Functional inequalities in Littlewood-Paley-Stein theory and paraproducts of martingales

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Functional Inequalities in the Littlewood-Paley-Stein Theory and Martingale Paraproducts

Inégalités fonctionelles dans la theorie de Littlewood-Paley-Stein et paraproduits de martingales

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Résumé

Cette thèse est consacrée à l'étude de certains aspects de la théorie de Littlewood-Paley-Stein et de celles des martingales dans différents contextes, notamment pour les fonctions à valeurs vectorielles et non commutatives.

Elle se compose de trois parties. Dans la première, nous établissons une équivalence ponctuelle entre la g-fonction carrée de Littlewood-Paley-Stein et la fonction carrée de martingales. Nos arguments reposent sur la construction d'un semi-groupe de diffusion symétrique spécifique associé à une filtration de martingales. Nous étendons également cette équivalence aux cas vectoriel et non commutatif. En conséquence, nous déterminons l'ordre d'une des meilleures constantes dans l'inégalité de Littlewood-Paley-Stein scalaire.

La deuxième partie se concentre sur le scénario à valeurs vectorielles. Nous montrons l'équivalence entre la norme L^p $(1 \leq p < +\infty)$ entre la q-variante vectorielle de l'intégrale de Lusin et celle de la g-fonction de Littlewood-Paley-Stein du semi-groupe dont le générateurs est un opérateur sectoriel satisfaisant à certaines conditions. Les outils principaux sont les espaces de tente à valeurs vectorielles et la fonction carrée intrinsèque introduite par Wilson. En particulier, nous obtenons l'ordre optimal de la meilleure constante correspondante de l'inégalité de Littlewood-Paley-Stein dans L^p à valeurs vectorielles dans un espace de Banach de type q $(1 < q \leq 2)$ de martingale pour p tendant vers 1.

La dernière partie porte sur la décomposition bilinéaire de la multiplication ponctuelle des éléments de l'espace de Hardy de martingales H^1 et de son espace dual BMO. Cette décomposition bilinéaire continue est étendue à l'espace de Hardy de martingales H^p $(0 < p < 1)$ et à son espace dual. Nos décompositions reposent sur les paraproduits de martingale. En conséquence, nous obtenons des résultats analogues pour les martingales dyadiques sur les espaces de type homogène grâce à la construction d'un système dyadique.

Mots-clefs

Inégalités de Littlewood-Paley-Stein ; Ordre optimal des meilleurs constants ; Espaces de Hardy et BMO ; Type et cotype de martingales ; Espaces de tente à valeurs vectorielles ; Paraproduits de martingales ; Fonction carée intrinsèque ; Espaces de Hardy de Musielak-Orlicz ; Martingales.

Abstract

This thesis is devoted to the study of certain aspects of Littlewood-Paley-Stein theory and martingale theory in various contexts, particularly for vector-valued and noncommutative functions.

It is composed of three parts. In the first part, we establish a pointwise equivalence between the Littlewood-Paley-Stein g-function and the square function of martingales. Our arguments are based on the construction of a specific symmetric diffusion semigroup associated with a martingale filtration. We also extend this equivalence to the vector-valued and noncommutative cases. Consequently, we determine the order of one of the best constants in the scalar Littlewood-Paley-Stein inequality.

The second part focuses on the vector-valued scenario. We demonstrate the equivalence between the L^p norm $(1 \leq p < \infty)$ of the q-variant of vector-valued Lusin area integral and that of the Littlewood-Paley-Stein g-function of the semigroup whose generator satisfies certain conditions. The main tools used are vector-valued tent spaces and the intrinsic square function introduced by Wilson. In particular, we obtain the optimal order of the corresponding best constant in the Littlewood-Paley-Stein inequality in vector-valued L^p spaces where the underlying Banach space is of martingale type q $(1 < q \leq 2)$ for p tending to 1.

The final part deals with the bilinear decomposition of pointwise multiplication of elements in the martingale Hardy space H^1 and its dual space BMO. This continuous bilinear decomposition is extended to the martingale Hardy space H^p $(0 < p < 1)$ and its dual space. Our decompositions rely on martingale paraproducts. Consequently, we obtain analogous results for dyadic martingales on spaces of homogeneous type through the construction of a dyadic system.

Keywords

Littlewood-Paley-Stein inequalities; Optimal order of best constants; Hardy and BMO spaces; Martingale paraproducts; Martingale type and cotype; Vector-valued tent spaces; Intrinsic square function; Musielak-Orlicz Hardy space; Martingales.

Contents

Chapter 0

Introduction

The Littlewood-Paley theory was initiated by Littlewood and Paley in the 1930s (cf. [58, 59, 60]) and its extensions have given rise to important tools in the field of harmonic analysis. They have been instrumental in the study of Fourier series, Fourier transforms, and various other mathematical and analytical problems. The development of classical Littlewood-Paley theory has inspired various other fields in harmonic analysis such as singular integral theory introduced by Calderón and Zygmund and the investigations of various H^p spaces (cf. [27, 40, 89]). One remarkable extension of the classical Littlewood-Paley theory was carried out by Stein who developed the theory considerably, widening its applicability both in the classical setting involving $\mathbb{R}^d$ and in abstract situations involving, among other things, Lie groups, symmetric diffusion semigroups (or so-called symmetric Markovian semigroups) and martingales (cf. [87]). The modern version of Littlewood-Paley theory is nowadays widely called Littlewood-Paley-Stein theory.

Based on the beautiful work of Stein, Cowling [31] presented an elegant alternative approach to Stein's theory for symmetric submarkovian semigroups via bounded holomorphic functional calculus. The various counterparts of Littlewood-Paley-Stein theory have been developed during the last thirty years. The noncommutative extension was studied for maximal function inequalities in [55, 67] and for square function inequalities in [53]. On the other hand, motivated by the Banach space geometry [82, 83], Xu established the vector-valued Littlewood-Paley-Stein theory for Poisson semigroup on the unit circle in [100], and later with his coauthors, extended this theory to symmetric Markovian semigroups in [63, 101]. Afterwards, Betancor *et al* developed this theory in some special cases which are not Markovian (cf. [9, 11, 10, 7]), such as Schrödinger, Hermite, Laguerre semigroups etc (see also [1, 6, 8, 46, 48, 75, 90] for related results). In a recent remarkable paper [102], Xu has investigated for the first time the vector-valued Littlewood-Paley-Stein inequalities for semigroups of regular contractions $\left\{e^{-tL}\right\}_{t>0}$ on $L^p(\Omega)$ for a *fixed* $1 < p < \infty$. That is, for a Banach space X of martingale cotype q ($2 \le q < \infty$), he showed that X is of Lusin cotype q relative to $\left\{e^{-t\sqrt{L}}\right\}_{t>0}$. More precisely, there exists a constant $C > 0$ such that

$$\|G_{q,\sqrt{L}}(f)\|_p \le C\|f\|_{L^p(X)}, \quad \forall f \in L^p(\Omega) \otimes X, \tag{.1}$$

where

$$G_{q,\sqrt{L}}(f)(x) = \left(\int_0^\infty \left\|t\sqrt{L}e^{-t\sqrt{L}}(f)(x)\right\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \tag{.2}$$

More importantly, by deeply exploring holomorphic functional calculus, Fendler's dilation, Calderón-Zygmund theory and Wilson's intrinsic square functions, Xu was able to obtain the sharp bounds depending on the martingale cotype constant. The derivation of these sharp bounds enables him to resolve an open problem posed by Naor and Young in [72]. More precisely, let $\mathsf{L}_{c,q,p}^{\sqrt{L}}(X)$ be the least constant C in (.1)—the *Lusin cotype constant* of X, and $\mathsf{M}_{c,q}(X)$ the martingale cotype q constant of X, he obtained

$$\mathsf{L}_{c,q,p}^{\sqrt{L}}(X) \lesssim \max\left\{p^{\frac{1}{q}}, p'\right\} \mathsf{M}_{c,q}(X) \tag{.3}$$

with the order $\max\left\{p^{\frac{1}{q}}, p'\right\}$ being sharp. We refer the reader to Section 1.4 for the definition of $\mathsf{M}_{c,q}(X)$ and the martingale type constant $\mathsf{M}_{t,q}(X)$.

By duality, the reverse inequality of (.1) also holds under the condition that X is of martingale type q $(1 < q \leq 2)$, namely,

$$\|f - \mathbf{F}(f)\|_{L^p(X)} \leq C\|G_{q,\sqrt{L}}(f)\|_p, \quad \forall\, f \in L^p(\Omega) \otimes X. \tag{.4}$$

Here $\mathbf{F}$ is the vector-valued extension of the projection from $L^p(\Omega)$ onto the fixed point space of $\{e^{-tL}\}_{t>0}$. Let $\mathsf{L}_{t,q,p}^{\sqrt{L}}(X)$ be the least possible constant in (.4), then the resulting *type* bounds satisfy

$$\mathsf{L}_{t,q,p}^{\sqrt{L}}(X) \lesssim \max\left\{p, p'^{\frac{1}{q'}}\right\} \mathsf{M}_{t,q}(X). \tag{.5}$$

Nevertheless the order $\max\left\{p, p'^{\frac{1}{q'}}\right\}$ is very likely to be suboptimal as suggested by the special case when $L = \Delta$—the Laplacian on $\mathbb{R}^d$, $q = 2$ and $X = \mathbb{C}$. In this case,

$$\mathsf{M}_{t,2}(\mathbb{C}) = 1, \text{ and } \sqrt{p} \lesssim \mathsf{L}_{t,2,p}^{\sqrt{\Delta}}(\mathbb{C}) \lesssim p, \tag{.6}$$

see for instance [103, Theorem 1]. The sharpness of (.6) when $p \to 1$ is essentially equivalent to the fact that the $L^1(\mathbb{R}^d)$-norm of the classical g-function controls that of the Lusin square function, which dominates $L^1(\mathbb{R})$-norm of the function itself. The relation between the aforementioned quantities involves the deep theory of Hardy and BMO spaces. Other than this special case, the problem of determining the optimal order of $\mathsf{L}_{t,q,p}^{\sqrt{L}}(X)$ in (.5) has been left open widely even in the case $L = \Delta$ when X is a Banach space of martingale type q, see e.g. Remark 1.3, Problem 1.8 and Problem 8.4 in the aforementioned paper [102]. Note also that the optimal order of $\mathsf{L}_{t,q,p}^{\sqrt{L}}(X)$ as $p \to \infty$ for a specific semigroup seems much harder to be determined, and actually the special case $\mathsf{L}_{t,2,p}^{\sqrt{\Delta}}(\mathbb{C})$ remains open (cf. [103, Problem 5]).

Motivated by all this pioneering work, the first theme of this thesis focuses on the study of optimal order of the best constants in the Littewood-Paley-Stein inequalities, in the scalar-valued, vector-valued and noncommutative settings. For the reader's convenience, we divide this theme into two parts, where the first part is mainly on the optimal order of $\mathsf{L}_{t,q,p}^{\sqrt{L}}(X)$ as $p \to \infty$ for symmetric diffusion semigroups, and the second focuses on the case when $p \to 1$ in the vector-valued setting. The first part is a joint work with Hao Zhang in [104], and the second is a joint work with Guixiang Hong and Hao Zhang still in progress.

Another theme of this thesis concerns the study of martingale paraproducts. This concept that first emerged in the theory of paradifferential operators (cf. [20]) has become an important tool in harmonic analysis. For example, Coiffman and Semmes in [25] proved the celebrate David-Journé $T1$ theorem (first established in [32]) by dyadic martingale paraproducts. Later, Nazarov, Treil and Volberg applied dyadic martingale paraproducts to establish the $T1$ theorem and the Tb theorem on non-homogeneous spaces in [73], etc. On the other hand, the dyadic shift representation of singular integral operators (see for instance, [41, 78, 81, 92]) is an important technique to establish the boundedness of singular integral operators via martingale transforms. It is noteworthy that Petermichl determined the sharp weighted bound of Hilbert transform and Riesz transform in [79] and [80]. The dyadic shift representation was also applied by Hytönen in [52] to fully resolve the famous A_2 conjecture. In these situations, dyadic martingale paraproducts naturally appear.

We shall adapt the martingale paraproducts to give a bilinear decomposition between the elements of martingale Hardy spaces and their dual space and we intend to derive the analogous results on space of homogeneous type from that in the martingale setting. This will be the main theme of the third part of this thesis. This is a joint work with Odysseas Bakas, Yujia Zhai and Hao Zhang in [5].

The remaining part of the introduction will be devoted to a detailed discussion on the background, motivations and main results. After the introduction, we will introduce the definitions and notation in Chapter 1, and in Chapter 2–5, we will present our work in detail. The presentation will be separated into three parts, and each is devoted to one of the topics highlighted above.

I Optimal Lusin type constant as p tending to ∞

This part follows the recent investigation of the Littlewood-Paley-Stein theory carried out by Xu. In his paper [102], Xu has developed a new powerful method to study the vector-valued Littlewood-Paley-Stein theory for semigroups. One interesting benefit of this new method is the fact that it often yields the optimal orders of the relevant best constants.

Let $\left\{e^{-tL}\right\}_{t>0}$ be a symmetric diffusion semigroup on a measure space $(\Omega, \mathcal{F}, \mu)$ and $\left\{e^{-t\sqrt{L}}\right\}_{t>0}$ its subordinated Poisson semigroup defined as follows

$$e^{-t\sqrt{L}}(f) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-\frac{t^2}{4s}L}(f)\, \mathrm{d}s.$$

Then the Littlewood-Paley g-function of $f \in L^p(\Omega)$ associated with $\left\{e^{-t\sqrt{L}}\right\}_{t>0}$ defined in (.2) can be written as (in this case $X = \mathbb{C}$ and $q = 2$)

$$G_{2,\sqrt{L}}(f)(x) = \left(\int_0^\infty \left| t\sqrt{L} e^{-t\sqrt{L}}(f)(x) \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{I.1}$$

Stein's celebrated Littlewood-Paley inequality states that for every $1 < p < \infty$ there exist two positive constants α_p and β_p such that

$$\alpha_p^{-1}\|f - \mathbf{F}(f)\|_p \leq \left\|G_{2,\sqrt{L}}(f)\right\|_p \leq \beta_p\|f\|_p, \quad \forall\, f \in L^p(\Omega),$$

where $\mathbf{F}$ is the projection from $L^p(\Omega)$ onto the fixed point subspace of $\{e^{-t\sqrt{L}}\}_{t>0}$ of $L^p(\Omega)$. We recall the best possible constants α_p and β_p are denoted as in (.3) and (.5) by taking $X = \mathbb{C}$ and $q = 2$. For short, we will denote them by $\mathsf{L}_{c,p}^{\sqrt{L}}$ and $\mathsf{L}_{t,p}^{\sqrt{L}}$, respectively.

Recall the following estimate given by Xu in the aforementioned paper [102]:

$$\mathsf{L}_{c,p}^{\sqrt{L}} \lesssim \max\left\{p, p'^{\frac{1}{2}}\right\} \quad \text{and} \quad \mathsf{L}_{t,p}^{\sqrt{L}} \lesssim \max\left\{p^{\frac{1}{2}}, p'\right\}.$$

Moreover, he has shown that the estimate above on $\mathsf{L}_{c,p}^{\sqrt{L}}$ is optimal as $p \to 1$ and $p \to \infty$ since it is already so for the classical Poisson semigroup on $\mathbb{R}$.

In this part, we show that Xu's above estimate on $\mathsf{L}_{t,p}^{\sqrt{L}}$ is in fact optimal as $p \to \infty$, namely, $\mathsf{L}_{t,p}^{\sqrt{L}} \gtrsim p$ as $p \to \infty$. In fact, we will prove the stronger inequality $\mathsf{L}_{t,p}^{L} \gtrsim p$ as $p \to \infty$ for a symmetric diffusion semigroup $\left\{e^{-tL}\right\}_{t>0}$.

Our argument relies on the deep relationship between martingale inequalities and Littlewood-Paley-Stein inequalities inspired by a result of Neveu. In [74], Neveu establishes a profound link between the martingale theory and the ergodic theory. Indeed, he shows that Doob's maximal inequality for martingale can be obtained from Dunford-Schwartz's maximal ergodic theorem. We will prove the analogous result for the square function.

We recall the definition of symmetric diffusion semigroups in Stein's sense [87, Chapter 3, Section 1].

Definition I.1. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. $\{\mathsf{T}_t\}_{t>0}$ is called a symmetric diffusion semigroup on $(\Omega, \mathcal{F}, \mu)$ if $\{\mathsf{T}_t\}_{t>0}$ satisfies the following conditions:

(a) T_t is a contraction on $L^p(\Omega)$ for every $1 \leq p \leq \infty$;

(b) $\mathsf{T}_t\mathsf{T}_s = \mathsf{T}_{t+s}$ for positive t and s;

(c) $\lim_{t\to 0^+} \mathsf{T}_t(f) = f$ in $L^2(\Omega)$ for every $f \in L^2(\Omega)$;

(d) T_t is positive (i.e. positivity preserving);

(e) T_t is selfadjoint on $L^2(\Omega)$;

(f) $\mathsf{T}_t(1) = 1$.

It is well-known that such symmetric diffusion semigroup has a unique infinitesimal generator, denoted by L. Then T_t can be written as $\mathsf{T}_t = e^{-tL}$. There are many important examples such as the classical heat semigroup $\{\mathbf{H}_t\}_{t<0}$ (with generator Δ) and classical Poisson semigroup $\{\mathbf{P}_t\}_{t>0}$ (with generator $\sqrt{\Delta}$) on Eulcidean space $\mathbb{R}^d$.

Recall that any symmetric diffusion semigroup $\{\mathsf{T}_t\}_{t>0}$ is analytic on $L^p(\Omega)$ for $1 < p < \infty$ which implies that $\partial_t \mathsf{T}_t$ is well-defined (see [87, Chapter 3, Section 2]). Thus we can also express the Littlewood-Paley g-function of f associated with $\{\mathsf{T}_t\}_{t>0}$ as

$$G_{2,\mathsf{T}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathsf{T}_t(f)(x)|^2 \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},$$

which will be more applicable to the later computation. We often call $G_{2,\mathsf{T}}(f)$ the g-function for short.

It is a classical fact (see e.g. [63]) that the orthogonal projection $\mathbf{F}$ on $L^2(\Omega)$ onto the fixed point subspace of $\{\mathsf{T}_t\}_{t>0}$ extends to a contractive projection on $L^p(\Omega)$ for $1 \le p \le \infty$. Then $\mathbf{F}$ is also positive and $\mathbf{F}\left(L_p(\Omega)\right)$ is the fixed point subspace of $\{\mathsf{T}_t\}_{t>0}$ on $L^p(\Omega)$. Recall that Stein's celebrated extension of the classical Littlewood-Paley inequality [87] asserts that for every symmetric diffusion semigroup $\{\mathsf{T}_t\}_{t>0}$ and every $1 < p < \infty$

$$\|f - \mathbf{F}(f)\|_{L_p(\Omega)} \approx_p \|G_{2,\mathsf{T}}(f)\|_{L_p(\Omega)}, \quad \forall\, f \in L^p(\Omega). \tag{I.2}$$

Stein proves the above inequality (I.2) by virtue of Burkholder-Gundy's martingale inequality and complex interpolation. He establishes a close connection between semigroup theory and martingale theory and studies semigroups through martingale theory via Rota's dilation. We will proceed in a reverse way by studying martingales by semigroups.

We consider the square function in the martingale setting. Throughout this part of work we will work with a fixed probability space $(\Omega, \mathcal{F}, \mu)$ and a sequence of σ-algebras

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}$$

such that $\sigma\left(\cup_{n=1}^\infty \mathcal{F}_n\right) = \mathcal{F}$. For a random variable $f \in L^1(\Omega, \mathcal{F}, \mu)$ and $n \in \mathbb{N}^*$, we will set

$$\mathbb{E}_n(f) = \mathbb{E}\left(f \mid \mathcal{F}_n\right), \quad d\mathbb{E}_n(f) = \mathbb{E}_n(f) - \mathbb{E}_{n-1}(f),$$

where $\mathbb{E}_0 = 0$ as convention. The martingale square function of the martingale $\{\mathbb{E}_n(f)\}_{n\ge1}$ is

$$S(f) = \left(\sum_{n=1}^\infty |d\mathbb{E}_n(f)|^2 \right)^{\frac{1}{2}}.$$

See [43] for more information on martingale inequalities. We will call $S(f)$ martingale square function for short.

We introduce the most important operator of the proof associated with martingales. Given a strictly increasing sequence $\{a_n\}_{n\ge0}$ with $a_0 = 0$ and $\lim\limits_{n\to\infty} a_n = 1$, define

$$T = \sum_{n=1}^\infty (a_n - a_{n-1})\mathbb{E}_n. \tag{I.3}$$

Then T is a positive contraction on $L^p(\Omega, \mathcal{F}, \mu)$ for $1 \le p \le \infty$. In particular, for $p = 2$, T is positive and selfadjoint on $L^2(\Omega, \mathcal{F}, \mu)$. We use T to produce a symmetric diffusion semigroup.

Theorem I.2. *If we denote by T^t the operator obtained by continuous functional calculus, then for $t > 0$*

$$T^t = \sum_{n=1}^{\infty} (1 - a_{n-1})^t d\mathbb{E}_n = \sum_{n=1}^{\infty} \left[(1 - a_{n-1})^t - (1 - a_n)^t \right] \mathbb{E}_n, \tag{I.4}$$

and T^t is a positive contraction for any $t > 0$ on $L^p(\Omega, \mathcal{F}, \mu)$ $(1 \leq p \leq \infty)$. Moreover, $\{T^t\}_{t>0}$ is a symmetric diffusion semigroup.

The above semigroup $\{T^t\}_{t>0}$ is exactly our desired semigroup generated by a martingale filtration. Then we select some specific sequence of $\{a_n\}_{n\geq 0}$ and use this particular semigroup to obtain the best order of $\mathsf{L}^L_{t,p}$. Below is one of our principal theorem, and it is the square function analogue of Neveu's result [74]. Note that $\mathbf{F} = \mathbb{E}_1$.

Theorem I.3. *If $a_n = 1 - \mathrm{e}^{-16^{n+1}}$ for $n \geq 1$, then the semigroup $\{T^t\}_{t>0}$ defined in Theorem I.2 satisfies*

$$\sqrt{\frac{7}{60}}\, S(f - \mathbf{F}(f)) \leq G_{2,T}(f) \leq \sqrt{\frac{23}{60}}\, S(f - \mathbf{F}(f)), \quad \forall\, f \in L^1(\Omega, \mathcal{F}, \mu). \tag{I.5}$$

Consequently, for $1 \leq p \leq \infty$

$$\sqrt{\frac{7}{60}}\, \|S(f - \mathbf{F}(f))\|_p \leq \|G_{2,T}(f)\|_p \leq \sqrt{\frac{23}{60}}\, \|S(f - \mathbf{F}(f))\|_p, \quad \forall\, f \in L^p(\Omega, \mathcal{F}, \mu).$$

Remark I.4. The choice of $\{a_n\}_{n\geq 1}$ is not unique. There are many other sequences of $\{a_n\}_{n\geq 1}$ satisfying the above pointwise inequalities with different constants, of course. Indeed, through appropriate choices of $\{a_n\}_{n\geq 1}$, the universal constants in inequality (I.5) can be replaced by $1/2 - \varepsilon$ and $1/2 + \varepsilon$ respectively for any $0 < \varepsilon < 1/2$.

The following corollary indicates that (I.5) also holds for the subordinated Poisson semigroups if we take another choice of $\{a_n\}_{n\geq 1}$.

Corollary I.5. *If $a_n = 1 - \mathrm{e}^{-16^{2(n+1)}}$ for $n \geq 1$, then the subordinated Poisson semigroup $\{P_t\}_{t>0}$ associated with the semigroup $\{T^t\}_{t>0}$ defined in Theorem I.2 satisfies*

$$\sqrt{\frac{7}{60}}\, S(f - \mathbf{F}(f)) \leq G_{2,P}(f) \leq \sqrt{\frac{23}{60}}\, S(f - \mathbf{F}(f)), \quad \forall\, f \in L^1(\Omega, \mathcal{F}, \mu).$$

Theorem I.3 shows that for any martingale, there exists a symmetric diffusion semigroup such that their corresponding square functions are equivalent. In this way, we can use the Littlewood-Paley-Stein inequality (I.2) to show the analogous inequality for the martingale square function. This means that the Burkholder-Gundy square function inequality for martingales can be deduced from the Littlewood-Paley-Stein inequality for semigroups. Moreover, the optimal constants in the martingale square function inequality can be applied to the setting of semigroups.

Recently, Xu has shown the vector-valued Littlewood-Paley-Stein inequality for analytic symmetric diffusion semigroups via holomorphic functional calculus (see [102]). His

method is optimal in the sense that it offers the optimal orders of the best constants. For the martingale cotype case, he shows that the classical Poisson semigroup meets the optimal orders (see [102, Proposition 8.5]). Unfortunately, the optimal order of the reverse Littlewood-Paley-Stein inequality was left unsolved in [102]. In the following, we will utilize Theorem I.3 to give the optimal order of $\mathsf{L}_{t,p}^{L}$ as $p \to \infty$.

To this end, we recall the following Burkholder-Gundy inequality [23, Theorem 3.1].

Theorem I.6. *Let $1 < p < \infty$ and $p^* = \max\{p, p/(p-1)\}$. Then*

$$(p^*-1)^{-1} \|S(f)\|_p \le \|f\|_p \le (p^*-1)\|S(f)\|_p.$$

In particular,

$$\begin{aligned} \|f\|_p &\ge (p-1)\|S(f)\|_p \quad \text{if} \quad 1 < p \le 2, \\ \|f\|_p &\le (p-1)\|S(f)\|_p \quad \text{if} \quad 2 \le p < \infty. \end{aligned}$$

Moreover, the constant $p-1$ is best possible.

Using Theorem I.3 and Theorem I.6, we immediately get the following corollary, which solves a problem left open by Xu in [102, Remark 1.3].

Corollary I.7. *For any symmetric diffusion semigroup $\{\mathsf{T}_t\}_{t>0}$, and for $p \ge 2$*

$$\|f - \mathbf{F}(f)\|_p \lesssim p\|G_{2,\mathsf{T}}(f)\|_p, \quad \forall\, f \in L^p(\Omega), \tag{I.6}$$

and p is the optimal order as $p \to \infty$.

Proof of Corollary I.7. Indeed, (I.6) has already been proved in [102, Theorem 8.1]. Then it suffices to show that p is the optimal order of $\mathsf{L}_p^{\mathsf{T}}$, which is an immediate consequence of Theorem I.3 and Theorem I.6. □

Theorem I.3 can be extended to the vector-valued and noncommutative settings. We first consider the vector-valued case. Given a Banach space X, let $L^p(\Omega; X)$ denote the L^p-space of strongly measurable p-integrable functions from Ω to X. It is a well-known fact that if T is a positive bounded operator on $L^p(\Omega)$ with $1 \le p \le \infty$, then $T \otimes \mathrm{Id}_X$ is bounded on $L^p(\Omega; X)$ with the same norm. For notational convenience, throughout this article, we will denote $T \otimes \mathrm{Id}_X$ by T too. Recall that $\mathbf{F}$ is the projection from $L^p(\Omega; X)$ onto the fixed point subspace of $\{T_t\}_{t>0}$. Thus $\{T_t\}_{t>0}$ is also a semigroup of contractions on $L^p(\Omega; X)$ for any Banach space X with $\mathbf{F}\left(L^p(\Omega; X)\right)$ as its fixed point subspace (see e.g. [63]).

For an X-valued L^p-martingale $\{\mathbb{E}_n(f)\}_{n\ge 1}$, define the corresponding q-variant of the martingale square function for $1 \le q < \infty$ as follows

$$S_q(f) = \left(\sum_{n=1}^{\infty} \|d\mathbb{E}_n(f)\|_X^q\right)^{\frac{1}{q}}, \quad \forall f \in L^p(\Omega; X).$$

Given a strictly increasing sequence $\{a_n\}_{n\ge 0}$ with $a_0 = 0$ and $\lim_{n\to\infty} a_n = 1$ as before, let T^t be defined by (I.3).

Then $\{T^t\}_{t>0}$ extended to $L^p(\Omega;X)$ remains to be a strongly continuous semigroup of contractions on $L^p(\Omega;X)$, and (I.4) remains valid in the X-valued case. Similarly as the scalar-valued case, the vector-valued g-function of f defined in (.2) can also be written as

$$G_{q,T}(f) = \left(\int_0^\infty \left\| t\partial_t T^t(f) \right\|_X^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \quad \forall\, f \in L^p(\Omega;X).$$

The following theorem is the vector-valued analogue of Theorem I.3:

Theorem I.8. *Let X be a Banach space and $1 \le p,\ q < \infty$. Then there exist a sequence $\{a_n\}_{n\ge 0}$ and two universal positive constants, c and C such that*

$$cS_q\left(f - \mathbf{F}(f)\right) \le G_{q,T}(f) \le CS_q\left(f - \mathbf{F}(f)\right). \tag{I.7}$$

for any $f \in L^p(\Omega;X)$.

Similarly, there exists a sequence $\{a_n\}_{n\ge 0}$ such that (I.7) still holds for the subordinated Poisson semigroups of $\{T^t\}_{t>0}$. The proof is an analogue of the argument for Corollary I.5. We leave it to the interested readers. We now turn to the noncommutative setting. We introduce the definitions of square functions for noncommutative martingales and noncommutative semigroups.

Let $\mathcal{M}$ denote a finite von Neumann algebra equipped with a normal faithful normalized trace τ, and $(\mathcal{M}_n)_{n\ge 1}$ an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ whose union is w*-dense in $\mathcal{M}$. For $1 \le p \le \infty$ we denote by $L^p(\mathcal{M},\tau)$, or simply $L^p(\mathcal{M})$ the usual noncommutative L^p-space associated with $(\mathcal{M},\tau)$. As usual, $L^p(\mathcal{M}_n) = L^p\left(\mathcal{M}_n, \tau|_{\mathcal{M}_n}\right)$ is naturally identified as a subspace of $L^p(\mathcal{M})$. It is well-known that there exists a unique normal faithful conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ such that $\tau \circ \mathcal{E}_n = \tau$. Moreover, $\mathcal{E}_n$ extends to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_n)$, for every $1 \le p < \infty$, which is still denoted by $\mathcal{E}_n$. Similarly, we denote by $d\mathcal{E}_n = \mathcal{E}_n - \mathcal{E}_{n-1}$ for $n \ge 1$ the martingale difference with $\mathcal{E}_0 = 0$ as convention.

A noncommutative martingale with respect to $(\mathcal{M}_n)_{n\ge 1}$ is a sequence $x = (x_n)_{n\ge 1}$ in $L^1(\mathcal{M})$ such that

$$x_n = \mathcal{E}_n(x_{n+1}), \quad \forall\, n \ge 1.$$

The difference sequence of x is $(d_n x)_{n\ge 1}$, where $d_n x = x_n - x_{n-1}$ (with $x_0 = 0$ by convention). Then we define L^p-martingales and bounded L^p-martingales, as usual. In the sequel, we will fix $(\mathcal{M},\tau)$ and $(\mathcal{M}_n)_{n\ge 1}$ as above and all noncommutative martingales will be with respect to $(\mathcal{M}_n)_{n\ge 1}$.

Recall that $|\cdot|$ stands for the usual (right) modulus of operators, i.e. $|a| = (a^*a)^{\frac{1}{2}}$. Define the column and row square functions respectively for $x \in L_1(\mathcal{M})$

$$S_c(x) = \left(\sum_{n\ge 1} |d_n x|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S_r(x) = \left(\sum_{n\ge 1} |d_n x^*|^2 \right)^{\frac{1}{2}}.$$

We refer to [55], [85] and [86] for more information.

Now we provide the definition of noncommtative symmetric diffusion semigroup as follows.

Definition I.9. Let $\{\mathsf{T}_t\}_{t>0}$ be a semigroup of operators on $\mathcal{M}$. We say that $\{\mathsf{T}_t\}_{t>0}$ is a noncommutative symmetric diffusion semigroup if it satisfies the following conditions:

(a) Each $\mathsf{T}_t : \mathcal{M} \to \mathcal{M}$ is a unital, normal and completely positive;

(b) For any $x \in \mathcal{M}$, $\mathsf{T}_t(x) \to x$ in the w^*-topology of $\mathcal{M}$ when $t \to 0^+$;

(c) Each $\mathsf{T}_t : \mathcal{M} \to \mathcal{M}$ is selfadjoint. Namely, for any $x, y \in \mathcal{M}$

$$\tau(\mathsf{T}_t(x)y) = \tau(x\mathsf{T}_t(y)).$$

(d) The extension of each $\mathsf{T}_t : \mathcal{M} \to \mathcal{M}$ from $L^p(\mathcal{M})$ to $L^p(\mathcal{M})$ for $1 \le p < \infty$ is completely contractive.

It is well-known that such a semigroup extends to a semigroup of contractions on $L^p(\mathcal{M})$ for any $1 \le p < \infty$, and that $\{\mathsf{T}_t\}_{t>0}$ is a selfadjoint semigroup on $L^2(\mathcal{M})$. Moreover, $\{\mathsf{T}_t\}_{t>0}$ is strongly continuous on $L^p(\mathcal{M})$ for any $1 \le p < \infty$.

Define

$$G_{2,\mathsf{T},c}(x) = \left(\int_0^\infty |t\partial_t \mathsf{T}_t(x)|^2 \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}$$

and

$$G_{2,\mathsf{T},r}(x) = \left(\int_0^\infty |t\partial_t \mathsf{T}_t(x)^*|^2 \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

We call $G_{2,\mathsf{T},c}(x)$ and $G_{2,\mathsf{T},r}(x)$ the column and row g-functions, respectively. We refer the reader to [53] for more information.

Given a strictly increasing sequence $\{a_n\}_{n\ge 0}$ with $a_0 = 0$ and $\lim_{n\to\infty} a_n = 1$ as before, analogously we define the mapping T^t $(t > 0)$ as follows

$$T^t = \sum_{n=1}^{\infty} (1 - a_{n-1})^t d\mathcal{E}_n = \sum_{n=1}^{\infty} \left[(1 - a_{n-1})^t - (1 - a_n)^t \right] \mathcal{E}_n. \tag{I.8}$$

Note that $\mathbf{F} = \mathcal{E}_1$ as well.

Theorem I.10. *If $a_n = 1 - \mathrm{e}^{-16^{n+1}}$ for $n \ge 1$, then $\{T^t\}_{t>0}$ defined in (I.8) is a noncommutative symmetric diffusion semigroup. Moreover, it satisfies for $x \in L_1(\mathcal{M})$*

$$\sqrt{\frac{7}{60}}\, S_c(x - \mathbf{F}(x)) \le G_{2,T,c}(x) \le \sqrt{\frac{23}{60}}\, S_c(x - \mathbf{F}(x)) \tag{I.9}$$

and similarly

$$\sqrt{\frac{7}{60}}\, S_r(x - \mathbf{F}(x)) \le G_{2,T,r}(x) \le \sqrt{\frac{23}{60}}\, S_r(x - \mathbf{F}(x)). \tag{I.10}$$

We have the same result as Corollary I.5 in the noncommutative setting. The corresponding proof is similar to that of Corollary I.5 as well. So we omit these details.

II Optimal Lusin type constant as p tending to 1

In this part, we will determine the optimal order of $\mathsf{L}^L_{t,q,p}(X)$ which appeared in (.5) as $p \to 1$ for a large class of approximation identities $\{e^{-tL}\}_{t>0}$ on $\mathbb{R}^d$, and thus answer the questions mentioned in [102, Remark 1.3 and Problem 1.8]. Moreover, our result will assert that the Lusin type of X relative to this class of approximation identities implies the martingale type of X, and thus partially resolve [102, Problem A.1 and Conjecture A.4].

Let L be a sectorial operator of type α ($0 \le \alpha < \pi/2$) on $L^2(\mathbb{R}^d)$, and thus it generates a holomorphic semigroup e^{-zL} with $0 \le |\mathrm{Arg}(z)| < \pi/2 - \alpha$. One may find these concepts in Section 3.1. Then it is well-known (see e.g. [102]) that the semigroup $\{e^{-tL}\}_{t>0}$ extends to $L^p(\mathbb{R}^d; X)$ ($1 \le p \le \infty$) under our assumptions below. The resulting semigroup is still denoted by $\{e^{-tL}\}_{t>0}$ without causing confusion. Partially inspired by [36, Section 6.2.2], the kind of approximation identity $\{e^{-tL}\}_{t>0}$ that we will be interested in the present paper is assumed to have kernel $K(t,x,y)$ satisfying the following three assumptions: there exist positive constants $0 < \beta, \gamma \le 1$ and c such that for any $t > 0$, $x, y, h \in \mathbb{R}^d$,

$$|K(t,x,y)| \le \frac{ct^\beta}{(t+|x-y|)^{d+\beta}}, \tag{II.1}$$

$$|K(t,x+h,y) - K(t,x,y)| + |K(t,x,y+h) - K(t,x,y)| \le \frac{c|h|^\gamma t^\beta}{(t+|x-y|)^{d+\beta+\gamma}} \tag{II.2}$$

whenever $2|h| \le t + |x-y|$, and

$$\int_{\mathbb{R}^d} K(t,x,y)\,\mathrm{d}x = \int_{\mathbb{R}^d} K(t,x,y)\,\mathrm{d}y = 1. \tag{II.3}$$

Let $1 < q < \infty$, the q-variant of Lusin area integral associated with L is defined as follows: for $f \in C_c(\mathbb{R}^d) \otimes X$,

$$S_{q,L}(f)(x) = \left(\int_0^\infty \int_{|y-x|<t} \left\| tLe^{-tL}(f)(y) \right\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \right)^{\frac{1}{q}}.$$

Recall that the q-variant of g-function associated with L defined in (.2) is

$$G_{q,L}(f)(x) = \left(\int_0^\infty \left\| tLe^{-tL}(f)(y) \right\|_X^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

Our main result reads as below.

Theorem II.1. *Let L be a sectorial operator of type α ($0 \le \alpha < \pi/2$) on $L^2(\mathbb{R}^d)$ satisfying* (II.1), (II.2) *and* (II.3). *Let $1 \le p < \infty$ and $1 < q < \infty$. For <u>any</u> Banach space X and $f \in C_c(\mathbb{R}^d) \otimes X$, there holds*

$$p^{-\frac{1}{q}} \|S_{q,\sqrt{\Delta}}(f)\|_p \lesssim_{\gamma,\beta} \|S_{q,L}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,\sqrt{\Delta}}(f)\|_p, \tag{II.4}$$

$$p^{-\frac{1}{q}} \|S_{q,L}(f)\|_p \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,L}(f)\|_p. \tag{II.5}$$

Moreover, the orders in both (II.4) *and* (II.5) *are* optimal *as $p \to 1$.*

When $X = \mathbb{C}$ and $q = 2$, the equivalence (II.4) in the case $1 < p < \infty$ without explicit orders follows from the classical Littlewood-Paley theory which in turn relies on Calderón-Zygmund theory; while the case $p = 1$ is deduced from the holomorphic functional calculus, Calderón-Zygmund theory and the theory of Hardy and BMO spaces associated with differential operators (cf. [36, Theorem 6.10]). Our estimate (II.4) for any Banach space X, any $1 \leq p < \infty$ and any $1 < q < \infty$ goes much beyond this and its proof provides a new approach to the mentioned scalar case with optimal orders as $p \to 1$. Indeed, the arbitrariness of X presents a surprise and usually one expects certain property of Banach space geometry to be imposed on the square function inequalities. For the technical side, the arbitrariness of X prevents us from the use of (vector-valued) Calderón-Zygmund theory. Instead, we will make use of vector-valued Wilson's intrinsic square functions as a media to relate Δ and L, and then exploit the vector-valued tent space theory such as interpolation, duality as well as atomic decomposition. Even though both of these two tools have been developed or applied in the literature, they need to be taken care of in the present setting. For instance, because our L's are usually not translation invariant or of scaling structure, we have to introduce Wilson's intrinsic square functions via nice functions of two variables satisfying (3.3.1), (3.3.2) and (3.3.3); to avoid the use of Calderón-Zygmund theory to deal with Wilson's intrinsic square functions (cf. [95, 103]), we prove the boundedness of a linear operator $\mathcal{K}$ on vector-valued tent spaces (see Lemma 3.2.5); last but not the least, since our interested X is arbitrary, one cannot establish the basic theory of vector-valued tent space using Calderón-Zygmund theory as in [51, 56, 57, 47], and we shall adapt the classical arguments (cf. [26]), see Section 3.2 for details.

After all the preparing work, the equivalence (II.4) will be an immediate consequence of Theorem 3.3.1, where we collect all the intermediate estimates involving vector-valued Wilson's square functions.

Regarding another equivalence (II.5), in the special situation $X = \mathbb{C}$ and $q = 2$ and $L = \sqrt{\Delta}$, the equivalence for $1 < p < \infty$ without optimal orders comes from the classical Littlewood-Paley theory while the case $p = 1$ constitutes one essential part of the famous real variable theory on Hardy spaces (cf. [38, 39, 40]); in particular the upper estimate of (II.5) follows from harmonicity of Poisson integrals or Calderón-Zygmund theory. Again, the arbitrariness of X excludes the use of vector-valued Calderón-Zygmund theory and there is an obvious lack of harmonicity related to general L. To surmount these difficulties, in addition to the application of Theorem 3.3.1—Wilson's intrinsic square functions, we will fully develop the duality theory between vector-valued Hardy and BMO type spaces in Section 3.4; the latter is inspired by Mei's duality arguments [66] (see also [97, 103]). In turn, part of the theory of vector-valued Hardy and BMO spaces will be deduced from vector-valued tent spaces, and the projection π_L (see Lemma 3.2.6) will play a key role in passing from the results about tent spaces to those on Hardy/BMO spaces.

Together with the related results in [63, 75] where the authors showed the Lusin type q of a Banach space X relative to $\{e^{-t\sqrt{\Delta}}\}_{t>0}$ is equivalent to the martingale type q of X (see Section 3.5), our vector-valued tent space theory and Theorem II.1 imply the following result, resolving partially [102, Problem 1.8, Problem A.1 and Conjecture A.4] (see Remark 3.5.4).

Theorem II.2. *Let L be a sectorial operator of type α ($0 \leq \alpha < \pi/2$) on $L^2(\mathbb{R}^d)$ satisfying* (II.1)*,* (II.2) *and* (II.3)*. Let $1 < q \leq 2$. The followings are equivalent*

(i) *X is of martingale type q;*

(ii) *for all $f \in C_c(\mathbb{R}^d) \otimes X$ with $1 < p \leq 2$, there holds*

$$\|f\|_{L^p(X)} \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p.$$

Combining the main result in [75], a much stronger result than Theorem II.2 involving the case $p = 1, \infty$ will be presented in Corollary 3.5.3.

III Multiplication between elements in martingale Hardy spaces and their dual spaces

The pointwise product of a function in the classical Hardy space $H^1(\mathbb{R}^n)$ and a function of bounded mean oscillation on $\mathbb{R}^n$ need not be in $L^1(\mathbb{R}^n)$; see e.g. [88, Chapter IV, Section 6.2]. However, using Fefferman's duality theorem [38] and the fact that the pointwise product of a BMO-function and a C_0^∞-function is in $BMO(\mathbb{R}^n)$, Bonami, Iwaniec, Jones and Zinsmeister defined in [17] the product $f \times g$ of a function $f \in H^1(\mathbb{R}^n)$ and a function $g \in BMO(\mathbb{R}^n)$ as a distribution given by

$$\langle f \times g, \phi \rangle := \langle g \cdot \phi, f \rangle, \quad \phi \in C_0^\infty(\mathbb{R}^n), \tag{III.1}$$

where in the right-hand side of (III.1) the duality between $f \in H^1(\mathbb{R}^n)$ and $g \cdot \phi \in BMO(\mathbb{R}^n)$ is employed. Moreover, it is shown in [17] that for any fixed $f \in H^1(\mathbb{R}^n)$ there exist two linear continuous operators S_f from $BMO(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and T_f from $BMO(\mathbb{R}^n)$ to a weighted Hardy–Orlicz space such that

$$f \times g = S_f(g) + T_f(g)$$

for all $g \in BMO(\mathbb{R}^n)$; see [17, Theorem 1.6].

In [16], using wavelet analysis, Bonami, Grellier and Ky showed that there exist two bilinear continuous operators S from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and T from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $H^{\log}(\mathbb{R}^n)$ such that

$$f \times g = S(f, g) + T(f, g)$$

for all $f \in H^1(\mathbb{R}^n)$ and for all $g \in BMO(\mathbb{R}^n)$; see [16, Theorem 1.1]. The Musielak Hardy–Orlicz space $H^{\log}(\mathbb{R}^n)$ is defined as the class consisting of all distributions h on $\mathbb{R}^n$ whose grand maximal function $\mathcal{M}h$ satisfies

$$\int_{\mathbb{R}^n} \frac{|\mathcal{M}h(x)|}{\log(e + |x|) + \log(e + |\mathcal{M}h(x)|)} dx < \infty$$

and is smaller than the weighted Hardy–Orlicz space appearing in [17]. The Musielak Hardy–Orlicz space $H^{\log}(\mathbb{R}^n)$ is optimal. More specifically, it is sharp when $n = 1$ (see [19]) and when $n > 1$, $H^{\log}(\mathbb{R}^n)$ is optimal in the sense that the smallest Banach X containing $H^{\log}(\mathbb{R}^n)$ has the same dual with $H^{\log}(\mathbb{R}^n)$, i.e. $X^* \cong (H^{\log}(\mathbb{R}^n))^*$, and X is the smallest Banach space containing products (see [15]).

In addition, continuous bilinear decomposition theorems for products of elements in $H^p(\mathbb{R}^n)$, for $0 < p < 1$, and their dual spaces were established in [14].

Using the theory of wavelets on spaces of homogeneous type, which was developed by Auscher and Hytönen in [3, 2], the aforementioned results have been extended to spaces of homogeneous type by Liu, Yang and Yuan [61] and Xing, Yang and Liang [99]. More precisely, in [61] and [99], continuous bilinear decompositions for products between elements in atomic Hardy spaces $H^p_{\rm at}(\Omega)$ (in the sense of Coifman and Weiss [27]) and their dual spaces were established in the case where $p \in (\frac{n}{n+1}, 1]$. Here n is defined as the dimension of the homogeneous space Ω.

Recently, in [4], a dyadic variant of the aforementioned results of Bonami, Grellier, and Ky was established; see [4, Theorem 24], which in turn was used to deduce a periodic version of [16, Theorem 1.1]; see [4, Theorem 28].

Motivated by [4], the first aim of this part is concerned with the study of multiplication between Hardy spaces and their dual spaces for martingales on a probability space Ω. More specifically, we study multiplications between functions in the martingale Hardy space $H^1(\Omega)$ and its dual space $BMO(\Omega)$ as stated in our first result, Theorem III.1. We also investigate the case $0 < p < 1$, namely multiplication between elements in $H^p(\Omega)$ and their dual spaces, the so-called martingale Lipschitz spaces $\Lambda_1(\alpha_p)$ with $\alpha_p := \frac{1}{p} - 1$, see Theorem III.2. Since the dual space $(H^p(\Omega))^*$ could be $\{0\}$ for some irregular martingales, we choose to consider only regular martingales where every $\sigma-$algebra $\mathcal{F}_k$ in the corresponding filtration is generated by countably many atoms.

We would like to mention that Aline Bonami, Yong Jiao, Guangheng Xie, Dachun Yang, and Dejian Zhou have independently obtained Theorem III.1, and derived from it interesting applications on the boundedness of operators involving commutators in [18]. We would like to remark that Theorem III.1 in both works rely heavily on a priori estimates for functions with finite martingale expansions. Nevertheless, the results for general martingales which concern limiting arguments are interpreted differently. In [18], the authors regarded $f \cdot g$ as a discrete process and in particular, a semi-martingale. They further provided a neat characterization of the semi-martingale as the sum of a martingale and a process with bounded variation.

With a different perspective from [18], we view the product $f \cdot g$ as a pointwise limit, which is indeed a function. Furthermore, the bilinear operators arising from the decomposition have boundedness properties described in Theorem III.1, where all the target spaces are complete spaces by definition. In the study of the bilinear operators, we consider Cauchy sequences of functions lying in the target spaces of our interest. This is because the completion of a space can be identified with equivalence classes of Cauchy sequences with respect to the appropriate norm. The appearance of sequences of functions establishes the connection between [18] and our results, whereas the norm also plays a crucial role for us to clarify the convergence of the Cauchy sequences.

