f|_{\infty,S} \lesssim \varepsilon_0. \tag{5.47}$$

Next, we recall from Proposition 5.3

$$\lambda \nabla_{e_3} f = 2\eta' - 2\eta + \mathbf{O}_1^1 \cdot \Gamma_w + r\Gamma_w \cdot \Gamma_w. \tag{5.48}$$

Applying Lemma 5.4, we obtain

$$|\eta'_K(e'_A) - \eta_K(e_A)| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Applying Proposition 5.15 and (5.47), we deduce

$$\begin{aligned} |r^{-\frac{2}{p}} \nabla_{e_3} f|_{p,S} &\lesssim |r^{-\frac{2}{p}} (\check{\eta}', \check{\eta})|_{p,S} + M |r^{-1-\frac{2}{p}} \Gamma_w|_{p,S} + |r^{1-\frac{2}{p}} \Gamma_w \cdot \Gamma_w|_{p,S} + \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{M\varepsilon}{r^{\frac{s+3}{2}}} + \frac{\varepsilon^2}{r^s} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned} \tag{5.49}$$

Finally, we recall from Proposition 5.3

$$\frac{1}{2} \nabla_{\lambda^{-1} e_4} f = -\frac{1}{4} \operatorname{tr} \chi f + \mathbf{O}_1^1 \cdot \Gamma_w + r\Gamma_w \cdot \Gamma_w,$$

which implies from (5.47)

$$\sup_{\mathcal{K}'} r^{\frac{s+1}{2}} |\nabla_{e_4} f| \lesssim \sup_{\mathcal{K}'} r^{\frac{s-1}{2}} |f| + \sup_{\mathcal{K}'} \left( \frac{M\varepsilon}{r} + \frac{\varepsilon^2}{r^{\frac{s-3}{2}}} \right) \lesssim \varepsilon_0. \tag{5.50}$$

Combining (5.47), (5.49) and (5.50), we deduce

$$\sup_{\mathcal{K}'} r^{\frac{s-1}{2}} |\mathfrak{d}^{\leq 1} f| \lesssim \varepsilon_0.$$

This concludes the proof of Proposition 5.16. □

<sup>21</sup>Recall that  $d_1^* d_1 f = d_1^*(\operatorname{div} f, \operatorname{curl} f) = -\nabla \operatorname{div} f + * \nabla \operatorname{curl} f$ .

### 5.2.11 Estimate for $\mathring{\mathcal{O}}_{0,1}(\check{\eta})$

**Proposition 5.17.** *We have the following estimate for  $q = 0, 1$ :*

$$|r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\check{\eta}|_{p,S_0(\underline{u})} \lesssim \varepsilon_0. \quad (5.51)$$

*Proof.* We recall from Proposition 5.3

$$\underline{\eta} = \underline{\eta}' - \frac{1}{4} \operatorname{tr} \underline{\chi} f + \mathbf{O}_1^1 \cdot \Gamma_w + r \Gamma_w \cdot \Gamma_w. \quad (5.52)$$

Recall that we have from Lemma 5.4

$$|\underline{\eta}_K(e_A) - \underline{\eta}'_K(e'_A)| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Thus, we obtain

$$\begin{aligned} |\check{\eta}_A| &\lesssim |\check{\eta}'_A| + |\underline{\eta}_K(e_A) - \underline{\eta}'_K(e'_A)| + \frac{M}{r} |\Gamma_w| + r |\Gamma_w \cdot \Gamma_w| \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{M}{r} \frac{\varepsilon}{r^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r^s} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}. \end{aligned} \quad (5.53)$$

Differentiating (5.52) by  $e_B$ , we obtain

$$e_B(\underline{\eta}_A) = e'_B(\underline{\eta}'_A) + \mathbf{O}_2^0 \cdot f^{(1)} + \mathbf{O}_2^1 \cdot \Gamma_w^{(1)} + \Gamma_w \cdot \Gamma_w^{(1)}.$$

Applying Lemma 5.4, we have

$$|e_B(\underline{\eta}_K)_A - e'_B(\underline{\eta}'_K)_A| \lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \quad (5.54)$$

Hence, we obtain

$$\begin{aligned} &|r^{-\frac{2}{p}} e_B(\check{\eta}_A)|_{p,S} \\ &\lesssim |r^{-\frac{2}{p}} e'_B(\check{\eta}'_A)|_{p,S} + |e_B(\underline{\eta}_K)_A - e'_B(\underline{\eta}'_K)_A| + r^{-2} |f^{(1)}|_{\infty,S} + M |r^{-2-\frac{2}{p}} \Gamma_w^{(1)}|_{p,S} + |r^{-\frac{2}{p}} \Gamma_w \cdot \Gamma_w^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}} + \frac{M\varepsilon}{r^{\frac{s+5}{2}}} + \frac{\varepsilon^2}{r^{s+1}} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

Combining with (5.53), we infer for  $q = 0, 1$

$$|r^{\frac{s+1}{2}-\frac{2}{p}}(r\nabla)^q\check{\eta}|_{p,S} \lesssim \varepsilon_0.$$

This concludes the proof of Proposition 5.17.  $\square$

### 5.2.12 Estimate for $\mathcal{O}_{0,1}(\check{\eta})$

**Proposition 5.18.** *We have the following estimate for  $q = 0, 1$ :*

$$|r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\check{\eta}|_{p,S} \lesssim \varepsilon_0. \quad (5.55)$$

*Proof.* We have from Propositions 2.34, 5.12, 5.13 and 5.17 that

$$\begin{aligned}
 |r^{2-\frac{2}{p}}[\underline{\mu}]|_{p,S_0(\underline{u})} &\lesssim |r^{2-\frac{2}{p}}\check{\underline{\mu}}|_{p,S_0(\underline{u})} + |r^{1-\frac{2}{p}}(\widetilde{\text{tr}}\underline{\chi}, \widetilde{\text{tr}}\underline{\chi})|_{p,S_0(\underline{u})} \\
 &\lesssim |r^{2-\frac{2}{p}}\text{div}\check{\underline{\eta}}|_{p,S_0(\underline{u})} + |r^{2-\frac{2}{p}}\check{\rho}|_{p,S_0(\underline{u})} + |r^{1-\frac{2}{p}}(\widetilde{\Omega}\text{tr}\underline{\chi}, \widetilde{\Omega^{-1}}\text{tr}\underline{\chi})|_{p,S_0(\underline{u})} \\
 &\quad + M^2|r^{-1-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S_0(\underline{u})} + |r^{2-\frac{2}{p}}\Gamma_b \cdot \Gamma_g|_{p,S_0(\underline{u})} \\
 &\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}}.
 \end{aligned} \tag{5.56}$$

We recall from Proposition 2.36

$$\nabla_3[\check{\underline{\mu}}] + \text{tr}\underline{\chi}[\check{\underline{\mu}}] = O(r^{-1})\check{\rho} + \mathbf{O}_3^1 \cdot \Gamma_b^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_b^{(1)}.$$

Applying Lemma 5.5 and (5.56), we obtain

$$\begin{aligned}
 |r^{2-\frac{2}{p}}[\check{\underline{\mu}}]|_{p,S} &\lesssim |r^{2-\frac{2}{p}}[\check{\underline{\mu}}]|_{p,S_0(\underline{u})} + \int_{u_0(\underline{u})}^u \left( |r^{1-\frac{2}{p}}\check{\rho}|_{p,S} + \frac{M}{r^2}|r^{1-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_b \cdot \Gamma_b^{(1)}|_{p,S} \right) du \\
 &\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \int_{u_0(\underline{u})}^u \left( \frac{\varepsilon_0}{r^2|u|^{\frac{s-3}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{|u|^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r^2|u|^{s-1}} \right) du \\
 &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}.
 \end{aligned}$$

We have from Proposition 2.34

$$\begin{aligned}
 \text{div}\check{\underline{\eta}} &= -[\check{\underline{\mu}}] - \check{\rho} + \mathbf{O}_3^2 \cdot \Gamma_b^{(1)} + \mathbf{O}_1^0 \cdot (\widetilde{\text{tr}}\underline{\chi}, \widetilde{\text{tr}}\underline{\chi}, r^{-1}\check{\underline{\Omega}}) + \Gamma_g \cdot \Gamma_b, \\
 \text{curl}\check{\underline{\eta}} &= \check{\sigma} + \mathbf{O}_3^2 \cdot \Gamma_b^{(1)} + \Gamma_g \cdot \Gamma_b.
 \end{aligned} \tag{5.57}$$

Applying Proposition 2.15, we deduce for  $q = 0, 1$ :

$$\begin{aligned}
 |r^{2+q-\frac{2}{p}}\nabla^q\check{\underline{\eta}}|_{p,S} &\lesssim |r^{3-\frac{2}{p}}([\check{\underline{\mu}}], \check{\rho})|_{p,S} + |r^{2-\frac{2}{p}}(\widetilde{\text{tr}}\underline{\chi}, \widetilde{\text{tr}}\underline{\chi}, r^{-1}\check{\underline{\Omega}})|_{p,S} + M^2|r^{-\frac{2}{p}}\Gamma_b^{(1)}|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_b|_{p,S} \\
 &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \frac{M^2}{r|u|^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{|u|^{s-2}} \\
 &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.
 \end{aligned}$$

This concludes the proof of Proposition 5.18. □

### 5.2.13 Estimate for $\mathcal{O}_{0,1,2}(\check{\underline{\Omega}})$

**Proposition 5.19.** *We have the following estimates:*

$$|r^{1-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\check{\underline{\Omega}}|_{p,S} \lesssim \varepsilon_0, \quad q = 0, 1, 2. \tag{5.58}$$

*Proof.* Recall that

$$\eta + \underline{\eta} = 2\nabla \log \Omega,$$

and

$$\eta_K + \underline{\eta}_K = 2\nabla \log \Omega_K.$$

Taking the difference, we obtain

$$\Delta \log \left( \frac{\Omega}{\Omega_K} \right) = \frac{1}{2} \text{div}(\check{\underline{\eta}} + \check{\underline{\eta}}),$$

Applying Propositions 2.15, 5.15 and 5.18, we have for  $q = 1, 2$ :

$$\left| r^{1+q-\frac{2}{p}} \nabla^q \left( \log \left( \frac{\Omega}{\Omega_K} \right) \right) \right|_{p,S} (u, \underline{u}) \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \quad (5.59)$$

Applying Proposition 5.10, we deduce that

$$\left| r \log \left( \frac{\Omega}{\Omega_K} \right) \right|_{\infty,S} \leq \left| r \left( \frac{\Omega}{\Omega_K} - 1 \right) \right|_{\infty,S} \lesssim |r\check{\Omega}|_{\infty,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}$$

so that (5.59) also holds true for  $q = 0$ . Hence, we have

$$\left\| \log \frac{\Omega}{\Omega_K} \right\|_{W^{2,4}(S)} \lesssim \frac{\varepsilon_0}{r^{\frac{1}{2}} |u|^{\frac{s-3}{2}}}.$$

We recall that the Sobolev space  $W^{2,4}(S)$  forms a Banach algebra, i.e.

$$\|uv\|_{W^{2,4}(S)} \lesssim \|u\|_{W^{2,4}(S)} \|v\|_{W^{2,4}(S)}, \quad \forall u, v \in W^{2,4}(S), \quad (5.60)$$

see for example Theorem 4.39 in [1]. Notice that

$$\check{\Omega} = \Omega_K \left( \frac{\Omega}{\Omega_K} - 1 \right) = \Omega_K \left( \exp \left( \log \left( \frac{\Omega}{\Omega_K} \right) \right) - 1 \right) = \Omega_K \sum_{k=1}^{\infty} \frac{1}{k!} \left( \log \frac{\Omega}{\Omega_K} \right)^k. \quad (5.61)$$

Combining (5.59), (5.60) and (5.61), we deduce

$$|(r\nabla)^{\leq 2} \check{\Omega}|_{4,S} \lesssim \sum_{k=1}^{\infty} \left| (r\nabla)^{\leq 2} \log \left( \frac{\Omega}{\Omega_K} \right) \right|_{4,S}^k \lesssim \frac{\varepsilon_0}{r^{\frac{1}{2}} |u|^{\frac{s-3}{2}}},$$

which implies for  $p \in [2, 4]$ :

$$|r^{1-\frac{2}{p}} (r\nabla)^q \check{\Omega}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}, \quad q = 0, 1, 2.$$

This concludes the proof of Proposition 5.19.  $\square$

#### 5.2.14 Estimate for $\mathring{O}_1(\widetilde{\Omega^{-1} \text{tr } \chi})$

**Proposition 5.20.** *We have the following estimate:*

$$|r^{\frac{s+3}{2}-\frac{2}{p}} \nabla(\widetilde{\Omega^{-1} \text{tr } \chi})|_{p,S_0(\underline{u})} \lesssim \varepsilon_0. \quad (5.62)$$

*Proof.* We have from Proposition 5.3

$$\Omega'^{-1} \text{tr } \underline{\chi}' = \Omega^{-1} \text{tr } \underline{\chi}.$$

Differentiating it by  $e_A$  and applying (5.5), we obtain

$$e_A(\Omega^{-1} \text{tr } \underline{\chi}) = e'_A(\Omega'^{-1} \text{tr } \underline{\chi}') - \frac{\lambda}{2} f_A e'_3(\Omega'^{-1} \text{tr } \underline{\chi}').$$

Applying Propositions 2.16, 5.16 and Lemma 5.4, we infer

$$\begin{aligned} |r^{-\frac{2}{p}} e_A(\widetilde{\Omega^{-1} \text{tr } \chi})|_{p,S} &\lesssim |r^{-\frac{2}{p}} e'_A(\widetilde{\Omega'^{-1} \text{tr } \chi}')|_{p,S} + |e_A(\Omega^{-1} \text{tr } \underline{\chi})_K - e'_A(\Omega^{-1} \text{tr } \underline{\chi})'_K| + |r^{-2-\frac{2}{p}} f_A|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s+3}{2}}}. \end{aligned}$$

This concludes the proof of Proposition 5.20.  $\square$

**5.2.15 Estimate for  $\mathcal{O}_1(\widetilde{\Omega \operatorname{tr} \chi})$**

**Proposition 5.21.** *We have the following estimates:*

$$|r^{3-\frac{2}{p}}|u|^{\frac{s-3}{2}}\nabla\widetilde{\Omega \operatorname{tr} \chi}|_{p,S} \lesssim \varepsilon_0. \tag{5.63}$$

*Proof.* Notice that we have

$$\widetilde{\Omega \operatorname{tr} \chi} = \Omega^2(\Omega^{-1} \operatorname{tr} \chi) - \Omega_K^2(\Omega_K^{-1} \operatorname{tr}_K \underline{\chi}_K) = \widetilde{\Omega}^2(\Omega^{-1} \operatorname{tr} \chi) + \Omega_K^2(\widetilde{\Omega}^{-1} \operatorname{tr} \underline{\chi}),$$

which implies

$$\nabla\widetilde{\Omega \operatorname{tr} \chi} = \mathbf{O}_1^0 \cdot \nabla\widetilde{\Omega} + \mathbf{O}_0^0 \cdot \widetilde{\operatorname{tr} \chi} + \Gamma_g \cdot \Gamma_g^{(1)}.$$

Hence, we have from Propositions 5.10 and 5.20

$$\begin{aligned} |r^{3-\frac{2}{p}}\nabla\widetilde{\Omega \operatorname{tr} \chi}|_{p,S_0(\underline{u})} &\lesssim |r^{3-\frac{2}{p}}\nabla\widetilde{\Omega}^{-1} \operatorname{tr} \underline{\chi}|_{p,S_0(\underline{u})} + |r^{2-\frac{2}{p}}\nabla\widetilde{\Omega}|_{p,S_0(\underline{u})} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S_0(\underline{u})} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

Next, we recall from Proposition 2.34

$$\nabla_3(\widetilde{\Omega \operatorname{tr} \chi}) + \operatorname{tr} \underline{\chi} \widetilde{\Omega \operatorname{tr} \chi} = \mathbf{O}_1^0 \cdot \widetilde{\Omega \omega} + \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_b \cdot \Gamma_b.$$

Differentiating it by  $r\nabla$  and applying Corollary 2.43, we deduce

$$\nabla_3(r\nabla\widetilde{\Omega \operatorname{tr} \chi}) + \operatorname{tr} \underline{\chi} r\nabla\widetilde{\Omega \operatorname{tr} \chi} = \mathbf{O}_0^0 \cdot \nabla\widetilde{\Omega \omega} + \Gamma_g^{(1)} \cdot \mathbf{O}_2^1 + \Gamma_b \cdot \Gamma_b^{(1)}.$$

Applying Lemma 5.5 and Proposition 5.9, we infer

$$\begin{aligned} &|r^{3-\frac{2}{p}}\nabla\widetilde{\Omega \operatorname{tr} \chi}|_{p,S} \\ &\lesssim |r^{3-\frac{2}{p}}\nabla\widetilde{\Omega \operatorname{tr} \chi}|_{p,S_0(\underline{u})} + \int_{u_0(\underline{u})}^u \left( |r^{2-\frac{2}{p}}\nabla\widetilde{\Omega \omega}|_{p,S} + \frac{M}{r^2}|r^{2-\frac{2}{p}}\Gamma_g|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_b \cdot \Gamma_b^{(1)}|_{p,S} \right) du \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u})}^u \left( \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}} + \frac{M}{r^2} \frac{\varepsilon}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{|u|^{s-1}} \right) du \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

This concludes the proof of Proposition 5.21. □

**5.2.16 Estimates for  $\mathcal{O}_{0,1}(\check{\chi})$  and  $\mathcal{O}_{0,1}(\check{\underline{\chi}})$**

**Proposition 5.22.** *We have the following estimates for  $q = 0, 1$ :*

$$\begin{aligned} |r^{2-\frac{2}{p}}|u|^{\frac{s-3}{2}}(r\nabla)^q\check{\chi}|_{p,S} &\lesssim \varepsilon_0, \\ |r^{1-\frac{2}{p}}|u|^{\frac{s-1}{2}}(r\nabla)^q\check{\underline{\chi}}|_{p,S} &\lesssim \varepsilon_0. \end{aligned} \tag{5.64}$$

*Proof.* We recall from Proposition 2.34

$$\operatorname{div} \check{\chi} = -\check{b} + \mathbf{O}_0^0 \cdot \nabla\widetilde{\operatorname{tr} \chi} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_g^{(1)} + \Gamma_g \cdot \Gamma_g.$$

Applying Propositions 2.15, 5.12, 5.15 and 5.18, we deduce for  $q = 0, 1$ :

$$\begin{aligned} |r^{2+q-\frac{2}{p}}\nabla^q\check{\chi}|_{p,S} &\lesssim |r^{3-\frac{2}{p}}\nabla\widetilde{\operatorname{tr} \chi}|_{p,S} + |r^{2-\frac{2}{p}}\check{\zeta}|_{p,S} + |r^{3-\frac{2}{p}}\check{b}|_{p,S} + \frac{M^2}{r^2}|r^{2-\frac{2}{p}}\Gamma_g^{(1)}|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$



Next, we recall from Proposition 2.34

$$\operatorname{div} \check{\underline{\chi}} = \check{\underline{\beta}} + \mathbf{O}_0^0 \cdot \nabla \operatorname{tr} \check{\underline{\chi}} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_b^{(1)} + \Gamma_b \cdot \Gamma_g.$$

Applying Propositions 2.15, 5.21, 5.15 and 5.18, we deduce for  $q = 0, 1$ :

$$\begin{aligned} |r^{1+q-\frac{2}{p}} \nabla^q \check{\underline{\chi}}|_{p,S} &\lesssim |r^{2-\frac{2}{p}} \nabla \operatorname{tr} \check{\underline{\chi}}|_{p,S} + |r^{1-\frac{2}{p}} \check{\zeta}|_{p,S} + |r^{2-\frac{2}{p}} \check{\underline{\beta}}|_{p,S} + \frac{M^2}{r^2} |r^{1-\frac{2}{p}} \Gamma_b^{(1)}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_g|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}}. \end{aligned}$$

This concludes the proof of Proposition 5.22.  $\square$

### 5.2.17 Estimates for $\mathcal{O}_{0,1}(\check{\underline{b}})$

**Proposition 5.23.** *We have the following estimate for  $q = 0, 1$ :*

$$|r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (r\nabla)^q \check{\underline{b}}^A|_{p,S} \lesssim \varepsilon_0. \quad (5.65)$$

*Proof.* We have from Proposition 2.40

$$\nabla_4(\check{\underline{b}}^A) = \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_g + \Gamma_g \cdot \Gamma_g. \quad (5.66)$$

Applying Lemma 5.5, we deduce from Propositions 5.15 and 5.18<sup>22</sup>

$$\begin{aligned} |r^{-\frac{2}{p}} \check{\underline{b}}^A|_{p,S} &\lesssim \int_{\underline{u}}^{\underline{u}_*} \left( |r^{-1-\frac{2}{p}} \check{\zeta}|_{p,S} + M^2 |r^{-3-\frac{2}{p}} \Gamma_g|_{p,S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p,S} \right) d\underline{u} \\ &\lesssim \int_{\underline{u}}^{\underline{u}_*} \left( \frac{\varepsilon_0}{r^3 |u|^{\frac{s-3}{2}}} + \frac{M^2 \varepsilon}{r^5 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^4 |u|^{s-3}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}} + \frac{M^2 \varepsilon}{r^4 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^3 |u|^{s-3}} \\ &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}}, \end{aligned}$$

which implies (5.65) in the case  $q = 0$ . Differentiating (5.66) by  $r\nabla$  and applying Corollary 2.43, we obtain

$$\nabla_4(r\nabla \check{\underline{b}}^A) = \mathbf{O}_1^0 \cdot r\nabla \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_g^{(1)} + \Gamma_g \cdot \Gamma_g^{(1)}.$$

Applying Lemma 5.5, we deduce from Propositions 5.15 and 5.18

$$\begin{aligned} |r^{-\frac{2}{p}} (r\nabla) \check{\underline{b}}^A|_{p,S} &\lesssim \int_{\underline{u}}^{\underline{u}_*} \left( |r^{-\frac{2}{p}} \nabla \check{\zeta}|_{p,S} + M^2 |r^{-3-\frac{2}{p}} \Gamma_g^{(1)}|_{p,S} + |r^{-\frac{2}{p}} \Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}}, \end{aligned}$$

which implies (5.65) in the case  $q = 1$ . This concludes the proof of Proposition 5.23.  $\square$

<sup>22</sup>Recall that  $\check{\underline{b}}^A = 0$  on  $\underline{C}_*$  and  $\zeta = \frac{\eta-\eta}{2}$ .

5.2.18 Estimate for  $\text{Osc}(x)$

**Proposition 5.24.** *We have the following estimates*

$$\sup_{\mathcal{K}'} r^{\frac{s-1}{2}} |x'^A - x^A| \lesssim \varepsilon_0. \tag{5.67}$$

*Proof.* We have from (5.3) and Proposition 2.28

$$\begin{aligned} (\Omega' e'_4 - \Omega' e'_3)(x'^A - x^A) &= -\underline{b}'^A - \Omega' e'_4(x^A) + \Omega' e'_3(x^A) \\ &= -\underline{b}'^A - \Omega \left( e_4 + f^B e_B + \frac{1}{4} |f|^2 e_3 \right) x^A + \frac{\Omega'^2}{\Omega^2} \underline{b}^A \\ &= \underline{b}^A - \underline{b}'^A - \Omega f^B e_B(x^A) - \frac{1}{4} |f|^2 \underline{b}^A + \left( \frac{\Omega'^2}{\Omega^2} - 1 \right) \underline{b}^A \\ &= \underline{b}^A - \underline{b}'^A + \mathbf{O}_1^0 \cdot f + \mathbf{O}_1^1 \cdot \Gamma_w + r \Gamma_w \cdot \Gamma_w. \end{aligned} \tag{5.68}$$

Notice from Lemma 5.4 that

$$|b'^A - b^A| \lesssim |\check{\underline{b}}'^A| + |\check{\underline{b}}^A| + |(b'_K)^A - b_K^A| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Hence, we obtain from Proposition 5.16

$$|(\Omega' e'_4 - \Omega' e'_3)(x'^A - x^A)| \lesssim r^{-1} |f| + \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} + \frac{M}{r} |\Gamma_w| + r |\Gamma_w \cdot \Gamma_w| \lesssim \frac{\varepsilon_0}{r^{\frac{s+1}{2}}}.$$

Integrating it along  $e'_4 - e'_3$ , we deduce<sup>23</sup>

$$|x'^A - x^A|_{\infty, S'} \lesssim \int_w^{u_*} \frac{\varepsilon_0}{r^{\frac{s+1}{2}}} dw \lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}}.$$

This concludes the proof of Proposition 5.24. □

5.2.19 Estimates for  $\mathcal{O}_{0,1}(\check{\gamma})$  and  $\mathcal{O}_{0,1}(\check{\epsilon})$

**Proposition 5.25.** *We have the following estimates for  $q = 0, 1$ :*

$$|r^{-1+q-\frac{2}{p}} |u|^{\frac{s-3}{2}} \nabla^q \check{\gamma}_{AB}|_{p,S} \lesssim \varepsilon_0, \tag{5.69}$$

$$|r^{-1+q-\frac{2}{p}} |u|^{\frac{s-3}{2}} \nabla^q \check{\epsilon}_{AB}|_{p,S} \lesssim \varepsilon_0. \tag{5.70}$$

*Proof.* We recall from Proposition 2.40

$$\nabla_4(\check{\gamma}_{AB}) - \text{tr} \chi(\check{\gamma}_{AB}) = \mathbf{O}_{-2}^0 \cdot (\check{\chi}, \widetilde{\text{tr}} \check{\chi}) + \mathbf{O}_{-1}^0 \cdot \check{\Omega} + r^3 \Gamma_g \cdot \Gamma_g.$$

Differentiating it by  $r\nabla$  and applying Corollary 2.43, we deduce

$$\nabla_4(r\nabla \check{\gamma}_{AB}) - \text{tr} \chi(r\nabla \check{\gamma}_{AB}) = \mathbf{O}_{-3}^0 \cdot \nabla(\check{\chi}, \widetilde{\text{tr}} \check{\chi}) + \mathbf{O}_{-2}^0 \cdot \nabla \check{\Omega} + r^3 \Gamma_g \cdot \Gamma_g^{(1)}.$$

Applying Lemma 5.5 and Propositions 5.12, 5.19 and 5.22, we obtain for  $q = 0, 1$

$$\begin{aligned} &|r^{-2-\frac{2}{p}} (r\nabla)^q \check{\gamma}_{AB}|_{p,S} \\ &\lesssim |r^{-2-\frac{2}{p}} (r\nabla)^q \check{\gamma}_{AB}|_{p, S_*(u)} + \int_{\underline{u}}^{u_*} \left( |r^{-\frac{2}{p}} (r\nabla)^q (\check{\chi}, \widetilde{\text{tr}} \check{\chi}, r^{-1} \check{\Omega})|_{p,S} + |r^{-1-\frac{2}{p}} \Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r |u|^{\frac{s-3}{2}}} + \int_{\underline{u}}^{u_*} \left( \frac{\varepsilon_0}{r^2 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^3 |u|^{s-3}} \right) d\underline{u} \\ &\lesssim \frac{\varepsilon_0}{r |u|^{\frac{s-3}{2}}}, \end{aligned}$$

which implies (5.69). The proof of (5.70) is similar and left to the reader. This concludes the proof of Proposition 5.25. □

<sup>23</sup>Recall that  $x'^A = x^A$  on  $\underline{C}_* \cap \Sigma_0$  which is chosen in Section 2.1.5.

### 5.2.20 Estimate for $\text{Osc}(r)$

**Proposition 5.26.** *We have the following estimate:*

$$\sup_{\mathcal{K}'} r^{\frac{s-3}{2}} |r' - r| \lesssim \varepsilon_0. \quad (5.71)$$

*Proof.* Notice that we have

$$\begin{aligned} \det \gamma_{AB} - \det(\gamma_K)_{AB} &= (\gamma_{11}\gamma_{22} - \gamma_{12}^2) - ((\gamma_K)_{11}(\gamma_K)_{22} - (\gamma_K)_{12}^2) \\ &= \check{\gamma}_{11}(\gamma_K)_{22} + (\gamma_K)_{11}\check{\gamma}_{22} - 2\check{\gamma}_{12}(\gamma_K)_{12} + \check{\gamma}_{11}\check{\gamma}_{22} - \check{\gamma}_{12}^2 \\ &= \mathbf{O}_{-2}^0 \cdot \check{\gamma}_{AB} + r^6 \Gamma_g \cdot \Gamma_g. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} 4\pi |r^2 - r_K^2| &= \left| \int_{\mathbb{S}^2} \left( \sqrt{\det \gamma_{AB}} - \sqrt{\det(\gamma_K)_{AB}} \right) d\sigma_{\mathbb{S}^2} \right| \\ &\lesssim \int_{\mathbb{S}^2} \left| \frac{\det \gamma_{AB} - \det(\gamma_K)_{AB}}{\sqrt{\det \gamma_{AB}} + \sqrt{\det(\gamma_K)_{AB}}} \right| d\sigma_{\mathbb{S}^2} \\ &\lesssim |\check{\gamma}_{AB}| + r^4 |\Gamma_g \cdot \Gamma_g| \\ &\lesssim \frac{r\varepsilon_0}{|u|^{\frac{s-3}{2}}} + r^4 \frac{\varepsilon^2}{r^4 |u|^{s-3}} \\ &\lesssim \frac{r\varepsilon_0}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

which implies

$$\sup_{\mathcal{K}} |r - r_K| \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \quad (5.72)$$

Similarly, we have

$$\sup_{\mathcal{K}'} |r' - r'_K| \lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}}. \quad (5.73)$$

Proceeding as in (5.9), we obtain for any  $p \in \mathcal{K}'$

$$\begin{aligned} |r'_K(p) - r_K(p)| &= |r_{Kerr}(\Phi'(p)) - r_{Kerr}(\Phi(p))| \\ &\lesssim |\partial_u(r_{Kerr})| |u' - u| + |\partial_A(r_{Kerr})| |x'^A - x^A| \\ &\lesssim |u' - u| \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}}, \end{aligned}$$

where we used Proposition 5.11. Combining with (5.72) and (5.73), we infer

$$\sup_{\mathcal{K}'} r^{\frac{s-3}{2}} |r' - r| \lesssim \varepsilon_0. \quad (5.74)$$

This concludes the proof of Proposition 5.26.  $\square$

### 5.3 Estimate for $\mathcal{O}_\Gamma$

**Proposition 5.27.** *We have the following estimates:*

$$\begin{aligned} |r^{2-\frac{2}{p}} |u|^{\frac{s-3}{2}} (\check{J}, \check{L})|_{p,S} &\lesssim \varepsilon_0, \\ |r^{1-\frac{2}{p}} |u|^{\frac{s-1}{2}} \check{J}|_{p,S} &\lesssim \varepsilon_0. \end{aligned} \quad (5.75)$$

*Proof.* We recall from Lemma 2.23

$$\begin{aligned}\Gamma_{\underline{u}A}^B &= \frac{1}{2}\gamma^{BC}\partial_{\underline{u}}(\gamma_{AC}), \\ \Gamma_{\underline{u}A}^B &= \frac{1}{2}\gamma^{BC}\left(\partial_{\underline{u}}(\gamma_{AC}) - \frac{\partial}{\partial x^A}(\gamma_{CD}b^D) + \frac{\partial}{\partial x^C}(\gamma_{AD}b^D)\right) + \frac{b^B}{4\Omega^2}\left(2\frac{\partial}{\partial x^A}(\Omega^2) - \frac{\partial}{\partial \underline{u}}(\gamma_{AD}b^D)\right), \\ \Gamma_{BC}^A &= \frac{1}{2}\gamma^{AD}\left(\frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} - \frac{\partial\gamma_{BC}}{\partial x^D}\right) + \frac{b^A}{4\Omega^2}\frac{\partial\gamma_{BC}}{\partial \underline{u}}.\end{aligned}$$

Hence, we have from Definition 2.24

$$\check{J}_A^B = \mathbf{O}_2^0 \cdot \partial_{\underline{u}}(\check{\gamma}_{AB}) + \mathbf{O}_3^0 \cdot \check{\gamma}_{AB} + r^{-1}\Gamma_g \cdot \partial_{\underline{u}}(\check{\gamma}_{AB}).$$

We have from Proposition 2.40

$$\partial_{\underline{u}}(\check{\gamma}_{AB}) = \text{tr } \chi(\check{\gamma}_{AB}) + \mathbf{O}_{-2}^0 \cdot (\widetilde{\text{tr } \chi}, \check{\chi}, r^{-1}\check{\Omega}) + r^3\Gamma_g \cdot \Gamma_g,$$

which implies from Propositions 5.12, 5.22, 5.19 and 5.25

$$|r^{-\frac{2}{p}}\partial_{\underline{u}}(\check{\gamma}_{AB})|_{p,S} \lesssim |r^{-1-\frac{2}{p}}\check{\gamma}_{AB}|_{p,S} + |r^{2-\frac{2}{p}}(\widetilde{\text{tr } \chi}, \check{\chi}, r^{-1}\check{\Omega})|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Thus, we infer from Proposition 5.25

$$|r^{2-\frac{2}{p}}\check{J}|_{p,S} \lesssim |r^{-1-\frac{2}{p}}\check{\gamma}_{AB}|_{p,S} + |r^{-\frac{2}{p}}\partial_{\underline{u}}(\check{\gamma}_{AB})|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Next, we have from Definition 2.24

$$\check{L}_{BC}^A = \mathbf{O}_3^2 \cdot \partial_{\underline{u}}(\check{\gamma}_{AB}) + \mathbf{O}_2^0 \cdot \check{\vartheta}^{\leq 1}\check{\gamma}_{AB} + \mathbf{O}_{-1}^0 \cdot \check{b}^A + r^2\Gamma_g \cdot \Gamma_g^{(1)}.$$

Thus, we deduce from Propositions 5.25 and 5.23

$$\begin{aligned}|r^{1-\frac{2}{p}}\check{L}_{BC}^A|_{p,S} &\lesssim \frac{M^2}{r^2}|r^{-\frac{2}{p}}\partial_{\underline{u}}(\check{\gamma}_{AB})|_{p,S} + |r^{-1-\frac{2}{p}}\check{\vartheta}^{\leq 1}\check{\gamma}_{AB}|_{p,S} + |r^{2-\frac{2}{p}}\check{b}^A|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{s-3}} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}},\end{aligned}$$

which implies

$$|r^{2-\frac{2}{p}}\check{L}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Finally, we have from Definition 2.24

$$\check{J}_A^B = \mathbf{O}_2^0 \cdot (\partial_{\underline{u}}(\check{\gamma}_{AB}), \partial_{\underline{u}}(\check{\gamma}_{AB}), r^{-1}\check{\vartheta}^{\leq 1}\check{\gamma}_{AB}) + \mathbf{O}_3^2 \cdot \check{\vartheta}^{\leq 1}\check{\Omega} + \mathbf{O}_0^0 \cdot \check{\vartheta}^{\leq 1}\check{b}^A + \mathbf{O}_1^2 \cdot \partial_{\underline{u}}(\check{b}^A) + r\Gamma_g \cdot \Gamma_g^{(1)}.$$

We recall from Propositions 2.40, 5.12, 5.19, 5.22, 5.23 and 5.25

$$\begin{aligned}|r^{-\frac{2}{p}}\partial_{\underline{u}}(\check{\gamma}_{AB})|_{p,S} &\lesssim |r^{-1-\frac{2}{p}}\check{\gamma}_{AB}| + |r^{2-\frac{2}{p}}(\check{\chi}, \widetilde{\text{tr } \chi}, r^{-1}\check{\Omega}, \check{b}^A)|_{p,S} + M^2|r^{-\frac{2}{p}}\Gamma_g|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \frac{r\varepsilon_0}{|u|^{\frac{s-1}{2}}} + \frac{M^2\varepsilon}{r^2|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{s-3}} \\ &\lesssim \frac{r\varepsilon_0}{|u|^{\frac{s-1}{2}}},\end{aligned}$$

and

$$\begin{aligned}
|r^{-\frac{2}{p}}\partial_{\underline{u}}(\check{b}^A)|_{p,S} &\lesssim |r^{-1-\frac{2}{p}}\check{\zeta}|_{p,S} + M^2|r^{-3-\frac{2}{p}}\Gamma_g|_{p,S} + |r^{-\frac{2}{p}}\Gamma_g \cdot \Gamma_g|_{p,S} \\
&\lesssim \frac{\varepsilon_0}{r^3|u|^{\frac{s-3}{2}}} + \frac{M^2\varepsilon}{r^5|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^4|u|^{s-3}} \\
&\lesssim \frac{\varepsilon_0}{r^3|u|^{\frac{s-3}{2}}}.
\end{aligned}$$

Combining the above estimates, we deduce from Propositions 5.19, 5.23 and 5.25

$$\begin{aligned}
|r^{1-\frac{2}{p}}\check{J}|_{p,S} &\lesssim |r^{-1-\frac{2}{p}}(\partial_u(\check{\gamma}_{AB}), \partial_{\underline{u}}(\check{\gamma}_{AB}), r^{-1}\check{\phi}^{\leq 1}\check{\gamma}_{AB})|_{p,S} + M^2|r^{-2-\frac{2}{p}}\check{\phi}^{\leq 1}\check{\Omega}|_{p,S} \\
&\quad + |r^{1-\frac{2}{p}}\check{\phi}^{\leq 1}\check{b}^A|_{p,S} + M^2|r^{-\frac{2}{p}}\check{b}^A|_{p,S} + |r^2\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \\
&\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}} + \frac{M^2\varepsilon_0}{r^3|u|^{\frac{s-3}{2}}} + \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \frac{M^2\varepsilon_0}{r^2|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^2|u|^{s-3}} \\
&\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}}.
\end{aligned}$$

This concludes the proof of Proposition 5.27.  $\square$

In view of Propositions 5.7-5.27, we obtain (5.2). This concludes the proof of Theorem M3.

## 6 Initialization and extension (Theorems M0, M2 and M4)

### 6.1 Control of the initial data layer (Theorem M0)

The goal of this section is to prove Theorem M0, which we recall below for convenience.

**Theorem M0.** *Assume that*<sup>24</sup>

$$\mathcal{O}' \leq \varepsilon_0, \quad \mathfrak{R}' \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \text{Osc} \leq \varepsilon. \quad (6.1)$$

Then, we have

$$\mathfrak{R}_0 \lesssim \varepsilon_0. \quad (6.2)$$

*Proof.* We denote for  $e_A$  an orthogonal frame on  $S$

$$R := \{\alpha_{AB}, \beta_A, \rho, \sigma, \underline{\beta}_A, \underline{\alpha}_{AB}\},$$

and for  $e'_A$  an orthogonal frame on  $S'$

$$R' := \{\alpha'_{AB}, \beta'_A, \rho', \sigma', \underline{\beta}'_A, \underline{\alpha}'_{AB}\}.$$

The transformation formulae for curvature in Proposition 5.3 can be written as follows:

$$R = R' + f \cdot R' + \text{l.o.t.}, \quad (6.3)$$

which implies

$$\check{R} = \check{R}' + R_K - R'_K + f \cdot (\check{R}' + R'_K) + \text{l.o.t.}$$

We have from Lemma 5.4

$$|R_K - R'_K| \lesssim \frac{M\varepsilon}{r^{\frac{s+5}{2}}}.$$

<sup>24</sup>Recall that quantities with prime are associated to the foliation of the initial data layer, see (5.7) and (5.8).

Hence, we obtain

$$|\check{R}| \lesssim |\check{R}'| + |f \cdot \check{R}'| + \frac{M}{r^3}|f| + |R_K - R'_K| \lesssim |\check{R}'| + \frac{M\varepsilon}{r^{\frac{s+5}{2}}},$$

where we used  $\text{Osc}(f) \leq \varepsilon$ . Then, we have

$$\int_{\Sigma_0 \setminus K} r^s |\check{R}|^2 \lesssim \int_{\Sigma_0 \setminus K} r^s |\check{R}'|^2 + r^s \left( \frac{M^2 \varepsilon^2}{r^{s+5}} \right) \lesssim \mathfrak{R}_0'^2 + \frac{M^2 \varepsilon^2}{R_0^2} \lesssim \varepsilon_0^2.$$

Moreover, we have from (6.3)

$$\partial R = (\partial R)' + \partial f \cdot R' + f \cdot (\partial R)' + \text{l.o.t.}, \quad (6.4)$$

where

$$\partial R := \{(\nabla_\mu \alpha)_{AB}, (\nabla_\mu \beta)_A, \nabla_\mu \rho, \nabla_\mu \sigma, (\nabla_\mu \underline{\beta})_A, (\nabla_\mu \underline{\alpha})_{AB}\}, \quad \mu = 1, 2, 3, 4.$$

We have from Lemma 5.4

$$|(\partial R)_K - (\partial R)'_K| \lesssim \frac{M\varepsilon}{r^{\frac{s+7}{2}}}.$$

Moreover, we have from Proposition 2.22

$$\widetilde{\partial R} = \partial \check{R} + (\partial - \partial_K)R_K = \partial \check{R} + (\check{J}, \check{L}, \check{J}) \cdot \mathbf{O}_3^1,$$

which implies

$$\begin{aligned} \int_{\Sigma_0 \setminus K} r^{s+2} |\partial \check{R}|^2 &\lesssim \int_{\Sigma_0 \setminus K} r^{s+2} |\widetilde{\partial R}|^2 + r^{s+2} \frac{M^2}{r^6} |\Gamma_w^{(1)}|^2 \\ &\lesssim \int_{\Sigma_0 \setminus K} r^{s+2} |\widetilde{\partial R}|^2 + \int_{R_0}^\infty \frac{M^2}{r^{4-s}} |\Gamma_w^{(1)}|_{2,S}^2 \\ &\lesssim \int_{\Sigma_0 \setminus K} r^{s+2} |\widetilde{\partial R}|^2 + \int_{R_0}^\infty \frac{M^2}{r^{4-s}} \frac{\varepsilon^2}{r^{s-1}} \\ &\lesssim \int_{\Sigma_0 \setminus K} r^{s+2} |\widetilde{\partial R}|^2 + \varepsilon_0^2. \end{aligned}$$

Similarly, we have

$$\int_{\Sigma_0 \setminus K} r^{s+2} |(\widetilde{\partial R})'|^2 \lesssim \int_{\Sigma_0 \setminus K} r^{s+2} |\partial' \check{R}'|^2 + \varepsilon_0^2 \lesssim \varepsilon_0^2.$$

Hence, we obtain from (6.4) and  $\text{Osc}(f) \leq \varepsilon$  that

$$\begin{aligned} \int_{\Sigma_0 \setminus K} r^{s+2} |\partial \check{R}|^2 &\lesssim \varepsilon_0^2 + \int_{\Sigma_0 \setminus K} r^{s+2} |\widetilde{\partial R}|^2 \\ &\lesssim \varepsilon_0^2 + \int_{\Sigma_0 \setminus K} r^{s+2} |(\widetilde{\partial R})'|^2 + r^{s+2} \frac{M^2 \varepsilon^2}{r^{s+7}} + |\partial f|^2 r^{s+2} |R'|^2 + |f|^2 r^{s+2} |\partial' R'|^2 \\ &\lesssim \varepsilon_0^2 + \int_{\Sigma_0 \setminus K} r^s |\check{R}'|^2 + r^{s+2} |(\widetilde{\partial R})'|^2 + r^{s+2} |f|^2 |(\partial' R')_K|^2 \\ &\lesssim \varepsilon_0^2 + \int_{\Sigma_0 \setminus K} r^{s+2} \frac{\varepsilon^2}{r^{s-1}} \frac{M^2}{r^8} \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

Thus, we deduce

$$\mathfrak{R}_0^2 = \sum_R \int_{\Sigma_0 \setminus K} r^s |\mathfrak{d}^{\leq 1} \check{R}|^2 \lesssim \sum_R \int_{\Sigma_0 \setminus K} r^s |\check{R}|^2 + r^{s+2} |\partial \check{R}|^2 \lesssim \varepsilon_0^2.$$

This concludes the proof of Theorem M0.  $\square$

## 6.2 Control of the last slice (Theorem M2)

The goal of this section is to prove Theorem M2 which we recall below for convenience.

**Theorem M2.** *Let  $\underline{C}_*$  endowed with the geodesic type foliation of Definition 6.1. We assume that*

$$\mathcal{O}'(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0, \quad \underline{\mathcal{R}}(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon.$$

Then, we have

$$\mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0.$$

### 6.2.1 Geodesic type foliation on $\underline{C}_*$

**Definition 6.1.** *A foliation  $(u, \underline{u}_*)$  on the last slice  $\underline{C}_*$  is called a geodesic type foliation if it satisfies the following conditions:*

$$\check{\Omega} = 0, \quad \check{\underline{b}}^A = 0. \quad (6.5)$$

**Remark 6.2.** *Notice that (6.5) is equivalent to*

$$e_3(u) = \Omega_K^{-1}, \quad e_3(x^A) = \Omega_K^{-1} \underline{b}_K^A.$$

*In particular, the choice of the foliation is prescribed by  $\check{\Omega} = 0$  while  $\check{\underline{b}}^A = 0$  corresponds in fact to the transport of the  $x^A$  coordinates. The local existence of geodesic type foliations as in Definition 6.1 is completely analogous to the one for the standard geodesic foliation, i.e.  $\Omega = \frac{1}{2}$ , with a transport of the  $x^A$  coordinates by  $e_3(x^A) = 0$ .*

**Lemma 6.3.** *Let  $\underline{C}_*$  endowed with the geodesic type foliation of Definition 6.1. Then, we have*

$$\check{\underline{\omega}} = 0, \quad \text{on } \underline{C}_*.$$

*Proof.* We have from Lemma 2.21 and (6.5)

$$\Omega e_3 - \Omega_K (e_3)_K = \check{\underline{b}}^A \partial_{x^A} = 0,$$

which implies from (6.5)

$$e_3 = (e_3)_K. \quad (6.6)$$

Hence, we have from (6.5) and (6.6)

$$\underline{\omega} = -\frac{1}{2} \nabla_3 (\log \Omega) = -\frac{1}{2} \nabla_{(e_3)_K} (\log \Omega_K) = \underline{\omega}_K.$$

This concludes the proof of Lemma 6.3. □

**Remark 6.4.** *Under the assumption  $\mathcal{O}^*(\underline{C}_*) \leq \varepsilon$ , we have on  $\underline{C}_*$*

$$\begin{aligned} \wp^{\leq 1} \widetilde{\text{tr}} \underline{\chi} &\in \Gamma_g, & \wp^{\leq 1} \check{\gamma} &\in r\Gamma_g, & \wp^{\leq 1} (\check{\epsilon}) &\in r\Gamma_g, \\ \check{L} &\in \Gamma_g, & \check{J} &\in \Gamma_b, & r\check{\underline{\mu}} &\in \Gamma_g. \end{aligned}$$

**Proposition 6.5.** *We have the following linearized equations on  $\underline{C}_*$ :*

$$\begin{aligned}
\Omega \nabla_3 (\widetilde{\Omega \operatorname{tr} \chi}) + \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} \widetilde{\Omega \operatorname{tr} \chi} &= -2\Omega^2 \check{\mu} + \Gamma_g \cdot \mathbf{O}_3^2 + \Gamma_g \cdot \Gamma_g, \\
\operatorname{div}(\check{\chi}) &= -\check{b} + \mathbf{O}_0^0 \cdot \nabla \widetilde{\operatorname{tr} \chi} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_g + \Gamma_g \cdot \Gamma_g, \\
\nabla_3 \widetilde{\Omega \operatorname{tr} \chi} + \operatorname{tr} \underline{\chi} \widetilde{\Omega \operatorname{tr} \chi} &= \Gamma_g \cdot \mathbf{O}_2^1 + \Gamma_b \cdot \Gamma_b, \\
\operatorname{div}(\check{\chi}) &= \check{\beta} + \mathbf{O}_0^0 \cdot \nabla \widetilde{\operatorname{tr} \chi} + \mathbf{O}_1^0 \cdot \check{\zeta} + \mathbf{O}_3^2 \cdot \Gamma_b + \Gamma_b \cdot \Gamma_g, \\
\Omega \nabla_3 [\check{\mu}] + \Omega \operatorname{tr} \underline{\chi} [\check{\mu}] &= \Omega \operatorname{tr} \underline{\chi} \check{\rho} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\beta}) + \mathbf{O}_3^1 \cdot \Gamma_g + \mathbf{O}_0^0 \cdot \check{\eta} \cdot \check{\beta} + r^{-1} \Gamma_b \cdot \Gamma_b.
\end{aligned} \tag{6.7}$$

We also have the following equation:

$$\begin{aligned}
&\nabla_3 (r \Delta \widetilde{\Omega \operatorname{tr} \chi}) + \frac{3}{2} \operatorname{tr} \underline{\chi} (r \Delta \widetilde{\Omega \operatorname{tr} \chi}) \\
&= \mathbf{O}_2^2 \cdot \nabla^2 \check{\chi} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r \Gamma_b \cdot \nabla^2 \check{\chi} + \mathbf{O}_{-1}^0 \cdot |\nabla \check{\chi}|^2 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)}.
\end{aligned} \tag{6.8}$$

*Proof.* Notice that (6.7) follows directly from Proposition 2.34 and Remark 6.4. Next, we have from Proposition A.1

$$\begin{aligned}
\Omega \nabla_3 \widetilde{\Omega \operatorname{tr} \chi} + (\Omega \operatorname{tr} \underline{\chi}) \widetilde{\Omega \operatorname{tr} \chi} &= -4\Omega \operatorname{tr} \underline{\chi} \widetilde{\Omega \omega} + \mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \chi}] + \operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \chi}], \\
\mathfrak{L}[\nabla_3 \widetilde{\operatorname{tr} \chi}] &:= -2\widetilde{\Omega \chi}(\Omega_K \widehat{\chi}_K) - 4\Omega_K \omega_K \widetilde{\Omega \operatorname{tr} \chi} - \check{b}^A \nabla_{\partial_{x^a}} \operatorname{tr}_K \underline{\chi}_K, \\
\operatorname{Err}[\nabla_3 \widetilde{\operatorname{tr} \chi}] &:= \frac{1}{2} (\widetilde{\Omega \operatorname{tr} \chi})^2 - |\widetilde{\Omega \chi}|^2.
\end{aligned}$$

Recalling from Lemma 6.3 that  $\check{\Omega} = 0$ ,  $\check{\omega} = 0$  and  $\check{b}^A = 0$  on  $\underline{C}_*$ , we obtain

$$\Omega \nabla_3 \widetilde{\Omega \operatorname{tr} \chi} + (\Omega \operatorname{tr} \underline{\chi}) \widetilde{\Omega \operatorname{tr} \chi} = \mathbf{O}_3^2 \cdot \check{\chi} + \mathbf{O}_2^1 \cdot \widetilde{\operatorname{tr} \chi} + \frac{1}{2} (\widetilde{\Omega \operatorname{tr} \chi})^2 - |\widetilde{\Omega \chi}|^2. \tag{6.9}$$

Differentiating it by  $\nabla$  and applying Proposition 2.41, we deduce

$$\begin{aligned}
\Omega \nabla_3 (\nabla \widetilde{\Omega \operatorname{tr} \chi}) + (\Omega \operatorname{tr} \underline{\chi}) \nabla \widetilde{\Omega \operatorname{tr} \chi} &= \mathbf{O}_3^2 \cdot \nabla \check{\chi} + \mathbf{O}_2^1 \cdot \nabla \widetilde{\operatorname{tr} \chi} + \mathbf{O}_0^0 \cdot \nabla |\check{\chi}|^2 + r^{-1} \Gamma_g \cdot \Gamma_g + [\Omega \nabla_3, \nabla] \widetilde{\Omega \operatorname{tr} \chi} \\
&= \mathbf{O}_3^2 \cdot \nabla \check{\chi} + \mathbf{O}_3^1 \cdot \Gamma_g + \mathbf{O}_0^0 \cdot \nabla |\check{\chi}|^2 - \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} \nabla \widetilde{\Omega \operatorname{tr} \chi} + r^{-1} \Gamma_b \cdot \Gamma_g,
\end{aligned}$$

which implies

$$\Omega \nabla_3 (\nabla \widetilde{\Omega \operatorname{tr} \chi}) + \frac{3}{2} \Omega \operatorname{tr} \underline{\chi} \nabla \widetilde{\Omega \operatorname{tr} \chi} = \mathbf{O}_3^2 \cdot \nabla \check{\chi} + \mathbf{O}_3^1 \cdot \Gamma_g + \mathbf{O}_0^0 \cdot \nabla |\check{\chi}|^2 + r^{-1} \Gamma_b \cdot \Gamma_g.$$

Differentiating it by  $r \operatorname{div}$  and applying Corollary 2.43, we infer

$$\Omega \nabla_3 (r \Delta \widetilde{\Omega \operatorname{tr} \chi}) + \frac{3}{2} \Omega \operatorname{tr} \underline{\chi} (r \Delta \widetilde{\Omega \operatorname{tr} \chi}) = \mathbf{O}_2^2 \cdot \nabla^2 \check{\chi} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r \Gamma_b \cdot \nabla^2 \check{\chi} + \mathbf{O}_{-1}^0 \cdot |\nabla \check{\chi}|^2 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)}.$$

This concludes the proof of Proposition 6.5.  $\square$

### 6.2.2 Estimate for $\mathcal{O}_1^*$

In the remainder of Section 6.2, we denote

$$S := S(u, \underline{u}_*), \quad S_* := \underline{C} \cap \Sigma_0. \tag{6.10}$$

**Proposition 6.6.** *For a geodesic type foliation on  $\underline{C}_*$  satisfying*

$$\mathcal{O}'(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0, \quad \underline{\mathcal{R}}(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon, \tag{6.11}$$

we have

$$\mathcal{O}_1^* \lesssim \varepsilon_0. \tag{6.12}$$



*Proof.* Notice that the assumption  $\mathcal{O}'(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0$  controls the initial data on the last sphere  $S_*$ . The idea of the proof is to transport the estimates in the direction of  $\nabla_3$  along  $\underline{C}_*$  using Proposition 6.5 and Lemma 5.5. The proof is similar to Section 5, and in fact easier since  $\check{\Omega} = 0$ ,  $\check{\omega} = 0$  and  $\check{b}^A = 0$  on  $\underline{C}_*$ .

**Step 1.** Estimate for  $\mathcal{O}_{0,1,2}^*(\widetilde{\text{tr}} \underline{\chi})$ .

The estimates for  $\mathcal{O}_0^*(\widetilde{\text{tr}} \underline{\chi})$  and  $\mathcal{O}_1^*(\widetilde{\text{tr}} \underline{\chi})$  follow directly from Propositions 5.13 and 5.21 in the particular case  $\check{\Omega} = 0$  and  $\check{\omega} = 0$ . Then, we focus on  $\mathcal{O}_2^*(\widetilde{\text{tr}} \underline{\chi})$ .

We recall from (6.8)

$$\nabla_3(r\Delta\widetilde{\text{tr}} \underline{\chi}) + \frac{3}{2} \text{tr} \underline{\chi}(r\Delta\widetilde{\text{tr}} \underline{\chi}) = \mathbf{O}_2^2 \cdot \nabla^2 \check{\underline{\chi}} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r\Gamma_b \cdot \nabla^2 \check{\underline{\chi}} + \mathbf{O}_{-1}^0 \cdot |\nabla \check{\underline{\chi}}|^2 + r^{-1}\Gamma_b \cdot \Gamma_g^{(1)}.$$

Moreover, we have from Proposition 6.5 and Remark 6.4

$$\nabla^2 \check{\underline{\chi}} = \nabla \check{\underline{\beta}} + r^{-2}\Gamma_g^{(1)}, \quad \nabla \check{\underline{\chi}} = \check{\underline{\beta}} + r^{-1}\Gamma_g.$$

Hence, we obtain

$$\nabla_3(r\Delta\widetilde{\text{tr}} \underline{\chi}) + \frac{3}{2} \text{tr} \underline{\chi}(r\Delta\widetilde{\text{tr}} \underline{\chi}) = \mathbf{O}_2^2 \cdot \nabla \check{\underline{\beta}} + r\Gamma_b \cdot \nabla \check{\underline{\beta}} + \mathbf{O}_{-1}^0 \cdot |\check{\underline{\beta}}^{(0)}|^2 + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_g^{(1)}.$$

We recall from Proposition 2.37

$$\nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr} \underline{\chi}(\check{\rho}, \check{\sigma}) = -d_1 \check{\underline{\beta}} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}} + \Gamma_g \cdot \check{\underline{\alpha}}.$$

We deduce

$$\nabla_3 \Xi + \frac{3}{2} \text{tr} \underline{\chi} \Xi = \mathbf{O}_3^2 \cdot \check{\underline{\alpha}} + \mathbf{O}_{-1}^0 \cdot |\check{\underline{\beta}}^{(0)}|^2 + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + \Gamma_g \cdot \check{\underline{\alpha}} + r^{-1}\Gamma_b \cdot \Gamma_g^{(1)},$$

where  $\Xi$  is the following renormalized quantity

$$\Xi = r\Delta\widetilde{\text{tr}} \underline{\chi} + \mathbf{O}_2^2 \cdot (\check{\rho}, \check{\sigma}) + r\Gamma_b \cdot (\check{\rho}, \check{\sigma}).$$

Applying Lemma 5.5, we deduce for  $p \in [2, 4]$

$$\begin{aligned} & |r^{3-\frac{2}{p}} \Xi|_{p,S} \\ & \lesssim |r^{3-\frac{2}{p}} \Xi|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u M^2 |r^{-\frac{2}{p}} \check{\underline{\alpha}}|_{p,S} + |r^{4-\frac{2}{p}} |\check{\underline{\beta}}^{(0)}|^2|_{p,S} + M |r^{-\frac{2}{p}} \Gamma_g^{(1)}| + |r^{3-\frac{2}{p}} \Gamma_g \cdot \check{\underline{\alpha}}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} \\ & \lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u \frac{M^2 \varepsilon_0}{r |u|^{\frac{s+1}{2}}} + \frac{M}{r^2 |u|^{\frac{s-3}{2}}} + \frac{\varepsilon_0}{|u|^{s-1}} + \frac{\varepsilon^2}{r |u|^{s-2}} + |r^{4-\frac{2}{p}} |\check{\underline{\beta}}^{(0)}|^2|_{p,S} \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u r^2 |r^{\frac{3}{4}} \check{\underline{\beta}}^{(0)}|_{8,S}^2 \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u |r \check{\underline{\beta}}^{(1)}|_{2,S}^2 \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{|u|^{s-2}} \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used the Sobolev inequality  $|r^{-\frac{1}{4}}U|_{8,S} \lesssim |r^{-1}U^{(1)}|_{2,S}$  for any  $S$ -tangent tensor  $U$ . Hence, we infer from Proposition 2.15

$$|r^{4-\frac{2}{p}}\nabla^2\widetilde{\Omega\text{tr}\underline{\chi}}|_{p,S} \lesssim |r^{3-\frac{2}{p}}\underline{\Xi}|_{p,S} + \frac{M^2}{r^2}|r^{3-\frac{2}{p}}(\check{\rho}, \check{\sigma})|_{p,S} + |r^{4-\frac{2}{p}}\Gamma_b \cdot (\check{\rho}, \check{\sigma})|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Thus, we obtain

$$\mathcal{O}_{0,1,2}^*(\widetilde{\text{tr}\underline{\chi}}) \lesssim \varepsilon_0. \quad (6.13)$$

**Step 2.** Estimate for  $\mathcal{O}_{0,1}^*(\widetilde{[\underline{\mu}]})$ .

The estimate for  $\mathcal{O}_0^*(\widetilde{[\underline{\mu}]})$  follows directly from Proposition 5.18 in the particular case  $\check{\Omega} = 0$  and  $\check{\omega} = 0$ . Next, we recall from (6.7)

$$\Omega\nabla_3[\widetilde{\underline{\mu}}] + \Omega\text{tr}\underline{\chi}[\widetilde{\underline{\mu}}] = \Omega\text{tr}\underline{\chi}\check{\rho} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\underline{\beta}}) + \mathbf{O}_3^1 \cdot \Gamma_g + \mathbf{O}_0^0 \cdot \check{\eta} \cdot \check{\underline{\beta}} + r^{-1}\Gamma_b \cdot \Gamma_b.$$

Differentiating it by  $r\nabla$  and applying Corollary 2.43, we infer

$$\nabla_3(r\nabla[\widetilde{\underline{\mu}}]) + \text{tr}\underline{\chi}(r\nabla[\widetilde{\underline{\mu}}]) = \mathbf{O}_0^0 \cdot (\nabla\check{\rho}, \nabla\check{\underline{\beta}}) + \mathbf{O}_0^0 \cdot \check{\rho} \cdot \check{\underline{\beta}} + r\Gamma_g \cdot \nabla\check{\underline{\beta}} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_b^{(1)}.$$

We deduce

$$\nabla_3\underline{U} + \text{tr}\underline{\chi}\underline{U} = \mathbf{O}_0^0 \cdot \check{\rho} \cdot \check{\underline{\beta}} + \mathbf{O}_3^1 \cdot \Gamma_g^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_b^{(1)},$$

where the renormalized quantity  $\underline{U}$  is given by

$$\underline{U} = r\nabla[\widetilde{\underline{\mu}}] + \mathbf{O}_0^0 \cdot (\check{b}, \check{\rho}, \check{\sigma}) + r\Gamma_g \cdot (\check{\rho}, \check{\sigma}).$$

Applying Lemma 5.5, we have

$$\begin{aligned} |r^{2-\frac{2}{p}}\underline{U}|_{p,S} &\lesssim |r^{2-\frac{2}{p}}\underline{U}|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}}\check{\rho} \cdot \check{\underline{\beta}}|_{p,S} + M|r^{-1-\frac{2}{p}}\Gamma_g^{(1)}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_b \cdot \Gamma_b^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \frac{M\varepsilon}{r^2|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{s-2}} + \int_{u_0(\underline{u}_*)}^u |r^{\frac{3}{2}}\check{\rho} \cdot \check{\underline{\beta}}|_{4,S} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u r^2|r^{-\frac{1}{4}}\check{\rho}|_{8,S}|r^{-\frac{1}{4}}\check{\underline{\beta}}|_{8,S} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u |\check{\rho}^{(1)}|_{2,S}|\check{\underline{\beta}}^{(1)}|_{2,S} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + r^{-2} \left( \int_{u_0(\underline{u}_*)}^u |r\check{\rho}^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \left( \int_{u_0(\underline{u}_*)}^u |r\check{\underline{\beta}}^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^2|u|^{s-2}} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}, \end{aligned}$$

where we used the Sobolev inequality  $|r^{-\frac{1}{4}}U|_{8,S} \lesssim |r^{-1}U^{(1)}|_{2,S}$  for any  $S$ -tangent tensor  $U$ . Hence, we obtain

$$|r^{3-\frac{2}{p}}\nabla[\widetilde{\underline{\mu}}]|_{p,S} \lesssim |r^{2-\frac{2}{p}}\underline{U}|_{p,S} + |r^{2-\frac{2}{p}}(\check{b}, \check{\rho}, \check{\sigma})|_{p,S} + |r^{3-\frac{2}{p}}\Gamma_g \cdot (\check{\rho}, \check{\sigma})|_{p,S} \lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}.$$

Thus, we have

$$\mathcal{O}_{0,1}^*(\widetilde{[\underline{\mu}]}) \lesssim \varepsilon_0. \quad (6.14)$$

**Step 3.** Estimate for  $\mathcal{O}_{0,1}^*(\widetilde{\text{tr } \chi})$ .

We recall from (6.7) that

$$\Omega \nabla_3(\widetilde{\Omega \text{tr } \chi}) = -2\widetilde{\Omega^2[\underline{\mu}]} + \Gamma_g \cdot \mathbf{O}_3^2 + \Gamma_g \cdot \Gamma_g. \quad (6.15)$$

We recall from (2.28) and (2.29)

$$[\underline{\mu}] + [\underline{\mu}] = \frac{1}{2} \text{tr } \chi \text{tr } \underline{\chi} - 2\Delta \log \Omega + \widehat{\chi} \cdot \widehat{\chi} - 2\rho,$$

which implies

$$\widetilde{\Omega^2[\underline{\mu}]} = -\widetilde{\Omega^2[\underline{\mu}]} + \frac{1}{2} \widetilde{\Omega \text{tr } \chi} \Omega \text{tr } \underline{\chi} + \mathbf{O}_1^0 \cdot \widetilde{\text{tr } \underline{\chi}} - 2\Omega^2 \check{\rho} + \mathbf{O}_3^2 \cdot \Gamma_b + \Gamma_g \cdot \Gamma_b.$$

Injecting it into (6.15), we deduce

$$\Omega \nabla_3(\widetilde{\Omega \text{tr } \chi}) + \Omega \text{tr } \underline{\chi} \widetilde{\Omega \text{tr } \chi} = 2\widetilde{\Omega^2[\underline{\mu}]} + 4\Omega^2 \check{\rho} + \mathbf{O}_1^0 \cdot \widetilde{\text{tr } \underline{\chi}} + \mathbf{O}_3^2 \cdot \Gamma_b + \Gamma_b \cdot \Gamma_g. \quad (6.16)$$

Applying Lemma 5.5, we infer

$$\begin{aligned} & |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \chi}|_{p,S} \\ & \lesssim |r^{2-\frac{2}{p}} \widetilde{\Omega \text{tr } \chi}|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}} \widetilde{[\underline{\mu}]}|_{p,S} + |r^{2-\frac{2}{p}} \check{\rho}|_{p,S} + |r^{1-\frac{2}{p}} \widetilde{\text{tr } \underline{\chi}}|_{p,S} + M^2 |r^{-1-\frac{2}{p}} \Gamma_b|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_g|_{p,S} \\ & \lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \int_{u_0(\underline{u}_*)}^u \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \frac{M^2 \varepsilon}{r^2 |u|^{\frac{s-1}{2}}} + \frac{\varepsilon^2}{r|u|^{s-2}} \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

Next, we differentiate (6.16) by  $\nabla$  and apply Proposition 2.41 to deduce

$$\begin{aligned} & \Omega \nabla_3(\nabla \widetilde{\Omega \text{tr } \chi}) + \Omega \text{tr } \underline{\chi} (\nabla \widetilde{\Omega \text{tr } \chi}) \\ & = 2\Omega^2 \nabla \widetilde{[\underline{\mu}]} + 4\Omega^2 \nabla \check{\rho} - \Omega \underline{\chi} \cdot \nabla \widetilde{\Omega \text{tr } \chi} + \mathbf{O}_1^0 \cdot \nabla \widetilde{\text{tr } \underline{\chi}} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)} \\ & = 2\Omega^2 \nabla \widetilde{[\underline{\mu}]} + 4\Omega^2 \nabla \check{\rho} - \frac{1}{2} \Omega \text{tr } \underline{\chi} (\nabla \widetilde{\Omega \text{tr } \chi}) + \mathbf{O}_1^0 \cdot \nabla \widetilde{\text{tr } \underline{\chi}} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)}, \end{aligned}$$

which implies from  $\check{\Omega} = 0$  that

$$\nabla_3(\nabla \widetilde{\text{tr } \chi}) + \frac{3}{2} \text{tr } \underline{\chi} (\nabla \widetilde{\text{tr } \chi}) = 2\nabla \widetilde{[\underline{\mu}]} + 4\nabla \check{\rho} + \mathbf{O}_1^0 \cdot \nabla \widetilde{\text{tr } \underline{\chi}} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)}. \quad (6.17)$$

Next, we define  $\chi^\dagger$  by the following transport equation:

$$\Omega \nabla_3 \chi^\dagger + \frac{1}{2} \Omega \text{tr } \underline{\chi} \chi^\dagger = 4\Omega \check{\sigma} \quad \text{on } \underline{C}_*, \quad \chi^\dagger = 0 \quad \text{on } S_*. \quad (6.18)$$

Differentiating it by  ${}^* \nabla$  and applying Proposition 2.41, we infer

$$\begin{aligned} \Omega \nabla_3 {}^* \nabla \chi^\dagger + \frac{1}{2} \Omega \text{tr } \underline{\chi} ({}^* \nabla \chi^\dagger) + \frac{1}{2} {}^* \nabla (\Omega \text{tr } \underline{\chi}) \chi^\dagger & = 4 {}^* \nabla (\Omega \check{\sigma}) + [\Omega \nabla_3, {}^* \nabla] \chi^\dagger \\ & = 4 {}^* \nabla (\Omega \check{\sigma}) - \frac{1}{2} \Omega \text{tr } \underline{\chi} ({}^* \nabla \chi^\dagger) - \Omega \widehat{\chi} \cdot {}^* \nabla \chi^\dagger, \end{aligned}$$

which implies<sup>25</sup>

$$\nabla_3^* \nabla \chi^\dagger + \operatorname{tr} \underline{\chi}^* \nabla \chi^\dagger = 4^* \nabla \check{\sigma} + \mathbf{O}_4^2 \cdot \Gamma_g^{(1)} + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)}. \quad (6.19)$$

We recall from Proposition 2.37

$$\nabla_3 \check{b} + \frac{3}{2} \operatorname{tr} \underline{\chi} \check{b} = \frac{1}{2} \operatorname{tr} \underline{\chi} \check{b} + \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + r^{-1} \Gamma_g \cdot \Gamma_b^{(1)}. \quad (6.20)$$

Combining (6.17), (6.19) and (6.20), we deduce

$$\nabla_3 X + \operatorname{tr} \underline{\chi} X = -2 \operatorname{tr} \underline{\chi} \check{b} + 2 \nabla [\underline{\mu}] + \mathbf{O}_1^0 \cdot \nabla \widetilde{\operatorname{tr} \underline{\chi}} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + r^{-1} \Gamma_b \cdot \Gamma_g^{(1)},$$

where

$$X := \nabla \widetilde{\operatorname{tr} \underline{\chi}} + {}^* \nabla \chi^\dagger - 4 \check{b}.$$

Applying Lemma 5.5 and (6.14), we obtain

$$\begin{aligned} & |r^{3-\frac{2}{p}} X|_{p,S} \\ & \lesssim |r^{3-\frac{2}{p}} X|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}} \check{b}|_{p,S} + |r^{3-\frac{2}{p}} \nabla [\underline{\mu}]|_{p,S} + |r^{3-\frac{2}{p}} \nabla \widetilde{\operatorname{tr} \underline{\chi}}|_{p,S} + M |r^{-\frac{2}{p}} \Gamma_g^{(1)}|_{p,S} + |r^{2-\frac{2}{p}} \Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} \\ & \lesssim \frac{\varepsilon_0}{r^{\frac{s-3}{2}}} + \int_{u_0(\underline{u}_*)}^u \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}} + \frac{M\varepsilon}{r^2|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r|u|^{\frac{s-1}{2}}} \\ & \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}. \end{aligned}$$

Then, we have from Proposition 2.15

$$|r^{3-\frac{2}{p}} \nabla (\widetilde{\operatorname{tr} \underline{\chi}}, \chi^\dagger)|_{p,S} \lesssim |r^{3-\frac{2}{p}} X|_{p,S} + |r^{3-\frac{2}{p}} \check{b}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Next, applying Lemma 5.5 to (6.18), we deduce

$$|r^{1-\frac{2}{p}} \chi^\dagger|_{p,S} \lesssim \int_{u_0(\underline{u}_*)}^u |r^{1-\frac{2}{p}} \check{\sigma}|_{p,S} \lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}.$$

Hence, we improve the assumption  $\chi^\dagger \in \Gamma_g$  and obtain

$$\mathcal{O}_{0,1}(\widetilde{\operatorname{tr} \underline{\chi}}) \lesssim \varepsilon_0. \quad (6.21)$$

**Step 4.** Estimates for  $\mathcal{O}_{0,1}^*(\check{\zeta})$ ,  $\mathcal{O}_{0,1}^*(\check{\chi})$  and  $\mathcal{O}_{0,1}^*(\check{\underline{\chi}})$ .

We have from (6.14)

$$|r^{4-\frac{2}{p}} \nabla [\underline{\mu}]|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}},$$

which implies from (2.29) and (6.21)

$$|r^{4-\frac{2}{p}} \nabla \underline{\mu}|_{p,S} \lesssim |r^{4-\frac{2}{p}} \nabla [\underline{\mu}]|_{p,S} + |r^{3-\frac{2}{p}} \nabla (\widetilde{\operatorname{tr} \underline{\chi}}, \widetilde{\operatorname{tr} \underline{\chi}})|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

We proceed as in Proposition 5.18 to deduce

$$|r^{2-\frac{2}{p}} (r \nabla)^q \check{\eta}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}, \quad q = 0, 1.$$

<sup>25</sup>We assume here that  $\chi^\dagger \in \Gamma_g$ . This assumption will be improved at the end of Step 3.

