



HAL
open science

Limites d'échelle de marches branchantes critiques à valeurs dans des arbres, et du nombre de Horton-Strahler d'arbres de Galton-Watson

Robin Khanfir

► **To cite this version:**

Robin Khanfir. Limites d'échelle de marches branchantes critiques à valeurs dans des arbres, et du nombre de Horton-Strahler d'arbres de Galton-Watson. Probabilités [math.PR]. Sorbonne Université, 2023. Français. NNT : 2023SORUS499 . tel-04572268

HAL Id: tel-04572268

<https://theses.hal.science/tel-04572268>

Submitted on 10 May 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



SORBONNE UNIVERSITÉ
École Doctorale de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation

THÈSE DE DOCTORAT
Discipline : Mathématiques

présentée par

Robin KHANFIR

**Limites d'échelle de marches branchantes critiques à
valeurs dans des arbres, et du nombre de Horton-Strahler
d'arbres de Galton-Watson**

sous la direction de **Thomas DUQUESNE** et de **Shen LIN**

Rapporteurs :

M. Jean-François DELMAS École des Ponts ParisTech
Mme. Christina GOLDSCHMIDT University of Oxford

Soutenue le 06 novembre 2023 devant le jury composé de :

M. Nicolas BROUTIN	Sorbonne Université	Examineur
M. Nicolas CURIEN	Université Paris-Saclay	Président du jury
M. Jean-François DELMAS	École des Ponts ParisTech	Rapporteur
M. Thomas DUQUESNE	Sorbonne Université	Directeur
Mme. Christina GOLDSCHMIDT	University of Oxford	Rapporteuse
M. Yueyun HU	Université Paris 13	Examineur
M. Shen LIN	Sorbonne Université	Co-directeur

Laboratoire de Probabilités, Statistique et Modélisation (LPSM, UMR 8001)
Sorbonne Université
4 place Jussieu, Paris CEDEX 05, FRANCE

Remerciements – Acknowledgements

Dans ces quelques lignes, je tiens à exprimer la gratitude que je ressens envers les personnes qui m'ont aidé, épaulé, et accompagné au cours de cette thèse. Maintenant que ces années de travail s'achèvent et que je m'appête à travailler à l'étranger pendant quelques temps, je réalise à quel point votre présence et vos actions ont joué un rôle important dans l'aboutissement de ce projet.

Mes premiers remerciements vont à mes directeurs de thèse, Thomas Duquesne et Shen Lin, pour m'avoir guidé sur un sujet de recherche aussi passionnant. La confiance que vous m'avez accordée, de part la variété des questions que vous m'avez proposées et la liberté que vous m'avez laissée dans le choix de ces dernières, me fut précieuse. Il me tarde de continuer à travailler à vos côtés. Thomas, plus le temps passe et plus je prends conscience de tout ce que j'ai appris grâce à toi. Pour tous tes conseils, tes coups de main, tes relectures, nos conversations, ton humour, et tous les efforts que tu as fournis pour moi, je te suis infiniment reconnaissant. Shen, tu n'as jamais manqué une occasion de me rendre service ou de m'aider, et toujours avec beaucoup de bienveillance. Je tiens à te remercier pour ta sollicitude et ta gentillesse.

J'éprouve un grand honneur à ce que Jean-François Delmas et Christina Goldschmidt aient accepté de rapporter le présent manuscrit. Vous avez accompli cette tâche avec le plus grand soin, alors même que les délais étaient serrés, en me fournissant des retours détaillés et instructifs. Je vous en remercie chaleureusement. Ma gratitude va également à Nicolas Broutin, Nicolas Curien, et Yueyun Hu pour constituer le reste de mon jury. Je ressens une grande fierté à pouvoir présenter mes résultats devant ces mathématiciens que je tiens en haute estime et dont les travaux ont continuellement inspiré mes recherches.

Je tiens à rendre hommage aux enseignants et professeurs dont la formation et l'influence ont fini par me donner les capacités et la drôle d'idée de suivre une carrière dans la recherche mathématique. M. Demoron, Mme. Lods, M. Tosel, M. Cerf, merci beaucoup, je ne regrette rien ! I warmly thank Pr. Pitman for showing me a glimpse of what it means to be a mathematician, and for leaving me with a lasting memory of Berkeley.

I would like to acknowledge Louigi Addario-Berry and Jackie Castreje's kindness in agreeing to postpone the start of my postdoc while helping me prepare for my departure. Your patience has enabled me to complete this thesis in good condition. Thank you very much, I look forward to meeting you in person in Montreal.

Je souhaiterais maintenant dédier quelques mots aux membres du LPSM. Si le laboratoire est un environnement de travail si stimulant et agréable, c'est grâce à vous. Je commence par remercier toute l'équipe du secrétariat pour ses compétences administratives qui me font cruellement défaut. Je salue plus particulièrement Imen Sisaber qui m'a sorti d'un labyrinthe d'incompréhension face aux réservations de missions. Bien sûr, j'envoie un grand merci à Hugues Moretto-Viry pour sa réactivité légendaire et sa facilité déconcertante à me tirer de n'importe quel pépin informatique.

Ensuite, je voudrais exprimer ma reconnaissance envers les permanents pour faire vivre le laboratoire, que ce soit par l'organisation des séminaires (et de pots), en partageant leur expérience, ou simplement par de simples discussions amicales. Je remercie notamment Nicolas Broutin, Quentin Berger, et Olivier Zindy pour nos échanges, leur accompagnement, et surtout leur sympathie.

J'ai eu la chance de partager la vie de thésard avec des personnes géniales : mes camarades doctorants. Je vous dis merci du fond du cœur pour la myriade de souvenirs que vous m'avez

offerts. À défaut de pouvoir les retranscrire à l'écrit, j'espère vous aider à les remémorer à travers ces quelques lignes.

Mes premières pensées sont dirigées vers ceux avec qui j'ai passé le plus de temps, c'est-à-dire mes compagnons de bureau. Merci Armand pour ton assurance, tu as rempli ton rôle d'aîné avec brio. À Jérémy pour avoir toléré mes gamineries avec bienveillance, alors qu'il te fallait aller chercher ton gosse en fin de journée. Merci Yoan pour ton rire si reconnaissable mais si contagieux, pour ton bordel qui éclipsait le mien, et pour ton explication particulièrement convaincante sur l'origine du symbole du laplacien. Ton charisme et ton talent m'ont poussé à me dépasser, ton humour et ton empathie m'ont permis de décompresser. À Antonio pour t'être autant investi dans le labo tout en me pardonnant ma désinvolture. Ta répartie, ton style, et ta franchise sont telles que ton simple changement de couloir aura créé un vide. Merci à Guillaume pour nos joutes verbales, pour tes gaffes mythiques, et pour les fléchettes-ventouses. Par contre, pas touche à ma chaise avant mon départ. Merci Tristan d'avoir partagé avec moi la procrastination active devant des sodokus, la rage post-TD, et la hype de One Piece. Comme avec le Sleuh, tu es quelqu'un de moteur et je te fais confiance pour mener à bien tous tes projets. À Sonia pour s'être laissée faite traquenarder à l'aïkido, pour monopoliser le titre de la personne la plus beauif du labo, et pour ne pas savoir que les pingouins sont des oiseaux. Tu vas gérer pour la suite, et comme tu le sais, j'ai toujours raison. À Mathieu T. pour avoir enfin repris mon rôle de benjamin du bureau pour me permettre de faire l'ancien : tu as un truc gamin, tu iras loin. Enfin, merci au ficus de clore tout débat sur l'identité du meilleur bureau.

Malgré la supériorité évidente du bureau 203, il y a quand même quelques gens assez sympas ailleurs. Pour commencer par nos voisins directs, merci à Loïc de m'avoir permis de rencontrer le chill sous forme humaine. À Lucas I. pour m'avoir rappelé que s'il y avait une vie sociale avant le Covid, alors il y en aurait une après. Merci Arthur de t'être servi de mon esprit de compétition/contradiction pour me motiver à ta manière : désolé, la soutenance n'est pas en décembre ! À Jean-David pour mettre constamment à l'épreuve les limites de l'absurde, sans forcer et le sourire aux lèvres. Merci Pierre B. de m'avoir fait profiter de ta compagnie alors que ça te demandait pas mal de temps et d'efforts, j'espère avoir participé à te convaincre que ça en valait le coup. À Benjamin, ses clopes, son café, et ses calculs sous le soleil des calanques.

À l'autre bout du couloir, je salue le bureau de la glacière aux merveilles. Merci à Lucas B., qui porte la casquette comme personne, pour son flegme et pour avoir tout risqué en me suivant jusqu'aux tréfonds du Grand Séminaire. À Émilien dont le charisme rend impossible le fait de deviner qu'il est mon acolyte de la team chômage, je te dédie la fameuse slide que j'ai pu coder grâce à ton aide : des conseils efficaces, sans accrocs, mais donnés de manière fun, ça te ressemble parfaitement. David, I must confess that I am not the most intelligent man in Jussieu, but I am confident that you are the most epic guy here! Thank you for the hugs, the noodles, and the *yeeeeeehs*. Merci Nicolas, mon partenaire de vac... séminaire privilégié, pour ta poésie, tes chehs tout en douceur, et pour m'avoir appris que le Brownien, c'est avant tout les potes qu'on s'est faits en chemin. À Thomas J. pour avoir complimenté mon planteur sans relâche, mon égo te remercie platement.

À l'étage inférieur, je tiens d'abord à rendre hommage à Farf', un agent du chaos à tous les niveaux. On partage notre haine de la ligne 7, notre sens du sarcasme, et notre entrain pour l'apéro. Quoiqu'il m'arrive, je sais que je pourrai en rire avec toi comme par magie, et réciproquement. Merci au chad officiel du labo, Lucas D., qui arrive à faire autant de sport alors qu'il est déjà lesté avec un cœur en or. À l'altruisme de Bastien, qui me propose toujours de partager sa nourriture alors qu'il en a presque deux fois plus besoin que moi. On dit que le propre des grandes personnes est de laisser leur marque même après leur départ, et ton influence, William, est présente jusque

dans la forme de ce manuscrit. À Alexandra et son château, qui nous ont certainement protégé de dangereux ennemis.

Enfin, il me reste encore quelques membres de Jussieu à saluer. Merci à Ariane et Ludovic, mes collaborateurs du GTT, pour avoir trouvé des questions à poser pendant les exposés de stats. To Nicolaï, his naps and his *hellos*. Merci à Ons d'avoir fermé les yeux devant mes nombreux bluffs alors qu'on préparait nos présentations du Sleuh. À Yazid pour m'aider à gérer Sasila et pour son pot de thèse. Merci à Alexandre, Elias, Léo, Maxence, Maya, Rolland, Ulysse et à tous les nouveaux de s'être intégrés assez vite pour qu'on ait eu l'occasion de rigoler ensemble. Pour finir, à Arnaud, Jesús, Laurent, Lucas J., et Roman que je croisais moins souvent dans les allées de Jussieu mais avec qui j'avais toujours plaisir à discuter.

Je pense sincèrement que si j'ai pu aller au bout de l'effort de longue haleine que représente la thèse, c'est en grande partie grâce à l'équilibre que m'ont offert mes proches en dehors de la recherche. Vous m'avez permis de garder le sourire et la tête haute, et je vous en suis infiniment reconnaissant.

À mes camarades de promo, ancien collocs, et amis : Assil, Béranger, Dolby, Lucas G.-G., Malachi, et Raphaël. On a traversé Villejuif Plage ensemble, et je pense qu'il en faut moins pour lier des personnes à vie. Merci à Alia de m'avoir entraîné dans la préparation du N4 : je n'ai pas réussi l'examen mais j'ai gagné une amie. Aucune pression, mais j'attends le pot de thèse *tardif* avec impatience. À Alexis et Maëli, merci d'être à la fois une source d'énergie et de douceur, vous êtes géniaux. Merci à Yolaine et Pierre-Franck et à tous les membres de la Fleur et le Sabre – Domenico, Dominique, Erwan, Eugénie, Francesco, Hilton, Léa, Marie-Agathe, Maxence P., Maylis, Pierre S., Stan, Sylvia, et tous les autres – de m'avoir aidé à progresser, à relâcher la pression, et à donner le meilleur de moi-même, le tout dans la bonne humeur. Je vous dis, comme d'habitude, à bientôt sur les tatamis !

Un grand merci à mes anciens co-colles, Mathieu H., Éva, et Héloïse. Vous êtes tous les trois des modèles d'humour et de gentillesse. Nos amitiés précieuses, alors même que nous avons formé nos groupes de colles sur un coup de tête, me faisaient presque croire au destin. À Anthony, Augustin, Bruno, Cécile, Charles D., David T., Erwan, Félix, Lucas C., Mathieu V.-R., et Paul, vous avez fait mentir tous les préjugés sur la prépa et fait de cette période une des meilleures de ma vie. C'est à chaque fois un véritable bonheur de vous revoir, et je sens que ce sera le cas encore pour de très longues années.

Merci du fond du cœur à RobLep, Romain, Sasila, Sofiane, et Sohaïb de préserver notre amitié malgré la distance, le manque de temps, et nos différences de carrière. Je sais, et vous me prouvez aujourd'hui, que vous serez présents dans les moments importants pour moi et c'est évidemment réciproque. Bref, on a déjà fait les 400 coups avant le BAC, ils nous en restent au moins autant à faire ! Mais en attendant, profitez bien du buffet !

Merci aux amis et à la famille de Lola pour si bien l'entourer et pour m'accorder autant d'attention et de gentillesse. En particulier, merci à Mona et Nelson, à Charlène et Mélanie, et à Joël, Plume, et Montsé pour les voyages, les fêtes, et les repas en pagaille. On s'amuse beaucoup avec vous, donc promis, on reviendra vite !

J'aimerais remercier infiniment Titie, Mamie, les taties, les tontons, les cousins grands comme petits, Odile, et tous les membres de ma grande famille en général. Vous êtes mon pilier, le simple fait de savoir que je pourrais compter sur vous quoi qu'il arrive m'emplit de courage. Maman, Papa, et Nanie, je vous remercie pour tout. Votre soutien absolu, vos innombrables attentions, vos conseils avisés, votre confiance tranquille... Je vous dois tant que je ne pourrais tout lister ici. Je n'aurais pu rêver mieux, vous me manquez déjà.

Je dédie ces dernières lignes à la femme qui partage mon quotidien. Lola, merci de m'épauler, de m'inspirer, de me faire rire, de me faire me sentir bien. Merci d'avoir été une oreille attentive, un regard patient, et une complice de plaisanteries selon ce dont j'avais besoin. Je suis heureux quand je te redécouvre chaque jour, quand je constate à quel point on se complète, quand on se taquine librement. Avec toi, je suis épanoui, j'ai la sensation de pouvoir accomplir l'impossible, et je le ressens dans mon travail. Grâce à toi, je ne m'inquiète pas une seconde pour l'avenir. Je ne ressens que l'excitation de vivre de nouvelles aventures ensemble.

Limites d'échelle de marches branchantes critiques à valeurs dans des arbres, et du nombre de Horton-Strahler d'arbres de Galton-Watson

Résumé

Dans cette thèse, on étudie des phénomènes discrets de branchement aléatoire et on cherche à les mettre en relation avec des structures métriques fractales continues. Les arbres de Galton-Watson, qui décrivent l'histoire généalogique d'une population asexuée dont les individus se reproduisent sous la même loi et indépendamment les uns des autres, constituent notre modèle principal.

Dans un premier temps, on se consacre à l'étude d'une marche aléatoire (dite biaisée critique) sur un arbre infini (appelé l'environnement) et indexée par un arbre de Galton-Watson critique conditionné à être grand (appelé la généalogie). On suppose que la loi de reproduction de la généalogie est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. On traite d'une part du cas où l'environnement est un arbre régulier enraciné, et d'autre part du cas où il s'agit d'un arbre de Galton-Watson sur-critique modifié de façon à être de profondeur infinie. Sous une certaine hypothèse de moments pour l'environnement, on montre que le nombre de points visités par la marche croît linéairement, et à vitesse déterministe, en fonction de la taille de la généalogie lorsque cette dernière tend vers l'infini. En outre, on prouve que le sous-arbre des points visités par la marche branchante admet une limite d'échelle. Auparavant introduit dans le contexte de l'étude des cartes planaires aléatoires, cet espace métrique limite est le cactus brownien (réfléchi) avec mécanisme de branchement α -stable. La comparaison de cette étude nouvelle avec les travaux antérieurs sur les marches aléatoires indexées par le temps ou à valeurs dans un réseau euclidien illustre l'influence des branchements de la généalogie et de l'environnement.

Dans un second temps, on s'intéresse à la complexité de branchement des arbres de Galton-Watson en étudiant leurs nombres de Horton-Strahler. Cet outil combinatoire, aussi appelé fonction de registre, a été originellement introduit en hydrogéologie mais a été redécouvert et appliqué par de nombreuses autres disciplines scientifiques par la suite. Ici, on donne un équivalent asymptotique déterministe du nombre de Horton-Strahler d'un arbre de Galton-Watson critique conditionné par la taille et dont la loi de reproduction est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in [1, 2]$. Cette estimation ne dépend que de α lorsque $\alpha \neq 1$, mais les cas $\alpha = 1$ sont modèle-dépendants et sujets à des comportements plus complexes. On examine ensuite les fluctuations du nombre de Horton-Strahler chez la famille spécifique des arbres de Galton-Watson stables, qui contient l'arbre de Galton-Watson critique binaire. On est alors amené à introduire une variante continue du nombre de Horton-Strahler, et on montre que celle-ci converge après recentrage vers une caractéristique métrique de la limite d'échelle des arbres. On étudie les propriétés de cette quantité nouvelle qui joue le rôle d'un analogue du nombre de Horton-Strahler pour les arbres continus.

Mots-clés : arbres de Galton-Watson ; limite d'échelle ; marche branchante ; marche aléatoire sur un arbre ; trace d'une marche aléatoire ; nombre de Horton-Strahler ; fonction de registre ; serpent brownien ; cactus brownien ; arbres stables ; arbres réels aléatoires.

Scaling limits of tree-valued critical branching random walks, and of the Horton-Strahler number of Galton-Watson trees

Abstract

In this thesis, we study discrete random branching phenomena and seek to relate them to continuum fractal metric structures. Galton-Watson trees, which describe the genealogical history of an asexual population whose individuals reproduce under the same law and independently of each other, are our main model.

In the first part, we focus on the study of a random walk (said critical biased) on an infinite tree (called the environment) and indexed by a critical Galton-Watson tree conditioned to be large (called the genealogy). The offspring distribution of the genealogy is assumed to be in the domain of attraction of a stable law of index $\alpha \in (1, 2]$. We both consider the case where the environment is a regular rooted tree and the case where it is a supercritical Galton-Watson tree modified to have an infinite depth. Under some hypothesis of moments for the environment, we show that the number of points visited by the random walk grows linearly, and at a deterministic speed, with respect to the size of the genealogy when the latter tends to infinity. Furthermore, we prove that the subtree of points visited by the branching random walk admits a scaling limit. Previously introduced in the context of the study of random planar maps, this limit metric space is the (reflected) Brownian cactus with α -stable branching mechanism. Comparison of this new study with earlier work about random walks indexed by a linear time or taking values in a Euclidean lattice illustrates the influence of the branching nature of the genealogy and the environment.

In the second part, we study the branching complexity of Galton-Watson trees by considering their Horton-Strahler numbers. This combinatorial tool, also known as the register function, was originally introduced in hydrogeology but was subsequently rediscovered and applied by many other scientific disciplines. Here, we give a deterministic asymptotic equivalent of the Horton-Strahler number of a size-conditioned critical Galton-Watson tree whose offspring distribution is in the domain of attraction of a stable law of index $\alpha \in [1, 2]$. This estimate depends only on α when $\alpha \neq 1$, but the $\alpha = 1$ cases are model-dependent and subject to more complex behaviors. We then examine the fluctuations of the Horton-Strahler number for the specific family of stable Galton-Watson trees, which contains the binary critical Galton-Watson tree. To do so, we introduce a continuous variant of the Horton-Strahler number that converges after recentering towards a metric characteristic of the scaling limit of the trees. We study the properties of this new quantity, which acts as a continuum analog of the Horton-Strahler number.

Keywords: Galton-Watson trees; scaling limit; branching random walk; random walk on a tree; range of a random walk; Horton-Strahler number; register function; Brownian snake; Brownian cactus; stable trees; random real trees.

Contents

1	Introduction	13
1.1	Arbres de Galton-Watson	14
1.1.1	Mots et arbres	14
1.1.2	Arbres de Galton-Watson conditionnés par la taille	15
1.1.3	Limites d'échelles par contour d'arbres de Galton-Watson critiques	16
1.1.4	Arbres réels compacts enracinés mesurés	19
1.1.5	Arbres stables continus, et arbres de Galton-Watson stables	21
1.2	Marches branchantes critiques	23
1.2.1	Modèle et motivations	23
1.2.2	Le serpent brownien et la Integrated Super-Brownian Excursion	24
1.3	Marche aléatoire biaisée sur un arbre	28
1.3.1	Modèle, motivations, et quelques résultats connus	28
1.3.2	Arbres de Galton-Watson de profondeur infinie	30
1.4	Nombre de Horton-Strahler	33
1.4.1	Quelques définitions équivalentes	33
1.4.2	Un bref historique général	35
1.4.3	Quelques résultats probabilistes sur le nombre de Horton-Strahler	36
1.4.4	Irrégularité des fluctuations du nombre de Horton-Strahler	37
1.5	Contributions de cette thèse	39
1.5.1	Marches branchantes critiques et récurrentes nulles sur un arbre régulier	39
1.5.2	Marches branchantes biaisées critiques sur un arbre de Galton-Watson invariant	43
1.5.3	Nombre de Horton-Strahler des arbres de Galton-Watson critiques : estimées	46
1.5.4	Nombre de Horton-Strahler des arbres de Galton-Watson stables : fluctuations	49
2	Scaling limits of tree-valued branching random walks	57
2.1	Introduction	58
2.2	Tree with a possibly infinite line of ancestors.	62
2.3	Tree-valued branching random walks.	68
2.3.1	Definitions.	68
2.3.2	Metric properties of the range of free branching random walks.	69
2.3.3	A coupling between \mathbb{W}_b - and \mathbb{W}_b^* -valued branching random walks.	72
2.3.4	Estimates.	76
2.3.5	Law invariance.	81

2.3.6	Proof of Theorem 2.1.1.	83
2.4	Snake metrics and the Brownian cactus.	85
2.4.1	Pseudo-metrics on a closed interval.	85
2.4.2	Scaling limit of the range of biased RWs on trees.	90
2.4.3	Snake metrics.	92
2.4.4	Stable Lévy trees.	97
2.4.5	One-dimensional reflected Brownian snakes.	99
2.4.6	Reflected Brownian cactus with stable branching mechanism.	101
2.5	Scaling limit of the range of the BRW.	102
2.5.1	Continuous interpolation of discrete snakes.	102
2.5.2	Invariance principle for discrete snakes.	107
2.5.3	Proof of Theorem 2.1.2.	108
3	Scaling limit of the range of tree-valued branching random walks in random environment	111
3.1	Introduction	112
3.2	Tree-valued branching RWs and snakes: definitions, previous results	117
3.2.1	Rooted ordered trees and forests, coding functions, discrete snakes.	117
3.2.2	Trees with an infinite line of ancestors.	121
3.2.3	Biased random walks on trees	126
3.2.4	Tree-valued branching RWs	130
3.2.5	Stable continuous height processes.	132
3.2.6	Real trees, convergences of metric spaces and pseudometrics.	137
3.2.7	Continuous snakes and related metrics.	139
3.2.8	Tree-valued branching walks and snake-pseudometrics	143
3.3	Proof of Theorem 3.1.1	146
3.3.1	Preliminary results on critical biased RWs on GW-trees	146
3.3.2	Estimates on critical biased branching RWs.	148
3.3.3	Proof of Propositions 3.2.26 and 3.2.32.	151
3.3.4	Proof of Theorem 3.1.1	156
3.4	Limit of the τ_∞ -indexed branching RW and its snakes	158
3.4.1	Harmonic coordinates and related estimates.	158
3.4.2	Statements of the result, overview of its proof	165
3.4.3	Quenched tightness of the height-snake of the τ_∞ -indexed branching RWs	169
3.4.4	Proof of Proposition 3.4.10	174
3.4.5	Proof of Proposition 3.4.14	180
3.5	Proof of Theorem 3.1.2	183
3.5.1	Limit of the τ_n -indexed RW and its snakes.	183
3.5.2	Proof of Theorem 3.1.2	186
3.A	Proof of Lemma 3.2.33	189
3.B	Proof of Lemma 3.2.37	190
3.C	Proof of Proposition 3.4.5	191
4	The Horton-Strahler number of Galton-Watson trees with possibly infinite variance	195
4.1	Introduction	196
4.2	Framework and tools	200
4.2.1	Galton-Watson trees	200
4.2.2	Limit theorems.	203
4.3	Distribution of the Horton-Strahler number of Galton-Watson trees	208

4.3.1	Alternative definitions of the Horton–Strahler number and basic results . . .	208
4.3.2	Tail estimates of joint laws	211
4.3.3	Tail estimates.	215
4.4	Proofs of Theorems 4.1.2, 4.1.3 and 4.1.4	219
4.4.1	Proof of Theorem 4.1.2	219
4.4.2	Proof of Theorem 4.1.3	220
4.4.3	Proof of Theorem 4.1.4	223
5	Fluctuations of the Horton–Strahler number of stable Galton–Watson trees	227
5.1	Introduction	228
5.2	Framework, notations, and definitions	235
5.2.1	Topological framework	235
5.2.2	Discrete trees as sets of words	238
5.2.3	Real trees	242
5.2.4	Grafting rooted compact metric spaces	245
5.2.5	Stable Lévy processes	247
5.3	Preliminary tools	249
5.3.1	Marchal’s algorithm	249
5.3.2	Limit theorems for stable Galton–Watson trees	250
5.3.3	An estimate for the height function of a Galton–Watson tree with edge lengths	253
5.4	The weighted Horton–Strahler number	255
5.4.1	The classic Horton–Strahler number	255
5.4.2	First properties of the weighted Horton–Strahler number	256
5.4.3	The weighted Horton–Strahler number of stable Galton–Watson weighted trees	257
5.5	The weighted Horton pruning of trees with edge lengths	260
5.5.1	Definition and invariance	260
5.5.2	Some applications	264
5.6	Proof of Theorem 5.1.3	266
5.7	A first description of the limit tree in Theorem 5.1.3	270
5.7.1	Proof of Theorem 5.7.3	272
5.8	The Strahler dilation of rooted compact real trees	277
5.8.1	Definition and properties of the Strahler dilation	277
5.8.2	Proof of Theorem 5.1.4 (ii)	282
5.9	Proofs of Theorems 5.1.5 and 5.1.6	287
5.9.1	Proof of Theorem 5.1.5	287
5.9.2	Proof of Theorem 5.1.6	289
5.10	The Strahler dilation of the Brownian tree: proof of Theorem 5.1.7	292

Chapter 1

Introduction

Dans cette introduction, on présente d'abord les objets mathématiques rencontrés au cours de cette thèse. On motive historiquement leurs définitions tout en donnant quelques exemples de travaux où ils apparaissent. De plus, on expose les techniques classiques et les résultats connus les concernant que nous sommes utiles par la suite. La section 1.1 présente les arbres de Galton-Watson. La section 1.2 concerne les marches branchantes indexées par des arbres de Galton-Watson critiques. La section 1.3 traite des marches aléatoires biaisées sur des arbres. La section 1.4 porte sur le nombre de Horton-Strahler. Ces sections ne contiennent pas de nouveaux résultats.

Les contributions de ce travail de thèse sont énoncées et commentées dans la section 1.5. Cette dernière est divisée en quatre sous-sections qui présentent respectivement les résultats démontrés dans les chapitres 2, 3, 4, et 5, eux-mêmes basés sur les quatre articles suivants.

- [60] : SCALING LIMITS OF TREE-VALUED BRANCHING RANDOM WALKS, écrit en collaboration avec Thomas Duquesne, Shen Lin & Niccolò Torri. Publié dans *Electronic Journal of Probability*.
- [59] : SCALING LIMITS OF THE RANGE OF TREE-VALUED BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT, écrit en collaboration avec Thomas Duquesne. Travail en préparation.
- [103] : THE HORTON-STRAHLER NUMBER OF GALTON-WATSON TREES WITH POSSIBLY INFINITE VARIANCE. Soumis à publication.
- [102] : FLUCTUATIONS OF THE HORTON-STRAHLER NUMBER OF STABLE GALTON-WATSON TREES. Soumis à publication.

Sauf mention explicite du contraire, toutes les variables aléatoires considérées sont définies sur un même espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. On note \mathbb{E} l'espérance associée à la probabilité \mathbb{P} . Toujours sauf mention explicite du contraire, tous les espaces topologiques rencontrés sont munis de leur tribu borélienne. Tout au long de ce manuscrit, on utilise les notations suivantes

$$\mathbb{R}_+ = [0, \infty), \quad \mathbb{N} = \{0, 1, 2, 3, \dots\}, \quad \text{et} \quad \mathbb{N}^* = \{1, 2, 3, \dots\}.$$

1.1 Arbres de Galton-Watson

1.1.1 Mots et arbres

Dans cette section, on introduit le formalisme d'Ulam qui fournit un cadre pratique pour définir les arbres de Galton-Watson, que l'on voit comme des arbres aléatoires enracinés et ordonnés. En général, un arbre (discret) est un graphe connexe et acyclique. Cependant, dans ce manuscrit, on considère exclusivement des arbres planaires (i.e. enracinés et ordonnés), qui peuvent se voir intuitivement comme des arbres généalogiques. On adopte le formalisme d'Ulam comme présenté par Neveu [133]. Soit $\mathbb{N}^* = \{1, 2, 3, \dots\}$ l'ensemble des entiers strictement positifs et soit \mathbb{U} l'ensemble des mots écrits avec l'alphabet \mathbb{N}^* :

$$\mathbb{U} = \bigcup_{n \in \mathbb{N}} (\mathbb{N}^*)^n, \quad \text{avec la convention} \quad (\mathbb{N}^*)^0 = \{\emptyset\}. \quad (1.1.1)$$

L'ensemble \mathbb{U} est dénombrable et muni de la topologie discrète. Soit $u = (u_1, \dots, u_n) \in (\mathbb{N}^*)^n \subset \mathbb{U}$, on pose $|u| = n$ la *hauteur* (aussi appelée la *génération*) de u . Le mot vide \emptyset est le seul élément de \mathbb{U} de hauteur nulle. On note \leq l'*ordre lexicographique*, qui est un ordre total sur \mathbb{U} . Par exemple, $\emptyset < (1) < (1, 1) < (1, 2) < (2)$. Pour $v = (v_1, \dots, v_m) \in \mathbb{U}$, la notation $u * v$ désigne la *concaténation* $(u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ de u et v . En particulier, on écrit $u * \emptyset = \emptyset * u = u$. Si $|u| = n \geq 1$ alors on note $\overleftarrow{u} = (u_1, \dots, u_{n-1})$ le *parent* de u . On dit également que u est un *enfant* de v lorsque $\overleftarrow{u} = v$. L'*ordre généalogique* \preceq est un ordre partiel sur \mathbb{U} défini par $u \preceq v \iff \exists w \in \mathbb{U}, v = u * w$. Dans ce cas, on dit que u est un *ancêtre* de v . On appelle le *plus récent ancêtre commun* de u et v leur ancêtre commun de hauteur maximale noté $u \wedge v \in \mathbb{U}$. On peut munir \mathbb{U} d'une structure de graphe dont les arêtes sont les paires $\{\overleftarrow{u}, u\}$ où $u \in \mathbb{U} \setminus \{\emptyset\}$. Sa distance de graphe s'exprime alors comme suit :

$$\forall u, v \in \mathbb{U}, \quad d_{\text{gr}}(u, v) = |u| + |v| - 2|u \wedge v|. \quad (1.1.2)$$

Définition 1.1.1 Un *arbre* est un sous-ensemble t de \mathbb{U} qui satisfait les conditions suivantes :

- (a) $\emptyset \in t$, qui joue le rôle de la racine de t ,
- (b) pour tout $u \in t$, si $u \neq \emptyset$ alors $\overleftarrow{u} \in t$,
- (c) pour tout $u \in t$, il existe un entier $k_u(t) \in \mathbb{N}$ tel que $u * (i) \in t \iff 1 \leq i \leq k_u(t)$, que l'on appelle le nombre d'enfants de u dans t .

On dote l'espace \mathbb{T} de tous les arbres de la topologie de la *convergence locale*, engendrée par les ensembles $\{t \in \mathbb{T} : A \subset t\}$ où A varie parmi les parties finies de \mathbb{U} . L'espace \mathbb{T} est polonais. \square

Soit t un arbre. Son nombre de sommets (ou cardinal) est aussi appelé sa *taille* et est noté $\#t$. On autorise $\#t$ à être infini. On dit que $u \in t$ est une *feuille* de t lorsque $k_u(t) = 0$. Pour tout $u \in t$, on définit le *sous-arbre de t issu de u* en posant

$$\theta_u t = \{v \in \mathbb{U} : u * v \in t\},$$

ce qui donne bien un arbre au sens de la définition 1.1.1.

Ce formalisme permet de donner la définition récursive suivante des arbres de Galton-Watson.

Définition 1.1.2 Soit $\mu = (\mu(k))_{k \in \mathbb{N}}$ une mesure de probabilité sur \mathbb{N} . Un *arbre de Galton-Watson de loi de reproduction μ* est un arbre aléatoire τ dont la loi est l'unique mesure de probabilité sur \mathbb{T} qui satisfait les propriétés suivantes.

- (a) La loi de $k_{\emptyset}(\tau)$ est μ .

(b) Pour tout $k \in \mathbb{N}^*$ tel que $\mu(k) > 0$, sous la probabilité conditionnelle $\mathbb{P}(\cdot | k_{\mathcal{O}}(\tau) = k)$, les arbres aléatoires $\theta_{(1)}\tau, \dots, \theta_{(k)}\tau$ sont indépendants et ont la même loi que τ sous \mathbb{P} . \square

Intuitivement, un arbre de Galton-Watson décrit la généalogie d'une population dont les individus se reproduisent indépendamment les uns des autres et selon une distribution commune. Dans la suite, on suppose toujours que μ est *non-triviale*, ce qui signifie que

$$\mu(0) + \mu(1) < 1.$$

Dans les cas où $\mu(0) + \mu(1) = 1$, l'arbre de Galton-Watson est dépourvu de branchement : il forme une ligne de longueur géométrique si $\mu(1) < 1$ ou infinie si $\mu(1) = 1$. On rappelle de plus que

$$\mathbb{P}(\#\tau = \infty) > 0 \quad \text{si et seulement si} \quad m := \sum_{k \in \mathbb{N}} k\mu(k) > 1.$$

On dit que la loi de reproduction μ est *sur-critique* lorsque $m > 1$, qu'elle est *sous-critique* lorsque $m < 1$, et qu'elle est *critique* lorsque $m = 1$. L'existence et l'unicité de la loi d'un arbre de Galton-Watson ont été prouvées par Neveu [133]. De plus, Otter [135] a explicité que si τ est un arbre de Galton-Watson, alors pour tout arbre fini t ,

$$\mathbb{P}(\tau = t) = \prod_{u \in t} \mu(k_u(t)). \quad (1.1.3)$$

Néanmoins, cette formule ne décrit complètement la loi de τ que si μ est critique ou sous-critique.

Les arbres de Galton-Watson interviennent de trois manières distinctes dans ce travail :

- en tant qu'indexation de marches branchantes,
- en tant qu'environnement sur lequel celles-ci évoluent,
- en tant qu'objet principal d'intérêt au regard de leur nombre de Horton-Strahler (qui quantifie leur complexité de branchement).

Avant de présenter ces sujets, on rappelle les méthodes et les objets classiques intervenant dans l'étude des arbres de Galton-Watson.

1.1.2 Arbres de Galton-Watson conditionnés par la taille

Soit μ une mesure de probabilité sur \mathbb{N} telle que $\mu(1) < 1$ et $\mu(0) > 0$, et soit τ un arbre de Galton-Watson de loi de reproduction μ . La formule (1.1.3) justifie qu'il existe une infinité d'entiers $n \in \mathbb{N}$ tels que $\mathbb{P}(\#\tau = n) > 0$. Pour chacun de ces entiers, on se donne un arbre aléatoire τ_n dont la loi sous \mathbb{P} est celle de τ sous $\mathbb{P}(\cdot | \#\tau = n)$:

$$\tau_n \stackrel{d}{=} \tau \text{ sous } \mathbb{P}(\cdot | \#\tau = n). \quad (1.1.4)$$

Pour plusieurs classes combinatoires d'arbres à n sommets, la loi uniforme peut être réalisée en tant que celle de τ_n en choisissant une loi de reproduction particulière. Donnons quelques exemples.

- Si μ est géométrique de paramètre $1/2$, c'est-à-dire $\mu(k) = 2^{-1-k}$ pour tout $k \in \mathbb{N}$, la loi de τ_n est uniforme sur l'ensemble des arbres (planaires) de taille n .
- Si μ est binaire et critique, c'est-à-dire $\mu(0) = \mu(2) = 1/2$, la loi de τ_n est uniforme sur l'ensemble des arbres binaires de taille n , parfois appelés arbres de Catalan.
- Si μ est la loi Poisson de paramètre 1, c'est-à-dire $\mu(k) = e^{-1}/k!$ pour tout $k \in \mathbb{N}$, la loi de τ_n est uniforme sur l'ensemble des arbres de Cayley (i.e. étiquetés non-ordonnés) de taille n .

On réfère à Le Gall [78, Section 1.5] pour plus de précisions. Dans cette thèse, les résultats impliquant les arbres de Galton-Watson conditionnés par leur taille supposent presque toujours les conditions suivantes sur la loi de reproduction, conditions qui assurent d'une régularité asymptotique de ces arbres conditionnés.

- (a) μ est critique et non-triviale, ce qui veut dire $\mu(1) < 1$ et $\sum_{k \in \mathbb{N}} k\mu(k) = 1$.
- (b) μ est apériodique, autrement dit le sous-groupe additif engendré par le support de μ est \mathbb{Z} .
- (c) μ est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in [1, 2]$.

Comme déjà énoncé, (a) implique qu'un arbre de Galton-Watson τ de loi de reproduction μ est fini presque sûrement. De plus, on peut démontrer que $\mathbb{P}(\#\tau = n)$ décroît de manière polynomiale lorsque μ est critique, alors que cette probabilité décroît à vitesse exponentielle quand μ est sous-critique. Comme défendu par Aldous [8], le cas critique paraît donc plus naturel pour étudier τ_n .

Bien que rarement nécessaire, l'hypothèse (b) évite des complications techniques car elle implique que $\mathbb{P}(\#\tau = n) > 0$ à partir d'un certain rang.

On peut traduire l'hypothèse (c) comme suit. Soit $(X_n)_{n \in \mathbb{N}}$ une marche aléatoire sur \mathbb{Z} , de valeur initiale $X_0 = 0$ et dont la loi de sauts est donnée par $\mathbb{P}(X_1 = k) = \mu(k+1)$ pour tout entier $k \geq -1$. L'hypothèse (c) est satisfaite si et seulement si il existe deux suites déterministes (a_n) et (b_n) de nombres réels avec $0 < a_n \rightarrow \infty$ telles que

$$\frac{X_n + b_n}{a_n} \xrightarrow{d} X_1, \quad \text{où} \quad \mathbb{E}[\exp(-\lambda X_1)] = \begin{cases} \exp(\lambda^\alpha) & \text{si } \alpha \in (1, 2], \\ \exp(\lambda \ln \lambda) & \text{si } \alpha = 1, \end{cases} \quad (1.1.5)$$

pour tout nombre réel $\lambda > 0$. En particulier, si la variance σ^2 de μ est finie alors (c) est vérifiée avec $\alpha = 2$, $b_n = 0$, et $a_n = \sigma\sqrt{n/2}$ pour tout $n \in \mathbb{N}^*$. D'après [23, Theorem 8.3.1], la condition (c) est satisfaite si et seulement si il existe une fonction à variations lentes L telle que

$$\mu([n, \infty)) \sim n^{-\alpha} L(n) \quad \text{si } \alpha \in [1, 2), \quad \text{et} \quad -1 + \sum_{k=0}^n k^2 \mu(k) \sim 2L(n) \quad \text{si } \alpha = 2. \quad (1.1.6)$$

Rappelons qu'une fonction mesurable et localement bornée $L : (0, \infty) \rightarrow (0, \infty)$ est dite à variations lentes lorsque pour tout $\lambda > 0$, $L(\lambda x) \sim L(x)$ quand $x \rightarrow \infty$.

Cas où $\alpha \in (1, 2]$. Si (1.1.5) a lieu alors des résultats standards garantissent que l'on peut choisir $b_n = 0$ et que $n^{-1/\alpha} a_n$ coïncide avec une fonction à variations lentes. Voir, par exemple, [74, Chapter XVII.5]. Comme rappelé dans la section suivante, sous les hypothèses (a), (b), et (c) avec $\alpha \in (1, 2]$, les arbres τ_n admettent une limite d'échelle en loi.

Cas où $\alpha = 1$. Si (1.1.5) a lieu alors on peut choisir $b_n = n\mathbb{E}[X_1 \mathbf{1}_{\{X_1 > a_n\}}]$ et na_n varie lentement. On renvoie à [74, Chapter IX.8, Equation (8.15)] par exemple. Les cas 1-stables, moins réguliers, ont été étudiés par Kortchemski & Richier [108] qui ont montré qu'il est possible de donner un sens à une limite d'échelle pour les τ_n en utilisant des arbres à boucles. Les arbres à boucles ne seront pas abordés dans cette thèse, mais on réfère à Curien & Kortchemski [42], Marzouk [131], Blanc-Renaudie [24], et à [101] pour plus d'informations à leur sujet et une étude de leurs convergences.

On renvoie à Bingham, Goldies & Teugels [23] et Feller [74] pour plus de détails sur les domaines d'attraction de lois stables.

1.1.3 Limites d'échelles par contour d'arbres de Galton-Watson critiques

Afin d'étudier le comportement asymptotique d'arbres aléatoires, il est souvent commode de les coder par des fonctions à valeurs réelles et de tirer partie des nombreux résultats d'analyse

stochastique et des théorèmes limites des processus réels. Dans ce qui suit, on rappelle la définition d'une de ces fonctions codantes et on présente les processus qui apparaissent en tant que ses limites d'échelle.

Codage d'un arbre discret par sa fonction de contour. Soit t un arbre *fini*. En tant qu'arbre planaire, on peut le plonger dans le demi-plan supérieur de façon à ce que la racine se trouve sur la frontière. On imagine alors une particule qui explore l'arbre en partant de la racine, dans le sens des aiguilles d'une montre, et en revenant sur ses pas le moins possible. Elle traverse chaque arête exactement deux fois (une première fois en s'éloignant de la racine, une deuxième fois en s'en rapprochant), si bien qu'il lui faut $2(\#t - 1)$ étapes pour revenir à la racine après avoir terminé son exploration. On appelle *exploration par contour de t* , notée $c(t) = (c_i(t); 0 \leq i \leq 2(\#t - 1))$, la suite des positions de la particule sur l'arbre. Plus formellement, on peut décrire $c(t)$ récursivement de la façon suivante : $c_0(t) = \emptyset$, $c_1(t) = (1)$ si $\#t \geq 2$, et pour tout $1 \leq i \leq 2(\#t - 1) - 1$, on a

- (i) si $c_{i-1}(t) = \overleftarrow{c_i(t)}$ et $k_{c_i(t)}(t) = 0$ alors $c_{i+1}(t) = \overleftarrow{c_i(t)}$,
- (ii) si $c_{i-1}(t) = \overleftarrow{c_i(t)}$ et $k_{c_i(t)}(t) \geq 1$ alors $c_{i+1}(t) = c_i(t) * (1)$,
- (iii) si $c_{i-1}(t) = c_i(t) * (j)$ avec $j < k_{c_i(t)}(t)$ alors $c_{i+1}(t) = c_i(t) * (j + 1)$,
- (iv) si $c_{i-1}(t) = c_i(t) * (j)$ avec $j = k_{c_i(t)}(t)$ alors $c_{i+1}(t) = \overleftarrow{c_i(t)}$.

On associe à l'arbre t un processus à valeurs réelles en considérant les hauteurs successives prises par l'exploration par contour. En d'autres termes, on pose $C_i(t) = |c_i(t)|$ pour tout entier $0 \leq i \leq 2(\#t - 1)$, puis on étend $C(t)$ en une fonction $(C_s(t))_{s \geq 0}$ sur \mathbb{R}_+ par interpolation linéaire sur $[0, 2(\#t - 1)]$ et en posant $C_s(t) = 0$ pour $s > 2(\#t - 1)$. La fonction continue $C(t) = (C_s(t))_{s \geq 0}$ est appelée la *fonction de contour de t* . Elle caractérise entièrement l'arbre t , voir Le Gall [78, Section 2], et permet de retrouver aisément sa distance de graphe (1.1.2) :

$$\forall 0 \leq i \leq j \leq 2(\#t - 1), \quad d_{\text{gr}}(c_i(t), c_j(t)) = C_i(t) + C_j(t) - 2 \inf_{[i,j]} C(t). \quad (1.1.7)$$

Limite d'échelle des fonctions de contour d'arbres de Galton-Watson conditionnés.

Cas où μ est à variance finie. Aldous [9] montre que la limite d'échelle des fonctions de contour d'arbres de Galton-Watson conditionnés à être grands et dont la loi de reproduction a une variance finie est l'*excursion brownienne standard* $\mathbf{e} = (\mathbf{e}_s)_{s \in [0,1]}$, dont une définition possible est

$$\forall s \in [0, 1], \quad \mathbf{e}_s = (D - G)^{-1/2} |B_{G+s(D-G)}|, \quad (1.1.8)$$

où $(B_s)_{s \geq 0}$ est un mouvement brownien réel standard issu de 0, et où $G = \sup\{s < 1 : B_s = 0\}$ et $D = \inf\{s > 1 : B_s = 0\}$. Plus précisément, on a l'énoncé suivant.

Théorème 1.1.3 (Aldous [9]) Soit μ une mesure de probabilité sur \mathbb{N} de variance σ^2 , et soit τ un arbre de Galton-Watson de loi de reproduction μ . On suppose que μ est critique et aperiodique, et que $\sigma^2 \in (0, \infty)$. Alors, la convergence en loi suivante

$$\left(\frac{\sigma}{2\sqrt{n}} C_{2ns}(\tau) \right)_{s \in [0,1]} \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \xrightarrow{d} \mathbf{e}$$

a lieu sur l'espace $\mathbb{C}([0, 1], \mathbb{R})$ des fonctions continues muni de la topologie de la convergence uniforme.

Cas où μ est dans le domaine d'attraction d'une loi stable. Duquesne [50] a généralisé le théorème 1.1.3 sous l'hypothèse que μ est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Le processus limite est alors l'excursion normalisée du processus de hauteur α -stable que l'on présente ci-après.

Soit $X = (X_s)_{s \geq 0}$ le processus canonique sur l'espace $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ des fonctions càdlàg muni de la topologie de Skorokhod usuelle. On suppose que sous \mathbb{P} , X est un processus de Lévy stable spectralement positif d'indice $\alpha \in (1, 2]$, et normalisé de sorte que

$$\forall \lambda, s \geq 0, \quad \mathbb{E}[\exp(-\lambda X_s)] = \exp(-s\lambda^\alpha).$$

Lorsque $\alpha = 2$, le processus $\frac{1}{\sqrt{2}}X$ est un mouvement brownien standard. Si $\alpha \in (1, 2)$, alors X n'a pas de composante brownienne et est caractérisé par sa mesure de Lévy :

$$\Pi(dr) = \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} r^{-1-\alpha} \mathbf{1}_{(0, \infty)}(r) dr,$$

où Γ est la fonction gamma usuelle. De plus, X jouit d'une propriété de mise à l'échelle : pour tout $\lambda > 0$, le processus $(\lambda^{-1/\alpha} X_{\lambda s}; s \geq 0)$ a la même loi que X . Plus d'informations sur les processus de Lévy stables sont présentées par Bertoin [19, Chapter VIII]. Pour tout $s \geq 0$, on pose $I_s = \inf_{[0, s]} X$ pour définir le processus infimum de X . L'absence de sauts négatifs de X implique la continuité de I . Par des arguments standards de théorie des fluctuations (voir Bertoin [19, Chapter VI]), $X - I$ est un processus de Markov fort pour lequel le point 0 est régulier et instantané. D'après [19, Theorem VII.1], on peut et on va choisir $-I$ en tant que temps local de $X - I$ en 0. On note $(g_j, d_j), j \in J$, les intervalles d'excursions de $X - I$ au dessus 0, et on pose $\omega_s^j = X_{\min(g_j + s, d_j)} - X_{g_j}$ pour tous $j \in J$ et $s \geq 0$. Comme $\inf X = -\infty$ presque sûrement d'après [19, Chapter VIII], la mesure ponctuelle aléatoire

$$\mathcal{N} = \sum_{j \in J} \delta_{(-I_{g_j}, \omega^j)}$$

est une mesure de Poisson sur $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ d'intensité $ds \mathbf{N}_\alpha(d\omega)$, où \mathbf{N}_α est une mesure sigma-finie sur $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ appelée la *mesure d'excursion*.

Le Gall & Le Jan [81] (voir aussi Duquesne & Le Gall [61]) ont construit un processus continu et positif $H = (H_s)_{s \geq 0}$ tel que pour tout $s \in \mathbb{R}_+$, la convergence suivante a lieu en \mathbb{P} -probabilité et en \mathbf{N}_α -mesure :

$$H_s = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{X_r < I_s + \varepsilon\}} dr, \quad \text{où } I_s^r = \inf_{[r, s]} X \text{ pour tout } r \in [0, s]. \quad (1.1.9)$$

Le processus H est appelé le *processus de hauteur stable d'indice α* . La valeur H_s est une fonctionnelle mesurable de X apparentée au temps local au temps s au supremum du processus dual $(X_s - X_{s-r})_{r \in [0, s]}$. Lorsque $\alpha = 2$, on a de manière explicite $H = X - I$ sous \mathbb{P} , et $H = X$ sous \mathbf{N}_2 . Cependant, X et H sont distincts quand $\alpha \in (1, 2)$, le premier étant un processus à sauts alors que le deuxième est continu. De plus, comme cas particulier de Duquesne & Le Gall [61, Theorem 1.4.4], on a le résultat de régularité suivant.

Proposition 1.1.4 *Pour tout $\gamma \in (0, 1 - 1/\alpha)$, H est localement γ -hölderien \mathbb{P} -presque sûrement. Pour tout $\gamma \in (0, 1 - 1/\alpha)$, H est γ -hölderien \mathbf{N}_α -presque partout.*

Il est possible de prouver à partir de (1.1.9), voir Duquesne & Le Gall [61, Chapter 1], que sous \mathbb{P} , les ensembles des zéros de $X - I$ et de H sont égaux presque sûrement. Sous \mathbf{N}_α , on définit alors la *durée de vie* commune de X et de H comme

$$\zeta = \sup\{s \in \mathbb{R}_+ : X_s > 0\} = \sup\{s \in \mathbb{R}_+ : H_s > 0\}.$$

Un argument élémentaire de la théorie des fluctuations, voir par exemple [19, Chapter VIII.4], permet d'obtenir que $\mathbf{N}_\alpha(\zeta = 0) = 0$ et que

$$\forall \ell > 0, \quad \mathbf{N}_\alpha(\zeta > \ell) = \frac{1}{\Gamma(1 - 1/\alpha)} \ell^{-1/\alpha}. \quad (1.1.10)$$

On observe grâce à (1.1.9) et à la propriété de mise à l'échelle de X que pour tout $\lambda > 0$, les processus $(\lambda^{1/\alpha-1}H_{\lambda s}; s \geq 0)$ et H ont la même loi sous \mathbb{P} . Il s'ensuit que la loi du processus $(\zeta^{1/\alpha-1}H_{\zeta s}; s \geq 0)$ sous la mesure de probabilité $\mathbf{N}_\alpha(\cdot | \zeta > \ell)$ ne dépend pas de $\ell > 0$. Celle-ci est appelée la loi de l'*excursion normalisée du processus de hauteur α -stable* et on la note par $\mathbf{N}_\alpha(dH | \zeta = 1)$ ici. Dans le cas $\alpha = 2$, $\mathbf{N}_2(dH | \zeta = 1)$ est la loi de $\sqrt{2}e$, où e désigne l'excursion brownienne standard. Enfin, (1.1.10) permet de prouver que pour toute fonction mesurable et positive $F : \mathbb{C}(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}_+$, on a la formule de désintégration

$$\mathbf{N}_\alpha[F(H)] = \frac{1}{\alpha\Gamma(1 - 1/\alpha)} \int_{\mathbb{R}_+} \mathbf{N}_\alpha[F(\ell^{1-1/\alpha}H_{s/\ell}; s \geq 0) | \zeta = 1] \frac{d\ell}{\ell^{1+1/\alpha}}. \quad (1.1.11)$$

On peut finalement énoncer le théorème de Duquesne [50] mentionné plus haut.

Théorème 1.1.5 (Duquesne [50]) *Soit μ une mesure de probabilité sur \mathbb{N} , et soit τ un arbre de Galton-Watson de loi de reproduction μ . On suppose que μ est critique et non-triviale, apériodique, et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Alors, il existe une suite déterministe de réels strictement positifs $a_n \rightarrow \infty$, donnée par (1.1.5), telle que la convergence en loi suivante*

$$\left(\frac{a_n}{n} C_{2n_s}(\tau) \right)_{s \in [0,1]} \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \xrightarrow{d} H \text{ sous } \mathbf{N}_\alpha(dH | \zeta = 1)$$

a lieu sur l'espace $\mathbb{C}([0, 1], \mathbb{R})$ des fonctions continues muni de la topologie de la convergence uniforme.

Mentionnons que le théorème 1.1.5 reste valide sans l'hypothèse d'apériodicité si on restreint la convergence à la suite des entiers n tels que $\mathbb{P}(\#\tau = n) > 0$.

1.1.4 Arbres réels compacts enracinés mesurés

On peut exprimer les théorèmes limites sur les fonctions de contour énoncés ci-dessus de manière intrinsèque en termes de convergences d'espaces métriques compacts.

Distance de Gromov-Hausdorff-Prokhorov. On dit que (E, d, a, μ) est un *espace métrique compact enraciné mesuré* lorsque (E, d) est un espace métrique compact, muni d'un point distingué $a \in E$ que l'on nomme sa *racine*, et d'une mesure borélienne finie μ sur E . De même, on dit que (E, d, a) est un *espace métrique compact enraciné*. Deux espaces métriques compacts enracinés mesurés (E_1, d_1, a_1, μ_1) et (E_2, d_2, a_2, μ_2) sont *équivalents* lorsque il existe une isométrie bijective ϕ de E_1 vers E_2 telle que $\phi(a_1) = a_2$ et telle que la mesure image de μ_1 par ϕ est égale à μ_2 . On note \mathbb{K} l'ensemble des classes d'équivalence d'espaces métriques compacts enracinés mesurés. Tant que cela reste clair selon le contexte, on désignera indifféremment un espace métrique compact enraciné mesuré (E, d, a, μ) et sa classe d'équivalence dans \mathbb{K} par son ensemble sous-jacent E .

Soit (E_1, d_1, a_1, μ_1) et (E_2, d_2, a_2, μ_2) deux espaces métriques compacts enracinés mesurés, la *distance de Gromov-Hausdorff-Prokhorov* entre eux deux est

$$d_{\text{GHP}}(E_1, E_2) = \inf_{\phi_1, \phi_2} \max(d(\phi_1(a_1), \phi_2(a_2)), d_{\text{Haus}}^E(\phi_1(E_1), \phi_2(E_2)), d_{\text{Prok}}^E(\mu_1 \circ \phi_1^{-1}, \mu_2 \circ \phi_2^{-1}))$$

où l'infimum est pris sur tous les plongements isométriques $\phi_1 : E_1 \rightarrow E$ et $\phi_2 : E_2 \rightarrow E$ dans un même espace métrique polonais (E, d) , où d_{Haus}^E est la distance de Hausdorff sur les sous-ensembles compacts de E , et où d_{Prok}^E est la distance de Prokhorov sur les mesures boréliennes finies de E . Il est clair que $d_{\text{GHP}}(E_1, E_2)$ ne dépend que des classes d'équivalences de E_1 et E_2 dans \mathbb{K} . Abraham, Delmas & Hoscheit [4, Theorem 2.5] ont démontré que d_{GHP} induit une distance sur \mathbb{K} que l'on note de la même façon et que l'espace métrique $(\mathbb{K}, d_{\text{GHP}})$ est séparable et complet.

Il peut être commode d'utiliser une autre expression de la distance de Gromov-Hausdorff-Prokhorov ne faisant pas appel à de tierces espaces métriques. Une *correspondance* entre E_1 et E_2 est un sous-ensemble \mathcal{R} de $E_1 \times E_2$ tel que $(a_1, a_2) \in \mathcal{R}$ et qui vérifie que pour tous $x_1 \in E_1$ et $y_2 \in E_2$, il existe $x_2 \in E_2$ et $y_1 \in E_1$ tels que (x_1, x_2) et (y_1, y_2) sont dans \mathcal{R} . La *distorsion* de la correspondance \mathcal{R} est définie par

$$\text{dis}(\mathcal{R}) = \sup \{ |d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2) \in \mathcal{R} \text{ et } (y_1, y_2) \in \mathcal{R} \}.$$

Soit ν une mesure borélienne finie sur $E_1 \times E_2$, la *divergence* (ou discrepancy en anglais) de ν par rapport à μ_1 et μ_2 est

$$\text{Dsp}(\nu ; \mu_1, \mu_2) = \sup_{\substack{B_1 \subset E_1 \\ \text{borélien}}} |\nu(B_1 \times E_2) - \mu_1(B_1)| + \sup_{\substack{B_2 \subset E_2 \\ \text{borélien}}} |\nu(E_1 \times B_2) - \mu_2(B_2)|.$$

Khezeli [105, Theorem 3.5] montre qu'on peut exprimer d_{GHP} comme suit :

$$d_{\text{GHP}}(E_1, E_2) = \inf_{\mathcal{R}, \nu} \max \left(\frac{1}{2} \text{dis}(\mathcal{R}), \text{Dsp}(\nu ; \mu_1, \mu_2) + \nu((E_1 \times E_2) \setminus \mathcal{R}) \right), \quad (1.1.12)$$

où l'infimum est pris sur toutes les correspondances compactes \mathcal{R} entre E_1 et E_2 , et toutes les mesures boréliennes finies ν sur $E_1 \times E_2$.

Codages d'arbres réels par des excursions. Les espaces métriques apparaissant en tant que limites d'échelle d'arbres (discrets) sont également acycliques et connexes, c'est-à-dire sont des arbres réels, définis comme suit.