Theorem III.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $\{\mathcal{F}_k\}_{k\geq 1}$.*

1. *Suppose that $f \in H^1(\Omega)$ and $g \in BMO(\Omega)$ have finite martingale expansions. Then the pointwise multiplication can be decomposed as*

$$f \cdot g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g),$$

where Π_1, Π_2 *and* Π_3 *are continuous bilinear operators satisfying*

$$\Pi_1 : H^1(\Omega) \times BMO(\Omega) \to L^1(\Omega),$$
$$\Pi_2 : H^1(\Omega) \times BMO(\Omega) \to H^1(\Omega),$$
$$\Pi_3 : H^1(\Omega) \times BMO(\Omega) \to H^\Phi(\Omega).$$

2. *For general* $f \in H^1(\Omega)$ *and* $g \in BMO(\Omega)$*, the above decomposition still holds in the pointwise sense where* Π_1 *and* Π_2 *are continuous bilinear operators satisfying*

$$\Pi_1 : H^1(\Omega) \times BMO(\Omega) \to L^1(\Omega),$$
$$\Pi_2 : H^1(\Omega) \times BMO(\Omega) \to H^1(\Omega).$$

Here Π_3 *is the pointwise limit of a sequence which is Cauchy in* $H^\Phi(\Omega)$.

In Theorem III.1, $H^\Phi(\Omega)$ is a martingale Hardy–Orlicz space defined in terms of the growth function $\Phi(t)$; see Definition 1.3.16 and (1.3.4) below. We shall refer to the terms $\Pi_2(f,g)$ and $\Pi_3(f,g)$ as the martingale paraproducts.

Theorem III.1 can be regarded as an extension of [4, Theorem 24] to the general case of martingales.

For $0 < p < 1$, if $f \in H^p(\Omega)$, $g \in \Lambda_1(\alpha_p)$ and $f_0 = g_0 = 0$, then their product can be regarded as a continuous linear functional on $L^\infty(\Omega) \cap \Lambda_1(\alpha_p)$. To be more precise, for any $h \in L^\infty(\Omega) \cap \Lambda_1(\alpha_p)$, define

$$\langle f \times g, h\rangle := \langle h \cdot g, f\rangle,$$

where in the right-hand side the duality between $H^p(\Omega)$ and $\Lambda_1(\alpha_p)$ is invoked. Note that $h \cdot g$ belongs to $\Lambda_1(\alpha_p)$ since h is a pointwise multiplier on $\Lambda_1(\alpha_p)$ (see [71]).

Our following theorem establishes a continuous bilinear decomposition for products between elements in $H^p(\Omega)$ and functions in the dual space $\Lambda_1(\alpha_p)$ when $0 < p < 1$.

Theorem III.2. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space equipped with the filtration* $\{\mathcal{F}_k\}_{k\geq 1}$*, where* $\mathcal{F}_k$ *is generated by countably many atoms for any* $k \geq 1$.

If $H^p(\Omega)$ $(0 < p < 1)$ *are martingale Hardy spaces, then there exist continuous bilinear operators* $\Pi_1 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to L^1(\Omega)$, $\Pi_2 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to H^1(\Omega)$ *and* $\Pi_3 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to H^p(\Omega)$ *such that*

$$f \times g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g)$$

for all $f \in H^p(\Omega)$ *and* $g \in \Lambda_1(\alpha_p)$.

One important property of martingale Hardy spaces which we heavily rely on in the proof of Theorems III.1 and III.2 is that they admit the atomic decomposition. As a consequence, estimates for a general element in martingale Hardy spaces can be reduced to the corresponding estimates on atoms. It is noteworthy that the martingale atomic decomposition in our setting are defined in the sense of duality (see Definition 4.1.3), which is different from the one usually defined in the probabilistic setup ([93]) and is inspired by the definition of atomic Hardy spaces defined on spaces of homogenous type. We would like to emphasize that the martingale Hardy spaces and the martingale atomic

Hardy spaces in our definition are complete whereas the martingale Hardy spaces in [93] are not for $0 < p < 1$.

The second part of this topic is method-oriented and aims at applying results in the martingale setting to spaces of homogeneous type. In particular, we establish the connection between spaces of homogenous type and its dyadic variant, namely dyadic martingales on spaces of homogeneous type which are first constructed in [49]. Such connection extends Mei's results in [65] and is interesting in its own right.

We further study analogues of Theorems III.1 and III.2 for the case of dyadic martingales on spaces of homogeneous type, and derive from them results for products between elements in Hardy spaces and those in their duals on spaces of homogeneous type. Compared with the probability setting, the case of spaces of homogeneous type is more difficult to deal with since backward martingales arise, and the underlying measures on spaces of homogeneous type may be infinite.

The structure of the thesis is outlined as follows. In Chapter 1, we set down notation and give some background on holomorphic functional calculus, martingales, spaces of homogeneous type and geometric properties of Banach spaces. The subsequent contents of the thesis are divided into three distinct parts, each containing detailed proofs of the aforementioned results.

The first part of this thesis is presented in Chapter 2. Our focus is on the optimal Lusin type constant as $p \to \infty$. To achieve this, we establish the pointwise equivalence of g-functions and martingale square functions. Specially, we present Theorem I.3, Theorem I.8 and Theorem I.10 for scalar-valued, vector-valued and noncommutative settings, respectively. In Section 2.1, we prove Theorem I.2, Theorem I.3 and Corollary I.5. Subsequently, in Section 2.2 we prove the analogous result, namely Theorem I.8, for the vector-valued case. Finally, we provide a brief overview of the proof of Theorem I.10.

The second part, detailed in Chapter 3, is dedicated to exploring the optimal Lusin type constant in vector-valued Littlewood-Paley-Stein inequality as $p \to 1$. In Section 3.1, we outline our primary assumptions and provide definitions of vector-valued tent, Hardy, and BMO spaces. Following this, Section 3.2 revisits some fundamental properties of vector-valued tent spaces. Additionally, we introduce two crucial operators, $\mathcal{K}$ and π_L, which will serve as key tools in our analysis. In Section 3.3, we extend Wilson's intrinsic functions (cf. [95]) to the vector-valued setting. The subsequent Section 3.4 is dedicated to the proof of Theorem II.1 and Theorem II.2. Finally, in Section 3.5, we derive the optimal Lusin type constants and the characterization of martingale type.

The third part consists of Chapters 4 and 5, focusing on the multiplication between elements in martingale Hardy spaces and their dual spaces. Section 4.1 is dedicated to the elaboration of martingale atomic Hardy spaces and their close relationship with martingale Hardy spaces, playing a pivotal role in our argument. In section 4.2, we prove Theorem III.1. Moving on to Section 4.3, we introduce a characterization of martingale Lipschitz spaces $\Lambda_1(\alpha_p)$—of independent interest (refer to Theorem 4.3.4 and Remark 4.3.5 below)—and subsequently present the proof of Theorem III.2. Chapter 5 concerns spaces of homogeneous type. For reader convenience, Section 1.5 revisits definitions and facts concerning Hardy spaces and Lipschitz spaces on spaces of homogeneous type, following the framework of Coifman and Weiss [27]. In section 5.1, we offer new proofs for certain results from [27] based on the martingale method and the existence of dyadic martingales

on homogeneous spaces. In section 5.2, we establish analogues of Theorems III.1 and III.2 for dyadic martingales on spaces of homogeneous type, as seen in Theorem 5.2.7 below. In the last section of this part, we apply Theorem 5.2.7 to obtain a decomposition of products of functions in Hardy spaces and their dual spaces on spaces of homogeneous type.

Throughout this thesis, X is a Banach space and $1 < q < \infty$. We denote X^* the dual Banach space of X and q' the Hölder conjugate of q. Additionally, the positive real interval $\mathbb{R}_+ = (0, \infty)$ is equipped with the measure $\mathrm{d}t/t$ without providing additional explanations. The terms "homogeneous spaces" and "spaces of homogeneous type" will be used interchangeably.

The following notation is employed: $A \lesssim B$ (resp. $A \lesssim_\varepsilon B$) indicates that $A \leq CB$ (resp. $A \leq C_\varepsilon B$) for some absolute positive constant C (resp. if we want to emphasize that the constant C_ε depends on ε). $A \approx B$ or $A \approx_\varepsilon B$ means that these inequalities as well as their inverses hold. For $1 \leq p \leq \infty$, we also denote by $\|\cdot\|_p$ the norm $\|\cdot\|_{L^p(\mathbb{R}^d)}$ and by $\|\cdot\|_{L^p(X)}$ the norm $\|\cdot\|_{L^p(\mathbb{R}^d;X)}$.

Chapter 1

Preliminaries

In this chapter, we provide notation and background that will be used in the thesis.

1.1 Notation

We will consider sums and intersections of quasi-normed spaces. For the convenience of the reader we recall these notions.

Definition 1.1.1. Let $(X_1, \|\cdot\|_{X_1}), (X_2, \|\cdot\|_{X_2})$ be two quasi-normed spaces and let X be a topological vector space X such that $X_1, X_2 \subset X$ continuously.

1. $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$ is the intersection of X_1 and X_2, where

$$\|x\|_{X_1 \cap X_2} := \max\{\|x\|_{X_1}, \|x\|_{X_2}\}$$

   for all $x \in X_1 \cap X_2$;

2. $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ is the sum of X_1 and X_2, where

$$\|x\|_{X_1+X_2} := \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2} : \ x = x_1 + x_2, \ x_1 \in X_1, \ x_2 \in X_2\}$$

   for all $x \in X_1 + X_2$.

For convenience, the sum $X_1 + X_2 + \cdots + X_n$ and the intersection $X_1 \cap X_2 \cap \cdots \cap X_n$ will also be denoted by $\sum_{k=1}^n X_k$ and $\bigcap_{k=1}^n X_k$, respectively.

Note that $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$ and $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ are both quasi-normed spaces. Moreover, if $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ are Banach spaces, then $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$ and $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ are both Banach spaces.

1.2 Functional calculus

We briefly introduce some preliminaries around the holomorphic functional calculus (cf. [64]). Let $0 \leq \alpha < \pi$. Define the closed sector in the complex plane $\mathbb{C}$ as

$$S_\alpha = \{z \in \mathbb{C} : |\arg z| \leq \alpha\},$$

and S_α^0 is denoted as the interior of S_α. Let $\gamma > \alpha$ and denote by $H(S_\gamma^0)$ the space of all holomorphic functions on S_γ^0. Define

$$H_\infty(S_\gamma^0) = \left\{ b \in H(S_\gamma^0) : \|b\|_\infty < \infty \right\},$$

where $\|b\|_\infty = \sup\left\{ |b(z)| : z \in S_\gamma^0 \right\}$ and

$$\Psi(S_\gamma^0) = \left\{ \psi \in H(S_\gamma^0) : \exists\, s > 0 \text{ s.t. } |\psi(z)| \leq c|z|^s(1+|z|^{2s})^{-1} \right\}.$$

A densely defined closed operator L acting on a Banach space Y is called a *sectorial operator of type* α if for each $\gamma > \alpha$, $\sigma(L) \subset S_\gamma$ and

$$\sup\left\{ \|z(z\mathrm{Id} - L)^{-1}\|_{B(Y)} : z \notin S_\gamma \right\} < \infty,$$

where $\|\cdot\|_{B(Y)}$ denotes the operator norm and Id the identity operator.

Assume that L is a sectorial operator of type α. Let $0 \leq \alpha < \theta < \gamma < \pi$ and Γ be the boundary of S_θ oriented in the positive sense. For $\psi \in \Psi(S_\gamma^0)$, we define the operator $\psi(L)$ as

$$\psi(L) = \frac{1}{2\pi i}\int_\Gamma \psi(z)(z\mathrm{Id} - L)^{-1}\,\mathrm{d}z.$$

By Cauchy's theorem, this integral converges absolutely in $B(Y)$ and it is clear that the definition is independent of the choice of θ. For every $t > 0$, denote by $\psi_t(z) = \psi(tz)$ for $z \in S_\gamma^0$, we have $\psi_t \in \Psi(S_\gamma^0)$. Set

$$h(z) = \int_0^\infty \psi(tz)\,\frac{\mathrm{d}t}{t}, \quad z \in S_\gamma^0.$$

One gets that h is a constant on S_γ^0, hence by the convergence lemma (cf. [30, Lemma 2.1]),

$$h(L)x = \int_0^\infty \psi(tL)x\,\frac{\mathrm{d}t}{t} = cx, \quad x \in \mathscr{D}(L) \cap \mathrm{im}(L).$$

By applying a limiting argument, the above identity extends to $\overline{\mathrm{im}(L)}$. In particular, take $\psi(z) = z^2e^{-2z}$, then

$$\int_0^\infty -tLe^{-tL}(-tLe^{-tL})x\,\frac{\mathrm{d}t}{t} = \frac{1}{4}x, \quad x \in \overline{\mathrm{im}(L)}, \tag{1.2.1}$$

which will be useful later. We refer the reader to [45] for more information on functional calculus.

1.3 Martingales

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. Given a filtration which consists of a sequence of σ-algebras

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{F}$$

such that $\sigma(\cup_{k=0}^\infty \mathcal{F}_k) = \mathcal{F}$, for a random variable $f \in L^1(\Omega, \mathcal{F}, P)$ and $k \in \mathbb{N}$, we set

$$f_k = \mathbb{E}\left(f \mid \mathcal{F}_k\right), \quad d_k f = f_k - f_{k-1},$$

where we adopt the convention that $f_{-1} = 0$. We shall also denote f_k by $\mathbb{E}_k(f)$. A sequence $f = \{f_k\}_{k\geq 0}$ is called a martingale with respect to $\{\mathcal{F}_k\}_{k\geq 0}$ if $f_k = \mathbb{E}\left(f_{k+1} \mid \mathcal{F}_k\right)$ for every $k \geq 0$, and $\{d_k f\}_{k\geq 1}$ is called the martingale difference sequence of $f = \{f_k\}_{k\geq 0}$. To simplify notation, we write $L^p(\Omega)$ instead of $L^p(\Omega, \mathcal{F}, P)$, $0 < p < \infty$.

In particular, if $f \in L^1(\Omega)$, then $\{f_k\}_{k\geq 0}$ is a martingale induced by f, where $f_k = \mathbb{E}_k(f)$ for all $k \geq 0$. In this case, we identify f with $\{f_k\}_{k\geq 0}$.

Definition 1.3.1. (Regular filtration) A filtration is regular if there exists a constant $C > 0$ such that for all $k \geq 1, F_k \in \mathcal{F}_k$, there exists a $G_k \in \mathcal{F}_{k-1}$ satisfying

$$F_k \subset G_k, \quad P(G_k) \leq C \cdot P(F_k).$$

In addition, a martingale $f = \{f_k\}_{k\geq 0}$ with respect to such a regular filtration is called a regular martingale.

Remark 1.3.2. Suppose that for a positive random variable $f \in L^1(\Omega)$ the corresponding martingale $\{f_k\}_{k\geqslant 0}$ is regular. Then for any $k \geq 1$

$$f_k \leqslant A \cdot f_{k-1},$$

where $A > 0$ is a constant that depends only on the constant C of Definition 1.3.1.

See [62] for more information about regular filtrations and martingales.

Definition 1.3.3. Let $f = \{f_k\}_{k\geq 0}$ and $\{d_k f\}_{k\geq 1}$ be as above, we define:

1. the maximal function
$$f^* := \sup_{k\geqslant 0} |f_k|;$$
2. the square function
$$S(f) := \left(|f_0|^2 + \sum_{k=1}^{\infty} |d_k f|^2\right)^{\frac{1}{2}};$$
3. the conditional square function
$$s(f) := \left(|f_0|^2 + \sum_{k=1}^{\infty} \mathbb{E}_{k-1}|d_k f|^2\right)^{\frac{1}{2}}.$$

1.3.1 Martingale Hardy, BMO and Lipchitz spaces

There are several types of martingale Hardy spaces, which are defined in terms of maximal functions, square functions and conditional square functions. For the sake of convenience, we will assume from now on that $f_0 = 0$.

Definition 1.3.4. For $1 \leq p < \infty$, the martingale Hardy spaces $h^p(\Omega), H^p(\Omega), H^p_m(\Omega)$ are defined as follows

$$\begin{aligned} h^p(\Omega) &:= \{f \in L^1(\Omega) : \|f\|_{h^p(\Omega)} := \|s(f)\|_p < \infty\}, \\ H^p(\Omega) &:= \{f \in L^1(\Omega) : \|f\|_{H^p(\Omega)} := \|S(f)\|_p < \infty\}, \\ H^p_m(\Omega) &:= \{f \in L^1(\Omega) : \|f\|_{H^p_m(\Omega)} := \|f^*\|_p < \infty\}, \end{aligned}$$

respectively.

Remark 1.3.5. For general martingales, $H^p(\Omega) = H^p_m(\Omega)$ for $1 \leq p < \infty$ (see [22], [33], [94]). For regular martingale filtrations, one has $H^p(\Omega) = h^p(\Omega) = H^p_m(\Omega)$ for $1 \leq p < \infty$.

The following result is the atomic decomposition of $H^1(\Omega)$, which follows from the noncommutative result in [77]. In particular, it reveals the relationship between $H^1(\Omega)$ and $h^1(\Omega)$ and shows that $H^1(\Omega) \neq h^1(\Omega)$ for general martingales.

Theorem 1.3.6. *We have $H^1(\Omega) = h^1(\Omega) + h^1_d(\Omega)$, where h^1_d denotes the diagonal Hardy space of martingale differences*

$$h^1_d(\Omega) := \left\{ h \in L^1(\Omega) : \|h\|_{h^1_d(\Omega)} := \sum_{k=1}^{\infty} \|d_k h\|_1 < \infty \right\}.$$

We also provide a definition of the martingale Hardy spaces for $0 < p < 1$.

Definition 1.3.7. Let $0 < p < 1$, we first introduce the following quasi-norms. For $f \in L^2(\Omega)$,

$$\|f\|_{h^p(\Omega)} := \|s(f)\|_p; \quad \|f\|_{H^p(\Omega)} := \|S(f)\|_p; \quad \|f\|_{H^p_m(\Omega)} := \|s(f)\|_p.$$

We then define the martingale Hardy spaces $h^p(\Omega)$, $H^p(\Omega)$ and $H^p_m(\Omega)$ as the completion of $L^2(\Omega)$ under the corresponding quasi-norms $\|\cdot\|_{h^p(\Omega)}$, $\|\cdot\|_{H^p(\Omega)}$ and $\|\cdot\|_{H^p_m(\Omega)}$ respectively.

In the current setting where Ω has finite measure, the condition $\|s(f)\|_p < \infty$ for $0 < p < 1$ is automatically satisfied for $f \in L^2(\Omega)$, and similar observations hold true for $H^p(\Omega)$ and $H^p_m(\Omega)$. We provide the definitions for $h^p(\Omega), H^p(\Omega)$ and $H^p_m(\Omega)$ with $0 < p < 1$ in a unified fashion with the general setting described in Section 5.1.

Remark 1.3.8. For regular martingale filtrations, $H^p(\Omega) = h^p(\Omega) = H^p_m(\Omega)$ when $0 < p < 1$. In particular, for any function $f \in L^2(\Omega)$,

$$\|S(f)\|_p \approx_p \|s(f)\|_p \approx_p \|f^*\|_p.$$

See [93], [94] and [62] for the proof of the above inequality. Since $H^p(\Omega)$, $h^p(\Omega)$ and $H^p_m(\Omega)$ are defined as the completion of $L^2(\Omega)$ with respect to the appropriate quasi-norm respectively, one can conclude the equivalence between the aforementioned martingale Hardy spaces.

We shall now provide the definitions of the martingale BMO and bmo spaces, which are the dual spaces of $H^1(\Omega)$ and $h^1(\Omega)$, respectively (see Theorem 1.3.13 below).

Definition 1.3.9. Assume that $f \in L^2(\Omega)$. Consider the the following semi-norms

$$\|f\|_{BMO(\Omega)} := \sup_{n \geqslant 1} \|\mathbb{E}_n |f - f_{n-1}|^2\|_\infty^{\frac{1}{2}},$$

and

$$\|f\|_{bmo(\Omega)} := \sup_{n \geqslant 0} \|\mathbb{E}_n |f - f_n|^2\|_\infty^{\frac{1}{2}}.$$

It is clear that $\|f\|_{BMO(\Omega)} = 0$ if and only if $f = f_0$, and similarly $\|f\|_{bmo(\Omega)} = 0$ if and only if $f = f_0$.

We now define the martingale BMO and bmo spaces as follows:

$$BMO(\Omega) := \{f \in L^2(\Omega) : \|f\|_{BMO(\Omega)} < \infty, \mathbb{E}_0 f = 0\},$$

and

$$bmo(\Omega) := \{f \in L^2(\Omega) : \|f\|_{bmo(\Omega)} < \infty, \mathbb{E}_0 f = 0\}$$

We observe that $BMO(\Omega)$ and $bmo(\Omega)$ are both Banach spaces.

Remark 1.3.10. We note that another natural definition of $BMO(\Omega)$ as a normed space would be as a quotient space defined by

$$\{f \in L^2(\Omega) : \|f\|_{BMO(\Omega)} < \infty\}/\mathcal{K}_{\mathcal{F}_0},$$

where $\mathcal{K}_{\mathcal{F}_0} := \{\mathcal{F}_0 \text{ measurable functions}\}$.

For regular martingales, $BMO(\Omega) = bmo(\Omega)$. The following result is the so-called martingale John–Nirenberg inequality and can be found in [43].

Theorem 1.3.11. *There exists a sufficiently small constant $\kappa > 0$ such that for any $f \in BMO(\Omega)$ with $\|f\|_{BMO(\Omega)} \leq \kappa$, we have*

$$\mathbb{E}\left(e^{|f|}\right) \leqslant 1.$$

Remark 1.3.12. From the martingale John–Nirenberg inequality, we have for any $1 \leq p < \infty$,

$$\|f\|_{BMO(\Omega)} \approx_p \sup_{n \geqslant 1} \|\mathbb{E}_n |f - f_{n-1}|^p\|_\infty^{\frac{1}{p}}.$$

However, the above John–Nirenberg inequality fails for $bmo(\Omega)$ in the general martingale setting.

For the following duality theorem, see [43], [62], [94].

Theorem 1.3.13. $(H^1(\Omega))^* = BMO(\Omega)$ *and* $(h^1(\Omega))^* = bmo(\Omega)$.

The following proposition, which can be found in [28] and [43], is a consequence of Theorems 1.3.6 and 1.3.13 and it gives a description of the relationship between $BMO(\Omega)$ and $bmo(\Omega)$. In particular, it implies that $BMO(\Omega) \subsetneqq bmo(\Omega)$ for general martingales.

Proposition 1.3.14. *Assume f is a martingale BMO function. Then*

$$\|f\|_{BMO(\Omega)} \approx \|f\|_{bmo(\Omega)} + \sup_{k \geq 1} \|d_k f\|_\infty. \tag{1.3.1}$$

We shall now recall the definition of martingale Lipschitz spaces.

Definition 1.3.15. For $\alpha \in (0, \infty)$ and $q \in [1, \infty)$, the martingale Lipschitz space is defined by

$$\Lambda_q(\alpha) := \tag{1.3.2}$$
$$\left\{ f \in L^2(\Omega) : \|f\|_{\Lambda_q(\alpha)} = \sup_{n \geq 0} \sup_{A \in \mathcal{F}_n} P(A)^{-\frac{1}{q}-\alpha} \left(\int_A |f - f_n|^q dP \right)^{\frac{1}{q}} < \infty,\ \mathbb{E}_0 f = 0 \right\}.$$

As in case of BMO spaces, Lipschitz spaces can also be defined in terms of quotient spaces.

For $\alpha_p := \frac{1}{p} - 1 > 0$, one has $(h^p(\Omega))^* = \Lambda_2(\alpha_p)$. Furthermore, if the martingale filtration is regular, one has $\Lambda_1(\alpha_p) = \Lambda_2(\alpha_p)$; see [93].

1.3.2 Musielak–Orlicz-type spaces

We shall first recall some definitions and properties of Orlicz-type spaces and Musielak–Orlicz-type spaces. In what follows, $(\Omega, \mathcal{F}, \mu)$ denotes a σ-finite measure space.

A function $\Phi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is strictly positive on $(0,\infty)$, non-decreasing, unbounded and $\Phi(0) = 0$. A measurable function $\Psi : \Omega \times [0,\infty) \to [0,\infty)$ is called a Musielak–Orlicz function if for all $x \in \Omega$, $\Psi(x,\cdot)$ is an Orlicz function.

The Musielak–Orlicz-type space $L^\Psi(\Omega)$ is the set consisting of all measurable functions f on Ω such that

$$\int_\Omega \Psi(x, \lambda^{-1}|f(x)|)\,\mathrm{d}\mu < \infty$$

for some $\lambda > 0$. We equip $L^\Psi(\Omega)$ with the Luxembourg quasi-norm

$$\|f\|_{L^\Psi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \Psi(x, \lambda^{-1}|f(x)|)\,\mathrm{d}\mu \leqslant 1\right\}, \quad f \in L^\Psi(\Omega).$$

Let $p \in \mathbb{R}$. A Musielak–Orlicz function is said to be of uniformly lower type (respectively, upper type) p if there exists a positive constant C such that

$$\Psi(x, st) \leqslant Cs^p \Psi(x,t)$$

for all $x \in \Omega$, $t \geqslant 0$ and $s \in (0,1)$ (respectively, $s \in [1,\infty)$). In particular, if Ψ is of uniformly lower type p with $0 < p < 1$ and of uniformly upper type 1 then

$$\Psi(x, ct) \approx_c \Psi(x,t) \quad \text{for all } c > 0. \tag{1.3.3}$$

In the sequel, $\Psi(x,t)$ is always assumed to be of uniformly lower type p with $0 < p < 1$ and of uniformly upper type 1, and to be continuous in the t variable. For more information on Musielak–Orlicz spaces, we refer the reader to [16] and [105].

We end this subsection with the definition of martingale Musielak–Orlicz Hardy spaces and the generalized Hölder inequality.

Definition 1.3.16. For $f \in L^1(\Omega)$, we define the quasi-norm of f with respect to Ψ, described in Section 1.3.2, by

$$\|f\|_{H^\Psi(\Omega)} := \|S(f)\|_{L^\Psi(\Omega)}.$$

The martingale Musielak–Orlicz Hardy space $H^\Psi(\Omega)$ is defined as the completion of the space

$$H_0^\Psi(\Omega) := \{f \in L^1(\Omega) : \|f\|_{H^\Psi(\Omega)} < \infty\}$$

under the quasi-norm $\|\cdot\|_{H^\Psi(\Omega)}$.

Moreover, If Ψ is replaced by an Orlicz function Φ, the corresponding Hardy–Orlicz space $H^\Phi(\Omega)$ is defined in an analogous way.

To obtain the generalized Hölder inequality, we shall introduce a particular Orlicz space $L^\Phi(\Omega)$, where

$$\Phi(t) := \frac{t}{\log(e+t)}, \quad t \geq 0. \tag{1.3.4}$$

Note that Φ is an Orlicz function of uniformly lower type p ($0 < p < 1$) and upper type 1, which guarantees that the vector space $L^\Phi(\Omega)$ is a quasi-normed space. Note that $L^1(\Omega) \subset L^\Phi(\Omega)$.

Remark 1.3.17. Let Φ be as in (1.3.4). It follows from [69] that for any $f \in L^1(\Omega)$ such that the induced martingale $\{f_k\}_{k\geq 0}$ is regular, one has

$$\|f\|_{H^\Phi(\Omega)} \approx \|f\|_{H^\Phi_m(\Omega)} \approx \|f\|_{h^\Phi(\Omega)},$$

where

$$\|f\|_{H^\Phi_m(\Omega)} := \|f^*\|_{L^\Phi(\Omega)} \quad \text{and} \quad \|f\|_{h^\Phi(\Omega)} := \|s(f)\|_{L^\Phi(\Omega)}.$$

Hence,

$$H^\Phi_0(\Omega) = H^\Phi_{m,0}(\Omega) = h^\Phi_0(\Omega),$$

where $H^\Phi_{m,0}(\Omega) := \{f \in L^1(\Omega) : \|f\|_{H^\Phi_m(\Omega)} < \infty\}$ and $h^\Phi_0(\Omega) := \{f \in L^1(\Omega) : \|f\|_{h^\Phi(\Omega)} < \infty\}$. One thus deduces that

$$H^\Phi(\Omega) = H^\Phi_m(\Omega) = h^\Phi(\Omega),$$

where $H^\Phi_m(\Omega)$ denotes the completion of $H^\Phi_{m,0}(\Omega)$ with respect to $\|\cdot\|_{H^\Phi_m(\Omega)}$ and $h^\Phi(\Omega)$ denotes the completion of $h^\Phi_0(\Omega)$ with respect to $\|\cdot\|_{h^\Phi(\Omega)}$.

The following lemma is a variant of [17, Proposition 2.1] in the martingale setting.

Lemma 1.3.18. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $f \in L^1(\Omega)$ and $g \in BMO(\Omega)$. Then $f \cdot g \in L^\Phi(\Omega)$ and*

$$\|f \cdot g\|_{L^\Phi(\Omega)} \lesssim \|f\|_1 \|g\|_{BMO(\Omega)}. \tag{1.3.5}$$

Proof. The proof is similar to the proof of the corresponding Euclidean result and we shall only outline it here for the convenience of the reader. By [17, Lemma 2.1], one has

$$\frac{st}{M + \log(e + st)} \leq e^{t-M} + s. \tag{1.3.6}$$

for all $M \geq 0, s \geq 0, t \geq 0$.

When $\|f\|_1 = 0$ or $\|g\|_{BMO(\Omega)} = 0$, (1.3.5) trivially holds. Assume $g \in BMO(\Omega)$ with $\|g\|_{BMO(\Omega)} > 0$ and $f \in L^1(\Omega)$ with $\|f\|_1 > 0$. Let κ be the constant in Theorem 1.3.11, $M = 0$, $t = \frac{\kappa|g(x)|}{\|g\|_{BMO(\Omega)}}$ and $s = \frac{|f(x)|}{\|f\|_1}$. Then by Theorem 1.3.11 and (1.3.6), we have

$$\int_\Omega \Phi\left(\frac{|f(x) \cdot g(x)|}{\kappa^{-1}\|f\|_1 \|g\|_{BMO(\Omega)}}\right) dP \leq \int_\Omega e^{\frac{\kappa|g(x)|}{\|g\|_{BMO(\Omega)}}} dP + \left\|\frac{f}{\|f\|_1}\right\|_1 \leq 2. \tag{1.3.7}$$

Hence, from (1.3.3) we conclude

$$\|f \cdot g\|_{L^\Phi(\Omega)} \lesssim \kappa^{-1}\|f\|_1 \|g\|_{BMO(\Omega)},$$

which completes the proof of the lemma. □

We shall refer to (1.3.5) as the generalized Hölder inequality.

1.4 Geometric properties of Banach space

We define the modulus of convexity and modulus of smoothness of a Banach space X by (see, e.g. [82])

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{a-b}{2}\right\|_X : a, b \in X, \|a\|_X = \|B\|_X = 1, \|a-b\|_X = \varepsilon\right\}, \quad 0 < \varepsilon < 2.$$

$$\rho_X(t) = \sup\left\{\frac{\|a+tb\|_X + \|a-tb\|_X}{2} : a, b \in X, \|a\|_X = \|b\|_X = 1\right\}, \quad t > 0.$$

X is called *q-uniform convex* if $\delta_X(\varepsilon) \geq c\varepsilon^q$ for some positive constants c and q; X is called *q-uniform smooth* if $\rho_X(t) \leq ct^q$ for some positive constants c and q. We refer the reader to [84] for more information.

These geometric properties of Banach space can be characterized by martingale inequalities. Recall that X is said to be of *martingale cotype* q (with $2 \leq q < \infty$) if there exists a positive constant c such that for every finite X-valued L^q-martingale $(f_n)_{n\geq 0}$, the following inequality holds

$$\sum_{n\geq 1} \mathbb{E}\|f_n - f_{n-1}\|_X^q \leq c^q \sup_{n\geq 0} \mathbb{E}\|f_n\|_X^q,$$

where $\mathbb{E}$ denotes the underlying expectation; and the least constant c is called the martingale cotype constant, denoted by $\mathsf{M}_{c,q}(X)$. While X is said to be of *martingale type* q (with $1 < q \leq 2$) if the reverse inequalities holds with c^{-1} in place of c and the corresponding martingale type constant is denoted by $\mathsf{M}_{t,q}(X)$. Pisier's famous renorming theorem shows that X is of martingale cotype (respectively, type) q if and only if X admits an equivalent q-uniform convex (respectively, smooth) norm. We refer the reader to [82, 83, 84] for more details.

By [63], the one sided Littlewood-Paley-Stein inequality can also describe the underlying Banach space X. We define X to be of *Lusin cotype* q (with $2 \leq q < \infty$) related to $\left\{e^{-t\sqrt{\Delta}}\right\}_{t>0}$ if there exists a positive constant C_p such that the following inequality holds (for some $1 < p < \infty$),

$$\left\|\left(\int_0^\infty \left\|t\sqrt{\Delta}e^{-t\sqrt{\Delta}}(f)(x)\right\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\right\|_p \leq C_p\|f\|_{L^p(\mathbb{R}^d;X)}, \quad \forall\, f \in L^p(\mathbb{R}^d;X),$$

X is of *Lusin type* q(with $1 < q \leq 2$) if the reverse inequality holds. In [63], Martínez *et al* showed that X is of Lusin cotype (respectively, type) q if and only if X is q-uniform convex (respectively, smooth).

1.5 Homogeneous spaces

In this section, we introduce some fundamental concepts and important theorems for homogeneous spaces, which can be found in [27]. We also refer the reader to the recent survey [76]. We begin with the definition of homogeneous spaces. Recall that d is a quasi-metric on Ω if

(1) $d(x,y) \geq 0$, $\forall\, x, y \in \Omega$, and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$, $\forall\, x, y \in \Omega$;

(3) there exists a constant $A_0 \geq 1$ such that

$$d(x,y) \leqslant A_0(d(x,z) + d(z,y)), \quad \forall\, x, y, z \in \Omega. \tag{1.5.1}$$

Denote by $B(x,r) := \{y \in \Omega : d(y,x) < r\}$ the open ball centered at x with radius r. In this paper, all quasi-metric spaces are assumed to have the doubling property: there exists a positive integer $A_1 \in \mathbb{N}$ such that for every $x \in \Omega$ and for every $r > 0$, the ball $B(x,r)$ can be covered by at most A_1 balls $B(x_i, \frac{r}{2})$ for some $x_i \in \Omega$.

Definition 1.5.1. A σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ equipped with a quasi-metric d is called a homogeneous space if μ is a Borel measure of homogeneous type:

$$0 < \mu\left(B(x,2r)\right) \leqslant 2^{C_\mu}\mu\left(B(x,r)\right) < \infty, \quad \forall\, x \in \Omega, r > 0, \tag{1.5.2}$$

where the constant C_μ is independent of x and r.

In [27], Coifman and Weiss defined Hardy spaces on homogeneous spaces by regarding their elements as linear functionals acting on some appropriate quasi-normed spaces. In order to state the definition of Coifman and Weiss, we need to introduce the notions of atoms, BMO and Lipschitz spaces on homogeneous spaces.

Definition 1.5.2. If $0 < p \leqslant 1 \leqslant q \leq \infty$ and $p < q$, we say that a measurable function a is a (p,q)-atom if

(1) $\operatorname{supp}(a) \subset B$ where B is a ball;

(2) $\|a\|_q \leqslant (\mu(B))^{\frac{1}{q}-\frac{1}{p}}$;

(3) $\int_\Omega a \,\mathrm{d}\mu = \int_B a \,\mathrm{d}\mu = 0$.

Definition 1.5.3. For a locally integrable function f, we define the BMO semi-norm of f by

$$\|f\|_{BMO(\mu)} := \sup_B \frac{1}{\mu(B)} \int_B |f - f_B| \,\mathrm{d}\mu,$$

where $f_B := \frac{1}{\mu(B)} \int_B f \,\mathrm{d}\mu$ and the supremum runs over all balls B.

In order to consider normed spaces, one needs to quotient out the kernel space (the functions f for which the semi-norm $\|f\|_{BMO(\mu)}$ equal to 0) which is the space of constant functions and the BMO space is defined by

$$BMO(\mu) := \{f \in L^1_{\mathrm{loc}}(\mu) : \|f\|_{BMO} < \infty\}$$

as a quotient space modulo constant functions.

Definition 1.5.4. For $\alpha > 0$, a locally integrable function l is called a Lipschitz function if

$$|l(x) - l(y)| \leqslant C_\alpha \left(\mu(B)\right)^\alpha \text{ for any } x, y \in \Omega \text{ and any ball } B \text{ containing } x, y. \tag{1.5.3}$$

Moreover, we define the semi-norm

$$\|l\|_{\mathcal{L}_\alpha(\mu)} := \inf\{C_\alpha : |l(x) - l(y)| \leqslant C_\alpha \left(\mu(B)\right)^\alpha, \ \forall\, x, y \in B\}, \tag{1.5.4}$$

where the infimum runs over all balls B. We notice that the kernel space is indeed the space of constant functions. Similarly as the BMO space, we define the Lipschitz space as a quotient space modulo constant functions:

$$\mathcal{L}_\alpha(\mu) := \{l \in L^1_{\mathrm{loc}}(\mu) : \|l\|_{\mathcal{L}_\alpha(\mu)} < \infty\}.$$

It is well-known that each BMO function can be regarded as a continuous linear functional on the vector space generated by finite linear combinations of $(1, q)$-atoms for $1 < q \leq \infty$ (cf. [27]), which gives rise to the following definition of the atomic Hardy space.

Definition 1.5.5. We define the atomic Hardy space $H^{1,q}_{\mathrm{at}}(\mu)$ $(1 < q \leq \infty)$ as follows:

$$H^{1,q}_{\mathrm{at}}(\mu) := \left\{ f \in (BMO(\mu))^* : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a } (1,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j| < \infty \right\} \tag{1.5.5}$$

endowed with the norm

$$\|f\|_{H^{1,q}_{\mathrm{at}}(\mu)} := \inf\left\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a } (1,q)\text{-atom} \right\}.$$

Similarly, each Lipschitz function $l \in \mathcal{L}_{\alpha_p}(\mu)$ can be also regarded as a continuous linear functional of the vector space generated by finite linear combinations of (p, q)-atoms where $0 < p < 1 \leq q \leq \infty$ and $\alpha_p = 1/p - 1$ (cf. [27]).

Definition 1.5.6. We define the atomic Hardy spaces $H^{p,q}_{\mathrm{at}}(\mu)$ as follows:

$$H^{p,q}_{\mathrm{at}}(\mu) := \left\{ f \in \left(\mathcal{L}_{\alpha_p}(\mu)\right)^* : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a } (p,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j|^p < \infty \right\} \tag{1.5.6}$$

endowed with the quasi-norm

$$\|f\|_{H^{p,q}_{\mathrm{at}}(\mu)} := \inf\left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a } (p,q)\text{-atom} \right\}.$$

Although the Hardy spaces vary with p and q according to the above definitions, the following theorem, which can be found in [27], shows that the Hardy spaces actually depend only on p. Consequently, this enables us to define the Hardy spaces $H^p_{\mathrm{at}}(\mu)$ for $0 < p \leqslant 1$ to be any one of the spaces $H^{p,q}_{\mathrm{at}}(\mu)$ for $0 < p < q \leqslant \infty, 1 \leqslant q \leq \infty$.

Theorem 1.5.7. *$H^{p,q}_{\mathrm{at}}(\mu) = H^{p,\infty}_{\mathrm{at}}(\mu)$ whenever $0 < p \leq 1 \leqslant q \leqslant \infty$ and $p < q$.*

We end this section with the following duality theorem obtained in [27].

Theorem 1.5.8. *$\left(H^1_{\mathrm{at}}(\mu)\right)^* = BMO(\mu)$, and $\left(H^p_{\mathrm{at}}(\mu)\right)^* = \mathcal{L}_{\alpha_p}(\mu)$ for $0 < p < 1$.*

The proofs of Theorem 1.5.7 and Theorem 1.5.8 that appeared in [27] are very technical. In the following sections, by employing martingale methods, we give very neat proofs of these facts. Our approach is based on the fact that $H^p_{\mathrm{at}}(\mu)$ for $0 < p \leq 1$ is the finite sum of several dyadic martingale Hardy spaces.

Chapter 2

Optimal Lusin type constant as p tending to ∞

This chapter is devoted to the proofs of the main theorems mentioned in Chapter 0, Section I. We prove them separately in the subsequent three sections, addressing the scalar-valued, vector-valued, and noncommutative cases. Our approach relies on the properties of the operator associated with a martingale filtration, as outlined in, for instance, (I.3).

2.1 Proofs of Theorem I.2, Theorem I.3 and Corollary I.5

We will need the following elementary lemmas.

Lemma 2.1.1. *If S is a positive injective continuous operator on a Hilbert space $\mathcal{H}$, then for $x \in \mathcal{H}$*

$$\lim_{t\to 0^+} \|S^t(x) - x\| = 0.$$

Proof. By the spectral decomposition, there exists a resolution of the identity $\{E_\lambda\}_{0\leq\lambda\leq\|S\|}$ such that

$$S = \int_0^{\|S\|} \lambda \mathrm{d}E_\lambda.$$

Since S is injective, $E_0 = 0$. Hence for $x \in \mathcal{H}$

$$x = \int_{(0,\|S\|]} \mathrm{d}E_\lambda(x).$$

Thus

$$\begin{aligned}\|S^t(x) - x\|^2 &= \left\| \int_{(0,\|S\|]} (\lambda^t - 1)\mathrm{d}E_\lambda(x) \right\|^2 \\ &= \int_{(0,\|S\|]} |\lambda^t - 1|^2 \mathrm{d}\langle E_\lambda(x), x\rangle.\end{aligned}$$

By the Lebesgue dominated convergence theorem and the fact that $|\lambda^t - 1| \to 0$ as $t \to 0^+$ for $\lambda \neq 0$, we deduce the desired limit. □

Lemma 2.1.2 (Gershgorin circle theorem)**.** *Let $A = (a_{i,j})_{1\le i,j\le n}$ be a complex $n \times n$ matrix. If λ is an eigenvalue of A, then*

$$\lambda \in \bigcup_{k=1}^{n} D\left(a_{kk}, \sum_{j=1, j\neq k}^{n} |a_{kj}|\right),$$

where $D\left(a_{kk}, \sum\limits_{j=1,j\neq k}^{n} |a_{kj}|\right)$ is the closed disc of the complex plane centered at a_{kk} with radius $\sum\limits_{j=1,j\neq k}^{n} |a_{kj}|$.

Proof. Assume that λ is an eigenvalue of A. Then there exists a nonzero vector $x = (x_1, \cdots, x_n)$ such that $Ax = \lambda x$. Choose i such that $|x_i| = \max_{1\le j\le n}|x_j| > 0$. Since

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i,$$

we obtain by the triangle inequality

$$|\lambda - a_{ii}| = \left|\sum_{j=1, j\neq i}^{n} a_{ij}\frac{x_j}{x_i}\right| \le \sum_{j=1, j\neq i}^{n} |a_{ij}|,$$

which implies the desired result. □

Proof of Theorem I.2. It is easy to prove the first part of Theorem I.2. Indeed, we can rewrite T as

$$T = \sum_{n=1}^{\infty}(1 - a_{n-1})d\mathbb{E}_n.$$

Since

$$\begin{aligned} T &= \lim_{n\to\infty}\sum_{k=1}^{n}(a_k - a_{k-1})E_k = \lim_{n\to\infty}\sum_{k=1}^{n}(a_k - a_{k-1})\sum_{i=1}^{k} d\mathbb{E}_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} d\mathbb{E}_i \sum_{k=i}^{n}(a_k - a_{k-1}) = \lim_{n\to\infty}\sum_{i=1}^{n}(a_n - a_{i-1})d\mathbb{E}_i \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n}(1 - a_{i-1})d\mathbb{E}_i - (1 - a_n)\mathbb{E}_n\right] \\ &= \sum_{n=1}^{\infty}(1 - a_{n-1})d\mathbb{E}_n. \end{aligned}$$

Then by the continuous functional calculus and $d\mathbb{E}_p d\mathbb{E}_q = 0$ for $p \neq q$, we obtain

$$T^t = \sum_{n=1}^{\infty}(1 - a_{n-1})^t d\mathbb{E}_n, \quad t > 0. \tag{2.1.1}$$

Hence we conclude that T^t is a positive contraction for any $t > 0$ on $L^p(\Omega, \mathcal{F}, \mu)$ $(1 \le p \le \infty)$. Actually, for $t > 0$

$$T^t = \sum_{n=1}^{\infty}\left[(1 - a_{n-1})^t - (1 - a_n)^t\right]\mathbb{E}_n. \tag{2.1.2}$$

It suffices to show that $\{T^t\}_{t>0}$ is a symmetric diffusion semigroup. T^t satisfies the conditions (a) and (d) (see, for instance, Definition I.1) since T^t is a positive contraction on $L^p(\Omega, \mathcal{F}, \mu)$ $(1 \leq p \leq \infty)$. The condition (b) is obvious as T^t is generated by T via continuous functional calculus. It remains to prove that T^t satisfies the conditions (c), (e), (f).

For the condition (e), T^t is selfadjoint on $L^2(\Omega, \mathcal{F}, \mu)$ since all conditional expectations are selfadjoint. As for the condition (f),

$$T^t(1) = \sum_{n=1}^{\infty}(1 - a_{n-1})^t d\mathbb{E}_n(1) = (1 - a_0)^t d\mathbb{E}_1(1) = 1.$$

Note that the conditional expectations are projections on $L^2(\Omega, \mathcal{F}, \mu)$.

In order to show that the semigroup $\{T^t\}_{t>0}$ satisfies the condition (c), it suffices to prove that T is injective on $L^2(\Omega, \mathcal{F}, \mu)$ by Lemma 2.1.1. This is because for $f \in L^2(\Omega, \mathcal{F}, \mu)$

$$\|T(f)\|_2^2 = \sum_{n=1}^{\infty}(1 - a_{n-1})^2 \|d\mathbb{E}_n(f)\|_2^2 \tag{2.1.3}$$

and

$$\|f\|_2^2 = \lim_{n\to\infty} \|\mathbb{E}_n(f)\|_2^2 = \sum_{n=1}^{\infty} \|d\mathbb{E}_n(f)\|_2^2. \tag{2.1.4}$$

Thus we obtain that $\|T(f)\|_2 = 0$ if and only if $\|f\|_2 = 0$, which implies T is injective on $L^2(\Omega, \mathcal{F}, \mu)$. □

Remark 2.1.3. From (2.1.3) and (2.1.4), we can also obtain

$$\mathbf{F}(f) = \mathbb{E}_1(f), \quad f \in L^p(\Omega, \mathcal{F}, \mu),\ 1 \leq p \leq \infty.$$

Note that

$$S(f - \mathbf{F}(f)) = \left(\sum_{n=2}^{\infty} |d\mathbb{E}_n(f)|^2\right)^{\frac{1}{2}}.$$

Proof of Theorem I.3. We will calculate the derivative $\partial_t T^t(f)$. To simplify our presentation, we assume that $f \in L^p(\Omega, \mathcal{F}_n, \mu)$ for a fixed $n \geq 2$ since f can be approximated by a sequence of f_n on $L^p(\Omega, \mathcal{F}_n, \mu)(1 \leq p < \infty, n \in \mathbb{N}^*)$.

Since $f \in L^p(\Omega, \mathcal{F}_n, \mu)$, $d\mathbb{E}_m(f) = 0$ if $m > n$. Then for $t > 0$

$$T^t(f) = \sum_{k=1}^{n}(1 - a_{k-1})^t d\mathbb{E}_k(f),$$

which implies

$$\partial_t T^t(f) = \sum_{k=1}^{n}(1 - a_{k-1})^t \ln(1 - a_{k-1}) d\mathbb{E}_k(f) = \sum_{k=2}^{n}(1 - a_{k-1})^t \ln(1 - a_{k-1}) d\mathbb{E}_k(f).$$

Hence

$$\begin{aligned}
&\int_0^\infty \left|t\partial_t T^t(f)\right|^2 \frac{\mathrm{d}t}{t} = \int_0^\infty t\left|\partial_t T^t(f)\right|^2 \mathrm{d}t \\
&= \int_0^\infty t\left|\sum_{k=2}^n (1-a_{k-1})^t \ln(1-a_{k-1}) d\mathbb{E}_k(f)\right|^2 \mathrm{d}t \\
&= \sum_{i,j=2}^n \int_0^\infty t(1-a_{i-1})^t(1-a_{j-1})^t \ln(1-a_{i-1})\ln(1-a_{j-1}) d\mathbb{E}_i(f) d\mathbb{E}_j(f)\mathrm{d}t \\
&= \sum_{i,j=2}^n \ln(1-a_{i-1})\ln(1-a_{j-1}) d\mathbb{E}_i(f) d\mathbb{E}_j(f) \int_0^\infty t(1-a_{i-1})^t(1-a_{j-1})^t \mathrm{d}t \\
&= \sum_{i,j=2}^n \frac{\ln(1-a_{i-1})\ln(1-a_{j-1})}{\ln^2\left[(1-a_{i-1})(1-a_{j-1})\right]} d\mathbb{E}_i(f) d\mathbb{E}_j(f). \qquad (2.1.5)
\end{aligned}$$

We write $y_i = -\ln(1-a_{i-1})$ $(i \geq 2)$. Then $\{y_i\}_{i\geq 2}$ is a strictly increasing positive sequence which tends to infinity as $i \to \infty$. Let $B = (b_{ij})_{1\leq i,j\leq n-1}$ be the real symmetric matrix given by

$$b_{ij} = \frac{y_{i+1}y_{j+1}}{(y_{i+1}+y_{j+1})^2}.$$

We need to determine the lower and upper bounds of the eigenvalues of B.