Recalling that

$$2\check{\zeta} = \check{\eta} - \underline{\check{\eta}}, \quad \check{\eta} + \underline{\check{\eta}} = \Delta \overline{\log \Omega} = 0,$$

we obtain

$$|r^{2-\frac{2}{p}}(r\nabla)^q \check{\zeta}|_{p,S} = |r^{2-\frac{2}{p}}(r\nabla)^q \underline{\check{\eta}}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}, \quad q = 0, 1.$$

Hence, we have

$$\mathcal{O}_{0,1}^*(\check{\zeta}) \lesssim \varepsilon_0. \quad (6.22)$$

Finally, we apply Proposition 5.22 in the particular case of  $\check{\Omega} = 0$  to deduce

$$\mathcal{O}_{0,1}^*(\check{\chi}) + \mathcal{O}_{0,1}^*(\underline{\check{\chi}}) \lesssim \varepsilon_0. \quad (6.23)$$

Combining the estimates (6.13), (6.14), (6.21), (6.22) and (6.23), this concludes the proof of Proposition 6.6.  $\square$

### 6.2.3 Estimate for $\mathcal{O}_\gamma^*$

**Proposition 6.7.** *For a geodesic type foliation on  $\underline{C}_*$  in the sense of Definition 6.1, we assume that*

$$\mathcal{O}'(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0, \quad \underline{\mathcal{R}}(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon. \quad (6.24)$$

Then, we have

$$\mathcal{O}_\gamma^* \lesssim \varepsilon_0. \quad (6.25)$$

*Proof.* We recall from Proposition 2.40

$$\nabla_3(\check{\gamma}_{AB}) - \text{tr} \underline{\check{\chi}}(\check{\gamma}_{AB}) = \mathbf{O}_{-2}^0 \cdot (\underline{\check{\chi}}, \widetilde{\text{tr}} \underline{\check{\chi}}) + \mathbf{O}_3^2 \cdot \check{\gamma}_{AB}. \quad (6.26)$$

Differentiating it by  $\nabla$  and applying Corollary 2.43, we deduce

$$\begin{aligned} \Omega \nabla_3(\nabla \check{\gamma}_{AB}) - \Omega \text{tr} \underline{\check{\chi}}(\nabla \check{\gamma}_{AB}) &= \nabla(\Omega \text{tr} \underline{\check{\chi}}) \check{\gamma}_{AB} + \mathbf{O}_{-2}^0 \cdot (\nabla \underline{\check{\chi}}, \nabla \widetilde{\text{tr}} \underline{\check{\chi}}) + \mathbf{O}_3^2 \cdot \nabla \check{\gamma}_{AB} + [\Omega \nabla_3, \nabla] \check{\gamma}_{AB} \\ &= -\frac{1}{2} \Omega \text{tr} \underline{\check{\chi}} \nabla \check{\gamma}_{AB} + \mathbf{O}_{-2}^0 \cdot (\nabla \underline{\check{\chi}}, \nabla \widetilde{\text{tr}} \underline{\check{\chi}}) + \mathbf{O}_1^2 \cdot \Gamma_g + r^2 \Gamma_g \cdot \Gamma_g, \end{aligned}$$

which implies<sup>26</sup>

$$\nabla_3(\nabla \check{\gamma}_{AB}) - \frac{1}{2} \text{tr} \underline{\check{\chi}}(\nabla \check{\gamma}_{AB}) = \mathbf{O}_{-2}^0 \cdot (\nabla \underline{\check{\chi}}, \nabla \widetilde{\text{tr}} \underline{\check{\chi}}) + \mathbf{O}_1^2 \cdot \Gamma_g + r^2 \Gamma_g \cdot \Gamma_g. \quad (6.27)$$

Applying Lemma 5.5, we obtain

$$\begin{aligned} &|r^{-1-\frac{2}{p}} \nabla \check{\gamma}_{AB}|_{p,S} \\ &\lesssim |r^{-1-\frac{2}{p}} \nabla \check{\gamma}_{AB}|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u \left( |r^{1-\frac{2}{p}} \nabla(\underline{\check{\chi}}, \widetilde{\text{tr}} \underline{\check{\chi}})|_{p,S} + M^2 |r^{-2-\frac{2}{p}} \Gamma_g|_{p,S} + |r^{1-\frac{2}{p}} \Gamma_g \cdot \Gamma_g|_{p,S} \right) du \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \int_{u_0(\underline{u}_*)}^u \left( \frac{\varepsilon_0}{r|u|^{\frac{s-1}{2}}} + \frac{M^2}{r^4|u|^{\frac{s-3}{2}}} + \frac{\varepsilon^2}{r^3|u|^{s-3}} \right) du \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}. \end{aligned}$$

<sup>26</sup>We use the convention introduced in Remark 6.4.

Differentiating (6.27) by  $r\nabla$  and applying Corollary 2.43, we infer

$$\nabla_3(r\nabla^2\check{\gamma}_{AB}) - \frac{1}{2}\text{tr}\underline{\chi}(r\nabla^2\check{\gamma}_{AB}) = \mathbf{O}_{-3}^0 \cdot (\nabla^2\check{\underline{\chi}}, \nabla^2\check{\text{tr}}\underline{\chi}) + \mathbf{O}_1^2 \cdot \Gamma_g^{(1)} + r^2\Gamma_g \cdot \Gamma_g^{(1)}.$$

Next, we have from (6.7)

$$\nabla^2\check{\underline{\chi}} = \nabla\check{\underline{\beta}} + \mathbf{O}_0^0 \cdot \nabla^2\check{\text{tr}}\underline{\chi} + \mathbf{O}_1^0 \cdot \nabla\check{\zeta} + \mathbf{O}_4^2 \cdot \Gamma_b^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_g^{(1)}.$$

Hence, we obtain

$$\nabla_3(r\nabla^2\check{\gamma}_{AB}) - \frac{1}{2}\text{tr}\underline{\chi}(r\nabla^2\check{\gamma}_{AB}) = \mathbf{O}_{-3}^0 \cdot (\nabla\check{\underline{\beta}}, \nabla^2\check{\text{tr}}\underline{\chi}, r^{-1}\nabla\check{\zeta}) + \mathbf{O}_1^2 \cdot \Gamma_b^{(1)} + r^2\Gamma_b \cdot \Gamma_g^{(1)}.$$

We recall from Proposition 2.37

$$\nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2}\text{tr}\underline{\chi}(\check{\rho}, \check{\sigma}) = -d_1\check{\underline{\beta}} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}} + \Gamma_g \cdot \check{\underline{\alpha}}.$$

We obtain

$$\nabla_3G - \frac{1}{2}\text{tr}\underline{\chi}G = \mathbf{O}_{-3}^0 \cdot (\nabla^2\check{\text{tr}}\underline{\chi}, r^{-1}\nabla\check{\zeta}, r^{-1}\check{\rho}, r^{-1}\check{\sigma}) + \mathbf{O}_0^1 \cdot \Gamma_g^{(1)} + \mathbf{O}_0^2 \cdot \check{\underline{\alpha}} + r^3\Gamma_g \cdot \check{\underline{\alpha}} + r^2\Gamma_b \cdot \Gamma_g^{(1)},$$

where the renormalized quantity  $G$  is given by

$$G = r\nabla^2\check{\gamma}_{AB} + \mathbf{O}_{-3}^0 \cdot (\check{\rho}, \check{\sigma}).$$

Applying Lemma 5.5, we infer

$$\begin{aligned} |r^{-1-\frac{2}{p}}G|_{p,S} &\lesssim |r^{-1-\frac{2}{p}}G|_{p,S_*} + \int_{u_0(\underline{u}_*)}^u |r^{2-\frac{2}{p}}\nabla^2\check{\text{tr}}\underline{\chi}|_{p,S} + |r^{1-\frac{2}{p}}(\nabla\check{\zeta}, \check{\rho}, \check{\sigma})|_{p,S} \\ &\quad + \int_{u_0(\underline{u}_*)}^u M|r^{-1-\frac{2}{p}}\Gamma_g^{(1)}|_{p,S} + M^2|r^{-1-\frac{2}{p}}\check{\underline{\alpha}}|_{p,S} + |r^{2-\frac{2}{p}}\Gamma_g \cdot \check{\underline{\alpha}}|_{p,S} + |r^{1-\frac{2}{p}}\Gamma_b \cdot \Gamma_g^{(1)}|_{p,S} \\ &\lesssim \frac{\varepsilon_0}{r^{\frac{s-1}{2}}} + \int_{u_0(\underline{u}_*)}^u \frac{\varepsilon_0}{r^2|u|^{\frac{s-3}{2}}} + \frac{M\varepsilon}{r^3|u|^{\frac{s-3}{2}}} + \frac{M^2\varepsilon}{r^2|u|^{\frac{s+1}{2}}} + \frac{\varepsilon^2}{r|u|^{s-1}} + \frac{\varepsilon^2}{r^2|u|^{s-2}} \\ &\lesssim \frac{\varepsilon_0}{r|u|^{\frac{s-3}{2}}}. \end{aligned}$$

Hence, we deduce

$$|r^{1-\frac{2}{p}}\nabla^2\check{\gamma}_{AB}|_{p,S} \lesssim |r^{-\frac{2}{p}}G|_{p,S} + |r^{3-\frac{2}{p}}(\check{\rho}, \check{\sigma})|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.$$

Thus, we have

$$\mathcal{O}_{0,1,2}^*(\check{\gamma}) \lesssim \varepsilon_0. \tag{6.28}$$

The estimate for  $\mathcal{O}_{0,1,2}(\check{\varepsilon})$  is similar. This concludes the proof of Proposition 6.7.  $\square$

### 6.2.4 Estimate for $\mathcal{O}_\Gamma^*$

**Proposition 6.8.** *For a geodesic type foliation on  $\underline{C}_*$  in the sense of Definition 6.1, we assume that*

$$\mathcal{O}'(\underline{C}_* \cap \Sigma_0) \leq \varepsilon_0, \quad \underline{\mathcal{R}}(\underline{C}_*) \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \leq \varepsilon. \tag{6.29}$$

Then, we have

$$\mathcal{O}_\Gamma^* \lesssim \varepsilon_0. \tag{6.30}$$

*Proof.* We recall from Lemma 2.23

$$\begin{aligned}\Gamma_{uA}^B &= \frac{1}{2}\gamma^{BC} \left( \partial_u(\gamma_{AC}) - \frac{\partial}{\partial x^A}(\gamma_{CD}b^D) + \frac{\partial}{\partial x^C}(\gamma_{AD}b^D) \right) + \frac{b^B}{4\Omega^2} \left( 2\frac{\partial}{\partial x^A}(\Omega^2) - \frac{\partial}{\partial \underline{u}}(\gamma_{AD}b^D) \right), \\ \Gamma_{BC}^A &= \frac{1}{2}\gamma^{AD} \left( \frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} - \frac{\partial\gamma_{BC}}{\partial x^D} \right) + \frac{b^A}{4\Omega^2} \frac{\partial\gamma_{BC}}{\partial \underline{u}}.\end{aligned}$$

Applying Definition 2.24, and since  $\check{\Omega} = 0$  and  $\check{b} = 0$ , we infer

$$\begin{aligned}\check{J} &= \mathbf{O}_0^0 \cdot (\partial_u \check{\gamma}, \nabla \check{\gamma}, \partial_{\underline{u}} \check{\gamma}, r^{-1} \check{\gamma}) + r\Gamma_g \cdot \Gamma_b, \\ \check{L} &= \mathbf{O}_0^0 \cdot (\nabla \check{\gamma}, \partial_{\underline{u}} \check{\gamma}, r^{-1} \check{\gamma}) + r\Gamma_g \cdot \Gamma_g,\end{aligned}$$

where we used Remark 6.4. Applying Proposition 2.40, we deduce

$$\begin{aligned}\check{J} &= \mathbf{O}_0^0 \cdot (\widetilde{\text{tr}} \check{\chi}, \check{\chi}, \widetilde{\text{tr}} \check{\chi}, \check{\chi}, r^{-1} \check{\gamma}, \nabla \check{\gamma}) + r\Gamma_g \cdot \Gamma_b, \\ \check{L} &= \mathbf{O}_0^0 \cdot (\widetilde{\text{tr}} \check{\chi}, \check{\chi}, r^{-1} \check{\gamma}, \nabla \check{\gamma}) + r\Gamma_g \cdot \Gamma_g.\end{aligned}$$

Differentiating it by  $r\nabla$ , we obtain

$$\begin{aligned}r\nabla \check{J} &= \mathbf{O}_0^0 \cdot (r\nabla \widetilde{\text{tr}} \check{\chi}, r\nabla \check{\chi}, r\nabla \widetilde{\text{tr}} \check{\chi}, r\nabla \check{\chi}, \nabla \check{\gamma}, r\nabla^2 \check{\gamma}) + r\Gamma_g \cdot \Gamma_b^{(1)}, \\ r\nabla \check{L} &= \mathbf{O}_0^0 \cdot (r\nabla \widetilde{\text{tr}} \check{\chi}, r\nabla \check{\chi}, \nabla \check{\gamma}, r\nabla^2 \check{\gamma}) + r\Gamma_g \cdot \Gamma_g^{(1)}.\end{aligned}$$

Combining with Propositions 6.6 and 6.7, we deduce for  $q = 0, 1$

$$\begin{aligned}|r^{1-\frac{2}{p}}(r\nabla)^q \check{J}|_{p,S} &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}} + |r^{2-\frac{2}{p}}\Gamma_g \cdot \Gamma_b^{(1)}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-1}{2}}}, \\ |r^{2-\frac{2}{p}}(r\nabla)^q \check{L}|_{p,S} &\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}} + |r^{3-\frac{2}{p}}\Gamma_g \cdot \Gamma_g^{(1)}|_{p,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-3}{2}}}.\end{aligned}$$

This concludes the proof of Proposition 6.8.  $\square$

Combining Propositions 6.6, 6.7 and 6.8, this concludes the proof of Theorem M2.

### 6.3 Extension argument (Theorem M4)

In this section, we prove Theorem M4 which we recall below for convenience.

**Theorem M4.** *We consider the spacetime  $\mathcal{K}$  and its double null foliation  $(u, \underline{u})$  which satisfies the assumptions:*

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0, \quad \text{Osc} \lesssim \varepsilon_0, \quad \mathcal{O}^*(\underline{C}_*) \lesssim \varepsilon_0. \quad (6.31)$$

*Then, we can extend the spacetime  $\mathcal{K} = V(u_0, \underline{u}_*)$  and the double null foliation to a new spacetime  $\widehat{\mathcal{K}} = \widehat{V}(u_0, \underline{u}_* + \nu)$ , where  $\nu > 0$  is sufficiently small, and an associated double null foliation  $(\widehat{u}, \underline{u})$ . Moreover, the new foliation  $(\widehat{u}, \underline{u})$  is geodesic on the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$  and the new corresponding norms satisfy*

$$\widehat{\mathcal{O}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}} \lesssim \varepsilon_0, \quad \widehat{\text{Osc}} \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}^*(\underline{C}_{**}) \lesssim \varepsilon_0. \quad (6.32)$$

*Proof.* We define the region

$$\mathcal{A}_0 := \{p \in \Sigma_0 / w(p) \in [\underline{u}_*, \underline{u}_* + \nu]\}.$$

For  $\nu > 0$  small enough, we consider the future domain of dependence  $\mathcal{K}'_0$  of  $\mathcal{A}_0$ , with

$$\mathcal{K}'_0 := V'(u_*, \underline{u}_* + \nu) \subset \mathcal{K}'.$$

The boundary of  $\mathcal{K}'_0$  contains, see Figure 4.7:

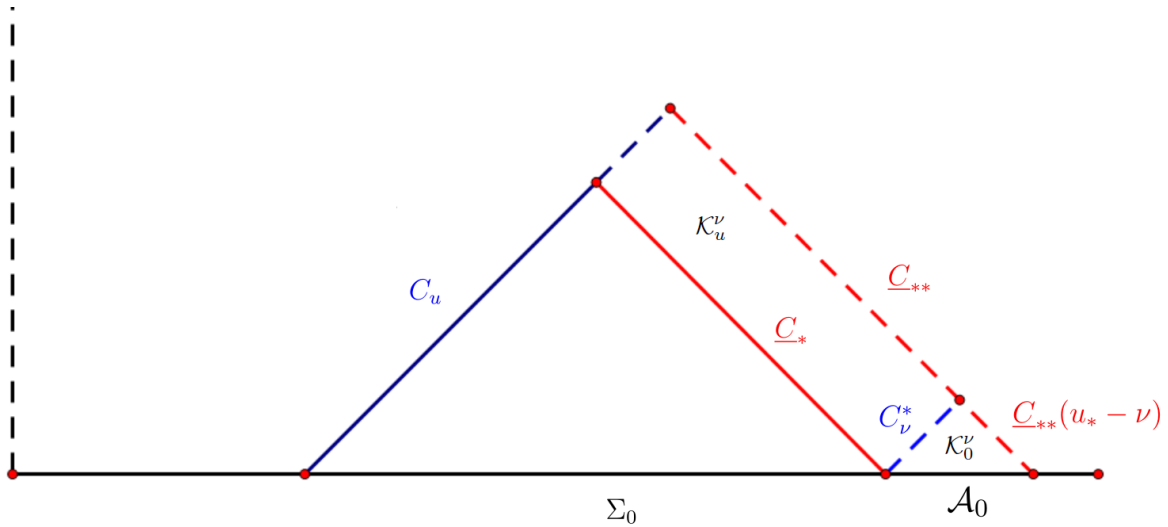


Figure 4.7: Extension argument

1. the part  $\mathcal{A}_0$  of  $\Sigma_0$ ;
2. a part of an ingoing cone denoted by

$$\underline{C}_{**}(u_* - \nu) := \underline{C}(u_* + \nu; [u_* - \nu, u_*]),$$

which is a part of the new last slice  $\underline{C}_{**} := \underline{C}_{\underline{u}_* + \nu}$ , where

$$u_* - \nu := u|_{\underline{C}_{\underline{u}_* + \nu} \cap \Sigma_0}, \quad u_* = u|_{\underline{C}_{\underline{u}_*} \cap \Sigma_0};$$

3. a part of the outgoing cone emanating from  $S_0(u_*) = \underline{C}_{\underline{u}_*} \cap \Sigma_0$  and in  $V'(u_*, \underline{u}_* + \nu)$ , denoted by  $C_\nu^*$ .

For the double null foliation  $(u', \underline{u})$  in  $\mathcal{K}_0^\nu$ , we have:

$$\mathcal{O}'(\mathcal{K}_0^\nu) \lesssim \varepsilon_0, \quad \mathcal{R}'(\mathcal{K}_0^\nu) \lesssim \varepsilon_0, \tag{6.33}$$

for  $\nu > 0$  small enough by local existence. By the local existence theorem for the characteristic Cauchy problem in [44], we can extend the spacetime  $\mathcal{K} \cup \mathcal{K}_0^\nu$  to the future domain of dependence of  $C_\nu^* \cup \underline{C}_*$ . Moreover, we extend the outgoing cones of  $\mathcal{K}$  to this new domain, construct the ingoing cones from  $C_\nu^*$  and denote  $(\tilde{u}, \underline{u})$  the global foliation in  $\tilde{\mathcal{K}} := \mathcal{K} \cup \mathcal{K}_u^\nu \cup \mathcal{K}_0^\nu$ . As a consequence of Theorem 1 of [44], we have

$$\tilde{\mathcal{O}}(\mathcal{K}_u^\nu) \lesssim \varepsilon_0, \quad \tilde{\mathcal{R}}(\mathcal{K}_u^\nu) \lesssim \varepsilon_0. \tag{6.34}$$

Combining (6.31), (6.33) and (6.34), we obtain

$$\tilde{\mathcal{O}}(\tilde{\mathcal{K}}) \lesssim \varepsilon_0, \quad \tilde{\mathcal{R}}(\tilde{\mathcal{K}}) \lesssim \varepsilon_0.$$

The foliation  $(\tilde{u}, \underline{u})$  on the new last slice  $\underline{C}_{**}$  of  $V(u_0, \underline{u}_* + \nu)$  is in general not of the geodesic type of Definition 6.1. Starting from  $S_{**} := \underline{C}_{**} \cap \Sigma_0$ , we construct a geodesic type foliation  $(\hat{u}, \underline{u}_*)$  on  $\underline{C}_{**}(-\underline{u}_* - \nu < \hat{u} < u_1)$ , where  $u_1$  is sufficiently close to  $-\underline{u}_* - \nu$ . We also denote

$$\widehat{V}(u_1) := \widehat{V}(u_1, \underline{u}_* + \nu) := J^-(\widehat{S}(u_1, \underline{u}_* + \nu)).$$



Proceeding as in Section 2.1.5, we can construct a double null foliation  $(\widehat{u}, \underline{u})$  in  $\widehat{V}(u_1)$ . Let  $U$  the set of  $u_1 \leq u_0$  such that we have the following estimates:

$$\widehat{\mathcal{O}}(\widehat{V}(u_1)) \leq \varepsilon, \quad \widehat{\mathcal{R}}(\widehat{V}(u_1)) \leq \varepsilon, \quad \widehat{\mathcal{Osc}}(\widehat{V}(u_1)) \leq \varepsilon, \quad \widehat{\mathcal{O}}^*(\underline{\mathcal{C}}_{**} \cap \widehat{V}(u_1)) \leq \varepsilon. \quad (6.35)$$

We also denote

$$u_2 := \sup U. \quad (6.36)$$

Assume that  $u_2 < u_0$ . Applying Theorems M0–M3 in  $\widehat{V}(u_2)$ , we obtain

$$\widehat{\mathcal{O}}(\widehat{V}(u_2)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}}(\widehat{V}(u_2)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{Osc}}(\widehat{V}(u_2)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}^*(\underline{\mathcal{C}}_{**} \cap \widehat{V}(u_2)) \lesssim \varepsilon_0.$$

Thus, by the continuity of the norms, we obtain  $u_2 + \delta \in U$  for  $\delta > 0$  small enough, a contradiction with (6.36). Hence, we have  $u_2 = u_0$  which implies

$$\widehat{\mathcal{O}}(\widehat{V}(u_0)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{R}}(\widehat{V}(u_0)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{Osc}}(\widehat{V}(u_0)) \lesssim \varepsilon_0, \quad \widehat{\mathcal{O}}^*(\underline{\mathcal{C}}_{**} \cap \widehat{V}(u_0)) \lesssim \varepsilon_0.$$

This concludes the proof of Theorem M4.  $\square$

## A Proof of results in Section 2.8

### A.1 Proof of Proposition 2.34

We prove below a more precise version of Proposition 2.34.

**Proposition A.1.** *We have the following linearized null structure equations:*

$$\begin{aligned} \Omega \nabla_4 \widetilde{\Omega \text{tr } \chi} + (\Omega \text{tr } \chi) \widetilde{\Omega \text{tr } \chi} &= -4\Omega \text{tr } \chi \widetilde{\Omega \omega} + \mathfrak{L}[\nabla_4 \widetilde{\text{tr } \chi}] + \text{Err}[\nabla_4 \widetilde{\text{tr } \chi}], \\ \mathfrak{L}[\nabla_4 \widetilde{\text{tr } \chi}] &:= -2\widetilde{\Omega \chi}(\Omega_K \widehat{\chi}_K) - 4\Omega_K \omega_K \widetilde{\Omega \text{tr } \chi}, \\ \text{Err}[\nabla_4 \widetilde{\text{tr } \chi}] &:= \frac{1}{2}(\widetilde{\Omega \text{tr } \chi})^2 - |\widetilde{\Omega \chi}|^2, \\ \\ \Omega \nabla_3 \widetilde{\Omega \text{tr } \underline{\chi}} + (\Omega \text{tr } \underline{\chi}) \widetilde{\Omega \text{tr } \underline{\chi}} &= -4\Omega \text{tr } \underline{\chi} \widetilde{\Omega \omega} + \mathfrak{L}[\nabla_3 \widetilde{\text{tr } \underline{\chi}}] + \text{Err}[\nabla_3 \widetilde{\text{tr } \underline{\chi}}], \\ \mathfrak{L}[\nabla_3 \widetilde{\text{tr } \underline{\chi}}] &:= -2\widetilde{\Omega \underline{\chi}}(\Omega_K \widehat{\underline{\chi}}_K) - 4\Omega_K \omega_K \widetilde{\Omega \text{tr } \underline{\chi}} - \check{b}^A \partial_A(\Omega_K \text{tr}_K \underline{\chi}_K), \\ \text{Err}[\nabla_3 \widetilde{\text{tr } \underline{\chi}}] &:= \frac{1}{2}(\widetilde{\Omega \text{tr } \underline{\chi}})^2 - |\widetilde{\Omega \underline{\chi}}|^2, \\ \\ \Omega \nabla_4(\widetilde{\Omega \text{tr } \chi}) &= -2\widetilde{\Omega^2[\mu]} + \mathfrak{L}[\nabla_4(\widetilde{\text{tr } \chi})] + \text{Err}[\nabla_4(\widetilde{\text{tr } \chi})], \\ \mathfrak{L}[\nabla_4(\widetilde{\text{tr } \chi})] &:= 4\Omega_K \eta_K(\widetilde{\Omega \underline{\eta}}), \\ \text{Err}[\nabla_4(\widetilde{\text{tr } \chi})] &:= 2|\widetilde{\Omega \underline{\eta}}|^2, \\ \\ \Omega \nabla_3(\widetilde{\Omega \text{tr } \chi}) &= -2\widetilde{\Omega^2[\mu]} + \mathfrak{L}[\nabla_3(\widetilde{\text{tr } \chi})] + \text{Err}[\nabla_3(\widetilde{\text{tr } \chi})], \\ \mathfrak{L}[\nabla_3(\widetilde{\text{tr } \chi})] &:= 4\Omega_K \eta_K(\widetilde{\Omega \underline{\eta}}) + \check{b}^A \partial_A(\Omega_K \text{tr}_K \chi_K), \\ \text{Err}[\nabla_3(\widetilde{\text{tr } \chi})] &:= 2|\widetilde{\Omega \underline{\eta}}|^2. \end{aligned}$$

We also have the linearized Codazzi equations:

$$\begin{aligned}\operatorname{div}(\check{\chi}) &= \frac{1}{2}\nabla\widetilde{\operatorname{tr}\chi} + \frac{1}{2}\operatorname{tr}\chi\check{\zeta} - \check{b} + \mathfrak{L}[\operatorname{div}(\check{\chi})] + \operatorname{Err}[\operatorname{div}(\check{\chi})], \\ \mathfrak{L}[\operatorname{div}(\check{\chi})] &:= \frac{1}{2}\widetilde{\operatorname{tr}\chi}\zeta_K - \check{\zeta}\cdot\widehat{\chi}_K - \zeta_K\cdot\check{\chi} + (\gamma^{AB} - (\gamma_K)^{AB})\nabla_A(\widehat{\chi}_K)_{B\bullet} - \gamma_K^{AB}\check{L}_{AB}^C(\widehat{\chi}_K)_{C\bullet}, \\ \operatorname{Err}[\operatorname{div}(\check{\chi})] &:= -\check{\zeta}\cdot\check{\chi}, \\ \operatorname{div}(\check{\underline{\chi}}) &= \frac{1}{2}\nabla\widetilde{\operatorname{tr}\underline{\chi}} - \frac{1}{2}\operatorname{tr}\underline{\chi}\check{\zeta} + \check{\underline{\beta}} + \mathfrak{L}[\operatorname{div}(\check{\underline{\chi}})] + \operatorname{Err}[\operatorname{div}(\check{\underline{\chi}})], \\ \mathfrak{L}[\operatorname{div}(\check{\underline{\chi}})] &:= -\frac{1}{2}\widetilde{\operatorname{tr}\underline{\chi}}\zeta_K + \check{\zeta}\cdot\widehat{\underline{\chi}}_K + \zeta_K\cdot\check{\underline{\chi}} + (\gamma^{AB} - (\gamma_K)^{AB})\nabla_A(\widehat{\underline{\chi}}_K)_{B\bullet} - \gamma_K^{AB}\check{L}_{AB}^C(\widehat{\underline{\chi}}_K)_{C\bullet}, \\ \operatorname{Err}[\operatorname{div}(\check{\underline{\chi}})] &:= \check{\zeta}\cdot\check{\underline{\chi}}.\end{aligned}$$

The linearized mass aspect functions are given by:

$$\begin{aligned}\operatorname{div}\check{\eta} &= -\check{\mu} - \check{\rho} + \mathfrak{L}[\operatorname{div}(\check{\eta})] + \operatorname{Err}[\operatorname{div}(\check{\eta})], \\ \mathfrak{L}[\operatorname{div}(\check{\eta})] &= (\gamma^{AB} - (\gamma_K)^{AB})\nabla_A(\eta_K)_B - (\gamma_K)^{AB}\check{L}_{AB}^C(\eta_K)_C + \frac{1}{2}\widehat{\chi}_K\cdot\check{\underline{\chi}} + \frac{1}{2}\check{\underline{\chi}}\cdot\widehat{\underline{\chi}}_K, \\ \operatorname{Err}[\operatorname{div}(\check{\eta})] &= \frac{1}{2}\widehat{\chi}\cdot\check{\underline{\chi}},\end{aligned}$$

$$\begin{aligned}\operatorname{div}\check{\underline{\eta}} &= -\check{\underline{\mu}} - \check{\underline{\rho}} + \mathfrak{L}[\operatorname{div}(\check{\underline{\eta}})] + \operatorname{Err}[\operatorname{div}(\check{\underline{\eta}})], \\ \mathfrak{L}[\operatorname{div}(\check{\underline{\eta}})] &= (\gamma^{AB} - (\gamma_K)^{AB})\nabla_A(\underline{\eta}_K)_B - (\gamma_K)^{AB}\check{L}_{AB}^C(\underline{\eta}_K)_C + \frac{1}{2}\widehat{\underline{\chi}}_K\cdot\check{\underline{\chi}} + \frac{1}{2}\check{\underline{\chi}}\cdot\widehat{\underline{\chi}}_K, \\ \operatorname{Err}[\operatorname{div}(\check{\underline{\eta}})] &= \frac{1}{2}\widehat{\underline{\chi}}\cdot\check{\underline{\chi}}.\end{aligned}$$

The linearized torsion equations are given by:

$$\begin{aligned}\operatorname{curl}\check{\eta} &= \check{\sigma} + \mathfrak{L}[\operatorname{curl}\check{\eta}] + \operatorname{Err}[\operatorname{curl}\check{\eta}], \\ \mathfrak{L}[\operatorname{curl}\check{\eta}] &:= \frac{1}{2}\check{\underline{\chi}}\wedge\widehat{\chi}_K + \frac{1}{2}\widehat{\underline{\chi}}_K\wedge\check{\underline{\chi}} - (\epsilon^{AB} - \epsilon_K^{AB})\nabla_A(\eta_K)_B + (\epsilon_K)^{AB}\check{L}_{AB}^C(\eta_K)_C, \\ \operatorname{Err}[\operatorname{curl}\check{\eta}] &:= \frac{1}{2}\check{\underline{\chi}}\wedge\check{\underline{\chi}}, \\ \operatorname{curl}\check{\underline{\eta}} &= -\check{\underline{\sigma}} + \mathfrak{L}[\operatorname{curl}\check{\underline{\eta}}] + \operatorname{Err}[\operatorname{curl}\check{\underline{\eta}}], \\ \mathfrak{L}[\operatorname{curl}\check{\underline{\eta}}] &:= -\frac{1}{2}\check{\underline{\chi}}\wedge\widehat{\chi}_K - \frac{1}{2}\widehat{\underline{\chi}}_K\wedge\check{\underline{\chi}} - (\epsilon^{AB} - \epsilon_K^{AB})\nabla_A(\underline{\eta}_K)_B + (\epsilon_K)^{AB}\check{L}_{AB}^C(\underline{\eta}_K)_C, \\ \operatorname{Err}[\operatorname{curl}\check{\underline{\eta}}] &:= -\frac{1}{2}\check{\underline{\chi}}\wedge\check{\underline{\chi}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\Omega\nabla_4(\check{\underline{\omega}}) &= \frac{1}{2}\check{\underline{\omega}}^2\rho + \mathfrak{L}[\nabla_4\check{\underline{\omega}}] + \operatorname{Err}[\nabla_4\check{\underline{\omega}}], \\ \mathfrak{L}[\nabla_4\check{\underline{\omega}}] &:= \frac{3}{2}(\check{\underline{\omega}} - \check{\underline{\omega}})\cdot(\Omega_K\eta_K - \Omega_K\underline{\eta}_K) - \frac{1}{4}(\check{\underline{\omega}} - \check{\underline{\omega}})\cdot(\Omega_K\eta_K + \Omega_K\underline{\eta}_K) \\ &\quad - \frac{1}{4}(\Omega_K\eta_K - \Omega_K\underline{\eta}_K)\cdot(\check{\underline{\omega}} + \check{\underline{\omega}}) - \frac{1}{4}(\check{\underline{\omega}} + \check{\underline{\omega}})\cdot(\eta_K + \underline{\eta}_K), \\ \operatorname{Err}[\nabla_4\check{\underline{\omega}}] &:= \frac{3}{4}|\check{\underline{\omega}} - \check{\underline{\omega}}|^2 - \frac{1}{4}(\check{\underline{\omega}} - \check{\underline{\omega}})\cdot(\check{\underline{\omega}} + \check{\underline{\omega}}) - \frac{1}{8}|\check{\underline{\omega}} + \check{\underline{\omega}}|^2,\end{aligned}$$

$$\begin{aligned}
\Omega \nabla_3(\widetilde{\Omega\omega}) &= \frac{1}{2}\widetilde{\Omega^2\rho} + \mathfrak{L}[\nabla_3\widetilde{\omega}] + \text{Err}[\nabla_3\widetilde{\omega}], \\
\mathfrak{L}[\nabla_3\widetilde{\omega}] &:= \frac{3}{2}(\widetilde{\Omega\eta} - \widetilde{\Omega\underline{\eta}}) \cdot (\Omega_K\eta_K - \Omega_K\underline{\eta}_K) + \frac{1}{4}(\widetilde{\Omega\eta} - \widetilde{\Omega\underline{\eta}}) \cdot (\Omega_K\eta_K + \Omega_K\underline{\eta}_K) \\
&\quad + \frac{1}{4}(\Omega_K\eta_K - \Omega_K\underline{\eta}_K) \cdot (\widetilde{\Omega\eta} + \widetilde{\Omega\underline{\eta}}) - \frac{1}{4}(\widetilde{\Omega\eta} + \widetilde{\Omega\underline{\eta}}) \cdot (\eta_K + \underline{\eta}_K) + \check{b}^A \nabla_{\partial_A}(\Omega_K\omega_K), \\
\text{Err}[\nabla_3\widetilde{\omega}] &:= \frac{3}{4}|\widetilde{\Omega\eta} - \widetilde{\Omega\underline{\eta}}|^2 + \frac{1}{4}(\widetilde{\Omega\eta} - \widetilde{\Omega\underline{\eta}}) \cdot (\widetilde{\Omega\eta} + \widetilde{\Omega\underline{\eta}}) - \frac{1}{8}|\widetilde{\Omega\eta} + \widetilde{\Omega\underline{\eta}}|^2.
\end{aligned}$$

**Remark A.2.** Combining Propositions A.1 and 2.28 and Definitions 2.3 and 2.25, we obtain immediately Proposition 2.34.

*Proof of Proposition 2.34.* We recall from Corollary 2.18

$$\Omega \nabla_4(\Omega \text{tr } \chi) + \frac{1}{2}(\Omega \text{tr } \chi)^2 = -\Omega^2|\widehat{\chi}|^2 - 4\Omega\omega(\Omega \text{tr } \chi),$$

which implies

$$\Omega_K \nabla_{(e_4)_K}(\Omega_K \text{tr}_K \chi_K) + \frac{1}{2}(\Omega_K \text{tr}_K \chi_K)^2 = -\Omega_K^2|\widehat{\chi}_K|^2 - 4\Omega_K\omega_K(\Omega_K \text{tr}_K \chi_K).$$

Taking the difference, we infer

$$\begin{aligned}
\Omega \nabla_4(\widetilde{\Omega \text{tr } \chi}) &= -\frac{1}{2}(\Omega \text{tr } \chi)^2 - \Omega^2|\widehat{\chi}|^2 - 4\Omega\omega(\Omega \text{tr } \chi) + \frac{1}{2}(\Omega_K \text{tr}_K \chi_K)^2 + \Omega_K^2|\widehat{\chi}_K|^2 + 4\Omega_K\omega_K(\text{tr}_K \chi_K) \\
&= -\Omega \text{tr } \chi \widetilde{\Omega \text{tr } \chi} + \frac{1}{2}\widetilde{\Omega \text{tr } \chi}^2 - 2\widetilde{\Omega\widehat{\chi}}(\Omega_K\widehat{\chi}_K) - |\widetilde{\Omega\widehat{\chi}}|^2 - 4\Omega_K\omega_K\widetilde{\Omega \text{tr } \chi} \\
&\quad - 4\Omega_K \text{tr}_K \chi_K \widetilde{\Omega\omega} - 4\widetilde{\Omega\omega} \widetilde{\Omega \text{tr } \chi}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Omega \nabla_3(\widetilde{\Omega \text{tr } \underline{\chi}}) &= -\Omega \text{tr } \underline{\chi} \widetilde{\Omega \text{tr } \underline{\chi}} + \frac{1}{2}\widetilde{\Omega \text{tr } \underline{\chi}}^2 - 2(\widetilde{\Omega\widehat{\chi}})\Omega_K\widehat{\chi}_K - |\widetilde{\Omega\widehat{\chi}}|^2 - 4\Omega_K\omega_K\widetilde{\Omega \text{tr } \underline{\chi}} \\
&\quad - 4\Omega_K \text{tr}_K \underline{\chi}_K \widetilde{\Omega\omega} - 4\widetilde{\Omega\omega} \widetilde{\Omega \text{tr } \underline{\chi}} - \check{b}^A \partial_A(\Omega_K \text{tr}_K \underline{\chi}_K).
\end{aligned}$$

Next, we have from Corollary 2.18

$$\Omega \nabla_4(\Omega \text{tr } \underline{\chi}) = -2\Omega^2[\underline{\mu}] + 2\Omega^2|\underline{\eta}|^2,$$

which implies

$$\Omega_K \nabla_{(e_4)_K}(\Omega_K \text{tr}_K \underline{\chi}_K) = -2\Omega_K^2[\underline{\mu}]_K + 2\Omega_K^2|\underline{\eta}_K|^2.$$

Taking the difference, we infer

$$\Omega \nabla_4(\widetilde{\Omega \text{tr } \underline{\chi}}) = -2\widetilde{\Omega^2[\underline{\mu}]} + 4\Omega_K\eta_K(\widetilde{\Omega\underline{\eta}}) + 2|\widetilde{\Omega\underline{\eta}}|^2.$$

Similarly, we have

$$\Omega \nabla_3(\widetilde{\Omega \text{tr } \chi}) = -2\widetilde{\Omega^2[\underline{\mu}]} + 4\Omega_K\eta_K(\widetilde{\Omega\underline{\eta}}) + 2|\widetilde{\Omega\underline{\eta}}|^2 + \check{b}^A \partial_A(\Omega_K \text{tr}_K \chi_K).$$

Now we consider the Codazzi equation

$$\text{div } \widehat{\chi} = \frac{1}{2}(\nabla \text{tr } \chi + \text{tr } \chi \zeta) - \zeta \cdot \widehat{\chi} - \beta,$$

which implies

$$\operatorname{div}_K \widehat{\chi}_K = \frac{1}{2}(\nabla_K \operatorname{tr}_K \chi_K + \operatorname{tr}_K \chi_K \zeta_K) - \zeta_K \cdot \widehat{\chi}_K - \beta_K.$$

Taking the difference, we obtain from Proposition 2.22

$$\begin{aligned} \operatorname{div}(\widetilde{\chi}) &= \operatorname{div} \widehat{\chi} - \operatorname{div} \widehat{\chi}_K \\ &= \operatorname{div} \widehat{\chi} - \operatorname{div}_K \widehat{\chi}_K - (\operatorname{div} - \operatorname{div}_K) \widehat{\chi}_K \\ &= \frac{1}{2} \nabla \widetilde{\operatorname{tr} \chi} + \frac{1}{2} \operatorname{tr} \chi \check{\zeta} + \frac{1}{2} \widetilde{\operatorname{tr} \chi} \zeta_K - \check{\zeta} \cdot \widehat{\chi}_K - \zeta_K \cdot \check{\chi} - \check{\zeta} \cdot \check{\chi} - \check{b} - \widetilde{\gamma^{AB}} \nabla_A (\widehat{\chi}_K)_B \\ &\quad - \gamma_K^{AB} \check{L}_{AB}^D (\widehat{\chi}_K)_{\bullet D} - \gamma_K^{AB} \check{L}_{A\bullet}^D (\widehat{\chi}_K)_{BD}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \operatorname{div}(\underline{\check{\chi}}) &= \frac{1}{2} \nabla \widetilde{\operatorname{tr} \underline{\check{\chi}}} - \frac{1}{2} \operatorname{tr} \underline{\check{\chi}} \check{\zeta} - \frac{1}{2} \widetilde{\operatorname{tr} \underline{\check{\chi}}} \zeta_K + \check{\zeta} \cdot \widehat{\chi}_K + \zeta_K \cdot \check{\chi} + \check{\zeta} \cdot \check{\chi} + \check{\beta} - \widetilde{\gamma^{AB}} \nabla_A (\widehat{\chi}_K)_B \\ &\quad - \gamma_K^{AB} \check{L}_{AB}^D (\widehat{\chi}_K)_{\bullet D} - \gamma_K^{AB} \check{L}_{A\bullet}^D (\widehat{\chi}_K)_{BD}. \end{aligned}$$

Next, we recall from (2.28)

$$\mu = -\operatorname{div} \eta + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} - \rho.$$

Hence, we have

$$\mu_K = -\operatorname{div}_K \eta_K + \frac{1}{2} \widehat{\chi}_K \cdot \widehat{\chi}_K - \rho_K.$$

Taking the difference, we obtain

$$\begin{aligned} \check{\mu} &= -\operatorname{div} \check{\eta} + (\operatorname{div} - \operatorname{div}_K) \eta_K + \frac{1}{2} \widehat{\chi}_K \cdot \widehat{\chi} + \frac{1}{2} \check{\chi} \cdot \widehat{\chi}_K + \frac{1}{2} \widehat{\chi} \cdot \check{\chi} - \check{\rho} \\ &= -\operatorname{div} \check{\eta} - \check{\rho} + (\gamma^{AB} - (\gamma_K)^{AB}) \nabla_A (\eta_K)_B - (\gamma_K)^{AB} \check{L}_{AB}^C (\eta_K)_C \\ &\quad + \frac{1}{2} \widehat{\chi}_K \cdot \widehat{\chi} + \frac{1}{2} \check{\chi} \cdot \widehat{\chi}_K + \frac{1}{2} \widehat{\chi} \cdot \check{\chi}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \underline{\check{\mu}} &= -\operatorname{div} \underline{\check{\eta}} - \check{\rho} + (\gamma^{AB} - (\gamma_K)^{AB}) \nabla_A (\underline{\eta}_K)_B - (\gamma_K)^{AB} \check{L}_{AB}^C (\underline{\eta}_K)_C \\ &\quad + \frac{1}{2} \widehat{\chi}_K \cdot \widehat{\chi} + \frac{1}{2} \check{\chi} \cdot \widehat{\chi}_K + \frac{1}{2} \widehat{\chi} \cdot \check{\chi}. \end{aligned}$$

We consider the torsion equation

$$\operatorname{curl} \eta = -\operatorname{curl} \underline{\eta} = \sigma + \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi},$$

which implies

$$\operatorname{curl}_K \eta_K = -\operatorname{curl}_K \underline{\eta}_K = \sigma_K + \frac{1}{2} \widehat{\chi}_K \wedge \widehat{\chi}_K.$$

Taking the difference, we infer from Proposition 2.22 that

$$\begin{aligned} \operatorname{curl} \check{\eta} &= \operatorname{curl} \eta - \operatorname{curl} \eta_K = \operatorname{curl} \eta - \operatorname{curl}_K \eta_K - (\operatorname{curl} - \operatorname{curl}_K) \eta_K \\ &= \check{\sigma} + \frac{1}{2} \check{\chi} \wedge \check{\chi} + \frac{1}{2} \check{\chi} \wedge \widehat{\chi}_K + \frac{1}{2} \widehat{\chi}_K \wedge \check{\chi} - \widetilde{\epsilon^{AB}} \nabla_A (\eta_K)_B + \epsilon_K^{AB} \check{L}_{AB}^C (\eta_K)_C. \end{aligned}$$

Similarly, we have

$$-\operatorname{curl} \check{\eta} = \check{\sigma} + \frac{1}{2} \check{\chi} \wedge \check{\chi} + \frac{1}{2} \check{\chi} \wedge \widehat{\chi}_K + \frac{1}{2} \widehat{\chi}_K \wedge \check{\chi} + \widetilde{\epsilon}^{AB} \nabla_A (\underline{\eta}_K)_B - \epsilon_K^{AB} \check{L}_{AB}^C (\underline{\eta}_K)_C.$$

Finally, we recall from Proposition 2.16

$$\Omega \nabla_4 (\underline{\Omega} \omega) = \frac{1}{2} \Omega^2 \rho + \frac{3}{4} |\Omega \eta - \Omega \underline{\eta}|^2 - \frac{1}{4} (\Omega \eta - \Omega \underline{\eta}) \cdot (\Omega \eta + \Omega \underline{\eta}) - \frac{1}{8} |\Omega \eta + \Omega \underline{\eta}|^2,$$

which implies

$$\begin{aligned} \Omega_K \nabla_{(e_4)_K} \omega_K &= \frac{1}{2} \Omega_K^2 \rho_K + \frac{3}{4} |\Omega_K \eta_K - \Omega_K \underline{\eta}_K|^2 - \frac{1}{4} (\Omega_K \eta_K - \Omega_K \underline{\eta}_K) \cdot (\Omega_K \eta_K + \Omega_K \underline{\eta}_K) \\ &\quad - \frac{1}{8} |\Omega_K \eta_K + \Omega_K \underline{\eta}_K|^2. \end{aligned}$$

Taking the difference, we obtain

$$\begin{aligned} \Omega \nabla_4 (\widetilde{\Omega} \omega) &= \frac{3}{4} \Omega^2 |\eta - \underline{\eta}|^2 - \frac{1}{4} \Omega^2 (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8} \Omega^2 |\eta + \underline{\eta}|^2 + \frac{1}{2} \Omega^2 \rho \\ &\quad - \frac{3}{4} \Omega_K^2 |\eta_K - \underline{\eta}_K|^2 + \frac{1}{4} \Omega_K^2 (\eta_K - \underline{\eta}_K) \cdot (\eta_K + \underline{\eta}_K) + \frac{1}{8} \Omega_K^2 |\eta_K + \underline{\eta}_K|^2 - \frac{1}{2} \Omega_K^2 \rho_K \\ &= \frac{3}{4} |\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}|^2 + \frac{3}{2} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\Omega_K \eta_K - \Omega_K \underline{\eta}_K) - \frac{1}{4} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) \\ &\quad - \frac{1}{4} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\Omega_K \eta_K + \Omega_K \underline{\eta}_K) - \frac{1}{4} (\Omega_K \eta_K - \Omega_K \underline{\eta}_K) \cdot (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) - \frac{1}{8} |\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}|^2 \\ &\quad - \frac{1}{4} (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) \cdot (\eta_K + \underline{\eta}_K) + \frac{1}{2} \widetilde{\Omega}^2 \rho. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \Omega \nabla_3 (\widetilde{\Omega} \omega) &= \frac{3}{4} |\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}|^2 + \frac{3}{2} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\Omega_K \eta_K - \Omega_K \underline{\eta}_K) + \frac{1}{4} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) \\ &\quad + \frac{1}{4} (\widetilde{\Omega} \eta - \widetilde{\Omega} \underline{\eta}) \cdot (\Omega_K \eta_K + \Omega_K \underline{\eta}_K) + \frac{1}{4} (\Omega_K \eta_K - \Omega_K \underline{\eta}_K) \cdot (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) - \frac{1}{8} |\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}|^2 \\ &\quad - \frac{1}{4} (\widetilde{\Omega} \eta + \widetilde{\Omega} \underline{\eta}) \cdot (\eta_K + \underline{\eta}_K) + \frac{1}{2} \widetilde{\Omega}^2 \rho + \check{b}^A \nabla_{\partial_{x^a}} (\Omega_K \omega_K). \end{aligned}$$

This concludes the proof of Proposition A.1.  $\square$

## A.2 Proof of Proposition 2.36

We recall from Corollary 2.18 that

$$\begin{aligned} \Omega \nabla_4 [\mu] + \Omega \operatorname{tr} \chi [\mu] &= \Omega [F] + \Omega (\operatorname{tr} \chi \rho - 2\eta \cdot \beta), \\ [F] &:= \widehat{\chi} \cdot (\nabla \widehat{\otimes} \eta) + (\eta - \underline{\eta}) \cdot (\nabla \operatorname{tr} \chi + \operatorname{tr} \chi \zeta) + \frac{1}{2} \operatorname{tr} \chi (|\eta|^2 - |\underline{\eta}|^2) \\ &\quad - \frac{1}{2} (\operatorname{tr} \chi |\widehat{\chi}|^2 + \operatorname{tr} \chi (\widehat{\chi} \cdot \widehat{\chi})) + 2\eta \cdot \widehat{\chi} \cdot \underline{\eta}, \end{aligned}$$

which implies

$$\Omega_K (\nabla_K)_{(e_4)_K} [\mu]_K + \Omega_K \operatorname{tr}_K \chi_K [\mu]_K = \Omega_K [F]_K + \Omega_K (\operatorname{tr} \chi_K \rho_K - 2\eta_K \cdot \beta_K).$$

Taking the difference, we obtain

$$\begin{aligned} \Omega \nabla_4 [\widetilde{\mu}] + \Omega \operatorname{tr} \chi [\widetilde{\mu}] &= \widetilde{\Omega} [\widetilde{F}] + \Omega (\operatorname{tr} \chi \rho - 2\eta \cdot \beta) - \Omega_K (\operatorname{tr} \chi_K \rho_K - 2\eta_K \cdot \beta_K) \\ &= \widetilde{\Omega} [\widetilde{F}] + \widetilde{\Omega} \operatorname{tr} \chi \rho_K + \Omega \operatorname{tr} \chi \check{\rho} - 2\widetilde{\Omega} \eta \cdot \beta_K - 2\Omega_K \eta_K \cdot \check{b} - 2\widetilde{\Omega} \eta \cdot \check{b} \\ &= \Omega \operatorname{tr} \chi \check{\rho} + \widetilde{\Omega} [\widetilde{F}] + \mathbf{O}_3^1 \cdot \Gamma_g + \mathbf{O}_3^2 \cdot \check{b} - 2\widetilde{\Omega} \eta \cdot \check{b}. \end{aligned}$$

Next, we compute

$$\begin{aligned}\Omega[F] &= -2\widehat{\chi} \cdot d_2^* \eta + (\eta - \underline{\eta}) \cdot \nabla \operatorname{tr} \chi + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_1^0 + \Gamma_g) \cdot (\mathbf{O}_3^2 + \Gamma_g) \\ &\quad + (\mathbf{O}_1^0 + \Gamma_g) \cdot (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^2 + \Gamma_b) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^2 + \Gamma_g) \\ &= -2\widehat{\chi} \cdot d_2^* \eta + (\eta - \underline{\eta}) \cdot \nabla \operatorname{tr} \chi + \mathbf{O}_7^4 + \mathbf{O}_4^2 \cdot \Gamma_b + r^{-1} \Gamma_g \cdot \Gamma_b.\end{aligned}$$

Hence, we deduce from Proposition 2.22 that

$$\begin{aligned}\widetilde{\Omega}[F] &= -2\widetilde{\chi} \cdot (d_2^*)_K \eta_K - 2\widehat{\chi}_K \cdot \widetilde{d}_2^* \eta - 2\widetilde{\chi} \cdot \widetilde{d}_2^* \eta + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla_K \operatorname{tr}_K \chi_K + (\eta_K - \underline{\eta}_K) \cdot \nabla \operatorname{tr} \widetilde{\chi} \\ &\quad + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \widetilde{\chi} + \mathbf{O}_4^2 \cdot \Gamma_b + r^{-1} \Gamma_g \cdot \Gamma_b \\ &= \mathbf{O}_3^2 \cdot \widetilde{d}_2^* \eta - 2\widetilde{\chi} \cdot \widetilde{d}_2^* \eta + \mathbf{O}_3^2 \cdot \nabla \operatorname{tr} \widetilde{\chi} + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \widetilde{\chi} + \mathbf{O}_4^2 \cdot \Gamma_b + r^{-1} \Gamma_g \cdot \Gamma_b \\ &= \mathbf{O}_3^2 \cdot (d_2^* \widetilde{\eta} + \check{L} \cdot \mathbf{O}_3^2) - 2\widetilde{\chi} \cdot (d_2^* \widetilde{\eta} + \check{L} \cdot \mathbf{O}_3^2) + \mathbf{O}_3^2 \cdot \nabla \operatorname{tr} \widetilde{\chi} + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \widetilde{\chi} + \mathbf{O}_4^2 \cdot \Gamma_b + r^{-1} \Gamma_g \cdot \Gamma_b \\ &= \mathbf{O}_3^2 \cdot (d_2^* \widetilde{\eta}) - 2\widetilde{\chi} \cdot (d_2^* \widetilde{\eta}) + \mathbf{O}_6^4 \cdot \check{L} + r^{-1} \Gamma_g \cdot \check{L} + \mathbf{O}_3^2 \cdot \nabla \operatorname{tr} \widetilde{\chi} + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \widetilde{\chi} + \mathbf{O}_4^2 \cdot \Gamma_b + r^{-1} \Gamma_g \cdot \Gamma_b.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\Omega \nabla_4 \widetilde{\mu} + \Omega \operatorname{tr} \chi \widetilde{\mu} &= \Omega \operatorname{tr} \chi \check{\rho} + \mathbf{O}_3^2 \cdot (d_2^* \widetilde{\eta}, \nabla \operatorname{tr} \widetilde{\chi}, \check{b}) + \mathbf{O}_3^1 \cdot (\Gamma_g, \check{L}) - 2\widetilde{\chi} \cdot (d_2^* \widetilde{\eta}) + r^{-1} \Gamma_g \cdot \check{L} \\ &\quad + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \widetilde{\chi} + r^{-1} \Gamma_g \cdot \Gamma_b - 2\widetilde{\Omega} \widetilde{\eta} \cdot \check{b}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Omega \nabla_3 \widetilde{\mu} + \Omega \operatorname{tr} \underline{\chi} \widetilde{\mu} &= \Omega \operatorname{tr} \underline{\chi} \check{\rho} + \mathbf{O}_3^2 \cdot (d_2^* \widetilde{\eta}, \nabla \operatorname{tr} \underline{\chi}, \check{\beta}) + \mathbf{O}_3^1 \cdot (\Gamma_g, \check{L}) - 2\widetilde{\chi} \cdot (d_2^* \widetilde{\eta}) + r^{-1} \Gamma_b \cdot \check{L} \\ &\quad + (\widetilde{\eta} - \underline{\widetilde{\eta}}) \cdot \nabla \operatorname{tr} \underline{\chi} + r^{-1} \Gamma_b \cdot \Gamma_b + 2\widetilde{\Omega} \widetilde{\eta} \cdot \check{\beta}.\end{aligned}$$

This concludes the proof of Proposition 2.36.

### A.3 Proof of Propositions 2.37 and 2.38

We first prove the following lemma.

**Lemma A.3.** *Let  $k, k' = 0, 1, 2$ ,  $\lambda_0$  a real number and  $\nabla$  a first order elliptic operator. Then, we have the following properties:*

1. Assume that  $\psi, h \in \mathfrak{s}_k$  and  $\psi' \in \mathfrak{s}_{k'}$  satisfy the following equation:

$$\nabla_3 \psi + \lambda_0 \operatorname{tr} \underline{\chi} \psi = \nabla \psi' + h. \quad (\text{A.1})$$

Then, we have

$$\nabla_3 \not\partial \psi + \lambda_0 \operatorname{tr} \underline{\chi} \not\partial \psi = \nabla \not\partial \psi' + h^{[1]}, \quad (\text{A.2})$$

where

$$h^{[1]} = \not\partial h + (\Gamma_g + \mathbf{O}_3^2) h + (\Gamma_g + \mathbf{O}_3^2) \cdot \psi'^{(1)} + (\Gamma_b + \mathbf{O}_3^2) \cdot \psi^{(1)} + \Gamma_b^{(1)} \cdot \psi.$$

2. Assume that  $\psi, h \in \mathfrak{s}_k$  and  $\psi' \in \mathfrak{s}_{k'}$  satisfy the following equation:

$$\nabla_4 \psi + \lambda_0 \operatorname{tr} \chi \psi = \nabla \psi' + h. \quad (\text{A.3})$$

Then, we have

$$\nabla_4 \not\partial \psi + \lambda_0 \operatorname{tr} \chi \not\partial \psi = \nabla \not\partial \psi' + h^{[1]}, \quad (\text{A.4})$$

where

$$h^{[1]} = \not\partial h + (\Gamma_g + \mathbf{O}_3^2) h + (\Gamma_g + \mathbf{O}_3^2) \cdot (\psi'^{(1)}, \psi^{(1)}) + \Gamma_g^{(1)} \cdot \psi.$$

*Proof.* We have from (A.1)

$$\Omega \nabla_3 \psi + \lambda_0 \Omega \operatorname{tr} \underline{\chi} \psi = \Omega \nabla \psi' + \Omega h.$$

Differentiating it by  $\not\partial$ , we obtain

$$\Omega \nabla_3(\not\partial \psi) + \lambda_0 r \nabla(\Omega \operatorname{tr} \underline{\chi}) \psi + \lambda_0 \Omega \operatorname{tr} \underline{\chi}(\not\partial \psi) = r \nabla \Omega \cdot \nabla \psi' + \Omega \nabla(\not\partial \psi') + \not\partial(\Omega h) + [\Omega \nabla_3, \not\partial] \psi,$$

which implies from Corollary 2.43

$$\begin{aligned} & \Omega \nabla_3(\not\partial \psi) + \lambda_0 \Omega \operatorname{tr} \underline{\chi}(\not\partial \psi) \\ &= \Omega \nabla(\not\partial \psi') + (\Gamma_g + \mathbf{O}_3^2) \cdot \nabla \psi' + (\Gamma_g^{(1)} + \mathbf{O}_3^2) \cdot \psi + (\Gamma_g + \mathbf{O}_3^2) h + \Omega \not\partial h \\ &+ (\Gamma_b + \mathbf{O}_3^2) \cdot \not\partial \psi + (\Gamma_b^{(1)} + \mathbf{O}_3^2) \cdot \psi \\ &= \Omega \nabla(\not\partial \psi') + (\Gamma_g + \mathbf{O}_3^2) h + (\Gamma_b + \mathbf{O}_3^2) \cdot \psi'^{(1)} + (\Gamma_g + \mathbf{O}_3^2) \cdot \psi^{(1)} + \Gamma_b^{(1)} \cdot \psi + \Omega \not\partial h. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \nabla_3(\not\partial \psi) + \lambda_0 \operatorname{tr} \underline{\chi}(\not\partial \psi) \\ &= \nabla(\not\partial \psi') + (\Gamma_g + \mathbf{O}_3^2) h + (\Gamma_g + \mathbf{O}_3^2) \cdot \psi'^{(1)} + (\Gamma_b + \mathbf{O}_3^2) \cdot \psi^{(1)} + \Gamma_b^{(1)} \cdot \psi + \not\partial h, \end{aligned} \quad (\text{A.5})$$

which implies (A.2). The proof of (A.4) is similar. This concludes the proof of Lemma A.3.  $\square$

We next prove the following lemma concerning the linearization of Bianchi equations.

**Lemma A.4.** *Let  $k, k' = 0, 1, 2$ ,  $\lambda_0$  a real number and  $\nabla$  a first order elliptic operator. Then, we have the following properties:*

1. *Assume that  $\psi, h \in \mathfrak{s}_k$  and  $\psi' \in \mathfrak{s}_{k'}$  satisfy the following equation:*

$$\nabla_3 \psi + \lambda_0 \operatorname{tr} \underline{\chi} \psi = \nabla \psi' + h. \quad (\text{A.6})$$

*Then, we have*

$$\nabla_3 \check{\psi} + \lambda_0 \operatorname{tr} \underline{\chi} \check{\psi} = \nabla \psi' + \check{h} + \check{J} \cdot \psi_K + \check{L} \cdot \psi'_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \psi_K. \quad (\text{A.7})$$

2. *Assume that  $\psi, h \in \mathfrak{s}_k$  and  $\psi' \in \mathfrak{s}_{k'}$  satisfy the following equation:*

$$\nabla_4 \psi + \lambda_0 \operatorname{tr} \chi \psi = \nabla \psi' + h. \quad (\text{A.8})$$

*Then, we have*

$$\nabla_4 \check{\psi} + \lambda_0 \operatorname{tr} \chi \check{\psi} = \nabla \psi' + \check{h} + \check{J} \cdot \psi_K + \check{L} \cdot \psi'_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \psi_K. \quad (\text{A.9})$$

*Proof.* Assume that we have

$$\nabla_3 \psi + \lambda_0 \operatorname{tr} \underline{\chi} \psi = \nabla \psi' + h.$$

Then, we obtain

$$(\nabla_K)_{(e_3)_K} \psi_K + \lambda_0 \operatorname{tr}_K \underline{\chi}_K \psi_K = \nabla_K \psi'_K + h_K.$$

Taking the difference, we deduce

$$\nabla_3 \psi - (\nabla_K)_{(e_3)_K} \psi_K + \lambda_0 (\operatorname{tr} \underline{\chi} \psi - \operatorname{tr}_K \underline{\chi}_K \psi_K) = \nabla \psi' - \nabla_K \psi'_K + \check{h}. \quad (\text{A.10})$$

Recall from (2.40) that

$$\begin{aligned} e_3 - (e_3)_K &= \Omega^{-1}(\Omega e_3 - \Omega_K(e_3)_K) + \left(\frac{\Omega_K}{\Omega} - 1\right)(e_3)_K \\ &= \Omega^{-1}\check{b}^A \partial_{x^A} + \Omega^{-1}\check{\Omega}(e_3)_K. \end{aligned}$$

Thus, we obtain

$$(\nabla_K)_{e_3 - (e_3)_K} \psi_K = \Gamma_g \cdot \partial_{x^A}(\psi_K) + r\Gamma_g \cdot (\nabla_K)_{(e_3)_K} \psi_K = \Gamma_g \cdot \mathfrak{d}\psi_K.$$

Combining with (A.10), we deduce

$$\begin{aligned} \nabla_3 \check{\psi} + \lambda_0 \operatorname{tr} \underline{\chi} \check{\psi} &= \nabla_3(\psi - \psi_K) + \lambda_0 \operatorname{tr} \underline{\chi}(\psi - \psi_K) \\ &= ((\nabla_K)_{(e_3)_K} - \nabla_3) \psi_K + \lambda_0(\operatorname{tr}_K \underline{\chi}_K - \operatorname{tr} \underline{\chi})\psi_K + \nabla \check{\psi}' + (\nabla - \nabla_K)\psi'_K + \check{h} \\ &= ((\nabla_K)_{(e_3)_K - e_3} + (\nabla_K - \nabla)) \psi_K - \lambda_0 \widetilde{\operatorname{tr} \underline{\chi}} \psi_K + \nabla \check{\psi}' + (\nabla - \nabla_K)\psi'_K + \check{h} \\ &= \nabla \psi' + \check{h} - (\nabla - \nabla_K)_{e_3} \psi_K + (\nabla - \nabla_K)\psi'_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \psi_K. \end{aligned}$$

Recalling Lemma 2.41, we have

$$\begin{aligned} (\nabla - \nabla_K)_{e_3} \psi_K &= \check{J} \cdot \psi_K + \check{L} \cdot \psi_K, \\ (\nabla - \nabla_K)\psi'_K &= \check{L} \cdot \psi_K. \end{aligned}$$

Thus, we obtain

$$\nabla_3 \check{\psi} + \lambda_0 \operatorname{tr} \underline{\chi} \check{\psi} = \nabla \psi' + \check{h} + (\check{J}, \check{L}) \cdot \psi_K + \check{L} \cdot \psi'_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \psi_K,$$

which implies (A.7). The proof of (A.9) is similar. This concludes the proof of Lemma A.4.  $\square$

We are now ready to prove the following proposition, which implies directly Propositions 2.37 and 2.38.

**Proposition A.5.** *We have the following linearized Bianchi equations:*

$$\begin{aligned} \nabla_4 \check{\underline{\alpha}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \check{\underline{\alpha}} &= \nabla \widehat{\otimes} \check{\underline{\beta}} + \mathfrak{L}[\nabla_4 \check{\underline{\alpha}}] + \operatorname{Err}[\nabla_4 \check{\underline{\alpha}}], \\ \mathfrak{L}[\nabla_4 \check{\underline{\alpha}}] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}} + \mathbf{O}_3^2 \cdot (\check{\underline{\beta}}, \check{\underline{\rho}}, \check{\underline{\sigma}}), \\ \operatorname{Err}[\nabla_4 \check{\underline{\alpha}}] &:= \Gamma_g \cdot (\check{\underline{\alpha}}, \check{\underline{\beta}}) + \Gamma_b \cdot (\check{\underline{\rho}}, \check{\underline{\sigma}}), \\ \nabla_4 \check{\underline{\vartheta}} \check{\underline{\alpha}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \check{\underline{\vartheta}} \check{\underline{\alpha}} &= -\nabla \widehat{\otimes} \check{\underline{\vartheta}} \check{\underline{\beta}} + \mathfrak{L}[\nabla_4 \check{\underline{\vartheta}} \check{\underline{\alpha}}] + \operatorname{Err}[\nabla_4 \check{\underline{\vartheta}} \check{\underline{\alpha}}], \\ \mathfrak{L}[\nabla_4 \check{\underline{\vartheta}} \check{\underline{\alpha}}] &= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\underline{\beta}}^{(1)}, \check{\underline{\rho}}^{(1)}, \check{\underline{\sigma}}^{(1)}), \\ \operatorname{Err}[\nabla_4 \check{\underline{\vartheta}} \check{\underline{\alpha}}] &= \Gamma_g^{(1)} \cdot (\check{\underline{\alpha}}, \check{\underline{\beta}}) + \Gamma_b^{(1)} \cdot (\check{\underline{\rho}}, \check{\underline{\sigma}}) + \Gamma_g \cdot (\check{\underline{\alpha}}^{(1)}, \check{\underline{\beta}}^{(1)}) + \Gamma_b \cdot (\check{\underline{\rho}}^{(1)}, \check{\underline{\sigma}}^{(1)}), \\ \nabla_3 \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\beta}} &= -\operatorname{div} \check{\underline{\alpha}} + \mathfrak{L}[\nabla_3 \check{\underline{\beta}}] + \operatorname{Err}[\nabla_3 \check{\underline{\beta}}], \\ \mathfrak{L}[\nabla_3 \check{\underline{\beta}}] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}} + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}}, \\ \operatorname{Err}[\nabla_3 \check{\underline{\beta}}] &:= \Gamma_b \cdot \check{\underline{\beta}} + \Gamma_g \cdot \check{\underline{\alpha}}, \\ \nabla_3 \check{\underline{\vartheta}} \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\vartheta}} \check{\underline{\beta}} &= -\operatorname{div} \check{\underline{\vartheta}} \check{\underline{\alpha}} + \mathfrak{L}[\nabla_3 \check{\underline{\vartheta}} \check{\underline{\beta}}] + \operatorname{Err}[\nabla_3 \check{\underline{\vartheta}} \check{\underline{\beta}}], \end{aligned}$$



$$\begin{aligned}\mathfrak{L}[\nabla_3 \check{\rho}] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\rho}^{(1)} + \mathbf{O}_3^2 \cdot \check{\alpha}^{(1)}, \\ \text{Err}[\nabla_3 \check{\rho}] &:= \Gamma_b^{(1)} \cdot \check{\rho} + \Gamma_b \cdot \check{\rho}^{(1)} + \Gamma_g^{(1)} \cdot \check{\alpha} + \Gamma_g \cdot \check{\alpha}^{(1)},\end{aligned}$$

$$\begin{aligned}\nabla_4 \check{\beta} + \text{tr} \chi \check{\beta} &= -\nabla \check{\rho} + {}^* \nabla \check{\sigma} + \mathfrak{L}[\nabla_4 \check{\beta}] + \text{Err}[\nabla_4 \check{\beta}], \\ \mathfrak{L}[\nabla_4 \check{\beta}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\beta} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}, \check{b}), \\ \text{Err}[\nabla_4 \check{\beta}] &:= \Gamma_g \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_b \cdot \check{b},\end{aligned}$$

$$\begin{aligned}\nabla_4 \check{\rho} + \text{tr} \chi \check{\rho} &:= -\nabla \check{\rho} + {}^* \nabla \check{\sigma} + \mathfrak{L}[\nabla_4 \check{\rho}] + \text{Err}[\nabla_4 \check{\rho}], \\ \mathfrak{L}[\nabla_4 \check{\rho}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\rho}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}, \check{b}^{(1)}), \\ \text{Err}[\nabla_4 \check{\rho}] &:= \Gamma_g^{(1)} \cdot (\check{\rho}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\rho}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \Gamma_b^{(1)} \cdot \check{b} + \Gamma_b \cdot \check{b}^{(1)},\end{aligned}$$

$$\begin{aligned}\nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr} \chi(\check{\rho}, \check{\sigma}) &= -d_1 \check{\beta} + \mathfrak{L}[\nabla_3 \check{\rho}] + \text{Err}[\nabla_3 \check{\rho}], \\ \mathfrak{L}[\nabla_3 \check{\rho}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}, \check{\beta}), \\ \text{Err}[\nabla_3 \check{\rho}] &:= \Gamma_g \cdot (\check{\alpha}, \check{\beta}),\end{aligned}$$

$$\begin{aligned}\nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr} \chi(\check{\rho}, \check{\sigma}) &= -d_1 \check{\rho} + \mathfrak{L}[\nabla_3(\check{\rho}, \check{\sigma})] + \text{Err}[\nabla_3(\check{\rho}, \check{\sigma})], \\ \mathfrak{L}[\nabla_3(\check{\rho}, \check{\sigma})] &:= \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}^{(1)}, \check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}), \\ \text{Err}[\nabla_3(\check{\rho}, \check{\sigma})] &:= \Gamma_g^{(1)} \cdot (\check{\alpha}, \check{\beta}) + \Gamma_g \cdot (\check{\alpha}^{(1)}, \check{\beta}^{(1)}) + \Gamma_b \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \Gamma_b^{(1)} \cdot (\check{\rho}, \check{\sigma}),\end{aligned}$$

$$\begin{aligned}\nabla_4(\check{\rho}, -\check{\sigma}) + \frac{3}{2} \text{tr} \chi(\check{\rho}, -\check{\sigma}) &= d_1 \check{b} + \mathfrak{L}[\nabla_4 \check{\rho}] + \text{Err}[\nabla_4 \check{\rho}], \\ \mathfrak{L}[\nabla_4 \check{\rho}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}, \check{b}), \\ \text{Err}[\nabla_4 \check{\rho}] &:= \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b},\end{aligned}$$

$$\begin{aligned}\nabla_4(\check{\rho}, -\check{\sigma}) + \frac{3}{2} \text{tr} \chi(\check{\rho}, -\check{\sigma}) &= d_1 \check{\rho} + \mathfrak{L}[\nabla_4(\check{\rho}, -\check{\sigma})] + \text{Err}[\nabla_4(\check{\rho}, -\check{\sigma})], \\ \mathfrak{L}[\nabla_4(\check{\rho}, -\check{\sigma})] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}^{(1)}, \check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}), \\ \text{Err}[\nabla_4(\check{\rho}, -\check{\sigma})] &:= \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot (\check{b}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \check{b} + \text{tr} \chi \check{b} &= \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \mathfrak{L}[\nabla_3 \check{b}] + \text{Err}[\nabla_3 \check{b}], \\ \mathfrak{L}[\nabla_3 \check{b}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b} + \mathbf{O}_3^2 \cdot (\check{\beta}, \check{\rho}, \check{\sigma}), \\ \text{Err}[\nabla_3 \check{b}] &:= \Gamma_b \cdot \check{b} + \Gamma_g \cdot (\check{\beta}, \check{\rho}, \check{\sigma}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \check{\rho} + \text{tr} \chi \check{\rho} &= \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \mathfrak{L}[\nabla_3 \check{\rho}] + \text{Err}[\nabla_3 \check{\rho}], \\ \mathfrak{L}[\nabla_3 \check{\rho}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}), \\ \text{Err}[\nabla_3 \check{\rho}] &:= \Gamma_b^{(1)} \cdot \check{b} + \Gamma_b \cdot \check{b}^{(1)} + \Gamma_g^{(1)} \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_4 \check{b} + 2 \operatorname{tr} \chi \check{b} &= \operatorname{div} \check{\alpha} + \mathfrak{L}[\nabla_4 \check{b}] + \operatorname{Err}[\nabla_4 \check{b}], \\ \mathfrak{L}[\nabla_4 \check{b}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}, \check{\alpha}), \\ \operatorname{Err}[\nabla_4 \check{b}] &:= \Gamma_g \cdot (\check{b}, \check{\alpha}),\end{aligned}$$

$$\begin{aligned}\nabla_4 \check{\beta} + 2 \operatorname{tr} \chi \check{\beta} &= \operatorname{div} \check{\beta} \check{\alpha} + \mathfrak{L}[\nabla_4 \check{\beta}] + \operatorname{Err}[\nabla_4 \check{\beta}], \\ \mathfrak{L}[\nabla_4 \check{\beta}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}), \\ \operatorname{Err}[\nabla_4 \check{\beta}] &:= \Gamma_g^{(1)} \cdot (\check{b}, \check{\alpha}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}),\end{aligned}$$

$$\begin{aligned}\nabla_3 \check{\alpha} + \frac{1}{2} \operatorname{tr} \chi \check{\alpha} &= \nabla \widehat{\otimes} \check{b} + \mathfrak{L}[\nabla_3 \check{\alpha}] + \operatorname{Err}[\nabla_3 \check{\alpha}], \\ \mathfrak{L}[\nabla_3 \check{\alpha}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}, \check{b}), \\ \operatorname{Err}[\nabla_3 \check{\alpha}] &:= \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b},\end{aligned}$$

$$\begin{aligned}\nabla_3 \check{\alpha} + \frac{1}{2} \operatorname{tr} \chi \check{\alpha} &= \nabla \widehat{\otimes} \check{\beta} + \mathfrak{L}[\nabla_3 \check{\alpha}] + \operatorname{Err}[\nabla_3 \check{\alpha}], \\ \mathfrak{L}[\nabla_3 \check{\alpha}] &:= \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}, \check{b}^{(1)}), \\ \operatorname{Err}[\nabla_3 \check{\alpha}] &:= \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot \check{b} + \Gamma_g \cdot \check{b}^{(1)}.\end{aligned}$$

*Proof.* We recall from Proposition 2.19

$$\begin{aligned}\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + h[\underline{\alpha}_4], \\ h[\underline{\alpha}_4] &:= 4\omega \underline{\alpha} - 3(\widehat{\chi} \rho - {}^* \widehat{\chi} \sigma) + (\zeta - 4\eta) \widehat{\otimes} \underline{\beta}.\end{aligned}$$

Applying Lemma A.3, we deduce

$$\nabla_4 \check{\alpha} + \frac{1}{2} \operatorname{tr} \chi \check{\alpha} = -\nabla \widehat{\otimes} \check{\beta} + h[\underline{\alpha}_4]^{[1]},$$

where

$$\begin{aligned}h[\underline{\alpha}_4]^{[1]} &= \check{\beta} h[\underline{\alpha}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\alpha}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\underline{\alpha}^{(1)}, \underline{\beta}^{(1)}) + \Gamma_g^{(1)} \cdot \underline{\alpha} \\ &= \check{\beta} h[\underline{\alpha}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\alpha}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\check{\underline{\alpha}}^{(1)}, \check{\underline{\beta}}^{(1)}) + \Gamma_g^{(1)} \cdot \check{\underline{\alpha}} \\ &\quad + \Gamma_g \cdot \mathbf{O}_5^3 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_g^{(1)} \cdot \mathbf{O}_5^3.\end{aligned}$$