Définition 1.1.6 Un espace métrique (T, d) est un *arbre réel* lorsque pour tous $x, y \in T$:

- (a) il existe une unique isométrie $g_{x,y} : [0, d(x, y)] \rightarrow T$ telle que $g_{x,y}(0) = x$ et $g_{x,y}(d(x, y)) = y$, que l'on appelle la *géodésique de x à y* ,
- (b) toutes les fonctions continues injectives $\phi : [0, 1] \rightarrow T$ avec $\phi(0) = x$ et $\phi(1) = y$ partagent la même image que l'on note $\llbracket x, y \rrbracket = \phi([0, 1]) = g_{x,y}([0, d(x, y)])$.

□

Alternativement, un espace métrique connexe (T, d) est un arbre réel si et seulement si

$$d(x_1, x_2) + d(x_3, x_4) \leq \max (d(x_1, x_3) + d(x_2, x_4), d(x_1, x_4) + d(x_2, x_3)) \quad (1.1.13)$$

pour tous $x_1, x_2, x_3, x_4 \in T$. L'inégalité (1.1.13) est appelée l'inégalité des quatre points. Voir Buneman [30], et on renvoie à Evans [70] pour de plus amples informations. Un espace métrique compact enraciné mesuré (T, d, ρ, μ) est un *arbre réel compact enraciné mesuré* lorsque (T, d) est un arbre réel. On note $\mathbb{T}_{\mathbb{R}}$ le sous-espace de \mathbb{K} formé des classes d'équivalence des arbres réels compacts enracinés mesurés. L'inégalité (1.1.13) permet de montrer que $\mathbb{T}_{\mathbb{R}}$ est un sous-espace fermé de $(\mathbb{K}, d_{\text{GHP}})$, donc $(\mathbb{T}_{\mathbb{R}}, d_{\text{GHP}})$ est aussi séparable et complet. Voir Evans, Pitman & Winter [71, Theorem 1].

De la même manière que la fonction de contour caractérise un arbre (discret), une fonction continue code naturellement un arbre réel. Soit $\ell \in \mathbb{R}_+$ et soit $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ une fonction continue. Pour tous $s_2 \geq s_1 \geq 0$, on pose par analogie avec (1.1.7)

$$d_f(s_1, s_2) = d_f(s_2, s_1) = f(s_1) + f(s_2) - 2 \inf_{[s_1, s_2]} f. \quad (1.1.14)$$

Duquesne & Le Gall [57, Theorem 2.1] montrent que d_f induit un arbre réel. Plus précisément, Evans & Winter [72, Lemma 3.1] prouvent que d_f est une pseudo-distance continue sur \mathbb{R}_+ qui satisfait l'inégalité des quatre points (1.1.13). Ainsi, écrire $s_1 \sim_f s_2$ lorsque $d_f(s_1, s_2) = 0$ définit une relation d'équivalence \sim_f sur \mathbb{R}_+ . Sur l'espace quotient $\mathcal{T}_f^\ell = [0, \ell] / \sim_f$, d_f induit une vraie distance que l'on note d_f^ℓ . Alors, $(\mathcal{T}_f^\ell, d_f^\ell)$ est un arbre réel compact. On note \mathfrak{p}_f^ℓ la projection canonique de $[0, \ell]$ sur \mathcal{T}_f^ℓ , puis on pose $\rho_f^\ell = \mathfrak{p}_f^\ell(0)$, et on définit μ_f^ℓ comme la mesure image par \mathfrak{p}_f^ℓ de la mesure de Lebesgue sur $[0, \ell]$. On appelle

$$(\mathcal{T}_f^\ell, d_f^\ell, \rho_f^\ell, \mu_f^\ell) \quad (1.1.15)$$

l'arbre réel (compact enraciné mesuré) codé par (f, ℓ) . L'application $(f, \ell) \mapsto \mathcal{T}_f^\ell \in \mathbb{T}_\mathbb{R}$ est continue pour la topologie de la convergence uniforme sur les compacts d'une part, et pour la distance de Gromov-Hausdorff-Prokhorov d'autre part, car (1.1.12) permet de vérifier que

$$d_{\text{GHP}}(\mathcal{T}_{f_1}^{\ell_1}, \mathcal{T}_{f_2}^{\ell_2}) \leq |\ell_1 - \ell_2| + 2 \sup_{0 \leq s \leq \ell_1 \vee \ell_2} |f_1(s) - f_2(s)|. \quad (1.1.16)$$

1.1.5 Arbres stables continus, et arbres de Galton-Watson stables

Tout arbre fini t muni de la distance de graphe d_{gr} donnée par (1.1.2), de sa racine \emptyset , et de sa mesure de comptage $\sum_{u \in t} \delta_u$ devient un espace métrique compact enraciné mesuré, mais qui n'est pas un arbre réel. En effet, un tel espace ne peut être connexe que s'il est réduit à un unique point. Néanmoins, remplacer chaque arête de t par un segment réel de longueur $\lambda > 0$ définit un vrai arbre réel compact. D'après (1.1.7) et (1.1.14), celui-ci est exactement l'arbre réel codé par une version mise à l'échelle (en temps et en espace) de la fonction de contour $C(t)$. Par continuité de l'application de codage d'un arbre réel par une fonction continue, on obtient que sous les hypothèses du théorème 1.1.5 de Duquesne [50], si τ est un arbre de Galton-Watson de loi de reproduction μ alors la convergence en loi

$$\left(\tau, \frac{a_n}{n} d_{\text{gr}}, \emptyset, \frac{1}{n} \sum_{u \in \tau} \delta_u \right) \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \xrightarrow{d} \mathcal{T}_H^1 \text{ sous } \mathbf{N}_\alpha(\text{dH} | \zeta = 1) \quad (1.1.17)$$

a lieu sur l'espace \mathbb{K} muni de la distance de Gromov-Hausdorff-Prokhorov, où $\alpha \in (1, 2]$ et où $\mathbf{N}_\alpha(\text{dH} | \zeta = 1)$ est la loi de l'excursion normalisée du processus de hauteur α -stable. La convergence (1.1.17) reste valide pour la sous-suite des entiers tels que $\mathbb{P}(\#\tau = n) > 0$, même quand μ n'est pas a périodique.

Arbres stables. L'espace aléatoire limite dans (1.1.17) est appelé *l'arbre α -stable* : autrement dit, un arbre α -stable est une variable aléatoire sur $\mathbb{T}_\mathbb{R}$ qui a la même loi que \mathcal{T}_H^1 sous $\mathbf{N}_\alpha(\text{dH} | \zeta = 1)$. À un facteur $\frac{1}{\sqrt{2}}$ sur les distances près, l'arbre 2-stable est l'arbre réel \mathcal{T}_{2e}^1 codé par le double de l'excursion brownienne standard e , qui est *l'arbre brownien* introduit par Aldous [7, 8, 9] sous le nom de *Continuum Random Tree* (voir aussi Le Gall [80]).

Les arbres stables entretiennent des liens étroits avec les processus de coalescence et de fragmentation (voir Bertoin [20] pour une introduction générale). En effet, les arbres stables sont

des membres de la classe des arbres de fragmentation : des arbres réels aléatoires qui décrivent la généalogie sous-jacente d'un processus de fragmentation autosimilaire (voir Haas & Miermont [89]) et qui apparaissent notamment en tant que limites d'échelle des arbres de Markov (voir Haas & Miermont [90]). Plus précisément, Haas, Pitman & Winkel [150] montrent que les arbres stables sont exactement les arbres de fragmentation invariants par réenracinement uniforme. On renvoie à Goldschmidt & Haas [84] pour une étude fine des processus de fragmentation stables s'appuyant sur les arbres stables. On mentionne également les liens des arbres stables avec les graphes stables inhomogènes, mis en évidence par Conchon-Kerjan & Goldschmidt [37] et Goldschmidt, Haas & Sénizergues [85] par exemple.

Si σ est un point d'un arbre α -stable \mathcal{T}_{nr} , alors le nombre de composantes connexes de $\mathcal{T}_{nr} \setminus \{\sigma\}$ appartient à $\{1, 2, 3\}$ si $\alpha = 2$, ou à $\{1, 2, \infty\}$ si $\alpha \in (1, 2)$ (voir Duquesne & Le Gall [57, Theorem 4.6]). Enfin, les arbres stables présentent des propriétés fractales : leur dimension de Hausdorff est égale à $\frac{\alpha}{\alpha-1}$ par exemple, comme prouvé indépendamment par Duquesne & Le Gall [57] et par Haas & Miermont [88]. De plus, un arbre stable est un arbre continuum au sens de Aldous [9] car sa mesure, dite *de masse*, est diffuse et supportée par l'ensemble de ses feuilles (i.e. les points dont le retrait ne déconnecte pas l'arbre). Duquesne & Le Gall [57, Section 4] justifient que la mesure de masse ne dépend que de la métrique. Elle peut même être décrite géométriquement, soit en tant que mesure de Hausdorff dans le cas $\alpha = 2$, soit en tant que mesure de packing dans les cas $\alpha \in (1, 2)$. Voir respectivement Duquesne & Le Gall [58] et Duquesne [54]. Enfin, les lois de la hauteur et du diamètre d'un arbre stable sont calculées par Duquesne & Minmin [62].

Arbres de Galton-Watson stables. Soit $(\mathcal{T}_{nr}, d_{nr}, \rho_{nr}, \mu_{nr})$ un arbre α -stable, on tire de façon indépendante et selon la mesure de probabilité μ_{nr} une suite de points $(\sigma_j)_{j \geq 1}$ de \mathcal{T}_{nr} . Pour tout $n \in \mathbb{N}^*$, l'arbre réel compact $\bigcup_{1 \leq j \leq n} \llbracket \rho_{nr}, \sigma_j \rrbracket$ a un nombre fini de feuilles. En se concentrant sur sa forme combinatoire et en oubliant les longueurs de ses branches, on peut donc le voir comme un arbre planaire t_n au sens de la définition 1.1.1. Alors, t_n est distribué comme un arbre de Galton-Watson conditionné à avoir n feuilles et dont la loi de reproduction μ_α est caractérisée par la fonction génératrice

$$\forall s \in [0, 1], \quad \varphi_\alpha(s) := \sum_{k \in \mathbb{N}} s^k \mu_\alpha(k) = s + \frac{1}{\alpha}(1-s)^\alpha. \quad (1.1.18)$$

On renvoie à Duquesne & Le Gall [61, Theorem 3.3.3] pour une preuve de cet énoncé et une construction précise de t_n . En outre, la suite $(t_n)_{n \geq 1}$ admet une description combinatoire directe sans passer par l'arbre α -stable. C'est une chaîne de Markov sur l'espace des arbres discrets dont les probabilités de transitions sont données par l'algorithme de Rémy [139] dans le cas $\alpha = 2$ ou de Marchal [128] dans les cas $\alpha \in (1, 2)$. Sans rentrer dans les détails, on obtient t_{n+1} en attachant une nouvelle arête soit au milieu d'une arête de t_n soit directement sur un sommet de t_n , le choix se faisant selon des poids qui ne dépendent que de α et des degrés des sommets de t_n . Ces algorithmes sont un outil efficace dans l'étude des arbres stables car Curien & Haas [41] prouvent que, après renormalisation, $(t_n)_{n \geq 1}$ converge presque sûrement vers \mathcal{T}_{nr} pour la distance de Gromov-Hausdorff-Prokhorov. Par exemple, Curien & Haas [41] montrent que si $1 < \alpha_1 \leq \alpha_2 \leq 2$ alors un multiple d'un arbre α_2 -stable se plonge isométriquement dans un arbre α_1 -stable.

Définition 1.1.7 Soit $\alpha \in (1, 2]$, un *arbre de Galton-Watson α -stable* est un arbre de Galton-Watson de loi de reproduction μ_α donnée par (1.1.18). En particulier, si $\alpha = 2$ alors μ_2 est la loi de reproduction binaire critique, c'est-à-dire $\mu_2(0) = \mu_2(2) = \frac{1}{2}$. \square

On observe $\mu_\alpha(1) = 0$ et que μ_α est critique, c'est-à-dire $\sum_{k \in \mathbb{N}} k \mu_\alpha(k) = 1$. L'arbre de Galton-Watson α -stable joue le rôle d'objet de référence parmi tous les arbres de Galton-Watson critiques dont la loi de reproduction est dans le domaine d'attraction d'une loi α -stable, en tant que

marginales fini-dimensionnelles de leur limite d'échelle universelle. Les arbres de Galton-Watson stables héritent ainsi de plusieurs propriétés de stabilité ou d'invariance, mentionnons par exemple les travaux de Duquesne & Winkel [63] et de Kovchegov, Xu & Zaliapin [109]. On peut en outre citer de nombreuses autres apparitions de ces arbres : Zolotarev [152], Neveu [134], Le Jan [120], ou Abraham & Delmas [1]. On se restreint nous-même au cas des arbres de Galton-Watson stables pour étudier les fluctuations du nombre de Horton-Strahler dans le chapitre 5 de cette thèse.

1.2 Marches branchantes critiques

1.2.1 Modèle et motivations

Une marche branchante est un modèle simple d'une population qui explore un environnement tout en se reproduisant. Soit un arbre t vu comme l'arbre généalogique d'une population dont l'ancêtre est représenté par la racine \emptyset , et dont chaque individu $u \in t$ choisit sa position Y_u dans un espace polonais E uniquement à partir de celle de son parent \overleftarrow{u} . Plus précisément, la loi de la marche branchante $(Y_u)_{u \in t}$ de position initiale $z \in E$ et de noyau de transition $p(x, dy)$ est donnée par

$$\mathbb{E}[F(Y_u; u \in t)] = \int_{E^t} F(y_u; u \in t) \delta_z(dy_\emptyset) \prod_{v \in t \setminus \{\emptyset\}} p(y_{\overleftarrow{v}}, dy_v) \quad (1.2.1)$$

pour toute fonction $F : E^t \rightarrow \mathbb{R}$ mesurable et bornée.

Depuis les années 1970, les marches branchantes ont fait l'objet de nombreux travaux, en lien notamment avec les superprocessus, les solutions d'ondes progressives pour des équations semi-linéaires aux dérivées partielles (FKPP), et divers modèles de mécanique statistique (champ libre Gaussien, cascades de Mandelbrot, processus de contact, modèle du votant...). Sans pouvoir donner une liste exhaustive de ces liens, on renvoie au livre de Shi [140] pour une vue d'ensemble. Pour citer toutefois quelques travaux où la généalogie est infinie, on mentionne l'étude des particules extrêmes de marches branchantes à valeurs réelles de Biggins [21] et Bramson [27, 26], mais également les résultats de Gouëzel, Hueter, Lalley, Sellke [86, 94, 114, 115] sur les marches branchantes à valeurs dans des espaces hyperboliques, ainsi que de Benjamini & Müller [16] et de Liggett [122] pour une étude des marches branchantes sur des arbres.

Ici, on étudie des marches branchantes dans un régime critique où l'arbre indexant la marche $(Y_u)_{u \in \tau_n}$ est un arbre de Galton-Watson critique conditionné à avoir n sommets, avec n grand. Notre objet d'intérêt principal est l'ensemble $\mathcal{R}_n = \{Y_u : u \in \tau_n\}$ des sites visités, aussi appelé la *trace* (ou *range* en anglais) de la marche branchante. Plus précisément, on cherche à répondre aux questions suivantes.

- (A) Combien de sites différents la marche branchante visite-elle ? Autrement dit, $\#\mathcal{R}_n$ converge-t-il vers une limite non-triviale après mise à l'échelle ?
- (B) À quoi ressemble la trace en tant qu'espace métrique compact ? Admet-elle une limite d'échelle non dégénérée ?

Dans cette thèse, on se concentre sur l'étude nouvelle d'une marche branchante critique sur un arbre sur-critique infini. En revanche, dans le cas où l'espace est \mathbb{Z}^k et les transitions sont celles de la marche simple aux plus proches voisins, la question (A) a été traitée par Le Gall & Lin [83, 82]. Dans ce même cadre, une réponse à la question (B) en termes de fonctions de contour est apportée par Janson & Marckert [97] et par Marzouk [130]. Les objets limites apparaissant dans leurs travaux interviennent également dans nos résultats : on les présente dans la section suivante.

Cependant, on peut dès déjà donner les réponses aux questions (A) et (B) posées pour la marche aléatoire $(X_n)_{n \in \mathbb{N}}$ (non-branchante) simple aux plus proches voisins sur \mathbb{Z}^k . D'après le théorème

de Donsker, la limite d'échelle de sa trace $\{X_j : 0 \leq j \leq n\}$ jusqu'au temps n est l'image d'un mouvement brownien k -dimensionnel jusqu'au temps 1. En dimension $k = 1$, le nombre de points de la trace correspond au diamètre de celle-ci, ce qui permet directement de répondre à la question (A). De plus, le cas des dimensions $k \geq 2$ est traité par Dvoretzky & Erdős [65]. On résume les réponses apportées à la question (A) dans le théorème ci-dessous.

Théorème 1.2.1 (Dvoretzky & Erdős [65]) *Soit $(X_n)_{n \in \mathbb{N}}$ une marche aléatoire (non-branchante) simple aux plus proches voisins sur \mathbb{Z}^k . On pose $R_n = \#\{X_j : 0 \leq j \leq n\}$ pour tout $n \in \mathbb{N}$. On a les convergences suivantes.*

- (i) *Si $k \geq 3$, alors on a $\frac{1}{n}R_n \rightarrow c_k$ presque sûrement, où $c_k \in (0, 1)$ est la probabilité que la marche aléatoire (X_n) ne revienne jamais à son point de départ.*
- (ii) *Si $k = 2$, alors on a $\frac{\ln n}{n}R_n \rightarrow \pi$ presque sûrement.*
- (iii) *Si $k = 1$, alors la convergence $\frac{1}{\sqrt{n}}R_n \xrightarrow{d} \sup_{[0,1]} B - \inf_{[0,1]} B$ a lieu en loi, où B est un mouvement brownien réel standard.*

La méthode de Dvoretzky & Erdős [65] pour prouver (i) et (ii) s'appuie sur des estimées de moments d'ordre 1 et 2 et se généralise à des marches aléatoires dont les lois de saut sont centrées et de variance finie. Le point (i) du théorème peut aussi être obtenu comme conséquence du théorème ergodique sous-additif (voir par exemple Spitzer [144]). Cette approche permet en fait de montrer que pour tout $k \in \mathbb{N}^*$ et toute marche aléatoire $(X_n)_{n \in \mathbb{N}}$ sur \mathbb{Z}^k , on a que $\frac{1}{n}\#\{X_j : 0 \leq j \leq n\}$ converge presque sûrement vers la probabilité de non-retour de la marche à l'origine, cette dernière étant nulle si et seulement si la marche est récurrente.

1.2.2 Le serpent brownien et la Integrated Super-Brownian Excursion

Le théorème de Donsker et le théorème 1.1.5 de Duquesne donnent l'intuition que la limite d'échelle d'une marche branchante critique simple sur \mathbb{Z}^k devrait être un mouvement brownien indexé par un arbre stable. Cette intuition, bien que correcte, requiert une construction formelle.

On note \mathbf{C}_0^k l'espace des fonctions continues de \mathbb{R}_+ vers \mathbb{R}^k , muni de la distance suivante

$$\forall f, g \in \mathbf{C}_0^k, \quad d_\infty^{\text{loc}}(f, g) = \sum_{n \in \mathbb{N}^*} 2^{-n} \min(1, \sup_{r \in [0, n]} \|f(r) - g(r)\|), \quad (1.2.2)$$

où $\|\cdot\|$ désigne la norme euclidienne sur \mathbb{R}^k , qui induit la topologie de la convergence uniforme sur les compacts. L'espace métrique $(\mathbf{C}_0^k, d_\infty^{\text{loc}})$ est séparable et complet. De plus, on désigne par $\mathbb{C}([0, 1], \mathbb{R})$ l'espace des fonctions $h = (h_s)_{s \in [0, 1]}$ continues à valeurs dans \mathbb{R} , et par $\mathbb{C}([0, 1], \mathbf{C}_0^k)$ l'espace des fonctions $w = (w_s)_{s \in [0, 1]}$ continues à valeurs dans \mathbf{C}_0^k . On dote alors l'espace produit $\mathbb{C}([0, 1], \mathbb{R}) \times \mathbb{C}([0, 1], \mathbf{C}_0^k)$ d'une structure d'espace métrique séparable et complet en le munissant de la distance suivante : pour tous $(h, w), (h', w') \in \mathbb{C}([0, 1], \mathbb{R}) \times \mathbb{C}([0, 1], \mathbf{C}_0^k)$, on pose

$$d_\Sigma((h, w), (h', w')) = \sup_{s \in [0, 1]} |h_s - h'_s| + \sup_{s \in [0, 1]} d_\infty^{\text{loc}}(w_s, w'_s). \quad (1.2.3)$$

Introduits par Le Gall [77], les *serpents* sont des processus à valeurs dans l'ensemble des trajectoires continues arrêtées qui offrent une paramétrisation commode de processus indexés par un arbre réel.

Définition 1.2.2 Un *serpent* à valeurs dans \mathbb{R}^k est un élément (h, w) de $\mathbb{C}([0, 1], \mathbb{R}) \times \mathbb{C}([0, 1], \mathbf{C}_0^k)$ qui satisfait les conditions suivantes.

- (a) Pour tout $s \in [0, 1]$, on a $h_s \geq 0$ et $h_0 = 0$.
- (b) Pour tout $s \in [0, 1]$ et tout $r \in [h_s, \infty)$, on a $w_s(r) = w_s(h_s) =: \hat{w}_s$.

(c) Pour tous $0 \leq s_1 \leq s_2 \leq 1$ et tout $0 \leq r \leq \inf_{[s_1, s_2]} h$, on a $w_{s_1}(r) = w_{s_2}(r)$.

Le couple de fonctions (h, \widehat{w}) est appelée l'*extrémité* du serpent (h, w) . L'ensemble Σ^k des serpents à valeurs dans \mathbb{R}^k est un sous-espace fermé de $\mathbb{C}([0, 1], \mathbb{R}) \times \mathbb{C}([0, 1], \mathbf{C}_0^k)$ pour la distance d_Σ . \square

Par un léger abus de langage, la fonction w à valeurs dans \mathbf{C}_0^k est elle aussi parfois appelée un serpent. Prenons quelques instants pour expliciter pourquoi un serpent est une paramétrisation d'une fonction continue définie sur un arbre réel. Soit (h, w) un serpent à valeurs dans \mathbb{R}^k . On rappelle que d_h est la pseudo-distance sur $[0, 1]$ codée par h comme dans (1.1.14) et que $(\mathcal{T}_h^1, d_h^1, \rho_h^1, \mu_h^1)$ est l'arbre réel compact enraciné mesuré codé par $(h, 1)$ comme défini par (1.1.15). La propriété (c) implique que

$$\forall s_1, s_2 \in [0, 1], \quad \text{si } d_h(s_1, s_2) = 0 \quad \text{alors} \quad \widehat{w}_{s_1} = \widehat{w}_{s_2}. \quad (1.2.4)$$

Par continuité de la projection canonique $p_h^1 : [0, 1] \rightarrow \mathcal{T}_h^1$, \widehat{w} induit alors une fonction \widetilde{w} continue sur \mathcal{T}_h^1 en posant $\widetilde{w}(p_h^1(s)) = \widehat{w}_s$ pour tout $s \in [0, 1]$. Réciproquement, si $h : [0, 1] \rightarrow \mathbb{R}$ est une fonction continue qui vérifie la condition (a), alors toute fonction continue $\widetilde{w} : \mathcal{T}_h^1 \rightarrow \mathbb{R}^k$ induit un serpent (h, w) en posant

$$\forall s \in [0, 1], \forall r \in \mathbb{R}_+, \quad w_s(r) = \widetilde{w}\left(g_{\rho_h^1, p_h^1(s)}(r \wedge h_s)\right),$$

où $g_{x,y} : [0, d_h^1(x, y)] \rightarrow \mathcal{T}_h^1$ désigne la géodésique de x à y sur \mathcal{T}_h^1 (voir la définition 1.1.6).

Après ces considérations déterministes, on présente les *serpents browniens* introduits par Le Gall [77] (voir aussi Le Gall [79] et Duquesne & Le Gall [61]), qui paramétrisent des processus analogues au mouvement brownien mais indexés par des arbres réels. Ces objets sont des processus de Markov inhomogènes sur l'espace \mathbf{C}_0^k dont on construit les noyaux de transition ci-après. Pour tout $f \in \mathbf{C}_0^k$ et tous $0 \leq a \leq b$, on note $R_{a,b}(f, df')$ la loi sur \mathbf{C}_0^k du processus $F(\cdot)$ caractérisé par

- $F(r) = f(r)$ pour tout $r \in [0, a]$,
- le processus $(F(a+r))_{r \geq 0}$ a la même loi que $(f(a) + B_{r \wedge (b-a)})_{r \geq 0}$, où B est un mouvement brownien k -dimensionnel standard.

On fixe une fonction $h : [0, 1] \rightarrow \mathbb{R}_+$ continue avec $h(0) = 0$. Pour tous $0 \leq s_1 \leq s_2 \leq 1$, on écrit $m_h(s_1, s_2) = \inf_{[s_1, s_2]} h$ pour alléger les notations. On dit qu'un processus $(W_s^h)_{s \in [0, 1]}$ à valeurs dans \mathbf{C}_0^k est un *serpent brownien k -dimensionnel dirigé par h* lorsque $W_0^h(r) = 0$ pour tout $r \in \mathbb{R}_+$ et lorsque pour tous $0 \leq s_1 \leq \dots \leq s_p \leq 1$, la loi de $(W_{s_k}^h)_{1 \leq k \leq p}$ est

$$R_{m_h(0, s_1), h(s_1)}(W_0, dw_1) R_{m_h(s_1, s_2), h(s_2)}(w_1, dw_2) \dots R_{m_h(s_{p-1}, s_p), h(s_p)}(w_{p-1}, dw_p). \quad (1.2.5)$$

D'après Duquesne & Le Gall [61, Proposition 4.4.1] (voir aussi Duquesne & Le Gall [57] pour une autre approche), si h est höldérienne alors il existe un serpent brownien k -dimensionnel dirigé par h qui est presque sûrement continu par rapport à la distance d_∞^{loc} donnée par (1.2.2). Autrement dit, il existe alors une variable aléatoire (h, W^h) sur l'espace Σ^k des serpents à valeurs dans \mathbb{R}^k telle que $W_0^h(r) = 0$ pour tout $r \in \mathbb{R}_+$ et dont les marginales de dimensions finies sont caractérisées par (1.2.5). Avec les notations de la définition 1.2.2, on constate que $(\widehat{W}_s^h)_{s \in [0, 1]}$ est un processus continu gaussien centré à valeurs dans \mathbb{R}^k tel que $\widehat{W}_0^h = 0$ et dont les covariances sont données

$$\forall s_1, s_2 \in [0, 1], \quad \mathbb{E}\left[(\widehat{W}_{s_1}^h - \widehat{W}_{s_2}^h)(\widehat{W}_{s_1}^h - \widehat{W}_{s_2}^h)^\top\right] = d_h(s_1, s_2) \cdot \text{Id}_k,$$

où Id_k est la matrice identité de taille k . Cette régularité permet de considérer des serpents browniens dirigés par des fonctions aléatoires. De la section 1.1.3, on rappelle que $\mathbf{N}_\alpha(dH \mid \zeta = 1)$ est la loi de l'excursion normalisée du processus de hauteur α -stable.

Définition 1.2.3 Soit $\alpha \in (1, 2]$. Un *serpent brownien k -dimensionnel avec mécanisme de branchement α -stable* est une variable aléatoire (H, W) sur l'espace des serpents à valeurs dans \mathbb{R}^k telle que la loi de H est $\mathbf{N}_\alpha(dH \mid \zeta = 1)$ et telle que conditionnellement à H , W est un serpent brownien k -dimensionnel dirigé par H . En reprenant les mêmes notations que dans la définition 1.2.2, on appelle le processus (H, \widehat{W}) l'*extrémité du serpent brownien k -dimensionnel avec mécanisme de branchement α -stable*. \square

Par le résultat de Duquesne & Le Gall [61, Proposition 4.4.1] présenté plus haut, la proposition 1.1.4 entraîne l'existence d'un tel objet. On renvoie à la section 2.4.5 du chapitre 2 pour plus de détails.

Soit (H, W) serpent brownien k -dimensionnel avec mécanisme de branchement α -stable avec $\alpha = 2$. La mesure d'occupation de $(\widehat{W}_s)_{s \in [0,1]}$ est notée \mathcal{I} : c'est une mesure aléatoire telle que pour toute fonction mesurable et bornée $g : \mathbb{R}^k \rightarrow \mathbb{R}$, on a

$$\int_{\mathbb{R}^k} g(x) \mathcal{I}(dx) = \int_{[0,1]} g(\widehat{W}_s) ds.$$

Il s'agit d'une mesure presque sûrement diffuse dont le support topologique est la *trace* du serpent $\text{supp } \mathcal{I} = \{\widehat{W}_s : s \in [0, 1]\}$. De plus, $\text{supp } \mathcal{I}$ est un compact de \mathbb{R}^k dont la dimension de Hausdorff est $\min(k, 4)$ d'après Dawson & Hochberg [43], ce qui indique le caractère critique de la dimension 4. La mesure aléatoire \mathcal{I} est la *Integrated Super-Brownian Excursion* (abrégé en ISE) qui a été très étudiée depuis Aldous [10]. En tant qu'exemple de référence parmi les super-processus de Dawson-Watanabe, elle apparaît comme limites de systèmes de particules en interaction et d'objets en mécanique statistique comme le processus de contact, le modèle du votant, les lattice trees, ou les agrégats de percolation. Mentionnons par exemple les travaux de Cox, Durrett & Perkins [38], Derbez & Slade [46], Hara & Slade [92], Slade [143], Ben Arous, Cabezas & Fribergh [15], et Cabezas, Fribergh, Holmes & Perkins [33]. La mesure aléatoire ISE \mathcal{I} , de même que la mesure d'occupation totale du super-mouvement brownien, admet plusieurs descriptions géométriques à partir de son support, selon la dimension k et le paramètre de stabilité α . Dans le cas brownien $\alpha = 2$, Dawson, Iscoe & Perkins [44] montrent que la mesure de Lebesgue sur \mathbb{R}^k de $\text{supp } \mathcal{I}$ est nulle si et seulement si $k \geq 4$, puis que \mathcal{I} correspond à une mesure de Hausdorff explicite de $\text{supp } \mathcal{I}$ en dimension sur-critique $k \geq 5$. Le Gall [119] donne un résultat analogue pour la dimension critique $k = 4$ (toujours avec $\alpha = 2$). Duquesne [55] prouve que si $\alpha = 2$ et $k \geq 5$ alors \mathcal{I} coïncide aussi avec une mesure de packing exacte de son support. Bien qu'il n'y ait probablement pas de mesure de Hausdorff exacte dans les cas $\alpha \in (1, 2)$, Duquesne & Duhalde [56] montrent que \mathcal{I} est une mesure de packing explicite de $\text{supp } \mathcal{I}$ quand $k > 2\alpha/(\alpha - 1)$. Enfin, Croydon [39] prouve que le serpent brownien avec mécanisme de branchement 2-stable est injectif en tant que fonction définie sur l'arbre brownien dès que $k \geq 8$.

On est maintenant prêt à énoncer les résultats de Janson & Marckert, Marzouk, et Le Gall & Lin mentionnés plus haut. Soit μ une mesure de probabilité sur \mathbb{N} de variance $\sigma^2 \in (0, \infty]$, et soit τ un arbre de Galton-Watson de loi de reproduction μ . Conditionnellement à τ , soit $Y = (Y_u)_{u \in \tau}$ une marche branchante simple aux plus proches voisins sur \mathbb{Z}^k , indexée par τ , et de position initiale $Y_\emptyset = (0, \dots, 0)$. On note $\mathcal{R} = \{Y_u : u \in \tau\}$ la trace de τ . On rappelle que $c(\tau) = (c_i(\tau) ; 0 \leq i \leq 2(\#\tau - 1))$ désigne l'exploration par contour de τ , comme définie en section 1.1.3. On pose $\widehat{W}_i(Y) = Y_{c_i(\tau)}$ pour tout entier $0 \leq i \leq 2(\#\tau - 1)$, et on étend $\widehat{W}(Y)$ en une fonction $(\widehat{W}_s(Y))_{s \geq 0}$ sur \mathbb{R}_+ par interpolation linéaire sur $[0, 2(\#\tau - 1)]$ et en posant $\widehat{W}_s(Y) = (0, \dots, 0)$ pour tout $s > 2(\#\tau - 1)$. On appelle $\widehat{W}(Y)$ la *fonction de contour spatial* de la marche branchante Y . Les travaux de Janson & Marckert [97] et de Marzouk [130] impliquent le résultat suivant.

Théorème 1.2.4 (Janson & Marckert [97], Marzouk [130]) *On suppose que μ est critique et non-triviale, apériodique, et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Alors il existe une suite déterministe de réels strictement positifs $a_n \rightarrow \infty$, donnée par (1.1.5), telle que la convergence en loi suivante*

$$\left(\frac{a_n}{n} C_{2n.s}(\tau), \sqrt{\frac{a_n}{n}} \widehat{W}_{2n.s}(Y) \right)_{s \in [0,1]} \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n) \xrightarrow{d} (H_s, \widehat{W}_s)_{s \in [0,1]} \quad (1.2.6)$$

a lieu sur l'espace $\mathbb{C}([0, 1], \mathbb{R} \times \mathbb{R}^k)$ des fonctions continues muni de la topologie de la convergence uniforme, où (H, \widehat{W}) est l'extrémité d'un serpent brownien k -dimensionnel avec mécanisme de branchement α -stable

Janson & Marckert [97] ont d'abord traité le cas $\alpha = 2$ sous des hypothèses de moments exponentiels pour la loi de reproduction μ , puis Marzouk [130] a prouvé le cas général $\alpha \in (1, 2]$ sans autre hypothèse supplémentaire. Par souci de simplicité, on a ici restreint l'énoncé du théorème 1.2.4 au cas particulier de la marche branchante simple aux plus proches voisins, mais les théorèmes de Janson & Marckert [97] et de Marzouk [130] incluent des marches branchantes dont les déplacements spatiaux sont plus généraux. Plus précisément, la convergence (1.2.6) reste valide si la loi de saut est centrée, de variance égale à 1, et satisfait une certaine hypothèse de moment (voir Marzouk [130] pour un commentaire sur l'optimalité de ces hypothèses).

Le théorème 1.1.5 de Duquesne donne exactement la convergence de la première marginale dans (1.2.6). De plus, la convergence des lois fini-dimensionnelles pour la seconde marginale s'obtient sans réel problème en appliquant le théorème de Donsker conditionnellement à la généalogie τ . Ainsi, la difficulté technique principale de la preuve est de vérifier la tension pour la fonction de contour spatial. La méthode utilisée est de contrôler des moments de $\widehat{W}(Y)$ conditionnellement à τ , puis d'obtenir ou d'utiliser des moments pour la fonction de contour de τ via une estimée de Gittenberger pour Janson & Marckert [97] et une estimée de type Hölder pour Marzouk [130]. Le théorème 1.2.4 permet de justifier que la limite d'échelle de la trace de la marche branchante est la trace $\{\widehat{W}_s : s \in [0, 1]\}$ du serpent brownien, ce qui répond à la question (B) posée dans la section 1.2.1. La réponse à la question (A) est donnée par le résultat suivant de Le Gall & Lin [83, 82].

Théorème 1.2.5 (Le Gall & Lin [83, 82]) *On garde les notations précédentes et on rappelle que $\mathcal{R} = \{Y_u : u \in \tau\}$ est la trace de la marche branchante Y . On suppose que μ est la loi géométrique de paramètre $1/2$, de sorte que $\mu(j) = 2^{-1-j}$ pour tout $j \in \mathbb{N}$. On a les convergences suivantes.*

- (i) *Si $k \geq 5$, alors on a $\mathbb{P}(|\frac{1}{n}\#\mathcal{R} - c_k| > \varepsilon \mid \#\tau = n) \rightarrow 0$ pour tout $\varepsilon > 0$, où $c_k \in (0, 1)$ est une constante déterministe ne dépendant que de la dimension k .*
- (ii) *Si $k = 4$, alors on a $\mathbb{P}(|\frac{\ln n}{n}\#\mathcal{R} - 8\pi^2| > \varepsilon \mid \#\tau = n) \rightarrow 0$ pour tout $\varepsilon > 0$.*
- (iii) *Si $k \leq 3$, alors la loi de $n^{-k/4}\#\mathcal{R}$ sous $\mathbb{P}(\cdot \mid \#\tau = n)$ converge faiblement vers la loi de $2^{k/4} \text{Leb}_k(\text{supp } \mathcal{I})$, où $\text{Leb}_k(\text{supp } \mathcal{I}) > 0$ est la mesure de Lebesgue sur \mathbb{R}^k du support de la mesure aléatoire ISE sur \mathbb{R}^k .*

Ce résultat est à comparer avec le théorème 1.2.1 : la dimension critique pour le comportement de la trace est maintenant $k = 4$ au lieu de $k = 2$ comme pour la marche aléatoire non-branchante. Cela peut s'expliquer assez facilement de manière heuristique. Pour tout $k \in \mathbb{N}^*$, le théorème central limite montre que la marche aléatoire simple sur \mathbb{Z}^k atteint typiquement des sommets jusqu'à distance \sqrt{n} de l'origine en moins de n pas. Ainsi, la marche peut raisonnablement atteindre $n^{k/2}$ sommets distincts en n pas, et si $k > 2$ alors il y a assez de sommets disponibles pour que la taille de la trace ne soit limitée que par n . Dans le cas de la marche branchante, le même argument tient en remplaçant le nombre de pas par la hauteur maximale de l'indexation, qui est ici de l'ordre

\sqrt{n} d'après le théorème 1.1.3 d'Aldous. Pour savoir si le nombre de sommets disponibles est un facteur limitant, il suffit de comparer $n^{k/4}$ et n , et on retrouve alors la dimension critique $k = 4$. Le Gall & Lin [83, 82] ont généralisé les différentes convergences du théorème 1.2.5 pour des lois de reproduction et de saut plus générales, sous certaines hypothèses de moments et de symétrie. En particulier, ils prouvent le résultat suivant grâce à une application non-triviale du théorème ergodique sous-additif.

Théorème 1.2.6 (Le Gall & Lin [83]) *On garde les notations précédentes et on rappelle que $\mathcal{R} = \{Y_u : u \in \tau\}$ est la trace de la marche branchante Y . On suppose que μ est critique et non-triviale, apériodique, et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Alors, il existe une constante déterministe $c_{\mu,k} \in [0, 1)$ qui ne dépend que de μ et k telle que la convergence suivante*

$$\frac{1}{n} \#\mathcal{R} \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n) \longrightarrow c_{\mu,k}$$

a lieu en probabilité. En outre, si $k > \frac{2\alpha}{\alpha-1}$ alors $c_{\mu,k} > 0$.

De manière similaire au théorème 1.2.1 (i) de Dvoretzky & Erdős [65], la constante $c_{\mu,k}$ peut s'interpréter comme la probabilité de non-retour à l'origine d'une marche branchante modifiée de manière à être invariante pour une certaine transformation. L'argument heuristique précédent permet de retrouver la condition suffisante $k > 2\alpha/(\alpha - 1)$. D'après le théorème 1.1.5 de Duquesne, la hauteur maximale de l'indexation est d'ordre $n/a_n \simeq n^{1-1/\alpha}$ sous les hypothèses du théorème 1.2.6. On est donc amené à comparer $n^{k/2-k/2\alpha}$ et n pour savoir si le nombre de sommets disponibles est un facteur limitant. Le comportement du nombre de sites distincts visités par une marche branchante sur un arbre sur-critique, comme étudiée dans cette thèse, est analogue au régime sur-critique sur \mathbb{Z}^k . En effet, le nombre de sommets dans la boule de rayon n centrée en la racine d'un arbre sur-critique croît géométriquement en n et domine donc toute puissance de n .

1.3 Marche aléatoire biaisée sur un arbre

1.3.1 Modèle, motivations, et quelques résultats connus

Soit un arbre t (au sens de la définition 1.1.1) non réduit à sa racine et soit $\lambda > 0$. La *marche aléatoire λ -biaisée sur t* est la chaîne de Markov $(X_n)_{n \in \mathbb{N}}$ sur t dont les probabilités de transitions sont données par

$$p_t^\lambda(x, \overleftarrow{x}) = \frac{\lambda}{\lambda + k_x(t)} \quad \text{et} \quad p_t^\lambda(\overleftarrow{x}, x) = \mathbf{1}_{\{\overleftarrow{x} \neq \emptyset\}} \frac{1}{\lambda + k_{\overleftarrow{x}}(t)} + \mathbf{1}_{\{\overleftarrow{x} = \emptyset\}} \frac{1}{k_{\emptyset}(t)} \quad (1.3.1)$$

pour tout $x \in t \setminus \{\emptyset\}$. Par défaut, la position initiale de la marche est la racine de l'arbre, c'est-à-dire $X_0 = \emptyset$.

Ce modèle a été introduit formellement par Lyons [123], mais était déjà apparu de manière indépendante dans l'étude d'agrégations aléatoires (Krug [113]) et au sein de simulations à la Monte-Carlo de marches auto-évitantes (Berretti & Sokal [18]). Expliquons brièvement pourquoi ces probabilités de transitions sont particulièrement intéressantes à considérer pour une marche aléatoire sur un arbre. Tout d'abord, la marche aléatoire λ -biaisée est réversible, donc on peut la décrire par analogie avec un réseau électrique (voir Lyons & Peres [127]). On vérifie alors que les conductances des arêtes prennent l'expression agréable et naturelle $c(\overleftarrow{x}, x) = \lambda^{1-|x|}$ pour tout $x \in t \setminus \{\emptyset\}$. Néanmoins, deux autres modèles pourraient sembler plus naturels a priori :

- la marche aléatoire simple sur l'arbre vu comme graphe, dont les transitions sont données par $p(x, \overleftarrow{x}) = 1/(k_x(t) + 1)$ et $p(\overleftarrow{x}, x) = 1/(k_{\overleftarrow{x}}(t) + 1)$,

- la marche aléatoire simple du point de vue des hauteurs et indifférente aux branches, dont les transitions sont données par $p(x, \overleftarrow{x}) = 1/2$ et $p(\overleftarrow{x}, x) = 1/2k_{\overleftarrow{x}}(t)$.

La première marche aléatoire sera transiente sur des arbres où un sommet “moyen” a strictement plus d’un enfant, comme les arbres de Galton-Watson sur-critiques, ce qui empêche l’émergence d’un régime “critique” où la marche serait nullement récurrente. De plus, elle ne tient pas compte de la structure hiérarchique de l’arbre. A contrario, la deuxième marche aléatoire ne tient pas compte de la structure de ramification de l’arbre, ce qui se traduit par une dépendance de la conductance d’une arête donnée aux degrés de sommets lointains. La marche aléatoire λ -biaisée semble fournir un bon compromis entre ces deux modèles, et il se trouve que son comportement apporte des informations non-triviales sur l’arbre.

En effet, Lyons [123] montre qu’il existe une quantité $\text{br}(t) \in [0, \infty]$ appelée le *nombre de branchement* telle que la marche aléatoire λ -biaisée est récurrente si $\lambda > \text{br}(t)$ et est transiente si $\lambda < \text{br}(t)$. Cette propriété caractérise $\text{br}(t)$, mais ce dernier s’exprime également comme

$$\text{br}(t) = \sup \left\{ \lambda \in \mathbb{R}_+ : \inf_{\Pi} \sum_{x \in \Pi} \lambda^{-|x|} > 0 \right\}$$

avec la convention $\sup \emptyset = 0$, où l’infimum est pris sur tous les sous-ensembles Π de $t \setminus \{\emptyset\}$ tels que la composante connexe de la racine \emptyset dans $t \setminus \Pi$ (en tant que graphe) est finie. De plus, le nombre de branchement admet aussi la définition métrique suivante. On appelle l’ensemble

$$\text{Bd}(t) = \{(x_n)_{n \in \mathbb{N}} \in t^{\mathbb{N}} : x_0 = \emptyset, x_n = \overleftarrow{x_{n+1}}\}$$

le *bord à l’infini* de l’arbre t et on le munit d’une distance ultramétrique en posant $d_{\text{Bd}}((x_n), (y_n)) = \lim_{n \rightarrow \infty} e^{-|x_n \wedge y_n|}$ pour tous $(x_n), (y_n) \in \text{Bd}(t)$. Alors, $\ln \text{br}(t)$ est égal à la dimension de Hausdorff de $(\text{Bd}(t), d_{\text{Bd}})$. Voir Lyons [123] et Lyons & Peres [127, Chapter 1] pour plus de détails.

Le nombre de branchement peut être interprété comme le nombre moyen d’enfants d’un sommet. Cette intuition est renforcée par Lyons [123, Proposition 6.4] qui affirme que si T est un arbre de Galton-Watson sur-critique de loi de reproduction ν de moyenne $m := \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty)$, alors $\text{br}(T) = m$ presque sûrement sur l’évènement $\{\#T = \infty\}$ de non-extinction. De plus, toujours sur l’évènement de non-extinction, si $\lambda > m$ alors la marche est récurrente positive, et si $\lambda = m$ alors la marche est récurrente nulle. Enfin, on sait d’après Lyons, Pemantle & Peres [126, 124] que si $\lambda < m$ alors la marche s’échappe à vitesse positive, c’est-à-dire qu’il existe une constante déterministe $v(\lambda) > 0$ telle que si $\#T = \infty$ alors $|X_n|/n \rightarrow v(\lambda)$ presque sûrement.

Dans cette thèse, on se concentre sur la marche branchante biaisée critique sur un arbre de Galton-Watson sur-critique. Cette étude nouvelle fait écho à plusieurs travaux antérieurs sur la marche aléatoire (non-branchante). Duquesne [52] commence par décrire la limite d’échelle de la trace de marches légèrement sur-critiques sur un arbre régulier déterministe : c’est un arbre réel codé par un couple de mouvements browniens avec drift, qui se trouvent chacun être la limite du processus des hauteurs prises par la marche. Par ailleurs, Duquesne [52] énonce aussi une loi des grands nombres pour le nombre de sites visités par ces marches. Sur un arbre de Galton-Watson aléatoire et sous une hypothèse de moments exponentiels, Peres & Zeitouni [137] prouvent un théorème central limite pour le processus des hauteurs prises par la marche biaisée critique. Celui-ci est généralisé par Dembo & Sun [45] lorsque l’environnement est un arbre de Galton-Watson multi-type et sous une hypothèse de moments plus faible. Enfin, Aidékon & de Raphélis [6] étendent le résultat de Peres & Zeitouni sous une simple hypothèse de moment d’ordre 2. De plus, ils montrent que la limite d’échelle de la trace de la marche est alors l’arbre réel codé par le processus limite, ce qui rappelle le résultat de Duquesne [52] mentionné plus haut. Énonçons précisément le théorème de Aidékon & de Raphélis [6].

Soit ν une mesure de probabilité sur \mathbb{N} sur-critique et de variance finie, on pose

$$m = \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty) \quad \text{et} \quad \eta^2 = \frac{m(m-1)}{\sum_{k \in \mathbb{N}} k(k-1)\nu(k)} \in (0, \infty).$$

Pour tout arbre t , on définit P_t comme la loi de la marche aléatoire m -biaisée sur t , notée $(X_n)_{n \in \mathbb{N}}$ de manière canonique. De plus, on étend la suite $(|X_n|)_{n \in \mathbb{N}}$ des hauteurs prises par la marche en une fonction continue $(|X_s|)_{s \geq 0}$ sur \mathbb{R}_+ par interpolation linéaire. On note $\mathcal{R}_n = \{X_k : k \leq n\}$ l'ensemble des sommets de t visités par la marche jusqu'au temps n . C'est un sous-arbre de t , enraciné en \emptyset , et muni de la distance de graphe d_{gr} définie par (1.1.2).

Théorème 1.3.1 (Aïdékon & de Raphélis [6]) *Soit T un arbre de Galton-Watson de loi de reproduction ν . Presque sûrement sur l'évènement $\{\#T = \infty\}$, la convergence suivante sous P_T*

$$\left(\frac{1}{\sqrt{\eta^2 n}} |X_{ns}| \right)_{s \in [0,1]} \xrightarrow{d} (|B_s|)_{s \in [0,1]}, \quad (1.3.2)$$

a lieu en loi sur l'espace $\mathcal{C}([0,1], \mathbb{R})$ des fonctions continues muni de la topologie de la convergence uniforme, où B est un mouvement brownien standard. Dans ce même cas, conjointement avec (1.3.2), la convergence

$$\left(\mathcal{R}_n, \frac{1}{\sqrt{\eta^2 n}} d_{\text{gr}}, \emptyset, 0 \right) \xrightarrow{d} (\mathcal{T}_{|B|}^1, d_{|B|}^1, \rho_{|B|}^1, 0) \quad (1.3.3)$$

a lieu en loi sur \mathbb{K} muni de la distance de Gromov-Hausdorff-Prokhorov, où $\mathcal{T}_{|B|}^1$ est défini par (1.1.15).

Dans cette thèse, on obtient un résultat similaire pour la marche branchante biaisée critique. Pour prouver le théorème 1.3.1, Aïdékon & de Raphélis [6] montrent que la collection des temps locaux d'une excursion de la marche (au dessus de la racine) forme un arbre de Galton-Watson multi-type. Cette propriété repose sur le fait que pour tout sommet $x \neq \emptyset$ de l'arbre, les excursions au dessus de x sont linéairement ordonnées et indépendantes, et ne contiennent pas de temps d'atteinte de la racine. Cela n'est plus vrai pour une marche branchante, en particulier car une excursion peut s'achever par extinction de la généalogie, sans jamais revenir à son point de départ. L'approche sur laquelle on s'appuie dans ce travail ressemble plutôt à celle suivie par Peres & Zeitouni [137] et Dembo & Sun [45], qui utilise une mesure invariante sur un espace d'arbres de profondeur infinie. Ces outils, que l'on présente ci-après, nous permettent également d'obtenir une loi des grands nombres pour la taille de la trace, à la fois similaire au théorème 1.2.6 de Le Gall & Lin [83] et au résultat de Duquesne [52] mentionné plus haut.

1.3.2 Arbres de Galton-Watson de profondeur infinie

Soit T un arbre de Galton-Watson de loi de reproduction ν sur-critique et telle que $m \in (1, \infty)$, où $m = \sum_{k \in \mathbb{N}} k\nu(k)$. De manière informelle, si $(X_n)_{n \in \mathbb{N}}$ est une marche aléatoire m -biaisée sur T , alors l'arbre obtenu en recentrant T sur X_n sous $\mathbb{P}(\cdot | \#T = \infty)$ converge localement en loi vers un arbre ayant une lignée ancestrale infinie et invariant pour la marche (en un sens que nous précisons à la fin de cette section). Afin de tirer partie de cette idée, on présente la notion d'arbres de profondeur infinie via le formalisme des mots bilatéraux.

Un *mot bilatéral* est une suite d'entiers strictement positifs, indexée par les entiers relatifs plus petits qu'un certain $n \in \mathbb{Z}$. Ainsi, l'ensemble $\bar{\mathcal{U}}$ des mots bilatéraux est défini par

$$\bar{\mathcal{U}} = \{(u_i)_{i \leq n} : n \in \mathbb{Z}, u_k \in \mathbb{N}^* \text{ pour tout } k \in \mathbb{Z} \text{ avec } k \leq n\}.$$

On note que $\emptyset \notin \overline{\mathbb{U}}$. Pour travailler sur $\overline{\mathbb{U}}$, on adopte une terminologie et des notations similaires à celles utilisées sur l'ensemble \mathbb{U} des mots finis, défini par (1.1.1), et sur l'ensemble \mathbb{T} des arbres. Soit $u = (u_i)_{i \leq n} \in \overline{\mathbb{U}}$, on dit que $|u| = n \in \mathbb{Z}$ est la *hauteur relative* de u . On insiste sur le fait que $|u|$ peut prendre des valeurs négatives. On note $\overleftarrow{u} = (u_i)_{i \leq n-1}$ le *parent* de u , de sorte que $|\overleftarrow{u}| = |u| - 1$. On peut définir la concaténation du mot bilatéral $u \in \overline{\mathbb{U}}$ avec un mot fini $u' = (u'_1, \dots, u'_m) \in \mathbb{U}$ comme suit :

$$u * u' = (a_i)_{i \leq n+m} \in \overline{\mathbb{U}}, \quad \text{où} \quad a_i = \begin{cases} u_i & \text{si } i \leq n, \\ u'_{i-n} & \text{si } n+1 \leq i \leq n+m. \end{cases}$$

Ainsi, $|u * u'| = |u| + |u'|$ et $u * \emptyset = u$. Soit $v \in \overline{\mathbb{U}}$. Un mot bilatéral $w \in \overline{\mathbb{U}}$ est un *ancêtre commun* à u et v lorsque l'on peut trouver $u', v' \in \mathbb{U}$ tels que $u = w * u'$ et $v = w * v'$. On observe que deux mots bilatéraux u et v n'ont pas toujours d'ancêtres communs mais lorsque c'est le cas, on note $u \wedge v$ leur *plus récent ancêtre commun* qui est leur ancêtre commun de hauteur relative maximale. On note que pour tous $w \in \overline{\mathbb{U}}$ et $u', v' \in \mathbb{U}$, on a $(w * u') \wedge (w * v') = w * (u' \wedge v')$. Enfin, on étend l'ordre lexicographique aux paires de mots bilatéraux admettant un ancêtre commun en posant $w * u' \leq w * v' \iff u' \leq v'$, pour tous $w \in \overline{\mathbb{U}}$ et $u', v' \in \mathbb{U}$.

Définition 1.3.2 Un *arbre de profondeur infinie* est un sous-ensemble T de $\overline{\mathbb{U}}$ qui satisfait les conditions suivantes :

- (a) pour tout $u \in T$, on a $\overleftarrow{u} \in \overline{\mathbb{U}}$,
- (b) pour tous $u, v \in T$, il existe un ancêtre commun à u et v (et donc $u \wedge v \in T$),
- (c) pour tout $u \in T$, il existe un entier $k_u(T) \in \mathbb{N}$ tel que $u * (i) \in T \iff 1 \leq i \leq k_u(T)$.

Un *arbre pointé de profondeur infinie* (T, r) est un arbre de profondeur infinie T muni d'un sommet $r \in T$. On note $\overline{\mathbb{T}}^\bullet$ l'ensemble des arbres pointés de profondeur infinie. \square

Par exemple, si $t \in \mathbb{T}$ est un arbre et si $\varpi = (1)_{i \leq 0}$ désigne le mot bilatéral uniquement écrit avec des 1 et de hauteur relative nulle, alors $\varpi * t = \{\varpi * v : v \in t\}$ est un arbre de profondeur infinie. Un arbre de profondeur infinie T admet une structure naturelle de graphe où les arêtes sont les $\{\overleftarrow{u}, u\}$ pour $u \in T$, et dont la distance de graphe prend la même expression que (1.1.2) :

$$\forall u, v \in T, \quad d_{\text{gr}}(u, v) = |u| + |v| - 2|u \wedge v|. \quad (1.3.4)$$

Pour tout $u \in T$, l'ensemble $\theta_u T = \{v \in \mathbb{U} : u * v \in T\}$ est un arbre au sens de la définition 1.1.1 que l'on appelle le *sous-arbre de T issu de u* . Soit (T, r) un arbre pointé de profondeur infinie, on définit sa *p -troncature* en posant

$$[T, r]_p = (\theta_{r_p} T, r'_p),$$

où $r_p \in T$ et $r'_p \in \mathbb{U}$ sont caractérisés par $r = r_p * r'_p$ et $|r_p| = |r| - p$. Par définition, $r'_p \in \theta_{r_p} T$ et $[T, r]_p$ est un élément de $\overline{\mathbb{T}} \times \mathbb{U}$. On munit $\overline{\mathbb{T}}^\bullet$ de la *topologie de la convergence locale* en écrivant $(T_n, r_n) \longrightarrow (T, r)$ sur $\overline{\mathbb{T}}^\bullet$ lorsque $|r_n| = |r|$ à partir d'un certain rang et $[T_n, r_n]_p \longrightarrow [T, r]_p$ sur $\overline{\mathbb{T}} \times \mathbb{U}$ pour tout $p \in \mathbb{N}$. On vérifie alors facilement que $\overline{\mathbb{T}}^\bullet$ est un espace polonais (voir la section 2.2 dans le chapitre 2).

On se donne maintenant quelques opérations afin de manipuler les arbres pointés de profondeur infinie. Pour tout $l \in \mathbb{Z}$, on définit le *l -décalage* d'un mot bilatéral $u = (u_i)_{i \leq n}$ et d'un arbre de profondeur infinie T en posant

$$\varphi_l(u) = (u_{i+l})_{i \leq n-l} \quad \text{et} \quad \varphi_l(T) = \{\varphi_l(v) : v \in T\}.$$

Notons que $|\varphi_l(u)| = |u| - l$. On recentre un arbre pointé de profondeur infinie (T, r) par

$$\text{cent}(T, r) = (\varphi_{|r|}(T), \varphi_{|r|}(r)),$$

de façon à ce que son point distingué devienne de hauteur relative nulle. Enfin, on remarque que l'ordre lexicographique est un ordre total sur T , et que si $\{u \in T : r < u\}$ est non vide alors il admet un élément minimal pour \leq . On désigne alors cet élément minimal par $\text{succ}_T(r)$ dans ce cas, et on pose $\text{succ}_T(r) = r$ sinon. On définit l'application successeur par

$$\text{succ}(T, r) = \text{cent}(T, \text{succ}_T(r)).$$

On présente maintenant les *arbres de Galton-Watson de profondeur infinie*, dont la loi est motivée par le principe de Many-to-One formulé à la proposition 1.3.4.

Définition 1.3.3 (Arbre de Galton-Watson infini) Soit μ une mesure de probabilité sur \mathbb{N} telle que $\mu(0) + \mu(1) < 1$ et d'espérance $\mathfrak{m}_\mu = \sum_{k \in \mathbb{N}} k\mu(k)$ finie. Soit $\mathbf{T} = (T, o)$ une variable aléatoire sur $\overline{\mathbb{T}}^\bullet$ telle que $|o| = 0$ presque sûrement. En notant $o = (J_{-i})_{i \leq 0}$, on utilise les notations $o(n) = (J_{-i})_{i \leq -n}$ pour tout $n \in \mathbb{N}$, de sorte que $o(0) = o$ et $o(n+1)$ est le parent de $o(n)$. On pose également

$$\text{Sp} = \{o(n) : n \in \mathbb{N}^*\} \quad \text{et} \quad \partial\text{Sp} = \{u \in T \setminus \text{Sp} : \overleftarrow{u} \in \text{Sp}\}.$$

Alors, \mathbf{T} est un *arbre pointé de Galton-Watson infini de loi de reproduction μ* lorsque les conditions suivantes sont satisfaites.