Let λ be an eigenvalue of B. By Lemma 2.1.2 we have

$$\lambda \in \bigcup_{k=1}^{n-1} D\left(b_{kk}, \sum_{j=1, j\neq k}^{n-1} |b_{kj}|\right).$$

However $b_{kk} = \frac{1}{4}$ and $y_i = 16^i$ for $i \geq 2$, thus for $k \neq j$

$$b_{kj} = \frac{y_{k+1}y_{j+1}}{(y_{k+1}+y_{j+1})^2} = \left(\frac{y_{k+1}}{y_{j+1}} + \frac{y_{j+1}}{y_{k+1}} + 2\right)^{-1} \leq 16^{-|k-j|}.$$

Thus for every $1 \leq k \leq n-1$,

$$\sum_{j=1,j\neq k}^{n-1} |b_{kj}| \leq \sum_{j=1,j\neq k}^{n-1} 16^{-|k-j|} \leq 2\sum_{i=1}^{\infty} 16^{-i} = \frac{2}{15}.$$

We then get

$$\frac{7}{60} \leq \lambda \leq \frac{23}{60}.$$

Thus B satisfies (I denoting the identity matrix)

$$\frac{7}{60} I \leq B \leq \frac{23}{60} I.$$

Therefore by (2.1.5),

$$\frac{7}{60}\sum_{k=2}^n |d\mathbb{E}_n(f)|^2 \leq \int_0^\infty \left|t\partial_t T^t(f)\right|^2 \frac{\mathrm{d}t}{t} \leq \frac{23}{60}\sum_{k=2}^n |d\mathbb{E}_n(f)|^2$$

which implies

$$\sqrt{\frac{7}{60}}\, S(f-\mathbf{F}(f)) \leq G_{2,T}(f) \leq \sqrt{\frac{23}{60}}\, S(f-\mathbf{F}(f)). \qquad (2.1.6)$$

□

Remark 2.1.4. By virtue of the special semigroups induced by martingales in Theorem I.2 and with the help of Theorem I.3, we get a lower bound of optimal orders of best constants of Littlewood-Paley-Stein inequality for symmetric diffusion semigroups via inequalities of martingale square functions. However, to show that this lower bound given by martingale inequalities is exactly the optimal order of best constants for Littlewood-Paley-Stein inequality is much more delicate. For more details, see [102].

Proof of Corollary I.5. Firstly, we calculate P_t for $t > 0$ as follows

$$\begin{aligned} P_t &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} T^{\frac{t^2}{4s}} \, \mathrm{d}s \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \left(\sum_{n=1}^\infty (1-a_{n-1})^{\frac{t^2}{4s}} d\mathbb{E}_n \right) \mathrm{d}s \\ &= \sum_{n=1}^\infty \left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} (1-a_{n-1})^{\frac{t^2}{4s}} \, \mathrm{d}s \right) d\mathbb{E}_n \\ &= \sum_{n=1}^\infty e^{-t\sqrt{-\ln(1-a_{n-1})}} d\mathbb{E}_n \\ &= d\mathbb{E}_1 + \sum_{n=2}^\infty e^{-16^n t} d\mathbb{E}_n. \end{aligned}$$

Therefore, by Theorem I.3, the desired inequalities are deduced. □

2.2 Proof of Theorem I.8

The proof of Theorem I.8 is more complicated than the one of Theorem I.3. In contrast to Theorem I.3, the choice of the sequence $\{a_n\}_{n\geq 0}$ is not explicit for Theorem I.8.

To make our proof easier and neater, in the sequel we denote $b_n = -\ln(1-a_n)$ for convenience. Then $b_0 = 0$ and $\lim\limits_{n\to\infty} b_n = +\infty$. Let $1 \leq p, q < \infty$. We can also write $G_{q,T}(f)$ in the following form

$$G_{q,T}(f) = \left(\int_0^\infty \left\| \sum_{k=1}^\infty t b_k e^{-t b_k} d\mathbb{E}_{k+1}(f) \right\|_X^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \quad \forall f \in L^p(\Omega; X). \tag{2.2.1}$$

Hence to find a suitable sequence $\{a_n\}_{n\geq 1}$ that satisfies Theorem I.8, we should search for the existence of the sequence $\{b_k\}_{k\geq 0}$.

Fixing M sufficiently large, we denote

$$G_{q,T,M}(f) = \left(\int_0^M \left\| \sum_{k=1}^\infty t b_k e^{-t b_k} d\mathbb{E}_{k+1}(f) \right\|_X^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

Now we are about to prove

$$G_{q,T}(f) \approx_{p,q} S_q\left(f - \mathrm{F}(f)\right), \quad \forall f \in L^p(\Omega; X). \tag{2.2.2}$$

Indeed, we have a stronger result that the constants in (2.2.2) are universal.

Proof of Theorem I.8. Take a strictly decreasing sequence $\{t_n\}_{n\geq 0}$ with $\lim\limits_{n\to\infty} t_n = 0$ and $t_0 = M$, then

$$G_{q,T,M}(f) = \left(\sum_{n=1}^{\infty}\int_{t_n}^{t_{n-1}} \left\|\sum_{k=1}^{\infty} tb_k e^{-tb_k} d\mathbb{E}_{k+1}(f)\right\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

For $n \geq 1$ let

$$R_n = \left(\int_{t_n}^{t_{n-1}} \left\|\sum_{k=1}^{\infty} tb_k e^{-tb_k} d\mathbb{E}_{k+1}(f)\right\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}},$$

and for $k \geq 1$

$$R_{n,k} = \left(\int_{t_n}^{t_{n-1}} \left\|tb_k e^{-tb_k} d\mathbb{E}_{k+1}(f)\right\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

Considering the interval (t_n, t_{n-1}) equipped with the measure $\frac{\mathrm{d}t}{t}$, by the triangle inequality, we get

$$\begin{aligned} R_n &= \left\|\sum_{k=1}^{\infty} tb_k e^{-tb_k} d\mathbb{E}_{k+1}(f)\right\|_{L^q((t_n,t_{n-1}),\frac{\mathrm{d}t}{t};X)} \\ &\geq \left|\left\|tb_n e^{-tb_n} d\mathbb{E}_{n+1}(f)\right\|_{L^q((t_n,t_{n-1}),\frac{\mathrm{d}t}{t};X)} - \left(\sum_{k\neq n,k\geq 1} \left\|tb_n e^{-tb_n} d\mathbb{E}_{n+1}(f)\right\|_{L^q((t_n,t_{n-1}),\frac{\mathrm{d}t}{t};X)}\right)\right| \\ &= \left|R_{n,n} - \left(\sum_{k\neq n,k\geq 1} R_{n,k}\right)\right|. \end{aligned}$$

Thus we deduce

$$G_{q,M}^T(f) = \left(\sum_{n=1}^{\infty} R_n^q\right)^{\frac{1}{q}} \geq \left(\sum_{n=1}^{\infty}\left|R_{n,n} - \left(\sum_{k\neq n,k\geq 1} R_{n,k}\right)\right|^q\right)^{\frac{1}{q}}.$$

To get the desired control, we are concerned with making $\sum\limits_{k\neq n,k\geq 1} R_{n,k}$ sufficiently small.

We take l_k and m_k $(k \geq 1)$ such that

$$\begin{aligned} \int_0^{l_k} t^{q-1}e^{-t}\mathrm{d}t &= \frac{1}{2^{q^2(k+2)}}\Gamma(q), \\ \int_{m_k}^{\infty} t^{q-1}e^{-t}\mathrm{d}t &= \frac{1}{2^{q^2(k+2)}}\Gamma(q), \end{aligned}$$

where $\Gamma(q) = \int_0^{\infty} t^{q-1}e^{-t}\mathrm{d}t$, as $q \geq 1$. Denote $l_0 = m_0 = 1$. It is easy to verify that $l_k < m_k$ $(k \geq 1)$, $\lim\limits_{k\to\infty} l_k = 0$ and $\lim\limits_{k\to\infty} m_k = +\infty$.

Then $\{t_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are just defined by the following formulas

$$\begin{aligned} &t_0 = M, \quad b_0 = 0, \\ &t_k b_k q = l_k, \quad t_{k-1} b_k q = m_k, \qquad k \geq 1. \end{aligned}$$

By elementary calculations, it follows that

$$t_k = M \prod_{j=0}^{k} \frac{l_j}{m_j} \qquad \text{for } k \geq 0,$$
$$b_k = \frac{1}{Mq} \prod_{j=1}^{k} \frac{m_j}{l_{j-1}} \quad \text{for } k \geq 1 \text{ and } b_0 = 0.$$

It is obvious that $\{t_k\}_{k\geq 0}$ is decreasing, $\{b_k\}_{k\geq 0}$ is increasing, and

$$\lim_{k\to\infty} t_k = 0, \quad \lim_{k\to\infty} b_k = \infty.$$

Then

$$\begin{aligned}
|R_{n,n}|^q &= \int_{t_n}^{t_{n-1}} \left\| t b_n e^{-t b_n} d\mathbb{E}_{n+1}(f) \right\|_X^q \frac{\mathrm{d}t}{t} \\
&= \int_{t_n b_n q}^{t_{n-1} b_n q} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{n+1}(f)\|_X^q}{q^q} \\
&= \int_{l_n}^{m_n} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{n+1}(f)\|_X^q}{q^q} \\
&= \left(1 - \frac{2}{2^{q^2(n+2)}}\right) \frac{\Gamma(q)}{q^q} \|d\mathbb{E}_{n+1}(f)\|_X^q .
\end{aligned}$$

On the other hand, for $1 \leq k < n$,

$$\begin{aligned}
|R_{n,k}|^q &= \int_{t_n b_k q}^{t_{n-1} b_k q} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{k+1}(f)\|_X^q}{q^q} \\
&\leq \int_{0}^{t_{n-1} b_{n-1} q} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{k+1}(f)\|_X^q}{q^q} \\
&= \int_{0}^{l_{n-1}} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{k+1}(f)\|_X^q}{q^q} \\
&= \frac{\Gamma(q)}{2^{q^2(n+1)} q^q} \|d\mathbb{E}_{k+1}(f)\|_X^q .
\end{aligned}$$

Similarly, for $k > n$

$$\begin{aligned}
|R_{n,k}|^q &\leq \int_{m_k}^{\infty} t^{q-1} e^{-t} \mathrm{d}t \frac{\|d\mathbb{E}_{k+1}(f)\|_X^q}{q^q} \\
&= \frac{\Gamma(q)}{2^{q^2(k+2)} q^q} \|d\mathbb{E}_{k+1}(f)\|_X^q .
\end{aligned}$$

Let $r_n, r_{n,k} > 0$ be defined as follows

$$\begin{aligned}
|r_n|^q &= \left(1 - \frac{2}{2^{q^2(n+2)}}\right) \frac{\Gamma(q)}{q^q} \|d\mathbb{E}_{n+1}(f)\|_X^q , \\
|r_{n,k}|^q &= \frac{\Gamma(q)}{2^{q^2(n+1)} q^q} \|d\mathbb{E}_{k+1}(f)\|_X^q \text{ when } n > k \geq 1, \\
|r_{n,k}|^q &= \frac{\Gamma(q)}{2^{q^2(k+2)} q^q} \|d\mathbb{E}_{k+1}(f)\|_X^q \text{ when } n < k, \\
r_{k,k} &= 0.
\end{aligned}$$

Denote $r := \{r_n\}_{n\geq 1},\ r^{(k)} := \{r_{n,k}\}_{n\geq 1}$. We estimate the ℓ_q-norms of r and $r^{(k)}$. By the definition of r, we have

$$\|r\|_{\ell_q} = \left(\sum_{n=1}^{\infty}\left(1-\frac{2}{2^{q^2(n+2)}}\right)\frac{\Gamma(q)}{q^q}\left\|d\mathbb{E}_{n+1}(f)\right\|_X^q\right)^{\frac{1}{q}} \geq \left(\left(1-\frac{2}{2^{3q^2}}\right)\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)).$$

It remains to estimate $\left\|\sum\limits_{k\geq 1}^{\infty} r^{(k)}\right\|_{\ell_q}$. We consider the measure space $(\mathbb{N}^2, 2^{\mathbb{N}^2}, \mu)$ where

$$\mu\left(\{(n,k)\}\right) = \begin{cases} \left(\frac{\Gamma(q)}{2^{q^2(n+1)}q^q}\right)^{\frac{1}{q}} & , \text{ if } \quad 1\leq k<n \\ 0 & , \text{ if } \quad k=n \\ \left(\frac{\Gamma(q)}{2^{q^2(k+2)}q^q}\right)^{\frac{1}{q}} & , \text{ if } \quad k>n \end{cases}.$$

It is very easy to verify that μ is a finite measure. Assume that ν is the counting measure on $\mathbb{N}$. Define

$$\begin{aligned} f : \mathbb{N}^2 &\longrightarrow \mathbb{R}_+ \\ (n,k) &\longmapsto \left\|d\mathbb{E}_{k+1}(f)\right\|_X. \end{aligned}$$

Then we deduce by the Hölder inequality

$$\begin{aligned} \left\|\sum_{k\geq 1}^{\infty} r^{(k)}\right\|_{\ell_q} &= \left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} r_{n,k}\right|^q\right)^{\frac{1}{q}} \\ &= \left(\int_{n\in\mathbb{N}}\left|\int_{k\in\mathbb{N}} f(n,k)\mathrm{d}\mu\right|^q \mathrm{d}\nu\right)^{\frac{1}{q}} \\ &\leq \left(\int_{n\in\mathbb{N}}\left|\int_{k\in\mathbb{N}} f(n,k)^q\mathrm{d}\mu\right|\left(\int_{k\in\mathbb{N}} 1\mathrm{d}\mu\right)^{q-1}\mathrm{d}\nu\right)^{\frac{1}{q}} \\ &\leq \left(\int_{n\in\mathbb{N}}\int_{k\in\mathbb{N}} f(n,k)^q\mathrm{d}\mu\mathrm{d}\nu\right)^{\frac{1}{q}} \sup_{n\geq 1}\left(\int_{k\in\mathbb{N}} 1\mathrm{d}\mu\right)^{\frac{q-1}{q}}. \end{aligned}$$

However, for $n\geq 1$

$$\begin{aligned} \left(\int_{k\in\mathbb{N}} 1\mathrm{d}\mu\right)^{\frac{q-1}{q}} &\leq \left(n\left(\frac{\Gamma(q)}{2^{q^2(n+1)}q^q}\right)^{\frac{1}{q}} + \sum_{k>n}\left(\frac{\Gamma(q)}{2^{q^2(k+2)}q^q}\right)^{\frac{1}{q}}\right)^{\frac{q-1}{q}} \\ &= \left(\frac{\Gamma(q)}{q^q}\right)^{\frac{q-1}{q^2}}\left(\frac{n}{2^{q(n+1)}} + \sum_{k>n}\frac{1}{2^{q(k+2)}}\right)^{\frac{q-1}{q}} \\ &\leq \left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{q-1}{q^2}}. \end{aligned}$$

Moreover, in the same way

$$\begin{aligned} \left(\int_{n\in\mathbb{N}}\int_{k\in\mathbb{N}} f(n,k)^q\mathrm{d}\mu\mathrm{d}\nu\right)^{\frac{1}{q}} &= \left(\sum_{k=1}^{\infty}\left(k\left(\frac{\Gamma(q)}{2^{q^2(k+2)}q^q}\right)^{\frac{1}{q}} + \sum_{n>k}\left(\frac{\Gamma(q)}{2^{q^2(n+1)}q^q}\right)^{\frac{1}{q}}\right)\left\|d\mathbb{E}_{k+1}(f)\right\|_X^q\right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q^2}} S_q(f-\mathbf{F}(f)). \end{aligned}$$

Thus we obtain

$$\left\|\sum_{k\geq 1}^{\infty} r^{(k)}\right\|_{\ell_q} \leq \frac{1}{2}\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)).$$

By the triangle inequality again, then it follows

$$\begin{aligned}
G_{q,T,M}(f) &\geq \left(\sum_{n=1}^{\infty}\left|R_{n,n}-\left(\sum_{k\neq n,k\geq 1} R_{n,k}\right)\right|^q\right)^{\frac{1}{q}} \geq \left(\sum_{n=1}^{\infty}\left|r_n-\left(\sum_{k\geq 1} r_{n,k}\right)\right|^q\right)^{\frac{1}{q}} \\
&= \left\|r-\left(\sum_{k=1}^{\infty} r^{(k)}\right)\right\|_{\ell_q} \geq \|r\|_{\ell_q} - \left\|\sum_{k=1}^{\infty} r^{(k)}\right\|_{\ell_q} \\
&\geq \left(\left(1-\frac{2}{2^{3q^2}}\right)\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)) - \frac{1}{2}\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)) \\
&\geq \left(\frac{1}{2}-\frac{2}{2^{3q^2}}\right)\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)) \\
&\geq \frac{1}{4}\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} S_q(f-\mathbf{F}(f)).
\end{aligned}$$

Since $\left(\frac{\Gamma(q)}{q^q}\right)^{\frac{1}{q}} \gtrsim 1$ for all $q \geq 1$, it is easy to deduce that

$$G_{q,T}(f) \geq G_{q,T,M}(f) \gtrsim S_q(f-\mathrm{F}(f)). \tag{2.2.3}$$

For the reversing inequality, we have by the convexity of norms

$$\begin{aligned}
G_{q,T}(f)^q &= \int_0^{\infty}\left\|\sum_{k=1}^{\infty} tb_k e^{-tb_k} d\mathbb{E}_{k+1}(f)\right\|_X^q \frac{\mathrm{d}t}{t} \\
&\leq \int_0^{\infty} t^{q-1}\left(\sum_{k=1}^{\infty} b_k e^{-tb_k}\right)^{q-1}\sum_{k=1}^{\infty} b_k e^{-tb_k}\left\|d\mathbb{E}_{k+1}(f)\right\|_X^q \mathrm{d}t.
\end{aligned}$$

To this end, we choose a positive number $N > 2$ such that $b_{k+1}/b_k = m_{k+1}/l_k > N$ for all $k \geq 1$. Then we have $b_k \leq N(N-1)^{-1}(b_k - b_{k-1})$, so

$$\sum_{k=1}^{\infty} b_k e^{-tb_k} \leq \frac{N}{N-1}\sum_{k\geq 1}\int_{b_{k-1}}^{b_k} e^{-tx}\mathrm{d}x \leq \frac{N}{N-1}\int_0^{\infty} e^{-tx}\mathrm{d}x \leq \frac{N}{(N-1)t}.$$

Then

$$\begin{aligned}
G_{q,T}(f)^q &\leq \left(\frac{N}{N-1}\right)^{q-1}\sum_{k=1}^{\infty}\left[\left\|d\mathbb{E}_{k+1}(f)\right\|_X^q \int_0^{\infty} b_k e^{-tb_k}\mathrm{d}t\right] \\
&\leq \left(\frac{N}{N-1}\right)^{q-1}\sum_{k=1}^{\infty}\left\|d\mathbb{E}_{k+1}(f)\right\|_X^q \\
&\leq \left(\frac{N}{N-1}\right)^{q-1} S_q(f-\mathrm{F}(f))^q.
\end{aligned} \tag{2.2.4}$$

Combine (2.2.3) and (2.2.4), and we finally obtain

$$S_q(f - \mathrm{F}(f)) \approx G_{q,T}(f). \tag{2.2.5}$$

This completes the proof. □

Recall that a Banach space X is of martingale cotype q ($q \geq 2$) if there exists a positive constant c such that every finite X-valued L^q-martingale $\{\mathbb{E}_n(f)\}_{n\geq 1}$ satisfies the following inequality

$$\mathbb{E}[S_q(f)^q] \leq c^q \sup_{n\geq 1} \mathbb{E}\,\|E_n(f)\|_X^q\,, \tag{2.2.6}$$

where $\mathbb{E}$ denotes the underlying expectation. See [82] for more information about martingale cotype. Pisier's famous renorming theorem asserts that X is of martingale cotype q iff X admits an equivalent norm that is q-convex. Besides, he also proved that X is of martingale cotype q iff all X-valued Walsh-Paley martingales satisfy (2.2.6). See [83] for more details.

Corollary 2.2.1. *Assume $2 \leq q < \infty$. If for every symmetric diffusion semigroup $\{T_t\}_{t>0}$ and for every $1 < p < \infty$ there exists a constant c such that*

$$\left\| \left(\int_0^\infty \|t\partial_t T_t(f)\|_X^q \, \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L_p(\Omega)} \leq c\|f\|_{L_p(\Omega;X)}, \quad \forall f \in L_p(\Omega;X). \tag{2.2.7}$$

Then X is of martingale cotype q.

Proof. It is an immediate consequence of Theorem I.8 and the definition of martingale cotype. □

Remark 2.2.2. Corollary 2.2.1 was first shown by Xu in [100, Theorem 3.1]. There he constructed a highly lacunary Fourier series to show that all Walsh-Paley martingales satisfied (2.2.6) only by using the Poisson semigroup on the unit disk. Hence, from his proof, X is martingale cotype q iff the Poisson semigroup satisfies (2.2.7).

Remark 2.2.3. Here we use Theorem I.8 to give an alternative proof of "if" part of [102, Theorem A] or [63, Theorem 2.1]. However, our proof needs much additional and stronger conditions on semigroups.

Remark 2.2.4. From Theorem I.8, the best constants of martingale inequalities can be dominated by the best constants of the Littlewood-Paley-Stein inequality. Accordingly, we can use martingale inequities to obtain the Littlewood-Paley-Stein inequality, which is due to Stein [87, Chapter IV.5].

2.3 Proof of Theorem I.10

We end this part with the proof of Theorem I.10. Although it is similar to that of Theorem III.1, we give some details for reader's convenience.

Note that for $1 < p < \infty$, any noncommutative symmetric diffusion semigroup $\{\mathsf{T}_t\}_{t>0}$ is analytic. See [53] for details. Then the column and row square functions are well-defined.

Proof of Theorem I.10. At first, we show that $\{T^t\}_{t>0}$ is a noncommutative symmetric diffusion semigroup by the same method as Theorem I.2. More precisely, the condition (a) (see Definition I.9) follows from the fact that all conditional expectations on $\mathcal{M}$ are unital normal and completely positive. As for the condition (b), observe that $T_t(x) \to x$ for $x \in \mathcal{M}_n$ ($\forall n \geq 1$) as $t \to 0^+$ in the w^*-topology of $\mathcal{M}$, and then use the fact that $\cup_{n\geq 1}\mathcal{M}_n$ is w^*-dense in $\mathcal{M}$.

The condition (c) is due to the fact that all conditional expectations on $\mathcal{M}$ are selfadjoint. The condition (d) follows from the fact that any conditional expectation can extend to a completely contractive operator from $L^p(\mathcal{M})$ to $L^p(\mathcal{M})$ for $1 \leq p < \infty$. Hence, we conclude that $\{T^t\}_{t>0}$ is a noncommutative symmetric diffusion semigroup.

Then it suffices to prove (I.9) and (I.10). Observe that if we have

$$\frac{7}{60}S_c(x - \mathbf{F}(x))^2 \leq G_{2,T,c}(x)^2 \leq \frac{23}{60}S_c(x - \mathbf{F}(x))^2, \tag{2.3.1}$$

then the desired inequality follows immediately since the square root function is operator increasing. We only need to consider the case where $x \in L^p(\mathcal{M}_n)$ for a fixed arbitrary $n \in \mathbb{N}$. By the same calculation as (2.1.5), we obtain

$$G_{2,T,c}(x)^2 = \sum_{i,j=2}^{n} \frac{\ln(1-a_{i-1})\ln(1-a_{j-1})}{\ln^2\left[(1-a_{i-1})(1-a_{j-1})\right]} d_i x^* d_j x.$$

Since $d_i x^*, d_j x$ and $d_i x^* d_j x$ ($2 \leq i, j \leq n$) are all measurable operators affiliated to $\mathcal{M}$, they have a common domain denoted by $\mathcal{K}$ which is a dense subspace of $\mathcal{H}$.

For any $\xi \in \mathcal{K}$,

$$\begin{aligned}\langle G_{2,T,c}(x)^2(\xi), \xi\rangle &= \sum_{i,j=2}^{n} \frac{\ln(1-a_{i-1})\ln(1-a_{j-1})}{\ln^2\left[(1-a_{i-1})(1-a_{j-1})\right]} \langle d_i x^* d_j x(\xi), \xi\rangle \\ &= \sum_{i,j=2}^{n} \frac{\ln(1-a_{i-1})\ln(1-a_{j-1})}{\ln^2\left[(1-a_{i-1})(1-a_{j-1})\right]} \langle d_j x(\xi), d_i x(\xi)\rangle,\end{aligned}$$

which implies that

$$\frac{7}{60}\sum_{i=2}^{n}\langle d_i x(\xi), d_i x(\xi)\rangle \leq \langle G_{2,T,c}(x)^2(\xi), \xi\rangle \leq \frac{23}{60}\sum_{i=2}^{n}\langle d_i x(\xi), d_i x(\xi)\rangle.$$

Using the fact that $\langle d_i x(\xi), d_i x(\xi)\rangle = \langle d_i x^* d_i x(\xi), (\xi)\rangle$, we obtain the desired inequality (2.3.1). As for the row square function, it can be deduced by the same way. □

We introduce the norms in the Hardy spaces of martingales defined in [85] and [86]. Let $1 \leq p \leq \infty$ and $x = (x_n)_{n\geq 1}$ be an L^p-martingale. Set, for $p \geq 2$

$$\|x\|_{\mathcal{H}^p} = \max\left\{\|S_c(x)\|_p, \|S_r(x)\|_p\right\}$$

and for $p < 2$

$$\|x\|_{\mathcal{H}^p} = \inf\left\{\|S_c(y)\|_p + \|S_r(z)\|_p\right\},$$

where the infimum runs over all decompositions $x = y + z$ of x as sums of two L_p martingales.

Similarly, we introduce the semigroup analogues of these norms. Let $\|\cdot\|_{p,F}$ denote the p-norm of semigroup square function defined as follows: for $x \in L^p(\mathcal{M})$

$$\|x\|_{p,F} = \max\left\{\|G_{2,T,c}(x)\|_p,\ \|G_{2,T,r}(x)\|_p\right\} \quad \text{if} \quad 2 \le p < \infty,$$

and

$$\|x\|_{p,F} = \inf\left\{\|G_{2,T,c}(y)\|_p + \|G_{2,T,r}(z)\|_p : x = y + z\right\} \quad \text{if} \quad 1 \le p < 2.$$

Note that our definition of $\|x\|_{p,F}$ is different from that of $\|x\|_{p,F}$ in [53, Chapter 6] when $1 \le p < 2$.

Remark 2.3.1. It should be noted that when $1 \le p < \infty$, we have the same equivalence for the p-norm of their corresponding square functions like the classical setting for the noncommutative symmetric diffusion semigroup in Theorem I.10, i.e.

$$\sqrt{\frac{7}{60}}\,\|x\|_{\mathcal{H}_p} \le \|x\|_{p,F} \le \sqrt{\frac{23}{60}}\,\|x\|_{\mathcal{H}_p}, \quad \forall x \in L^p(\mathcal{M}),$$

which follows from (I.9) and (I.10).

Chapter 3

Optimal Lusin type constant as p tending to 1

In this chapter, we are devoted to the vector-valued Littlewood-Paley theory. First we provide the main assumptions and a rigorous introduction of vector-valued tent, Hardy and BMO spaces. Then we recall the basic properties of tent space and give two important operators $\mathcal{K}$ and π_L, which will be key tools in our argument. Next, we extend Wilson's intrinsic functions (cf. [95]) to the vector-valued setting. In Section 3.4, we prove Theorem II.1 and Theorem II.2. At last, in Section 3.5, we obtain the optimal Lusin type constants and the characterization of martingale type.

3.1 Preliminaries

3.1.1 Main assumptions

Throughout this part, we assume L is a sectorial operator of type α ($0 \leq \alpha < \pi/2$) on $L^2(\mathbb{R}^d)$ such that the kernels $\{K(t,x,y)\}_{t>0}$ of $\{e^{-tL}\}_{t>0}$ satisfying assumptions (II.1), (II.2) and (II.3) with $\beta > 0, 0 < \gamma \leq 1$. It is well-known that such L generates a holomorphic semigroup e^{-zL} with $0 \leq |\mathrm{Arg}(z)| < \pi/2 - \alpha$ (cf. [45, Chapter 3, 3.2]). Let $\{k(t,x,y)\}_{t>0}$ be the kernels of $\{-tLe^{-tL}\}_{t>0}$ and it is easy to see

$$k(t,x,y) = t\partial_t K(t,x,y).$$

The following lemma has been justified in [36, Lemma 6.9]

Lemma 3.1.1. *Let L be an operator satisfying* (II.1) *and* (II.2) *with $\beta > 0$, $0 < \gamma \leq 1$. Then*

(i) *there exist constants $0 < \beta_1 < \beta$, $0 < \gamma_1 < \gamma$ and $c > 0$ such that*

$$|k(t,x,y)| \leq \frac{ct^{\beta_1}}{(t+|x-y|)^{d+\beta_1}}, \tag{3.1.1}$$

and

$$|k(t,x+h,y) - k(t,x,y)| + |k(t,x,y+h) - k(t,x,y)| \leq \frac{c|h|^{\gamma_1}t^{\beta_1}}{(t+|x-y|)^{d+\beta_1+\gamma_1}}$$

whenever $2|h| \le t + |x-y|$*;*

(ii) *for* $\alpha < \theta < \pi/2$*, there exist positive constants* $0 < \beta_2 < \beta$*,* $0 < \gamma_2 < \gamma$ *and* $c > 0$ *such that for any* $|\arg z| < \pi/2 - \theta$*,*

$$|K(z,x,y)| \le \frac{c|z|^{\beta_2}}{(|z| + |x-y|)^{d+\beta_2}} \tag{3.1.2}$$

and

$$|K(z,x+h,y) - K(z,x,y)| + |K(z,x,y+h) - K(z,x,y)| \le \frac{c|h|^{\gamma_2}|z|^{\beta_2}}{(|z| + |x-y|)^{d+\beta_2+\gamma_2}}$$

whenever $2|h| \le |z| + |x-y|$.

Remark 3.1.2. By [29, Lemma 2.5], the estimate (3.1.2) implies that for all $k \in \mathbb{N}$, $t > 0$ and almost everywhere $x, y \in \mathbb{R}^d$,

$$\left|t^k \partial_t^k K(t,x,y)\right| \le \frac{ct^{\beta_2}}{(t + |x-y|)^{d+\beta_2}}. \tag{3.1.3}$$

Convention. To simplify the notations, we will write below γ, β instead of γ_1, β_1 and γ_2, β_2 appeared in Lemma 3.1.1, and it should not cause any confusion.

One can verify that each e^{-tL} is a regular operator on $L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$ (see Proposition B.0.1 in Appendix B). Then the semigroup $\{e^{-tL}\}_{t>0}$ extends to $L^p(\mathbb{R}^d; X)$ (see e.g. [102]). The resulting semigroup is still denoted by $\{e^{-tL}\}_{t>0}$ without causing confusion. To well define the vector-valued BMO type spaces, we need more notations. For $\varepsilon > 0$, define

$$\mathcal{N}_\varepsilon = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X) : \exists\, c > 0 \text{ such that } \int_{\mathbb{R}^d} \frac{\|f(x)\|_X}{(1+|x|)^{d+\varepsilon}}\, \mathrm{d}x \le c \right\},$$

equipped with norm defined as the infimum of all the possible constant c. Then $\mathcal{N}_\varepsilon$ is a Banach space (cf. [36]). For a given generator L, let $\Theta(L) = \sup\{\beta_2 > 0 : (3.1.2) \text{ holds}\}$. Then we define

$$\mathcal{N} = \begin{cases} \mathcal{N}_{\Theta(L)}, & \text{if } \Theta(L) < \infty; \\ \bigcup_{0<\varepsilon<\infty} \mathcal{N}_\varepsilon, & \text{if } \Theta(L) = \infty. \end{cases}$$

It is clear that $L^p(\mathbb{R}^d; X) \subset \mathcal{N}$ for all $1 \le p \le \infty$. Moreover, By the definition of $\mathcal{N}$ and Remark 3.1.2, we know that the operators e^{-tL} and tLe^{-tL} are well-defined on $\mathcal{N}$.

Denote by $\mathbf{F}_L$ the fixed point space of $\{e^{-tL}\}_{t>0}$ on $\mathcal{N}$, namely

$$\mathbf{F}_L = \left\{ f \in \mathcal{N} : e^{-tL}(f) = f,\ \forall\, t > 0 \right\}.$$

It is well-known that $\mathbf{F}_L$ coincides with the null space of $\{tLe^{-tL}\}_{t>0}$, and the resulting quotient space is defined as $\mathcal{N}_L := \mathcal{N}/\mathbf{F}_L$. For $1 \le p < \infty$, the fixed point subspace of $L^p(\mathbb{R}^d; X)$ is $\mathbf{F}_L \cap L^p(\mathbb{R}^d; X) = \{0\}$ (cf. [36, Theorem 6.10], see also Proposition B.0.2 in Appendix B). Then the projection from $L^p(\mathbb{R}^d; X)$ to the fixed point subspace for all $1 \le p < \infty$ is indeed 0. See e.g. [63, 102] for more information on this projection.

Remark 3.1.3. Let L^* be the adjoint operator of L. Then L^* is also a sectorial operator with the kernels of $\{e^{-tL^*}\}_{t>0}$ satisfying (II.1), (II.2) and (II.3) (cf. [36, Theorem 6.10]).

3.1.2 Vector-valued tent, Hardy and BMO spaces

In this subsection, we introduce several spaces including vector-valued tent spaces, vector-valued Hardy and BMO spaces associated with a generator L.

Vector-valued Tent spaces

We first introduce vector-valued tent spaces. We denote by $\mathbb{R}^{d+1}_+$ the usual upper half-space in $\mathbb{R}^{d+1}$ i.e. $\mathbb{R}^d \times (0,\infty)$. Let $\Gamma(x) = \{(y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < t\}$ denote the standard cone with vertex at x. For any closed subset $F \subset \mathbb{R}^d$, define $\mathcal{R}(F) = \bigcup_{x\in F} \Gamma(x)$. If $O \subset \mathbb{R}^d$ is an open subset, then the tent over O, denoted by $\widehat{O}$, is given as $\widehat{O} = \Big(\mathcal{R}(O^C)\Big)^C$.

For any strongly measurable function $f : \mathbb{R}^{d+1}_+ \to X$, we define two operators as follows:

$$\mathcal{A}_q(f)(x) = \left(\int_{\Gamma(x)} \|f(y,t)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)^{\frac{1}{q}}, \quad \mathcal{C}_q(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|}\int_{\widehat{B}} \|f(y,t)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{\frac{1}{q}},$$

where the supremum runs over all balls B in $\mathbb{R}^d$.

Definition 3.1.4. Let $1 \le p < \infty$ and $1 < q < \infty$. The vector-valued tent space $T^p_q(\mathbb{R}^{d+1}_+; X)$ is defined as the subspace consisting of all the strongly measurable functions $f : \mathbb{R}^{d+1}_+ \to X$ such that

$$\|f\|_{T^p_q(X)} := \|\mathcal{A}_q(f)\|_p < \infty,$$

and $T^\infty_q(\mathbb{R}^{d+1}_+; X)$ is defined as the subspace of all the strongly measurable functions $g : \mathbb{R}^{d+1}_+ \to X$ such that

$$\|g\|_{T^\infty_q(X)} := \|\mathcal{C}_q(g)\|_\infty < \infty.$$

Let $C_c(\mathbb{R}^{d+1}_+) \otimes X$ be the space of finite linear combinations of elements from $C_c(\mathbb{R}^{d+1}_+)$ and X. The following density follows from the standard arguments (see e.g. [47]), and we omit the details.

Lemma 3.1.5. *Let X be a Banach space and $1 < q < \infty$. Then $C_c(\mathbb{R}^{d+1}_+) \otimes X$ is norm dense in $T^p_q(\mathbb{R}^{d+1}_+; X)$ for $1 \le p < \infty$, and weak-$*$ dense in $\Big(T^1_{q'}(\mathbb{R}^{d+1}_+; X^*)\Big)^*$.*

Vector-valued Hardy spaces

Given a function $f \in \mathcal{N}_L$, the q-variant of Lusin area integral function of f associated with L is defined by

$$S_{q,L}(f)(x) = \left(\int_{\Gamma(x)} \|tLe^{-tL}(f)(y)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)^{\frac{1}{q}};$$

and the q-variant of Littlewood-Paley g-function is defined by

$$G_{q,L}(f)(x) = \left(\int_0^\infty \|tLe^{-tL}(f)(x)\|_X^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

Definition 3.1.6. Let $1 \le p < \infty$ and $1 < q < \infty$. We define the vector-valued Hardy space $H^p_{q,L}(\mathbb{R}^d; X)$ associated with L as

$$H^p_{q,L}(\mathbb{R}^d; X) = \left\{ f \in \mathcal{N}_L : S_{q,L}(f) \in L^p(\mathbb{R}^d) \right\},$$

equipped with the norm

$$\|f\|_{H^p_{q,L}(X)} = \|S_{q,L}(f)\|_p.$$

It can be verified that $H^p_{q,L}(\mathbb{R}^d; X)$ is a Banach space from the definition of $\mathcal{N}_L$. The space $H^p_{q,L}(\mathbb{R}^d; X)$ has deep connection with the vector-valued tent space, namely, a strongly measurable function $f \in \mathcal{N}_L$ belongs to $H^p_{q,L}(\mathbb{R}^d; X)$ if and only if $\mathcal{Q}(f) \in T^p_q(\mathbb{R}^{d+1}_+; X)$ where $\mathcal{Q}(f)(x,t) = -tLe^{-tL}(f)(x)$. Moreover,

$$\|f\|_{H^p_{q,L}(X)} = \|\mathcal{Q}(f)\|_{T^p_q(X)}.$$

Vector-valued BMO spaces

Definition 3.1.7. Let $1 \le p \le \infty$ and $1 < q < \infty$. We define the vector-valued BMO space $BMO^p_{q,L}(\mathbb{R}^d; X)$ associated with L as

$$BMO^p_{q,L}(\mathbb{R}^d; X) = \{ f \in \mathcal{N}_L : \|\mathcal{C}_q(\mathcal{Q}(f))\|_p < \infty \}$$

equipped with the norm

$$\|f\|_{BMO^p_{q,L}(X)} = \|\mathcal{C}_q(\mathcal{Q}(f))\|_p.$$

In particular, for $p = \infty$, we denote it by $BMO_{q,L}(\mathbb{R}^d; X)$ for short.

It can be verified that $BMO^p_{q,L}(\mathbb{R}^d; X)$ equipped the the norm $\|\cdot\|_{BMO^p_{q,L}(X)}$ is a Banach space from the definition of $\mathcal{N}_L$.

The vector-valued Hardy and BMO spaces enjoy the similar relationship as the scalar-valued ones (see e.g. [26]). We collect them below with a brief explanation.

Lemma 3.1.8. *Let X be any fixed Banach space and $1 < q < \infty$. One has for $f \in C_c(\mathbb{R}^{d+1}_+) \otimes X$,*

$$\|\mathcal{C}_q(f)\|_p \lesssim \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \|\mathcal{A}_q(f)\|_p, \quad q < p \le \infty, \tag{3.1.4}$$

and

$$\|\mathcal{A}_q(f)\|_p \lesssim q^{\frac{p}{q}} \|\mathcal{C}_q(f)\|_p, \quad 1 \le p < \infty. \tag{3.1.5}$$

Therefore, we have for $1 \le p \le q$,

$$BMO^p_{q,L}(\mathbb{R}^d; X) \subset H^p_{q,L}(\mathbb{R}^d; X)$$

and for $q < p < \infty$,

$$H^p_{q,L}(\mathbb{R}^d; X) = BMO^p_{q,L}(\mathbb{R}^d; X)$$

with equivalent norms.

Proof. Given an X-valued function f defined on $\mathbb{R}^{d+1}_+$, we consider the scalar-valued function $\widetilde{f}(x,t) = \|f(x,t)\|_X$. Then one may apply (3.1.4) and (3.1.5) in the case $X = \mathbb{C}$ for $\widetilde{f}$ (see e.g. [26, Theorem 3]) to obtain (3.1.4) and (3.1.5) for general X. Thus by using the operator $\mathcal{Q}$, for any $f \in BMO^p_{q,L}(\mathbb{R}^d; X)$ $(1 \le p \le q)$, we get

$$\|f\|_{H^p_{q,L}(X)} = \|\mathcal{A}_q(\mathcal{Q}(f))\|_p \lesssim q^{\frac{p}{q}} \|\mathcal{C}_q(\mathcal{Q}(f))\|_p = \|f\|_{BMO^p_{q,L}(X)},$$

and the same argument holds for $q < p < \infty$. □

Remark 3.1.9. In particular, $BMO_{q,L}(\mathbb{R}^d; X)$ is closely related to the Carleson measure. Recall that a scalar-valued measure μ defined on $\mathbb{R}^{d+1}_+$ is a Carleson measure if there exists a constant c such that for all balls B in $\mathbb{R}^d$,

$$|\mu(\widehat{B})| \le c|B|,$$

where $\widehat{B}$ is the tent over B. The norm is defined as

$$\|\mu\|_c = \sup_B |B|^{-1} |\mu(\widehat{B})|,$$

where the supremum runs over all the balls in $\mathbb{R}^d$.

For a vector-valued function $f \in \mathcal{N}_L$, we define a measure $\mu_{q,f}$ as

$$\mu_{q,f}(x,t) = \frac{\|\mathcal{Q}(f)(x,t)\|^q_X \mathrm{d}x\mathrm{d}t}{t}.$$

Then f belongs to $BMO_{q,L}(\mathbb{R}^d; X)$ if and only if $\mu_{q,f}$ is a Carleson measure, and moreover

$$\|f\|_{BMO_{q,L}(X)} = \|\mu_{q,f}\|_c^{\frac{1}{q}}.$$

3.2 Theory of vector-valued tent spaces and two key linear operators

In this section, we will first present the basic theory of vector-valued tent spaces such as interpolation, duality and atomic decomposition, and then introduce two important linear operators $\mathcal{K}$ and π_L which enable us to exploit the basic theory of tent spaces to investigate in later sections vector-valued Wilson's square functions and Theorem II.1.

Note that if the underlying Banach space X has some geometric property such as UMD, then the vector-valued tent space theory have been established in the literature [51, 56, 57]. In the present paper, we observe that the theory of vector-valued tent space holds for any Banach space; and this is quite essential for the applications in the present paper.

3.2.1 Basic theory of vector-valued tent spaces

We begin this subsection by providing a concise overview of interpolation theory in the context of vector-valued tent spaces.

To apply the theory of vector-valued tent spaces, we will need the atomic decomposition of $T^1_q(\mathbb{R}^{d+1}_+; X)$. Recall that a strongly measurable function $a : \mathbb{R}^{d+1}_+ \to X$ is called an (X, q)-atom if

1. $\operatorname{supp} a \subset \widehat{B}$ where B is a ball in $\mathbb{R}^d$;
2. $\left(\int_{\mathbb{R}^{d+1}_+} \|a(x,t)\|_X^q \frac{\mathrm{d}x\mathrm{d}t}{t}\right)^{\frac{1}{q}} \leq |B|^{\frac{1}{q}-1}$.

Lemma 3.2.1. *Let X be any fixed Banach space and $1 < q < \infty$. For each $f \in T_q^1(\mathbb{R}^{d+1}_+; X)$, there exists a sequence of complex numbers $\{\lambda_k\}_{k\geq 1}$ and (X,q)-atoms a_k such that*

$$f = \sum_{k\geq 1} \lambda_k a_k, \quad \|f\|_{T_q^1(X)} \approx \sum_{k\geq 1} |\lambda_k|.$$

Proof. Let a be an (X,q)-atom and $\operatorname{supp} a \subset \widehat{B}$ where $B = B(c_B, r_B)$ with center c_B and radius r_B. If $\Gamma(x) \cap \widehat{B} \neq \emptyset$, there exists $(y,t) \in \Gamma(x) \cap \widehat{B}$. Then we have $|x - c_B| \leq |x-y| + |y - c_B| < t + r_B < 2r_B$. By Hölder's inequality and Fubini's theorem,

$$\|a\|_{T_q^1(X)} = \int_{2B} \left(\int_{\Gamma(x)} \|a(y,t)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)^{\frac{1}{q}} \mathrm{d}x \lesssim |2B|^{1-\frac{1}{q}} \left(\int_{\mathbb{R}^{d+1}_+} \|a(y,t)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t}\right)^{\frac{1}{q}} \lesssim 1.$$

Therefore any (X,q)-atom belongs to $T_q^1(\mathbb{R}^{d+1}_+; X)$.

Let $0 < \lambda < 1/2$. We define two sequences of open sets $\{O_k\}_{k\in\mathbb{Z}}$ and $\{O_k^*\}_{k\in\mathbb{Z}}$ as

$$O_k = \left\{x \in \mathbb{R}^d : \mathcal{A}_q(f)(x) > 2^k\right\}, \quad O_k^* = \left\{x \in \mathbb{R}^d : M(\mathbb{1}_{O_k})(x) > 1 - \lambda\right\},$$

where $M(\mathbb{1}_{O_k})$ is the centered Hardy-Littlewood maximal function. It is clear that both O_k and O_k^* have finite measure. Additionally, the following properties hold: $O_{k+1} \subset O_k$, $O_{k+1}^* \subset O_k^*$ and $|O_k^*| \leq C_\lambda |O_k|$ (see e.g. [26]).

We follow a similar construction as in [56]. The Vitali covering lemma and [57, Lemma 4.2] assert that for each O_k^*, there exist disjoint balls $B_k^j \subset O_k^*$ $(j \geq 1)$ such that

$$\widehat{O}_k^* \subset \bigcup_{j\geq 1} \widehat{5B}_k^j, \quad \sum_{j\geq 1} |B_k^j| \leq |O_k^*|.$$

With this setup, we proceed to define a family of functions χ_k^j by the partition of unity:

$$0 \leq \chi_k^j \leq 1, \ \sum_{j\geq 1} \chi_k^j = 1 \text{ on } \widehat{O}_k^* \ \text{ and } \ \operatorname{supp} \chi_k^j \subset \widehat{5B}_k^j.$$

Therefore

$$f = \sum_{k\in\mathbb{Z}} f_k = \sum_{k\in\mathbb{Z}} \sum_{j\geq 1} \chi_k^j f_k = \sum_{k\in\mathbb{Z}} \sum_{j\geq 1} \lambda_k^j a_k^j,$$

where

$$f_k = f\mathbb{1}_{\widehat{O}_k^* \setminus \widehat{O}_{k+1}^*}, \ \lambda_k^j = |5B_k^j|^{\frac{1}{q'}} \left(\int_{5B_k^j} \mathcal{A}_q(f_k)^q(x)\,\mathrm{d}x\right)^{\frac{1}{q}}, \ a_k^j = \frac{\chi_k^j f_k}{\lambda_k^j}. \tag{3.2.1}$$

Now we only need to show that each a_k^j is an (X,q)-atom and

$$\sum_{k\in\mathbb{Z}} \sum_{j\geq 1} |\lambda_k^j| \lesssim \|f\|_{T_q^1(X)}.$$

It is clear that $\operatorname{supp} a_k^j \subset \widehat{5B_k^j}$. Furthermore,

$$\begin{aligned}\|a\|_{L^q(\mathbb{R}_+^{d+1};X)}^q &\leq |5B_k^j|^{1-q}\|\mathcal{A}_q(f_k)\mathbb{1}_{5B_k^j}\|_q^{-q}\left(\int_{\widehat{5B_k^j}}\|f_k(y,t)\|_X^q\,\frac{\mathrm{d}y\mathrm{d}t}{t}\right)\\ &\leq |5B_k^j|^{1-q}\|\mathcal{A}_q(f_k)\mathbb{1}_{5B_k^j}\|_q^{-q}\left(\int_{5B_k^j}(\mathcal{A}_q(f_k)(x))^q\,\mathrm{d}x\right)\\ &= |5B_k^j|^{1-q}.\end{aligned}$$

Hence each a_k^j is an (X,q)-atom.