We compute

$$\begin{aligned}h[\underline{\alpha}_4] &= 4\omega \underline{\alpha} - 3(\widehat{\chi} \rho - {}^* \widehat{\chi} \sigma) + (\zeta - 4\eta) \widehat{\otimes} \underline{\beta} \\ &= (\mathbf{O}_2^1 + \Gamma_g)(\mathbf{O}_5^3 + \check{\underline{\alpha}}) + (\mathbf{O}_3^2 + \Gamma_b) \cdot (\mathbf{O}_3^1 + \check{\rho} + \check{\sigma}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{\underline{\beta}}) \\ &= \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_5^3 + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}} + \Gamma_g \cdot \check{\underline{\alpha}} + \mathbf{O}_6^3 + \Gamma_b \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}) + \Gamma_b \cdot (\check{\rho}, \check{\sigma}) \\ &\quad + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{\underline{\beta}} + \Gamma_g \cdot \check{\underline{\beta}} \\ &= \mathbf{O}_6^3 + \Gamma_b \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}} + \mathbf{O}_3^2 \cdot (\check{\underline{\beta}}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\underline{\alpha}}, \check{\underline{\beta}}) + \Gamma_b \cdot (\check{\rho}, \check{\sigma}),\end{aligned} \tag{A.11}$$

which implies

$$\begin{aligned}h[\underline{\alpha}_4]^{[1]} &= \mathbf{O}_6^3 + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\underline{\beta}}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\check{\underline{\alpha}}, \check{\underline{\beta}}) + \Gamma_b^{(1)} \cdot (\check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\underline{\alpha}}^{(1)}, \check{\underline{\beta}}^{(1)}) + \Gamma_b \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}).\end{aligned}$$

Applying Lemma A.4, we obtain

$$\begin{aligned}\nabla_4 \underline{\dot{\alpha}} + \frac{1}{2} \operatorname{tr} \chi \underline{\dot{\alpha}} &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]} + \check{J} \cdot \underline{\alpha}_K + \check{L} \cdot \underline{\beta}_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \underline{\alpha}_K \\ &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]} + \Gamma_g^{(1)} \cdot \mathbf{O}_5^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \Gamma_g \cdot \mathbf{O}_5^3 \\ &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2,\end{aligned}$$

and similarly

$$\nabla_4 \underline{\check{\alpha}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\check{\alpha}} = -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]^{[1]}} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2.$$

Hence, we infer

$$\begin{aligned}\nabla_4 \underline{\dot{\alpha}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\dot{\alpha}} &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 \\ &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\dot{\alpha}} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}, \underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}) + \Gamma_b \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nabla_4 \underline{\check{\alpha}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\check{\alpha}} &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \widehat{h[\underline{\alpha}_4]^{[1]}} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 \\ &= -\nabla \widehat{\otimes} \underline{\check{\beta}} + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \underline{\dot{\alpha}}^{(1)} + \mathbf{O}_3^2 \cdot (\underline{\check{\beta}}^{(1)}, \underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\underline{\dot{\alpha}}, \underline{\check{\beta}}) + \Gamma_b^{(1)} \cdot (\underline{\check{\rho}}, \underline{\check{\sigma}}) + \Gamma_g \cdot (\underline{\dot{\alpha}}^{(1)}, \underline{\check{\beta}}^{(1)}) + \Gamma_b \cdot (\underline{\check{\rho}}^{(1)}, \underline{\check{\sigma}}^{(1)}).\end{aligned}$$

Next, we recall from Proposition 2.19

$$\begin{aligned}\nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -\operatorname{div} \underline{\alpha} + h[\underline{\beta}_3], \\ h[\underline{\beta}_3] &:= -2\underline{\omega} \underline{\beta} + (2\underline{\zeta} - \underline{\eta}) \cdot \underline{\alpha}.\end{aligned}$$

Applying Lemma A.3, we deduce

$$\nabla_3 \underline{\check{\beta}} + 2 \operatorname{tr} \underline{\chi} \underline{\check{\beta}} = -\operatorname{div} \underline{\check{\alpha}} + h[\underline{\beta}_3]^{[1]},$$

where

$$\begin{aligned}h[\underline{\beta}_3]^{[1]} &= \check{\rho} h[\underline{\beta}_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\beta}_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot \underline{\alpha}^{(1)} + (\Gamma_b + \mathbf{O}_3^2) \cdot \underline{\beta}^{(1)} + \Gamma_b^{(1)} \cdot \underline{\beta} \\ &= \check{\rho} h[\underline{\beta}_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\beta}_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot \underline{\dot{\alpha}}^{(1)} + (\Gamma_b + \mathbf{O}_3^2) \cdot \underline{\check{\beta}}^{(1)} + \Gamma_b^{(1)} \cdot \underline{\check{\beta}} \\ &\quad + \Gamma_b \cdot \mathbf{O}_5^3 + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2.\end{aligned}$$

We compute

$$\begin{aligned}h[\underline{\beta}_3] &= -2\underline{\omega} \underline{\beta} + (2\underline{\zeta} - \underline{\eta}) \cdot \underline{\alpha} \\ &= (\mathbf{O}_2^1 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \underline{\check{\beta}}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_5^3 + \underline{\dot{\alpha}}) \\ &= \mathbf{O}_6^3 + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}} + \Gamma_b \cdot \underline{\check{\beta}} + \mathbf{O}_8^5 + \Gamma_g \cdot \mathbf{O}_5^3 + \mathbf{O}_3^2 \cdot \underline{\dot{\alpha}} + \Gamma_g \cdot \underline{\dot{\alpha}} \\ &= \mathbf{O}_6^3 + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}} + \mathbf{O}_3^2 \cdot \underline{\dot{\alpha}} + \Gamma_b \cdot \underline{\check{\beta}} + \Gamma_g \cdot \underline{\dot{\alpha}},\end{aligned}$$

which implies

$$h[\underline{\beta}_3]^{[1]} = \mathbf{O}_6^3 + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \underline{\check{\beta}}^{(1)} + \mathbf{O}_3^2 \cdot \underline{\dot{\alpha}}^{(1)} + \Gamma_b^{(1)} \cdot \underline{\check{\beta}} + \Gamma_b \cdot \underline{\check{\beta}}^{(1)} + \Gamma_g^{(1)} \cdot \underline{\dot{\alpha}} + \Gamma_g \cdot \underline{\dot{\alpha}}^{(1)}.$$

Applying Lemma A.4, we obtain

$$\begin{aligned}\nabla_3 \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\beta}} &= -\operatorname{div} \check{\underline{\alpha}} + \widetilde{h[\underline{\beta}_3]} + \check{\underline{J}} \cdot \underline{\beta}_K + \check{\underline{L}} \cdot \check{\underline{\alpha}}_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \underline{\beta}_K \\ &= -\operatorname{div} \check{\underline{\alpha}} + \widetilde{h[\underline{\beta}_3]} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \Gamma_g^{(1)} \cdot \mathbf{O}_5^3 + \Gamma_g \cdot \mathbf{O}_4^2 \\ &= -\operatorname{div} \check{\underline{\alpha}} + \widetilde{h[\underline{\beta}_3]} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2,\end{aligned}$$

and similarly

$$\nabla_3 \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\beta}} = -\operatorname{div} \check{\underline{\alpha}} + \widetilde{h[\underline{\beta}_3]}^{[1]} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2.$$

Hence, we deduce

$$\nabla_3 \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\beta}} = -\operatorname{div} \check{\underline{\alpha}} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}} + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}} + \Gamma_b \cdot \check{\underline{\beta}} + \Gamma_g \cdot \check{\underline{\alpha}}.$$

Similarly, we have

$$\begin{aligned}\nabla_3 \check{\underline{\beta}} + 2 \operatorname{tr} \underline{\chi} \check{\underline{\beta}} &= -\operatorname{div} \check{\underline{\alpha}} + \Gamma_b^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}}^{(1)} + \mathbf{O}_3^2 \cdot \check{\underline{\alpha}}^{(1)} \\ &\quad + \Gamma_b^{(1)} \cdot \check{\underline{\beta}} + \Gamma_b \cdot \check{\underline{\beta}}^{(1)} + \Gamma_g^{(1)} \cdot \check{\underline{\alpha}} + \Gamma_g \cdot \check{\underline{\alpha}}^{(1)}.\end{aligned}$$

Next, we recall from Proposition 2.19

$$\begin{aligned}\nabla_4 \underline{\beta} + \operatorname{tr} \underline{\chi} \underline{\beta} &= -\nabla \rho + {}^* \nabla \sigma + h[\underline{\beta}_4], \\ h[\underline{\beta}_4] &:= 2\omega \underline{\beta} + 2\widehat{\underline{\chi}} \cdot \underline{\beta} - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma).\end{aligned}$$

Applying Lemma A.3, we infer

$$\nabla_4 \check{\underline{\beta}} + \operatorname{tr} \underline{\chi} \check{\underline{\beta}} = -\nabla \check{\underline{\rho}} + {}^* \nabla \check{\underline{\sigma}} + h[\underline{\beta}_4]^{[1]},$$

where

$$\begin{aligned}h[\underline{\beta}_4]^{[1]} &= \check{\underline{\rho}} h[\underline{\beta}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\beta}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\underline{\beta}^{(1)}, \rho^{(1)}, \sigma^{(1)}) + \Gamma_g^{(1)} \cdot \underline{\beta}^{(1)} \\ &= \check{\underline{\rho}} h[\underline{\beta}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\underline{\beta}_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\check{\underline{\beta}}^{(1)}, \check{\underline{\rho}}^{(1)}, \check{\underline{\sigma}}^{(1)}) + \Gamma_g^{(1)} \cdot \check{\underline{\beta}}^{(1)} \\ &\quad + \Gamma_g \cdot \mathbf{O}_4^2 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2.\end{aligned}$$

We compute

$$\begin{aligned}h[\underline{\beta}_4] &= (\mathbf{O}_2^1 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{\underline{\beta}}) + (\mathbf{O}_3^2 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \check{\underline{b}}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^1 + \check{\underline{\rho}} + \check{\underline{\sigma}}) \\ &= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}} + \Gamma_g \cdot \check{\underline{\beta}} + \mathbf{O}_7^4 + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{\underline{b}} + \Gamma_b \cdot \check{\underline{b}} \\ &\quad + \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\underline{\rho}}, \check{\underline{\sigma}}) + \Gamma_g \cdot (\check{\underline{\rho}}, \check{\underline{\sigma}}) \\ &= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}} + \mathbf{O}_3^2 \cdot (\check{\underline{\rho}}, \check{\underline{\sigma}}, \check{\underline{b}}) + \Gamma_g \cdot (\check{\underline{\beta}}, \check{\underline{\rho}}, \check{\underline{\sigma}}) + \Gamma_b \cdot \check{\underline{b}},\end{aligned}$$

which implies

$$\begin{aligned}h[\underline{\beta}_4]^{[1]} &= \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\underline{\rho}}^{(1)}, \check{\underline{\sigma}}^{(1)}, \check{\underline{b}}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\check{\underline{\beta}}, \check{\underline{\rho}}, \check{\underline{\sigma}}) + \Gamma_g \cdot (\check{\underline{\beta}}^{(1)}, \check{\underline{\rho}}^{(1)}, \check{\underline{\sigma}}^{(1)}) + \Gamma_b^{(1)} \cdot \check{\underline{b}} + \Gamma_b \cdot \check{\underline{b}}^{(1)}.\end{aligned}$$

Applying Lemma A.4, we obtain

$$\begin{aligned}\nabla_4 \check{\underline{\beta}} + \operatorname{tr} \underline{\chi} \check{\underline{\beta}} &= -\nabla \check{\underline{\rho}} + {}^* \nabla \check{\underline{\sigma}} + \widetilde{h[\underline{\beta}_4]} + \check{\underline{J}} \cdot \underline{\beta}_K + \check{\underline{L}} \cdot (\rho_K, \sigma_K) + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \underline{\beta}_K \\ &= -\nabla \check{\underline{\rho}} + {}^* \nabla \check{\underline{\sigma}} + \widetilde{h[\underline{\beta}_4]} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1.\end{aligned}$$

Hence, we infer

$$\nabla_4 \check{\underline{\beta}} + \text{tr} \chi \check{\underline{\beta}} = -\nabla \check{\rho} + {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\underline{\beta}} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}, \check{b}) + \Gamma_g \cdot (\check{\underline{\beta}}, \check{\rho}, \check{\sigma}) + \Gamma_b \cdot \check{b}.$$

Similarly, we have

$$\begin{aligned} \nabla_4 \check{\check{\underline{\beta}}} + \text{tr} \chi \check{\check{\underline{\beta}}} &= -\nabla \check{\check{\rho}} + {}^* \nabla \check{\check{\sigma}} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\check{\underline{\beta}}}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\check{\rho}}^{(1)}, \check{\check{\sigma}}^{(1)}, \check{b}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\check{\check{\underline{\beta}}}, \check{\check{\rho}}, \check{\check{\sigma}}) + \Gamma_g \cdot (\check{\check{\underline{\beta}}}^{(1)}, \check{\check{\rho}}^{(1)}, \check{\check{\sigma}}^{(1)}) + \Gamma_b^{(1)} \cdot \check{b} + \Gamma_b \cdot \check{b}^{(1)}. \end{aligned}$$

Next, we recall from Proposition 2.19

$$\begin{aligned} \nabla_3(\rho, \sigma) + \frac{3}{2} \text{tr} \chi(\rho, \sigma) &= -d_1 \underline{\beta} + h[\rho_3, \sigma_3], \\ h[\rho_3, \sigma_3] &= -\frac{1}{2} \widehat{\chi} \cdot (\underline{\alpha}, -{}^* \underline{\alpha}) + \zeta \cdot (\underline{\beta}, -{}^* \underline{\beta}) - 2\eta \cdot (\underline{\beta}, -{}^* \underline{\beta}). \end{aligned}$$

Applying Lemma A.3, we obtain

$$\nabla_3 \check{\rho}(\rho, \sigma) + \frac{3}{2} \text{tr} \chi \check{\rho}(\rho, \sigma) = -d_1 \check{\underline{\beta}} + h[\rho_3, \sigma_3]^{[1]},$$

where

$$\begin{aligned} h[\rho_3, \sigma_3]^{[1]} &= \check{\rho} h[\rho_3, \sigma_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\rho_3, \sigma_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot (\rho^{(1)}, \sigma^{(1)}) + (\Gamma_g + \mathbf{O}_3^2) \cdot \underline{\beta}^{(1)} + \Gamma_b^{(1)} \cdot (\rho, \sigma) \\ &= \check{\rho} h[\rho_3, \sigma_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\rho_3, \sigma_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + (\Gamma_g + \mathbf{O}_3^2) \cdot \check{\underline{\beta}}^{(1)} + \Gamma_b^{(1)} \cdot (\check{\rho}, \check{\sigma}) \\ &\quad + \Gamma_b \cdot \mathbf{O}_3^1 + \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1. \end{aligned}$$

We compute

$$\begin{aligned} h[\rho_3, \sigma_3] &= (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_5^3 + \underline{\alpha}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{\underline{\beta}}) \\ &= \mathbf{O}_8^5 + \Gamma_g \cdot \mathbf{O}_5^3 + \mathbf{O}_3^2 \cdot \underline{\alpha} + \Gamma_g \cdot \underline{\alpha} + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{\underline{\beta}} + \Gamma_g \cdot \check{\underline{\beta}} \\ &= \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot (\underline{\alpha}, \check{\underline{\beta}}) + \Gamma_g \cdot (\underline{\alpha}, \check{\underline{\beta}}), \end{aligned}$$

which implies

$$\begin{aligned} h[\rho_3, \sigma_3]^{[1]} &= \mathbf{O}_6^3 + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\alpha}^{(1)}, \check{\underline{\beta}}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\underline{\alpha}, \check{\underline{\beta}}) + \Gamma_g \cdot (\underline{\alpha}^{(1)}, \check{\underline{\beta}}^{(1)}) + \Gamma_b \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \Gamma_b^{(1)} \cdot (\check{\rho}, \check{\sigma}). \end{aligned}$$

Applying Lemma A.4, we deduce

$$\begin{aligned} \nabla_3(\check{\rho}, \check{\sigma}) + \frac{3}{2} \text{tr} \chi(\check{\rho}, \check{\sigma}) &= -d_1 \check{\underline{\beta}} + \widetilde{h[\rho_3, \sigma_3]} + \check{L} \cdot \underline{\beta}_K + \Gamma_g \cdot (\rho_K, \sigma_K) \\ &= -d_1 \check{\underline{\beta}} + \widetilde{h[\rho_3, \sigma_3]} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 \\ &= -d_1 \check{\underline{\beta}} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\alpha}, \check{\underline{\beta}}) + \Gamma_g \cdot (\underline{\alpha}, \check{\underline{\beta}}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nabla_3(\check{\check{\rho}}, \check{\check{\sigma}}) + \frac{3}{2} \text{tr} \chi(\check{\check{\rho}}, \check{\check{\sigma}}) &= -d_1 \check{\check{\underline{\beta}}} + \Gamma_b^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\underline{\alpha}^{(1)}, \check{\check{\underline{\beta}}}^{(1)}, \check{\check{\rho}}^{(1)}, \check{\check{\sigma}}^{(1)}) \\ &\quad + \Gamma_g^{(1)} \cdot (\underline{\alpha}, \check{\check{\underline{\beta}}}) + \Gamma_g \cdot (\underline{\alpha}^{(1)}, \check{\check{\underline{\beta}}}^{(1)}) + \Gamma_b \cdot (\check{\check{\rho}}^{(1)}, \check{\check{\sigma}}^{(1)}) + \Gamma_b^{(1)} \cdot (\check{\check{\rho}}, \check{\check{\sigma}}). \end{aligned}$$

We have from Proposition 2.19

$$\begin{aligned}\nabla_4(\rho, -\sigma) + \frac{3}{2} \operatorname{tr} \chi(\rho, -\sigma) &= d_1 \beta + h[\rho_4, \sigma_4], \\ h[\rho_4, \sigma_4] &= -\frac{1}{2} \widehat{\chi} \cdot (\alpha, * \alpha) + (\zeta + 2\underline{\eta}) \cdot (\beta, * \beta).\end{aligned}$$

Applying Lemma A.3, we have

$$\nabla_4 \mathfrak{P}(\rho, -\sigma) + \frac{3}{2} \operatorname{tr} \chi \mathfrak{P}(\rho, -\sigma) = d_1 \mathfrak{P} \beta + h[\rho_4, \sigma_4]^{[1]},$$

where

$$\begin{aligned}h[\rho_4, \sigma_4]^{[1]} &= \mathfrak{P} h[\rho_4, \sigma_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\rho_4, \sigma_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\beta^{(1)}, \rho^{(1)}, \sigma^{(1)}) + \Gamma_g^{(1)} \cdot (\rho, \sigma) \\ &= \mathfrak{P} h[\rho_4, \sigma_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\rho_4, \sigma_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \Gamma_g^{(1)} \cdot (\check{\rho}, \check{\sigma}) \\ &\quad + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1.\end{aligned}$$

We compute

$$\begin{aligned}h[\rho_4, \sigma_4] &= (\mathbf{O}_3^2 + \Gamma_b) \cdot (\mathbf{O}_5^3 + \check{\alpha}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{b}) \\ &= \mathbf{O}_8^5 + \Gamma_b \cdot \mathbf{O}_5^3 + \mathbf{O}_3^2 \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha} + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{b} + \Gamma_g \cdot \check{b} \\ &= \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot (\check{\alpha}, \check{b}) + \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b},\end{aligned}$$

which implies

$$\begin{aligned}h[\rho_4, \sigma_4]^{[1]} &= \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}^{(1)}, \check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) \\ &\quad + \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot (\check{b}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}).\end{aligned}$$

Applying Lemma A.4, we deduce

$$\begin{aligned}\nabla_4(\check{\rho}, -\check{\sigma}) + \frac{3}{2} \operatorname{tr} \chi(\check{\rho}, -\check{\sigma}) &= d_1 \check{b} + h[\widetilde{\rho_4}, \widetilde{\sigma_4}] + \check{J} \cdot (\rho_K, \sigma_K) + \check{L} \cdot \beta_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1}(\rho_K, \sigma_K) \\ &= d_1 \check{b} + h[\widetilde{\rho_4}, \widetilde{\sigma_4}] + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 \\ &= d_1 \check{b} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}, \check{b}) + \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nabla_4(\check{\rho}, -\check{\sigma}) + \frac{3}{2} \operatorname{tr} \chi(\check{\rho}, -\check{\sigma}) &= d_1 \check{\rho} \beta + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\alpha}^{(1)}, \check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) \\ &\quad + \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot (\check{b}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}).\end{aligned}$$

Next, we recall from Proposition 2.19

$$\begin{aligned}\nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= \nabla \rho + * \nabla \sigma + h[\beta_3], \\ h[\beta_3] &= 2\underline{\omega} \beta + 2\widehat{\chi} \cdot \underline{\beta} + 3(\eta \rho + * \eta \sigma).\end{aligned}$$

Applying Lemma A.3, we deduce

$$\nabla_3 \mathfrak{P} \beta + \operatorname{tr} \underline{\chi} \mathfrak{P} \beta = \nabla \mathfrak{P} \rho + * \nabla \mathfrak{P} \sigma + h[\beta_3]^{[1]},$$

where

$$\begin{aligned}h[\beta_3]^{[1]} &= \mathfrak{P} h[\beta_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\beta_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot \beta^{(1)} + (\Gamma_g + \mathbf{O}_3^2) \cdot (\rho^{(1)}, \sigma^{(1)}) + \Gamma_g^{(1)} \cdot \beta \\ &= \mathfrak{P} h[\beta_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\beta_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot \check{b}^{(1)} + (\Gamma_g + \mathbf{O}_3^2) \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}) + \Gamma_g^{(1)} \cdot \check{b} \\ &\quad + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2.\end{aligned}$$

We compute

$$\begin{aligned}
h[\beta_3] &= (\mathbf{O}_2^1 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \check{b}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{\beta}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^1 + \check{\rho} + \check{\sigma}) \\
&= \mathbf{O}_6^3 + \Gamma_b \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{b} + \Gamma_b \cdot \check{b} + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{\beta} + \Gamma_g \cdot \check{\beta} \\
&\quad + \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\rho}, \check{\sigma}) \\
&= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b} + \mathbf{O}_3^2 \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_b \cdot \check{b} + \Gamma_g \cdot (\check{\beta}, \check{\rho}, \check{\sigma}),
\end{aligned}$$

which implies

$$\begin{aligned}
h[\beta_3]^{[1]} &= \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}, \check{b}^{(1)}) \\
&\quad + \Gamma_b^{(1)} \cdot \check{b} + \Gamma_b \cdot \check{b}^{(1)} + \Gamma_g^{(1)} \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}).
\end{aligned}$$

Applying Lemma A.4, we infer

$$\begin{aligned}
\nabla_3 \check{b} + \text{tr } \underline{\chi} \check{b} &= \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \widetilde{h[\beta_3]} + \check{L} \cdot (\rho_K, \sigma_K) + \check{J} \cdot \beta_K + \Gamma_g \cdot \mathfrak{d}^{\leq 1} \beta_K \\
&= \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \widetilde{h[\beta_3]} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1.
\end{aligned}$$

Hence, we obtain

$$\nabla_3 \check{b} + \text{tr } \underline{\chi} \check{b} = \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b} + \mathbf{O}_3^2 \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_b \cdot \check{b} + \Gamma_g \cdot (\check{\beta}, \check{\rho}, \check{\sigma}).$$

Similarly, we have

$$\begin{aligned}
\nabla_3 \check{\beta} + \text{tr } \underline{\chi} \check{\beta} &= \nabla \check{\rho} + {}^* \nabla \check{\sigma} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{b}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}) \\
&\quad + \Gamma_b^{(1)} \cdot \check{b} + \Gamma_b \cdot \check{b}^{(1)} + \Gamma_g^{(1)} \cdot (\check{\beta}, \check{\rho}, \check{\sigma}) + \Gamma_g \cdot (\check{\beta}^{(1)}, \check{\rho}^{(1)}, \check{\sigma}^{(1)}).
\end{aligned}$$

Next, we recall from Proposition 2.19

$$\begin{aligned}
\nabla_4 \beta + 2 \text{tr } \chi \beta &= \text{div } \alpha + h[\beta_4] \\
h[\beta_4] &= -2\omega\beta + (2\zeta + \eta)\alpha.
\end{aligned}$$

Applying Lemma A.3, we deduce

$$\nabla_4 \check{\beta} + 2 \text{tr } \chi \check{\beta} = \text{div } \check{\beta} \alpha + h[\beta_4]^{[1]},$$

where

$$\begin{aligned}
h[\beta_4]^{[1]} &= \check{\beta} h[\beta_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\beta_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\beta^{(1)}, \alpha^{(1)}) + \Gamma_g^{(1)} \cdot \beta \\
&= \check{\beta} h[\beta_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\beta_4] + (\Gamma_g + \mathbf{O}_3^2) \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}) + \Gamma_g^{(1)} \cdot \check{b} \\
&\quad + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2.
\end{aligned}$$

We compute

$$\begin{aligned}
h[\beta_4] &= (\mathbf{O}_2^1 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{b}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_5^3 + \check{\alpha}) \\
&= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot \check{b} + \Gamma_g \cdot \check{b} + \mathbf{O}_8^5 + \Gamma_g \cdot \mathbf{O}_5^3 + \mathbf{O}_3^2 \cdot \check{\alpha} + \Gamma_g \cdot \check{\alpha} \\
&= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}, \check{\alpha}) + \Gamma_g \cdot (\check{b}, \check{\alpha}),
\end{aligned}$$

which implies

$$h[\beta_4]^{[1]} = \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}) + \Gamma_g^{(1)} \cdot (\check{b}, \check{\alpha}).$$

Applying Lemma A.4, we obtain

$$\begin{aligned}\nabla_4 \check{b} + 2 \operatorname{tr} \chi \check{b} &= \operatorname{div} \check{\alpha} + \widehat{h[\beta_4]} + \check{J} \cdot \beta_K + \check{L} \cdot \alpha_K + \Gamma_g \cdot \beta_K \\ &= \operatorname{div} \check{\alpha} + \widehat{h[\beta_4]} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 \\ &= \operatorname{div} \check{\alpha} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}, \check{\alpha}) + \Gamma_g \cdot (\check{b}, \check{\alpha}).\end{aligned}$$

Similarly, we have

$$\nabla_4 \check{\beta} + 2 \operatorname{tr} \chi \check{\beta} = \operatorname{div} \check{\beta} \alpha + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \mathbf{O}_2^1 \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}) + \Gamma_g^{(1)} \cdot (\check{b}, \check{\alpha}) + \Gamma_g \cdot (\check{b}^{(1)}, \check{\alpha}^{(1)}).$$

Finally, we recall from Proposition 2.19

$$\begin{aligned}\nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \alpha &= \nabla \widehat{\otimes} \beta + h[\alpha_3], \\ h[\alpha_3] &= 4\underline{\omega} \alpha - 3(\widehat{\chi} \rho + {}^* \widehat{\chi} \sigma) + (\zeta + 4\eta) \widehat{\otimes} \beta.\end{aligned}$$

Applying Lemma A.3, we obtain

$$\nabla_3 \check{\beta} \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \check{\beta} \alpha = \nabla \widehat{\otimes} \check{\beta} \beta + h[\alpha_3]^{[1]},$$

where

$$\begin{aligned}h[\alpha_3]^{[1]} &= \check{\beta} h[\alpha_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\alpha_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot \alpha^{(1)} + (\Gamma_g + \mathbf{O}_3^2) \cdot \beta^{(1)} + \Gamma_b^{(1)} \cdot \alpha \\ &= \check{\beta} h[\alpha_3] + (\Gamma_g + \mathbf{O}_3^2) \cdot h[\alpha_3] + (\Gamma_b + \mathbf{O}_3^2) \cdot \check{\alpha}^{(1)} + (\Gamma_g + \mathbf{O}_3^2) \cdot \check{b}^{(1)} + \Gamma_b^{(1)} \cdot \check{\alpha} \\ &\quad + \Gamma_b \cdot \mathbf{O}_5^3 + \mathbf{O}_8^5 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_7^4 + \Gamma_b^{(1)} \cdot \mathbf{O}_5^3.\end{aligned}$$

We compute

$$\begin{aligned}h[\alpha_3] &= (\mathbf{O}_2^1 + \Gamma_b) \cdot (\mathbf{O}_5^3 + \check{\alpha}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_3^1 + \check{\rho} + \check{\sigma}) + (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{b}) \\ &= \mathbf{O}_7^4 + \Gamma_b \cdot \mathbf{O}_5^3 + \mathbf{O}_2^1 \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha} + \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}) \\ &\quad + \mathbf{O}_7^4 + \Gamma_g \cdot \mathbf{O}_4^2 + \mathbf{O}_3^2 \cdot \check{b} + \Gamma_g \cdot \check{b} \\ &= \mathbf{O}_6^3 + \Gamma_g \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}, \check{b}) + \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b},\end{aligned}$$

which implies

$$\begin{aligned}h[\alpha_3]^{[1]} &= \mathbf{O}_6^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}, \check{b}^{(1)}) \\ &\quad + \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot \check{b} + \Gamma_g \cdot \check{b}^{(1)}.\end{aligned}$$

Applying Lemma A.4, we infer

$$\begin{aligned}\nabla_3 \check{\alpha} + \frac{1}{2} \operatorname{tr} \underline{\chi} \check{\alpha} &= \nabla \widehat{\otimes} \check{b} + \widehat{h[\alpha_3]} + \check{J} \cdot \alpha_K + \check{L} \cdot \beta_K + \Gamma_g \cdot \alpha_K \\ &= \nabla \widehat{\otimes} \check{b} + \widehat{h[\alpha_3]} + \Gamma_b^{(1)} \cdot \mathbf{O}_5^3 + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 + \Gamma_g \cdot \mathbf{O}_5^3 \\ &= \nabla \widehat{\otimes} \check{b} + \widehat{h[\alpha_3]} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^2 \\ &= \nabla \widehat{\otimes} \check{b} + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha} + \mathbf{O}_3^2 \cdot (\check{\rho}, \check{\sigma}, \check{b}) + \Gamma_b \cdot \check{\alpha} + \Gamma_g \cdot \check{b}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nabla_3 \check{\beta} \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \check{\beta} \alpha &= \nabla \widehat{\otimes} \check{\beta} \beta + \Gamma_g^{(1)} \cdot \mathbf{O}_3^1 + \mathbf{O}_2^1 \cdot \check{\alpha}^{(1)} + \mathbf{O}_3^2 \cdot (\check{\rho}^{(1)}, \check{\sigma}^{(1)}, \check{b}^{(1)}) \\ &\quad + \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)} + \Gamma_g^{(1)} \cdot \check{b} + \Gamma_g \cdot \check{b}^{(1)}.\end{aligned}$$

This concludes the proof of Proposition A.5.  $\square$



#### A.4 Proof of Proposition 2.40

**Proposition A.6.** *We have the following equations:*

$$\begin{aligned}\Omega\nabla_4(\check{\gamma}_{AB}) &= \Omega \operatorname{tr} \chi \check{\gamma}_{AB} + \widetilde{\Omega \operatorname{tr} \chi}(\gamma_K)_{AB} + 2\Omega\check{\chi}_{AB} + 2\check{\Omega}(\widehat{\chi}_K)_{AB}, \\ \Omega\nabla_4(\check{\Omega}) &= -2\Omega(\check{\Omega}\omega) - 2\check{\Omega}(\Omega_K\omega_K), \\ \Omega\nabla_4(\check{b}^A) &= -4\left(\check{\Omega}^2\zeta_B\gamma^{AB} + \Omega_K^2\check{\zeta}_B\gamma^{AB} + \Omega_K^2(\zeta_K)_B\gamma^{AB}\right), \\ \Omega\nabla_4(\check{\epsilon}_{AB}) &= \Omega \operatorname{tr} \chi \check{\epsilon}_{AB} + \widetilde{\Omega \operatorname{tr} \chi}(\epsilon_K)_{AB}.\end{aligned}$$

In the case  $\check{\Omega} = 0$  and  $\check{b} = 0$ , we have the following equations:

$$\begin{aligned}\Omega\nabla_3(\check{\gamma}_{AB}) - \Omega \operatorname{tr} \underline{\chi} \check{\gamma}_{AB} &= \widetilde{\Omega \operatorname{tr} \underline{\chi}}(\gamma_K)_{AB} + 2\Omega\underline{\check{\chi}}_{AB} - \partial_A(\underline{b}_K^C)\check{\gamma}_{BC} - \partial_B(\underline{b}_K^C)\check{\gamma}_{AC}, \\ \Omega\nabla_3(\check{\epsilon}_{AB}) - \Omega \operatorname{tr} \underline{\chi} \check{\epsilon}_{AB} &= \widetilde{\Omega \operatorname{tr} \underline{\chi}}(\epsilon_K)_{AB} - \partial_A(\underline{b}_K^C)\check{\epsilon}_{BC} - \partial_B(\underline{b}_K^C)\check{\epsilon}_{AC}.\end{aligned}$$

**Remark A.7.** *Note that Proposition 2.40 follows directly from Propositions 2.28 and A.6 and Definitions 2.3 and 2.25.*

*Proof of Proposition A.6.* Firstly, recall from Proposition 2.20 that

$$\Omega\nabla_4(\gamma_{AB}) = 2\Omega\chi_{AB},$$

which implies

$$\begin{aligned}\Omega\nabla_4(\check{\gamma}_{AB}) &= \Omega\nabla_4(\gamma_{AB} - (\gamma_K)_{AB}) \\ &= \Omega\nabla_4(\gamma_{AB}) - \Omega\nabla_{(e_4)_K}(\gamma_K)_{AB} \\ &= 2\Omega\chi_{AB} - 2\Omega_K(\chi_K)_{AB} \\ &= \Omega \operatorname{tr} \chi \check{\gamma}_{AB} + \widetilde{\Omega \operatorname{tr} \chi}(\gamma_K)_{AB} + 2\Omega\check{\chi}_{AB} - 2\Omega_K(\widehat{\chi}_K)_{AB} \\ &= \Omega \operatorname{tr} \chi \check{\gamma}_{AB} + \widetilde{\Omega \operatorname{tr} \chi}(\gamma_K)_{AB} + 2\Omega\check{\chi}_{AB} + 2\check{\Omega}(\widehat{\chi}_K)_{AB}.\end{aligned}$$

Next, we compute

$$\begin{aligned}\Omega\nabla_4(\check{\Omega}) &= \Omega\nabla_4(\Omega - \Omega_K) \\ &= \Omega\nabla_4\Omega - \Omega_K\nabla_{(e_4)_K}\Omega_K \\ &= -2\omega\Omega^2 + 2\omega_K\Omega_K^2 \\ &= -2\check{\Omega}\omega\Omega - 2\check{\Omega}(\Omega_K\omega_K).\end{aligned}$$

Recall from Proposition 2.20 that

$$\nabla_4(\underline{b}^A) = -4\Omega\zeta^A = -4\Omega\zeta_B\gamma^{AB}.$$

Hence, we have

$$\begin{aligned}\Omega\nabla_4\check{b}^A &= \Omega\nabla_4(\underline{b}^A) - \Omega_K\nabla_{(e_4)_K}(\underline{b}_K^A) \\ &= -4\Omega^2\zeta_B\gamma^{AB} + 4\Omega_K^2(\zeta_K)_B(\gamma_K)^{AB} \\ &= -4\left(\check{\Omega}^2\zeta_B\gamma^{AB} + \Omega_K^2\check{\zeta}_B\gamma^{AB} + \Omega_K^2(\zeta_K)_B\gamma^{AB}\right).\end{aligned}$$

Next, we recall from Proposition 2.20

$$\Omega\nabla_4(\epsilon_{AB}) = \Omega \operatorname{tr} \chi \epsilon_{AB},$$

which implies

$$\begin{aligned}\Omega\nabla_4(\check{\epsilon}_{AB}) &= \Omega\nabla_4(\epsilon_{AB}) - \Omega_K\nabla_{(e_4)_K}(\epsilon_K)_{AB} \\ &= \Omega\operatorname{tr}\chi\epsilon_{AB} - \Omega_K\operatorname{tr}_K\chi_K(\epsilon_K)_{AB} \\ &= \Omega\operatorname{tr}\chi\check{\epsilon}_{AB} + \widetilde{\Omega\operatorname{tr}\chi}(\epsilon_K)_{AB}.\end{aligned}$$

Then, we recall from Proposition 2.20

$$\Omega e_3(\gamma_{AB}) = 2\Omega\underline{\chi}_{AB} - \partial_A(\underline{b}^C)\gamma_{BC} - \partial_B(\underline{b}^C)\gamma_{AC},$$

which implies

$$\Omega_K(e_3)_K((\gamma_K)_{AB}) = 2\Omega_K(\underline{\chi}_K)_{AB} - \partial_A(\underline{b}_K^C)(\gamma_K)_{BC} - \partial_B(\underline{b}_K^C)(\gamma_K)_{AC}.$$

Thus, we infer

$$\begin{aligned}\Omega e_3(\check{\gamma}_{AB}) &= \Omega e_3(\gamma_{AB}) - \Omega_K(e_3)_K(\gamma_K)_{AB} - \widetilde{\Omega}e_3(\gamma_K)_{AB} \\ &= 2\Omega\underline{\chi}_{AB} - \partial_A(\underline{b}^C)\gamma_{BC} - \partial_B(\underline{b}^C)\gamma_{AC} - 2\Omega_K(\underline{\chi}_K)_{AB} \\ &\quad + \partial_A(\underline{b}_K^C)(\gamma_K)_{BC} + \partial_B(\underline{b}_K^C)(\gamma_K)_{AC} - \check{\underline{b}}^C\partial_C(\gamma_K)_{AB} \\ &= \Omega\operatorname{tr}\underline{\chi}\gamma_{AB} - \Omega_K\operatorname{tr}_K\underline{\chi}_K(\gamma_K)_{AB} + 2(\Omega\underline{\check{\chi}}_{AB} - \Omega_K(\underline{\check{\chi}}_K)_{AB}) \\ &\quad - \partial_A(\underline{b}^C)\gamma_{BC} - \partial_B(\underline{b}^C)\gamma_{AC} + \partial_A(\underline{b}_K^C)(\gamma_K)_{BC} + \partial_B(\underline{b}_K^C)(\gamma_K)_{AC} - \check{\underline{b}}^C\partial_C(\gamma_K)_{AB} \\ &= \Omega\operatorname{tr}\underline{\chi}\check{\gamma}_{AB} + \widetilde{\Omega\operatorname{tr}\underline{\chi}}(\gamma_K)_{AB} + 2\Omega\underline{\check{\chi}}_{AB} + 2\widetilde{\Omega}(\underline{\check{\chi}}_K)_{AB} \\ &\quad - \partial_A(\check{\underline{b}}^C)\gamma_{BC} - \partial_A(\underline{b}_K^C)\check{\gamma}_{BC} - \partial_B(\check{\underline{b}}^C)\gamma_{AC} - \partial_B(\underline{b}_K^C)\check{\gamma}_{AC} - \check{\underline{b}}^C\partial_C(\gamma_K)_{AB}.\end{aligned}$$

In the particular case  $\widetilde{\Omega} = 0$  and  $\check{\underline{b}} = 0$ , we deduce

$$\Omega e_3(\check{\gamma}_{AB}) - \Omega\operatorname{tr}\underline{\chi}\check{\gamma}_{AB} = \widetilde{\Omega\operatorname{tr}\underline{\chi}}(\gamma_K)_{AB} + 2\Omega\underline{\check{\chi}}_{AB} - \partial_A(\underline{b}_K^C)\check{\gamma}_{BC} - \partial_B(\underline{b}_K^C)\check{\gamma}_{AC}.$$

Similarly, we have from Proposition 2.20

$$\Omega\nabla_3(\epsilon_{AB}) = \Omega\operatorname{tr}\underline{\chi}\epsilon_{AB} - \partial_A(\underline{b}^C)\epsilon_{BC} - \partial_B(\underline{b}^C)\epsilon_{BC},$$

which yields

$$\Omega_K(e_3)_K(\epsilon_K)_{AB} = \Omega_K\operatorname{tr}_K\underline{\chi}_K(\epsilon_K)_{AB} - \partial_A(\underline{b}_K^C)(\epsilon_K)_{BC} - \partial_B(\underline{b}_K^C)(\epsilon_K)_{BC}.$$

Hence, we have

$$\begin{aligned}\Omega\nabla_3(\check{\epsilon}_{AB}) &= \Omega e_3(\epsilon_{AB}) - \Omega_K(e_3)_K(\epsilon_K)_{AB} - \widetilde{\Omega}e_3(\epsilon_K)_{AB} \\ &= \Omega\operatorname{tr}\underline{\chi}\epsilon_{AB} - \partial_A(\underline{b}^C)\epsilon_{BC} - \partial_B(\underline{b}^C)\epsilon_{BC} - \Omega_K\operatorname{tr}_K\underline{\chi}_K(\epsilon_K)_{AB} \\ &\quad + \partial_A(\underline{b}_K^C)(\epsilon_K)_{BC} + \partial_B(\underline{b}_K^C)(\epsilon_K)_{BC} - \check{\underline{b}}^C\partial_C(\epsilon_K)_{AB} \\ &= \Omega\operatorname{tr}\underline{\chi}\check{\epsilon}_{AB} + \widetilde{\Omega\operatorname{tr}\underline{\chi}}(\epsilon_K)_{AB} - \partial_A(\check{\underline{b}}^C)\epsilon_{BC} - \partial_A(\underline{b}_K^C)\check{\epsilon}_{BC} \\ &\quad - \partial_B(\check{\underline{b}}^C)\epsilon_{AC} - \partial_B(\underline{b}_K^C)\check{\epsilon}_{AC} - \check{\underline{b}}^C\partial_C(\gamma_K)_{AB},\end{aligned}$$

which implies from  $\widetilde{\Omega} = 0$  and  $\check{\underline{b}} = 0$

$$\Omega\nabla_3(\check{\epsilon}_{AB}) = \Omega\operatorname{tr}\underline{\chi}\check{\epsilon}_{AB} + \widetilde{\Omega\operatorname{tr}\underline{\chi}}(\epsilon_K)_{AB} - \partial_A(\underline{b}_K^C)\check{\epsilon}_{BC} - \partial_B(\underline{b}_K^C)\check{\epsilon}_{AC}.$$

This concludes the proof of Proposition A.6.  $\square$

## B Proof of the case $s \in (3, 4)$

In Sections 4–6, we have provided the proof of Theorem 3.2 in the case  $s \in [4, 6]$ . In this appendix, we outline the proof in the case  $s \in (3, 4)$ , which is similar to the case of  $s \in [4, 6]$ .

### B.1 Fundamental norms

The definitions of some of the norms are different from Section 3.1. In the sequel, we only mention the norms that differ from the ones in Section 3.1.

We define the  $\mathcal{R}_0^S[\check{b}]$  and  $\mathcal{R}_0^S[\check{\rho}, \check{\sigma}]$  as follows:

$$\begin{aligned}\mathcal{R}_0^S[\check{b}] &:= \sup_{\mathcal{K}} \sup_{p \in [2, 4]} |r^{\frac{s+3}{2} - \frac{2}{p}} \check{b}|_{p, S}, \\ \mathcal{R}_0^S[\check{\rho}, \check{\sigma}] &:= \sup_{\mathcal{K}} \sup_{p \in [2, 4]} |r^{\frac{s+2}{2} - \frac{2}{p}} |u|^{\frac{1}{2}} (\check{\rho}, \check{\sigma})|_{p, S}.\end{aligned}$$

The following flux of curvature are different from the case  $s \in [4, 6]$  for  $q = 0, 1$ :

$$\begin{aligned}\mathcal{R}_q[\check{b}](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q \check{b}\|_{2, C_{\underline{u}}^V}, \\ \mathcal{R}_q[(\check{\rho}, \check{\sigma})](u, \underline{u}) &:= \|r^{\frac{s}{2}} (r\nabla)^q (\check{\rho}, \check{\sigma})\|_{2, \underline{C}_{\underline{u}}^V}.\end{aligned}$$

The other norms are the same as in the case  $s \in [4, 6]$ .

### B.2 Estimates for Ricci coefficients and curvature components

We discuss the curvature estimates of Section 4 in this case. Recall that we have four Bianchi pairs:  $(\check{\alpha}, \check{b})$ ,  $(\check{b}, (\check{\rho}, \check{\sigma}))$ ,  $((\check{\rho}, \check{\sigma}), \check{\beta})$  and  $(\check{\beta}, \check{\alpha})$ . In Section 4, we took respectively  $p = s, 4, 2, 0$ , to estimate the Bianchi pairs. In the case  $s \in (3, 4)$ , we take  $p = s, s, 2, 0$  to estimate  $\mathcal{R}$ . The method is then exactly the same as in Section 4.

Concerning the control of Ricci coefficients, we can proceed by the same method as in Section 5. Recall that Proposition 5.9 used the fact that there exists a constant  $\delta > 0$  such that

$$\sup_{p \in [2, 4]} |r^{3+\delta - \frac{2}{p}} \check{b}|_{p, S} \lesssim \varepsilon_0.$$

This still holds true since we have  $\frac{s+3}{2} > 3$  for  $s > 3$ . The other propositions still hold true since we only used  $s > 3$  in their proofs.

### B.3 Conclusion

The proof of Theorems M0, M2 and M4 remain exactly the same since all the arguments also apply to the case  $s > 3$ . Consequently, we have Theorem 3.2 in this case. Hence, we deduce that Theorem 3.2 holds true for  $s \in (3, 6]$ .

## C Proof of the case $s > 6$

In this appendix, we prove Theorem 3.2 in the case  $s > 6$ . Then, we compare the result to the peeling decay for curvature components obtained in [7, 30].

### C.1 Fundamental norms

The definitions of  $\mathcal{R}$ -norms for  $\check{\alpha}$  and  $\check{b}$  are different from Section 3.1.1. We denote

$$s_0 := \min \left\{ s, \frac{29}{4} \right\}. \quad (\text{C.1})$$

We define for  $q = 0, 1$ :

$$\begin{aligned} \mathcal{R}_q[\check{b}] &:= |u|^{\frac{s-4}{2}} \|r^2 (r\nabla)^q \check{b}\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[\check{b}] &:= \|r^3 |u|^{\frac{s-6}{2}} (r\nabla)^q \check{b}\|_{2, \underline{C}_u^V}, \\ \mathcal{R}_q[\check{\alpha}] &:= \|r^3 |u|^{\frac{s-6}{2}} (r\nabla)^q \check{\alpha}\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[\check{\alpha}] &:= |u|^{\frac{s-s_0}{2}} \|r^{\frac{s_0}{2}} (r\nabla)^q \check{\alpha}\|_{2, \underline{C}_u^V}. \end{aligned} \quad (\text{C.2})$$

We also define:

$$\underline{\mathcal{R}}_0^S[\check{b}] := \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{4-\frac{2}{p}} |u|^{\frac{s-5}{2}} \check{b}|_{p,S}. \quad (\text{C.3})$$

The norms  $\mathcal{R}_0^S[\check{\alpha}]$  are defined as follows:

$$\begin{aligned} \underline{\mathcal{R}}_0^S[\check{\alpha}] &:= \sup_{\mathcal{K}} |r^{\frac{s+1}{2}} \check{\alpha}|_{2,S}, & s \in (6, 7), \\ \mathcal{R}_0^S[\check{\alpha}] &:= \sup_{\mathcal{K}} |r^4 (\log r)^{-\frac{1}{2}} \check{\alpha}|_{2,S}, & s = 7, \\ \mathcal{R}_0^S[\check{\alpha}] &:= \sup_{\mathcal{K}} |r^4 |u|^{\frac{s-7}{2}} \check{\alpha}|_{2,S}, & s > 7. \end{aligned} \quad (\text{C.4})$$

We also define:

$$\underline{\mathcal{R}}_0^S[\check{\alpha}] := \sup_{\mathcal{K}} \sup_{p \in [2,4]} |r^{1-\frac{2}{p}} |u|^{\frac{s+1}{2}} \check{\alpha}|_{p,S}. \quad (\text{C.5})$$

All the other norms are defined as in Section 3.1.1.

### C.2 Curvature estimates

In Section 4, we took respectively  $p = s, 4, 2, 0$ , to estimate the Bianchi pairs  $(\check{\alpha}, \check{b})$ ,  $(\check{b}, (\check{\rho}, -\check{\sigma}))$ ,  $((\check{\rho}, \check{\sigma}), \check{\beta})$  and  $(\check{\beta}, \check{\alpha})$ . In the case  $s > 6$ , we take respectively  $p = 6, 4, 2, 0$  for the Bianchi pairs. Proceeding as in Section 4, we deduce from Propositions 4.9–4.12 that

$$\sum_{q=0}^1 \left( \mathcal{R}_q[\check{\alpha}] + \mathcal{R}_q[\check{b}] + \underline{\mathcal{R}}_q[\check{b}] + \mathcal{R}_q[\check{\rho}, \check{\sigma}] + \mathcal{R}_q[\check{\beta}] + \underline{\mathcal{R}}_q[\check{\rho}, \check{\sigma}] + \underline{\mathcal{R}}_q[\check{\beta}] + \mathcal{R}_q[\check{\alpha}] \right) \lesssim \varepsilon_0, \quad (\text{C.6})$$

which implies from Proposition 2.45 that

$$\mathcal{R}_0^S[\check{b}] + \mathcal{R}_0^S[\check{\rho}, \check{\sigma}] + \mathcal{R}_0^S[\check{\beta}] \lesssim \varepsilon_0. \quad (\text{C.7})$$

In view of (C.7), it remains to estimate  $\check{\alpha}$  and  $\check{\alpha}$ . Next, we state the following *Teukolsky equation* first derived by Teukolsky in the linearized setting in [58].

**Proposition C.1.** *We have the following Teukolsky equations for  $\alpha$  and  $\underline{\alpha}$ :*

$$\begin{aligned} \Omega \nabla_3 (\Omega \nabla_4 (r \Omega^2 \alpha)) + 2r \Omega^2 d_2^* d_2 (\Omega^2 \alpha) &= -\frac{2\Omega}{r} \nabla_3 (r \Omega^2 \alpha) + \mathfrak{I}^{\leq 1} \left( (\mathbf{O}_3^2 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \beta) \right), \\ \Omega \nabla_4 (\Omega \nabla_3 (r \Omega^2 \underline{\alpha})) + 2r \Omega^2 d_2^* d_2 (\Omega^2 \underline{\alpha}) &= \frac{2\Omega}{r} \nabla_4 (r \Omega^2 \underline{\alpha}) + \mathfrak{I}^{\leq 1} \left( (\mathbf{O}_3^2 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \underline{\alpha}) \right). \end{aligned} \quad (\text{C.8})$$

*Proof.* See for example Propositions 3.4.6 and 3.4.7 in [11]. □

**Lemma C.2.** *We define the following quantities:*

$$\begin{aligned}\hat{\alpha} &:= \frac{1}{r^4} \nabla_4(r^5 \alpha) \in \mathfrak{s}_2, & \phi &:= rd_2 \alpha \in \mathfrak{s}_1, \\ \underline{\hat{\alpha}} &:= \frac{1}{r^4} \nabla_3(r^5 \underline{\alpha}) \in \mathfrak{s}_2, & \underline{\phi} &:= rd_2 \underline{\alpha} \in \mathfrak{s}_1.\end{aligned}\tag{C.9}$$

Then, we have the following equations:

$$\begin{aligned}\nabla_3 \hat{\alpha} &= -2d_2^* \phi + \frac{4\alpha}{r} + \mathfrak{p}^{\leq 1} \left( (\mathbf{O}_2^1 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{b}) \right), \\ \nabla_4 \phi + \frac{5}{2} \operatorname{tr} \chi \phi &= d_2 \hat{\alpha} + \mathfrak{p}^{\leq 1} \left( (\mathbf{O}_2^1 + \Gamma_g) \cdot (\mathbf{O}_4^2 + \check{b}) \right),\end{aligned}\tag{C.10}$$

and

$$\begin{aligned}\nabla_4 \underline{\hat{\alpha}} &= -2d_2^* \underline{\phi} + \frac{4\underline{\alpha}}{r} + \mathfrak{p}^{\leq 1} \left( (\mathbf{O}_2^1 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \hat{\alpha}) \right), \\ \nabla_3 \underline{\phi} + \frac{5}{2} \operatorname{tr} \underline{\chi} \underline{\phi} &= d_2 \underline{\hat{\alpha}} + \mathfrak{p}^{\leq 1} \left( (\mathbf{O}_2^1 + \Gamma_b) \cdot (\mathbf{O}_4^2 + \hat{\alpha}) \right).\end{aligned}\tag{C.11}$$

*Proof.* It follows directly from Lemma C.3 in [53].<sup>27</sup>  $\square$

**Corollary C.3.** *We have the following linearized equations:*

$$\begin{aligned}\nabla_3 \check{\alpha} &= -2d_2^* \check{\phi} + \frac{4\check{\alpha}}{r} + \mathfrak{L}[\alpha] + \operatorname{Err}[\alpha], \\ \nabla_4 \check{\phi} + \frac{5}{2} \operatorname{tr} \chi \check{\phi} &= d_2 \check{\alpha} + \mathfrak{L}[\alpha] + \operatorname{Err}[\alpha], \\ \mathfrak{L}[\alpha] &= \mathbf{O}_4^2 \cdot \Gamma_g^{(1)} + \mathbf{O}_2^1 \cdot \check{b}^{(1)}, \\ \operatorname{Err}[\alpha] &= \Gamma_g^{(1)} \cdot \check{b} + \Gamma_g \cdot \check{b}^{(1)},\end{aligned}$$

and

$$\begin{aligned}\nabla_4 \check{\underline{\alpha}} &= -2d_2^* \check{\underline{\phi}} + \frac{4\check{\underline{\alpha}}}{r} + \mathfrak{L}[\underline{\alpha}] + \operatorname{Err}[\underline{\alpha}], \\ \nabla_3 \check{\underline{\phi}} + \frac{5}{2} \operatorname{tr} \underline{\chi} \check{\underline{\phi}} &= d_2 \check{\underline{\alpha}} + \mathfrak{L}[\underline{\alpha}] + \operatorname{Err}[\underline{\alpha}], \\ \mathfrak{L}[\underline{\alpha}] &= \mathbf{O}_4^2 \cdot \Gamma_b^{(1)} + \mathbf{O}_2^1 \cdot \hat{\alpha}^{(1)}, \\ \operatorname{Err}[\underline{\alpha}] &= \Gamma_b^{(1)} \cdot \hat{\alpha} + \Gamma_b \cdot \hat{\alpha}^{(1)}.\end{aligned}$$

*Proof.* It follows directly from Lemmas C.2 and 2.22.  $\square$

**Lemma C.4.** *We have the following identity for any real number  $p$ :*

$$\begin{aligned}& \operatorname{Div}(r^p |\check{\alpha}|^2 e_3) + 2\operatorname{Div}(r^p |\check{\phi}|^2 e_4) + (p+2)r^{p-1} |\check{\alpha}|^2 + 2(8-p)r^{p-1} |\check{\phi}|^2 \\ &= 4r^p d_1(\check{\alpha} \cdot \check{\phi}) + 8r^{p-1} \check{\alpha} \cdot \check{\alpha} + 2r^p (\check{\alpha} + 2\check{\phi}) \cdot (\mathfrak{L}[\alpha] + \operatorname{Err}[\alpha]) + r^p (\mathbf{O}_2^1 + \Gamma_b) \cdot (|\check{\alpha}|^2, |\check{\phi}|^2).\end{aligned}\tag{C.12}$$

*Proof.* Applying Lemma 4.2 with  $\psi_{(1)} = \check{\alpha}$ ,  $\psi_{(2)} = \check{\phi}$ ,  $a_{(1)} = 0$ ,  $a_{(2)} = \frac{5}{2}$ ,  $h_{(1)} = \frac{4\check{\alpha}}{r} + \mathfrak{L}[\alpha] + \operatorname{Err}[\alpha]$ ,  $h_{(2)} = \mathfrak{L}[\alpha] + \operatorname{Err}[\alpha]$  and  $k = 2$ , we infer

$$\begin{aligned}& \operatorname{Div}(r^p |\check{\alpha}|^2 e_3) + 2\operatorname{Div}(r^p |\check{\phi}|^2 e_4) - \left(1 + \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\check{\alpha}|^2 + 2 \left(4 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\check{\phi}|^2 \\ &= 4r^p d_1(\check{\alpha} \cdot \check{\phi}) + 8r^{p-1} \check{\alpha} \cdot \check{\alpha} + 2r^p (\check{\alpha} + 2\check{\phi}) \cdot (\mathfrak{L}[\alpha] + \operatorname{Err}[\alpha]) + r^p (\mathbf{O}_2^1 + \Gamma_b) \cdot (|\check{\alpha}|^2, |\check{\phi}|^2),\end{aligned}$$

which concludes the proof of Lemma C.4.  $\square$

<sup>27</sup>Note that  $\Gamma_a$  in [53] decays better than  $\mathbf{O}_2^1 + \Gamma_b$ .

We recall the following Poincaré inequality.

**Proposition C.5.** *For any  $\alpha \in \mathfrak{s}_2$ , we have the following Poincaré inequality:*

$$|rd_2\alpha|_{2,S}^2 \geq c_2|\alpha|_{2,S}^2,$$

for sphere  $S = S(u, \underline{u})$ , where  $c_2$  is a constant satisfying:

$$c_2 = 2 - O(\varepsilon). \tag{C.13}$$

*Proof.* See for example Proposition 9.3.2 in [11]. □

**Proposition C.6.** *We have the following estimate:*

$$\int_{C_{\underline{u}}^V} r^{s_0} |\check{\alpha}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\check{\alpha}^{\leq 1}|^2 + \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\alpha}^{\leq 1}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

*Proof.* Integrating (C.12) with  $p = s_0$  and proceeding as in Proposition 4.9, we obtain

$$\begin{aligned} & \int_{C_{\underline{u}}} r^{s_0} |\check{\alpha}|^2 + \int_{\underline{C}_{\underline{u}}} r^{s_0} |\check{\alpha}|^2 + \int_V r^{s_0-1} ((s_0 + 2)|\check{\alpha}|^2 + 2(8 - s_0)|\check{\alpha}|^2 - 8\check{\alpha} \cdot \check{\alpha}) \\ & \lesssim \int_{\Sigma_0 \cap V} r^{s_0} (|\check{\alpha}|^2 + |\check{\alpha}|^2) + \int_V r^{s_0} |(\check{\alpha}, \check{\alpha})| |\mathfrak{L}[\alpha] + \text{Err}[\alpha]| + r^{s_0} (\mathbf{O}_2^1 + \Gamma_b) (|\check{\alpha}|^2 + |\check{\alpha}|^2). \end{aligned}$$

First, we have

$$\int_{\Sigma_0 \cap V} r^{s_0} (|\check{\alpha}|^2 + |\check{\alpha}|^2) \lesssim |u|^{s_0-s} \int_{\Sigma_0 \cap V} r^s (|\check{\alpha}|^2 + |\check{\alpha}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

Next, we estimate

$$\begin{aligned} & \int_V r^{s_0} |(\check{\alpha}, \check{\alpha})| |\mathfrak{L}[\alpha]| \\ & \lesssim \int_V r^{s_0} |(\check{\alpha}, \check{\alpha})| \left| \mathbf{O}_4^2 \cdot \Gamma_g^{(1)} + \mathbf{O}_2^1 \cdot \check{b}^{(1)} \right| \\ & \lesssim \int_V M^2 r^{s_0-4} |(\check{\alpha}, \check{\alpha})| |\Gamma_g^{(1)}| + M r^{s_0-2} |(\check{\alpha}, \check{\alpha})| |\check{b}^{(1)}| \\ & \lesssim M^2 \int_{-\underline{u}}^u \frac{du}{r^{4-\frac{s_0}{2}}} \left( \int_{C_{\underline{u}}^V} r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{C_{\underline{u}}^V} |\Gamma_g^{(1)}|^2 \right)^{\frac{1}{2}} + M^2 \int_{|u|}^u \frac{du}{r^{5-\frac{s_0}{2}}} \left( \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_{\underline{u}}^V} r^2 |\Gamma_g^{(1)}|^2 \right)^{\frac{1}{2}} \\ & \quad + M \int_{-\underline{u}}^u \frac{du}{r^{4-\frac{s_0}{2}}} \left( \int_{C_{\underline{u}}^V} r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{C_{\underline{u}}^V} r^4 |\check{b}^{(1)}|^2 \right)^{\frac{1}{2}} + M \int_{|u|}^u \frac{du}{r^{5-\frac{s_0}{2}}} \left( \int_{\underline{C}_{\underline{u}}^V} r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_{\underline{u}}^V} r^6 |\check{b}^{(1)}|^2 \right)^{\frac{1}{2}} \\ & \lesssim M^2 \int_{-\underline{u}}^u \frac{du}{r^{4-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \left( \int_{|u|}^u du |\Gamma_g^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} + M^2 \int_{|u|}^u \frac{du}{r^{5-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \left( \int_{-\underline{u}}^u du |r\Gamma_g^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \\ & \quad + M \int_{-\underline{u}}^u \frac{du}{r^{4-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} + M \int_{|u|}^u \frac{du}{r^{5-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-6}{2}}} \\ & \lesssim M^2 \int_{-\underline{u}}^u \frac{du}{|u|^{4-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-2}{2}}} + M^2 \int_{|u|}^u \frac{du}{r^{5-\frac{s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-s_0}{2}}} \frac{\varepsilon}{|u|^{\frac{s-4}{2}}} + \frac{\varepsilon^2 M}{|u|^{s-s_0+1}} \\ & \lesssim \frac{\varepsilon^2 M^2}{|u|^{s-s_0+2}} + \frac{\varepsilon^2 M}{|u|^{s-s_0+1}} \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}. \end{aligned}$$

Proceeding as in Proposition C.5 in [53], we infer

$$\int_V r^{s_0} |(\check{\alpha}, \check{\phi})| |\text{Err}[\alpha]| \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

We have from Proposition 2.22

$$\check{\phi} = \widetilde{rd_2\alpha} = rd_2\check{\alpha} + (rd_2 - r_K(d_2)_K)\alpha_K = rd_2\check{\alpha} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^3.$$

Moreover, we have from Proposition C.5

$$\int_V r^{s_0-1} \left( (s_0+2)|\check{\alpha}|^2 + 2(8-s_0)|\check{\phi}|^2 - 8\check{\alpha} \cdot \check{\alpha} \right) \gtrsim \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}^{\leq 1}\check{\alpha}|^2) - \int_V r^{s_0-1} |\Gamma_g^{(1)} \cdot \mathbf{O}_4^3|^2,$$

where we used  $6 < s_0 \leq \frac{29}{4}$  and  $c_2 = 2 - O(\varepsilon)$ . Finally, we have

$$\int_V r^{s_0} (\mathbf{O}_2^1 + \Gamma_b) (|\check{\alpha}|^2 + |\check{\phi}|^2) \lesssim (\varepsilon + \mathcal{E}_0) \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}|^2).$$

Combining the above estimates, we deduce

$$\begin{aligned} & \int_{C_u} r^{s_0} |\check{\alpha}|^2 + \int_{\underline{C}_u} r^{s_0} |\check{\phi}^{\leq 1}\check{\alpha}|^2 + \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}^{\leq 1}\check{\alpha}|^2) \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}} + (\varepsilon + \mathcal{E}_0) \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}|^2) + \int_{\underline{C}_u^V} r^{s_0} |\Gamma_g^{(1)} \cdot \mathbf{O}_4^3|^2 + \int_V r^{s_0-1} |\Gamma_g^{(1)} \cdot \mathbf{O}_4^3|^2 \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}} + (\varepsilon + \mathcal{E}_0) \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}|^2) + M^6 \int_{-u}^u \frac{du}{r^{8-s_0}} |\Gamma_g^{(1)}|_{2,S}^2 + M^6 \int_{-u}^u du \int_{|u|}^u \frac{du}{r^{9-s_0}} |\Gamma_g^{(1)}|_{2,S}^2 \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}} + (\varepsilon + \mathcal{E}_0) \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}|^2) + \frac{M^6 \varepsilon^2}{|u|^{6+s-s_0}} \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}} + (\varepsilon + \mathcal{E}_0) \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}|^2), \end{aligned}$$

which implies for  $\varepsilon$  and  $\mathcal{E}_0$  small enough

$$\int_{C_u} r^{s_0} |\check{\alpha}|^2 + \int_{\underline{C}_u} r^{s_0} |\check{\phi}^{\leq 1}\check{\alpha}|^2 + \int_V r^{s_0-1} (|\check{\alpha}|^2 + |\check{\phi}^{\leq 1}\check{\alpha}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s-s_0}}.$$

This concludes the proof of Proposition C.6.  $\square$

**Proposition C.7.** *We have the following estimate:*

$$\underline{\mathcal{R}}_0[\check{\alpha}] + \underline{\mathcal{R}}_1[\check{\alpha}] + \mathcal{R}_0^S[\check{\alpha}] \lesssim \varepsilon_0. \quad (\text{C.14})$$

*Proof.* We have from Propositions 2.15 and C.6

$$\underline{\mathcal{R}}_0[\check{\alpha}]^2 + \underline{\mathcal{R}}_1[\check{\alpha}]^2 \lesssim |u|^{s-s_0} \int_{\underline{C}_u^V} |\check{\alpha}|^2 + |\check{\phi}\check{\alpha}|^2 \lesssim \varepsilon_0^2. \quad (\text{C.15})$$

Recalling that

$$\check{\alpha} = \frac{1}{r^4} \nabla_4(r^5 \check{\alpha}) + \mathbf{O}_4^3 \cdot \Gamma_g^{(1)},$$

we infer from Lemma 2.7 that

$$\begin{aligned}
\Omega e_4 \left( \int_S |r^4 \check{\alpha}|^2 \right) &= \int_S 2\Omega \nabla_4(r^4 \check{\alpha}) \cdot (r^4 \check{\alpha}) + \Omega \operatorname{tr} \chi |r^4 \check{\alpha}|^2 \\
&= \int_S 2\Omega r^4 \check{\alpha} \cdot (\nabla_4(r^4 \check{\alpha}) + r^3 \check{\alpha}) + \Omega \left( \operatorname{tr} \chi - \frac{2}{r} \right) |r^4 \check{\alpha}|^2 \\
&= \int_S 2\Omega r^4 \check{\alpha} \cdot (\nabla_4(r^5 \check{\alpha}) r^{-1} + e_4(r^{-1}) r^5 \check{\alpha} + r^3 \check{\alpha}) + (\mathbf{O}_2^1 + \Gamma_g) |r^4 \check{\alpha}|^2 \\
&= \int_S 2\Omega r^4 \check{\alpha} \cdot (\nabla_4(r^5 \check{\alpha}) r^{-1}) + (\mathbf{O}_2^1 + \Gamma_g) |r^4 \check{\alpha}|^2 \\
&= \int_S 2\Omega r^4 \check{\alpha} \cdot r^3 \check{\alpha} + (\mathbf{O}_2^1 + \Gamma_g) |r^4 \check{\alpha}|^2 + (r^4 \check{\alpha}) \cdot \mathbf{O}_1^3 \cdot \Gamma_g^{(1)}.
\end{aligned}$$

Hence, we deduce

$$|e_4(|r^4 \check{\alpha}|_{2,S}^2)| \lesssim |r^4 \check{\alpha}|_{2,S} |r^3 \check{\alpha}|_{2,S} + (M + \varepsilon) r^{-2} |r^4 \check{\alpha}|_{2,S}^2 + |r^4 \check{\alpha}|_{2,S} |\mathbf{O}_1^3 \cdot \Gamma_g^{(1)}|_{2,S},$$

which implies

$$|e_4(|r^4 \check{\alpha}|_{2,S})| \lesssim |r^3 \check{\alpha}|_{2,S} + (M + \varepsilon) r^{-2} |r^4 \check{\alpha}|_{2,S} + |\mathbf{O}_1^3 \cdot \Gamma_g^{(1)}|_{2,S}. \quad (\text{C.16})$$

Integrating (C.16) along  $C_u$  and applying the initial assumption and Proposition C.6, we infer

$$\begin{aligned}
|r^4 \check{\alpha}|_{2,S} &\lesssim \frac{\varepsilon_0}{r^{\frac{s-7}{2}}} + \int_{|u|}^u \frac{d\underline{u}}{|u|} |r^3 \check{\alpha}|_{2,S} + \int_{|u|}^u \frac{(M + \varepsilon) d\underline{u}}{r^2} |r^4 \check{\alpha}|_{2,S} + \int_{|u|}^u \frac{d\underline{u}}{r^2 |u|^{\frac{s-3}{2}}} M^3 \\
&\lesssim \frac{\varepsilon_0}{|u|^{\frac{s-7}{2}}} + \left( \int_{|u|}^u r^{6-s_0} d\underline{u} \right)^{\frac{1}{2}} \left( \int_{|u|}^u r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}} + \int_{|u|}^u \frac{(M + \varepsilon) d\underline{u}}{r^2} |r^4 \check{\alpha}|_{2,S}.
\end{aligned}$$

Applying Gronwall inequality, we obtain

$$|r^4 \check{\alpha}|_{2,S} \lesssim \frac{\varepsilon_0}{|u|^{\frac{s-7}{2}}} + \left( \int_{|u|}^u r^{6-s_0} d\underline{u} \right)^{\frac{1}{2}} \left( \int_{|u|}^u r^{s_0} |\check{\alpha}|^2 \right)^{\frac{1}{2}}. \quad (\text{C.17})$$

Notice that we have

$$\int_{|u|}^u r^{6-s_0} d\underline{u} = \begin{cases} \frac{1}{7-s_0} (\underline{u}^{7-s_0} - |u|^{7-s_0}), & s_0 \in (6, 7), \\ \log \left( \frac{u}{|u|} \right), & s_0 = 7, \\ \frac{1}{s_0-7} (|u|^{7-s_0} - \underline{u}^{7-s_0}), & s_0 > 7. \end{cases}$$

Hence, we obtain from (C.17)

$$|r^4 \alpha|_{2,S} \lesssim \begin{cases} \varepsilon_0 r^{\frac{7-s}{2}}, & s \in (6, 7), \\ \varepsilon_0 (\log r)^{\frac{1}{2}}, & s = 7, \\ \varepsilon_0 |u|^{\frac{7-s}{2}}, & s > 7, \end{cases} \quad (\text{C.18})$$

which implies

$$\mathcal{R}_0^S[\check{\alpha}] \lesssim \varepsilon_0. \quad (\text{C.19})$$

Combining (C.15) and (C.19), this concludes the proof of Proposition C.7.  $\square$



**Proposition C.8.** *We have the following estimate:*

$$\underline{\mathcal{R}}_0^S[\check{\alpha}] \lesssim \varepsilon_0.$$

*Proof.* We recall from Corollary C.3

$$\begin{aligned} \nabla_4 \check{\alpha} &= -2d_2^* \check{\alpha} + \frac{4\check{\alpha}}{r} + \mathfrak{L}[\alpha] + \text{Err}[\alpha], \\ \nabla_3 \check{\alpha} + \frac{5}{2} \text{tr} \chi \check{\alpha} &= d_2 \check{\alpha} + \mathfrak{L}[\alpha] + \text{Err}[\alpha], \\ \mathfrak{L}[\alpha] &= \mathbf{O}_4^2 \cdot \Gamma_b^{(1)} + \mathbf{O}_2^1 \cdot \check{\alpha}^{(1)}, \\ \text{Err}[\alpha] &= \Gamma_b^{(1)} \cdot \check{\alpha} + \Gamma_b \cdot \check{\alpha}^{(1)}. \end{aligned}$$

Applying Lemma 4.2 with  $\psi_{(1)} = \check{\alpha}$ ,  $\psi_{(2)} = \check{\alpha}$ ,  $a_{(1)} = \frac{5}{2}$ ,  $a_{(2)} = 0$ ,  $h_{(1)} = \mathfrak{L}[\alpha] + \text{Err}[\alpha]$ ,  $h_{(2)} = \frac{4\check{\alpha}}{r} + \mathfrak{L}[\alpha] + \text{Err}[\alpha]$ ,  $k = 2$  and  $p = -2$ , we obtain

$$\begin{aligned} &2\mathbf{Div}(r^{-2}|\check{\alpha}|^2 e_3) + \mathbf{Div}(r^{-2}|\check{\alpha}|^2 e_4) + 10r^{-2} \text{tr} \chi |\check{\alpha}|^2 \\ &= 2r^{-2} d_1(\check{\alpha} \cdot \check{\alpha}) + r^{-2}(\check{\alpha}, \check{\alpha}) \cdot (\mathfrak{L}[\alpha] + \text{Err}[\alpha]) + 8r^{-3} \check{\alpha} \cdot \check{\alpha} + r^{-2}(\mathbf{O}_2^1 + \Gamma_b)(|\check{\alpha}|^2, |\check{\alpha}|^2). \end{aligned}$$

We compute for  $c_2$  defined in Proposition C.5

$$\begin{aligned} 2c_2 \mathbf{Div}(r^{-2}|\check{\alpha}|^2 e_3) &= 2c_2 \nabla_3(r^{-2}|\check{\alpha}|^2) + 2c_2 r^{-2} |\check{\alpha}|^2 \mathbf{Div}(e_3) \\ &= 2c_2 \nabla_3(r^{-12}|r^5 \check{\alpha}|^2) + 2c_2 r^{-2} |\check{\alpha}|^2 (\text{tr} \chi - 2\omega) \\ &= 20c_2 r^{-3} |\check{\alpha}|^2 + 4c_2 r^{-3} \check{\alpha} \cdot \check{\alpha} + r^{-2}(\mathbf{O}_2^1 + \Gamma_b) |\check{\alpha}|^2 + \mathbf{O}_7^3 \cdot \Gamma_b^{(1)} \cdot \check{\alpha}, \end{aligned}$$

where we used

$$\begin{aligned} \check{\alpha} &= \frac{1}{r^4} \nabla_3(r^5 \alpha) - \frac{1}{r_K^4} (\nabla_K)_{(e_3)_K} (r_K^5 \alpha_K) \\ &= \frac{1}{r^4} \nabla_3(r^5 \alpha) - \frac{1}{r^4} \nabla_3(r^5 \alpha_K) + \frac{1}{r^4} \nabla_3(r^5 \alpha_K) - \frac{1}{r_K^4} (\nabla_K)_{(e_3)_K} (r_K^5 \alpha_K) \\ &= \frac{1}{r^4} \nabla_3(r^5 \check{\alpha}) + \mathbf{O}_4^3 \cdot \Gamma_b^{(1)}. \end{aligned} \tag{C.20}$$

Combining the above identities, we deduce

$$\begin{aligned} &2\mathbf{Div}(r^{-2}(|\check{\alpha}|^2 - c_2 |\check{\alpha}|^2) e_3) + \mathbf{Div}(r^{-2}|\check{\alpha}|^2) + 10r^{-2} \text{tr} \chi |\check{\alpha}|^2 + 20c_2 r^{-3} |\check{\alpha}|^2 \\ &= 2r^{-2} d_1(\check{\alpha} \cdot \check{\alpha}) + (8 - 4c_2) r^{-3} \check{\alpha} \cdot \check{\alpha} + r^{-2}(\check{\alpha}, \check{\alpha}) \cdot (\mathfrak{L}[\alpha] + \text{Err}[\alpha]) + \mathbf{O}_7^3 \cdot \Gamma_b^{(1)} \cdot \check{\alpha} + r^{-2}(\mathbf{O}_2^1 + \Gamma_b) |\check{\alpha}|^2. \end{aligned}$$

Integrating it in  $V$ , we obtain from Proposition C.5

$$\begin{aligned} \int_{C_u^V} r^{-2} (|\check{\alpha}|^2 - c_2 |\check{\alpha}|^2) + \int_{C_u^V} r^{-2} |\check{\alpha}|^2 &\lesssim \int_{\Sigma_0 \cap V} r^{-2} (|\check{\alpha}|^2 + |\check{\alpha}|^2) + \int_V r^{-3} |\check{\alpha}|^2 + \varepsilon r^{-3} |\check{\alpha}| |\check{\alpha}| \\ &\quad + \int_V r^{-2} |(\check{\alpha}, \check{\alpha})| |\mathfrak{L}[\alpha] + \text{Err}[\alpha]| + |\mathbf{O}_7^3 \cdot \Gamma_b^{(1)}| |\check{\alpha}| + r^{-2} (\mathbf{O}_2^1 + \Gamma_b) |\check{\alpha}|^2. \end{aligned}$$

We first have from Proposition 2.22

$$\check{\alpha} = r \widetilde{d_2 \alpha} = r d_2 \check{\alpha} + (r d_2 - r_K (d_2)_K) \alpha_K = r d_2 \check{\alpha} + \Gamma_g^{(1)} \cdot \mathbf{O}_4^3.$$

Noticing that

$$\int_{C_u^V} r^{-2} |\Gamma_g^{(1)} \cdot \mathbf{O}_4^3|^2 \lesssim \int_{|u|}^u du \frac{M^6}{r^{12}} |r \Gamma_g^{(1)}|_{2,S}^2 \lesssim M^6 \int_{|u|}^u du \frac{\varepsilon^2}{r^{12} |u|^{s-3}} \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}},$$

Hence, we obtain from Proposition C.5

$$\int_{C_u^V} r^{-2}(|\check{\underline{\alpha}}|^2 - c_2|\check{\underline{\alpha}}|^2) \geq \int_{C_u^V} r^{-2}(|rd_2\check{\underline{\alpha}}|^2 - c_2|\check{\underline{\alpha}}|^2) - \int_{C_u^V} r^{-2}|\Gamma_g^{(1)} \cdot \mathbf{O}_4^3|^2 \gtrsim -\frac{\varepsilon_0^2}{|u|^{s+2}}.$$

Next, we have

$$\int_{\Sigma_0 \cap V} r^{-2}(|\check{\underline{\alpha}}|^2 + |\check{\underline{\alpha}}|^2) \lesssim \frac{1}{|u|^{s+2}} \int_{\Sigma_0 \cap V} r^s(|\check{\underline{\alpha}}|^2 + |\check{\underline{\alpha}}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}.$$

Then, we compute

$$\begin{aligned} \int_V r^{-3}|\check{\underline{\alpha}}|^2 + \varepsilon \int_V r^{-3}|\check{\underline{\alpha}}||\check{\underline{\alpha}}| &\lesssim \int_{|u|}^u \frac{du}{r^3} \int_{C_u^V} |\check{\underline{\alpha}}|^2 + \varepsilon \int_{|u|}^u r^{-2} \left( \int_{C_u^V} r^{-2}|\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} du \\ &\lesssim \varepsilon_0^2 \int_{|u|}^u \frac{du}{r^3|u|^s} + \varepsilon \int_{|u|}^u r^{-2} \left( \frac{\varepsilon^2}{|u|^{s+2}} \right)^{\frac{1}{2}} \left( \frac{\varepsilon_0^2}{|u|^s} \right)^{\frac{1}{2}} du \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}. \end{aligned}$$

We also have

$$\begin{aligned} &\int_V r^{-2}|\check{\underline{\alpha}}, \check{\underline{\alpha}}||\mathfrak{L}[\underline{\alpha}]| \\ &\lesssim \int_V r^{-2}|\check{\underline{\alpha}}, \check{\underline{\alpha}}| \left| \mathbf{O}_4^2 \cdot \Gamma_b^{(1)} + \mathbf{O}_2^1 \cdot \check{\underline{\alpha}}^{(1)} \right| \\ &\lesssim \int_V M^2 r^{-6}|\check{\underline{\alpha}}||\Gamma_b^{(1)}| + M^2 r^{-6}|\check{\underline{\alpha}}||\Gamma_b^{(1)}| + Mr^{-4}|\check{\underline{\alpha}}||\check{\underline{\alpha}}^{(1)}| + Mr^{-4}|\check{\underline{\alpha}}||\check{\underline{\alpha}}^{(1)}| \\ &\lesssim M^2 \int_{|u|}^u \frac{du}{r^5} \left( \int_{C_u^V} r^{-2}|\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\Gamma_b^{(1)}|^2 \right)^{\frac{1}{2}} + M^2 \int_{|u|}^u \frac{du}{r^5} \left( \int_{C_u^V} |\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |r^{-1}\Gamma_b^{(1)}|^2 \right)^{\frac{1}{2}} \\ &\quad + M \int_{|u|}^u \frac{du}{r^3} \left( \int_{C_u^V} r^{-2}|\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\check{\underline{\alpha}}^{(1)}|^2 \right)^{\frac{1}{2}} + M \int_{|u|}^u \frac{du}{r^4} \left( \int_{C_u^V} |\check{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{C_u^V} |\check{\underline{\alpha}}^{(1)}|^2 \right)^{\frac{1}{2}} \\ &\lesssim M^2 \int_{|u|}^u \frac{du}{r^5} \frac{\varepsilon}{|u|^{\frac{s+2}{2}}} \left( \int_{-u}^u du |\Gamma_b^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} + M^2 \int_{|u|}^u \frac{du}{r^5} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \left( \int_{-u}^u du |r^{-1}\Gamma_b^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \\ &\quad + M \int_{|u|}^u \frac{du}{r^3} \frac{\varepsilon}{|u|^{\frac{s+2}{2}}} \frac{\varepsilon}{|u|^{\frac{s}{2}}} + M \int_{|u|}^u \frac{du}{r^4} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \\ &\lesssim \frac{M^2 \varepsilon^2}{r^4 |u|^s} + \frac{M \varepsilon^2}{r^2 |u|^{s+1}} + \frac{M \varepsilon^2}{r^3 |u|^s} \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}. \end{aligned}$$

We proceed as in Proposition C.7 in [53] to deduce

$$\int_V r^{-2}|\check{\underline{\alpha}}, \check{\underline{\alpha}}||\text{Err}[\underline{\alpha}]| \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}.$$

Moreover, we have

$$\begin{aligned}
\int_V |\dot{\underline{\alpha}}| |\mathbf{O}_7^3 \cdot \Gamma_b^{(1)}| &\lesssim M^3 \int_{|u|}^u \frac{du}{r^7} \left( \int_{\underline{C}_u^V} |\dot{\underline{\alpha}}|^2 \right)^{\frac{1}{2}} \left( \int_{\underline{C}_u^V} |\Gamma_b^{(1)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim M^3 \int_{|u|}^u \frac{du}{r^7} \frac{\varepsilon}{|u|^{\frac{s}{2}}} \left( \int_{-u}^u du |\Gamma_b^{(1)}|_{2,S}^2 \right)^{\frac{1}{2}} \\
&\lesssim M^3 \varepsilon \int_{|u|}^u \frac{du}{r^7 |u|^{\frac{s}{2}}} \frac{\varepsilon}{|u|^{\frac{s-2}{2}}} \\
&\lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}.
\end{aligned}$$

Finally, we have

$$\int_V r^{-2} (\mathbf{O}_2^1 + \Gamma_b) |\dot{\underline{\alpha}}|^2 \lesssim \int_V (\varepsilon + \varepsilon_0) r^{-3} |\dot{\underline{\alpha}}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}.$$

Combining the above estimates, we deduce

$$\int_{\underline{C}_u^V} r^{-2} |\check{\underline{\alpha}}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}},$$

which implies from (C.20) and (4.13)

$$\int_{\underline{C}_u^V} r^{-2} |\nabla_3 \dot{\underline{\alpha}}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{s+2}}. \quad (\text{C.21})$$

Combining (C.21) and (C.2) and applying Proposition 2.45, we deduce

$$\underline{\mathcal{R}}_0^S[\check{\underline{\alpha}}] \lesssim \varepsilon_0.$$

This concludes the proof of Proposition C.8.  $\square$

Combining Propositions C.7 and C.8 with (C.2), this concludes the proof of Theorem M1 in the case  $s > 6$ . Notice that Theorems M0, M2, M3, M4 also hold true in the case  $s > 6$  since we only used  $s > 3$  in their proofs<sup>28</sup>. Combining with Section B.3, this concludes the proof of Theorem 3.2 for  $s > 3$  as stated.