- (a) Les variables aléatoires $(J_{-n}, k_{o(n+1)}(\mathbf{T}))$ pour $n \in \mathbb{N}$ sont indépendantes et de même loi, donnée par $\mathbb{P}(J_{-n} = j ; k_{o(n+1)}(\mathbf{T}) = k) = \mathbf{1}_{\{1 \leq j \leq k\}} \mu(k) / \mathfrak{m}_\mu$ pour tous $j, k \in \mathbb{N}^*$.
- (b) Conditionnellement à $(J_{-n}, k_{o(n+1)}(\mathbf{T}))_{n \in \mathbb{N}}$, les arbres $(\theta_u \mathbf{T} ; u \in \partial\text{Sp})$ sont des arbres de Galton-Watson indépendants de loi de reproduction μ .

□

Là où on peut voir un arbre de Galton-Watson comme la généalogie issue d'un individu, un arbre pointé de Galton-Watson infini représente l'arbre généalogique passé et futur d'un individu présent : tous les membres de sa famille se reproduisent de manière indépendante, ses ancêtres selon la loi de reproduction biaisée par la taille $\bar{\mu}(k) = k\mu(k) / \mathfrak{m}_\mu$, ses descendants (y compris lui-même) et ses cousins selon μ . Les arbres pointés de Galton-Watson infinis sont liés aux arbres de Galton-Watson via la proposition suivante.

Proposition 1.3.4 Soit μ une mesure de probabilité sur \mathbb{N} d'espérance $\mathfrak{m}_\mu = \sum_{k \in \mathbb{N}} k\mu(k)$ finie et telle que $\mu(0) + \mu(1) < 1$. Soit τ un arbre de Galton-Watson et soit $\mathbf{T} = (T, o)$ un arbre pointé de Galton-Watson infini, tous deux de loi de reproduction μ . Alors les assertions suivantes sont vérifiées.

- (i) Pour toute fonction mesurable et positive $F : \mathbb{N} \times T \times U \rightarrow \mathbb{R}_+$, on a

$$\mathbb{E} \left[\sum_{u \in \tau} F(|u|, \tau, u) \right] = \sum_{n \in \mathbb{N}} \mathfrak{m}_\mu^n \mathbb{E} [F(n, [\mathbf{T}, o]_n)].$$

- (ii) On suppose $\mathfrak{m}_\mu = 1$ et que μ est apériodique. On note $\varpi = (1)_{i \leq 0} \in \overline{U}$. Si \mathbf{u} est uniforme sur les sommets de τ , alors en loi pour la convergence locale sur $\overline{\mathbb{T}}^\bullet$, on a

$$\text{cent}(\varpi * \tau, \varpi * \mathbf{u}) \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n) \xrightarrow{d} \mathbf{T}.$$

- (iii) Si $\mathfrak{m}_\mu = 1$ alors la loi de \mathbf{T} est invariante par l'opération succ .

Le point (i) est appelé le principe de *Many-To-One*, ou encore la *décomposition spinale en un point*. C'est une identité classique et souvent utile pour étudier les arbres de Galton-Watson (voir Duquesne [51] par exemple). Le point (ii) est une conséquence de (i) (voir la proposition 2.2.9 du chapitre 2). Il semble être bien connu. En effet, les arbres pointés de Galton-Watson infinis sont une variante des arbres de Galton-Watson biaisés par la taille introduits par Kesten [100] et qui apparaissent en tant que limites locales (vues de la racine) d'arbres de Galton-Watson critiques conditionnés à être grands. Pour des résultats précis dans cette veine, on renvoie à Lyons, Pemantle & Peres [125], Aldous & Pitman [11], Duquesne [51], et Abraham & Delmas [2]. On prouve le point (iii) en tant que proposition 2.2.10 dans le chapitre 2 à partir du point (ii), mais on peut trouver dans Le Gall & Lin [82, Proposition 2] une autre preuve de cette invariance.

Remarquons que l'on a choisi que le point distingué o d'un arbre pointé de Galton-Watson infini se reproduit selon μ alors que son parent se reproduit selon la loi de reproduction biaisée par la taille $\bar{\mu} = k\mu(k)/m_\mu$. Il ne s'agit que d'une question de convention, et il aurait été tout à fait raisonnable que o se reproduise selon $\bar{\mu}$ alors que ses enfants se reproduisent selon μ . Afin d'effacer l'importance de ce choix, il est naturel de définir un arbre pointé de profondeur infinie aléatoire dont le point distingué se reproduit selon la moyenne de μ et $\bar{\mu}$.

Définition 1.3.5 (Arbre de Galton-Watson invariant) Soit ν une mesure de probabilité sur \mathbb{N} telle que $m = \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty)$. Une variable aléatoire $\mathbf{T} = (T, o)$ sur \mathbb{T}^\bullet est un *arbre pointé de Galton-Watson invariant de loi de reproduction ν* lorsque pour toute fonction mesurable bornée $F : \mathbb{T}^\bullet \rightarrow \mathbb{R}$,

$$\mathbb{E}[F(\mathbf{T})] = \mathbb{E}\left[\frac{m + k_{o'}(\mathbf{T}')}{2m} F(\mathbf{T}')\right],$$

où $\mathbf{T}' = (T', o')$ un arbre pointé de Galton-Watson infini de loi de reproduction ν comme à la définition 1.3.3. \square

L'adjectif "invariant" se justifie par le fait suivant. Par analogie avec (1.3.1), on appelle *marche aléatoire λ -biaisée sur un arbre pointé de profondeur infinie* (T, r) la chaîne de Markov $(X_n)_{n \in \mathbb{N}}$ sur T , de position initiale $X_0 = r$, et dont les probabilités de transitions sont données par

$$\forall x \in T, \quad p_T^\lambda(x, \overleftarrow{x}) = \frac{\lambda}{\lambda + k_x(T)} \quad \text{et} \quad p_T^\lambda(\overleftarrow{x}, x) = \frac{1}{\lambda + k_{\overleftarrow{x}}(T)}. \quad (1.3.5)$$

Soit $\mathbf{T} = (T, o)$ un arbre pointé de Galton-Watson invariant de loi de reproduction ν et conditionnellement à \mathbf{T} , soit $(X_n)_{n \in \mathbb{N}}$ une marche aléatoire m -biaisée sur \mathbf{T} (comme définie ci-dessus avec $\lambda = m$). Alors, la loi de \mathbf{T} est réversible au sens où on a l'égalité en loi

$$(T, X_0, X_1) \stackrel{d}{=} (\varphi_{|X_1|}(T), \varphi_{|X_1|}(X_1), \varphi_{|X_1|}(X_0)). \quad (1.3.6)$$

En particulier, la loi de $\text{cent}(T, X_n)$ ne dépend pas de n . Cette invariance est un ingrédient important des preuves de Peres & Zeitouni [137] et de Aïdékon & de Raphélys [6], et est utilisée de manière cruciale dans cette thèse au chapitre 3.

1.4 Nombre de Horton-Strahler

1.4.1 Quelques définitions équivalentes

Le nombre de Horton-Strahler d'un arbre (planaire) fini est un entier positif qui quantifie sa complexité de branchement. On en donne une première définition formelle ci-après.

Définition 1.4.1 Soit un arbre fini t au sens de la définition 1.1.1. Son *nombre de Horton-Strahler* $S(t) \in \mathbb{N}$ est défini comme suit.

- (a) Si t est réduit à sa racine, alors $\mathcal{S}(t) = 0$.
- (b) Si la racine de t a $k = k_{\emptyset}(t) \geq 1$ enfants, alors $\mathcal{S}(t)$ est le maximum des nombres de Horton-Strahler des sous-arbres $\theta_{(1)}t, \dots, \theta_{(k)}t$ issus de ces enfants, plus 1 si ce maximum n'est pas atteint de manière unique. Autrement dit, on a la relation de récurrence

$$\mathcal{S}(t) = \max_{1 \leq i \leq k_{\emptyset}(t)} \mathcal{S}(\theta_{(i)}t) + \mathbf{1}_{\{2 \leq \#\operatorname{argmax}_{1 \leq i \leq k_{\emptyset}(t)} \mathcal{S}(\theta_{(i)}t)\}}. \quad (1.4.1)$$

□

Définition par la hauteur maximale d'un arbre binaire parfait plongé. On peut également définir le nombre de Horton-Strahler en termes de plongement d'arbres binaires parfaits de la manière suivante. Soient deux arbres finis t, t' , on dit que une application injective $\phi : t \rightarrow t'$ est un *plongement* lorsque $\phi(u \wedge v) = \phi(u) \wedge \phi(v)$ pour tous $u, v \in t$. Pour tout $n \in \mathbb{N}$, on note

$$\mathbb{W}_{2,n} = \bigcup_{0 \leq k \leq n} \{1, 2\}^k, \quad \text{où } \{1, 2\}^0 = \{\emptyset\},$$

l'arbre binaire parfait de hauteur n , défini comme l'ensemble des mots de longueur au plus n écrits avec l'alphabet $\{1, 2\}$. Alors, pour tout arbre fini t , on a

$$\mathcal{S}(t) = \max \{n \in \mathbb{N} : \exists \phi : \mathbb{W}_{2,n} \rightarrow t \text{ plongement}\}. \quad (1.4.2)$$

Cette identité semble être bien connue mais on en donne une courte preuve dans la section 4.3.1 du chapitre 4. En substance, on peut trouver ce plongement optimal en suivant les branches de l'arbre sur lesquelles le nombre de Horton-Strahler diminue au plus d'une unité à chaque arête traversée.

Définition par le nombre minimal de registres pour évaluation. En informatique théorique, Ershov [68] a observé que le nombre de registres nécessaires à l'évaluation d'une expression arithmétique est égal à 1 plus le nombre de Horton-Strahler de son arbre syntaxique. Ainsi, $1 + \mathcal{S}$ est souvent appelé la *fonction de registre* dans ce domaine. Donnons quelques précisions.

Une *expression arithmétique* est soit de la forme d'une variable x_i avec $i \in \mathbb{N}$, soit de la forme $\operatorname{op}_{1 \leq j \leq k} e_j$ où $k \in \mathbb{N}^*$ et $\operatorname{op} \in \{\Sigma, \Pi\}$, et où les e_j sont des expressions arithmétiques sur des ensembles de variables disjoints. On peut alors représenter une telle expression par un arbre, dit *syntaxique*, dont chaque sommet avec $k \geq 1$ enfants correspond à une somme ou un produit, et dont chaque feuille (i.e. sommet sans enfant) correspond à une variable x_i . Voir la figure 1.1 pour un exemple concret.

Pour évaluer une expression arithmétique de la forme x_i , il suffit de stocker la valeur de x_i dans un unique registre avant de la renvoyer. Voici la stratégie la plus économe en mémoire pour évaluer $e_1 \operatorname{op} e_2$: on commence par évaluer l'expression demandant le plus grand nombre de registres, disons $N(e_1)$ registres pour e_1 , de conserver le résultat dans un registre maintenant libre, d'utiliser $N(e_2)$ autres registres pour calculer e_2 , et d'effectuer l'opération $e_1 \operatorname{op} e_2$ pour finir. Le nombre de registres utilisés est alors $\max(N(e_1), 1 + N(e_2))$ et on retrouve la formule (1.4.1). Le cas $\operatorname{op}_{1 \leq j \leq k} e_j$ avec $k \geq 3$ se traite de la même manière. On renvoie à la figure 1.1 pour un exemple.

Définition par l'élagage de Horton. Une autre définition alternative fait appel aux opérations de *réduction des lignes*, notée Prun_0 , et de *l'élagage de Horton*, notée Prun , dit *Horton pruning* en anglais. Soit t un arbre fini, on construit l'arbre $\operatorname{Prun}_0(t)$ en enlevant les sommets $u \in t$ tels que $k_u(t) = 1$, puis en fusionnant les arêtes adjacentes en une seule, et en effaçant l'unique arête issue de la racine si elle existe. Dit de manière informelle, l'arbre $\operatorname{Prun}_0(t)$ garde la même forme que t mais n'a pas de sommets ayant un seul enfant. Si $\operatorname{Prun}_0(t)$ n'est pas réduit à sa racine, on

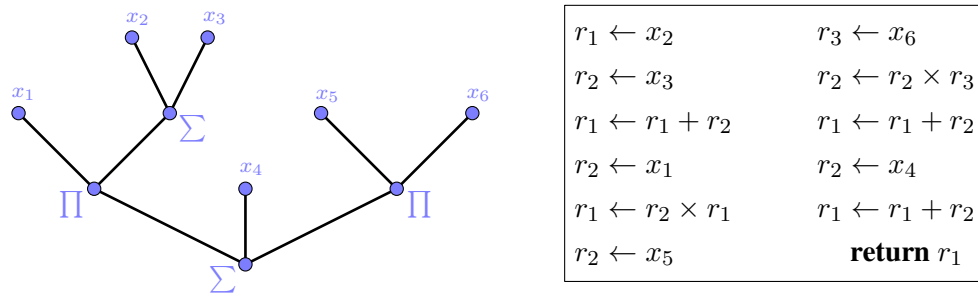


Figure 1.1: A gauche : l'arbre syntaxique, de nombre de Horton-Strahler égal à 2, représentant l'expression arithmétique $x_1 \cdot (x_2 + x_3) + x_4 + x_5 \cdot x_6$. A droite : un protocole d'évaluation de cette expression utilisant 3 registres r_1, r_2, r_3 .

obtient l'arbre $\text{Prun}(t)$ en appliquant la réduction des lignes Prun_0 sur t , puis en effaçant toutes les feuilles (et leurs arêtes parentales) de $\text{Prun}_0(t)$, puis en appliquant Prun_0 une nouvelle fois. Voir la figure 1.2 pour un exemple. Alors, le nombre de Horton-Strahler est le nombre minimal d'élagages de Horton requis pour transformer t en l'arbre $\{\emptyset\}$ réduit à sa racine :

$$\mathcal{S}(t) = \min \{n \in \mathbb{N} : \text{Prun}_n(t) = \{\emptyset\}\}, \quad (1.4.3)$$

où Prun_n désigne la n -ième itération de Prun pour tout $n \in \mathbb{N}^*$. On renvoie à Kovchegov & Zaliapin [111, Section 2.3] pour une preuve et plus de détails.

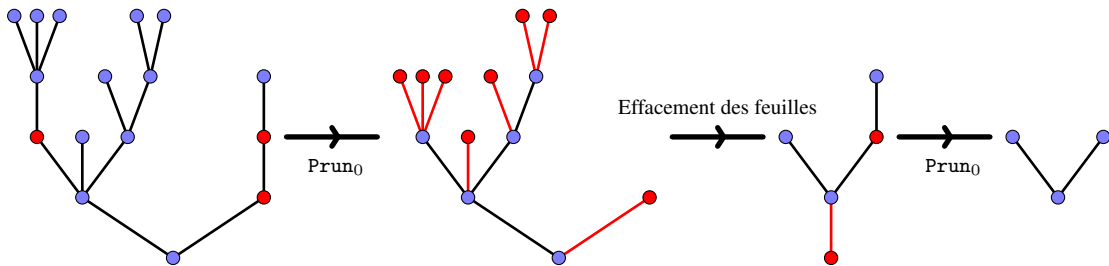


Figure 1.2: Exemple d'un élagage de Horton. Avant chaque étape, les éléments à effacer ou à enlever sont indiqués en rouge.

1.4.2 Un bref historique général

Le nombre de Horton-Strahler a été introduit en hydrogéologie, d'abord par Horton [93] en 1945 sous une forme légèrement différente et ensuite par Strahler [146] en 1952 avec la définition 1.4.1. Leur motivation était de formuler des lois quantitatives empiriques pour des réseaux de cours d'eau en fonction de leur structure de branchement. En effet, on peut modéliser un réseau fluvial par un arbre dont la racine correspond à l'embouchure et dont les feuilles représentent les sources. Depuis ces travaux fondateurs, de nombreuses caractéristiques physiques des cours d'eau ont pu être décrites à l'aide du nombre de Horton-Strahler : voir Peckham [136], Fac-Beneda [73], Chavan & Srinivas [34], et Bamuffeh et al. [14] entre autres. Aujourd'hui, il est encore fréquemment utilisé en hydrogéologie pour classer les fleuves (Moussa & Bocquillon [132]), ou estimer le risque de crues soudaines (Bamuffeh et al. [14]).

De par son caractère simple et naturel, le nombre de Horton-Strahler apparaît indépendamment (parfois sous d'autres noms) dans des disciplines scientifiques variées souhaitant décrire des structures ramifiées. En anatomie, on le retrouve dans l'étude des systèmes vasculaire et respiratoire. La biologie l'applique aux arbres (les organismes végétaux, ici). On croise le nombre de Horton-Strahler en physique pour comprendre l'émergence de motifs arborescents fractals, comme avec les phénomènes de digitation visqueuse ou de claquage électrique. Quelques travaux de chimie moléculaire l'appliquent pour prédire le repliement d'une protéine à partir de son code ADN. C'est aussi un outil intéressant pour l'analyse des réseaux sociaux. Enfin, l'informatique théorique fait grand usage du nombre de Horton-Strahler vu comme fonction de registre pour optimiser la manipulation de certaines structures de données. On renvoie à Viennot [148] pour une vue d'ensemble de ces diverses applications, ainsi que pour un algorithme de synthèse d'images d'arbres basé sur le nombre de Horton-Strahler.

Le nombre de Horton-Strahler montre un large éventail d'occurrences en mathématiques. Sans être exhaustif, on peut le croiser en logique mathématique, en théorie du langage formel, en algèbre, en combinatoire, en topologie, ou en théorie de l'approximation. Le lecteur pourra trouver une liste plus complète de ces applications dans l'article de Esparza, Luttenberger & Schlund [69]. La plupart des travaux probabilistes impliquant le nombre de Horton-Strahler se concentre sur sa distribution pour des classes d'arbres aléatoires, ou sur des propriétés d'invariance et d'attraction par rapport à l'élagage de Horton (on présente quelques-uns de ces résultats ci-dessous). Enfin, l'étude de ce nombre dans cette thèse a été originellement motivée par son rôle en tant que coût d'adaptation d'estimées pour des marches aléatoires (classiques) à des marches branchantes.

1.4.3 Quelques résultats probabilistes sur le nombre de Horton-Strahler

Pour tout $n \in \mathbb{N}^*$, soit T_n un arbre planaire binaire (aussi appelé un arbre de Catalan) de taille $2n - 1$ de loi uniforme. On a vu en section 1.1.2 que T_n a la même loi qu'un arbre de Galton-Watson conditionné à avoir $2n - 1$ sommets et dont la loi de reproduction est donnée par $\mu_2(0) = \mu_2(2) = 1/2$. De manière indépendante, Flajolet, Raoult & Vuillemin [75] et Kemp [98] ont étudié la loi du nombre de Horton-Strahler $\mathcal{S}(T_n)$ de T_n et ont établi le développement asymptotique suivant :

$$\mathbb{E}[\mathcal{S}(T_n)] = \log_4 n + D(\log_4 n) + o(1), \quad (1.4.4)$$

où $\log_b x = \ln x / \ln b$ désigne le logarithme de x en base b , et où D est une fonction 1-périodique continue dont la décomposition en série de Fourier est explicite. Devroye & Kruszewski [47] ont complété ce résultat en montrant que $\mathcal{S}(T_n)$ est concentré autour de sa valeur moyenne :

$$\forall m \geq 0, \quad \sup_{n \geq 0} \mathbb{P}(|\mathcal{S}(T_n) - \mathbb{E}[\mathcal{S}(T_n)]| \geq m) \leq 4^{1-m}. \quad (1.4.5)$$

En particulier, on obtient directement que $\frac{1}{\log_4 n} \mathcal{S}(T_n)$ converge en probabilité vers 1. Drmota & Prodinger [49] ont étendu (1.4.4) et (1.4.5) aux arbres planaires k -aires uniformes pour $k \geq 3$. Beaucoup plus récemment, Brandenberger, Devroye & Reddad [28] ont généralisé toutes les convergences de premier ordre précédemment obtenues au cadre des arbres de Galton-Watson comme suit.

Théorème 1.4.2 (Brandenberger, Devroye & Reddad [28]) *Soit μ une mesure de probabilité sur \mathbb{N} et soit τ un arbre de Galton-Watson de loi de reproduction μ . Si μ est critique et non-triviale, apériodique, et de variance finie, alors la convergence suivante a lieu en probabilité :*

$$\frac{1}{\log_4 n} \mathcal{S}(\tau) \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1.$$

A notre connaissance, aucun résultat de convergence au second ordre pour le nombre de Horton-Strahler n'est connu. On en discute la difficulté dans la section suivante.

Dans une direction parallèle, Burd, Waymire & Winn [31] se sont intéressés aux propriétés d'invariance et d'attraction des arbres de Galton-Watson critiques sous l'action de l'élagage de Horton, noté Prun et défini en section 1.4.1. Ils montrent notamment que si τ_2 est un arbre de Galton-Watson de loi de reproduction μ_2 (voir la définition 1.1.7), alors l'arbre $\text{Prun}(\tau_2)$ sous $\mathbb{P}(\cdot | \tau_2 \neq \{\emptyset\})$ est encore un arbre de Galton-Watson de loi de reproduction μ_2 (binaire planaire). Ils déduisent alors que la loi de $\mathcal{S}(\tau_2)$ est géométrique de paramètre $1/2$. Enfin, Burd, Waymire & Winn [31] prouvent que si τ est un arbre de Galton-Watson dont la loi de reproduction μ est supposée critique, non-triviale, et à support borné, alors pour tout arbre t , on a

$$\mathbb{P}(\text{Prun}_n(\tau) = t | \mathcal{S}(\tau) \geq n) \longrightarrow \mathbb{P}(\tau_2 = t),$$

où on rappelle que l'on note Prun_n la n -ième itération de Prun pour tout $n \in \mathbb{N}^*$.

Kovchegov & Zaliapin [111, 110] ont élargi leur étude à des lois de reproduction de variance potentiellement infinie. Ils montrent que si la loi de reproduction d'un arbre de Galton-Watson τ est critique et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$, alors τ est invariant par élagage de Horton si et seulement si sa loi de reproduction est μ_α comme dans la définition 1.1.7. En outre, si τ_α est un arbre de Galton-Watson α -stable alors la loi de $\mathcal{S}(\tau_\alpha)$ est géométrique de paramètre $1/\alpha$:

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(\mathcal{S}(\tau_\alpha) \geq n) = (1 - \frac{1}{\alpha})^n. \quad (1.4.6)$$

Soit μ une mesure de probabilité sur \mathbb{N} et soit τ un arbre de Galton-Watson de loi de reproduction μ . Kovchegov & Zaliapin [111, 110] prouvent que si μ est critique, non-triviale, et dans le domaine d'attraction d'une loi α -stable, alors pour tout arbre t ,

$$\mathbb{P}(\text{Prun}_n(\tau) = t | \mathcal{S}(\tau) \geq n) \longrightarrow \mathbb{P}(\tau_\alpha = t).$$

On présente un dernier résultat qui semble encore mal compris. Par calculs de fonctions génératrices, Flajolet, Raoult & Vuillemin [75] et Kemp [98] ont indépendamment révélé un lien entre le nombre de Horton-Strahler des arbres planaires binaires et le \log_2 de la hauteur des arbres planaires généraux. Leurs énoncés respectifs ne coïncident pas à cause d'une légère erreur de calcul, mais Françon [76] retrouve le résultat correct ci-dessous grâce à une méthode purement combinatoire.

Pour tous $n, p \in \mathbb{N}$, il y a autant d'arbres t de taille n tels que $\lfloor \log_2(1 + |u|) \rfloor \leq p$ pour tout $u \in t$, que d'arbres binaires t_2 de taille $2n - 1$ tels que $\mathcal{S}(t_2) \leq p$.

Grâce aux observations de la section 1.1.2 et à la formule (1.1.3), on peut reformuler ce résultat en une égalité en loi comme suit.

Si τ est un arbre de Galton-Watson de loi de reproduction géométrique de paramètre $1/2$, et si τ_2 est un arbre de Galton-Watson de loi de reproduction μ_2 , alors on a

$$\left(2\#\tau - 1, \lfloor \log_2(1 + \max_{u \in \tau} |u|) \rfloor \right) \stackrel{d}{=} (\#\tau_2, \mathcal{S}(\tau_2)). \quad (1.4.7)$$

1.4.4 Irrégularité des fluctuations du nombre de Horton-Strahler

Comme dans la section précédente, on note T_n un arbre de Catalan de taille $2n - 1$ de loi uniforme. L'estimée (1.4.5) permet de voir que la suite des variables aléatoires $\mathcal{S}(T_n) - \mathbb{E}[\mathcal{S}(T_n)]$ est tendue. Néanmoins, le développement asymptotique (1.4.4) indique que la loi de $\mathcal{S}(T_n)$ est soumise à des oscillations déterministes. Pire, ces variations semblent impacter négativement les applications du

nombre de Horton-Strahler dans le monde réel. En hydrogéologie par exemple, l'extraction d'un réseau fluvial en un arbre combinatoire nécessite de choisir un seuil à partir duquel on accepte de considérer un cours d'eau, et il y a été observé plusieurs fois qu'une différence légère sur ce choix peut engendrer des écarts importants sur le nombre de Horton-Strahler résultant. On renvoie entre autres à Moussa & Bocquillon [132] ou à Chavan & Srinivas [34]. Comme l'avertit Bamufleh et al. [14], ces écarts peuvent avoir des conséquences sur les mesures hydrologiques concrètes proposées aux décideurs politiques.

Ces observations tendent à suggérer l'inexistence d'une limite d'échelle pour le nombre de Horton-Strahler $\mathcal{S}(T_n)$ des arbres de Catalan. Donnons en outre un argument mathématique heuristique allant dans ce sens à l'aide d'un modèle plus simple mais apparenté. On rappelle la définition 1.1.7 de l'arbre de Galton-Watson 2-stable τ_2 et que, comme vu à la section 1.1.2, T_n est distribué comme τ_2 sous $\mathbb{P}(\cdot | \#\tau_2 = 2n - 1)$. On construit ensuite un arbre aléatoire T'_n en collant $n + 1$ arbres de Galton-Watson 2-stables indépendants le long d'une ligne de longueur n , comme illustré par la figure 1.3. En adoptant la terminologie de la section 1.3.2, T'_n est la n -troncature d'un arbre pointé de Galton-Watson infini de loi de reproduction μ_2 . Soit $(G_i)_{i \in \mathbb{N}}$ une suite de variables aléatoires indépendantes et géométriques de paramètre $1/2$. Grâce à la définition 1.4.1 et à la formule (1.4.6), on observe que $\mathcal{S}(T'_n)$ a la même loi que S_n , où la suite (S_i) est définie récursivement par

$$S_0 = G_0 \quad \text{et} \quad S_{i+1} = \max(G_{i+1}, S_i) + \mathbf{1}_{\{G_{i+1} = S_i\}} \quad \text{pour tout } i \in \mathbb{N}.$$

On peut aisément se convaincre qu'il existe un évènement A_n dépendant des $(G_i)_{i \in \mathbb{N}}$ tel que

$$S_n = M_n + \mathbf{1}_{A_n}, \quad \text{où} \quad M_n = \max_{0 \leq i \leq n} G_i.$$

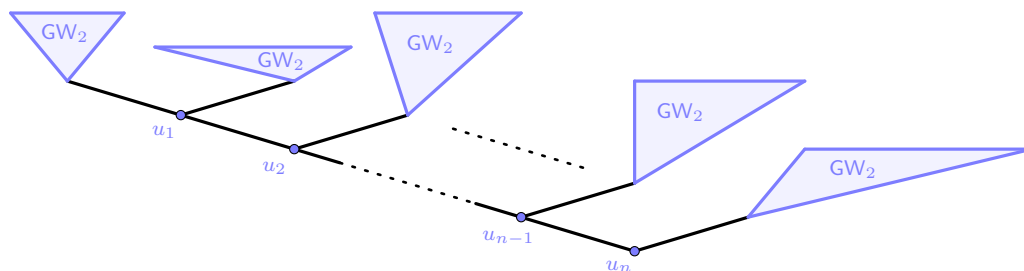


Figure 1.3: L'arbre binaire aléatoire T'_n .

Il est assez difficile de décrire l'évènement A_n de manière exacte : par exemple, il est réalisé lorsque $(G_i)_{0 \leq i \leq n} = (1, 1, 2, 3, \dots, n)$ mais pas quand $(G_i)_{0 \leq i \leq n} = (1, 2, 1, 3, \dots, n)$. Toutefois, se concentrer sur le terme M_n suffit déjà à révéler un comportement irrégulier. En effet, on sait qu'il n'existe pas de suites déterministes (a_n) et (b_n) telles que $(M_n - b_n)/a_n$ converge en loi vers une limite non-dégénérée. C'est une application classique du théorème des valeurs extrêmes qui est couramment utilisé en statistiques, voir Arnold, Balakrishnan & Nagaraja [12, page 217] par exemple. On renvoie aussi à Eisenberg [67] pour plus d'informations sur la loi de M_n . Les statistiques d'ordres de variables géométriques indépendantes font apparaître d'autres curiosités qui tendent à complexifier le comportement de S_n . Par exemple, Bruss & O'Conneide [29] montrent que la probabilité que le maximum des $(G_i)_{0 \leq i \leq n}$ soit atteint au moins deux fois (auquel cas l'évènement A_n est clairement réalisé) ne converge pas.

Ces phénomènes semblent provenir de la nature discrète du modèle. En effet, si on considère plutôt une suite $(E_i)_{i \in \mathbb{N}}$ de variables aléatoires indépendantes et exponentielles de moyenne

$(\ln 2)^{-1}$, de sorte que la partie entière de E_i a la même loi que G_i , alors on calcule aisément que

$$\forall x \in \mathbb{R}, \quad \mathbb{P}\left(\max_{0 \leq i \leq n} E_i - \log_2 n \leq x\right) \longrightarrow \exp(-2^{-x}),$$

et on reconnaît que la limite est la fonction de répartition de la loi de Gumbel de position 0 et d'échelle $(\ln 2)^{-1}$. Cette observation suggère de modifier la définition du nombre de Horton-Strahler pour obtenir une quantité continue dont la loi serait exponentielle pour un arbre de Galton-Watson 2-stable.

1.5 Contributions de cette thèse

1.5.1 Marches branchantes critiques et récurrentes nulles sur un arbre régulier

Cette section présente les résultats de l'article [60], écrit en collaboration avec Thomas Duquesne, Shen Lin & Niccolò Torri, correspondant au chapitre 2 de ce manuscrit.

Rappelons la loi générale (1.2.1) d'une marche branchante. Soit $b \in \mathbb{N}$ avec $b \geq 2$. On étudie ici une marche branchante sur l'arbre b -régulier infini

$$\mathbb{W}_b = \bigcup_{n \in \mathbb{N}} \{1, \dots, b\}^n, \quad \text{où} \quad \{1, \dots, b\}^0 = \{\emptyset\}.$$

On considère une marche branchante $(Y_u)_{u \in \tau}$ à valeurs dans \mathbb{W}_b dont la loi est décrite comme suit : l'arbre τ indexant la marche branchante est un arbre de Galton-Watson de loi de reproduction μ non-triviale et critique, c'est-à-dire que $\mu(1) \neq 1$ et $\sum_{k \in \mathbb{N}} k\mu(k) = 1$; conditionnellement à la généalogie τ , $(Y_u)_{u \in \tau}$ est une marche branchante dont les déplacements spatiaux sont ceux de la marche b -biaisée sur \mathbb{W}_b , c'est-à-dire que pour toute fonction mesurable $F : \mathbb{W}_b^T \rightarrow \mathbb{R}_+$, on a

$$\mathbb{E}[F(Y_u; u \in \tau) \mid \tau] = \int_{\mathbb{W}_b^T} F(y_u; u \in \tau) \delta_{\emptyset}(dy_{\emptyset}) \prod_{v \in \tau \setminus \{\emptyset\}} p(y_{\overline{v}}, dy_v),$$

où les quantités $p(x, y)$ désignent les probabilités de transition de la marche b -biaisée sur \mathbb{W}_b données par (1.3.1) et qui s'expriment ici par

$$p(x, \overline{x}) = \frac{1}{2} \quad \text{et} \quad p(\overline{x}, x) = \mathbf{1}_{\{\overline{x} \neq \emptyset\}} \frac{1}{2b} + \mathbf{1}_{\{\overline{x} = \emptyset\}} \frac{1}{b}$$

pour tout $x \in \mathbb{W}_b \setminus \{\emptyset\}$. On observe en particulier que $Y_{\emptyset} = \emptyset$ et que le processus $(|Y_u|)_{u \in \tau}$ des hauteurs prises par $(Y_u)_{u \in \tau}$ est une marche branchante simple sur \mathbb{N} (réfléchi en 0).

L'indexation de notre marche branchante est un arbre de Galton-Watson critique conditionné à être grand. Plus précisément, on rappelle (1.1.5) et les hypothèses présentées en section 1.1.2, et on considère une mesure de probabilité μ sur \mathbb{N} comme suit.

Hypothèse 1.5.1 *La mesure de probabilité μ est critique et non-triviale, aperiodique et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Il existe alors une suite $(a_n)_{n \in \mathbb{N}}$ de réels strictement positifs telle que $n^{-1/\alpha} a_n$ coïncide avec une fonction à variations lentes, et telle que si $(L_n)_{n \in \mathbb{N}}$ est une marche aléatoire sur \mathbb{Z} partant de $L_0 = 0$ et de sauts donnés par $\mathbb{P}(L_1 = k) = \mu(k + 1)$ pour tout entier $k \geq -1$ alors on a la convergence en loi suivante :*

$$\frac{1}{a_n} L_n \xrightarrow{d} X_1, \quad \text{où} \quad \mathbb{E}[\exp(-\lambda X_1)] = \exp(\lambda^\alpha) \text{ pour tout } \lambda > 0. \quad (1.5.1)$$

Notre premier résultat est une loi des grands nombres sur le nombre de points visités par la marche branchante, ce qui fait écho au théorème 1.2.6 de Le Gall & Lin.

Théorème 1.5.1 Soit τ un arbre de Galton-Watson de loi de reproduction μ dont on suppose qu'elle satisfait l'hypothèse 1.5.1. On liste les sommets de τ dans l'ordre lexicographique sous la forme $\emptyset = u_0 < u_1 < \dots < u_{\#\tau-1}$. Conditionnellement à τ , soit $(Y_u)_{u \in \tau}$ une marche branchante de position initiale $\emptyset \in \mathbb{W}_{\mathfrak{b}}$ et dont les transitions sont celles de la marche aléatoire \mathfrak{b} -biaisée sur $\mathbb{W}_{\mathfrak{b}}$. Alors, il existe une constante déterministe $c_{\mu, \mathfrak{b}} \in (0, 1)$ qui ne dépend que de μ et de \mathfrak{b} telle que

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} \max_{1 \leq k \leq n} |\#\{Y_{u_j} : 0 \leq j < k\} - c_{\mu, \mathfrak{b}} k| > \varepsilon \mid \#\tau = n \right) = 0.$$

En particulier, $\lim_{n \rightarrow \infty} \mathbb{P}(|\frac{1}{n} \#\mathcal{R} - c_{\mu, \mathfrak{b}}| > \varepsilon \mid \#\tau = n) = 0$ pour tout $\varepsilon > 0$, où \mathcal{R} désigne la trace $\{Y_u : u \in \tau\}$ de la marche branchante.

La preuve suit dans les grandes lignes la méthode utilisée par Le Gall & Lin [83]. On se ramène d'abord par un couplage spécifique à l'étude de la marche branchante sur l'arbre pointé de Galton-Watson invariant de loi de reproduction $\delta_{\mathfrak{b}}$ et indexée par l'arbre pointé de Galton-Watson infini de loi de reproduction μ , voir les définitions 1.3.3 et 1.3.5. Ce modèle est invariant par les opérations simultanées de successeur sur l'indexation et de recentrage en la position de la marche sur l'environnement. Puis, la loi des grands nombres pour la taille de la trace s'obtient par une application du théorème ergodique sous-additif, et la constante $c_{\mu, \mathfrak{b}}$ s'interprète alors comme une probabilité de non-retour à l'origine pour le modèle invariant. Enfin, la convergence uniforme suit grâce à des arguments de type Dini.

Notre second résultat est une limite d'échelle pour la trace de la marche branchante. Commençons par construire l'objet limite. Soit (h, w) un serpent à valeurs dans \mathbb{R} au sens de la définition 1.2.2 dont l'extrémité est notée (h, \hat{w}) pour rappel. Pour tous $0 \leq s_1 \leq s_2 \leq 1$, on pose $m_h(s_1, s_2) = \inf_{[s_1, s_2]} h$ et

$$M_{h,w}(s_1, s_2) = M_{h,w}(s_2, s_1) = \min \left(\inf_{m_h(s_1, s_2) \leq r} w_{s_1}(r), \inf_{m_h(s_1, s_2) \leq r} w_{s_2}(r) \right)$$

pour alléger les notations. On définit alors

$$\forall s_1, s_2 \in [0, 1], \quad d_{h,w}(s_1, s_2) = \hat{w}_{s_1} + \hat{w}_{s_2} - 2M_{h,w}(s_1, s_2), \quad (1.5.2)$$

et on prouve que $d_{h,w}$ satisfait les propriétés suivantes (voir le lemme 2.4.22 du chapitre 2).

Proposition 1.5.2 L'application $d_{h,w} : [0, 1]^2 \rightarrow \mathbb{R}$ est une pseudo-distance continue qui satisfait l'inégalité (1.1.13) dite des quatre points.

Par conséquent, l'application $d_{h,w}$ induit une vraie distance (toujours notée $d_{h,w}$) sur l'espace quotient $\mathcal{T}_{h,w} := [0, 1] / \{d_{h,w} = 0\}$. De plus, l'espace métrique $(\mathcal{T}_{h,w}, d_{h,w})$ est un arbre réel compact. En notant la projection canonique $\mathfrak{p}_{h,w} : [0, 1] \rightarrow \mathcal{T}_{h,w}$, on pose $\rho_{h,w} = \mathfrak{p}_{h,w}(0)$ et on désigne par $\mu_{h,w}$ la mesure image par $\mathfrak{p}_{h,w}$ de la mesure de Lebesgue sur $[0, 1]$. Autrement dit, on pose pour toute fonction mesurable et bornée $g : \mathcal{T}_{h,w} \rightarrow \mathbb{R}$,

$$\int_{\mathcal{T}_{h,w}} g \, d\mu_{h,w} = \int_{[0,1]} g \circ \mathfrak{p}_{h,w}(s) \, ds.$$

On appelle *arbre-serpent* codé par (h, w) l'arbre réel compact enraciné mesuré comme suit :

$$(\mathcal{T}_{h,w}, d_{h,w}, \rho_{h,w}, \mu_{h,w}). \quad (1.5.3)$$

La pseudo-distance $d_{h,w}$ et l'arbre-serpent $\mathcal{T}_{h,w}$ admettent une description similaire à celle de l'arbre réel codé par une fonction continue sur un segment. Pour alléger les notations, on note

$(\mathcal{T}_h, d_h, \rho_h, \mu_h)$ l'arbre réel compact enraciné mesuré codé par $(h, 1)$ comme défini par (1.1.15), et $\rho_h : [0, 1] \rightarrow \mathcal{T}_h$ la projection canonique. Comme expliqué au cours de la section 1.2.2, grâce à la propriété (1.2.4), le serpent (h, w) induit une fonction \tilde{w} continue sur \mathcal{T}_h en posant $\tilde{w}(\rho_h(s)) = \hat{w}_s$ pour tout $s \in [0, 1]$. Soit $s_1, s_2 \in [0, 1]$, on pose $\sigma_1 = \rho_h(s_1)$ et $\sigma_2 = \rho_h(s_2)$, et on a alors l'identité

$$d_{h,w}(s_1, s_2) = \tilde{w}(\sigma_1) + \tilde{w}(\sigma_2) - 2 \inf_{\sigma \in [\sigma_1, \sigma_2]} \tilde{w}(\sigma), \quad (1.5.4)$$

où $[\sigma_1, \sigma_2]$ est l'image de la géodésique de σ_1 à σ_2 sur \mathcal{T}_h (voir la définition 1.1.6). On renvoie à la remarque 2.4.21 du chapitre 3 pour plus de détails. Ainsi, on peut voir $d_{h,w}$ comme une pseudo-distance continue sur \mathcal{T}_h , et $\mathcal{T}_{h,w}$ comme un espace métrique quotient de \mathcal{T}_h . Dans le cas particulier où l'arbre réel \mathcal{T}_h est simplement un segment métrique, la formule (1.5.4) devient l'expression classique (1.1.14).

On prouve dans la section 2.4.3 du chapitre 2 que le codage (1.5.2) d'arbres-serpent bénéficie d'une propriété de continuité. Plus précisément, si une suite de serpents réels (h_n, w_n) converge vers (h, w) par rapport à la distance d_Σ donnée par (1.2.3), alors d_{h_n, w_n} converge vers $d_{h,w}$ uniformément sur $[0, 1]^2$. Cela implique alors que \mathcal{T}_{h_n, w_n} converge vers $\mathcal{T}_{h,w}$ pour la distance d_{GHP} de Gromov-Hausdorff-Prokhorov donnée par (1.1.12). En d'autres termes, l'application $(h, w) \in \Sigma^1 \mapsto \mathcal{T}_{h,w} \in \mathbb{K}$ est continue, où Σ^1 désigne l'espace des serpents réels muni de d_Σ et \mathbb{K} désigne l'espace des classes d'équivalence d'espaces métriques compacts enracinés mesurés muni de d_{GHP} .

Soit $\alpha \in (1, 2]$ et soit (H, W) un serpent brownien unidimensionnel avec mécanisme de branchement α -stable comme dans la définition 1.2.3. On rappelle que l'arbre réel \mathcal{T}_H codé par H est un arbre α -stable, et que (H, W) est une variable aléatoire sur Σ^1 . D'après l'argument précédent, les arbres-serpent $\mathcal{T}_{H,W}$ et $\mathcal{T}_{H,|W|}$ définis par (1.5.3) sont des variables aléatoires sur \mathbb{K} . Il se trouve que ce genre d'objets a déjà été construit par Curien, Le Gall & Miermont [40] (voir aussi Le Gall [116]) dans le cadre de l'étude de cartes planaires aléatoires. Ils l'ont baptisé "Brownian cactus", un terme que l'on reprend ici.

Définition 1.5.3 On appelle $\mathcal{T}_{H,W}$ le *cactus brownien normalisé avec mécanisme de branchement α -stable*, et $\mathcal{T}_{H,|W|}$ le *cactus brownien réfléchi normalisé avec mécanisme de branchement α -stable*.

On mentionne maintenant quelques propriétés géométriques de ces objets et on renvoie à la section 2.4.6 du chapitre 2 pour plus de détails. La mesure $\mu_{H,W}$ (resp. $\mu_{H,|W|}$) de $\mathcal{T}_{H,W}$ (resp. $\mathcal{T}_{H,|W|}$) est diffuse et supportée par l'ensemble de ses feuilles (i.e. les points dont le retrait ne déconnecte pas le cactus). Contrairement aux arbres stables d'indice $\alpha \in (1, 2)$, les branchements des cactus browniens sont essentiellement binaires quelque soit la valeur de α : si σ est un point de $\mathcal{T}_{H,W}$ (resp. $\mathcal{T}_{H,|W|}$) distinct de la racine, alors le nombre de composantes connexes de $\mathcal{T}_{H,W} \setminus \{\sigma\}$ (resp. $\mathcal{T}_{H,|W|} \setminus \{\sigma\}$) appartient à $\{1, 2, 3\}$. On sait aussi que $\mathcal{T}_{H,W} \setminus \{\rho_{H,W}\}$ est connexe, alors que $\mathcal{T}_{H,|W|} \setminus \{\rho_{H,|W|}\}$ a une infinité de composantes connexes. Enfin, Curien, Le Gall & Miermont [40, Corollary 5.3] calculent que la dimension de Hausdorff du cactus brownien normalisé avec mécanisme de branchement 2-stable est égale à 4. On peut finalement énoncer notre résultat.

Théorème 1.5.4 Soit τ un arbre de Galton-Watson de loi de reproduction μ dont on suppose qu'elle satisfait l'hypothèse 1.5.1, et soit (a_n) comme dans (1.5.1). On note $\mathbf{m} = \sum_{u \in \tau} \delta_u$ la mesure de comptage sur τ . Conditionnellement à τ , soit $(Y_u)_{u \in \tau}$ une marche branchante de position initiale $\emptyset \in \mathbb{W}_b$ et dont les transitions sont celles de la marche aléatoire b -biaisée sur \mathbb{W}_b . On note $\mathcal{R} = \{Y_u : u \in \tau\}$ la trace de la marche branchante, et $\mathbf{m}_{\text{occ}} = \sum_{u \in \tau} \delta_{Y_u}$ sa mesure d'occupation. Les ensembles τ et \mathcal{R} sont tous deux munis de la distance de graphe d_{gr} donnée par

(1.1.2), et du point distingué \emptyset . Comme ci-dessus, on désigne par $(\mathcal{T}_H; \mathcal{T}_{H,|W|})$ la paire formée de l'arbre α -stable et du cactus brownien réfléchi normalisé avec mécanisme de branchement α -stable. Alors, la convergence jointe suivante

$$\left(\left(\tau, \frac{a_n}{n} \mathbf{d}_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m} \right); \left(\mathcal{R}, \sqrt{\frac{a_n}{n}} \mathbf{d}_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}_{\text{occ}} \right) \right) \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n) \\ \xrightarrow{d} \left((\mathcal{T}_H, d_H, \rho_H, \mu_H); (\mathcal{T}_{H,|W|}, d_{H,|W|}, \rho_{H,|W|}, \mu_{H,|W|}) \right) \quad (1.5.5)$$

a lieu en loi sur l'espace produit \mathbb{K}^2 , où \mathbb{K} est muni de la distance de Gromov-Hausdorff-Prokhorov donnée par (1.1.12). De plus, si on note $\mathbf{m}_{\text{count}} = \sum_{x \in \mathcal{R}} \delta_x$ la mesure de comptage sur \mathcal{R} , alors

$$\forall \varepsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P} \left(\mathbf{d}_{\text{Prok}}^{(n)} \left(\frac{c_{\mu, \mathbf{b}}}{n} \mathbf{m}_{\text{occ}}, \frac{1}{n} \mathbf{m}_{\text{count}} \right) > \varepsilon \mid \#\tau = n \right) = 0,$$

où $c_{\mu, \mathbf{b}}$ est comme dans le théorème 1.5.1 et où $\mathbf{d}_{\text{Prok}}^{(n)}$ désigne la distance de Prokhorov sur les mesures boréliennes finies de $(\mathcal{R}, \sqrt{a_n/n} \mathbf{d}_{\text{gr}})$. Cela implique que la convergence en loi suivante a lieu sur \mathbb{K} conjointement avec (1.5.5) :

$$\left(\mathcal{R}, \sqrt{\frac{a_n}{n}} \mathbf{d}_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}_{\text{count}} \right) \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n) \xrightarrow{d} (\mathcal{T}_{H,|W|}, d_{H,|W|}, \rho_{H,|W|}, c_{\mu, \mathbf{b}} \cdot \mu_{H,|W|}).$$

Remarquons que la convergence de la première marginale énoncée par le théorème est exactement la convergence (1.1.17) déduite du théorème 1.1.5 de Duquesne. La première étape de la preuve de (1.5.5) est une analyse discrète de la structure de la trace dans le cas où \mathbf{b} vaut l'infini. Dans ce cas, la marche ne traverse jamais deux fois la même arête dans le même sens, et on montre alors que la trace \mathcal{R} est un arbre-serpent discret codé par un serpent $(h(Y), w(Y))$ résultant du processus $(|Y_u|)_{u \in \tau}$ des hauteurs prises par Y . De manière un peu plus explicite, on a

$$h_{i/N}(Y) = |c_i(\tau)|, \quad \widehat{w}_{i/N}(Y) = |Y_{c_i(\tau)}|, \quad \text{et} \quad \mathbf{d}_{\text{gr}}(Y_{c_i(\tau)}, Y_{c_j(\tau)}) = d_{h(Y), w(Y)} \left(\frac{i}{N}, \frac{j}{N} \right)$$

pour tous entiers $0 \leq i, j \leq N$, où $N = 2(\#\tau - 1)$, où $d_{h(Y), w(Y)}$ est définie par (1.5.2), et où on rappelle de la section 1.1.3 que $(c_i(\tau); 0 \leq i \leq 2(\#\tau - 1))$ désigne l'exploration par contour de τ . Comme $(|Y_u|)_{u \in \tau}$ a la même loi que la valeur absolue d'une marche branchante simple sur \mathbb{Z} , le théorème 1.2.4 de Marzouk et la propriété de continuité du codage d'arbres-serpent permettent de prouver qu'après renormalisation, $d_{h(Y), w(Y)}$ converge en loi (pour la topologie de la convergence uniforme sur $[0, 1]^2$) vers la pseudo-distance $d_{H,|W|}$ codée par le serpent brownien réfléchi.

La seconde étape de la preuve consiste à justifier que même lorsque \mathbf{b} est fini, la trace de la marche branchante est très proche de l'arbre-serpent discret rencontré précédemment. En effet, la contraction envoyant l'arbre régulier "libre" sur l'arbre \mathbf{b} -aire préserve les hauteurs et n'influence que localement la géométrie de la trace. Plus précisément, la différence des distances avant et après contraction est presque une loi géométrique de paramètre $1/\mathbf{b}$. Grâce à cette idée de couplage, on prouve qu'avec probabilité tendant vers 1 sous $\mathbb{P}(\cdot \mid \#\tau = n)$ quand $n \rightarrow \infty$, on a

$$\sup_{u, v \in \tau} \left| |Y_u| + |Y_v| - 2 \min_{u' \in \llbracket u, v \rrbracket_\tau} |Y_{u'}| - \mathbf{d}_{\text{gr}}(Y_u, Y_v) \right| \leq n^2 \mathbf{b}^{-(\ln n)^2},$$

où $\llbracket u, v \rrbracket_\tau$ désigne le plus court chemin de u à v sur l'arbre τ vu comme un graphe. À quelques détails près, cela conclut la preuve de (1.5.5). Pour finir, les derniers points du théorème 1.5.4 sont des conséquences du théorème 1.5.1. Il est intéressant d'observer que contrairement au théorème 1.5.1, la convergence en loi (1.5.5) du théorème 1.5.4 ne fait pas intervenir le paramètre de régularité \mathbf{b} de l'environnement.

L'argument de contraction de la seconde étape de la preuve ne repose pas vraiment sur la régularité de l'environnement, car on peut prouver que pour toute marche aléatoire λ -biaisée $(X_i)_{i \in \mathbb{N}}$ sur un arbre t infini (de transitions données par (1.3.1)), si X est récurrente et si $\lambda > 1$ alors on a

$$\forall i, \ell \in \mathbb{N}, \quad \mathbb{P}\left(\left||X_0| + |X_i| - 2 \min_{0 \leq j \leq i} |X_j| - d_{\text{gr}}(X_0, X_i)\right| \geq 2\ell\right) \leq j \frac{\lambda - 1}{\lambda^\ell - 1}. \quad (1.5.6)$$

De ce fait, on peut raisonnablement assimiler la marche non-branchante $(X_i)_{i \leq n}$ à l'exploration par contour de sa trace $\{X_j : j \leq n\}$ afin de récupérer la métrique de la trace à partir des hauteurs prises par la marche via la formule (1.1.7). Cette approximation nous permet de retrouver la limite d'échelle de la trace de la marche aléatoire (non-branchante) présentée par le théorème 1.3.1 de Aïdékon & de Raphaélis à partir du théorème central limite pour le processus des hauteurs de la marche, qui avait déjà été obtenu par Peres & Zeitouni [137] sous une hypothèse de moments exponentiels. De même, on déduit d'un résultat de Dembo & Sun [45] que la limite d'échelle de la trace de la marche aléatoire biaisée récurrente nulle sur un arbre de Galton-Watson multi-type est l'arbre réel codé par le mouvement brownien réfléchi sur $[0, 1]$. On renvoie à la section 2.4.2 du chapitre 2 pour un énoncé précis.

1.5.2 Marches branchantes biaisées critiques sur un arbre de Galton-Watson invariant

Cette section présente les résultats de l'article [59], écrit en collaboration avec Thomas Duquesne, correspondant au chapitre 3 de ce manuscrit.

Ici, on expose une généralisation des théorèmes 1.5.1 et 1.5.4 de la section 1.5.1 en environnement aléatoire, lorsque le déplacement spatial de la marche branchante est celui d'une marche aléatoire biaisée critique sur un arbre de Galton-Watson \mathbf{T} de loi de reproduction sur-critique ν qui est invariant (voir la définition 1.3.5). On commence par donner une loi des grands nombres pour l'image de la marche branchante, ce résultat ayant lieu en moyennant sur l'environnement (*annealed*). Puis, on énonce un théorème limite pour l'image de la marche branchante (vue comme un arbre aléatoire), ce résultat ayant lieu conditionnellement à l'environnement \mathbf{T} (*quenched*). L'espace métrique limite est alors un cactus brownien normalisé avec mécanisme de branchement stable (voir la définition 1.5.3) qui est indépendant de \mathbf{T} .

L'environnement de la marche aléatoire branchante que l'on considère est, comme déjà mentionné, un arbre pointé de Galton-Watson invariant $\mathbf{T} = (\mathbf{T}, o)$ de loi de reproduction ν comme à la définition 1.3.5. On fait les hypothèses suivantes sur ν .

$$m := \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty) \quad \text{et} \quad \eta^2 := \frac{m(m-1)}{\sum_{k \in \mathbb{N}} k(k-1)\nu(k)} \in (0, \infty), \quad (1.5.7)$$

ce qui implique en particulier que ν admet un moment d'ordre 2 et que \mathbf{T} n'a pas de feuilles.

On rappelle la définition générale (1.2.1) de la loi d'une marche branchante. On considère une marche branchante $(Y_u)_{u \in \tau}$ à valeurs dans \mathbf{T} dont la loi est décrite comme suit : l'arbre τ indexant la marche branchante est un arbre de Galton-Watson de loi de reproduction μ non-triviale et critique, c'est-à-dire que $\mu(1) \neq 1$ et $\sum_{k \in \mathbb{N}} k\mu(k) = 1$, et indépendant de $\mathbf{T} = (\mathbf{T}, o)$; conditionnellement à l'environnement \mathbf{T} et à la généalogie τ , $(Y_u)_{u \in \tau}$ est une marche branchante dont les déplacements spatiaux sont ceux de la marche aléatoire m -biaisée sur \mathbf{T} , c'est-à-dire que pour toute fonction mesurable $F : \mathbf{T}^\tau \rightarrow \mathbb{R}_+$, on a

$$\mathbb{E}[F(Y_u; u \in \tau) \mid \mathbf{T}, \tau] = \int_{\mathbf{T}^\tau} F(y_u; u \in \tau) \delta_o(dy_\emptyset) \prod_{v \in \tau \setminus \{\emptyset\}} p_{\mathbf{T}}(y_v, dy_v) \quad (1.5.8)$$

où, conditionnellement à \mathbf{T} , les quantités $p_{\mathbf{T}}(x, y)$ désignent les probabilités de transition de la marche m -biaisée sur \mathbf{T} données par (1.3.5) et qui s'expriment ici par

$$\forall x, y \in \mathbf{T}, \quad p_{\mathbf{T}}(x, y) = \mathbf{1}_{\{\overleftarrow{x}=y\}} \frac{m}{m + k_x(\mathbf{T})} + \mathbf{1}_{\{\overrightarrow{y}=x\}} \frac{1}{m + k_x(\mathbf{T})}. \quad (1.5.9)$$

On observe en particulier que $Y_{\emptyset} = o$.

Plus précisément, la généalogie de la marche branchante est un arbre de Galton-Watson de loi de reproduction vérifiant l'hypothèse 1.5.1, indépendant de \mathbf{T} , et conditionné à être grand. Sous ces conditions, on a un résultat analogue au théorème 1.5.1.

Théorème 1.5.5 *Soit $\mathbf{T} = (\mathbf{T}, o)$ un arbre pointé de Galton-Watson invariant de loi de reproduction ν . Indépendamment de \mathbf{T} , soit τ un arbre de Galton-Watson de loi de reproduction μ . Soit $(Y_u)_{u \in \tau}$ la marche branchante à valeurs dans \mathbf{T} définie par (1.5.8). On suppose que μ satisfait l'hypothèse 1.5.1. On suppose également que ν satisfait (1.5.7) et qu'elle possède un moment fini d'ordre $\mathbf{a} > 1$. La suite $\emptyset = u_0 < u_1 < \dots < u_{\#\tau-1}$ désigne les sommets de τ listés dans l'ordre lexicographique. Alors, il existe une constante déterministe $c_{\mu, \nu} \in (0, 1)$ qui ne dépend que de μ et de ν telle que*

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} \max_{1 \leq k \leq n} |\#\{Y_{u_j} : 0 \leq j < k\} - c_{\mu, \nu} k| > \varepsilon \mid \#\tau = n \right) = 0.$$

En particulier, $\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} \#\mathcal{R} - c_{\mu, \nu} > \varepsilon \mid \#\tau = n \right) = 0$ pour tout $\varepsilon > 0$, où \mathcal{R} désigne la trace $\{Y_u : u \in \tau\}$ de la marche branchante.

La preuve repose essentiellement sur les mêmes arguments que celle du théorème 1.5.1. L'idée est d'introduire une marche branchante $(Y_u^*)_{u \in \tau^*}$ sur le même arbre \mathbf{T} et avec les mêmes probabilités de transitions (1.5.9) que $(Y_u)_{u \in \tau}$, mais indexée cette fois par un arbre pointé de Galton-Watson infini $\tau^* = (\tau^*, \rho)$ de loi de reproduction μ , comme à la définition 1.3.3. Le point-clef est que la loi de cette marche branchante est invariante et ergodique par la transformation qui consiste à recentrer τ^* en l'individu $\rho^+ := \text{succ}(\tau^*, \rho)$ venant juste après ρ dans l'ordre lexicographique sur τ^* et à recentrer les positions spatiales Y_u^* en $\varphi_{|Y_{\rho^+}^*|}(Y_u)$. On renvoie à la section 1.3.2 pour les définitions de ces transformations et à la proposition 3.2.32 du chapitre 3 pour un énoncé précis de cette invariance. La suite de la démonstration se résume en une application du théorème ergodique sous-additif globalement inspirée de la preuve du théorème 1.2.6 de Le Gall & Lin [83].

Comme notre preuve repose sur la propriété d'invariance (1.3.6) de l'environnement, elle ne permet pas d'obtenir directement un résultat vrai conditionnellement à \mathbf{T} . Les arguments de couplage qui seraient nécessaires pour obtenir un tel résultat quenched sont rendus difficiles par l'absence d'homogénéité due à la dépendance des trajectoires en l'environnement. De même, le cas d'un environnement avec racine pose des problèmes de couplage qui sont discutés en fin de section.