According to [26, Lemma 5], it is known that $\mathcal{A}_q(f_k)$ is supported in $O_k^* \setminus O_{k+1}$, then we deduce that $\mathcal{A}_q(f_k)(x) \leq 2^{k+1}$ by definition. Thus

$$\sum_{k\in\mathbb{Z}}\sum_{j\geq 1}|\lambda_k^j| \leq \sum_{k\in\mathbb{Z}}\sum_{j\geq 1}|5B_k^j|^{\frac{1}{q'}}2^{k+1}|5B_k^j|^{\frac{1}{q}} \leq \sum_{k\in\mathbb{Z}}2^{k+1}|O_k^*| \leq \sum_{k\in\mathbb{Z}}2^{k+1}C_\lambda|O_k|.$$

However, $\mathcal{A}_q(f)(x) > 2^{(k+m)}$ on O_{k+m}, then

$$2^k|O_k| = \int_{O_k}2^k\,\mathrm{d}x = \sum_{m=0}^{\infty}\int_{O_{k+m}\setminus O_{k+m+1}}2^k\,\mathrm{d}x \leq \sum_{m=0}^{\infty}2^{-m}\int_{O_{k+m}\setminus O_{k+m+1}}\mathcal{A}_q(f)(x)\,\mathrm{d}x.$$

Hence

$$\sum_{k\in\mathbb{Z}}2^{k+1}C_\lambda|O_k| \leq \sum_{m=0}^{\infty}\sum_{k\in\mathbb{Z}}2^{-m+1}C_\lambda\int_{O_{k+m}\setminus O_{k+m+1}}\mathcal{A}_q(f)(x)\,\mathrm{d}x \lesssim \|f\|_{T_q^1(X)}.$$

We complete the proof. □

Remark 3.2.2. From the atomic decomposition of $T_q^1(\mathbb{R}_1^{d+1};X)$—Lemma 3.2.1, one may conclude a molecule decomposition of the corresponding Hardy space. This might have further applications, and we include it in Appendix A.

The following lemma is the complex interpolation theory of vector-valued tent spaces.

Lemma 3.2.3. *Let X be any fixed Banach space, $1 < q < \infty$ and $1 \leq p_1 < p < p_2 < \infty$ such that $1/p = (1-\theta)/p_1 + \theta/p_2$ with $0 \leq \theta \leq 1$. Then*

$$[T_q^{p_1}(\mathbb{R}_+^{d+1};X), T_q^{p_2}(\mathbb{R}_+^{d+1};X)]_\theta = T_q^p(\mathbb{R}_+^{d+1};X),$$

with equivalent norms, where $[\cdot,\cdot]_\theta$ is the complex interpolation space. More precisely, for $f \in C_c(\mathbb{R}_+^{d+1}) \otimes X$, one has

$$\|f\|_{T_q^p(X)} \lesssim \|f\|_{[T_q^{p_1}(\mathbb{R}_+^{d+1};X),T_q^{p_2}(\mathbb{R}_+^{d+1};X)]_\theta} \lesssim p^{\frac{1}{q}}\|f\|_{T_q^p(X)}.$$

Proof. Denote by E the Banach space $L^q(\mathbb{R}_+^{d+1};X)$ equipped with the measure $\mathrm{d}x\mathrm{d}t/t^{d+1}$. At the beginning of the proof, we introduce two important operators, which allow us immerse $T_q^p(\mathbb{R}_+^{d+1};X)$ into $L^p(\mathbb{R}^d;E)$. Let $f \in T_q^p(\mathbb{R}_+^{d+1};X)$, we define an operator as follows

$$i(f)(x,y,t) = \mathbb{1}_{\Gamma(x)}(y,t)f(y,t).$$

Then it is clear that $\|i(f)\|_{L^p(E)} = \|f\|_{T_q^p(X)}$. Denote by $\widetilde{T}_q^p$ the range of the operator i. Now we introduce another operator N given by

$$N(f)(x,y,t) = \mathbb{1}_{\Gamma(x)}(y,t)\frac{1}{w_d t^d}\int_{|z-y|<t} f(z,y,t)\,\mathrm{d}z,$$

where w_d is the volume of the d-dimensional unit ball. It is known that N is a continuous projection from $L^p(\mathbb{R}^d;E)$ onto itself with range $\widetilde{T}_q^p$ for $1<p<\infty$ (cf. [47]). Consider the maximal operator

$$M_1(f)(x,y,t) = \sup_{x\in B}\frac{1}{|B|}\int_B \|f(z,y,t)\|_X\,\mathrm{d}z.$$

It is known from the maximal inequalities (see e.g. [88, Chapter II]) that M_1 is bounded on $L^p\left(\mathbb{R}^d; L^q(\mathbb{R}^{d+1};\mathrm{d}y\mathrm{d}t/t^{d+1})\right)$ for $1<p<\infty$; in particular, we view $\|f\|_X$ as a scalar-valued function in $L^p\left(\mathbb{R}^d; L^q(\mathbb{R}^{d+1};\mathrm{d}y\mathrm{d}t/t^{d+1})\right)$, then

$$\|M_1(f)\|_{L^p\left(\mathbb{R}^d;L^q\left(\mathbb{R}^{d+1}_+,\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)\right)} = \|M_1(\|f\|_X)\|_{L^p\left(\mathbb{R}^d;L^q\left(\mathbb{R}^{d+1}_+,\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)\right)} \lesssim p^{\frac{1}{q}}\|f\|_{L^p(E)},\quad q\le p<\infty.$$

Then we deduce from the definition of N that

$$\|N(f)(x,y,t)\|_X \le \mathbb{1}_{\Gamma(x)}(y,t)\frac{1}{|B(y,t)|}\int_{B(y,t)}\|f(z,y,t)\|_X\,\mathrm{d}z \le M_1(f)(x,y,t).$$

Therefore

$$\|N(f)\|_{L^p(E)} \le \|M_1(f)\|_{L^p\left(\mathbb{R}^d;L^q\left(\mathbb{R}^{d+1}_+;\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)\right)} \lesssim p^{\frac{1}{q}}\|f\|_{L^p(E)},\quad q\le p<\infty.$$

We denote by F the Banach space $L^{q'}(\mathbb{R}^{d+1}_+;X^*)$ equipped with the measure $\mathrm{d}x\mathrm{d}t/t^{d+1}$. Then it is clear that $F\subset E^*$ and F is norming for E. For $1<p<q$, we have

$$\begin{aligned}
\|N(f)\|_{L^p(E)} &= \sup_g\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d+1}_+}\langle N(f)(x,y,t),g(x,y,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\mathrm{d}x\right|\\
&= \sup_g\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d+1}_+}\langle f(x,y,t),N(g)(x,y,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\mathrm{d}x\right|\\
&\le \|f\|_{L^p(E)}\|N(g)\|_{L^{p'}(F)} \lesssim p'^{\frac{1}{q'}}\|f\|_{L^p(E)}\|g\|_{L^{p'}(F)},
\end{aligned}$$

where the supremum is taken over all g in the unit ball of $L^{p'}(\mathbb{R}^d;F)$. We conclude

$$\|N(f)\|_{L^p(E)} \lesssim \max\left\{p^{\frac{1}{q}}, p'^{\frac{1}{q'}}\right\}\|f\|_{L^p(E)},\quad 1<p<\infty. \tag{3.2.2}$$

Now we turn to the interpolation theory. The proof of the case $1<p_1<p_2<\infty$ follows from [47] by virtue of the immersion i and projection N.

For $p_1=1$, we adapt the classical argument as in [26, Lemma 4, Lemma 5]. Since the immersion i is an isometry, the exactness of the exponent θ of complex interpolation functor reads that

$$\begin{aligned}
&\|i(f)\|_{[L^1(\mathbb{R}^d;E),L^{p_2}(\mathbb{R}^d;E)]_\theta}\\
\le\ &\|i\|^{1-\theta}_{T_q^1(\mathbb{R}^{d+1}_+;X)\to L^1(\mathbb{R}^d;E)}\|i\|^{\theta}_{T_q^{p_2}(\mathbb{R}^{d+1}_+;X)\to L^{p_2}(\mathbb{R}^d;E)}\|f\|_{[T_q^1(\mathbb{R}^{d+1}_+;X),T_q^{p_2}(\mathbb{R}^{d+1}_+;X)]_\theta}\\
\le\ &\|f\|_{[T_q^1(\mathbb{R}^{d+1}_+;X),T_q^{p_2}(\mathbb{R}^{d+1}_+;X)]_\theta}.
\end{aligned}$$

By the interpolation theory of vector-valued L^p spaces, (see e.g. [13]), we have

$$\|i(f)\|_{[L^1(\mathbb{R}^d;E),L^{p_2}(\mathbb{R}^d;E)]_\theta} = \|i(f)\|_{L^p(E)} = \|f\|_{T_q^p(X)}.$$

Thus

$$\|f\|_{T_q^p(X)} \leq \|f\|_{[T_q^1(\mathbb{R}_+^{d+1};X),T_q^{p_2}(\mathbb{R}_+^{d+1};X)]_\theta}.$$

For the reverse direction, let $f \in T_q^p(\mathbb{R}_+^{d+1};X)$ and $\|f\|_{T_q^p(X)} = 1$. We define the interpolation functor F as

$$F(z) = \sum_{k\in\mathbb{Z}} 2^{k(\alpha(z)p-1)} f_k,$$

where $\alpha(z) = 1 - z + z/p_2$ and f_k is defined in (3.2.1). We have $F(\theta) = f$. Then the proof is indeed the same as in [26, Lemma 5], we omit the details. □

We now provide a characterization of $T_q^p(\mathbb{R}_+^{d+1};X)$-norm. It belongs to the norming subspace theory of vector-valued L^p-spaces, see e.g. [35, Chapter II, Section 4]. The proof is in spirit the same as the scalar-valued case (cf. [47, Theorem 2.4] and [26, Theorem 1], but we include a proof here to provide explicit orders for later applications.

Let $(\Omega,\mathcal{F},\mu)$ be a measure space. Recall that a Banach space X has the *Radon-Nikodým property* with respect to $(\Omega,\mathcal{F},\mu)$ if for each μ-continuous vector-valued measure $\nu : \mathcal{F} \to X$ of bounded variation, there exists $g \in L^1(\Omega;X)$ with respect to the measure μ such that

$$\nu(E) = \int_E g \,\mathrm{d}\mu, \quad \forall\, E \in \mathcal{F}.$$

In the following context, we call a Banach space has the Radon-Nikodým property for short when there is no ambiguity. We refer readers to [34, Chapter III] for more details.

Lemma 3.2.4. *Let X be any fixed Banach space and $1 < q < \infty$. The space $T_{q'}^{p'}(\mathbb{R}_+^{d+1};X^*)$ is isomorphically identified as a subspace of the dual space of $T_q^p(\mathbb{R}_+^{d+1};X)$. Moreover, it is norming for $T_q^p(\mathbb{R}_+^{d+1};X)$ in the following sense,*

$$\|f\|_{T_q^p(X)} \lesssim \max\left\{p^{\frac{1}{q}}, p'^{\frac{1}{q'}}\right\} \sup_g \left|\int_{\mathbb{R}_+^{d+1}} \langle f(x,t), g(x,t)\rangle_{X\times X^*} \,\frac{\mathrm{d}x\mathrm{d}t}{t}\right| \quad 1 < p < \infty, \quad (3.2.3)$$

where the supremum is taken over all $g \in C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ such that $\|\mathcal{A}_{q'}(g)\|_{p'} \leq 1$; and similarly,

$$\|f\|_{T_q^p(X)} \lesssim \left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}} \sup_g \left|\int_{\mathbb{R}_+^{d+1}} \langle f(x,t), g(x,t)\rangle_{X\times X^*} \,\frac{\mathrm{d}x\mathrm{d}t}{t}\right| \quad 1 \leq p < q, \quad (3.2.4)$$

where the supremum is taken over all $g \in C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ such that $\|\mathcal{C}_{q'}(g)\|_{p'} \leq 1$. Furthermore, if X^* has the Radon-Nikodým property, then

$$T_{q'}^{p'}(\mathbb{R}_+^{d+1};X^*) = \left(T_q^p(\mathbb{R}_+^{d+1};X)\right)^*, \quad 1 \leq p < \infty.$$

Proof. We adopt the notation used in the proof of Lemma 3.2.3. We first prove the estimate (3.2.3).

For any $g \in C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ and $f \in T_q^p(\mathbb{R}_+^{d+1}; X)$, we denote by

$$g(f) = \int_{\mathbb{R}_+^{d+1}} \langle f(x,t), g(x,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}x\mathrm{d}t}{t}.$$

Thus we have

$$\begin{aligned} |g(f)| &= \left| \int_{\mathbb{R}_+^{d+1}} \left\langle f(y,t), g(y,t) \left(w_d^{-1} \int_{|x-y|<t} 1 \, \mathrm{d}x \right) \right\rangle_{X \times X^*} \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \right| \\ &= w_d^{-1} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^{d+1}} \langle i(f)(x,y,t), i(g)(x,y,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \mathrm{d}x \right|. \end{aligned} \tag{3.2.5}$$

Since $i(f) \in \widetilde{T}_q^p$, we have $N(i(f)) = i(f)$. Then we deduce that

$$\begin{aligned} \|f\|_{T_q^p(X)} = \|i(f)\|_{L^p(E)} &= \sup_g \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^{d+1}} \langle i(f)(x,y,t), g(x,y,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \mathrm{d}x \right| \\ &= \sup_g \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^{d+1}} \langle N(i(f))(x,y,t), g(x,y,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \mathrm{d}x \right| \\ &= \sup_g \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^{d+1}} \langle i(f)(x,y,t), N(g)(x,y,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \mathrm{d}x \right| \end{aligned} \tag{3.2.6}$$

where the supremum is taken over all g in the unit ball of $L^{p'}(\mathbb{R}^d; F)$. Notice that $N(g) = i\left(i^{-1}(N(g))\right)$ and by (3.2.2)

$$\|i^{-1}(N(g))\|_{T_{q'}^{p'}(X^*)} = \|N(g)\|_{L^{p'}(F)} \lesssim \max\left\{ p^{\frac{1}{q}}, {p'}^{\frac{1}{q'}} \right\} \|g\|_{L^{p'}(F)}.$$

Consequently, combining (3.2.5) and (3.2.6), we obtain

$$\|f\|_{T_q^p(X)} \lesssim \max\left\{ p^{\frac{1}{q}}, {p'}^{\frac{1}{q'}} \right\} \sup_g |g(f)|, \quad 1 < p < \infty,$$

where the supremum is taken over all $g \in C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ such that $\|\mathcal{A}_{q'}(g)\|_{p'} \leq 1$. Here we adapt a limiting argument. Since the unit ball of $C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ is contained in the unit ball of $T_{q'}^{p'}(\mathbb{R}_+^{d+1}; X^*)$, and its closure contains the unit sphere, then we obtain that the unit ball of $C_c(\mathbb{R}_+^{d+1}) \otimes X^*$ is still norming for $T_q^p(\mathbb{R}_+^{d+1}; X)$.

Now we deal with the estimate (3.2.4) in the case $1 < p < q$. Let $g \in L^{p'}(\mathbb{R}^d; F)$. By definition we have

$$\begin{aligned} \|i^{-1}(N(g))(y,t)\|_{X^*}^{q'} &\leq \left(\frac{1}{|B(y,t)|} \int_{B(y,t)} \|g(z,y,t)\|_{X^*} \, \mathrm{d}z \right)^{q'} \\ &\leq \frac{1}{|B(y,t)|} \int_{|z-y|<t} \|g(z,y,t)\|_{X^*}^{q'} \, \mathrm{d}z. \end{aligned}$$

For a ball B in $\mathbb{R}^d$, we observe

$$\begin{aligned} \int_{\widehat{B}} \|i^{-1}(N(g))(y,t)\|_{X^*}^{q'} \, \frac{\mathrm{d}y\mathrm{d}t}{t} &\lesssim \int_{\widehat{B}} \int_{|z-y|<t} \|g(z,y,t)\|_{X^*}^{q'} \, \mathrm{d}z \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \\ &\leq \int_{2B} \int_{\mathbb{R}_+^{d+1}} \|\mathbb{1}_{\widehat{B}}(y,t) g(z,y,t)\|_{X^*}^{q'} \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \mathrm{d}z \\ &= \int_{2B} H^{q'}(z) \, \mathrm{d}z, \end{aligned}$$

where

$$H(z)=\left(\int_{\mathbb{R}^{d+1}_+}\|\mathbb{1}_{\widehat{B}}(y,t)g(z,y,t)\|_{X^*}^{q'}\,\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right)^{\frac{1}{q'}}.$$

Then we have

$$\mathcal{C}_{q'}\left[i^{-1}(N(g))\right](x)\lesssim\left(\mathcal{M}(H^{q'})(x)\right)^{\frac{1}{q'}},$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Therefore when $q'<p'\le\infty$, we obtain

$$\|\mathcal{C}_{q'}\left[i^{-1}(N(g))\right]\|_{p'}\lesssim\|\mathcal{M}(H^{q'})^{\frac{1}{q'}}\|_{p'}\lesssim\left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}}\|H\|_{p'}\le\left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}}\|g\|_{L^{p'}(F)}.$$

Thus we observe

$$\|f\|_{T^p_q(X)}\lesssim\left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}}\sup_g|g(f)|,\quad 1<p<q,$$

with the supremum being taken over all $g\in C_c(\mathbb{R}^{d+1}_+)\otimes X^*$ such that $\|\mathcal{C}_{q'}(g)\|_{p'}\le 1$.

For the endpoint case $p=1$ of (3.2.4), because of the failure of vector-valued Calderón-Zygmund theory, the above arguments adapted from [47, Theorem 2.4] do not work any more. Instead, one may exploit the classical arguments in [26, Theorem 1] and utilize the atomic decomposition of $T^1_q(\mathbb{R}^{d+1}_+;X)$, we leave the details to the interested reader.

When the Banach space X^* has the Radon-Nikodým property, one gets $F=E^*$ (cf. e.g. [50, Theorem 1.3.10]). Then the duality follows from then an analogous argument in [47] for $1<p<\infty$. Again, the duality in the case $p=1$ can be deduced as in the scalar-valued case [26, Theorem 1], and we leave the details to the interested reader. □

3.2.2 The two linear operators $\mathcal{K}$ and π_L

Let $\mathcal{K}:\mathbb{R}^{d+1}_+\times\mathbb{R}^{d+1}_+\to\mathbb{R}$ be a reasonable real-valued function such that for $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$, the linear operator $\mathcal{K}$ is well defined as below,

$$\mathcal{K}(f)(x,t):=\int_{\mathbb{R}^{d+1}_+}\mathcal{K}_{t,s}(x,y)f(y,s)\,\frac{\mathrm{d}y\mathrm{d}s}{s}.$$

Lemma 3.2.5. *Let X be any fixed Banach space and $1<q<\infty$. Assume that the kernel $\mathcal{K}_{t,s}(x,y)$ satisfies the following estimation: there exist positive constants κ, ε, C such that*

$$|\mathcal{K}_{t,s}(x,y)|\le\frac{C\min\left\{\frac{s}{t},\frac{t}{s}\right\}^{\varepsilon}\min\left\{\frac{1}{t},\frac{1}{s}\right\}^{d}}{\left(1+\min\left\{\frac{1}{t},\frac{1}{s}\right\}|x-y|\right)^{d+\kappa}}.\tag{3.2.7}$$

Then the linear operator $\mathcal{K}$ initially defined on $C_c(\mathbb{R}^{d+1}_+)\otimes X$ extends to a bounded linear operator on $T^p_q(\mathbb{R}^{d+1}_+;X)$ for $1\le p<\infty$. More precisely,

$$\|\mathcal{K}(f)\|_{T^p_q(X)}\lesssim_{\varepsilon,\kappa}p^{\frac{1}{q}}\|f\|_{T^p_q(X)},\quad\forall\,f\in T^p_q(\mathbb{R}^{d+1}_+;X),\ 1\le p<\infty.$$

Furthermore, for any $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$, we have

$$\|\mathcal{C}_q\left(\mathcal{K}(f)\right)\|_p\lesssim_{\varepsilon,\kappa}\|\mathcal{C}_q(f)\|_p,\quad 1\le p\le\infty.$$

Proof. Fix $f \in C_c(\mathbb{R}_+^{d+1}) \otimes X$. Without loss of generality, we can assume $\kappa < \varepsilon$ from (3.2.7). We first deal with the case $p = q$. By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} \|\mathcal{K}(f)(x,t)\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t} &= \int_{\mathbb{R}_+^{d+1}} \left\| \int_{\mathbb{R}_+^{d+1}} \mathcal{K}_{t,s}(y,w) f(w,s) \frac{\mathrm{d}w\mathrm{d}s}{s} \right\|_X^q \frac{\mathrm{d}y\mathrm{d}t}{t} \\ &\leq \int_{\mathbb{R}_+^{d+1}} \left(\int_{\mathbb{R}_+^{d+1}} |\mathcal{K}_{t,s}(y,w)| \frac{\mathrm{d}w\mathrm{d}s}{s} \right)^{\frac{q}{q'}} \\ &\quad \cdot \left(\int_{\mathbb{R}_+^{d+1}} |\mathcal{K}_{t,s}(y,w)| \|f(w,s)\|_X^q \frac{\mathrm{d}w\mathrm{d}s}{s} \right) \frac{\mathrm{d}y\mathrm{d}t}{t}.\end{aligned}$$

We obtain that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} |\mathcal{K}_{t,s}(y,w)| \frac{\mathrm{d}w\mathrm{d}s}{s} &\leq \int_0^t \int_{\mathbb{R}^d} \frac{Cs^\varepsilon t^{-\varepsilon} t^{-d}}{(1+t^{-1}|y-w|)^{d+\kappa}} \frac{\mathrm{d}w\mathrm{d}s}{s} \\ &\quad + \int_t^\infty \int_{\mathbb{R}^d} \frac{Ct^\varepsilon s^{-\varepsilon} s^{-d}}{(1+s^{-1}|y-w|)^{d+\kappa}} \frac{\mathrm{d}w\mathrm{d}s}{s} \\ &\lesssim_{\varepsilon,\kappa} \int_{\mathbb{R}^d} \frac{Ct^{-d}}{(1+t^{-1}|y-w|)^{d+\kappa}} \mathrm{d}w + \int_t^\infty t^\varepsilon s^{-\varepsilon-1} \mathrm{d}s \\ &\lesssim_{\varepsilon,\kappa} 1.\end{aligned}$$

It is clear that in the assumption of $\mathcal{K}_{t,s}(y,w)$, (w,s) plays the same role as (y,t). Thus

$$\|\mathcal{K}(f)\|_{L^q(\mathbb{R}_+^{d+1};X)}^q \lesssim_{\varepsilon,\kappa} \int_{\mathbb{R}_+^{d+1}} \|f(w,s)\|_X^q \frac{\mathrm{d}w\mathrm{d}s}{s} = \|f\|_{L^q(\mathbb{R}_+^{d+1};X)}^q. \tag{3.2.8}$$

Then the case $p = q$ is done since $\|f\|_{T_q^q(X)} \approx \|f\|_{L^q(\mathbb{R}_+^{d+1};X)}$. Moreover, from the proof we observe that $\mathcal{K}$ is always bounded on $L^p(\mathbb{R}_+^{d+1};X)$ for $1 \leq p \leq \infty$.

For $1 \leq p < q$, by the interpolation—Lemma 3.2.3, it suffices to show the case $p = 1$. By the atomic decomposition—Lemma 3.2.1, It suffices to show that

$$\|\mathcal{K}(a)\|_{T_q^1(X)} \lesssim_{\varepsilon,\kappa} 1, \tag{3.2.9}$$

where a is an (X,q)-atom with $\mathrm{supp}(a) \subset \widehat{B}$ and $B = B(c_B, r_B)$. One can write

$$\begin{aligned}\|\mathcal{A}_q[\mathcal{K}(a)]\|_1 &= \int_{4B} \mathcal{A}_q[\mathcal{K}(a)](x)\,\mathrm{d}x + \int_{(4B)^C} \mathcal{A}_q[\mathcal{K}(a)](x)\,\mathrm{d}x \\ &= I + II.\end{aligned}$$

From (3.2.8) we obtain

$$\|\mathcal{A}_q[\mathcal{K}(a)]\|_q^q \lesssim_{\varepsilon,\kappa} \int_{\mathbb{R}_+^{d+1}} \|a(w,s)\|_X^q \frac{\mathrm{d}w\mathrm{d}s}{s} \leq |B|^{1-q}. \tag{3.2.10}$$

Then we can estimate the term I:

$$I \leq |4B|^{\frac{1}{q'}} \|\mathcal{A}_q[\mathcal{K}(a)]\|_q \lesssim_{\varepsilon,\kappa} 1. \tag{3.2.11}$$

Now we handle the second term II. By Hölder's inequality, we observe

$$\begin{aligned}(\mathcal{A}_q[\mathcal{K}(a)](x))^q &\le \int_0^\infty \int_{|y-x|<t} \left(\int_{\widehat{B}} |\mathcal{K}_{t,s}(y,w)|^{q'} \frac{\mathrm{d}w\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \cdot \left(\int_{\widehat{B}} \|a(w,s)\|_X^q \frac{\mathrm{d}w\mathrm{d}s}{s}\right) \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \\ &\lesssim |B|^{1-q} \int_0^\infty \int_{|y-x|<t} \left(\int_{\widehat{B}} |\mathcal{K}_{t,s}(y,w)|^{q'} \frac{\mathrm{d}w\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \\ &= |B|^{1-q} \int_0^{r_B} \int_{|y-x|<t} \left(\int_{\widehat{B}} |\mathcal{K}_{t,s}(y,w)|^{q'} \frac{\mathrm{d}w\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \\ &\quad + |B|^{1-q} \int_{r_B}^\infty \int_{|y-x|<t} \left(\int_{\widehat{B}} |\mathcal{K}_{t,s}(y,w)|^{q'} \frac{\mathrm{d}w\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \\ &=: J_1 + J_2.\end{aligned}$$

When $x \in (4B)^C,\ w \in B$, we have

$$r_B < |x-w| \le |x-y| + |y-w| < t + |y-w|,$$

hence

$$|x-c_B| \le |x-w| + |w-c_B| < 2(t+|y-w|) \le 2(\max\{t,s\} + |y-w|).$$

Therefore we observe from (3.2.7) that

$$\begin{aligned}|\mathcal{K}_{t,s}(y,w)| &\lesssim_{\varepsilon,\kappa} \frac{\min\left\{\frac{s}{t},\frac{t}{s}\right\}^{\varepsilon} \min\left\{\frac{1}{t},\frac{1}{s}\right\}^{d}}{(\max\{t,s\}+|y-w|)^{d+\kappa} \min\left\{\frac{1}{t},\frac{1}{s}\right\}^{d+\kappa}} \\ &\lesssim_{\varepsilon,\kappa} \frac{\min\left\{\frac{s}{t},\frac{t}{s}\right\}^{\varepsilon} \min\left\{\frac{1}{t},\frac{1}{s}\right\}^{-\kappa}}{|x-c_B|^{d+\kappa}} = \frac{\min\{s^\varepsilon t^{\kappa-\varepsilon}, t^\varepsilon s^{\kappa-\varepsilon}\}}{|x-c_B|^{d+\kappa}}.\end{aligned}$$

Then

$$\begin{aligned}J_1 &\lesssim_{\varepsilon,\kappa} \frac{|B|^{1-q}|B|^{\frac{q}{q'}}}{|x-c_B|^{q(d+\kappa)}} \int_0^{r_B} \left(\int_0^{r_B} \min\left\{s^{q'\varepsilon}t^{q'(\kappa-\varepsilon)}, t^{q'\varepsilon}s^{q'(\kappa-\varepsilon)}\right\} \frac{\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}t}{t} \\ &= \frac{1}{|x-c_B|^{q(d+\kappa)}} \int_0^{r_B} \left(\int_0^t t^{q'(\kappa-\varepsilon)} s^{q'\varepsilon} \frac{\mathrm{d}s}{s} + \int_t^{r_B} s^{q'(\kappa-\varepsilon)} t^{q'\varepsilon} \frac{\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}t}{t} \\ &\lesssim_{\varepsilon,\kappa} \frac{r_B^{q\kappa}}{|x-c_B|^{q(d+\kappa)}}.\end{aligned}$$

For J_2, since $t \ge r_B \ge s$, and $|x-c_B| < 2(t+|y-w|)$,

$$\begin{aligned}J_2 &\lesssim_{\varepsilon,\kappa} \frac{|B|^{1-q}|B|^{\frac{q}{q'}}}{|x-c_B|^{q(d+\kappa)}} \int_{r_B}^\infty \left(\int_0^{r_B} s^{q'\varepsilon} t^{q'(\kappa-\varepsilon)} \frac{\mathrm{d}s}{s}\right)^{\frac{q}{q'}} \frac{\mathrm{d}t}{t} \\ &\lesssim_{\varepsilon,\kappa} \frac{r_B^{q\kappa}}{|x-c_B|^{q(d+\kappa)}}.\end{aligned}$$

Thus

$$\mathcal{A}_q[\mathcal{K}(a)](x) \lesssim_{\varepsilon,\kappa} \frac{r_B^\kappa}{|x-c_B|^{d+\kappa}}, \quad x \in (4B)^C.$$

Since

$$\int_{(4B)^C} \frac{r_B^\kappa}{|x-c_B|^{d+\kappa}}\,\mathrm{d}x = \sum_{m=2}^{\infty}\int_{2^{m+1}B\setminus 2^mB} \frac{r_B^\kappa}{|x-c_B|^{d+\kappa}}\,\mathrm{d}x \le \sum_{m=2}^{\infty}\int_{2^{m+1}B} \frac{r_B^\kappa}{2^{m(d+\kappa)}r_B^{d+\kappa}}\,\mathrm{d}x$$
$$\lesssim \sum_{m=2}^{\infty}\frac{(2^{m+1}r_B)^d}{2^{m(d+\kappa)}r_B^d} \lesssim \sum_{m=2}^{\infty} 2^{-m\kappa} \lesssim_\kappa 1,$$

we obtain

$$II \lesssim_{\varepsilon,\kappa} 1. \tag{3.2.12}$$

For the case $q<p<+\infty$, we denote by $\mathcal{K}^*$ the adjoint operator. It is clear that the kernel of $\mathcal{K}^*$ has the same estimation as that of $\mathcal{K}$. For $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$, we obtain from Lemma 3.2.4 that

$$\begin{aligned}\|\mathcal{K}(f)\|_{T_q^p(X)} &\lesssim p^{\frac{1}{q}}\sup_g\left|\int_{\mathbb{R}^{d+1}_+}\langle \mathcal{K}(f)(x,t),g(x,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right|\\ &= p^{\frac{1}{q}}\sup_g\left|\int_{\mathbb{R}^{d+1}_+}\langle f(x,t),\mathcal{K}^*(g)(x,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right|\\ &\le p^{\frac{1}{q}}\|f\|_{T_q^p(X)}\|\mathcal{K}^*(g)\|_{T_{q'}^{p'}(X^*)} \lesssim p^{\frac{1}{q}}\|f\|_{T_q^p(X)},\end{aligned}$$

where the supremum is taken over all $g\in C_c(\mathbb{R}^{d+1}_+)\otimes X$ such that $\|\mathcal{A}_{q'}(g)\|_{p'}\le 1$. Consequently, we observe that $\mathcal{K}$ extends to a bounded linear operator on $T_q^p(\mathbb{R}^{d+1}_+;X)$ for $1\le p<\infty$. More precisely,

$$\|\mathcal{K}(f)\|_{T_q^p(X)} \lesssim_{\varepsilon,\kappa} p^{\frac{1}{q}}\|f\|_{T_q^p(X)},\quad \forall\, f\in T_q^p(\mathbb{R}^{d+1}_+;X).$$

Now we prove the second assertion of this lemma. Fix $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$, take a ball B in $\mathbb{R}^d$, we can write

$$\begin{aligned}\left(\int_{\widehat{B}}\|\mathcal{K}(f)(x,t)\|_X^q\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right)^{\frac{1}{q}} &= \sup_g\left|\int_{\widehat{B}}\langle \mathcal{K}(f)(x,t),g(x,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right|\\ &= \sup_g\left|\int_{\widehat{B}}\langle f(x,t),\mathcal{K}^*(g)(x,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right|\\ &\le \sup_g\|\mathcal{K}^*(g)\|_{L^{q'}(\widehat{B};X^*)}\left(\int_{\widehat{B}}\|f(x,t)\|_X^q\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right)^{\frac{1}{q}},\end{aligned}$$

where the supremum is taken over all g in the unit ball of $L^{q'}(\widehat{B};X^*)$. From (3.2.8) we know that

$$\|\mathcal{K}^*(g)\|_{L^{q'}(\mathbb{R}^{d+1}_+;X^*)} \lesssim_{\varepsilon,\kappa} \|g\|_{L^{q'}(\widehat{B};X^*)}.$$

Thus for any $x\in\mathbb{R}^d$,

$$\begin{aligned}\left(\mathcal{C}_q\left[\mathcal{K}(f)\right](x)\right)^q &= \sup_{x\in B}\frac{1}{|B|}\int_{\widehat{B}}\|\mathcal{K}(f)(x,t)\|_X^q\,\frac{\mathrm{d}x\mathrm{d}t}{t}\\ &\lesssim_{\varepsilon,\kappa}\sup_{x\in B}\frac{1}{|B|}\int_{\widehat{B}}\|f(x,t)\|_X^q\,\frac{\mathrm{d}x\mathrm{d}t}{t} = \left(\mathcal{C}_q(f)(x)\right)^q.\end{aligned}$$

Therefore we obtain

$$\|\mathcal{C}_q\left[\mathcal{K}(f)\right]\|_p \lesssim_{\varepsilon,\kappa} \|\mathcal{C}_q\left(f\right)\|_p, \quad 1 \le p \le \infty.$$

The proof is completed. □

Now we come to the second important linear operator, which will relate the tent space $T_q^p(\mathbb{R}^{d+1}_+;X)$ to the Hardy space $H^p_{q,L}(\mathbb{R}^d;X)$.

Recall the operator $\mathcal{Q}(f)(x,t) = -tLe^{-tL}(f)(x)$. Define the operator π_L acting on $C_c(\mathbb{R}^{d+1}_+)\otimes X$ as

$$\pi_L(f)(x) = \int_0^\infty \mathcal{Q}(f(\cdot,t))(x,t)\,\frac{\mathrm{d}t}{t}, \quad \forall\, x\in\mathbb{R}^d.$$

It is easy to verify that π_L is well-defined. The following lemma asserts that π_L extends to a bounded linear operator from $T_q^p(\mathbb{R}^{d+1}_+;X)$ to $H^p_{q,L}(\mathbb{R}^d;X)$. We will denote it by π_L as well.

Lemma 3.2.6. *Let X be any fixed Banach space and $1<q<\infty$. The operator π_L initially defined on $C_c(\mathbb{R}^{d+1}_+)\otimes X$ extends to a bounded linear operator from $T_q^p(\mathbb{R}^{d+1}_+;X)$ to $H^p_{q,L}(\mathbb{R}^d;X)$ for $1\le p<\infty$. More precisely,*

$$\|\pi_L(f)\|_{H^p_{q,L}(X)} \lesssim_\beta p^{\frac{1}{q}}\|f\|_{T^p_q(X)}, \quad \forall\, f\in T_q^p(\mathbb{R}^{d+1}_+;X),\ 1\le p<\infty.$$

Furthermore, for any $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$, we have

$$\|\pi_L(f)\|_{BMO^p_{q,L}(X)} \lesssim_\beta \|\mathcal{C}_q(f)\|_p, \quad 1\le p\le\infty.$$

Proof. Let $f\in C_c(\mathbb{R}^{d+1}_+)\otimes X$. Recall that $k(t,x,y)$ is the kernel of the operator $\mathcal{Q}$, then

$$\begin{aligned}\mathcal{Q}[\pi_L(f)](x,t) &= \int_{\mathbb{R}^d} k(t,y,z)\pi_L(f)(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^d} k(t,y,z)\left(\int_{\mathbb{R}^{d+1}_+} k(s,z,w)f(w,s)\,\frac{\mathrm{d}w\mathrm{d}s}{s}\right)\mathrm{d}z\\ &= \int_{\mathbb{R}^{d+1}_+}\left(\int_{\mathbb{R}^d} k(t,y,z)k(s,z,w)\,\mathrm{d}z\right)f(w,s)\,\frac{\mathrm{d}w\mathrm{d}s}{s}.\end{aligned} \tag{3.2.13}$$

We denote by

$$\Phi_{t,s}(y,w) = \int_{\mathbb{R}^d} k(t,y,z)k(s,z,w)\,\mathrm{d}z.$$

Note that $k(t,\cdot,\cdot)$ is the kernel of the operator $\mathcal{Q}=-te^{-tL}$, thus $\Phi_{t,s}(\cdot,\cdot)$ is the kernel of $-tLe^{-tL}\circ(-sLe^{-sL}) = tsL^2e^{-(t+s)L}$. On the other hand, $\partial_r^2(e^{-rL})|_{r=t+s} = L^2e^{-(t+s)L}$ which has the kernel $\partial_r^2K(r,\cdot,\cdot)|_{r=t+s}$. Then by (3.1.3), we obtain

$$|\Phi_{t,s}(y,w)| \lesssim_{d,\beta} \frac{ts}{(t+s)^{2-\beta}(t+s+|y-w|)^{d+\beta}} \lesssim_\beta \frac{\min\left\{\frac{s}{t},\frac{t}{s}\right\}\min\left\{\frac{1}{t},\frac{1}{s}\right\}^d}{\left(1+\min\left\{\frac{1}{t},\frac{1}{s}\right\}|x-y|\right)^{d+\beta}}.$$

Denote by

$$\mathcal{P} = 4\mathcal{Q}\circ\pi_L. \tag{3.2.14}$$

From Lemma 3.2.5, we conclude that $\mathcal{P}$ initially defined on $C_c(\mathbb{R}^{d+1}_+) \otimes X$ extends to a bounded linear operator on $T^p_q(\mathbb{R}^{d+1}_+; X)$. Moreover,

$$\|\mathcal{P}(f)\|_{T^p_q(X)} \lesssim_\beta p^{\frac{1}{q}} \|f\|_{T^p_q(X)}.$$

Therefore

$$\|\pi_L(f)\|_{H^p_{q,L}(X)} = 4^{-1}\|\mathcal{P}(f)\|_{T^p_q(X)} \lesssim_\beta p^{\frac{1}{q}}\|f\|_{T^p_q(X)},$$

which is the desired assertion.

For the second part, we obtain the desired assertion from Lemma 3.2.5 immediately. □

Remark 3.2.7. One can verify that $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$, thus $\mathcal{P}$ serves as a projection from $T^p_q(\mathbb{R}^{d+1}_+; X)$ onto itself. Indeed, we can also obtain this lemma under the assumption that L is a sectotrial operator satisfying only (3.1.2).

3.3 vector-valued intrinsic square functions

In this section, we begin with the introduction of vector-valued intrinsic square functions, originally presented by Wilson in [96] in the case of convolution operators. We then proceed to compare them with the q-variant of Lusin area integral associated with a generator L.

Recall that L is assumed to be a sectorial operator of type α ($0 \le \alpha < \pi/2$) satisfying assumptions (II.1), (II.2) and (II.3) with $\beta > 0, 0 < \gamma \le 1$. Define $\mathcal{H}_{\gamma,\beta}$ as the family of functions $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$|\varphi(x,y)| \le \frac{1}{(1+|x-y|)^{d+\beta}}, \tag{3.3.1}$$

$$|\varphi(x+h,y) - \varphi(x,y)| + |\varphi(x,y+h) - \varphi(x,y)| \le \frac{|h|^\gamma}{(1+|x-y|)^{d+\beta+\gamma}} \tag{3.3.2}$$

whenever $2|h| \le 1 + |x-y|$ and

$$\int_{\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}y = 0. \tag{3.3.3}$$

For $\varphi \in \mathcal{H}_{\gamma,\beta}$, define $\varphi_t(x,y) = t^{-d}\varphi(t^{-1}x, t^{-1}y)$.

Let $f \in C_c(\mathbb{R}^d) \otimes X$. We define

$$A_{\gamma,\beta}(f)(x,t) = \sup_{\varphi \in \mathcal{H}_{\gamma,\beta}} \left\| \int_{\mathbb{R}^d} \varphi_t(x,y) f(y)\,\mathrm{d}y \right\|_X, \quad \forall\,(x,t) \in \mathbb{R}^{d+1}_+.$$

Then the intrinsic square functions of f are defined as

$$S_{q,\gamma,\beta}(f)(x) = \left(\int_{\Gamma(x)} (A_{\gamma,\beta}(f)(y,t))^q \, \frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}} \right)^{\frac{1}{q}},$$

and

$$G_{q,\gamma,\beta}(f)(x) = \left(\int_0^\infty (A_{\gamma,\beta}(f)(x,t))^q \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

Theorem 3.3.1. *Let X be any fixed Banach space, $1 < q < \infty$ and $1 \le p < \infty$. Let L be any fixed sectorial operator L satisfying* (II.1), (II.2) *and* (II.3). *For any $f \in C_c(\mathbb{R}^d) \otimes X$, we have*

$$S_{q,\gamma,\beta}(f)(x) \approx_{\gamma,\beta} G_{q,\gamma,\beta}(f)(x), \tag{3.3.4}$$

$$S_{q,L}(f)(x) \lesssim S_{q,\gamma,\beta}(f)(x), \quad G_{q,L}(f)(x) \lesssim G_{q,\gamma,\beta}(f)(x), \tag{3.3.5}$$

and

$$\|S_{q,\gamma,\beta}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,L}(f)\|_p. \tag{3.3.6}$$

Remark 3.3.2. The following g-function version of (3.3.6) holds also

$$\|G_{q,\gamma,\beta}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{2}{q}} \|G_{q,L}(f)\|_p. \tag{3.3.7}$$

But its proof is much more involved and depends in turn on Theorem II.1 that will be concluded in the next section.

As in the classical case [95], the assertions (3.3.4) and (3.3.5) can be deduced easily from the following facts on $\mathcal{H}_{\gamma,\beta}$.

Lemma 3.3.3. *Let $\varphi \in \mathcal{H}_{\gamma,\beta}$. The following properties hold:*

(i) *if $t \ge 1$, then $t^{-d-\gamma}\varphi_t \in \mathcal{H}_{\gamma,\beta}$;*

(ii) *if $|z| \le 1$, $t \ge 1$, then $(2t)^{-d-\gamma-\beta}\left(\varphi^{(z)}\right)_t \in \mathcal{H}_{\gamma,\beta}$, where $\varphi^{(z)}(x,y) = \varphi(x-z,y)$.*

Proof. The proof is similar to the case of Wilson [95], while the present setting is non-convolutive, let us give the sketch. The claim (i) is easy by definition. For the claim (ii), notice that

$$2^{-1}(1+|x-y|) \le 1 + |(x-z)-y| \le 2(1+|x-y|).$$

By definition, we have

$$|\varphi^{(z)}(x,y)| = |\varphi(x-z,y)| \le \frac{1}{(1+|(x-z)-y|)^{d+\beta}} \le \frac{2^{d+\beta}}{(1+|x-y|)^{d+\beta}}.$$

and

$$\begin{aligned} |\varphi^{(z)}(x+h,y) - \varphi^{(z)}(x,y)| &= |\varphi(x-z+h,y) - \varphi(x-z,y)| \\ &\le \frac{|h|^\gamma}{(1+|(x-z)-y|)^{d+\beta+\gamma}} \\ &\le \frac{2^{d+\beta+\gamma}|h|^\gamma}{(1+|x-y|)^{d+\beta+\gamma}}. \end{aligned}$$

The same Hölder continuity estimation holds for the variable y. Thus we obtain that $2^{-d-\beta-\gamma}\varphi^{(z)} \in \mathcal{H}_{\gamma,\beta}$. Then the claim (ii) follows from the claim (i). □

With Lemma 3.3.3, the assertions (3.3.4) and (3.3.5) will follow easily. The most challenging part of Theorem 3.3.4 lies in (3.3.6). In addition to the interpolation and duality theory on the (vector-valued) tent space that have been built in Section 3.2, the following pointwise estimate is another technical part in the proof of estimate (3.3.6).

Recall that $k(t,x,y)$ is the kernel of the operator $\mathcal{Q}$. Let $\theta \in \mathcal{H}_{\gamma,\beta}$, define

$$\mathcal{L}^\theta_{t,s}(y,w) = \int_{\mathbb{R}^d} \theta_t(y,z)k(s,z,w)\,\mathrm{d}z.$$

Lemma 3.3.4. *Let $\nu = 2^{-1}\min\{\gamma,\beta\}$ and $\zeta = (d+2^{-1}\beta)(d+\beta)^{-1}$, then*

$$\sup_{\theta\in\mathcal{H}_{\gamma,\beta}} |\mathcal{L}^{\theta}_{t,s}(y,w)| \lesssim_{\gamma,\beta} \frac{\min\left\{\frac{s}{t},\frac{t}{s}\right\}^{(1-\zeta)\nu}\min\left\{\frac{1}{t},\frac{1}{s}\right\}^{d}}{\left(1+\min\left\{\frac{1}{t},\frac{1}{s}\right\}|y-w|\right)^{d+\frac{1}{2}\beta}}.$$

Proof. To estimate the kernel $\mathcal{L}^{\theta}_{t,s}(y,w)$, we follow a similar argument presented in [44, Chapter 8, 8.6.3].

Let $\theta\in\mathcal{H}_{\gamma,\beta}$, we have

$$|\theta_t(y,z)| \le \frac{t^{-d}}{(1+t^{-1}|y-z|)^{d+\beta}}, \quad \forall\, y,z\in\mathbb{R}^d,\ t>0. \tag{3.3.8}$$

For $2|z-z'|<t+|y-z|$, we have $t^{-1}|z-z'|<1+t^{-1}|y-z|$, then

$$\begin{aligned}|\theta_t(y,z)-\theta_t(y,z')| &\le \frac{t^{-d-\gamma}|z-z'|^{\gamma}}{(1+t^{-1}|y-z|)^{d+\beta+\gamma}} \le \frac{\min\left\{(t^{-1}|z-z'|)^{\gamma},(1+t^{-1}|y-z|)^{\gamma}\right\}}{t^d(1+t^{-1}|y-z|)^{d+\beta+\gamma}}\\ &\lesssim \frac{\min\{1,(t^{-1}|z-z'|)^{\gamma}\}}{t^d}.\end{aligned}$$

For $2|z-z'|\ge t+|y-z|$, we have $t^{-1}|z-z'|\ge 1/2$, then

$$|\theta_t(y,z)-\theta_t(y,z')| \le |\theta_t(y,z)|+|\theta_t(y,z')| \le 2t^{-d} \lesssim \frac{\min\{1,(t^{-1}|z-z'|)^{\gamma}\}}{t^d}.$$

Hence

$$|\theta_t(y,z)-\theta_t(y,z')| \lesssim \frac{\min\{1,(t^{-1}|z-z'|)^{\gamma}\}}{t^d}, \quad \forall\, y,z,z'\in\mathbb{R}^d,\ t>0.$$

On the other hand, Lemma 3.1.1 asserts that there exists a positive constant C_k such that $C_k^{-1}(k(s,\cdot,\cdot))_{s^{-1}}\in\mathcal{H}_{\gamma,\beta}$ (see also the Convention afterwards). Thus, one gets for all $w,z,z'\in\mathbb{R}^d,\ s>0$,

$$|k(s,z,w)| \lesssim \frac{C_k s^{-d}}{(1+s^{-1}|z-w|)^{d+\beta}}, \quad |k(s,z,w)-k(s,z',w)| \lesssim \frac{C_k\min\{1,(s^{-1}|z-z'|)^{\gamma}\}}{s^d}.$$

Now we start to deal with the kernel $\mathcal{L}^{\theta}_{t,s}(y,w)$. By symmetry, it suffices to handle the case $s\le t$. First we observe the following estimate,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{s^{-d}\min\{1,(t^{-1}|u|)^{\gamma}\}}{(1+s^{-1}|u|)^{d+\beta}}\,\mathrm{d}u &= \int_{|u|<t}\frac{s^{-d}(t^{-1}|u|)^{\gamma}}{(1+s^{-1}|u|)^{d+\beta}}\,\mathrm{d}u + \int_{|u|>t}\frac{s^{-d}}{(1+s^{-1}|u|)^{d+\beta}}\,\mathrm{d}u\\ &\le \int_{|v|<t/s}\left(\frac{s}{t}\right)^{\gamma}\frac{|v|^{\gamma}}{(1+|v|)^{d+\beta}}\,\mathrm{d}v + \int_{|u|>t}s^{\beta}|u|^{-d-\beta}\,\mathrm{d}u\\ &=: J_1+J_2\end{aligned}$$

Taking $\nu=2^{-1}\min\{\gamma,\beta\}$, and we have $|v|^{\gamma}<(t/s)^{\gamma-\nu}|v|^{\nu}$. Then we obtain

$$J_1 \le \left(\frac{s}{t}\right)^{\nu}\int_{\mathbb{R}^d}\frac{|v|^{\nu}}{(1+|v|)^{d+\beta}}\,\mathrm{d}v \lesssim_{\gamma,\beta}\left(\frac{s}{t}\right)^{\nu}.$$

For J_2, we have

$$J_2 \lesssim \int_t^\infty s^\beta r^{-\beta-1}\,\mathrm{d}r \lesssim_\beta \left(\frac{s}{t}\right)^\beta \le \left(\frac{s}{t}\right)^\nu.$$

Thus for $s \le t$, by the vanishing property (3.3.3) of $k(s,\cdot,w)$, one gets

$$\begin{aligned}|\mathcal{L}^\theta_{t,s}(y,w)| &\le \left|\int_{\mathbb{R}^d} [\theta_t(y,z)-\theta_t(y,w)]\,k(s,z,w)\,\mathrm{d}z\right| \\ &\le C_k \int_{\mathbb{R}^d} \frac{\min\{1,(t^{-1}|z-w|)^\gamma\}}{t^d}\frac{s^{-d}}{(1+s^{-1}|z-w|)}\,\mathrm{d}z \\ &\lesssim_{\gamma,\beta} t^{-d}\left(\frac{s}{t}\right)^\nu \le \min\left\{\frac{1}{t},\frac{1}{s}\right\}^d \min\left\{\frac{s}{t},\frac{t}{s}\right\}^\nu.\end{aligned}$$

On the other hand,

$$|\mathcal{L}^\theta_{t,s}(y,w)| \le \int_{\mathbb{R}^d} |\theta_t(y,z)||k(s,z,w)|\,\mathrm{d}z \lesssim_\beta \frac{\min\left\{\frac{1}{t},\frac{1}{s}\right\}^d}{\left(1+\min\left\{\frac{1}{t},\frac{1}{s}\right\}|y-w|\right)^{d+\beta}}.$$

Let $\zeta = (d+2^{-1}\beta)(d+\beta)^{-1}$, we then get

$$|\mathcal{L}^\theta_{t,s}(y,w)| = |\mathcal{L}^\theta_{t,s}(y,w)|^{1-\zeta}|\mathcal{L}^\theta_{t,s}(y,w)|^\zeta \lesssim_{\gamma,\beta} \frac{\min\left\{\frac{s}{t},\frac{t}{s}\right\}^{(1-\zeta)\nu}\min\left\{\frac{1}{t},\frac{1}{s}\right\}^d}{\left(1+\min\left\{\frac{1}{t},\frac{1}{s}\right\}|y-w|\right)^{d+\frac{1}{2}\beta}}.$$

It is clear that the estimation of $\mathcal{L}^\theta_{t,s}(y,w)$ is independent of the choice of θ, and thus the desired estimate is obtained. □

Now let us prove Theorem 3.3.1.