### C.3 Peeling decay for curvature components

As a consequence of Theorem M1 in the case  $s > 7$ , the curvature components satisfy the following decay:

$$\begin{aligned}
|r^{5-\frac{2}{p}} |u|^{\frac{s-7}{2}} \check{\underline{\alpha}}|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{4-\frac{2}{p}} |u|^{\frac{s-5}{2}} \check{\underline{b}}|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{3-\frac{2}{p}} |u|^{\frac{s-3}{2}} (\check{\underline{\rho}}, \check{\underline{\sigma}})|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{2-\frac{2}{p}} |u|^{\frac{s-1}{2}} \check{\underline{\beta}}|_{p,S} &\lesssim \varepsilon_0, \\
\sup_{p \in [2,4]} |r^{1-\frac{2}{p}} |u|^{\frac{s+1}{2}} \check{\underline{\alpha}}|_{p,S} &\lesssim \varepsilon_0.
\end{aligned} \quad (\text{C.22})$$

<sup>28</sup>See the discussion in Section B.2.

**Remark C.9.** Assume  $s > 7$  and that the initial data in  $\mathcal{K}_{(0)}$  has sufficient regularity properties. Then, commuting  $r\nabla$  with the Bianchi equations in Propositions 2.37 and 2.38 and the Teukolsky equation (C.8), proceeding as in Sections 4 and C.2, we deduce the following strong peeling properties:

$$\begin{aligned} |\check{\alpha}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^5 |u|^{\frac{s-7}{2}}}, & |\check{b}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^4 |u|^{\frac{s-5}{2}}}, \\ |\check{\rho}, \check{\sigma}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^3 |u|^{\frac{s-3}{2}}}, & |\check{\beta}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r^2 |u|^{\frac{s-1}{2}}}, & |\check{\alpha}|_{\infty, S(u, \underline{u})} &\lesssim \frac{\varepsilon_0}{r |u|^{\frac{s+1}{2}}}, \end{aligned} \quad (\text{C.23})$$

which recovers the results obtained in [7] by the vectorfield method. See also Theorem 1 in [30] and Remark C.8 in [53] for the particular case of Minkowski stability.



## Chapter 5

# Construction of GCM hypersurfaces in perturbations of Kerr

## 1 Introduction

### 1.1 Stability of Kerr conjecture

In this paper, we construct general covariant modulated (GCM) hypersurfaces, which play a central role in the proof of the nonlinear stability of Kerr family for  $|a| \ll m$ .

**Conjecture** (Stability of Kerr conjecture). *Vacuum initial data sets, sufficiently close to Kerr initial data, have a maximal development with complete future null infinity<sup>1</sup> and with domain of outer communication which approaches (globally) a nearby Kerr solution.*

The paper builds on the strategy laid out in [33] in the context of the nonlinear stability of Schwarzschild for axially symmetric polarized perturbations. The central new idea of [33] was the introduction and construction of GCM spheres and GCM hypersurfaces on which specific geometric quantities take Schwarzschildian values. This was made possible by taking into account the full general covariance of the Einstein vacuum equations. The construction, however, also made essential use of the polarization assumption.

The goal of this, and its companion papers [34] and [35], is to get rid of the symmetry restriction in the GCM procedure and thus remove an essential obstruction in extending the result in [33] to a full stability proof of the Kerr family.

### 1.2 Stability of Schwarzschild in the polarized case

#### 1.2.1 GCM admissible spacetimes in [33]

In [33], Klainerman and Szeftel proved the nonlinear stability of the Schwarzschild space under axially symmetric polarized perturbations. The final spacetime in [33] was constructed as the limit of a continuous family of finite GCM admissible spacetimes as represented in Figure 5.1 below, whose future boundaries consist of the union  $\mathcal{A} \cup \underline{\mathcal{C}}_* \cup \mathcal{C}_* \cup \Sigma_*$  where  $\mathcal{A}$  and  $\Sigma_*$  are spacelike,  $\underline{\mathcal{C}}_*$  is incoming null, and  $\mathcal{C}_*$  outgoing null. The boundary  $\mathcal{A}$  is chosen so that, in the limit when  $\mathcal{M}$  converges to the final state, it is included inside the black hole region of the limit spacetime. The spacetime  $\mathcal{M}$  also contains a timelike hypersurface  $\mathcal{T}$  which divides  $\mathcal{M}$  into an exterior region we call  ${}^{(ext)}\mathcal{M}$  and an interior one  ${}^{(int)}\mathcal{M}$ . Both  ${}^{(ext)}\mathcal{M}$  and  ${}^{(int)}\mathcal{M}$  are foliated by 2-surfaces as follows.

- (i) The far region  ${}^{(ext)}\mathcal{M}$  is foliated by a geodesic foliation  $S(u, s)$  induced by an outgoing optical function  $u$  initialized on  $\Sigma_*$  with  $s$  the affine parameter along the null geodesic generators of  ${}^{(ext)}\mathcal{M}$ . We denote by  $r = r(u, s)$  the area radius of  $S(u, s)$ . On the boundary  $\Sigma_*$  of  ${}^{(ext)}\mathcal{M}$  we also assume that  $r$  is sufficiently large.
- (ii) The near region  ${}^{(int)}\mathcal{M}$  is foliated by a geodesic foliation induced by an incoming optical function  $\underline{u}$  initialized at  $\mathcal{T}$  such that its level sets on  $\mathcal{T}$  coincide with those of  $u$ .

To prove convergence to the final state one has to establish precise decay estimates for all Ricci and curvature coefficients decomposed relative to the null geodesic frames associated to the foliations in  ${}^{(ext)}\mathcal{M}$  and  ${}^{(int)}\mathcal{M}$ . The decay properties of both Ricci and curvature coefficients in  ${}^{(ext)}\mathcal{M}$  depend heavily on the choice of the boundary  $\Sigma_*$  as well as on the choice of the cuts of the optical function  $u$  on it. As such, the central idea of [33] was the introduction and construction of GCM hypersurfaces on which specific geometric quantities take Schwarzschildian values.

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<sup>1</sup>This means, roughly, that observers which are far away from the black hole may live forever.

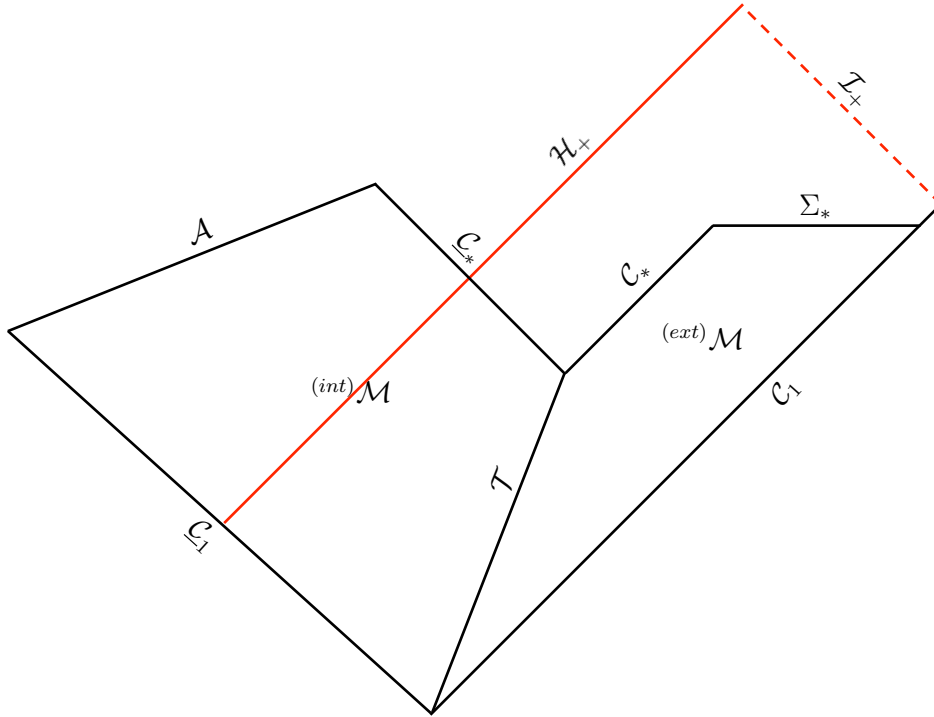


Figure 5.1: The GCM admissible space-time  $\mathcal{M}$  of [33]

### 1.2.2 The role played by GCM admissible spacetimes

As mentioned above the final spacetime was constructed as the limit of a continuous family of finite GCM admissible spacetimes. At every stage one assumes that all Ricci and curvature coefficients of a fixed GCM admissible spacetime  $\mathcal{M}$  verify precise bootstrap assumptions. One makes use of the GCM admissibility properties of  $\Sigma_*$  and the smallness of the initial conditions to show that all the bounds of the Ricci and curvature coefficients of  $\mathcal{M}$  depend only on the size of the initial data and thus, in particular, improve the bootstrap assumptions. This allows to extend the spacetime to a larger one  $\mathcal{M}'$  in which the bootstrap assumptions are still valid. To make sure that the extended spacetime is admissible, one has to construct a new GCM hypersurface  $\tilde{\Sigma}_*$  in  $\mathcal{M}' \setminus \mathcal{M}$  and use it to define a new extended GCM admissible spacetime  $\tilde{\mathcal{M}}$ . The goal of the present work is to extend the construction of  $\Sigma_*$  of Section 9.8 of [33], for polarized perturbations of Schwarzschild, to the case of general perturbations of Kerr.

### 1.3 Review of the main results of [34]

The main building block of our GCM hypersurface are the GCM spheres constructed in [34] which we now review.

#### 1.3.1 Background space

As in [34], we consider spacetime regions  $\mathcal{R}$  foliated by a geodesic foliation  $S(u, s)$  induced by an outgoing optical function  $u$  with  $s$  a properly normalized affine parameter along the null geodesic generators of  $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$  where  $\mathbf{g}$  is the spacetime metric. We denote by  $r = r(u, s)$  the area radius of  $S(u, s)$  and let  $(e_3, e_4, e_1, e_2)$  be an adapted null frame with  $e_4$  proportional to  $L$  and  $e_1, e_2$  tangent to spheres  $S = S(u, s)$ , see Section 2.8. The main assumptions made in [34] were that the Ricci and curvature coefficients, relative to the adapted null frame, have the same asymptotics in powers of  $r$  as in Schwarzschild space. Note that these assumptions hold true in the far region of Kerr and is expected to hold true for the far region of realistic perturbations



of Kerr. The actual size of the perturbation from Kerr is measured with respect to a small parameter  $\overset{\circ}{\varepsilon} > 0$ , see sections 2.8.2 and 2.8.3 for precise definitions.

### 1.3.2 Null frame transformation

In general, two null frames  $(e_3, e_4, e_1, e_2)$  and  $(e'_3, e'_4, e'_1, e'_2)$  are related by a frame transformation of the following form:<sup>2</sup>

$$\begin{aligned} e'_4 &= \lambda \left( e_4 + f^b e_b + \frac{1}{4} |f|^2 e_3 \right), \\ e'_a &= \left( \delta_{ab} + \frac{1}{2} \underline{f}_a f_b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left( \frac{1}{2} f_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \\ e'_3 &= \lambda^{-1} \left( \left( 1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16} |f|^2 |\underline{f}|^2 \right) e_3 + \left( \underline{f}^b + \frac{1}{4} |f|^2 f^b \right) e_b + \frac{1}{4} |f|^2 e_4 \right), \end{aligned} \quad (1.1)$$

where the scalar  $\lambda$  and the 1-forms  $f$  and  $\underline{f}$  are called the transition coefficients from  $(e_3, e_4, e_1, e_2)$  to  $(e'_3, e'_4, e'_1, e'_2)$ .

### 1.3.3 Basis of $\ell = 1$ modes

We introduce the following generalization of the  $\ell = 1$  spherical harmonics of the standard sphere<sup>3</sup>.

**Definition 1.1.** *On a sphere  $S$ , an  $\overset{\circ}{\varepsilon}$ -approximated basis of  $\ell = 1$  modes is a triplet of functions  $J^{(p)}$  on  $S$  verifying*

$$\begin{aligned} (r^2 \Delta + 2) J^{(p)} &= O(\overset{\circ}{\varepsilon}), \quad p = 0, +, -, \\ \frac{1}{|S|} \int_S J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\overset{\circ}{\varepsilon}), \quad p, q = 0, +, -, \\ \frac{1}{|S|} \int_S J^{(p)} &= O(\overset{\circ}{\varepsilon}), \quad p = 0, +, -, \end{aligned} \quad (1.2)$$

where  $\overset{\circ}{\varepsilon} > 0$  is a sufficiently small constant.

**Remark 1.2.** *For simplicity, throughout this paper,  $J^{(p)}$  is called a basis of  $\ell = 1$  modes.*

Assuming the existence of such a basis  $J^{(p)}$ ,  $p \in \{-, 0, +\}$ , we define, for a scalar function  $h$ ,

$$(h)_{\ell=1}^S := \left\{ \int_S h J^{(p)}, \quad p = -, 0, + \right\}. \quad (1.3)$$

A scalar function  $h$  is said to be supported on  $\ell \leq 1$  modes, i.e.  $(f)_{\ell \geq 2}^S = 0$ , if there exist constants  $A_0, B_-, B_0, B_+$  such that

$$h = A_0 + B_- J^{(-)} + B_0 J^{(0)} + B_+ J^{(+)}. \quad (1.4)$$

<sup>2</sup>See Lemma 2.13 for a precise statement.

<sup>3</sup>Recall that on the standard sphere  $\mathbb{S}^2$ , in spherical coordinates  $(\theta, \varphi)$ , these are  $J^{(0, \mathbb{S}^2)} = \cos \theta$ ,  $J^{(+, \mathbb{S}^2)} = \sin \theta \cos \varphi$ ,  $J^{(-, \mathbb{S}^2)} = \sin \theta \sin \varphi$ .

### 1.3.4 Definition of GCM spheres

The null expansions  $\kappa := \text{tr } \chi$  and  $\underline{\kappa} := \text{tr } \underline{\chi}$  relative to the adapted null frame  $(e_3, e_4, e_1, e_2)$  are defined by

$$\text{tr } \chi := \mathbf{g}^{ab} \chi_{ab}, \quad \text{tr } \underline{\chi} := \mathbf{g}^{ab} \underline{\chi}_{ab},$$

where

$$\chi_{ab} := \mathbf{g}(\mathbf{D}_{e_a} e_4, e_b), \quad \underline{\chi}_{ab} := \mathbf{g}(\mathbf{D}_{e_a} e_3, e_b).$$

The mass aspect function  $\mu$  is defined by

$$\mu := -\text{div } \zeta - \rho + \frac{1}{2} \widehat{\chi} \cdot \widehat{\underline{\chi}},$$

where the shears  $\widehat{\chi}, \widehat{\underline{\chi}}$ , the torsion  $\zeta$  and the curvature components  $\rho$  are defined by

$$\begin{aligned} \widehat{\chi}_{ab} &:= \chi_{ab} - \frac{1}{2} \delta_{ab} \kappa, & \widehat{\underline{\chi}}_{ab} &:= \underline{\chi}_{ab} - \frac{1}{2} \delta_{ab} \underline{\kappa}, \\ \zeta_a &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_a} e_4, e_3), & \rho &:= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4). \end{aligned}$$

In an outgoing geodesic foliation of Schwarzschild spacetime, we have:

$$\kappa = \frac{2}{r}, \quad \underline{\kappa} = -\frac{2\Upsilon}{r}, \quad \mu = \frac{2m}{r^3}, \quad (1.5)$$

where  $\Upsilon = 1 - \frac{2m}{r}$  and  $r, m$  denote the area radius and Hawking mass of  $S$ , i.e.

$$r := \sqrt{\frac{|S|}{4\pi}}, \quad \frac{2m}{r} := 1 + \frac{1}{16\pi} \int_{\mathbf{S}} \kappa \underline{\kappa}. \quad (1.6)$$

The idea to construct GCM spheres is to mimic the condition (1.5) in the perturbed spacetimes. More precisely, the GCM spheres are topological spheres  $\mathbf{S}$  embedded in  $\mathcal{R}$  endowed with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  adapted to  $\mathbf{S}$  (i.e.  $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$  tangent to  $\mathbf{S}$ ), relative to which the null expansions  $\kappa^{\mathbf{S}} = \text{tr } \chi^{\mathbf{S}}$ ,  $\underline{\kappa}^{\mathbf{S}} = \text{tr } \underline{\chi}^{\mathbf{S}}$  and mass aspect function  $\mu^{\mathbf{S}}$  satisfy:

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (1.7)$$

where  $r^{\mathbf{S}}$  and  $m^{\mathbf{S}}$  denote the area radius and Hawking mass of  $\mathbf{S}$ .

### 1.3.5 Deformations of spheres and frame transformations

The construction of GCM spheres in [34] was obtained by deforming a given sphere  $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$  of the background foliation of  $\mathcal{R}$ . An  $O(\delta)$  deformation of  $\overset{\circ}{S}$  is defined by a map  $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S} \subset \mathcal{R}$  of the form

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \left( \overset{\circ}{u} + U(y^1, y^2), \overset{\circ}{s} + S(y^1, y^2), y^1, y^2 \right) \quad (1.8)$$

with  $(U, S)$  smooth functions on  $\overset{\circ}{S}$ , vanishing at a fixed point of  $\overset{\circ}{S}$ , of size proportional to the small constant  $\delta$  and  $(y^1, y^2)$  are spherical coordinates on  $\overset{\circ}{S}$ . Given such a deformation we identify, at any point on  $\mathbf{S}$ , two important null frames.

1. The null frame  $(e_3, e_4, e_1, e_2)$  of the background foliation of  $\mathcal{R}$ .
2. A null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  obtained from (1.1) adapted to  $\mathbf{S}$ , (i.e.  $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$  tangent to  $\mathbf{S}$ ).

**Remark 1.3.** Throughout this paper, we denote  $(f, \underline{f}, \lambda)$  the transition coefficients from the background frame  $(e_3, e_4, e_1, e_2)$  of  $\mathcal{R}$  to the null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  adapted to  $\mathbf{S}$ .

### 1.3.6 GCM spheres with $\ell = 1$ modes in [34]

Here is a short version of the main result in [34].

**Theorem 1.4** (Existence of GCM spheres in [34]). *Let  $\mathcal{R}$  be fixed spacetime region, endowed with an outgoing geodesic foliation  $S(u, s)$ , verifying specific asymptotic assumptions<sup>4</sup> expressed in terms of two parameters  $0 < \overset{\circ}{\delta} \leq \overset{\circ}{\varepsilon}$ . In particular we assume that the GCM quantities of the background spheres in  $\mathcal{R}$ , i.e.*

$$\kappa - \frac{2}{r}, \quad \left( \frac{\kappa}{r} + \frac{2\Upsilon}{r} \right)_{\ell \geq 2}, \quad \left( \mu - \frac{2m}{r^3} \right)_{\ell \geq 2}, \quad (1.9)$$

are small with respect to the parameter  $\overset{\circ}{\delta}$ . Let  $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$  be a fixed sphere of the foliation with  $\overset{\circ}{r}$  and  $\overset{\circ}{m}$  denoting respectively its area radius and Hawking mass, with  $\overset{\circ}{r}$  sufficiently large. Then, for any fixed triplets  $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$  verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \overset{\circ}{\delta}, \quad (1.10)$$

there exists a unique sphere  $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ , together with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ , which is GCM, i.e.  $\mathbf{S}$  is a deformation of  $\overset{\circ}{S}$ , such that<sup>5</sup>

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \frac{\kappa^{\mathbf{S}}}{r^{\mathbf{S}}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (1.11)$$

and

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (1.12)$$

where  $(f, \underline{f}, \lambda)$  denote the transition coefficients of the transformation (1.1) from the background frame of  $\mathcal{R}$  to the frame adapted to  $\mathbf{S}$ .

**Remark 1.5.** *The conditions (1.9), (1.11) and (1.12) depend on the definition of  $\ell = 1$  modes respectively on  $\overset{\circ}{S}$  and  $\mathbf{S}$ . In [34], once a choice of  $\ell = 1$  modes on  $\overset{\circ}{S}$  is made, it is then extended to  $\mathbf{S}$  using the background foliation. As a consequence, the GCM spheres of Theorem 1.4 depend on the particular choice of  $\ell = 1$  modes on  $\overset{\circ}{S}$ .*

## 1.4 First version of the main theorem

The goal of this paper is to construct hypersurfaces which are suitable concatenations of the spheres of Theorem 1.4. We give below a simplified version of our main theorem, see Theorem 4.1 for the precise version.

**Theorem 1.6** (Existence of GCM hypersurfaces, first version). *Let  $\mathcal{R}$  be fixed spacetime region, endowed with an outgoing geodesic foliation  $S(u, s)$ , verifying same assumptions as Theorem 1.4. Assume in addition that  $e_3(J^{(p)})$ ,  $(\operatorname{div} \eta)_{\ell=1}$ ,  $(\operatorname{div} \underline{\xi})_{\ell=1}$ ,  $r - s$  and  $e_3(r) - e_3(s)$  are small with respect to the parameter  $\overset{\circ}{\delta}$ .*

Let  $\mathbf{S}_0$  be a fixed sphere included in the region  $\mathcal{R}$ , let a pair of triplets  $\Lambda_0, \underline{\Lambda}_0 \in \mathbb{R}^3$  such that

$$|\Lambda_0|, |\underline{\Lambda}_0| \lesssim \overset{\circ}{\delta},$$

<sup>4</sup>Compatible with small perturbations of Kerr.

<sup>5</sup>Note that the GCM conditions (1.11) require a choice of  $\ell = 1$  modes on  $\mathbf{S}$ , see Remark 1.5.

and let  $J^{(\mathbf{S}_0,p)}$  a basis of  $\ell = 1$  modes on  $\mathbf{S}_0$ , such that we have on  $\mathbf{S}_0$

$$\kappa^{\mathbf{S}_0} - \frac{2}{r^{\mathbf{S}_0}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}_0} + \frac{2\Upsilon^{\mathbf{S}_0}}{r^{\mathbf{S}_0}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}_0} - \frac{2m^{\mathbf{S}_0}}{(r^{\mathbf{S}_0})^3} \right)_{\ell \geq 2} = 0,$$

and

$$(\operatorname{div}^{\mathbf{S}_0} f)_{\ell=1} = \Lambda_0, \quad (\operatorname{div}^{\mathbf{S}_0} \underline{f})_{\ell=1} = \underline{\Lambda}_0,$$

where  $(f, \underline{f})$  denote the transition coefficients of the transformation (1.1) from the background frame of  $\mathcal{R}$  to the frame adapted to  $\mathbf{S}_0$ .

Then, there exists a unique, local, smooth, spacelike hypersurface  $\Sigma_0$  passing through  $\mathbf{S}_0$ , a scalar function  $u^{\mathbf{S}}$  defined on  $\Sigma_0$ , whose level surfaces are topological spheres denoted by  $\mathbf{S}$ , a smooth collection of triplets of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  and a triplet of functions  $J^{(\mathbf{S},p)}$  defined on  $\Sigma_0$  verifying,

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \quad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0, \quad J^{(\mathbf{S},p)}|_{\mathbf{S}_0} = J^{(\mathbf{S}_0,p)},$$

such that the following conditions are verified:

1. The surfaces  $\mathbf{S}$  of constant  $u^{\mathbf{S}}$ , together with a null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ , verify

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left( \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left( \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (1.13)$$

and

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda^{\mathbf{S}}, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}^{\mathbf{S}}, \quad (1.14)$$

for the triplets of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  and  $\ell = 1$  modes  $J^{(\mathbf{S},p)}$ .

2. The following transversality conditions

$$\xi^{\mathbf{S}} = 0, \quad \omega^{\mathbf{S}} = 0, \quad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \quad (1.15)$$

and

$$e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0, \quad e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1 \quad (1.16)$$

are assumed on  $\Sigma_0$ .

3. We have, for some constant  $c_0$ ,

$$u^{\mathbf{S}} + r^{\mathbf{S}} = c_0, \quad \text{along } \Sigma_0. \quad (1.17)$$

4. Let  $\nu^{\mathbf{S}}$  be the unique vectorfield tangent to the hypersurface  $\Sigma_0$ , normal to  $\mathbf{S}$ , and normalized by  $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$ . Let  $b^{\mathbf{S}}$  be the unique scalar function on  $\Sigma_0$  such that  $\nu^{\mathbf{S}}$  is given by

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}. \quad (1.18)$$

The following normalization condition holds true

$$\overline{b^{\mathbf{S}}} = -1 - \frac{2m_{(0)}}{r^{\mathbf{S}}}, \quad (1.19)$$

where  $\overline{b^{\mathbf{S}}}$  is the average value of  $b^{\mathbf{S}}$  over  $\mathbf{S}$  and  $m_{(0)}$  is a constant.

5. We have the following identities on  $\Sigma_0$ :

$$(\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} = 0, \quad (\operatorname{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} = 0. \quad (1.20)$$

6. The transition functions  $(f, \underline{f}, \lambda)$ , area radius  $r^{\mathbf{S}}$  and Hawking mass  $m^{\mathbf{S}}$  verify appropriate estimates.

**Remark 1.7.** Theorem 1.6 is the generalization of Theorem 9.52 in [33] in the absence of symmetry. It plays a central role in the proof of Theorems M6 and M7 in [36], see sections 8.4 and 8.5 in [36].

**Remark 1.8.** We provide below more explanations for the statements 1-5 in Theorem 1.6:

1. Since we concatenate a family of GCM spheres  $\mathbf{S}(\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}})$  emanating from  $\mathbf{S}_0$  to construct the GCM hypersurfaces  $\Sigma_0$ , by Theorem 1.4, we have automatically (1.13) and (1.14) on every  $\mathbf{S}$ .
2. The transversality conditions (1.15) and (1.16) are consistent with a local extension by an outgoing geodesic foliation initialized on  $\Sigma_0$ , see item 1 in Remark 4.12. The role of these transversality conditions is to make sense of  $\eta^{\mathbf{S}}$  and  $\underline{\xi}^{\mathbf{S}}$  on  $\Sigma_0$ , see item 5 below.
3.  $u^{\mathbf{S}}$  should be chosen to be constant on the GCM spheres foliating  $\Sigma_0$ . The choice (1.17) is simple and fulfills this condition but other choices making  $\Sigma_0$  spacelike are possible.
4. The value  $\overline{b^{\mathbf{S}}}$  is free and should be prescribed. Note that the choice (1.19) coincides with the value for the hypersurface  $\{u + r = c_0\}$  in Schwarzschild spacetime.
5. In (1.20),  $\eta^{\mathbf{S}}$  and  $\underline{\xi}^{\mathbf{S}}$  are defined intrinsically on  $\Sigma_0$  by

$$\eta_a^{\mathbf{S}} = \frac{1}{2} \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_4^{\mathbf{S}}, e_a^{\mathbf{S}}), \quad \underline{\xi}_a^{\mathbf{S}} = \frac{1}{2} \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_3^{\mathbf{S}}, e_a^{\mathbf{S}}) + b^{\mathbf{S}} \zeta_a^{\mathbf{S}}.$$

These definitions are consistent with the standard ones provided  $\Sigma_0$  satisfies (1.15) which is equivalent to extending  $\Sigma_0$  locally by an outgoing geodesic foliation, see item 2 in Remark 4.12. In the sequel, (1.20) will be enforced thanks to a special choice of  $\Lambda^{\mathbf{S}}$  and  $\underline{\Lambda}^{\mathbf{S}}$ , see Section 1.5 for more explanations.

**Remark 1.9.** As in Section 9.8 of [33],  $\Sigma_0$  is chosen to be spacelike. One may wonder whether  $\Sigma_0$  could be chosen to be null<sup>6</sup>. The reason for choosing it to be spacelike is that it allows more flexibility: all spheres foliating  $\Sigma_0$  in Theorem 1.6 are GCM spheres, while only one could be a GCM sphere on a null hypersurface.

## 1.5 Sketch of the proof of the main theorem

The idea of the proof is to construct  $\Sigma_0$  as a 1-parameter union of GCM spheres.

**Step 1.** For every background sphere  $S(u, s)$  in  $\mathcal{R}$ , every pair of triplets  $(\Lambda, \underline{\Lambda})$  and every triplet of functions  $\tilde{J}^{(p)}$  satisfying

$$\sum_{p=0,+,-} \|J^{(p)} - \tilde{J}^{(p)}\|_{\mathfrak{h}_{s_{max}}(S(u,s))} \lesssim r \delta,$$

<sup>6</sup>In the context of the stability of Minkowski, the last slice in the original proof by Christodoulou and Klainerman in [10] is spacelike, while it is null in the proof by Klainerman and Nicolò in [29] in the case of the exterior of an outgoing null cone.

where  $J^{(p)}$  is the  $\ell = 1$  modes on  $S(u, s)$  and  $\mathfrak{h}_{s_{max}}(S(u, s))$  denotes the Sobolev space on  $S(u, s)$  of order  $s_{max}$ , we construct by Theorem 1.4 a unique GCM sphere  $\mathbf{S}[u, s, \Lambda, \underline{\Lambda}, \tilde{J}^{(p)}]$ , as a deformation of  $S(u, s)$  with  $\ell = 1$  modes in the definition of (1.11) and (1.12) computed w.r.t.  $\tilde{J}^{(p)}$ . In particular, (1.11) and (1.12) are verified and we have  $\mathbf{S}_0 = \mathbf{S}[\overset{\circ}{u}, \overset{\circ}{s}, \Lambda_0, \underline{\Lambda}_0, \tilde{J}^{(p)}]$ , provided we choose  $\tilde{J}^{(p)}|_{\mathbf{S}_0} = J^{(\mathbf{S}_0, p)}$ .

**Step 2.** Given  $(\Psi(s), \Lambda(s), \underline{\Lambda}(s))$  such that

$$\Psi(\overset{\circ}{s}) = \overset{\circ}{u}, \quad \Lambda(\overset{\circ}{s}) = \Lambda_0, \quad \underline{\Lambda}(\overset{\circ}{s}) = \underline{\Lambda}_0,$$

We construct, relying on Step 1 and a Banach fixed-point argument, see Theorem 4.10, a family of basis of  $\ell = 1$  modes  $\tilde{J}(s)$  and of GCM spheres  $\mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]$  verifying

$$\nu^{\mathbf{S}}(\tilde{J}(s)) = 0 \quad \text{on } \Sigma, \quad \tilde{J}^{(p)}(\overset{\circ}{s}) = J^{(\mathbf{S}_0, p)},$$

where the hypersurface  $\Sigma$  is given by

$$\Sigma = \bigcup_s \mathbf{S}(s) = \bigcup_s \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)],$$

and where  $\nu^{\mathbf{S}}$  is the unique vectorfield tangent to  $\Sigma$  with  $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$  and normal to  $\mathbf{S}(s)$ , see Figure 5.2 for a geometric description.

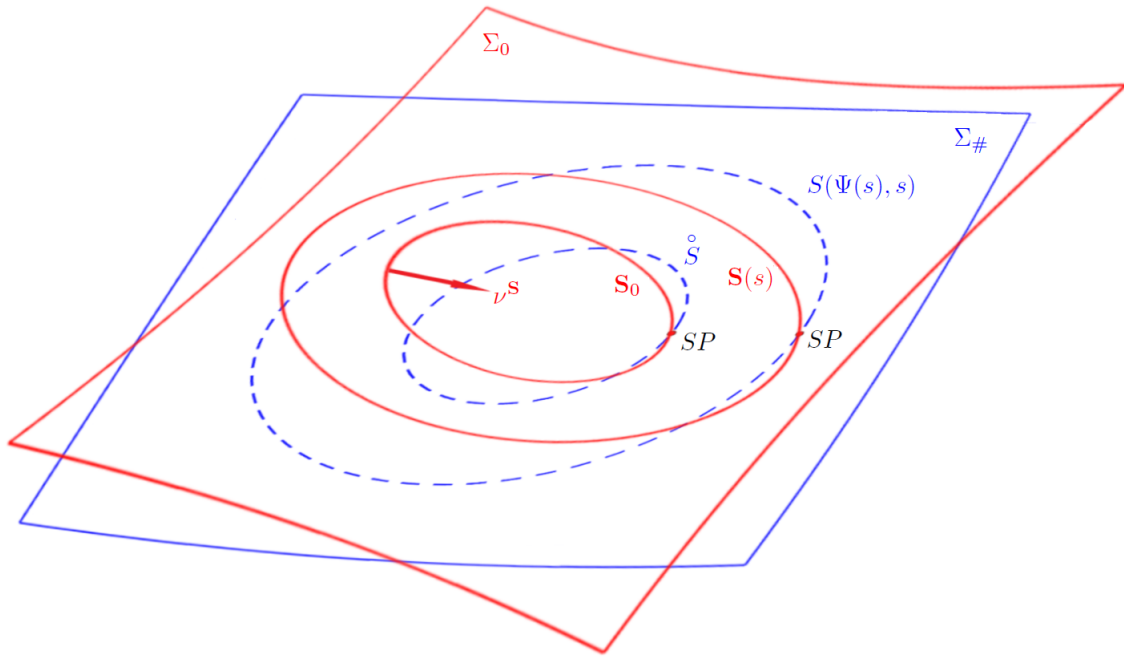


Figure 5.2: The GCM hypersurface  $\Sigma_0$  as a deformation of  $\Sigma_{\#} = \{u = \Psi(s)\}$

**Step 3.** We then derive for  $(\Psi, \Lambda, \underline{\Lambda})$  an ODE system of the following type:

$$\begin{aligned} \frac{1}{\Psi'(s)} \Lambda'(s) &= (\text{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} - \frac{1}{2} r^{-1} \Lambda(s) - \frac{1}{2} r^{-1} \underline{\Lambda}(s) + \text{l.o.t.}, \\ \frac{1}{\Psi'(s)} \underline{\Lambda}'(s) &= (\text{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} + \text{l.o.t.}, \\ \Psi'(s) &= -1 - \frac{1}{2} \left( \overline{b^{\mathbf{S}}} + 1 + \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} \right) + \text{l.o.t.}, \end{aligned} \tag{1.21}$$

where l.o.t. denote lower order terms, see Section 4.4 for the precise statement.

**Step 4.** We look for a special choice  $(\check{\Psi}(s), \check{\Lambda}(s), \check{\underline{\Lambda}}(s))$  such that the additional GCM conditions (1.19) and (1.20) are verified. These conditions lead, in view of Step 3, to an ODE system for  $(\check{\Psi}(s), \check{\Lambda}(s), \check{\underline{\Lambda}}(s))$ , with prescribed initial conditions at  $\mathring{s}$  which allows us to uniquely determine the desired hypersurface  $\Sigma_0$ .

**Remark 1.10.** *The proof of Theorem 1.6 is largely analogous to that of Theorem 9.52 in [33]. Below, we compare the proof in this paper and that in Section 9.8 of [33].*

- *In Step 1 and Step 2, we show that, in general, one can choose the approximate basis of  $\ell = 1$  modes so that they are transported along the normal direction to the GCM spheres  $\mathbf{S}(s)$  on  $\Sigma$ . This contrasts with [33] where the basis of  $\ell = 1$  modes is fixed by the polarized symmetry.*
- *Once the choice of  $\ell = 1$  modes is made, in Step 3, we derive the system of ODEs (1.21). Note that the coefficients of linear terms of  $\Lambda(s)$  and  $\underline{\Lambda}(s)$  on the R.H.S. of (1.21) are different from that of (9.8.74) in [33], which is due to the different choice of  $\ell = 1$  modes.<sup>7</sup>*
- *Steps 1-3 are significantly more involved than the corresponding in [33] due the absence of symmetry, while Step 4 is similar to that in [33].*

## 1.6 Structure of the paper

The structure of the paper is as follows:

- In Section 2, we review the geometric set-up of [34].
- In Section 3, we review the main result of [34] on the construction of GCM spheres.
- In Section 4, we prove our main theorem concerning the construction of GCM hypersurfaces.

## 1.7 Acknowledgements

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# 2 Geometric set-up

## 2.1 Ricci and curvature coefficients

As in [34], we consider given a vacuum spacetime  $\mathcal{R}$  with metric  $\mathbf{g}$  endowed with an outgoing geodesic foliation by spheres  $S(u, s)$  of fixed  $(u, s)$ , where  $u$  is an outgoing optical function<sup>8</sup> with  $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$  its null geodesic generator and  $s$  chosen such that

$$L(s) = \frac{1}{\varsigma}, \quad L(\varsigma) = 0.$$

Let  $e_4 = \varsigma L$  and  $e_3$  the unique null vectorfield orthogonal to  $S(u, s)$  and such that  $\mathbf{g}(e_3, e_4) = -2$ . We then let  $(e_1, e_2)$  be an orthogonal basis of the tangent space of  $S(u, s)$ . The corresponding connection coefficients relative to the null frame  $(e_3, e_4, e_1, e_2)$  are denoted by  $\chi, \underline{\chi}, \xi, \underline{\xi}, \omega, \underline{\omega}, \eta, \underline{\eta}, \zeta$

<sup>7</sup>More precisely, the basis of  $\ell = 1$  modes in [33], fixed by polarized symmetry, is not transported along the vectorfield  $\nu^{\mathbf{S}}$ .

<sup>8</sup>That is  $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ .

and the null components of the curvature tensor by  $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \star\rho$ . For the convenience of the reader we recall their definition below:

$$\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a e_3, e_b), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b), \\
\underline{\xi}_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_a), & \xi_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_a), \\
\underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \\
\underline{\eta}_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_a), & \eta_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_a), \\
\underline{\zeta}_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_a} e_4, e_3),
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\alpha_{ab} &= \mathbf{R}(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= \mathbf{R}(e_a, e_3, e_b, e_3), \\
\beta_a &= \frac{1}{2} \mathbf{R}(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2} \mathbf{R}(e_a, e_3, e_3, e_4), \\
\rho &= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4), & \star\rho &= \frac{1}{4} \star\mathbf{R}(e_3, e_4, e_3, e_4),
\end{aligned} \tag{2.2}$$

where  $\star\mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ . The null second fundamental forms  $\chi, \underline{\chi}$  are further decomposed in their traces  $\kappa = \text{tr } \chi$  and  $\underline{\kappa} = \text{tr } \underline{\chi}$ , and traceless parts  $\widehat{\chi}$  and  $\widehat{\underline{\chi}}$ :

$$\begin{aligned}
\kappa &:= \text{tr } \chi = \delta^{ab} \chi_{ab}, & \widehat{\chi}_{ab} &:= \chi_{ab} - \frac{1}{2} \delta_{ab} \kappa, \\
\underline{\kappa} &:= \text{tr } \underline{\chi} = \delta^{ab} \underline{\chi}_{ab}, & \widehat{\underline{\chi}}_{ab} &:= \underline{\chi}_{ab} - \frac{1}{2} \delta_{ab} \underline{\kappa}.
\end{aligned} \tag{2.3}$$

We define the horizontal covariant operator  $\nabla$  as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3.$$

We also define  $\nabla_4 X$  and  $\nabla_3 X$  to be the horizontal projections:

$$\begin{aligned}
\nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_4) e_3, \\
\nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_3) e_3 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_4) e_4.
\end{aligned}$$

As a direct consequence of (2.1), we have the Ricci formulas:

$$\begin{aligned}
\mathbf{D}_a e_b &= \nabla_a e_b + \frac{1}{2} \chi_{ab} e_3 + \frac{1}{2} \underline{\chi}_{ab} e_4, \\
\mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \underline{\zeta}_a e_3, \\
\mathbf{D}_a e_4 &= \chi_{ab} e_b - \zeta_a e_4, \\
\mathbf{D}_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4, \\
\mathbf{D}_4 e_a &= \nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_4, \\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b, \\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\underline{\eta}_b e_b, \\
\mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_b e_b, \\
\mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_b e_b.
\end{aligned} \tag{2.4}$$



We recall, see Lemma 2.4 in [34], that the geodesic nature of the foliation implies

$$\omega = \xi = 0, \quad \underline{\eta} = -\zeta, \quad \varsigma = \frac{2}{e_3(u)}.$$

We denote

$$\underline{\Omega} := e_3(r), \quad z := e_3(u) = \frac{2}{\varsigma}. \quad (2.5)$$

The area radius  $r = r(u, s)$  is defined such that the volume of  $S$  is given by  $4\pi r^2$ . The Hawking mass  $m = m(u, s)$  of  $S = S(u, s)$  is given by the formula

$$\frac{2m}{r} = 1 + \frac{1}{16\pi} \int_S \kappa_{\underline{K}}. \quad (2.6)$$

The Gauss curvature of  $S$  is denoted by  $K$ . It verifies the Gauss equation

$$K = -\rho - \frac{1}{4} \kappa_{\underline{K}} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi}. \quad (2.7)$$

The mass aspect function  $\mu$  is defined by

$$\mu := -\operatorname{div} \zeta - \rho + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi}. \quad (2.8)$$

As in [34], we define the renormalized quantities<sup>9</sup>

$$\begin{aligned} \widetilde{\operatorname{tr} \chi} &:= \operatorname{tr} \chi - \frac{2}{r}, & \widetilde{\operatorname{tr} \underline{\chi}} &:= \operatorname{tr} \underline{\chi} + \frac{2\Upsilon}{r}, & \widetilde{\omega} &:= \omega - \frac{m}{r^2}, \\ \check{K} &:= K - \frac{1}{r^2}, & \check{\rho} &:= \rho + \frac{2m}{r^3}, & \check{\mu} &:= \mu - \frac{2m}{r^3}, \\ \check{\underline{\Omega}} &:= \underline{\Omega} + \Upsilon, & \check{\varsigma} &:= \varsigma - 1, & \check{z} &:= z - 2, \end{aligned} \quad (2.9)$$

where

$$\Upsilon := 1 - \frac{2m}{r},$$

and the sets

$$\begin{aligned} \Gamma_g &:= \left\{ \check{\kappa}, \widehat{\chi}, \zeta, \check{\underline{\kappa}}, r\check{\mu}, r\check{\rho}, r^*\rho, r\beta, r\alpha, r\check{K} \right\}, \\ \Gamma_b &:= \left\{ \eta, \widehat{\underline{\chi}}, \check{\omega}, \check{\underline{\xi}}, r\check{\beta}, \underline{\alpha}, r^{-1}\check{\underline{\Omega}}, r^{-1}\check{\varsigma}, r^{-1}\check{z}, r^{-1}(e_3(r) + \Upsilon) \right\}. \end{aligned} \quad (2.10)$$

**Remark 2.1.** *The renormalized Ricci coefficients are divided into two groups  $\Gamma_g$  and  $\Gamma_b$  according to their  $r$ -weights. See Remark 4.24 for more details.*

## 2.2 Basic equations for the renormalized quantities

We recall some of the null structure equations and Bianchi identities for the renormalized quantities, see Proposition 5.1.17 in [36].

<sup>9</sup>Renormalized quantities are obtained by subtracting their Schwarzschild values.

**Proposition 2.2.** *In an outgoing geodesic foliation, the following null structure equations hold true*

$$\begin{aligned} \nabla_4 \check{\kappa} + \frac{2}{r} \check{\kappa} &= \Gamma_g \cdot \Gamma_g, \\ \nabla_4 \widehat{\chi} + \frac{2}{r} \widehat{\chi} &= -\alpha + \Gamma_g \cdot \Gamma_g, \\ \nabla_4 \zeta + \frac{2}{r} \zeta &= -\beta + \Gamma_g \cdot \Gamma_g, \\ \nabla_4 \check{\kappa} + \frac{1}{r} \check{\kappa} &= -2 \operatorname{div} \zeta + 2\check{\rho} + \frac{\Upsilon}{r} \check{\kappa} + \Gamma_b \cdot \Gamma_g, \\ \nabla_4 \widehat{\chi} + \frac{1}{r} \widehat{\chi} &= \frac{\Upsilon}{r} \widehat{\chi} - \nabla \widehat{\otimes} \zeta + \Gamma_b \cdot \Gamma_g, \end{aligned}$$

and

$$\begin{aligned} \nabla_3 \check{\kappa} - \frac{\Upsilon}{r} \check{\kappa} &= 2 \operatorname{div} \eta + 2\check{\rho} - \frac{1}{r} \check{\kappa} + \frac{4}{r} \check{\omega} + \frac{2m}{r^2} \check{\kappa} + \frac{2}{r^2} \check{y} + \Gamma_b \cdot \Gamma_b, \\ \nabla_3 \check{\kappa} - \frac{2\Upsilon}{r} \check{\kappa} &= 2 \operatorname{div} \underline{\xi} + \frac{4\Upsilon}{r} \check{\omega} - \frac{2m}{r^2} \check{\kappa} - \left( \frac{2}{r^2} - \frac{8m}{r^3} \right) \check{y} + \Gamma_b \cdot \Gamma_b, \\ \nabla_3 \widehat{\chi} - \frac{2\Upsilon}{r} \widehat{\chi} &= -\alpha - \frac{2m}{r^2} \widehat{\chi} + \nabla \widehat{\otimes} \underline{\xi} + \Gamma_b \cdot \Gamma_b, \\ \nabla_3 \zeta - \frac{\Upsilon}{r} \zeta &= -\beta - 2\nabla \check{\omega} + \frac{\Upsilon}{r} (\eta + \zeta) + \frac{1}{r} \underline{\xi} + \frac{2m}{r} (\zeta - \eta) + \Gamma_b \cdot \Gamma_b, \\ \nabla_3 \widehat{\chi} - \frac{\Upsilon}{r} \widehat{\chi} &= \nabla \widehat{\otimes} \eta - \frac{1}{r} \widehat{\chi} + \frac{2m}{r^2} \widehat{\chi} + \Gamma_b \cdot \Gamma_b. \end{aligned}$$

Also,

$$\begin{aligned} \operatorname{curl} \eta &= * \rho + \Gamma_b \cdot \Gamma_g, \\ \operatorname{curl} \underline{\xi} &= \Gamma_b \cdot \Gamma_b, \\ \check{\mu} &= -\operatorname{div} \zeta - \check{\rho} + \Gamma_b \cdot \Gamma_g. \end{aligned}$$

The Bianchi identities are given by

$$\begin{aligned} \nabla_3 \alpha - \frac{\Upsilon}{r} \alpha &= \nabla \widehat{\otimes} \beta + \frac{4m}{r^2} \alpha + \frac{6m}{r^3} \widehat{\chi} + \Gamma_b \cdot (\alpha, \beta) + r^{-1} \Gamma_g \cdot \Gamma_g, \\ \nabla_4 \beta + \frac{4}{r} \beta &= -\operatorname{div} \alpha + r^{-1} \Gamma_g \cdot \Gamma_g, \\ \nabla_3 \beta - \frac{2\Upsilon}{r} \beta &= (\nabla \rho + * \nabla^* \rho) + \frac{2m}{r^2} \beta - \frac{6m}{r^3} \eta + r^{-1} \Gamma_b \cdot \Gamma_g, \\ \nabla_4 \check{\rho} + \frac{3}{r} \check{\rho} &= \operatorname{div} \beta + \frac{3m}{r^3} \check{\kappa} + r^{-1} \Gamma_b \cdot \Gamma_g, \\ \nabla_3 \check{\rho} - \frac{3\Upsilon}{r} \check{\rho} &= -\operatorname{div} \underline{\beta} + \frac{3m}{r^3} \check{\kappa} - \frac{6m}{r^4} \check{\omega} - \frac{1}{2} \widehat{\chi} \cdot \underline{\alpha} + r^{-1} \Gamma_b \cdot \Gamma_b, \\ \nabla_4 * \rho + \frac{3}{r} * \rho &= -\operatorname{curl} \beta + r^{-1} \Gamma_b \cdot \Gamma_g, \\ \nabla_3 * \rho - \frac{3\Upsilon}{r} * \rho &= -\operatorname{curl} \underline{\beta} - \frac{1}{2} \widehat{\chi} \cdot * \underline{\alpha} + r^{-1} \Gamma_b \cdot \Gamma_b. \end{aligned}$$

*Proof.* The proof follows immediately from the standard null structure equations and Bianchi identities (see Propositions 7.3.2 and 7.4.1 in [10]) and the definition (2.9) of the renormalized quantities, see Proposition 5.1.17 in [36] for more details.  $\square$

### 2.3 Commutation lemmas

We recall the following commutation lemmas.

**Lemma 2.3.** *For any tensor on a sphere  $S$ , the following commutation formulas hold true*

$$\begin{aligned} [\nabla_3, \nabla_a]f &= -\frac{1}{2} \operatorname{tr} \underline{\chi} \nabla_a f + (\eta_a - \zeta_a) \nabla_3 f - \widehat{\chi}_{ab} \nabla_b f + \xi_a \nabla_4 f + (\underline{F}[f])_a, \\ [\nabla_4, \nabla_a]f &= -\frac{1}{2} \operatorname{tr} \chi \nabla_a f + (\underline{\eta}_a + \zeta_a) \nabla_4 f - \widehat{\chi}_{ab} \nabla_b f + \xi_a \nabla_3 f + (F[f])_a, \end{aligned}$$

where the tensors  $F[f]$  and  $\underline{F}[f]$  have the following schematic form

$$F[f] = (\beta, \chi \cdot \underline{\eta}, \underline{\chi} \cdot \xi) \cdot f, \quad \underline{F}[f] = (\underline{\beta}, \underline{\chi} \cdot \eta, \chi \cdot \underline{\xi}) \cdot f.$$

*Proof.* See Lemma 7.3.3 in [10]. □

**Lemma 2.4.** *The following commutation formulas hold true for any tensor  $f$  on a sphere  $S$ :*

$$\begin{aligned} [\nabla_3, \nabla]f &= \frac{\Upsilon}{r} \nabla f + \Gamma_b \cdot \nabla_3 f + r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leq 1} f, \\ [\nabla_4, \nabla]f &= -\frac{1}{r} \nabla f + r^{-1} \Gamma_g \cdot \mathfrak{d}^{\leq 1} f, \\ [\nabla_\nu, \nabla]f &= \frac{2}{r} \nabla f + \Gamma_b \cdot \nabla_\nu f + r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leq 1} f, \\ [\nabla_\nu, \Delta]f &= \frac{4}{r} \nabla f + r^{-1} \mathfrak{d}^{\leq 1} (\Gamma_b \cdot \nabla_\nu f + r^{-1} \Gamma_b \cdot \mathfrak{d} f), \end{aligned}$$

where

$$\mathfrak{d} \in \{\nabla_3, r\nabla_4, \mathfrak{d}\}, \quad \mathfrak{d} := r\nabla, \quad \mathfrak{d}^{\leq 1} f := (f, \mathfrak{d}f). \quad (2.11)$$

*Proof.* See Lemma 5.1.19 in [36]. □

### 2.4 Hodge operators

We recall the following Hodge operators acting on 2–surface  $S$  (see Chapter 2 in [10]):

**Definition 2.5.** *We define the following Hodge operators:*

1. The operator  $\mathfrak{d}_1$  takes any 1–form  $f$  into the pair of functions  $(\operatorname{div} f, \operatorname{curl} f)$ .
2. The operator  $\mathfrak{d}_2$  takes any symmetric traceless 2–tensor  $f$  into the 1–form  $\operatorname{div} f$ .
3. The operator  $\mathfrak{d}_1^*$  takes any pair of scalars  $(h, {}^*h)$  into the 1–form  $-\nabla h + {}^*\nabla {}^*h$ .
4. The operator  $\mathfrak{d}_2^*$  takes any 1–form  $f$  into the symmetric traceless 2–tensor  $-\frac{1}{2} \nabla \widehat{\otimes} f$ .

One can easily check that  $\mathfrak{d}_k^*$  is the formal adjoint on  $L^2(S)$  of  $\mathfrak{d}_k$  for  $k = 1, 2$ . Moreover,

$$\begin{aligned} \mathfrak{d}_1^* \mathfrak{d}_1 &= -\Delta_1 + K, & \mathfrak{d}_1 \mathfrak{d}_1^* &= -\Delta_0, \\ \mathfrak{d}_2^* \mathfrak{d}_2 &= -\frac{1}{2} \Delta_2 + K, & \mathfrak{d}_2 \mathfrak{d}_2^* &= -\frac{1}{2} (\Delta_1 + K), \end{aligned} \quad (2.12)$$

where  $\Delta_k$ ,  $k = 0, 1, 2$  is the Laplace operator on scalars, 1–forms and symmetric traceless 2–tensors.

### 2.5 Definition of $\ell = 1$ modes

**Definition 2.6.** On a sphere  $S$ , a (suitable  $\overset{\circ}{\varepsilon}$ -approximation of) basis of  $\ell = 1$  modes is a triplet of functions  $J^{(p)}$  on  $S$  satisfying<sup>10</sup>

$$\begin{aligned} (r^2\Delta + 2)J^{(p)} &= O(\overset{\circ}{\varepsilon}), & p = 0, +, -, \\ \frac{1}{|S|} \int_S J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\overset{\circ}{\varepsilon}), & p, q = 0, +, -, \\ \frac{1}{|S|} \int_S J^{(p)} &= O(\overset{\circ}{\varepsilon}), & p = 0, +, -, \end{aligned} \tag{2.13}$$

where  $\overset{\circ}{\varepsilon} > 0$  is a sufficiently small constant.

**Remark 2.7.** For simplicity, throughout this paper,  $J^{(p)}$  is called a basis of  $\ell = 1$  modes.

**Proposition 2.8.** For a basis of  $\ell = 1$  modes  $J^{(p)}$  defined in Definition 2.6, we have

$$\not{d}_2^* \not{d}_1^* J^{(p)} = r^{-2} O(\overset{\circ}{\varepsilon}). \tag{2.14}$$

*Proof.* See Lemma 5.2.8 in [36]. □

**Definition 2.9.** Given a scalar function  $f$  defined on a sphere  $S$ , we define the  $\ell = 1$  modes of  $f$  by the triplet

$$(f)_{\ell=1} := \left\{ \int_S f J^{(p)}, \quad p = 0, +, - \right\}.$$

### 2.6 Elliptic estimates

For a tensor  $f$  on a sphere  $S$ , we define the following norms for any integer  $k \geq 0$

$$\begin{aligned} \|f\|_{\mathfrak{h}_k(S)} &:= \sum_{j=0}^k \|\not{d}^j f\|_{L^2(S)}, \\ \|f\|_{\infty, k} &:= \sum_{j=0}^k \|\not{d}^j f\|_{L^\infty(S)}, \end{aligned} \tag{2.15}$$

Under the assumptions

$$\|\Gamma_g\|_{\infty, k} \leq \varepsilon r^{-2}, \quad \|\Gamma_b\|_{\infty, k} \leq \varepsilon r^{-1}, \tag{2.16}$$

where  $\varepsilon > 0$  is sufficiently small, we have the following coercive properties of the operators  $\not{d}_1$ ,  $\not{d}_2$ ,  $\not{d}_1^*$ .

**Lemma 2.10.** Consider a sphere  $S$  such that its Gauss curvature  $K$  satisfies

$$\left\| K - \frac{1}{r^2} \right\|_{\mathfrak{h}_{s_{max}-1}(S)} \lesssim \overset{\circ}{\varepsilon}, \tag{2.17}$$

Then, the following estimates hold for all integer  $k \leq s_{max}$ :

---

<sup>10</sup>The properties (2.13) of the scalar functions  $J^{(p)}$  are motivated by the fact that the  $\ell = 1$  spherical harmonics on the standard sphere  $\mathbb{S}^2$ , given by  $J^{(0, \mathbb{S}^2)} = \cos \theta$ ,  $J^{(+, \mathbb{S}^2)} = \sin \theta \cos \varphi$ ,  $J^{(-, \mathbb{S}^2)} = \sin \theta \sin \varphi$ , satisfy (2.13) with  $\overset{\circ}{\varepsilon} = 0$ . Note also that on  $\mathbb{S}^2$ , there holds

$$\int_{\mathbb{S}^2} (\cos \theta)^2 = \int_{\mathbb{S}^2} (\sin \theta \cos \varphi)^2 = \int_{\mathbb{S}^2} (\sin \theta \sin \varphi)^2 = \frac{4\pi}{3}, \quad |\mathbb{S}^2| = 4\pi.$$

1. If  $f$  is a 1-form

$$\|f\|_{\mathfrak{h}_{k+1}(S)} \lesssim r \|\not{d}_1 f\|_{\mathfrak{h}_k(S)}. \quad (2.18)$$

2. If  $v$  is a symmetric traceless 2-tensor

$$\|v\|_{\mathfrak{h}_{k+1}(S)} \lesssim r \|\not{d}_2 v\|_{\mathfrak{h}_k(S)}. \quad (2.19)$$

3. If  $(h, {}^*h)$  is a pair of scalars

$$\|(h - \bar{h}, {}^*h - \overline{{}^*h})\|_{\mathfrak{h}_{k+1}(S)} \lesssim r \|\not{d}_1^*(h, {}^*h)\|_{\mathfrak{h}_k(S)}. \quad (2.20)$$

*Proof.* See Lemma 2.20 in [34].  $\square$

**Lemma 2.11.** *On a fixed sphere  $S$  for which (2.17) holds, we have for any 1-form  $f$  and integer  $0 \leq k \leq s_{max}$ :*

$$\|f\|_{\mathfrak{h}_{k+1}(S)} \lesssim r \|\not{d}_2^* f\|_{\mathfrak{h}_k(S)} + |(f)_{\ell=1}|, \quad (2.21)$$

where the  $\ell = 1$  modes are given by Definition 2.9.

*Proof.* See Lemma 2.19 in [34] for the case  $k = 0$ . Assume that (2.21) holds for  $1 \leq k \leq n - 1$ , we have from standard elliptic regularity that

$$\begin{aligned} \|f\|_{\mathfrak{h}_{n+1}(S)} &\lesssim r^2 \|\Delta f\|_{\mathfrak{h}_{n-1}(S)} \\ &\lesssim r^3 \|\not{d}_2^* \Delta f\|_{\mathfrak{h}_{n-2}(S)} + r^2 |(\Delta f)_{\ell=1}| \\ &\lesssim r^3 \|\Delta \not{d}_2^* f\|_{\mathfrak{h}_{n-2}(S)} + r^3 \|[\Delta, \not{d}_2^*] f\|_{\mathfrak{h}_{n-2}(S)} + r^2 \left| \int_S (\Delta f) J^{(p)} \right| \\ &\lesssim r \|\not{d}_2^* f\|_{\mathfrak{h}_n(S)} + \|f\|_{\mathfrak{h}_n(S)} + r^2 \left| \int_S f(\Delta J^{(p)}) \right| \\ &\lesssim r \|\not{d}_2^* f\|_{\mathfrak{h}_n(S)} + |(f)_{\ell=1}|, \end{aligned}$$

where we used (2.21) in the case of  $k = n - 1$  and (2.13) at the last step. By induction, we deduce (2.21) for  $k = n$ . This concludes the proof of Lemma 2.11.  $\square$

The following corollary will be useful in Section 4.3.

**Corollary 2.12.** *On a fixed sphere  $S$  for which (2.17) holds for  $\varepsilon$  small enough, we have for any symmetric traceless 2-tensor  $v$  and integer  $0 \leq k \leq s_{max} - 1$ :*

$$\|v\|_{\mathfrak{h}_{k+2}(S)} \lesssim r^2 \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)}, \quad (2.22)$$

$$\|v\|_{\mathfrak{h}_{k+2}(S)} \lesssim r^2 \left\| \left( \not{d}_2^* \not{d}_2 + \frac{1}{r^2} \right) v \right\|_{\mathfrak{h}_k(S)}. \quad (2.23)$$

*Proof.* By Lemma 2.11, we have

$$\begin{aligned} \|\not{d}_2 v\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)} + |(\not{d}_1 \not{d}_2 v)_{\ell=1}| \\ &\lesssim r \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)} + \left| \int_S (\not{d}_1 \not{d}_2 v) J^{(p)} \right| \\ &\lesssim r \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)} + \left| \int_S v(\not{d}_2^* \not{d}_1^* J^{(p)}) \right| \\ &\lesssim r \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)} + \varepsilon \|v\|_{\infty, k}, \end{aligned}$$

where we have used integration by parts and (2.14). Then, we apply Lemma 2.10 to conclude

$$\|v\|_{\mathfrak{h}_{k+2}(S)} \lesssim r \|\not{d}_2 v\|_{\mathfrak{h}_{k+1}(S)} \lesssim r^2 \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)} + \overset{\circ}{\varepsilon} r \|v\|_{\infty, k},$$

which implies, together with Sobolev and  $\overset{\circ}{\varepsilon} > 0$  small enough,

$$\|v\|_{\mathfrak{h}_{k+2}(S)} \lesssim r^2 \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_k(S)}.$$

This concludes (2.22).

Since  $\not{d}_2^* \not{d}_2$  is a positive operator on  $L^2(S)$ , the case  $k = 0$  of (2.23) follows immediately from (2.22). Assuming that (2.23) holds true for  $0 \leq k \leq n - 1$ , we have

$$\begin{aligned} \|v\|_{\mathfrak{h}_{n+2}(S)} &\lesssim r^2 \|\not{d}_2^* \not{d}_2 v\|_{\mathfrak{h}_n(S)} \\ &\lesssim r^2 \left\| \left( \not{d}_2^* \not{d}_2 + \frac{1}{r^2} \right) v \right\|_{\mathfrak{h}_n(S)} + \|v\|_{\mathfrak{h}_n(S)} \\ &\lesssim r^2 \left\| \left( \not{d}_2^* \not{d}_2 + \frac{1}{r^2} \right) v \right\|_{\mathfrak{h}_n(S)}, \end{aligned}$$

where we used (2.23) for  $k = n - 2$ . By induction, we obtain (2.23) for all  $0 \leq k \leq s_{max} - 1$ . This concludes the proof of Corollary 2.12.  $\square$

### 2.7 General frame transformations

We recall below Lemma 3.1 in [34].

**Lemma 2.13.** *Given a null frame  $(e_3, e_4, e_1, e_2)$ , a general null transformation from the null frame  $(e_3, e_4, e_1, e_2)$  to another null frame  $(e'_3, e'_4, e'_1, e'_2)$  can be written in the form,*

$$\begin{aligned} e'_4 &= \lambda \left( e_4 + f^b e_b + \frac{1}{4} |f|^2 e_3 \right), \\ e'_a &= \left( \delta_{ab} + \frac{1}{2} \underline{f}_a \underline{f}_b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left( \frac{1}{2} f_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \quad a = 1, 2, \\ e'_3 &= \lambda^{-1} \left( \left( 1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16} |f|^2 |\underline{f}|^2 \right) e_3 + \left( \underline{f}^b + \frac{1}{4} |f|^2 \underline{f}^b \right) e_b + \frac{1}{4} |f|^2 e_4 \right), \end{aligned} \tag{2.24}$$

where  $\lambda$  is a scalar,  $f$  and  $\underline{f}$  are horizontal 1-forms on the sphere generated by  $(e'_1, e'_2)$ . The dot product and magnitude  $|\cdot|$  are taken with respect to the standard Euclidean norm of  $\mathbb{R}^2$ .<sup>11</sup> We call  $(f, \underline{f}, \lambda)$  the transition coefficients of the change of frame. We denote

$$F := (f, \underline{f}, \overset{\circ}{\lambda}), \quad \overset{\circ}{\lambda} := \lambda - 1.$$

We recall some of the identities of Proposition 3.3 in [34].

**Proposition 2.14.** *Under a general transformation of type (2.24), the Ricci coefficients transform as follows:*

- The transformation formula for  $\xi$  is given by

$$\begin{aligned} \lambda^{-2} \xi' &= \xi + \frac{1}{2} \nabla_{\lambda^{-1} e'_4} f + \frac{1}{4} \text{tr} \chi f + \omega f + \text{Err}(\xi, \xi'), \\ \text{Err}(\xi, \xi') &= \frac{1}{2} f \cdot \widehat{\chi} + \frac{1}{4} |f|^2 \eta + \frac{1}{2} (f \cdot \zeta) f - \frac{1}{4} |f|^2 \underline{\eta} \\ &\quad + \lambda^{-2} \left( \frac{1}{2} (f \cdot \xi') \underline{f} + \frac{1}{2} (f \cdot \underline{f}) \xi' \right) + l.o.t. \end{aligned} \tag{2.25}$$

<sup>11</sup>Here  $f$  and  $\underline{f}$  are 1-forms defined on a sphere, we use an orthonormal basis of the sphere to define the dot product and magnitude w.r.t. the standard Euclidean norm of  $\mathbb{R}^2$ .