On continue à désigner par $\mathbf{T} = (\mathbf{T}, o)$ un arbre pointé de Galton-Watson invariant de loi de reproduction ν . On rappelle que pour tout $x \in \mathbf{T}$, la hauteur relative de x est notée $|x| \in \mathbb{Z}$, de sorte que $|\overleftarrow{x}| = |x| - 1$ et $|o| = 0$. Pour alléger l'écriture (puisque l'on énonce un résultat de convergence en loi conditionnelle), on utilise la notation τ_n introduite avec (1.1.4) : l'arbre aléatoire τ_n sous \mathbb{P} a la même loi que l'arbre de Galton-Watson τ de loi de reproduction μ sous $\mathbb{P}(\cdot \mid \#\tau = n)$. De plus, on suppose que \mathbf{T} et τ_n sont indépendants. On rappelle également de la section 1.1.3 que l'on note $(c_j(\tau_n); 0 \leq j \leq 2(n-1))$ l'exploration par contour de τ_n . Pour tout entier $0 \leq j \leq 2(n-1)$, on pose $C_j^{(n)} = |c_j(\tau_n)|$ et $\widehat{W}_j^{(n)} = |Y_{c_j(\tau_n)}|$, où $(Y_u)_{u \in \tau_n}$ est la marche branchante à valeurs dans \mathbf{T} définie comme précédemment par (1.5.8). Pour tout $j \geq 2(n-1)$, on pose $C_j^{(n)} = \widehat{W}_j^{(n)} = 0$

et on étend $C^{(n)}$ et $\widehat{W}^{(n)}$ en des fonctions continues sur \mathbb{R}_+ par interpolation linéaire entre les temps entiers. On montre tout d'abord le théorème limite suivant, qui a lieu conditionnellement à l'environnement (quenched).

Théorème 1.5.6 *On garde les notations ci-dessus et on se place sous les mêmes hypothèses que dans le théorème 1.5.5. Soient $\alpha \in (1, 2]$ et (a_n) comme dans (1.5.1), et soit η^2 comme dans (1.5.7). On suppose de plus que ν admet un moment fini d'ordre $\mathfrak{a} > 1 + \frac{4\alpha}{\alpha-1}$. Soit (H, W) le serpent brownien unidimensionnel avec mécanisme de branchement α -stable, dont l'extrémité est notée (H, \widehat{W}) , comme à la définition 1.2.3. Alors conditionnellement à l'environnement $\mathbf{T} = (\mathbb{T}, o)$, on a presque sûrement la convergence en loi suivante*

$$\left(\frac{a_n}{n} C_{2ns}^{(n)}, \sqrt{\frac{a_n}{\eta^2 n}} \widehat{W}_{2ns}^{(n)} \right)_{s \in [0,1]} \text{ sous } \mathbb{P}(\cdot | \mathbf{T}) \xrightarrow{d} (H, \widehat{W}) \quad (1.5.10)$$

sur l'espace $\mathbb{C}([0, 1], \mathbb{R}^2)$ des fonctions continues muni de la topologie de la convergence uniforme.

Ce résultat fait à la fois écho au théorème 1.2.4 de Marzouk et au théorème 1.3.1 de Aidékon & de Raphélis. Dans le cas d'un environnement régulier où $\nu = \delta_b$ avec un entier $b \geq 2$, le processus $(|Y_u|)_{u \in \tau}$ des hauteurs relatives prises par la marche est une marche branchante simple aux plus proches voisins sur \mathbb{Z} . Cependant, en environnement aléatoire, $(|Y_u|)_{u \in \tau}$ n'est plus une marche branchante car les transitions depuis une position x dépendent du nombre d'enfants de x sur l'arbre.

Pour montrer (1.5.10), on adapte l'approche de Peres & Zeitouni [137] consistant à utiliser les coordonnées harmoniques. Plus précisément, on peut construire une famille de variables aléatoires $(S_x)_{x \in \mathbb{T}}$ telle que si $(X_n)_{n \in \mathbb{N}}$ est une marche aléatoire \mathfrak{m} -biaisée sur \mathbf{T} alors $(S_{X_n})_{n \in \mathbb{N}}$ est une martingale conditionnellement à $\mathbf{T} = (\mathbb{T}, o)$. En utilisant l'inégalité de Rosenthal pour les martingales comme dans Marzouk [130] et un théorème central limite conditionnel, on obtient la convergence conditionnellement à \mathbf{T} de la marche branchante en coordonnées harmoniques $(S_{Y_u})_{u \in \tau_n}$ (parcourue selon l'exploration par contour de τ_n et correctement renormalisée). Enfin, un argument spécifique basé sur l'invariance de \mathbf{T} montre que $(S_{Y_u})_{u \in \tau_n}$ et $(|Y_u|)_{u \in \tau_n}$ sont asymptotiquement proches à une constante déterministe multiplicative près.

On reprend les notations précédentes et on note $\mathcal{R}_n = \{Y_u : u \in \tau_n\}$ la trace de la marche branchante indexée par l'arbre conditionné τ_n . Les ensembles τ_n et \mathcal{R}_n sont munis des distances de graphe d_{gr} respectivement données par (1.1.2) et (1.3.4). On introduit également la mesure de comptage sur τ_n et la mesure d'occupation induite sur \mathcal{R}_n :

$$\mathfrak{m}_n = \sum_{u \in \tau_n} \delta_u \quad \text{et} \quad \mathfrak{m}_{\text{occ}}^{(n)} = \sum_{u \in \tau_n} \delta_{Y_u}.$$

On obtient alors le théorème limite métrique suivant, qui complète le théorème 1.5.6.

Théorème 1.5.7 *On se place sous les hypothèses du théorème 1.5.6 et on garde les notations précédentes. On désigne par $(\mathcal{T}_H; \mathcal{T}_{H,W})$ la paire formée de l'arbre α -stable et du cactus brownien normalisé avec mécanisme de branchement α -stable, respectivement définis par (1.1.15) et (1.5.3) à partir de (H, W) . Alors conditionnellement à l'environnement $\mathbf{T} = (\mathbb{T}, o)$, la convergence en loi jointe suivante*

$$\left(\left(\tau_n, \frac{a_n}{n} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathfrak{m}_n \right); \left(\mathcal{R}_n, \sqrt{\frac{a_n}{\eta^2 n}} d_{\text{gr}}, o, \frac{1}{n} \mathfrak{m}_{\text{occ}}^{(n)} \right) \right) \text{ sous } \mathbb{P}(\cdot | \mathbf{T}) \\ \xrightarrow{d} \left((\mathcal{T}_H, d_H, \rho_H, \mu_H); (\mathcal{T}_{H,W}, d_{H,W}, \rho_{H,W}, \mu_{H,W}) \right)$$

a lieu sur l'espace produit \mathbb{K}^2 , où \mathbb{K} est muni de la distance de Gromov-Hausdorff-Prokhorov (1.1.12), et conjointement avec (1.5.10).

Ce résultat en environnement aléatoire est analogue au théorème 1.5.4. Néanmoins, ici, le cactus brownien n'est pas réfléchi. Cette différence est due à l'absence de réflexions lorsque la marche branchante se déplace sur un arbre de profondeur infinie, et non pas sur un arbre enraciné. Il est intéressant de souligner que les objets limites des théorèmes 1.5.6 et 1.5.7 ne dépendent pas de l'aléa contenu dans l'environnement $\mathbf{T} = (\mathbb{T}, o)$. Pour prouver le théorème 1.5.7, on utilise l'estimée (1.5.6) afin de montrer que la trace \mathcal{R}_n est très proche de l'arbre-serpent discret induit par $(C^{(n)}, \widehat{W}^{(n)})$. Puis, le théorème 1.5.6 et la propriété générale de continuité du codage d'arbres-serpent présentée au cours de la section 1.5.1 permettent de conclure.

Les théorèmes de cette section concernent la marche branchante sur un arbre pointé de Galton-Watson invariant. Par absolue continuité, ils restent valides si l'environnement est remplacé par un arbre pointé de Galton-Watson infini de loi de reproduction ν (voir la définition 1.3.3). Néanmoins, l'obtention de résultats analogues pour la marche branchante sur un arbre de Galton-Watson (avec racine) fait l'objet d'un travail encore en cours car elle est bien plus délicate. Tout d'abord, la réflexion de la marche à la racine brise la propriété de martingale associée aux coordonnées harmoniques, ce qui empêche de simplement adapter la preuve du théorème 1.5.6. On perd aussi l'invariance utilisée pour montrer le théorème 1.5.5.

Pour le travail en environnement régulier, on parvient à éviter ces difficultés en couplant les marches sur l'arbre avec racine et sur l'arbre de profondeur infinie, de façon à ce que leurs déplacements coïncident en dehors des réflexions à la racine. Cet argument s'appuie sur l'homogénéité des transitions, qui est perdue dans le cas non-régulier. Pour étudier la marche aléatoire (non-branchante) sur un arbre, Peres & Zeitouni [137] développent un autre couplage qui identifie les excursions au dessus des sommets visités pour la première fois. Il paraît ardu d'adapter leur méthode à la marche branchante car la structure des excursions y est plus complexe. En effet, alors que les excursions d'une marche aléatoire sont ordonnées linéairement, celles d'une marche branchante s'agencent de manière arborescente. En particulier, le nombre d'excursions au sortir d'une précédente est aléatoire et dépend fortement de l'environnement déjà exploré, ce qui va à l'encontre de l'objectif du couplage.

1.5.3 Nombre de Horton-Strahler des arbres de Galton-Watson critiques : estimées

Cette section présente les résultats de l'article [103], correspondant au chapitre 4 de ce manuscrit.

Rappelons la définition 1.4.1 du nombre de Horton-Strahler noté \mathcal{S} , ainsi que les hypothèses présentées en section 1.1.2 avec notamment (1.1.5) et (1.1.6). On considère ici un arbre de Galton-Watson dont la loi de reproduction μ est supposé critique et non-triviale, c'est-à-dire

$$\mu(1) < 1 \quad \text{et} \quad \sum_{k \in \mathbb{N}} k \mu(k) = 1,$$

et on cherche à décrire le comportement asymptotique de son nombre de Horton-Strahler de diverses manières. Les fonctions suivantes jouent un rôle privilégié dans cette analyse :

$$\forall s \in [0, 1], \quad \varphi(s) = \sum_{k \in \mathbb{N}} s^k \mu(k) \quad \text{et} \quad \psi(s) = \varphi(1-s) - (1-s). \quad (1.5.11)$$

On commence à relier le nombre de Horton-Strahler à d'autres quantités simples de l'arbre grâce à plusieurs inégalités, regroupées dans la proposition ci-dessous. Ces estimées (excepté le point (iv)) sont valables sous la seule hypothèse que μ est critique et non-triviale.

Proposition 1.5.8 *Soit τ un arbre de Galton-Watson dont la loi de reproduction μ est critique et non-triviale. Pour tout $n \geq -1$, on pose $\mathbb{P}(\mathcal{S}(\tau) > n) = q_n$. On rappelle l'expression (1.5.11) de ψ .*

(i) On rappelle que la taille $\#\tau$ de τ est son nombre de sommets. Alors, on a

$$\forall n \in \mathbb{N}, \quad \psi'(q_{n-1})\mathbb{E}[\#\tau \mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}}] \leq 2.$$

(ii) On note $|\tau| = \max_{u \in \tau} |u|$ la hauteur de τ . Alors, on a

$$\forall \lambda > 0, \quad \limsup_{n \rightarrow \infty} \mathbb{P}(\psi'(q_{n-1})|\tau| \leq \lambda \mid \mathcal{S}(\tau) = n) \leq 1 - e^{-\lambda}.$$

(iii) On note encore $|\tau|$ la hauteur de τ . Pour tous $n, m \in \mathbb{N}$, si $2\psi'(\mathbb{P}(|\tau| \geq n/2 - 1)) \leq \psi'(q_m)$ alors on a

$$\mathbb{P}(\mathcal{S}(\tau) \leq m \mid |\tau| \geq n) \leq \exp\left(-\frac{1}{8}n\psi'(q_m)\right).$$

(iv) On note encore $|\tau|$ la hauteur de τ . Si μ est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$, alors il existe une constante $C_\mu \in (0, \infty)$ qui ne dépend que de μ telle que

$$\forall n, m \in \mathbb{N}, \quad \mathbb{P}(\mathcal{S}(\tau) \leq m \mid |\tau| \geq n) \leq C_\mu \exp\left(-\frac{1}{8}n\psi'(q_m)\right).$$

(v) On note $\Delta(\tau) = \max_{u \in \tau} k_u(\tau)$ le degré sortant maximal de τ . Pour tout $n \in \mathbb{N}^*$, si $\mathbb{P}(\Delta(\tau) \geq n) > 0$ alors on a

$$\forall m \in \mathbb{N}^*, \quad \mathbb{P}(\mathcal{S}(\tau) \leq m \mid \Delta(\tau) \geq n) \leq \exp(-nq_m).$$

La preuve de cette proposition repose sur les propriétés de convexité de φ et ψ d'une part, et sur le principe de Many-To-One (énoncé plus haut comme point (i) de la proposition 1.3.4) d'autre part. On applique la proposition 1.5.8 plusieurs fois par la suite, ainsi que dans le papier [102] et le chapitre 5 présentés dans la section suivante.

Notre seconde contribution consiste à étendre le théorème 1.4.2 de Brandenberger, Devroye & Reddad en déterminant l'ordre de grandeur du nombre de Horton-Strahler d'un arbre de Galton-Watson critique conditionné à être grand pour des lois de reproduction avec une variance infinie. On en profite aussi pour estimer les queues de distribution du nombre de Horton-Strahler qui apparaissent dans les bornes données par la proposition 1.5.8. Pour ce faire, on travaille sous l'hypothèse que la loi de reproduction est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in [1, 2]$. On expose d'abord nos résultats dans les cas $\alpha \in (1, 2]$, puis on discute des cas $\alpha = 1$ qui présentent des phénomènes plus complexes.

Les cas α -stables avec $\alpha \in (1, 2]$. Ici, nos résultats ne font intervenir que l'indice de stabilité $\alpha \in (1, 2]$. Tout d'abord, la proposition suivante montre que les queues de distributions suivent une décroissance exponentielle.

Proposition 1.5.9 *Soit τ un arbre de Galton-Watson de loi de reproduction μ . Si μ est critique, non-triviale, et dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$, alors on a*

$$-\log_{\frac{\alpha}{\alpha-1}} \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n,$$

où $\log_b x = \ln x / \ln b$ désigne le logarithme de x en base b .

Cette proposition étend le travail de Brandenberger, Devroye & Reddad [28] qui traite du cas $\alpha = 2$ sous l'hypothèse plus forte que la variance de μ est finie. La preuve consiste en une analyse élémentaire d'une relation de récurrence. Ensuite, on rappelle que si μ est apériodique alors $\mathbb{P}(\#\tau = n) > 0$ pour tout n assez grand.

Théorème 1.5.10 Soit τ un arbre de Galton-Watson de loi de reproduction μ . On suppose que μ est critique, non-triviale, et apériodique. Si μ est dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$, alors la convergence suivante a lieu en probabilité :

$$\frac{\alpha \mathcal{S}(\tau)}{\log_{\frac{\alpha}{\alpha-1}} n} \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1. \quad (1.5.12)$$

C'est une généralisation directe du théorème 1.4.2 de Brandenberger, Devroye & Reddad [28]. Pour prouver ce résultat, on s'appuie sur le théorème 1.1.5 de Duquesne pour contrôler la hauteur de τ , puis sur la proposition 1.5.8 pour en déduire une estimation de $\mathcal{S}(\tau)$ sous $\mathbb{P}(\cdot | \#\tau \geq n)$. On complète la preuve grâce à une relation d'absolue continuité donnée par Kortchemski [106] entre $\mathbb{P}(\cdot | \#\tau \geq n)$ et $\mathbb{P}(\cdot | \#\tau = n)$. Mentionnons que la convergence (1.5.12) reste valide quand μ n'est pas apériodique quitte à se restreindre aux entiers n tels que $\mathbb{P}(\#\tau = n) > 0$.

Les cas 1-stables. Ces cas plus complexes sont modèle-dépendants au sens où ils font intervenir le comportement asymptotique de la loi de reproduction μ de manière précise. Les fonctions suivantes, exprimées à partir de la fonction ψ définie par (1.5.11), joue un rôle important :

$$\forall s \in (0, 1), \quad \Upsilon(s) = \int_s^1 \frac{dr}{r \ln \Lambda(r)}, \quad \text{où} \quad \Lambda(s) = \frac{s\psi'(s)}{s\psi'(s) - \psi(s)}. \quad (1.5.13)$$

Avant d'énoncer nos résultats, prenons quelques instants pour rappeler des formulations équivalentes de l'hypothèse que μ est dans le domaine d'attraction d'une loi stable d'indice 1 : d'après (1.1.6), il existe une fonction à variations lentes L telle que $\int_1^\infty y^{-1} L(y) dy < \infty$ et

$$\mu([n, \infty)) \sim \frac{L(n)}{n}. \quad (1.5.14)$$

Des arguments standards (voir [23, Theorem 8.1.6] par exemple) impliquent alors que

$$\psi(s) \sim_{0+} s\ell(1/s), \quad \text{où} \quad \ell(x) = \int_x^\infty \frac{L(y)}{y} dy \text{ pour tout } x > 0. \quad (1.5.15)$$

De plus, ℓ est une fonction à variations lentes.

Soit $(X_n)_{n \in \mathbb{N}}$ une marche aléatoire sur \mathbb{Z} de valeur initiale $X_0 = 0$ et dont la loi des sauts est donnée par $\mathbb{P}(X_1 = k) = \mu(k+1)$ pour tout entier $k \geq -1$. D'après [74, Chapter IX.8], il existe une suite (a_n) de nombres réels strictement positifs tendant vers l'infini telle qu'on a la convergence en loi

$$\frac{X_n + b_n}{a_n} \xrightarrow{d} X_1, \quad \text{où} \quad b_n = n\mathbb{E}[X_1 \mathbf{1}_{\{|X_1| > a_n\}}] \quad \text{et} \quad \mathbb{E}[e^{-\lambda X_1}] = e^{\lambda \ln \lambda}, \quad (1.5.16)$$

pour tous $n \in \mathbb{N}$ et $\lambda > 0$. La distribution de X_1 est une loi de Cauchy (aussi dite 1-stable) spectralement positive. L'expression de sa transformée de Fourier est donnée pour tout $u \in \mathbb{R}$ par $\mathbb{E}[\exp(iuX_1)] = \exp(-\frac{\pi}{2}|u| - iu \ln |u|)$. On mentionne que l'on a nécessairement

$$a_n \sim nL(a_n) \quad \text{et} \quad b_n \sim n\ell(a_n),$$

où L est donnée par (1.5.14) et ℓ par (1.5.15). On renvoie à la proposition 4.2.12 du chapitre 4 pour plus de détails.

Proposition 1.5.11 Soit μ une mesure de probabilité sur \mathbb{N} . On suppose que μ est critique et est dans le domaine d'attraction d'une loi stable d'indice 1. On a alors

$$\Lambda(s) \sim_{0+} \frac{\ell(1/s)}{L(1/s)},$$

où Λ , L , et ℓ sont respectivement données par (1.5.13), (1.5.14), et (1.5.15). De plus, si τ est un arbre de Galton-Watson de loi de reproduction μ , alors on a

$$\Upsilon\left(\mathbb{P}(\mathcal{S}(\tau) \geq n)\right) \sim n,$$

où Υ est donnée par (1.5.13).

Ainsi, contrairement aux cas où $\alpha \in (1, 2]$ pour lesquels on prouve $\Upsilon(s) \sim_{0+} \log_{\frac{\alpha}{\alpha-1}} 1/s$, le comportement asymptotique de Υ dépend de L quand $\alpha = 1$. Donnons quelques exemples concrets.

- (a) Si $L(n) \sim (\ln n)^{-1-\kappa}$ avec $\kappa \in (0, \infty)$, alors $\Upsilon(s) \sim_{0+} \frac{\ln 1/s}{\ln \ln 1/s}$ et $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n \ln n$.
- (b) Si $L(n) \sim \exp(-(\ln n)^\kappa)$ avec $\kappa \in (0, 1)$, alors on a $\Upsilon(s) \sim_{0+} \frac{1}{1-\kappa} \frac{\ln 1/s}{\ln \ln 1/s}$, et donc on a $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim (1-\kappa)n \ln n$.
- (c) Si $L(n) \sim \exp(-\ln n / \ln \ln n)$, alors $\Upsilon(s) \sim_{0+} \frac{\ln 1/s}{\ln \ln \ln 1/s}$ et $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n \ln \ln n$.

Théorème 1.5.12 Soit τ un arbre de Galton-Watson de loi de reproduction μ . On suppose que μ est critique et dans le domaine d'attraction d'une loi stable d'indice 1. Alors, la convergence suivante

$$\frac{\mathcal{S}(\tau)}{\Upsilon\left(\frac{1}{b_n}\right)} \text{ sous } \mathbb{P}(\cdot | \#\tau \geq n) \longrightarrow 1,$$

a lieu en probabilité, où Υ est donnée par (1.5.13) et $(b_n)_{n \in \mathbb{N}}$ par (1.5.16).

Comme souligné par Kortchemski & Richier [108] et Berger [17], il est déraisonnable d'espérer pouvoir contrôler τ sous $\mathbb{P}(\cdot | \#\tau = n)$ avec la seule hypothèse que $n\mu([n, \infty)) \sim L(n)$ avec une fonction à variations lentes L , comme dans (1.5.14). À la place, on travaille sous l'hypothèse plus forte que $n^2\mu(n) \sim L(n)$, qui implique la précédente et que μ est apériodique.

Théorème 1.5.13 Soit τ un arbre de Galton-Watson de loi de reproduction μ . On suppose que μ est critique et qu'il existe une fonction à variations lentes L telle que $n^2\mu(n) \sim L(n)$. Alors, la convergence suivante

$$\frac{\mathcal{S}(\tau)}{\Upsilon\left(\frac{1}{b_n}\right)} \text{ sous } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1,$$

a lieu en probabilité, où Υ est donnée par (1.5.13) et $(b_n)_{n \in \mathbb{N}}$ par (1.5.16).

Nos preuves des théorèmes 1.5.12 et 1.5.13 reposent sur plusieurs résultats de Kortchemski & Richier [108] et de Berger [17] qui décrivent le comportement asymptotique d'une excursion positive de la marche aléatoire (X_n) . La proposition 1.5.8 joue également un rôle. Afin de fixer les idées, on explicite l'équivalent asymptotique $\Upsilon\left(\frac{1}{b_n}\right)$ pour les trois exemples concrets précédents.

- (a) Si $L(n) \sim (\ln n)^{-1-\kappa}$ avec $\kappa \in (0, \infty)$, alors $\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{\ln n}{\ln \ln n}$.
- (b) Si $L(n) \sim \exp(-(\ln n)^\kappa)$ avec $\kappa \in (0, 1)$, alors $\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{1}{1-\kappa} \frac{\ln n}{\ln \ln n}$.
- (c) Si $L(n) \sim \exp(-\ln n / \ln \ln n)$, alors $\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{\ln n}{\ln \ln \ln n}$.

1.5.4 Nombre de Horton-Strahler des arbres de Galton-Watson stables : fluctuations

Cette section présente les résultats de l'article [102], correspondant au chapitre 5 de ce manuscrit.

On rappelle la définition 1.4.1 du nombre de Horton-Strahler \mathcal{S} . Pour tout $x \in \mathbb{R}_+$, on note $\lfloor x \rfloor$ la partie entière de x , avec $\lfloor 2.9 \rfloor = \lfloor 2.1 \rfloor = \lfloor 2 \rfloor = 2$ par exemple. On fixe les notations

$$\alpha \in (1, 2], \quad \beta = \frac{1}{\alpha-1}, \quad \gamma = \ln \frac{\alpha}{\alpha-1} \quad \text{et} \quad \delta = \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} = e^{\gamma(\alpha-1)} \in (1, 2]. \quad (1.5.17)$$

Ici, on étudie les fluctuations du nombre de Horton-Strahler d'un arbre de Galton-Watson α -stable, de définition 1.1.7, conditionné à être grand. On rappelle que sa loi de reproduction μ_α est caractérisée par l'expression (1.1.18) de sa fonction génératrice :

$$\forall s \in [0, 1], \quad \varphi_\alpha(s) := \sum_{k \in \mathbb{N}} s^k \mu_\alpha(k) = s + \frac{1}{\alpha}(1-s)^\alpha. \quad (1.1.18)$$

On sait que μ_α est critique avec $\mu_\alpha(1) = 0$ et que μ_α est dans le domaine d'attraction d'une loi α -stable. Plus précisément, si $(X_n)_{n \in \mathbb{N}}$ est une marche aléatoire sur \mathbb{Z} de position initiale $X_0 = 0$ et dont les sauts sont tels que $\mathbb{P}(X_1 = k) = \mu_\alpha(k+1)$ pour tout entier $k \geq -1$, alors on a

$$\frac{1}{a_n} X_n \xrightarrow{d} X_1, \quad \text{où} \quad a_n = \alpha^{-1/\alpha} n^{1/\alpha} \quad \text{et} \quad \mathbb{E}[\exp(-\lambda X_1)] = \exp(\lambda^\alpha) \quad (1.5.18)$$

pour tous $n \in \mathbb{N}^*$ et $\lambda \in (0, \infty)$. On peut le vérifier grâce à (1.1.18) et au théorème de continuité de la transformée de Laplace bilatère (voir [151, Appendix A] par exemple). On sait que si $\alpha = 2$ alors $\mu_2(0) = \mu_2(2) = 1/2$, et dériver (1.1.18) donne que si $\alpha \in (1, 2)$ alors

$$\mu_\alpha(0) = \frac{1}{\alpha}, \quad \mu_\alpha(1) = 0, \quad \mu_\alpha(2) = \frac{\alpha-1}{2}, \quad \text{et} \quad \mu_\alpha(k) = \frac{1}{k!} \prod_{i=1}^{k-1} |\alpha - i| = \frac{(\alpha-1)\Gamma(k-\alpha)}{\Gamma(2-\alpha)k!} \quad (1.5.19)$$

pour tout entier $k \geq 3$, où Γ est la fonction Gamma usuelle. En particulier, μ_α est apériodique si et seulement si $\alpha \in (1, 2)$. Ainsi, la convergence (1.1.17) déduite du théorème 1.1.5 de Duquesne a lieu tant que $n-1 \in \lfloor \alpha \rfloor \mathbb{N}$.

Nombre de Horton-Strahler pondéré. Motivé par les observations et arguments de la section 1.4.4, on introduit une version continue du nombre de Horton-Strahler pour les arbres dont les feuilles sont dotées de poids à valeurs dans $[0, 1)$. Soit t un arbre fini au sens de la définition 1.1.1. On note $\partial t = \{u \in t : k_u(t) = 0\}$ l'ensemble des *feuilles* de t .

Définition 1.5.14 On dit que $\mathbf{t} = (t, (w_v)_{v \in \partial t})$ est un *arbre pondéré* lorsque t est un arbre fini et que $w_v \in [0, 1)$ pour tout $v \in \partial t$. On dote l'espace \mathbb{T}_w de tous les arbres pondérés de la topologie produit : autrement dit, $(t^n, (w_v^n)_{v \in \partial t^n}) \rightarrow \mathbf{t}$ sur \mathbb{T}_w si et seulement si $t^n = t$ à partir d'un certain rang et $w_v^n \rightarrow w_v$ pour tout $v \in \partial t$. L'espace \mathbb{T}_w est polonais. \square

Si $\mathbf{t} = (t, (w_v)_{v \in \partial t})$ est un arbre pondéré, alors pour tout $u \in t$, on pose

$$\theta_u \mathbf{t} = (\theta_u t, (w_{u*v})_{v \in \partial \theta_u t})$$

qui est un arbre pondéré bien défini car $\partial \theta_u t = \{v \in \theta_u t : u * v \in \partial t\}$.

Définition 1.5.15 (Nombre de Horton-Strahler pondéré) Soit \mathbf{t} un arbre pondéré. Son *nombre de Horton-Strahler pondéré* $\mathcal{S}(\mathbf{t}) \in [0, \infty)$ est défini comme suit.

- (a) Si $\mathbf{t} = (\{\emptyset\}, w_\emptyset)$ est réduit à sa racine avec poids w_\emptyset , alors $\mathcal{S}(\mathbf{t}) = w_\emptyset$.
- (b) On note $k = k_\emptyset(t)$ le nombre d'enfants de la racine de t . Si $k = 1$, alors $\mathcal{S}(\mathbf{t}) = \mathcal{S}(\theta_{(1)} \mathbf{t})$. Si $k \geq 2$, on note $\mathcal{S}(\theta_{(i_1)} \mathbf{t}) \geq \mathcal{S}(\theta_{(i_2)} \mathbf{t}) \geq \dots \geq \mathcal{S}(\theta_{(i_k)} \mathbf{t})$ les nombres de Horton-Strahler pondérés des sous-arbres issus de ces enfants listés dans l'ordre décroissant et on a alors $\mathcal{S}(\mathbf{t}) = \max(\mathcal{S}(\theta_{(i_1)} \mathbf{t}), 1 + \mathcal{S}(\theta_{(i_2)} \mathbf{t}))$. Ces conditions sont résumées dans l'équation

$$\mathcal{S}(\mathbf{t}) = \max_{1 \leq i, j \leq k_\emptyset(\tau)} \max(\mathcal{S}(\theta_{(i)} \mathbf{t}), \mathcal{S}(\theta_{(j)} \mathbf{t}), \mathbf{1}_{\{i \neq j\}} + \min(\mathcal{S}(\theta_{(i)} \mathbf{t}), \mathcal{S}(\theta_{(j)} \mathbf{t}))). \quad (1.5.20)$$

□

On renvoie à la figure 1.4 pour un exemple d'arbre pondéré et du calcul de son nombre de Horton-Strahler pondéré. Le nombre de Horton-Strahler pondéré est lié au nombre de Horton-Strahler classique par la relation

$$\forall \mathbf{t} \in \mathbb{T}_w, \quad \lfloor \mathcal{S}(\mathbf{t}) \rfloor = \mathcal{S}(t) \quad \text{où} \quad \mathbf{t} = (t, (w_v)_{v \in \partial t}),$$

qui est une conséquence de l'exigence que les poids soient à valeurs dans $[0, 1)$ (voir la proposition 5.4.3 dans le chapitre 5). Ainsi, se contenter d'étudier le nombre de Horton-Strahler pondéré n'entraîne aucune perte d'information sur le nombre de Horton-Strahler habituel.

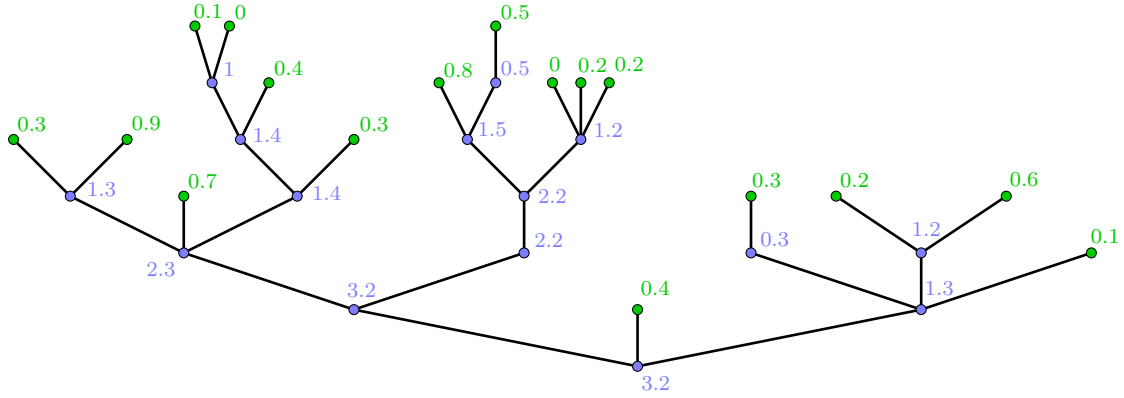


Figure 1.4: Un arbre pondéré \mathbf{t} tel que $\mathcal{S}(\mathbf{t}) = 3.2$. Les feuilles et leurs poids sont en vert. À côté de chaque sommet est indiqué le nombre de Horton-Strahler pondéré du sous-arbre qui en est issu.

Arbres pondérés de Galton-Watson stables. Afin de tirer parti du nombre de Horton-Strahler pondéré, on spécifie les poids que l'on met sur les feuilles d'un arbre de Galton-Watson α -stable.

Définition 1.5.16 Soit W une variable aléatoire sur $[0, 1)$ dont la loi est caractérisée par

$$\forall r \in [0, 1], \quad \mathbb{P}(W \leq r) = \frac{1 - e^{-\gamma r}}{1 - e^{-\gamma}}. \quad (1.5.21)$$

Un *arbre pondéré de Galton-Watson α -stable* est un arbre pondéré aléatoire $\tau = (\tau, (W_v)_{v \in \partial \tau})$ tel que τ est un arbre de Galton-Watson α -stable et conditionnellement à τ , les poids $(W_v)_{v \in \partial \tau}$ sont indépendants et de même loi que W . □

On souligne que si E est une variable aléatoire exponentielle de moyenne $1/\gamma$ alors sa partie entière $\lfloor E \rfloor$ et sa partie fractionnaire $E - \lfloor E \rfloor$ sont indépendantes, $\lfloor E \rfloor$ est géométrique de paramètre $1/\alpha$, et la loi de $E - \lfloor E \rfloor$ est donnée par (1.5.21). Le choix des poids proposé par la définition 1.5.16 se justifie par les résultats suivants.

Proposition 1.5.17 Soit $\tau = (\tau, (W_v)_{v \in \partial \tau})$ un arbre pondéré de Galton-Watson α -stable.

- (i) La loi de $\mathcal{S}(\tau)$ est exponentielle de moyenne $1/\gamma$, c'est-à-dire $\mathbb{P}(\mathcal{S}(\tau) \geq x) = e^{-\gamma x}$ pour tout $x \in \mathbb{R}_+$.
- (ii) Pour tout $x \in \mathbb{R}_+ \setminus \mathbb{N}$, la loi de τ sous $\mathbb{P}(\cdot \mid |\mathcal{S}(\tau) - x| < \varepsilon)$ converge faiblement vers une mesure de probabilité sur l'espace des arbres finis quand $\varepsilon \rightarrow 0^+$. On note cette dernière $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$.

(iii) La fonction $x \in \mathbb{R}_+ \setminus \mathbb{N} \mapsto \mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ est continue pour la topologie de la convergence faible des mesures de probabilité sur l'espace des arbres finis.

On observe que le point (i) de la proposition 1.5.17 permet de retrouver la loi du nombre de Horton-Strahler d'un arbre de Galton-Watson stable, telle qu'exprimée par (1.4.6). Par le théorème de densité de Lebesgue, les autres points impliquent que $x \in \mathbb{R}_+ \setminus \mathbb{N} \mapsto \mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ est une version de la loi conditionnelle de τ sachant $\mathcal{S}(\tau)$, c'est-à-dire que pour tout arbre fini t et toute fonction $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ mesurable bornée, on a

$$\mathbb{E}[\mathbf{1}_{\{\tau=t\}}g(\mathcal{S}(\tau))] = \int_{\mathbb{R}_+} \mathbb{P}(\tau = t \mid \mathcal{S}(\tau) = x)g(x)\gamma e^{-\gamma x} dx.$$

De plus, on prouve que ces lois admettent une limite d'échelle lorsque $x \rightarrow \infty$. Comme vu en section 1.1.4, on rappelle que l'espace \mathbb{K} des classes d'équivalence d'espaces métriques compacts enracinés mesurés est muni de la distance de Gromov-Hausdorff-Prokhorov d_{GHP} , et que $\mathbb{T}_{\mathbb{R}}$ désigne son sous-espace formé des arbres réels compacts enracinés mesurés (voir la définition 1.1.6).

Théorème 1.5.18 Soit τ un arbre pondéré de Galton-Watson α -stable. L'arbre sous-jacent τ est muni de la distance de graphe d_{gr} donnée par (1.1.2), de sa racine notée \emptyset , et de sa mesure de comptage $\sum_{u \in \tau} \delta_u$. Il existe une variable aléatoire $(\mathcal{T}_{\text{hs}}, d_{\text{hs}}, \rho_{\text{hs}}, \mu_{\text{hs}})$ sur $\mathbb{T}_{\mathbb{R}}$ telle que la convergence

$$\left(\tau, e^{-\gamma(\alpha-1)x} d_{\text{gr}}, \emptyset, e^{-\gamma\alpha x} \sum_{u \in \tau} \delta_u \right) \text{ sous } \mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x) \xrightarrow[x \rightarrow \infty, x \notin \mathbb{N}]{d} (\mathcal{T}_{\text{hs}}, d_{\text{hs}}, \rho_{\text{hs}}, \mu_{\text{hs}})$$

a lieu en loi sur l'espace \mathbb{K} muni de la distance de Gromov-Hausdorff-Prokhorov.

On décrit la loi de l'arbre réel limite plus loin, mais ce théorème est avant tout un résultat d'existence. Pour le prouver, on introduit une variante pondéré de l'élagage de Horton présenté en section 1.4.1 : au lieu d'effacer toutes les feuilles, on n'efface que celles dont le poids est strictement inférieur à un seuil prédéterminé $r \in [0, 1]$. Cette opération soustrait exactement r au nombre de Horton-Strahler pondéré, et elle laisse invariante (en un certain sens) la loi d'un arbre pondéré de Galton-Watson stable. Voir le théorème 5.5.4 du chapitre 5 pour un résultat précis. On complète alors la preuve du théorème en vérifiant la réalisation d'un critère de Cauchy, ce qui explique pourquoi la loi limite reste encore implicite à ce stade.

Dilatation de Strahler. On reprend les notations de la section 1.1.3 : $\mathbf{N}_\alpha(d\text{H})$ est la mesure d'excursion (au dessus de 0) du processus de hauteur α -stable H , ζ est la durée de vie de H , et $\mathbf{N}_\alpha(d\text{H} \mid \zeta = 1)$ est la loi de l'excursion normalisée du processus de hauteur α -stable. Jusqu'à la fin de cette section, on allège les notations en écrivant $(\mathcal{T}_H, d_H, \rho_H, \mu_H) = \mathcal{T}_H^\zeta$ sous \mathbf{N}_α , où on rappelle de (1.1.15) que \mathcal{T}_H^ζ est l'arbre réel codé par (H, ζ) . Au regard du théorème 1.5.18 et de la convergence (1.1.17), on peut raisonnablement s'attendre à ce qu'il existe une fonctionnelle $\mathfrak{S} : \mathbb{T}_{\mathbb{R}} \rightarrow [0, \infty]$, jouant le rôle d'un analogue métrique du nombre de Horton-Strahler (pondéré), telle que la loi de \mathcal{T}_{hs} puisse s'interpréter comme celle de \mathcal{T}_H sous \mathbf{N}_α conditionnellement à ce que $\mathfrak{S}(\mathcal{T}_H)$ soit fixé. On construit ci-après une telle application.

Soit $(T, d, \rho, \mu) \in \mathbb{T}_{\mathbb{R}}$ un arbre réel compact enraciné mesuré. Par analogie avec le formalisme des arbres (discrets), on voit T comme un arbre généalogique dont l'ancêtre commun est ρ . Avec les notations de la définition 1.1.6, pour tous $\sigma_1, \sigma_2 \in T$, on rappelle que $\llbracket \sigma_1, \sigma_2 \rrbracket$ désigne l'image de l'unique géodésique de σ_1 à σ_2 sur T . Par la définition 1.1.6 (b), on vérifie facilement que $\llbracket \rho, \sigma_1 \rrbracket$, $\llbracket \rho, \sigma_2 \rrbracket$, et $\llbracket \sigma_1, \sigma_2 \rrbracket$ ont exactement un point en commun que l'on note

$$\{\sigma_1 \wedge \sigma_2\} = \llbracket \rho, \sigma_1 \rrbracket \cap \llbracket \rho, \sigma_2 \rrbracket \cap \llbracket \sigma_1, \sigma_2 \rrbracket. \quad (1.5.22)$$

Notons que la notation \wedge dépend de (T, d, ρ) , et en particulier du choix de la racine ρ . Soit

$$\mathbb{W}_2 = \bigcup_{n \in \mathbb{N}} \{1, 2\}^n, \quad \text{où} \quad \{1, 2\}^0 = \{\emptyset\},$$

l'arbre binaire régulier infini. Toujours par analogie avec le formalisme discret, et plus particulièrement avec (1.4.2), on dit qu'une application injective $\varphi : \mathbb{W}_2 \rightarrow T$ est un *plongement* lorsque $\varphi(u) \wedge \varphi(v) = \varphi(u \wedge v)$ pour tous $u, v \in \mathbb{W}_2$, la notation \wedge ayant un sens différent mais analogue à gauche et à droite de l'égalité. Si $\varphi : \mathbb{W}_2 \rightarrow T$ est un plongement, alors on pose $\varphi(\overline{\emptyset}) = \rho$ par convention. On rappelle de (1.5.17) que $\delta = \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1}$. Finalement, on définit

$$\mathfrak{S}_\delta(T, d, \rho, \mu) = \sup_{\substack{\varphi: \mathbb{W}_2 \rightarrow T \\ \text{plongement}}} \liminf_{n \rightarrow \infty} \frac{1}{n+1} \inf_{\substack{u \in \mathbb{W}_2 \\ |u|=n}} \sum_{\substack{v \leq u}} \delta^{1+|v|} d(\varphi(\overline{v}), \varphi(v)), \quad (1.5.23)$$

et on nomme $\mathfrak{S}_\delta(T, d, \rho, \mu)$ la *dilatation de Strahler en base δ* de (T, d, ρ, μ) . On souligne la similarité entre (1.5.23) et l'expression (1.4.2) du nombre de Horton-Strahler en termes de supremum sur l'ensemble des plongements partant d'arbres binaires parfaits. De plus, on observe que pour tout $\lambda \in (0, \infty)$ et pour tout $(T, d, \rho, \mu) \in \mathbb{T}_\mathbb{R}$, on a $\mathfrak{S}_\delta(T, \lambda d, \rho, \mu) = \lambda \mathfrak{S}_\delta(T, d, \rho, \mu)$. Ainsi, la dilatation de Strahler est une caractéristique métrique non-triviale des arbres réels compacts enracinés.

Théorème 1.5.19 *L'application $\mathfrak{S}_\delta : \mathbb{T}_\mathbb{R} \rightarrow [0, \infty]$ est mesurable et si \mathcal{T}_{hs} est l'espace limite du théorème 1.5.18, alors $\mathfrak{S}_\delta(\mathcal{T}_{\text{hs}}) = 1$ presque sûrement.*

La preuve de l'égalité $\mathfrak{S}_\delta(\mathcal{T}_{\text{hs}}) = 1$ s'appuie sur une décomposition auto-similaire de \mathcal{T}_{hs} qui fait apparaître explicitement le plongement optimal pour (1.5.23). Voir le théorème 5.7.3 du chapitre 5 pour une formulation exacte de ce résultat technique. On énonce maintenant les liens attendus entre \mathcal{T}_{hs} et \mathcal{T}_H sous \mathbf{N}_α .

Théorème 1.5.20 *Soit $(\mathcal{T}_{\text{hs}}, d_{\text{hs}}, \rho_{\text{hs}}, \mu_{\text{hs}})$ l'espace limite du théorème 1.5.18. On note encore \mathfrak{S}_δ la dilatation de Strahler en base δ comme définie par (1.5.23). Pour toutes fonctions $F : \mathbb{T}_\mathbb{R} \rightarrow \mathbb{R}_+$ et $g : [0, \infty] \rightarrow \mathbb{R}_+$ mesurables et positives, on a l'identité suivante*

$$\mathbf{N}_\alpha[F(\mathcal{T}_H)g(\mathfrak{S}_\delta(\mathcal{T}_H))] = \beta \alpha^{-\beta} \int_{\mathbb{R}_+} \mathbb{E}\left[F(\mathcal{T}_{\text{hs}}, x d_{\text{hs}}, \rho_{\text{hs}}, \alpha^\beta x^{\alpha\beta} \mu_{\text{hs}})\right] g(x) \frac{dx}{x^{1+\beta}},$$

où on rappelle que $\beta(\alpha - 1) = 1$ d'après (1.5.17). En particulier, on a $\mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_H) = 0) = 0$ et $\mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_H) > x) = (\alpha x)^{-\beta}$ pour tout $x > 0$.

D'après ce résultat, la dilatation de Strahler en base δ de \mathcal{T}_H est strictement positive et finie. On peut prouver (voir la proposition 5.8.8 dans le chapitre 5) que le paramètre δ est le seul à vérifier cette propriété et joue donc le rôle d'une base critique. En plus de donner la loi de $\mathfrak{S}_\delta(\mathcal{T}_H)$ sous la mesure sigma-finie $\mathbf{N}_\alpha(d_H)$, le théorème 1.5.20 fournit une version régulière (continue) de la loi conditionnelle de \mathcal{T}_H sachant $\mathfrak{S}_\delta(\mathcal{T}_H)$ en termes de redimensionnement de l'arbre réel aléatoire \mathcal{T}_{hs} . Réciproquement, on peut retrouver la loi de \mathcal{T}_{hs} en divisant les distances sur \mathcal{T}_H par $\mathfrak{S}_\delta(\mathcal{T}_H)$: pour tout $y \in (0, \infty)$ et pour toute fonction $F : \mathbb{T}_\mathbb{R} \rightarrow \mathbb{R}$ continue et bornée, on a

$$\mathbb{E}[F(\mathcal{T}_{\text{hs}})] = \mathbf{N}_\alpha\left[F(\mathcal{T}_H, \mathfrak{S}_\delta(\mathcal{T}_H)^{-1} d_H, \rho_H, \alpha^{-\beta} \mathfrak{S}_\delta(\mathcal{T}_H)^{-\alpha\beta} \mu_H) \mid \mathfrak{S}_\delta(\mathcal{T}_H) > y\right].$$

En outre, l'intuition initiale sur la loi de \mathcal{T}_{hs} est justifiée par l'identité suivante :

$$\mathbb{E}[F(\mathcal{T}_{\text{hs}})] = \lim_{\varepsilon \rightarrow 0^+} \mathbf{N}_\alpha\left[F(\mathcal{T}_H, \alpha^{1/\alpha} d_H, \rho_H, \mu_H) \mid |\mathfrak{S}_\delta(\mathcal{T}_H) - \alpha^{-1/\alpha}| < \varepsilon\right].$$

La preuve du théorème 1.5.20 est basée sur une comparaison bayésienne de la convergence (1.1.17) avec le théorème 1.5.18, qui est rendue possible grâce à l'homogénéité de \mathfrak{S}_δ et au théorème 1.5.19.

La même méthode permet de montrer que si τ est un arbre pondéré de Galton-Watson α -stable, alors la limite d'échelle de $\delta^{\mathcal{S}(\tau)}$ sous $\mathbb{P}(\cdot \mid \#\tau \geq n)$ est égale à la dilatation de Strahler en base δ de la limite d'échelle de l'arbre τ . Néanmoins, cet argument n'est pas suffisant pour étudier le comportement asymptotique sous le conditionnement local $\mathbb{P}(\cdot \mid \#\tau = n)$ car $\mathbf{N}_\alpha(\zeta = 1) = 0$. Pour éviter ce problème, on utilise un algorithme dû à Marchal [128] qui génère une suite imbriquée d'arbres de Galton-Watson α -stables conditionnés à avoir n feuilles, ce qui permet de profiter d'une propriété de monotonie du nombre de Horton-Strahler. En outre, on se sert d'un théorème de Curien & Haas [41] qui affirme que cette suite converge presque sûrement vers un arbre α -stable après renormalisation, ainsi que de résultats de Kortchemski [107] reliant les conditionnements par le nombre de feuilles et par la taille. On prouve finalement le théorème ci-dessous, et on atteint donc notre but initial.

Théorème 1.5.21 *Soit τ un arbre pondéré de Galton-Watson α -stable, on reprend les notations du théorème 1.5.18. On note encore \mathfrak{S}_δ la dilatation de Strahler en base $\delta = (\frac{\alpha}{\alpha-1})^{\alpha-1}$ comme donnée par (1.5.23). Comme dans (1.5.18), on pose $a_n = \alpha^{-1/\alpha} n^{1/\alpha}$ pour tout $n \in \mathbb{N}$. On rappelle que si $\alpha \in (1, 2)$ alors $\lfloor \alpha \rfloor = 1$, et si $\alpha = 2$ alors $\lfloor \alpha \rfloor = 2$. Alors, la convergence jointe suivante*

$$\left(\left(\tau, \frac{a_n}{n} \mathbf{d}_{\text{gr}}, \emptyset, \frac{1}{n} \sum_{u \in \tau} \delta_u \right); \frac{a_n}{n} \delta^{\mathcal{S}(\tau)} \right) \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n+1) \\ \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]d} \left((\mathcal{T}_H, d_H, \rho_H, \mu_H); \mathfrak{S}_\delta(\mathcal{T}_H) \right) \text{ sous } \mathbf{N}_\alpha(dH \mid \zeta = 1)$$

a lieu en loi sur l'espace $\mathbb{K} \times (0, \infty)$, où \mathbb{K} est muni de la distance de Gromov-Hausdorff-Prokhorov.

Le théorème 1.5.20 et la formule (1.1.11) de désintégration impliquent que $\mathfrak{S}_\delta(\mathcal{T}_H)$ est aussi à valeurs dans $(0, \infty)$ sous $\mathbf{N}_\alpha(dH \mid \zeta = 1)$. En appliquant $\log_\delta(\cdot) = \ln(\cdot)/\ln(\delta)$ sur la seconde marginale dans le théorème 1.5.21, on obtient la convergence en loi

$$\mathcal{S}(\tau) - \frac{\alpha-1}{\alpha} \log_\delta n - \frac{1}{\alpha} \log_\delta \alpha \text{ sous } \mathbb{P}(\cdot \mid \#\tau = n+1) \\ \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]d} \log_\delta \mathfrak{S}_\delta(\mathcal{T}_H) \text{ sous } \mathbf{N}_\alpha(dH \mid \zeta = 1) \quad (1.5.24)$$

sur \mathbb{R} . Puisque $\mathcal{S}(\tau) = \lfloor \mathcal{S}(\tau) \rfloor$, cela permet d'une part de retrouver la "loi des grands nombres" donnée par le théorème 1.5.10 dans le cas spécifique où $\mu = \mu_\alpha$. D'autre part, l'évolution de la partie fractionnaire de $\frac{\alpha-1}{\alpha} \log_\delta n$ fait apparaître le même phénomène périodique que dans l'estimation asymptotique (1.4.4) prouvée par Flajolet, Raoult & Vuillemin [75] et Kemp [98].

Le théorème 1.5.21 indique qu'il est possible de retrouver avec une précision arbitraire asymptotiquement le nombre de Horton-Strahler pondéré d'un grand arbre pondéré de Galton-Watson stable à partir de la seule structure métrique de la limite d'échelle. Cela est remarquable pour deux raisons. D'une part, les nombres de Horton-Strahler classiques et pondérés sont des quantités combinatoires plutôt que métriques. Par exemple, greffer un arbre binaire parfait de hauteur $(\ln n)^2$ sur une feuille d'un arbre de Galton-Watson stable à n sommets augmente considérablement le nombre de Horton-Strahler, d'après (1.4.2), mais ne modifie pas la limite d'échelle. On rappelle de la section 1.1.5 que Duquesne & Le Gall [57, Section 4] montrent un phénomène similaire pour la masse totale.

D'autre part, le nombre de Horton-Strahler pondéré dépend des poids aux feuilles, et donc d'un aléa indépendant de l'arbre. Bien que l'influence de ceux-ci se résume à la partie fractionnaire, cette contribution d'ordre constant est non-négligeable au sein de la convergence (1.5.24). La disparition

de cette dépendance à la limite s'explique par le choix spécifique de la loi des poids, introduits pour enlever les phénomènes arithmétiques parasites du nombre de Horton-Strahler entier, qui est donc un aspect intrinsèque de l'arbre de Galton-Watson stable. De même, l'indice $\mathfrak{S}_\delta(\mathcal{T}_H)$ est une nouvelle caractéristique métrique de l'arbre stable dont les propriétés pourraient renseigner sur sa géométrie.

Notre dernier résultat porte spécifiquement sur le cas $\alpha = 2$, où $\frac{1}{\sqrt{2}}H$ sous $\mathbf{N}_\alpha(dH | \zeta = 1)$ a la même loi que l'excursion brownienne standard donnée par (1.1.8).

Théorème 1.5.22 *Ici $\alpha = 2$. Alors sous $\mathbf{N}_2(dH | \zeta = 1)$, il y a égalité en loi entre les deux variables aléatoires suivantes :*

$$2\mathfrak{S}_2(\mathcal{T}_H) \stackrel{d}{=} \sup_{s \in [0,1]} H_s = \sup_{\sigma \in \mathcal{T}_H} d_H(\rho_H, \sigma).$$

Ce résultat fait écho à l'identité en loi (1.4.7) car l'arbre brownien est à la fois la limite d'échelle des arbres uniformes à n sommets et des arbres binaires uniformes à n feuilles. Notre preuve, purement calculatoire, adapte la méthode de Flajolet, Raoult & Vuillemin [75] et de Kemp [98]. Ces calculs exacts ne peuvent être étendus aux cas $\alpha \in (1, 2)$. Il serait souhaitable de trouver une preuve plus directe du théorème 1.5.22 afin de comprendre en profondeur cette identité en loi et de savoir si elle reste valide pour les autres arbres stables.

Chapter 2

Scaling limits of tree-valued branching random walks

This chapter is adapted from the paper [60], written in collaboration with Thomas Duquesne, Shen Lin & Niccolò Torri, which is published in Electronic Journal of Probability.

Abstract

We consider a branching random walk (BRW) taking its values in the b -ary rooted tree \mathbb{W}_b (i.e. the set of finite words written in the alphabet $\{1, \dots, b\}$, with $b \geq 2$). The BRW is indexed by a critical Galton–Watson tree conditioned to have n vertices; its offspring distribution is aperiodic and is in the domain of attraction of a γ -stable law, $\gamma \in (1, 2]$. The jumps of the BRW are those of a nearest-neighbour null-recurrent random walk on \mathbb{W}_b (reflection at the root of \mathbb{W}_b and otherwise: probability $1/2$ to move closer to the root of \mathbb{W}_b and probability $1/(2b)$ to move away from it to one of the b sites above). We denote by $\mathcal{R}_b(n)$ the range of the BRW in \mathbb{W}_b which is the set of all sites in \mathbb{W}_b visited by the BRW. We first prove a law of large numbers for $\#\mathcal{R}_b(n)$ and we also prove that if we equip $\mathcal{R}_b(n)$ (which is a random subtree of \mathbb{W}_b) with its graph-distance d_{gr} , then there exists a scaling sequence $(a_n)_{n \in \mathbb{N}}$ satisfying $a_n \rightarrow \infty$ such that the metric space $(\mathcal{R}_b(n), a_n^{-1} d_{\text{gr}})$, equipped with its normalized empirical measure, converges to the reflected Brownian cactus with γ -stable branching mechanism: namely, a random compact real tree that is a variant of the Brownian cactus introduced by Curien, Le Gall & Miermont in [40].

Contents

2.1	Introduction	58
2.2	Tree with a possibly infinite line of ancestors.	62
2.3	Tree-valued branching random walks.	68
2.3.1	Definitions.	68
2.3.2	Metric properties of the range of free branching random walks.	69
2.3.3	A coupling between \mathbb{W}_b - and \mathbb{W}_b^* -valued branching random walks.	72
2.3.4	Estimates.	76
2.3.5	Law invariance.	81
2.3.6	Proof of Theorem 2.1.1.	83
2.4	Snake metrics and the Brownian cactus.	85

2.4.1	Pseudo-metrics on a closed interval.	85
2.4.2	Scaling limit of the range of biased RWs on trees.	90
2.4.3	Snake metrics.	92
2.4.4	Stable Lévy trees.	97
2.4.5	One-dimensional reflected Brownian snakes.	99
2.4.6	Reflected Brownian cactus with stable branching mechanism.	101
2.5	Scaling limit of the range of the BRW.	102
2.5.1	Continuous interpolation of discrete snakes.	102
2.5.2	Invariance principle for discrete snakes.	107
2.5.3	Proof of Theorem 2.1.2.	108

2.1 Introduction

Since the seventies, branching random walk (BRW) is an area of research that is intensively studied and is linked to traveling wave solutions of semi-linear partial differential equations (FKPP) or various models of statistical mechanics (Generalized random energy model, Mandelbrot's cascades, Gaussian free field): we refer to the book of Shi [140] for an overview of this topic; we also refer to the works of Gouëzel, Huerter, Lalley and Sellke [115, 94, 114, 86] for the study of BRW in hyperbolic spaces and to Liggett [122] and to Benjamini & Müller [16] for branching random walks on trees. In most of the previous works, BRWs are indexed by an infinite supercritical Galton–Watson tree (GW-trees) and questions focus on various survival events or extremal behaviours of BRWs.

In this chapter, we consider instead a BRW that takes its values in the b -ary tree \mathbb{W}_b and that is indexed by a critical Galton–Watson tree conditioned to have n vertices. The jumps of the BRW are those of a nearest-neighbour null-recurrent RW on \mathbb{W}_b . Namely, at the root of \mathbb{W}_b (that is denoted by \emptyset), it is reflected and elsewhere, with probability $1/2$, it jumps to the neighbour closer to the root of \mathbb{W}_b and with probability $1/(2b)$, it moves further from the root of \mathbb{W}_b and it jumps to one of the b sites above. We study the range $\mathcal{R}_b(n)$ of this BRW when $n \rightarrow \infty$. More precisely, we first show that $\frac{1}{n} \#\mathcal{R}_b(n)$ converges in probability to a constant; this law of large numbers is the analogue of the results due to Le Gall & Lin [83, 82] who treat the cases of \mathbb{Z}^d -valued BRWs that are indexed by critical GW-trees conditioned to have n vertices. We then prove that $\mathcal{R}_b(n)$, seen as a subtree of the b -ary tree, converges, when it is suitably rescaled, to a continuum random tree called the reflected Brownian cactus with γ -stable branching mechanism: namely, a random compact real tree that is a variant of the Brownian cactus introduced by Curien, Le Gall & Miermont in [40].