Proof. The pointwise estimate (3.3.4) follows from Lemma 3.3.3 (ii). Indeed, for $|x-y| < t$, let $w = (x-y)/t$; then for any $\varphi \in \mathcal{H}_{\gamma,\beta}$, we have $2^{-d-\beta-\gamma}\varphi^{(w)} \in \mathcal{H}_{\gamma,\beta}$. Hence

$$\begin{aligned}A_{\gamma,\beta}(f)(x,t) &= \sup_{\varphi\in\mathcal{H}_{\gamma,\beta}}\left\|\int_{\mathbb{R}^d}\varphi_t(x,z)f(z)\,\mathrm{d}z\right\|_X \\ &\le 2^{d+\beta+\gamma}\sup_{\varphi^{(w)}\in\mathcal{H}_{\gamma,\beta}}\left\|\int_{\mathbb{R}^d}\left(\varphi^{(w)}\right)_t(x,z)f(z)\,\mathrm{d}z\right\|_X \\ &= 2^{d+\beta+\gamma}A_{\gamma,\beta}(f)(y,t).\end{aligned}$$

Exchanging x and y and taking $-w$ in place of w, the reverse inequality is also true. Then (3.3.4) follows immediately.

Now we turn to the pointwise estimates (3.3.5). Lemma 3.1.1 asserts that there exists a positive constant C_k such that $C_k^{-1}(k(t,\cdot,\cdot))_{t^{-1}} \in \mathcal{H}_{\gamma,\beta}$ (see also the Convention afterwards). Consequently, for all $x \in \mathbb{R}^d,\ t>0$, we have

$$\begin{aligned}\|\mathcal{Q}(f)(x,t)\|_X &= \left\|\int_{\mathbb{R}^d}k(t,x,y)f(y)\,\mathrm{d}y\right\|_X = C_k\left\|\int_{\mathbb{R}^d}\left(C_k^{-1}(k(t,x,y))_{t^{-1}}\right)_t f(y)\,\mathrm{d}y\right\|_X \\ &\le C_k\sup_{\varphi\in\mathcal{H}_{\gamma,\beta}}\left\|\int_{\mathbb{R}^d}\varphi_t(x,y)f(y)\,\mathrm{d}y\right\|_X = C_kA_{\gamma,\beta}(f)(x,t). \qquad (3.3.9)\end{aligned}$$

Then the estimates (3.3.5) follows trivially.

Below we explain the proof of (3.3.6). Let $h \in C_c(\mathbb{R}^{d+1}_+) \otimes X$, we have

$$\begin{aligned}
A_{\gamma,\beta}[\pi_L(h)](y,t) &= \sup_{\theta\in\mathcal{H}_{\gamma,\beta}} \left\| \int_{\mathbb{R}^d} \theta_t(y,z) \left(\int_{\mathbb{R}^{d+1}_+} k(s,z,w) h(w,s) \frac{\mathrm{d}w\mathrm{d}s}{s} \right) \mathrm{d}z \right\|_X \\
&= \sup_{\theta\in\mathcal{H}_{\gamma,\beta}} \left\| \int_{\mathbb{R}^{d+1}_+} \left(\int_{\mathbb{R}^d} \theta_t(y,z) k(s,z,w)\, \mathrm{d}z \right) h(w,s) \frac{\mathrm{d}w\mathrm{d}s}{s} \right\|_X \\
&= \sup_{\theta\in\mathcal{H}_{\gamma,\beta}} \left\| \int_{\mathbb{R}^{d+1}_+} \mathcal{L}^\theta_{t,s}(y,w) h(w,s) \frac{\mathrm{d}w\mathrm{d}s}{s} \right\|_X \\
&\le \int_{\mathbb{R}^{d+1}_+} \left(\sup_{\theta\in\mathcal{H}_{\gamma,\beta}} |\mathcal{L}^\theta_{t,s}(y,w)| \right) \|h(w,s)\|_X \frac{\mathrm{d}w\mathrm{d}s}{s} \\
&=: \mathcal{L}(\|h\|_X)(y,t),
\end{aligned}$$

where the linear operator $\mathcal{L}$ has the kernel

$$\mathcal{L}_{t,s}(y,w) = \sup_{\theta\in\mathcal{H}_{\gamma,\beta}} |\mathcal{L}^\theta_{t,s}(y,w)|.$$

Then by Lemma 3.3.4 and Lemma 3.2.5 in the case $X = \mathbb{C}$, one obtaines

$$\|\mathcal{L}(\|h\|_X)\|_{T^p_q(\mathbb{C})} \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|\|h\|_X\|_{T^p_q(\mathbb{C})} = p^{\frac{1}{q}} \|h\|_{T^p_q(X)}, \quad 1 \le p < \infty.$$

Therefore

$$\|A_{\gamma,\beta}[\pi_L(h)]\|_{T^p_q(\mathbb{C})} \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|h\|_{T^p_q(X)}, \quad 1 \le p < \infty.$$

Let $f \in C_c(\mathbb{R}^d) \otimes X$, then we have $\mathcal{Q}(f) \in T^p_q(\mathbb{R}^{d+1}_+; X)$; moreover from the formula (1.2.1) and Remark **??**, the following Calderón identity holds

$$f = 4 \int_0^\infty \mathcal{Q}\left[\mathcal{Q}(f)(\cdot,t)\right](\cdot,t) \frac{\mathrm{d}t}{t}. \tag{3.3.10}$$

Therefore, one has that for $1 \le p < \infty$,

$$\begin{aligned}
\|S_{q,\gamma,\beta}(f)\|_p &= \|A_{\gamma,\beta}(f)\|_{T^p_q(\mathbb{C})} = 4\|A_{\gamma,\beta}\left[\pi_L(\mathcal{Q}(f))\right]\|_{T^p_q(\mathbb{C})} \\
&\lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|\mathcal{Q}(f)\|_{T^p_q(X)} = p^{\frac{1}{q}} \|S_{q,L}(f)\|_p,
\end{aligned}$$

which is the desired inequality. □

Remark 3.3.5. For any $f \in C_c(\mathbb{R}^d) \otimes X$, by Lemma 3.2.5, we also obtain

$$\|A_{\gamma,\beta}(f)\|_{T^\infty_q(\mathbb{C})} = 4\|A_{\gamma,\beta}\left[\pi_L(\mathcal{Q}(f))\right]\|_{T^\infty_q(\mathbb{C})} \lesssim_{\gamma,\beta} \|\mathcal{Q}(f)\|_{T^\infty_q(X)} = \|f\|_{BMO_{q,L}(X)}.$$

Together with the pointwise estimate (3.3.9), one gets the BMO–version of Theorem II.1: Let L be a generator as in Theorem II.1, then

$$\|f\|_{BMO_{q,L}(X)} \approx_{\gamma,\beta} \|f\|_{BMO_{q,\sqrt{\Delta}}(X)}. \tag{3.3.11}$$

3.4 Proof of the main Theorem

As pointed out in the introduction, the equivalence (II.4) in Theorem II.1 is an easy consequence of Theorem 3.3.1; but for another equivalence (II.5), we need to develop fully Mei's duality arguments between vector-valued Hardy and BMO type spaces [66]. This will be accomplished in the present section by combining the theory of vector-valued tent spaces and vector-valued Wilson's square functions—Theorem 3.3.1.

First of all, based on the duality between tent spaces—Lemma 3.2.4, the boundedness of the projection π_L—Lemma 3.2.6—yields the following vector-valued Fefferman-Stein duality theorem.

Theorem 3.4.1. *Let X be any fixed Banach space and $1 < q < \infty$. Let L be as in Theorem II.1. Both the spaces $BMO^{p'}_{q',L^*}(\mathbb{R}^d;X^*)$ and $H^{p'}_{q',L^*}(\mathbb{R}^d;X^*)$ are isomorphically identified as subspaces of the dual space of $H^p_{q,L}(\mathbb{R}^d;X)$. Moreover, they are norming for $H^p_{q,L}(\mathbb{R}^d;X)$ in the following sense,*

$$\|f\|_{H^p_{q,L}} \lesssim_\beta \max\left\{p^{\frac{1}{q}}p'^{\frac{1}{q'}}, p'^{\frac{2}{q'}}\right\} \sup_g \left|\int_{\mathbb{R}^d} \langle f(x), g(x)\rangle_{X\times X^*}\, \mathrm{d}x\right|, \quad 1<p<\infty,$$

where the supremum is taken over all $g \in C_c(\mathbb{R}^d)\otimes X^*$ such that $\|g\|_{H^{p'}_{q',L^*}(X^*)} \le 1$, and similarly,

$$\|f\|_{H^p_{q,L}} \lesssim_\beta \left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}} \sup_g \left|\int_{\mathbb{R}^d} \langle f(x), g(x)\rangle_{X\times X^*}\, \mathrm{d}x\right|, \quad 1\le p<q,$$

where the supremum is taken over all $g \in C_c(\mathbb{R}^d)\otimes X^*$ such that $\|g\|_{BMO^{p'}_{q',L^*}(X^*)} \le 1$. Furthermore, if X^* has the Radon-Nikodým property. Then

$$BMO^{p'}_{q',L^*}(X^*) = \left(H^p_{q,L}(\mathbb{R}^d;X)\right)^*, \quad 1\le p<q;$$
$$H^{p'}_{q',L^*}(X^*) = \left(H^p_{q,L}(\mathbb{R}^d;X)\right)^*, \quad 1<p<\infty.$$

Remark 3.4.2. Indeed, we can also obtain this duality theorem under the assumption that L is a sectotrial operator satisfying only (3.1.2), see Remark 3.2.7.

The more essential auxiliary result is the following duality property, which is inspired by [66, Theorem 2.4] (see also [97, 103]).

Proposition 3.4.3. *Let X be any fixed Banach space and $1 \le p < q$. Let L be any fixed sectorial operator satisfying* (II.1), (II.2) *and* (II.3). *Then for any $f \in C_c(\mathbb{R}^d)\otimes X$ and $g \in C_c(\mathbb{R}^d)\otimes X^*$, one has*

$$\left|\int_{\mathbb{R}^d} \langle f(x), g(x)\rangle_{X\times X^*}\, dx\right| \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p^{\frac{p}{q}} \|S_{q,L}(f)\|_p^{1-\frac{p}{q}} \|g\|_{BMO^{p'}_{q',L^*}(X^*)}. \tag{3.4.1}$$

Proof. Fixing $f \in C_c(\mathbb{R}^d)\otimes X$ and $g \in C_c(\mathbb{R}^d)\otimes X^*$, we consider truncated versions of $G_{q,L}(f)(x)$ as follows:

$$G(x,t) := \left(\int_t^\infty \|\mathcal{Q}(f)(x,s)\|_X^q \frac{dxds}{s}\right)^{\frac{1}{q}}, \quad x\in\mathbb{R}^d, t>0.$$

By approximation, we can assume that $G(x,t)$ is strictly positive. The operator $-tL^*e^{-tL^*}$ is denoted by $\mathcal{Q}^*$. By the Calderón identity—(3.3.10), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}\langle f(x),g(x)\rangle_{X\times X^*}\,\mathrm{d}x\right| &= 4\left|\int_{\mathbb{R}^{d+1}_+}\langle \mathcal{Q}(f)(x,t),\mathcal{Q}^*(g)(x,t)\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right| \\
&= 4\left|\int_{\mathbb{R}^{d+1}_+}\left\langle G^{\frac{p-q}{q}}(x,t)\mathcal{Q}(f)(x,t),G^{\frac{q-p}{q}}(x,t)\mathcal{Q}^*(g)(x,t)\right\rangle_{X\times X^*}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right| \\
&\lesssim \left(\int_{\mathbb{R}^{d+1}_+}G^{p-q}(x,t)\|\mathcal{Q}(f)(x,t)\|_X^q\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\quad\cdot\left(\int_{\mathbb{R}^{d+1}_+}G^{\frac{q-p}{q-1}}(x,t)\|\mathcal{Q}^*(g)(x,t)\|_{X^*}^{q'}\,\frac{\mathrm{d}x\mathrm{d}t}{t}\right)^{\frac{1}{q'}} \\
&= I\cdot II.
\end{aligned}$$

The term I is estimated as below,

$$\begin{aligned}
I^q &= -\int_{\mathbb{R}^d}\int_0^\infty G^{p-q}(x,t)\partial_t\left(G^q(x,t)\right)\,\mathrm{d}t\mathrm{d}x \\
&= -q\int_{\mathbb{R}^d}\int_0^\infty G^{p-1}(x,t)\partial_t G(x,t)\,\mathrm{d}t\mathrm{d}x \\
&\le -q\int_{\mathbb{R}^d}\int_0^\infty G^{p-1}(x,0)\partial_t G(x,t)\,\mathrm{d}t\mathrm{d}x \\
&= q\int_{\mathbb{R}^d}G^p(x,0)\,\mathrm{d}t\mathrm{d}x = q\|G_{q,L}(f)\|_p^p,
\end{aligned}$$

since $G(x,t)$ is decreasing in t, and $G(x,0)=G_{q,L}(f)(x)$.

For the term II, we introduce two more variants of $S_{q,\gamma,\beta}(f)$ (cf. [103]). The first is defined similarly to $G(\cdot,t)$:

$$S(x,t)=\left(\int_t^\infty\int_{|y-x|<s-\frac{t}{2}}(A_{\gamma,\beta}(f)(y,s))^q\,\frac{\mathrm{d}y\mathrm{d}s}{s^{d+1}}\right)^{\frac{1}{q}},\quad x\in\mathbb{R}^d,\ t>0.$$

To introduce the second one, let $\mathscr{D}_k$ be the family of dyadic cubes in $\mathbb{R}^d$ of side length 2^{-k}, that is,

$$\mathscr{D}_k=\left\{2^{-k}\prod_{j=1}^d[m_j,m_j+1):m_j\in\mathbb{Z},k\in\mathbb{Z}\right\}.$$

Denote c_Q as the center of a cube Q. Then, we define

$$\mathcal{S}(x,k)=\left(\int_{\sqrt{d}2^{-k}}^\infty\int_{|y-c_Q|<s}(A_{\gamma,\beta}(f)(y,s))^q\,\frac{\mathrm{d}y\mathrm{d}s}{s^{d+1}}\right)^{\frac{1}{q}},\quad \text{if } x\in Q\in\mathscr{D}_k,\ k\in\mathbb{Z}.$$

By definition, we have the following properties,

(i) $\mathcal{S}(\cdot,k)$ is increasing in k,

(ii) $\mathcal{S}(\cdot,k)$ is constant on every cube $Q\in\mathscr{D}_k$,

(iii) $\mathcal{S}(x,-\infty)=0$ and $\mathcal{S}(x,\infty)=S(x,0)=S_{q,\gamma,\beta}(f)(x)$.

If $s \geq t \geq \sqrt{d}2^{-k}$ and $x \in Q \in \mathscr{D}_k$, then $B(x, s - \frac{t}{2}) \subset B(c_Q, s)$, where $B(x,t)$ denotes the ball with center x and radius t. This implies

$$S(x,t) \leq \mathcal{S}(x,k), \quad x \in Q \in \mathscr{D}_k \text{ whenever } t \geq \sqrt{d}2^{-k}.$$

Using (3.3.4) and (3.3.5) we have

$$G_{q,L}(f)(x) \lesssim G_{q,\gamma,\beta}(f)(x) \approx_{\gamma,\beta} S_{q,\gamma,\beta}(f)(x),$$

and similarly,

$$G(x,t) \lesssim_{\gamma,\beta} S(x,t). \tag{3.4.2}$$

Now we proceed to estimate the term B based on these observations. Applying (3.4.2) to II, we have

$$\begin{aligned}
II^{q'} &\lesssim_{\gamma,\beta} \int_{\mathbb{R}^{d+1}_+} S^{\frac{q-p}{q-1}}(x,t) \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}x\mathrm{d}t}{t} \\
&= \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_k} \int_Q \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} S^{\frac{q-p}{q-1}}(x,t) \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_k} \int_Q \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \mathcal{S}^{\frac{q-p}{q-1}}(x,k) \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k} D(x,j) \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x
\end{aligned}$$

where $D(x,j) = \mathcal{S}^{\frac{q-p}{q-1}}(x,j) - \mathcal{S}^{\frac{q-p}{q-1}}(x,j-1)$. Then $D(x,j)$ is constant on every cube $Q \in \mathscr{D}_j$. Thus

$$\begin{aligned}
II^{q'} &\lesssim_{\gamma,\beta} \int_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} D(x,j) \left(\sum_{k=j}^{\infty} \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \right) \mathrm{d}x \\
&= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_j} \int_Q D(x,j) \int_0^{\sqrt{d}2^{-j+1}} \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x \\
&= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_j} D(x,j) \mathbb{1}_Q(x) \int_Q \int_0^{2\sqrt{d}\ell(Q)} \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x,
\end{aligned}$$

where $\ell(Q)$ denotes the length of Q. There exists a ball B such that $Q \subset B$, $Q \times (0, 2\sqrt{d}\ell(Q)] \subset \widehat{B}$ and $|B| \lesssim |Q|$. Then we deduce that

$$\int_Q \int_0^{2\sqrt{d}\ell(Q)} \|\mathcal{Q}^*(g)(x,t)\|^{q'}_{X^*} \frac{\mathrm{d}t}{t} \mathrm{d}x \leq \inf_{y \in B} \left\{ \mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](y) \right\}^{q'} |B| \lesssim \inf_{y \in Q} \left\{ \mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](y) \right\}^{q'} |Q|.$$

Therefore

$$\begin{aligned}
II^{q'} &\lesssim_{\gamma,\beta} \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_j} D(x,j) \mathbb{1}_Q(x) \inf_{y \in Q} \left\{ \mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](y) \right\}^{q'} |Q| \\
&\leq \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_j} \int_Q D(x,j) \left(\mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](x) \right)^{q'} \mathrm{d}x \leq \int_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} D(x,j) \left(\mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](x) \right)^{q'} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathcal{S}^{\frac{q-p}{q-1}}(x,\infty) \left(\mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right](x) \right)^{q'} \mathrm{d}x = \|S^{\frac{q-p}{q-1}}_{q,\gamma,\beta}(f)\|_r \left\| \left(\mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right] \right)^{q'} \right\|_{r'} \\
&= \|S_{q,\gamma,\beta}(f)\|_p^{\frac{q-p}{q-1}} \|\mathcal{C}_{q'} \left[\mathcal{Q}^*(g) \right]\|^{q'}_{p'},
\end{aligned}$$

where $1/r = 1 - q'/p' = (q-p)/(qp-p)$.

Combining the estimates of I and II with Theorem 3.3.1, we get the desired assertion. □

Finally, we arrive at the proof of our main theorem.

Proof of Theorm II.1. The first part (II.4) of Theorem II.1 is a consequence of Theorem 3.3.1. Indeed, suppose L be a generator such that the kernels of the generating semigroup satisfy (II.1) , (II.2) and (II.3) with $0 < \beta, \gamma \leq 1$, then the classical Poisson semigroup generated by $\sqrt{\Delta}$ satisfy obviously the same assumptions. Then

$$\|S_{q,L}(f)\|_p \lesssim \|S_{q,\gamma,\beta}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,\sqrt{\Delta}}(f)\|_p, \quad 1 \leq p < \infty.$$

Similarly we obtain

$$\|S_{q,\sqrt{\Delta}}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,L}(f)\|_p, \quad 1 \leq p < \infty.$$

As for another part (II.5), one side is easy by Theorem 3.3.1,

$$\|G_{q,L}(f)\|_p \lesssim \|G_{q,\gamma,\beta}(f)\|_p \approx_{\gamma,\beta} \|S_{q,\gamma,\beta}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}} \|S_{q,L}(f)\|_p, \quad 1 \leq p < \infty.$$

For the reverse direction, by Theorem 3.4.1 and Proposition 3.4.3, we have for $1 \leq p < (1+q)/2$,

$$\begin{aligned}
\|f\|_{H^p_{q,L}(X)} &\lesssim \left(\frac{p(q-1)}{q-p}\right)^{\frac{1}{q'}} \sup_g \left| \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle_{X \times X^*} \, \mathrm{d}x \right| \\
&\lesssim_{\gamma,\beta} \sup_g \|G_{q,L}(f)\|_p^{\frac{p}{q}} \|S_{q,L}(f)\|_p^{1-\frac{p}{q}} \|g\|_{BMO^{p'}_{q',L^*}(X^*)} \\
&\lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p^{\frac{p}{q}} \|S_{q,L}(f)\|_p^{1-\frac{p}{q}},
\end{aligned}$$

where the supremum is taken over all $g \in C_c(\mathbb{R}^d) \otimes X^*$ such that its $BMO^{p'}_{q',L^*}(X^*)$-norm is not more than 1. Hence

$$\|S_{q,L}(f)\|_p \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p, \quad 1 \leq p < \frac{1+q}{2}.$$

Now we deal with the case $(1+q)/2 \leq p < \infty$. Let $f \in C_c(\mathbb{R}^d) \otimes X$, we deduce from Theorem 3.4.1 that

$$\begin{aligned}
\|f\|_{H^p_{q,L}(X)} &\lesssim \max\left\{ p'^{\frac{1}{q'}} p^{\frac{1}{q}}, p'^{\frac{2}{q'}} \right\} \sup_h \left| \int_{\mathbb{R}^d} \langle f(x), h(x) \rangle_{X \times X^*} \, \mathrm{d}x \right| \\
&\lesssim p^{\frac{1}{q}} \sup_h \left| \int_{\mathbb{R}^{d+1}_+} \langle \mathcal{Q}(f)(x,t), \mathcal{Q}^*(h)(x,t) \rangle_{X \times X^*} \, \frac{\mathrm{d}x\mathrm{d}t}{t} \right| \\
&\lesssim p^{\frac{1}{q}} \sup_h \|G_{q,L}(f)\|_p \|G_{q',L^*}(h)\|_{p'} \\
&\lesssim_{\gamma,\beta} p^{\frac{1}{q}} p'^{\frac{1}{q'}} \sup_h \|G_{q,L}(f)\|_p \|S_{q',L^*}(h)\|_{p'} \\
&\lesssim p^{\frac{1}{q}} \|G_{q,L}(f)\|_p,
\end{aligned}$$

where the supremum is taken over all $h \in C_c(\mathbb{R}^d) \otimes X^*$ such that its $H^{p'}_{q',L^*}(X^*)$-norm is not more than 1.

Combining the estimations above we conclude that

$$p^{-\frac{1}{q}}\|S_{q,L}(f)\|_p \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_p \lesssim_{\gamma,\beta} p^{\frac{1}{q}}\|S_{q,L}(f)\|_p, \quad 1 \le p < \infty.$$

We complete the proof. □

3.5 Applications

In this section, we first recall the previous related results in [75, 63]. These, together with the tent space theory and Theorem II.1, will enable us to obtain the optimal Lusin type constants and the characterization of martingale type. In particular, this resolves partially Problem 1.8, Problem A.1 and Conjecture A.4 in the recent paper due to Xu [102].

Some notions or notations need to be presented. We first introduce the vector-valued atomic Hardy space $H^1_{\rm at}(\mathbb{R}^d;X)$. A measurable function $a \in L^\infty(\mathbb{R}^d;X)$ is called an X-valued atom if

$$\mathrm{supp}(a) \subset B,\ \int_{\mathbb{R}^d} a(x)\,\mathrm{d}x = 0,\ \|a\|_{L^\infty(X)} \le |B|^{-1},$$

where B is a ball in $\mathbb{R}^d$. The atomic Hardy space $H^1_{\rm at}(\mathbb{R}^d;X)$ is defined as the function space consisting of all functions f which admits an expression of the form

$$f = \sum_{j=1}^\infty \lambda_j a_j, \quad \sum_{j=1}^\infty |\lambda_j| < \infty,$$

where a_j is an X-valued atom. The norm of $H^1_{\rm at}(\mathbb{R}^d;X)$ is defined as

$$\|f\|_{H^1_{\rm at}(X)} = \inf\left\{\sum_{j=1}^\infty |\lambda_j| : f(x) = \sum_{j=1}^\infty \lambda_j a_j(x)\right\}.$$

This is a Banach space.

The BMO space $BMO(\mathbb{R}^d;X)$ is defined as the space of all $f \in L^1_{\rm loc}(\mathbb{R}^d;X)$ equipped with the semi-norm

$$\|f\|_{BMO(X)} = \sup_B \frac{1}{|B|}\int_B \|f - f_B\|_X\,\mathrm{d}x < \infty,$$

where the supremum runs over all the balls in $\mathbb{R}^d$ and f_B represents the average of f over B. $BMO(\mathbb{R}^d;X)$ is a Banach space modulo constants.

It is well-known that $BMO(\mathbb{R}^d;X^*)$ is isomorphically identified as a subspace of the dual space of $H^1_{\rm at}(\mathbb{R}^d;X)$ (cf. [21]) and it is norming for $H^1_{\rm at}(\mathbb{R}^d;X)$ in the following sense

$$\|f\|_{H^1_{\rm at}(X)} \approx \sup\left\{|\,\langle f,g\rangle\,| : g \in BMO(\mathbb{R}^d;X^*),\ \|g\|_{BMO(X)} \le 1\right\}$$

with universal constants. Furthermore, if the Banach space X^* has the Radon-Nikodým property, then (cf. [12])

$$(H^1_{\rm at}(\mathbb{R}^d;X))^* = BMO(\mathbb{R}^d;X^*), \tag{3.5.1}$$

with equivalent norms.

We recall the definitions of martingale type and cotype mentioned in Section 1.4. X is of *martingale cotype* q (with $2 \leq q < \infty$) if there exists a positive constant c such that every finite X-valued L^q-martingale $(f_n)_{n\geq 0}$, the following inequality holds

$$\sum_{n\geq 1} \mathbb{E}\|f_n - f_{n-1}\|_X^q \leq c^q \sup_{n\geq 0} \mathbb{E}\|f_n\|_X^q,$$

where $\mathbb{E}$ denotes the underlying expectation; and the least constant c is called the martingale cotype constant, denoted as $\mathsf{M}_{c,q}(X)$. While X is said to be of *martingale type* q (with $1 < q \leq 2$) if the reverse inequalities holds with c^{-1} in place of c and the corresponding martingale type constant is denoted by $\mathsf{M}_{t,q}(X)$.

Let X be a Banach space and $1 < q \leq 2$. The authors in [63] showed that the assertion that X is of martingale type q is equivalent to the one that for any $1 < p < \infty$, there exists a constant c_p such that for any $f \in C_c(\mathbb{R}^d) \otimes X$,

$$\|f\|_{L^p(X)} \leq c_p \|S_{q,\sqrt{\Delta}}(f)\|_p. \tag{3.5.2}$$

Later on, in [75] the authors investigated the relationships between $H^1_{\mathrm{at}}(\mathbb{R}^d; X)$ and $H^1_{q,\sqrt{\Delta}}(\mathbb{R}^d; X)$ as well as the ones between $BMO(\mathbb{R}^d; X)$ and $BMO_{q,\sqrt{\Delta}}(\mathbb{R}^d; X)$, and provided insights into the geometric properties of the underlying Banach space X.

Theorem 3.5.1. *Let X be a Banach space and $1 < q \leq 2$. The followings are equivalent*

(i) *X is of martingale type q;*

(ii) *there exists a constant c such that for any $f \in C_c(\mathbb{R}^d) \otimes X$,*

$$\|f\|_{H^1_{\mathrm{at}}(X)} \lesssim \|S_{q,\sqrt{\Delta}}(f)\|_1;$$

(iii) *there exist a constant c such that for any $f \in C_c(\mathbb{R}^d) \otimes X$,*

$$\|f\|_{BMO(X)} \lesssim \|f\|_{BMO_{q,\sqrt{\Delta}}(X)}.$$

The following theorem follows from the interpolation theory between vector-valued tent spaces—Lemma 3.2.3—and the boundedness of the projection π_L—Lemma 3.2.6. See for instance the general interpolation theory of complemented subspaces (cf. [91, Section 1.17]), and we omit the details.

Theorem 3.5.2. *Let X be any fixed Banach space, $1 < q < \infty$ and $1 \leq p_1 < p < p_2 < \infty$ such that $1/p = (1-\theta)/p_1 + \theta/p_2$ with $0 \leq \theta \leq 1$. Let L be as in Theorem II.1. Then*

$$[H^{p_1}_{q,L}(\mathbb{R}^d; X), H^{p_2}_{q,L}(\mathbb{R}^d; X)]_\theta = H^p_{q,L}(\mathbb{R}^d; X),$$

with equivalent norms, where $[\cdot,\cdot]_\theta$ is the complex interpolation space. More precisely, for $f \in C_c(\mathbb{R}^d) \otimes X$, one has

$$\|f\|_{H^p_{q,L}(X)} \lesssim \|f\|_{[H^{p_1}_{q,L}(\mathbb{R}^d;X), H^{p_2}_{q,L}(\mathbb{R}^d;X)]_\theta} \lesssim p^{\frac{2}{q}} \|f\|_{H^p_{q,L}(X)}.$$

Now we are at the position to give the applications.

Corollary 3.5.3. *Let X be a Banach space and $1 < q \leq 2$. Let L be as in Theorem II.1. The followings are equivalent*

(i) *X is of martingale type q;*

(ii) *for any $f \in C_c(\mathbb{R}^d) \otimes X$,*

$$\|f\|_{H^1_{\rm at}(X)} \lesssim_{\gamma,\beta} \|G_{q,L}(f)\|_1;$$

(iii) *for any $1 < p < \infty$ and $f \in C_c(\mathbb{R}^d) \otimes X$,*

$$\|f\|_{L^p(X)} \lesssim_{\gamma,\beta} p\|G_{q,L}(f)\|_p;$$

(iv) *for any $f \in C_c(\mathbb{R}^d) \otimes X$,*

$$\|f\|_{BMO(X)} \lesssim_{\gamma,\beta} \|f\|_{BMO_{q,L}(X)}.$$

Proof. (i)⇔(ii). This follows immediately from Theorem II.1 and Theorem 3.5.1.

(iii)⇒(i). This is deduced from Theorem II.1 and (3.5.2).

(i)⇒(iii). In the case $1 < p < q$, by Theorem II.1, it suffices to show that

$$\|f\|_{L^p(X)} \lesssim \|S_{q,\sqrt{\Delta}}\|_p. \tag{3.5.3}$$

Keeping in mind (3.5.2) and Theorem 3.5.1 (ii), we consider

$$[H^1_{q,\sqrt{\Delta}}(\mathbb{R}^d;X), H^q_{q,\sqrt{\Delta}}(\mathbb{R}^d;X)]_\theta \subset [H^1_{\rm at}(\mathbb{R}^d;X), L^q(\mathbb{R}^d;X)]_\theta;$$

then combining Theorem 3.5.2 with the interpolation between $H^1_{\rm at}(\mathbb{R}^d;X)$ and $L^q(\mathbb{R}^d;X)$ (cf. [13, Theorem A]), one gets for any $f \in C_c(\mathbb{R}^d) \otimes X$, $1/p = 1 - \theta + \theta/q$,

$$\|f\|_{L^p(X)} \lesssim \|f\|_{[H^1_{\rm at}(\mathbb{R}^d;X), L^q(\mathbb{R}^d;X)]_\theta} \lesssim \|f\|_{[H^1_{q,\sqrt{\Delta}}(\mathbb{R}^d;X), H^q_{q,\sqrt{\Delta}}(\mathbb{R}^d;X)]_\theta} \lesssim \|f\|_{H^p_{q,\sqrt{\Delta}}(X)}.$$

This is the desired (3.5.3). Combining it with the related result for $q \leq p < \infty$ in [102], we conclude

$$\|f\|_{L^p(X)} \lesssim_{\gamma,\beta} p\|G_{q,L}(f)\|_p, \quad 1 < p < \infty.$$

(i)⇔(iv). This follows from Remark 3.3.5 and Theorem 3.5.1 (iii).

□

Remark 3.5.4. (1). Taking $L = \sqrt{\Delta}$ in the assertion (iii), we get

$$\mathsf{L}^{\sqrt{\Delta}}_{t,q,p}(X) \lesssim p\mathsf{M}_{t,q}(X), \quad 1 < p < \infty,$$

where the order is optimal as p tends to 1. This solves partially [102, Problem 1.8].

(2). The implication (iii)→(i) says that a Banach space X which is of Lusin type q relative to $\{e^{-tL}\}_{t>0}$ implies the martingale type q for a large class of generators L. This answers partially [102, Problem A.1 and Conjecture A.4].

Chapter 4

Bilinear decompositions on probability spaces

In this chapter, we first show the equivalence of different atomic Hardy spaces. Then we prove Theorem III.1. Next, we will present a characterization of martingale Lipschitz spaces $\Lambda_1(\alpha_p)$, which is needed to show Theorem III.2 (see Theorem 4.3.4 below). At last, we will prove Theorem III.2. Our method is mainly based on the atomic decomposition.

4.1 Atomic decomposition of Hardy spaces

An important aspect of martingale Hardy spaces is that they admit atomic decompositions. In this section we present atomic decompositions for several types of Hardy spaces and we then explain how they are related to each other and to the Hardy spaces defined in the previous section. The definition of atoms in the martingale setting is given below.

Definition 4.1.1. A random variable $a : \Omega \to \mathbb{C}$ is called a martingale simple (p, q)-atom $(0 < p \leq 1, 1 \leq q \leq \infty)$ if there exist $k \in \mathbb{N}$ and $A \in \mathcal{F}_k$ such that

1. $\mathbb{E}_k(a) = 0$;
2. $\operatorname{supp}(a) \subset A$;
3. $\|a\|_q \leq P(A)^{\frac{1}{q}-\frac{1}{p}}$,

where $\frac{1}{q} := 0$ when $q = \infty$ as convention.

Definition 4.1.2. We first define the space of functions which admit pointwise atomic decompositions by

$$\mathcal{H}^{1,2}_{\mathrm{at}}(\Omega) := \Bigg\{ f \in L^1(\Omega) : f = \sum_{j=0}^{\infty} \lambda_j a^j \text{ a.e.,} \quad \text{where } a^j \text{ is a simple } (1,2)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j| < \infty \Bigg\}.$$

For $f \in \mathcal{H}^{1,2}_{\mathrm{at}}(\Omega)$, we define the norm

$$\|f\|_{\mathcal{H}^{1,2}_{\mathrm{at}}(\Omega)} := \inf\left\{\sum_{j=0}^{\infty}|\lambda_j| : f = \sum_{j=0}^{\infty}\lambda_j a^j \text{ a.e., where } a^j \text{ is a simple } (1,2)\text{-atom}\right\}.$$

It is well-known that $h^1(\Omega) = \mathcal{H}^{1,2}_{\mathrm{at}}(\Omega)$ with equivalent norms by a similar argument in [93] and [94], see also [28]. Furthermore, if the martingale filtration is regular, we can obtain the same result by taking simple $(1,q)$-atoms $(1 < q \leq \infty)$ in place of simple $(1,2)$-atoms as well.

The following definition for martingale atomic Hardy spaces is inspired by the atomic Hardy space on homogeneous space (e.g. [27]).

Definition 4.1.3. We define the following martingale atomic Hardy spaces in the sense of duality.

$$H^{1,2}_{\mathrm{at}}(\Omega) := \Bigg\{ f \in (bmo(\Omega))^* : f = \sum_{j=0}^{\infty}\lambda_j a^j, \\ \text{where } a^j \text{ is a simple } (1,2)\text{-atom and } \sum_{j=0}^{\infty}|\lambda_j| < \infty \Bigg\} \quad (4.1.1)$$

and for $0 < p < 1$,

$$H^{p,2}_{\mathrm{at}}(\Omega) := \Bigg\{ f \in (\Lambda_2(\alpha_p))^* : f = \sum_{j=0}^{\infty}\lambda_j a^j, \\ \text{where } a^j \text{ is a simple } (p,2)\text{-atom and } \sum_{j=0}^{\infty}|\lambda_j|^p < \infty \Bigg\}. \quad (4.1.2)$$

The corresponding norm is identified by $(0 < p \leq 1)$:

$$\|f\|_{H^{p,2}_{\mathrm{at}}(\Omega)} := \inf\left\{\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{\frac{1}{p}} : f = \sum_{j=0}^{\infty}\lambda_j a^j, \text{ where } a^j \text{ is a simple } (p,2)\text{-atom}\right\}.$$

Here $f = \sum\limits_{j=0}^{\infty}\lambda_j a^j$ is in terms of w^*-topology.

One important property of the martingale atomic Hardy spaces is the completeness, for which we will provide a proof.

Lemma 4.1.4. *For $0 < p < 1$, $H^{p,2}_{\mathrm{at}}(\Omega)$ is complete with respect to the quasi-norm $\|\cdot\|_{H^{p,2}_{\mathrm{at}}(\Omega)}$, and $H^{1,2}_{\mathrm{at}}(\Omega)$ is a complete normed space.*

Proof. Let $(f^n)_{n\geq 1}$ be any Cauchy sequence in $H^{p,2}_{\mathrm{at}}(\Omega)$. Then there exists a strictly increasing sequence $(n_k)_{k\geq 1} \subset \mathbb{N}$ such that

$$\|f^{n_{k+1}} - f^{n_k}\|_{H^{p,2}_{\mathrm{at}}(\Omega)} < \frac{1}{2^k}.$$

Since $f^{n_{k+1}} - f^{n_k} \in H^{p,2}_{\mathrm{at}}(\Omega)$, there exist $a_{k,j}$ which are $(p,2)$-atoms and $(\mu_{k,j})_{j\geq 0} \in \ell_p$ such that

$$f^{n_{k+1}} - f^{n_k} = \sum_{j=0}^{\infty} \mu_{k,j} a_{k,j}$$

with

$$\|f^{n_{k+1}} - f^{n_k}\|_{H^{p,2}_{\mathrm{at}}(\Omega)} \approx_p \left(\sum_{j=0}^{\infty} |\mu_{k,j}|^p \right)^{\frac{1}{p}}.$$

Note that $f := \sum\limits_{k=1}^{\infty} \sum\limits_{j=0}^{\infty} \mu_{k,j} a_{k,j} \in H^{p,2}_{\mathrm{at}}(\Omega)$ since

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} |\mu_{k,j}|^p < \infty.$$

Also, one has

$$\|f^{n_k} - f\|_{H^{p,2}_{\mathrm{at}}(\Omega)} = \left\| \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \mu_{i,j} a_{i,j} \right\|_{H^{p,2}_{\mathrm{at}}(\Omega)} \leq \left(\sum_{i=k}^{\infty} \sum_{j=0}^{\infty} |\mu_{i,j}|^p \right)^{\frac{1}{p}} \lesssim \left(\sum_{i=k}^{\infty} \frac{1}{2^{ip}} \right)^{\frac{1}{p}}.$$

This implies that $(f^n)_{n\geq 1}$ converges to f in $H^{p,2}_{\mathrm{at}}(\Omega)$ and we conclude that $H^{p,2}_{\mathrm{at}}(\Omega)$ is complete. □

We would like to remark that the martingale Hardy spaces $h^p(\Omega)$ and the martingale atomic Hardy spaces $H^{p,2}_{\mathrm{at}}(\Omega)$ are defined differently from those defined in [93] where martingale sequences are concerned. It is proved in [93] that martingale Hardy spaces and the atomic martingale Hardy spaces defined for martingale sequences can be identified with each other when $0 < p \leq 1$.

We will show that with our definition, the martingale Hardy spaces $h^p(\Omega)$ can be identified with the martingale atomic Hardy spaces $H^{p,2}_{\mathrm{at}}(\Omega)$ as well. As a consequence, we will also see that the two definitions of martingale Hardy spaces $\mathcal{H}^{1,2}_{\mathrm{at}}(\Omega)$ and $H^{1,2}_{\mathrm{at}}(\Omega)$ coincide.

Proposition 4.1.5. *For $0 < p \leq 1$, $h^p(\Omega) = H^{p,2}_{\mathrm{at}}(\Omega)$.*

Proof. We will show that there exists a bijection between $H^{p,2}_{\mathrm{at}}(\Omega)$ and $h^p(\Omega)$ with equivalent norms.

We will first elaborate on how to define the map from a dense subspace of $h^p(\Omega)$, namely $L^2(\Omega)$ endowed with the quasi-norm $\|\cdot\|_{h^p(\Omega)}$, to $H^{p,2}_{\mathrm{at}}(\Omega)$.

If $f \in L^2(\Omega)$ and $0 < p < 1$, then by atomic decomposition in [94, Theorem 2.5], there exist a^k which are simple $(p,2)$-atoms and $(\mu_k)_{k\geq 0} \in \ell_p$ such that

$$\sum_{k=0}^{\infty} \mu_k \mathbb{E}_n(a^k) = f_n \quad \text{a.e. for all } n.$$

Furthermore, the following inequality holds:

$$\|f\|_{h^p(\Omega)} \approx_p \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

Let $g \in \Lambda_2(\alpha_p)$, then g induces a continuous linear functional on $\left(L^2(\Omega), \|\cdot\|_{h^p(\Omega)}\right)$ defined by

$$\varphi_g(f) = \mathbb{E}\left(\sum_{n=1}^{\infty} d_n f d_n g\right) \quad \text{for all } f \in L^2(\Omega).$$

Indeed, one can show that for $0 < p < 1$, if $f \in L^2(\Omega)$ and $g \in \Lambda_2(\alpha_p)$, then

$$\mathbb{E}\left(\sum_{n=1}^{\infty} |d_n f d_n g|\right) \lesssim_p \|f\|_{h^p(\Omega)} \|g\|_{\Lambda_2(\alpha_p)}.$$

See [62, Theorem 2.8.2].

Next, note that

$$f_n = \sum_{k=0}^{\infty} \mu_k \mathbb{E}_n(a^k)$$

also converges in $h^p(\Omega)$. Thus

$$\begin{aligned}\varphi_f(g) &:= \lim_{m\to\infty} \sum_{n=0}^{m} \mathbb{E}(d_n f d_n g) = \lim_{m\to\infty} \mathbb{E}(f_m g_m) = \lim_{m\to\infty} \mathbb{E}(f_m g) \\ &= \lim_{m\to\infty} \mathbb{E}(\sum_{k=0}^{\infty} \mu_k \mathbb{E}_m(a^k) \cdot g) = \lim_{m\to\infty} \sum_{k=0}^{\infty} \mu_k \mathbb{E}(\mathbb{E}_m(a^k) \cdot g).\end{aligned}$$

Note that

$$\sum_{k=0}^{\infty} |\mu_k| \cdot |\mathbb{E}(a^k \cdot g)| \lesssim \|g\|_{\Lambda_2(\alpha_p)} \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{\frac{1}{p}} \approx_p \|f\|_{h^p(\Omega)} \|g\|_{\Lambda_2(\alpha_p)}.$$

Therefore we can exchange the limit and sum so as to obtain

$$\varphi_f(g) = \sum_{k=0}^{\infty} \mu_k \lim_{n\to\infty} \mathbb{E}(\mathbb{E}_n a^k \cdot g) = \sum_{k=0}^{\infty} \mu_k \mathbb{E}(a^k \cdot g) \quad \text{for all } g \in \Lambda_2(\alpha_p).$$

Thus we conclude that φ_f admits an atomic decomposition in the sense of duality. Moreover,

$$\|\varphi_f\| \approx_p \|f\|_{h^p(\Omega)} \approx_p \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

which indicates that the map from $\left(L^2(\Omega), \|\cdot\|_{h^p(\Omega)}\right)$ to $H^{p,2}_{\mathrm{at}}(\Omega)$ is injective.

Now due to Lemma 4.1.4, the map defined on $\left(L^2(\Omega), \|\cdot\|_{h^p(\Omega)}\right)$ can be extended to $h^p(\Omega)$. More precisely, for any $f \in h^p(\Omega)$,

$$\varphi_f(g) := \lim_{m\to\infty} \varphi_{f^m}(g), \quad \forall\, g \in \Lambda_2(\alpha_p),$$

where $(f^m)_{m\geq 1}$ is a sequence of L^2 functions that are Cauchy with respect to the quasi-norm $\|\cdot\|_{h^p(\Omega)}$. We notice that $(\varphi_{f^m})_{m\geq 1}$ is a Cauchy sequence in $H^{p,2}_{\mathrm{at}}(\Omega)$ since

$$\|\varphi_{f^m} - \varphi_{f^n}\|_{H^{p,2}_{\mathrm{at}}(\Omega)} = \|\varphi_{f^m - f^n}\|_{H^{p,2}_{\mathrm{at}}(\Omega)} \approx_p \|f^m - f^n\|_{h^p(\Omega)},$$

which goes to 0 as $m, n \to \infty$. Lemma 4.1.4 thus implies that $\varphi_f \in H^{p,2}_{\mathrm{at}}(\Omega)$. It is not difficult to verify that the extended map is also injective.

We have shown that the map from $h^p(\Omega)$ to $H^{p,2}_{\mathrm{at}}(\Omega)$ is well-defined and injective. The case for $p = 1$ follows from a similar argument where we use $bmo(\Omega)$ instead of $\Lambda_2(\alpha_p)$. We omit the details.

To show that the map is also surjective, we claim that for any $\varphi \in H^{p,2}_{\mathrm{at}}(\Omega)$, there exists $f \in h^p(\Omega)$, such that $\varphi = \varphi_f$. We only give the proof for $0 < p < 1$ and the case $p = 1$ can be derived similarly.

For any $\varphi \in (\Lambda_2(\alpha_p))^*$, one has by definition that $\varphi = \sum_{j=0}^{\infty} \mu_j a^j$ in the weak sense where all a^j are $(p, 2)$-atoms and $(\mu_j)_{j\geq 0} \in \ell_p$. Define a sequence of L^2 functions $(f^k)_{k\geq 1}$ by

$$f^k = \sum_{j=0}^{k} \mu_j a^j.$$

Then for all $n > m \geq 0$,

$$\|f^n - f^m\|_{h^p(\Omega)} \leq \left(\sum_{j=m+1}^{n} |\mu_j|^p \right)^{\frac{1}{p}}.$$

Furthermore, the Cauchy sequence $(f^k)_{k\geq 1}$ can be identified with an element $f \in h^p(\Omega)$ such that $f^k \to f$ in $h^p(\Omega)$. Recall that for a general element in $f \in h^p(\Omega)$, the map is defined by

$$\varphi_f(g) = \lim_{k\to\infty} \varphi_{f^k}(g) = \sum_{j=0}^{\infty} \mu_j \mathbb{E}(a^j \cdot g),$$

for any $g \in \Lambda_2(\alpha_p)$, where $(f^k)_{k\geq 1}$ is the (equivalence class of) Cauchy sequences $(f^k)_{k\geq 1} \subset L^2(\Omega)$ that is Cauchy with respect to $\|\cdot\|_{h^p(\Omega)}$. This completes the proof of the proposition. □

Remark 4.1.6. From the proof of Proposition 4.1.5, we see that when $0 < p \leq 1$, for any $f \in H^{p,2}_{\mathrm{at}}(\Omega)$,

$$\|f\|_{H^{p,2}_{\mathrm{at}}(\Omega)} \approx_p \|f\|_{(\Lambda_2(\alpha_p))^*}.$$

Remark 4.1.7. Since $h^1(\Omega) = \mathcal{H}^{1,2}_{\mathrm{at}}(\Omega)$, by Proposition 4.1.5, we conclude that $\mathcal{H}^{1,2}_{\mathrm{at}}(\Omega) = H^{1,2}_{\mathrm{at}}(\Omega)$ with equivalent norms.