- The transformation formula for  $\underline{\xi}$  is given by

$$\begin{aligned}\lambda^2 \underline{\xi}' &= \underline{\xi} + \frac{1}{2} \lambda \nabla'_3 \underline{f} + \underline{\omega} \underline{f} + \frac{1}{4} \operatorname{tr} \underline{\chi} \underline{f} + \operatorname{Err}(\underline{\xi}, \underline{\xi}'), \\ \operatorname{Err}(\underline{\xi}, \underline{\xi}') &= \frac{1}{2} \underline{f} \cdot \widehat{\underline{\chi}} - \frac{1}{2} (\underline{f} \cdot \underline{\zeta}) \underline{f} + \frac{1}{4} |\underline{f}|^2 \underline{\eta} - \frac{1}{4} |\underline{f}|^2 \eta' + l.o.t.\end{aligned}\quad (2.26)$$

- The transformation formula for  $\operatorname{tr} \underline{\chi}$  is given by

$$\begin{aligned}\lambda^{-1} \operatorname{tr} \underline{\chi}' &= \operatorname{tr} \underline{\chi} + \operatorname{div}' \underline{f} + \underline{f} \cdot \underline{\eta} + \underline{f} \cdot \underline{\zeta} + \operatorname{Err}(\operatorname{tr} \underline{\chi}', \operatorname{tr} \underline{\chi}), \\ \operatorname{Err}(\operatorname{tr} \underline{\chi}', \operatorname{tr} \underline{\chi}) &= \underline{f} \cdot \underline{\xi} + \frac{1}{4} \underline{f} \cdot \underline{f} \operatorname{tr} \underline{\chi} + \underline{\omega} (\underline{f} \cdot \underline{f}) - \underline{\omega} |\underline{f}|^2 \\ &\quad - \frac{1}{4} |\underline{f}|^2 \operatorname{tr} \underline{\chi} - \frac{1}{4} (\underline{f} \cdot \underline{f}) \lambda^{-1} \operatorname{tr} \underline{\chi}' + l.o.t.\end{aligned}\quad (2.27)$$

- The transformation formula for  $\underline{\zeta}$  is given by

$$\begin{aligned}\zeta' &= \zeta - \nabla'(\log \lambda) - \frac{1}{4} \operatorname{tr} \underline{\chi} \underline{f} + \underline{\omega} \underline{f} - \underline{\omega} \underline{f} + \frac{1}{4} \underline{f} \operatorname{tr} \underline{\chi} + \operatorname{Err}(\zeta, \zeta'), \\ \operatorname{Err}(\zeta, \zeta') &= -\frac{1}{2} \widehat{\underline{\chi}} \cdot \underline{f} + \frac{1}{2} (\underline{f} \cdot \underline{\zeta}) \underline{f} - \frac{1}{2} (\underline{f} \cdot \underline{\eta}) \underline{f} + \frac{1}{4} \underline{f} (\underline{f} \cdot \underline{\eta}) + \frac{1}{4} \underline{f} (\underline{f} \cdot \underline{\zeta}) \\ &\quad + \frac{1}{4} \underline{f}^* (\underline{f} \wedge \underline{\eta}) + \frac{1}{4} \underline{f}^* (\underline{f} \wedge \underline{\zeta}) + \frac{1}{4} \underline{f} \operatorname{div}' \underline{f} + \frac{1}{4} \underline{f}^* \operatorname{curl}' \underline{f} + \frac{1}{2} \lambda^{-1} \underline{f} \cdot \widehat{\underline{\chi}}' \\ &\quad - \frac{1}{16} (\underline{f} \cdot \underline{f}) \underline{f} \lambda^{-1} \operatorname{tr} \underline{\chi}' + \frac{1}{16} \underline{f} \lambda^{-1} (\underline{f} \wedge \underline{f}) \operatorname{tr} \underline{\chi}' + l.o.t.\end{aligned}\quad (2.28)$$

- The transformation formula for  $\underline{\eta}$  is given by

$$\begin{aligned}\eta' &= \eta + \frac{1}{2} \lambda \nabla'_3 \underline{f} + \frac{1}{4} \underline{f} \operatorname{tr} \underline{\chi} - \underline{\omega} \underline{f} + \operatorname{Err}(\eta, \eta'), \\ \operatorname{Err}(\eta, \eta') &= \frac{1}{2} (\underline{f} \cdot \underline{f}) \eta + \frac{1}{2} \underline{f} \cdot \widehat{\underline{\chi}} + \frac{1}{2} \underline{f} (\underline{f} \cdot \underline{\zeta}) - (\underline{f} \cdot \underline{f}) \eta' + \frac{1}{2} \underline{f} (\underline{f} \cdot \eta') + l.o.t.\end{aligned}\quad (2.29)$$

- The transformation formula for  $\underline{\eta}$  is given by

$$\begin{aligned}\underline{\eta}' &= \underline{\eta} + \frac{1}{2} \nabla_{\lambda^{-1} e'_4} \underline{f} + \frac{1}{4} \operatorname{tr} \underline{\chi} \underline{f} - \underline{\omega} \underline{f} + \operatorname{Err}(\underline{\eta}, \underline{\eta}'), \\ \operatorname{Err}(\underline{\eta}, \underline{\eta}') &= \frac{1}{2} \underline{f} \cdot \widehat{\underline{\chi}} + \frac{1}{2} (\underline{f} \cdot \underline{\eta}) \underline{f} - \frac{1}{4} (\underline{f} \cdot \underline{\zeta}) \underline{f} - \frac{1}{4} |\underline{f}|^2 \lambda^{-2} \xi' + l.o.t.\end{aligned}\quad (2.30)$$

- The transformation formula for  $\underline{\omega}$  is given by

$$\begin{aligned}\lambda^{-1} \omega' &= \omega - \frac{1}{2} \lambda^{-1} e'_4 (\log \lambda) + \frac{1}{2} \underline{f} \cdot (\underline{\zeta} - \underline{\eta}) + \operatorname{Err}(\omega, \omega'), \\ \operatorname{Err}(\omega, \omega') &= -\frac{1}{4} |\underline{f}|^2 \underline{\omega} - \frac{1}{8} \operatorname{tr} \underline{\chi} |\underline{f}|^2 + \frac{1}{2} \lambda^{-2} \underline{f} \cdot \xi' + l.o.t.\end{aligned}\quad (2.31)$$

- The transformation formula for  $\underline{\omega}$  is given by

$$\begin{aligned}\lambda \omega' &= \underline{\omega} + \frac{1}{2} \lambda e'_3 (\log \lambda) - \frac{1}{2} \underline{f} \cdot \underline{\zeta} - \frac{1}{2} \underline{f} \cdot \underline{\eta} + \operatorname{Err}(\underline{\omega}, \underline{\omega}'), \\ \operatorname{Err}(\underline{\omega}, \underline{\omega}') &= \underline{f} \cdot \underline{f} \underline{\omega} - \frac{1}{4} |\underline{f}|^2 \underline{\omega} + \frac{1}{2} \underline{f} \cdot \underline{\xi} + \frac{1}{8} (\underline{f} \cdot \underline{f}) \operatorname{tr} \underline{\chi} - \frac{1}{8} |\underline{f}|^2 \operatorname{tr} \underline{\chi} \\ &\quad - \frac{1}{4} \lambda \underline{f} \cdot \nabla'_3 \underline{f} + \frac{1}{2} (\underline{f} \cdot \underline{f}) (\underline{f} \cdot \eta') - \frac{1}{4} |\underline{f}|^2 (\underline{f} \cdot \eta') + l.o.t.\end{aligned}\quad (2.32)$$

where, for the transformation formulas of the Ricci coefficients above, *l.o.t.* denote expressions of the type

$$l.o.t. = O((\underline{f}, \underline{f})^3) \Gamma + O((\underline{f}, \underline{f})^2) \check{\Gamma}$$

containing no derivatives of  $\underline{f}$ ,  $\underline{f}$ ,  $\Gamma$  and  $\check{\Gamma}$ .

*Proof.* See Appendix A of [34]. □

## 2.8 Background spacetime

### 2.8.1 Adapted coordinates

Recall that we consider given a vacuum spacetime  $\mathcal{R}$  and endowed with of  $(u, s)$  foliation, see Section 2.1. A coordinate system  $(u, s, y^1, y^2)$  on  $\mathcal{R}$  is said to be adapted to an outgoing geodesic foliation as above if  $e_4(y^1) = e_4(y^2) = 0$ . In that case the spacetime metric can be written in the form, see Lemma 2.6 in [34],

$$\mathbf{g} = -2\varsigma dud s + \varsigma^2 \underline{\Omega} du^2 + g_{ab} (dy^a - \varsigma \underline{B}^a du) (dy^b - \varsigma \underline{B}^b du), \quad (2.33)$$

where

$$\underline{\Omega} = e_3(s), \quad \underline{B}^a = \frac{1}{2} e_3(y^a), \quad g_{ab} = \mathbf{g}(\partial_{y^a}, \partial_{y^b}). \quad (2.34)$$

Relative to these coordinates

$$\partial_s = e_4, \quad \partial_u = \varsigma \left( \frac{1}{2} e_3 - \frac{1}{2} \underline{\Omega} e_4 - \underline{B}^a \partial_{y^a} \right), \quad \partial_{y^a} = \sum_{c=1,2} Y_{(a)}^c e_c, \quad a = 1, 2, \quad (2.35)$$

with coefficients  $Y_{(a)}^b$  verifying

$$g_{ab} = \sum_{c=1,2} Y_{(a)}^c Y_{(b)}^c.$$

As a direct consequence of (2.35), we have<sup>12</sup>

$$e_4 = \partial_s, \quad e_3 = z \partial_u + \underline{\Omega} \partial_s + 2 \underline{B}^a \partial_{y^a}, \quad e_c = X_{(c)}^a \partial_{y^a}, \quad (2.36)$$

where  $X_{(a)}^c$  is defined by

$$\sum_{c=1,2} X_{(c)}^a Y_{(a)}^c = 1.$$

As in [34], we assume that  $\mathcal{R}$  is covered by two coordinate systems, i.e.  $\mathcal{R} = \mathcal{R}_N \cup \mathcal{R}_S$ , such that:

1. The North coordinate chart  $\mathcal{R}_N$  is given by the coordinates  $(u, s, y_N^1, y_N^2)$  with  $(y_N^1)^2 + (y_N^2)^2 < 2$  while the South coordinate chart  $\mathcal{R}_S$  is given by the coordinates  $(u, s, y_S^1, y_S^2)$  with  $(y_S^1)^2 + (y_S^2)^2 < 2$ .
2. The two coordinate charts intersect in the open equatorial region  $\mathcal{R}_{Eq} := \mathcal{R}_N \cap \mathcal{R}_S$  in which both coordinate systems are defined.
3. In  $\mathcal{R}_{Eq}$  the transition functions between the two coordinate systems are given by the smooth functions  $\varphi_{SN}$  and  $\varphi_{NS} = \varphi_{SN}^{-1}$ .

The metric coefficients for the two coordinate systems are given by

$$\begin{aligned} \mathbf{g} &= -2\varsigma dud s + \varsigma^2 \underline{\Omega} du^2 + g_{ab}^N (dy_N^a - \varsigma \underline{B}_N^a du) (dy_N^b - \varsigma \underline{B}_N^b du), \\ \mathbf{g} &= -2\varsigma dud s + \varsigma^2 \underline{\Omega} du^2 + g_{ab}^S (dy_S^a - \varsigma \underline{B}_S^a du) (dy_S^b - \varsigma \underline{B}_S^b du), \end{aligned}$$

where

$$\underline{\Omega} = e_3(s), \quad \underline{B}_N^a = \frac{1}{2} e_3(y_N^a), \quad \underline{B}_S^a = \frac{1}{2} e_3(y_S^a).$$

---

<sup>12</sup>Recall that  $z := \frac{z}{\varsigma}$  is defined in (2.5).



For a  $S(u, s)$ -tangent tensor  $f$ , we consider the following norms

$$\begin{aligned} \|f\|_\infty &:= \|f\|_{L^\infty(S)}, & \|f\|_2 &:= \|f\|_{L^2(S)}, \\ \|f\|_{\infty, k} &= \sum_{i=0}^k \|\mathfrak{d}^i f\|_\infty, & \|f\|_{2, k} &= \sum_{i=0}^k \|\mathfrak{d}^i f\|_2, \end{aligned} \quad (2.37)$$

where  $\mathfrak{d}^i$  stands for any combination of length  $i$  of operators of the form  $e_3, re_4, r\nabla$ .

Finally, we recall the following lemma for geodesic foliations.

**Lemma 2.15.** *For the background geodesic foliation of  $\mathcal{R}$  and any scalar function  $h$  defined on  $\mathcal{R}$ , we have*

$$\begin{aligned} e_4 \left( \int_S h \right) &= \int_S (e_4(h) + \kappa h), \\ e_3 \left( \int_S h \right) &= z \int_S (z^{-1}e_3(h) - z^{-1}\underline{\Omega}e_4(h) + z^{-1}\underline{\kappa}h - z^{-1}\underline{\Omega}\kappa h) + \underline{\Omega} \int_S (e_4(h) + \kappa h). \end{aligned} \quad (2.38)$$

In particular, we have for the area radius  $r$ :

$$e_4(r) = \frac{r}{2}\bar{\kappa}, \quad e_3(r) = \frac{r}{2} \left( z\overline{z^{-1}\underline{\kappa}} - z\overline{z^{-1}\underline{\Omega}\kappa} + \underline{\Omega}\bar{\kappa} \right). \quad (2.39)$$

*Proof.* Firstly, recall from (2.35) that we have

$$e_4 \left( \int_S h \right) = \partial_s \left( \int_S h \right) = \int_S \partial_s h + \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a \partial_s, \partial_b) h = \int_S (e_4(h) + \kappa h) \quad (2.40)$$

as stated. Next, recall from (2.35) that

$$\begin{aligned} \partial_u \left( \int_S h \right) &= \int_S \partial_u h + \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a \partial_u, \partial_b) h \\ &= \int_S z^{-1}e_3(h) - z^{-1}\underline{\Omega}e_4(h) - \frac{2}{z}\underline{B}^c \partial_c(h) + \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a(z^{-1}e_3), \partial_b) h \\ &\quad - \int_S \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a(z^{-1}\underline{\Omega}e_4), \partial_b) h + \mathbf{g}^{ab} \mathbf{g} \left( \mathbf{D}_a \left( \frac{2}{z}\underline{B}^c \partial_c \right), \partial_b \right) h \\ &= \int_S z^{-1}e_3(h) - z^{-1}\underline{\Omega}e_4(h) + z^{-1}\underline{\kappa}h - z^{-1}\underline{\Omega}\kappa h \\ &\quad - \int_S \frac{2}{z}\underline{B}^c \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a \partial_c, \partial_b) h + \nabla_c \left( \frac{2}{z}\underline{B}^c h \right). \end{aligned}$$

Recall that the divergence theorem implies that

$$\int_S \nabla_c \left( \frac{2}{z}\underline{B}^c h \right) = 0.$$

On the other hand

$$\begin{aligned} \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_a \partial_c, \partial_b) &= \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_c \partial_a, \partial_b) = \frac{1}{2} \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_c \partial_a, \partial_b) + \frac{1}{2} \mathbf{g}^{ab} \mathbf{g}(\mathbf{D}_c \partial_b, \partial_a) \\ &= \frac{1}{2} \mathbf{g}^{ab} \mathbf{D}_c(\mathbf{g}_{ab}) = \frac{1}{2} \mathbf{D}_c(\mathbf{g}^{ab} \mathbf{g}_{ab}) = 0. \end{aligned}$$

Thus, we obtain

$$\partial_u \left( \int_S h \right) = \int_S z^{-1}e_3(h) - z^{-1}\underline{\Omega}e_4(h) + z^{-1}\underline{\kappa}h - z^{-1}\underline{\Omega}\kappa h.$$

Together with (2.35) and (2.40), we infer

$$\begin{aligned} e_3 \left( \int_S h \right) &= z \partial_u \left( \int_S h \right) + \underline{\Omega} e_4 \left( \int_S h \right) \\ &= z \int_S (z^{-1} e_3(h) - z^{-1} \underline{\Omega} e_4(h) + z^{-1} \underline{\kappa} h - z^{-1} \underline{\Omega} \kappa h) + \underline{\Omega} \int_S (e_4(h) + \kappa h) \end{aligned}$$

as stated. Taking  $h = 1$ , we obtain (2.39). This concludes the proof of Lemma 2.15.  $\square$

### 2.8.2 Background spacetime region $\mathcal{R}$

In the following definition, we specify the background spacetime region  $\mathcal{R}$ .

**Definition 2.16.** *Let  $m_0 > 0$  a constant. Let  $\overset{\circ}{\varepsilon} > 0$  a sufficiently small constant, and let  $(\overset{\circ}{u}, \overset{\circ}{s}, \overset{\circ}{r})$  three real numbers with  $\overset{\circ}{r}$  sufficiently large so that*

$$\overset{\circ}{\varepsilon} \ll m_0, \quad \overset{\circ}{r} \gg m_0. \tag{2.41}$$

We define  $\mathcal{R}$  to be the region

$$\mathcal{R} := \left\{ |u - \overset{\circ}{u}| \leq \overset{\circ}{\varepsilon}, \quad |s - \overset{\circ}{s}| \leq \overset{\circ}{\varepsilon} \right\}, \tag{2.42}$$

such that assumptions **A1-A4** below with constant  $\overset{\circ}{\varepsilon}$  on the background foliation of  $\mathcal{R}$ , are verified.

### 2.8.3 Main assumptions for $\mathcal{R}$

Given an integer  $s_{max} \geq 5$ , we assume<sup>13</sup> the following:

**A1.** For  $k \leq s_{max}$

$$\begin{aligned} \|\Gamma_g\|_{k,\infty} &\leq \overset{\circ}{\varepsilon} r^{-2}, \\ \|\Gamma_b\|_{k,\infty} &\leq \overset{\circ}{\varepsilon} r^{-1}. \end{aligned} \tag{2.43}$$

**A2.** The Hawking mass  $m = m(u, s)$  of  $S(u, s)$  verifies

$$\sup_{\mathcal{R}} \left| \frac{m}{m_0} - 1 \right| \leq \overset{\circ}{\varepsilon}. \tag{2.44}$$

**A3.** In the region of their respective validity<sup>14</sup> we have

$$\underline{B}_N^a, \underline{B}_S^a \in r^{-1} \Gamma_b, \quad Z_N^a, Z_S^a \in \Gamma_b, \quad r^{-2} \check{g}_{ab}^N, r^{-2} \check{g}_{ab}^S \in r \Gamma_g, \tag{2.45}$$

where

$$\begin{aligned} \check{g}_{ab}^N &= g_{ab}^N - \frac{4r^2}{(1 + (y_N^1)^2 + (y_N^2)^2)} \delta_{ab}, \\ \check{g}_{ab}^S &= g_{ab}^S - \frac{4r^2}{(1 + (y_S^1)^2 + (y_S^2)^2)} \delta_{ab}, \\ Z^c &= \underline{B}^a Y_{(a)}^c. \end{aligned}$$

<sup>13</sup>In view of (2.43), we will often replace  $\Gamma_g$  by  $r^{-1} \Gamma_b$ .

<sup>14</sup>That is the quantities on the left verify the same estimates as those for  $\Gamma_b$ , respectively  $\Gamma_g$ .

**A4.** We assume the existence of a smooth family of scalar functions  $J^{(p)} : \mathcal{R} \rightarrow \mathbb{R}$ , for  $p = 0, +, -$ , verifying the following properties

(a) On the sphere  $\mathring{S}$  of the background foliation, there holds (2.13) with  $\varepsilon = \mathring{\varepsilon}$ , i.e.

$$\begin{aligned} \left( (\mathring{r})^2 \mathring{\Delta} + 2 \right) J^{(p)} &= O(\mathring{\varepsilon}), \quad p = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\mathring{\varepsilon}), \quad p, q = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} &= O(\mathring{\varepsilon}), \quad p = 0, +, -. \end{aligned} \quad (2.46)$$

(b) We extend  $J^{(p)}$  from  $\mathring{S}$  to  $\mathcal{R}$  by  $\partial_s J^{(p)} = \partial_u J^{(p)} = 0$ , i.e.

$$J^{(p)}(u, s, y^1, y^2) = J^{(p)}(\mathring{u}, \mathring{s}, y^1, y^2). \quad (2.47)$$

**Remark 2.17.** We note that the assumptions **A1**, **A2**, **A3**, **A4**, are expected to be valid in the far regions, i.e.  $r$  large, of a perturbed Kerr. In particular, they hold in far regions of Kerr, see Lemma 2.10 in [34].

## 2.9 Deformation of surfaces in $\mathcal{R}$

**Definition 2.18.** We say that  $\mathbf{S}$  is a deformation of  $\mathring{S}$  if there exist smooth scalar functions  $U, S$  defined on  $\mathring{S}$  and a map  $\Psi : \mathring{S} \rightarrow \mathbf{S}$  verifying, on any coordinate chart  $(y^1, y^2)$  of  $\mathring{S}$ ,

$$\Psi(\mathring{u}, \mathring{s}, y^1, y^2) = \left( \mathring{u} + U(y^1, y^2), \mathring{s} + S(y^1, y^2), y^1, y^2 \right). \quad (2.48)$$

**Definition 2.19.** Given a deformation  $\Psi : \mathring{S} \rightarrow \mathbf{S}$  we say that a new frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  on  $\mathbf{S}$ , obtained from the standard frame  $(e_3, e_4, e_1, e_2)$  via the transformation (2.24), is  $\mathbf{S}$ -adapted if the vectorfields  $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$  are tangent to  $\mathbf{S}$  and the vectorfields  $e_3^{\mathbf{S}}, e_4^{\mathbf{S}}$  are orthogonal to  $\mathbf{S}$ .

**Definition 2.20.** Let  $\mathbf{S} \subset \mathcal{R}$  be a compact 2-sphere, which is a deformation of a leaf  $S(u, s)$  of the background geodesic foliation of  $\mathcal{R}$  and let  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  the null frame adapted to  $\mathbf{S}$ . Then, we denote

- by  $\chi^{\mathbf{S}}, \underline{\chi}^{\mathbf{S}}, \zeta^{\mathbf{S}}, \dots$ , the corresponding Ricci coefficients,
- by  $\alpha^{\mathbf{S}}, \beta^{\mathbf{S}}, \rho^{\mathbf{S}}, \dots$ , the corresponding curvature coefficients,
- by  $r^{\mathbf{S}}, m^{\mathbf{S}}, K^{\mathbf{S}}$  and  $\mu^{\mathbf{S}}$  respectively the corresponding area radius, Hawking mass, Gauss curvature and mass aspect function,
- by  $\nabla^{\mathbf{S}}$  the corresponding covariant derivative.

**Definition 2.21.** We will work with the following weighted Sobolev norms on  $\mathbf{S}$

$$\|f\|_{\mathfrak{h}_s(\mathbf{S})} := \sum_{i=0}^s \|(\mathfrak{d}^{\mathbf{S}})^i f\|_{L^2(\mathbf{S})}, \quad \mathfrak{d}^{\mathbf{S}} = r^{\mathbf{S}} \nabla^{\mathbf{S}}. \quad (2.49)$$

We will need the following lemmas.

**Lemma 2.22.** *Let  $\mathring{S} \subset \mathcal{R}$ . Let  $\Psi : \mathring{S} \rightarrow \mathbf{S}$  be a deformation generated by the functions  $(U, S)$  as in Definition 2.18 and denote by  $g^{\mathbf{S},\#}$  the pull back of the metric  $g^{\mathbf{S}}$  to  $\mathring{S}$ . Assume the bound*

$$\|(U, S)\|_{L^\infty(\mathring{S})} + r^{-1} \|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim \mathring{\delta}. \tag{2.50}$$

Then

1. We have

$$\|g^{\mathbf{S},\#} - \mathring{g}\|_{L^\infty} + r^{-1} \|g^{\mathbf{S},\#} - \mathring{g}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \mathring{\delta}r. \tag{2.51}$$

2. For any tensor  $h$  on  $\mathcal{R}$

$$\|h\|_{\mathfrak{h}_s(\mathbf{S})} \lesssim r \sup_{\mathcal{R}} \left( |\mathring{\rho}^{\leq s} h| + \mathring{\delta} |\mathring{\mathfrak{d}}^{\leq s} h| \right), \quad 0 \leq s \leq s_{max}. \tag{2.52}$$

3. If  $V \in \mathfrak{h}_s(\mathbf{S})$  and  $V^\#$  is its pull-back by  $\Psi$ , we have for all  $0 \leq s \leq s_{max}$ ,

$$\|V\|_{\mathfrak{h}_s(\mathbf{S})} = \|V^\#\|_{\mathfrak{h}_s(\mathring{S}, g^{\mathbf{S},\#})} = \|V^\#\|_{\mathfrak{h}_s(\mathring{S}, \mathring{g})} (1 + O(r^{-1} \mathring{\delta})). \tag{2.53}$$

4. As a corollary of (2.53) (choosing  $V = 1$  and  $s = 0$ ), we deduce

$$\frac{r^{\mathbf{S}}}{\mathring{r}} = 1 + O(r^{-1} \mathring{\delta}) \tag{2.54}$$

where  $r^{\mathbf{S}}$  is the area radius of  $\mathbf{S}$  and  $\mathring{r}$  that of  $\mathring{S}$ .

5. We also have

$$|m - m^{\mathbf{S}}| = O(\mathring{\delta}). \tag{2.55}$$

*Proof.* See Lemma 5.8, Proposition 5.10 and Corollary 5.17 in [34]. □

We also have the following lemma.

**Lemma 2.23.** *Under the same assumptions as in Lemma 2.22, the following estimate holds for a scalar function  $F$  defined on  $\mathcal{R}$ ,*

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{S}} F \right| \lesssim \mathring{\delta}r \left( \sup_{\mathcal{R}} |F| + r \sup_{\mathcal{R}} (|\partial_u F| + |\partial_s F|) \right).$$

*Proof.* See Corollary 5.9 in [34]. □

We introduce the following schematic presentation of the error terms which appear in various calculations below.

**Definition 2.24.** *We denote by  $Err_k$ ,  $k = 1, 2$ , error terms<sup>15</sup> which can be written schematically in the form,*

$$\begin{aligned} rErr_1 &= F \cdot (r\Gamma_b) + F^2 + F \cdot (r\nabla^{\mathbf{S}})F = F \cdot (r\Gamma_b) + F \cdot (r\nabla^{\mathbf{S}})^{\leq 1} F, \\ r^2Err_2 &= (r\nabla^{\mathbf{S}})^{\leq 1} (rErr_1) + F \cdot r\mathring{\partial}\Gamma_b, \end{aligned} \tag{2.56}$$

where

$$F := (f, \underline{f}, \mathring{\lambda}), \quad \mathring{\lambda} = \lambda - 1.$$

---

<sup>15</sup>Note however that the precise error terms differ in each particular case and that we only emphasize here their general structure.

We recall the following identities, see Corollary 4.6 in [34].

**Proposition 2.25.** *The following relations hold true for any adapted frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  to a given sphere  $\mathbf{S}$  connected to the reference frame  $(e_3, e_4, e_1, e_2)$  by the transition coefficients  $(f, \underline{f}, \lambda)$ ,*

$$\begin{aligned}
\operatorname{curl}^{\mathbf{S}} f &= -\operatorname{Err}_1[\operatorname{curl}^{\mathbf{S}} f], \\
\operatorname{curl}^{\mathbf{S}} \underline{f} &= -\operatorname{Err}_1[\operatorname{curl}^{\mathbf{S}} \underline{f}], \\
\operatorname{div}^{\mathbf{S}} f + \kappa \overset{\circ}{\lambda} - \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= \kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} - \left( \kappa - \frac{2}{r} \right) - \operatorname{Err}_1[\operatorname{div}^{\mathbf{S}} f] - \frac{2(r - r^{\mathbf{S}})^2}{r(r^{\mathbf{S}})^2}, \\
\operatorname{div}^{\mathbf{S}} \underline{f} - \underline{\kappa} \overset{\circ}{\lambda} + \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= \underline{\kappa}^{\mathbf{S}} + \frac{2}{r^{\mathbf{S}}} - \left( \underline{\kappa} + \frac{2}{r} \right) - \operatorname{Err}_1[\operatorname{div}^{\mathbf{S}} \underline{f}] + \frac{2(r - r^{\mathbf{S}})^2}{r(r^{\mathbf{S}})^2}, \\
\Delta^{\mathbf{S}} \overset{\circ}{\lambda} + V \overset{\circ}{\lambda} &= \mu^{\mathbf{S}} - \mu - \left( \underline{\omega} + \frac{1}{4} \underline{\kappa} \right) (\kappa^{\mathbf{S}} - \kappa) + \left( \omega + \frac{1}{4} \kappa \right) (\underline{\kappa}^{\mathbf{S}} - \underline{\kappa}) + \operatorname{Err}_2[\Delta^{\mathbf{S}} \overset{\circ}{\lambda}], \\
\Delta^{\mathbf{S}} \overset{\circ}{b} &= \frac{1}{2} \operatorname{div}^{\mathbf{S}} \left( \underline{f} - \Upsilon f + \operatorname{Err}_1[\Delta^{\mathbf{S}} \overset{\circ}{b}] \right),
\end{aligned} \tag{2.57}$$

where

$$\begin{aligned}
\overset{\circ}{b} &:= r - r^{\mathbf{S}}, \\
V &:= - \left( \frac{1}{2} \kappa \underline{\kappa} + \kappa \underline{\omega} + \underline{\kappa} \omega \right),
\end{aligned}$$

and the error terms  $\operatorname{Err}_1[\operatorname{curl}^{\mathbf{S}} f]$ ,  $\operatorname{Err}_1[\operatorname{curl}^{\mathbf{S}} \underline{f}]$ ,  $\operatorname{Err}_1[\operatorname{div}^{\mathbf{S}} f]$ ,  $\operatorname{Err}_1[\operatorname{div}^{\mathbf{S}} \underline{f}]$  and  $\operatorname{Err}_1[\Delta^{\mathbf{S}} \overset{\circ}{b}]$  are consistent with  $\operatorname{Err}_1$  in Definition 2.24 whereas the error term  $\operatorname{Err}_2[\Delta^{\mathbf{S}} \overset{\circ}{\lambda}]$  is consistent with  $\operatorname{Err}_2$  in Definition 2.24.

*Proof.* The first two identities in (2.57) follow from the fact that  ${}^{(a)}\kappa := \epsilon^{ab} \chi_{ab} = 0$  and  ${}^{(a)}\underline{\kappa} := \epsilon^{ab} \underline{\chi}_{ab} = 0$ . The third, fourth and fifth identities in (2.57) follow from the transformation formulae for  $\kappa, \underline{\kappa}$  and  $\zeta$ . See Lemma 4.3 in [34] for a detailed proof. The last identity in (2.57) is proven in Corollary 4.6 in [34].  $\square$

### 3 GCM spheres

#### 3.1 GCM spheres with $\ell = 1$ modes in [34]

We review below Theorem 6.1 of [34] on existence and uniqueness of GCM spheres in the context of an arbitrary choice of a  $\ell = 1$  modes on  $\overset{\circ}{S}$ , denoted  $J^{(p)}$ , which verify the assumptions **A4** given in Section 2.8.3.

**Theorem 3.1** (GCM spheres with  $\ell = 1$  modes in [34]). *Let  $m_0 > 0$  a constant. Let  $0 < \overset{\circ}{\delta} \leq \overset{\circ}{\epsilon}$  two sufficiently small constants, and let  $(\overset{\circ}{u}, \overset{\circ}{s}, \overset{\circ}{r})$  three real numbers with  $\overset{\circ}{r}$  sufficiently large so that*

$$\overset{\circ}{\epsilon} \ll m_0, \quad \overset{\circ}{r} \gg m_0.$$

Let a fixed spacetime region  $\mathcal{R}$ , as in Definition 2.16, together with a  $(u, s)$  outgoing geodesic foliation verifying the assumptions **A1-A4**. Let  $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$  be a fixed sphere from this foliation

with  $\overset{\circ}{r}$  and  $\overset{\circ}{m}$  denoting its area radius and Hawking mass. Assume that the GCM quantities  $\kappa, \underline{\kappa}, \mu$  of the background foliation verify the following:

$$\begin{aligned}\kappa &= \frac{2}{r} + \dot{\kappa}, \\ \underline{\kappa} &= -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_p \underline{C}^{(p)} J^{(p)} + \dot{\underline{\kappa}}, \\ \mu &= \frac{2m}{r^3} + M_0 + \sum_p M^{(p)} J^{(p)} + \dot{\mu},\end{aligned}\tag{3.1}$$

where the scalar functions  $\underline{C}_0 = \underline{C}_0(u, s)$ ,  $\underline{C}^{(p)} = \underline{C}^{(p)}(u, s)$ ,  $M_0 = M_0(u, s)$  and  $M^{(p)} = M^{(p)}(u, s)$ , defined on the spacetime region  $\mathcal{R}$ , depend only on the coordinates  $(u, s)$ , and where  $\dot{\kappa}$ ,  $\dot{\underline{\kappa}}$  and  $\dot{\mu}$  satisfy the following estimates

$$\sup_{\mathcal{R}} |\not\partial^{\leq s_{max}}(\dot{\kappa}, \dot{\underline{\kappa}})| \lesssim r^{-2} \overset{\circ}{\delta}, \quad \sup_{\mathcal{R}} |\not\partial^{\leq s_{max}} \dot{\mu}| \lesssim r^{-3} \overset{\circ}{\delta},\tag{3.2}$$

where  $\not\partial = r\nabla$ . Then for any fixed triplets  $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$  verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \overset{\circ}{\delta},\tag{3.3}$$

there exists a unique GCM sphere  $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ , which is a deformation of  $\overset{\circ}{S}$ , such that there exist constants  $\underline{C}_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S}, p)}, M_0^{\mathbf{S}}, M^{(\mathbf{S}, p)}$ ,  $p \in \{-, 0, +\}$  for which the following GCM conditions are verified

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S}, p)} J^{(p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S}, p)} J^{(p)}.\end{aligned}\tag{3.4}$$

Relative to these modes we also have

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}.\tag{3.5}$$

The resulting deformation has the following additional properties:

1. The triplet  $(f, \underline{f}, \dot{\lambda})$  verifies

$$\|(f, \underline{f}, \dot{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S})} \lesssim \overset{\circ}{\delta}.\tag{3.6}$$

2. The GCM constants  $\underline{C}_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S}, p)}, M_0^{\mathbf{S}}, M^{(\mathbf{S}, p)}$ ,  $p \in \{-, 0, +\}$  verify

$$\begin{aligned}|\underline{C}_0^{\mathbf{S}} - \overline{\underline{C}_0}^{\mathbf{S}}| + |\underline{C}^{(\mathbf{S}, p)} - \overline{\underline{C}^{(p)}}^{\mathbf{S}}| &\lesssim r^{-2} \overset{\circ}{\delta}, \\ |M_0^{\mathbf{S}} - \overline{M_0}^{\mathbf{S}}| + |M^{(\mathbf{S}, p)} - \overline{M^{(p)}}^{\mathbf{S}}| &\lesssim r^{-3} \overset{\circ}{\delta}.\end{aligned}\tag{3.7}$$

3. The volume radius  $r^{\mathbf{S}}$  verifies

$$\left| \frac{r^{\mathbf{S}}}{\overset{\circ}{r}} - 1 \right| \lesssim r^{-1} \overset{\circ}{\delta}.\tag{3.8}$$

4. The parameter functions  $U, S$  of the deformation verify

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim r\mathring{\delta}. \quad (3.9)$$

5. The Hawking mass  $m^{\mathbf{S}}$  of  $\mathbf{S}$  verifies the estimate

$$|m^{\mathbf{S}} - \mathring{m}| \lesssim \mathring{\delta}. \quad (3.10)$$

6. The well defined<sup>16</sup> Ricci and curvature coefficients of  $\mathbf{S}$  verify,

$$\begin{aligned} \|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} &\lesssim \mathring{\varepsilon}r^{-1}, \\ \|\Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} &\lesssim \mathring{\varepsilon}. \end{aligned} \quad (3.11)$$

7. The transition parameters  $f, \underline{f}, \lambda$  are continuously differentiable with respect to  $\Lambda, \underline{\Lambda}$  and

$$\begin{aligned} \frac{\partial f}{\partial \Lambda} &= O(r^{-1}), & \frac{\partial f}{\partial \underline{\Lambda}} &= O(\mathring{\delta}r^{-1}), & \frac{\partial f}{\partial \Lambda} &= O(\mathring{\delta}r^{-1}), & \frac{\partial f}{\partial \underline{\Lambda}} &= O(r^{-1}), \\ \frac{\partial \lambda}{\partial \Lambda} &= O(\mathring{\delta}r^{-1}), & \frac{\partial \lambda}{\partial \underline{\Lambda}} &= O(\mathring{\delta}r^{-1}). \end{aligned} \quad (3.12)$$

8. The parameter functions  $(U, S)$  of the deformation are continuously differentiable with respect to  $\Lambda, \underline{\Lambda}$  and

$$\frac{\partial U}{\partial \Lambda} = O(1), \quad \frac{\partial U}{\partial \underline{\Lambda}} = O(1), \quad \frac{\partial S}{\partial \Lambda} = O(1), \quad \frac{\partial S}{\partial \underline{\Lambda}} = O(1). \quad (3.13)$$

9. Relative to the coordinate system induced by  $\Psi$  the metric  $g^{\Lambda, \underline{\Lambda}}$  of  $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$  is continuous with respect to the parameters  $\Lambda, \underline{\Lambda}$  and verifies

$$\|\partial_{\Lambda} g^{\mathbf{S}}, \partial_{\underline{\Lambda}} g^{\mathbf{S}}\|_{L^{\infty}(\mathbf{S})} \lesssim O(r^2). \quad (3.14)$$

**Remark 3.2.** The conclusions of Theorem 3.1 still hold if we replace (3.2) with the weaker condition<sup>17</sup>

$$\|(\dot{\kappa}, \dot{\underline{\kappa}})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim r^{-1}\mathring{\delta}, \quad \|\dot{\mu}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim r^{-2}\mathring{\delta}, \quad (3.15)$$

for any deformed sphere  $\mathbf{S}$  with  $(U, S)$  satisfying (3.9), where (3.15) is uniform w.r.t such spheres. See Remark 6.2 in [34].

<sup>16</sup>Note that while the Ricci coefficients  $\kappa^{\mathbf{S}}, \underline{\kappa}^{\mathbf{S}}, \hat{\chi}^{\mathbf{S}}, \hat{\underline{\chi}}^{\mathbf{S}}, \zeta^{\mathbf{S}}$  as well as all curvature components and mass aspect function  $\mu^{\mathbf{S}}$  are well defined on  $\mathbf{S}$ , this is not the case of  $\eta^{\mathbf{S}}, \underline{\eta}^{\mathbf{S}}, \xi^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \omega^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}}$  which require the derivatives of the frame in the  $e_3^{\mathbf{S}}$  and  $e_4^{\mathbf{S}}$  directions.

<sup>17</sup>Note that (3.2) implies (3.15) in view of (2.52).

### 3.2 Linearized GCM equations

**Definition 3.3.** Let  $\mathbf{S} \subset \mathcal{R}$  a smooth  $O(\overset{\circ}{\varepsilon})$ -sphere. We say that  $F = (f, \underline{f}, \overset{\circ}{\lambda})$  verifies the inhomogeneous linearized GCM system on  $\mathbf{S}$  if the following holds true:

$$\begin{aligned}
\operatorname{curl}^{\mathbf{S}} f &= h_1 - \overline{h_1}^{\mathbf{S}}, \\
\operatorname{curl}^{\mathbf{S}} \underline{f} &= \underline{h_1} - \overline{\underline{h_1}}^{\mathbf{S}}, \\
\operatorname{div}^{\mathbf{S}} f + \frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda} - \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= h_2, \\
\operatorname{div}^{\mathbf{S}} \underline{f} + \frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda} + \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= \underline{\dot{C}}_0 + \sum_p \underline{\dot{C}}^{(p)} J^{(\mathbf{S}, p)} + \underline{h_2}, \\
\left( \Delta^{\mathbf{S}} + \frac{2}{(r^{\mathbf{S}})^2} \right) \overset{\circ}{\lambda} &= \dot{M}_0 + \sum_p \dot{M}^{(p)} J^{(\mathbf{S}, p)} + \frac{1}{2r^{\mathbf{S}}} \left( \underline{\dot{C}}_0 + \sum_p \underline{\dot{C}}^{(p)} J^{(\mathbf{S}, p)} \right) + h_3, \\
\Delta^{\mathbf{S}} \overset{\circ}{b} - \frac{1}{2} \operatorname{div}^{\mathbf{S}} (\underline{f} - f) &= h_4 - \overline{h_4}^{\mathbf{S}}, \quad \overset{\circ}{b} = b_0,
\end{aligned} \tag{3.16}$$

for some choice of constants  $\underline{\dot{C}}_0, \dot{M}_0, \underline{\dot{C}}^{(p)}, \dot{M}^{(p)}, b_0$ , and scalar functions  $h_1, h_2, h_3, h_4, \underline{h_1}, \underline{h_2}$ .

The following proposition provides a priori estimates for the linearized GCM system (3.16), which play a crucial role in the proof of Theorem 3.1 (see the proof of Theorem 6.1 in [34]).

**Proposition 3.4.** Assume  $\mathbf{S}$  is a given  $O(\overset{\circ}{\varepsilon})$ -sphere in  $\mathcal{R}$ . Assume given a solution  $(f, \underline{f}, \overset{\circ}{\lambda}, \underline{\dot{C}}_0, \dot{M}_0, \underline{\dot{C}}^{(p)}, \dot{M}^{(p)}, \overset{\circ}{b})$  of the system (3.16), and verifying

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}.$$

Then, the following a priori estimates are verified, for  $3 \leq s \leq s_{max} + 1$ ,

$$\begin{aligned}
& \left\| \left( f, \underline{f}, \overset{\circ}{\lambda} - \overline{\overset{\circ}{\lambda}}^{\mathbf{S}} \right) \right\|_{\mathfrak{h}_s(\mathbf{S})} + \sum_p \left( r^2 |\underline{\dot{C}}^{(p)}| + r^3 |\dot{M}^{(p)}| \right) \\
& \lesssim r \|(h_1 - \overline{h_1}^{\mathbf{S}}, \underline{h_1} - \overline{\underline{h_1}}^{\mathbf{S}}, h_2 - \overline{h_2}^{\mathbf{S}}, \underline{h_2} - \overline{\underline{h_2}}^{\mathbf{S}})\|_{\mathfrak{h}_{s-1}(\mathbf{S})} \\
& \quad + r^2 \|h_3 - \overline{h_3}^{\mathbf{S}}\|_{\mathfrak{h}_{s-2}(\mathbf{S})} + r \|h_4 - \overline{h_4}^{\mathbf{S}}\|_{\mathfrak{h}_{s-3}(\mathbf{S})} + |\Lambda| + |\underline{\Lambda}|,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
r^2 |\underline{\dot{C}}_0| + r^3 |\dot{M}_0| + r \left| \overset{\circ}{\lambda} \right| & \lesssim r \|(h_1 - \overline{h_1}^{\mathbf{S}}, \underline{h_1} - \overline{\underline{h_1}}^{\mathbf{S}}, h_2 - \overline{h_2}^{\mathbf{S}}, \underline{h_2} - \overline{\underline{h_2}}^{\mathbf{S}})\|_{\mathfrak{h}_{s-1}(\mathbf{S})} \\
& \quad + r^2 \|h_3 - \overline{h_3}^{\mathbf{S}}\|_{\mathfrak{h}_{s-2}(\mathbf{S})} + r \|h_4 - \overline{h_4}^{\mathbf{S}}\|_{\mathfrak{h}_{s-3}(\mathbf{S})} + |\Lambda| + |\underline{\Lambda}| + |b_0|.
\end{aligned} \tag{3.18}$$

*Proof.* See Proposition 4.13 in [34]. □

Proposition 3.4 will be used in Sections 4.5 and 4.6.

### 3.3 GCM spheres with more general $\ell = 1$ modes

We have the following corollary of Theorem 3.1.

**Corollary 3.5.** Under the same assumptions as in Theorem 3.1, assume moreover that there exists a triplet of scalar functions  $\tilde{J}^{(p)}$  verifying

$$\sum_p \left\| J^{(p)} - \tilde{J}^{(p)} \right\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})} \lesssim r \overset{\circ}{\delta}, \tag{3.19}$$



and

$$\partial_u \tilde{J}^{(p)}(u, s, y^1, y^2) = \partial_s \tilde{J}^{(p)}(u, s, y^1, y^2) = 0. \quad (3.20)$$

Then, for any fixed triplets  $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$  verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \overset{\circ}{\delta}, \quad (3.21)$$

there exists a unique GCM sphere  $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda}, \tilde{J}^{(p)})$ , which is a deformation of  $\overset{\circ}{S}$ , such that there exists constants  $\underline{C}_0^{\mathbf{S}}, M_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S}, p)}, M^{(\mathbf{S}, p)}$ ,  $p \in \{-, 0, +\}$  for which the following GCM conditions are verified

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S}, p)} \tilde{J}^{(p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S}, p)} \tilde{J}^{(p)}. \end{aligned} \quad (3.22)$$

Moreover,

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (3.23)$$

where  $(\operatorname{div}^{\mathbf{S}} f)_{\ell=1}, (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1}$  are defined with respect to  $\tilde{J}^{(p)}$ . Finally, the resulting deformation verifies all the properties 1-9 of Theorem 3.1.

*Proof.* Notice that (3.19), (3.20) and (2.46) implies that  $\tilde{J}^{(p)}$  satisfies Assumption **A4**. Then, it suffices to apply Theorem 3.1 with  $\tilde{J}^{(p)}$  replacing  $J^{(p)}$ .  $\square$

## 4 Construction of GCM hypersurfaces

We are ready to state the precise version of the main result of this paper.

**Theorem 4.1** (Existence of GCM hypersurfaces, version 2). *Let  $m_0 > 0$  a constant. Let  $0 < \overset{\circ}{\delta} \leq \overset{\circ}{\varepsilon}$  two sufficiently small constants, and let  $(\overset{\circ}{u}, \overset{\circ}{s}, \overset{\circ}{r})$  three real numbers with  $\overset{\circ}{r}$  sufficiently large so that*

$$\overset{\circ}{\varepsilon} \ll m_0, \quad \overset{\circ}{r} \gg m_0.$$

*Let a fixed space-time region  $\mathcal{R}$  as in Definition 2.16, together with a  $(u, s)$  outgoing geodesic foliation and a basis of  $\ell = 1$  modes  $J^{(p)}$  verifying assumptions **A1-A4** in Section 2.8.3. We further assume that*

$$\sup_{\mathcal{R}} r \left| \tilde{\mathfrak{d}}^{\leq s_{max}+1} e_3(J^{(p)}) \right| \lesssim \overset{\circ}{\delta}, \quad s_{max} \geq 5, \quad (4.1)$$

where

$$\tilde{\mathfrak{d}} := (e_3 - (z + \underline{\Omega})e_4, \mathfrak{d}), \quad \mathfrak{d} = r\nabla,$$

denotes the weighted derivatives tangent to the level hypersurfaces of  $u + s$ . In addition, we assume on  $\mathcal{R}$ , the estimates

$$\sup_{\mathcal{R}} \left| \tilde{\mathfrak{d}}^{\leq s_{max}+1} (\dot{\kappa}, \dot{\underline{\kappa}}) \right| \lesssim r^{-2} \overset{\circ}{\delta}, \quad \sup_{\mathcal{R}} \left| \tilde{\mathfrak{d}}^{\leq s_{max}+1} \dot{\mu} \right| \lesssim r^{-3} \overset{\circ}{\delta}, \quad (4.2)$$

and

$$|(\operatorname{div} \eta)_{\ell=1}| \lesssim \overset{\circ}{\delta}, \quad |(\operatorname{div} \underline{\xi})_{\ell=1}| \lesssim \overset{\circ}{\delta}. \quad (4.3)$$

We also assume

$$|r - s| + |e_3(r) - e_3(s)| \lesssim \overset{\circ}{\delta}, \quad (4.4)$$

as well as the existence of a constant  $m_{(0)}$  such that we have on  $\mathcal{R}$ ,

$$\left| \overline{z + \underline{\Omega}} - 1 - \frac{2m_{(0)}}{r} \right| \lesssim \overset{\circ}{\delta}, \quad (4.5)$$

where  $\overline{z + \underline{\Omega}}$  denotes the average of  $z + \underline{\Omega}$  on the sphere  $S(u, s)$ .

Let  $\mathbf{S}_0$  be a fixed sphere included in the region  $\mathcal{R}$ , let a pair of triplets  $\Lambda_0, \underline{\Lambda}_0 \in \mathbb{R}^3$  such that

$$|\Lambda_0|, |\underline{\Lambda}_0| \lesssim \overset{\circ}{\delta}, \quad (4.6)$$

and let  $J^{(\mathbf{S}_0, p)}$  be a basis of  $\ell = 1$  modes on  $\mathbf{S}_0$  satisfying

$$\|J^{(\mathbf{S}_0, p)} - J^{(p)}\|_{\mathfrak{h}_{s_{\max}+1}(\mathbf{S}_0)} \lesssim r \overset{\circ}{\delta}. \quad (4.7)$$

Assume that we have on  $\mathbf{S}_0$

$$\begin{aligned} \kappa^{\mathbf{S}_0} &= \frac{2}{r^{\mathbf{S}_0}}, \\ \underline{\kappa}^{\mathbf{S}_0} &= -\frac{2\Upsilon^{\mathbf{S}_0}}{r^{\mathbf{S}_0}} + \underline{C}_0^{\mathbf{S}_0} + \sum_p \underline{C}^{(\mathbf{S}_0, p)} J^{(\mathbf{S}_0, p)}, \\ \mu^{\mathbf{S}_0} &= \frac{2m^{\mathbf{S}_0}}{r^{\mathbf{S}_0}} + M_0^{\mathbf{S}_0} + \sum_p M^{(\mathbf{S}_0, p)} J^{(\mathbf{S}_0, p)}, \end{aligned} \quad (4.8)$$

as well as

$$(\operatorname{div} f_0)_{\ell=1} = \Lambda_0, \quad (\operatorname{div} \underline{f}_0)_{\ell=1} = \underline{\Lambda}_0, \quad (4.9)$$

with  $(f_0, \underline{f}_0)$  corresponding to the coefficients from the background frame to the frame adapted to  $\mathbf{S}_0$ , and the  $\ell = 1$  modes being taken w.r.t. the basis  $J^{(\mathbf{S}_0, p)}$ .

Then, there exists a unique, local, smooth, spacelike hypersurface  $\Sigma_0$  passing through  $\mathbf{S}_0$ , a scalar function  $w^{\mathbf{S}}$  defined on  $\Sigma_0$ , whose level surfaces are topological spheres denoted by  $\mathbf{S}$  with adapted frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ , a smooth collection of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  and a triplet of functions  $J^{(\mathbf{S}, p)}$  defined on  $\Sigma_0$  verifying

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \quad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0, \quad J^{(\mathbf{S}, p)}|_{\mathbf{S}_0} = J^{(\mathbf{S}_0, p)},$$

such that the following conditions are verified:

1. The following GCM conditions hold on  $\Sigma_0$

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S}, p)} J^{(\mathbf{S}, p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S}, p)} J^{(\mathbf{S}, p)}. \end{aligned} \quad (4.10)$$

2. Denoting  $r^{\mathbf{S}}$  to be the area radius of the spheres  $\mathbf{S}$  we have, for some constant  $c_0$ ,

$$u^{\mathbf{S}} + r^{\mathbf{S}} = c_0, \quad \text{along } \Sigma_0. \quad (4.11)$$

3. Let  $\nu^{\mathbf{S}}$  be the unique vectorfield tangent to the hypersurface  $\Sigma_0$ , normal to  $\mathbf{S}$ , and normalized by  $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$ . There exists a unique scalar function  $b^{\mathbf{S}}$  on  $\Sigma_0$  such that  $\nu^{\mathbf{S}}$  is given by

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}.$$

Then, the following normalization condition holds true on every sphere  $\mathbf{S}$

$$\overline{b^{\mathbf{S}}} = -1 - \frac{2m(0)}{r^{\mathbf{S}}}, \quad (4.12)$$

where  $\overline{b^{\mathbf{S}}}$  denotes the average of  $b^{\mathbf{S}}$  on the sphere  $\mathbf{S}$ .

4. The triplet of functions  $J^{(\mathbf{S},p)}$  verifies on  $\Sigma_0$ <sup>18</sup>

$$\nu^{\mathbf{S}}(J^{(\mathbf{S},p)}) = 0, \quad p = 0, +, -. \quad (4.13)$$

5. The following transversality conditions are assumed<sup>19</sup>

$$\xi^{\mathbf{S}} = 0, \quad \omega^{\mathbf{S}} = 0, \quad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \quad (4.14)$$

and

$$e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1, \quad e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0. \quad (4.15)$$

6. In view of (4.14), the Ricci coefficients  $\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}$  are well defined for every  $\mathbf{S} \subset \Sigma_0$ . They verify

$$(\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} = 0, \quad (\operatorname{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} = 0. \quad (4.16)$$

7. The transition coefficients from the background foliation to that of  $\Sigma_0$  verify

$$\|(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{\max}+1}(\mathbf{S})} + \|\mathfrak{d}(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{\max}}(\mathbf{S})} \lesssim \overset{\circ}{\delta}, \quad (4.17)$$

where

$$\mathfrak{d} \in \{\nabla_3^{\mathbf{S}}, r\nabla^{\mathbf{S}}, r\nabla_4^{\mathbf{S}}\}.$$

We state the following corollary.

**Corollary 4.2.** *Let  $\mathcal{R}$  be a fixed spacetime region with a background foliation verifying the assumption of Theorem 4.1, including (4.1)-(4.5). Also, assume given a GCM hypersurface  $\Sigma_0 \subset \mathcal{R}$  foliated by surfaces  $\mathbf{S}$  such that*

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(\mathbf{S},p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(\mathbf{S},p)}, \\ (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} &= (\operatorname{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} = 0, \quad \nu^{\mathbf{S}}(J^{(\mathbf{S},p)}) = 0, \quad p = 0, +, -, \end{aligned}$$

<sup>18</sup>A basis of  $\ell = 1$  modes verifying (4.13) can simplify the transport equations along  $\nu^{\mathbf{S}}$  for the  $\ell = 1$  modes of the transition functions  $(f, \underline{f})$ , see Lemma 4.34 for more details.

<sup>19</sup>A priori, the quantities  $\xi^{\mathbf{S}}, \omega^{\mathbf{S}}, \underline{\eta}^{\mathbf{S}}, e_4^{\mathbf{S}}(r^{\mathbf{S}})$  and  $e_4^{\mathbf{S}}(u^{\mathbf{S}})$  do not make sense on  $\Sigma_0$ . The conditions (4.14) and (4.15) are in fact consistent with a local extension of  $\Sigma_0$  by an outgoing geodesic foliation, see Remark 4.12.

where the  $\ell = 1$  modes are defined w.r.t.  $J^{(\mathbf{S},p)}$ . Assume moreover

$$|\tilde{\mathfrak{d}}^{\leq s_{max}}(\operatorname{div} \eta)_{\ell=1}| \lesssim \overset{\circ}{\delta}, \quad |\tilde{\mathfrak{d}}^{\leq s_{max}}(\operatorname{div} \xi)_{\ell=1}| \lesssim \overset{\circ}{\delta}, \quad (4.18)$$

and

$$|\tilde{\mathfrak{d}}^{\leq s_{max}}(e_3(r) - e_3(s))| \lesssim \overset{\circ}{\delta}, \quad (4.19)$$

where  $\tilde{\mathfrak{d}} \in \{e_3 - (z + \underline{\Omega})e_4, \emptyset\}$ . Then:

1. If we assume in addition that for a given sphere  $\mathbf{S}_0$  on  $\Sigma_0$ , the transition coefficients  $(f, \underline{f}, \lambda)$  from the background foliation to  $\mathbf{S}_0$  verify

$$\|(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}, \quad (4.20)$$

then

$$\|\mathfrak{d}^{\leq s_{max}+1}(f, \underline{f}, \overset{\circ}{\lambda})\|_{L^2(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}.$$

2. If we assume in addition that for a given sphere  $\mathbf{S}_0$  on  $\Sigma_0$ , the transition coefficients  $(f, \underline{f}, \lambda)$  from the background foliation to  $\mathbf{S}_0$  verify

$$\|f\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} + (r^{\mathbf{S}_0})^{-1} \|(f, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}, \quad (4.21)$$

then

$$\|\mathfrak{d}^{\leq s_{max}+1} f\|_{L^2(\mathbf{S}_0)} + (r^{\mathbf{S}_0})^{-1} \|\mathfrak{d}^{\leq s_{max}+1}(f, \overset{\circ}{\lambda})\|_{L^2(\mathbf{S}_0)} + \|\mathfrak{d}^{\leq s_{max}} \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(f, \overset{\circ}{\lambda})\|_{L^2(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}.$$

**Remark 4.3.** Theorem 4.1 is the generalization of Theorem 9.52 in [33] and Corollary 4.2 is the generalization of Corollary 9.53 in [33] in the absence of symmetry. Theorem 4.1 plays a central role in the proof of Theorems M6 and M7 in [36], see Section 8.4 and 8.5 in [36]. Note that Theorem 4.1 and Corollary 4.2 are restated as Theorem 8.1.10 and Corollary 8.1.11 in [36].

**Remark 4.4.** The proof of Theorem 4.1 and Corollary 4.2 is similar to the one of Theorem 9.52 in [33] and Corollary 9.53 in [33]. The main differences are:

- the absence of symmetry in Theorem 4.1 and Corollary 4.2 while [33] relies on axial polarization,
- the dependance of the construction on the choice of the basis of  $\ell = 1$  modes, while the basis of  $\ell = 1$  modes in [33] is fixed by the polarized symmetry.

The proof of Theorem 4.1 is given in Sections 4.1-4.5 and the proof of Corollary 4.2 is given in Section 4.6.

#### 4.1 Definition of a family of hypersurfaces $\Sigma$

As stated in Theorem 4.1, we assume given a spacetime region

$$\mathcal{R} = \{|u - \overset{\circ}{u}| \leq \overset{\circ}{\varepsilon}, |s - \overset{\circ}{s}| \leq \overset{\circ}{\varepsilon}\}$$

endowed with a background foliation such that the conditions **A1-A4** and (4.1)-(4.5) hold true. We also assume given a sphere  $\mathbf{S}_0$ , triplets  $(\Lambda_0, \underline{\Lambda}_0)$  and a basis of  $\ell = 1$  modes  $J^{(p, \mathbf{S}_0)}$  satisfying

(4.6)-(4.7). In view of the uniqueness in Corollary 3.5, note that  $\mathbf{S}_0$  is in fact the deformation sphere

$$\mathbf{S}_0 = \mathbf{S}[\overset{\circ}{u}, \overset{\circ}{s}, \Lambda_0, \underline{\Lambda}_0, J^{(p)}[\mathbf{S}_0]]$$

of the given sphere  $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$  of the background foliation provided by Corollary 3.5. Our goal is to construct, in a small neighborhood of  $\mathbf{S}_0$ , a spacelike hypersurface  $\Sigma_0$  passing through  $\mathbf{S}_0$  verifying all the desired properties of Theorem 4.1. To this end, we first define more general families of hypersurfaces within which we will make a suitable choice of  $\Sigma_0$  in Section 4.5. We start with the following definition.

**Definition 4.5.** *Under the same assumptions as in Theorem 4.1, let  $\Psi(s)$  be a real valued function,  $\Lambda(s)$ ,  $\underline{\Lambda}(s)$  triplet of functions satisfying*

$$\begin{aligned} |\Lambda(s)|, |\Lambda'(s)|, |\underline{\Lambda}(s)|, |\underline{\Lambda}'(s)|, |\Psi(s) + s - c_0|, |\Psi'(s) + 1| &\lesssim \overset{\circ}{\delta}, & s \in \overset{\circ}{I}, \\ \Lambda(\overset{\circ}{s}) = \Lambda_0, & \underline{\Lambda}(\overset{\circ}{s}) = \underline{\Lambda}_0, & \Psi(\overset{\circ}{s}) = \overset{\circ}{u}, \end{aligned} \quad (4.22)$$

where

$$c_0 := \overset{\circ}{u} + \overset{\circ}{s}, \quad \overset{\circ}{I} := [\overset{\circ}{s}, s_1],$$

with  $s_1$  verifying  $|s_1 - \overset{\circ}{s}| \lesssim \overset{\circ}{\varepsilon}$ . Then, we denote

$$\Sigma_{\#} := \bigcup_{s \in \overset{\circ}{I}} S(\Psi(s), s),$$

and  $\nu_{\#}$  is the unique tangent vector to  $\Sigma_{\#}$  normal to  $S(\Psi(s), s)$  with  $\mathbf{g}(\nu_{\#}, e_4) = -2$ . By definition,  $\Sigma_{\#}$  is the hypersurface  $\{u = \Psi(s), s \in \overset{\circ}{I}\}$ .

Let a triplet of functions  $\tilde{J}$  defined on  $\Sigma_{\#}$  such that

$$\sup_{s \in \overset{\circ}{I}} \sum_p \|\tilde{J}^{(p)}(s) - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(S(\Psi(s), s))} \lesssim r \overset{\circ}{\delta}, \quad \tilde{J}(s) := \tilde{J}|_{S(\Psi(s), s)}. \quad (4.23)$$

For every  $s \in \overset{\circ}{I}$ , we apply Corollary 3.5 to the background sphere  $S(\Psi(s), s)$  and the triplet  $\tilde{J}(s)$ <sup>20</sup> to obtain a GCM sphere  $\mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]$ . The deformation map is given by:

$$\begin{aligned} \Phi^{\tilde{J}}(s) : S(\Psi(s), s) &\longrightarrow \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)] \subset \mathcal{R}, \\ (\Psi(s), s, y^1, y^2) &\longrightarrow (\Psi(s) + U^{\tilde{J}}(\Psi(s), s, y^1, y^2), s + S^{\tilde{J}}(\Psi(s), s, y^1, y^2), y^1, y^2). \end{aligned}$$

The adapted null frame of  $\mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]$  is denoted by  $(e_3^{\tilde{J}}, e_4^{\tilde{J}}, e_1^{\tilde{J}}, e_2^{\tilde{J}})$  and the transition functions are denoted by

$$F^{\tilde{J}}(s) := \left( f^{\tilde{J}}(s), \underline{f}^{\tilde{J}}(s), \overset{\circ}{\lambda}^{\tilde{J}}(s) \right).$$

Moreover, we denote

$$F^{\tilde{J}, \#}(s) := \left( f^{\tilde{J}, \#}(s), \underline{f}^{\tilde{J}, \#}(s), \overset{\circ}{\lambda}^{\tilde{J}, \#}(s) \right) := (\Phi^{\tilde{J}}(s))^{\#} \left( f^{\tilde{J}}(s), \underline{f}^{\tilde{J}}(s), \overset{\circ}{\lambda}^{\tilde{J}}(s) \right).$$

<sup>20</sup>For every  $s$ , we extend  $\tilde{J}(s)$  to  $\mathcal{R}$  by (3.20) in order to apply Corollary 3.5.

Then, we define

$$\Sigma_{\tilde{J}} := \bigcup_{s \in \mathring{I}} \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]. \quad (4.24)$$

Let  $\nu^{\tilde{J}}$  the unique vectorfield tangent to the hypersurface  $\Sigma_{\tilde{J}}$  such that  $\mathbf{g}(\nu^{\tilde{J}}, e_4^{\tilde{J}}) = -2$  and normal to  $\mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]$ . We define

$$\nu_{\#}^{\tilde{J}} := \left( (\Phi^{\tilde{J}})^{-1} \right)_{\#} (\nu^{\tilde{J}}),$$

which is a tangent vectorfield on  $\Sigma_{\#}$ , where  $\Phi^{\tilde{J}} : \Sigma_{\#} \rightarrow \Sigma_{\tilde{J}}$  is defined by

$$\Phi^{\tilde{J}}|_{S(\Psi(s), s)} = \Phi^{\tilde{J}}(s).$$

In the sequel, we denote

$$S := S(\Psi(s), s), \quad \mathring{S} := S(\mathring{u}, \mathring{s}). \quad (4.25)$$

We will need the following four lemmas.

**Lemma 4.6.** For a triplet of functions  $\tilde{J}$  on  $\Sigma_{\#}$  satisfying (4.23), we define  $(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}}, \nu_{\#}^{\tilde{J}})$  as in Definition 4.5. Then, we have

$$\nu_{\#}^{\tilde{J}} = \left( 1 + A(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}, \#}) \right) \nu_{\#} + B^a(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}, \#}) \partial_{y^a}, \quad (4.26)$$

where

$$A(U, S, F) = O(\mathfrak{d}_{\#}^{\leq 1}(U, S), F), \quad B^a(U, S, F) = \mathring{\varepsilon} r^{-2} O(\mathfrak{d}_{\#}^{\leq 1}(U, S), F),$$

with

$$\mathfrak{d}_{\#} := \{ \emptyset, \nu_{\#} \}.$$

*Proof.* See Appendix A. □

**Lemma 4.7.** For a triplet of functions  $\tilde{J}$  on  $\Sigma_{\#}$  satisfying (4.23), we define  $(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}}, \nu_{\#}^{\tilde{J}})$  as in Definition 4.5. In addition, we assume that

$$\left\| \nu_{\#}^{\tilde{J}}(\tilde{J}) \right\|_{L^{\infty}(\Sigma_{\#})} \leq 1. \quad (4.27)$$

Then, we have

$$\| \nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}}) \|_{\mathfrak{h}_s(S)} \lesssim r \mathring{\delta}, \quad 0 \leq s \leq s_{max}. \quad (4.28)$$

*Proof.* See Appendix B. □

**Lemma 4.8.** For two triplets of functions  $\tilde{J}$  and  $\hat{J}$  on  $\Sigma_{\#}$  satisfying (4.23), we define  $(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}}, \nu_{\#}^{\tilde{J}})$  and respectively  $(U^{\hat{J}}, S^{\hat{J}}, F^{\hat{J}}, \nu_{\#}^{\hat{J}})$  as in Definition 4.5. In addition, we assume that

$$\left\| \nu_{\#}^{\tilde{J}}(\tilde{J}, \hat{J}) \right\|_{L^{\infty}(\Sigma_{\#})} \leq 1. \quad (4.29)$$

Then, we have

$$r^{-1} \| (U^{\tilde{J}} - U^{\hat{J}}, S^{\tilde{J}} - S^{\hat{J}}) \|_{\mathfrak{h}_{s+1}(S)} + \| F^{\tilde{J}, \#} - F^{\hat{J}, \#} \|_{\mathfrak{h}_{s+1}(S)} \lesssim \mathring{\varepsilon} r^{-1} \| \tilde{J} - \hat{J} \|_{\mathfrak{h}_s(S)}, \quad (4.30)$$

for all  $0 \leq s \leq s_{max}$  and

$$\| \nu_{\#}(U^{\tilde{J}} - U^{\hat{J}}, S^{\tilde{J}} - S^{\hat{J}}) \|_{\mathfrak{h}_{s+1}(S)} \lesssim \mathring{\varepsilon} \| \tilde{J} - \hat{J} \|_{\mathfrak{h}_s(S)} + \mathring{\varepsilon} \| \nu_{\#}(\tilde{J} - \hat{J}) \|_{\mathfrak{h}_s(S)}, \quad (4.31)$$

for all  $0 \leq s \leq s_{max} - 1$ .

*Proof.* See Appendix C.  $\square$

**Lemma 4.9.** *For a triplet of functions  $\tilde{J}$  satisfying (4.23), we define  $\nu_{\#}^{\tilde{J}}$  as in Definition 4.5. Then, for any scalar function  $h$  and  $0 \leq k \leq s_{max}$ , we have*

$$\sup_{S \subset \Sigma_{\#}} \|h\|_{\mathfrak{h}_k(S)} \leq (1 + O(\hat{\varepsilon})) \|h\|_{\mathfrak{h}_k(\overset{\circ}{S})} + \sup_{S \subset \Sigma_{\#}} \left\| \nu_{\#}^{\tilde{J}}(h) \right\|_{\mathfrak{h}_k(S)}. \quad (4.32)$$

*Proof.* See Appendix D.  $\square$

The following theorem will allow us to introduce a suitable family of hypersurfaces in Definition 4.11.

**Theorem 4.10.** *There exists a unique  $\tilde{J}$  satisfying (4.23) and verifying*

$$\begin{aligned} \nu^{\tilde{J}} \left( \left( (\Phi^{\tilde{J}})^{-1} \right)^{\#} \tilde{J} \right) &= 0, & \text{on } \Sigma_{\tilde{J}}, \\ \left( (\Phi^{\tilde{J}})^{-1} \right)^{\#} \tilde{J}^{(p)} &= J^{(\mathbf{S}_0, p)}, & \text{on } \mathbf{S}_0. \end{aligned} \quad (4.33)$$

*Proof.* The proof follows from a standard Banach fixed-point argument. More precisely, we construct a map  $T$  which sends a triplet of functions  $\tilde{J}$  on  $\Sigma_{\#}$  to another triplet of functions  $T(\tilde{J})$  on  $\Sigma_{\#}$ . By taking a suitable choice of the domain of  $T$ , we prove that  $T$  is a contraction map and hence admits a unique fixed point. Finally, we conclude that the fixed point of  $T$  satisfies (4.33).

**Step 1.** Construction of the map  $T$ .

For a triplet of functions  $\tilde{J}$  defined on  $\Sigma_{\#}$  satisfying (4.23), we define the triplet of functions  $T(\tilde{J})$  on  $\Sigma_{\#}$  by the following transport equation:

$$\begin{aligned} \nu_{\#}^{\tilde{J}}(T(\tilde{J})) &= 0, & \text{on } \Sigma_{\#}, \\ T(\tilde{J})^{(p)} &= J^{(p)}[\mathbf{S}_0], & \text{on } \overset{\circ}{S}, \end{aligned} \quad (4.34)$$

where  $J^{(p)}[\mathbf{S}_0]$  is the extension of  $J^{(\mathbf{S}_0, p)}$  to  $\mathcal{R}$  by  $\partial_u(J^{(p)}[\mathbf{S}_0]) = \partial_s(J^{(p)}[\mathbf{S}_0]) = 0$ .

We define the following norms for triplets of functions  $H^{(p)}$  on  $\Sigma_{\#}$  and  $1 \leq s \leq s_{max}$ :

$$\|H\|_{\mathcal{X}_s} := \sup_{S \subset \Sigma_{\#}} \sum_p \left( \|H^{(p)}\|_{\mathfrak{h}_s(S)} + \|\nu_{\#}(H^{(p)})\|_{\mathfrak{h}_{s-1}(S)} \right). \quad (4.35)$$

We denote  $H \in \mathcal{X}_s$  if  $\|H\|_{\mathcal{X}_s} < +\infty$ . Then  $\mathcal{X}_s$  is a Banach space for  $1 \leq s \leq s_{max}$ .

Recalling  $\partial_u(J^{(p)}) = \partial_s(J^{(p)}) = 0$ , we can rewrite (4.7) as follows:

$$\sum_p \|J^{(p)}[\mathbf{S}_0] - J^{(p)}\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} \leq C_0 \overset{\circ}{r} \overset{\circ}{\delta}, \quad (4.36)$$

where  $C_0$  is a given constant. We define

$$\mathcal{X} := \left\{ \tilde{J} \in \mathcal{X}_{s_{max}} / \|\tilde{J} - J\|_{\mathcal{X}_{s_{max}}} \leq 2C_0 \overset{\circ}{r} \overset{\circ}{\delta}, \quad \left\| \nu_{\#}^{\tilde{J}}(\tilde{J}) \right\|_{L^\infty(\Sigma_{\#})} \leq 1 \right\}. \quad (4.37)$$

By definition,  $\mathcal{X}$  is a closed subset of  $\mathcal{X}_{s_{max}-1}$ .

**Step 2.** Boundedness of  $T$ .

Notice that for  $\tilde{J} \in \mathcal{X}$ , (4.23) holds true. Thus,  $T$  is well-defined on  $\mathcal{X}$ . Moreover (4.27) holds, and we can thus apply Lemma 4.7 to obtain

$$\|\nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_{s_{max}}(S)} \lesssim r \mathring{\delta}. \quad (4.38)$$

Moreover, (3.6) and (3.9), together with Lemma 2.22<sup>21</sup>, imply

$$r^{-1} \|(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_{s_{max}+1}(S)} + \|(F^{\tilde{J}, \#})\|_{\mathfrak{h}_{s_{max}+1}(S)} \lesssim \mathring{\delta}. \quad (4.39)$$

Recalling (4.1), (4.26), (4.34), (4.38) and  $\|J\|_{\mathfrak{h}_{s_{max}+1}(S)} \lesssim r$ , we infer

$$\begin{aligned} \|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}}(S)} &= \|\nu_{\#}^{\tilde{J}}(J)\|_{\mathfrak{h}_{s_{max}}(S)} \\ &\lesssim \|\nu_{\#}(J)\|_{\mathfrak{h}_{s_{max}}(S)} + \mathring{\varepsilon} \mathring{\delta} r^{-2} \|\partial_{y^a}(J)\|_{\mathfrak{h}_{s_{max}}(S)} \\ &\lesssim \mathring{\delta} + \mathring{\varepsilon} \mathring{\delta} \lesssim \mathring{\delta}. \end{aligned} \quad (4.40)$$

Applying Lemma 4.9 with  $h = T(\tilde{J}) - J$  and recalling (4.36), for  $\mathring{\varepsilon}$  small enough and  $\mathring{r}$  large enough, we obtain

$$\begin{aligned} \|T(\tilde{J}) - J\|_{\mathfrak{h}_{s_{max}}(S)} &\leq (1 + O(\mathring{\varepsilon})) \|T(\tilde{J}) - J\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} + \sup_S \|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}}(S)} \\ &\leq \frac{3}{2} C_0 \mathring{\delta} \mathring{r}. \end{aligned} \quad (4.41)$$

Moreover, recalling (4.26), (4.38), (4.39)<sup>22</sup>, (4.40) and (4.41), we have

$$\begin{aligned} \|\nu_{\#}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}-1}(S)} &\lesssim \|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}-1}(S)} + \mathring{\varepsilon} r^{-2} \|\partial_{y^a}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}-1}(S)} \\ &\lesssim \mathring{\delta} + \mathring{\varepsilon} r^{-2} \mathring{\delta} \|T(\tilde{J}) - J\|_{\mathfrak{h}_{s_{max}}(S)} \lesssim \mathring{\delta}. \end{aligned}$$

Thus, for  $\mathring{r}$  large enough, we obtain

$$\|\nu_{\#}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}-1}(S)} \leq \frac{1}{2} C_0 \mathring{\delta} \mathring{r}. \quad (4.42)$$

From (4.41) and (4.42) we obtain

$$\|T(\tilde{J}) - J\|_{\mathfrak{h}_{s_{max}}(S)} + \|\nu_{\#}(T(\tilde{J}) - J)\|_{\mathfrak{h}_{s_{max}-1}(S)} \leq 2C_0 \mathring{\delta} \mathring{r}, \quad (4.43)$$

which implies

$$\|T(\tilde{J}) - J\|_{\mathcal{X}_{s_{max}}} \leq 2C_0 \mathring{\delta} \mathring{r}. \quad (4.44)$$

Moreover, (4.43) implies that

$$\|\mathfrak{d}_{\#}(T(\tilde{J}))\|_{\mathfrak{h}_{s_{max}-1}(S)} \lesssim r. \quad (4.45)$$

<sup>21</sup>Lemma 2.22 allows to compare the norms  $\|F^{\tilde{J}}\|_{\mathfrak{h}_{s_{max}+1}(S)}$  and  $\|F^{\tilde{J}, \#}\|_{\mathfrak{h}_{s_{max}+1}(S)}$ .  
<sup>22</sup>(4.38) and (4.39) imply the boundedness of  $A$  and  $B$  in (4.26).