This limit theorem for $\mathcal{R}_b(n)$ is related to earlier works on scaling limits of the range of tree-valued critical or near-critical biased random walks (RWs): in particular we refer to Duquesne [52] who deals with near-critical biased RWs on b -ary trees, to Peres & Zeitouni [137] who show that the distance to the root of a critical biased RW in a Galton–Watson environment is diffusive, to Dembo & Sun [45] who study the cases of critical biased RWs on N -type GW-trees, to Aidékon & de Raphélis [6] who improve Peres & Zeitouni's result and who show that the range of the same RW converges when suitably rescaled, to a variant of the Brownian CRT, and to Chen & Miermont [35] who show that rescaled Brownian bridges and loops in hyperbolic spaces converge to the Brownian CRT. Their work is based on a previous results due to Bougerol & Jeulin [25]. Independently, Stewart shows in his PhD Thesis [145] that the rescaled simple RW bridges on a d -regular tree ($d \geq 3$) converge to the Brownian CRT.

Let us describe more precisely the results that we obtain. We consider a (rooted and ordered) Galton–Watson tree τ with offspring distribution μ that satisfies the following: we fix $\gamma \in (1, 2]$ and

we assume

$$(\mathbf{H}) \quad \begin{cases} (\mathbf{H}_1) : & \mu(0) > 0 \text{ and } \sum_{k \in \mathbb{N}} k\mu(k) = 1, \\ (\mathbf{H}_2) : & \mu \text{ is aperiodic (namely, } \mu \text{ is not supported by a proper subgroup of } \mathbb{Z}), \\ (\mathbf{H}_3) : & \mu \text{ is in the domain of attraction of a } \gamma\text{-stable law.} \end{cases} \quad (2.1.1)$$

Note that (\mathbf{H}_1) implies that a.s. the total number of vertices $\#\tau$ is finite; (\mathbf{H}_2) implies that for all large enough integers n , $\mathbf{P}(\#\tau = n) > 0$. We translate (\mathbf{H}_3) into the following assertion: let $(L_n)_{n \in \mathbb{N}}$ be an i.i.d. sequence of \mathbb{N} -valued random variables such that $\mathbf{P}(L_n = k) = \mu(k)$, $k \in \mathbb{N}$ and let X be a real random variable whose law is spectrally positive γ -stable; it is characterised by its Laplace exponent: $\log \mathbf{E}[\exp(-\lambda X)] = \lambda^\gamma$, $\lambda \in [0, \infty)$. Then under (\mathbf{H}_3) , there exists a nondecreasing $\frac{\gamma-1}{\gamma}$ -regularly varying sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$\frac{a_n}{n} (L_1 + \dots + L_n - n) \xrightarrow[n \rightarrow \infty]{(\text{law})} X. \quad (2.1.2)$$

In particular, if the variance σ^2 of μ is finite then (\mathbf{H}_3) is verified with $\gamma = 2$, and (2.1.2) holds with $a_n = \sqrt{2n/\sigma^2}$. As we see below, under (\mathbf{H}) , τ behaves regularly when it is conditioned to be large: namely, we see that suitably rescaled versions of τ under $\mathbf{P}(\cdot | \#\tau = n)$ converge in distribution when $n \rightarrow \infty$.

Conveniently, we view τ as a family tree whose ancestor is the root and where siblings are ordered by birth-rank. The depth-first exploration of τ is the sequence of vertices $(u_k)_{0 \leq k < \#\tau}$ that is defined recursively as follows: u_0 is the root and for all $k \in \{0, \dots, \#\tau - 2\}$, let v be the most recent ancestor of u_k having at least one unexplored child (note that possibly $v = u_k$); then u_{k+1} is the unexplored child of v with least birth-rank. Our first result is the following law of large numbers for the size of the range of the \mathbb{W}_b -valued τ -indexed critical branching random walk (see Sections 2.2 and 2.3 for more precise definitions).

Theorem 2.1.1 *Let τ be a Galton–Watson tree with offspring distribution μ that satisfies (\mathbf{H}) . Recall that $(u_k)_{0 \leq k < \#\tau}$ stands for the depth-first exploration of τ . Conditionally given τ , let $(Y_v)_{v \in \tau}$ be a \mathbb{W}_b -valued τ -indexed critical branching random walk starting at $Y_{\text{root}} = \emptyset$. Then, there exists a constant $c_{\mu, b} \in (0, \infty)$ that only depends on μ and b such that*

$$\forall \varepsilon \in (0, \infty), \quad \lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{1}{n} \max_{1 \leq k \leq n} |\#\{Y_{u_l}; 0 \leq l < k\} - c_{\mu, b} k| > \varepsilon \mid \#\tau = n\right) = 0, \quad (2.1.3)$$

In particular, for all $\varepsilon \in (0, \infty)$, we get $\lim_{n \rightarrow \infty} \mathbf{P}(|\frac{1}{n} \#\mathcal{R}_b - c_{\mu, b}| > \varepsilon \mid \#\tau = n) = 0$, where \mathcal{R}_b stands for the range $\{Y_v; v \in \tau\}$ of the branching random walk.

Let τ , $(Y_v)_{v \in \tau}$ and $\mathcal{R}_b = \{Y_v; v \in \tau\}$ be as in Theorem 2.1.1. Observe that \mathcal{R}_b is a subtree of \mathbb{W}_b . Our second main result is a limit theorem for rescaled versions of the metric spaces $(\mathcal{R}_b, d_{\text{gr}})$ where d_{gr} stands for the graph-distance. To state it, let us first recall a limit theorem for (τ, d_{gr}) . Set $H_k(\tau) = d_{\text{gr}}(\text{root}, u_k)$ for all $k \in \{0, \dots, \#\tau - 1\}$, that is the height process of τ . Note that $H(\tau) = (H_k(\tau))_{k \in \{0, \dots, \#\tau - 1\}}$ entirely codes τ . Then, Theorem 3.1 in Duquesne [50] asserts the following: *assume (\mathbf{H}) as in (2.1.1) and let (a_n) be as in (2.1.2); then there is a nonnegative continuous process $H = (H_s)_{s \in [0, 1]}$ such that*

$$\left(\frac{1}{a_n} H_{\lfloor ns \rfloor}(\tau)\right)_{s \in [0, 1]} \quad \text{under } \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{} H, \quad (2.1.4)$$

weakly on $\mathbf{C}([0, 1], \mathbb{R})$. When $\gamma = 2$, H is the normalized Brownian excursion. When the variance of μ is finite, this result is due to Aldous (see Theorem 23 in Aldous [9]). When $\gamma \in (1, 2)$, H is

the normalized excursion of the γ -stable height process that is a local-time function of a γ -stable spectrally positive Lévy process.

The metric space that is the limit of $(\tau, \frac{1}{a_n} d_{\text{gr}})$ as $n \rightarrow \infty$ is derived from the normalized excursion of the γ -stable height process H as follows: for all $s_1, s_2 \in [0, 1]$, we set

$$d_H(s_1, s_2) = H_{s_1} + H_{s_2} - 2 \min_{s_1 \wedge s_2 \leq s \leq s_1 \vee s_2} H_s.$$

We easily check that a.s. d_H is a pseudo-metric on $[0, 1]$. We introduce the relation \sim_H on $[0, 1]$ by setting $s_1 \sim_H s_2$ if and only if $d_H(s_1, s_2) = 0$; clearly, \sim_H is an equivalence relation and the normalized γ -stable Lévy tree is taken as the quotient space $T_H = [0, 1] / \sim_H$, equipped with the distance induced by d_H that we keep denoting d_H . We denote by $p_H: [0, 1] \rightarrow T_H$ the canonical projection. Note that p_H is continuous; therefore T_H is compact and connected. Moreover, T_H is a real tree, namely, a metric space such that all pairs of points are joined by a unique simple arc that turns out to be a geodesic (see Definition 2.4.6 for more details). We set $r_H := p_H(0)$ that is viewed as the root of T_H and we equip T_H with the measure μ_H that is the image of the Lebesgue measure on $[0, 1]$ via p_H , namely, $\int_{T_H} f d\mu_H = \int_0^1 f(p_H(s)) ds$, for all continuous $f: T_H \rightarrow \mathbb{R}$. The convergence (2.1.4) then implies the following one.

$$\left(\tau, \frac{1}{a_n} d_{\text{gr}}, \text{root}, \frac{1}{n} \mathfrak{m}\right) \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{} (T_H, d_H, r_H, \mu_H) \quad (2.1.5)$$

where $\mathfrak{m} = \sum_{v \in \tau} \delta_v$ stands for the counting measure on τ . Here the convergence holds weakly on the space \mathbb{M} of isometry classes of pointed measured compact metric spaces equipped with the Gromov–Hausdorff–Prokhorov distance δ_{GHP} that makes it a Polish space, as proved in Abraham, Delmas & Hoscheit [4, Theorem 2.5]. (See (2.4.10) for a precise definition of δ_{GHP} and see (2.4.11) for more details.) For more details on Lévy trees see Le Gall & Le Jan [81] and Duquesne & Le Gall [61, 57] (see also Section 2.4.4).

The limit of rescaled versions of the metric spaces $(\mathcal{R}_b, d_{\text{gr}})$ is constructed as follows: as proved in Duquesne & Le Gall [57] (Lemma 6.4 p. 600, that is recalled in Lemma 2.4.32), conditionally given H , there exists a Hölder-continuous centered Gaussian process $\sigma \in T_H \mapsto W_\sigma \in \mathbb{R}$ whose covariance is characterised by $\mathbf{E}[|W_{\sigma_1} - W_{\sigma_2}|^2 \mid H] = d_H(\sigma_1, \sigma_2)$, for all $\sigma_1, \sigma_2 \in T_H$. Then, we set

$$\forall \sigma_1, \sigma_2 \in T_H, \quad d_{H,W}(\sigma_1, \sigma_2) = |W_{\sigma_1}| + |W_{\sigma_2}| - 2 \min_{\sigma \in [\sigma_1, \sigma_2]} |W_\sigma|,$$

where $[\sigma_1, \sigma_2]$ is the unique geodesic that joins σ_1 to σ_2 in T_H . In Lemma 2.4.22, we prove that $d_{H,W}$ is a pseudo-metric on T_H ; we then define the equivalence relation $\sim_{H,W}$ on T_H by setting $\sigma_1 \sim_{H,W} \sigma_2$ if and only if $d_{H,W}(\sigma_1, \sigma_2) = 0$ and we denote by $T_{H,W} = T_H / \sim_{H,W}$ the quotient metric space and we keep denoting by $d_{H,W}$ the resulting metric; we denote by $\pi_{H,W}: T_H \rightarrow T_{H,W}$ the canonical projection that is continuous. Thus $T_{H,W}$ is compact and connected, and $(T_{H,W}, d_{H,W})$ is a real tree (see Section 2.4.6, especially Proposition 2.4.34, for more properties of $T_{H,W}$). It turns out that this kind of spaces has been introduced in Curien, Le Gall & Miermont [40] (see also Le Gall [116] for a different purpose); they coined the name Brownian cactus, so we call $(T_{H,W}, d_{H,W})$ the normalized reflected Brownian cactus with γ -stable branching mechanism. We next set $r_{H,W} = \pi_{H,W}(r_H)$ that is viewed as the root of $T_{H,W}$ and we equip $T_{H,W}$ with the measure $\mu_{H,W}$ that is the image of μ_H via $\pi_{H,W}$: namely, $\int_{T_{H,W}} f d\mu_{H,W} = \int_{T_H} f(\pi_{H,W}(\sigma)) \mu_H(d\sigma)$, for all continuous $f: T_{H,W} \rightarrow \mathbb{R}$. Our second result is the following limit theorem.

Theorem 2.1.2 *Let τ be a Galton–Watson tree with offspring distribution μ that satisfies (H) as in (2.1.1). We denote by $\mathfrak{m} = \sum_{v \in \tau} \delta_v$ the counting measure on τ and we denote by d_{gr} the graph distance on τ . Conditionally given τ , let $(Y_v)_{v \in \tau}$ be a \mathbb{W}_b -valued τ -indexed critical branching random walk starting at $Y_{\text{root}} = \emptyset$. We denote by $\mathcal{R}_b = \{Y_v; v \in \tau\}$ the range of Y ,*

by $\mathbf{m}_{\text{occ}}^{\text{b}} = \sum_{v \in \tau} \delta_{Y_v}$ the occupation measure of Y and by d_{gr} the graph distance on \mathcal{R}_{b} . We denote by (T_H, d_H, r_h, μ_H) the normalized γ -stable Lévy tree and by $(T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})$ the normalized reflected Brownian cactus with γ -stable branching mechanism as defined above. Let $(a_n)_{n \in \mathbb{N}}$ be as in (2.1.2). Then, the following limit holds weakly on $(\mathbb{M}, \delta_{\text{GHP}})^2$

$$\begin{aligned} & \left((\tau, \frac{1}{a_n} d_{\text{gr}}, \text{root}, \frac{1}{n} \mathbf{m}), (\mathcal{R}_{\text{b}}, \frac{1}{\sqrt{a_n}} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}_{\text{occ}}^{\text{b}}) \right) \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \\ & \xrightarrow[n \rightarrow \infty]{(law)} \left((T_H, d_H, r_H, \mu_H), (T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W}) \right). \end{aligned} \quad (2.1.6)$$

Moreover, denote by $\mathbf{m}_{\text{count}}^{\text{b}} = \sum_{x \in \mathcal{R}_{\text{b}}} \delta_x$ the counting measure on \mathcal{R}_{b} ; then, for all $\varepsilon \in (0, \infty)$,

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(d_{\text{Prök}}^{(n)} \left(\frac{c_{\mu, \text{b}}}{n} \mathbf{m}_{\text{occ}}^{\text{b}}, \frac{1}{n} \mathbf{m}_{\text{count}}^{\text{b}} \right) > \varepsilon \mid \#\tau = n \right) = 0,$$

where $c_{\mu, \text{b}}$ is as in Theorem 2.1.1 and where $d_{\text{Prök}}^{(n)}$ stands for the Prokhorov distance on the space of finite measures on $(\mathcal{R}_{\text{b}}, \frac{1}{\sqrt{a_n}} d_{\text{gr}})$. It implies that the following limit holds jointly with (2.1.6):

$$\left(\mathcal{R}_{\text{b}}, \frac{1}{\sqrt{a_n}} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}_{\text{count}}^{\text{b}} \right) \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{(law)} (T_{H,W}, d_{H,W}, r_{H,W}, c_{\mu, \text{b}} \cdot \mu_{H,W}).$$

Theorem 2.1.1 is the analogue of Proposition 5 and Theorem 7 in Le Gall & Lin [82] established for \mathbb{Z}^d -valued BRWs. Let us mention that our strategy of proof is similar: we define a specific invariant shift for infinite tree-valued BRWs and we use the subadditive ergodic theorem; the constant $c_{\mu, \text{b}}$ is interpreted as the probability that the invariant BRW visits its starting point only once.

The proof of Theorem 2.1.2 is distinct from that of Theorem 2.1.1. On one hand, it relies on general arguments on weak limits of random metrics (see Proposition 2.4.4). As an application of these results, we prove that the range of critical biased RWs on N -type supercritical GW-trees converges to the tree coded by a reflected Brownian motion (see Corollary 2.4.16). This result is derived from a much more difficult result due to Dembo & Sun [45] that asserts that the distance from the root of the RW converges, when suitably rescaled, to a reflected Brownian motion. The same idea allows to recover previous scaling limits for the range of RWs on supercritical GW-trees such as in Duquesne [52] (critical biased RWs on b -ary trees) or in Aïdékon & de Raphélis [6] (biased RWs and RWs in random environment on single-type GW-trees). We refer to the end of Section 2.4.2 for more details. On the other hand, the proof of Theorem 2.1.2 uses limit theorems for discrete snakes that have been obtained by Janson & Marckert in [97] in the Brownian case and by Marzouk [130] in the stable cases.

Organization of the chapter

The chapter is organised as follows. In Section 2.2, we introduce notations on trees with an infinite line of ancestors that constitute a natural state-space for invariant tree-valued BRWs. In Section 2.3.1, we define the kinds of BRWs that we study; Section 2.3.2 is devoted to metric properties of the range of the so-called free BRWs. In Section 2.3.3, we state a coupling for BRWs that is a key argument in the proof of Theorem 2.1.1. In Section 2.3.4, we prove estimates that are used mostly to prove that $c_{\mu, \text{b}} > 0$. Theorem 2.1.1 is proved in Sections 2.3.5 and 2.3.6. In Section 2.4.1, we prove general convergence results for random metrics. In Section 2.4.2 we apply these results to get scaling limits for the range of RWs on N -type GW-trees (see Corollary 2.4.16). In Section 2.4.3 we introduce snake metrics and we prove specific results. In Sections 2.4.4 and 2.4.5, we recall definitions and properties on stable Lévy trees and Lévy snakes. Section 2.4.6 is devoted to basic properties of reflected Brownian cactuses. Theorem 2.1.2 is proved in Section 2.5.

2.2 Tree with a possibly infinite line of ancestors.

Words. Recall that \mathbb{N} stands for the set of nonnegative integers $\{0, 1, 2, \dots\}$ and that $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let A be a set with more than two elements that is viewed as an alphabet. We denote by \mathbb{W}_A the finite words written with alphabet A : namely,

$$\mathbb{W}_A = \bigcup_{n \in \mathbb{N}} A^n. \quad (2.2.1)$$

Here, A^0 is taken as $\{\emptyset\}$, \emptyset being the empty word. Let $u = (a_1, \dots, a_n) \in \mathbb{W}_A$ be distinct from \emptyset . We set $|u| = n$ that is the *height* of u , with the convention that $|\emptyset| = 0$. We next set $\overleftarrow{u} = (a_1, \dots, a_{n-1})$ that is interpreted as the *parent* of u (if $n = 1$, then $\overleftarrow{u} = \emptyset$). More generally for all $p \in \{1, \dots, n\}$, we set $u|_p = (a_1, \dots, a_p)$, with the convention: $u|_0 = \emptyset$. For all $v = (b_1, \dots, b_m) \in \mathbb{W}_A$, we set $u * v = (a_1, \dots, a_n, b_1, \dots, b_m)$ that is the *concatenation* of u with v , with the convention that $\emptyset * u = u * \emptyset = u$. We shall also define the most recent common ancestor of u and v in \mathbb{W}_A as $u \wedge v = u|_p = v|_p$ where $p = \max\{k \in \mathbb{N} : u|_k = v|_k\}$. We shall consider three cases:

- $A = \mathbb{N}^*$; in that case we use the notation $\mathbb{U} := \mathbb{W}_{\mathbb{N}^*}$, the letter U being for Ulam.
- $A = \{1, \dots, b\}$, b being an integer ≥ 2 ; we use the notation $\mathbb{W}_b := \mathbb{W}_{\{1, \dots, b\}}$; \mathbb{W}_b is the b -ary tree.
- $A = [0, 1]$; we call $\mathbb{W}_{[0,1]}$ the *free tree*.

Definition 2.2.1 Rooted ordered trees can be viewed as subsets $t \subset \mathbb{U}$ that satisfy the following.

- (a) $\emptyset \in t$.
- (b) If $u \in t \setminus \{\emptyset\}$, then $\overleftarrow{u} \in t$.
- (c) For all $u \in t$, there exists $k_u(t) \in \mathbb{N}$ such that $u*(i) \in t$ if and only if $1 \leq i \leq k_u(t)$.

We denote by \mathbb{T} the set of rooted ordered trees. □

The quantity $k_u(t)$ is interpreted as the *number of children of u* and $u*(i)$ is the i -th child of u , $1 \leq i \leq k_u(t)$. If $k_u(t) = 0$, then there is no child stemming from u and assertion (c) is empty. We next set the shift of t at u by $\theta_u t = \{v \in \mathbb{U} : u*v \in t\}$ that is also a rooted ordered tree: it is viewed as the *subtree of the descendants stemming from u* . Unless otherwise specified, all the random variables that are mentioned in this chapter are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 2.2.2 We equip \mathbb{T} with the sigma-field $\mathcal{F}(\mathbb{T})$ generated by the sets $\{t \in \mathbb{T} : u \in t\}$, $u \in \mathbb{T}$. A Galton–Watson tree with offspring distribution μ (a $GW(\mu)$ -tree, for short) is a $(\mathcal{F}, \mathcal{F}(\mathbb{T}))$ -measurable r.v. $\tau : \Omega \rightarrow \mathbb{T}$ that satisfies the following.

- $k_\emptyset(\tau)$ has law μ .
- For all $k \geq 1$ such that $\mu(k) > 0$, the subtrees $\theta_{(1)\tau}, \dots, \theta_{(k)\tau}$ under $\mathbf{P}(\cdot | k_\emptyset(\tau) = k)$ are independent with the same law as τ under \mathbf{P} . □

Recall that τ is a.s. finite if and only if μ is critical or subcritical: $\sum_{k \geq 1} k\mu(k) \leq 1$.

Bilateral words. We shall consider branching random walks seen from the spatial and genealogical position of a tagged individual. To that end, it is convenient to introduce ordered trees that are rooted possibly at a negative generation and we also introduce their local limits that may have an infinite line of ancestors. It is therefore convenient to introduce words indexed by possibly negative numbers.

To simplify, we set $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}^*$ and $\overleftarrow{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\}$. An A -bilateral word is a (possibly infinite) sequence $u = (a_k)_{k_1 < k \leq k_2}$ where $k_1 \in \overleftarrow{\mathbb{Z}}$ and $k_2 \in \mathbb{Z}$ are such that $k_2 \geq k_1$ and where

$a_k \in A$ for all $k_1 < k \leq k_2$. We denote by $\overline{\mathbb{W}}_A$ the set of bilateral words, including the empty word denoted by \emptyset (if $k_1 \geq k_2$, then we agree on $u = \emptyset$). If $u \neq \emptyset$, we introduce the following notation.

$$\begin{aligned} |u|_- &= k_1, \text{ be the depth of } u, \\ |u| &= k_2, \text{ be the relative height of } u \text{ (note that it may take negative values),} \\ \overleftarrow{u} &= (a_k)_{k_1 < k \leq k_2 - 1} \text{ be the parent of } u, \\ u_{|(l_1, l_2]} &= (a_k)_{k_1 \vee l_1 < k \leq k_2 \wedge l_2}, \\ \text{end}(u) &= a_{k_2}, \end{aligned} \tag{2.2.2}$$

for all $l_1 \in \overleftarrow{\mathbb{Z}}, l_2 \in \mathbb{Z}$ such that $l_1 \leq l_2$. To simplify, we also set

$$\forall l \in \mathbb{Z}, \quad u|_l = u_{|(-\infty, l]}. \tag{2.2.3}$$

Let us stress that a bilateral word has at most a finite number of letters a_k indexed by positive indices, $k \in \mathbb{N}^*$, while it can have infinitely many letters indexed by negative indices, $k \in \overleftarrow{\mathbb{Z}} \cap \mathbb{Z}_-$. Note that \emptyset has neither relative height nor depth. Note that if $l_1 \leq k_1$ and $k_2 \leq l_2$, then $u_{|(l_1, l_2]} = u$.

• *Shift.* For all $l \in \mathbb{Z}$, we denote the l -shift operator $\varphi_l: \overline{\mathbb{W}}_A \rightarrow \overline{\mathbb{W}}_A$ by:

$$\varphi_l(u) = (a_{k+l})_{k_1 - l < k \leq k_2 - l},$$

Note that $|\varphi_l(u)|_- = |u|_- - l$, that $|\varphi_l(u)| = |u| - l$ and that $\varphi_l(\emptyset) = \emptyset$. Clearly, $\varphi_l \circ \varphi_{l'} = \varphi_{l+l'}$, φ_l is bijective and $\varphi_l \circ \varphi_{-l}$ is the identity map.

• *Concatenation.* For all $u = (a_k)_{k_1 < k \leq k_2} \in \overline{\mathbb{W}}_A$ and all $v = (b_k)_{1 \leq k \leq k_3} \in \mathbb{W}_A$, we define

$$u * v = (c_l)_{k_1 < k \leq k_2 + k_3} \quad \text{where} \quad c_k = \begin{cases} a_k & \text{If } k_1 < k \leq k_2, \\ b_{k-k_2} & \text{If } k_2 < k \leq k_2 + k_3. \end{cases} \tag{2.2.4}$$

The bilateral word $u * v$ is the concatenation of a bilateral word u on the left with a null depth word v on the right. Note that $|u * v|_- = |u|_-$, that $|u * v| = |u| + |v|$ and that $u * \emptyset = u$.

• *Convergence in $\overline{\mathbb{W}}_A$.* Assume that (A, d_A) is a Polish space. We equip $\overline{\mathbb{W}}_A$ with the following local convergence.

Let $u^{(p)} \in \overline{\mathbb{W}}_A$, $p \in \mathbb{N}$; the sequence of words $u^{(p)}$ converges to u if $|u^{(p)}|_- \rightarrow |u|_-$ in $\overleftarrow{\mathbb{Z}}$ and if for all $l \in \mathbb{N}$ and for all $\varepsilon \in (0, 1)$, there exists $p_{l, \varepsilon} \in \mathbb{N}$ such that for all $p \geq p_{l, \varepsilon}$, $|u^{(p)}| = |u|$, $(-l) \vee |u^{(p)}|_- = (-l) \vee |u|_-$ and $\max_{(-l) \vee |u|_- \leq k \leq |u|} d_A(u_k^{(p)}, u_k) < \varepsilon$.

It is easy to see that this convergence corresponds to a Polish metric and we equip $\overline{\mathbb{W}}_A$ with the corresponding Borel sigma-field. Note that the shifts operators φ_l are homeomorphisms with respect to local convergence. If $A = \mathbb{N}^*$, we shall use the notation $\overline{\mathbb{U}} := \overline{\mathbb{W}}_{\mathbb{N}^*}$. If $A = \{1, \dots, b\}$, we shall use the notation $\overline{\mathbb{W}}_b := \overline{\mathbb{W}}_{\{1, \dots, b\}}$. Note that in these cases, \mathbb{N}^* and $\{1, \dots, b\}$ are equipped with the discrete topology.

Trees with infinite line of ancestors.

Definition 2.2.3 A non-empty subset $R \subset \overline{\mathbb{W}}_A$ is a subtree of $\overline{\mathbb{W}}_A$ if it satisfies the following.

- There exists $|R|_- \in \overleftarrow{\mathbb{Z}}$ such that $|v|_- = |R|_-$ for all $v \in R \setminus \{\emptyset\}$.
- For all $v \in R \setminus \{\emptyset\}$ such that $|v| > |R|_-$, $\overleftarrow{v} \in R$.
- For all u and v in R , there exists $l \in \mathbb{Z}$ such that $u|_l = v|_l$ (see (2.2.3)). □

We call $|R|_-$ the *depth of the subtree* R . If $|R|_- > -\infty$, then (b) implies that the empty word \emptyset is an element of R and $|R|_- + 1 = \min_{v \in R \setminus \{\emptyset\}} |v|$, (c) is always fulfilled since for all $l \leq |R|_-$ and all $v \in R$, we get $v|_l = \emptyset$. If $|R|_- = -\infty$, then $\emptyset \notin R$.

• *Common ancestor.* Recall from (2.2.3) the notation $u|_l$. Let $R \subset \overline{\mathbb{W}}_A$ be a subtree as in Definition 2.2.3. We define the *common ancestor* of $u, v \in R$ by

$$u \wedge v = u|_{b(u,v)} = v|_{b(u,v)} \quad \text{where} \quad b(u,v) := \max\{l \in \mathbb{Z} : u|_l = v|_l\}, \quad (2.2.5)$$

that is well-defined thanks to Definition 2.2.3 (c).

• *Graph distance on subtrees of $\overline{\mathbb{W}}_A$.* A subtree $R \subset \overline{\mathbb{W}}_A$ as in Definition 2.2.3 corresponds to the following graph-tree: its set of vertices is R and its set of edges is $\{\{v, \overleftarrow{v}\}; v \in R : |v| > |R|_-\}$. We easily observe that the graph distance d_{gr} on R is given by

$$\forall x, y \in R, \quad d_{\text{gr}}(x, y) = |x| + |y| - 2|x \wedge y|. \quad (2.2.6)$$

• *Ordered trees with a possibly infinite line of ancestors.* We next extend Definition 2.2.1 to ordered trees with a possibly infinite line of ancestors as follows.

Definition 2.2.4 Recall the notation $\overline{\mathbb{U}} := \overline{\mathbb{W}}_{\mathbb{N}^*}$. A subset $t \subset \overline{\mathbb{U}}$ is an ordered tree (with a possibly infinite line of ancestors) if it is a subtree of $\overline{\mathbb{U}}$ as in Definition 2.2.3 satisfying (a), (b) and (c) with $A = \mathbb{N}^*$ and if it furthermore satisfies the condition that any word has a finite number of children, that is,

(d) $\forall u \in t, k_u(t) := \#\{v \in t : \overleftarrow{v} = u\} < \infty$ and $\{1, \dots, k_u(t)\} = \{\text{end}(v); v \in t : \overleftarrow{v} = u\}$ if $k_u(t) \geq 1$.

We shall consider that the singleton $\{\emptyset\}$ is the only tree with one point. We denote by $\overline{\mathbb{T}}$ the set of ordered trees. \square

For all $k \in \overleftarrow{\mathbb{Z}}$, we set $\mathbb{T}_k = \{\{\emptyset\}\} \cup \{t \in \overline{\mathbb{T}} : |t|_- = k\}$. Note that $\mathbb{T}_0 = \overline{\mathbb{T}}$, where $\overline{\mathbb{T}}$ is as in Definition 2.2.1 and observe that $\varphi_{-k}(\overline{\mathbb{T}}) = \mathbb{T}_k$ when $k > -\infty$.

• *Lexicographical order and successor of a vertex.* Let $t \in \overline{\mathbb{T}}$. By Definition 2.2.3 (c), the vertices of t are totally ordered by the *lexicographical order* \leq_t that is formally defined as follows. Let $u, v \in t \setminus \{\emptyset\}$; recall from (2.2.5) the definition of $b(u, v)$ and from (2.2.2) the definition of $\text{end}(\cdot)$; then,

$$u \leq_t v \quad \text{if and only if} \quad \text{end}(u|_{b(u,v)+1}) \leq \text{end}(v|_{b(u,v)+1}). \quad (2.2.7)$$

Note that \leq_t actually depends on t : it is not defined on the whole set of bilateral words $\overline{\mathbb{U}}$ but only on t (indeed, to define \leq_t , branching points have to be well-defined, which requires possibly infinite words to share a prefix). If $\emptyset \in t$, then \emptyset is the \leq_t -least element of t . We denote by $<_t$ the strict order associated with \leq_t . We also introduce the following related notation: for all $u \in t$, the *successor* $\text{scc}(u)$ of u is defined as the \leq_t -least element of $\{v \in t : u <_t v\}$ if this set is not empty, otherwise we simply take $\text{scc}(u) = u$.

• *Subtree.* Let $t \in \overline{\mathbb{T}}$ and $u \in t$. The subtree $\theta_u t$ stemming from u is defined as follows.

$$\theta_u t = \{v \in \mathbb{U} : u * v \in t\}, \quad (2.2.8)$$

where we recall from (2.2.4) the definition of the concatenation $*$ of a bilateral word on the left with a null-depth word on the right. Note that $|\theta_u t|_- = 0$, namely: $\theta_u t \in \overline{\mathbb{T}}$, where $\overline{\mathbb{T}}$ is as in Definition 2.2.1.

Pointed labelled trees. To deal with branching random walks, we introduce labelled trees where the label of a vertex is viewed as its position in space. More precisely, let (E, d_E) be a Polish metric space. We define the space of *pointed E -labelled trees* as follows. For all $k \in \overline{\mathbb{Z}}$, we set:

$$\mathbb{T}_k^\bullet(E) = \{\partial\} \cup \{\mathbf{t} = (t, \circ; \mathbf{x} = (x_v)_{v \in t}) : t \in \mathbb{T}_k, \circ \in t, x_v \in E, v \in t\} \quad \text{and} \quad \overline{\mathbb{T}}^\bullet(E) = \bigcup_{k \in \overline{\mathbb{Z}}} \mathbb{T}_k^\bullet(E),$$

where ∂ stands for a cemetery point. Here, the label of $v \in t$ is $x_v \in E$ that is viewed as the spatial position of v . If there is no label, we simply write \mathbb{T}_k^\bullet and $\overline{\mathbb{T}}^\bullet$.

• *Shift operator on labelled trees.* Shift operators act naturally on the space of pointed E -labelled trees as follows: let $l \in \mathbb{Z}$; we set $\varphi_l(\partial) = \partial$ and for all $\mathbf{t} = (t, \circ; \mathbf{x}) \in \overline{\mathbb{T}}^\bullet(E) \setminus \{\partial\}$, we set

$$\varphi_l(\mathbf{t}) = (\varphi_l(t), \varphi_l(\circ); \mathbf{x} \circ \varphi_{-l}), \quad \text{where} \quad \mathbf{x} \circ \varphi_{-l} = (x_{\varphi_{-l}(w)})_{w \in \varphi_l(t)}.$$

• *Truncation.* We next define a natural truncation procedure for pointed labelled trees along the line of ancestors of the distinguished point.

Definition 2.2.5 Let $\mathbf{t} = (t, \circ; \mathbf{x} = (x_v)_{v \in t}) \in \overline{\mathbb{T}}^\bullet(E)$. Let $p \in \overline{\mathbb{Z}}$ and $q \in \mathbb{Z} \cup \{\infty\}$ be such that $p < q$. We define the following.

- If $(p, q] \cap (|\circ|_-, |\circ|] = \emptyset$, then we set $[\mathbf{t}]_p^q = \partial$. We also set $[\partial]_p^q = \partial$.
- If $(p, q] \cap (|\circ|_-, |\circ|] \neq \emptyset$, then we set $[\mathbf{t}]_p^q = (t', \circ'; \mathbf{x}')$ where

$$\circ' = \circ|_{(p,q]}, \quad t' = \{v|_{(p,q]}; v \in t : v|_p = \circ|_p\} \quad \text{and} \quad x'_{v'} = x_v,$$

where $v \in t$ is such that $v' = v|_{(p,q]}$ and $v|_p = \circ|_p$ (recall notation $v|_p$ from (2.2.3)).

We simply set $[\mathbf{t}]_p$ instead of $[\mathbf{t}]_p^\infty$. If $\mathbf{t} \in \mathbb{T}_k^\bullet(E)$, then note that $[\mathbf{t}]_p^q \in \mathbb{T}_{k \vee p}^\bullet(E)$. We use a similar notation for pointed trees without label. \square

• *Local convergence on $\overline{\mathbb{T}}^\bullet(E)$.* For all $\mathbf{t} = (t, \circ; \mathbf{x})$, $\mathbf{t}' = (t', \circ'; \mathbf{x}')$ in $\overline{\mathbb{T}}^\bullet(E)$, we first set

$$\Delta(\mathbf{t}, \mathbf{t}') = \mathbf{1}_{\{(t, \circ) \neq (t', \circ')\}} + \mathbf{1}_{\{(t, \circ) = (t', \circ')\}} \max_{v \in t} (1 \wedge d_E(x_v, x'_{v'}))$$

with $\Delta(\mathbf{t}, \partial) = 1$ and $\Delta(\partial, \partial) = 0$. We easily check that Δ is a metric on $\overline{\mathbb{T}}^\bullet(E)$. Then, we define the local convergence as follows.

Let $\mathbf{t}_n = (t^{(n)}, \circ^{(n)}; \mathbf{x}^{(n)}) \in \overline{\mathbb{T}}^\bullet(E)$, $n \in \mathbb{N}$; the sequence $(\mathbf{t}_n)_{n \geq 0}$ is said to converge locally to $\mathbf{t} = (t, \circ; \mathbf{x}) \in \overline{\mathbb{T}}^\bullet(E)$ if for all $\varepsilon \in (0, 1)$ and all $q \in \mathbb{N}$, there exists $n_{q, \varepsilon} \in \mathbb{N}$ such that for all integers $n \geq n_{q, \varepsilon}$, $\Delta([\mathbf{t}^{(n)}]_{-q}^q, [\mathbf{t}]_{-q}^q) < \varepsilon$ (when there is no label, it simply means that $[\mathbf{t}^{(n)}]_{-q}^q = [\mathbf{t}]_{-q}^q$).

Local convergence corresponds for instance to the following metric:

$$\forall \mathbf{t}, \mathbf{t}' \in \overline{\mathbb{T}}^\bullet(E), \quad \delta_{\text{loc}}(\mathbf{t}, \mathbf{t}') = \sum_{q \in \mathbb{N}} 2^{-q-1} \Delta([\mathbf{t}]_{-q}^q, [\mathbf{t}']_{-q}^q),$$

with $\delta_{\text{loc}}(\mathbf{t}, \partial) = 1$ and $\delta_{\text{loc}}(\partial, \partial) = 0$. We easily check that $(\overline{\mathbb{T}}^\bullet(E), \delta_{\text{loc}})$ is Polish and we note that shift operators are isometries.

Definition 2.2.6 Let $(t, \circ) \in \overline{\mathbb{T}}^\bullet$ be distinct from ∂ .

- (a) We set $\text{cent}(t, \circ) = \varphi_{|\circ|}(t, \circ)$ that is the *centering* map: it shifts trees so that their distinguished point is at relative height 0.

- (b) We next set $\text{scc}(t, \circ) = \text{cent}(t, \text{scc}(\circ))$ where we recall that $\text{scc}(\circ)$ stands for the vertex of t coming next in the lexicographical order as defined by (2.2.7). We call $\text{scc}(\cdot)$ the *successor map*.
- (c) Observe that there is a unique pointed tree $(t', \circ') \in \overline{\mathbb{T}}^\bullet$ and a unique one-to-one map $\psi: t' \rightarrow \{\circ|_p; p \leq |\circ|\} \cup \{v \in t: \circ <_t v\}$ such that $\psi(\varrho') = \varrho$, that is increasing with respect to the lexicographical order and that preserves the relative height; we set $[(t, \circ)]^+ = (t', \circ')$ that is called the *right-part* of (t, \circ) . Intuitively, the only difference between (t', ρ') and $\{\circ|_p; p \leq |\circ|\} \cup \{v \in t: \circ <_t v\}$ is that (t', ρ') respects the convention (c) in Definition 2.2.1 we have imposed on rooted ordered trees. We also set

$$\text{scc}^+(t, \circ) = [\text{scc}(t, \circ)]^+.$$

The map $\text{scc}^+(\cdot)$ is called the *right-successor*.

By convenience, we set $\text{cent}(\partial) = \text{scc}(\partial) = \text{scc}^+(\partial) = \partial$. \square

Note that

$$[\text{cent}(t, \circ)]^+ = \text{cent}([(t, \circ)]^+) \quad \text{and} \quad \text{scc}^+(t, \circ) = \text{scc}^+([(t, \circ)]^+). \quad (2.2.9)$$

Let us state a technical result about the continuity of the maps cent , $[\cdot]^+$, scc and scc^+ .

Lemma 2.2.7 *The maps $\text{cent}(\cdot)$ and $[\cdot]^+$ are locally continuous and the maps $\text{scc}(\cdot)$ and $\text{scc}^+(\cdot)$ are locally continuous at the pointed trees (t, \circ) such that $\text{scc}(\circ) \neq \circ$.*

Proof. Since shift-operators are $\delta_{1\circ\text{c}}$ -isometries, the continuity of cent follows from the continuity of $(t, \circ) \rightarrow |\circ|$ that is a direct consequence of the definition of local convergence; $[\cdot]^+$ is locally continuous because $[[\mathbf{t}]^q_{-q}]^+ = [[\mathbf{t}]^+]^q_{-q}$ and to complete the proof, it is then sufficient to prove that $(t, \circ) \mapsto (t, \text{scc}(\circ))$ is locally continuous at trees such that $\text{scc}(\varrho) \neq \varrho$.

To that end, let $(t_n, \circ_n) \rightarrow (t, \circ)$ locally in $\overline{\mathbb{T}}^\bullet$. Set $p_0 = |\text{scc}(\circ)|$ and suppose that $\circ \neq \text{scc}(\circ)$, which implies $|\circ \wedge \text{scc}(\circ)| = p_0 - 1$. Let $p \in \mathbb{N}$ be such that $-p < p_0 - 1 < |\circ| + 1 < p$; there is $n_p \in \mathbb{N}$ such that for all $n \geq n_p$, $(t'_n, \circ'_n) := [(t_n, \circ_n)]^p_{-p}$ is equal to $(t', \circ') := [(t, \circ)]^p_{-p}$; thus, the successor of \circ'_n in t'_n is equal to the successor of \circ' in t' . Consequently, $[(t_n, \text{scc}(\circ_n))]^p_{-p} = (t'_n, \text{scc}(\circ'_n)) = (t', \text{scc}(\circ')) = [(t, \text{scc}(\circ))]^p_{-p}$, which entails the desired result. \blacksquare

Infinite pointed Galton–Watson trees. For all $t \in \overline{\mathbb{T}}$ and for all $u \in t$, recall from (2.2.8) the definition of the subtree $\theta_u t \in \overline{\mathbb{T}}$.

Definition 2.2.8 Let $(r(j, k))_{k \geq j \geq 1}$ be a probability measure on the octant $\{(j, k) \in (\mathbb{N}^*)^2 : j \leq k\}$. Let μ be a probability measure on \mathbb{N} . Let $\tau^* = (\tau^*, \varrho) : \Omega \rightarrow \overline{\mathbb{T}}^\bullet$ be a Borel-measurable random pointed tree such that a.s. $|\varrho| = 0$ and $|\tau^*|_- = -\infty$. We introduce the following notation.

$$\forall p \in \mathbb{N}, \quad \varrho(p) = \varrho|_{]-\infty, -p]}, \quad \text{Sp} = \{\varrho(p); p \in \mathbb{N}\} \quad \text{and} \quad \partial \text{Sp} = \{u \in \tau^* \setminus \text{Sp} : \overleftarrow{u} \in \text{Sp}\}.$$

Then, τ^* is an *infinite pointed Galton–Watson tree with offspring distribution μ and dispatching measure r* if the r.v. $S := (\text{end}(\varrho(p)), k_{\varrho(p+1)}(\tau^*))_{p \in \mathbb{N}}$ are i.i.d. with law r and if conditionally given S , the subtrees $(\theta_u \tau; u \in \partial \text{Sp})$ are independent $\text{GW}(\mu)$ -trees.

We shall deal with the following special cases that are well-defined if $m_\mu := \sum_{k \in \mathbb{N}} k\mu(k)$ is a finite quantity.

- (i) If $r(j, k) = \mu(k)/m_\mu$, for all $k \geq j \geq 1$, then we say that τ^* is an *infinite pointed $\text{GW}(\mu)$ -tree* (an $\text{IPGW}(\mu)$ -tree for short).

(ii) If $r(j, k) = \mathbf{1}_{\{j=1\}} \bar{\mu}(k)$, for all $k \geq j \geq 1$, where $\bar{\mu}(k) = m_\mu^{-1} \sum_{l \geq k} \mu(l)$ for all $k \geq 1$, then we say that τ^* is the *right part of an infinite pointed GW(μ)-tree* (an $\text{IPGW}^+(\mu)$ -tree for short).

Note that if τ^* is an $\text{IPGW}(\mu)$ -tree, then $[\tau^*]^+$ is an $\text{IPGW}^+(\mu)$ -tree. \square

IPGW -trees are related to GW -trees via the *many-to-one* principle (or the *one-point decomposition of GW-trees*) that asserts the following: let μ be a probability distribution on \mathbb{N} such that $m_\mu := \sum_{k \in \mathbb{N}} k\mu(k) < \infty$. Let τ be a $\text{GW}(\mu)$ -tree and let τ^* be an $\text{IPGW}(\mu)$ -tree as in Definition 2.2.8. Then for all Borel-measurable functions $F: \mathbb{N} \times \bar{\mathbb{T}}^\bullet \rightarrow [0, \infty)$,

$$\mathbf{E} \left[\sum_{v \in \tau} F(|v|; \varphi_{|v|}(\tau, v)) \right] = \sum_{p \geq 0} m_\mu^p \mathbf{E} [F(p; [\tau^*]_{-p})], \quad (2.2.10)$$

where we recall from Definition 2.2.5 the notation $[\tau^*]_{-p}$ for the pointed tree τ^* truncated above the ancestor of ϱ at generation $-p$. Based on this identity, the following proposition shows that IPGW trees are local limits of critical GW -trees conditioned to be large and seen from a uniformly chosen vertex. This result is part of the folklore; its proof derives from (2.2.10) and it is left to the reader.

Proposition 2.2.9 *Let μ be a probability distribution on \mathbb{N} such that $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. We assume that μ is aperiodic. Let τ be a $\text{GW}(\mu)$ -tree; let \mathbf{u} be uniformly distributed on the set of vertices of τ . Let τ^* be an $\text{IPGW}(\mu)$ -tree as in Definition 2.2.8. Then*

$$\text{cent}(\tau, \mathbf{u}) \quad \text{under} \quad \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{} \tau^*$$

weakly on $\bar{\mathbb{T}}^\bullet$ with respect to local convergence.

We use the previous proposition to prove the following one.

Proposition 2.2.10 *Let μ be a probability measure on \mathbb{N} such that $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. Recall $\text{scc}(\cdot)$ and $\text{scc}^+(\cdot)$ from Definition 2.2.6. Then, the law of $\text{IPGW}(\mu)$ -trees (resp. $\text{IPGW}^+(\mu)$ -trees) is preserved by $\text{scc}(\cdot)$ (resp. by $\text{scc}^+(\cdot)$).*

Proof. Let us first mention that a different proof of the result for $\text{scc}^+(\cdot)$ is given in Le Gall & Lin [82, Proposition 2]. Then, note that the result for $\text{scc}^+(\cdot)$ is implied by the result for $\text{scc}(\cdot)$ by (2.2.9) and since the right-part of a $\text{IPGW}(\mu)$ -tree is an $\text{IPGW}^+(\mu)$ -tree. Let us prove the result for $\text{scc}(\cdot)$. Let $\tau^* = (\tau^*, \varrho)$ be an $\text{IPGW}(\mu)$ -tree and let τ be a $\text{GW}(\mu)$ -tree; let \mathbf{u} be uniformly distributed on the set of vertices of τ ; denote by v_* the last vertex of τ with respect to the lexicographical order. Set $\mathbf{u}' = \text{scc}(\mathbf{u})$ if $\mathbf{u} \neq v_*$ and $\mathbf{u}' = \emptyset$ if $\mathbf{u} = v_*$: clearly, \mathbf{u}' is uniformly distributed on τ and $\mathbf{P}(\text{scc}(\mathbf{u}) \neq \mathbf{u}' \mid \#\tau = n) = 1/n$. Let $F: \bar{\mathbb{T}}^\bullet \rightarrow [0, \infty)$ be locally continuous and bounded. By Definition 2.2.6 (b), $\text{scc}(\tau, \mathbf{u}) = \text{cent}(\tau, \text{scc}(\mathbf{u}))$. Thus,

$$|\mathbf{E}[F(\text{scc}(\tau, \mathbf{u})) \mid \#\tau = n] - \mathbf{E}[F(\text{cent}(\tau, \mathbf{u}')) \mid \#\tau = n]| \leq 2\|F\|_\infty/n.$$

First suppose that μ is aperiodic. Then the previous inequality and Proposition 2.2.9 imply that

$$\lim_{n \rightarrow \infty} \mathbf{E}[F(\text{scc}(\tau, \mathbf{u})) \mid \#\tau = n] = \lim_{n \rightarrow \infty} \mathbf{E}[F(\text{cent}(\tau, \mathbf{u}')) \mid \#\tau = n] = \mathbf{E}[F(\tau^*)].$$

Moreover, since it is clear that a.s. $\text{scc}(\varrho) \neq \varrho$, Lemma 2.2.7 and Proposition 2.2.9 entail

$$\lim_{n \rightarrow \infty} \mathbf{E}[F(\text{scc}(\tau, \mathbf{u})) \mid \#\tau = n] = \mathbf{E}[F(\text{scc}(\tau^*))],$$

which completes the proof when μ is aperiodic.

Let us consider a general μ . For all $\varepsilon \in (0, 1)$, we set $\mu_\varepsilon := \varepsilon\delta_1 + (1-\varepsilon)\mu$. Namely, μ_ε is a critical aperiodic offspring distribution. Let τ_ε^* be an infinite $\text{GW}(\mu_\varepsilon)$ -tree. We easily check that $\tau_\varepsilon^* \rightarrow \tau^*$ locally as $\varepsilon \rightarrow 0$. Then the local continuity of $\text{scc}(\cdot)$ (see Lemma 2.2.7) entails the desired result. \blacksquare

2.3 Tree-valued branching random walks.

2.3.1 Definitions.

Let E be a (Polish) space of labels. Let $(q(y, dy'))_{y \in E}$ be a transition kernel and let ϖ be a Borel probability measure on E . For all pointed tree $\mathbf{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$, we define the law $Q_{\varpi, \mathbf{t}}$ on $\overline{\mathbb{T}}^\bullet(E)$ of the q -branching random walk with genealogical tree t and such that ϖ is the law of the spatial position of the distinguished individual u .

To that end, we first assume that t is finite and we introduce the following notation: for any $v, w \in t$, we denote by $\llbracket v, w \rrbracket$ the set of vertices on the shortest path joining v to w in the tree t . If $w \neq v$, then we denote by \overleftarrow{w}^v the unique $v' \in \llbracket v, w \rrbracket$ that is at graph-distance 1 from w ; we call \overleftarrow{w}^v the v -parent of w .

Then, the r.v. $\Theta = (t, u; (Y_v)_{v \in t}) : \Omega \rightarrow \overline{\mathbb{T}}^\bullet(E)$ has law $Q_{\varpi, \mathbf{t}}$ if

$$\text{the joint law of } (Y_v)_{v \in t} \text{ is } \varpi(dy_u) \prod_{v \in t \setminus \{u\}} q(y_{\overleftarrow{v}^u}, dy_v), \quad (2.3.1)$$

The definition (2.3.1) can be extended to the case in which \mathbf{t} is infinite. For such purpose it is enough to note that $[\Theta]_p^q$ has law $Q_{\varpi, [(t, \varrho)]_p^q}$ (see Definition 2.2.5 for the truncation $[\cdot]_p^q$).

Let us note that

$$\mathbf{t}_n \xrightarrow[n \rightarrow \infty]{} \mathbf{t} \text{ locally in } \overline{\mathbb{T}}^\bullet \implies Q_{\varpi, \mathbf{t}_n} \xrightarrow[n \rightarrow \infty]{} Q_{\varpi, \mathbf{t}} \text{ weakly on } \overline{\mathbb{T}}^\bullet(E).$$

When $\varpi = \delta_y$ for some $y \in E$, we simply write $Q_{y, \mathbf{t}}$ instead of $Q_{\delta_y, \mathbf{t}}$. Since $y \mapsto q(y, A)$ is Borel-measurable for all Borel subsets A of E , it is easy to check that $y \mapsto Q_{y, \mathbf{t}}$ is also Borel measurable and that $Q_{\varpi, \mathbf{t}} = \int_E \varpi(dy) Q_{y, \mathbf{t}}$.

As an immediate consequence of the definition, we also get the following: fix $l \in \mathbb{Z}$ and set $(t', u') = (\varphi_l(t), \varphi_l(u))$.

$$\text{If } \Theta = (t, u; (Y_v)_{v \in t}) \text{ has law } Q_{\varpi, (t, u)}, \text{ then } (t', u'; (Y_{\varphi_{-l}(w)})_{w \in t'}) \text{ has law } Q_{\varpi, (t', u')}. \quad (2.3.2)$$

Definition 2.3.1 We fix $\mathbf{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$. We shall consider mostly the four following cases.

- (i) $E = \mathbb{W}_b$, the b -ary tree equipped with the local convergence; $\varpi = \delta_\emptyset$ and $q(x, dy) = p_b^+(x, dy)$ where for all measurable $f : \mathbb{W}_b \rightarrow [0, \infty)$,

$$\int_{\mathbb{W}_b} p_b^+(x, dy) f(y) = \begin{cases} \frac{1}{2} f(\overleftarrow{x}) + \frac{1}{2b} \sum_{1 \leq i \leq b} f(x*(i)) & \text{if } x \neq \emptyset \\ \frac{1}{b} \sum_{1 \leq i \leq b} f((i)) & \text{if } x = \emptyset. \end{cases}$$

We denote by $Q_{\mathbf{t}}^{+b}$ the law of the \mathbb{W}_b -valued branching random walk with transition kernel p_b^+ and with "initial" position \emptyset in \mathbb{W}_b .

- (ii) $E = \mathbb{W}_{[0,1]}$, the free tree equipped with the local convergence; $\varpi = \delta_\emptyset$ and $q(x, dy) = p^+(x, dy)$ where for all measurable $f : \mathbb{W}_{[0,1]} \rightarrow [0, \infty)$,

$$\int_{\mathbb{W}_{[0,1]}} p^+(x, dy) f(y) = \begin{cases} \frac{1}{2} f(\overleftarrow{x}) + \frac{1}{2} \int_0^1 ds f(x*(s)) & \text{if } x \neq \emptyset \\ \int_0^1 ds f((s)) & \text{if } x = \emptyset. \end{cases}$$

We denote by $Q_{\mathbf{t}}^+$ the law of the $\mathbb{W}_{[0,1]}$ -valued branching random walk with transition kernel p^+ and with "initial" position \emptyset in $\mathbb{W}_{[0,1]}$.

(iii) $E = \mathbb{W}_b^* := \{x \in \overline{\mathbb{W}}_b : |x|_- = -\infty\}$, equipped with the local convergence; we fix $o \in \mathbb{W}_b^*$ and we take $\varpi = \delta_o$ and $q(x, dy) = p_b(x, dy)$ where for all $x \in \mathbb{W}_b^*$ and for all measurable $f : \mathbb{W}_b^* \rightarrow [0, \infty)$,

$$\int_{\mathbb{W}_b^*} p_b(x, dy) f(y) = \frac{1}{2} f(\overleftarrow{x}) + \frac{1}{2b} \sum_{1 \leq i \leq b} f(x^*(i)). \quad (2.3.3)$$

We denote by $Q_{o,t}^b$ the law of the \mathbb{W}_b^* -valued branching random walk with transition kernel p_b and with "initial" position o in \mathbb{W}_b^* .

(iv) $E = \mathbb{W}_{[0,1]}^* := \{x \in \overline{\mathbb{W}}_{[0,1]} : |x|_- = -\infty\}$, equipped with the local convergence; we fix $o \in \mathbb{W}_{[0,1]}^*$; we take $\varpi = \delta_o$ and $q(x, dy) = p(x, dy)$, where for all $x \in \overline{\mathbb{W}}_{[0,1]}$ and for all measurable $f : \mathbb{W}_{[0,1]}^* \rightarrow [0, \infty)$,

$$\int_{\mathbb{W}_{[0,1]}^*} p(x, dy) f(y) = \frac{1}{2} f(\overleftarrow{x}) + \frac{1}{2} \int_0^1 ds f(x^*(s)).$$

We denote by $Q_{o,t}$ the law of the $\mathbb{W}_{[0,1]}^*$ -valued branching random walk with transition kernel p and with "initial" position o in $\mathbb{W}_{[0,1]}^*$; we shall refer to this branching random walk as the *free branching random walk*. \square

Remark 2.3.2 Note that if $\Theta = (t, u; (X_v)_{v \in t})$ has law $Q_{o,t}^b$ (or $Q_{o,t}$) then $|\Theta| := (t, u; (|X_v|)_{v \in t})$ is a \mathbb{Z} -valued branching random walk whose spatial motion is that of the simple symmetric random walk. Similarly, if Θ has law Q_t^{+b} (or Q_t^+), then $|\Theta|$ is an \mathbb{N} -valued branching random walk whose spatial motion is that of the simple symmetric random walk reflected at 0. \square

Definition 2.3.3 We define the *b-contraction map* $\Phi_b : \overline{\mathbb{W}}_{[0,1]} \rightarrow \overline{\mathbb{W}}_b$ as follows. For every $r \in (0, \infty)$, we set $\lceil r \rceil = \min\{k \in \mathbb{Z} : r \leq k\}$ and by convenience we take $\lceil 0 \rceil = 1$. Then, for all $x = (a_k)_{|x|_- < k \leq |x|} \in \overline{\mathbb{W}}_{[0,1]}$, we define

$$\Phi_b(x) = (\lceil ba_k \rceil)_{|x|_- < k \leq |x|} \in \overline{\mathbb{W}}_b.$$

The b-contraction map is measurable and preserves the depth and the relative height of words. \square

Remark 2.3.4 Note that Φ_b transforms respectively the kernel $p(x, dy)$ into $p_b(x, dy)$ and the kernel $p^+(x, dy)$ into $p_b^+(x, dy)$. It naturally extends to $\overline{\mathbb{W}}_{[0,1]}$ -labelled pointed trees as follows: if $\Theta = (t, \varrho; (x_v)_{v \in t}) \in \overline{\mathbb{T}}^\bullet(\overline{\mathbb{W}}_{[0,1]})$, we set $\Phi_b(\Theta) = (t, \varrho; (\Phi_b(x_v))_{v \in t})$. It is clear that $\Phi_b(\Theta)$ is a $\overline{\mathbb{W}}_b$ -labelled pointed tree and we easily check that the map is measurable. Moreover, if $\Theta = (t, \varrho; (X_v)_{v \in t})$ has law Q_t^+ (resp. $Q_{o,t}$) then $\Phi_b(\Theta)$ has law Q_t^{+b} (resp. $Q_{\Phi_b(o),t}^b$). \square

2.3.2 Metric properties of the range of free branching random walks.

We gather basic facts about the range of a free branching random walk in terms of the heights of the spatial positions in $\mathbb{W}_{[0,1]}^*$. Let $\mathfrak{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$ such that $|t|_- = -\infty$. For all $w \in t$, we recall that $\llbracket v, w \rrbracket$ is the shortest path (with respect to the graph-distance) that joins vertex v to vertex w . Moreover, we set $\llbracket v, w \rrbracket = \llbracket v, w \rrbracket \setminus \{w\}$, $\llbracket v, w \rrbracket = \llbracket v, w \rrbracket \setminus \{v\}$ and $\llbracket v, w \rrbracket = \llbracket v, v \rrbracket \setminus \{v, w\}$; we also denote by $\llbracket -\infty, v \rrbracket$ the lineage of v : namely, $\llbracket -\infty, v \rrbracket = \{v_l : l \leq |v|\}$.