Remark 4.1.8. Assume that the martingale filtration is regular. Let $0 < p \leq 1$ and $1 < q \leq \infty$. If $f = \sum_{j=0}^{\infty} \lambda_j a^j$ where each a^j is a simple (p, q)-atom and $(\lambda_j)_{j\geq 0} \in \ell_p$, then $f \in (bmo(\Omega))^*$ for $p = 1$, and $f \in (\Lambda_2(\alpha_p))^*$ for $0 < p < 1$. Hence we can define $H^{p,q}_{\mathrm{at}}(\Omega)$ by virtue of (p, q)-atoms as in Definition 4.1.3. Moreover, in this case, $h^p(\Omega) = H^{p,q}_{\mathrm{at}}(\Omega)$.

4.2 Bilinear decompositions on $H^1(\Omega) \times BMO(\Omega)$

In this section we prove Theorem III.1. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and let $f \in H^1(\Omega)$, $g \in BMO(\Omega)$. We will briefly describe the strategy of the proof. If we assume that f and g have finite martingale expansions, then we may write their pointwise product $f \cdot g$ as follows

$$f \cdot g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g), \tag{4.2.1}$$

where

$$\Pi_1(f,g) := \sum_{k=1}^{\infty} d_k f d_k g, \quad \Pi_2(f,g) := \sum_{k=1}^{\infty} f_{k-1} d_k g \quad \text{and} \quad \Pi_3(f,g) := \sum_{k=1}^{\infty} g_{k-1} d_k f.$$

We will mainly focus on proving the boundedness property for each operator arising in the decomposition for $f \in H^1(\Omega)$ and $g \in BMO(\Omega)$ with finite martingale expansions. To do so, we shall make use of the atomic decomposition of $H^1(\Omega)$. In Section 4.2.4, we present a direct way to deal with $\Pi_3(f,g)$, which avoids the use of the atomic decomposition.

We will now first elaborate on how to use a limiting argument to reduce the estimates to functions with finite martingale expansions. For any $f \in H^1(\Omega)$ and $g \in BMO(\Omega)$,

$$f \cdot g := \lim_{n\to\infty} f_n \cdot g_n = \lim_{n\to\infty} \Pi_1(f_n, g_n) + \lim_{n\to\infty} \Pi_2(f_n, g_n) + \lim_{n\to\infty} \Pi_3(f_n, g_n),$$

where $f_n := \mathbb{E}_n(f)$ and $g_n := \mathbb{E}_n(g)$ for each $n \in \mathbb{N}$. Here the limits denote the pointwise limits, and are well-defined on the product space $H^1(\Omega) \times BMO(\Omega)$. In particular, $(\Pi_1(f_n, g_n))_n$ is absolutely convergent almost everywhere:

$$\sum_{k=1}^{n} |d_k f d_k g| \le \left(\sum_{k=1}^{\infty} |d_k f|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |d_k g|^2\right)^{\frac{1}{2}} \le S(f) \cdot S(g).$$

The boundedness of Π_1 for general functions can be derived by Fatou's lemma and the corresponding estimate for functions with finite martingale expansions showed in Section 4.2.1:

$$\| \lim_{n\to\infty} \Pi_1(f_n, g_n)\|_1 \le \lim_{n\to\infty} \|\Pi_1(f_n, g_n)\|_1 \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}.$$

For the sequence $(\Pi_2(f_n, g_n))_{n\ge1}$, its pointwise limit is indeed the limit of the sequence under the norm $\|\cdot\|_{H^1(\Omega)}$ thanks to the characterization of $H^1(\Omega)$ in terms of the maximal function as described in Remark 1.3.5. The boundedness of Π_2 thus follows from the estimate for functions with finite martingale expansions, which is proved in Section 4.2.2.

As imposed by the decomposition, we claim that $\Pi_3(f,g)$ is the pointwise limit of the Cauchy sequence $(\Pi_3(f_n, g_n))_{n\ge1}$ in $H^\Phi(\Omega)$. To show that $\Pi_3(f_n, g_n) \in H^\Phi(\Omega)$ for each $n \in \mathbb{N}$, it suffices to show that $\Pi_3(f_n, g_n)$ can be identified with a Cauchy sequence of functions in $L^1(\Omega)$ with respect to $\|\cdot\|_{H^\Phi(\Omega)}$, due to the definition of $H^\Phi(\Omega)$ which is defined as the completion under the quasi-norm $\|\cdot\|_{H^\Phi(\Omega)}$ of the space

$$H_0^\Phi(\Omega) := \{f \in L^1(\Omega) : \|f\|_{H^\Phi(\Omega)} < \infty\}.$$

Since functions in $L^2(\Omega)$ with $f_0 = 0$ form a dense subspace of $H^1(\Omega)$, there exist Cauchy sequences $(f^k)_{k\ge1} \subset L^2(\Omega)$ converging to f in $H^1(\Omega)$. Section 4.2.3 and Section 4.2.4 will be devoted to proving the boundedness of Π_3 restricted on $L^2_{\text{fin}}(\Omega) \times BMO(\Omega)$ where

$$L^2_{\text{fin}}(\Omega) := \{f \in L^2(\Omega) : f \text{ has finite martingale expansion}, f_0 = 0\}. \tag{4.2.2}$$

It follows from the bilinearity and boundedness of Π_3 on $L^2_{\text{fin}}(\Omega) \times BMO(\Omega)$ that

$$\Big(\Pi_3\left((f^k)_n, g_n\right)\Big)_{k\ge1} \subset L^1(\Omega)$$

is a Cauchy sequence with respect to $\|\cdot\|_{H^\Phi(\Omega)}$. Moreover, one can verify that $(\Pi_3(f_n, g_n))_{n\geq 1}$ is a Cauchy sequence in $H^\Phi(\Omega)$ by showing

$$\lim_{k\to\infty} \|\Pi_3((f^k)_n, g_n) - \Pi_3((f^k)_m, g_m)\|_{H^\Phi(\Omega)} \to 0 \text{ as } n, m \to \infty.$$

Let $\widetilde{\Pi}_3(f, g)$ denote the limit of the Cauchy sequence $(\Pi_3(f_n, g_n))_{n\geq 1}$ under the quasi-norm $\|\cdot\|_{H^\Phi(\Omega)}$. One has from the boundedness of Π_3 on functions with finite martingale expansions that

$$\|\widetilde{\Pi}_3(f, g)\|_{H^\Phi(\Omega)} = \lim_{n\to\infty} \|\Pi_3(f_n, g_n)\|_{H^\Phi(\Omega)} \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)}.$$

It remains an open and interesting question how to compare the limit $\widetilde{\Pi}_3(f, g)$ with the pointwise limit $\Pi_3(f, g)$ of the sequence $(\Pi_3(f_n, g_n))_{n\geq 1}$.

Remark 4.2.1. Let $0 < p < 1$ be fixed, by definition of $L^\Phi(\Omega)$, we have for any $f \in L^\Phi(\Omega)$ and $\|f\|_{L^\Phi(\Omega)} = 1$,

$$\begin{aligned}
\int_\Omega |f|^p \,\mathrm{d}\mu &= \int_{\{|f|\leq 1\}} |f|^p \,\mathrm{d}\mu + \int_{\{|f|\geq 1\}} |f|^p \,\mathrm{d}\mu \\
&\leq \left(\int_{\{|f|\leq 1\}} \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu\right)^p \cdot \left(\int_{\{|f|\leq 1\}} \log(e + |f|) \,\mathrm{d}\mu\right)^{1-p} + \int_{\{|f|\geq 1\}} \frac{|f|}{|f|^{1-p}} \,\mathrm{d}\mu \\
&\lesssim \left(\int_{\{|f|\leq 1\}} \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu\right)^p + \int_{\{|f|\geq 1\}} \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu.
\end{aligned}$$

However, from $\|f\|_{L^\Phi(\Omega)} = 1$ we know that

$$\int_\Omega \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu \leq 1.$$

Hence

$$\|f\|_p^p \lesssim \left(\int_{\{|f|\leq 1\}} \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu\right)^p + \int_{\{|f|\geq 1\}} \frac{|f|}{\log(e + |f|)} \,\mathrm{d}\mu \lesssim 1.$$

Thus $L^\Phi(\Omega)$ continuously embeds into $L^p(\Omega)$. Moreover, we can also identify $H^\Phi(\Omega)$ as a complete subspace of $L^p(\Omega)$.

On the other hand, note that $H^1(\Omega)$ continuously embeds into $L^1(\Omega)$, and $BMO(\Omega)$ continuously embeds into $L^2(\Omega)$, thus we observe that

$$\Pi_3(f_n, g_n) = f_n g_n - \Pi_1(f_n, g_n) - \Pi_2(f_n, g_n)$$

also forms a Cauchy sequence in $L^{\frac{2}{3}}(\Omega)$. By Fatou's Lemma we obtain that the $L^{\frac{2}{3}}$-norm limit of $(\Pi_3(f_n, g_n))_{n\geq 1}$ and $\Pi_3(f, g)$ coincide. As shown above, $H^\Phi(\Omega)$ continuously embeds into $L^{\frac{2}{3}}(\Omega)$, then we obtain that $\widetilde{\Pi}_3(f, g)$ and $\Pi_3(f, g)$ coincide in $L^{\frac{2}{3}}(\Omega)$.

Proof of Theorem III.1. Thanks to the comments above, we will assume that $f \in H^1(\Omega)$ and $g \in BMO(\Omega)$ have finite martingale expansions. By Theorem 1.3.6, there exist two functions f^h and f^d such that $f = f^h + f^d$ and

$$\|f\|_{H^1(\Omega)} \approx \|f^h\|_{h^1(\Omega)} + \|f^d\|_{h_1^d(\Omega)}, \tag{4.2.3}$$

where $f^h \in h^1(\Omega)$ and $f^d \in h^1_d(\Omega)$. For such decomposition of f, since $f^h \in h^1(\Omega)$, there exist $(\lambda_j)_{j\geq 0} \in \ell_1$ and simple $(1,2)$-atoms $\{a^j\}_{j\geq 0}$ such that

$$(f^h)_n = \sum_{j=0}^{\infty} \lambda_j \mathbb{E}_n(a^j) \text{ a.e. for all } n \in \mathbb{N}, \text{ and } \|f^h\|_{h^1(\Omega)} \approx \sum_{j=0}^{\infty} |\lambda_j|, \tag{4.2.4}$$

where we assume $\operatorname{supp}(a^j) \subset A_{n_j}$ and $A_{n_j} \in \mathcal{F}_{n_j}$ with $P(A_{n_j}) > 0$ for $j \geq 1$. Then

$$d_n f^h = \sum_{j=0}^{\infty} \lambda_j d_n a^j, \ \forall n \in \mathbb{N}. \tag{4.2.5}$$

4.2.1 Estimates for $\Pi_1(f^h, g)$ and $\Pi_1(f^d, g)$

We are going to show that Π_1 is a bounded bilinear operator from $H^1(\Omega)\times BMO(\Omega)$ to $L^1(\Omega)$ with input functions having finite martingale expansions. In fact, the boundedness of Π_1 follows naturally from the duality between $H^1(\Omega)$ and $BMO(\Omega)$, i.e. Theorem 1.3.13 (see [43]). For the reader's convenience, we give a proof based on atomic decompositions.

We first focus on $\Pi_1(f^h, g)$, which can further be decomposed into atoms as described in (4.2.4). Note that from (4.2.5)

$$\|d_k f d_k g\|_1 = \left\| \sum_{j=0}^{\infty} \lambda_j d_k a^j d_k g \right\|_1 \leq \sum_{j=0}^{\infty} |\lambda_j| \left\| d_k a^j d_k g \right\|_1 .$$

This implies that

$$\left\| \Pi_1(f^h, g) \right\|_1 \leq \sum_{k=1}^{\infty} \|d_k f d_k g\|_1 \leq \sum_{j=0}^{\infty} |\lambda_j| \sum_{k=1}^{\infty} \left\| d_k a^j d_k g \right\|_1 . \tag{4.2.6}$$

It thus suffices to consider the boundedness of

$$\sum_{k=1}^{\infty} \left\| d_k a^j d_k g \right\|_1 ,$$

which can further be localized as $d_k a^j = \mathbb{1}_{A_{n_j}} d_k a^j$ when $k \geq n_j + 1$ since $A_{n_j} \in \mathcal{F}_{n_j}$, namely

$$\sum_{k=1}^{\infty} \left\| d_k a^j d_k g \right\|_1 = \sum_{k=n_j+1}^{\infty} \left\| \mathbb{1}_{A_{n_j}} d_k a^j d_k g \right\|_1 = \mathbb{E}\left(\sum_{k=n_j+1}^{\infty} \left| \mathbb{1}_{A_{n_j}} d_k a^j d_k g \right| \right).$$

Now, by applying the Cauchy-Schwarz inequality, we derive the estimate

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} \left| \mathbb{1}_{A_{n_j}} d_k a^j d_k g \right| \right) &\leq \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} |d_k a^j|^2 \right) \right]^{\frac{1}{2}} \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} \mathbb{1}_{A_{n_j}} |d_k g|^2 \right) \right]^{\frac{1}{2}} \\
&\leq \|a^j\|_2 \left[\mathbb{E}\mathbb{E}_{n_j}\left(\sum_{k=n_j+1}^{\infty} \mathbb{1}_{A_{n_j}} |d_k g|^2 \right) \right]^{\frac{1}{2}} \\
&\leq P(A_{n_j})^{-\frac{1}{2}} \left[\mathbb{E}\left(\mathbb{1}_{A_{n_j}} \mathbb{E}_{n_j}\left(\sum_{k=n_j+1}^{\infty} |d_k g|^2 \right) \right) \right]^{\frac{1}{2}} \\
&\leq P(A_{n_j})^{-\frac{1}{2}} \|g\|_{bmo(\Omega)} P(A_{n_j})^{\frac{1}{2}}
\end{aligned}$$

where the fourth inequality follows from the definition of the atom. Hence, we deduce from the definition of the $bmo-$norm that

$$\sum_{k=1}^{\infty} \left\| d_k a^j d_k g \right\|_1 \leq \|g\|_{bmo(\Omega)}. \tag{4.2.7}$$

By using (4.2.7) and (4.2.6), we have by Theorem 1.3.14

$$\begin{aligned}
\|\Pi_1(f,g)\|_1 &\leq \sum_{j=0}^{\infty} |\lambda_j| \|g\|_{bmo(\Omega)} + \left\| \sum_{k=1}^{\infty} d_k f^d d_k g \right\|_1 \\
&\lesssim \|f^h\|_{h^1} \|g\|_{bmo(\Omega)} + \left(\sup_{k\geq 1} \|d_k g\|_\infty \right) \left(\sum_{k=1}^{\infty} \|d_k f^d\|_1 \right) \\
&\lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h_1^d(\Omega)} \right) \|g\|_{BMO(\Omega)}.
\end{aligned}$$

This implies from (4.2.3) that

$$\|\Pi_1(f,g)\|_1 \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.8}$$

4.2.2 Estimates for $\Pi_2(f^h, g)$ and $\Pi_2(f^d, g)$

We are going to show that Π_2 is a bounded bilinear operator from $H^1(\Omega) \times BMO(\Omega)$ to $H^1(\Omega)$ with input functions having finite martingale expansions. Note that from (4.2.4)

$$\begin{aligned}
S(\Pi_2(f^h,g)) &= \left(\sum_{k=1}^{\infty} |f_{k-1} d_k g|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} \left| \sum_{j=0}^{\infty} \lambda_j \mathbb{E}_{k-1}(a^j) d_k g \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{j=0}^{\infty} |\lambda_j| \left(\sum_{k=1}^{\infty} \left| \mathbb{E}_{k-1}(a^j) d_k g \right|^2 \right)^{\frac{1}{2}} = \sum_{j=0}^{\infty} |\lambda_j| \, S(\Pi_2(a^j,g)).
\end{aligned} \tag{4.2.9}$$

Arguing as in section 4.2.1, we perform the localization on each term

$$\Pi_2(a^j,g) = \sum_{k=1}^{\infty} a_{k-1}^j d_k g = \sum_{k=n_j+2}^{\infty} \mathbb{1}_{A_{n_j}} a_{k-1}^j d_k g.$$

It is easy to verify that

$$d_k(\Pi_2(a^j,g)) = a_{k-1}^j d_k g, \quad k \geq n_j + 2 \ \text{ and } \ d_k(\Pi_2(a^j,g)) = 0, \quad 1 \leq k \leq n_j + 1.$$

We consider the corresponding square function

$$S\left(\Pi_2(a^j,g)\right) = \left(\sum_{k=n_j+2}^{\infty} \left(|a_{k-1}^j|^2 \mathbb{1}_{A_{n_j}} |d_k g|^2 \right) \right)^{\frac{1}{2}} \leq |(a^j)^*| \left(\sum_{k=n_j+2}^{\infty} \mathbb{1}_{A_{n_j}} \left(|d_k g|^2 \right) \right)^{\frac{1}{2}}.$$

Then by invoking the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\|\Pi_2(a^j,g)\|_{H^1(\Omega)} = \mathbb{E}\left[S\left(\Pi_2(a^j,g)\right) \right] &\leq \|(a^j)^*\|_2 \left[\mathbb{E}\left(\sum_{k=n_j+2}^{\infty} \mathbb{1}_{A_{n_j}} (|d_k g|^2) \right) \right]^{\frac{1}{2}} \\
&\leq \|a^j\|_2 \left[\mathbb{E}\left(\mathbb{1}_{A_{n_j}} \mathbb{E}_{n_j} \left(\sum_{k=n_j+2}^{\infty} |d_k g|^2 \right) \right) \right]^{\frac{1}{2}} \\
&\leq P(A_{n_j})^{-\frac{1}{2}} \|g\|_{BMO(\Omega)} P(A_{n_j})^{\frac{1}{2}}
\end{aligned}$$

and hence,

$$\|\Pi_2(a^j, g)\|_{H^1(\Omega)} \le \|g\|_{BMO(\Omega)}. \tag{4.2.10}$$

Similarly, by Theorem 1.3.14

$$\begin{aligned}
\left\|\Pi_2(f^d, g)\right\|_{H^1(\Omega)} &= \mathbb{E}\left[S\left(\Pi_2(\sum_{m=1}^{\infty} d_m f^d, g)\right)\right] \le \sum_{m=1}^{\infty} \mathbb{E}\left[S\left(\Pi_2(d_m f^d, g)\right)\right] \\
&= \sum_{m=1}^{\infty} \mathbb{E}\left[\mathbb{E}_m\left(\sum_{k=m+1}^{\infty} |d_m f^d|^2 |d_k g|^2\right)^{\frac{1}{2}}\right] \\
&= \sum_{m=1}^{\infty} \mathbb{E}\left[|d_m f^d| \mathbb{E}_m\left(\sum_{k=m+1}^{\infty} |d_k g|^2\right)^{\frac{1}{2}}\right] \\
&\le \sum_{m=1}^{\infty} \left(\|d_m f^d\|_1 \left\|\mathbb{E}_m\left(\sum_{k=m+1}^{\infty} |d_k g|^2\right)\right\|_{\infty}^{\frac{1}{2}}\right)
\end{aligned}$$

and hence,

$$\left\|\Pi_2(f^d, g)\right\|_{H^1(\Omega)} \le \|f^d\|_{h^1_d(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.11}$$

By using (4.2.9), (4.2.11) and (4.2.10), we have by Theorem 1.3.6

$$\begin{aligned}
\|\Pi_2(f, g)\|_{H^1(\Omega)} &\le \left\|\Pi_2(f^h, g)\right\|_{H^1(\Omega)} + \left\|\Pi_2(f^d, g)\right\|_{H^1(\Omega)} \\
&\le \sum_{j=0}^{\infty} |\lambda_j| \|g\|_{BMO(\Omega)} + \|f^d\|_{h^d_1(\Omega)} \|g\|_{BMO(\Omega)} \\
&\lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h^1_d(\Omega)}\right) \|g\|_{BMO(\Omega)}.
\end{aligned}$$

This implies from (4.2.3) that

$$\|\Pi_2(f, g)\|_{H^1(\Omega)} \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.12}$$

4.2.3 Estimates for $\Pi_3(f^h, g)$ and $\Pi_3(f^d, g)$

We are going to show that Π_3 is a bilinear operator with the boundedness property

$$\Pi_3 : \left(L^2_{\text{fin}}(\Omega), \|\cdot\|_{H^1(\Omega)}\right) \times BMO(\Omega) \to H^{\Phi}(\Omega)$$

so that it extends to $H^1(\Omega) \times BMO(\Omega) \to H^{\Phi}(\Omega)$ because $L^2_{\text{fin}}(\Omega)$ is dense in $H^1(\Omega)$. The treatment relies on the atomic decomposition, while in the next subsection we will present a direct proof without invoking the atomic decomposition. Though the latter appears to be shorter, the former is worth a detailed presentation because it is consistent with the argument for Π_1 and Π_2, and is also more compatible to dyadic martingales on homogeneous spaces discussed in Section 5.2.

To this end, we first deal with $\Pi_3(f^h, g)$. Note that

$$\begin{aligned} S(\Pi_3(f^h,g)) &= \left(\sum_{k=1}^{\infty}|g_{k-1}d_kf|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty}\left|\sum_{j=0}^{\infty}\lambda_j g_{k-1}d_k a^j\right|^2\right)^{\frac{1}{2}} \\ &\le \sum_{j=0}^{\infty}|\lambda_j|\left(\sum_{k=1}^{\infty}\left|g_{k-1}d_k a^j\right|^2\right)^{\frac{1}{2}} = \sum_{j=0}^{\infty}|\lambda_j|\left(\sum_{k=n_j+1}^{\infty}|g_{k-1}|^2|d_k a^j|^2\right)^{\frac{1}{2}} \\ &\le \sum_{j=0}^{\infty}|\lambda_j|\left(\sum_{k=n_j+1}^{\infty}|g_{k-1}-g_{n_j}|^2|d_k a^j|^2\right)^{\frac{1}{2}} + \sum_{j=0}^{\infty}|\lambda_j||g_{n_j}|S(a^j) \\ &=: I_1 + I_2. \end{aligned}$$

It thus suffices to handle I_1 and I_2. For I_1, we have

$$\begin{aligned} \mathbb{E}(I_1) &\le \sum_{j=0}^{\infty}|\lambda_j|\mathbb{E}\left(\sum_{k=n_j+1}^{\infty}\mathbb{1}_{A_{n_j}}|g_{k-1}-g_{n_j}|^2|d_k a^j|^2\right)^{\frac{1}{2}} \\ &\le \sum_{j=0}^{\infty}|\lambda_j|\left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty}\mathbb{1}_{A_{n_j}}|g_{k-1}-g|^2|d_k a^j|^2\right)^{\frac{1}{2}} + \mathbb{E}\left(\sum_{k=n_j+1}^{\infty}\mathbb{1}_{A_{n_j}}|g-g_{n_j}|^2|d_k a^j|^2\right)^{\frac{1}{2}}\right] \\ &\le \sum_{j=0}^{\infty}|\lambda_j|\left\{P(A_{n_j})^{\frac{1}{2}}\left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty}|g_{k-1}-g|^2|d_k a^j|^2\right)\right]^{\frac{1}{2}} + \mathbb{E}\left(\mathbb{1}_{A_{n_j}}|g-g_{n_j}|S(a^j)\right)\right\} \\ &\le \sum_{j=0}^{\infty}|\lambda_j|\left\{P(A_{n_j})^{\frac{1}{2}}\left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty}|d_k a^j|^2\mathbb{E}_k(|g_{k-1}-g|^2)\right)\right]^{\frac{1}{2}} + \|a^j\|_2 P(A_{n_j})^{\frac{1}{2}}\|g\|_{BMO(\Omega)}\right\} \\ &\le 2\sum_{j=0}^{\infty}|\lambda_j|P(A_{n_j})^{\frac{1}{2}}\|g\|_{BMO(\Omega)}\|a^j\|_2 \end{aligned}$$

and so,

$$\mathbb{E}(I_1) \lesssim \|f^h\|_{h^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{4.2.13}$$

Next, we obtain an estimate for I_2. To this end, notice that

$$\begin{aligned} I_2 &\le \left(\sum_{j=0}^{\infty}\mathbb{1}_{A_{n_j}}|\lambda_j|S(a^j)\right)\cdot|g| + \sum_{j=0}^{\infty}|\lambda_j|\mathbb{1}_{A_{n_j}}|g_{n_j}-g|S(a^j) \\ &=: I_3 + I_4. \end{aligned}$$

Since a^j is a simple $(1,2)$-atom, we have $\|\mathbb{1}_{A_{n_j}}S(a^j)\|_1 \le 1$ and

$$\left\|\sum_{j=0}^{\infty}\mathbb{1}_{A_{n_j}}|\lambda_j|S(a^j)\right\|_1 \le \sum_{j=0}^{\infty}|\lambda_j| \lesssim \|f^h\|_{h^1(\Omega)}.$$

By Lemma 1.3.18, we have

$$\|I_3\|_{L^\Phi(\Omega)} \lesssim \left\|\sum_{j=0}^{\infty}\mathbb{1}_{A_{n_j}}|\lambda_j|S(a^j)\right\|_1\|g\|_{BMO(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{4.2.14}$$

The following estimate is implicit in the proof of (4.2.13):

$$\mathbb{E}(I_4) \leq \sum_{j=0}^{\infty} |\lambda_j| P(A_{n_j})^{\frac{1}{2}} \|g\|_{BMO(\Omega)} \|a^j\|_2 \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.15}$$

By combining (4.2.14) and (4.2.15), we deduce that

$$\|I_2\|_{L^\Phi(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.16}$$

In conclusion, by (4.2.13) and (4.2.16) we get

$$\|\Pi_3(f^h, g)\|_{H^\Phi(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.17}$$

It remains to deal with $\Pi_3(f^d, g)$. Note that

$$\begin{aligned} S(\Pi_3(f^d, g)) &= \left(\sum_{k=1}^{\infty} |g_{k-1}|^2 |d_k f^d|^2 \right)^{\frac{1}{2}} \leq \sum_{k=1}^{\infty} |g_{k-1}| |d_k f^d| \\ &\leq \sum_{k=1}^{\infty} |g_{k-1} - g| |d_k f^d| + |g| \left(\sum_{k=1}^{\infty} |d_k f^d| \right). \end{aligned}$$

By Lemma 1.3.18,

$$\left\| g \left(\sum_{k=1}^{\infty} |d_k f^d| \right) \right\|_{L^\Phi(\Omega)} \lesssim \left(\sum_{k=1}^{\infty} \|d_k f^d\|_1 \right) \|g\|_{BMO(\Omega)} = \|f^d\|_{h^1_d(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.18}$$

For the remaining term, we have

$$\begin{aligned} \mathbb{E} \left(\sum_{k=1}^{\infty} |g_{k-1} - g| |d_k f^d| \right) &= \mathbb{E} \left(\sum_{k=1}^{\infty} |d_k f^d| \mathbb{E}_k |g_{k-1} - g| \right) \\ &\leq \|g\|_{BMO(\Omega)} \left(\sum_{k=1}^{\infty} \|d_k f^d\|_1 \right) \end{aligned}$$

and so

$$\mathbb{E} \left(\sum_{k=1}^{\infty} |g_{k-1} - g| |d_k f^d| \right) \leq \|f^d\|_{h^1_d(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.19}$$

Hence, by (4.2.18) and (4.2.19), we get

$$\|\Pi_3(f^d, g)\|_{H^\Phi(\Omega)} \lesssim \|f^d\|_{h^1_d(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.20}$$

By (4.2.17) and (4.2.20), we obtain

$$\|\Pi_3(f, g)\|_{H^\Phi(\Omega)} \lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h^1_d(\Omega)} \right) \|g\|_{BMO(\Omega)}.$$

Thus we conclude from (4.2.3) that

$$\|\Pi_3(f, g)\|_{H^\Phi(\Omega)} \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.21}$$

This completes the proof of Theorem III.1 □

4.2.4 Another method for handling $\Pi_3(f, g)$

In this section we present a different method for dealing with $\Pi_3(f, g)$, which is much neater and simpler than the one presented above. The proof relies on the following theorem which has been shown in [43].

Theorem 4.2.2. *If $g \in BMO(\Omega)$ with $g_0 = 0$, then $(g^*)_0 \lesssim \|g\|_{BMO(\Omega)}$ and $g^* \in BMO(\Omega)$. Moreover, $\|g^*\|_{BMO(\Omega)} \lesssim \|g\|_{BMO(\Omega)}$.*

We begin with a pointwise estimate for $S(\Pi_3(f, g))$. Towards this aim, note that $d_k(\Pi_3(f, g)) = g_{k-1} d_k f$, which implies that

$$S(\Pi_3(f, g)) = \left(\sum_{k=1}^{\infty} |g_{k-1}|^2 |d_k f|^2 \right)^{\frac{1}{2}} \leq |g^*| S(f) \leq J_1 + J_2,$$

where

$$J_1 := |g^* - (g^*)_0| S(f) \quad \text{and} \quad J_2 := S(f) \|g\|_{BMO(\Omega)}.$$

Clearly,

$$\|J_2\|_{L^\Phi(\Omega)} \lesssim \|J_2\|_1 \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.22}$$

By Theorem 4.2.2 and Lemma 1.3.18, we deduce that

$$\|J_1\|_{L^\Phi(\Omega)} \lesssim \|g^*\|_{BMO(\Omega)} \|S(f)\|_1 \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{4.2.23}$$

Recall that $S(\Pi_3(f, g)) \leq J_1 + J_2$. By combining (4.2.22) with (4.2.23), we conclude that

$$\|\Pi_3(f, g)\|_{H^\Phi(\Omega)} = \|S(\Pi_3(f, g))\|_{L^\Phi(\Omega)} \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)},$$

as desired.

4.3 Bilinear decompositions on $H^p(\Omega) \times \Lambda_1(\alpha_p)$ for $0 < p < 1$

In this section, we give a proof of Theorem III.2. Arguing as in the proof of Theorem III.1, it suffices to establish appropriate estimates for the bilinear operators Π_1, Π_2 and Π_3.

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. If we consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \mathcal{F}$ for all $k \geq 1$, then $H^p(\Omega) = L^p(\Omega)$ for $0 < p < \infty$. It is well-known that $(L^p(\Omega))^* \neq \{0\}$ if and only if the probability space $(\Omega, \mathcal{F}, P)$ contains at least one atom with non-zero measure when $0 < p < 1$. This means that $(H^p(\Omega))^* = \{0\}$ may occur. Therefore, we choose to deal with regular martingales where every $\mathcal{F}_k$ is generated by countably many atoms.

To prove Theorem III.2, we start with the following lemma, which holds for general martingales that are not necessarily regular. Its proof is based on Doob's maximal inequality combined with the fact that on a probability space if a sublinear operator is of weak-type $(1, 1)$ then it maps $L^1(\Omega)$ to $L^p(\Omega)$ for $p \in (0, 1)$. This shall be familiar to the experts in the area, but we will enclose the proof here for the sake of completeness.

Lemma 4.3.1. *For any $0 < p < 1$, we have*

$$\|f\|_{H^p_m(\Omega)} \le \Big(\frac{1}{1-p}\Big)^{\frac{1}{p}} \|f\|_1$$

for all $f \in L^1(\Omega)$. In particular, $L^1(\Omega)$ can be regarded as a subspace of $H^p_m(\Omega)$.

Proof. By Doob's maximal inequality, for any $f \in L^1(\Omega)$ and for any $\lambda > 0$ we have

$$P(f^* > \lambda) \le \frac{1}{\lambda} \int_{\{f^* > \lambda\}} |f| dP. \tag{4.3.1}$$

Without loss of generality, we may assume $\|f\|_1 \le 1$. Then

$$\begin{aligned}
\|f^*\|_p^p &= \int_\Omega |f^*|^p dP = p \int_0^\infty P(|f^*| > \lambda) \lambda^{p-1} d\lambda \\
&= p \int_0^1 P(f^* > \lambda) \lambda^{p-1} d\lambda + p \int_1^\infty P(f^* > \lambda) \lambda^{p-1} d\lambda \\
&\le p \int_0^1 \lambda^{p-1} d\lambda + p \int_1^\infty \frac{1}{\lambda} \left(\int_{\{f^* > \lambda\}} |f| dP \right) \lambda^{p-1} d\lambda \\
&= 1 + p \int_{\{f^* > 1\}} |f| \left(\int_1^{f^*} \lambda^{p-2} d\lambda \right) dP \\
&= 1 + \frac{p}{1-p} \int_{\{f^* > 1\}} |f| \left(1 - |f^*|^{p-1} \right) dP \\
&\le 1 + \frac{p}{1-p} \int_{\{f^* > 1\}} |f| dP \le \frac{1}{1-p}.
\end{aligned}$$

This implies that for any $f \in L^1(\Omega)$, one has the desired result. □

For regular martingales, $L^1(\Omega)$ can be regarded as a subspace of $H^p_m(\Omega) = H^p(\Omega) = h^p(\Omega)$. In what follows, the martingales are always assumed to be regular and every $\mathcal{F}_k$ is generated by countable atoms.

Corollary 4.3.2. *For $0 < p < 1$ and $1 \le q \le \infty$, $H^p(\Omega) = H^{p,q}_{\rm at}(\Omega)$.*

Proof. By considering the aforementioned atomic decomposition of $H^p(\Omega)$ and Definition 4.1.2, we have $H^p(\Omega) = H^{p,\infty}_{\rm at}(\Omega)$. It is easy to see $H^{p,\infty}_{\rm at}(\Omega) \subset H^{p,q}_{\rm at}(\Omega) \subset H^{p,1}_{\rm at}(\Omega)$. It thus suffices to show that $H^{p,1}_{\rm at}(\Omega) \subset H^p(\Omega)$. By Lemma 4.3.1, if a is a simple $(p,1)$-atom, then

$$\|a\|_{H^p(\Omega)} \lesssim_p \|a\|_1,$$

which implies that $a \in H^p(\Omega)$. Hence, $H^{p,1}_{\rm at}(\Omega) \subset H^p(\Omega)$ and so, $H^p(\Omega) = H^{p,q}_{\rm at}(\Omega)$. □

4.3.1 Characterization of martingale Lipschitz spaces

In this subsection, we give a characterization of martingale Lipschitz spaces that appears to be useful in our argument.

Recall the definition of martingale Lipschitz spaces mentioned in (1.3.2). The following result is classical and had been obtained by Meyers in [68] using the definition of $\Lambda_1(\alpha_p)$. Here we provide a proof based on Lemma 4.3.1 and duality.

Corollary 4.3.3. *For any $g \in \Lambda_1(\alpha_p)$ with $g_0 = 0$, we have $\|g\|_\infty \lesssim_p \|g\|_{\Lambda_1(\alpha_p)}$.*

Proof. By duality and Lemma 4.3.1, for any $f \in L^2(\Omega)$ which is a dense subspace of $L^1(\Omega)$,

$$|\mathbb{E}\Big(fg\Big)| \lesssim_p \|f\|_{H^p}\|g\|_{\Lambda_1(\alpha_p)} \lesssim_p \|f\|_1\|g\|_{\Lambda_1(\alpha_p)}.$$

The above estimate together with the fact $\Big(L^1\Big(\Omega\Big)\Big)^* = L^\infty(\Omega)$ yields

$$\|g\|_\infty \lesssim_p \|g\|_{\Lambda_1(\alpha_p)},$$

which finishes the proof. □

By virtue of Corollary 4.3.3, we have the following property of martingale Lipschitz spaces.

Theorem 4.3.4. *If $g \in \Lambda_1(\alpha_p)$, we have $\|\mathbb{1}_A \cdot |g - g_n|\|_\infty \lesssim_p P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)}$, for any $n \in \mathbb{N}$ and any $A \in \mathcal{F}_n$.*

Proof. Note that when $P(A) = 0$, the desired result holds trivially. Fix $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ with $P(A) \neq 0$. For $k \geq 0$, let $\mathcal{F}_k^A := \{B \in \mathcal{F}_{k+n} : B \subset A\}$. Note that the union $\mathcal{F}^A$ of all $\mathcal{F}_k^A$ is exactly $\{B \in \mathcal{F} | B \subset A\}$. Hence, if we define

$$P_A(B) := \frac{P(B)}{P(A)} \quad (B \in \mathcal{F}^A)$$

then $(A, \mathcal{F}^A, P_A)$ is a probability space. Note that for any $g \in L^1(A, \mathcal{F}^A, P_A)$ one has

$$\mathbb{E}(g|\mathcal{F}_k^A) = \mathbb{1}_A \cdot \mathbb{E}(g|\mathcal{F}_{k+n}).$$

Denote $\mathbb{E}(\cdot|\mathcal{F}_k^A)$ by $\mathbb{E}_k^A$. It is easy to verify $\{\mathbb{E}_k^A(g)\}_{k\geq 0}$ is also a regular martingale on $(A, \mathcal{F}^A, P_A)$. If $g \in \Lambda_1(\alpha_p)$, then for $B \in \mathcal{F}_k^A$ with $P(B) \neq 0$,

$$\begin{aligned} P_A(B)^{-1-\alpha_p}\left(\int_B |g - \mathbb{E}_k^A(g)|dP_A\right) &= P(A)^{\alpha_p}\left(P(B)^{-1-\alpha_p}\left(\int_B |g - g_{k+n}|dP\right)\right) \\ &\leq P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)} \end{aligned}$$

which, by Corollary 4.3.3, implies that

$$\|\mathbb{1}_A \cdot |g - g_n|\|_\infty = \|\mathbb{1}_A \cdot |g - \mathbb{E}_0^A(g)|\|_\infty \lesssim_p P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)}.$$

This completes the proof of the theorem. □

Remark 4.3.5. By Theorem 4.3.4 and (1.3.2), we conclude that for $g \in \Lambda_1(\alpha_p)$ we have the characterization

$$\|g\|_{\Lambda_1(\alpha_p)} \approx_p \sup_{n\geq 0}\sup_{A\in\mathcal{F}_n} P(A)^{-\alpha_p}\|\mathbb{1}_A \cdot |g - g_n|\|_\infty. \tag{4.3.2}$$

Remark 4.3.6. By Remark 4.3.5, we see for $q \in [1, \infty)$

$$\Lambda_1(\alpha_p) = \Lambda_q(\alpha_p).$$

4.3.2 Proof of Theorem III.2

We first note that $L^2_{\text{fin}}(\Omega)$ defined by (4.2.2) is dense in $(L^2(\Omega), \|\cdot\|_{H^p(\Omega)})$ and hence, $L^2_{\text{fin}}(\Omega)$ can be regarded as a dense subspace of $H^p(\Omega)$. Recall that for any $f \in H^p(\Omega)$ and $g \in \Lambda_1(\alpha_p)$, the product $f \times g$ is defined in the sense of duality: for any $h \in L^\infty(\Omega) \cap \Lambda_1(\alpha_p)$,

$$\langle f \times g, h\rangle := \langle h \cdot g, f\rangle.$$

It follows from the definition of $H^p(\Omega)$ and the duality between $H^p(\Omega)$ and $\Lambda_1(\alpha_p)$ that for any (equivalence class) $f \in H^p(\Omega)$ and for any (Cauchy) sequence of functions $(f^m)_{m\geq 1}$ in $L^2_{\text{fin}}(\Omega)$ that corresponding to f one has

$$\langle h \cdot g, f\rangle = \lim_{m\to\infty} \langle h \cdot g, f^m\rangle.$$

Consider a function f_m in the sequence $(f^m)_{m\geq 1}$, with $(f^m)_{m\geq 1}$ being in the equivalent class of f. For $n \in \mathbb{N}$, we write

$$\langle h \cdot g, f^m\rangle = \langle h \cdot g, f^m - (f^m)_n\rangle + \langle h \cdot (g - g_n), (f^m)_n\rangle + \langle h \cdot g_n, (f^m)_n\rangle.$$

Since h is a pointwise multiplier of $\Lambda_1(\alpha_p)$ (cf. [71]), it follows that

$$\lim_{n\to\infty} |\langle h \cdot g, (f^m) - (f^m)_n\rangle| = 0 \text{ and } \lim_{n\to\infty} |\langle h \cdot (g - g_n), (f^m)_n\rangle| = 0.$$

Recall that in our setting $(f^m)_0 = g_0 = 0$. Hence

$$\langle h \cdot g, f^m\rangle = \lim_{n\to\infty} \langle h \cdot g_n, (f^m)_n\rangle = \lim_{n\to\infty} \langle h \cdot \sum_{k=1}^n d_k g, \sum_{k=1}^n d_k f^m\rangle$$

By employing the estimates proved in Subsections 4.3.2 and 4.3.2 below, it follows that that the limits

$$\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, d_k f^m\rangle, \quad \lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, (f^m)_{k-1}\rangle, \quad \lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot g_{k-1}, d_k f^m\rangle$$

exist and hence,

$$\langle h \cdot g, f^m\rangle = \lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, d_k f^m\rangle + \lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, (f^m)_{k-1}\rangle + \lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot g_{k-1}, d_k f^m\rangle.$$

Since $(f^m)_{m\geq 1}$ is Cauchy with respect to the H^p-norm, by appealing again to the estimates of Subsections 4.3.2 and 4.3.2 that the limits

$$\lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, d_k f^m\rangle, \quad \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, (f^m)_{k-1}\rangle, \quad \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot g_{k-1}, d_k f^m\rangle$$

exist. We may thus set

$$\langle \Pi_1(f,g), h\rangle := \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g, d_k f^m\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h, d_k g d_k f^m\rangle,$$

$$\langle \Pi_2(f,g), h\rangle := \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h \cdot d_k g (f^m)_{k-1}\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \sum_{k=1}^n \langle h, d_k g (f^m)_{k-1}\rangle,$$

and

$$\langle \Pi_3(f,g), h\rangle := \lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=1}^{n}\langle h\cdot g_{k-1}, d_k f^m\rangle = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=1}^{n}\langle h, g_{k-1} d_k f^m\rangle.$$

Note that, by appealing again to the estimates in Subsections 4.3.2 and 4.3.2, one can show that the value of $\langle \Pi_i(f,g), h\rangle$, $i = 1,2,3$, is independent of the choice of the sequence in $L^2_{\rm fin}(\Omega)$ that corresponding to f.

It thus suffices to prove the boundedness of the bilinear operators Π_1, Π_2 and Π_3 restricted on $\left(L^2_{\rm fin}(\Omega), \|\cdot\|_{H^p(\Omega)}\right) \times \Lambda_1(\alpha_p)$. Moreover, if $f \in L^2_{\rm fin}(\Omega)$ and $g \in \Lambda_1(\alpha_p)$, then

$$f\times g = \Pi_1(f,g) + \Pi_3(f,g) + \Pi_3(f,g) = \sum_{k=1}^{\infty} d_k g d_k f + \sum_{k=1}^{\infty} d_k g f_{k-1} + \sum_{k=1}^{\infty} g_{k-1} d_k f$$

in the sense of duality from the above calculation. Hence, when we restrict ourselves to the case $L^2_{\rm fin}(\Omega) \times \Lambda_1(\alpha_p)$, the above product becomes the usual multiplication between measurable functions and Π_1, Π_2 and Π_3 coincide with the corresponding operators in the $p = 1$ case.

As in the proof of Theorem III.1, we divide the proof into three parts.

Estimates for $\Pi_1(f,g)$ and $\Pi_3(f,g)$

The boundedness of Π_1 from $H^p(\Omega)\times\Lambda_1(\alpha_p)$ to $L^1(\Omega)$ follows directly from the duality between $H^p(\Omega)$ and $\Lambda_1(\alpha_p)$. We include the sketch of the proof here for the sake of completeness. Similarly as the argument in Section 4.2.1, it suffices to consider the case where $f = a$ where a is a simple $(p,2)$-atom with the support $A \in \mathcal{F}_n$ and $P(A) > 0$. Then $d_k a = \mathbb{1}_A d_k a$ when $k \geq n+1$.

By applying the Cauchy-Schwarz inequality, we derive the estimate

$$\begin{aligned}
\|\Pi_1(a,g)\|_1 &= \mathbb{E}\left(\left|\sum_{k=n+1}^{\infty} \mathbb{1}_A d_k a d_k g\right|\right)\\
&\leq \left[\mathbb{E}\left(\sum_{k=n+1}^{\infty}|d_k a|^2\right)\right]^{\frac12}\left[\mathbb{E}\left(\sum_{k=n+1}^{\infty}\mathbb{1}_A|d_k g|^2\right)\right]^{\frac12} \leq \|a\|_2\left[\mathbb{E}\mathbb{E}_n\left(\sum_{k=n+1}^{\infty}\mathbb{1}_A|d_k g|^2\right)\right]^{\frac12}\\
&\leq P(A)^{\frac12-\frac1p}\left[\mathbb{E}\left(\mathbb{1}_A\mathbb{E}_n\left(\sum_{k=n+1}^{\infty}|d_k g|^2\right)\right)\right]^{\frac12} = P(A)^{\frac12-\frac1p}\left(\mathbb{E}\left(\mathbb{1}_A\mathbb{E}_n\left(|g-g_n|^2\right)\right)\right)^{\frac12}\\
&\leq P(A)^{\frac12-\frac1p}P(A)^{\frac12+\alpha_p}\|g\|_{\Lambda_2(\alpha_p)} = \|g\|_{\Lambda_2(\alpha_p)} \lesssim_p \|g\|_{\Lambda_1(\alpha_p)}
\end{aligned}$$

where the fourth inequality follows from the definition of the atom.

If $f \in L^2_{\rm fin}(\Omega)$, then it admits an atomic decomposition, namely, there exist a^j which are simple $(p,2)$-atoms and $(\lambda_j)_{j\geq 0} \in \ell^p$ such that,

$$f_n = \sum_{j=0}^{\infty}\lambda_j \mathbb{E}_n(a^j), \qquad \text{a.e. for all } n \in \mathbb{N},$$

and

$$\|f\|_{H^p(\Omega)} \approx_p \left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{\frac1p}.$$

We have

$$\Pi_1(f_n, g_n) = \sum_{k=1}^{n} d_k g d_k f = \sum_{j=0}^{\infty} \lambda_j \sum_{k=1}^{n} d_k g d_k a^j = \sum_{j=0}^{\infty} \lambda_j \Pi_1(a^j, g).$$

Thus

$$\|\Pi_1(f,g)\|_1 \leq \sum_{j=0}^{\infty} |\lambda_j| \|\Pi_1(a^j, g)\|_1 \lesssim_p \|g\|_{\Lambda_1(\alpha_p)} \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} \approx_p \|f\|_{H^p(\Omega)} \|g\|_{\Lambda_1(\alpha_p)}.$$

We shall also prove that Π_3 is a bounded bilinear operator from $H^p(\Omega) \times \Lambda_1(\alpha_p)$ to $H^p(\Omega)$. Without loss of generality, we consider $f \in L^2_{\text{fin}} \subset H^p(\Omega)$. Note that

$$S(\Pi_3(f,g))^2 = \sum_{k=1}^{\infty} |g_{k-1}|^2 |d_k f|^2 \leq (g^*)^2 S(f)^2. \tag{4.3.3}$$

Hence we conclude from Corollary 4.3.3 and the L^∞ boundedness of the maximal function (see. e.g. [43, Lemma I.4.2]) that

$$\|\Pi_3(f,g)\|^p_{H^p(\Omega)} \lesssim \|g^*\|^p_\infty \mathbb{E}(S(f)^p) \leq \|g\|^p_\infty \|f\|^p_{H^p(\Omega)} \lesssim_p \|f\|^p_{H^p(\Omega)} \|g\|^p_{\Lambda_1(\alpha_p)}, \tag{4.3.4}$$

as desired.

Estimates for $\Pi_2(f,g)$.

To show that Π_2 is a bounded bilinear operator from $H^p(\Omega) \times \Lambda_1(\alpha_p)$ to $H^1(\Omega)$, it suffices to show the boundedness restricted on the dense subspace $L^2_{\text{fin}}(\Omega)$.