Next, recalling (4.26), (4.34), (4.35), (4.43), (4.45) and applying Lemma 4.8, we infer

$$\begin{aligned}
 \left| \nu_{\#}^{T(\tilde{J})}(T(\tilde{J})) \right| &= \left| \nu_{\#}^{T(\tilde{J})}(T(\tilde{J})) - \nu_{\#}^{\tilde{J}}(T(\tilde{J})) \right| \\
 &\lesssim \left| A(U^{T(\tilde{J})}, S^{T(\tilde{J})}, F^{T(\tilde{J}),\#}) - A(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J},\#}) \right| |\nu_{\#}(T(\tilde{J}))| \\
 &\quad + \left| B(U^{T(\tilde{J})}, S^{T(\tilde{J})}, F^{T(\tilde{J}),\#}) - B(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J},\#}) \right| |\partial_{y^a}(T(\tilde{J}))| \\
 &\lesssim \left| \mathfrak{d}_{\#}^{\leq 1}(U^{T(\tilde{J})} - U^{\tilde{J}}, S^{T(\tilde{J})} - S^{\tilde{J}}) \right| + \left| F^{T(\tilde{J}),\#} - F^{\tilde{J},\#} \right| \\
 &\lesssim \mathring{\varepsilon} r^{-1} \|T(\tilde{J}) - \tilde{J}\|_{\mathfrak{h}_2(S)} + \mathring{\varepsilon} r^{-1} \|\nu_{\#}(T(\tilde{J}) - \tilde{J})\|_{\mathfrak{h}_2(S)} \\
 &\lesssim \mathring{\varepsilon} r^{-1} \|T(\tilde{J}) - J\|_{\mathcal{X}_{s_{max}}} \lesssim \mathring{\delta},
 \end{aligned}$$

which implies for  $\mathring{\delta}$  small enough

$$\left\| \nu_{\#}^{T(\tilde{J})}(T(\tilde{J})) \right\|_{L^\infty(\Sigma_{\#})} \leq 1. \tag{4.46}$$

Notice that (4.44) and (4.46) imply  $T(\tilde{J}) \in \mathcal{X}$ . Hence, we have  $T : \mathcal{X} \rightarrow \mathcal{X}$ .

**Step 3.**  $T$  is a contraction map.

For two triplets of functions  $\tilde{J}$  and  $\hat{J}$  in  $\mathcal{X}$ , we have

$$\|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-1}(S)} = \|\nu_{\#}^{\tilde{J}}(T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-1}(S)} = \|(\nu_{\#}^{\tilde{J}} - \nu_{\#}^{\hat{J}})(T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-1}(S)}.$$

Recall from (4.26) that

$$\begin{aligned}
 \nu_{\#}^{\tilde{J}} - \nu_{\#}^{\hat{J}} &= O\left(\mathfrak{d}_{\#}^{\leq 1}(U^{\tilde{J}} - U^{\hat{J}}, S^{\tilde{J}} - S^{\hat{J}}), F^{\tilde{J},\#} - F^{\hat{J},\#}\right) \nu_{\#} \\
 &\quad + \frac{\mathring{\varepsilon}}{r^2} O\left(\mathfrak{d}_{\#}^{\leq 1}(U^{\tilde{J}} - U^{\hat{J}}, S^{\tilde{J}} - S^{\hat{J}}), F^{\tilde{J},\#} - F^{\hat{J},\#}\right) \partial_{y^a}.
 \end{aligned}$$

Applying Lemma 4.8, (4.35) and (4.45), we obtain

$$\begin{aligned}
 \|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-1}(S)} &\lesssim \|\mathfrak{d}_{\#}^{\leq 1}(U^{\tilde{J}} - U^{\hat{J}}, S^{\tilde{J}} - S^{\hat{J}}), F^{\tilde{J},\#} - F^{\hat{J},\#}\|_{\mathfrak{h}_{s_{max}-1}(S)} \\
 &\lesssim \mathring{\varepsilon} \left( \|\tilde{J} - \hat{J}\|_{\mathfrak{h}_{s_{max}-1}(S)} + \|\nu_{\#}(\tilde{J} - \hat{J})\|_{\mathfrak{h}_{s_{max}-2}(S)} \right) \\
 &\lesssim \mathring{\varepsilon} \|\tilde{J} - \hat{J}\|_{\mathcal{X}_{s_{max}-1}}.
 \end{aligned} \tag{4.47}$$

Applying Lemma 4.9 with  $h = T(\tilde{J}) - T(\hat{J})$  and Recalling that  $T(\tilde{J}) = T(\hat{J})$  on  $\mathring{S}$ , we obtain

$$\|T(\tilde{J}) - T(\hat{J})\|_{\mathfrak{h}_{s_{max}-1}(S)} \lesssim \mathring{\varepsilon} \|\tilde{J} - \hat{J}\|_{\mathcal{X}_{s_{max}-1}}. \tag{4.48}$$

Moreover, we have from (4.26), (4.35) and (4.47)

$$\begin{aligned}
 \|\nu_{\#}(T(\tilde{J}) - T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-2}(S)} &\lesssim \|\nu_{\#}^{\tilde{J}}(T(\tilde{J}) - T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-2}(S)} + \mathring{\varepsilon} r^{-2} \|T(\tilde{J}) - T(\hat{J})\|_{\mathfrak{h}_{s_{max}-1}(S)} \\
 &\lesssim \mathring{\varepsilon} \|\tilde{J} - \hat{J}\|_{\mathcal{X}_{s_{max}-1}} + \mathring{\varepsilon} \|T(\tilde{J}) - T(\hat{J})\|_{\mathfrak{h}_{s_{max}-1}(S)}.
 \end{aligned}$$

Combining with (4.48), we obtain

$$\|T(\tilde{J}) - T(\hat{J})\|_{\mathfrak{h}_{s_{max}-1}(S)} + \|\nu_{\#}(T(\tilde{J}) - T(\hat{J}))\|_{\mathfrak{h}_{s_{max}-2}(S)} \lesssim \mathring{\varepsilon} \|\tilde{J} - \hat{J}\|_{\mathcal{X}_{s_{max}-1}}.$$

Hence, by (4.35) and for  $\varepsilon^\circ$  small enough, we have

$$\|T(\tilde{J}) - T(\hat{J})\|_{\mathcal{X}_{s_{max}-1}} \leq \frac{1}{2} \|\tilde{J} - \hat{J}\|_{\mathcal{X}_{s_{max}-1}}, \tag{4.49}$$

which implies that  $T$  is a contraction map on  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}_{s_{max}-1}})$ .

**Step 4.** Fixed point of  $T$ .

Since  $\mathcal{X}$  is a closed subset of  $\mathcal{X}_{s_{max}-1}$ , we deduce that  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}_{s_{max}-1}})$  is a complete metric space. Then, applying Banach fixed-point theorem to  $T : \mathcal{X} \rightarrow \mathcal{X}$ , we obtain a unique fixed point  $\tilde{J} \in \mathcal{X}$  of  $T$ , i.e.

$$T(\tilde{J}) = \tilde{J}. \tag{4.50}$$

Injecting (4.50) into (4.34), we obtain

$$\begin{aligned} \nu_{\#}^{\tilde{J}}(\tilde{J}) &= 0, & \text{on } \Sigma_{\#}, \\ \tilde{J}^{(p)} &= J^{(p)}[\mathbf{S}_0], & \text{on } \overset{\circ}{S}. \end{aligned}$$

Finally, we have

$$\nu^{\tilde{J}}\left((\Phi^{\tilde{J}})^{-1}\#(\tilde{J})\right) = (\Phi^{\tilde{J}})_{\#}(\nu_{\#}^{\tilde{J}}\left((\Phi^{\tilde{J}})^{-1}\#(\tilde{J})\right)) = \nu_{\#}^{\tilde{J}}(\tilde{J}) = 0,$$

which implies (4.33). This concludes the proof of Theorem 4.10. □

**Definition 4.11.** Let  $P(s) := (\Psi(s), s, \Lambda(s), \underline{\Lambda}(s))$  be a curve satisfying (4.22). By Theorem 4.10, we define a triplet of functions  $\tilde{J}^{(p)}$  and a hypersurface of the form

$$\Sigma = \bigcup_{s \geq \overset{\circ}{s}} \mathbf{S}[P(s)] = \bigcup_{s \geq \overset{\circ}{s}} \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}^{(p)}]. \tag{4.51}$$

We define the following family of triplets on  $\Sigma$ :

$$J^{(\mathbf{S}, p)} := (\Phi^{-1})_{\#}(\tilde{J}^{(p)}). \tag{4.52}$$

According to the construction in (4.51), there is an  $\mathbf{S}$ -adapted null frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  on every GCM sphere  $\mathbf{S}[P(s)]$ .

We also assume the transversality conditions:

$$\xi^{\mathbf{S}} = 0, \quad \omega^{\mathbf{S}} = 0, \quad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \quad \text{on } \Sigma, \tag{4.53}$$

and

$$e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1, \quad e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0, \quad \text{on } \Sigma, \tag{4.54}$$

where  $u^{\mathbf{S}}$  is defined as

$$u^{\mathbf{S}} := c_0 - r^{\mathbf{S}} \quad \text{on } \Sigma, \quad u^{\mathbf{S}}|_{\mathbf{S}_0} = \overset{\circ}{u}. \tag{4.55}$$

**Remark 4.12.** Let us provide below justifications for introducing the transversality conditions (4.53) and (4.54):

1. (4.53) and (4.54) are consistent with a local extension by an outgoing geodesic foliation initialized on  $\Sigma$ . The use of transversality conditions instead of a local extension is chosen here to have intrinsic definitions on  $\Sigma$ .

2. The role of the transversality conditions (4.53) is to make sense of the Ricci coefficients  $\eta^{\mathbf{S}}$ ,  $\underline{\xi}^{\mathbf{S}}$  and  $\underline{\omega}^{\mathbf{S}}$  on  $\Sigma$  through the formulae:<sup>23</sup>

$$\begin{aligned} \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_4^{\mathbf{S}}, e_a^{\mathbf{S}}) &= 2\eta_a^{\mathbf{S}} + 2b^{\mathbf{S}} \xi_a^{\mathbf{S}} = 2\eta_a^{\mathbf{S}}, \\ \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_3^{\mathbf{S}}, e_a^{\mathbf{S}}) &= 2\underline{\xi}_a^{\mathbf{S}} + 2b^{\mathbf{S}} \underline{\eta}_a^{\mathbf{S}} = 2\underline{\xi}_a^{\mathbf{S}} - 2b^{\mathbf{S}} \zeta_a^{\mathbf{S}}, \\ \mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}} e_3^{\mathbf{S}}, e_4^{\mathbf{S}}) &= 4\underline{\omega}^{\mathbf{S}} - 4b^{\mathbf{S}} \omega^{\mathbf{S}} = 4\underline{\omega}^{\mathbf{S}}. \end{aligned} \quad (4.56)$$

3. The role of the transversality conditions (4.54) is to make sense of  $e_3^{\mathbf{S}}(r^{\mathbf{S}})$  and  $e_3^{\mathbf{S}}(u^{\mathbf{S}})$  on  $\Sigma$  through the formulae:

$$\begin{aligned} e_3^{\mathbf{S}}(r^{\mathbf{S}}) &= \nu^{\mathbf{S}}(r^{\mathbf{S}}) - b^{\mathbf{S}} e_4^{\mathbf{S}}(r^{\mathbf{S}}) = \nu^{\mathbf{S}}(r^{\mathbf{S}}) - b^{\mathbf{S}}, \\ e_3^{\mathbf{S}}(u^{\mathbf{S}}) &= \nu^{\mathbf{S}}(u^{\mathbf{S}}) - b^{\mathbf{S}} e_4^{\mathbf{S}}(u^{\mathbf{S}}) = \nu^{\mathbf{S}}(u^{\mathbf{S}}). \end{aligned} \quad (4.57)$$

We have the following proposition.

**Proposition 4.13.** *Let  $\Sigma$  a hypersurface given by Definition 4.11. Then, the following properties hold:*

1. The curve  $\gamma(s) = (\Psi(s), s, 0, 0)$  of South Poles of the background spheres  $S(\Psi(s), s)$  verifies  $\gamma(s) \subset \Sigma$ ,  $s \in \mathring{I}$ .
2. On  $\mathbf{S}$ , the following GCM conditions hold

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(\mathbf{S},p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(\mathbf{S},p)}, \end{aligned} \quad (4.58)$$

where  $J^{(\mathbf{S},p)}$  is defined by (4.52). Moreover,

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (4.59)$$

where  $(f, \underline{f}, \lambda)$  are the transition parameters of the frame transformation from the background frame  $(e_3, e_4, e_1, e_2)$  to the adapted frame  $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$  and the  $\ell = 1$  modes are defined w.r.t.  $J^{(\mathbf{S},p)}$ . The triplets of constants  $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$  depend smoothly on the surfaces  $\mathbf{S}$  and

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \quad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0.$$

3. There are two maps  $\Xi_S : \mathcal{R}_S \rightarrow \mathbf{S}$  and  $\Xi_N : \mathcal{R}_N \rightarrow \mathbf{S}$  given by

$$\begin{aligned} \Xi_{S,N} : (u, s, y_{S,N}^1, y_{S,N}^2) &\mapsto \\ (u + U_{S,N}(y_{S,N}^1, y_{S,N}^2, u, s, \Lambda, \underline{\Lambda}), s + S_{S,N}(y_{S,N}^1, y_{S,N}^2, u, s, \Lambda, \underline{\Lambda}), y_{S,N}^1, y_{S,N}^2) \end{aligned} \quad (4.60)$$

with  $U_S, S_S$  vanishing at South Pole and verify the following transition condition

$$\Xi_N = \Xi_S \circ \varphi_{NS},$$

where  $\varphi_{NS}$  is the transition map between the 2 coordinates charts. Using these two maps, we can define a map  $\Xi : S(u, s) \rightarrow \mathbf{S}$  by

$$\Xi = \Xi_S \text{ on } \mathcal{R}_S, \quad \Xi = \Xi_N \text{ on } \mathcal{R}_N. \quad (4.61)$$

<sup>23</sup>Note that the L.H.S. of (4.56) are well defined on  $\Sigma$  since  $\nu^{\mathbf{S}}$  is tangent to  $\Sigma$ .

4. Let  $\nu^{\mathbf{S}}$  be the unique vectorfield tangent to the hypersurface  $\Sigma$ , normal to  $\mathbf{S}$ , and normalized by  $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$ . There exists a unique scalar function  $b^{\mathbf{S}}$  on  $\Sigma$  such that  $\nu^{\mathbf{S}}$  is given by

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}.$$

5. The transversality conditions hold:

$$\xi^{\mathbf{S}} = 0, \quad \omega^{\mathbf{S}} = 0, \quad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \quad \text{on } \Sigma,$$

and

$$e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1, \quad e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0, \quad \text{on } \Sigma.$$

6. The Ricci coefficients  $\kappa^{\mathbf{S}}, \underline{\kappa}^{\mathbf{S}}, \widehat{\chi}^{\mathbf{S}}, \underline{\widehat{\chi}}^{\mathbf{S}}, \zeta^{\mathbf{S}}$  are well defined on each sphere  $\mathbf{S}$  of  $\Sigma$ , and hence on  $\Sigma$ . The same holds true for all curvature coefficients  $\alpha^{\mathbf{S}}, \beta^{\mathbf{S}}, \rho^{\mathbf{S}}, \star\rho^{\mathbf{S}}, \underline{\beta}^{\mathbf{S}}, \underline{\alpha}^{\mathbf{S}}$ . Taking into account our transversality conditions (4.53) on  $\Sigma$ , we remark that all the Ricci coefficients are well defined on  $\Sigma$  including  $\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}$  and  $\underline{\omega}^{\mathbf{S}}$ , see item 2 in Remark 4.12.

7. The  $\ell = 1$  modes  $J^{(\mathbf{S},p)}$  verify:

$$\nu^{\mathbf{S}}(J^{(\mathbf{S},p)}) = 0, \quad J^{(\mathbf{S},p)}|_{\mathbf{S}_0} = J^{(\mathbf{S}_0,p)}, \quad p = 0, +, -.$$

8. The hypersurface  $\Sigma$  constructed above is a smooth hypersurface.

**Remark 4.14.** Since we always work on the hypersurface  $\Sigma$ , with no risk of confusion, we denote

$$\Xi_{S,N}(s, y_{S,N}^1, y_{S,N}^2) := \Xi_{S,N}(\Psi(s), s, y_{S,N}^1, y_{S,N}^2) \tag{4.62}$$

in the following context.

*Proof of Proposition 4.13.* Properties 1-7 are immediate consequences of Definition 4.11. So we only prove Property 8. For this purpose, we first prove the smoothness of the functions  $\Xi_p$  for  $p = S, N$ . Notice that

$$\begin{aligned} \partial_s \Xi_p(s, y_p^1, y_p^2) &= (\Psi'(s) + \partial_P U_p(\cdot) P'(s), 1 + \partial_P S_p(\cdot) P'(s), 0, 0), \\ \partial_{y_p^a} \Xi_p(s, y_p^1, y_p^2) &= (\partial_{y_p^a} U_p(\cdot), \partial_{y_p^a} S_p(\cdot), \delta_{a1}, \delta_{a2}), \end{aligned} \tag{4.63}$$

where

$$\begin{aligned} \partial_P U_p(\cdot) P'(s) &= \Psi'(s) \partial_u U_p(\cdot) + \partial_s U_p(\cdot) + \Lambda'(s) \partial_\Lambda U_p(\cdot) + \underline{\Lambda}'(s) \partial_{\underline{\Lambda}} U_p(\cdot), \\ \partial_P S_p(\cdot) P'(s) &= \Psi'(s) \partial_u S_p(\cdot) + \partial_s S_p(\cdot) + \Lambda'(s) \partial_\Lambda S_p(\cdot) + \underline{\Lambda}'(s) \partial_{\underline{\Lambda}} S_p(\cdot). \end{aligned}$$

Thus, in view of the smoothness of  $U_p, S_p$  w.r.t. the parameters  $\Lambda, \underline{\Lambda}$  and  $u, s$ , which are statements 7-9 in Theorem 3.1 and Corollary 6.11 in [34], we deduce that  $\Xi_p$  for  $p = S, N$  are smooth and they are immersions at every point. Recall that

$$\Xi_p(s, y_p^1, y_p^2) = (\Psi(s) + U(y_p^1, y_p^2, P(s)), s + S(y_p^1, y_p^2, P(s)), y_p^1, y_p^2).$$

According to (4.63) and the implicit function theorem, at every point  $(s, y_p^1, y_p^2)$ , its inverse map is locally given by

$$\Xi_p^{-1}(h(z_p, y_p^1, y_p^2), z_p, y_p^1, y_p^2) = (s(z_p, y_p^1, y_p^2), y_p^1, y_p^2), \tag{4.64}$$

where  $h(z_p, y_p^1, y_p^2), s(z_p, y_p^1, y_p^2)$  are smooth functions.

Now we recall the following result, see Theorem 2.1.2 in [4] for a proof.

**Theorem 4.15.**  *$V$  is a  $d$ -dimensional smooth submanifold of  $\mathbb{R}^n$  if for every point  $x \in V$ , there exists an open neighborhood  $U \subset \mathbb{R}^n$  of  $x$ , an open domain  $\Omega \subset \mathbb{R}^d$  and a map  $g : \Omega \rightarrow \mathbb{R}^n$  such that*

1.  *$g$  is a smooth map.*
2.  *$g$  is homeomorphism from  $\Omega$  to  $U \cap V$ .*
3.  *$g$  is an immersion at every point of  $\Omega$ .*

For every  $x \in \Sigma$ , without loss of generality, we assume that  $x$  can be written as

$$x = \Xi_S(s_0, y_0^1, y_0^2) = (\Psi(s_0) + U(y_0^1, y_0^2, P(s_0)), s_0 + S(y_0^1, y_0^2, P(s_0)), y_0^1, y_0^2)$$

in the South coordinate chart. We define

$$U_x := B(x, \varepsilon) := \left\{ q \in \mathbb{R}^4 \mid \|q - x\|_{\mathbb{R}^4} \leq \varepsilon \right\}.$$

Then, for every<sup>24</sup>

$$q = (\Psi(s) + U(y^1, y^2, P(s)), s + S(y^1, y^2, P(s)), y^1, y^2) \in U_x \cap \Sigma,$$

we have

$$|s + S(y^1, y^2, P(s)) - s_0 - S(y_0^1, y_0^2, P(s_0))|, |y^1 - y_0^1|, |y^2 - y_0^2| \leq \varepsilon.$$

Thus, in view of the smallness dependence of  $S$  w.r.t. the parameters  $(s, y^1, y^2)$ , we deduce that  $q \in \mathcal{N}$  where

$$\mathcal{N} := \{(s, y^1, y^2) \in \mathbb{R}^3 \mid |s - s_0|, |y^1 - y_0^1|, |y^2 - y_0^2| \leq 2\varepsilon\}.$$

For  $\varepsilon$  small enough,  $\Xi_S$  is a homeomorphism from  $\mathcal{N}$  to its image. We then denote

$$\Omega := \Xi_S^{-1}(U_x \cap \Sigma) \subset \mathcal{N},$$

where  $\Xi_S^{-1}$  is defined in (4.64). Hence, the map

$$\Xi_S|_{\Omega} : \Omega \mapsto U_x \cap \Sigma,$$

is smooth, homeomorphism and is an immersion. The similar conclusion holds for  $\Xi_N$ . Applying Theorem 4.15, we deduce that  $\Sigma$  is a smooth hypersurface as stated. This concludes the proof of Proposition 4.13.  $\square$

## 4.2 Estimates for the transition functions $(f, \underline{f}, \lambda)$

Let  $\Sigma$  be a smooth spacelike hypersurface defined as in Definition 4.11. We define the scalars<sup>25</sup>

$$z^{\mathbf{S}} := e_3^{\mathbf{S}}(u^{\mathbf{S}}), \quad \underline{\Omega}^{\mathbf{S}} := e_3^{\mathbf{S}}(r^{\mathbf{S}}), \quad (4.65)$$

and the renormalized quantities

$$\tilde{z}^{\mathbf{S}} := z^{\mathbf{S}} - 2, \quad \check{\underline{\Omega}}^{\mathbf{S}} := \underline{\Omega}^{\mathbf{S}} + \Upsilon^{\mathbf{S}}. \quad (4.66)$$

We define the following quantities along  $\Sigma$

$$B^{\mathbf{S}} := (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1}, \quad \underline{B}^{\mathbf{S}} := (\operatorname{div}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1}, \quad D^{\mathbf{S}} := \overline{\check{b}}^{\mathbf{S}} = \overline{b}^{\mathbf{S}} + 1 + \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}, \quad (4.67)$$

where the  $\ell = 1$  modes are defined w.r.t.  $J^{(\mathbf{S}, p)}$ .

The goal of this section is to prove Proposition 4.22. To this end, we first prove a series of lemmas which will be used throughout this paper.

<sup>24</sup>By taking  $\varepsilon$  small enough,  $q$  can be written in the South coordinate chart.

<sup>25</sup>Recall that (4.54) allows to make sense of  $e_3^{\mathbf{S}}(u^{\mathbf{S}})$  and  $e_3^{\mathbf{S}}(r^{\mathbf{S}})$ , see item 3 in Remark 4.12.

**Lemma 4.16.** *We have the following identities:*

$$\begin{aligned} e_a^{\mathbf{S}}(z^{\mathbf{S}}) &= (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})z^{\mathbf{S}}, \\ e_a^{\mathbf{S}}(\underline{\Omega}^{\mathbf{S}}) &= (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})\underline{\Omega}^{\mathbf{S}} - \underline{\xi}_a^{\mathbf{S}}, \\ b^{\mathbf{S}} &= -z^{\mathbf{S}} - \underline{\Omega}^{\mathbf{S}}. \end{aligned} \tag{4.68}$$

*Proof.* By Lemma 2.3, we have

$$[e_a, e_3] = \frac{1}{2}\underline{\kappa}e_a + \widehat{\chi}_{ab}e_b - \nabla_3e_a + (\zeta_a - \eta_a)e_3 - \underline{\xi}_ae_4.$$

Since  $e_a^{\mathbf{S}}(u^{\mathbf{S}}) = e_a^{\mathbf{S}}(r^{\mathbf{S}}) = 0$ ,  $e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1$  and  $e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0$ , we deduce that

$$\begin{aligned} e_a^{\mathbf{S}}(e_3^{\mathbf{S}}(u^{\mathbf{S}})) &= [e_a^{\mathbf{S}}, e_3^{\mathbf{S}}]u^{\mathbf{S}} = \left[ \frac{1}{2}\underline{\kappa}^{\mathbf{S}}e_a^{\mathbf{S}} + \widehat{\chi}_{ab}^{\mathbf{S}}e_b^{\mathbf{S}} - \nabla_{e_3^{\mathbf{S}}}e_a^{\mathbf{S}} + (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})e_3^{\mathbf{S}} - \underline{\xi}_a^{\mathbf{S}}e_4^{\mathbf{S}} \right] u^{\mathbf{S}} \\ &= (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})e_3^{\mathbf{S}}(u^{\mathbf{S}}), \\ e_a^{\mathbf{S}}(e_3^{\mathbf{S}}(r^{\mathbf{S}})) &= [e_a^{\mathbf{S}}, e_3^{\mathbf{S}}]r^{\mathbf{S}} = \left[ \frac{1}{2}\underline{\kappa}^{\mathbf{S}}e_a^{\mathbf{S}} + \widehat{\chi}_{ab}^{\mathbf{S}}e_b^{\mathbf{S}} - \nabla_{e_3^{\mathbf{S}}}e_a^{\mathbf{S}} + (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})e_3^{\mathbf{S}} - \underline{\xi}_a^{\mathbf{S}}e_4^{\mathbf{S}} \right] r^{\mathbf{S}} \\ &= (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})e_3^{\mathbf{S}}(r^{\mathbf{S}}) - \underline{\xi}_a^{\mathbf{S}}. \end{aligned}$$

We deduce, in view of the definition of  $z^{\mathbf{S}}$  and  $\underline{\Omega}^{\mathbf{S}}$ ,

$$\begin{aligned} e_a^{\mathbf{S}}(z^{\mathbf{S}}) &= e_a^{\mathbf{S}}(e_3^{\mathbf{S}}(u^{\mathbf{S}})) = (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})z^{\mathbf{S}}, \\ e_a^{\mathbf{S}}(\underline{\Omega}^{\mathbf{S}}) &= e_a^{\mathbf{S}}(e_3^{\mathbf{S}}(r^{\mathbf{S}})) = (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})\underline{\Omega}^{\mathbf{S}} - \underline{\xi}_a^{\mathbf{S}}, \end{aligned} \tag{4.69}$$

which implies the first two identities in (4.68).

Next, we prove the last identity in (4.68). In view of the definition of  $\nu^{\mathbf{S}}$  and  $z^{\mathbf{S}}$  we make use of (4.54) to deduce

$$\nu^{\mathbf{S}}(u^{\mathbf{S}}) = e_3^{\mathbf{S}}(u^{\mathbf{S}}) + b^{\mathbf{S}}e_4^{\mathbf{S}}(u^{\mathbf{S}}) = z^{\mathbf{S}}.$$

On the other hand, since  $u^{\mathbf{S}} = c_0 - r^{\mathbf{S}}$  along  $\Sigma$ , we have

$$z^{\mathbf{S}} = \nu^{\mathbf{S}}(u^{\mathbf{S}}) = -\nu^{\mathbf{S}}(r^{\mathbf{S}}) = -e_3^{\mathbf{S}}(r^{\mathbf{S}}) - b^{\mathbf{S}}e_4^{\mathbf{S}}(r^{\mathbf{S}}) = -\underline{\Omega}^{\mathbf{S}} - b^{\mathbf{S}}, \tag{4.70}$$

where we used (4.54) and (4.65). This concludes the proof of Lemma 4.16. □

**Lemma 4.17.** *For every scalar function  $h$ , we have the formula*

$$\nu^{\mathbf{S}}\left(\int_{\mathbf{S}} h\right) = z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left( \nu^{\mathbf{S}}(h) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})h \right). \tag{4.71}$$

*In particular*

$$\nu^{\mathbf{S}}(r^{\mathbf{S}}) = \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}(\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})},$$

where the average is with respect to  $\mathbf{S}$ .

*Proof.* See Lemma 5.2.2 in [36]. □

**Lemma 4.18.** *We have the following properties for the transition coefficients  $(f, \underline{f}, \lambda)$ :*

$$\begin{aligned} \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(f) &= 2(\eta^{\mathbf{S}} - \eta) - \frac{1}{2}\kappa(\underline{f} + b^{\mathbf{S}}f) + 2f\underline{\omega} + F \cdot \Gamma_b + l.o.t., \\ \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(\underline{f}) &= 2(\underline{\xi}^{\mathbf{S}} - \underline{\xi}) - \frac{1}{2}\underline{f}\underline{\kappa} - 2\underline{\omega}(\underline{f} - b^{\mathbf{S}}f) + b^{\mathbf{S}}\left(2\nabla^{\mathbf{S}}(\lambda) - \frac{1}{2}\underline{f}\kappa\right) \\ &\quad + F \cdot \Gamma_b + l.o.t., \\ \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(\lambda) &= 2(\underline{\omega}^{\mathbf{S}} - \underline{\omega}) - 2\underline{\omega}\overset{\circ}{\lambda} + F \cdot \Gamma_b + l.o.t., \end{aligned} \tag{4.72}$$

where l.o.t. denotes terms which are linear in  $\Gamma_g, \Gamma_b$  and linear and higher order in  $F$ .<sup>26</sup>

*Proof.* Schematically, from the transformation formulas for  $\eta, \underline{\xi}$  and  $\underline{\omega}$  in Proposition 2.14, we have

$$\begin{aligned}\nabla_3^{\mathbf{S}}(f) &= 2(\eta^{\mathbf{S}} - \eta) - \frac{1}{2}\underline{f}\kappa + 2\underline{\omega}f + F \cdot \Gamma_b + \text{l.o.t.}, \\ \nabla_3^{\mathbf{S}}(\underline{f}) &= 2(\underline{\xi}^{\mathbf{S}} - \underline{\xi}) - \frac{1}{2}\underline{f}\kappa - 2\underline{\omega}\underline{f} + F \cdot \Gamma_b + \text{l.o.t.}, \\ \nabla_3^{\mathbf{S}}(\lambda) &= 2(\underline{\omega}^{\mathbf{S}} - \underline{\omega}) - 2\underline{\omega}\overset{\circ}{\lambda} + F \cdot \Gamma_b + \text{l.o.t.},\end{aligned}$$

where  $F = (f, \underline{f}, \overset{\circ}{\lambda})$  and l.o.t. denotes terms which are linear in  $\Gamma_g, \Gamma_b$  and linear and higher order in  $F$ . Recall also that the  $e_4^{\mathbf{S}}$  derivatives of  $F$  are fixed by our transversality condition (4.14). More precisely we have from (4.14) and the transformation formulas of  $\xi, \omega, \eta$  and  $\zeta$  of Proposition 2.14

$$\begin{aligned}\nabla_4^{\mathbf{S}}(f) &= -\frac{1}{2}\kappa f + F \cdot \Gamma_b + \text{l.o.t.}, \\ \nabla_4^{\mathbf{S}}(\underline{f}) &= 2\nabla^{\mathbf{S}}(\lambda) + 2\underline{\omega}\underline{f} - \frac{1}{2}\underline{f}\kappa + F \cdot \Gamma_b + \text{l.o.t.}, \\ \nabla_4^{\mathbf{S}}(\lambda) &= F \cdot \Gamma_b + \text{l.o.t.}\end{aligned}\tag{4.73}$$

Using  $\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}}$ , we obtain (4.72). This concludes the proof of Lemma 4.18.  $\square$

**Lemma 4.19.** *We have, for any scalar function  $h$  on  $\mathcal{R}$ , and any  $1 \leq l \leq s$ ,*

$$\|(\nu^{\mathbf{S}})^l h\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \lesssim r \sup_{\mathcal{R}} \left( |\tilde{\mathfrak{d}}^{\leq s} h| + \overset{\circ}{\delta} |\mathfrak{d}^{\leq s} h| \right) + \|(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}})^{\leq l-1}(F, b^{\mathbf{S}} - b)\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s} h|,$$

where

$$\tilde{\mathfrak{d}} := (e_3 - (z + \underline{\Omega})e_4, \emptyset), \quad b := -z - \underline{\Omega}.$$

*Proof.* We denote

$$\nu_{\mathcal{R}} := e_3 + be_4.$$

By definition, we have

$$\nu^{\mathbf{S}}(h) - \nu_{\mathcal{R}}(h) = (e_3^{\mathbf{S}} - e_3)h + b^{\mathbf{S}}(e_4^{\mathbf{S}} - e_4)h + (b^{\mathbf{S}} - b)e_4(h).$$

Applying Lemma 2.13, we have

$$e_4^{\mathbf{S}}h - e_4h = O(F)(\mathfrak{d}^{\leq 1}h), \quad e_3^{\mathbf{S}}h - e_3h = O(F)(\mathfrak{d}^{\leq 1}h).$$

Thus, we obtain

$$\nu^{\mathbf{S}}(h) - \nu_{\mathcal{R}}(h) = O(F)\mathfrak{d}^{\leq 1}(h) + O(b^{\mathbf{S}} - b)\mathfrak{d}^{\leq 1}(h).$$

More generally, for  $1 \leq l \leq s$  we have

$$(\nu^{\mathbf{S}})^l(h) - (\nu_{\mathcal{R}})^l(h) = O\left((\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}})^{\leq l-1}(F, b^{\mathbf{S}} - b)\right)\mathfrak{d}^{\leq l}(h),$$

which implies

$$\|(\nu^{\mathbf{S}})^l(h) - (\nu_{\mathcal{R}})^l(h)\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \lesssim \|(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}})^{\leq l-1}(F, b^{\mathbf{S}} - b)\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s} h|. \tag{4.74}$$

<sup>26</sup>Here l.o.t. is different from that of Proposition 2.14.

Next, applying (2.52), we have

$$\begin{aligned} \|(\nu_{\mathcal{R}})^l(h)\|_{\mathfrak{h}_{s-l}(\mathbf{S})} &\lesssim r \sup_{\mathcal{R}} \left( |\mathfrak{d}^{\leq s-l}(\nu_{\mathcal{R}})^l(h)| + \overset{\circ}{\delta} |\mathfrak{d}^{\leq s-l}(\nu_{\mathcal{R}})^l(h)| \right) \\ &\lesssim r \sup_{\mathcal{R}} \left( |\tilde{\mathfrak{d}}^{\leq s}h| + \overset{\circ}{\delta} |\mathfrak{d}^{\leq s}h| \right). \end{aligned} \tag{4.75}$$

Combining (4.74) and (4.75), we deduce

$$\|(\nu^{\mathbf{S}})^l h\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \lesssim r \sup_{\mathcal{R}} \left( |\tilde{\mathfrak{d}}^{\leq s}h| + \overset{\circ}{\delta} |\mathfrak{d}^{\leq s}h| \right) + \|(\nabla_{\nu^{\mathbf{S}}})^{\leq l-1}(F, b^{\mathbf{S}} - b)\|_{\mathfrak{h}_{s-l}(\mathbf{S})} \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s}h|$$

as stated. This concludes the proof of Lemma 4.19.  $\square$

In the remainder of this section, we denote  $K := s_{max} + 1$  and

$$\nu := \nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}.$$

**Lemma 4.20.** *Under the assumption (4.2), we have for  $0 \leq l \leq K$*

$$\begin{aligned} \left\| \nu^l(\dot{\kappa}, \dot{\kappa}) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} &\lesssim r^{-1} \overset{\circ}{\delta} + r^{-1} \overset{\circ}{\varepsilon} |D^{\mathbf{S}}| + r^{-1} \overset{\circ}{\varepsilon} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}, \\ \left\| \nu^l(\dot{\mu}) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} &\lesssim r^{-2} \overset{\circ}{\delta} + r^{-2} \overset{\circ}{\varepsilon} |D^{\mathbf{S}}| + r^{-2} \overset{\circ}{\varepsilon} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}. \end{aligned} \tag{4.76}$$

*Proof.* Notice that we have from (3.13) and (4.22)

$$z^{\mathbf{S}} - z, b^{\mathbf{S}} - b = O(1),$$

which implies from Lemma 4.16 that

$$|b^{\mathbf{S}}|, |z^{\mathbf{S}}|, |\underline{\Omega}^{\mathbf{S}}| \lesssim 1. \tag{4.77}$$

Applying Lemma 4.16 and (4.77), we infer that  $(z^{\mathbf{S}} - \overline{z^{\mathbf{S}}}) - (z - \overline{z})$  and  $(\underline{\Omega}^{\mathbf{S}} - \overline{\underline{\Omega}^{\mathbf{S}}}) - (\underline{\Omega} - \overline{\underline{\Omega}})$  can be estimated by  $\eta^{\mathbf{S}} - \eta$  and  $\underline{\xi}^{\mathbf{S}} - \underline{\xi}$ . Thus, these quantities can be controlled by  $\nabla_{\nu}^{\mathbf{S}}(f, \underline{f})$  thanks to Lemma 4.18 and hence

$$\left\| (z^{\mathbf{S}} - \overline{z^{\mathbf{S}}}) - (z - \overline{z}), (\underline{\Omega}^{\mathbf{S}} - \overline{\underline{\Omega}^{\mathbf{S}}}) - (\underline{\Omega} - \overline{\underline{\Omega}}) \right\|_{\mathfrak{h}_K(\mathbf{S})} \lesssim \overset{\circ}{\delta} + r \|\nabla_{\nu}^{\mathbf{S}}(f, \underline{f})\|_{\mathfrak{h}_{K-1}(\mathbf{S})}. \tag{4.78}$$

Combining (4.78) and Lemma 4.16, we obtain

$$\left\| (b^{\mathbf{S}} - \overline{b^{\mathbf{S}}}) - (b - \overline{b}) \right\|_{\mathfrak{h}_K(\mathbf{S})} \lesssim r \overset{\circ}{\delta} + r \|\nabla_{\nu}^{\mathbf{S}}(f, \underline{f})\|_{\mathfrak{h}_{K-1}(\mathbf{S})}. \tag{4.79}$$

Recalling (4.5) and Lemma 2.22, we deduce

$$\overline{b^{\mathbf{S}}} - \overline{b} = D^{\mathbf{S}} - 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} + 1 + \frac{2m(0)}{r} = D^{\mathbf{S}} + r^{-1} O(\overset{\circ}{\delta}). \tag{4.80}$$

Thus, we have

$$\|b^{\mathbf{S}} - b\|_{\mathfrak{h}_K(\mathbf{S})} \lesssim r \overset{\circ}{\delta} + r |D^{\mathbf{S}}| + r \|\nabla_{\nu}^{\mathbf{S}}(f, \underline{f})\|_{\mathfrak{h}_{K-1}(\mathbf{S})}. \tag{4.81}$$

Similarly, we have

$$\|(\nabla_{\nu}^{\mathbf{S}})^{\leq l-1}(b^{\mathbf{S}} - b)\|_{\mathfrak{h}_{K-l-1}(\mathbf{S})} \lesssim r \overset{\circ}{\delta} + r |D^{\mathbf{S}}| + r \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^j(f, \underline{f})\|_{\mathfrak{h}_{K-j}(\mathbf{S})}. \tag{4.82}$$



Applying Proposition 2.2, we deduce

$$\begin{aligned} e_4 \dot{\kappa} &= -\frac{2}{r} \dot{\kappa} + \Gamma_g \cdot \Gamma_g = r^{-3} O(\overset{\circ}{\varepsilon}), \\ e_4 \dot{\underline{\kappa}} &= -\frac{1}{r} \dot{\underline{\kappa}} - 2 \operatorname{div} \zeta + 2 \check{\rho} + \Gamma_b \cdot \Gamma_g = r^{-3} O(\overset{\circ}{\varepsilon}), \\ e_4 \dot{\mu} &= e_4 (-\operatorname{div} \zeta - \check{\rho} + \Gamma_b \cdot \Gamma_g) = r^{-4} O(\overset{\circ}{\varepsilon}). \end{aligned}$$

Recalling (4.2), we obtain

$$\mathfrak{d}^{\leq s}(\dot{\kappa}, \dot{\underline{\kappa}}) = r^{-2} O(\overset{\circ}{\varepsilon}), \quad \mathfrak{d}^{\leq s} \dot{\mu} = r^{-3} O(\overset{\circ}{\varepsilon}). \quad (4.83)$$

Applying Lemma 4.19 with  $h = (\dot{\kappa}, \dot{\underline{\kappa}})$  and  $s = K$ , combining with (4.2), (4.82) and (4.83), we infer

$$\begin{aligned} \|\nu^l(\dot{\kappa}, \dot{\underline{\kappa}})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} &\lesssim r \sup_{\mathcal{R}} \left( |\tilde{\mathfrak{d}}^{\leq s}(\dot{\kappa}, \dot{\underline{\kappa}})| + \overset{\circ}{\delta} |\mathfrak{d}^{\leq s}(\dot{\kappa}, \dot{\underline{\kappa}})| \right) + \|(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}})^{\leq l-1}(F, b^{\mathbf{S}} - b)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s}(\dot{\kappa}, \dot{\underline{\kappa}})|, \\ &\lesssim r^{-1} \overset{\circ}{\delta} + r^{-2} \overset{\circ}{\varepsilon} \left( r \overset{\circ}{\delta} + r |D^{\mathbf{S}}| + r \sum_{j=1}^l \|(\nabla_{\nu^{\mathbf{S}}})^j F\|_{\mathfrak{h}_{K-j}(\mathbf{S})} \right) \\ &\lesssim r^{-1} \overset{\circ}{\delta} + r^{-1} \overset{\circ}{\varepsilon} |D^{\mathbf{S}}| + r^{-1} \overset{\circ}{\varepsilon} \sum_{j=1}^l \|(\nabla_{\nu^{\mathbf{S}}})^j F\|_{\mathfrak{h}_{K-j}(\mathbf{S})}, \end{aligned}$$

which implies the first estimate in (4.76). The second estimate in (4.76) is similar. This concludes the proof of Lemma 4.20.  $\square$

**Lemma 4.21.** *In view of the GCM conditions (3.4), the system (2.57) can be written in the following form:*

$$\begin{aligned} \operatorname{curl}^{\mathbf{S}} f &= h_1 - \overline{h_1}^{\mathbf{S}}, \\ \operatorname{curl}^{\mathbf{S}} \underline{f} &= \underline{h_1} - \overline{\underline{h_1}}^{\mathbf{S}}, \\ \operatorname{div}^{\mathbf{S}} f + \frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda} - \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= h_2, \\ \operatorname{div}^{\mathbf{S}} \underline{f} + \frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda} + \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{b} &= \dot{\underline{C}}_0 + \sum_p \dot{\underline{C}}^{(p)} J^{(\mathbf{S}, p)} + \underline{h_2}, \\ \left( \Delta^{\mathbf{S}} + \frac{2}{(r^{\mathbf{S}})^2} \right) \overset{\circ}{\lambda} &= \dot{M}_0 + \sum_p \dot{M}^{(p)} J^{(\mathbf{S}, p)} + \frac{1}{2r^{\mathbf{S}}} \left( \dot{\underline{C}}_0 + \sum_p \dot{\underline{C}}^{(p)} J^{(\mathbf{S}, p)} \right) + h_3, \\ \Delta^{\mathbf{S}} \overset{\circ}{b} - \frac{1}{2} \operatorname{div}^{\mathbf{S}} (\underline{f} - f) &= h_4 - \overline{h_4}^{\mathbf{S}}, \quad \overline{\overset{\circ}{b}}^{\mathbf{S}} = b_0, \end{aligned} \quad (4.84)$$

where

$$\overset{\circ}{b} := r - r^{\mathbf{S}},$$

and

$$\begin{aligned} &\left\| \nu^l(h_1, \overline{h_1}, h_2, \overline{h_2}, h_4) \right\|_{\mathfrak{h}_s(\mathbf{S})} + r \left\| \nu^l(h_3) \right\|_{\mathfrak{h}_s(\mathbf{S})} \\ &\lesssim r^{-1} \overset{\circ}{\delta} + (\overset{\circ}{\varepsilon} r^{-1} + r^{-2}) \left( \left\| (\nabla_{\nu^{\mathbf{S}}})^l(F) \right\|_{\mathfrak{h}_s(\mathbf{S})} + r^{-1} \left\| \nu^l(\overset{\circ}{b}) \right\|_{\mathfrak{h}_s(\mathbf{S})} \right) \\ &+ r^{-1} \left( \left\| (\nabla_{\nu^{\mathbf{S}}})^{\leq l-1}(F) \right\|_{\mathfrak{h}_s(\mathbf{S})} + r^{-1} \left\| \nu^{\leq l-1} \overset{\circ}{b} \right\|_{\mathfrak{h}_s(\mathbf{S})} \right) + r^{-2} \overset{\circ}{\delta} \left\| \nu^{\leq l-1}(b^{\mathbf{S}}) \right\|_{\mathfrak{h}_s(\mathbf{S})}. \end{aligned} \quad (4.85)$$

*Proof.* See Remark 4.10 in [34] for the proof of (4.84) and the explicit expressions of  $(h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4)$ . Remark that (4.85) is a direct consequence of GCM conditions (3.4), Remark 4.11 in [34] and Lemma 4.20.  $\square$

**Proposition 4.22.** *We have the following estimate*

$$\|F\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S})} + \|\nabla_\nu^{\mathbf{S}} F\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}|. \quad (4.86)$$

*Proof.* In view of the construction of  $\Sigma$  and (3.6), for every  $\mathbf{S} \subset \Sigma$ , we have

$$\|F\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S})} \lesssim \overset{\circ}{\delta}. \quad (4.87)$$

To derive the remaining tangential derivatives of  $F$  along  $\Sigma_0$ , we commute the GCM system (4.84) with respect to  $\nabla_\nu^{\mathbf{S}}$ . Applying  $\nu(J^{(\mathbf{S},p)}) = 0$  and  $|\nu(r^{\mathbf{S}})| \lesssim 1^{27}$ , we obtain

$$\begin{aligned} \operatorname{curl}^{\mathbf{S}} \nabla_\nu^{\mathbf{S}}(f) &= \nabla_\nu^{\mathbf{S}}(h_1 - \bar{h}_1^{\mathbf{S}}) + [\operatorname{curl}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]f, \\ \operatorname{curl}^{\mathbf{S}} \nabla_\nu^{\mathbf{S}}(\underline{f}) &= \nabla_\nu^{\mathbf{S}}(\bar{h}_1 - \underline{\bar{h}}_1^{\mathbf{S}}) + [\operatorname{curl}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]\underline{f}, \\ \operatorname{div}^{\mathbf{S}} \nabla_\nu^{\mathbf{S}}(f) + \frac{2}{r^{\mathbf{S}}} \nabla_\nu^{\mathbf{S}}(\overset{\circ}{\lambda}) - \frac{2}{(r^{\mathbf{S}})^2} \nabla_\nu^{\mathbf{S}}(\overset{\circ}{b}) &= \nabla_\nu^{\mathbf{S}}(h_2) + [\operatorname{div}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]f + r^{-2}O(F) + r^{-3}O(\overset{\circ}{b}), \\ \operatorname{div}^{\mathbf{S}} \nabla_\nu^{\mathbf{S}}(\underline{f}) + \frac{2}{r^{\mathbf{S}}} \nabla_\nu^{\mathbf{S}}(\overset{\circ}{\lambda}) + \frac{2}{(r^{\mathbf{S}})^2} \nabla_\nu^{\mathbf{S}}(\overset{\circ}{b}) &= \nu(\dot{\underline{C}}_0) + \sum_p \nu(\dot{\underline{C}}^{(p)})J^{(\mathbf{S},p)} + \nabla_\nu^{\mathbf{S}}(\bar{h}_2) \\ &\quad + [\operatorname{div}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]\underline{f} + r^{-2}O(F) + r^{-3}O(\overset{\circ}{b}), \\ \left(\Delta^{\mathbf{S}} + \frac{2}{(r^{\mathbf{S}})^2}\right) (\nabla_\nu^{\mathbf{S}} \overset{\circ}{\lambda}) &= \nu(\dot{M}_0) + \sum_p \nu(\dot{M}^{(p)})J^{(\mathbf{S},p)} \\ &\quad + \frac{1}{2r^{\mathbf{S}}} \left( \nu(\dot{\underline{C}}_0) + \sum_p \nu(\dot{\underline{C}}^{(p)})J^{(\mathbf{S},p)} \right) + \nabla_\nu^{\mathbf{S}}(h_3) \\ &\quad + [\Delta^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]\overset{\circ}{\lambda} + r^{-3}O(F) + r^{-2}O(\dot{\underline{C}}_0, \dot{\underline{C}}^{(p)}), \\ \Delta^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}} \overset{\circ}{b}) - \frac{1}{2} \operatorname{div}^{\mathbf{S}} \left( \nabla_\nu^{\mathbf{S}} \underline{f} - \nabla_\nu^{\mathbf{S}} f \right) &= \nabla_\nu^{\mathbf{S}}(h_4 - \bar{h}_4^{\mathbf{S}}) + [\Delta^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]\overset{\circ}{b} + [\operatorname{div}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}](F), \\ b_0^{(1)} &:= \nu \overset{\circ}{b}^{\mathbf{S}}. \end{aligned} \quad (4.88)$$

Recalling that

$$F = r^{-1}O(\overset{\circ}{\delta}), \quad \overset{\circ}{b} = O(\overset{\circ}{\delta}),$$

and using Lemma 2.4, we have

$$\left([\operatorname{div}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]F, [\operatorname{curl}^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]F\right) = r^{-3}O(\overset{\circ}{\delta}) + r^{-1}O(\overset{\circ}{\varepsilon})\nabla_\nu^{\mathbf{S}}(F),$$

and

$$\begin{aligned} [\Delta^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]F &= r^{-4}O(\overset{\circ}{\delta}) + r^{-1}\mathfrak{P}^{\leq 1}(\Gamma_b \cdot \nabla_\nu^{\mathbf{S}}F) = r^{-4}O(\overset{\circ}{\delta}) + r^{-2}O(\overset{\circ}{\varepsilon})\mathfrak{P}^{\leq 1}(\nabla_\nu^{\mathbf{S}}(F)), \\ [\Delta^{\mathbf{S}}, \nabla_\nu^{\mathbf{S}}]\overset{\circ}{b} &= r^{-3}O(\overset{\circ}{\delta}) + r^{-1}\mathfrak{P}^{\leq 1}(\Gamma_b \cdot \nabla_\nu^{\mathbf{S}}\overset{\circ}{b}) = r^{-3}O(\overset{\circ}{\delta}) + r^{-2}O(\overset{\circ}{\varepsilon})\mathfrak{P}^{\leq 1}(\nabla_\nu^{\mathbf{S}}(\overset{\circ}{b})). \end{aligned}$$

Moreover, applying Proposition 3.4 to (4.84), we deduce

$$(\dot{\underline{C}}_0, \dot{\underline{C}}^{(p)}) = r^{-2}O(\overset{\circ}{\delta}). \quad (4.89)$$

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<sup>27</sup>Notice that we have from (3.13) and (4.22) that  $\nu(r^{\mathbf{S}}) = O(1)$ .

Thus, applying Proposition 3.4 to (4.88), in view of (4.85), we infer

$$\begin{aligned} \left\| \nabla_{\nu}^{\mathbf{S}}(f, \underline{f}), \nu(\overset{\circ}{\lambda}) - \overline{\nu(\overset{\circ}{\lambda})}^{\mathbf{S}} \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} &\lesssim \overset{\circ}{\delta} + \left| (\operatorname{div}^{\mathbf{S}} \nabla_{\nu}^{\mathbf{S}} f)_{\ell=1} \right| + \left| (\operatorname{div}^{\mathbf{S}} \nabla_{\nu}^{\mathbf{S}} \underline{f})_{\ell=1} \right| \\ &+ (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right), \end{aligned} \quad (4.90)$$

and

$$\begin{aligned} r \left| \overline{\nu(\overset{\circ}{\lambda})}^{\mathbf{S}} \right| &\lesssim \overset{\circ}{\delta} + \left| (\operatorname{div}^{\mathbf{S}} \nabla_{\nu}^{\mathbf{S}} f)_{\ell=1} \right| + \left| (\operatorname{div}^{\mathbf{S}} \nabla_{\nu}^{\mathbf{S}} \underline{f})_{\ell=1} \right| + |b_0^{(1)}| \\ &+ (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right). \end{aligned} \quad (4.91)$$

To estimate the  $\ell = 1$  modes of  $(\nabla_{\nu}^{\mathbf{S}})(f, \underline{f})$ , we make use of the equations (4.72) to obtain

$$\begin{aligned} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})(f) J^{(\mathbf{S}, p)} &= 2 \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} - \operatorname{div}^{\mathbf{S}} \eta) J^{(\mathbf{S}, p)} + \text{l.o.t.} \\ &= 2B^{\mathbf{S}} - \int_{\mathbf{S}} (\operatorname{div} \eta) J^{(\mathbf{S}, p)} + \int_{\mathbf{S}} (\operatorname{div} \eta - \operatorname{div}^{\mathbf{S}} \eta) J^{(\mathbf{S}, p)} + \text{l.o.t.} \\ &= 2B^{\mathbf{S}} + O(\overset{\circ}{\delta}), \end{aligned}$$

where we used (4.3), (4.67), Lemmas 2.22 and 2.23 at the last step. Similarly, we have

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})(\underline{f}) J^{(\mathbf{S}, p)} = 2\underline{B}^{\mathbf{S}} + O(\overset{\circ}{\delta}),$$

Thus, we deduce from (4.90) that

$$\begin{aligned} \left\| \nabla_{\nu}^{\mathbf{S}}(f, \underline{f}), \nu(\overset{\circ}{\lambda}) - \overline{\nu(\overset{\circ}{\lambda})}^{\mathbf{S}} \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} &\lesssim \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right), \end{aligned} \quad (4.92)$$

and

$$\begin{aligned} r \left| \overline{\nu(\overset{\circ}{\lambda})}^{\mathbf{S}} \right| &\lesssim \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |b_0^{(1)}| \\ &+ (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right). \end{aligned} \quad (4.93)$$

In the sequel, we denote for  $1 \leq l \leq s_{max} + 1$ .

$$\mathcal{E}_l := \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}| + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| (\nabla_{\nu}^{\mathbf{S}})^l(F) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \left\| \nu^l(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \right), \quad (4.94)$$

Next, we consider  $b_0^{(1)} = \overline{\nu(r - r^{\mathbf{S}})}^{\mathbf{S}}$ . Applying Lemmas 2.15, 4.17, (2.24), (4.2), (3.6) and (4.70), we infer

$$\begin{aligned} \nu(r - r^{\mathbf{S}}) &= e_3(r) + b^{\mathbf{S}} e_4(r) - \nu(r^{\mathbf{S}}) + O(\overset{\circ}{\delta}) \\ &= \frac{r}{2} \left( \overline{\Omega \bar{\kappa}} + z z^{-1} \overline{\underline{\kappa}} - z z^{-1} \overline{\Omega \kappa} \right) + b^{\mathbf{S}} \frac{r}{2} \bar{\kappa} - \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} (z^{\mathbf{S}})^{-1} (\overline{\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}} \bar{\kappa}^{\mathbf{S}}})^{\mathbf{S}} + O(\overset{\circ}{\delta}) \\ &= \underline{\Omega} + \frac{r z}{2} \overline{z^{-1} \underline{\kappa}} - z z^{-1} \overline{\Omega} + b^{\mathbf{S}} - \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} (z^{\mathbf{S}})^{-1} \underline{\kappa}^{\mathbf{S}} + z^{\mathbf{S}} (z^{\mathbf{S}})^{-1} (\overline{\underline{\Omega}^{\mathbf{S}} + z^{\mathbf{S}}})^{\mathbf{S}} + O(\overset{\circ}{\delta}) \\ &= \underline{\Omega} + \frac{r z}{2} \overline{z^{-1} \underline{\kappa}} - z z^{-1} \overline{\Omega} - \underline{\Omega}^{\mathbf{S}} - \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} (z^{\mathbf{S}})^{-1} \underline{\kappa}^{\mathbf{S}} + z^{\mathbf{S}} (z^{\mathbf{S}})^{-1} \underline{\Omega}^{\mathbf{S}} + O(\overset{\circ}{\delta}). \end{aligned}$$

Recall from Lemma 2.22 and Proposition 2.14 that

$$r - r^{\mathbf{S}} = O(\overset{\circ}{\delta}), \quad \underline{\kappa}^{\mathbf{S}} - \underline{\kappa} = O(\overset{\circ}{\delta}). \quad (4.95)$$

Notice that (4.81) and (4.92) imply

$$|b - b^{\mathbf{S}}| \lesssim \overset{\circ}{\delta} + |D^{\mathbf{S}}| + \|\nabla_{\nu}^{\mathbf{S}}(f, \underline{f})\|_{\mathfrak{h}_{K-1}(\mathbf{S})} = O(\mathcal{E}_1). \quad (4.96)$$

Recall that Lemma 2.23 allows us to compare the average on  $\mathbf{S}$  and  $\mathbf{S}$ . Thus, we may assume from now on that all averages are on  $\mathbf{S}$ . We obtain

$$\begin{aligned} \nu(r - r^{\mathbf{S}}) &= \underline{\Omega} + \frac{rz}{2} \overline{z^{-1}\underline{\kappa}} - \overline{zz^{-1}\underline{\Omega}} - \underline{\Omega}^{\mathbf{S}} - \frac{r}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}\underline{\kappa}} + z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}\underline{\Omega}^{\mathbf{S}}} + O(\overset{\circ}{\delta}) \\ &= -b + \frac{rz}{2} \overline{z^{-1}\underline{\kappa}} + \overline{zz^{-1}b} + b^{\mathbf{S}} - \frac{rz^{\mathbf{S}}}{2} \overline{(z^{\mathbf{S}})^{-1}\underline{\kappa}} - z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}b^{\mathbf{S}}} + O(\mathcal{E}_1) \\ &= \overline{zz^{-1} \left( \frac{r}{2}\underline{\kappa} + b \right)} - z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} \left( \frac{r}{2}\underline{\kappa} + b \right)} + O(\mathcal{E}_1). \end{aligned}$$

We denote

$$\frac{r}{2}\underline{\kappa} + b := -2 + h_0, \quad h_0 = r\Gamma_b = O(\overset{\circ}{\varepsilon}).$$

Thus, we deduce

$$\nu(r - r^{\mathbf{S}}) = -2 \left( \overline{zz^{-1}} - z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}} \right) + O(\overset{\circ}{\varepsilon}) \overline{z - z^{\mathbf{S}}} + O(\mathcal{E}_1). \quad (4.97)$$

Taking the average of (4.97) on  $\mathbf{S}$ , recalling that  $\overline{ab} = \overline{a\bar{b}} + \overline{(a - \bar{a})(b - \bar{b})}$  and applying (4.78), we obtain

$$\begin{aligned} \overline{\nu(r - r^{\mathbf{S}})} &= -2 \left( \overline{\overline{zz^{-1}} - z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}}} \right) + O(\overset{\circ}{\varepsilon}) |z - z^{\mathbf{S}}| + O(\mathcal{E}_1) \\ &= 2 \left( \overline{\overline{(z - \bar{z})(z^{-1} - \bar{z}^{-1})} - \overline{(z^{\mathbf{S}} - \bar{z}^{\mathbf{S}}) \left( (z^{\mathbf{S}})^{-1} - \overline{(z^{\mathbf{S}^{-1}})} \right)}} \right) + O(\overset{\circ}{\varepsilon}) |z - z^{\mathbf{S}}| + O(\mathcal{E}_1) \\ &= O(\overset{\circ}{\varepsilon}) \overline{z - z^{\mathbf{S}}} + O(\mathcal{E}_1). \end{aligned}$$

On the other hand, we apply (4.2), (4.4), (4.70), (4.96) and Lemma 2.15 to obtain

$$\begin{aligned} \nu(r - r^{\mathbf{S}}) &= e_3(r) + b^{\mathbf{S}} e_4(r) - b^{\mathbf{S}} - \underline{\Omega}^{\mathbf{S}} + O(\overset{\circ}{\delta}) = e_3(r) + b \frac{r}{2} \bar{\kappa} + z^{\mathbf{S}} + O(\mathcal{E}_1) \\ &= e_3(r) - e_3(s) + \underline{\Omega} + b + z^{\mathbf{S}} + O(\overset{\circ}{\delta}) = e_3(r) - e_3(s) + z^{\mathbf{S}} - z + O(\mathcal{E}_1) \\ &= z^{\mathbf{S}} - z + O(\mathcal{E}_1). \end{aligned}$$

Thus, we deduce

$$\overline{z^{\mathbf{S}} - z} = \overline{\nu(r - r^{\mathbf{S}})} + O(\mathcal{E}_1) = O(\overset{\circ}{\varepsilon}) \overline{(z^{\mathbf{S}} - z)} + O(\mathcal{E}_1), \quad (4.98)$$

which implies

$$\overline{z - z^{\mathbf{S}}} = O(\mathcal{E}_1), \quad \overline{\nu(r - r^{\mathbf{S}})} = O(\mathcal{E}_1),$$

and hence, recalling (4.94) and Lemma 2.23, we obtain

$$\begin{aligned} b_0^{(1)} &= \overline{\nu(r - r^{\mathbf{S}})} + O(\overset{\circ}{\delta}) \\ &\lesssim \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}| \\ &\quad + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right). \end{aligned} \quad (4.99)$$

Finally, we recall the equation of  $\overset{\circ}{b}$  in (4.88). By (4.85) and Lemma 2.10, we have

$$\left\| \nu(\overset{\circ}{b}) - \overline{\nu(\overset{\circ}{b})}^{\mathbf{S}} \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \lesssim r\overset{\circ}{\delta} + r \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + (\overset{\circ}{\varepsilon} + r^{-1}) \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})}. \quad (4.100)$$

Thus, from (4.92), (4.93) (4.99) and (4.100), we conclude that

$$\begin{aligned} & \left\| \nabla_{\nu}^{\mathbf{S}}(f, \underline{f}, \overset{\circ}{\lambda}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \\ & \lesssim \overset{\circ}{\delta} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}| + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \left\| \nabla_{\nu}^{\mathbf{S}}(F) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + r^{-1} \left\| \nu(\overset{\circ}{b}) \right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \right). \end{aligned}$$

For  $\overset{\circ}{\varepsilon}$  and small enough and  $\overset{\circ}{r}$  large enough, we deduce (4.86). This concludes the proof of Proposition 4.22.  $\square$

### 4.3 Extrinsic properties of $\Sigma$

The goal of this section is to prove the following proposition.

**Proposition 4.23.** *We assume that*

$$|B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}| \leq \overset{\circ}{\varepsilon}^{\frac{2}{3}}. \quad (4.101)$$

Then, the following estimates hold true for all  $k \leq s_{max}$ :

1. The Ricci coefficients  $\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}}$  verify

$$\begin{aligned} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} & \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}|, \\ \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} & \lesssim \overset{\circ}{\varepsilon} + |\underline{B}^{\mathbf{S}}|, \\ \|\underline{\omega}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} & \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}|. \end{aligned}$$

2. The scalar  $b^{\mathbf{S}}$  verifies

$$(r^{\mathbf{S}})^{-1} \|\check{b}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |D^{\mathbf{S}}|.$$

3. We also have

$$\begin{aligned} (r^{\mathbf{S}})^{-1} \|\check{\underline{\Omega}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} & \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}|, \\ (r^{\mathbf{S}})^{-1} \|\check{\underline{z}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} & \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}|. \end{aligned}$$

**Remark 4.24.** *We split the Ricci and curvature coefficients as follows.*

$$\begin{aligned} \Gamma_g^{\mathbf{S}} & = \{ \check{\kappa}^{\mathbf{S}}, \check{\chi}^{\mathbf{S}}, \zeta^{\mathbf{S}}, \check{\underline{\kappa}}^{\mathbf{S}}, r\check{\rho}^{\mathbf{S}}, r^*\check{\rho}^{\mathbf{S}}, r\check{\beta}^{\mathbf{S}}, r\check{\alpha}^{\mathbf{S}}, r\check{\mu}^{\mathbf{S}}, r\check{K}^{\mathbf{S}} \}, \\ \Gamma_{b,w}^{\mathbf{S}} & = \{ \check{\chi}^{\mathbf{S}}, r\check{\beta}^{\mathbf{S}}, \check{\underline{\alpha}}^{\mathbf{S}} \}, \\ \Gamma_{b,i}^{\mathbf{S}} & = \{ \eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}} \}. \end{aligned}$$

Since all the terms in  $\Gamma_g^{\mathbf{S}} \cup \Gamma_{b,w}^{\mathbf{S}}$  are well-defined on a sphere, as a direct consequence of Corollary 3.5, see also (3.11), we have the following estimates for  $k \leq s_{max}$ .

$$\|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} r^{-1}, \quad \|\Gamma_{b,w}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon}. \quad (4.102)$$

By the null structure equations and Bianchi equations introduced in Proposition 2.2, we obtain immediately<sup>28</sup>

$$\|\nabla_4^{\mathbf{S}} \Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_{k-1}(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} r^{-2}, \quad \|\nabla_4^{\mathbf{S}} \Gamma_{b,w}^{\mathbf{S}}\|_{\mathfrak{h}_{k-1}(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} r^{-1}. \tag{4.103}$$

**Remark 4.25.** *In the sequel, unless stated otherwise, we admit the following conventions:*

- For a scalar quantity  $h^{\mathbf{S}}$  defined on  $\mathbf{S}$ , we denote  $\overline{h^{\mathbf{S}}}$  for its average over the sphere  $\mathbf{S}$ ;
- For a background scalar quantity  $h$ , we denote  $\overline{h}$  for its average over the background sphere  $S(u, s)$ ;
- For any scalar quantity  $h$ , we denote  $\overline{h^{\mathbf{S}}}$  for its average over the sphere  $\mathbf{S}$ .

Recalling that  $\delta \leq \overset{\circ}{\varepsilon}$ , applying Proposition 4.22, (4.101) and Lemma 4.18, we deduce the following estimates for  $k \leq s_{max}$

$$\|\Gamma_{b,i}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon}^{\frac{2}{3}}. \tag{4.104}$$

Notice that we have

$$\|\nabla_3^{\mathbf{S}} \Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_{k-1}(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon}^{\frac{2}{3}} r^{-1}, \quad \|\nabla_3^{\mathbf{S}} \Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_{k-1}(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon}^{\frac{2}{3}}, \tag{4.105}$$

as a direct consequence of (4.104) and Proposition 2.2.

To prove Proposition 4.23, we need the following lemmas.

**Lemma 4.26.** *Under the GCM conditions (4.58) on  $\Sigma$ , we have*

$$\begin{aligned} \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \mu^{\mathbf{S}}) &= r^{-5} O(\overset{\circ}{\varepsilon}), \\ \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\nabla^{\mathbf{S}} \kappa^{\mathbf{S}}) &= 0, \\ \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} ((\not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} + 2K^{\mathbf{S}}) \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \underline{\kappa}^{\mathbf{S}}) &= r^{-6} O(\overset{\circ}{\varepsilon}). \end{aligned}$$

**Lemma 4.27.** *The following estimates hold true for  $k \leq s_{max} - 5$*

$$\begin{aligned} \|\not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1}, \\ \|\not{d}_2^{\mathbf{S},*} \underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1}, \\ \|\not{d}_1^{\mathbf{S},*} \underline{\omega}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1} + r^{-1} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})}. \end{aligned} \tag{4.106}$$

**Lemma 4.28.** *We have the following identity for  $\overline{\omega}^{\mathbf{S}}$ .*

$$\overline{\omega}^{\mathbf{S}} = \overline{\omega}^{\mathbf{S}} - \frac{m^{\mathbf{S}}}{(r^{\mathbf{S}})^2} = -\frac{1}{2r^{\mathbf{S}}} \overline{\underline{\Omega}^{\mathbf{S}}} + \frac{r^{\mathbf{S}}}{2} \left( \overline{\underline{\mu}^{\mathbf{S}}} - \overline{\eta^{\mathbf{S}} \cdot \eta^{\mathbf{S}}} + \frac{\overline{b^{\mathbf{S}}}}{2} \overline{\widehat{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}}} \right).$$

**Lemma 4.29.** *We have the following estimate for  $k \leq s_{max}$ :*

$$\left\| (\check{b}^{\mathbf{S}}, \check{z}^{\mathbf{S}}, \check{\underline{\Omega}}^{\mathbf{S}}) \right\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r \overset{\circ}{\varepsilon}^{\frac{2}{3}}. \tag{4.107}$$

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<sup>28</sup>As explained in Remark 4.12, the transversality conditions are consistent with extension by local outgoing geodesic foliation initialized on  $\Sigma$ . Hence, the equations in Proposition 2.2 are valid on  $\Sigma$ .

*Proof of Lemma 4.26.* Recalling that

$$\check{\mu}^{\mathbf{S}} = M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(\mathbf{S},p)}, \quad (4.108)$$

we infer

$$\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}} = \sum_p M^{(\mathbf{S},p)} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)}).$$

Therefore,

$$\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}}) = \nu^{\mathbf{S}} (M^{(\mathbf{S},p)}) \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} + M^{(\mathbf{S},p)} \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)}). \quad (4.109)$$

Firstly, we estimate  $\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)})$ . Recall that  $\nu^{\mathbf{S}} (J^{(\mathbf{S},p)}) = 0$ . By Lemma 2.4, we have

$$\begin{aligned} \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)}) &= [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*}] J^{(\mathbf{S},p)} \\ &= [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \not\partial_2^{\mathbf{S},*}] \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} + \not\partial_2^{\mathbf{S},*} [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \not\partial_1^{\mathbf{S},*}] J^{(\mathbf{S},p)} \\ &= \frac{2}{r^{\mathbf{S}}} \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} + \Gamma_b^{\mathbf{S}} \cdot \not\partial \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} \\ &\quad + \not\partial_2^{\mathbf{S},*} \left( \frac{2}{r^{\mathbf{S}}} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} + \Gamma_b^{\mathbf{S}} \cdot \not\partial J^{(\mathbf{S},p)} \right) \\ &= \frac{4}{r^{\mathbf{S}}} \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)} + (r^{\mathbf{S}})^{-2} O(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Recall that (see Lemma 2.8 and also Corollary 5.2.10 in [36]).

$$|\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} J^{(\mathbf{S},p)}| \lesssim \frac{\circ}{r^2} \varepsilon, \quad M^{(\mathbf{S},p)} \in r^{-1} \Gamma_g^{\mathbf{S}}.$$

According to (4.109), we deduce

$$\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}}) = (r^{\mathbf{S}})^{-5} O(\varepsilon^{\circ}),$$

which implies the first estimate. The third estimate is similar and left to the reader. Finally, the second estimate follows immediately from the GCM condition  $\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}$ . This concludes the proof of Lemma 4.26.  $\square$

*Proof of Lemma 4.27.* We denote

$$\begin{aligned} C_1 &= \nabla_3^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}}), \\ C_2 &= \nabla_3^{\mathbf{S}} (\nabla \kappa^{\mathbf{S}}), \\ C_3 &= \nabla_3^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_2^{\mathbf{S}} + 2K^{\mathbf{S}}) \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \underline{\kappa}^{\mathbf{S}}. \end{aligned} \quad (4.110)$$

Recalling that  $\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}$ , we obtain in view of Lemma 4.26

$$\begin{aligned} C_1 &= -b^{\mathbf{S}} \nabla_4^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}}) + r^{-5} O(\varepsilon^{\circ}), \\ C_2 &= -b^{\mathbf{S}} \nabla_4^{\mathbf{S}} (\nabla^{\mathbf{S}} \kappa^{\mathbf{S}}), \\ C_3 &= -b^{\mathbf{S}} \nabla_4^{\mathbf{S}} \left( M^{(\mathbf{S},p)} (\not\partial_2^{\mathbf{S}} \not\partial_2^{\mathbf{S},*} + 2K^{\mathbf{S}}) \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \underline{\kappa}^{\mathbf{S}} \right) + r^{-6} O(\varepsilon^{\circ}). \end{aligned}$$

We claim that

$$\begin{aligned} \|\nabla_4^{\mathbf{S}} (\not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \mu^{\mathbf{S}})\|_{\mathfrak{h}_{k-3}(\mathbf{S})} &\lesssim \varepsilon^{\circ} r^{-5}, \\ \|\nabla_4^{\mathbf{S}} (\nabla^{\mathbf{S}} \kappa^{\mathbf{S}})\|_{\mathfrak{h}_{k-2}(\mathbf{S})} &\lesssim \varepsilon^{\circ} r^{-3}, \\ \|\nabla_4^{\mathbf{S}} \left( (\not\partial_2^{\mathbf{S},*} \not\partial_2^{\mathbf{S}} + 2K^{\mathbf{S}}) \not\partial_2^{\mathbf{S},*} \not\partial_1^{\mathbf{S},*} \underline{\kappa}^{\mathbf{S}} \right)\|_{\mathfrak{h}_{k-5}(\mathbf{S})} &\lesssim \varepsilon^{\circ} r^{-6}. \end{aligned} \quad (4.111)$$

Indeed, making use of Lemma 2.4 and (4.103), we have

$$\begin{aligned}\nabla_4^{\mathbf{S}} \nabla^{\mathbf{S}} \kappa^{\mathbf{S}} &= [\nabla_4^{\mathbf{S}}, \nabla^{\mathbf{S}}] \check{\kappa}^{\mathbf{S}} + \nabla^{\mathbf{S}} \nabla_4^{\mathbf{S}} \check{\kappa}^{\mathbf{S}} \\ &= -r^{-1} \nabla^{\mathbf{S}} \check{\kappa}^{\mathbf{S}} + r^{-1} \Gamma_g^{\mathbf{S}} \cdot \mathfrak{d}^{\leq 1} \check{\kappa}^{\mathbf{S}} + r^{-1} \mathfrak{d} \nabla_4^{\mathbf{S}} (\Gamma_g^{\mathbf{S}}) \\ &= r^{-4} O(\overset{\circ}{\varepsilon}),\end{aligned}$$

which implies the second estimate in (4.111). The first and third estimates in (4.111) are similar and left to the reader.