We next decompose a free branching random walk by first describing the heights of the vertices as a \mathbb{Z} -valued branching random walk $(\mathbf{h}_v)_{v \in t}$ and then explaining how to embed $(\mathbf{h}_v)_{v \in t}$ randomly in $\mathbb{W}_{[0,1]}^*$. More specifically, for all $v \in t$, let $\mathbf{h}_v \in \mathbb{Z}$ be such that

$$\forall v \in t, \quad |\mathbf{h}_v - \mathbf{h}_{\overleftarrow{v}}| = 1 \quad \text{and} \quad \inf_{w \in \llbracket v, u \rrbracket} \mathbf{h}_w \xrightarrow{|v| \rightarrow -\infty} -\infty. \quad (2.3.4)$$

Let $(U_v)_{v \in t}$ be a family of independent r.v. that are uniformly distributed on $[0, 1]$. With $v \in t$, we associate a spatial position in $\mathbb{W}_{[0,1]}^*$ as follows. Since \mathbf{h} takes arbitrary negative values on the lineage of v , for all integers $k \leq \mathbf{h}_v$,

$$\text{there is a unique } v(k) \in \llbracket -\infty, v \rrbracket \text{ such that } \mathbf{h}_{v(k)} = \min_{w \in \llbracket v(k), v \rrbracket} \mathbf{h}_w = k > \mathbf{h}_{v(k)}^{\leftarrow}. \quad (2.3.5)$$

Namely, $\{\overleftarrow{v(k)}; k \leq \mathbf{h}_v\}$ is the set of vertices in the lineage of v where \mathbf{h} reaches a new infimum. In particular, note that $v = v(\mathbf{h}_v)$. Then, we set

$$\forall v \in t, \quad X_v = (U_{v(k)})_{k \leq \mathbf{h}_v}. \quad (2.3.6)$$

By construction $|X_v| = \mathbf{h}_v$, $v \in t$. Then, $(t, u; (X_v)_{v \in t})$ is a $\mathbb{W}_{[0,1]}^*$ -valued branching random walk satisfying

$$X_v = \begin{cases} X_{\overleftarrow{v}} * (U_v) & \text{if } |X_v| - |X_{\overleftarrow{v}}| = 1, \\ \overleftarrow{X_{\overleftarrow{v}}} & \text{if } |X_v| - |X_{\overleftarrow{v}}| = -1. \end{cases}$$

We view $(X_v)_{v \in t}$ as a version of free branching random walk conditionally given the relative heights $(\mathbf{h}_v)_{v \in t}$ of the spatial positions. The following proposition is a key point to analyse the metric of the range of free branching random walks.

Proposition 2.3.5 *Let $\mathbf{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$, $(\mathbf{h}_v, U_v, X_v)_{v \in t}$ be as above. Then, $\{X_v; v \in t\}$ is a subtree of $\mathbb{W}_{[0,1]}^*$ as in Definition 2.2.3 and if we denote by d_{gr} its graph-distance, we get*

$$\mathbf{P}\text{-a.s. for all } v, w \in t, \quad d_{\text{gr}}(X_v, X_w) = |X_v| + |X_w| - 2 \min_{v' \in \llbracket v, w \rrbracket} |X_{v'}|. \quad (2.3.7)$$

Proof. We fix $v, w \in t$. By (2.2.6) we know that $d_{\text{gr}}(X_v, X_w) = |X_v| + |X_w| - 2|X_v \wedge X_w|$. Since t is countable, to prove (2.3.7) it is enough to show that a.s. $|X_v \wedge X_w| = \min_{v' \in \llbracket v, w \rrbracket} |\mathbf{h}_{v'}|$. We prove it in two steps.

Step 1. Recall from (2.3.5) the definition of the $v(k)$, $k \leq \mathbf{h}_v$ and define similarly the $w(k)$, $k \leq \mathbf{h}_w$. We then set

$$k = \max \{ \ell \leq \mathbf{h}_v : v(\ell) \in \llbracket -\infty, v \wedge w \rrbracket \} \quad \text{and} \quad m = \max \{ \ell \leq \mathbf{h}_w : w(\ell) \in \llbracket -\infty, v \wedge w \rrbracket \}.$$

Let us prove that $k = \min_{\llbracket v, v \wedge w \rrbracket} \mathbf{h}$. If $k < \mathbf{h}_v$, then $v(k+1) \in \llbracket v \wedge w, v \rrbracket$ and $\overleftarrow{v}(k+1) \in \llbracket v \wedge w, v \rrbracket$; then $\mathbf{h}_{\overleftarrow{v}(k+1)} = \mathbf{h}_{v(k+1)} - 1 = k = \min_{\llbracket v, v \wedge w \rrbracket} \mathbf{h}$. If $k = \mathbf{h}_v$, then, by (2.3.5), it holds that $\mathbf{h}_v = \mathbf{h}_{v(k)} = \min_{\llbracket v, v(k) \rrbracket} \mathbf{h}$ and since $v(k) \in \llbracket -\infty, v \wedge w \rrbracket$, we also get $k = \min_{\llbracket v, v \wedge w \rrbracket} \mathbf{h}$. Similarly, we get $m = \min_{\llbracket w, v \wedge w \rrbracket} \mathbf{h}$. By definition (2.3.6),

$$X_v = X_{v(k)} * (U_{v(k+1)}, \dots, U_{v(\mathbf{h}_v)}) \quad \text{and} \quad X_w = X_{w(m)} * (U_{w(m+1)}, \dots, U_{w(\mathbf{h}_w)}) \quad (2.3.8)$$

with the observation that $X_v = X_{v(k)}$ (resp. $X_w = X_{w(m)}$) if $k = \mathbf{h}_v$ (resp. if $m = \mathbf{h}_w$). Without loss of generality, we can assume that $k \leq m$. Then, $v(k-i) = w(k-i)$, for all $i \in \mathbb{N}$ and

$$X_{w(m)} = X_{v(k)} * (U_{w(k+1)}, \dots, U_{w(m)}). \quad (2.3.9)$$

Step 2. We conclude the proof by proving that

$$X_v \wedge X_w = X_{v(k)}. \quad (2.3.10)$$

Suppose first that $k < m$ and that $k < \mathbf{h}_v$. Then $w(k+1) \in \llbracket -\infty, v \wedge w \rrbracket$ but $v(k+1) \in \llbracket v \wedge w, v \rrbracket$. Thus, $w(k+1) \neq v(k+1)$ and since the $(U_{v'})_{v' \in t}$ are independent with a diffuse law, a.s. $U_{w(k+1)} \neq$

$U_{v(k+1)}$. By (2.3.8) and (2.3.9), we get (2.3.10). If $k < m$ and $k = \mathbf{h}_v$, then $X_v = X_{v(k)}$ and we immediately get (2.3.10) by (2.3.8) and (2.3.9).

Next, suppose that $k = m$. Thus $X_{w(m)} = X_{v(k)}$ by (2.3.9); if $k = m < \mathbf{h}_v \wedge \mathbf{h}_w$, then $v(k+1) \in \llbracket v \wedge w, v \rrbracket$ and $w(k+1) = w(m+1) \in \llbracket v \wedge w, w \rrbracket$; therefore $w(k+1) \neq v(k+1)$, which implies that a.s. $U_{w(k+1)} \neq U_{v(k+1)}$ and (2.3.10) consequently. If $k = m$ and $k = \mathbf{h}_v$ (resp. $m = \mathbf{h}_w$), then $X_v = X_{v(k)} = X_{w(m)}$ (resp. $X_w = X_{w(m)} = X_{v(k)}$) and (2.3.8) and (2.3.9) also entail (2.3.10). This completes the proof of (2.3.10). \blacksquare

We shall use Proposition 2.3.5 under the following form.

Corollary 2.3.6 *Let $t' \in \mathbb{T}$ be a finite rooted ordered tree. Let $(t', \emptyset; (Y_v)_{v \in t'})$ be a $\mathbb{W}_{[0,1]}$ -valued branching random walk with law $Q_{t'}^+$ as in Definition 2.3.1 (ii). Then, a.s. for all $v, w \in t'$, we get $d_{\text{gr}}(Y_v, Y_w) = |Y_v| + |Y_w| - 2 \min_{v' \in \llbracket v, w \rrbracket} |Y_{v'}|$.*

Proof. For all $k \in \mathbb{Z}_-$ we set $\varrho_k = (a_n)_{n \leq k}$, where $a_n = 1$ for all $n \in \mathbb{Z}_-$, and we also define $t = \{\varrho_0 * u; u \in t'\} \cup \{\varrho_k; k \in \mathbb{Z}_-\}$. Then, $t \in \overline{\mathbb{T}}^\bullet$, $|t|_- = -\infty$ and $t' = \theta_{\varrho_0} t$. Let $(\mathbf{h}'_v)_{v \in t'}$ be distributed as an \mathbb{N} -valued branching random walk whose initial position \mathbf{h}'_\emptyset is 0 and whose transition kernel $q(y, dy')$ is that of the simple symmetric random walk on \mathbb{N} reflected at 0: namely, $q(0, dy') = \delta_1$ and $q(y, dy') = \frac{1}{2}(\delta_{y-1}(dy') + \delta_{y+1}(dy'))$, for all integers $y' \geq 1$ (see (2.3.1)). We next set $\mathbf{h}_{\varrho_0 * v} = \mathbf{h}'_v$, for all $v \in t'$ and $\mathbf{h}_{\varrho_k} = k$, for all $k \in \mathbb{Z}_-$. Clearly \mathbf{h} satisfies (2.3.4). We assume that $(U_v)_{v \in t}$ is independent from $(\mathbf{h}_v)_{v \in t}$ and we define $(X_v)_{v \in t}$ as in (2.3.6). For all $v \in t'$, we finally set $Y_v = \theta_{X_{\varrho_0}} X_{\varrho_0 * v}$. Then, it is easy to see that $(t', \emptyset, (Y_v)_{v \in t'})$ has law $Q_{t'}^+$. Since $\mathbf{h}_{\varrho_0 * v} \geq 0$, for all $v \in t'$, we get $d_{\text{gr}}(X_{\varrho_0 * v}, X_{\varrho_0 * w}) = d_{\text{gr}}(Y_v, Y_w)$ and $|X_{\varrho_0 * v}| = |Y_v|$, for all $v, w \in t'$, which implies the desired result by (2.3.7). \blacksquare

We next consider subranges of free branching random walks. More precisely, let $\mathbf{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$ be such that $|t|_- = -\infty$; for all $v \in t$, let $\mathbf{h}_v \in \mathbb{Z}$ satisfy (2.3.4); let $(U_v)_{v \in t}$ be a family of independent r.v. that are uniformly distributed on $[0, 1]$ and let $(X_v)_{v \in t}$ be derived from $(U_v)_{v \in t}$ as specified in (2.3.5) and (2.3.6). Let $\mathbf{a} \subset t$ and $v_0 \in \mathbf{a}$ be such that

$$\forall v \in \mathbf{a} \setminus \{v_0\}, \quad \overleftarrow{v} \in \mathbf{a}, \quad v \text{ is a descendent of } v_0 \text{ (namely } v = v_0 * (\theta_{v_0} v)) \text{ and } \mathbf{h}_v \geq \mathbf{h}_{v_0}. \quad (2.3.11)$$

Observe that for all $v \in \mathbf{a}$, X_{v_0} is a prefix of X_v and it makes sense to define “the subrange”:

$$\mathcal{R}(\mathbf{a}) = \{\theta_{X_{v_0}} X_v; v \in \mathbf{a}\}$$

that is a subtree of $\mathbb{W}_{[0,1]}^*$ whose elements have a null depth. We define the following.

$$\forall v, w \in t, \quad d(v, w) = \mathbf{h}_v + \mathbf{h}_w - 2 \min_{\llbracket v, w \rrbracket} \mathbf{h}. \quad (2.3.12)$$

Note that $\llbracket v, w \rrbracket \subset \mathbf{a}$; thus, the pseudo-metric d on $\mathbf{a} \times \mathbf{a}$ only depends on \mathbf{a} and on $(\mathbf{h}_v - \mathbf{h}_{v_0})_{v \in \mathbf{a}}$. We define the relation \sim on \mathbf{a} by setting $v \sim w$ if and only if $d(v, w) = 0$ and we introduce

$$T(\mathbf{a}) = \mathbf{a} / \sim, \quad \text{proj} : \mathbf{a} \rightarrow T(\mathbf{a}), \text{ the canonical projection,} \quad r = \text{proj}(v_0) \quad (2.3.13)$$

and we keep denoting d the (genuine) metric induced by d on $T(\mathbf{a})$. If $\mathbf{t} = (t, \varrho) \in \overline{\mathbb{T}}^\bullet$ and $(\mathbf{h}_v, U_v, X_v)_{v \in t}$ are as in Proposition 2.3.5, then (2.3.7) implies that $(T(\mathbf{a}), d)$ is a graph-tree that is isometric to the subtree $\mathcal{R}(\mathbf{a})$. More precisely, let $x \in T(\mathbf{a})$ and let $v, w \in \mathbf{a}$ such that $\text{proj}(v) = \text{proj}(w) = x$; by (2.3.7), we get a.s. $X_v = X_w$ and it makes sense to set $Z_x = \theta_{X_{v_0}} X_v$. Then (2.3.7) asserts that $Z : T(\mathbf{a}) \rightarrow \mathcal{R}(\mathbf{a})$ is an isometry:

$$\forall x, y \in T(\mathbf{a}), \quad d(x, y) = |Z_x| + |Z_y| - 2|Z_x \wedge Z_y| = d_{\text{gr}}(Z_x, Z_y). \quad (2.3.14)$$

Thus, the graph-metric of the subtree $\mathcal{R}(\mathbf{a})$ only depends on \mathbf{a} and on $(\mathbf{h}_v - \mathbf{h}_{v_0})_{v \in \mathbf{a}}$. Next, the conditional law of $\mathcal{R}(\mathbf{a})$ given $T(\mathbf{a})$ is characterized as follows. For all $x \in T(\mathbf{a}) \setminus \{r\}$, let V_x be the unique real number of $[0, 1]$ such that

$$Z_x = \overleftarrow{Z}_{x^*}(V_x). \quad (2.3.15)$$

We easily check the following.

$$\text{Conditionally given } T(\mathbf{a}), \text{ the } V_x \text{ are i.i.d. } [0, 1]\text{-uniform r.v.} \quad (2.3.16)$$

Recall from Definition 2.3.3 the \mathfrak{b} -contraction map $\Phi_{\mathfrak{b}} : \overline{\mathbb{W}}_{[0,1]} \rightarrow \overline{\mathbb{W}}_{\mathfrak{b}}$. Then, first note that $|Z_x \wedge Z_y| \leq |\Phi_{\mathfrak{b}}(Z_x) \wedge \Phi_{\mathfrak{b}}(Z_y)|$, where common ancestors are taken respectively in $\overline{\mathbb{W}}_{[0,1]}$ and in $\overline{\mathbb{W}}_{\mathfrak{b}}$. Moreover, (2.3.16) implies that $\mathbf{P}(|\Phi_{\mathfrak{b}}(Z_x) \wedge \Phi_{\mathfrak{b}}(Z_y)| - |Z_x \wedge Z_y| \geq k) \leq \mathfrak{b}^{-k}$, for all $k \in \mathbb{N}$. This inequality combined with the argument of the proof of Corollary 2.3.6 implies the following lemma that will be used in Theorem 2.1.2.

Lemma 2.3.7 *Let $t \in \mathbb{T}$ be a finite rooted ordered tree. Let $(t, \emptyset; (Y_v)_{v \in t})$ be a $\overline{\mathbb{W}}_{[0,1]}$ -valued branching random walk with law Q_t^+ as in Definition 2.3.1 (ii). Recall the \mathfrak{b} -contraction map $\Phi_{\mathfrak{b}} : \overline{\mathbb{W}}_{[0,1]} \rightarrow \overline{\mathbb{W}}_{\mathfrak{b}}$ from Definition 2.3.3 and recall that $(t, \emptyset, (\Phi_{\mathfrak{b}}(Y_v))_{v \in t})$ has law $Q_t^{+\mathfrak{b}}$ as in Definition 2.3.1 (i). We denote the graph distance on $\overline{\mathbb{W}}_{[0,1]}$ and $\overline{\mathbb{W}}_{\mathfrak{b}}$ in the same way by d_{gr} . Then, for all $v, w \in t$, there exists an \mathbb{N} -valued r.v. $G_{v,w}$ such that*

$$2G_{v,w} = d_{\text{gr}}(X_v, X_w) - d_{\text{gr}}(\Phi_{\mathfrak{b}}(X_v), \Phi_{\mathfrak{b}}(X_w)) \quad \text{and} \quad \mathbf{P}(G_{v,w} \geq k) \leq \mathfrak{b}^{-k}, \quad k \in \mathbb{N}. \quad (2.3.17)$$

2.3.3 A coupling between $\overline{\mathbb{W}}_{\mathfrak{b}}$ - and $\overline{\mathbb{W}}_{\mathfrak{b}}^*$ -valued branching random walks.

This section is devoted to the proof of the following proposition.

Proposition 2.3.8 *Let $t \in \mathbb{T}$ be a finite rooted ordered tree as in Definition 2.2.1 (namely, $|t|_- = 0$). We set $\mathfrak{t} = (t, \emptyset)$. For all $n \in \{0, \dots, \#t - 1\}$, we set $t_n = \{v \in t : v \leq_t v_n\}$ where v_n is the n -th smallest vertex of t with respect to the lexicographical order \leq_t . Then, there exists two branching random walks $\Theta_{\mathfrak{b}} = (t, \emptyset; (Y_v)_{v \in t})$ and $\Theta_{\mathfrak{b}}^+ = (t, \emptyset; (Y_v^+)_{v \in t})$ that satisfy the following.*

- (a) $\Theta_{\mathfrak{b}}^+$ is a $\overline{\mathbb{W}}_{\mathfrak{b}}$ -valued branching random walk with law $Q_{\mathfrak{t}}^{+\mathfrak{b}}$ as in Definition 2.3.1 (i).
- (b) $\mathfrak{o}_{\mathfrak{b}} := Y_{\emptyset}$ is a \mathbb{Z}_- -indexed sequence of independent r.v. that are uniformly distributed on $\{1, \dots, \mathfrak{b}\}$ and conditionally given $\mathfrak{o}_{\mathfrak{b}}$, $\Theta_{\mathfrak{b}}$ is a $\overline{\mathbb{W}}_{\mathfrak{b}}$ -valued branching random walk with law $Q_{\mathfrak{o}_{\mathfrak{b}}, \mathfrak{t}}^{\mathfrak{b}}$ as in Definition 2.3.1 (iii).
- (c) For all $c \in \mathbb{N}^*$, there exists an event B_c such that $\mathbf{P}(B_c) \leq 2\mathfrak{b}^{-c}(\#t)^3$ and such that on $\Omega \setminus B_c$,

$$\forall n \in \{0, \dots, \#t - 1\}, \quad |\#\{Y_v; v \in t_n\} - \#\{Y_v^+; v \in t_n\}| \leq \#\{v \in t : |Y_v^+| \leq c + 1\}. \quad (2.3.18)$$

Overlap of independent trees. We first prove a result concerning the overlap of independent trees that are randomly embedded in the \mathfrak{b} -ary tree $\overline{\mathbb{W}}_{\mathfrak{b}}$. More precisely, let $(T_u, r(u))$, $u \in S$ be a finite family of rooted graph-trees (not necessarily ordered) equipped with their graph-distance d_{gr} . To simplify notation, we set $|x| = d_{\text{gr}}(r(u), x)$, for all $x \in T_u$. Let $x, y \in T_u$; recall that $\llbracket x, y \rrbracket$ stands for the shortest path joining x to y and that $x \wedge y$ is the most recent common ancestor of x and y in T_u rooted at $r(u)$. Namely, $\llbracket r(u), x \rrbracket \cap \llbracket r(u), y \rrbracket = \llbracket r(u), x \wedge y \rrbracket$. Let $x \in T_u \setminus \{r(u)\}$; for all $i \in \{1, \dots, |x|\}$ we denote by x_i the unique ancestor of x at height i . Let V_x^u , $x \in T_u$, $u \in S$ be independent uniform r.v. on $\{1, \dots, \mathfrak{b}\}$. Then, we define the random word:

$$Z_x^u = (V_{x_1}^u, V_{x_2}^u, \dots, V_{x_{|x|-1}}^u, V_x^u)$$

We also set $Z_{r(u)}^u = \emptyset$ and we introduce the following random subsets of the b -ary tree \mathbb{W}_b :

$$\forall u \in S, \quad \mathbf{R}_u := \{Z_x^u; x \in \mathbf{T}_u\}.$$

For all $c \in \mathbb{N}^*$, we also set $M_c = \sum_{u \in S} \#\{x \in \mathbf{T}_u : |x| \leq c\}$.

Lemma 2.3.9 *We keep the notation from above. Let $c \in \mathbb{N}^*$. Let $w_u \in \mathbb{W}_b$, $u \in S$. Then, there exists an event A_c of probability $\mathbf{P}(A_c) \leq b^{-c} M_c^2$ such that on $\Omega \setminus A_c$,*

$$0 \leq \sum_{u \in S} \#\mathbf{R}_u - \#\left(\bigcup_{u \in S} w_u * (\mathbf{R}_u)\right) \leq M_c. \quad (2.3.19)$$

Proof. Let u and u' be distinct elements of S ; let $x \in \mathbf{T}_u$ and $y \in \mathbf{T}_{u'}$ be such that $|x| = |y| = c$. We introduce the event $A(x, y) = \{(w_u * Z_x^u) \wedge (w_{u'} * Z_y^{u'}) \in \{w_u * Z_x^u, w_{u'} * Z_y^{u'}\}\}$. Note that on $A(x, y)$, $w_u \wedge w_{u'} \in \{w_u, w_{u'}\}$; so without loss of generality, we can suppose that $|w_u| \leq |w_{u'}|$ and we easily check that $\mathbf{P}(A(x, y) | Z_y^{u'}) = b^{-c}$ and thus $\mathbf{P}(A(x, y)) = b^{-c}$. We then set for all $u \in S$,

$$\mathbf{T}_u(c) = \{x \in \mathbf{T}_u : |x| = c\} \quad \text{and} \quad A_c = \bigcup \{A(x, y); x \in \mathbf{T}_u(c), y \in \mathbf{T}_{u'}(c), u, u' \in S, \text{ distinct}\}.$$

Thus, $\mathbf{P}(A_c) \leq b^{-c} \sum_{u, u' \in S, u \neq u'} \#\mathbf{T}_u(c) \#\mathbf{T}_{u'}(c) \leq \frac{1}{2} b^{-c} (\sum_{u \in S} \#\mathbf{T}_u(c))^2 \leq b^{-c} M_c^2$.

The first inequality in (2.3.19) is true everywhere on Ω . For the second inequality in (2.3.19), we argue deterministically on $\Omega \setminus A_c$. Let $u \in S$. We first set $R_u(c) = \{w_u * Z_x^u; x \in \mathbf{T}_u(c)\}$. Let $u' \in S$ be distinct from u ; suppose that $W \in R_u(c)$ and that $W' \in R_{u'}(c)$; by definition of A_c , we have $W \wedge W' \notin \{W, W'\}$ and $\theta_W(w_u * \mathbf{R}_u) \cap \theta_{W'}(w_{u'} * \mathbf{R}_{u'}) = \emptyset$. Moreover, if $W, W' \in R_u(c)$ are distinct, since $|W| = |W'| = c + |w_u|$, we also get $W \wedge W' \notin \{W, W'\}$ and thus $\theta_W(w_u * \mathbf{R}_u) \cap \theta_{W'}(w_u * \mathbf{R}_u) = \emptyset$. Consequently, on $\Omega \setminus A_c$, the subsets $\theta_W(w_u * \mathbf{R}_u)$, $W \in R_u(c)$, $u \in S$, are pairwise disjoint.

Now observe that $(w_u * \mathbf{R}_u) \setminus (\bigcup_{W \in R_u(c)} \theta_W(w_u * \mathbf{R}_u)) = \{w_u * Z_x^u; |x| < c\}$ and note that $\#(\{w_u * Z_x^u; |x| < c\}) \leq \#(\{x \in \mathbf{T}_u : |x| < c\})$. Thus, on $\Omega \setminus A_c$, we get

$$\begin{aligned} \left(\sum_{u \in S} \#\mathbf{R}_u\right) - M_c &= \sum_{u \in S} (\#\mathbf{R}_u - \#\{x \in \mathbf{T}_u : |x| < c\}) \\ &\leq \sum_{u \in S} (\#(w_u * \mathbf{R}_u) - \#(\{w_u * Z_x^u; |x| < c\})) \\ &\leq \sum_{u \in S} \#\left(\bigcup_{W \in R_u(c)} \theta_W(w_u * \mathbf{R}_u)\right) \\ &= \#\left(\bigcup_{u \in S} \bigcup_{W \in R_u(c)} \theta_W(w_u * \mathbf{R}_u)\right) \leq \#\left(\bigcup_{u \in S} w_u * (\mathbf{R}_u)\right), \end{aligned}$$

that implies the desired result. ■

Path coupling. We first state the following elementary coupling.

Lemma 2.3.10 *Let $K : \mathbb{N} \rightarrow \mathbb{N}$ be such that $K(0) = 0$ and $K(2p + 1) = K(2p + 2) = 2p + 2$, for all $p \in \mathbb{N}$. Let $(\mathbf{h}_n)_{n \in \mathbb{N}}$ be a \mathbb{Z} -valued simple symmetric random walk such that a.s. $\mathbf{h}_0 = 0$. For all $n \in \mathbb{N}$, we set $I_n = \inf_{0 \leq k \leq n} \mathbf{h}_k$ and $\mathbf{h}_n^+ = \mathbf{h}_n + K(-I_n)$. Then, \mathbf{h}^+ is an \mathbb{N} -valued simple symmetric random walk reflected at 0.*

Proof. For all $n \geq 1$, set $\xi_n = \mathbf{h}_n - \mathbf{h}_{n-1}$; the r.v. are i.i.d. and uniform on $\{1, -1\}$. Observe that $\mathbf{h}_n^+ = \mathbf{h}_n - I_n + \mathbf{1}_{\{I_n \text{ odd}\}}$. It is easy to check that $\mathbf{h}_{n+1}^+ - \mathbf{h}_n^+ = \xi_{n+1} \mathbf{1}_{\{\mathbf{h}_n^+ \geq 1\}} + \mathbf{1}_{\{\mathbf{h}_n^+ = 0\}}$, which implies the desired result. ■

The next lemma state the branching random walk version of this coupling.

Lemma 2.3.11 *Let $t \in \mathbb{T}$ be a rooted ordered tree as in Definition 2.2.1 (namely, $|t|_- = 0$). Let $(t, \emptyset; (\mathbf{h}_u)_{u \in t})$ be a \mathbb{Z} -valued branching random walk whose spatial motion is that of a simple symmetric random walk on \mathbb{Z} and whose initial position is $\mathbf{h}_\emptyset = 0$. Recall the function K from Lemma 2.3.10 and set*

$$\forall u \in t, \quad I_u = \min_{u' \in \llbracket \emptyset, u \rrbracket} \mathbf{h}_{u'} \quad \text{and} \quad \mathbf{h}_u^+ = \mathbf{h}_u + K(-I_u). \quad (2.3.20)$$

Then, $(t, \emptyset; (\mathbf{h}_u^+)_{u \in t})$ is a branching random walk whose spatial motion is that of an \mathbb{N} -valued simple symmetric random walk reflected at 0 and whose initial position is $\mathbf{h}_\emptyset^+ = 0$.

Proof. We denote by $q_+(x, dy)$ the transition kernel of the \mathbb{N} -valued simple symmetric random walk reflected at 0: namely, $q_+(0, dy) = \delta_1(dy)$ and $q_+(x, dy) = \frac{1}{2}(\delta_{x-1}(dy) + \delta_{x+1}(dy))$, if $x \geq 1$. For all $n \in \mathbb{N}$, we set $t_{|n} = \{u \in t : |u| \leq n\}$ and we assume the following property

(P_n) : *the spatial motion of the branching random walk $\Theta_n = (t_{|n}, \emptyset; (\mathbf{h}_u^+)_{u \in t_{|n}})$ is that of a \mathbb{N} -valued simple symmetric random walk reflected at 0 with initial position $\mathbf{h}_\emptyset^+ = 0$.*

We next set $S = \{u \in t : |u| = n + 1\}$ and $\xi_u = \mathbf{h}_u - \mathbf{h}_{\leftarrow u}$ for all $u \in S$; the r.v. $(\xi_u)_{u \in S}$ are independent from Θ_n and they are also i.i.d. and uniform on $\{1, -1\}$; by definition of \mathbf{h}^+ , we also get $\mathbf{h}_u^+ = \mathbf{h}_u^+ + \xi_u \mathbf{1}_{\{\mathbf{h}_u^+ \geq 1\}} + \mathbf{1}_{\{\mathbf{h}_u^+ = 0\}}$. This entails that the r.v. \mathbf{h}_u^+ , $u \in S$ are conditionally independent given Θ_n and that the conditional law of \mathbf{h}_u^+ is $q_+(\mathbf{h}_u^+, dy)$. This shows that (P_n) implies (P_{n+1}) , which recursively proves the lemma since (P_0) holds true trivially. \blacksquare

The coupling of Θ_b and Θ_b^+ . Let $t \in \mathbb{T}$ be a finite rooted ordered tree as in Definition 2.2.1 (namely, $|t|_- = 0$). We shall use the notation $\mathbf{t} = (t, \emptyset)$ for the pointed tree. We denote by $\varrho = (\dots, 1, \dots, 1)$ the infinite sequence of 1 indexed by \mathbb{Z}_- . For all $l \in \mathbb{Z}_-$, ϱ_l is the infinite sequence of 1 indexed by the integers $\leq l$. We then set

$$t^* = \{\varrho_l; l \in \mathbb{Z}_-\} \cup \{\varrho^* u; u \in t\}.$$

Then we define $\mathbf{t}^* := (t^*, \varrho) \in \overline{\mathbb{T}}^\bullet$ and we observe that $|t^*|_- = -\infty$.

Let $(t^*, \varrho; (\mathbf{h}_v^*)_{v \in t^*})$ be a \mathbb{Z} -valued branching random walk whose transition kernel is that of a simple symmetric random walk on \mathbb{Z} and whose “initial” position is $\mathbf{h}_\emptyset^* = 0$. Let $(U_v^*)_{v \in t^*}$ be independent r.v. that are uniformly distributed on $[0, 1]$ and that are independent from $(\mathbf{h}_v^*)_{v \in t^*}$. We define $(X_v^*)_{v \in t^*}$ as in (2.3.5) and (2.3.6): namely, $X_v^* = (U_{v(k)}^*)_{k \leq \mathbf{h}_v^*}$, where $v(k) \in \llbracket -\infty, v \rrbracket$, $\mathbf{h}_{v(k)}^* = k$ and where $\{\overleftarrow{v}(k); k \leq \mathbf{h}_v^*\}$ is the set of vertices of $\llbracket -\infty, v \rrbracket$ where \mathbf{h}^* reaches a new strict infimum. We set $\mathbf{o} = X_\emptyset^*$ that is a \mathbb{Z}_- -indexed sequence of mutually independent r.v. that are uniformly distributed on $[0, 1]$. We easily see that conditionally given \mathbf{o} , $\Theta^* = (t^*, \varrho; (X_v^*)_{v \in t^*})$ has law $Q_{\mathbf{o}, t^*}$ as in Definition 2.3.1 (iv). Then we set the following

$$\forall u \in t, \quad U_u = U_{\varrho^* u}^*, \quad X_u = X_{\varrho^* u}^*, \quad \mathbf{h}_u = \mathbf{h}_{\varrho^* u}^*, \quad Y_u = \Phi_b(X_u),$$

and we also set $\Theta = (t, \emptyset; (X_u)_{u \in t})$ and $\Theta_b = (t, \emptyset; (Y_u)_{u \in t})$, where we recall the definition of the b -contraction map Φ_b from Remark 2.3.3. Clearly, conditionally given \mathbf{o} , Θ has law $Q_{\mathbf{o}, t}$ as in Definition 2.3.1 (iv) and by Remark 2.3.3, the conditional law of Θ_b is $Q_{\Phi_b(\mathbf{o}), t}^b$ as in Definition 2.3.1 (iii). Namely, Θ_b is distributed as in Proposition 2.3.8 (b).

Let $(U'_u)_{u \in t}$ be i.i.d. r.v. that are uniformly distributed on $[0, 1]$. We suppose that $(U'_u)_{u \in t}$ is independent from $(U_v^*)_{v \in t^*}$ (and thus from $(U_u)_{u \in t}$) and from $(\mathbf{h}_v^*)_{v \in t^*}$. We next construct Θ_b^+ thanks to $(U'_u, U_u, \mathbf{h}_u)_{u \in t}$ as follows: For all $u \in t$, we define I_u and \mathbf{h}_u^+ by (2.3.20) in Lemma 2.3.11 and we introduce the following notation.

$$\forall p \in \mathbb{N}, \quad S_p = \{u \in t : -p = I_u < I_{\leftarrow u}\}, \quad S = \bigcup_{p \in \mathbb{N}} S_p, \quad U_u^+ = \begin{cases} U'_u & \text{if } u \in \bigcup_{p \in \mathbb{N}} S_{2p+1}, \\ U_u & \text{otherwise.} \end{cases} \quad (2.3.21)$$

The (disjoint) union S is clearly a finite set, since t is a finite tree. Then, $(U_u^+)_{u \in t}$ are i.i.d. $[0, 1]$ -uniform r.v. that are independent from $(\mathbf{h}_u^+)_{u \in t}$. Recall from Lemma 2.3.11 that $(t, \emptyset; (\mathbf{h}_u^+)_{u \in t})$ is a branching random walk whose spatial motion is that of an \mathbb{N} -valued simple symmetric random walk reflected at 0 and whose initial position is $\mathbf{h}_\emptyset^+ = 0$. In particular, $\mathbf{h}_u^+ \geq 0$ for all $u \in t$. We then define $(X_u^+)_{u \in t}$ and $(Y_u^+)_{u \in t}$ as follows.

- If $\mathbf{h}_u^+ = 0$, then we set $X_u^+ = Y_u^+ = \emptyset$.
- If $\mathbf{h}_u^+ \geq 1$, then for all $i \in \{1, \dots, \mathbf{h}_u^+\}$, we denote by $u(i)$ is the unique $v \in \llbracket \emptyset, u \rrbracket$ such that $\mathbf{h}_v^+ < i = \mathbf{h}_v^+ = \min_{\llbracket v, u \rrbracket} \mathbf{h}^+$ and we set

$$X_u^+ = (U_{u(1)}^+, \dots, U_{u(\mathbf{h}_u^+)}^+) \quad \text{and} \quad Y_u^+ = \Phi_{\mathbf{b}}(X_u^+). \quad (2.3.22)$$

We set $\Theta^+ = (t, \emptyset; (X_u^+)_{u \in t})$ and $\Theta_{\mathbf{b}}^+ = (t, \emptyset; (Y_u^+)_{u \in t})$. Then, observe that $\mathbf{h}_u^+ = |X_u^+|$ and that $X_u^+ = X_{\frac{u}{\mathbf{h}_u^+}}^+ * (U_u^+)$ if $\mathbf{h}_u^+ = \mathbf{h}_{\frac{u}{\mathbf{h}_u^+}}^+ + 1$ and note that X_u^+ is the parent of $X_{\frac{u}{\mathbf{h}_u^+}}^+$ if $\mathbf{h}_u^+ = \mathbf{h}_{\frac{u}{\mathbf{h}_u^+}}^+ - 1$. Thus, it proves that Θ^+ as law Q_t^+ as in Definition 2.3.1 (ii) and by Remark 2.3.3, $\Theta_{\mathbf{b}}^+$ is distributed as in Proposition 2.3.8 (a) (namely its law is $Q_t^{+\mathbf{b}}$ as in Definition 2.3.1 (i)).

Proof of Proposition 2.3.8. We keep the previous notations. We fix $n \in \{0, \dots, \#t - 1\}$. Recall that $t_n = \{v \in t : v \leq_t v_n\}$ where v_n is the n -th smallest vertex of t in the lexicographical order \leq_t on t . Note that $t_n \in \mathbb{T}$, namely it is a rooted ordered tree with null depth. Recall from (2.3.21) the definition of S . For all $u \in S$, we set

$$\mathbf{a}_u = \{v \in t_n : v := u * \theta_u v \text{ and } \min_{w \in \llbracket u, v \rrbracket} \mathbf{h}_w = \mathbf{h}_u\} \subseteq t \text{ and } \mathcal{R}(\mathbf{a}_u) = \{\theta_{X_u} X_v; v \in \mathbf{a}_u\} \subseteq \overline{\mathbb{W}}_{[0,1]}.$$

Note that \mathbf{a}_u depends on n and that if $u \notin t_n$, then \mathbf{a}_u and $\mathcal{R}(\mathbf{a}_u)$ are empty. If $u \in t_n$, then the previous definitions make sense because \mathbf{a}_u satisfies (2.3.11) (with $u = v_0$) and X_u is necessarily a prefix of X_v since v is a descendent of u and $I_v = I_u$. Observe that $\mathcal{R}(\mathbf{a}_u)$ is a subtree of $\overline{\mathbb{W}}_{[0,1]}$ as in Definition 2.2.3.

Recall from (2.3.12) the notation $d(v, w) = \mathbf{h}_v + \mathbf{h}_w - 2 \min_{\llbracket v, w \rrbracket} \mathbf{h}$, for all $v, w \in t$, that is a pseudo-metric on t . We denote by $v \sim w$ the equivalence relation $d(v, w) = 0$ and as in (2.3.13), we set for all $u \in S$,

$$T(\mathbf{a}_u) = \mathbf{a}_u / \sim, \quad \text{proj} : \mathbf{a}_u \rightarrow T(\mathbf{a}_u), \text{ the canonical projection,} \quad r(u) = \text{proj}(u).$$

Note that $T(\mathbf{a}_u)$ only depends on $(\mathbf{h}_v - I_u)_{v \in \mathbf{a}_u}$ and that $\mathcal{R}(\mathbf{a}_u)$ only depends on $(\mathbf{h}_v - I_u; U_v)_{v \in \mathbf{a}_u}$. Let $x \in T(\mathbf{a}_u)$; by (2.3.7) in Lemma 2.3.5, for all $v, w \in \mathbf{a}_u$ such that $\text{proj}(v) = \text{proj}(w) = x$, we get $X_v = X_w$ and it makes sense to set $Z_x^u = \theta_{X_u} X_v$; then, $Z^u : T(\mathbf{a}_u) \rightarrow \mathcal{R}(\mathbf{a}_u)$ is an isometry given in (2.3.14). As in (2.3.15), for all $x \in T(\mathbf{a}_u) \setminus \{r(u)\}$ we denote by V_x^u the unique real number of $[0, 1]$ such that $Z_x^u = \overleftarrow{Z}_x^u * (V_x^u)$.

Then by (2.3.16), for all $u \in S$, conditionally given \mathbf{h} , the V_x^u , $x \in T(\mathbf{a}_u)$ are i.i.d. $[0, 1]$ -uniform r.v. Since the subsets $(\mathbf{a}_u)_{u \in S}$ are pairwise disjoint and since $\mathcal{R}(\mathbf{a}_u)$ only depends on $(\mathbf{h}_v - I_u; U_v)_{v \in \mathbf{a}_u}$, the $(\mathcal{R}(\mathbf{a}_u))_{u \in S}$ are conditionally independent given \mathbf{h} . Thus, conditionally given \mathbf{h} , the V_x^u , $x \in T(\mathbf{a}_u) \setminus \{r(u)\}$, $u \in S$, are i.i.d. $[0, 1]$ -uniform r.v.

Next observe that

$$\{X_v; v \in t_n\} = \bigcup_{u \in S} X_u * (\mathcal{R}(\mathbf{a}_u)) \quad (2.3.23)$$

Therefore, using (2.3.21) and noting that $(X_u)_{u \in S}$ only depends on \mathbf{h} and on $(U_{\varrho_l}^*)_{l \in \mathbb{Z}_-}$ that is independent from $(U_u)_{u \in t}$, we conclude that, conditionally given \mathbf{h} and $(X_u)_{u \in S}$, the V_x^u ,

$x \in T(\mathbf{a}_u) \setminus \{r(u)\}$, $u \in S$, are i.i.d. $[0, 1]$ -uniform r.v. From the coupling defined in (2.3.21) and (2.3.22) we also derive easily the following.

$$\{X_v^+; v \in t_n\} = \bigcup_{u \in S} X_u' * (\mathcal{R}(\mathbf{a}_u)), \text{ where for all } u \text{ in } S, X_u' := \begin{cases} U_u' & \text{if } u \in \bigcup_{p \in \mathbb{N}} S_{2p+1}, \\ \emptyset & \text{otherwise.} \end{cases} \quad (2.3.24)$$

Since the $(U_u')_{u \in t}$ are independent from \mathbf{h} and from $(U_u)_{u \in t}$, conditionally given \mathbf{h} and $(U_u')_{u \in S}$, the r.v. V_x^u , $x \in T(\mathbf{a}_u) \setminus \{r(u)\}$, $u \in S$, are i.i.d. $[0, 1]$ -uniform. Denote by \mathcal{G} the sigma-field generated by \mathbf{h} and by $(X_u, U_u')_{u \in S}$. Therefore, we have proved the following.

$$\text{Conditionally given } \mathcal{G}, \text{ the } V_x^u, x \in T(\mathbf{a}_u) \setminus \{r(u)\}, u \in S, \text{ are i.i.d. } [0, 1]\text{-uniform r.v.} \quad (2.3.25)$$

Then we set

$$\forall u \in S, \mathbf{R}_u = \Phi_{\mathbf{b}}(\mathcal{R}(\mathbf{a}_u)) \quad \text{and} \quad \forall c \in \mathbb{N}, M_c(n) = \sum_{u \in S} \#\{x \in T(\mathbf{a}_u) : d(r(u), x) \leq c\}.$$

Note that $M_c(n) \leq \#t_n \leq \#t$. By (2.3.25), conditionally given \mathcal{G} we can apply Lemma 2.3.9 to $T_u := T(\mathbf{a}_u)$ and $w_u := \Phi_{\mathbf{b}}(X_u)$, $u \in S$, to get an event $A_c(n)$ such that $\mathbf{P}(A_c(n)) \leq \mathbf{b}^{-c}(\#t)^2$ and such that on $\Omega \setminus A_c(n)$,

$$0 \leq \sum_{u \in S} \#\mathbf{R}_u - \#\{Y_v; v \in t_n\} \stackrel{\text{by (2.3.23)}}{=} \sum_{u \in S} \#\mathbf{R}_u - \#\left(\bigcup_{u \in S} \Phi_{\mathbf{b}}(X_u) * (\mathbf{R}_u)\right) \leq M_c(n).$$

Similarly, conditionally given \mathcal{G} , we apply Lemma 2.3.9 to $T_u := T(\mathbf{a}_u)$ and to $w_u := \Phi_{\mathbf{b}}(X_u')$, $u \in S$, to get an event $A'_c(n)$ such that $\mathbf{P}(A'_c(n)) \leq \mathbf{b}^{-c}(\#t)^2$ and such that on $\Omega \setminus A'_c(n)$,

$$0 \leq \sum_{u \in S} \#\mathbf{R}_u - \#\{Y_v^+; v \in t_n\} \stackrel{\text{by (2.3.24)}}{=} \sum_{u \in S} \#\mathbf{R}_u - \#\left(\bigcup_{u \in S} \Phi_{\mathbf{b}}(X_u') * (\mathbf{R}_u)\right) \leq M_c(n).$$

Then, we set $B_c(n) = A_c(n) \cup A'_c(n)$; thus, $\mathbf{P}(B_c(n)) \leq 2\mathbf{b}^{-c}(\#t)^2$ and on $\Omega \setminus B_c(n)$, we get $|\#\{Y_v; v \in t_n\} - \#\{Y_v^+; v \in t_n\}| \leq M_c(n)$. Now observe that if $x \in T(\mathbf{a}_u)$, then $\mathbf{h}_x^+ = |X_x^+| = |X_x'| + |Z_x^u| \leq 1 + d(r(u), x)$. This implies $M_c(n) \leq \#\{v \in t; |Y_v^+| \leq c + 1\}$. Thus, on $\Omega \setminus B_c(n)$, we get

$$|\#\{Y_v; v \in t_n\} - \#\{Y_v^+; v \in t_n\}| \leq \#\{v \in t; |Y_v^+| \leq c + 1\}.$$

We completes the proof of Proposition 2.3.8 by taking $B_c = \bigcup_{0 \leq n < \#t} B_c(n)$. \blacksquare

2.3.4 Estimates.

The goal of this section is to establish Proposition 2.3.16 below. To that end, we first state preliminary estimates. Recall that $\mathbb{W}_{\mathbf{b}}^* = \{x \in \overline{\mathbb{W}}_{\mathbf{b}} : |x|_- = -\infty\}$. We denote by $(Y_n)_{n \in \mathbb{N}}$ the canonical process on the space $(\mathbb{W}_{\mathbf{b}}^*)^{\mathbb{N}}$ equipped with product topology (that is Polish) and with the corresponding Borel sigma-field. In this section, let us denote by P_{\circ} (instead of $Q_{\circ, \mathbb{N}}^{\mathbf{b}}$) the law of a Markov chain on $\mathbb{W}_{\mathbf{b}}^*$ with transition kernel $p_{\mathbf{b}}$ as defined in (2.3.3) and whose initial position is \circ . The following result only contains some standard estimates.

Lemma 2.3.12 *Let $\circ \in \mathbb{W}_{\mathbf{b}}^*$ be such that $|\circ| = 0$. For all $k, p \in \mathbb{N}$, we set*

$$\circ(p) = \circ|_{-\infty, -p], \quad \text{Sp} = \{\circ(p); p \in \mathbb{N}\}, \quad \mathbf{r}_0 = 0 \quad \text{and} \quad \mathbf{r}_{k+1} = \inf\{n > \mathbf{r}_k : Y_n \in \text{Sp}\}. \quad (2.3.26)$$

The following holds true.

- (i) P_o -almost surely for all $k \in \mathbb{N}$, $\mathbf{r}_k < \infty$ and there exists $Z_k \in \mathbb{N}$ such that $o(Z_k) = Y_{\mathbf{r}_k}$. Moreover, $(Z_k)_{k \in \mathbb{N}}$ is an \mathbb{N} -valued birth-and-death Markov chain whose transition probabilities $(\rho(p, q))_{p, q \in \mathbb{N}}$ are given as follows: for all $p \in \mathbb{N}^*$,

$$\rho(p, p+1) = \frac{1}{2}, \quad \rho(p, p-1) = \frac{1}{2b}, \quad \rho(p, p) = \frac{b-1}{2b}, \quad (2.3.27)$$

and $\rho(0, 0) = \rho(0, 1) = 1/2$. Then, Z is transient which implies that $|o \wedge Y_n| \rightarrow -\infty$ almost surely and that $(Y_n)_{n \in \mathbb{N}}$ is transient under P_o .

- (ii) For all $x, y \in \mathbb{W}_b^*$, we set $G_b(x, y) = \sum_{n \in \mathbb{N}} P_x(Y_n = y)$. Then, for all $p \in \mathbb{N}$,

$$G_b(x, y) = \frac{2b}{b-1} b^{|x \wedge y| - |y|}. \quad (2.3.28)$$

- (iii) For all $y \in \mathbb{W}_b^*$, we set $H_y = \inf\{n \in \mathbb{N} : Y_n = y\}$, with the convention that $\inf \emptyset = \infty$. For all $s \in [0, \infty)$, we set $g(s) = -\log(1 - \sqrt{1 - e^{-2s}})$. Then $E_x[e^{-s(1+H_x)}] = \exp(-g(s))$ for all $x \in \mathbb{W}_b^*$. Moreover, there exists a constant $C \in (1, \infty)$ such that

$$P_o\text{-a.s. for all } k \in \mathbb{N} \text{ and for all } s \in [0, 1], \quad E_o[e^{-s(\mathbf{r}_{k+1} - \mathbf{r}_k)} | Z] \geq e^{-C\sqrt{s}}. \quad (2.3.29)$$

Proof. Since under P_o , $(|Y_n|)_{n \in \mathbb{N}}$ is distributed as a simple symmetric random walk on \mathbb{Z} , we easily see that P_o -a.s. for all $k \in \mathbb{N}$, $\mathbf{r}_k < \infty$. The strong Markov property at the stopping times \mathbf{r}_k implies that $(Z_k)_{k \in \mathbb{N}}$ is an \mathbb{N} -valued birth-and-death Markov chain whose transition probabilities $\rho(p, q)$ are given by (2.3.27) which easily implies that $Z_k \rightarrow \infty$ which entails (i).

Let us prove (ii). For all $x \in \mathbb{W}_b^*$, we introduce the stopping times $H_x = \inf\{n \in \mathbb{N} : Y_n = x\}$ and $H_x^o = \inf\{n \in \mathbb{N}^* : Y_n = x\}$. First observe that for all $x, y \in \mathbb{W}_b^*$, $P_x(H_y < \infty) = P_{x \wedge y}(H_y < \infty)$. By adapting the argument of (i), the height of the process Y restricted to $]-\infty, y]$ is a birth-and-death process with transition $\rho(\cdot, \cdot)$ and the Gambler's ruin estimate implies $P_{x \wedge y}(H_y < \infty) = b^{|x \wedge y| - |y|}$. Similarly, we get $P_y(H_y^o = \infty) = (b-1)/2b$. The Markov property at resp. H_y and H_y^o implies resp. that $G_b(x, y) = P_x(H_y < \infty)G_b(y, y)$ and that $G_b(y, y) = 1 + P_y(H_y^o < \infty)G_b(y, y)$, which entails (2.3.28).

Let us prove (iii). Note that under P_x , $(|Y_n|)_{n \in \mathbb{N}}$ is distributed as a simple symmetric random walk on \mathbb{Z} with initial position $|x|$. Then $H_x^<$ is the first time $(|Y_n|)_{n \in \mathbb{N}}$ reaches $|x|-1$; therefore, the law of $H_x^<$ does not depend on x and we get $E_x[\exp(-s(H_x^< + 1))] = \exp(-g(s))$ by well-known arguments. Next, let us work conditionally given Z ; we set $x = Y_{\mathbf{r}_{k+1}}$; on the event $\{Z_k = Z_{k+1}\}$, $\overleftarrow{x} = o(Z_k)$ and $\mathbf{r}_{k+1} - \mathbf{r}_k - 1$ is the time at which the shifted random walk $(Y_{\mathbf{r}_{k+1}+n})_{n \in \mathbb{N}}$ returns for the first time to \overleftarrow{x} ; thus, by Markov at \mathbf{r}_k and by the previous argument, P_o -a.s. on the event $\{Z_k = Z_{k+1}\}$, $E_o[\exp(-s(\mathbf{r}_{k+1} - \mathbf{r}_k)) | Z] = \exp(-g(s))$. If $Z_k \neq Z_{k+1}$, then $\mathbf{r}_{k+1} - \mathbf{r}_k = 1$. Consequently, $E_o[\exp(-s(\mathbf{r}_{k+1} - \mathbf{r}_k)) | Z] = \exp(-g(s)\mathbf{1}_{\{Z_k = Z_{k+1}\}} - s\mathbf{1}_{\{Z_k \neq Z_{k+1}\}})$, which implies (2.3.29) because there exists $C \in (1, \infty)$ such that $g(s) \leq C\sqrt{s}$ for all $s \in [0, 1]$ and since $s \leq \sqrt{s}$ for all $s \in [0, 1]$. \blacksquare

Proposition 2.3.13 Let $o \in \mathbb{W}_b^*$ such that $|o| = 0$. Recall that P_o stands for the canonical law of \mathbb{W}_b^* -valued Markov chains with transition kernel p_b defined by (2.3.3) and with initial position o . Let $f : [0, \infty) \rightarrow [0, 1]$ be such that $\sum_{p \in \mathbb{N}} \sqrt{f(p)} < \infty$.

$$P_o\text{-a.s.} \quad \sum_{n \in \mathbb{N}} f(-|o \wedge Y_n|) < \infty.$$

Proof. Recall from (2.3.26) in Lemma 2.3.12 notations $o(p)$, $\mathbf{S}p$ and \mathbf{r}_k and recall from Lemma 2.3.12 (i) the definition of the \mathbb{N} -valued birth-and-death process $(Z_k)_{k \in \mathbb{N}}$. Let $\ell \in \mathbb{N}^*$. First observe that

$$W_\ell := \sum_{0 \leq n < \mathbf{r}_\ell} f(-|o \wedge Y_n|) = \sum_{0 \leq k < \ell} \sum_{\mathbf{r}_k \leq n < \mathbf{r}_{k+1}} f(Z_k) = \sum_{0 \leq k < \ell} (\mathbf{r}_{k+1} - \mathbf{r}_k) f(Z_k).$$

By the Markov property, the sequence of random times $\mathbf{r}_{k+1} - \mathbf{r}_k$ are conditionally independent given Z . Thus by (2.3.29) in Lemma 2.3.12 (iii), there is $C \in (1, \infty)$ such that for all $s \in [0, 1]$,

$$\begin{aligned} E_o[e^{-sW_\ell} | Z] &= \prod_{0 \leq k < \ell} \mathbf{E}[e^{-s(\mathbf{r}_{k+1} - \mathbf{r}_k)f(Z_k)} | Z] \geq \exp\left(-C \sum_{0 \leq k < \ell} \sqrt{sf(Z_k)}\right) \\ &\geq \exp\left(-C\sqrt{s} \sum_{p \in \mathbb{N}} N_p \sqrt{f(p)}\right), \text{ where } N_p = \#\{k \in \mathbb{N} : Z_k = p\}. \end{aligned} \quad (2.3.30)$$

Next observe that $E_o[N_p] = G_b(o, o(p)) = 2b/(b-1)$ by (2.3.28) in Lemma 2.3.12 (ii). Set $W = \sum_{n \in \mathbb{N}} f(-|o \wedge Y_n|)$; by letting $\ell \rightarrow \infty$ in (2.3.30) and by Jensen's inequality, we get

$$E_o[e^{-sW}] \geq \exp\left(-C\sqrt{s} \sum_{p \in \mathbb{N}} \mathbf{E}[N_p] \sqrt{f(p)}\right) = \exp\left(-C'\sqrt{s} \sum_{p \in \mathbb{N}} \sqrt{f(p)}\right),$$

where $C' = 2bC/(b-1)$. Since $\sum_{p \in \mathbb{N}} \sqrt{f(p)} < \infty$, we get $\lim_{s \rightarrow 0^+} \mathbf{E}[e^{-sW}] = 1$ which entails that P_o -a.s. $W < \infty$. This completes the proof of the proposition. \blacksquare

Lemma 2.3.14 *Let μ be a probability measure on \mathbb{N} such that $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. Let $\tau = (\tau, \emptyset)$ be a random pointed tree such that a.s. $|\tau|_- = 0$, $k_\emptyset(\tau) = 1$ and such that $\theta_{(1)}\tau$ is a $GW(\mu)$ -tree. Let $x \in \mathbb{W}_b^*$ and let $\Theta = (\tau, \emptyset; (X_v)_{v \in \tau})$ be a random \mathbb{W}_b^* -valued branching random walk that has law $Q_{x, \tau}^b$ conditionally given τ , as defined in Definition 2.3.1 (iii). For all $y \in \mathbb{W}_b^*$ we set*

$$\xi(x, y) = \mathbf{P}(y \notin \{X_v; v \in \tau \setminus \{\emptyset\}\}), \quad (2.3.31)$$

which turns out to be strictly positive. Then, we get $1 - \xi(x, y) \leq G_b(x, y) = \frac{2b}{b-1} b^{|x \wedge y| - |y|}$.

Proof. By a simple union bound we first get the following.

$$\mathbf{P}(y \in \{X_v; v \in \tau \setminus \{\emptyset\}\} | \tau) \leq \mathbf{E}\left[\sum_{v \in \tau \setminus \{\emptyset\}} \mathbf{1}_{\{X_v = y\}} \mid \tau\right] \leq \sum_{v \in \tau \setminus \{\emptyset\}} Q_{x, \tau}^b(X_v = y).$$

By definition of branching random walks, $Q_{x, \tau}^b(X_v = y) = P_x(Y_{|v|} = y)$, where P_x stands for the canonical law of the random walk that starts at x in \mathbb{W}_b^* and whose transition kernel is p_b as defined in (2.3.3) in Definition 2.3.1 (iii). Thus, $\mathbf{P}(y \in \{X_v; v \in \tau \setminus \{\emptyset\}\}) \leq \sum_{n \geq 1} \mathbf{E}[N_n] P_x(Y_n = y)$, where $N_n = \#\{v \in \tau : |v| = n\}$. Since the offspring distribution μ is critical, $\mathbf{E}[N_n] = 1$. Therefore, recalling Lemma (2.3.12) (ii), we have that the right member is smaller or equal to $G_b(x, y)$, which concludes the proof. \blacksquare

Lemma 2.3.15 *Let μ be a probability measure on \mathbb{N} such that $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. Let us suppose that there exists $\beta \in (0, \infty)$ such that $\sum_{k \in \mathbb{N}^*} \mu(k) k(\log k)^{1+\beta} < \infty$. For all $r \in [0, 1]$, we set $\psi(r) = \sum_{k \in \mathbb{N}} r^k \bar{\mu}(k+1)$, where we recall that for all $k \in \mathbb{N} \setminus \{0\}$, $\bar{\mu}(k)$ stands for $\sum_{\ell \geq k} \mu(\ell)$. Then, there exists $C_{\beta, \mu} \in (0, \infty)$ such that*

$$\forall r \in (0, 1/2), \quad 0 \leq 1 - \psi(1-r) \leq \frac{C_{\beta, \mu}}{(\log 1/r)^{1+\beta}}. \quad (2.3.32)$$

Proof. Let Z be an $\mathbb{N} \setminus \{0\}$ -valued random variable distributed according to $\bar{\mu}$. Observe that $C := \mathbf{E}[(\log Z)^{1+\beta}] = \sum_{l \geq k \geq 1} \mu(l) (\log k)^{1+\beta} \leq \sum_{k \geq 1} \mu(k) k (\log k)^{1+\beta} < \infty$. Then, for all $t > 0$ and for all $r \in (0, 1/2)$, we get

$$\begin{aligned} 1 - \psi(1-r) &= \mathbf{E}[(1 - (1-r)^{Z-1}) \mathbf{1}_{\{Z \leq t+1\}}] + \mathbf{E}[(1 - (1-r)^{Z-1}) \mathbf{1}_{\{Z > t+1\}}] \\ &\leq 1 - (1-r)^t + \mathbf{P}(Z > t+1) \\ &\leq 2rt + \frac{\mathbf{E}[(\log Z)^{1+\beta}]}{(\log(1+t))^{1+\beta}} = 2rt + C(\log(1+t))^{-1-\beta}. \end{aligned}$$

We choose $1 + t = r^{-1}(\log 1/r)^{-1-\beta}$ and we easily get (2.3.32). \blacksquare

Proposition 2.3.16 *Let μ be a probability measure on \mathbb{N} such that*

$$\sum_{k \in \mathbb{N}} k\mu(k) = 1 \quad \text{and} \quad \exists \beta \in (1, \infty) \text{ such that } \sum_{k \in \mathbb{N}} k(\log k)^{1+\beta} \mu(k) < \infty. \quad (2.3.33)$$

Let $\circ \in \mathbb{W}_b^$ be such that $|\circ| = 0$ and let $\Theta := (\tau^*, \varrho; (X_v)_{v \in \tau^*})$ be a $\overline{\mathbb{W}}_b$ -valued branching random walk whose distribution is the following: $\tau^* = (\tau^*, \circ)$ is an IPGW $^+(\mu)$ -tree as in Definition 2.2.8 (ii) and conditionally given τ^* , Θ has law Q_{\circ, τ^*}^b as in Definition 2.3.1 (iii). Then, with positive probability $(X_v)_{v \in \tau}$ visits its initial position only once. Namely,*

$$\kappa_{\mu, b} := \mathbf{P}(\circ \notin \{X_v; v \in \tau^* \setminus \{\varrho\}\}) > 0. \quad (2.3.34)$$

Proof. Let us recall the following notation

$$\forall p \in \mathbb{N}, \quad \varrho(p) = \varrho|_{]-\infty, -p]}, \quad Y_p = X_{\varrho(p)}, \quad \mathbf{Sp} = \{\varrho(p); p \in \mathbb{N}\} \quad \text{and} \quad \partial \mathbf{Sp} = \{u \in \tau^* \setminus \mathbf{Sp} : \overleftarrow{u} \in \mathbf{Sp}\}.$$

By Definition 2.2.8 (ii) of the right part of an infinite pointed GW(μ)-tree, the subtrees $(\theta_u \tau^*)_{u \in \partial \mathbf{Sp}}$ are i.i.d. GW(μ)-trees and the r.v. $(k_{\varrho(p)}(\tau^*))_{p \in \mathbb{N}}$ are independent, $k_{\circ}(\tau^*)$ has law μ and for all $p \geq 1$, $k_{\varrho(p)}(\tau^*)$ has law $\bar{\mu}$ that is defined by $\bar{\mu}(k) = \sum_{l \geq k} \mu(l)$, $k \in \mathbb{N}^*$ and $\bar{\mu}(0) = 0$. For all $u \in \partial \mathbf{Sp}$, we define $\Theta_u := (\tau_u; \emptyset; (X_v^u)_{v \in \tau_u})$ as follows:

– τ_u is the unique tree such that $k_{\emptyset}(\tau_u) = 1$ and $\theta_{(1)}\tau_u = \theta_u \tau^*$ (see (2.2.8) for the definition of $\theta_u \tau^*$).