By a similar argument in Section 4.3.2, one can show that it suffices to focus on Π_2 acting on atoms. In particular,

$$S(\Pi_2(a^j, g)) = \left(\sum_{k=n_j+1}^{\infty} \mathbb{1}_{A_{n_j}} |a^j_{k-1}|^2 |d_k g|^2 \right)^{\frac{1}{2}} \leq |(a^j)^*| \left(\sum_{k=n_j+1}^{\infty} \mathbb{1}_{A_{n_j}} |d_k g|^2 \right)^{\frac{1}{2}}. \tag{4.3.5}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[S(\Pi_2(a^j, g)) \right] &\leq \|(a^j)^*\|_\infty \left[\mathbb{E} \left(\mathbb{1}_{A_{n_j}} \sum_{k=n_j+1}^{\infty} |d_k g|^2 \right)^{\frac{1}{2}} \right] \\
&\leq \|a^j\|_\infty \left(P(A_{n_j}) \left(\mathbb{E} \sum_{k=n_j+1}^{\infty} |d_k g|^2 \right) \right)^{\frac{1}{2}} \\
&\leq P(A_{n_j})^{-\frac{1}{p}} \left(P(A_{n_j}) \|g\|^2_{\Lambda_2(\alpha_p)} P(A_{n_j})^{1+2\alpha_p} \right)^{\frac{1}{2}} \\
&= \|g\|_{\Lambda_2(\alpha_p)} P(A_{n_j})^{-\frac{1}{p}} P(A_{n_j})^{1+\alpha_p} \\
&\leq \|g\|_{\Lambda_2(\alpha_p)} \lesssim_p \|g\|_{\Lambda_1(\alpha_p)},
\end{aligned}$$

where the last inequality follows from the condition that $\alpha_p = 1/p - 1$. Follow a similar argument of the estimate of $\Pi_1(f,g)$, as a consequence of the above estimates, we have that

$$\|\Pi_2(f,g)\|^p_{H^1(\Omega)} \leq \sum_{j=1}^{\infty} |\lambda_j|^p \left[\mathbb{E} S(\Pi_2(a^j, g)) \right]^p \lesssim_p \|f\|^p_{H^p(\Omega)} \|g\|^p_{\Lambda_1(\alpha_p)}. \tag{4.3.6}$$

This completes the proof of the theorem.

Chapter 5

Bilinear decompositions on homogeneous spaces

This chapter is devoted to extending Theorem III.1 and Theorem III.2 to spaces of homogeneous type. Our approach is on the basis of the existence of dyadic systems on spaces of homogeneous type developed by Hytönen and Kairema in [49]. We also show that $H^p_{\rm at}(\mu)$ for $0 < p \leq 1$ is some finite sum of several dyadic martingale Hardy spaces on homogeneous spaces.

5.1 Dyadic systems on homogeneous spaces

In this section, we start with introducing dyadic systems on homogeneous spaces, which first appeared in the work of Hytönen and Kairema [49]. With the help of these dyadic structures, we then show that $H^p_{\rm at}(\mu)$ is exactly the finite sum of martingale Hardy spaces associated with some adjacent dyadic martingales, which extends Mei's result [65] to homogeneous spaces.

The following theorem concerning the existence of dyadic structures is due to Hytönen and Kairema [49].

Theorem 5.1.1. *Let Ω be a homogeneous space. Suppose that the constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$ satisfy*

$$12A_0^3 C_0 \delta \leq c_0,$$

where A_0 is specified in the definition of quasi-metric, see (1.5.1).

Given a set of reference points $\{z^k_\alpha\}_\alpha,\ \alpha \in \mathcal{A}_k$ (an index set), for every $k \in \mathbb{Z}$, with the properties that

$$d(z^k_\alpha, z^k_\beta) \geq c_0\delta^k,\ (\alpha \neq \beta) \quad \min_\alpha d(x, z^k_\alpha) < C_0\delta^k,\ \text{for all } x \in \Omega,$$

one can construct families of sets $\tilde{Q}^k_\alpha \subseteq Q^k_\alpha \subseteq \bar{Q}^k_\alpha$, called open, half-open and closed dyadic

cubes respectively, such that:

$$\tilde{Q}_\alpha^k \text{ and } \bar{Q}_\alpha^k \text{are the interior and closure of } Q_\alpha^k; \tag{5.1.1}$$

$$\text{if } k \le l, \text{ then either } Q_\beta^l \subseteq Q_\alpha^k \text{ or } Q_\beta^l \cap Q_\alpha^k = \emptyset; \tag{5.1.2}$$

$$X = \bigcup_\alpha Q_\alpha^k \text{ (disjoint union) for all } k \in \mathbb{Z}; \tag{5.1.3}$$

$$B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k) =: B(Q_\alpha^k),$$
$$\text{where } c_1 = (3A_0^2)^{-1}c_0 \text{ and } C_1 = 2A_0C_0; \tag{5.1.4}$$

$$\text{if } k \le l \text{ and } Q_\beta^l \subseteq Q_\alpha^k, \text{ then } B(Q_\beta^l) \subseteq B(Q_\alpha^k). \tag{5.1.5}$$

The open and closed cubes $\tilde{Q}_\alpha^k$ and $\bar{Q}_\alpha^k$ depend only on the points z_β^l for $l \ge k$. The half-open cubes Q_α^k depend on z_β^l for $l \ge \min(k, k_0)$, where $k_0 \in \mathbb{Z}$ is a preassigned number entering the construction.

It is obvious that the construction of the above dyadic systems is not unique, and it depends on the set of the reference points $\{z_\alpha^k\}_\alpha$. We denote this dyadic system by $\mathscr{D} = \{Q_\alpha^k\}_{k,\alpha}$. Let $\mathcal{F}_k = \sigma(\{Q_\alpha^k\}_\alpha)$ be the σ-algebra generated by $\{Q_\alpha^k\}_\alpha$. Then it is clear that

$$\cdots \subset \mathcal{F}_{k-1} \subset \mathcal{F}_k \subset \cdots,$$

which implies that $\{\mathcal{F}_k\}_{k\in\mathbb{Z}}$ is a filtration generated by atoms. Let $\mathcal{F} = \sigma(\cup_{k\in\mathbb{Z}}\mathcal{F}_k)$. Note that each Q_α^k is an atom of $\mathcal{F}_k$.

Remark 5.1.2. The standard dyadic grid on the real line is a dyadic system given by

$$\mathcal{F}_k = \{[2^{-k}m, 2^{-k}(m+1)) : m \in \mathbb{Z}\} \quad \text{for all } k \in \mathbb{Z}.$$

Similarly, an example of a dyadic system on $\mathbb{R}^n$ is given by the family of standard dyadic cubes in $\mathbb{R}^n$.

The following theorem ensures that there exist enough dyadic cubes to cover all balls on homogeneous spaces, which can be found in [49].

Theorem 5.1.3. *Given a set of reference points $\{z_\alpha^k\}, k \in \mathbb{Z}, \alpha \in \mathcal{A}_k$, suppose that there exists constant $\delta \in (0,1)$ that satisfies $96A_0^6\delta \le 1$. Then there exists a finite collection of families $\mathscr{D}^t, t = 1, 2, \cdots, K = K(A_0, A_1, \delta) < \infty$, where each $\mathscr{D}^t$ is a collection of dyadic cubes, associated to dyadic points $\{z_\alpha^k\}, k \in \mathbb{Z}, \alpha \in \mathcal{A}_k$, with the properties* (5.1.1)-(5.1.5) *in Theorem 5.1.1.*

In addition, the following property is satisfied:

$$\text{for every } B(x,r) \subseteq \Omega, \text{ there exist } t \text{ and } Q \in \mathscr{D}^t \text{ such that } B(x,r) \subseteq Q \text{ and } \mathrm{diam}(Q) \le Cr. \tag{5.1.6}$$

The constant $C < \infty$ in (5.1.6) *only depends on the quasi-metric constant A_0 and the parameter δ.*

We will provide the definition of the martingale BMO and Lipschitz spaces with respect to the dyadic system $\mathscr{D}$, which can be perceived as the dyadic variants of the BMO and Lipschitz spaces introduced in Definition 1.5.3 and 1.5.4.

Definition 5.1.4. For $0 < p < 1$, $q = 1$ or 2 and $\alpha_p = 1/p - 1$, define

$$BMO^{\mathscr{D}}(\mu) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega,\mu) \ : \ \|f\|_{BMO^{\mathscr{D}}(\mu)} := \sup_{Q\in\mathscr{D}} \frac{1}{\mu(Q)} \int_Q |f - f_Q| \,\mathrm{d}\mu < \infty \right\},$$

$$\Lambda_q^{\mathscr{D}}(\alpha_p) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega,\mu) \ : \ \|f\|_{\Lambda_q^{\mathscr{D}}(\alpha_p)} := \sup_{Q\in\mathscr{D}} \mu(Q)^{-\frac{1}{q}-\alpha_p} \left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu \right)^{\frac{1}{q}} < \infty \right\},$$

as quotient spaces modulo constant functions.

Note that $\Lambda_1^{\mathscr{D}}(\alpha_p) = \Lambda_2^{\mathscr{D}}(\alpha_p)$, for which we will provide a brief explanation, and more details can be found in [93, Corollary 1]. One first observes that $\Lambda_2^{\mathscr{D}}(\alpha_p) \subset \Lambda_1^{\mathscr{D}}(\alpha_p)$ by Hölder's inequality. For the reverse direction, for any $f \in \Lambda_1^{\mathscr{D}}(\alpha_p)$ and a cube $Q \in \mathscr{D}$. By the case of probability space, we obtain

$$\begin{aligned} \mu(Q)^{-\frac{1}{2}-\alpha_p} \left(\int_Q |f - f_Q|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} &\le \sup_{R\subset Q\in\mathscr{D}} \mu(R)^{-\frac{1}{2}-\alpha_p} \left(\int_R |f - f_R|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} \\ &\lesssim_p \sup_{R\subset Q\in\mathscr{D}} \mu(R)^{-1-\alpha_p} \int_R |f - f_R| \,\mathrm{d}\mu \le \|f\|_{\Lambda_1^{\mathscr{D}}(\alpha_p)}. \end{aligned}$$

This inequality holds independent of Q. Thus we obtain

$$\|f\|_{\Lambda_2^{\mathscr{D}}(\alpha_p)} \lesssim \|f\|_{\Lambda_1^{\mathscr{D}}(\alpha_p)}.$$

Therefore $\Lambda_1^{\mathscr{D}}(\alpha_p) \subset \Lambda_2^{\mathscr{D}}(\alpha_p)$.

The martingale BMO and Lipschitz spaces are closely related to the BMO and Lipschitz spaces (Definitions 1.5.3 and 1.5.4). In particular, the following theorem is proved in [49], which is an extension of a result due to Mei [65]:

$$BMO(\mu) = \bigcap_{t=1}^{K} BMO^{\mathscr{D}^t}(\mu). \tag{5.1.7}$$

We will now establish an analogous result for $\mathcal{L}_{\alpha_p}(\mu)$ $(0 < p < 1)$.

Theorem 5.1.5. *Let* $0 < p < 1$,

$$\mathcal{L}_{\alpha_p}(\mu) = \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p).$$

Proof. By Theorem 5.1.1, for any $Q \in \mathscr{D}^t$ (and $t = 1, 2, \cdots, K$), there exists a ball B such that $Q \subset B$ and $\mu(B) \lesssim \mu(Q)$. If $f \in \mathcal{L}_{\alpha_p}(\mu)$, then for any $x, y \in Q$, we have

$$|f(x) - f(y)| \le \|f\|_{\mathcal{L}_{\alpha_p}(\mu)} \mu(B)^{\alpha_p} \lesssim \|f\|_{\mathcal{L}_{\alpha_p}(\mu)} \mu(Q)^{\alpha_p}.$$

We thus have

$$\begin{aligned} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)} &\le \sup_{Q\in\mathscr{D}^t} \mu(Q)^{-\frac{1}{2}-\alpha_p} \left(\mu(Q)^{-2} \int_Q \left(\int_Q |f(x) - f(y)| \,\mathrm{d}\mu(y) \right)^2 \mathrm{d}\mu(x) \right)^{\frac{1}{2}} \\ &\le \sup_{Q\in\mathscr{D}^t} \mu(Q)^{-\frac{1}{2}-\alpha_p} \left(\int_Q \|f\|^2_{\mathcal{L}_{\alpha_p}(\mu)} \mu\left(Q\right)^{2\alpha_p} \,\mathrm{d}\mu \right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{\mathcal{L}_{\alpha_p}(\mu)}, \end{aligned}$$

which yields

$$\mathcal{L}_{\alpha_p}(\mu) \subset \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p). \tag{5.1.8}$$

Conversely, let $f \in \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p)$. By Theorem 4.3.4, for $Q \in \mathscr{D}^t$,

$$|f(x) - f_Q| \lesssim \mu(Q)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)} \quad \forall x \in Q,$$

which implies that for any $x, y \in Q$,

$$|f(x) - f(y)| \lesssim \mu(Q)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)}. \tag{5.1.9}$$

By Theorem 5.1.3, for any ball $B \subset \Omega$, there exist t and $Q \in \mathscr{D}^t$ such that $B \subset Q$ and $\mu(Q) \lesssim \mu(B)$. Then for any $x, y \in B$, by (5.1.9)

$$|f(x) - f(y)| \lesssim \mu(B)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)}.$$

Thus

$$\|f\|_{\mathcal{L}_{\alpha_p}(\mu)} \lesssim \sum_{t=1}^{K} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)},$$

which implies

$$\bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p) \subset \mathcal{L}_{\alpha_p}(\mu). \tag{5.1.10}$$

The theorem follows from (5.1.8) and (5.1.10). □

As in the probabilistic setting, one can also provide the definition of different martingale Hardy spaces defined on homogeneous spaces. For $f \in L^1_{\mathrm{loc}}(\Omega, \mathcal{F}, \mu)$, the martingale maximal function, the square function and the conditional square function of f associated with $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ are defined by

$$f^* := \max_{k\in\mathbb{Z}} |f_k|, \quad S(f) := \left(\sum_{k\in\mathbb{Z}} |d_k f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s(f) := \left(\sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}|d_k f|^2\right)^{\frac{1}{2}},$$

respectively.

Definition 5.1.6. Let $0 < p \le 1$. We first define

$$H^p_{m,\mathscr{D},0}(\mu) := \{f \in L^1(\Omega) : f^* \in L^p(\Omega)\}.$$

For $f \in H^p_{m,\mathscr{D},0}(\mu)$, we define the quasi-norm

$$\|f\|_{H^p_{m,\mathscr{D}}(\mu)} := \|f^*\|_p.$$

The martingale Hardy space $H^p_{m,\mathscr{D}}(\mu)$ is defined as the completion of $H^p_{m,\mathscr{D},0}(\mu)$ with respect to the quasi-norm $\|\cdot\|_{H^p_{m,\mathscr{D}}(\mu)}$.

Similarly, we define $H^p_{\mathscr{D}}(\mu)$ and $h^p_{\mathscr{D}}(\mu)$ by the square functions and the conditional square functions respectively, with the additional assumption that

$$\lim_{n\to-\infty} \int_\Omega \sup_{k\le n} |f_k|^p \, \mathrm{d}\mu = 0. \tag{5.1.11}$$

From (5.1.11), we have

$$\lim_{n\to-\infty} \sup_{k\le n} |f_k| = 0.$$

One natural question is whether the martingale Hardy spaces defined on homogeneous spaces admit atomic decompositions, to which we will provide a positive answer.

Definition 5.1.7. If $0 < p \leq 1 \leq q \leq \infty$ and $p < q$, we say that a measurable function a is a simple (p, q)-atom with respect to the dyadic system $\mathscr{D}$ (abbreviated as dyadic simple (p, q)-atom if the underlying dyadic system is fixed) if

(1) $\operatorname{supp}(a) \subset Q$ where $Q \in \mathscr{D}$ is a dyadic cube;

(2) $\|a\|_q \leq (\mu(B))^{\frac{1}{q}-\frac{1}{p}}$;

(3) $\int_\Omega a \,\mathrm{d}\mu = \int_Q a \,\mathrm{d}\mu = 0$.

Definition 5.1.8. We define the martingale atomic Hardy spaces $H^{p,q}_{\mathrm{at},\mathscr{D}}(\mu)$ ($0 < p < 1 \leq q \leq \infty$ or $p = 1, 1 < q \leq \infty$) analogous to Definition 1.5.5 and 1.5.6, namely,

$$H^{1,q}_{\mathrm{at},\mathscr{D}}(\mu) := \Bigg\{ f \in \Big(BMO^{\mathscr{D}}(\mu)\Big)^* : f = \sum_{j=0}^{\infty} \lambda_j a^j, \\ \text{where } a^j \text{ is a simple } (1,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j| < \infty \Bigg\} \quad (5.1.12)$$

where $f = \sum\limits_{j=0}^{\infty} \lambda_j a^j$ is in terms of w^*-topology. And for $0 < p < 1$,

$$H^{p,q}_{\mathrm{at},\mathscr{D}}(\mu) := \Bigg\{ f \in \Big(\Lambda^{\mathscr{D}}_2(\alpha_p)\Big)^* : f = \sum_{j=0}^{\infty} \lambda_j a^j, \\ \text{where } a^j \text{ is a simple } (p,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j|^p < \infty \Bigg\} \quad (5.1.13)$$

where $f = \sum\limits_{j=0}^{\infty} \lambda_j a^j$ is in terms of w^*-topology. The norm is identified as for $0 < p \leq 1$,

$$\|f\|_{H^{p,q}_{\mathrm{at},\mathscr{D}}} = \inf \left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a simple } (p,q)\text{-atom} \right\}.$$

Remark 5.1.9. Since every dyadic simple (p, q)-atom is locally supported, by Corollary 4.3.2, we conclude that for $0 < p < 1 \leq q \leq \infty$ or $p = 1, 1 < q \leq \infty$

$$H^{p,q}_{\mathrm{at},\mathscr{D}}(\mu) = H^{p,\infty}_{\mathrm{at},\mathscr{D}}(\mu).$$

Thus we are only concerned with $H^p_{\mathrm{at},\mathscr{D}}(\mu) := H^{p,\infty}_{\mathrm{at},\mathscr{D}}(\mu)$.

The following proposition states that the martingale Hardy spaces introduced in Definition 5.1.6 admit atomic decompositions.

Proposition 5.1.10. *For $0 < p \leq 1$, the martingale Hardy spaces defined above are mutually equivalent. Namely, $H^p_{\mathscr{D}}(\mu) = H^p_{m,\mathscr{D}}(\mu) = h^p_{\mathscr{D}}(\mu) = H^p_{\mathrm{at},\mathscr{D}}(\mu)$.*

Proof. Let $p \in (0,1]$ be fixed. First, we show $H^p_{\mathscr{D}}(\mu) = H^p_{m,\mathscr{D}}(\mu)$. Suppose that $f \in H^p_{m,\mathscr{D}}(\mu)$. Then for any $n > 0$, by a well-known inequality of Burkholder–Davis–Gundy,

$$\int_\Omega \left(|f_{-n}|^2 + \sum_{k=-n+1}^{n} |d_k f|^2 \right)^{\frac{p}{2}} \mathrm{d}\mu \lesssim \int_\Omega \sup_{-n\le k\le n} |f_k|^p \,\mathrm{d}\mu \lesssim \int_\Omega |f^*|^p \,\mathrm{d}\mu.$$

Now by letting $n \to \infty$ and by Fatou's lemma, we conclude that

$$\|S(f)\|_p \lesssim \|f^*\|_p.$$

Thus $H^p_{m,\mathscr{D}}(\mu) \subset H^p_{\mathscr{D}}(\mu)$.

Conversely, if $f \in H^p_{\mathscr{D}}(\mu)$, then for $n > 0$,

$$\int_\Omega \sup_{-n\le k\le n} |f_k|^p \,\mathrm{d}\mu \lesssim \int_\Omega \left(|f_{-n}|^2 + \sum_{k=-n+1}^{n} |d_k f|^2 \right)^{\frac{p}{2}} \mathrm{d}\mu,$$

and hence

$$\int_\Omega \sup_{-n\le k\le n} |f_k|^p \,\mathrm{d}\mu \lesssim \int_\Omega \sup_{k\le -n} |f_k|^p \,\mathrm{d}\mu + \int_\Omega |S(f)|^p \,\mathrm{d}\mu < \infty. \tag{5.1.14}$$

Then by letting $n \to \infty$ and applying Fatou's lemma, we obtain $\|f^*\|_p < \infty$ and

$$\|f^*\|_p \lesssim \|S(f)\|_p.$$

Therefore, $H^p_{\mathscr{D}}(\mu) \subset H^p_{m,\mathscr{D}}(\mu)$ and $H^p_{m,\mathscr{D}}(\mu) = H^p_{\mathscr{D}}(\mu)$.

One shows $H^p_{m,\mathscr{D}}(\mu) = h^p_{\mathscr{D}}(\mu)$ in a completely analogous way. To show $h^p_{\mathscr{D}}(\mu) = H^p_{\mathrm{at},\mathscr{D}}(\mu)$, it suffices to show that if $f \in L^1_{\mathrm{loc}}(\Omega) \cap h^p_{\mathscr{D}}(\mu)$, then $f \in H^p_{\mathrm{at},\mathscr{D}}(\mu)$. One can mimic the standard way in the proof of [94, Theorem 2.5] to show that, and we omit the details. □

We now describe the duality between martingale atomic Hardy spaces and martingale BMO and Lipschitz spaces, namely

$$\left(H^1_{\mathrm{at},\mathscr{D}}(\mu)\right)^* = BMO^{\mathscr{D}}(\mu), \tag{5.1.15}$$

and for $0 < p < 1$,

$$\left(H^p_{\mathrm{at},\mathscr{D}}(\mu)\right)^* = \Lambda^{\mathscr{D}}_q(\alpha_p). \tag{5.1.16}$$

We will provide a sketch of the proof for the duality between $H^1_{\mathrm{at},\mathscr{D}}(\mu)$ and $BMO^{\mathscr{D}}(\mu)$. The case $p < 1$ follows an analogous strategy. More details can be found in the argument for Theorem 4 in [93]. By definition of $H^1_{\mathrm{at},\mathscr{D}}(\mu)$, we know that any function $g \in BMO^{\mathscr{D}}(\mu)$ induces a linear functional on $H^1_{\mathrm{at},\mathscr{D}}(\mu)$. In turn, let ℓ be a linear functional on $H^1_{\mathrm{at},\mathscr{D}}(\mu)$, the norm is denoted by $\|\ell\|$. Let $Q_n \in \mathcal{F}_{-n}$ be a finite union of cubes. Recall that $BMO(Q_n)$ is the martingale BMO space in the Definition 1.3.9. Denote by

$$H^1_{\mathrm{at}}(Q_n) = \left\{ f \in H^1_{\mathrm{at},\mathscr{D}}(\mu) : \operatorname{supp} f \subset Q_n \right\}.$$

For any $f \in H^1_{\mathrm{at},\mathscr{D}}(\mu)$ supported in Q_n, and $\mathbb{E}_n(f) = 0$. We know that $f \in H^1_{\mathrm{at}}(Q_n)$. Then we obtain from the probability case that there exists a function $g \in BMO(Q_n)$ such that

$$\ell(f) = \int_{Q_n} f \mathbb{1}_{Q_n} g \,d\mu = \int_\Omega f g \mathbb{1}_{Q_n} \,\mathrm{d}\mu. \tag{5.1.17}$$

Moreover, we have $\|g\mathbb{1}_{Q_n}\|_{BMO(Q_n)} \lesssim \|\ell\|$. The function g is uniquely determined up to a constant. We follow an analogous argument of [27, Section 3]. Let $\{Q_n\}_{n\geq 1}$ be an increasing sequence where $Q_n \in \mathcal{F}_{-n}$ such that $\mathbb{1}_{Q_n}$ converges to 1 pointwisely. Then we obtain a function $g_n = g$ on each cube Q_n such that (5.1.17) holds. Thus the condition $\mathbb{E}_1(g\mathbb{1}_{Q_1}) = 0$ gives a function g such that (5.1.17) holds for each cube Q in $\mathscr{D}$. Moreover,

$$\frac{1}{\mu(Q)}\int_Q |g - g_Q|\,\mathrm{d}\mu \leq \|g\mathbb{1}_Q\|_{BMO(Q)} \lesssim \|\ell\|.$$

Thus we conclude $g \in BMO^{\mathscr{D}}(\mu)$, which implies the linear functional ℓ can be expressed as a function in $BMO^{\mathscr{D}}(\mu)$.

By virtue of Proposition 5.1.10 and Theorem 5.1.3, we have the following theorem, which reveals the relation between atomic Hardy spaces and martingale (atomic) Hardy spaces and extends Mei's result in [65].

Theorem 5.1.11. *For $0 < p \leq 1$, we have*

$$H^p_{\mathrm{at}}(\mu) = \sum_{t=1}^K H^p_{\mathrm{at},\mathscr{D}^t}(\mu) = \sum_{t=1}^K H^p_{\mathscr{D}^t}(\mu) = \sum_{t=1}^K H^p_{m,\mathscr{D}^t}(\mu) = \sum_{t=1}^K h^p_{\mathscr{D}^t}(\mu). \tag{5.1.18}$$

Proof. Let $p \in (0,1]$ be fixed. In view of Proposition 5.1.10, it suffices to show $H^p_{\mathrm{at}}(\mu) = \sum_{t=1}^K H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$. We prove it via comparing the atoms. Let a be a (p,∞)-atom in $H^p_{\mathrm{at}}(\mu)$. Then there exists a ball B such that

$$\mathrm{supp}(a) \subset B,\ \|a\|_\infty \leq (\mu(B))^{-\frac{1}{p}},\ \int_B a(x)\,\mathrm{d}\mu = 0.$$

By Theorem 5.1.3, there exist t and a cube $Q \in \mathscr{D}^t$ such that $B \subset Q$, and $\mu(Q) \lesssim \mu(B)$. Then

$$\mathrm{supp}(a) \subset B \subset Q,\ \|a\|_\infty \leq (\mu(B))^{-\frac{1}{p}} \lesssim (\mu(Q))^{-\frac{1}{p}},\ \int_Q a\,\mathrm{d}\mu = 0,$$

which implies that a is a constant multiple of a simple (p,∞)-atom in $H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$.

Now we derive the desired result from atoms. We only prove the case $0 < p < 1$ while the case $p = 1$ follows a similar argument. By definition, for any $f \in H^p_{\mathrm{at}}(\mu)$, there exists a sequence of real numbers $\{\lambda_j\}_{j\geq 0}$ and (p,∞)-atoms a_j such that for any reasonable $g \in \mathcal{L}_{\alpha_p}(\mu)$,

$$\langle f, g\rangle = \sum_{j=0}^\infty \lambda_j \langle a^j, g\rangle. \tag{5.1.19}$$

It is clear that this series is absolutely convergent.

The aforementioned result shows that we can separate the set of atoms $\{a^j\}$ into K parts, the t-th part is denoted by $\left\{a^t_{n_j}\right\}$. Each part $\left\{a^t_{n_j}\right\}$ only consists of simple (p,∞)-atoms associated with dyadic system $\mathscr{D}^t$ $(1 \leq t \leq K)$ respectively. Then we define the following functions

$$f^t = \sum_{j=0}^\infty \lambda^t_{n_j} a^t_{n_j}, \quad t = 1, 2, \cdots, K.$$

The series converges w^* in $\left(\Lambda_2^{\mathscr{D}^t}(\alpha_p)\right)^*$, respectively. Here $\lambda_{n_j}^t$ is the coefficient accompanying with the atom $a_{n_j}^t$ in (5.1.19). Then we can rearrange the series in (5.1.19). We obtain

$$\langle f, g\rangle = \sum_{t=1}^{K}\sum_{j=0}^{\infty}\lambda_{n_j}^t\langle a_{n_j}^t, g\rangle = \sum_{t=1}^{K}\langle f^t, g\rangle,$$

since $g \in \Lambda_2^{\mathscr{D}^t}(\alpha_p)$ for all $1 \le t \le K$ due to Theorem 5.1.5. Moreover,

$$\|f\|_{H^p_{\mathrm{at}}(\mu)} \approx \left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{\frac{1}{p}} \approx \sum_{t=1}^{K}\left(\sum_{j=0}^{\infty}|\lambda_{n_j}^t|^p\right)^{\frac{1}{p}} \approx \sum_{t=1}^{K}\|f^t\|_{H^p_{\mathrm{at},\mathscr{D}^t}(\mu)}.$$

Thus we conclude

$$H^p_{\mathrm{at}}(\mu) \subset \sum_{t=1}^{K} H^p_{\mathrm{at},\mathscr{D}^t}(\mu). \tag{5.1.20}$$

For any $t = 1, 2, \cdots, K$ and for any given simple (p,∞)-atom b in $H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$, there exists $Q \in \mathscr{D}^t$ such that

$$\mathrm{supp}(b) \subset Q,\ \|b\|_\infty \le (\mu(Q))^{-\frac{1}{p}},\ \int_Q b\,\mathrm{d}\mu = 0.$$

By Theorem 5.1.1, there exists a ball B such that $Q \subset B$ and $\mu(Q) \gtrsim \mu(B)$. Hence

$$\mathrm{supp}(b) \subset Q \subset B,\ \|b\|_\infty \le (\mu(Q))^{-\frac{1}{p}} \lesssim (\mu(B))^{-\frac{1}{p}},\ \int_B b\,\mathrm{d}\mu = 0,$$

which implies that a multiple of b is also a (p,∞)-atom in $H^p_{\mathrm{at}}(\mu)$. Thus by applying a similar argument as before, we conclude

$$\sum_{t=1}^{K} H^p_{\mathrm{at},\mathscr{D}^t}(\mu) \subset H^p_{\mathrm{at}}(\mu). \tag{5.1.21}$$

We combine (5.1.20) and (5.1.21) to complete the proof of the theorem. □

Remark 5.1.12. Theorem 1.5.7 follows immediately from Corollary 4.3.2, Proposition 5.1.10 and Theorem 5.1.11, which provides an alternative approach to the original proof by Coifman and Weiss in [27].

Remark 5.1.13. Theorem 5.1.11 and Theorem 5.1.5 give a different proof of Theorem 1.5.8 originally established by Coifman and Weiss [27]:

$$(H^p_{\mathrm{at}}(\mu))^* = \left(\sum_{t=1}^{K} H^p_{\mathrm{at},\mathscr{D}^t}(\mu)\right)^* = \bigcap_{t=1}^{K}(H^p_{\mathrm{at},\mathscr{D}^t}(\mu))^* = \bigcap_{t=1}^{K}\Lambda_2^{\mathscr{D}^t}(\alpha_p) = \mathcal{L}_{\alpha_p}(\mu).$$

5.2 Bilinear decompositions for dyadic martingales on homogeneous spaces

In this section, we focus on bilinear decompositions arising in the study of products between elements in spaces of dyadic martingales on homogeneous spaces introduced in

the previous section. In the setting of homogeneous spaces, due to their quasi-metrics and measures, the dyadic martingales behave worse than martingales in probability spaces and the underlying analysis is more intricate.

In Section 5.2.1 we prove appropriate generalized Hölder-type inequalities on homogeneous spaces (see Lemmas 5.2.2 and 5.2.4 below). We then introduce a class of pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $BMO^{\mathscr{D}}_+(\mu)$; see Theorem 5.2.5 below. Using Theorem 5.2.5, we define products between dyadic martingale Hardy spaces on homogeneous spaces and their duals and finally, in Section 5.2.2 we establish analogues of the results of Sections 4.2 and 4.3 in the setting of homogeneous spaces.

5.2.1 A generalized Hölder-type inequality

Let $0 < p \le 1$ and $\mathscr{D}$ be a dyadic system constructed as in Theorem 5.1.1. The martingale Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathscr{D}}(\mu)$ consist of all measurable functions f on $(\Omega, \mathcal{F}, \mu)$ such that $s(f) \in L^{\Psi_p}(\Omega)$ where $O \in \Omega$ is a fixed point, and

$$\Psi_1(x,t) := \frac{t}{\log\left(e + d(x,O)\right) + \log(e+t)},$$
$$\Psi_p(x,t) := \frac{t}{1 + \{t[1+\mu(B(O,d(x,O)))]\}^{1-p}} \quad (0 < p < 1).$$

Note that $L^{\Psi_p}(\Omega)$ is a quasi-normed space.

Let $M := (C_\mu + 1)\log\left(e + d(x,O)\right)$. By (1.3.6) we obtain

$$\Psi_1(x,st) \lesssim (e+d(x,O))^{-(C_\mu+1)}e^t + s \lesssim w(x)e^t + s, \quad \text{for all } x \in \Omega, s, t > 0, \tag{5.2.1}$$

where $w : \Omega \to \mathbb{R}_+$ is a weight function with

$$w(x) \lesssim \min\left\{1, d(x,O)^{-(C_\mu+1)}\right\}. \tag{5.2.2}$$

Let $Q^0 \in \mathcal{F}_0$ be the dyadic cube such that $O \in Q^0$. For $g \in BMO^{\mathscr{D}}(\mu)$, define

$$\|g\|_{BMO^{\mathscr{D}}_+(\mu)} := \sup_{\alpha \in \mathcal{A}_0} \frac{|g_{Q^0_\alpha} - g_{Q^0}|}{\log\left(e + d(z^0_\alpha, O)\right)} + |g_{Q^0}| + \|g\|_{BMO^{\mathscr{D}}(\mu)},$$

where $Q^0_\alpha \in \mathcal{F}_0$ is a dyadic cube with its center z^0_α and $\mathcal{A}_0$ is the index set in Theorem 5.1.1. Denote by $BMO^{\mathscr{D}}_+(\mu)$ the space consisting of all $g \in BMO^{\mathscr{D}}(\mu)$ such that $\|g\|_{BMO^{\mathscr{D}}_+(\mu)} < \infty$. It is not difficult to verify that $\|\cdot\|_{BMO^{\mathscr{D}}_+(\mu)}$ is a norm on $BMO^{\mathscr{D}}_+(\mu)$.

Remark 5.2.1. Note that if $g \in BMO(\mu)$, then $g \in BMO^{\mathscr{D}}_+(\mu)$. Moreover,

$$\|g\|_{BMO^{\mathscr{D}}_+(\mu)} \lesssim \|g\|_{BMO(\mu)} + |g_{Q^0}|.$$

The proof is similar to the Euclidean setting, and we will sketch the proof. For $Q^0 \in \mathcal{F}_0$, there exists a ball B_1 such that $Q^0 \subset B_1$ and $|B_1| \lesssim |Q^0|$. Similar construction can be performed with respect to Q^0_α, and we will denote the corresponding ball by B_2. Therefore we have $|g_{Q^0_\alpha} - g_{Q^0}| \approx |g_{B_2} - g_{B_1}|$. Moreover, we can normalize the radius of B_1 and B_2 to be 1. Denote by $d = d(z^0_\alpha, O)$. Then there exists a ball B with radius $2d$ such that $B_1, B_2 \subset B$ and

$$|g_{B_2} - g_{B_1}| \le |g_{B_2} - g_B| + |g_B - g_{B_1}| \lesssim (1 + \log d)\|g\|_{BMO(\mu)}.$$

Combining this inequality into the definition of $BMO^{\mathscr{D}}_+(\mu)$, we obtain the desired result.

We now introduce the following generalized Hölder inequality for $L^1(\Omega,\mathcal{F},\mu)$ and $BMO_+^{\mathscr{D}}(\mu)$.

Lemma 5.2.2. *If $f\in L^1(\Omega,\mathcal{F},\mu)$ and $g\in BMO_+^{\mathscr{D}}(\mu)$, then $f\cdot g\in L^{\Psi_1}(\Omega)$. Moreover,*

$$\|fg\|_{L^{\Psi_1}(\Omega)}\lesssim \|f\|_1\|g\|_{BMO_+^{\mathscr{D}}(\mu)}. \tag{5.2.3}$$

Proof. Without loss of generality, assume $\|f\|_1\le 1$, $\|g\|_{BMO_+^{\mathscr{D}}(\mu)}\le 1$ and $g_{Q^0}=0$. It suffices to show that

$$\int_\Omega \Psi_1(x,|f(x)g(x)|)\,\mathrm{d}\mu\lesssim 1.$$

Let $S_k:=B(O,C_0\delta^k)\setminus B(O,C_0\delta^{k+1})$ for $k<0$ and $S_0:=B(O,C_0)$, where $\delta\in(0,1)$ is the constant in Theorem 5.1.1. Then for each $k\le 0$, there exists a finite index subset $\mathcal{B}_k\subset\mathcal{A}_0$ such that $B(O,C_0\delta^k)\subset\bigcup\limits_{\alpha\in\mathcal{B}_k}Q_\alpha^0$ (where $Q_\alpha^0\in\mathcal{F}_0$) and

$$\sum_{\alpha\in\mathcal{B}_k}\mu\left(Q_\alpha^0\right)=\mu\left(\bigcup_{\alpha\in\mathcal{B}_k}Q_\alpha^0\right)\le\mu\left(B(O,2A_0C_0\delta^k)\right)\lesssim\delta^{C_\mu k}.$$

Let $s=\nu^{-1}|f(x)|,t=\nu|g(x)|$ in (5.2.1). One has

$$\begin{aligned}\int_\Omega \Psi_1(x,|f(x)g(x)|)\,\mathrm{d}\mu&=\sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0}\Psi_1(x,|f(x)g(x)|)\,\mathrm{d}\mu\\&\lesssim\sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0}w(x)e^{\nu|g(x)|}\,\mathrm{d}\mu+\nu^{-1}\|f\|_1.\end{aligned}$$

Therefore,

$$\int_\Omega \Psi_1(x,|f(x)g(x)|)\,\mathrm{d}\mu\lesssim T_1+\nu^{-1}\|f\|_1, \tag{5.2.4}$$

where

$$T_1:=\sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0}w(x)e^{\nu\left|g(x)-g_{Q_\alpha^0}\right|}e^{\nu\left|g_{Q_\alpha^0}\right|}\,\mathrm{d}\mu.$$

Let $\nu:=\frac{\min\{\kappa,1\}}{2}>0$ (where κ is defined in Theorem 1.3.11). By (5.2.2) and Theorem 4.3.1, one has

$$\begin{aligned}T_1&\lesssim\sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\frac{\mu(Q_\alpha^0)\,(e+d(z_\alpha^0,O))^{\frac12}}{\delta^{(k+1)(C_\mu+1)}}\\&\lesssim\sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\frac{\mu(Q_\alpha^0)\delta^{\frac{k}{2}}}{\delta^{(k+1)(C_\mu+1)}}\lesssim\sum_{k=-\infty}^{0}\frac{\delta^{C_\mu k}\delta^{\frac{k}{2}}}{\delta^{C_\mu k+k}}\\&\lesssim\sum_{k=-\infty}^{0}\delta^{-\frac12 k},\end{aligned}$$

and hence

$$T_1\lesssim 1. \tag{5.2.5}$$

We combine (5.2.4), (5.2.5) and the fact that $\nu^{-1}\|f\|_1\lesssim 1$, and the proof is complete. □

We consider the case $0 < p < 1$. Define

$$\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} := \sup_{\alpha\in\mathcal{A}_0} \frac{|g_{Q^0_\alpha} - g_{Q^0}|}{1+\mu\left\{B\left(O, d(z^0_\alpha, O)\right)\right\}^{\alpha_p}} + |g_{Q^0}| + \|g\|_{\Lambda^{\mathscr{D}}_1(\alpha_p)},$$

Denote by $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ the space consisting of all $g \in \Lambda^{\mathscr{D}}_1(\alpha_p)$ such that $\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} < \infty$. It is easy to verify that $\|\cdot\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}$ is a norm on $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$.

Remark 5.2.3. If we consider the dyadic martingales on $\mathbb{R}^n$, by taking appropriate cubes Q^0 one can show that if $g \in \Lambda^{\mathscr{D}}_1(\alpha_p)$, then $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. Note that if $g \in \mathcal{L}_{\alpha_p}(\mu)$, then $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. Moreover,

$$\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \lesssim \|g\|_{\mathcal{L}_{\alpha_p}(\mu)} + |g_{Q^0}|.$$

The proof is similar to the case of $BMO^{\mathscr{D}}_+(\mu)$ (see Remark 5.2.1). We omit the details.

Next we present a generalized Hölder inequality for $L^p(\Omega, \mathcal{F}, \mu)$ and $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ for $0 < p < 1$.

Lemma 5.2.4. *If $f \in L^p(\Omega, \mathcal{F}, \mu)$ and $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ for $0 < p < 1$, then $f \cdot g \in L^{\Psi_p}(\mu)$. Moreover,*

$$\|fg\|_{L^{\Psi_p}(\Omega)} \lesssim \|f\|_p \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}. \tag{5.2.6}$$

Proof. Without loss of generality, assume $\|f\|_p \le 1$, $\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \le 1$ and $g_{Q^0} = 0$. It suffices to show that

$$\int_\Omega \Psi_p(x, |f(x)g(x)|)\,\mathrm{d}\mu \lesssim 1.$$

Take the same family of sets $\{S_k\}_{k\le 0}$ as above. From Theorem 4.3.4, we know that for $x \in Q^0_\alpha$,

$$\begin{aligned}
|g(x)| = |g(x) - g_{Q^0}| &\le \left|g(x) - g_{Q^0_\alpha}\right| + \left|g_{Q^0_\alpha} - g_{Q^0}\right| \\
&\le \left(\mu(Q^0_\alpha)\right)^{\alpha_p} + \mu\left\{B\left(O, d(z^0_\alpha, O)\right)\right\}^{\alpha_p} + 1 \\
&\lesssim \mu\left(B(O, 2A_0C_0\delta^k)\right)^{\alpha_p} + 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_\Omega \Psi_p(x, |f(x)g(x)|)\,\mathrm{d}\mu &= \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k} \int_{S_k\cap Q^0_\alpha} \frac{|g(x)||f(x)|}{1+\{|g(x)||f(x)|\,[1+\mu\,(B(O,d(x,O))]\}^{1-p}}\,\mathrm{d}\mu \\
&\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k} \int_{S_k\cap Q^0_\alpha} \frac{|g(x)|^p|f(x)|^p}{1+\mu\,(B(O,d(x,O))^{1-p}}\,\mathrm{d}\mu \\
&\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k} \int_{S_k\cap Q^0_\alpha} \frac{\mu\left(B(O,2A_0C_0\delta^k)\right)^{\alpha_p p}+1}{\{1+\mu\,(B(O,C_0\delta^{k+1})\}^{1-p}}|f(x)|^p\,\mathrm{d}\mu \\
&\lesssim 1,
\end{aligned}$$

which finishes the proof. □

We are now about to present the analogues of the results in Sections 4.2 and 4.3 concerning bilinear decompositions for dyadic martingales on homogeneous spaces. We first need to define the product between martingale Hardy spaces and their dual spaces first. As in the probability setting, we regard the product in the sense of distribution as follows: for $0 < p < 1$,

$$\langle f \times g, h\rangle := \langle h \cdot g, f\rangle, \quad f \in H^p_{\mathrm{at},\mathscr{D}}(\mu),\ g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p),$$

where h is a test function such that $h \cdot g$ is in $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. For $p = 1$, we may define the product between $H^1_{\mathrm{at},\mathscr{D}}(\mu)$ and $BMO^{\mathscr{D}}_+(\mu)$ analogously. To this end, we need to introduce some pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $BMO^{\mathscr{D}}_+(\mu)$.

Denote the space of test functions by $\mathcal{H}(\alpha_p)$ $(0 < p \le 1)$, and a measurable function h is a test function if it satisfies the following properties:

$$|h(x)| \lesssim \frac{1}{(1+\mu(B(O,d(x,O)))^{\alpha_p})\log(e+d(x,O))}, \quad \forall x \in \Omega, \tag{5.2.7}$$

and

$$|h(y)-h(z)| \lesssim \frac{\mu(B)^{\alpha_p}}{(1+\mu[B(O,1+r+d(c_B,O))]^{\alpha_p})\log(e+r+d(c_B,O))} \tag{5.2.8}$$

whenever y, z are both contained in a ball B with center c_B and radius $r \le \frac{d(c_B,O)}{2A_0} + 1$.

It is obvious that $\mathcal{H}(\alpha_p) \subset L^\infty(\Omega)$. The following theorem shows that if $h \in \mathcal{H}(\alpha_p)$, then h is a pointwise multiplier of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$.

Theorem 5.2.5. *For $0 < p < 1$ and any dyadic system $\mathscr{D}$, $\mathcal{H}(\alpha_p)$ is a space of pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. For $p = 1$, $\mathcal{H}(0)$ is a space of pointwise multipliers of $BMO^{\mathscr{D}}_+(\mu)$. More precisely, for any $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $h \in \mathcal{H}(\alpha_p)$, we have*

$$\|g\cdot h\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \left(\|h\|_{L^\infty(\Omega)} + 1\right),$$

and for any $g \in BMO^{\mathscr{D}}_+(\mu)$ and $h \in \mathcal{H}(0)$, we have

$$\|g\cdot h\|_{BMO^{\mathscr{D}}_+(\mu)} \lesssim \|g\|_{BMO^{\mathscr{D}}_+(\mu)} \left(\|h\|_{L^\infty(\Omega)} + 1\right).$$

Proof. First, we consider the case $0 < p < 1$. Assume that $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $h \in \mathcal{H}(\alpha_p)$. According to [70], it suffices to show that

$$\sup_Q \frac{|g_Q|}{\mu(Q)^{\alpha_p+1}} \left(\int_Q |h(x) - h_Q|\,\mathrm{d}x\right) < \infty, \tag{5.2.9}$$

where Q runs over all dyadic cubes in $\mathscr{D}$.

If $Q \subset Q^0_\beta$ for some $\beta \in \mathcal{A}_0$, there exists a collection of cubes $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_N = Q^0_\beta$ such that there exists a universal constant $0 < \delta' < 1$ with $\mu(Q_{k-1}) \le \delta'\mu(Q_k)$. Hence

$$\begin{aligned}
|g_Q - g_{Q^0_\beta}| &\le \sum_{k=1}^N |g_{Q_k} - g_{Q_{k-1}}| \lesssim \sum_{k=1}^N \mu(Q_k)^{\alpha_p}\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \\
&\lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \sum_{k=1}^N \int_{\mu(Q_{k-1})}^{\mu(Q_k)} t^{\alpha_p-1}dt \\
&\lesssim \mu(Q^0_\beta)^{\alpha_p}\|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}.
\end{aligned}$$

Similarly, if $Q_\beta^0 \subset Q$, we have

$$|g_Q - g_{Q_\beta^0}| \lesssim \mu(Q)^{\alpha_p} \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}.$$

By Theorem 5.1.1, there exists a ball B, with center c_B and radius r, such that $Q \subset B$ and $\mu(B) \lesssim \mu(Q)$.

If $Q_\beta^0 \subset Q$ and $r > \frac{d(O,c_B)}{2A_0} + 1$, for any $x \in B(O,r)$, we have $d(c_B,x) \le A_0(d(c_B,O) + d(O,x)) < (2A_0^2 + A_0)r$. Then $\mu(Q) \gtrsim \mu(B) \gtrsim C_\mu^{-(2A_0^2+A_0)} \mu\left(B(O,r)\right) \gtrsim 1$. Similarly, we also have $d(z_\beta^0, O) < (2A_0^2 + A_0)r$ and $\mu\left\{B\left(O, d(z_\beta^0, O)\right)\right\} \lesssim \mu(B) \lesssim \mu(Q)$. Thus

$$\begin{aligned}
\frac{|g_Q|}{\mu(Q)^{\alpha_p+1}} \left(\int_Q |h(x) - h_Q| \, \mathrm{d}x \right) &\lesssim \frac{|g_Q - g_{Q_\beta^0}| + |g_{Q_\beta^0} - g_{Q^0}| + |g_{Q^0}|}{\mu(Q)^{\alpha_p}} \cdot \|h\|_{L^\infty(\Omega)} \\
&\lesssim \frac{\mu(Q)^{\alpha_p} + \mu\left\{B\left(O, d(z_\beta^0, O)\right)\right\}^{\alpha_p} + 1}{\mu(Q)^{\alpha_p}} \cdot \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \|h\|_{L^\infty(\Omega)} \\
&\lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \|h\|_{L^\infty(\Omega)}.
\end{aligned}$$

If $Q_\beta^0 \subset Q$ and $r \le \frac{d(O,c_B)}{2A_0} + 1$, for any $x \in B$, we have $d(x,O) \le A_0(d(O,c_B) + r)$, then $\mu(Q) \lesssim \mu\left(B(O, A_0(d(O,c_B) + r))\right)$. Thus

$$\begin{aligned}
\frac{|g_Q|}{\mu(Q)^{\alpha_p+1}} \left(\int_Q |h(x) - h_Q| \, \mathrm{d}x \right) &\lesssim \frac{|g_Q - g_{Q_\beta^0}| + |g_{Q_\beta^0} - g_{Q^0}| + |g_{Q^0}|}{\mu(Q)^{\alpha_p+1}} \\
&\qquad \cdot \frac{\mu(B)^{\alpha_p+1}}{\left(1 + \mu[B(O, 1 + r + d(O,c_B))]\right)^{\alpha_p}} \\
&\lesssim \frac{\left(\mu(Q)^{\alpha_p} + \mu\left\{B\left(O, d(z_\beta^0, O)\right)\right\}^{\alpha_p} + 1\right) \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}}{\left(1 + \mu[B(O, 1 + r + d(O,c_B))]\right)^{\alpha_p}} \\
&\lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}.
\end{aligned}$$

If $Q \subset Q_\beta^0$, from Theorem 5.1.1, we can choose C_0 sufficiently small such that $C_1 = 2A_0C_0 \le 1$, then $r \le C_1 \le \frac{d(c_B,O)}{2A_0} + 1$. For any $x \in Q_\beta^0$, we have $d(O,x) \le A_0(d(O,z_\beta^0) + C_1)$. Then

$$\mu(Q_\beta^0) \lesssim \mu\left\{B\left(O, A_0(d(O,z_\beta^0) + C_1)\right)\right\}.$$

By a calculation similar to the one presented above, we get the desired result.