Writing  $b^{\mathbf{S}} = \check{b}^{\mathbf{S}} - 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}$ , recall the bootstrap assumptions (4.104) and making use of product estimates we deduce

$$\begin{aligned}\|C_1\|_{\mathfrak{h}_{k-3}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-4}, \\ \|C_2\|_{\mathfrak{h}_{k-2}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-3}, \\ \|C_3\|_{\mathfrak{h}_{k-5}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-5}.\end{aligned}$$

Next, we estimate the error terms

$$\begin{aligned}E_k &:= r^{-1} \|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \\ &\lesssim r^{-2} \overset{\circ}{\varepsilon} + r^{-1} \|\Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \|\Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \\ &\lesssim r^{-1} \overset{\circ}{\varepsilon}, \quad \forall k \leq s_{max}.\end{aligned}$$

We cite the following identities, see Propositions 5.1.21 and 5.1.22 in [36]:

$$\begin{aligned}2 \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} &= -\not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} \nabla_3^{\mathbf{S}} \nabla^{\mathbf{S}} \kappa^{\mathbf{S}} + \frac{2}{r^{\mathbf{S}}} \nabla_3^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \mu^{\mathbf{S}} - \frac{4}{r^{\mathbf{S}}} \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \operatorname{div}^{\mathbf{S}} \underline{\beta}^{\mathbf{S}} \\ &\quad + (r^{\mathbf{S}})^{-5} (\not{\vartheta}^{\mathbf{S}})^{\leq 4} \Gamma_g^{\mathbf{S}} + (r^{\mathbf{S}})^{-4} (\not{\vartheta}^{\mathbf{S}})^{\leq 4} (\Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}), \\ 2 \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \underline{\xi}^{\mathbf{S}} &= \nabla_3^{\mathbf{S}} (\not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} + 2K^{\mathbf{S}}) \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \underline{\kappa}^{\mathbf{S}} + \frac{2}{r^{\mathbf{S}}} \nabla_3^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \mu^{\mathbf{S}} \\ &\quad - \frac{4}{r^{\mathbf{S}}} \not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \operatorname{div}^{\mathbf{S}} \underline{\beta}^{\mathbf{S}} + (r^{\mathbf{S}})^{-5} (\not{\vartheta}^{\mathbf{S}})^{\leq 4} \Gamma_g^{\mathbf{S}} + (r^{\mathbf{S}})^{-4} (\not{\vartheta}^{\mathbf{S}})^{\leq 4} (\Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}), \\ 2 \nabla^{\mathbf{S}} \underline{\omega}^{\mathbf{S}} &= \frac{1}{r^{\mathbf{S}}} \underline{\xi}^{\mathbf{S}} - \nabla_3^{\mathbf{S}} \underline{\zeta}^{\mathbf{S}} - \underline{\beta}^{\mathbf{S}} + \frac{1}{r^{\mathbf{S}}} \eta^{\mathbf{S}} + (r^{\mathbf{S}})^{-1} \Gamma_g^{\mathbf{S}} + \Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}.\end{aligned}$$

Recalling that

$$\not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} = 2 \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} + 2K,$$

we deduce

$$\begin{aligned}\not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} &= 2 \not{d}_2^{\mathbf{S},*} (\not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} + K) \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} \\ &= 2 \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} + \frac{2}{r^2} \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} \\ &\quad + 2 \check{K} \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} + 2 \not{d}_2^{\mathbf{S},*} (\check{K}) \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}.\end{aligned}$$

Notice that

$$\begin{aligned}2 \check{K} \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} &= O(\overset{\circ}{\varepsilon} r^{-3}) \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}, \\ 2 \not{d}_2^{\mathbf{S},*} (\check{K}) \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}} &= O(\overset{\circ}{\varepsilon} r^{-4}) \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}},\end{aligned}$$

where we used (4.102). Thus, by Corollary 2.12, we obtain

$$\begin{aligned}\|\not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim r^4 \|\not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + r^2 \|\not{d}_2^{\mathbf{S},*} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \\ &\lesssim r^4 \|\not{d}_2^{\mathbf{S},*} \not{d}_1^{\mathbf{S},*} \not{d}_1^{\mathbf{S}} \not{d}_2^{\mathbf{S}} \not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + O(\overset{\circ}{\varepsilon}) \|\not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+2}(\mathbf{S})},\end{aligned}$$



which implies

$$\|\underline{\not{d}}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} \lesssim r^4 \|\underline{\not{d}}_2^{\mathbf{S},*} \underline{\not{d}}_1^{\mathbf{S},*} \underline{\not{d}}_1^{\mathbf{S}} \underline{\not{d}}_2^{\mathbf{S}} \underline{\not{d}}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})}.$$

Using (4.102), (4.103) and (4.104), we deduce

$$\|\underline{\not{d}}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} \lesssim r \|C_2\|_{\mathfrak{h}_{k+3}(\mathbf{S})} + r^3 \|C_1\|_{\mathfrak{h}_k(\mathbf{S})} + \overset{\circ}{\varepsilon} r^{-1} + E_{4+k} \lesssim \overset{\circ}{\varepsilon} r^{-1}.$$

Similarly, we have

$$\|\underline{\not{d}}_2^{\mathbf{S},*} \underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} \lesssim r^4 \|C_3\|_{\mathfrak{h}_k(\mathbf{S})} + r^3 \|C_1\|_{\mathfrak{h}_{k+3}(\mathbf{S})} + \overset{\circ}{\varepsilon} r^{-1} + E_{4+k} \lesssim \overset{\circ}{\varepsilon} r^{-1}.$$

Finally, recalling that  $r\underline{\beta}^{\mathbf{S}} \in \Gamma_b^{\mathbf{S}}$  is well-defined on a sphere, we obtain

$$\begin{aligned} \|\underline{\not{d}}_1^{\mathbf{S},*} \underline{\omega}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim r^{-1} \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \overset{\circ}{\varepsilon} + E_{4+k} \\ &\lesssim r^{-1} \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \overset{\circ}{\varepsilon} \end{aligned}$$

as stated. This concludes the proof of Lemma 4.27.  $\square$

*Proof of Lemma 4.28.* We recall the following null structure equation, see Proposition 7.4.1 in [10],

$$\nabla_3^{\mathbf{S}} \kappa^{\mathbf{S}} + \frac{1}{2} \underline{\kappa}^{\mathbf{S}} \kappa^{\mathbf{S}} - 2 \underline{\omega}^{\mathbf{S}} \kappa^{\mathbf{S}} = 2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} - \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} + 2(\eta^{\mathbf{S}})^2 + 2\rho^{\mathbf{S}}.$$

Making use of the GCM condition  $\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}$  we deduce

$$\begin{aligned} \underline{\omega}^{\mathbf{S}} &= \frac{1}{2\kappa^{\mathbf{S}}} \left[ \nabla_3^{\mathbf{S}} \kappa^{\mathbf{S}} + \frac{1}{2} \underline{\kappa}^{\mathbf{S}} \kappa^{\mathbf{S}} - 2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} + \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} - 2(\eta^{\mathbf{S}})^2 - 2\rho^{\mathbf{S}} \right] \\ &= \frac{1}{2\kappa^{\mathbf{S}}} \left[ \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} \left( \frac{2}{r^{\mathbf{S}}} \right) - b^{\mathbf{S}} \nabla_4^{\mathbf{S}} \kappa^{\mathbf{S}} + \frac{1}{2} \underline{\kappa}^{\mathbf{S}} \kappa^{\mathbf{S}} - 2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} + \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} - 2(\eta^{\mathbf{S}})^2 - 2\rho^{\mathbf{S}} \right] \\ &= \frac{1}{2\kappa^{\mathbf{S}}} \left[ -2 \frac{e_3^{\mathbf{S}}(r^{\mathbf{S}}) + b^{\mathbf{S}}}{(r^{\mathbf{S}})^2} + b^{\mathbf{S}} \left( \frac{1}{2} (\kappa^{\mathbf{S}})^2 + |\widehat{\chi}^{\mathbf{S}}|^2 \right) + \frac{1}{2} \underline{\kappa}^{\mathbf{S}} \kappa^{\mathbf{S}} - 2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} + \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} - 2(\eta^{\mathbf{S}})^2 - 2\rho^{\mathbf{S}} \right] \\ &= \frac{1}{2\kappa^{\mathbf{S}}} \left[ -\frac{2e_3^{\mathbf{S}}(r^{\mathbf{S}})}{(r^{\mathbf{S}})^2} + b^{\mathbf{S}} |\widehat{\chi}^{\mathbf{S}}|^2 + \frac{1}{2} \underline{\kappa}^{\mathbf{S}} \kappa^{\mathbf{S}} - 2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} + \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} - 2(\eta^{\mathbf{S}})^2 - 2\rho^{\mathbf{S}} \right] \\ &= -\frac{e_3^{\mathbf{S}}(r^{\mathbf{S}})}{2r^{\mathbf{S}}} + \frac{1}{4} b^{\mathbf{S}} r^{\mathbf{S}} |\widehat{\chi}^{\mathbf{S}}|^2 + \frac{1}{4} \underline{\kappa}^{\mathbf{S}} + \frac{r^{\mathbf{S}}}{4} \left[ -2 \operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}} + \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} - 2(\eta^{\mathbf{S}})^2 - 2\rho^{\mathbf{S}} \right], \end{aligned}$$

where we used

$$e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1, \quad \underline{\Omega}^{\mathbf{S}} = e_3^{\mathbf{S}}(r^{\mathbf{S}}), \quad e_4^{\mathbf{S}}(\kappa^{\mathbf{S}}) = -\frac{1}{2} (\kappa^{\mathbf{S}})^2 - |\widehat{\chi}^{\mathbf{S}}|^2.$$

Taking the average over  $\mathbf{S}$ , and recalling that  $\mu^{\mathbf{S}} = -\operatorname{div}^{\mathbf{S}} \zeta^{\mathbf{S}} - \rho^{\mathbf{S}} + \frac{1}{2} \underline{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}}$  and (4.116), we obtain

$$\overline{\underline{\omega}^{\mathbf{S}}} = -\frac{1}{2r^{\mathbf{S}}} \overline{\underline{\Omega}^{\mathbf{S}}} + \frac{r^{\mathbf{S}}}{2} \overline{\mu^{\mathbf{S}} - (\eta^{\mathbf{S}})^2 + \frac{b^{\mathbf{S}}}{2} |\widehat{\chi}^{\mathbf{S}}|^2},$$

or

$$\overline{\underline{\omega}^{\mathbf{S}}} = \overline{\underline{\omega}^{\mathbf{S}}} - \frac{m^{\mathbf{S}}}{(r^{\mathbf{S}})^2} = -\frac{1}{2r^{\mathbf{S}}} \overline{\underline{\Omega}^{\mathbf{S}}} + \frac{r^{\mathbf{S}}}{2} \left( \overline{\mu^{\mathbf{S}}} - \overline{\eta^{\mathbf{S}} \cdot \eta^{\mathbf{S}}} + \frac{b^{\mathbf{S}}}{2} \overline{\widehat{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}}} \right).$$

This concludes the proof of Lemma 4.28.  $\square$

*Proof of Lemma 4.29.* We have from Proposition 4.22, (4.79) and (4.3) that

$$\left\| (\check{b}^{\mathbf{S}} - \check{\bar{b}}^{\mathbf{S}}) - (\check{b} - \check{\bar{b}}) \right\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\delta + r|D^{\mathbf{S}}| \lesssim r\varepsilon^{\frac{2}{3}},$$

which implies

$$\|\check{b}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\varepsilon^{\frac{2}{3}} + r|D^{\mathbf{S}}| + \|\check{b} - \check{\bar{b}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\varepsilon^{\frac{2}{3}}. \tag{4.112}$$

We have from Lemma 4.28 and (4.104)

$$\left| \check{\underline{\Omega}}^{\mathbf{S}} \right| \lesssim r|\underline{\omega}^{\mathbf{S}}| + r^2|\check{\underline{\mu}}^{\mathbf{S}}| + r^2|\Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}| \lesssim \varepsilon^{\frac{2}{3}}.$$

Next, we have from Proposition 4.22, (4.78) and (4.3) that

$$\left\| (\check{\underline{\Omega}}^{\mathbf{S}} - \check{\bar{\underline{\Omega}}}^{\mathbf{S}}) - (\check{\underline{\Omega}} - \check{\bar{\underline{\Omega}}}) \right\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\delta + r|D^{\mathbf{S}}| \lesssim r\varepsilon^{\frac{2}{3}},$$

which implies

$$\|\check{\underline{\Omega}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\varepsilon^{\frac{2}{3}} + \|\check{\underline{\Omega}} - \check{\bar{\underline{\Omega}}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\varepsilon^{\frac{2}{3}}. \tag{4.113}$$

Finally, we have from (4.68) that

$$\begin{aligned} \check{z}^{\mathbf{S}} &= z^{\mathbf{S}} - 2 = -b^{\mathbf{S}} - \underline{\Omega}^{\mathbf{S}} - 2 \\ &= -b^{\mathbf{S}} - 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} - \underline{\Omega}^{\mathbf{S}} - 1 + \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} \\ &= -\check{b}^{\mathbf{S}} - \check{\underline{\Omega}}^{\mathbf{S}}, \end{aligned} \tag{4.114}$$

which implies from (4.112) and (4.113)

$$\|\check{z}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \|\check{b}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\Omega}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim r\varepsilon^{\frac{2}{3}}. \tag{4.115}$$

Combining (4.112), (4.113) and (4.115), this concludes the proof of Lemma 4.29.  $\square$

We are now ready to prove Proposition 4.23.

*Proof of Proposition 4.23.* We start with the estimate of  $\check{\underline{\Omega}}^{\mathbf{S}}$ . We first note that the GCM condition  $\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}$  together with the definition of the Hawking mass

$$\frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} = 1 + \frac{1}{16\pi} \int_{\mathbf{S}} \kappa^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}}$$

implies that,

$$\underline{\kappa}^{\mathbf{S}} = -\frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}, \tag{4.116}$$

where  $\Upsilon^{\mathbf{S}} = 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}$ . Thus, in view of Lemma 4.17, we deduce

$$\begin{aligned} \check{\underline{\Omega}}^{\mathbf{S}} + \check{b}^{\mathbf{S}} - 2 &= \underline{\Omega}^{\mathbf{S}} + b^{\mathbf{S}} = \nu^{\mathbf{S}}(r^{\mathbf{S}}) = \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}} \kappa^{\mathbf{S}})} \\ &= \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} \left( \check{\underline{\kappa}}^{\mathbf{S}} - \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} + \left( \check{b}^{\mathbf{S}} - 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} \right) \frac{2}{r^{\mathbf{S}}} \right)} \\ &= \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} \left( \check{\underline{\kappa}}^{\mathbf{S}} - \frac{4}{r^{\mathbf{S}}} + \frac{2\check{b}^{\mathbf{S}}}{r^{\mathbf{S}}} \right)} \\ &= \frac{r^{\mathbf{S}}}{2} z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} \check{\underline{\kappa}}^{\mathbf{S}}} - 2z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1}} + z^{\mathbf{S}} \overline{(z^{\mathbf{S}})^{-1} \check{b}^{\mathbf{S}}}. \end{aligned}$$

Multiplying by  $(z^{\mathbf{S}})^{-1}$  and taking the average over  $\mathbf{S}$ , we infer

$$\overline{\check{\underline{\Omega}}^{\mathbf{S}}(z^{\mathbf{S}})^{-1}} = \frac{r^{\mathbf{S}}}{2} \overline{(z^{\mathbf{S}})^{-1} \check{\underline{\kappa}}^{\mathbf{S}}}.$$

We write

$$\frac{1}{z^{\mathbf{S}}} = \frac{1}{2 + \check{z}^{\mathbf{S}}} = \frac{1}{2} + O(\check{z}^{\mathbf{S}}),$$

and hence, recalling that  $\overline{\check{\underline{\kappa}}^{\mathbf{S}}} = 0$ , we infer

$$\overline{\check{\underline{\Omega}}^{\mathbf{S}}} = \overline{O(\check{z}^{\mathbf{S}}) \check{\underline{\Omega}}^{\mathbf{S}}} + r^{\mathbf{S}} \overline{O(\check{z}^{\mathbf{S}}) \check{\underline{\kappa}}^{\mathbf{S}}}. \quad (4.117)$$

Combining Lemma 4.16 and (4.117), we obtain

$$\begin{aligned} e_a^{\mathbf{S}}(\check{\underline{\Omega}}^{\mathbf{S}}) &= (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}}) \check{\underline{\Omega}}^{\mathbf{S}} - \check{\underline{\xi}}_a^{\mathbf{S}}, \\ \overline{\check{\underline{\Omega}}^{\mathbf{S}}} &= \overline{O(\check{z}^{\mathbf{S}}) \check{\underline{\Omega}}^{\mathbf{S}}} + r \Gamma_g^{\mathbf{S}}. \end{aligned}$$

Applying Lemma 4.29 and (4.102), we deduce for  $k \leq s_{max} - 1$ ,

$$\begin{aligned} r^{-1} \|\check{\underline{\Omega}}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\xi}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + r^2 \|\check{z}^{\mathbf{S}} \cdot \check{\underline{\Omega}}^{\mathbf{S}}\|_{L^\infty(\mathbf{S})} \\ &\lesssim \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\xi}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \mathring{\varepsilon}. \end{aligned} \quad (4.118)$$

Applying Lemma 2.10 to the equation

$$e_a^{\mathbf{S}}(z^{\mathbf{S}}) = (\zeta^{\mathbf{S}} - \eta^{\mathbf{S}}) z^{\mathbf{S}},$$

we derive

$$\begin{aligned} r^{-1} \|z^{\mathbf{S}} - \overline{z^{\mathbf{S}}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \|(\zeta^{\mathbf{S}} - \eta^{\mathbf{S}}) z^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \\ &\lesssim \|(\zeta^{\mathbf{S}} - \eta^{\mathbf{S}}) \check{z}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \mathring{\varepsilon} \\ &\lesssim \mathring{\varepsilon} + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})}. \end{aligned} \quad (4.119)$$

Next, we estimate  $\overline{\check{z}^{\mathbf{S}}}$ . For this, we have from (4.114)

$$\overline{\check{z}^{\mathbf{S}}} = -\overline{\check{b}^{\mathbf{S}}} - \overline{\check{\underline{\Omega}}^{\mathbf{S}}} = -D^{\mathbf{S}} - \overline{\check{\underline{\Omega}}^{\mathbf{S}}}.$$

Then, applying (4.118), we obtain

$$|\overline{\check{z}^{\mathbf{S}}}| \lesssim |D^{\mathbf{S}}| + \|\check{\underline{\Omega}}^{\mathbf{S}}\|_{L^\infty} \lesssim |D^{\mathbf{S}}| + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\xi}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \mathring{\varepsilon}.$$

By (4.119) and the above estimate, we conclude

$$r^{-1} \|\check{z}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} \lesssim |D^{\mathbf{S}}| + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\xi}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \mathring{\varepsilon}. \quad (4.120)$$

Finally, according to (4.114), we have

$$\begin{aligned} r^{-1} \|\check{b}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim r^{-1} \|\check{z}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} + r^{-1} \|\check{\underline{\Omega}}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} \\ &\lesssim |D^{\mathbf{S}}| + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\check{\underline{\xi}}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \mathring{\varepsilon}. \end{aligned} \quad (4.121)$$

It remains to estimate  $(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\check{\omega}}^{\mathbf{S}})$ . Recall from Lemma 4.27 that,  $\forall k \leq s_{max} - 5$ , we have

$$\begin{aligned} \|\not{d}_2^{\mathbf{S},*} \eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1}, \\ \|\not{d}_2^{\mathbf{S},*} \underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1}, \\ \|\not{d}_1^{\mathbf{S},*} \underline{\omega}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} r^{-1} + r^{-1} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})} + r^{-1} \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+4}(\mathbf{S})}. \end{aligned}$$

In view of the definition of  $B^{\mathbf{S}}$  and  $\underline{B}^{\mathbf{S}}$  and Lemma 2.11, we deduce

$$\begin{aligned} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+5}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |(\text{curl}^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1}|, \\ \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+5}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |\underline{B}^{\mathbf{S}}| + |(\text{curl}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1}|. \end{aligned} \tag{4.122}$$

Recall from Proposition 2.2 that

$$\begin{aligned} \text{curl}^{\mathbf{S}} \eta^{\mathbf{S}} &= * \rho^{\mathbf{S}} + \Gamma_b^{\mathbf{S}} \cdot \Gamma_g^{\mathbf{S}}, \\ \text{curl}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}} &= \Gamma_b^{\mathbf{S}} \cdot \Gamma_b^{\mathbf{S}}. \end{aligned}$$

Applying (4.102) and (4.104), we obtain

$$\begin{aligned} \text{curl}^{\mathbf{S}} \eta^{\mathbf{S}} &= r^{-3} O(\overset{\circ}{\varepsilon}), \\ \text{curl}^{\mathbf{S}} \underline{\xi}^{\mathbf{S}} &= r^{-2} O(\overset{\circ}{\varepsilon}). \end{aligned}$$

Thus, (4.122) implies

$$\begin{aligned} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_{k+5}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}|, \\ \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_{k+5}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |\underline{B}^{\mathbf{S}}|. \end{aligned}$$

Next, we have from Lemma 4.28 that

$$\begin{aligned} |\overline{\underline{\omega}}^{\mathbf{S}}| &\lesssim (r^{\mathbf{S}})^{-1} \|\overline{\underline{\check{\omega}}}^{\mathbf{S}}\|_{L^\infty(\mathbf{S})} + \Gamma_g^{\mathbf{S}} + r^{\mathbf{S}} \|\Gamma_b^{\mathbf{S}}\|_{L^\infty(\mathbf{S})}^2 \\ &\lesssim \overset{\circ}{\varepsilon} r^{-1} + r^{-1} \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + r^{-1} \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})}. \end{aligned}$$

Together with the above estimate for  $\not{d}_1^{\mathbf{S},*} \underline{\omega}^{\mathbf{S}}$  and Lemma 2.10, we obtain

$$\begin{aligned} \|\underline{\check{\omega}}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + r |\overline{\underline{\omega}}^{\mathbf{S}}| \\ &\lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}|, \quad k \leq s_{max} - 1, \end{aligned}$$

which concludes the statement 1 of Proposition 4.23.

We can then go back to the preliminary estimates (4.118), (4.120) and (4.121) obtained above to derive

$$\begin{aligned} r^{-1} \|\overline{\underline{\check{\omega}}}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}|, \\ r^{-1} \|\check{\zeta}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |D^{\mathbf{S}}| + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}|, \\ r^{-1} \|\check{\theta}^{\mathbf{S}}\|_{\mathfrak{h}_{k+1}(\mathbf{S})} &\lesssim \overset{\circ}{\varepsilon} + |D^{\mathbf{S}}| + \|\eta^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\underline{\xi}^{\mathbf{S}}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \overset{\circ}{\varepsilon} + |B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}|. \end{aligned}$$

This concludes the remaining statements 2 and 3 of Proposition 4.23. □

#### 4.4 ODE system for $(\Psi, \Lambda, \underline{\Lambda})$

The goal of this section is to prove the following proposition which provides an ODE system for  $(\Psi, \Lambda, \underline{\Lambda})$ .

**Proposition 4.30.** *Let  $\Sigma$  a smooth spacelike hypersurface as in Definition 4.11. We define  $r(s), B(s), \underline{B}(s)$  and  $D(s)$  on  $\mathbf{S} := \mathbf{S}(s) = \mathbf{S}[P(s)]$  as*

$$r(s) := r^{\mathbf{S}}, \quad B(s) := B^{\mathbf{S}}, \quad \underline{B}(s) := \underline{B}^{\mathbf{S}}, \quad D(s) := \bar{b}^{\mathbf{S}} + 1 + \frac{2m(0)}{r^{\mathbf{S}}}, \quad (4.123)$$

where  $B^{\mathbf{S}}, \underline{B}^{\mathbf{S}}$  are given by (4.67).

Then we have the following equations for the functions  $D(s), r(s)$  and the triplets  $\Lambda(s), \underline{\Lambda}(s), B(s), \underline{B}(s)$ :

$$\begin{aligned} \frac{1}{-1 + \psi'(s)} \Lambda'(s) &= B(s) + G(\Lambda, \underline{\Lambda}, \psi)(s) + N(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \\ \frac{1}{-1 + \psi'(s)} \underline{\Lambda}'(s) &= \underline{B}(s) + \underline{G}(\Lambda, \underline{\Lambda}, \psi)(s) + \bar{N}(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \\ \psi'(s) &= -\frac{1}{2} D(s) + H(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)(s) + M(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \end{aligned}$$

where

$$\psi(s) := \Psi(s) + s - c_0, \quad (4.124)$$

and  $G, \underline{G}, H, N, \bar{N}, M$  satisfy the following properties:

- $G, \underline{G}$  are  $O(1)$ -Lipschitz functions of  $(\Lambda, \underline{\Lambda}, \psi)$ ;
- $H$  is a  $O(1)$ -Lipschitz function of  $(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)$ ;
- $M, N, \bar{N}$  are  $O(\varepsilon^{\frac{1}{2}})$ -Lipschitz functions of  $(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)$ .

Before proving this proposition, we first study some properties of the derivation  $\frac{d}{ds}$ . Recall that, according to Section 4.1, we have

$$\Sigma = \bigcup_{p=\{S, N\}} \left\{ \Xi_p(s, y_p^1, y_p^2) \mid s \geq \overset{\circ}{s}, (y_p^1)^2 + (y_p^2)^2 < 2 \right\}, \quad (4.125)$$

where the maps  $\Xi_p(s, y_p^1, y_p^2)$ ,  $p = S, N$  are defined in (4.62).

**Definition 4.31.** *We define the vectorfield  $X_p$  along  $\Sigma$  as the push forward of  $\frac{d}{ds}$  by  $\Xi_p$ , i.e, for every scalar function  $h$ , we define*

$$X_p(h) := \frac{d}{ds} h(\Xi_p(s, y_p^1, y_p^2)). \quad (4.126)$$

**Proposition 4.32.** *We have the following expression for  $X_p$*

$$X_p|_{(s, y_p^1, y_p^2)} = \left( \Psi'(s) + \check{A}_p(s, y_p^1, y_p^2) \right) \partial_u + \left( 1 + \check{B}_p(s, y_p^1, y_p^2) \right) \partial_s,$$

where

$$\begin{aligned} \check{A}_p(s, y_p^1, y_p^2) &:= \partial_P U_p(y_p^1, y_p^2, P(s)) P'(s), \\ \check{B}_p(s, y_p^1, y_p^2) &:= \partial_P S_p(y_p^1, y_p^2, P(s)) P'(s), \end{aligned}$$

with

$$\begin{aligned} \partial_P U_p(\cdot) P'(s) &= \Psi'(s) \partial_u U_p(\cdot) + \partial_s U_p(\cdot) + \Lambda'(s) \cdot \partial_{\Lambda} U_p(\cdot) + \underline{\Lambda}'(s) \cdot \partial_{\underline{\Lambda}} U_p(\cdot) + \partial_J U_p \cdot J'(s), \\ \partial_P S_p(\cdot) P'(s) &= \Psi'(s) \partial_u S_p(\cdot) + \partial_s S_p(\cdot) + \Lambda'(s) \cdot \partial_{\Lambda} S_p(\cdot) + \underline{\Lambda}'(s) \cdot \partial_{\underline{\Lambda}} S_p(\cdot) + \partial_J S_p \cdot J'(s). \end{aligned}$$

*Proof.* Clearly,

$$\partial_s \Xi_p(s, y_p^1, y_p^2) = (\Psi'(s) + \partial_P U_p(y_p^1, y_p^2, P(s))P'(s), 1 + \partial_P S_p(y_p^1, y_p^2, P(s))P'(s), 0, 0).$$

Given a function on  $\Sigma$ , we infer

$$\begin{aligned} \frac{d}{ds} h(\Xi_p(s, y_p^1, y_p^2)) &= (\Psi'(s) + \partial_P U_p(y_p^1, y_p^2, P(s))P'(s)) \partial_u h \\ &\quad + (1 + \partial_P S_p(y_p^1, y_p^2, P(s))P'(s)) \partial_s h \\ &= X_p(h), \end{aligned}$$

as stated. □

**Lemma 4.33.** *Let  $\gamma(s)$  be the curve of the South Poles of the background<sup>29</sup> that intersect the spheres foliating  $\Sigma$ , i.e.*

$$\gamma(s) = (\Psi(s), s, 0, 0) \tag{4.127}$$

in the south coordinates chart.

Then, for a function  $h$  defined on  $\gamma(s)$ :

$$h(s) := h(\Psi(s), s, 0, 0),$$

we have (recall  $\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}}$ )

$$X|_{\gamma(s)}(h) = C(s)\nu^{\mathbf{S}}|_{\gamma(s)}(h), \tag{4.128}$$

where

$$C(s) = \frac{\lambda}{z} \Psi'(s) \Big|_{\gamma(s)} + F \cdot \Gamma_b + O(F^2), \tag{4.129}$$

and

$$F := (f, \underline{f}, \overset{\circ}{\lambda}), \quad \overset{\circ}{\lambda} = \lambda - 1.$$

Moreover, we have the following expression for  $b^{\mathbf{S}}$ :

$$b^{\mathbf{S}}|_{\gamma(s)} = \frac{z - \underline{\Omega}\Psi'(s)}{\lambda^2 \Psi'(s)} \Big|_{\gamma(s)} + F \cdot \Gamma_b + O(F^2). \tag{4.130}$$

Here  $f, \underline{f}, \lambda$  are the transition coefficients and  $z, \underline{\Omega}$  correspond to the background foliation.

*Proof.* Note that  $U(0, 0, P(s)) = S(0, 0, P(s)) = 0$ , so we have

$$\check{A}(s, 0, 0) = \check{B}(s, 0, 0) = 0$$

in Proposition 4.32. Thus, we have

$$X|_{\gamma(s)} = \Psi'(s)\partial_u + \partial_s.$$

Note that  $X$  is tangent to  $\Sigma$ . Thus, we have

$$\mathfrak{g}(X, N^{\mathbf{S}})|_{\gamma(s)} = 0, \quad N^{\mathbf{S}} = e_3^{\mathbf{S}} - b^{\mathbf{S}}e_4^{\mathbf{S}},$$

---

<sup>29</sup>By the construction of GCM spheres in Theorem 3.1, the South poles of GCM spheres coincide with that of background spheres. Hence, we have  $\gamma(s) \subset \Sigma$ .

since the vectorfield  $N^{\mathbf{S}}$  is normal to  $\Sigma$ . Now, applying (2.35), (2.24) and  $Z^c \in \Gamma_b^{30}$ , and recalling that  $z = \frac{2}{\zeta}$ , we have

$$\begin{aligned} \mathbf{g}(X, N^{\mathbf{S}}) &= \mathbf{g}\left(\Psi'(s)\partial_u + \partial_s, e_3^{\mathbf{S}} - b^{\mathbf{S}}e_4^{\mathbf{S}}\right) \\ &= \mathbf{g}\left(\Psi'(s)\left(\frac{1}{z}e_3 - \frac{\underline{\Omega}}{z}e_4 - \frac{2}{z}Z^ce_c\right) + e_4, \lambda^{-1}e_3 + \lambda^{-1}\underline{f}^ce_c - b^{\mathbf{S}}(\lambda e_4 + \lambda f^ce_c)\right) + O(F^2) \\ &= \mathbf{g}\left(\frac{\Psi'}{z}e_3 + \left(1 - \frac{\underline{\Omega}\Psi'}{z}\right)e_4 - \frac{2\Psi'Z^c}{z}e_c, \lambda^{-1}e_3 - \lambda b^{\mathbf{S}}e_4 + (\lambda^{-1}\underline{f}^c - \lambda b^{\mathbf{S}}f^c)e_c\right) + O(F^2) \\ &= -\frac{2}{\lambda}\left(1 - \frac{\underline{\Omega}\Psi'}{z}\right) + 2\lambda b^{\mathbf{S}}\frac{\Psi'}{z} + F \cdot \Gamma_b + O(F^2). \end{aligned}$$

Since  $\mathbf{g}(X, N^{\mathbf{S}})|_{\gamma(s)} = 0$ , we deduce

$$\left[-\frac{2}{\lambda}\left(1 - \frac{\underline{\Omega}\Psi'}{z}\right) + 2\lambda b^{\mathbf{S}}\frac{\Psi'}{z}\right]\Big|_{\gamma(s)} = F \cdot \Gamma_b + O(F^2),$$

which implies

$$b^{\mathbf{S}}|_{\gamma(s)} = \frac{z - \underline{\Omega}\Psi'}{\lambda^2\Psi'}\Big|_{\gamma(s)} + F \cdot \Gamma_b + O(F^2),$$

as stated in (4.130).

In view of  $\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}}$ , (2.24) and (2.36), we have

$$\begin{aligned} \nu^{\mathbf{S}} &= e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}} \\ &= \lambda^{-1}(e_3 + \underline{f}^be_b) + b^{\mathbf{S}}\lambda(e_4 + f^be_b) + O(F^2) \\ &= \lambda^{-1}e_3 + \lambda b^{\mathbf{S}}e_4 + (\lambda^{-1}\underline{f}^c + \lambda b^{\mathbf{S}}f^c)e_c + O(F^2) \\ &= \lambda^{-1}(z\partial_u + \underline{\Omega}\partial_s + 2\underline{B}^a\partial_{y^a}) + \lambda b^{\mathbf{S}}\partial_s + (\lambda^{-1}\underline{f}^c + \lambda b^{\mathbf{S}}f^c)X_{(c)}^a\partial_{y^a} + O(F^2) \\ &= \frac{z}{\lambda}\partial_u + \left(\frac{\underline{\Omega}}{\lambda} + \lambda b^{\mathbf{S}}\right)\partial_s + \left(2\lambda^{-1}\underline{B}^a + (\lambda^{-1}\underline{f}^c + \lambda b^{\mathbf{S}}f^c)X_{(c)}^a\right)\partial_{y^a} + O(F^2). \end{aligned} \tag{4.131}$$

Applying (4.131) to a function  $h(s) = h(\Psi(s), s, 0, 0)$ , we obtain

$$\nu^{\mathbf{S}}|_{\gamma(s)}(h) = \frac{z}{\lambda}\partial_u(h) + \left(\frac{\underline{\Omega}}{\lambda} + \lambda b^{\mathbf{S}}\right)\partial_s(h). \tag{4.132}$$

Recalling (4.130) and comparing (4.132) to  $X|_{\gamma(s)} = \Psi'(s)\partial_u + \partial_s$ , we obtain

$$X|_{\gamma(s)}(h) = C(s)\nu^{\mathbf{S}}|_{\gamma(s)}(h),$$

where

$$C(s) = \frac{\lambda}{z}\Psi'(s)\Big|_{\gamma(s)} + F \cdot \Gamma_b + O(F^2),$$

as stated in (4.128). This concludes the proof of Lemma 4.33.  $\square$

Now, we derive the following equations for  $\Lambda(s)$  and  $\underline{\Lambda}(s)$ .

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<sup>30</sup>See mentioned in Assumption **A3**.

**Lemma 4.34.** *We have the following identities*

$$\begin{aligned} \nu^{\mathbf{S}}|_{\gamma(s)}\Lambda(s) &= \int_{\mathbf{S}} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} - \frac{4}{r^{\mathbf{S}}}\Lambda(s) + E(s), \\ \nu^{\mathbf{S}}|_{\gamma(s)}\underline{\Lambda}(s) &= \int_{\mathbf{S}} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} \underline{f})J^{(\mathbf{S},p)} - \frac{4}{r^{\mathbf{S}}}\underline{\Lambda}(s) + \underline{E}(s), \end{aligned} \tag{4.133}$$

with error terms<sup>31</sup>

$$\begin{aligned} E(s) &= \int_{\mathbf{S}} \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \\ &\quad + z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) - \frac{4}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}} f + \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) \operatorname{div}^{\mathbf{S}} f \right) J^{(\mathbf{S},p)}, \\ \underline{E}(s) &= \int_{\mathbf{S}} \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} \underline{f})J^{(\mathbf{S},p)} \\ &\quad + z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} \underline{f}) - \frac{4}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}} \underline{f} + \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) \operatorname{div}^{\mathbf{S}} \underline{f} \right) J^{(\mathbf{S},p)}. \end{aligned}$$

*Proof.* According to Lemma 4.17 we have

$$\nu^{\mathbf{S}} \left( \int_{\mathbf{S}} h \right) = z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left( \nu^{\mathbf{S}}(h) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})h \right).$$

Thus, choosing  $h = (\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)}$ , we infer

$$\begin{aligned} \nu^{\mathbf{S}}|_{\gamma(s)}(\Lambda) &= \nu^{\mathbf{S}} \left( \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \right) \Big|_{\gamma(s)} \\ &= z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left( \nu^{\mathbf{S}}((\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)}) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \right). \end{aligned}$$

Recalling that  $\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}$ , we have

$$\begin{aligned} \underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}b^{\mathbf{S}} - \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} + \check{\kappa}^{\mathbf{S}} \\ &= \frac{2}{r^{\mathbf{S}}} \left( b^{\mathbf{S}} + 1 + \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} \right) - \frac{2}{r^{\mathbf{S}}}(-\Upsilon^{\mathbf{S}} + 2) - \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} + \check{\kappa}^{\mathbf{S}} \\ &= \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} - \frac{4}{r^{\mathbf{S}}} + \check{\kappa}^{\mathbf{S}}. \end{aligned}$$

Since  $\nu^{\mathbf{S}}(J^{(\mathbf{S},p)}) = 0$ , we infer

$$\begin{aligned} \nu^{\mathbf{S}}|_{\gamma(s)}(\Lambda) &= \int_{\mathbf{S}} \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \right) \\ &\quad + z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \right) \\ &= \int_{\mathbf{S}} \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} + \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} - \frac{4}{r^{\mathbf{S}}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} \right) \\ &\quad + z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) - \frac{4}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}} f + \left( \frac{2}{r^{\mathbf{S}}}\check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) \operatorname{div}^{\mathbf{S}} f \right) J^{(\mathbf{S},p)} \\ &= \int_{\mathbf{S}} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f)J^{(\mathbf{S},p)} - \frac{4}{r^{\mathbf{S}}}\Lambda(s) + E(s), \end{aligned}$$

as stated. The proof for  $\underline{\Lambda}(s)$  is similar and left to the reader. This concludes the proof of Lemma 4.34.  $\square$

<sup>31</sup>For a scalar function  $h$  defined on  $\mathbf{S}$ ,  $h - h|_{\gamma(s)}$  is the scalar function define on  $\mathbf{S}$  by subtracting its value at South pole  $\gamma(s)$ .



Recall that in Proposition 4.23 we have made the auxiliary assumption (4.101), i.e.

$$|B^{\mathbf{S}}| + |\underline{B}^{\mathbf{S}}| + |D^{\mathbf{S}}| \leq \varepsilon^{\frac{2}{3}}.$$

Moreover, recalling the definition (4.67) of  $D^{\mathbf{S}}$  and (4.123) of  $D(s)$  and applying Assumption **A1** and (4.5), we have

$$\left| \frac{2m}{r} - \frac{2m_{(0)}}{r} \right| \lesssim \left| \overline{\check{z} + \check{\Omega}} \right| + \delta \lesssim \varepsilon + \delta \lesssim \varepsilon.$$

Recall from (2.55) that

$$|m - m^{\mathbf{S}}| \lesssim \delta,$$

thus, we obtain

$$|D^{\mathbf{S}} - D(s)| = \frac{2}{r^{\mathbf{S}}} |m^{\mathbf{S}} - m_{(0)}| \lesssim \frac{1}{r} |m - m_{(0)}| \lesssim \varepsilon,$$

so we have the estimate

$$|B(s)| + |\underline{B}(s)| + |D(s)| \lesssim \varepsilon^{\frac{2}{3}}.$$

The following notation is very useful in the proof of Proposition 4.30.

**Definition 4.35.** For a function  $h$  of  $(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)$ , we denote

$$h = \text{Good},$$

if

$$h = O(\varepsilon), \quad \frac{\partial h}{\partial v} = O(\varepsilon^{\frac{1}{2}}), \quad \forall v \in \{B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi\}.$$

**Lemma 4.36.** We have

$$\mathfrak{d}^{\leq 1}(F \cdot \Gamma_b) = r^{-2} \text{Good}, \quad F \cdot F = r^{-2} \text{Good}, \quad \text{Err}_1 = r^{-2} \text{Good}, \quad (4.134)$$

where  $\text{Err}_1$  is defined in Definition 2.24 and  $\text{Good}$  is defined in Definition 4.35. We also have

$$\nu^{\mathbf{S}}(\text{div}^{\mathbf{S}} f) \cdot \Gamma_b = r^{-3} \text{Good}. \quad (4.135)$$

Finally, for the following background quantities, we have

$$\left( \int_{S(\Psi(s), s)} (\text{div } \eta) J^{(p)}, \int_{S(\Psi(s), s)} (\text{div } \underline{\xi}) J^{(p)}, \int_{S(\Psi(s), s)} \Delta(\check{\kappa}) J^{(p)} \right) = \text{Good}, \quad (4.136)$$

and

$$\overline{z + \underline{\Omega}} - 1 - \frac{2m_{(0)}}{r} = \text{Good}. \quad (4.137)$$

*Proof.* Notice that (4.134) is a direct consequence of Definitions 2.24, 4.35, Assumption **A1** and (3.6).

Since  $\operatorname{div} \eta$ ,  $\operatorname{div} \underline{\xi}$ ,  $\Delta(\check{\kappa})$  and  $J^{(p)}$  only depend on the background foliation, the quantities in (4.136) only depend on  $\psi$ . Recalling (4.104), we have

$$\begin{aligned} \partial_\psi \left( \int_{S(\Psi(s),s)} (\operatorname{div} \eta) J^{(p)} \right) &= \partial_u \left( \int_{S(u,s)} (\operatorname{div} \eta) J^{(p)} \right) \\ &= \partial_u \left( \int_{\mathbb{S}} \sqrt{\det \mathbf{g}(s)} (\operatorname{div} \eta) J^{(p)} dy^1 dy^2 \right) \\ &= \int_{\mathbb{S}} \partial_u (\sqrt{\det \mathbf{g}(s)} J^{(p)}) (\operatorname{div} \eta) dy^1 dy^2 + \int_{S(u,s)} \partial_u (\operatorname{div} \eta) J^{(p)} \\ &= r \mathfrak{d}(\Gamma_b) + r \mathfrak{d}^{\leq 2}(\Gamma_b) = O(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Recalling from (4.3) that  $(\operatorname{div} \eta)_{\ell=1} = O(\delta^\circ)$ , we conclude

$$\int_{S(u,s)} (\operatorname{div} \eta) J^{(p)} = \text{Good}.$$

Similarly, using (4.3) and (4.2), we have

$$\int_{S(u,s)} (\operatorname{div} \underline{\xi}) J^{(p)} = \text{Good}, \quad \int_{S(u,s)} (\Delta \check{\kappa}) J^{(p)} = \text{Good}.$$

This concludes (4.136).

Next, Recall that (4.5) implies

$$\overline{z + \underline{\Omega}} - 1 - \frac{2m(0)}{r} = O(\delta^\circ).$$

Since  $z, \underline{\Omega}, r$  only depends on the background foliation,  $\overline{z + \underline{\Omega}} - 1 - \frac{2m(0)}{r}$  only depends on  $\psi$ . We have

$$\partial_u \left( \overline{z + \underline{\Omega}} - 1 - \frac{2m(0)}{r} \right) = r \mathfrak{d}^{\leq 1} \Gamma_b = O(\varepsilon^{\frac{1}{2}}),$$

which implies (4.137).

We have from Proposition 4.22 that

$$\|\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(F)\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \delta^\circ + |\mathbf{B}^{\mathbf{S}}| + |\underline{\mathbf{B}}^{\mathbf{S}}| + |\mathbf{D}^{\mathbf{S}}|, \quad (4.138)$$

Applying Lemma 2.4, (4.134) and (4.138), we have

$$\begin{aligned} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) &= [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}] f + \operatorname{div}^{\mathbf{S}}(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} f) \\ &= \frac{2}{r^{\mathbf{S}}} (\operatorname{div}^{\mathbf{S}} f) + \Gamma_b^{\mathbf{S}} \cdot \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} f + r^{-1} \Gamma_b^{\mathbf{S}} \cdot \mathfrak{d}^{\leq 1} f + r^{-1} \mathfrak{d}^{\leq 1}(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} f) \\ &= \frac{2}{r^{\mathbf{S}}} (\operatorname{div}^{\mathbf{S}} f) + r^{-1} \mathfrak{d}^{\leq 1}(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} f) + r^{-2} \text{Good}. \end{aligned}$$

Then, we infer

$$\begin{aligned} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) \cdot \Gamma_b &= r^{-1} \mathfrak{d}^{\leq 1}(f) \cdot \Gamma_b + r^{-1} \mathfrak{d}^{\leq 1}(\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}} f) \cdot \Gamma_b + r^{-3} \text{Good} \\ &= r^{-3} \text{Good}, \end{aligned}$$

where we used (4.134) and (4.138). This concludes the proof of (4.135), and hence concludes the proof of Lemma 4.36.  $\square$

We are now ready to prove Proposition 4.30.

*Proof of Proposition 4.30. Step 1* Firstly, we prove the following identities:

$$\begin{aligned} \int_{\mathbf{S}(s)} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S}, p)} &= 2B(s) + 3r^{-1}\Lambda(s) - r^{-1}\underline{\Lambda}(s) + \Lambda(s)O(r^{-2}) + \text{Good}, \\ \int_{\mathbf{S}(s)} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} \underline{f}) J^{(\mathbf{S}, p)} &= 2\underline{B}(s) + 5r^{-1}\underline{\Lambda}(s) + r^{-1}\Lambda(s) + O(r^{-2})(\Lambda(s) + \underline{\Lambda}(s)) + \text{Good}. \end{aligned} \quad (4.139)$$

In order to prove the first identity in (4.139), we write

$$\nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) = [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}]f + \operatorname{div}^{\mathbf{S}}(\nabla_{\nu^{\mathbf{S}}} f),$$

According to (4.72), Proposition 4.23 and Lemma 4.36, we have

$$\begin{aligned} \operatorname{div}^{\mathbf{S}}(\nabla_{\nu^{\mathbf{S}}} f) &= \operatorname{div}^{\mathbf{S}} \left( 2(\eta^{\mathbf{S}} - \eta) - \frac{1}{2}\kappa(\underline{f} + b^{\mathbf{S}}f) + 2f\underline{\omega} \right) + r^{-1}\mathfrak{d}^{\leq 1}(F \cdot \Gamma_b) + \text{l.o.t.} \\ &= 2\operatorname{div}^{\mathbf{S}}\eta^{\mathbf{S}} - 2\operatorname{div}^{\mathbf{S}}\eta - \frac{1}{r}\operatorname{div}^{\mathbf{S}}\underline{f} - \frac{b^{\mathbf{S}}}{r}\operatorname{div}^{\mathbf{S}}f + \frac{2m}{r^2}\operatorname{div}^{\mathbf{S}}f + r^{-1}\mathfrak{d}^{\leq 1}(F \cdot \Gamma_b) + \text{l.o.t.} \\ &= 2\operatorname{div}^{\mathbf{S}}\eta^{\mathbf{S}} - 2\operatorname{div}^{\mathbf{S}}\eta - \frac{1}{r}\operatorname{div}^{\mathbf{S}}\underline{f} + \frac{1}{r}\operatorname{div}^{\mathbf{S}}f + \frac{4m}{r^2}\operatorname{div}^{\mathbf{S}}f + r^{-3}\text{Good}. \end{aligned}$$

Also, by Lemma 2.4 and (4.138), we have

$$\begin{aligned} [\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}]f &= \frac{2}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}}f + \Gamma_b \cdot \nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}f + r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1}f + \text{l.o.t.} \\ &= \frac{2}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}}f + r^{-3}\text{Good}. \end{aligned}$$

Recalling from Lemma 2.22 that  $|r^{\mathbf{S}} - r| \lesssim \overset{\circ}{\delta}$ , we obtain

$$\nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) = 2\operatorname{div}^{\mathbf{S}}\eta^{\mathbf{S}} - 2\operatorname{div}^{\mathbf{S}}\eta - \frac{1}{r}\operatorname{div}^{\mathbf{S}}\underline{f} + \frac{3}{r}\operatorname{div}^{\mathbf{S}}f + \frac{4m}{r^2}\operatorname{div}^{\mathbf{S}}f + r^{-3}\text{Good}. \quad (4.140)$$

Thus, recalling from Lemma 2.22 that  $|m^{\mathbf{S}} - m| \lesssim \overset{\circ}{\delta}$ , we obtain

$$\int_{\mathbf{S}(s)} \nu^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S}, p)} = 2B(s) - 2 \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}}\eta) J^{(\mathbf{S}, p)} + \left( \frac{1}{3r} + \frac{4m}{r^2} \right) \Lambda(s) - \frac{1}{r}\underline{\Lambda}(s) + r^{-1}\text{Good}.$$

We claim that

$$\left| \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}}\eta) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \eta) J^{(p)} \right| \lesssim \varepsilon^{\frac{1}{2}\overset{\circ}{\delta}}. \quad (4.141)$$

Indeed,

$$\begin{aligned} &\left| \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}}\eta) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \eta) J^{(p)} \right| \\ &\lesssim \left| \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}}\eta - \operatorname{div} \eta) J^{(\mathbf{S}, p)} \right| + \left| \int_{\mathbf{S}(s)} \operatorname{div} \eta (J^{(\mathbf{S}, p)} - J^{(p)}) \right| \\ &\quad + \left| \int_{\mathbf{S}(s)} (\operatorname{div} \eta) J^{(p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \eta) J^{(p)} \right|. \end{aligned}$$

The first term can be estimated by Lemma 2.22. The second term can be estimated by (4.23). The last term can be estimated by Lemma 2.23. Hence,

$$\left| \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \eta) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \eta) J^{(p)} \right| \lesssim \varepsilon^{\frac{1}{2} \circ} \delta.$$

Recalling its structure and Definition 4.35, we obtain

$$\int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \eta) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \eta) J^{(p)} = \text{Good}.$$

Applying (4.136), we deduce

$$\int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \eta) J^{(\mathbf{S}, p)} = \text{Good}. \tag{4.142}$$

Therefore,

$$\int_{\mathbf{S}(s)} \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S}, p)} = 2B(s) + 3r^{-1} \Lambda(s) - r^{-1} \underline{\Lambda}(s) + \Lambda(s) O(r^{-2}) + \text{Good}.$$

Similarly, starting with

$$\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(\underline{f}) = 2(\underline{\xi}^{\mathbf{S}} - \underline{\xi}) - \frac{1}{2} \underline{f} \underline{\kappa} - 2\underline{\omega}(f - b^{\mathbf{S}} f) + b^{\mathbf{S}} \left( 2\nabla^{\mathbf{S}}(\lambda) - \frac{1}{2} \underline{f} \underline{\kappa} \right) + F \cdot \Gamma_b + \text{l.o.t.},$$

we deduce

$$\begin{aligned} \int_{\mathbf{S}(s)} \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \underline{f}) J^{(\mathbf{S}, p)} &= 2\underline{B}(s) - 2 \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \underline{\xi}) J^{(\mathbf{S}, p)} + 4r^{-1} \underline{\Lambda}(s) + 2 \int_{\mathbf{S}(s)} (\Delta^{\mathbf{S}} \lambda) J^{(\mathbf{S}, p)} \\ &\quad + O(r^{-2})(\Lambda(s) + \underline{\Lambda}(s)) + r^{-1} \text{Good}. \end{aligned}$$

Similarly to (4.141), we can prove

$$\left| \int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \underline{\xi}) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \underline{\xi}) J^{(p)} \right| \lesssim \varepsilon^{\frac{1}{2} \circ} \delta, \tag{4.143}$$

and thus,

$$\int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \underline{\xi}) J^{(\mathbf{S}, p)} - \int_{S(\Psi(s), s)} (\operatorname{div} \underline{\xi}) J^{(p)} = \text{Good}.$$

Applying (4.136), we obtain

$$\int_{\mathbf{S}(s)} (\operatorname{div}^{\mathbf{S}} \underline{\xi}) J^{(\mathbf{S}, p)} = \text{Good}.$$

Next, we recall the transformation formula for  $\kappa$  in Proposition 2.14:

$$\lambda^{-1} \kappa^{\mathbf{S}} = \kappa + \operatorname{div}^{\mathbf{S}} f + F \cdot \Gamma_b + \text{l.o.t.}$$

Applying Definition 4.35 and Lemma 4.36, recalling  $\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}$  and from Lemma 2.22 that  $|r - r^{\mathbf{S}}| \lesssim \varepsilon^{\frac{1}{2} \circ} \delta$ , we infer

$$\begin{aligned} \overset{\circ}{\lambda} &= \lambda - 1 = \frac{\kappa^{\mathbf{S}}}{\kappa + \operatorname{div}^{\mathbf{S}} f + F \cdot \Gamma_b + \text{l.o.t.}} - 1 \\ &= \frac{\frac{2}{r^{\mathbf{S}}}}{\frac{2}{r} + \check{\kappa} + \operatorname{div}^{\mathbf{S}} f + \text{Good}} - 1 \\ &= \frac{r}{r^{\mathbf{S}}} \left( 1 - \frac{r}{2} \check{\kappa} - \frac{r}{2} \operatorname{div}^{\mathbf{S}} f + \text{Good} \right) - 1 \\ &= \frac{r - r^{\mathbf{S}}}{r^{\mathbf{S}}} - \frac{r^2}{2r^{\mathbf{S}}} \check{\kappa} - \frac{r^2}{2r^{\mathbf{S}}} \operatorname{div}^{\mathbf{S}} f + \text{Good} \\ &= \frac{r - r^{\mathbf{S}}}{r^{\mathbf{S}}} - \frac{r^{\mathbf{S}}}{2} \check{\kappa} - \frac{r^{\mathbf{S}}}{2} \operatorname{div}^{\mathbf{S}} f + \text{Good}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbf{S}} (\Delta^{\mathbf{S}} \overset{\circ}{\lambda}) J^{(\mathbf{S},p)} &= \frac{1}{r^{\mathbf{S}}} \int_{\mathbf{S}} \Delta^{\mathbf{S}} (r - r^{\mathbf{S}}) J^{(\mathbf{S},p)} - \frac{r^{\mathbf{S}}}{2} \int_{\mathbf{S}} \Delta^{\mathbf{S}} (\check{\kappa}) J^{(\mathbf{S},p)} \\ &\quad - \frac{r^{\mathbf{S}}}{2} \int_{\mathbf{S}} \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(f) J^{(\mathbf{S},p)} + \text{Good}. \end{aligned} \quad (4.144)$$

Similarly as (4.141) and applying (4.136), we obtain

$$\frac{r^{\mathbf{S}}}{2} \int_{\mathbf{S}} \Delta^{\mathbf{S}} (\check{\kappa}) J^{(\mathbf{S},p)} = \text{Good}.$$

Applying Proposition 2.8 and Lemma 2.22, we infer

$$\begin{aligned} \int_{\mathbf{S}} \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(f) J^{(\mathbf{S},p)} &= \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(f) \Delta^{\mathbf{S}} J^{(\mathbf{S},p)} \\ &= -\frac{2}{(r^{\mathbf{S}})^2} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}}(f) J^{(\mathbf{S},p)} + \text{Good} \\ &= -\frac{2}{r^2} \Lambda(s) + \text{Good}. \end{aligned}$$

Recalling (2.57), we infer

$$\begin{aligned} \int_{\mathbf{S}} \Delta^{\mathbf{S}} (r - r^{\mathbf{S}}) J^{(\mathbf{S},p)} &= \frac{1}{2} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \left( \underline{f} - \Upsilon f + \operatorname{Err}_1[\Delta^{\mathbf{S}} \overset{\circ}{b}] \right) J^{(\mathbf{S},p)} \\ &= \frac{1}{2} \underline{\Lambda}(s) - \frac{1}{2} \Lambda(s) + O(r^{-2}) \Lambda(s) + r \text{Good}. \end{aligned}$$

Now, from (4.144) we deduce

$$\int_{\mathbf{S}} (\Delta^{\mathbf{S}} \overset{\circ}{\lambda}) J^{(\mathbf{S},p)} = \frac{1}{2r} \underline{\Lambda}(s) + \frac{1}{2r} \Lambda(s) + O(r^{-2}) \Lambda(s) + \text{Good}.$$

Finally, we obtain

$$\int_{\mathbf{S}(s)} \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \underline{f}) J^{(\mathbf{S},p)} = 2\underline{B}(s) + 5r^{-1} \underline{\Lambda}(s) + r^{-1} \Lambda(s) + O(r^{-2}) (\Lambda(s) + \underline{\Lambda}(s)) + \text{Good}.$$

This concludes the proof of (4.139).

**Step 2** Now, we prove that the error terms  $E(s), \underline{E}(s)$  defined in Lemma 4.34 satisfy

$$E(s) = \text{Good}, \quad \underline{E}(s) = \text{Good}. \quad (4.145)$$

Recall that

$$\begin{aligned} E(s) &= \int_{\mathbf{S}} \left( \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S},p)} \\ &\quad + z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) - \frac{4}{r^{\mathbf{S}}} \operatorname{div}^{\mathbf{S}} f + \left( \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) \operatorname{div}^{\mathbf{S}} f \right) J^{(\mathbf{S},p)}. \end{aligned}$$

Since

$$\left( \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) = r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leq 1}(f), \quad (4.146)$$

we obtain from Lemma 4.36

$$\int_{\mathbf{S}} \left( \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S},p)} = \text{Good}. \quad (4.147)$$

Also, since  $z^{\mathbf{S}} = 2 + r\Gamma_b$  and

$$\int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( -\frac{4}{r^{\mathbf{S}}} \operatorname{div}^{\mathbf{S}} f + \left( \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) \operatorname{div}^{\mathbf{S}} f \right) J^{(\mathbf{S},p)} = r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leq 1}(f),$$

we obtain from Lemma 4.36 that

$$z^{\mathbf{S}}|_{\gamma(s)} \int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \left( -\frac{4}{r^{\mathbf{S}}} + \frac{2}{r^{\mathbf{S}}} \check{b}^{\mathbf{S}} + \check{\kappa}^{\mathbf{S}} \right) (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S},p)} = \text{Good}. \quad (4.148)$$

Next, recalling  $z^{\mathbf{S}} = 2 + r\Gamma_b$  and applying Lemma 4.36, we have

$$\int_{\mathbf{S}} \left( (z^{\mathbf{S}})^{-1} - (z^{\mathbf{S}})^{-1}|_{\gamma(s)} \right) \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S},p)} = \text{Good}. \quad (4.149)$$

Combining (4.147), (4.148) and (4.149), we deduce

$$E(s) = \text{Good}$$

as stated. The proof for  $\underline{E}(s)$  is similar and left to the reader. This concludes the proof of (4.145).

**Step 3** Recall from (4.133) that we have

$$\begin{aligned} \nu^{\mathbf{S}}|_{\gamma(s)} \Lambda(s) &= \int_{\mathbf{S}} \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) J^{(\mathbf{S},p)} - \frac{4}{r^{\mathbf{S}}} \Lambda(s) + E(s), \\ \nu^{\mathbf{S}}|_{\gamma(s)} \underline{\Lambda}(s) &= \int_{\mathbf{S}} \nu^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \underline{f}) J^{(\mathbf{S},p)} - \frac{4}{r^{\mathbf{S}}} \underline{\Lambda}(s) + \underline{E}(s). \end{aligned}$$

Applying (4.128), (4.139) and (4.145), we obtain

$$\begin{aligned} \frac{1}{\Psi'(s)} \Lambda'(s) &= \frac{C(s)}{\Psi'(s)} (2B(s) - r^{-1} \Lambda(s) - r^{-1} \underline{\Lambda}(s) + O(r^{-2}) \Lambda(s) + \text{Good}), \\ \frac{1}{\Psi'(s)} \underline{\Lambda}'(s) &= \frac{C(s)}{\Psi'(s)} (2\underline{B}(s) + r^{-1} \Lambda(s) + r^{-1} \underline{\Lambda}(s) + O(r^{-2}) (\Lambda(s) + \underline{\Lambda}(s)) + \text{Good}). \end{aligned}$$

Recall that  $z = 2 + O(\overset{\circ}{\varepsilon})$  and also  $\lambda = 1 + O(r^{-1} \overset{\circ}{\delta})$ , thus, in view of (4.129)

$$\frac{C(s)}{\Psi'(s)} = \left( \frac{\lambda}{z} \right) \Big|_{\gamma(s)} + O(\overset{\circ}{\delta}) = \frac{1 + O(r^{-1} \overset{\circ}{\delta})}{2 + O(\overset{\circ}{\varepsilon})} + O(\overset{\circ}{\delta}) = \frac{1}{2} + O(\overset{\circ}{\varepsilon}).$$

Hence,

$$\begin{aligned} \frac{1}{\Psi'(s)} \Lambda'(s) &= \left( \frac{1}{2} + O(\overset{\circ}{\varepsilon}) \right) (2B(s) - r^{-1} \Lambda(s) - r^{-1} \underline{\Lambda}(s) + \Lambda(s) O(r^{-2}) + \text{Good}) \\ &= B(s) - \frac{1}{2} r^{-1} \Lambda(s) - \frac{1}{2} r^{-1} \underline{\Lambda}(s) + \Lambda(s) O(r^{-2}) + \text{Good}, \end{aligned}$$

where we used

$$O(\overset{\circ}{\varepsilon}) B(s) = \text{Good}.$$

Recalling that  $\Psi(s) = -s + \psi(s) + c_0$ , we obtain

$$\frac{1}{-1 + \psi'(s)} \Lambda'(s) = B(s) + G(\Lambda, \underline{\Lambda}, \psi)(s) + N(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s)$$

where

$$G(\Lambda, \underline{\Lambda}, \psi)(s) = -\frac{1}{2}r^{-1}\Lambda(s) - \frac{1}{2}r^{-1}\underline{\Lambda}(s) + O(r^{-2})\Lambda(s), \quad N = \text{Good},$$

verify the properties mentioned in Proposition 4.30.

In the same manner, we derive

$$\frac{1}{-1 + \psi'(s)} \underline{\Lambda}'(s) = \underline{B}(s) + \underline{G}(\Lambda, \underline{\Lambda}, \psi)(s) + \overline{N}(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s),$$

where

$$\underline{G}(\Lambda, \underline{\Lambda}, \psi)(s) = \frac{1}{2}r^{-1}\Lambda(s) + \frac{1}{2}r^{-1}\underline{\Lambda}(s) + O(r^{-2})(\Lambda(s) + \underline{\Lambda}(s)), \quad \overline{N} = \text{Good},$$

verify the properties mentioned in Proposition 4.30.

**Step 4** It remains to derive the equation of  $\psi'(s)$ . In view of (4.130) and Lemma 4.36, we have

$$\Psi'(s) = \left[ \frac{z}{\lambda^2 b^{\mathbf{S}} + \underline{\Omega}} \right] \Big|_{\gamma(s)} + \text{Good}. \quad (4.150)$$

By Definition 4.35,  $z^{\mathbf{S}} = 2 + r\Gamma_b$ ,  $b^{\mathbf{S}} = -1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} + r\Gamma_b$  and  $\underline{\Omega} = -\Upsilon + r\Gamma_b$ , we have

$$\begin{aligned} \frac{z}{\lambda^2 b^{\mathbf{S}} + \underline{\Omega}} &= \frac{z}{b^{\mathbf{S}} + (\lambda^2 - 1)b^{\mathbf{S}} + \underline{\Omega}} = \frac{z}{b^{\mathbf{S}} + \underline{\Omega}} \left( \frac{1}{1 + \frac{(\lambda^2 - 1)b^{\mathbf{S}}}{b^{\mathbf{S}} + \underline{\Omega}}} \right) \\ &= \frac{z}{b^{\mathbf{S}} + \underline{\Omega}} - \frac{2z\overset{\circ}{\lambda} b^{\mathbf{S}}}{(b^{\mathbf{S}} + \underline{\Omega})^2} + \text{Good} \\ &= \frac{z}{b^{\mathbf{S}} + \underline{\Omega}} - \frac{4\overset{\circ}{\lambda} \left(-1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}\right)}{\left(-2 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} + \frac{2m}{r}\right)^2} + \text{Good} \\ &= \frac{z}{b^{\mathbf{S}} + \underline{\Omega}} + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) + \text{Good}. \end{aligned}$$

Hence,

$$\psi'(s) = \Psi'(s) + 1 = \left[ \frac{b^{\mathbf{S}} + z + \underline{\Omega}}{b^{\mathbf{S}} + \underline{\Omega}} \right] \Big|_{\gamma(s)} + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) + \text{Good}.$$

Recall that

$$\overline{b^{\mathbf{S}}} = D(s) - 1 - \frac{2m_{(0)}}{r^{\mathbf{S}}}.$$

Applying (4.137), we infer<sup>32</sup>

$$\begin{aligned} (b^{\mathbf{S}} + z + \underline{\Omega}) \Big|_{\gamma(s)} &= (b^{\mathbf{S}} - \overline{b^{\mathbf{S}}}) \Big|_{\gamma(s)} + \overline{b^{\mathbf{S}}} + (z + \underline{\Omega} - \overline{z + \underline{\Omega}}) \Big|_{\gamma(s)} + \overline{z + \underline{\Omega}} \\ &= D(s) - \left( \frac{2m_{(0)}}{r^{\mathbf{S}}} - \frac{2m_{(0)}}{r} \right) \Big|_{\gamma(s)} + (\check{b}^{\mathbf{S}} - \overline{b^{\mathbf{S}}}) \Big|_{\gamma(s)} \\ &\quad + (z + \underline{\Omega} - \overline{z + \underline{\Omega}}) \Big|_{\gamma(s)} + \text{Good} \\ &= D(s) + H_1(s) + H_2(s) + H_3(s) + \text{Good}, \end{aligned}$$

<sup>32</sup>Recall that  $\overline{b^{\mathbf{S}}}$  is the average of  $b^{\mathbf{S}}$  over  $\mathbf{S}$  while  $\overline{z + \underline{\Omega}}$  is the average of  $z + \underline{\Omega}$  on  $S(\Psi(s), s)$ .

where

$$\begin{aligned} H_1(\Lambda, \underline{\Lambda}, \psi)(s) &:= \left( \frac{2m_{(0)}}{r} - \frac{2m_{(0)}}{r^{\mathbf{S}}} \right) \Big|_{\gamma(s)}, \\ H_2(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)(s) &:= (\check{b}^{\mathbf{S}} - \bar{b}^{\mathbf{S}}) \Big|_{\gamma(s)}, \\ H_3(s) &:= (z + \underline{\Omega} - \overline{z + \underline{\Omega}}) \Big|_{\gamma(s)}. \end{aligned}$$

From (3.14) and Assumption **A1**, we deduce that  $H_1$  is  $O(1)$ -Lipschitz on  $(\Lambda, \underline{\Lambda}, \psi)$ . We then consider  $H_2 = (b^{\mathbf{S}} - \bar{b}^{\mathbf{S}}) \Big|_{\gamma(s)}$ . Applying Lemma 4.16, we obtain

$$e_a^{\mathbf{S}}(b^{\mathbf{S}}) = (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})b^{\mathbf{S}} + \underline{\xi}_a^{\mathbf{S}}.$$

Taking another choice of  $(\tilde{\Lambda}, \tilde{\underline{\Lambda}}, \tilde{\psi})$ , we construct by Definition 4.11 another hypersurface  $\tilde{\Sigma}$ . Denoting  $\tilde{X}$  the associated quantity of  $X$  on  $\tilde{\Sigma}$ , we have for  $a = 1, 2$ :<sup>33</sup>

$$e_a^{\mathbf{S}}(b^{\mathbf{S}}) - e_a^{\tilde{\mathbf{S}}}(\tilde{b}^{\tilde{\mathbf{S}}}) = (\zeta_a^{\mathbf{S}} - \eta_a^{\mathbf{S}})(b^{\mathbf{S}} - \tilde{b}^{\tilde{\mathbf{S}}}) + \left( (\zeta_a^{\mathbf{S}} - \tilde{\zeta}_a^{\tilde{\mathbf{S}}}) - (\eta_a^{\mathbf{S}} - \tilde{\eta}_a^{\tilde{\mathbf{S}}}) \right) \tilde{b}^{\tilde{\mathbf{S}}} + \underline{\xi}_a^{\mathbf{S}} - \tilde{\underline{\xi}}_a^{\tilde{\mathbf{S}}}.$$

Applying Lemma 2.13, (3.12) and elliptic estimate, we deduce

$$(b^{\mathbf{S}} - \bar{b}^{\mathbf{S}}) - (\tilde{b}^{\tilde{\mathbf{S}}} - \bar{\tilde{b}}^{\tilde{\mathbf{S}}}) = O(r) \left( \eta_a^{\mathbf{S}} - \tilde{\eta}_a^{\tilde{\mathbf{S}}}, \underline{\xi}_a^{\mathbf{S}} - \tilde{\underline{\xi}}_a^{\tilde{\mathbf{S}}} \right) + O(1)(\Lambda - \tilde{\Lambda}, \underline{\Lambda} - \tilde{\underline{\Lambda}}, \psi - \tilde{\psi}).$$

which implies that  $b^{\mathbf{S}} - \bar{b}^{\mathbf{S}}$  is  $O(r)$ -dependent on  $\eta^{\mathbf{S}}$  and  $\underline{\xi}^{\mathbf{S}}$ . Next, we proceed as in the proofs of Lemma 4.27 and Proposition 4.23 to deduce

$$\begin{aligned} \mathcal{H}_2^{\mathbf{S},*} \eta_{ab}^{\mathbf{S}} - \widetilde{\mathcal{H}}_2^{\tilde{\mathbf{S}},*} \tilde{\eta}_{ab}^{\tilde{\mathbf{S}}} &= O(r^{-1})(\Lambda - \tilde{\Lambda}, \underline{\Lambda} - \tilde{\underline{\Lambda}}, \psi - \tilde{\psi}), \\ \text{curl}^{\mathbf{S}} \eta^{\mathbf{S}} - \widetilde{\text{curl}}^{\tilde{\mathbf{S}}} \tilde{\eta}^{\tilde{\mathbf{S}}} &= O(r^{-1})(\Lambda - \tilde{\Lambda}, \underline{\Lambda} - \tilde{\underline{\Lambda}}, \psi - \tilde{\psi}). \end{aligned}$$

Applying Lemmas 2.11, 2.13 and 2.22, we deduce that

$$(b^{\mathbf{S}} - \bar{b}^{\mathbf{S}}) - (\tilde{b}^{\tilde{\mathbf{S}}} - \bar{\tilde{b}}^{\tilde{\mathbf{S}}}) = O(1)(B - \tilde{B}, \underline{B} - \tilde{\underline{B}}, \Lambda - \tilde{\Lambda}, \underline{\Lambda} - \tilde{\underline{\Lambda}}, \psi - \tilde{\psi}).$$

Thus,  $H_2$  is  $O(1)$ -Lipschitz on  $(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)$  but independent on  $D$ . Moreover,

$$\begin{aligned} H_1 &= \frac{2m_{(0)}(r^{\mathbf{S}} - r)}{rr^{\mathbf{S}}} = r^{-2}O(\overset{\circ}{\delta}), \\ H_2 &= b^{\mathbf{S}} - \bar{b}^{\mathbf{S}} = \check{b}^{\mathbf{S}} - \bar{\check{b}}^{\mathbf{S}} = O(\overset{\circ}{\varepsilon}), \\ H_3 &= r\Gamma_b = O(\overset{\circ}{\varepsilon}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (b^{\mathbf{S}} + \underline{\Omega}) \Big|_{\gamma(s)} &= D(s) - 1 - \frac{2m_{(0)}}{r^{\mathbf{S}}} + (b^{\mathbf{S}} \Big|_{\gamma(s)} - \bar{b}^{\mathbf{S}}) + \check{\underline{\Omega}} \Big|_{\gamma(s)} - \Upsilon \Big|_{\gamma(s)} \\ &= D(s) - 2 + H_2(s) + H_4(s), \end{aligned}$$

where

$$H_4(\Lambda, \underline{\Lambda}, \psi)(s) := \left( \frac{2m}{r} - \frac{2m_{(0)}}{r^{\mathbf{S}}} \right) \Big|_{\gamma(s)} + \check{\underline{\Omega}} \Big|_{\gamma(s)}.$$

<sup>33</sup>Recall that we extended  $r^{\mathbf{S}}$  and  $u^{\mathbf{S}}$  to  $\mathcal{R}$  in Definition 4.11. Thus,  $z^{\mathbf{S}}$  and  $\underline{\Omega}^{\mathbf{S}}$  are well defined in  $\mathcal{R}$  by (4.65) and hence  $b^{\mathbf{S}}$  is also well defined in  $\mathcal{R}$  by  $b^{\mathbf{S}} = -z^{\mathbf{S}} - \underline{\Omega}^{\mathbf{S}}$ .



By Assumption **A2**, (3.8), (3.10) and (3.11), we have

$$H_4(s) = O(\overset{\circ}{\delta}) + r\Gamma_b = O(\overset{\circ}{\varepsilon}).$$

Remark that  $H_4(s)$  is  $O(1)$ -Lipschitz on  $(\Lambda, \underline{\Lambda}, \psi)$ .

Then, we infer

$$\begin{aligned} \psi'(s) &= \frac{b^{\mathbf{S}} + z + \underline{\Omega}}{b^{\mathbf{S}} + \underline{\Omega}} \Big|_{\gamma(s)} + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) + \text{Good} \\ &= \frac{D(s) + H_1(s) + H_2(s) + H_3(s) + \text{Good}}{D(s) - 2 + H_2(s) + H_4(s)} + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) + \text{Good} \\ &= (D(s) + H_1(s) + H_2(s) + H_3(s) + \text{Good}) \left(-\frac{1}{2} + O(D(s), H_2(s), H_4(s))\right) \\ &\quad + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) + \text{Good} \\ &= -\frac{1}{2}D(s) - \frac{1}{2}(H_1(s) + H_2(s) + H_3(s)) + O(\mathbf{H}(s)^2) + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right) \\ &\quad + O(D(s)\mathbf{H}(s)) + O(D(s)^2) + \text{Good}, \end{aligned}$$

where

$$\mathbf{H}(s) := (H_1(s), H_2(s), H_3(s), H_4(s))$$

is  $O(1)$ -Lipschitz on  $(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)$ . Thus, we can write

$$\psi'(s) = -\frac{1}{2}D(s) + H(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)(s) + M(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s),$$

where

$$\begin{aligned} H(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)(s) &= -\frac{1}{2}(H_1(s) + H_2(s) + H_3(s)) + O(\mathbf{H}(s)^2) + \overset{\circ}{\lambda} \left(1 + \frac{2m}{r}\right), \\ M(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s) &= O(D(s)\mathbf{H}(s)) + O(D(s)^2) + \text{Good}, \end{aligned}$$

verify the properties stated in Proposition 4.30. This concludes the proof of Proposition 4.30.  $\square$

#### 4.5 Proof of Theorem 4.1

**Step 1.** From Proposition 4.30, we have the following closed system of equations in  $(\Lambda, \underline{\Lambda}, \psi)$ :

$$\begin{aligned} \frac{1}{-1 + \psi'(s)} \Lambda'(s) &= B(s) + G(\Lambda, \underline{\Lambda}, \psi)(s) + N(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \\ \frac{1}{-1 + \psi'(s)} \underline{\Lambda}'(s) &= \underline{B}(s) + \underline{G}(\Lambda, \underline{\Lambda}, \psi)(s) + \overline{N}(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \\ \psi'(s) &= -\frac{1}{2}D(s) + H(B, \underline{B}, \Lambda, \underline{\Lambda}, \psi)(s) + M(B, \underline{B}, D, \Lambda, \underline{\Lambda}, \psi)(s), \end{aligned} \tag{4.151}$$

with initial conditions

$$\psi(\overset{\circ}{s}) = 0, \quad \Lambda(\overset{\circ}{s}) = \Lambda_0, \quad \underline{\Lambda}(\overset{\circ}{s}) = \underline{\Lambda}_0, \tag{4.152}$$

see Proposition 4.30 for the properties of  $G, \underline{G}, H, N, \overline{N}, M$ .