– Since $\overleftarrow{u} \in \mathbf{Sp}$, there exists $p \in \mathbb{N}$ such that $\overleftarrow{u} = \varrho(p)$ and we set $X_{\varrho}^u = Y_p$.

– For all $v \in \theta_u \tau$, $X_{(1)*v}^u = X_{u*v}$ (recall from (2.2.4) the definition of the concatenation $u * v$).

Namely, Θ_u is the restriction of the branching walk Θ to the subtree stemming from u , including the spatial position of \overleftarrow{u} . Conditionally given $Y = (Y_p)_{p \in \mathbb{N}}$ and $(k_{\varrho(p)}(\tau^*))_{p \in \mathbb{N}}$, the branching random walks Θ_u , $u \in \partial \mathbf{Sp}$, are independent; moreover, conditionally given (Y_p, τ_u) , Θ_u has law Q_{Y_p, τ_u}^d ; here τ_u stands for (τ_u, \emptyset) and p is such that $\overleftarrow{u} = \varrho(p)$. Then, for all $q \in \mathbb{N}^*$, we set

$$A_q = \{\circ \notin \{X_v; v \in \tau^* \setminus \{\varrho\}\} : |v \wedge \varrho| \geq -q\} \quad \text{and} \quad H_{\circ}^q = \inf\{p \geq 1 : Y_p = \circ\}.$$

The event A_q decreases as $q \rightarrow \infty$ to the event that the branching walk $(X_v)_{v \in \tau^*}$ only visits \circ once. The previous independence properties then imply the following.

$$\mathbf{P}(A_q | Y, \tau^*) = \mathbf{1}_{\{H_{\circ}^q > q\}} \prod_{u \in \partial \mathbf{Sp} : |u| > -q} Q_{Y_{|\overleftarrow{u}|}, \tau_u}^b(\circ \notin \{X_v^u; v \in \tau_u \setminus \{\emptyset\}\})$$

$$\text{and thus} \quad \mathbf{P}(A_q | Y, \mathbf{Sp} \cup \partial \mathbf{Sp}) = \mathbf{1}_{\{H_{\circ}^q > q\}} \prod_{u \in \partial \mathbf{Sp} : |u| > -q} \xi(Y_{|\overleftarrow{u}|}, \circ),$$

where we recall the notation $\xi(x, y)$ from (2.3.31) in Lemma 2.3.14. Since Y and the r.v. $k_{\varrho(p)}(\tau^*)$ are independent, we get

$$\begin{aligned} \mathbf{P}(A_q | Y) &= \mathbf{1}_{\{H_{\circ}^q > q\}} \mathbf{E}[\xi(\circ, \circ)^{k_{\circ}(\tau^*)}] \prod_{1 \leq p \leq q} \mathbf{E}[\xi(Y_p, \circ)^{k_{\varrho(p)}(\tau^*)-1} | Y] \\ &= \mathbf{1}_{\{H_{\circ}^q > q\}} \phi(\xi(\circ, \circ)) \prod_{1 \leq p \leq q} \psi(\xi(Y_p, \circ)), \end{aligned}$$

where we have set

$$\forall r \in [0, 1], \quad \phi(r) = \sum_{k \in \mathbb{N}} r^k \mu(k) \quad \text{and} \quad \psi(r) = \sum_{k \in \mathbb{N}} r^k \bar{\mu}(k+1) = \frac{1-\phi(r)}{1-r}.$$

Notice that as $q \rightarrow \infty$, the event $\{H_0^o > q\}$ decreases to the event $\{H_0^o = \infty\}$, which has strictly positive probability by Lemma 2.3.12 (i). Then, we easily see that (2.3.34) holds true if

$$\sum_{p \in \mathbb{N}} (1 - \psi(\xi(Y_p, o))) < \infty. \quad (2.3.35)$$

But by Lemma 2.3.14, $1 - \xi(Y_p, o) \leq \frac{2b}{b-1} b^{|Y_p \wedge o|}$ since $|o| = 0$. By (2.3.32) in Lemma 2.3.15, there are two constants $C, C' \in (0, \infty)$ such that if $-|Y_p \wedge o| \geq C$, we get

$$1 - \psi(\xi(Y_p, o)) \leq \frac{C_{\beta, \mu}}{(\log(\frac{b-1}{2b} b^{-|Y_p \wedge o|}))^{1+\beta}} \leq \frac{C'}{(1 - |Y_p \wedge o|)^{1+\beta}}$$

By Proposition 2.3.13 with $f(x) = (1+x)^{-1-\beta}$, if $\beta > 1$, then a.s. $\sum_{p \geq 0} (1 - |Y_p \wedge o|)^{-1-\beta} < \infty$, which implies (2.3.35) since $|Y_p \wedge o| \rightarrow -\infty$, as $p \rightarrow \infty$ as stated in Lemma 2.3.12 (i). This completes the proof of the proposition. \blacksquare

We conclude this section with a general estimate for recurrent biased random walks on a (deterministic) rooted ordered tree T that is infinite. More precisely, we fix $\lambda \in (1, \infty)$ and we denote by $(Y_n)_{n \in \mathbb{N}}$ the λ -biased RW whose transition probabilities are given for all $x, y \in T$ by

$$\mathbf{P}(Y_{n+1} = y \mid Y_n = x) = \begin{cases} \frac{\lambda}{\lambda + k_x(T)} & \text{if } y = \overleftarrow{x} \text{ and } x \neq \emptyset, \\ \frac{1}{\lambda + k_x(T)} & \text{if } x = \overleftarrow{y} \text{ and } x \neq \emptyset, \\ \frac{1}{k_\emptyset(T)} & \text{if } x = \overleftarrow{y} = \emptyset \end{cases} \quad (2.3.36)$$

and $\mathbf{P}(Y_{n+1} = y \mid Y_n = x) = 0$ otherwise. Here, recall that $k_x(T)$ is the number of children of x in T . We also recall that $|x|$ is the height of x in T , that $x \wedge y$ is the most recent common ancestor of x and y and that d_{gr} stands for the graph-distance on T : $d_{\text{gr}}(x, y) = |x| + |y| - 2|x \wedge y|$.

Lemma 2.3.17 *We keep the above notations. We assume that Y is recurrent. Then, for all $\ell, n_1, n_2 \in \mathbb{N}$ such that $n_1 < n_2$, we get*

$$\mathbf{P}\left(|Y_{n_1}| + |Y_{n_2}| - 2 \min_{n_1 \leq n \leq n_2} |Y_n| \geq 2\ell + d_{\text{gr}}(Y_{n_1}, Y_{n_2})\right) \leq (n_2 - n_1) \frac{\lambda - 1}{\lambda^\ell - 1}. \quad (2.3.37)$$

Proof. For all $x \in T$, we set $H_x^o = \inf\{n \in \mathbb{N}^* : Y_n = x\}$ and $\pi(x) = (\lambda + k_x(T))\lambda^{-|x|}$ if $x \neq \emptyset$ and $\pi(\emptyset) = k_\emptyset(T)$. Note that π is an invariant measure and standard arguments on electrical networks and random walks on graphs imply that for all distinct $x, y \in T$, $\pi(x)\mathbf{P}(H_x^o < H_y^o \mid Y_0 = y)$ is the effective conductance between x and y (see Lyons & Peres [127] p. 25). If $y \in \llbracket \emptyset, x \rrbracket$, then it easily implies $\mathbf{P}(H_x^o < H_y^o \mid Y_0 = y) = (\lambda - 1)/(\lambda^{|x| - |y| + 1} - 1)$. Similarly, we prove that $\mathbf{P}(H_x^o < H_\emptyset^o \mid Y_0 = \emptyset) = k_\emptyset(T)^{-1}(\lambda - 1)/(\lambda^{|x|} - 1)$. Thus, for all distinct $x, y \in T$ such that $y \in \llbracket \emptyset, x \rrbracket$,

$$\mathbf{P}(H_x^o < H_y^o \mid Y_0 = y) \leq (\lambda - 1)/(\lambda^{|x| - |y|} - 1).$$

We next define the following sequence of stopping times $(\sigma_k)_{k \in \mathbb{N}}$ by setting $\sigma_0 = 0$ and $\sigma_{k+1} = \inf\{n > \sigma_k : |Y_n| \leq (|Y_0| - k - 1)_+\}$ that are a.s. finite since Y is recurrent. We then consider the (possibly empty) event $A_k = \{\exists n \in \{\sigma_k, \dots, \sigma_{k+1} - 1\} : Y_n \in \llbracket \emptyset, Y_0 \rrbracket \text{ and } |Y_n| \geq \ell + (|Y_0| - k - 1)_+\}$. We fix $z \in T$ such that $|z| \geq \ell + (|z| - k - 1)_+$, and let $x, y \in \llbracket \emptyset, z \rrbracket$ be such that $|x| = \ell + (|z| - k - 1)_+$ and $|y| = (|z| - k - 1)_+$. By the strong Markov property at time σ_k and by the previous inequality for hitting times we get $\mathbf{P}(A_k \cap \{Y_0 = z\}) \leq (\lambda - 1)/(\lambda^\ell - 1)\mathbf{P}(Y_0 = z)$. Since on A_k , we have $|Y_0| \geq \ell + (|Y_0| - k - 1)_+$, the previous inequality implies $\mathbf{P}(A_k) \leq (\lambda - 1)/(\lambda^\ell - 1)$.

Next observe that $|Y_0| + |Y_n| - 2 \min_{0 \leq m \leq n} |Y_m| - d_{\text{gr}}(Y_0, Y_n) = 2(|Y_0 \wedge Y_n| - \min_{0 \leq m \leq n} |Y_m|)$ and that

$$\{ |Y_0 \wedge Y_n| - \min_{0 \leq m \leq n} |Y_m| \geq \ell \} \subset A_0 \cup A_1 \cup \dots \cup A_{n-1} .$$

Thus, $\mathbf{P}(|Y_0| + |Y_n| - 2 \min_{0 \leq m \leq n} |Y_m| \geq 2\ell + d_{\text{gr}}(Y_0, Y_n)) \leq n(\lambda-1)/(\lambda^\ell-1)$ and we get (2.3.37) by the Markov property at time n_1 . \blacksquare

2.3.5 Law invariance.

In this section, we first define a successor map for $\overline{\mathbb{W}}_{[0,1]}$ -labelled trees that centers the spatial positions *and* the genealogical tree at the individual coming next in the lexicographical order, generalizing Definition 2.2.6 for the pointed $\overline{\mathbb{W}}_{[0,1]}$ -labelled trees. We then show that free branching random walks are invariant in law under this successor map.

Definition 2.3.18 Let $\Theta = (t, \varrho; \mathbf{x} = (x_v)_{v \in t})$ be a labelled tree in $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{W}}_{[0,1]})$ or in $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{W}}_b)$.

- (a) We set $\text{cent}(\Theta) = (\varphi_{|\varrho|}(t), \varphi_{|\varrho|}(\varrho); (\varphi_{|x_{\varrho}|}(x_{\varphi_{-|\varrho|}(v)}))_{v \in \varphi_{|\varrho|}(t)})$. We keep calling $\text{cent}(\cdot)$ the *centering map*. As already mentioned, the spatial marks are also shifted.
- (b) We next set $\text{scc}(\Theta) = \text{cent}(t, \text{scc}(\varrho); \mathbf{x})$ where we recall that $\text{scc}(\varrho)$ stands for the vertex of t coming next in the lexicographical order as defined by (2.2.7). We keep calling $\text{scc}(\cdot)$ the *successor map*.
- (c) Recall from Definition 2.2.6 (c) the right-part $[(t, \circ)]^+$ of (t, \circ) . To simplify the notation, we set $(t', \varrho') = [(t, \varrho)]^+$ and we recall from Definition 2.3.18 (c) that there exists a unique one-to-one map $\psi: t' \rightarrow \{\varrho_{|p}; p \leq |\varrho|\} \cup \{v \in t: \varrho <_t v\}$ such that $\psi(\varrho') = \varrho$ that preserves the relative height and that is increasing with respect to the lexicographical order; then, we set $[\Theta]^+ = (t', \varrho', \mathbf{x}' = (x'_v)_{v \in t'})$ where $x'_v = x_{\psi(v)}$, for all $v \in t'$. We keep calling $[\Theta]^+$ the *right-part* of Θ .
- (d) We set $\text{scc}^+(\Theta) = [\text{scc}(\Theta)]^+$; we keep calling $\text{scc}^+(\cdot)$ the *right-successor map*. \square

We next explain a way to generate a free branching random walk from an i.i.d. field that is suited to the successor map. To that end we fix $\mathbf{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$ such that $|t|_- = -\infty$. Let \circ be a \mathbb{Z}_- -indexed sequence of mutually independent uniform r.v. on $[0, 1]$. Let Θ be a $\overline{\mathbb{W}}_{[0,1]}$ -valued branching random walk that has conditional law $Q_{\circ, \mathbf{t}}$ given \circ as in Definition 2.3.1 (iv). To simplify we denote by $\mathbf{Q}_{\mathbf{t}}$ the (unconditional) law of Θ . Let $\varepsilon_v, U_v, v \in t$ be independent r.v. such that U_v is uniformly distributed on $[0, 1]$ and $\mathbf{P}(\varepsilon_v = 1) = \mathbf{P}(\varepsilon_v = -1) = 1/2$. From the field of i.i.d. r.v. $\Theta_0 := (t, u; (\varepsilon_v, U_v)_{v \in t})$, we now explain how to construct a branching random walk that has law $\mathbf{Q}_{\mathbf{t}}$.

Recall that $\llbracket v, w \rrbracket$ stands for the shortest path (with respect to the graph-distance) that joins v and w in t (along with the previous notation $\llbracket v, w \rrbracket, \llbracket v, w \rrbracket$ and $\llbracket v, w \rrbracket$, introduced at the beginning of Section 2.3.2). Recall also that $\llbracket -\infty, v \rrbracket$ stands for the lineage of v : namely, $\llbracket -\infty, v \rrbracket = \{v_{|l}; l \leq |v|\}$. Recall from (2.2.5) the definition of the most recent common ancestor $v \wedge w$ in t . We first set

$$\forall v \in t, \quad H_v = \sum_{w \in \llbracket u, u \wedge v \rrbracket} (-\varepsilon_w) + \sum_{w \in \llbracket u \wedge v, v \rrbracket} \varepsilon_w ,$$

with the convention that $H_u = 0$. Note that H takes arbitrary negative values on the lineage of v . Namely H satisfies (2.3.4). It is easy to see that $(t, u; (H_v)_{v \in t})$ is a \mathbb{Z} -valued branching random walk whose spatial motion is that of a simple symmetric random walk on \mathbb{Z} and whose initial position is a.s. $H_u = 0$. Recall from (2.3.5) the notation $v(k)$ for all integers $k \leq H_v$: namely, $v(k) \in \llbracket -\infty, v \rrbracket$ is such that $H_{v(k)} = \min_{w \in \llbracket v(k), v \rrbracket} H_w = k > H_{v(k)}$.

As in (2.3.6), for all $v \in t$ we set $X_v = (U_{v(k)})_{k \leq H_v}$. We denote X_u by \mathfrak{o} . It shows that there exists a deterministic map F such that

$$F(\Theta_0) := (t, u; (X_v)_{v \in t}), \text{ which has law } \mathbf{Q}_t. \quad (2.3.38)$$

Let $u' \in t$ and set $F(t, u'; (\varepsilon_v, U_v)_{v \in t}) := (t, u', (X'_v)_{v \in t})$. Then, set $H'_v = |X'_v|$, $v \in t$. We deterministically check that

$$\forall v \in t, \quad H'_v - H_v = -H_{u'} \quad \text{and} \quad X'_v = \varphi_{|X_{u'}|}(X_v). \quad (2.3.39)$$

We next define $[\Theta_0]^+$ as in Definition 2.3.18 (c): to simplify notation, we set $(t', u') = [(t, u)]^+$ and we recall from Definition 2.2.6 (c) that there exists a unique one-to-one map ψ from t' to the set $\{u_p; p \leq |u|\} \cup \{v \in t : u <_t v\}$ such that $\psi(u') = u$ that preserves the relative height and that is increasing with respect to the lexicographical order. Then, we simply define $[\Theta_0]^+ = (t', u', (\varepsilon'_v, U'_v)_{v \in t'})$ where $(\varepsilon'_v, U'_v) = (\varepsilon_{\psi(v)}, U_{\psi(v)})$ for all $v \in t'$ and also set

$$\widetilde{\text{scc}}^+(\Theta_0) = [\text{scc}(t, u); (\varepsilon_v, U_v)_{v \in t}]^+. \quad (2.3.40)$$

By (2.3.39) and since X_v is a deterministic function of $((\varepsilon_w, U_w), w \in]-\infty, u] \cup]-\infty, v])$, it is easy to check deterministically

$$F(\widetilde{\text{scc}}^+(\Theta_0)) = \text{scc}^+(F(\Theta_0)). \quad (2.3.41)$$

Then, (2.3.2), (2.3.38), (2.3.39) and (2.3.41) immediately entail the following lemma.

Lemma 2.3.19 *Let $\mathfrak{t} = (t, u) \in \overline{\mathbb{T}}^\bullet$. We assume that $|t|_- = -\infty$. Let $\Theta = (t, u; (X_v)_{v \in t})$ have law \mathbf{Q}_t . Then $\text{scc}(\Theta)$ has law $\mathbf{Q}_{\text{scc}(\mathfrak{t})}$ and $\text{scc}^+(\Theta)$ has law $\mathbf{Q}_{\text{scc}^+(\mathfrak{t})}$.*

The next result is the key point in the proofs of the various laws of large numbers for the range of branching random walks that we prove in the next section.

Theorem 2.3.20 *Let μ be a probability measure on \mathbb{N} such that $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. Let $\mathfrak{o} := (U_k)_{k \in \mathbb{Z}_-}$ be a sequence of independent uniform r.v. on $\{1, \dots, \mathfrak{b}\}$. Let $\Theta = (\tau^*, \varrho; (X_v)_{v \in \tau^*})$ be a $\overline{\mathbb{W}}_{\mathfrak{b}}$ -valued branching random walk whose distribution is the following: $\tau^* = (\tau^*, \mathfrak{o})$ is an IPGW $^+(\mu)$ -tree as in Definition 2.2.8 (ii) and conditionally given τ^* and \mathfrak{o} , Θ has law $Q_{\mathfrak{o}, \tau^*}^{\mathfrak{b}}$ as in Definition 2.3.1 (iii). Then*

$$\text{scc}^+(\Theta) \stackrel{(\text{law})}{=} \Theta, \quad (2.3.42)$$

where we recall that scc^+ stands for the right-successor map as in Definition 2.3.18 (d).

Next, denote by $(v_n)_{n \in \mathbb{N}}$ the sequence of the vertices that are not direct ancestors of \mathfrak{o} listed in the lexicographical order. Namely, $v_0 = \mathfrak{o}$, $v_n <_{\tau^*} v_{n+1}$ and $\{v_n; n \in \mathbb{N}\} = \{v \in \tau^* : \varrho \leq_{\tau^*} v\}$. Then,

$$\mathbf{P}\text{-a.s.} \quad \frac{1}{n} \#\{X_{v_k}; 1 \leq k \leq n\} \xrightarrow{n \rightarrow \infty} c_{\mu, \mathfrak{b}} := \mathbf{P}(\mathfrak{o} \notin \{X_{v_n}; n \geq 1\}). \quad (2.3.43)$$

Furthermore, if μ satisfies (2.3.33), then $c_{\mu, \mathfrak{b}} > 0$.

Proof. Let $\Theta_0 := (\tau^*, \mathfrak{o}; (\varepsilon_v, U_v)_{v \in \tau^*})$ be distributed as follows: conditionally given τ^* , $\varepsilon_v, U_v, v \in t$ are independent r.v. such that U_v is uniformly distributed on $[0, 1]$ and ε_v is uniformly distributed on $\{-1, 1\}$. Then, we set $F(\Theta_0) = \Theta' = (\tau^*, \mathfrak{o}; (X'_v)_{v \in \tau^*})$ as in (2.3.38). Conditionally given τ^* , Θ' has law \mathbf{Q}_{τ^*} and Remarks 2.3.3 allows to take $\Theta = \Phi_{\mathfrak{b}}(\Theta')$ that has the desired law. Lemma 2.3.19 combined with proposition 2.2.10 imply that $\text{scc}^+(\Theta')$ has the same law as Θ' and since obviously $\Phi_{\mathfrak{b}}(\text{scc}^+(\Theta')) = \text{scc}^+(\Theta)$, we get (2.3.42).

For all $n \geq m \geq 0$, we set $R_{m,n}(\Theta) = \#\{X_{v_k}; m < k \leq n\}$ and we denote by scc_ℓ^+ the ℓ -th iterate of scc^+ . Observe that $R_{m,n}(\text{scc}_\ell^+(\Theta)) = R_{m+\ell,n+\ell}(\Theta)$, with an obvious notation. Then, the r.v. $(R_{m,n}(\Theta^+))_{m \geq n \geq 1}$ satisfy Assumptions (1.7), (1.8) and (1.9) of Liggett's version of Kingman's subadditive ergodic theorem (see Theorem 1.10 in [121] p. 1280) that asserts that there exists a $[0, 1]$ -valued r.v. R such that $R_{0,n}(\Theta)/n \rightarrow R$ almost surely (and in L^1).

We next prove that R is a.s. constant. Recall from (2.3.40) the definition of $\widetilde{\text{scc}}^+(\Theta_0)$ and we denote by $\widetilde{\text{scc}}_\ell^+$ the ℓ -th iterate of $\widetilde{\text{scc}}^+$: namely, $\widetilde{\text{scc}}_\ell^+(\Theta_0) = [(\tau^*, v_\ell); (\varepsilon_v, U_v)_{v \in \tau^*}]^+$. Denote by \mathcal{G}_ℓ the sigma-field generated by $\widetilde{\text{scc}}_\ell^+(\Theta_0)$. By (2.3.41), $\text{scc}_\ell^+(\Theta) = \Phi_d(F(\widetilde{\text{scc}}_\ell^+(\Theta_0)))$, so the r.v. $(R_{\ell,\ell+n}(\Theta))_{n \in \mathbb{N}}$ are \mathcal{G}_ℓ -measurable. Note that the sigma-fields \mathcal{G}_ℓ decrease in ℓ . Next, we set $\mathcal{G} = \bigcap_{\ell \in \mathbb{N}} \mathcal{G}_\ell$ that can be viewed as the tail sigma-field of the subtrees grafted on the infinite line of ancestors; since additional marks (ε_v, U_v) are i.i.d., Kolmogorov's zero-one law applies and \mathcal{G} is \mathbf{P} -trivial. Furthermore, the subadditivity for the $R_{m,n}$ entail that a.s. $R = \lim_{n \rightarrow \infty} n^{-1} R_{\ell,\ell+n}(\Theta)$, for all ℓ . Thus, R is \mathcal{G} -measurable which implies that it is a.s. constant.

Consequently, $R = \lim_{n \rightarrow \infty} \mathbf{E}[R_{0,n}(\Theta)]/n$. By an elementary argument,

$$\mathbf{E}[R_{0,n}(\Theta)] = \sum_{1 \leq k \leq n} \mathbf{P}(X_{v_k} \notin \{X_{v_{k+1}}, \dots, X_{v_n}\}).$$

Since the law of Θ is preserved by the map $\text{scc}^+(\cdot)$, we get $\mathbf{P}(X_{v_k} \notin \{X_{v_{k+1}}, \dots, X_{v_n}\}) = \mathbf{P}(X_{v_1} \notin \{X_{v_2}, \dots, X_{v_{n-k}}\})$ and thus

$$\frac{1}{n} \mathbf{E}[R_{0,n}(\Theta)] = \frac{1}{n} \sum_{0 \leq k \leq n-1} \mathbf{P}(X_{v_1} \notin \{X_{v_2}, \dots, X_{v_{n-k}}\}) \xrightarrow{n \rightarrow \infty} c_{\mu, \mathfrak{b}}$$

by Cesàro. Finally, observe that $c_{\mu, \mathfrak{b}} \geq \kappa_{\mu, \mathfrak{b}}$, where $\kappa_{\mu, \mathfrak{b}}$ is defined in (2.3.34) in Proposition 2.3.16 that completes the proof of the theorem. \blacksquare

2.3.6 Proof of Theorem 2.1.1.

We fix $\gamma \in (1, 2]$. Let τ be a $\text{GW}(\mu)$ -tree whose offspring distribution μ satisfies (\mathbf{H}) in (2.1.1). We set $\tau = (\tau, \emptyset)$. Let $\mathfrak{o} := (U_k)_{k \in \mathbb{Z}_-}$ be a sequence of independent uniform r.v. on $\{1, \dots, \mathfrak{b}\}$. Let the $\mathbb{W}_{\mathfrak{b}}$ -valued branching random walk $\Theta_d = (\tau, \emptyset; (Y_v)_{v \in \tau})$ have conditional law $Q_{\mathfrak{o}, \tau}^{\mathfrak{b}}$ given τ and \mathfrak{o} as in Definition 2.3.1 (iii). We also introduce the $\mathbb{W}_{\mathfrak{b}}$ -valued branching random walk $\Theta_b^+ = (\tau, \emptyset; (Y_v^+)_{v \in \tau})$ that has conditional law $Q_{\tau}^{+, \mathfrak{b}}$ given τ as in Definition 2.3.1 (i). For all integers $0 \leq k < \#\tau$, we denote

$$R_k(\Theta_b) = \#\{Y_v; v \in \tau: v \leq_\tau v_k\} \quad \text{and} \quad R_k(\Theta_b^+) = \#\{Y_v^+; v \in \tau: v \leq_\tau v_k\}, \quad (2.3.44)$$

where v_k stands for the k -th smallest vertex of τ with respect to the lexicographical order \leq_τ . Arguing as in Le Gall & Lin [82], we derive the following result from Theorem 2.3.20.

Proposition 2.3.21 (adapted from Theorem 7 [82]) *Let $\gamma \in (1, 2]$ and let μ satisfy (\mathbf{H}) as in (2.1.1). We keep the above notation and we recall from (2.3.43) the definition of $c_{\mu, \mathfrak{b}}$ and from (2.3.44) the definition of $R_n(\Theta_b)$. Then,*

$$\forall \varepsilon \in (0, \infty), \quad \mathbf{P}\left(\sup_{t \in [0, 1]} \frac{1}{n} |R_{[nt]}(\Theta_b) - c_{\mu, \mathfrak{b}} nt| > \varepsilon \mid \#\tau = n\right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.3.45)$$

Proof. The proof can be adapted verbatim from the way Proposition 6 and Theorem 7 pp. 284-289 in [82] are deduced from Proposition 3 and Theorem 4 pp. 280-284 in [82]. Here our Theorem 2.3.20

plays the role of Proposition 3 and Theorem 4 in [82] and we only give a brief sketch of the proof. Following the same arguments as in Proposition 6 in [82], we first prove that for any fixed $s \in [0, 1]$,

$$\forall \varepsilon \in (0, \infty), \quad \mathbf{P}\left(\frac{1}{n}|R_{[ns]}(\Theta_b) - c_{\mu,b} ns| > \varepsilon \mid \#\tau > n\right) \xrightarrow{n \rightarrow \infty} 0.$$

From this limit and the same absolute continuity argument as in the proof of Theorem 7 of [82], we get for any fixed $s \in [0, 1]$,

$$\forall \varepsilon \in (0, \infty), \quad \mathbf{P}\left(\frac{1}{n}|R_{[ns]}(\Theta_b) - c_{\mu,b} ns| > \varepsilon \mid \#\tau = n\right) \xrightarrow{n \rightarrow \infty} 0.$$

We then get (2.3.45) thanks to the following variant of the second Dini's theorem. Let us mention that in [82], the case $\gamma = 2$ is treated under the stronger assumption that the variance of μ is finite. Nonetheless, this is not needed in these proofs since the only ingredients that are invoked, namely the local limit theorem and the invariance principles for the height process of a forest of GW-trees and of a size-conditioned GW-tree, still hold under the sole assumption that μ is in the domain of attraction of a 2-stable law. \blacksquare

Lemma 2.3.22 *For all $n \in \mathbb{N}$, let $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$ be a probability space on which a nondecreasing right-continuous process $(X_t^{(n)})_{t \in [0, \infty)}$ is defined. Let $(x(t))_{t \in [0, \infty)}$ be a (deterministic) continuous function. Suppose that for all $t \in [0, \infty)$, the real valued r.v. $X_t^{(n)}$ under \mathbf{P}_n tends to $x(t)$ in law. Then $\lim_{n \rightarrow \infty} \mathbf{P}_n(\sup_{t \in [0, p]} |X_t^{(n)} - x(t)| > \varepsilon) = 0$ for all $\varepsilon \in (0, \infty)$ and all $p \in \mathbb{N}$.*

Proof of Lemma 2.3.22. We fix $p \in \mathbb{N}$. Set $w_{n,q} = \max_{0 \leq k \leq p2^q} |X_{k2^{-q}}^{(n)} - x(k2^{-q})|$ for all $n, q \in \mathbb{N}$. Since the convergence in law to a constant implies the convergence in probability, we easily get for all q that $\lim_{n \rightarrow \infty} \mathbf{P}_n(w_{n,q} > \varepsilon) = 0$. Next set $v_q = \max\{|x(t) - x(s)|; s, t \in [0, p] : |s - t| \leq 2^{-q}\}$ that tends to 0 as $q \rightarrow \infty$ since x is continuous. By monotonicity of $X_t^{(n)}$, it is then easy to see that for all $t \in [0, p]$, that $|X_t^{(n)} - x(t)| \leq 2v_q + 3w_{n,q}$, which easily implies the desired result. \blacksquare

End of the proof of Theorem 2.1.1. We keep the notation from above and recall from (2.3.43) the definition of $c_{\mu,b}$ and from (2.3.44) the definition of $R_n(\Theta_b^+)$. We now derive Theorem 2.1.1 from (2.3.45) and Proposition 2.3.8 that allows to change the state space from $\overline{\mathbb{W}}_b$ to \mathbb{W}_b .

For all positive integers n , we set $c_n = 4 \log_b n$ and $L_n = \#\{v \in \tau : |Y_v^+| \leq c_n\}$. We also set $N_n = \#\{v \in \tau : |v| = n\}$ and $C_n = N_0 + N_1 + \dots + N_n$. Let $(S_n)_{n \in \mathbb{N}}$ be a simple symmetric random walk on \mathbb{Z} with initial position $S_0 = 0$. By definition of Θ_b^+ , for all $v \in \tau$, we get $\mathbf{E}[f(|Y_v^+|) \mid \tau] = \mathbf{E}[f(|S_{|v|}) \mid \tau]$. Then, for all $b_n \in \mathbb{N}^*$ we get

$$\mathbf{E}[L_n \mid \tau] = \sum_{k \in \mathbb{N}} N_k \mathbf{P}(|S_k| \leq c_n) \leq C_{b_n} + \sigma(b_n, \frac{c_n}{\sqrt{b_n}}) \#\tau, \quad (2.3.46)$$

where we have set $\sigma(p, s) = \sup_{k \geq p} \mathbf{P}(|S_k| k^{-\frac{1}{2}} \leq s)$ for all $p \in \mathbb{N}^*$ and all $s \in (0, \infty)$. Observe that any fixed p , $\lim_{s \rightarrow 0^+} \sigma(p, s) = 0$. We now take $b_n = c_n^4$.

We next claim that for all $\eta \in (0, \infty)$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{1}{n} C_{b_n} \geq \eta \mid \#\tau = n\right) = 0. \quad (2.3.47)$$

Proof of (2.3.47). For all $\varepsilon \in (0, \infty)$ and all $y \in \mathbb{R}$, we set $f_\varepsilon(y) = (1 - \varepsilon^{-1}(|y| - \varepsilon)_+)_+$ that is continuous and such that $\mathbf{1}_{[-\varepsilon, \varepsilon]} \leq f_\varepsilon \leq \mathbf{1}_{[-2\varepsilon, 2\varepsilon]}$. Recall from the introduction that $(H_k)_{0 \leq k < \#\tau}$ stands for the height process of the tree τ and recall the convergence (2.1.4). Then note that

$$\frac{1}{n} C_{a_n \varepsilon} \leq Q_\varepsilon^n := \int_0^1 f_\varepsilon\left(\frac{1}{a_n} H_{[ns]}\right) ds.$$

By (2.1.4), Q_ε^n under $\mathbf{P}(\cdot | \#\tau = n)$ tends in law to $Q_\varepsilon := \int_0^1 f_\varepsilon(H_s) ds$. We recall here that a.s. for all $s \in (0, 1)$, $H_s > 0$ (see Duquesne [50] or Section 2.4.4). Thus $\lim_{\varepsilon \rightarrow 0^+} Q_\varepsilon = 0$ and we get for all $\eta \in (0, \infty)$

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \mathbf{P}\left(\frac{1}{n} C_{a_n \varepsilon} \geq \eta \mid \#\tau = n\right) &\leq \limsup_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \mathbf{P}\left(Q_\varepsilon^n \geq \eta \mid \#\tau = n\right) \\ &\leq \limsup_{\varepsilon \rightarrow 0^+} \mathbf{P}\left(Q_\varepsilon \geq \eta\right) = 0. \end{aligned} \quad (2.3.48)$$

Since $(a_n)_{n \in \mathbb{N}}$ is a $\frac{\gamma-1}{\gamma}$ -regularly varying sequence, $b_n/a_n \rightarrow 0$ and (2.3.48) entails (2.3.47). \square

Since $\frac{1}{n} C_{b_n} \in (0, 1]$, it also implies that $\mathbf{E}[\frac{1}{n} C_{b_n} | \#\tau = n] \rightarrow 0$ as $n \rightarrow \infty$. By (2.3.46) we get $\lim_{n \rightarrow \infty} \mathbf{E}[\frac{1}{n} L_n | \#\tau = n] = 0$. We next apply Proposition 2.3.8 to $c = \lfloor 4 \log_b n \rfloor + 1$: (2.3.18) implies the following.

$$\begin{aligned} &\mathbf{P}\left(\sup_{s \in [0,1]} \frac{1}{n} |R_{[ns]}(\Theta_b^+) - c_{\mu,b} ns| > 2\varepsilon \mid \#\tau = n\right) \\ &\leq 2bn^{-1} + \mathbf{P}\left(\frac{1}{n} L_n > \varepsilon \mid \#\tau = n\right) + \mathbf{P}\left(\sup_{s \in [0,1]} \frac{1}{n} |R_{[ns]}(\Theta_b) - c_{\mu,b} ns| > \varepsilon \mid \#\tau = n\right), \end{aligned}$$

which implies (2.1.3) in Theorem 2.1.1 by (2.3.45) since $\mathbf{E}[\frac{1}{n} L_n | \#\tau = n] \rightarrow 0$ as $n \rightarrow \infty$. \blacksquare

2.4 Snake metrics and the Brownian cactus.

2.4.1 Pseudo-metrics on a closed interval.

Definition 2.4.1 Let $\zeta \in (0, \infty)$. We introduce the following spaces.

- (a) We denote by $\mathbf{C}([0, \zeta]^2, \mathbb{R})$ the space of continuous functions from $[0, \zeta]^2$ to \mathbb{R} that is a Banach space when equipped with uniform norm $\|\cdot\|$.
- (b) We denote by $\mathbf{M}([0, \zeta])$ the set of continuous pseudo-metrics on $[0, \zeta]$. Namely, it is the set of $d \in \mathbf{C}([0, \zeta]^2, \mathbb{R})$ such that for all $s_1, s_2, s_3 \in [0, \zeta]$,

$$d(s_1, s_2) \geq 0, \quad d(s_1, s_1) = 0, \quad d(s_1, s_2) = d(s_2, s_1) \quad \text{and} \quad d(s_1, s_3) \leq d(s_1, s_2) + d(s_2, s_3).$$

- (c) We denote by $\mathbf{MT}([0, \zeta])$ the space of continuous pseudo-metrics $d \in \mathbf{M}([0, \zeta])$ such that for all $s_1, s_2, s_3, s_4 \in [0, \zeta]$,

$$d(s_1, s_2) + d(s_3, s_4) \leq \max(d(s_1, s_3) + d(s_2, s_4); d(s_1, s_4) + d(s_2, s_3)). \quad (2.4.1)$$

We call the latter inequality the *four points inequality*. \square

We easily check that $\mathbf{MT}([0, \zeta])$ and $\mathbf{M}([0, \zeta])$ are closed subsets of $(\mathbf{C}([0, \zeta]^2, \mathbb{R}), \|\cdot\|)$. We shall need the following compactness criterion in $\mathbf{M}([0, \zeta])$.

Lemma 2.4.2 For all $\eta \in (0, \infty)$ and all $d \in \mathbf{M}([0, \zeta])$, we set

$$q_\eta(d) = \max \{d(s, s'); s, s' \in [0, \zeta] : |s - s'| \leq \eta\}.$$

Let \mathcal{D} be a subset of $\mathbf{M}([0, \zeta])$. Then, the closure of \mathcal{D} is compact if and only if $\sup_{d \in \mathcal{D}} q_\eta(d) \rightarrow 0$ as $\eta \rightarrow 0^+$.

Proof. For all $f \in \mathbf{C}([0, \zeta]^2, \mathbb{R})$ and all $\eta \in (0, \infty)$, set

$$\omega_\eta(f) = \max \{|f(s_1, s'_1) - f(s_2, s'_2)|; s_1, s_2, s'_1, s'_2 \in [0, \zeta] : |s_1 - s_2| \vee |s'_1 - s'_2| \leq \eta\}$$

that is the η -modulus of uniform continuity of f . The Arzelà-Ascoli Theorem asserts that the closure of $\mathcal{A} \subset \mathbf{C}([0, \zeta]^2, \mathbb{R})$ is $\|\cdot\|$ -compact if and only if $\sup_{f \in \mathcal{A}} \omega_\eta(f) \rightarrow 0$ as $\eta \rightarrow 0+$ and $\sup_{f \in \mathcal{A}} |f(0, 0)| < \infty$. The desired result follows from the easy remark that $q_\eta(d) \leq \omega_\eta(d) \leq 2q_\eta(d)$ (here, the second inequality is a consequence of the triangle inequality) and from the obvious fact that $d(0, 0) = 0$ for any $d \in \mathcal{D}$. ■

This lemma immediately implies the following tightness criterion.

Proposition 2.4.3 *Let $(\mathbf{d}_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbf{M}([0, \zeta])$ -valued random variables. The sequence of their laws is tight on $(\mathbf{C}([0, \zeta]^2, \mathbb{R}), \|\cdot\|)$ if and only if $\lim_{\eta \rightarrow 0+} \sup_{n \in \mathbb{N}} \mathbf{P}(q_\eta(\mathbf{d}_n) > \varepsilon) = 0$ for all $\varepsilon \in (0, \infty)$.*

We now prove the following specific proposition that is used later in the proof of Theorem 2.1.2.

Proposition 2.4.4 *Let $\mathbf{d}_n, \mathbf{d}_n^*$, $n \in \mathbb{N}$, be $\mathbf{M}([0, \zeta])$ -valued r.v. such that*

- (i) $\mathbf{d}_n \rightarrow \mathbf{d}$ weakly on $(\mathbf{C}([0, \zeta]^2, \mathbb{R}), \|\cdot\|)$;
- (ii) For all $n \in \mathbb{N}$ and for all $s, s' \in [0, \zeta]$, a.s. $\mathbf{d}_n^*(s, s') \leq \mathbf{d}_n(s, s')$;
- (iii) For all $s, s' \in [0, \zeta]$, $|\mathbf{d}_n^*(s, s') - \mathbf{d}_n(s, s')| \rightarrow 0$ in probability.

Then, $(\mathbf{d}_n^, \mathbf{d}_n) \rightarrow (\mathbf{d}, \mathbf{d})$ weakly on $(\mathbf{C}([0, \zeta]^2, \mathbb{R}), \|\cdot\|)^2$.*

Proof. By (i) and Proposition 2.4.3, for all $\varepsilon \in (0, \infty)$, $\lim_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \mathbf{P}(q_\eta(\mathbf{d}_n) > \varepsilon) = 0$. Since \mathbf{d}_n and \mathbf{d}_n^* are continuous, (ii) actually implies that a.s. for all $n \in \mathbb{N}$ and for all $s, s' \in [0, \zeta]$, $\mathbf{d}_n^*(s, s') \leq \mathbf{d}_n(s, s')$, which immediately entails $q_\eta(\mathbf{d}_n^*) \leq q_\eta(\mathbf{d}_n)$. Thus, for all $\varepsilon \in (0, \infty)$, it holds that $\lim_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \mathbf{P}(q_\eta(\mathbf{d}_n^*) > \varepsilon) = 0$. By Proposition 2.4.3, the laws of the r.v. $(\mathbf{d}_n^*, \mathbf{d}_n)$ are tight in $\mathbf{C}([0, \zeta]^2, \mathbb{R})^2$ and we get the desired result because (i) and (iii) easily entail the weak convergence of the finite dimensional marginals of $(\mathbf{d}_n^*, \mathbf{d}_n)$ to those of (\mathbf{d}, \mathbf{d}) . ■

Induced metric spaces. Let $d \in \mathbf{M}([0, \zeta])$. We define the relation \sim_d on $[0, \zeta]$ as follows: for all $s_1, s_2 \in [0, \zeta]$, $s_1 \sim_d s_2$ if and only if $d(s_1, s_2) = 0$. Clearly \sim_d is an equivalence relation and we define the quotient space:

$$E_d = [0, \zeta] / \sim_d, \quad \text{proj}_d : [0, \zeta] \rightarrow E_d, \quad \text{the canonical projection, } r_d = \text{proj}_d(0). \quad (2.4.2)$$

We keep denoting d the metric induced by d on E_d . Since d is continuous on $[0, \zeta]^2$, proj_d is continuous and (E_d, d, r_d) is a pointed compact and connected metric space. We also equip E_d with the pushforward measure μ_d of the Lebesgue measure on $[0, \zeta]$ via the canonical projection: namely, for all nonnegative measurable functions f on E_d ,

$$\int_{E_d} f(x) \mu_d(dx) = \int_0^\zeta f(\text{proj}_d(s)) ds. \quad (2.4.3)$$

Note that μ_d is a finite measure with total mass ζ .

Remark 2.4.5 Let (E_d, d, r_d, μ_d) be the compact metric space corresponding to the pseudo-metric $d \in \mathbf{M}([0, \zeta])$. Let $a, b \in (0, \infty)$. Set $d'(s_1, s_2) = ad(s_1/b, s_2/b)$, $s_1, s_2 \in [0, b\zeta]$. Then, we easily check that $d' \in \mathbf{M}([0, b\zeta])$ and $(E_{d'}, d', r_{d'}, \mu_{d'})$ is isometric to $(E_d, ad, r_d, b\mu_d)$. □

Real trees. When $d \in \mathbf{MT}([0, \zeta])$, the resulting space E_d is a real tree. More precisely, real trees are metric spaces that extend the definition of graph-trees; they are defined as follows.

Definition 2.4.6 let (T, d) be a metric space; it is a *real tree* if and only if the following holds true.

- (a) For any $\sigma_1, \sigma_2 \in T$, there is a unique isometry $f : [0, d(\sigma_1, \sigma_2)] \rightarrow T$ such that $f(0) = \sigma_1$ and $f(d(\sigma_1, \sigma_2)) = \sigma_2$. Then, we set $\llbracket \sigma_1, \sigma_2 \rrbracket := f([0, d(\sigma_1, \sigma_2)])$.
- (b) For any continuous injective function $g : [0, 1] \rightarrow T$, such that $g(0) = \sigma_1$ and $g(1) = \sigma_2$, $g([0, 1]) = \llbracket \sigma_1, \sigma_2 \rrbracket$. \square

It turns out that the four points inequality is a metric characterisation of real trees. More precisely, if (T, d) is a connected metric space, then (T, d) is a real tree if and only if for any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in T$,

$$d(\sigma_1, \sigma_2) + d(\sigma_3, \sigma_4) \leq (d(\sigma_1, \sigma_3) + d(\sigma_2, \sigma_4)) \vee (d(\sigma_1, \sigma_4) + d(\sigma_2, \sigma_3)).$$

We refer to Evans [70] or to Dress, Moulton & Terhalle [48] for a detailed account on this property.

Let us introduce some notation about real trees. Let (T, d) be a compact pointed real tree. We distinguish a point $r \in T$ that is viewed as a *root*. We then define the *length* measure $\text{Length}(\cdot)$ on T as the one-dimensional Hausdorff measure: namely, it is the unique Borel measure such that $\text{Length}(\llbracket \sigma, \sigma' \rrbracket) = d(\sigma, \sigma')$, for all $\sigma, \sigma' \in T$. Let us next introduce branch points: let $\sigma_1, \sigma_2, \sigma_3 \in T$; then the geodesic paths $\llbracket \sigma_1, \sigma_2 \rrbracket$, $\llbracket \sigma_1, \sigma_3 \rrbracket$ and $\llbracket \sigma_2, \sigma_3 \rrbracket$ have exactly one point in common that is called the branch point of $\sigma_1, \sigma_2, \sigma_3$ and that is denoted by $\text{br}(\sigma_1, \sigma_2, \sigma_3)$; namely

$$\{\text{br}(\sigma_1, \sigma_2, \sigma_3)\} = \llbracket \sigma_1, \sigma_2 \rrbracket \cap \llbracket \sigma_1, \sigma_3 \rrbracket \cap \llbracket \sigma_2, \sigma_3 \rrbracket. \quad (2.4.4)$$

If we view T as a family tree whose ancestor is r , then $\text{br}(\sigma_1, \sigma_2, r)$ is the most recent common ancestor of σ_1 and σ_2 and we use the following notation

$$\sigma_1 \wedge \sigma_2 = \text{br}(\sigma_1, \sigma_2, r).$$

We next introduce the (extended) *degree* of any point $\sigma \in T$ as follows.

$$\deg(\sigma) \text{ is the (possibly infinite) number of connected components of the open set } T \setminus \{\sigma\}. \quad (2.4.5)$$

We say that σ is a *branch point* if $\deg(\sigma) \geq 3$; we say that σ is a *leaf* if $\deg(\sigma) = 1$ and we say that σ is *simple* if $\deg(\sigma) = 2$. We shall use the following notation for the set of leaves of T

$$\text{Lf}(T) := \{\sigma \in T \setminus \{r\} : \deg(\sigma) = 1\}.$$

Definition 2.4.7 Let μ be a finite Borel measure on T ; then (T, d, r, μ) is a *continuum real tree* in the sense of Aldous [9] if T is compact and if

$$\mu \text{ is diffuse, the topological support of } \mu \text{ is } T \text{ and } \mu(T \setminus \text{Lf}(T)) = 0. \quad (2.4.6)$$

Lemma 2.4.8 Let $\zeta \in (0, \infty)$ and $d \in \text{MT}([0, \zeta])$. To simplify we denote by (T, d, r, μ) the pointed and measured compact real tree induced by the pseudo-metric d and we denote by $p : [0, \zeta] \rightarrow T$ the canonical projection. We set

$$U = \{s \in (0, \zeta) : \forall s' \in [0, \zeta] \setminus \{s\}, d(0, s) + d(s, s') > d(0, s')\}. \quad (2.4.7)$$

We assume that $[0, \zeta] \setminus U$ is Lebesgue-negligible. Then, T is a continuum real tree as defined in (2.4.6).

Proof. By construction and since p is continuous, the topological support of μ is T . We next show that $p(U) \subset \text{Lf}(T)$. Let $s \in (0, \zeta)$ be such that $p(s) \neq r$ and such that $\deg(p(s)) \geq 2$. Let s' be such that $p(s')$ belongs to a connected component of $T \setminus \{p(s)\}$ that does not contain

r . Then, $p(s) \in]r, p(s')[$, which implies that $d(p(s'), r) = d(r, p(s)) + d(p(s), p(s'))$, namely, $d(0, s') = d(0, s) + d(s, s')$. Thus, $s \notin U$. This proves that $p(U) \subset \text{Lf}(T)$. Since $[0, \zeta] \setminus U$ is Lebesgue negligible, $T \setminus \text{Lf}(T)$ is μ -negligible.

It remains to prove that μ is diffuse. Let $s \in [0, \zeta]$. If $s \notin U$, then $p(s) \in T \setminus \text{Lf}(T)$ that is μ -negligible; thus $\mu(\{p(s)\}) = 0$. Assume now that $s \in U$ and suppose that s' is such that $d(s, s') = 0$. The triangle inequality for d implies that $d(0, s) = d(0, s')$. Since $s \in U$, we get $s = s'$. Thus $p^{-1}(\{p(s)\}) = \{s\}$ and $\mu(\{p(s)\}) = 0$, which completes the proof of the lemma. \blacksquare

Example 2.4.9 (*Real trees spanned by graph-trees*) Let T be a discrete graph-tree with a special vertex ρ viewed as a root. We denote by d_{gr} the graph-distance on T . The real tree spanned by T is obtained by joining neighbouring vertices of T by a unit-length segment of the real line with its own metric. Formally, we can take $\tilde{T} = \{(\rho, 1)\} \cup \bigcup_{v \in T \setminus \{\rho\}} \{v\} \times (0, 1]$ and for all $\sigma = (v, s)$ and $\sigma' = (v', s')$ in \tilde{T} , we set:

$$\tilde{d}_{\text{gr}}^T(\sigma, \sigma') = \begin{cases} d_{\text{gr}}(v, v') + s + s' - 2 & \text{if } v \wedge v' \notin \{v, v'\}, \\ d_{\text{gr}}(v, v') + s - s' & \text{if } v' = v \wedge v' \text{ and } v \neq v', \\ |s - s'| & \text{if } v = v', \end{cases}$$

where $v \wedge v'$ is the most recent common ancestor of v and v' when we view T as a family tree whose ancestor is ρ . Clearly, $(\tilde{T}, \tilde{d}_{\text{gr}}^T)$ is a real tree and if we identify T with $T \times \{1\} \subset \tilde{T}$, we easily check that \tilde{d}_{gr}^T extends d_{gr} . Note that $(\tilde{T}, \tilde{d}_{\text{gr}}^T)$ is compact if and only if T is finite. \square

Example 2.4.10 (*Real trees coded by continuous functions*) Let $\zeta \in (0, \infty)$ and let $h: [0, \zeta] \rightarrow \mathbb{R}$ be continuous. For all $s_1, s_2 \in [0, \zeta]$, we set

$$m_h(s_1, s_2) = \inf_{s \in [s_1 \wedge s_2, s_1 \vee s_2]} h(s) \quad \text{and} \quad d_h(s_1, s_2) = h(s_1) + h(s_2) - 2m_h(s_1, s_2). \quad (2.4.8)$$

We easily check that $d_h \in \text{MT}([0, \zeta])$ and to simplify we denote by (T_h, d_h, r_h, μ_h) the induced metric space $(E_{d_h}, d_h, r_{d_h}, \mu_{d_h})$ as defined in (2.4.2) and (2.4.3); we also denote by $p_h: [0, \zeta] \rightarrow T_h$ the canonical projection (instead of proj_{d_h}); (T_h, d_h, r_h, μ_h) is a pointed measured compact real tree that is the *tree coded by the function h* .

Let $h': [0, \zeta] \rightarrow \mathbb{R}$ be continuous. Then observe that

$$\forall s_1, s_2 \in [0, \zeta], \quad |d_h(s_1, s_2) - d_{h'}(s_1, s_2)| \leq 4 \sup_{s \in [0, \zeta]} |h(s) - h'(s)|, \quad (2.4.9)$$

which shows the continuity in $\text{MT}([0, \zeta])$ of the application $h \mapsto d_h$. \square

Remark 2.4.11 Let (T_h, d_h, r_h) be the real tree coded by the continuous function $h: [0, \zeta] \rightarrow \mathbb{R}$ and the canonical projection p_h as explained above. Suppose that h is nonnegative and that $h(0) = 0$. Then $d_h(0, s) = h(s)$ and for all $s_1, s_2 \in [0, \zeta]$, we easily get $m_h(s_1, s_2) = d_h(r_h, p_h(s_1) \wedge p_h(s_2))$.

However, let $d \in \text{MT}([0, \zeta])$ and set $h(s) = d(0, s)$, $s \in [0, \zeta]$ that is a continuous function. The real tree $T = [0, \zeta] / \sim_d$ is in general different from the real tree T_h coded by h : namely $d \neq d_h$. But let us mention that all compact real trees can be coded by (many) continuous functions (see Duquesne [53] for more details on the coding of real trees). \square

Gromov-Hausdorff-Prokhorov metric. Let (E_1, d_1, r_1, μ_1) and (E_2, d_2, r_2, μ_2) be two pointed measured compact metric spaces: here μ_1 and μ_2 are finite measures on the respective Borel sigma-fields of E_1 and E_2 , and $r_1 \in E_1$ and $r_2 \in E_2$ are distinguished points. The pointed

Gromov-Hausdorff-Prokhorov distance (the *GHP-distance* for short) between E_1 and E_2 is defined by

$$\delta_{\text{GHP}}(E_1, E_2) = \inf \left\{ d_E^{\text{Haus}}(\phi_1(E_1), \phi_2(E_2)) + d_E(\phi_1(r_1), \phi_2(r_2)) + d_E^{\text{Prok}}(\mu_1 \circ \phi_1^{-1}, \mu_2 \circ \phi_2^{-1}) \right\}. \quad (2.4.10)$$

Here, the infimum is taken over all Polish spaces (E, d_E) and all couples of isometric embeddings $\phi_i : E_i \hookrightarrow E$, $i \in \{1, 2\}$; d_E^{Haus} stands for the Hausdorff distance on the space of compact subsets of E (namely, $d_E^{\text{Haus}}(K_1, K_2) = \inf \{ \varepsilon \in (0, \infty) : K_1 \subset K_2^{(\varepsilon)} \text{ and } K_2 \subset K_1^{(\varepsilon)} \}$, where $A^{(\varepsilon)} = \{y \in E : d_E(y, A) \leq \varepsilon\}$ for all non-empty $A \subset E$); for all $i \in \{1, 2\}$, $\mu_i \circ \phi_i^{-1}$ stands for the pushforward measure of μ_i via ϕ_i ; d_E^{Prok} stands for the Prokhorov distance on the space of finite Borel measures on E , namely

$$d_E^{\text{Prok}}(\mu, \nu) = \inf \{ \varepsilon \in (0, \infty) : \mu(K) \leq \nu(K^{(\varepsilon)}) + \varepsilon \text{ and } \nu(K) \leq \mu(K^{(\varepsilon)}) + \varepsilon, \forall K \subset E \text{ compact} \}.$$

Remark 2.4.12 Let (E, d) be a Polish space and let $a, b, c \in (0, \infty)$. We denote by d_{Prok} (resp. d_{Prok}^a) the Prokhorov distance on the space of finite Borel measures on (E, d) (resp. (E, ad)) and we denote by d_{Haus} (resp. d_{Haus}^a) the Hausdorff distance on the space of the compact subsets of (E, d) (resp. (E, ad)). First note that $d_{\text{Haus}}^a = ad_{\text{Haus}}$. Then, observe that $d_{\text{Prok}}(b\mu, c\mu) = |b-c|\mu(E)$ and that $(a \wedge b)d_{\text{Prok}}(\mu, \nu) \leq d_{\text{Prok}}^a(b\mu, b\nu) \leq (a \vee b)d_{\text{Prok}}(\mu, \nu)$.