Combining the above estimates, we finish our proof for $0 < p < 1$. The case for $p = 1$ is similar. □

Remark 5.2.6. Note that Theorem 5.2.5 holds for any dyadic system. Then from Theorem 5.1.5 and (5.1.7), we conclude that $\mathcal{H}(\alpha_p)$ is a space of pointwise multipliers of $\mathcal{L}_{\alpha_p}(\mu)$ and $\mathcal{H}(0)$ is a space of pointwise multipliers of $BMO(\mu)$.

5.2.2 Bilinear decompositions

We now state the theorem concerning bilinear decompositions for dyadic martingales on homogenous spaces.

Theorem 5.2.7. (1) *There exist continuous bilinear operators* $\Pi_1 : H^1_{\mathscr{D}}(\mu)\times BMO^{\mathscr{D}}_+(\mu) \to L^1(\Omega)$, $\Pi_2 : H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}_+(\mu) \to H^1_{\mathscr{D}}(\mu)$ *and* $\Pi_3 : H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}_+(\mu) \to H^{\Psi_1}_{\mathscr{D}}(\mu)$ *such that*

$$f \times g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g)$$

for all $f \in H^1_{\mathscr{D}}(\mu)$ *and* $g \in BMO^{\mathscr{D}}_+(\mu)$.
(2) *Let* $0 < p < 1$. *There exist continuous bilinear operators* $\Pi_1 : H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_{1,+}(\alpha_p) \to L^1(\Omega)$, $\Pi_2 : H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_{1,+}(\alpha_p) \to H^1_{\mathscr{D}}(\mu)$ *and* $\Pi_3 : H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_{1,+}(\alpha_p) \to H^{\Psi_p}_{\mathscr{D}}(\mu)$ *such that*

$$f \times g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g)$$

for all $f \in H^p_{\mathscr{D}}(\mu)$ *and* $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$.

Denote by $\mathcal{H}^p_{\mathscr{D},\mathrm{fin}}(\mu)$ $(0 < p \le 1)$ the linear space consisting of all functions which can be written as a finite sum of simple (p,∞)-atoms. Thus if $f \in \mathcal{H}^p_{\mathscr{D},\mathrm{fin}}(\mu)$, f is locally supported, $f \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\int_\Omega f \,\mathrm{d}\mu = 0$. Note that $\mathcal{H}^p_{\mathscr{D},\mathrm{fin}}(\mu)$ is dense in $H^p_{\mathscr{D}}(\mu)$ with respect to the norm $\|\cdot\|_{H^p_{\mathscr{D}}(\mu)}$.

By the argument detailed in Section 4.3.2, we shall only consider the case where $f \in \mathcal{H}^p_{\mathscr{D},\mathrm{fin}}(\mu)$. Then the product $f \times g$ as a distribution is given by the usual multiplication $f \cdot g \in L^1(\Omega)$, and we can write

$$f \cdot g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{5.2.10}$$

where

$$\Pi_1(f,g) := \sum_{k=1}^{\infty} d_k f d_k g, \quad \Pi_2(f,g) := \sum_{k=1}^{\infty} f_{k-1} d_k g \quad \text{and} \quad \Pi_3(f,g) := \sum_{k=1}^{\infty} g_{k-1} d_k f.$$

Proof. For Π_1 and Π_2, we can argue as in the corresponding part of the proof of Theorem III.1. As for Π_3, we can also argue as in the corresponding part of the proof Theorem III.1 using atomic decompositions, where in the homogeneous setting one needs to apply Lemma 5.2.2 and Lemma 5.2.4. We omit the details. □

We would like to remark the simplified proof in Section 4.2.4 fails because Theorem 4.2.2 is no longer valid when $g^* \equiv \infty$.

Remark 5.2.8. For Π_1 and Π_2, the condition $H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}_+(\mu)$ and $H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ can be in fact replaced by $H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}(\mu)$ and $H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_1(\alpha_p)$, respectively.

5.3 Applications to Homogeneous spaces

In the first part of this section we show that $H^{\Psi_p}_{\mathscr{D}}(\mu)$ admits an atomic decomposition for $0 < p < 1$, which allows us to integrate several adjacent dyadic systems on homogeneous spaces.

For a given dyadic system $\mathscr{D}$ on Ω, we define the dyadic $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}$-atom as follows.

Definition 5.3.1. Let $0 < p \le 1$. A measurable function a is said to be an $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}$-atom if

(i) $\mathrm{supp}(a) \subset Q$ where $Q \in \mathscr{D}$ is a cube;

(ii) $\int_\Omega a \, \mathrm{d}\mu = 0$;

(iii) $\|a\|_\infty \leq \|\mathbb{1}_Q\|^{-1}_{L^{\Psi_p}(\Omega)}$.

The atomic dyadic martingale Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)$ are defined in a way analogous to Definition 1.5.5 and 1.5.6. We first introduce the space $BMO^{\mathscr{D}}_{\Psi_p}(\mu)$, which is a subspace of continuous linear functionals on finite sums of atoms.

Definition 5.3.2. Let $\mathscr{D}$ denote a dyadic system. For a locally integrable function g, we define the semi-norm

$$\|g\|_{BMO^{\mathscr{D}}_{\Psi_p}(\mu)} := \sup_{k\in\mathbb{Z}} \sup_{Q\in\mathcal{F}_k} \frac{1}{\|\mathbb{1}_Q\|_{L^{\Psi_p}(\Omega)}} \int_Q |g(x) - g_k(x)| \, \mathrm{d}x.$$

We define the space $BMO^{\mathscr{D}}_{\Psi_p}(\mu)$ associated with a dyadic system $\mathscr{D}$ by

$$BMO^{\mathscr{D}}_{\Psi_p}(\mu) := \{g \in L^1_{\mathrm{loc}}(\mu) : \|g\|_{BMO^{\mathscr{D}}_{\Psi_p}(\mu)} < \infty\}$$

as a quotient space modulo constant functions.

Definition 5.3.3. Let $0 < p \leq 1$. We define the atomic Musielak–Orlicz martingale Hardy spaces $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)$ as follows:

$$H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu) := \Bigg\{ f \in \left(BMO^{\mathscr{D}}_{\Psi_p}(\mu)\right)^* : \ f = \sum_{i=0}^{\infty} \lambda_i a_i, \\ \text{where } a_i \text{ is an } H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)\text{-atom supported on a cube } Q_i. \Bigg\},$$

where $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$ is in terms of w^*-topology and

$$\sum_{i=0}^{\infty} \int_{Q_i} \Psi_p(x, |\lambda_i| \|a_i\|_\infty) \, \mathrm{d}\mu < \infty.$$

Moreover,

$$\|f\|_{H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)} := \inf \left\{ \rho > 0 : \sum_{i=0}^{\infty} \int_{Q_i} \Psi_p(x, \rho^{-1}|\lambda_i| \|a_i\|_\infty) \, \mathrm{d}\mu \leq 1 \right\}.$$

Arguing as in [98], one can show that for $0 < p < 1$

$$H^{\Psi_p}_{\mathscr{D}}(\mu) = H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu). \tag{5.3.1}$$

We shall now introduce the atomic Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathrm{at}}(\mu)$ on the homogeneous space Ω. First, we present the definition of atoms for $H^{\Psi_p}_{\mathrm{at}}(\mu)$.

Definition 5.3.4. A measurable function $a(x)$ is said to be an $H^{\Psi_p}_{\mathrm{at}}(\mu)$-atom if

(i) $\mathrm{supp}(a) \subset B$ where $B \subset \Omega$ is a ball;

(ii) $\int_\Omega a \, \mathrm{d}\mu = 0$;

(iii) $\|a\|_\infty \le \|\mathbb{1}_B\|^{-1}_{L^{\Psi_p}(\Omega)}$.

Definition 5.3.5. For a locally integrable function g, we define the semi-norm

$$\|g\|_{BMO_{\Psi_p}(\mu)} := \sup_B \frac{1}{\|\mathbb{1}_B\|_{L^{\Psi_p}(\Omega)}} \int_B |g(x) - g_B| \, \mathrm{d}x.$$

where B runs over all balls in Ω. We now define the space $BMO_{\Psi_p}(\mu)$ by

$$BMO_{\Psi_p}(\mu) := \{f \in L^1_{\mathrm{loc}}(\mu) : \|g\|_{BMO_{\Psi_p}(\mu)} < \infty\}$$

as a quotient space modulo constant functions.

Definition 5.3.6. Let $0 < p \le 1$. The atomic Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathrm{at}}(\mu)$ are defined as follows:

$$H^{\Psi_p}_{\mathrm{at}}(\mu) := \Bigg\{ f \in \Big(BMO_{\Psi_p}(\mu)\Big)^* : f = \sum_{i=0}^{\infty} \lambda_i a_i, \\ \text{where } a_i \text{ is an } H^{\Psi_p}_{\mathrm{at}}(\mu)\text{-atom supported on a ball } B_i \Bigg\},$$

where $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$ is in terms of w^*-topology and

$$\sum_{i=0}^{\infty} \int_{B_i} \Psi_p(x, |\lambda_i| \|a_i\|_\infty) \, \mathrm{d}\mu < \infty.$$

Moreover,

$$\|f\|_{H^{\Psi_p}_{\mathrm{at}}(\mu)} := \inf \left\{ \rho > 0 : \sum_{i=0}^{\infty} \int_{B_i} \Psi_p(x, \rho^{-1}|\lambda_i| \|a_i\|_\infty) \, \mathrm{d}\mu \le 1 \right\}.$$

Let $\mathscr{D}^t$ $(1 \le t \le K)$ be the adjacent systems of Theorem 5.1.3. By arguing as in the proof of Theorem 5.1.11, we have the following:

Lemma 5.3.7. *For $0 < p < 1$, $H^{\Psi_p}_{\mathrm{at}}(\mu) = H^{\Psi_p}_{\mathrm{at},\mathscr{D}^1}(\mu) + H^{\Psi_p}_{\mathrm{at},\mathscr{D}^2}(\mu) + \cdots + H^{\Psi_p}_{\mathrm{at},\mathscr{D}^K}(\mu)$.*

Proof. It suffices to show that any dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom a is a constant multiple of an $H^{\Psi_p}(\mu)$-atom, and any $H^{\Psi_p}(\mu)$-atom b is a constant multiple of a dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom.

If $B := B(x_0, r)$, then denote the ball $B(x_0, Dr)$ by DB for $D \ge 1$. Denote $d := d(x_0, O)$. In what follows, $C(D, p, A_0, C_\mu)$ denotes a constant that depends on D, p, A_0, C_μ and may differ from line to line. We first show that if

$$\int_B \frac{1}{1 + [1 + \mu(B(O, d(x, O)))]^{1-p}} \, \mathrm{d}\mu(x) = 1,$$

then

$$\int_{DB} \frac{1}{1 + [1 + \mu(B(O, d(x, O)))]^{1-p}} \, \mathrm{d}\mu(x) \le C(D, p, A_0, C_\mu). \tag{5.3.2}$$

Notice that

$$\begin{aligned}1&=\int_B\frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}}\,\mathrm{d}\mu(x)\\&\geq\frac{\mu(B)}{\sup\limits_{x\in B}\{1+[1+\mu(B(O,d(x,O)))]^{1-p}\}}\\&\geq\frac{\mu(B)}{1+[1+\mu(B(O,A_0(d+r)))]^{1-p}},\end{aligned}$$

which implies

$$\mu(B)\leq 1+[1+\mu(B(O,A_0(d+r)))]^{1-p}.$$

If $d\leq 2A_0Dr$, we have

$$\begin{aligned}\mu(B)&\leq 1+[1+\mu(B(O,A_0(2A_0D+1)r))]^{1-p}\\&\leq 1+\{1+\mu[B(x_0,A_0(A_0+1)(2A_0D+1)r)]\}^{1-p}\\&\leq 1+\left\{1+[A_0(A_0+1)(2A_0D+1)]^{C_\mu}\,\mu(B)\right\}^{1-p},\end{aligned}$$

and thus $\mu(B)\leq C(D,p,A_0,C_\mu)$. Then

$$\int_{DB}\frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}}\,\mathrm{d}\mu(x)\leq\mu(DB)\leq C(D,p,A_0,C_\mu).\tag{5.3.3}$$

If $d>2A_0Dr$, then

$$\begin{aligned}&\int_{DB}\frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}}\,\mathrm{d}\mu(x)\\\leq&\frac{\mu(DB)}{\inf\limits_{x\in DB}\{1+[1+\mu(B(O,d(x,O)))]^{1-p}\}}\\\leq&\frac{D^{C_\mu}\mu(B)}{1+[1+\mu(B(O,d/A_0-Dr))]^{1-p}},\\\leq&\frac{D^{C_\mu}\left\{1+\mu[B\left(O,[A_0+1/(2D)]d\right)]\right\}^{1-p}+D^{C_\mu}}{1+\{1+\mu[B(O,d/(2A_0))]\}^{1-p}}\\\leq&\frac{D^{C_\mu}\left\{1+[(2A_0+1/D)A_0]^{C_\mu}\,\mu[B(O,d/(2A_0))]\right\}^{1-p}}{1+\{1+\mu[B(O,d/(2A_0))]\}^{1-p}}+D^{C_\mu}.\end{aligned}$$

Hence

$$\int_{DB}\frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}}\,\mathrm{d}\mu(x)\leq C(D,p,A_0,C_\mu)\tag{5.3.4}$$

Combining (5.3.3) with (5.3.4), we get (5.3.2).

Assume a is an $H^{\Psi_p}(\mu)$-atom supported on B. By Theorem 5.1.3, there exist t and a cube $Q\in\mathscr{D}^t$ such that $B\subset Q$ and $\mathrm{diam}(Q)\leq Cr$, hence $B\subset Q\subset CB$.

Note that $\mathrm{supp}(a)\subset Q$, $\int_Q a(x)\,\mathrm{d}\mu(x)=0$ and

$$\|\mathbb{1}_Q\|_{L^{\Psi_p}(\mu)}\leq\|\mathbb{1}_{CB}\|_{L^{\Psi_p}(\mu)}\leq C(C,p,A_0,C_\mu)\|\mathbb{1}_B\|_{L^{\Psi_p}(\mu)},$$

which follows from (5.3.2). Thus

$$\|a\|_\infty \le \|\mathbb{1}_B\|^{-1}_{L^{\Psi_p}(\mu)} \lesssim \|\mathbb{1}_Q\|^{-1}_{L^{\Psi_p}(\mu)},$$

which implies a is a multiple of dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom supported on Q.

For any $t = 1, 2\cdots, K$, assume b is a dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom supported on Q^k_β. By Theorem 5.1.1, there exists two balls such that $B(z^k_\beta, c_1\delta^k) \subset Q^k_\beta \subset B(z^k_\beta, C_1\delta^k)$.

Thus $\mathrm{supp}(b) \subset B(z^k_\beta, C_1\delta^k)$, $\int_{B(z^k_\beta, C_1\delta^k)} b(x)\,\mathrm{d}\mu(x) = 0$ and

$$\|\mathbb{1}_{B(z^k_\beta, C_1\delta^k)}\|_{L^{\Psi_p}(\mu)} \le C\left(\frac{C_1}{c_1}, p, A_0, C_\mu\right)\|\mathbb{1}_{B(z^k_\beta, c_1\delta^k)}\|_{L^{\Psi_p}(\mu)} \lesssim \|\mathbb{1}_{Q^k_\beta}\|_{L^{\Psi_p}(\mu)},$$

which follows from (5.3.2). Therefore,

$$\|b\|_\infty \le \|\mathbb{1}_{Q^k_\beta}\|^{-1}_{L^{\Psi_p}(\mu)} \lesssim \|\mathbb{1}_{B(z^k_\beta, C_1\delta^k)}\|^{-1}_{L^{\Psi_p}(\mu)},$$

which implies b is a multiple of dyadic H^{Ψ_p}-atom supported on $B(z^k_\beta, C_1\delta^k)$. □

Remark 5.3.8. In [42], Fu, Ma and Yang defined another kind of Musielak–Orlicz Hardy spaces by grand maximal function and they also proved that these Musielak–Orlicz Hardy spaces are equivalent to $H^{\Psi_p}_{\mathrm{at}}(\mu)$ with respect to the corresponding norms when $p \in (\frac{C_\mu}{C_\mu+1}, 1]$.

Let $B_1 := B(O, 1)$. Define

$$\|g\|_{BMO_+(\mu)} := |g_{B_1}| + \|g\|_{BMO(\mu)}, \quad \text{for } g \in BMO(\mu),$$

and

$$\|g\|_{\mathcal{L}_{+,\alpha}(\mu)} := |g_{B_1}| + \|g\|_{\mathcal{L}_{\alpha_p}(\mu)}, \quad \text{for } g \in \mathcal{L}_{\alpha_p}(\mu).$$

Thus $\|\cdot\|_{BMO_+(\mu)}$ and $\|\cdot\|_{\mathcal{L}_{+,\alpha_p}(\mu)}$ are quasi-norms on $BMO(\mu)$ and $\mathcal{L}_{\alpha_p}(\mu)$, respectively.

Theorem 5.3.9. *Let $0 < p < 1$ and $f \in H^p_{\mathrm{at}}(\mu)$. There exist three linear continuous operators $\Pi^f_1 : \mathcal{L}_{\alpha_p}(\mu) \to L^1(\Omega)$, $\Pi^f_2 : \mathcal{L}_{\alpha_p}(\mu) \to H^1_{\mathrm{at}}(\mu)$ and $\Pi^f_3 : \mathcal{L}_{\alpha_p}(\mu) \to H^{\Psi_p}_{\mathrm{at}}(\mu)$ such that*

$$f \cdot g = \Pi^f_1(g) + \Pi^f_2(g) + \Pi^f_3(g) \quad \textit{for all } g \in \mathcal{L}_{\alpha_p}(\mu),$$

where $\mathcal{L}_{\alpha_p}(\mu)$ is endowed with the quasi-norm $\|\cdot\|_{\mathcal{L}_{+,\alpha_p}(\mu)}$.

Proof. Let $f \in H^p_{\mathrm{at}}(\mu)$. By Theorem 5.1.11 there exist $f^t \in H^p_{\mathscr{D}^t}(\mu)$ $(t = 1, 2, \cdots, K)$ such that $f = f^1 + f^2 + \cdots + f^K$, and

$$\sum_{t=1}^K \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)} \approx \|f\|_{H^p_{\mathrm{at}}(\mu)}.$$

Define $\Pi^f_i(g) := \sum_{t=1}^K \Pi_i(f^t, g)$ for $i = 1, 2, 3$ and $g \in \mathcal{L}_{\alpha_p}(\mu)$ (Π_i defined as in Theorem 5.2.7). Then

$$f \cdot g = \Pi^f_1(g) + \Pi^f_2(g) + \Pi^f_3(g).$$

By Theorem 5.2.7, Theorem 5.1.11 and Lemma 5.3.7, we have

$$\|\Pi_1^f(g)\|_1 \lesssim \sum_{t=1}^{K} \|\Pi_1(f^t,g)\|_1 \lesssim \sum_{t=1}^{K} \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)} \|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\mathrm{at}}(\mu)} \|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)},$$

$$\|\Pi_2^f(g)\|_{H^1_{\mathrm{at}}(\mu)} \lesssim \sum_{t=1}^{K} \|\Pi_2(f^t,g)\|_{H^1_{\mathscr{D}^t}(\mu)} \lesssim \sum_{t=1}^{K} \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)} \|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\mathrm{at}}(\mu)} \|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)},$$

$$\|\Pi_3^f(g)\|_{H^{\Psi_p}_{\mathrm{at}}(\mu)} \lesssim \sum_{t=1}^{K} \|\Pi_3(f^t,g)\|_{H^{\Psi_p}_{\mathscr{D}^t}(\mu)} \lesssim \sum_{t=1}^{K} \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)} \|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\mathrm{at}}(\mu)} \|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)}.$$

which finishes the proof. □

Remark 5.3.10. If the homogeneous space (Ω,μ) satisfies the reverse doubling condition, then Lemma 5.3.7 holds for $p=1$. Then we conclude the following.

Let $f \in H^1_{\mathrm{at}}(\mu)$. There exist three linear continuous operators $\Pi_1^f : BMO(\mu) \to L^1(\Omega)$, $\Pi_2^f : BMO(\mu) \to H^1_{\mathrm{at}}(\mu)$ and $\Pi_3^f : BMO(\mu) \to H^{\Psi_1}_{\mathrm{at}}(\mu)$ such that

$$f \cdot g = \Pi_1^f(g) + \Pi_2^f(g) + \Pi_3^f(g) \quad \text{for all } g \in BMO(\mu),$$

where $BMO(\mu)$ is endowed with the norm $\|\cdot\|_{BMO_+(\mu)}$.

At this point, we would like to mention that in the setting of Theorem 5.3.9 the corresponding bilinear decomposition is established in [61].

Appendix A: Molecule decomposition

From the atomic decomposition of $T^1_q(\mathbb{R}^{d+1}_+;X)$, we derive the following molecular decomposition for $H^1_{q,L}(\mathbb{R}^d;X)$ for any Banach space X, which might have further applications.

Theorem A.0.1. *Let X be any fixed Banach space and $1<q<\infty$. For any $f\in H^1_{q,L}(\mathbb{R}^d;X)$, there exist a sequence of complex numbers $\{\lambda_j\}_{j\geq 1}$ and corresponding molecules $\alpha_j=\pi_L(a_j)$ with $a_j(x,t)$ being an (X,q)-atom such that*

$$f=\sum_{j\geq 1}\lambda_j\alpha_j,\quad \|f\|_{H^1_{q,L}(X)}\approx\sum_{j\geq 1}|\lambda_j|.$$

Proof. Let $f\in H^1_{q,L}(\mathbb{R}^d;X)$. It follows that $\mathcal{Q}(f)\in T^1_q(\mathbb{R}^{d+1}_+;X)$. Hence $\mathcal{Q}(f)$ admits an atomic decomposition by Lemma 3.2.1. More precisely, there exist a sequence of complex numbers $\{c_j\}_{j\geq 1}$ and (X,q)-atoms a_j such that

$$\mathcal{Q}(f)=\sum_{j=1}^{\infty}c_ja_j,\quad \|f\|_{H^1_{q,L}(X)}=\|\mathcal{Q}(f)\|_{T^1_q(X)}\approx\sum_{j=1}^{\infty}|c_j|.$$

Then by Lemma 3.2.6, it follows that $\pi_L(a_j)=\alpha_j\in H^1_{q,L}(\mathbb{R}^d;X)$ for all $j\geq 1$. Recall below the Calderón identity—(3.3.10),

$$f(x)=4\int_0^{\infty}\mathcal{Q}[\mathcal{Q}(f)(\cdot,t)](x,t)\,\frac{\mathrm{d}t}{t}.$$

This further deduce that

$$f(x)=4\sum_{j=1}^{\infty}c_j\int_0^{\infty}\mathcal{Q}[a_j(\cdot,t)](x,t))\,\frac{\mathrm{d}t}{t}=4\sum_{j=1}^{\infty}c_j\alpha_j(x),$$

and thus we obtain the desired molecular decomposition. □

Appendix B: Clarifications

We clarify some facts in Section 3.1. Recall that a regular operator T on $L^p(\mathbb{R}^d)$ is *regular* if there exists a constant c such that

$$\|\sup_k T(f_k)\|_p \le c\Big\|\sup_k |f_k|\Big\|_p.$$

for any finite sequence $\{f_k\}_{k\ge1} \subset L^p(\mathbb{R}^d)$. The least possible constant c is defined as the regular norm.

Let L be a sectorial operator satisfying (3.1.2). We have the following propositions.

Proposition B.0.1. *Let $1 \le p \le \infty$. For any $t > 0$, e^{-tL} is a regular operator on $L^p(\mathbb{R}^d)$, whose regular norm is dominated by a universal constant.*

Proof. Let $1 \le p < \infty$. Now we verify that e^{-tL} is a regular operator on $L^p(\mathbb{R}^d)$. Note that

$$\begin{aligned}
\|\sup_k e^{-tL}(f_k)\|_p^p &= \int_{\mathbb{R}^d} \left|\sup_k \int_{\mathbb{R}^d} K(t,x,y) f_k(y)\,\mathrm{d}y\right|^p \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} |K(t,x,y)| \left(\sup_k |f_k(y)|\right) \mathrm{d}y\right]^p \mathrm{d}x \\
&\le c^p \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{t^{-d}}{(1+t^{-1}|x-y|)^{d+\beta}} \left(\sup_k |f_k(y)|\right) \mathrm{d}y\right]^p \mathrm{d}x \\
&\lesssim_\beta c^p \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{t^{-d}}{(1+t^{-1}|x-y|)^{d+\beta}} \left(\sup_k |f_k(y)|\right)^p \mathrm{d}y\right] \mathrm{d}x \\
&\lesssim_\beta c^p \int_{\mathbb{R}^d} \left(\sup_k |f_k(y)|\right)^p \mathrm{d}y = c^p \Big\|\sup_k |f_k|\Big\|_p^p.
\end{aligned}$$

This argument also holds for $p = \infty$. □

Denote by $\mathbf{F}_L^p$ the fixed point subspace of $\left\{e^{-tL}\right\}_{t>0}$ on $L^p(\mathbb{R}^d)$, namely,

$$\mathbf{F}_L^p = \left\{f \in L^p(\mathbb{R}^d) : e^{-tL}(f) = f,\ \forall\, t > 0\right\}.$$

Proposition B.0.2. *Let $1 \le p < \infty$, then $\mathbf{F}_L^p = \{0\}$.*

Proof. We provide the sketch of the proof. For $f \in L^p(\mathbb{R}^d)$, note that

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} K(t,x,y) f(y)\,\mathrm{d}y\right| &\le \int_{\mathbb{R}^d} \frac{|f(y)| t^{-d}}{(1+t^{-1}|x-y|)^{d+\beta}}\,\mathrm{d}y \\
&\le \|f\|_p \left(\int_{\mathbb{R}^d} \frac{t^{-p'd}}{(1+t^{-1}|x-y|)^{p'(d+\beta)}}\,\mathrm{d}y\right)^{\frac{1}{p'}} \\
&\lesssim_\beta t^{-\frac{d}{p}} \|f\|_p.
\end{aligned}$$

Thus the limit

$$\lim_{t\to\infty} e^{-tL}(f)(x) = \lim_{t\to\infty} \int_{\mathbb{R}^d} K(t,x,y) f(y)\,\mathrm{d}y$$

exists and

$$\lim_{t\to\infty} e^{-tL}(f)(x) = 0, \quad \forall\, x \in \mathbb{R}^d.$$

For $1 \le p < \infty$, it is easy to check that

$$\mathbf{F}_L^p = \left\{ \widetilde{f} : f \in L^p(\mathbb{R}^d),\ \widetilde{f}(x) = \lim_{t\to\infty} e^{-tL}(f)(x) \right\}.$$

Thus we obtain that the fixed point subspace of $L^p(\mathbb{R}^d)$ is indeed zero space by the above argument. □

Appendix C: Open problems

C.1: Problems on vecor-valued LPS

Problem C.0.1. It remains open to determine the optimal order of $\mathsf{L}^{\sqrt{\Delta}}_{t,q,p}(X)$ as $p \to \infty$.

Problem C.0.2. The characterization of BMO space by semigroups generated by operators satisfying (II.1), (II.2) and (II.3). We wish to define BMO similarly to [54, 24], namely by

$$\|f\|_{BMO} := \sup_t \|e^{-tL}|f - e^{-tL}(f)|^2\|_\infty^{\frac{1}{2}}.$$

Can we establish the analogous duality between this BMO space and the Hardy space H^1 defined by the Littlewood-Paley g-function? Furthermore, is it possible to extend this characterization to vector-valued case?

Problem C.0.3. In Theorem II.1, we need to assume that the sectorial operator L satisfying (II.1), (II.2) and (II.3). Can we remove or weaken the assumption (II.3)? Namely, is Theorem II.1 valid for submarkovian semigroups?

Problem C.0.4. In [37], Eskenazis, Mendel and Naor study the coarse embedding theory by metric space-valued martingale inequalities, it is thus naturally to consider the corresponding Littlewood-Paley-Stein theory in metric space for both scalar and vector valued functions..

C.2: Problems on continuous bilinear decompositions

Problem C.0.5. Let Ω be a probability space. Do we have a continuous bilinear decomposition of pointwise multiplication between $h^1(\Omega)$ and $bmo(\Omega)$?

Problem C.0.6. Let Ω be a homogeneous space and n its dimension. Are there continuous bilinear decompositions of multiplication between atomic Hardy spaces $H^p_{\rm at}(\Omega)$ and their duals for $0 < p < n/(n+1)$?

Bibliography

[1] I. Abu-Falahah, P. Stinga, and J. Torrea. Square functions associated to Schrödinger operators. *Stud. Math.*, 2(203):171–194, 2011.

[2] P. Auscher and T. Hytönen. Addendum to orthonormal bases of regular wavelets in spaces of homogeneous type. *Appl. Comput. Harmon. Anal.*, 39(3):568–569, 2015.

[3] P. Auscher and T. P. Hytönen. Orthonormal bases of regular wavelets in spaces of homogeneous type. *Appl. Comput. Harmon. Anal.*, 34(2):266–296, 2013.

[4] O. Bakas, S. Pott, S. R.-López, and A. Sola. Notes on $H^{\log}$: structural properties, dyadic variants, and bilinear H^1-BMO mappings. *Ark. Mat.*, 60(2):231–275, 2022.

[5] O. Bakas, Z. Xu, Y. Zhai, and H. Zhang. Multiplication between elements in martingale hardy spaces and their dual spaces. *arXiv:2301.08723*, 2023.

[6] J. J. Betancor, A. J. Castro, J. C. Fariña, and L. Rodríguez-Mesa. UMD Banach spaces and square functions associated with heat semigroups for Schrödinger, Hermite and Laguerre operators. *Math. Nachr.*, 289(4):410–435, 2016.

[7] J. J. Betancor, A. J. Castro, and L. Rodríguez-Mesa. Characterization of uniformly convex and smooth Banach spaces by using Carleson measures in Bessel settings. *J. Convex Anal.*, 20:763–811, 2013.

[8] J. J. Betancor, J. C. Fariña, V. Galli, and S. M. Molina. Uniformly convex and smooth Banach spaces and l_p-boundedness properties of Littlewood-Paley and area functions associated with semigroups. *J. Math. Anal. Appl.*, 482(1):123534, 2020.

[9] J. J. Betancor, J. C. Fariña, T. Martínez, and J. L. Torrea. Riesz transform and g-function associated with Bessel operators and their appropriate Banach spaces. *Isr. J. Math.*, 157(1):259–282, 2007.

[10] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, A. Sanabria, and J. L. Torrea. Lusin type and cotype for Laguerre g-functions. *Isr. J. Math.*, 182:1–30, 2011.

[11] J. J. Betancor, S. M. Molina, and L. Rodríguez-Mesa. Area littlewood–paley functions associated with hermite and laguerre operators. *Potential Anal.*, 4(34):345–369, 2011.

[12] O. Blasco. Hardy spaces of vector-valued functions: duality. *Trans. Am. Math. Soc.*, 308(2):495–507, 1988.

[13] O. Blasco. Interpolation between $H^1_{B_0}$ and $L^p_{B_1}$. *Stud. Math.*, 92(3):295–210, 1989.

[14] A. Bonami, J. Cao, L. D. Ky, L. Liu, D. Yang, and W. Yuan. Multiplication between Hardy spaces and their dual spaces. *J. Math. Pures Appl.*, 131:130–170, 2019.

[15] A. Bonami, J. Feuto, S. Grellier, and L. D. Ky. Atomic decomposition and weak factorization in generalized Hardy spaces of closed forms. *Bull. Sci. Math.*, 141(7):676–702, 2017.

[16] A. Bonami, S. Grellier, and L. D. Ky. Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets. *J. Math. Pures Appl.*, 97(3):230–241, 2012.

[17] A. Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister. On the product of functions in BMO and H^1. In *Annales de l'institut Fourier*, volume 57, pages 1405–1439, 2007.

[18] A. Bonami, Y. Jiao, G. Xie, D. Yang, and D. Zhou. Products and commutators of martingales in H^1 and BMO. *J. Math. Anal. Appl.*, 2023.

[19] A. Bonami and L. D. Ky. Factorization of some Hardy-type spaces of holomorphic functions. *C. R. Math. Acad. Sci. Paris.*, 352(10):817–821, 2014.

[20] J. M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.

[21] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. *Probability theory and harmonic analysis*, pages 1–19, 1986.

[22] D. L. Burkholder. Martingale transforms. *Ann. Math. Statist.*, 37(6):1494–1504, 1966.

[23] D. L. Burkholder. Sharp inequalities for martingales and stochastic integrals. *Astérisque*, 157(158):75–94, 1988.

[24] Z. Chen, Q. Xu, and Z. Yin. Harmonic analysis on quantum tori. *Comm. Math. Phys.*, 322(3):755–805, 2013.

[25] R. R. Coifman, P. W. Jones, and S. Semmes. Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves. *J. Am. Math. Soc.*, 2(3):553–564, 1989.

[26] R. R. Coifman, Y. Meyer, and E. M. Stein. Some new function spaces and their applications to harmonic analysis. *J. Funct. Anal.*, 62(2):304–335, 1985.

[27] R. R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Am. Math. Soc.*, 83(6):569–645, 1977.

[28] J. M. Conde-Alonso and J. Parcet. Atomic blocks for noncommutative martingales. *Ind. Univ. Math. J.*, pages 1425–1443, 2016.

[29] T. Coulhon and X. T. Duong. Maximal regularity and kernel bounds: Observations on a theorem by Hieber and Prüss. *Adv. Differ. Equ.*, 5:343–368, 2000.

[30] M. Cowling, I. Doust, A. Micintosh, and A. Yagi. Banach space operators with a bounded H_∞ functional calculus. *J. Aust. Math. Soc.*, 60(1):51–89, 1996.

[31] M. G. Cowling. Harmonic analysis on semigroups. *Ann. Math.*, pages 267–283, 1983.

[32] G. David and J. L. Journé. A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. Math.*, 120(2):371–397, 1984.

[33] B. Davis. On the integrability of the martingale square function. *Isr. J. Math.*, 8:187–190, 1970.

[34] J. Diestel and J. J. Uhl. *Vector Measures*. Mathematical surveys and monographs. American Mathematical Society, 1977.

[35] N. Dinculeanu. *Vector measures*. Elsevier, 2014.

[36] X. T. Duong and L. Yan. Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. *J. Am. Math. Soc.*, 18(4):943–973, 2005.

[37] A. Eskenazis, M. Mendel, and A. Naor. Nonpositive curvature is not coarsely universal. *Invent. math.*, 217(3):833–886, 2019.

[38] C. Fefferman. Characterizations of bounded mean oscillation. *Bull. Am. Math. Soc.*, 77(4):587–588, 1971.

[39] C. Fefferman and E. M. Stein. Some maximal inequalities. *Am. J. Math.*, 93(1):107–115, 1971.

[40] C. Fefferman and E. M. Stein. H_p spaces of several variables. *Acta Math.*, 129(1):137–195, 1972.

[41] T. Figiel. Singular integral operators: a martingale approach. In *Geometry of Banach spaces (Strobl, 1989)*, volume 158, pages 95–110, 1990.

[42] X. Fu, T. Ma, and D. Yang. Real-variable characterizations of Musielak-Orlicz Hardy spaces on spaces of homogeneous type. *Ann. Acad. Sci. Fenn. Math.*, 45(1):343–410, 2020.

[43] A. M. Garsia. *Martingale inequalities: Seminar notes on recent progress*. Addison Wesley Longman, 1973.

[44] L. Grafakos. *Modern fourier analysis*, volume 250. Springer, New York, 2009.

[45] M. Haase. *The Functional Calculus of Sectorial Operators*. Birkhäuser Basel, 1st edition, 2005.

[46] E. Harboure, J. L. Torrea, and B. Viviani. Vector-valued extensions of operators related to the Ornstein-Uhlenbeck semigroup. *J. Anal. Math.*, 91(1):1–29, 2003.

[47] E. Harboure, J. L. Torrea, and B. E. Viviani. A vector-valued approach to tent spaces. *J. Anal. Math.*, 56:125–140, 1991.

[48] T. Hytönen. Littlewood-Paley-Stein theory for semigroups in UMD spaces. *Rev. Mat. Iberoam.*, 23(3):973–1009, 2007.

[49] T. Hytönen and A. Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012.

[50] T. Hytönen, J. Van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces*, volume 12. Springer, Berlin, 2016.

[51] T. Hytönen, J. M. A. M. van Neerven, and P. Portal. Conical square function estimates in UMD banach spaces and applications to H_∞-functional calculi. *J. Anal. Math.*, 106:317–351, 2008.

[52] T. P. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. Math.*, 175(3):1473–1506, 2012.

[53] M. Junge, C. Le Merdy, and Q. Xu. *H^∞ functional calculus and square functions on noncommutative L_p-spaces*. Société mathématique de France, 2006.

[54] M. Junge and T. Mei. BMO spaces associated with semigroups of operators. *Math. Ann.*, 352(3):691–743, 2012.

[55] M. Junge and Q. Xu. Noncommutative maximal ergodic theorems. *J. Am. Math. Soc.*, 20(2):385–439, 2007.

[56] M. Kemppainen. The vector-valued tent spaces T_1 and T_∞. *J. Aust. Math. Soc.*, 97(1):107–126, 2014.

[57] M. Kemppainen. On vector-valued tent spaces and Hardy spaces associated with non-negative self-adjoint operators. *Glasg. Math. J.*, 58(3):689–716, 2016.

[58] J. E. Littlewood and R. E. A. C. Paley. Theorems on Fourier series and power series. *J. Lond. Math. Soc.*, 6(3):230–233, 1931.

[59] J. E. Littlewood and R. E. A. C. Paley. Theorems on Fourier series and power series (II). *Proc. Lond. Math. Soc.*, 42(1):52–89, 1937.

[60] J. E. Littlewood and R. E. A. C. Paley. Theorems on Fourier series and power series (III). *Proc. Lond. Math. Soc.*, 43(2):105–126, 1938.

[61] L. Liu, D. Yang, and W. Yuan. Bilinear decompositions for products of Hardy and Lipschitz spaces on spaces of homogeneous type. *Diss. Math.*, 533:1–93, 2018.

[62] R. Long. *Martingale Spaces and Inequalities*. Peking University Press, Beijing; Friedr. Vieweg & Sohn, Braunschweig, 1993.

[63] T. Martínez, J. Torrea, and Q. Xu. Vector-valued Littlewood-Paley-Stein theory for semigroups. *Adv. Math.*, 203(2):430–475, 2006.

[64] A. McIntosh. Operators which have an H_∞ functional calculus. In *Miniconference on operator theory and partial differential equations*, volume 14, pages 210–232. Australian National University, Mathematical Sciences Institute, 1986.

[65] T. Mei. BMO is the intersection of two translates of dyadic BMO. *C. R. Math. Acad. Sci. Paris*, 336(12):1003–1006, 2003.

[66] T. Mei. *Operator valued Hardy spaces*. American Mathematical Soc., 2007.

[67] C. Le Merdy and Q. Xu. Maximal theorems and square functions for analytic operators on l^p-spaces. *J. Lond. Math. Soc.*, 86:343–365, 2012.

[68] N. G. Meyers. Mean oscillation over cubes and hölder continuity. *Proc. Am. Math. Soc.*, 15:717–721, 1964.

[69] T. Miyamoto, E. Nakai, and G. Sadasue. Martingale Orlicz-Hardy spaces. *Math. Nachr.*, 285(5-6):670–686, 2012.

[70] E. Nakai. Pointwise multipliers on weighted BMO spaces. *Stud. Math.*, 125(1):35–56, 1997.

[71] E. Nakai and G. Sadasue. Pointwise multipliers on martingale Campanato spaces. *Stud. Math.*, 220:87–100, 2014.

[72] A. Naor and R. Young. Foliated corona decompositions. *Acta Math.*, 229(1):55–200, 2022.

[73] F. Nazarov, S. Treil, and A. Volberg. The *tb*-theorem on non-homogeneous spaces. *Acta Math.*, 190(2):151–239, 2003.

[74] J. Neveu. Deux remarques sur la théorie des martingales. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 3(2):122–127, 1964.

[75] C. Ouyang and Q. Xu. BMO functions and Carleson measures with values in uniformly convex spaces. *Can. J. Math.*, 62(4):827–844, 2010.

[76] M. C. Pereyra. Dyadic harmonic analysis and weighted inequalities: The sparse revolution. In A. Aldroubi et al., editors, *New Trends in Applied Harmonic Analysis. Volume 2. Harmonic Analysis, Geometric Measure Theory, and Applications*, Applied and Numerical Harmonic Analysis, pages 159–239. Birkhäuser, 2019.

[77] M. Perrin. A noncommutative Davis' decomposition for martingales. *J. Lond. Math. Soc.*, 80(3):627–648, 2009.

[78] S. Petermichl. Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(6):455–460, 2000.

[79] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical a_p characteristic. *Am. J. Math.*, 129(5):1355–1375, 2007.

[80] S. Petermichl. The sharp weighted bound for the Riesz transforms. *Proc. Am. Math. Soc.*, 136(4):1237–1249, 2008.

[81] S. Petermichl, S. Treil, and A. Volberg. Why the Riesz transforms are averages of the dyadic shifts? *Publ. Mat.*, pages 209–228, 2002.

[82] G. Pisier. Martingales with values in uniformly convex spaces. *Isr. J. Math.*, 20(3):326–350, 1975.

[83] G. Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis*, pages 167–241. Springer, 1986.

[84] G. Pisier. *Martingales in Banach spaces*, volume 155. Cambridge University Press, 2016.

[85] G. Pisier and Q. Xu. Non-commutative martingale inequalities. *Comm. Math. Phys.*, 189(3):667–698, 1997.

[86] G. Pisier and Q. Xu. Non-commutative L_p-spaces. In *Handbook of the geometry of Banach spaces*, volume 2, pages 1459–1517. 2003.

[87] E. M Stein. *Topics in Harmonic Analysis, Related to the Littlewood-Paley Theory.* Princeton University Press, 1970.

[88] E. M. Stein and T. S. Murphy. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, volume 3. Princeton University Press, 1993.

[89] E. M. Stein and G. Weiss. On the theory of harmonic functions of several variables. *Acta Math.*, 103:25–62, 1960.

[90] J. L. Torrea and C. Zhang. Fractional vector-valued Littlewood-Paley-Stein theory for semigroups. *Proc. R. Soc. Edinb. A Math.*, 144(3):637–667, 2014.

[91] H. Triebel. *Interpolation Theory, Function Spaces and Differential Operators.* North-Holland, Amsterdam, 1978.

[92] A. Vagharshakyan. Recovering singular integrals from Haar shifts. *Proc. Am. Math. Soc.*, 138(12):4303–4309, 2010.

[93] F. Weisz. Martingale Hardy spaces for $0 < p \leq 1$. *Probab. Theory Relat. Fields.*, 84(3):361–376, 1990.

[94] F. Weisz. *Martingale Hardy Spaces and Their Applications in Fourier Analysis*, volume 1568. Springer-Verlag, Berlin Heidelberg, 1994.

[95] J. M. Wilson. The intrinsic square function. *Rev. Mat. Iberoam.*, 23(3):771–791, 2007.

[96] J. M. Wilson. *Weighted Littlewood-Paley theory and exponential-square integrability.* Number 1924. Springer Science & Business Media, 2008.

[97] R. Xia, X. Xiong, and Q. Xu. Characterizations of operator-valued Hardy spaces and applications to harmonic analysis on quantum tori. *Adv. Math.*, 291:183–227, 2016.

[98] G. Xie, Y. Jiao, and D. Yang. Martingale Musielak-Orlicz Hardy spaces. *Sci. China Math.*, 62(8):1567–1584, 2019.

[99] F. Xing, Y. Liang, and D. Yang. Products of functions in $BMO(\mathcal{X})$ and $H^1_{\rm at}(\mathcal{X})$ via wavelets over spaces of homogeneous type. *J. Fourier Anal. Appl.*, 23(4):919–990, 2017.

[100] Q. Xu. Littlewood-Paley theory for functions with values in uniformly convex spaces. *J. Reine Angew. Math.*, 1998:195–226, 1998.

[101] Q. Xu. Vector-valued Littlewood-Paley-Stein theory for semigroups II. *Int. Math. Res. Not.*, 21:7769–7791, 2020.

[102] Q. Xu. Holomorphic functional calculus and vector-valued Littlewood-Paley-Stein theory for semigroups. *arXiv:2105.12175. To appear in J. Euro. Math. Soc.*, 2021.

[103] Q. Xu. Optimal orders of the best constants in the Littlewood-Paley inequalities. *J. Funct. Anal.*, page 109570, 2022.

[104] Z. Xu and H. Zhang. From the Littlewood-Paley-Stein inequality to the Burkholder-Gundy inequality. *Trans. Am. Math. Soc.*, 376(01):371–389, 2023.

[105] D. Yang, Y. Liang, and L. D. Ky. *Real-Variable Theory of Musielak-Orlicz Hardy Spaces*, volume 2182. Springer-Verlag, 2017.

Résumé :

Cette thèse est consacrée à l'étude de certains aspects de la théorie de Littlewood-Paley-Stein et de celles des martingales dans différents contextes, notamment pour les fonctions à valeurs vectorielles et non commutatives.

Elle se compose de trois parties. Dans la première, nous établissons une équivalence ponctuelle entre la g-fonction carrée de Littlewood-Paley-Stein et la fonction carrée de martingales. Nos arguments reposent sur la construction d'un semi-groupe de diffusion symétrique spécifique associé à une filtration de martingales. Nous étendons également cette équivalence aux cas vectoriel et non commutatif. En conséquence, nous déterminons l'ordre d'une des meilleures constantes dans l'inégalité de Littlewood-Paley-Stein scalaire.

La deuxième partie se concentre sur le scénario à valeurs vectorielles. Nous montrons l'équivalence entre la norme L^p $(1 \leq p < +\infty)$ entre la q-variante vectorielle de l'intégrale de Lusin et celle de la g-fonction de Littlewood-Paley-Stein du semi-groupe dont le générateurs est un opérateur sectoriel satisfaisant à certaines conditions. Les outils principaux sont les espaces de tente à valeurs vectorielles et la fonction carrée intrinsèque introduite par Wilson. En particulier, nous obtenons l'ordre optimal de la meilleure constante correspondante de l'inégalité de Littlewood-Paley-Stein dans L^p à valeurs vectorielles dans un espace de Banach de type q $(1 < q \leq 2)$ de martingale pour p tendant vers 1.

La dernière partie porte sur la décomposition bilinéaire de la multiplication ponctuelle des éléments de l'espace de Hardy de martingales H^1 et de son espace dual BMO. Cette décomposition bilinéaire continue est étendue à l'espace de Hardy de martingales H^p $(0 < p < 1)$ et à son espace dual. Nos décompositions reposent sur les paraproduits de martingale. En conséquence, nous obtenons des résultats analogues pour les martingales dyadiques sur les espaces de type homogène grâce à la construction d'un système dyadique.

Mots-clés : Inégalités de Littlewood-Paley-Stein ; Ordre optimal des meilleurs constants ; Espaces de Hardy et BMO ; Type et cotype de martingales ; Espaces de tente à valeurs vectorielles ; Paraproduits de martingales ; Fonction carée intrinsèque ; Espaces de Hardy de Musielak-Orlicz ; Martingales.

Abstract:

This thesis is devoted to the study of certain aspects of Littlewood-Paley-Stein theory and martingale theory in various contexts, particularly for vector-valued and non-commutative functions.

It is composed of three parts. In the first part, we establish a pointwise equivalence between the Littlewood-Paley-Stein g-function and the square function of martingales. Our arguments are based on the construction of a specific symmetric diffusion semigroup associated with a martingale filtration. We also extend this equivalence to the vector-valued and noncommutative cases. Consequently, we determine the order of one of the best constants in the scalar Littlewood-Paley-Stein inequality.

The second part focuses on the vector-valued scenario. We demonstrate the equivalence between the L^p norm $(1 \leq p < \infty)$ of the q-variant of vector-valued Lusin area integral and that of the Littlewood-Paley-Stein g-function of the semigroup whose generator satisfies certain conditions. The main tools used are vector-valued tent spaces and the intrinsic square function introduced by Wilson. In particular, we obtain the optimal order of the corresponding best constant in the Littlewood-Paley-Stein inequality in vector-valued L^p spaces where the underlying Banach space is of martingale type q $(1 < q \leq 2)$ for p tending to 1.

The final part deals with the bilinear decomposition of pointwise multiplication of elements in the martingale Hardy space H^1 and its dual space BMO. This continuous bilinear decomposition is extended to the martingale Hardy space H^p $(0 < p < 1)$ and its dual space. Our decompositions rely on martingale paraproducts. Consequently, we obtain analogous results for

dyadic martingales on spaces of homogeneous type through the construction of a dyadic system.

Keywords : Littlewood-Paley-Stein inequalities; Optimal order of best constants; Hardy and BMO spaces; Martingale paraproducts; Martingale type and cotype; Vector-valued tent spaces; Intrinsic square function; Musielak-Orlicz Hardy space; Martingales.

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