The system (4.151) is verified for all hypersurface  $\Sigma$  as in Definition 4.11, and hence for any choice of  $(\Lambda, \underline{\Lambda}, \psi)$ . We now make a suitable choice to obtain the GCM hypersurface  $\Sigma_0$ .

Consider in particular the system obtained from (4.151) by setting  $B, \underline{B}, D$  to zero, i.e.

$$\begin{aligned} \frac{1}{-1 + \check{\psi}(s)} \check{\Lambda}'(s) &= G(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) + \check{N}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s), \\ \frac{1}{-1 + \check{\psi}(s)} \check{\underline{\Lambda}}'(s) &= \underline{G}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) + \check{\overline{N}}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s), \\ \check{\psi}'(s) &= H(0, 0, \check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) + \check{M}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s), \end{aligned} \tag{4.153}$$

where

$$\begin{aligned} \check{N}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) &:= N(0, 0, 0, \check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s), \\ \check{\overline{N}}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) &:= \overline{N}(0, 0, 0, \check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s), \\ \check{M}(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s) &:= M(0, 0, 0, \check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})(s). \end{aligned}$$

We initialize the system at  $s = \overset{\circ}{s}$  as in (4.152), i.e.

$$\check{\psi}(\overset{\circ}{s}) = 0, \quad \check{\Lambda}(\overset{\circ}{s}) = \Lambda_0, \quad \check{\underline{\Lambda}}(\overset{\circ}{s}) = \underline{\Lambda}_0.$$

Proposition 4.30 implies that  $G, \underline{G}, H, \check{N}, \check{\overline{N}}, \check{M}$  are  $O(1)$ -Lipschitz functions of  $(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})$ . Applying Cauchy-Lipschitz theorem, we deduce that the system admit a unique solution  $(\check{\Lambda}(s), \check{\underline{\Lambda}}(s), \check{\psi}(s))$  defined in a small neighborhood  $\overset{\circ}{I}$  of  $\overset{\circ}{s}$ , which satisfies

$$|(\check{\Lambda}(s), \check{\underline{\Lambda}}(s), \check{\psi}(s))| \lesssim \overset{\circ}{\delta}.$$

The desired hypersurface  $\Sigma_0$  is given from this solution  $(\check{\Lambda}(s), \check{\underline{\Lambda}}(s), \check{\psi}(s))$  by:

$$\Sigma_0 := \bigcup_{s \in \overset{\circ}{I}} \mathbf{S}[\check{\Psi}(s), s, \check{\Lambda}(s), \check{\underline{\Lambda}}(s)].$$

**Step 2.** Next, we show that the functions  $B, \underline{B}, D$  vanish on the hypersurface  $\Sigma_0$  defined above. Since the system (4.151) is verified for all functions and triplets  $(\Lambda, \underline{\Lambda}, \psi)$  and in particular by  $(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})$ , and since  $(\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi})$  satisfies (4.153), we deduce, taking the difference of (4.151) and (4.153),

$$\begin{aligned} -\frac{1}{2} \check{D}(s) + H(\check{B}, \check{\underline{B}}, \check{P})(s) + M(\check{B}, \check{B}, \check{D}, \check{P})(s) &= H(0, 0, \check{P})(s) + M(0, 0, 0, \check{P})(s), \\ \check{B}(s) + N(\check{B}, \check{\underline{B}}, \check{D}, \check{P})(s) &= N(0, 0, 0, \check{P})(s), \\ \check{\underline{B}}(s) + \overline{N}(\check{B}, \check{\underline{B}}, \check{D}, \check{P})(s) &= \overline{N}(0, 0, 0, \check{P})(s), \end{aligned} \tag{4.154}$$

where

$$\check{P} := (\check{\Lambda}, \check{\underline{\Lambda}}, \check{\psi}).$$

The first equation and the properties of  $M$  and  $H$  stated in Proposition 4.30 imply that

$$\begin{aligned} |\check{D}(s)| &\lesssim |H(\check{B}, \check{\underline{B}}, \check{P})(s) - H(0, 0, \check{P})(s)| + |M(\check{B}, \check{\underline{B}}, \check{D}, \check{P})(s) - M(0, 0, 0, \check{P})(s)| \\ &\lesssim \sup_{\overset{\circ}{I}} |\check{B}(s)| + \sup_{\overset{\circ}{I}} |\check{\underline{B}}(s)| + \overset{\circ}{\varepsilon}^{\frac{1}{2}} \sup_{\overset{\circ}{I}} |\check{D}(s)|. \end{aligned}$$

Taking the supreme over  $\overset{\circ}{I}$ , for  $\overset{\circ}{\varepsilon}$  small enough, we obtain

$$\sup_{\overset{\circ}{I}} |\check{D}(s)| \lesssim \sup_{\overset{\circ}{I}} |\check{B}(s)| + \sup_{\overset{\circ}{I}} |\check{\underline{B}}(s)|. \tag{4.155}$$

The properties of  $N, \bar{N}$  imply, together with (4.155), that

$$\begin{aligned} |N(\check{B}, \check{\underline{B}}, \check{D}, \check{P})(s) - N(0, 0, 0, \check{P})(s)| &\lesssim \varepsilon^{\frac{1}{2}} \left( \sup_{\mathring{I}} |\check{B}(s)| + \sup_{\mathring{I}} |\check{\underline{B}}(s)| \right), \\ |\bar{N}(\check{B}, \check{\underline{B}}, \check{D}, \check{P})(s) - \bar{N}(0, 0, 0, \check{P})(s)| &\lesssim \varepsilon^{\frac{1}{2}} \left( \sup_{\mathring{I}} |\check{B}(s)| + \sup_{\mathring{I}} |\check{\underline{B}}(s)| \right). \end{aligned}$$

Hence, the second and third equations of (4.154) imply that

$$\sup_{\mathring{I}} |\check{B}(s)| + \sup_{\mathring{I}} |\check{\underline{B}}(s)| \lesssim \varepsilon^{\frac{1}{2}} \left( \sup_{\mathring{I}} |\check{B}(s)| + \sup_{\mathring{I}} |\check{\underline{B}}(s)| \right),$$

and thus, for  $\varepsilon$  small enough,

$$\check{B} = 0, \quad \check{\underline{B}} = 0, \quad \text{on } \Sigma_0. \quad (4.156)$$

Finally, (4.155) and (4.156) imply that

$$\check{D} = 0, \quad \text{on } \Sigma_0. \quad (4.157)$$

In view of the definition of  $B, \underline{B}$  and  $D$ , this implies that the desired identities (4.12) and (4.16) hold on  $\Sigma_0$ .

**Step 3.**  $\Sigma_0$  satisfies the statements 1,2,4,5 of Theorem 4.1 by Proposition 4.13. Also, it satisfies the statements 3,6 of Theorem 4.1 by Step 2. Thus, it only remains to derive the estimates in the statement 7 which we recall below for convenience:

$$\|(f, \underline{f}, \check{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{s})} + \|\mathfrak{d}(f, \underline{f}, \check{\lambda})\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} \lesssim \delta, \quad (4.158)$$

To this end, we first apply Proposition 4.22 to obtain

$$\|F\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{s})} + \|\nabla_{\nu}^{\mathbf{S}} F\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} \lesssim \delta, \quad (4.159)$$

where we used (4.156) and (4.157). Next, we recall the following null transformation formulae (4.73)

$$\begin{aligned} \lambda^{-2} \xi^{\mathbf{S}} &= \xi + \frac{1}{2} \nabla_{\lambda^{-1} e_4^{\mathbf{S}}}^{\mathbf{S}} f + r^{-1} O(F) + \text{l.o.t.}, \\ \zeta^{\mathbf{S}} &= \zeta - \nabla^{\mathbf{S}} (\log \lambda) + r^{-1} O(F) + \text{l.o.t.}, \\ \underline{\eta}^{\mathbf{S}} &= \underline{\eta} + \frac{1}{2} \nabla_{\lambda^{-1} e_4^{\mathbf{S}}}^{\mathbf{S}} \underline{f} + r^{-1} O(F) + \text{l.o.t.}, \\ \lambda^{-1} \omega^{\mathbf{S}} &= \omega - \frac{1}{2} \lambda^{-1} e_4^{\mathbf{S}} (\log \lambda) + r^{-1} O(F) + \text{l.o.t.}, \end{aligned}$$

where l.o.t. denotes terms decay better. Combining with (4.14), we deduce

$$\|\nabla_{e_4^{\mathbf{S}}}^{\mathbf{S}}(F)\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} \lesssim r^{-1} \|F\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{s})} \lesssim r^{-1} \delta. \quad (4.160)$$

Then, we have from (4.159) and (4.160)

$$\|\nabla_{e_3^{\mathbf{S}}}^{\mathbf{S}}(F)\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} \lesssim \|\nabla_{\nu^{\mathbf{S}}}^{\mathbf{S}}(F)\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} + \|b^{\mathbf{S}} \nabla_{e_4^{\mathbf{S}}}^{\mathbf{S}}(F)\|_{\mathfrak{h}_{s_{max}}(\mathbf{s})} \lesssim \delta. \quad (4.161)$$

Combining (4.159), (4.160) and (4.161), we obtain (4.158) as stated. This concludes the proof of Theorem 4.1.

### 4.6 Proof of Corollary 4.2

**Step 1.** Consider first the simpler case where

$$\|(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta},$$

so that the estimates (4.87) holds true for  $\mathbf{S}_0$ . We then proceed exactly as Proposition 4.22 to derive the estimates for  $\nabla_{\nu}^{\mathbf{S}_0}(f, \underline{f}, \overset{\circ}{\lambda})$  for our distinguished sphere  $\mathbf{S}_0$ . Note that  $\mathbf{S}_0$  can be viewed as a deformation of the unique background sphere sharing the same south pole. In the sequel, we denote

$$\mathbf{S} := \mathbf{S}_0, \quad \nu := \nu^{\mathbf{S}} = \nu^{\mathbf{S}_0}.$$

The proof of the estimates for  $(\nabla_{\nu}^{\mathbf{S}})^l(f, \underline{f}, \overset{\circ}{\lambda})$  in the cases  $l \geq 2$  is similar so we only provide a sketch:

1. Elliptic estimates.

We commute the GCM system (4.84) by  $(\nabla_{\nu}^{\mathbf{S}})^l(F)$  and obtain the analog of (4.88) for  $(\nabla_{\nu}^{\mathbf{S}})^l F$ . All the commutators can be estimated by Lemma 2.4 as before, remark that

$$\begin{aligned} \nabla_{\nu}^{\mathbf{S}}([\nabla_{\nu}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}]f) &= \nabla_{\nu}^{\mathbf{S}}\left(\frac{2}{r^{\mathbf{S}}}\operatorname{div}^{\mathbf{S}}f + \Gamma_b^{\mathbf{S}} \cdot \nabla_{\nu}^{\mathbf{S}}(f)\right) + \text{l.o.t.} \\ &= \nu\left(\frac{2}{r^{\mathbf{S}}}\right)(\operatorname{div}^{\mathbf{S}}f) + \frac{2}{r^{\mathbf{S}}}\nabla_{\nu}^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}}f) \\ &\quad + \nabla_{\nu}^{\mathbf{S}}(\Gamma_b^{\mathbf{S}}) \cdot \nabla_{\nu}^{\mathbf{S}}(f) + \Gamma_b^{\mathbf{S}} \cdot (\nabla_{\nu}^{\mathbf{S}})^2(f) + \text{l.o.t.} \end{aligned}$$

Thus, in the cases  $l \geq 2$ , we obtain the new terms  $(\nabla_{\nu}^{\mathbf{S}})^{\leq l-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})$  from the commutators. Notice also that

$$\begin{aligned} \nu^2\left(\frac{2}{r^{\mathbf{S}}}\overset{\circ}{\lambda}\right) &= \frac{2}{r^{\mathbf{S}}}\nu^2(\overset{\circ}{\lambda}) + 2\nu\left(\frac{2}{r^{\mathbf{S}}}\right)\nu(\overset{\circ}{\lambda}) + \nu^2\left(\frac{2}{r^{\mathbf{S}}}\right)\overset{\circ}{\lambda} \\ &= \frac{2}{r^{\mathbf{S}}}\nu^2(\overset{\circ}{\lambda}) + r^{-2}\nu(F) - 2\nu\left(\frac{b^{\mathbf{S}} + \underline{\Omega}^{\mathbf{S}}}{(r^{\mathbf{S}})^2}\right)\overset{\circ}{\lambda} \\ &= \frac{2}{r^{\mathbf{S}}}\nu^2(\overset{\circ}{\lambda}) + r^{-2}\nu(F) + r^{-2}\nu(b^{\mathbf{S}} + \underline{\Omega}^{\mathbf{S}})F + r^{-3}F. \end{aligned}$$

Recalling (4.70), in the cases  $l \geq 2$ , we have the new terms  $(\nabla_{\nu}^{\mathbf{S}})^{\leq l-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})$  in the R.H.S. of the analog of (4.88) for  $(\nabla_{\nu}^{\mathbf{S}})^l(F)$ . Applying Proposition 3.4, we infer

$$\begin{aligned} &\left\|(\nabla_{\nu}^{\mathbf{S}})^l(f, \underline{f}), \nu^l(\overset{\circ}{\lambda}) - \overline{\nu^l(\overset{\circ}{\lambda})}^{\mathbf{S}}\right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \lesssim \overset{\circ}{\delta} + \left|(\operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})^l f)_{\ell=1}\right| + \left|(\operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})^l \underline{f})_{\ell=1}\right| \\ &\quad + \overset{\circ}{\delta} r^{-1} \sum_{j=1}^l \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + \overset{\circ}{\delta} \sum_{j=1}^l \|\nu^{j-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\ &\quad + \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r^{-1} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\ &\quad + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \|(\nabla_{\nu}^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \right), \end{aligned} \tag{4.162}$$

and

$$\begin{aligned}
r \left| \overline{\nu^l(\lambda)}^{\mathbf{S}} \right| &\lesssim \delta + \left| (\operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})^l f)_{\ell=1} \right| + \left| (\operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}})^l \underline{f})_{\ell=1} \right| + |b_0^{(l)}| \\
&+ \delta r^{-1} \sum_{j=1}^l \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + \delta \sum_{j=1}^l \|\nu^{j-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\
&+ \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r^{-1} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\
&+ (\overset{\circ}{\varepsilon} + r^{-1}) \left( \|(\nabla_{\nu}^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \right),
\end{aligned} \tag{4.163}$$

where

$$b_0^{(l)} := \overline{\nu^l(\overset{\circ}{b})}^{\mathbf{S}}.$$

Applying Lemma 2.10 for the equation of  $\nu^l(\overset{\circ}{b})$ , we obtain

$$\begin{aligned}
\left\| \nu^l(\overset{\circ}{b}) - \overline{\nu^l(\overset{\circ}{b})}^{\mathbf{S}} \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} &\lesssim r\delta + r \|(\nabla_{\nu}^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + (\overset{\circ}{\varepsilon} + r^{-1}) \|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \\
&+ \delta r^{-1} \sum_{j=1}^l \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + \delta \sum_{j=1}^l \|\nu^{j-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\
&+ \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r^{-1} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})}.
\end{aligned} \tag{4.164}$$

2. Estimates for the  $\ell = 1$  modes of  $(\nabla_{\nu}^{\mathbf{S}})^l(f, \underline{f})$ .

Applying Lemmas 2.4, 4.17, 4.18 and  $B = 0$ , we infer, since  $\nu$  is tangent to  $\Sigma_0$ ,

$$\begin{aligned}
0 = \nu(B(s)) &= \nu \left( \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}}) J^{(\mathbf{S}, p)} \right) \\
&= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left( \nu \left( (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}}) J^{(\mathbf{S}, p)} \right) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}} \kappa^{\mathbf{S}}) (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}}) J^{(\mathbf{S}, p)} \right) \\
&= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left( \operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}} \eta^{\mathbf{S}}) + [\nabla_{\nu}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}] \eta^{\mathbf{S}} + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}} \kappa^{\mathbf{S}}) (\operatorname{div}^{\mathbf{S}} \eta^{\mathbf{S}}) \right) J^{(\mathbf{S}, p)}.
\end{aligned} \tag{4.165}$$

Recall from (4.18) that we have

$$\nu \left( \int_{\mathbf{S}} (\operatorname{div} \eta) J^{(p)} \right) = O(\overset{\circ}{\delta}).$$

Similarly, applying Lemmas 2.4, 2.22, 2.23, 4.17, 4.18 and

$$(b^{\mathbf{S}} - b, \underline{\kappa}^{\mathbf{S}} - \underline{\kappa}, z^{\mathbf{S}} - z, \underline{\Omega}^{\mathbf{S}} - \underline{\Omega}, \kappa^{\mathbf{S}} - \kappa) = O(\overset{\circ}{\delta}),$$

which is proved in Proposition 4.22, combining with (4.165) and the fact that  $J^{(\mathbf{S}, p)}$  verifies (4.23), we deduce

$$\int_{\mathbf{S}} \left( \operatorname{div}^{\mathbf{S}}(\nabla_{\nu}^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta)) + [\nabla_{\nu}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}](\eta^{\mathbf{S}} - \eta) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}} \kappa^{\mathbf{S}}) \operatorname{div}^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta) \right) J^{(\mathbf{S}, p)} = O(\overset{\circ}{\delta}).$$

Recalling from Lemma 4.18 that

$$\begin{aligned} \nabla_\nu^{\mathbf{S}}(f) &= \eta^{\mathbf{S}} - \eta + r^{-1}O(F) + \text{l.o.t.}, \\ (\nabla_\nu^{\mathbf{S}})^2(f) &= \nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta) + r^{-1}O(F)\nu(b^{\mathbf{S}}) + r^{-1}\nabla_\nu^{\mathbf{S}}(F) + \text{l.o.t.}, \end{aligned}$$

we deduce

$$(\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})(\text{div}^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta)) = (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\kappa^{\mathbf{S}})(\text{div}^{\mathbf{S}}\nabla_\nu^{\mathbf{S}}(f)) + r^{-3}O(\overset{\circ}{\delta}) = r^{-3}O(\overset{\circ}{\delta}),$$

and from Lemma 2.4 that

$$\begin{aligned} [\nabla_\nu^{\mathbf{S}}, \text{div}^{\mathbf{S}}](\eta^{\mathbf{S}} - \eta) &= \frac{2}{r^{\mathbf{S}}}\text{div}^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta) + \Gamma_b^{\mathbf{S}} \cdot \nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta) + \text{l.o.t.} \\ &= r^{-3}O(\overset{\circ}{\delta}) + O(\overset{\circ}{\varepsilon})r^{-1}(\nabla_\nu^{\mathbf{S}})^2(F) + r^{-3}O(\overset{\circ}{\delta})\nu(b^{\mathbf{S}}). \end{aligned}$$

Thus, we obtain

$$\int_{\mathbf{S}} (\text{div}^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}} - \eta))) J^{(\mathbf{S},p)} = O(\overset{\circ}{\delta}) + rO(\overset{\circ}{\varepsilon})(\nabla_\nu^{\mathbf{S}})^2(F) + O(\overset{\circ}{\delta})\nu(b^{\mathbf{S}}).$$

Hence, we have

$$\begin{aligned} \left| \int_{\mathbf{S}} \text{div}^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}})^2(f) J^{(\mathbf{S},p)} \right| &\lesssim \overset{\circ}{\delta} + r^{-1}\overset{\circ}{\delta}\|\nu(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + \|(\nabla_\nu^{\mathbf{S}})F\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \\ &\quad + \overset{\circ}{\varepsilon}\|(\nabla_\nu^{\mathbf{S}})^2(F)\|_{\mathfrak{h}_{K-2}(\mathbf{S})}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left| \int_{\mathbf{S}} \text{div}^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}})^2(\underline{f}) J^{(\mathbf{S},p)} \right| &\lesssim \overset{\circ}{\delta} + r^{-1}\overset{\circ}{\delta}\|\nu(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-1}(\mathbf{S})} + \|(\nabla_\nu^{\mathbf{S}})(F)\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \\ &\quad + \overset{\circ}{\varepsilon}\|(\nabla_\nu^{\mathbf{S}})^2(F)\|_{\mathfrak{h}_{K-2}(\mathbf{S})}. \end{aligned}$$

These are the desired estimates for  $l = 2$ . The cases of  $l \geq 3$  are similar. Hence, we have for  $l \geq 2$

$$\begin{aligned} &\left| (\text{div}^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}})^l(f))_{\ell=1} \right| + \left| (\text{div}^{\mathbf{S}}(\nabla_\nu^{\mathbf{S}})^l(\underline{f}))_{\ell=1} \right| \\ &\lesssim \overset{\circ}{\delta} + r^{-1}\overset{\circ}{\delta} \sum_{j=1}^l \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + \sum_{j=1}^l \|(\nabla_\nu^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \quad (4.166) \\ &\quad + \overset{\circ}{\varepsilon}\|(\nabla_\nu^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})}. \end{aligned}$$

### 3. Boundedness of $\nu^{\leq l-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})$ and $(\nabla_\nu^{\mathbf{S}})^{\leq l-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})$ .

Recall that the  $\ell = 1$  modes of  $\nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}})$  are estimated in the previous step. Then, we commute the equations in the proof of Lemmas 4.27 and 4.28 with  $\nu$  to deduce

$$\begin{aligned} &\| \nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}}), \nabla_\nu^{\mathbf{S}}(\underline{\xi}^{\mathbf{S}}), \nabla_\nu^{\mathbf{S}}(\underline{\omega}^{\mathbf{S}}) \|_{\mathfrak{h}_{K-1}(\mathbf{S})} \\ &\lesssim \overset{\circ}{\varepsilon} + r^{-1}\overset{\circ}{\delta} \sum_{j=1}^2 \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + \sum_{j=1}^2 \|(\nabla_\nu^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}. \quad (4.167) \end{aligned}$$

We commute equations (4.69) and (4.70) with  $\nu$  to obtain

$$\begin{aligned} &\| \nu(b^{\mathbf{S}}) - \overline{\nu(b^{\mathbf{S}})}, \nu(z^{\mathbf{S}}) - \overline{\nu(z^{\mathbf{S}})}, \nu(\underline{\Omega}^{\mathbf{S}}) - \overline{\nu(\underline{\Omega}^{\mathbf{S}})} \|_{\mathfrak{h}_{K-1}(\mathbf{S})} \\ &\lesssim r \| \nabla_\nu^{\mathbf{S}}(\eta^{\mathbf{S}}), \nabla_\nu^{\mathbf{S}}(\underline{\xi}^{\mathbf{S}}) \|_{\mathfrak{h}_{K-1}(\mathbf{S})} \quad (4.168) \\ &\lesssim \overset{\circ}{\varepsilon}r + \overset{\circ}{\delta} \sum_{j=1}^2 \|\nu^{j-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r \sum_{j=1}^2 \|(\nabla_\nu^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}. \end{aligned}$$

To estimate  $\overline{\nu(b^{\mathbf{S}})}$ , we apply Lemma 4.17 and  $D = 0$  to deduce

$$\nu(\overline{b^{\mathbf{S}}}) = -\nu\left(\frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}\right) = -\frac{1}{16\pi}\nu\left(\int_{\mathbf{S}}\kappa^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}}\right) = O(1).$$

On the other hand

$$\begin{aligned}\nu(\overline{b^{\mathbf{S}}}) &= \frac{1}{|S|}\nu\left(\int_{\mathbf{S}}b^{\mathbf{S}}\right) - \frac{\nu(|S|)}{|S|}\overline{b^{\mathbf{S}}} \\ &= \frac{1}{|S|}z^{\mathbf{S}}\int_{\mathbf{S}}(z^{\mathbf{S}})^{-1}\left(\nu(b^{\mathbf{S}}) + (\underline{\kappa}^{\mathbf{S}} + b^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}})b^{\mathbf{S}}\right) + O(1) \\ &= \frac{1}{|S|}z^{\mathbf{S}}\int_{\mathbf{S}}\overline{(z^{\mathbf{S}})^{-1}}\left(\nu(b^{\mathbf{S}})\right) + O(1).\end{aligned}$$

Combining these three estimates, we obtain

$$\overline{\nu(b^{\mathbf{S}})} = O(1). \quad (4.169)$$

To prove the boundedness of  $\overline{\nu(z^{\mathbf{S}})}$ , we start from

$$-z^{\mathbf{S}} = \underline{\Omega}^{\mathbf{S}} + b^{\mathbf{S}} = \nu(r^{\mathbf{S}}) = \frac{r^{\mathbf{S}}}{2}z^{\mathbf{S}}\overline{(z^{\mathbf{S}})^{-1}(\kappa^{\mathbf{S}} + b^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}})},$$

which implies

$$\int_{\mathbf{S}}(z^{\mathbf{S}})^{-1}(\kappa^{\mathbf{S}} + b^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}}) = -8\pi r^{\mathbf{S}}.$$

Hence, we have

$$\nu\left(\int_{\mathbf{S}}(z^{\mathbf{S}})^{-1}(\kappa^{\mathbf{S}} + b^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}})\right) = -8\pi\nu(r^{\mathbf{S}}) = O(1). \quad (4.170)$$

Then, applying Lemma 4.17 and (4.169), we can deduce

$$|\overline{\nu((z^{\mathbf{S}})^{-1})}| \lesssim 1 + \sum_{j=1}^2 \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})},$$

details are left to the reader. Then (4.168) and (4.70) imply that, for  $\overset{\circ}{\delta}$  small enough

$$r^{-1}\left\|\nu\left(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}}\right)\right\|_{\mathfrak{h}_{K-1}(\mathbf{S})} \lesssim 1 + \sum_{j=1}^2 \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}. \quad (4.171)$$

Similarly, we can prove that

$$\begin{aligned}&\left\|(\nabla_{\nu}^{\mathbf{S}})^{l-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}})\right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1}\left\|\nu^{l-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}})\right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \\ &\lesssim 1 + \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{S})}.\end{aligned} \quad (4.172)$$

Notice that (4.162), (4.166) and (4.172) imply

$$\begin{aligned}&\left\|(\nabla_{\nu}^{\mathbf{S}})^l(f, \underline{f}), \nu^l(\overset{\circ}{\lambda}) - \overline{\nu^l(\overset{\circ}{\lambda})}^{\mathbf{S}}\right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \\ &\lesssim \overset{\circ}{\delta} + \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r^{-1}\sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\ &+ (\overset{\circ}{\varepsilon} + r^{-1})\left(\|(\nabla_{\nu}^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1}\|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})}\right).\end{aligned} \quad (4.173)$$

4. Estimate for  $b_0^{(l)}$ .

We need to prove that

$$b_0^{(l)} := \overline{\nu^l(\overset{\circ}{b})}^{\mathbf{S}} = \overline{\nu^l(r - r^{\mathbf{S}})} = O(\overset{\circ}{\delta}). \tag{4.174}$$

Similar to (4.97), we can deduce

$$\begin{aligned} \nu^l(r - r^{\mathbf{S}}) &= -2\nu^{l-1} \left( z\overline{z^{-1}} - z^{\mathbf{S}}\overline{(z^{\mathbf{S}})^{-1}} \right) + O(\overset{\circ}{\varepsilon})\overline{\nu^{l-1}(z - z^{\mathbf{S}})} \\ &\quad + O(\mathcal{E}_l) + O\left( (\nabla_{\nu}^{\mathbf{S}})^{\leq l-1}(F) \right) + O(\nu^{\leq l-1}(r - r^{\mathbf{S}})), \end{aligned}$$

where  $\mathcal{E}_l$  is defined in (4.94). Similar to (4.98), we can obtain

$$\overline{\nu^{l-1}(z - z^{\mathbf{S}})} = O(\overset{\circ}{\varepsilon})\overline{\nu^{l-1}(z - z^{\mathbf{S}})} + O(\mathcal{E}_l).$$

Thus, we have

$$\overline{\nu^{l-1}(z - z^{\mathbf{S}})} = O(\mathcal{E}_l).$$

By iteration, we conclude (4.174). The details of the proof is tedious but straightforward and is similar to the case of  $l = 1$ . We leave to the reader. Notice that (4.163), (4.166), (4.172) and (4.174) imply

$$\begin{aligned} \left\| \overline{\nu^l(\overset{\circ}{\lambda})}^{\mathbf{S}} \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} &\lesssim \overset{\circ}{\delta} + \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(F)\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} + r^{-1} \sum_{j=1}^l \|(\nabla_{\nu}^{\mathbf{S}})^{j-1}(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j+1}(\mathbf{S})} \\ &\quad + (\overset{\circ}{\varepsilon} + r^{-1}) \left( \|(\nabla_{\nu}^{\mathbf{S}})^l(F)\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \right). \end{aligned} \tag{4.175}$$

5. Conclusion.

Combining all the above estimates, we obtain for  $\overset{\circ}{\varepsilon}$  small enough and  $\overset{\circ}{r}$  large enough,

$$\|(\nabla_{\nu}^{\mathbf{S}})^l F\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \|\nu^l(\overset{\circ}{b})\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \lesssim \overset{\circ}{\delta} + \sum_{j=0}^{l-1} \|(\nabla_{\nu}^{\mathbf{S}})^j F\|_{\mathfrak{h}_{K-j}(\mathbf{S})} + r^{-1} \sum_{j=0}^{l-1} \|\nu^j(\overset{\circ}{b})\|_{\mathfrak{h}_{K-j}(\mathbf{S})},$$

which, together with (4.87), yields by iteration the desired estimates for all tangential derivatives

$$\|(\nabla_{\nu}^{\mathbf{S}})^l F\|_{\mathfrak{h}_{K-l}(\mathbf{S})} \lesssim \overset{\circ}{\delta}, \quad \forall l \leq K. \tag{4.176}$$

Notice that (4.176) implies for  $1 \leq l \leq K$

$$\left\| (\nabla_{\nu}^{\mathbf{S}})^{l-1}(\eta^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}}) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} + r^{-1} \left\| \nu^{l-1}(b^{\mathbf{S}}, z^{\mathbf{S}}, \underline{\Omega}^{\mathbf{S}}) \right\|_{\mathfrak{h}_{K-l}(\mathbf{S})} = O(1). \tag{4.177}$$

Next, remark that the estimates for  $(\nabla_4^{\mathbf{S}})^l(F)$  follows directly from the transversality conditions (4.14) and the null transformation formulas (4.73). Then, the estimates for  $(\nabla_3^{\mathbf{S}})^l(F)$  follows directly from  $\nu = e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}}$  and the estimates for  $\nu^{\leq l}(b^{\mathbf{S}})$ .

This concludes the proof of the statement 1 of Corollary 4.2.

**Step 2.** It remains to prove Corollary 4.2 in the more difficult case where

$$\|f\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} + (r^{\mathbf{S}_0})^{-1} \|(f, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}. \tag{4.178}$$



In view of the control (4.178) for  $(f, \underline{f}, \overset{\circ}{\lambda})$ , we may apply Lemma 7.3 in [34], which yields

$$\left| \frac{r^{\mathbf{S}_0}}{\overset{\circ}{r}} - 1 \right| + \sup_{\mathbf{S}_0} \left| \frac{r^{\mathbf{S}_0}}{r} - 1 \right| \lesssim \overset{\circ}{\delta}$$

so that  $r$  and  $r^{\mathbf{S}_0}$  are comparable, and hence

$$\|f\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} + r^{-1} \|(\underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}. \quad (4.179)$$

Next, we introduce as in Proposition 4.22 the notations  $K = s_{max} + 1$  and  $\nu := \nu^{\mathbf{S}_0}$ . Note that  $\mathbf{S}_0$  can be viewed as a deformation of the unique background sphere sharing the same south pole. We proceed exactly as Step 1 to derive the desired estimates for our distinguished sphere  $\mathbf{S} := \mathbf{S}_0$ .

In the following, we revisit all the terms estimated in Step 1, and explain that we can obtain the same estimates for  $(\nabla_{\nu}^{\mathbf{S}})^l(F)$  when  $l \geq 1$  under the weaker assumption (4.179).

1. Commutators.

Applying Lemma 2.4, we have

$$[\nabla_{\nu}^{\mathbf{S}}, \nabla^{\mathbf{S}}]F = \frac{2}{r} \nabla^{\mathbf{S}}(F) + \Gamma_b^{\mathbf{S}} \nabla_{\nu}^{\mathbf{S}}(F) + r^{-1} \Gamma_b^{\mathbf{S}} \cdot \mathfrak{d}^{\leq 1}(F),$$

which implies<sup>34</sup>

$$\|[\nabla_{\nu}^{\mathbf{S}}, \nabla^{\mathbf{S}}]F\|_{\mathfrak{h}_l(\mathbf{S})} \lesssim \frac{1}{r^2} \|F\|_{\mathfrak{h}_{l+1}(\mathbf{S})} + \frac{\overset{\circ}{\varepsilon}}{r} \|\nabla_{\nu}^{\mathbf{S}}(F)\|_{\mathfrak{h}_l(\mathbf{S})}.$$

Similarly, by Lemma 2.4, we have

$$\|[\nabla_{\nu}^{\mathbf{S}}, \Delta^{\mathbf{S}}]F\|_{\mathfrak{h}_l(\mathbf{S})} \lesssim \frac{1}{r^3} \|F\|_{\mathfrak{h}_{l+2}(\mathbf{S})} + \frac{\overset{\circ}{\varepsilon}}{r^2} \|\nabla_{\nu}^{\mathbf{S}}(F)\|_{\mathfrak{h}_l(\mathbf{S})}.$$

Observe that the terms of  $F$  on the R.H.S. gain a power of  $r^{-1}$  when compared to the terms of  $\nabla_{\nu}^{\mathbf{S}}(F)$ , which is consistent with the anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$ <sup>35</sup>.

2. Coefficients  $(\underline{\dot{C}}_0, \underline{\dot{C}}^{(p)})$ .

Recalling (4.89), we have

$$(\underline{\dot{C}}_0, \underline{\dot{C}}^{(p)}) = r^{-1} O(\overset{\circ}{\delta}),$$

according to the anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$ . We have

$$\begin{aligned} & \nu \left( \frac{1}{2r^{\mathbf{S}}} \left( \underline{\dot{C}}_0 + \sum_p \underline{\dot{C}}^{(p)} J^{(\mathbf{S}, p)} \right) \right) \\ &= O\left(\frac{1}{r^2}\right) \left( \underline{\dot{C}}_0 + \sum_p \underline{\dot{C}}^{(p)} J^{(\mathbf{S}, p)} \right) + \frac{1}{2r^{\mathbf{S}}} \left( \nu(\underline{\dot{C}}_0) + \sum_p \nu(\underline{\dot{C}}^{(p)}) J^{(\mathbf{S}, p)} \right). \end{aligned}$$

Remark that the first term on the R.H.S. has order  $r^{-3} O(\overset{\circ}{\delta})$ , which is consistent with the estimates (4.85). Thus, we can obtain the same estimates as Step 1.<sup>36</sup>

<sup>34</sup>Recall that  $\nabla_4^{\mathbf{S}}(F)$  can be easily estimated by (4.73).

<sup>35</sup>In (4.21), the decay of  $(\underline{f}, \overset{\circ}{\lambda})$  lose a power of  $r^{-1}$  when compared to the decay of  $f$ , it is called anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$

<sup>36</sup>The second term on the R.H.S. appears on the L.H.S when applying Proposition 3.4.

3. Error terms in  $(h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4)$ .

We recall Definition 2.24:

$$\begin{aligned} r\text{Err}_1 &= F \cdot (r\Gamma_b) + F \cdot (r\nabla^{\mathbf{S}})^{\leq 1} F, \\ r^2\text{Err}_2 &= (r\nabla^{\mathbf{S}})^{\leq 1} (r\text{Err}_1) + F \cdot r\partial\Gamma_b. \end{aligned}$$

According to the anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$ , we lose one power of  $r^{-1}$  in  $F \cdot (r\Gamma_b)$ . But if we inspect all the error terms in (2.57) carefully (see Lemma 4.3 and Corollary 4.6 in [34] for their expressions), we can write that

$$\begin{aligned} \text{Err}_1 &= f \cdot \Gamma_b + (\underline{f}, \overset{\circ}{\lambda}) \cdot \Gamma_g + r^{-1} F \cdot F + F \cdot \nabla^{\mathbf{S}} F, \\ \text{Err}_2 &= (r\nabla^{\mathbf{S}})^{\leq 1} (r\text{Err}_1) + f \cdot r\partial\Gamma_g + (\underline{f}, \overset{\circ}{\lambda}) \cdot r\partial\Gamma_b. \end{aligned}$$

According to Assumption **A1** and (4.179), we have

$$\nu(\text{Err}_1) = \nu(F) \cdot \Gamma_b + r^{-2} O(\underline{f}, \overset{\circ}{\lambda}) + r^{-1} O(f) + r^{-1} \delta^{\leq 1} (F \cdot \nu(F)) + r^{-2} F \cdot F.$$

Thus, we gain a power of  $r^{-1}$  for  $(\underline{f}, \overset{\circ}{\lambda})$  when comparing to the other terms. The estimate for  $\nu(\text{Err}_2)$  is similar.

4. Nonlinear terms in  $(h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4)$ .

The nonlinear terms include (see Remark 4.10 in [34]):

$$\frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda}, \quad \frac{2}{(r^{\mathbf{S}})^2} \overset{\circ}{\lambda}, \quad \left(\kappa - \frac{2}{r^{\mathbf{S}}}\right) \overset{\circ}{\lambda}, \quad \left(\underline{\kappa} + \frac{2}{r^{\mathbf{S}}}\right) \overset{\circ}{\lambda}, \quad \left(V - \frac{2}{(r^{\mathbf{S}})^2}\right) \overset{\circ}{\lambda}, \quad (4.180)$$

and

$$\frac{2(r - r^{\mathbf{S}})^2}{r(r^{\mathbf{S}})^2}, \quad \underline{C}_0 + \sum_p \underline{C}^{(p)} J^{(\mathbf{S}, p)}, \quad \underline{M}_0 + \sum_p \underline{M}^{(p)} J^{(\mathbf{S}, p)}, \quad (4.181)$$

where

$$V := -\left(\frac{1}{2} \underline{\kappa} \underline{\kappa} + \underline{\kappa} \underline{\omega} + \underline{\kappa} \underline{\omega}\right).$$

We also have

$$\nu\left(\frac{2}{r^{\mathbf{S}}} \overset{\circ}{\lambda}\right) = -\frac{2\nu(r^{\mathbf{S}})}{(r^{\mathbf{S}})^2} \overset{\circ}{\lambda} + \frac{2}{r^{\mathbf{S}}} \nu(\overset{\circ}{\lambda}) = O\left(\frac{1}{r^2}\right) \overset{\circ}{\lambda} + O\left(\frac{1}{r}\right) \nu(\overset{\circ}{\lambda}).$$

Remark that the first term on the R.H.S. gains a power of  $r^{-1}$ . The estimates for the other terms in (4.180) are similar. The first term in (4.181) can be estimated as before. The other two terms in (4.181) can be estimated by Corollary 5.20 in [34], Recalling that their estimates gain a power of  $r^{-1}$  for  $\underline{f}$  and are independent on  $\overset{\circ}{\lambda}$ .

5. The remaining terms in  $(h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4)$ .

According to Remark 4.11 in [34], the remaining terms in  $(h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4)$  either depend on  $(\dot{\kappa}, \underline{\dot{\kappa}}, \dot{\mu})$  or contain an additional power of  $r^{-1}$  compared to the other terms. The latter can be ignored since we obtain the desired estimate despite the anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$ .

Notice that  $\nu^l(\dot{\kappa}, \underline{\dot{\kappa}}, \dot{\mu})$  can be estimated by Lemma 4.20 as before, Recalling that its proof gains a power of  $r^{-1}$  for  $\overset{\circ}{\delta}$ , which is consistent with the anomalous behavior of  $(\underline{f}, \overset{\circ}{\lambda})$ .

6. Estimate for  $|b_0|$ .

To estimate  $b_0$ , we proceed exactly as Proposition 4.22, just Recalling that (4.95) also holds under the weaker assumption for  $(\underline{f}, \lambda)$ .

In summary, we can obtain, as in Step 1, the following estimate

$$\sum_{j=1}^K \|(\nabla_{\nu}^{\mathbf{S}})^j(F)\|_{\mathfrak{h}_{K-j}(\mathbf{s}_0)} \lesssim \delta. \quad (4.182)$$

To complete the proof of statement 2, we use the transversality conditions (4.14) and the null transformation formulas (4.73) to deduce the estimates of  $\nabla_4^{\mathbf{S}}(F)$ , and then recover the estimates of  $\nabla_3^{\mathbf{S}}(F)$  from  $\nu = e_3^{\mathbf{S}} + b^{\mathbf{S}}e_4^{\mathbf{S}}$ . This concludes the proof of Corollary 4.2.

## A Proof of Lemma 4.6

We denote

$$X_0 := \Psi'(s)\partial_u + \partial_s.$$

Notice that  $X_0$  is tangent to  $\Sigma_{\#}$  and recall that  $\nu_{\#}$  is the unique tangent vectorfield to  $\Sigma_{\#}$ , normal to  $S$  such that  $\mathbf{g}(\nu_{\#}, e_4) = -2$ . Together with (2.35), we deduce

$$\begin{aligned} X_0 &= \Psi'(s)\partial_u + \partial_s = \Psi'(s) \left( \frac{1}{z}e_3 - \frac{\Omega}{z}e_4 - \frac{2}{z}\underline{B}^a\partial_{y^a} \right) + e_4 \\ &= \frac{\Psi'(s)}{z}e_3 + \left( 1 - \frac{\Omega\Psi'(s)}{z} \right) e_4 - \frac{2\Psi'(s)}{z}\underline{B}^a\partial_{y^a} \\ &= \frac{\Psi'(s)}{z}\nu_{\#} - \frac{2\Psi'(s)}{z}\underline{B}^a\partial_{y^a}, \end{aligned} \quad (A.1)$$

where

$$\nu_{\#} = e_3 + b_{\#}e_4, \quad b_{\#} = \frac{z}{\Psi'(s)} \left( 1 - \frac{\Omega\Psi'(s)}{z} \right). \quad (A.2)$$

Notice that  $\Phi_{\#}^{\tilde{J}}(X_0, \partial_{y^1}, \partial_{y^2})$  forms a basis of the tangent space of  $\Sigma_{\tilde{J}}$ . We have

$$\begin{aligned} \Phi_{\#}^{\tilde{J}}(X_0) &= \left( \Psi'(s) + X_0(U^{\tilde{J}}) \right) \partial_u + \left( 1 + X_0(S^{\tilde{J}}) \right) \partial_s, \\ \Phi_{\#}^{\tilde{J}}(\partial_{y^a}) &= \partial_{y^a}(U^{\tilde{J}})\partial_u + \partial_{y^a}(S^{\tilde{J}})\partial_s + \partial_{y^a}. \end{aligned}$$

By Lemma 2.13 and (2.35), we can express  $\nu^{\tilde{J}}$  as follows:

$$\begin{aligned} \nu^{\tilde{J}} &= e_3^{\tilde{J}} + b^{\tilde{J}}e_4^{\tilde{J}} \\ &= e_3 + b^{\tilde{J}}e_4 + O(F^{\tilde{J}})(e_1, e_2, e_3, e_4) \\ &= z\partial_u + \underline{\Omega}\partial_s + 2\underline{B}^a\partial_{y^a} + b^{\tilde{J}}\partial_s + O(F^{\tilde{J}})(e_1, e_2, e_3, e_4) \\ &= A_{\tilde{J}}\Phi_{\#}^{\tilde{J}}(X_0) + B_{\tilde{J}}^a\Phi_{\#}^{\tilde{J}}(\partial_{y^a}), \end{aligned}$$

for suitable  $A_{\tilde{J}}$  and  $B_{\tilde{J}}^a$ . In order to compute  $A_{\tilde{J}}$  and  $B_{\tilde{J}}^a$ , we write

$$\begin{aligned} A_{\tilde{J}}\Phi_{\#}^{\tilde{J}}(X_0) + B_{\tilde{J}}^a\Phi_{\#}^{\tilde{J}}(\partial_{y^a}) &= \left( A_{\tilde{J}}\Psi'(s) + A_{\tilde{J}}X_0(U^{\tilde{J}}) + B_{\tilde{J}}^a\partial_{y^a}(U^{\tilde{J}}) \right) \partial_u \\ &\quad + \left( A_{\tilde{J}} + A_{\tilde{J}}X_0(S^{\tilde{J}}) + B_{\tilde{J}}^a\partial_{y^a}(S^{\tilde{J}}) \right) \partial_s + B_{\tilde{J}}^a\partial_{y^a}. \end{aligned}$$

Comparing the coefficients of  $\partial_u$  and  $\partial_{y^a}$  with the above expression of  $\nu^{\tilde{J}}$ , we obtain

$$\begin{aligned} A_{\tilde{J}} &= \frac{z}{\Psi'(s)} + A_0(U^{\tilde{J}}, F^{\tilde{J}}), \\ B_{\tilde{J}}^a &= 2\underline{B}^a + B_0(F^{\tilde{J}}), \end{aligned}$$

where

$$A_0(U, F) = O(\not\partial(U), X_0(U), F), \quad B_0(F) = O(F). \tag{A.3}$$

Thus, we have

$$\nu_{\#}^{\tilde{J}} = A_{\tilde{J}}^{\#} X_0 + (B_{\tilde{J}}^a)^{\#} \partial_{y^a}, \tag{A.4}$$

where

$$A_{\tilde{J}}^{\#} = \frac{z \circ \Phi^{\tilde{J}}}{\Psi'(s)} + A_0 \circ \Phi^{\tilde{J}}, \quad (B_{\tilde{J}}^a)^{\#} = 2\underline{B}^a \circ \Phi^{\tilde{J}} + B_0 \circ \Phi^{\tilde{J}}. \tag{A.5}$$

In view of (A.1), (A.4) and (A.5), we infer

$$\begin{aligned} \nu_{\#}^{\tilde{J}} &= \left( \frac{z \circ \Phi^{\tilde{J}}}{\Psi'(s)} + A_0 \circ \Phi^{\tilde{J}} \right) X_0 + \left( 2\underline{B}^a \circ \Phi^{\tilde{J}} + B_0 \circ \Phi^{\tilde{J}} \right) \partial_{y^a} \\ &= \left( \frac{z \circ \Phi^{\tilde{J}}}{\Psi'(s)} + A_0 \circ \Phi^{\tilde{J}} \right) \left( \frac{\Psi'(s)}{z} \nu_{\#} - \frac{2\Psi'(s)}{z} \underline{B}^a \partial_{y^a} \right) + \left( 2\underline{B}^a \circ \Phi^{\tilde{J}} + B_0 \circ \Phi^{\tilde{J}} \right) \partial_{y^a} \\ &= \left( \frac{z \circ \Phi^{\tilde{J}}}{z} + \frac{\Psi'(s)}{z} A_0 \circ \Phi^{\tilde{J}} \right) \nu_{\#} + \left( 2\underline{B}^a \left( \frac{z - z \circ \Phi^{\tilde{J}}}{z} \right) + 2(\underline{B}^a \circ \Phi^{\tilde{J}} - \underline{B}^a) + D_0 \right) \partial_{y^a}, \end{aligned}$$

where

$$D_0 = -\frac{2\Psi'(s)}{z} \underline{B}^a (A_0 \circ \Phi^{\tilde{J}}) + (B_0 \circ \Phi^{\tilde{J}}). \tag{A.6}$$

Thus, we obtain

$$\begin{aligned} A &= \frac{z \circ \Phi^{\tilde{J}} - z}{z} + \frac{\Psi'(s)}{z} A_0 \circ \Phi^{\tilde{J}}, \\ B^a &= 2\underline{B}^a \left( \frac{z - z \circ \Phi^{\tilde{J}}}{z} \right) + 2(\underline{B}^a \circ \Phi^{\tilde{J}} - \underline{B}^a) + D_0. \end{aligned} \tag{A.7}$$

Applying Taylor formula, recalling  $\check{z} \in r\Gamma_b$ ,  $\underline{B} \in r^{-1}\Gamma_b$ , (A.3) and (A.6), we have

$$A(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J},\#}) = O(\mathfrak{d}_{\#}^{\leq 1}(U^{\tilde{J}}, S^{\tilde{J}}), F^{\tilde{J},\#}), \quad B(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J},\#}) = \frac{\mathring{\varepsilon}}{r^2} O(\mathfrak{d}_{\#}^{\leq 1}(U^{\tilde{J}}, S^{\tilde{J}}), F^{\tilde{J},\#}).$$

This concludes the proof of Lemma 4.6.

## B Proof of Lemma 4.7

Firstly, notice that (3.6), (3.9) and Lemma 2.22 imply

$$r^{-1} \|(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_{s_{max}+1}(S)} + \|(f^{\tilde{J},\#}, \underline{f}^{\tilde{J},\#}, \overset{\circ}{\lambda}^{\tilde{J},\#})\|_{\mathfrak{h}_{s_{max}+1}(S)} \lesssim \overset{\circ}{\delta}. \tag{B.1}$$

Recall that the GCM system (4.88) is defined on  $\mathbf{S} := \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \tilde{J}(s)]$ . We pull it back by  $\Phi^{\tilde{J}}$  to obtain a GCM system for  $\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(f^{\tilde{J}, \#}, \underline{f}^{\tilde{J}, \#}, \overset{\circ}{\lambda}^{\tilde{J}, \#})$  on  $S := S(\Psi(s), s)$ . Similar as (4.90) and (4.100), we have, for  $\overset{\circ}{\varepsilon}$  small enough and  $\overset{\circ}{r}$  large enough

$$\|\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(f^{\tilde{J}, \#}, \underline{f}^{\tilde{J}, \#})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta} + \left| \left( \operatorname{div}^{\mathbf{S}, \#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#} f^{\tilde{J}, \#} \right)_{\ell=1} \right| + \left| \left( \operatorname{div}^{\mathbf{S}, \#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#} \underline{f}^{\tilde{J}, \#} \right)_{\ell=1} \right|, \quad (\text{B.2})$$

where the  $\ell = 1$  modes are computed w.r.t.  $\tilde{J}^{(p)}(s)$ . Recall from Lemma 4.33 that

$$|\nu^{\tilde{J}}(\Lambda(s))| \lesssim |\Lambda'(s)| \lesssim \overset{\circ}{\delta}.$$

Denoting

$$\tilde{J}^{(\mathbf{S}, p)} := \left( (\Phi^{\tilde{J}})^{-1} \right)^{\#} (\tilde{J}^{(p)}),$$

we have from (4.27)

$$\left| \nu^{\tilde{J}}(\tilde{J}^{(\mathbf{S}, p)}) \right| = \left| \Phi_{\#}^{\tilde{J}}(\nu_{\#}^{\tilde{J}}) \left( \left( (\Phi^{\tilde{J}})^{-1} \right)^{\#} \tilde{J}^{(p)} \right) \right| = |\nu_{\#}^{\tilde{J}}(\tilde{J}^{(p)})| \leq 1, \quad \text{on } \Sigma_{\#}. \quad (\text{B.3})$$

Applying Lemmas 2.4, 4.17 and (B.3), we infer

$$\begin{aligned} \nu^{\tilde{J}}(\Lambda(s)) &= \nu^{\tilde{J}} \left( \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} \right) \\ &= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \left[ \nu^{\tilde{J}} \left( (\operatorname{div}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} \right) + (\kappa^{\mathbf{S}} + b^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}}) (\operatorname{div}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} \right] \\ &= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} \nu^{\tilde{J}} (\operatorname{div}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} + z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} (\operatorname{div}^{\mathbf{S}} f^{\tilde{J}}) \nu^{\tilde{J}}(\tilde{J}^{(\mathbf{S}, p)}) + r^{-1} O(\overset{\circ}{\delta}) \\ &= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} (\operatorname{div}^{\mathbf{S}} \nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} + z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} ([\nabla_{\nu_{\tilde{J}}}^{\mathbf{S}}, \operatorname{div}^{\mathbf{S}}] f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} + O(\overset{\circ}{\delta}) \\ &= z^{\mathbf{S}} \int_{\mathbf{S}} (z^{\mathbf{S}})^{-1} (\operatorname{div}^{\mathbf{S}} \nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} + \overset{\circ}{\varepsilon} r O(\nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) + O(\overset{\circ}{\delta}) \\ &= \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} + \overset{\circ}{\varepsilon} r O(\nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) + O(\overset{\circ}{\delta}), \end{aligned}$$

which implies

$$\begin{aligned} \left| \left( \operatorname{div}^{\mathbf{S}, \#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#} f^{\tilde{J}, \#} \right)_{\ell=1} \right| &= \sum_p \left| \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \nabla_{\nu_{\tilde{J}}}^{\mathbf{S}} f^{\tilde{J}}) \tilde{J}^{(\mathbf{S}, p)} \right| \\ &\lesssim |\nu^{\tilde{J}}(\Lambda(s))| + \overset{\circ}{\delta} + \overset{\circ}{\varepsilon} \|\nabla_{\nu_{\tilde{J}}}^{\mathbf{S}}(f^{\tilde{J}})\|_{\mathfrak{h}_s(\mathbf{S})} \\ &\lesssim \overset{\circ}{\delta} + \overset{\circ}{\varepsilon} \|\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(f^{\tilde{J}, \#})\|_{\mathfrak{h}_s(S)}. \end{aligned}$$

Similarly, we have

$$\left| \left( \operatorname{div}^{\mathbf{S}, \#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#} \underline{f}^{\tilde{J}, \#} \right)_{\ell=1} \right| \lesssim \overset{\circ}{\delta} + \overset{\circ}{\varepsilon} \|\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(\underline{f}^{\tilde{J}, \#})\|_{\mathfrak{h}_s(S)}.$$

Thus, (B.2) implies that for  $\overset{\circ}{\varepsilon}$  small enough

$$\|\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(f^{\tilde{J}, \#}, \underline{f}^{\tilde{J}, \#})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta}. \quad (\text{B.4})$$

Next, we recall that

$$\partial_{y^a} U^{\tilde{J}} = \left( \mathcal{U}(f^{\tilde{J}}, \underline{f}^{\tilde{J}}, \Gamma)_b Y_{(a)}^b \right)^{\#}, \quad \partial_{y^a} S^{\tilde{J}} = \left( \mathcal{S}(f^{\tilde{J}}, \underline{f}^{\tilde{J}}, \Gamma)_b Y_{(a)}^b \right)^{\#}, \quad (\text{B.5})$$

where

$$\mathcal{U}(f, \underline{f}, \Gamma) = f + O(\overset{\circ}{\varepsilon})(f, \underline{f}), \quad \mathcal{S}(f, \underline{f}, \Gamma) = \frac{1}{2} (-\Upsilon f + \underline{f}) + O(\overset{\circ}{\varepsilon})(f, \underline{f}),$$

see Proposition 5.14 and Remark 5.15 in [34]. Commuting (B.5) with  $\nu_{\#}^{\tilde{J}}$  and using Lemma 2.4, (B.1) and (B.4), we obtain

$$r^{-1} \|\mathfrak{D}(\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}}))\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta} + \|\nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S}, \#}(f^{\tilde{J}, \#}, \underline{f}^{\tilde{J}, \#})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta}. \quad (\text{B.6})$$

We deduce

$$r^{-1} \|\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta} + \left| \nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)} \right|, \quad (\text{B.7})$$

where  $\gamma(s)$  is the curve of South Poles of  $S(\Psi(s), s)$ .

Next, we estimate  $\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)}$ . Recall that we have

$$U^{\tilde{J}}(\gamma(s)) = 0, \quad S^{\tilde{J}}(\gamma(s)) = 0.$$

Then, we have

$$\nu_{SP}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)} = 0,$$

where  $\nu_{SP}$  is the tangent vector field along  $\gamma(s)$ . Notice that  $\nu_{SP}$  is parallel to  $X_0 = \Psi'(s)\partial_u + \partial_s$ . Combining with (A.1) and (B.1), we obtain

$$\nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)} = 2\underline{B}^a \partial_{y^a}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)} = O(\overset{\circ}{\delta}). \quad (\text{B.8})$$

Applying (4.26), (B.1) and (B.8), we infer

$$\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})|_{\gamma(s)} = O(\overset{\circ}{\delta}). \quad (\text{B.9})$$

Thus, (B.7) and (B.9) yield

$$r^{-1} \|\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta}. \quad (\text{B.10})$$

To conclude, it remains to prove (B.10) with  $\nu_{\#}^{\tilde{J}}$  replaced by  $\nu_{\#}$ . To this end, we consider the connected open component  $\varpi_{\lambda} \subset S$  centered at  $\gamma(s)$ , i.e.

$$\varpi_{\lambda} := \{p \in S / d(p, \gamma(s)) \leq \lambda\},$$

where  $d$  is the geodesic distance on  $S$  induced by  $\mathbf{g}$  and  $\lambda > 0$  is a constant. Define  $\Omega \subset (0, +\infty)$  the set of  $\lambda$  such that the following holds:

$$r^{-1} \|\nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(\varpi_{\lambda})} \leq \overset{\circ}{\delta}^{\frac{1}{2}}. \quad (\text{B.11})$$

Notice that (B.8) implies that (B.11) holds for  $\varpi_\lambda$  when  $\lambda$  small enough, thus  $\Omega$  is non-empty. Note also that  $\Omega$  is closed. Recalling (B.1) and (B.11), we have for  $\lambda \in \Omega$

$$\|A(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}, \#})\|_{\mathfrak{h}_s(\varpi_\lambda)} \lesssim r \delta^{\frac{1}{2}}, \quad \|B(U^{\tilde{J}}, S^{\tilde{J}}, F^{\tilde{J}, \#})\|_{\mathfrak{h}_s(\varpi_\lambda)} \lesssim \overset{\circ}{\varepsilon} \delta^{\frac{1}{2}} r^{-1}.$$

Together with (4.26), (B.1) and (B.10), we have for  $\overset{\circ}{\delta}$  small enough

$$\|\nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(\varpi_\lambda)} \lesssim \|\nu_{\#}^{\tilde{J}}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(\varpi_\lambda)} + \|\partial_{y^a}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(\varpi_\lambda)} \lesssim r \overset{\circ}{\delta} \leq \frac{1}{2} r \overset{\circ}{\delta}^{\frac{1}{2}}. \quad (\text{B.12})$$

Thus, we have  $\lambda + \varepsilon \in \Omega$  for  $\varepsilon$  small enough. By definition of  $\Omega$ , we obtain  $(0, \lambda + \varepsilon) \subset \Omega$ , which implies that  $\Omega$  is open in  $(0, +\infty)$ . Hence  $\Omega$  is a open-closed non-empty subset of  $(0, +\infty)$ , which implies  $\Omega = (0, +\infty)$ . Since  $S$  is compact, there exists a  $\lambda \in \Omega$  large enough such that  $\varpi_\lambda = S$ . Finally, injecting  $\varpi_\lambda = S$  into (B.12), we obtain

$$r^{-1} \|\nu_{\#}(U^{\tilde{J}}, S^{\tilde{J}})\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\delta}. \quad (\text{B.13})$$

This concludes the proof of Lemma 4.7.

## C Proof of Lemma 4.8

Notice that (4.30) is a direct consequence of Corollary 6.11 in [34]. Also, the proof of (4.31) is similar to (4.30) and Lemma 4.7, so we only provide a sketch.

We denote

$$\begin{aligned} \delta H^{\#} &:= H^{\tilde{J}, \#} - H^{\hat{J}, \#}, & H &\in \{f, \underline{f}, \overset{\circ}{\lambda}, \overset{\circ}{b}\}, \\ \delta h &:= h^{\tilde{J}} - h^{\hat{J}}, & h &\in \left\{ \overset{\circ}{C}_0, \overset{\circ}{C}^{(p)}, M_0, M^{(p)}, h_1, \bar{h}_1, h_2, \bar{h}_2, h_3, h_4 \right\}. \end{aligned}$$

We recall the following GCM system, see (C.1)-(C.5) in [34],

$$\begin{aligned} \operatorname{curl}^{\tilde{\mathbf{S}}, \#} \delta f &= \delta h_1, \\ \operatorname{curl}^{\tilde{\mathbf{S}}, \#} \delta \underline{f} &= \delta \bar{h}_1, \\ \operatorname{div}^{\tilde{\mathbf{S}}, \#} \delta f + \frac{2}{r \mathbf{S}} \delta \overset{\circ}{\lambda} - \frac{2}{(r \mathbf{S})^2} \delta \overset{\circ}{b} &= \delta h_2, \\ \operatorname{div}^{\tilde{\mathbf{S}}, \#} \delta \underline{f} + \frac{2}{r \mathbf{S}} \delta \overset{\circ}{\lambda} + \frac{2}{(r \mathbf{S})^2} \delta \overset{\circ}{b} &= \delta \bar{h}_2, \\ \left( \Delta^{\tilde{\mathbf{S}}, \#} + \frac{2}{(r \mathbf{S})^2} \right) \delta \overset{\circ}{\lambda} &= \delta M_0 + \sum_p \delta M^{(p)} J^{(p)} + \frac{1}{2r \mathbf{S}} \left( \delta \overset{\circ}{C}_0 + \sum_p \delta \overset{\circ}{C}^{(p)} J^{(p)} \right) \delta h_3, \\ \Delta^{\tilde{\mathbf{S}}, \#} \delta \overset{\circ}{b} - \frac{1}{2} \operatorname{div}^{\tilde{\mathbf{S}}, \#} (\delta \underline{f} - \delta f) &= \delta h_4, \end{aligned} \quad (\text{C.1})$$

and

$$(\operatorname{div}^{\tilde{\mathbf{S}}, \#} \delta f)_{\ell=1} = \delta \Lambda, \quad (\operatorname{div}^{\tilde{\mathbf{S}}, \#} \delta \underline{f})_{\ell=1} = \delta \underline{\Lambda}. \quad (\text{C.2})$$

See Appendix C in [34] for the structures of  $(\delta h_1, \delta \bar{h}_1, \delta h_2, \delta \bar{h}_2, \delta h_3, \delta h_4)$ . We can deduce the following analog of (4.85):

$$\|\nu_{\#}^{\tilde{J}}(\delta h_1, \delta \bar{h}_1, \delta h_2, \delta \bar{h}_2, \delta h_4)\|_{\mathfrak{h}_s(S)} + r \|\nu_{\#}^{\tilde{J}}(\delta h_3)\|_{\mathfrak{h}_s(S)} \lesssim \overset{\circ}{\varepsilon} r^{-2} \|(\nu_{\#}^{\tilde{J}})^{\leq 1}(\tilde{J} - \hat{J})\|_{\mathfrak{h}_s(S)}. \quad (\text{C.3})$$

We commute the GCM system (C.1) with  $\nu_{\#}^{\tilde{J}}$  to obtain the following analog of (B.2):

$$\begin{aligned} \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_{s+1}(S)} &\lesssim \mathring{\varepsilon} r^{-1} \|\tilde{J} - \hat{J}\|_{\mathfrak{h}_s(S)} + \mathring{\varepsilon} r^{-1} \|\nu_{\#}^{\tilde{J}}(\tilde{J} - \hat{J})\|_{\mathfrak{h}_s(S)} \\ &+ \left| \left( \operatorname{div}^{\tilde{\mathbf{S}},\#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S},\#}(\delta f) \right)_{\ell=1} \right| + \left| \left( \operatorname{div}^{\tilde{\mathbf{S}},\#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta \underline{f}) \right)_{\ell=1} \right|. \end{aligned} \tag{C.4}$$

Recalling (B.25) in [34] for a similar structure of  $(\delta\Lambda, \delta\underline{\Lambda})$ , we can deduce that

$$\begin{aligned} |\nu_{\#}^{\tilde{J}}(\delta\Lambda, \delta\underline{\Lambda})| &\lesssim \mathring{\delta} \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_s(S)} + \|(\delta f, \delta \underline{f})\|_{\mathfrak{h}_s(S)} \\ &+ r^{-1}(r^{-1} + \mathring{\varepsilon}) \left\| \nu_{\#}^{\tilde{J}}(\delta U, \delta S) \right\|_{\mathfrak{h}_s(S)} + r^{-1} \|(\delta U, \delta S)\|_{\mathfrak{h}_s(S)}. \end{aligned} \tag{C.5}$$

Then, we proceed as in (B.4) to obtain

$$\begin{aligned} &\left| \left( \operatorname{div}^{\tilde{\mathbf{S}},\#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\mathbf{S},\#}(\delta f) \right)_{\ell=1} \right| + \left| \left( \operatorname{div}^{\tilde{\mathbf{S}},\#} \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta \underline{f}) \right)_{\ell=1} \right| \\ &\lesssim \mathring{\varepsilon} \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_s(S)} + \|(\delta f, \delta \underline{f})\|_{\mathfrak{h}_s(S)} + r^{-1}(r^{-1} + \mathring{\varepsilon}) \left\| \nu_{\#}^{\tilde{J}}(\delta U, \delta S) \right\|_{\mathfrak{h}_s(S)} + r^{-1} \|(\delta U, \delta S)\|_{\mathfrak{h}_s(S)}. \end{aligned}$$

Injecting it in (C.4) and applying (4.30), for  $\mathring{\varepsilon}$  small enough we obtain

$$\begin{aligned} \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_{s+1}(S)} &\lesssim \mathring{\varepsilon} r^{-1} \|\tilde{J} - \hat{J}\|_{\mathfrak{h}_s(S)} + \mathring{\varepsilon} r^{-1} \|\nu_{\#}^{\tilde{J}}(\tilde{J} - \hat{J})\|_{\mathfrak{h}_s(S)} \\ &+ r^{-1}(r^{-1} + \mathring{\varepsilon}) \left\| \nu_{\#}^{\tilde{J}}(\delta U, \delta S) \right\|_{\mathfrak{h}_s(S)}. \end{aligned} \tag{C.6}$$

Then, relying (B.5) and using (4.30), we easily obtain the following analog of (B.6),

$$r^{-1} \|\mathfrak{P}(\nu_{\#}^{\tilde{J}}(\delta U, \delta S))\|_{\mathfrak{h}_s(S)} \lesssim \mathring{\varepsilon} r^{-1} \|\tilde{J} - \hat{J}\|_{\mathfrak{h}_s(S)} + \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_s(S)}. \tag{C.7}$$

Then, (C.6) and (C.7) imply, for  $\mathring{r}$  large enough and  $\mathring{\varepsilon}$  small enough

$$\begin{aligned} &r^{-1} \|\nu_{\#}^{\tilde{J}}(\delta U, \delta S)\|_{\mathfrak{h}_{s+1}(S)} + \left\| \nabla_{\nu_{\#}^{\tilde{J}}}^{\tilde{\mathbf{S}},\#}(\delta f, \delta \underline{f}) \right\|_{\mathfrak{h}_{s+1}(S)} \\ &\lesssim \mathring{\varepsilon} r^{-1} \|\tilde{J} - \hat{J}\|_{\mathfrak{h}_s(S)} + \mathring{\varepsilon} r^{-1} \|\nu_{\#}^{\tilde{J}}(\tilde{J} - \hat{J})\|_{\mathfrak{h}_s(S)} + r^{-1} \left| \nu_{\#}^{\tilde{J}}(\delta U, \delta S) \right|_{\gamma(s)}. \end{aligned} \tag{C.8}$$

Recalling (B.5), proceeding as (B.8) and (B.9), we obtain the following estimate:

$$\nu_{\#}^{\tilde{J}}(\delta U, \delta S)|_{\gamma(s)} = O(\mathring{\delta}). \tag{C.9}$$

Finally (C.8) and (C.9) yield (4.31). This concludes the proof of Lemma 4.8.

## D Proof of Lemma 4.9

For any integer  $k$ , applying Lemma 2.15 we infer

$$\begin{aligned} \nu_{\#} \left( \int_S |\mathfrak{P}^k h|^2 \right) &= (e_3 + b_{\#} e_4) \left( \int_S |\mathfrak{P}^k h|^2 \right) \\ &= z \int_S (z^{-1} e_3 (|\mathfrak{P}^k h|^2) - z^{-1} \underline{\Omega} e_4 (|\mathfrak{P}^k h|^2) + z^{-1} \underline{\kappa} |\mathfrak{P}^k h|^2 - z^{-1} \underline{\Omega} \kappa |\mathfrak{P}^k h|^2) \\ &+ \underline{\Omega} \int_S (e_4 (|\mathfrak{P}^k h|^2) + \kappa |\mathfrak{P}^k h|^2) + b_{\#} \int_S (e_4 (|\mathfrak{P}^k h|^2) + \kappa |\mathfrak{P}^k h|^2). \end{aligned}$$



Recalling from (A.2) that  $b_{\#} + \underline{\Omega} = \frac{z}{\underline{\Psi}(s)}$ , we obtain

$$\begin{aligned}
\nu_{\#} \left( \int_S |\wp^k h|^2 \right) &= z \int_S \left( z^{-1} e_3(|\wp^k h|^2) - z^{-1} \underline{\Omega} e_4(|\wp^k h|^2) + z^{-1} \underline{\kappa} |\wp^k h|^2 - z^{-1} \underline{\Omega} \kappa |\wp^k h|^2 \right) \\
&+ z \int_S z^{-1} (b_{\#} + \underline{\Omega}) \left( e_4(|\wp^k h|^2) + \kappa |\wp^k h|^2 \right) \\
&= z \int_S z^{-1} \left( \nu_{\#}(|\wp^k h|^2) + (\underline{\kappa} + b_{\#} \kappa) |\wp^k h|^2 \right) \\
&= z \int_S z^{-1} \nu_{\#}(|\wp^k h|^2) + O(r^{-1}) \|h\|_{\mathfrak{h}_k(S)}^2.
\end{aligned}$$

Applying Lemmas 2.4, 4.6, 4.7,  $\check{z} \in r\Gamma_b$  and the divergence theorem, we infer

$$\begin{aligned}
z \int_S z^{-1} \nu_{\#}(|\wp^k h|^2) &= z \int_S z^{-1} (1+A)^{-1} \nu_{\#}^{\check{J}}(|\wp^k h|^2) - z \int_S z^{-1} (1+A)^{-1} B^a \partial_{y^a} (|\wp^k h|^2) \\
&= z \int_S z^{-1} (1+A)^{-1} \nu_{\#}^{\check{J}}(|\wp^k h|^2) + z \int_S \operatorname{div} (z^{-1} (1+A)^{-1} B) |\wp^k h|^2 \\
&= z \int_S z^{-1} (1+A)^{-1} \nu_{\#}^{\check{J}}(|\wp^k h|^2) + O(\check{\delta}) \|h\|_{\mathfrak{h}_k(S)}^2.
\end{aligned}$$

Combining the last two identities and applying Lemma 2.4, we infer

$$\left| \nu_{\#} \left( \int_S |\wp^k h|^2 \right) \right| \lesssim \int_S \left| \nu_{\#}^{\check{J}}(|\wp^k h|^2) \right| + \|h\|_{\mathfrak{h}_k(S)}^2 \lesssim \left\| \nu_{\#}^{\check{J}}(h) \right\|_{\mathfrak{h}_k(S)} \|h\|_{\mathfrak{h}_k(S)} + \|h\|_{\mathfrak{h}_k(S)}^2,$$

which implies

$$\left| \nu_{\#} (\|h\|_{\mathfrak{h}_k(S)}) \right| \lesssim \left\| \nu_{\#}^{\check{J}}(h) \right\|_{\mathfrak{h}_k(S)} + \|h\|_{\mathfrak{h}_k(S)}.$$

Integrating it from  $S = \overset{\circ}{S}$  and recalling that  $|\overset{\circ}{I}| \lesssim \overset{\circ}{\varepsilon}$ , we infer

$$\|h\|_{\mathfrak{h}_k(S)} - \|h\|_{\mathfrak{h}_k(\overset{\circ}{S})} \lesssim \overset{\circ}{\varepsilon} \sup_S \left\| \nu_{\#}^{\check{J}}(h) \right\|_{\mathfrak{h}_k(S)} + \overset{\circ}{\varepsilon} \sup_S \|h\|_{\mathfrak{h}_k(S)}.$$

Taking the supremum over  $S \subset \Sigma_{\#}$ , for  $\overset{\circ}{\varepsilon}$  small enough, we have

$$\sup_{S \subset \Sigma_{\#}} \|h\|_{\mathfrak{h}_k(S)} \leq (1 + O(\overset{\circ}{\varepsilon})) \|h\|_{\mathfrak{h}_k(\overset{\circ}{S})} + \sup_{S \subset \Sigma_{\#}} \left\| \nu_{\#}^{\check{J}}(h) \right\|_{\mathfrak{h}_k(S)}.$$

This concludes the proof of Lemma 4.9.

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## Résumé

Cette thèse est consacrée à l'étude de la stabilité de solutions particulières des équations d'Einstein dans le vide: l'espace-temps de Minkowski et les trous noirs de Kerr. Plus précisément, cette thèse contient quatre résultats principaux. Le premier résultat concerne la stabilité de l'espace-temps de Minkowski à l'extérieur d'un cône de lumière sortant et contient une généralisation du résultat de Klainerman-Nicolò. Le deuxième résultat montre la stabilité non linéaire globale de l'espace de Minkowski sous une hypothèse de décroissance minimale sur les données initiales généralisant le résultat de Bieri. Le troisième résultat fournit une nouvelle preuve de la stabilité de Kerr dans les régions extérieures, obtenue par Caciotta-Nicolò. Le point commun à ces trois travaux est l'application de la méthode  $r^p$  introduite par Dafermos-Rodnianski, comme alternative à la méthode des champs de vecteurs, pour obtenir le contrôle des termes de courbure. Le dernier résultat fournit une construction d'hypersurfaces GCM, qui fait partie intégrante d'une série de cinq travaux (les quatre autres sont dues à Klainerman-Szeftel et Giorgi-Klainerman-Szeftel) établissant la stabilité de Kerr pour les petits moments angulaires.

**Mots clés:** Relativité Générale, stabilité de Minkowski, stabilité de Kerr, feuilletage double-nul, feuilletage maximal-nul, méthode de  $r^p$ , propriétés de peeling, sphères GCM, hypersurfaces GCM.

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## Abstract

This thesis is devoted to study the stability of particular solutions to Einstein vacuum equations: the Minkowski spacetime and Kerr black holes. More precisely, this thesis contains four main results. The first result concerns the stability of Minkowski spacetime outside an outgoing null cone and contains a generalization of the result of Klainerman-Nicolò. The second result shows the global nonlinear stability of the Minkowski space under the assumption of minimal decay on the initial data generalizing a result of Bieri. The third result provides a new proof of Kerr stability in external regions, obtained by Caciotta-Nicolò. The common feature of these three works is the application of the  $r^p$ -weighted estimate method introduced by Dafermos-Rodnianski, as an alternative to the vectorfield method, to obtain control of the curvature terms. The last result provides a construction of GCM hypersurfaces, which is a part of a sequence of five works (the four others are due to Klainerman-Szeftel and Giorgi-Klainerman-Szeftel) establishing Kerr stability for small angular momentum.

**Keywords:** General relativity, Minkowski stability, Kerr stability, double-null foliation, maximal-null foliation,  $r^p$ -weighted estimates, peeling properties, GCM spheres, GCM hypersurfaces.