Let (E_1, d_1, r_1, μ_1) and (E_2, d_2, r_2, μ_2) be two pointed measured compact metric spaces. We set $(E'_i, d'_i, r'_i, \mu'_i) = (E_i, ad_i, r_i, b\mu_i)$, $i \in \{1, 2\}$. Then, it is easy to check that $(a \wedge b)\delta_{\text{GHP}}(E_1, E_2) \leq \delta_{\text{GHP}}(E'_1, E'_2) \leq (a \vee b)\delta_{\text{GHP}}(E_1, E_2)$. \square

Example 2.4.13 Let T be a finite graph-tree that is equipped with its graph distance and with its counting measure: $\mathfrak{m} = \sum_{v \in T} \delta_v$. Let $(\tilde{T}, \tilde{d}_{\text{gr}}^T)$ be the compact real tree spanned by T (see Example 2.4.9). We equip \tilde{T} with its length measure Length . Up to obvious identifications, we can assume that $T \subset \tilde{T}$. Then, we easily get $d_{\text{Haus}}(T, \tilde{T}) \leq 1$, $d_{\text{Prok}}(\mathfrak{m}, \text{Length}) \leq 2$ and thus, $\delta_{\text{GHP}}(T, \tilde{T}) \leq 3$. \square

We next recall from Theorem 2.5 in Abraham, Delmas & Hoscheit [4] the following assertions: δ_{GHP} is a pseudo-metric (i.e. it is symmetric and it satisfies the triangle inequality) and $\delta_{\text{GHP}}(E_1, E_2) = 0$ if and only if E_1 and E_2 are *isometric*, namely if and only if there exists a bijective isometry $\phi : E_1 \rightarrow E_2$ such that $\phi(r_1) = r_2$ and such that $\mu_2 = \mu_1 \circ \phi^{-1}$. Denote by \mathbb{M} the isometry classes of pointed measured compact metric spaces. Then, Theorem 2.5 in Abraham, Delmas & Hoscheit [4] asserts that

$$(\mathbb{M}, \delta_{\text{GHP}}) \text{ is a complete and separable metric space.} \quad (2.4.11)$$

Proposition 2.4.14 Let $d, d' \in \mathbb{M}([0, \zeta])$ and let E_d and $E_{d'}$ be the induced pointed measured compact metric spaces as defined by (2.4.2) and (2.4.3). Then

$$\delta_{\text{GHP}}(E_d, E_{d'}) \leq \frac{3}{2} \|d - d'\|. \quad (2.4.12)$$

Proof. We use the notation from (2.4.2) and we set $\mathcal{R} = \{(\text{proj}_d(s), \text{proj}_{d'}(s)); s \in [0, \zeta]\}$ that is a relation on $E_d \times E_{d'}$ since for all $x \in E_d$ and all $x' \in E_{d'}$, $\mathcal{R} \cap (\{x\} \times E_{d'})$ and $\mathcal{R} \cap (E_d \times \{x'\})$ are not empty. By the triangle inequality for pseudo-metrics, we get

$$\text{dis}(\mathcal{R}) := \sup \{ |d(x, y) - d'(x', y')|; (x, x'), (y, y') \in \mathcal{R} \} \leq \|d - d'\|. \quad (2.4.13)$$

We next set $G = E_d \sqcup E_{d'}$ that is a disjoint union, and we define a metric δ on G as follows: δ coincides with d on $E_d \times E_d$ and with d' on $E_{d'} \times E_{d'}$ and for all $x \in E_d$ and all $x' \in E_{d'}$, we set

$$\delta(x, x') = \inf \{d(x, y) + \frac{1}{2} \text{dis}(\mathcal{R}) + d'(y', x'); (y, y') \in \mathcal{R}\}.$$

We easily check that δ is a separable compact metric on G . Thus, (G, δ) is Polish. The inclusions $E_d \hookrightarrow G$ and $E_{d'} \hookrightarrow G$ are isometries and since $(r_d, r_{d'}) = (\text{proj}_d(0), \text{proj}_{d'}(0)) \in \mathcal{R}$, we get

$$\delta_{\text{Haus}}(E_d, E_{d'}) \vee \delta(r_d, r_{d'}) \leq \frac{1}{2} \text{dis}(\mathcal{R}), \quad (2.4.14)$$

where δ_{Haus} stands for the Hausdorff distance of the space of compact subsets of G .

Let K be a closed (and thus compact) subset of G . We set $Q = E_d \cap K$ and $Q' = E_{d'} \cap K$ that are also compact subsets of G . Note that $\mu_d(K) = \mu_d(Q)$ and $\mu_{d'}(K) = \mu_{d'}(Q')$. We set $C = \text{proj}_d^{-1}(Q)$, $Q'_0 = \text{proj}_{d'}(C)$ and $C' = \text{proj}_{d'}^{-1}(Q'_0)$. Since proj_d and $\text{proj}_{d'}$ are continuous, C and C' are compact subsets of $[0, \zeta]$ and Q'_0 is a compact subset of $E_{d'}$. Denote by ℓ the Lebesgue measure on $[0, \infty)$. Observe that $C \subset C'$, which implies that $\mu_d(K) = \mu_d(Q) = \ell(C) \leq \ell(C') = \mu_{d'}(Q'_0)$. Let $x' \in Q'_0$. There exists $s \in C$ such that $\text{proj}_{d'}(s) = x'$. But note that $x := \text{proj}_d(s) \in Q$ and thus $(x, x') \in \mathcal{R}$. Set $\eta := \frac{1}{2} \text{dis}(\mathcal{R})$; then we have proved that $\mu_d(K) \leq \mu_{d'}(K^\eta)$ where $K^\eta = \{z \in G : \delta(z, K) \leq \eta\}$. Similarly, we prove that $\mu_{d'}(K) \leq \mu_d(K^\eta)$ which implies that $\delta_{\text{Prok}}(\mu_d, \mu_{d'}) \leq \eta$, where δ_{Prok} stands for the Prohorov distance on the space of finite Borel measures on G . This inequality combined with (2.4.14) implies that $\delta_{\text{GHP}}(E_d, E_{d'}) \leq \frac{3}{2} \text{dis}(\mathcal{R})$, which entails (2.4.12) by (2.4.13). \blacksquare

2.4.2 Scaling limit of the range of biased RWs on trees.

Let T be an infinite rooted ordered tree. We fix $\lambda \in (1, \infty)$ and we denote by $(Y_n)_{n \in \mathbb{N}}$ the λ -biased RW on T whose transition probabilities are given by (2.3.36). We make T a real tree by joining neighbouring vertices by a line isometric to $[0, 1]$ as explained in Example 2.4.9 and we keep denoting by $(T, d_{\text{gr}}, \emptyset)$ the resulting rooted real tree. We also denote simply by $(Y_s)_{s \in [0, \infty)}$ the continuous interpolation of Y : namely, for all $n \in \mathbb{N}$ and all $s \in [n, n+1]$, Y_s is the unique point of the line $[[Y_n, Y_{n+1}]]$ in the (spanned) real tree T such that $d_{\text{gr}}(Y_n, Y_s) = s - n$.

Let $\zeta \in (0, \infty)$. For all $n \in \mathbb{N}$, we then set

$$\mathcal{R}_n = \{Y_s ; s \in [0, \zeta n]\}$$

that is a random compact real tree. We equip \mathcal{R}_n with the occupation measure $m_{\text{occ}}^{(n)}$ induced by the RW, namely $\int_{\mathcal{R}_n} f(\sigma) m_{\text{occ}}^{(n)}(d\sigma) = \int_0^\zeta f(Y_{sn}) ds$.

Proposition 2.4.15 *We keep the above notation. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. We assume that $(Y_n)_{n \in \mathbb{N}}$ is recurrent, that $\lim_{n \rightarrow \infty} a_n^{-1} \log n = 0$ and that there exists a continuous random process $(H_s)_{s \in [0, \zeta]}$ such that*

$$\left(\frac{1}{a_n} d_{\text{gr}}(\emptyset, Y_{ns}) \right)_{s \in [0, \zeta]} \xrightarrow[n \rightarrow \infty]{(law)} (H_s)_{s \in [0, \zeta]} \quad (2.4.15)$$

weakly on $\mathbf{C}([0, \zeta], \mathbb{R})$. Then, jointly with (2.4.15), the following convergence

$$\left(\mathcal{R}_n, \frac{1}{a_n} d_{\text{gr}}, \emptyset, m_{\text{occ}}^{(n)} \right) \xrightarrow[n \rightarrow \infty]{(law)} (T_H, d_H, r_H, \mu_H)$$

holds weakly on the space of rooted measured compact metric spaces \mathbb{M} equipped with the Gromov-Hausdorff-Prokhorov distance. Here, (T_H, d_H, r_H, μ_H) stands for the real tree coded by H as in Example 2.4.10.

Proof. For all $s \in [0, \zeta]$, we set $H_s^{(n)} = \frac{1}{a_n} d_{\text{gr}}(\emptyset, Y_{ns})$. For all $s, s' \in [0, \zeta]$, we also set $d_n(s, s') = H_s^{(n)} + H_{s'}^{(n)} - 2 \min_{r \in [s \wedge s', s \vee s']} H_r^{(n)}$, that is the tree pseudo-distance coded by $H^{(n)}$ as in Example 2.4.10. Clearly, by (2.4.9), (2.4.15) implies that $d_n \rightarrow d_H$ weakly on $\mathbf{C}([0, \zeta]^2, \mathbb{R})$. Then we set $d_n^*(s, s') = a_n^{-1} d_{\text{gr}}(Y_{ns}, Y_{ns'})$. Thus $d_n^*(s, s') \leq d_n(s, s')$ and by (2.3.37) in Lemma 2.3.17, we get for all $\varepsilon > 0$ that

$$\begin{aligned} \mathbf{P}(d_n(s, s') - d_n^*(s, s') > \varepsilon) &\leq \mathbf{P}\left(|Y_{[ns]}| + |Y_{[ns']}| - 2 \min_{[ns] \leq k \leq [ns']} |Y_k| \geq \varepsilon a_n - 2 + d_{\text{gr}}(Y_{[ns]}, Y_{[ns']})\right) \\ &\leq ([ns'] - [ns]) \frac{\lambda - 1}{\lambda^{\frac{1}{2} \varepsilon a_n - 1} - 1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

By Proposition 2.4.4, $(d_n^*, d_n) \rightarrow (d_H, d_H)$ weakly on $\mathbf{C}([0, \zeta]^2, \mathbb{R})^2$. This implies the desired result by Proposition 2.4.14 and because the compact measured rooted tree induced by the pseudo-metric d_n^* is isometric to $(\mathcal{R}_n, \frac{1}{a_n} d_{\text{gr}}, \emptyset, m_{\text{occ}}^{(n)})$. \blacksquare

We then derive from the previous proposition (the easy part) and from a result due to Dembo & Sun [45] (the difficult part) that the scaling limit of the range of critical biased RWs on a supercritical multi-type GW-tree is the Brownian tree. More precisely, we consider a N -type GW-tree; for all $k \in \{1, \dots, N\}$ we denote by $\mu(k, \cdot)$ a probability measure on $\mathbb{W}_N = \bigcup_{n \in \mathbb{N}} \{1, \dots, N\}^n$, the set of finite words written in the alphabet $\{1, \dots, N\}$. For all $p \in (0, \infty)$, the p -th moment matrix $M_p = (m_p(k, \ell))_{1 \leq k, \ell \leq N}$ is given by

$$m_p(k, \ell) = \sum_{n \in \mathbb{N}} \sum_{1 \leq w_1, \dots, w_n \leq N} \mu(k, (w_1, \dots, w_n)) \left(\#\{j \in \{1, \dots, n\} : w_j = \ell\} \right)^p.$$

We assume the following.

- (a) There exists $p \in (4, \infty)$ such that for all $k, \ell \in \{1, \dots, N\}$, $m_p(k, \ell) < \infty$.
- (b) There exists an integer $n_0 \geq 1$, such that $M_1^{n_0}$ has only strictly positive entries.
- (c) Let λ be the Perron-Frobenius eigenvalue of M_1 . We assume that $\lambda > 1$.

Let $\mathbf{T} = (T, \emptyset, (a_x)_{x \in T})$ be a N -type GW(μ)-tree. Namely, T is a random rooted ordered tree, $a_x \in \{1, \dots, N\}$ is the type of vertex $x \in T$ and \mathbf{T} satisfies the following property: recall that $k_\emptyset(T)$ is the number of children of the root and recall that for all $j \in \{1, \dots, k_\emptyset(T)\}$, $\theta_{(j)} \mathbf{T}$ stands for the tree stemming from the j -th child of \emptyset (equipped with the types of corresponding vertices). Then, conditionally given a_\emptyset , the types $(a_{(1)}, \dots, a_{(k_\emptyset(T))})$ of the children of the root have conditional law $\mu(a_\emptyset, \cdot)$ and conditionally given $(a_{(1)}, \dots, a_{(k_\emptyset(T))})$, the subtrees $\theta_{(1)} \mathbf{T}, \dots, \theta_{(k_\emptyset(T))} \mathbf{T}$ are independent GW(μ)-trees.

Then, conditionally given T , let $(Y_n)_{n \in \mathbb{N}}$ be the λ -biased RW on T started at \emptyset and denote by $(Y_s)_{s \in [0, \infty)}$ its continuous interpolation as explained above. Under Assumptions (a), (b) and (c) on μ , Theorem 1.1 in Dembo & Sun [45] p. 3 asserts that there exists $\sigma \in (0, \infty)$, a constant that depends only on μ , such that a.s. on the non-extinction event $\{|T| = \infty\}$,

$$\left(\frac{1}{\sigma \sqrt{n}} d_{\text{gr}}(\emptyset, Y_{ns}) \right)_{s \in [0, \infty)} \xrightarrow[n \rightarrow \infty]{(\text{law})} (|B_s|)_{s \in [0, \infty)} \quad (2.4.16)$$

where $(B_s)_{s \in [0, \infty)}$ is a standard linear Brownian motion such that $B_0 = 0$ a.s. By Proposition 2.4.15, we immediately get the following new result.

Corollary 2.4.16 *We keep the notation as above. Let $\zeta \in (0, \infty)$. Let $\mathcal{R}_n = \{Y_s; s \in [0, n\zeta]\}$ be the range of Y up to time $n\zeta$ and let $m_{\text{occ}}^{(n)}$ be the occupation measure of Y on \mathcal{R}_n as introduced before. Under Assumptions (a), (b) and (c) on μ , a.s. on the event $\{|T| = \infty\}$, we get*

$$\left(\mathcal{R}_n, \frac{1}{\sigma \sqrt{n}} d_{\text{gr}}, \emptyset, m_{\text{occ}}^{(n)} \right) \xrightarrow[n \rightarrow \infty]{(\text{law})} (T_{|B|}, d_{|B|}, r_{|B|}, \mu_{|B|}) \quad (2.4.17)$$

weakly on the space of rooted measured compact metric spaces \mathbb{M} equipped with the Gromov-Hausdorff-Prokhorov distance. Here, $(T|_{B|}, d|_{B|}, r|_{B|}, \mu|_{B|})$ stands for the real tree coded by the reflected Brownian motion $(|B_s|)_{s \in [0, \zeta]}$.

In literature, the first scaling limit for the range of tree-valued RWs appears in Duquesne [52]: in this paper the tree is b-ary and the RW is slightly super-critical (see Theorem 2.1 [52] p. 2224; Lemma 3.7 [52] p. 2241 also contains a local law of large numbers for the range). When T is a supercritical single-type GW-tree, Peres & Zeitouni [137] have first proved (2.4.16) when the offspring distribution has exponential moments (see Theorem 1 [137], p. 596). Then, Aidékon & de Raphélis in [6] have proved (2.4.16) for supercritical single-type GW-tree under a second moment assumption and they also proved (2.4.17) in these cases (Theorem 1.1 [6], p. 645). In the same article, they extend (2.4.16) and (2.4.17) to RWs in random environment on GW-trees (see Theorem 6.1 [6], p. 660).

2.4.3 Snake metrics.

Snakes are path-valued processes that provide a nice parametrization of the spatial positions of a population whose genealogy is a continuum tree and that are scaling limits of branching random walks. Snake processes, and in particular the Brownian snake, has been introduced by Le Gall in [77] to study fine properties of super-Brownian motion. In this section, we first recall basic definitions on snakes, in a deterministic setting and in dimension 1. Then, we introduce a pseudo-metric derived from a snake, we study its continuity properties and we show that snake metrics actually yield real-trees. Finally, we prove elementary geometric properties of such real trees that are deterministic version of the cactus introduced by Curien, Le Gall & Miermont in [40].

Definition 2.4.17 We fix $\zeta \in (0, \infty)$ and we denote by $\mathbf{C}([0, \zeta], \mathbb{R})$ the space of the continuous functions from $[0, \zeta]$ to \mathbb{R} ; it is a Banach space when equipped with the uniform norm $\|\cdot\|_\infty$. We also denote by \mathbf{C}_0 the space of continuous functions from $[0, \infty)$ to \mathbb{R} that is a Polish space when equipped with the following metric:

$$\forall w, w' \in \mathbf{C}_0, \quad \delta_u(w, w') = \sum_{n \in \mathbb{N}} 2^{-n-1} \min(1, \sup_{r \in [0, n]} |w(r) - w'(r)|). \quad (2.4.18)$$

- (a) We denote by $\mathbf{C}([0, \zeta], \mathbf{C}_0)$ the space of the δ_u -continuous functions from $[0, \zeta]$ to \mathbf{C}_0 equipped with the distance Δ^* given by $\Delta^*(w, w') = \sup_{s \in [0, \zeta]} \delta_u(w_s(\cdot), w'_s(\cdot))$, for all $w, w' \in \mathbf{C}([0, \zeta], \mathbf{C}_0)$. We next equip the product space $\mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0)$ with the following distance: for all $(h, w), (h', w') \in \mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0)$,

$$\Delta((h, w), (h', w')) = \|h - h'\|_\infty + \Delta^*(w, w').$$

- (b) We denote by $\Sigma([0, \zeta])$ the space of the \mathbb{R} -valued *snakes*; namely, the elements (h, w) of $\mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0)$ that satisfy $h \geq 0$, $h(0) = h(\zeta) = 0$ and the following.

(b1) For all $s \in [0, \zeta]$ and for all $r \in [h(s), \infty)$, $w_s(r) = w_s(h(s)) =: \widehat{w}_s$.

(b2) For all $s_1, s_2 \in [0, \zeta]$ and for all $r \in [0, m_h(s_1, s_2)]$, $w_{s_1}(r) = w_{s_2}(r)$, where we recall from (2.4.8) the definition of $m_h(s_1, s_2)$.

We refer to (b2) as to the *snake property*. The function h is called the *lifetime process* and the function $(\widehat{w}_s)_{s \in [0, \zeta]}$ is called the *endpoint process* of the snake. \square

We easily check that $(\mathbf{C}([0, \zeta], \mathbf{C}_0), \Delta^*)$ and $(\mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0), \Delta)$ are Polish spaces and that $\Sigma([0, \zeta])$ is a Δ -closed subset.

The following lemma is used in the proof of Theorem 2.1.2.

Lemma 2.4.18 *Let $(h, w) \in \Sigma([0, \zeta])$. For all $\eta \in (0, \infty)$, we have $\omega_\eta(w) \leq 2\omega_\eta(\widehat{w})$, where*

$$\begin{aligned}\omega_\eta(w) &= \sup \{ \delta_u(w_s, w_{s'}) ; s, s' \in [0, \zeta] : |s - s'| \leq \eta \}, \\ \omega_\eta(\widehat{w}) &= \sup \{ |\widehat{w}_s - \widehat{w}_{s'}| ; s, s' \in [0, \zeta] : |s - s'| \leq \eta \}.\end{aligned}$$

Proof. For all $s \in [0, \zeta]$ and for all $r \in [0, h(s)]$ set $\alpha_{s,r} = \sup\{s' \in [0, s] : h(s') \leq r\}$ and $\beta_{s,r} = \inf\{s' \in [s, \zeta] : h(s') \leq r\}$. Then, Definition 2.4.17 (b2) implies that $w_s(r) = w_{s'}(r)$ for all $s' \in [\alpha_{s,r}, \beta_{s,r}]$ and in particular, $\widehat{w}_{\alpha_{s,r}} = \widehat{w}_{\beta_{s,r}} = w_s(r)$ since $r = h(\alpha_{s,r}) = h(\beta_{s,r})$.

Next, fix $s, s' \in [0, \zeta]$ such that $|s - s'| \leq \eta$. To simplify we set $m = m_h(s, s')$. Let $r_* \in [m, h(s)]$ be such that $|w_s(r_*) - w_s(m)| = \max_{r \in [m, h(s)]} |w_s(r) - w_s(m)|$. Suppose that $s \leq s'$ (resp. that $s' \leq s$), then it holds that $\beta_{s,r_*}, \beta_{s,m} \in [s, s']$ (resp. $\alpha_{s,r_*}, \alpha_{s,m} \in [s', s]$) and we get that $|w_s(r_*) - w_s(m)| = |\widehat{w}_{\beta_{s,r_*}} - \widehat{w}_{\beta_{s,m}}|$ (resp. $= |\widehat{w}_{\alpha_{s,r_*}} - \widehat{w}_{\alpha_{s,m}}|$). As a consequence, we have $\max_{r \in [m, h(s)]} |w_s(r) - w_s(m)| \leq \omega_\eta(\widehat{w})$. Similarly, $\max_{r \in [m, h(s')]} |w_{s'}(r) - w_{s'}(m)| \leq \omega_\eta(\widehat{w})$. By Definition 2.4.17 (b2), we get $w_s(m) = w_{s'}(m)$ and $\sup_{r \in [0, \infty)} |w_s(r) - w_{s'}(r)| \leq \max_{r \in [m, h(s)]} |w_s(r) - w_s(m)| + \max_{r \in [m, h(s')]} |w_{s'}(r) - w_{s'}(m)|$, which easily implies the desired result. ■

Definition 2.4.19 Let $\zeta \in (0, \infty)$ and let $(h, w) \in \Sigma([0, \zeta])$. Recall from (2.4.8) the definition of $m_h(\cdot, \cdot)$ and recall from Definition 2.4.17 (b1) the definition of \widehat{w} . For all $s_1, s_2 \in [0, \zeta]$, we set

$$\begin{aligned}M_{h,w}(s_1, s_2) &= \min \left(\min \{ w_{s_1}(r) ; r \in [m_h(s_1, s_2), h(s_1)] \}, \min \{ w_{s_2}(r) ; r \in [m_h(s_1, s_2), h(s_2)] \} \right) \\ \text{and } d_{h,w}(s_1, s_2) &= \widehat{w}_{s_1} + \widehat{w}_{s_2} - 2M_{h,w}(s_1, s_2).\end{aligned}\tag{2.4.19}$$

We call $d_{h,w}$ the *snake metric associated with (h, w)* (see Lemma 2.4.22 below). □

Lemma 2.4.20 *Let $\zeta \in [0, \infty)$ and for all $n \in \mathbb{N}$, let $(h_n, w^{(n)}) \in \Sigma([0, \zeta])$ that converges to (h, w) in $\mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0)$ equipped with Δ as in Definition 2.4.17 (a). Recall that $\|\cdot\|$ stands for the uniform norm on $\mathbf{C}([0, \zeta]^2, \mathbb{R})$. Then, $\lim_{n \rightarrow \infty} \|d_{h_n} - d_h\| = \lim_{n \rightarrow \infty} \|d_{h_n, w^{(n)}} - d_{h,w}\| = 0$ (see (2.4.8) and (2.4.19)).*

Proof. The first limit follows from (2.4.9). To prove the second one, we fix $\varepsilon \in (0, 1)$ and we set $a = 1 + \sup_{n \in \mathbb{N}} \max_{s \in [0, \zeta]} |h_n(s)|$. Let $n_1 \in \mathbb{N}$ such that $\Delta((h_n, w^{(n)}), (h, w)) \leq 2^{-a-2}\varepsilon$, for all $n \geq n_1$. Thus, for all $n \geq n_1$ and for all $s \in [0, \zeta]$,

$$\|w_s^{(n)} - w_s\|_\infty := \sup_{r \in [0, \infty)} |w_s^{(n)}(r) - w_s(r)| = \sup_{r \in [0, a]} |w_s^{(n)}(r) - w_s(r)| \leq \varepsilon.\tag{2.4.20}$$

Fix $s_1, s_2 \in [0, \zeta]$ and set $m_n = m_{h_n}(s_1, s_2)$ and $m = m_h(s_1, s_2)$. By (2.4.20), we get

$$\forall n \geq n_1, \forall j \in \{1, 2\}, \quad \left| \min_{r \in [m_n, h_n(s_j)]} w_{s_j}^{(n)}(r) - \min_{r \in [m, h(s_j)]} w_{s_j}(r) \right| \leq \|w_{s_j}^{(n)} - w_{s_j}\|_\infty \leq \varepsilon.\tag{2.4.21}$$

Next, for all uniformly continuous $g : [0, \infty) \rightarrow \mathbb{R}$, and for all $\eta \in (0, \infty)$, we use the notation $\omega(g, \eta) = \sup\{|g(s) - g(s')| ; s, s' \in [0, \infty) : |s - s'| \leq \eta\}$ for the η -uniform modulus of continuity of g . We recall that $\|h - h_n\|_\infty \leq \Delta((h_n, w^{(n)}), (h, w))$. Observe that for all $j \in \{1, 2\}$ and all $n \in \mathbb{N}$,

$$\left| \min_{r \in [m_n, h_n(s_j)]} w_{s_j}(r) - \min_{r \in [m, h(s_j)]} w_{s_j}(r) \right| \leq \omega(w_{s_j}, \|h - h_n\|_\infty).\tag{2.4.22}$$

By the definition (2.4.19) of $d_{h_n, w^{(n)}}$, by (2.4.20), (2.4.21) and (2.4.22), for all $n \geq n_1$, we get the following:

$$\left| d_{h_n, w^{(n)}}(s_1, s_2) - d_{h,w}(s_1, s_2) \right| \leq 4\varepsilon + 3\omega(w_{s_1}, \|h - h_n\|_\infty) + 3\omega(w_{s_2}, \|h - h_n\|_\infty).\tag{2.4.23}$$

Since w is uniformly δ_u -continuous on $[0, \zeta]$, there exist $\sigma_1, \dots, \sigma_p \in [0, \zeta]$ such that for all $s \in [0, \zeta]$, there exists $k \in \{1, \dots, p\}$, such that $\|w_s - w_{\sigma_k}\|_\infty \leq \varepsilon$, which implies the inequality $|\omega(w_s, \eta) - \omega(w_{\sigma_k}, \eta)| \leq 2\varepsilon$, for all $\eta \in (0, \infty)$. By (2.4.23), it implies that

$$\|d_{h_n, w^{(n)}} - d_{h, w}\| \leq 16\varepsilon + 6 \max_{1 \leq k \leq p} \omega(w_{\sigma_k}, \|h - h_n\|_\infty),$$

which implies $\limsup_{n \rightarrow \infty} \|d_{h_n, w^{(n)}} - d_{h, w}\| \leq 16\varepsilon$, for all $\varepsilon \in (0, \infty)$. It completes the proof of the lemma. \blacksquare

Remark 2.4.21 Let $(h, w) \in \Sigma([0, \zeta])$. Definition 2.4.17 (b) means that w is actually defined on the real tree (T_h, d_h, r_h, μ_h) coded by h (as defined in Remark 2.4.10). Indeed, let $s_1, s_2 \in [0, \zeta]$ be such that $d_h(s_1, s_2) = 0$; then $h(s_1) = h(s_2) = m_h(s_1, s_2)$ and Definition 2.4.17 (b) implies that $w_{s_1} = w_{s_2}$. Up to a slight abuse of notation, it therefore makes sense to define w on T_h as follows: for all $\sigma \in T_h$ and for all $s \in [0, \zeta]$ such that $\sigma = p_h(s)$, then

$$w_\sigma(\cdot) := w_s(\cdot) \quad \text{and} \quad \widehat{w}_\sigma := \widehat{w}_s, \quad (2.4.24)$$

where we recall that $p_h : [0, \zeta] \rightarrow T_h$ stands for the canonical projection. It is easy to check that $w : T_h \rightarrow \mathbf{C}_0$ is continuous. Moreover, Definition 2.4.17 (b) combined with the argument of the proof of Lemma 2.4.18 entails the following.

$$\forall \sigma \in T_h, \forall \gamma \in \llbracket r_h, \sigma \rrbracket, \quad w_\sigma(d_h(r_h, \gamma)) = \widehat{w}_\gamma.$$

We also get the following: let $\sigma_1, \sigma_2 \in T_h$ and $s_1, s_2 \in [0, \zeta]$ such that $p_h(s_i) = \sigma_i$, $i \in \{1, 2\}$; then

$$M_{h, w}(s_1, s_2) = \min_{\gamma \in \llbracket \sigma_1, \sigma_2 \rrbracket} \widehat{w}_\gamma \quad \text{and thus} \quad d_{h, w}(s_1, s_2) = \widehat{w}_{\sigma_1} + \widehat{w}_{\sigma_2} - 2 \min_{\gamma \in \llbracket \sigma_1, \sigma_2 \rrbracket} \widehat{w}_\gamma. \quad (2.4.25)$$

Up to a slight abuse of notation, it makes sense to view $M_{h, w}$ and $d_{h, w}$ as continuous functions from $T_h \times T_h$ to \mathbb{R} . \square

Lemma 2.4.22 Let $\zeta \in (0, \infty)$ and let $(h, w) \in \Sigma([0, \zeta])$. Let $d_{h, w}$ be the associated snake metric as in Definition 2.4.19. Then, $d_{h, w} \in \mathbf{MT}([0, \zeta])$. Namely, it is a continuous pseudo-metric on $[0, \zeta]$ that satisfies the four points inequality (2.4.1).

Proof. We first prove the continuity of $d_{h, w}$. To that end, for all $a, b \in [0, \infty)$ and all $s \in [0, \zeta]$ we set $\phi_{a, b}(s) = \min\{w_s(r) ; r \in [a \wedge b, a \vee b]\}$. Note that $|\phi_{a, b}(s) - \phi_{a, b}(s')| \leq \sup_{r \in [0, a \vee b]} |w_s(r) - w_{s'}(r)|$. Then, observe that for a fixed s , $(a, b) \mapsto \phi_{a, b}(s)$ is continuous. This easily implies that $(a, b, s) \mapsto \phi_{a, b}(s)$ is continuous. Therefore, $d_{h, w}$ is continuous on $[0, \zeta]^2$.

Let $s_1, s_2, s_3, s_4 \in [0, \zeta]$. We set $X_1 = M_{h, w}(s_1, s_2) + M_{h, w}(s_3, s_4)$, $X_2 = M_{h, w}(s_1, s_3) + M_{h, w}(s_2, s_4)$ and $X_3 = M_{h, w}(s_1, s_4) + M_{h, w}(s_2, s_3)$, so that $d_{h, w}$ satisfies the four points inequality (2.4.1) if $X_1 \geq \min\{X_2, X_3\}$, which is a consequence of

$$\#\{i \in \{1, 2, 3\} : X_i = \min(X_1, X_2, X_3)\} \geq 2. \quad (2.4.26)$$

To prove (2.4.26) it is convenient to work on (T_h, d_h, r_h) that is the pointed compact real tree coded by h as explained in Example 2.4.10: recall that $p_h : [0, \infty] \rightarrow T_h$ stands for the canonical projection and recall from (2.4.24) the definition of w_σ and \widehat{w}_σ for all $\sigma \in T_h$. For all $i \in \{1, \dots, 4\}$, we set $\sigma_i = p_h(s_i)$ and we recall from (2.4.25) that $M_{h, w}(\sigma_i, \sigma_j) = M_{h, w}(s_i, s_j)$. Recall from (2.4.4) the definition of branch points in T_h . Since (2.4.26) does not depend on a specific indexation of

the s_i , without loss of generality we can assume that $\gamma := \text{br}(\sigma_1, \sigma_2, \sigma_3) = \text{br}(\sigma_1, \sigma_2, \sigma_4)$, that $\gamma' = \text{br}(\sigma_3, \sigma_4, \sigma_1) = \text{br}(\sigma_3, \sigma_4, \sigma_2)$ and that

$$a := \min_{\sigma \in \llbracket \gamma, \sigma_1 \rrbracket} \widehat{w}_\sigma \leq b := \min_{\sigma \in \llbracket \gamma, \sigma_2 \rrbracket} \widehat{w}_\sigma, \quad d := \min_{\sigma \in \llbracket \gamma', \sigma_3 \rrbracket} \widehat{w}_\sigma \leq e := \min_{\sigma \in \llbracket \gamma', \sigma_4 \rrbracket} \widehat{w}_\sigma \quad \text{and} \quad a \leq d.$$

We also set $c := \min_{\sigma \in \llbracket \gamma, \gamma' \rrbracket} \widehat{w}_\sigma$. Then, $X_1 = a + d$, $X_2 = (a \wedge c) + (b \wedge c \wedge e)$ and $X_3 = (a \wedge c) + (b \wedge c \wedge d)$. We have four cases to consider; (i): if $c \leq a$, then $X_1 = a + d \geq X_2 = X_3 = 2c$; (ii): if $a \leq c \leq d$, then $X_1 = a + d \geq X_2 = X_3 = a + (b \wedge c)$; (iii): if $d \leq c$ and $b \leq d$, then $X_1 = a + d \geq X_2 = X_3 = a + b$; (iv): if $d \leq c$ and $d \leq b$, then $X_1 = X_3 = a + d \leq X_2 = a + (b \wedge c \wedge e)$. This proves (2.4.26) and it completes the proof of the lemma. \blacksquare

Definition 2.4.23 Let $\zeta \in (0, \infty)$ and let $(h, w) \in \Sigma([0, \zeta])$. Let $d_{h,w}$ be the associated snake metric (as in Definition 2.4.19). Since $d_{h,w} \in \text{MT}([0, \zeta])$ by Lemma 2.4.22, we denote by $T_{h,w} = [0, \zeta] / \sim_{d_{h,w}}$ the corresponding real tree and to simplify we denote by $p_{h,w} : [0, \zeta] \rightarrow T_{h,w}$ the canonical projection, by $r_{h,w} = p_{h,w}(0)$ the root of $T_{h,w}$, and by $\mu_{h,w}$ the measure on $T_{h,w}$ induced by the Lebesgue measure on $[0, \zeta]$ via $p_{h,w}$: namely,

$$\int_{T_{h,w}} f(x) \mu_{h,w}(dx) = \int_0^\zeta f(p_{h,w}(s)) ds,$$

for all continuous $f : T_{h,w} \rightarrow \mathbb{R}$. Since the pseudo-metric $d_{h,w} : [0, \zeta]^2 \rightarrow [0, \infty)$ is continuous, so is $p_{h,w}$ and $(T_{h,w}, d_{h,w}, r_{h,w}, \mu_{h,w})$ is a pointed measured compact real tree that we call the *snake tree* associated with (h, w) . By Lemma 2.4.20 and Proposition 2.4.14, $(h, w) \in \Sigma([0, \zeta]) \mapsto T_{h,w} \in \mathbb{M}$ is $(\Delta, \delta_{\text{GHP}})$ -continuous. \square

Remark 2.4.24 Let $(h, w) \in \Sigma([0, \zeta])$ and $\alpha, a, b \in (0, \infty)$. For all $s \in [0, b\zeta]$, we set $h'(s) = \alpha h(s/b)$ and $w'_s(r) = aw_{s/b}(r/\alpha)$. Then $(h', w') \in \Sigma([0, b\zeta])$ and thanks to Remark 2.4.5, we easily check that $(T_{h'}, d_{h'}, r_{h'}, \mu_{h'})$ is isometric to $(T_h, \alpha d_h, r_h, b\mu_h)$, and $(T_{h',w'}, d_{h',w'}, r_{h',w'}, \mu_{h',w'})$ is isometric to $(T_{h,w}, ad_{h,w}, r_{h,w}, b\mu_{h,w})$. \square

Remark 2.4.25 Let $(h, w) \in \Sigma([0, \zeta])$; let (T_h, d_h, r_h, μ_h) and $(T_{h,w}, d_{h,w}, r_{h,w}, \mu_{h,w})$ be the compact real trees coded by resp. h and (h, w) and recall that p_h and $p_{h,w}$ stand for the canonical projections from $[0, \zeta]$ to resp. T_h and $T_{h,w}$. Observe that it actually makes sense to define a function $y : T_h \rightarrow T_{h,w}$ by setting

$$\forall s \in [0, \zeta], \quad y(p_h(s)) = p_{h,w}(s). \quad (2.4.27)$$

It is easy to deduce from (2.4.25) that

$$d_{h,w}(y(\sigma), y(\sigma')) = \widehat{w}_\sigma + \widehat{w}_{\sigma'} - 2 \min_{\gamma \in \llbracket \sigma, \sigma' \rrbracket} \widehat{w}_\gamma, \quad (2.4.28)$$

It implies that y is continuous and surjective. Note that $\mu_{h,w}$ is the image of μ_h via y . \square

We next prove two results that deal with basic geometric properties of snake trees. The first one provides conditions for a snake tree to be a continuum real tree.

Lemma 2.4.26 Let $(h, w) \in \Sigma([0, \zeta])$. We set

$$U_{h,w} = \left\{ s \in [0, \zeta] : h(s) > 0 \quad \text{and} \quad \forall \varepsilon \in (0, h(s)), \quad \inf_{r \in [h(s)-\varepsilon, h(s)]} w_s(r) < \widehat{w}_s \right\} \quad (2.4.29)$$

and we assume that $[0, \zeta] \setminus U_{h,w}$ is Lebesgue-negligible. We also assume that the tree (T_h, d_h, r_h, μ_h) coded by h is a continuum real tree as defined in (2.4.6). Then, the snake tree $(T_{h,w}, d_{h,w}, r_{h,w}, \mu_{h,w})$ is also a continuum real tree.

Proof. We set $V = U_{h,w} \cap p_h^{-1}(\text{Lf}(T_h))$. Since $[0, \zeta] \setminus U_{h,w}$ is Lebesgue-negligible and since T_h is a continuum real tree, $[0, \zeta] \setminus V$ is Lebesgue-negligible. To conclude the proof we are going to show that $V \subset U$, where U is defined in (2.4.7) with d replaced by $d_{h,w}$, so that Lemma 2.4.8 implies the desired result. To that end, we fix $s \in V$ and $s' \in [0, \zeta]$ be distinct from s . Suppose that $h(s) = m_h(s, s')$; if there is $s'' \in (s \wedge s', s \vee s')$ such that $h(s'') > m_h(s, s')$, then we have $p_h(s) \in \llbracket r, p_h(s'') \llbracket$ which is impossible since $p_h(s)$ has to be a leaf of T_h . Therefore, if $h(s) = m_h(s, s')$, then $h(s'') = m_h(s, s')$ for all $s'' \in [s \wedge s', s \vee s']$; thus $[s \wedge s', s \vee s'] \subset p_h^{-1}(\{p_h(s)\})$ and μ_h would have an atom at $p_h(s)$, which is impossible since T_h is a continuum real tree. Thus, we have proved that $h(s) > m_h(s, s')$. Therefore $\min_{r \in [m_h(s, s'), h(s)]} w_s(r) < \widehat{w}_s$ since $s \in U_{h,w}$. Next, we set $b = \min\{w_s(r); r \in [0, m_h(s, s')]\}$; by Definition 2.4.17 (b), b is also equal to $\min\{w_{s'}(r); r \in [0, m_h(s, s')]\}$. We also set $a = \min\{w_s(r); r \in [m_h(s, s'), h(s)]\}$ and $a' = \min\{w_{s'}(r); r \in [m_h(s, s'), h(s')]\}$. Then,

$$\begin{aligned} d_{h,w}(0, s) + d_{h,w}(s, s') - d_{h,w}(0, s') &= 2\widehat{w}_s + 2M_{h,w}(0, s') - 2M_{h,w}(0, s) - 2M_{h,w}(s, s') \\ &= 2\widehat{w}_s - 2a + 2(a + (b \wedge a') - (b \wedge a) - (a \wedge a')). \end{aligned}$$

We next check that $q := a + (b \wedge a') - (b \wedge a) - (a \wedge a') \geq 0$. Indeed, if $a \leq a'$, $q = (b \wedge a') - (b \wedge a) \geq 0$; if $b \leq a' \leq a$, $q = a - a' \geq 0$; if $a' \leq b \leq a$, $q = a - b \geq 0$; if $a' \leq a \leq b$, $q = 0$. Since we have proved that $a < \widehat{w}_s$, we get $d_{h,w}(0, s) + d_{h,w}(s, s') > d_{h,w}(0, s')$. This implies that $V \subset U$, which completes the proof. \blacksquare

Recall from (2.4.5) the definition of the degree of a point in a real tree and recall that a point is a branch point if its degree is ≥ 3 . The following lemma provides conditions ensuring that snake trees have only binary branch points.

Lemma 2.4.27 *Let $(h, w) \in \Sigma([0, \zeta])$. Let T_h and $T_{h,w}$ be the compact real trees coded by resp. h and (h, w) . Recall from Remark 2.4.21 that w can actually be defined on T_h . Let \mathcal{D} be a countable dense set of points of T_h . We consider two cases. In the first case, we assume the following.*

- (a) $\widehat{w}_0 = 0$ and $\widehat{w}_s \in [0, \infty)$ for all $s \in [0, \zeta]$.
- (b) For all distinct $\sigma, \sigma' \in \mathcal{D}$ such that $\min_{\gamma \in \llbracket \sigma, \sigma' \llbracket} \widehat{w}_\gamma > 0$, there is at most one $\sigma_0 \in \llbracket \sigma, \sigma' \llbracket$ such that $\widehat{w}_{\sigma_0} = \min_{\gamma \in \llbracket \sigma, \sigma' \llbracket} \widehat{w}_\gamma$ and when there is one, it is never a branch point of T_h .

Then, for all $x \in T_{h,w} \setminus \{r_{h,w}\}$, $\deg(x) \in \{1, 2, 3\}$. In the second case, we assume the following

- (a') $\widehat{w}_0 = 0$.
- (b') For all distinct $\sigma, \sigma' \in \mathcal{D}$, there is at most one $\sigma_0 \in \llbracket \sigma, \sigma' \llbracket$ such that $\widehat{w}_{\sigma_0} = \min_{\gamma \in \llbracket \sigma, \sigma' \llbracket} \widehat{w}_\gamma$ and when there is one, it is never a branch point of T_h .

Then, for all $x \in T_{h,w}$, $\deg(x) \in \{1, 2, 3\}$.

Proof. Let us consider the first case. We assume (a) and (b). We shall argue by contradiction. To that end, suppose that $x_0 \in T_{h,w} \setminus \{r_{h,w}\}$ is such that $\deg(x_0) \geq 4$. Recall from (2.4.27) in Remark 2.4.25 the definition of the continuous surjective function $y : T_h \rightarrow T_{h,w}$. Then, choose $x_1, x_2, x_3 \in y(\mathcal{D})$ such that $r_{h,w}, x_1, x_2, x_3$ are in distinct connected components of $T_{h,w} \setminus \{x_0\}$. Let $\sigma_0 \in T_h$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{D}$ be such that $y(\sigma_i) = x_i$, $i \in \{0, 1, 2, 3\}$. We first claim that for all $i, j \in \{1, 2, 3\}$ distinct,

$$0 < d_{h,w}(r_{h,w}, x_0) = \widehat{w}_{\sigma_0} = \min_{\gamma \in \llbracket \sigma_0, \sigma_i \llbracket} \widehat{w}_\gamma \leq \inf_{\gamma \in \llbracket \sigma_i, \sigma_j \llbracket} \widehat{w}_\gamma. \quad (2.4.30)$$

Proof of (2.4.30). Note that $y(r_h) = r_{h,w}$. Since $x_0 \neq r_{h,w}$, by (a) and (2.4.28), we get $0 < d_{h,w}(x_0, r_{h,w}) = \widehat{w}_{\sigma_0} + \widehat{w}_{r_h} - 2 \min_{\llbracket r_h, \sigma_0 \llbracket} \widehat{w} = \widehat{w}_{\sigma_0}$. Fix $i \in \{1, 2, 3\}$. Since $x_0 \in \llbracket r_{h,w}, x_i \llbracket$, the identity (2.4.28) implies that $\widehat{w}_{\sigma_i} = d_{h,w}(r_{h,w}, x_i) = d_{h,w}(r_{h,w}, x_0) + d_{h,w}(x_0, x_i)$ and thus

$\widehat{w}_{\sigma_i} = 2\widehat{w}_{\sigma_0} + \widehat{w}_{\sigma_i} - 2 \min_{\gamma \in \llbracket \sigma_0, \sigma_i \rrbracket} \widehat{w}_\gamma$, which implies the second equality in (2.4.30). We complete the proof of (2.4.30) by noting that $\llbracket \sigma_i, \sigma_j \rrbracket \subset \llbracket \sigma_0, \sigma_i \rrbracket \cup \llbracket \sigma_0, \sigma_j \rrbracket$. \square

For all $i \in \{1, 2, 3\}$, let C_i be the connected component of $T_{h,w} \setminus \{x_0\}$ that contains x_i . By connectivity, there are $\gamma_1, \gamma_2 \in \llbracket \sigma_1, \sigma_2 \rrbracket$ such that $y(\gamma_1) = y(\gamma_2) = x_0$ and $\llbracket \sigma_i, \gamma_i \rrbracket \subset y^{-1}(C_i)$, $i \in \{1, 2\}$. Since $y(\gamma_i) = x_0$, $\widehat{w}_{\sigma_0} + \widehat{w}_{\gamma_i} - 2 \min_{\gamma \in \llbracket \sigma_0, \gamma_i \rrbracket} \widehat{w}_\gamma = 0$, which implies that $\widehat{w}_{\gamma_i} = \widehat{w}_{\sigma_0}$. By (2.4.30), we get $\widehat{w}_{\gamma_1} = \widehat{w}_{\gamma_2} = \min_{\gamma \in \llbracket \sigma_1, \sigma_2 \rrbracket} \widehat{w}_\gamma$. Then Assumption (b) implies that $\gamma_1 = \gamma_2$.

We next introduce $\beta = \text{br}(\sigma_1, \sigma_2, \sigma_3)$, the branch point of $\sigma_1, \sigma_2, \sigma_3$ as defined in (2.4.4). Without loss of generality we can assume that $\gamma_1 \in \llbracket \sigma_1, \beta \rrbracket$ and thus $\gamma_1 \in \llbracket \sigma_1, \sigma_3 \rrbracket$. By connectivity, there is $\gamma_3 \in \llbracket \sigma_1, \sigma_3 \rrbracket$ such that $y(\gamma_3) = x_0$ and $\llbracket \sigma_3, \gamma_3 \rrbracket \subset y^{-1}(C_3)$. It implies that $\widehat{w}_{\gamma_1} = \widehat{w}_{\gamma_3} = \widehat{w}_{\sigma_0} = \min_{\gamma \in \llbracket \sigma_1, \sigma_3 \rrbracket} \widehat{w}_\gamma$. Then Assumption (b) implies that $\gamma_3 = \gamma_1 = \gamma_2$. By definition, the subsets $\llbracket \sigma_i, \gamma_i \rrbracket$, $i \in \{1, 2, 3\}$ are pairwise disjoint and therefore γ_3 is a branch point of T_h (it is actually $\text{br}(\sigma_1, \sigma_2, \sigma_3)$) such that $\widehat{w}_{\gamma_3} = \min_{\llbracket \sigma_1, \sigma_3 \rrbracket} \widehat{w}_\gamma$ which is not possible by Assumption (b). This proves the first point of the lemma by contradiction. The exact same arguments can be used to prove the second case of the lemma. \blacksquare

2.4.4 Stable Lévy trees.

In this section that contains no new result, we briefly recall basic definitions and properties on stable Lévy trees. Lévy trees are a class of random compact metric spaces that have been introduced by Le Gall & Le Jan in [81] (and further studied in Le Gall & Duquesne [61]) as the genealogy of continuous state branching processes. Among stable trees, Aldous' continuum random tree corresponds to the Brownian case (see Aldous [9] and here below). Stable trees (and more generally Lévy trees) are the scaling limit of Galton–Watson trees as recalled in Introduction (see also below).

More precisely, let $\gamma \in (1, 2]$ be the index of a spectrally positive stable Lévy process $X = (X_s)_{s \in [0, \infty)}$: namely, the law of X is characterised by its Laplace exponent that is given by

$$\forall s, \lambda \in [0, \infty), \quad \mathbf{E}[\exp(-\lambda X_s)] = \exp(s\lambda^\gamma).$$

Note that X is a Brownian motion when $\gamma = 2$ and we shall refer to this case as the *Brownian case*. As shown in [81] (see also [61], Chapter 1), there exists a *continuous process* $H(X) = (H_s(X))_{s \in [0, \infty)}$ such that for any $s \in [0, \infty)$,

$$H_s(X) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_s^r < X_r < I_s^r + \varepsilon\}} dr, \quad (2.4.31)$$

where I_s^r stands for $\inf_{r \leq u \leq s} X_u$ and the convergence in (2.4.31) holds in probability. The process $H(X)$ is the γ -stable height process. Note that in the Brownian case, $H(X)$ is simply the reflected Brownian motion $s \mapsto 2(X_s - I_s^0)$. From the scaling property of X and from (2.4.31), we see that for any $r \in (0, \infty)$, $(r^{(\gamma-1)/\gamma} H_{t/r}(X))_{t \in [0, \infty)}$ has the same distribution as $H(X)$.

As in the discrete setting, the process $H(X)$ encodes a family of continuum real trees: each excursion of $H(X)$ above 0 corresponds to an excursion of X above its infimum (it is obvious in the Brownian case) and each excursion of $H(X)$ above 0 corresponds to a single continuum real tree of the family. The scaling property of $H(X)$ yields the following definition for the normalized excursion of $H(X)$ that is provided in Duquesne [50] (see p.1005).

Set $\ell_1 = \max\{s \in [0, 1] : X_s = I_s\}$, $r_1 = \min\{s \in [1, \infty) : X_s = I_s\}$ and $\zeta_1 = r_1 - \ell_1$. Then, we set:

$$\forall s \in [0, 1], \quad H_s := \zeta_1^{-(1-\frac{1}{\gamma})} H_{\ell_1 + s\zeta_1}(X).$$

The process H is taken as the definition of the normalized excursion of the γ -stable height process.

As shown in Theorem 1.4.4 in Le Gall & Duquesne [61], for all $\alpha \in (0, 1 - \frac{1}{\gamma})$, \mathbf{P} -a.s. $H(X)$ is α -locally Hölder-continuous. Thus, the same holds true for H : namely,

$$\text{for all } \alpha \in (0, 1 - \frac{1}{\gamma}), \text{ a.s. } H \text{ is } \alpha\text{-Hölder continuous.} \quad (2.4.32)$$

Normalized stable Lévy trees. We call *normalized γ -stable Lévy tree* the real tree

$$(T_H, d_H, r_H, \mu_H)$$

coded by the function H , as defined in (2.4.8). Recall from (2.4.9) that $h \mapsto d_h$ is a 4-Lipschitz function from $\mathbf{C}([0, 1], \mathbb{R})$ to $\mathbf{C}([0, 1]^2, \mathbb{R})$, both equipped with uniform norms. Thus, d_H is a measurable random element of $\mathbf{MT}([0, 1])$ and by Proposition 2.4.14, T_H is a measurable random element of the Polish space $(\mathbb{M}, \delta_{\text{GHP}})$ as mentioned in (2.4.11).

Let us briefly recall some geometric properties of T_H . One can prove that a.s. T_H is a continuum real tree (as defined in (2.4.6)) and that the set of its branch points is a countable dense set; moreover we recall from Theorem 4.6 in Le Gall & Duquesne [57] (p. 583) the following result.

$$\text{A.s. for all } \sigma \in T_H, \deg(\sigma) \in A_\gamma, \text{ where } A_\gamma = \begin{cases} \{1, 2, \infty\} & \text{if } \gamma \in (1, 2), \\ \{1, 2, 3\} & \text{if } \gamma = 2. \end{cases} \quad (2.4.33)$$

The contour of a discrete trees. We briefly recall how to code discrete trees by various functions. Let $t \in \mathbb{T}$ be a *finite* (rooted ordered) tree as in Definition 2.2.1. To simplify, we set $\#t = n$; since t is finite, we can list the vertices of t in the lexicographical order: $u_0 = \emptyset <_t u_1 <_t \dots <_t u_{n-1}$; for all $j \in \{0, \dots, n-1\}$, we set $H_j(t) = |u_j|$ that is the height of the j -th vertex of t . By convenience we also set $H_n(t) = 0$. The function $(H_k(t))_{0 \leq k \leq n}$ is the *discrete height function of t* . Note that $H(t)$ entirely encodes t .

We also introduce another coding function known as the *contour* (or the *depth-first exploration*) function of t . Informally, we embed t into the clockwise oriented half plane so that order on siblings corresponds to orientation; we think of a particle that visits the tree at unit speed, that starts at the root and that goes from the left to right, backtracking as less as possible; we denote by $v(k)$ the vertex visited at time k during this depth-first exploration. The particle crosses exactly twice each edge of t : once upwards and once downwards; so, the total time needed to go back to the root is $2n - 2$, that is twice the number of edges. Namely, $\{(v(k), v(k+1)); 0 \leq k \leq 2n-2\} = \{(u, \overleftarrow{u}), (\overleftarrow{u}, u); u \in t \setminus \{\emptyset\}\}$. Then, for all $k \in \{0, \dots, 2n-2\}$, we set $C_k(t) = |v(k)|$ and we also set $C_{2n-1}(t) = C_{2n}(t) = 0$. We call *contour function of t* the linear interpolation of the $C_k(t)$ that we still denote by $(C_s(t))_{s \in [0, 2n]}$.

The discrete height function and the contour function of t are related as follows. For all $0 \leq j \leq n$, we set $b_j = 2j - H_j(t)$; we easily check that (b_j) is an increasing sequence from 0 to $2n$. Moreover, for all $j < n-1$, observe that for all $s \in [b_j, b_{j+1}]$,

$$C_s(t) = \begin{cases} H_j(t) - (s - b_j) & \text{if } s \in [b_j, b_{j+1} - 1], \\ s - b_{j+1} + H_{j+1}(t) & \text{if } s \in [b_{j+1} - 1, b_{j+1}] \end{cases} \quad (2.4.34)$$

and that $C_s(t) = H_{n-1}(t) - (s - b_{n-1})$ if $s \in [b_{n-1}, b_n]$.

The pointed measured compact real tree coded by the contour function of t is described as follows: to simplify notation, we set $C_s = C_{2s}(t)$, $s \in [0, n]$ and we recall from Example 2.4.9 the definition of $(\tilde{t}, \tilde{d}_{\text{gr}}^t)$, the compact real tree spanned by the graph-tree t . It is easy to check that the pointed compact measured real tree (T_C, d_C, r_C, μ_C) coded by C is isometric to $(\tilde{t}, \tilde{d}_{\text{gr}}^t, \emptyset, \tilde{\mathfrak{m}})$ where $\tilde{\mathfrak{m}} = \delta_\emptyset + \text{Length}$, where δ_\emptyset stands for the Dirac mass at the root \emptyset .

We next recall the following limit theorem from Duquesne [50] (Theorem 3.1, p. 1006).

Theorem 2.4.28 (Theorem 3.1 [50]) *We fix $\gamma \in (1, 2]$. Let τ be a Galton–Watson tree whose offspring distribution satisfies **(H)** as in (2.1.1). Then, there exists a nondecreasing sequence of*

positive real numbers $(a_n)_{n \in \mathbb{N}}$ that tends to ∞ such that

$$\left(\frac{1}{a_n} H_{\lfloor ns \rfloor}(\tau); \frac{1}{a_n} C_{2ns}(\tau) \right)_{s \in [0,1]} \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \\ \xrightarrow[n \rightarrow \infty]{(\text{law})} (H_s, H_s)_{s \in [0,1]}$$

where H is the normalized excursion of the γ -stable height process.

2.4.5 One-dimensional reflected Brownian snakes.

In this section we briefly introduce normalized one-dimensional (reflected) Brownian snake with γ -stable branching mechanism (i.e. whose lifetime process is a γ -stable height process). For more details we refer to the monograph of Le Gall & Duquesne [61] (Chapter 4, pp. 107-149).

Let I be an interval of \mathbb{R} whose interior is not empty. Let $(Z_r^x)_{r \in [0, \infty)}$ be an I -valued continuous Markov process starting at $Z_0^x = x \in I$. We shall restrict to the two following cases:

- either I is \mathbb{R} and Z^x is a *Brownian motion*,
- or $I = [0, \infty)$ and Z^x is a *Brownian motion that is reflected at 0*.

Recall from Definition 2.4.17 that \mathbf{C}_0 stands for the space of continuous functions from $[0, \infty)$ to \mathbb{R} equipped with (Polish) topology of uniform convergence on all compact subsets. We let $w \in \mathbf{C}_0$ be I -valued and we fix two nonnegative real numbers $b \geq a$. We then denote by $R_{a,b}(w, dw')$ the law on \mathbf{C}_0 of the process $W(\cdot)$ that is defined as follows:

- for all $r \in [0, a]$, $W(r) = w(r)$;
- the process $(W(a+r))_{r \in [0, \infty)}$ has the same law as $(Z_{r \wedge (b-a)}^{w(a)})_{r \in [0, \infty)}$.

It is clear that $(a, b, w) \mapsto R_{a,b}(w, dw')$ is weakly continuous on \mathbf{C}_0 .

We next fix $\zeta \in (0, \infty)$ and $h \in \mathbf{C}([0, \zeta], [0, \infty))$ such that $h(0) = h(\zeta) = 0$. Recall from (2.4.8) the definition of $m_h(\cdot, \cdot)$. It is easy to check that one defines a \mathbf{C}_0 -valued process $(W_s(\cdot))_{s \in [0, \zeta]}$ by specifying its finite dimensional marginals as follows:

- $W_0(r) = 0$, $r \in [0, \infty)$.
- For all $0 \leq s_1 \leq \dots \leq s_p \leq \zeta$, $(W_{s_k})_{1 \leq k \leq p}$ has law

$$R_{m_h(0, s_1), h(s_1)}(W_0, dw_1) R_{m_h(s_1, s_2), h(s_2)}(w_1, dw_2) \dots R_{m_h(s_{p-1}, s_p), h(s_p)}(w_{p-1}, dw_p). \quad (2.4.35)$$

When the Z^x are Brownian motions, the resulting collection of \mathbf{C}_0 -valued r.v. $(W_s)_{s \in [0, \zeta]}$ is a *Brownian snake with lifetime process h and initial position 0*. When the Z^x are reflected Brownian motions, W is a *reflected Brownian snake with lifetime process h and initial position 0*. The following \mathbb{R} -valued process

$$\forall s \in [0, \zeta], \quad \widehat{W}_s = W_s(h(s)),$$

is the *endpoint process* of the snake W .

Remark 2.4.29 If $(W_s)_{s \in [0, \zeta]}$ is a Brownian snake with lifetime process h and initial position 0, then clearly $(|W_s|)_{s \in [0, \zeta]}$ is a reflected Brownian snake with lifetime process h and initial position 0. \square

Let us discuss the regularity of Brownian snakes. First, denote by M_{s_1, \dots, s_p}^h the law on \mathbf{C}_0^p defined by (2.4.35). It is easy to check that $(s_1, \dots, s_p; h) \in [0, \zeta]^p \times \mathbf{C}([0, \zeta], \mathbb{R}) \mapsto M_{s_1, \dots, s_p}^h$ is weakly continuous on \mathbf{C}_0^p . Although h is continuous, the process W may not be continuous in general. However, the arguments used in the proof of Proposition 4.4.1 in Le Gall & Duquesne [61] (Chapter 4 p. 127) show that if h is Hölder, then W has a continuous modification.

More precisely, for all $\zeta, c \in (0, \infty)$ and all $\alpha \in (0, 1]$, denote by $\text{Hö}1_{c,\alpha}([0, \zeta])$ the set of functions $h \in \mathbf{C}([0, \zeta], [0, \infty))$ such that $h(0) = h(\zeta) = 0$ and $|h(s) - h(s')| \leq c|s - s'|^\alpha$, for all $s, s' \in [0, \zeta]$. From the proof of Proposition 4.4.1 in Le Gall & Duquesne [61], we easily get the following lemma.

Lemma 2.4.30 *Let $\zeta \in (0, \infty)$ and let $h \in \text{Hö}1_{c,\alpha}([0, \zeta])$. Let $(W_s)_{s \in [0, \zeta]}$ be a Brownian snake (or a reflected Brownian snake) with lifetime process h and initial position 0. Then, for all $p \in (1, \infty)$,*

$$\forall s, s' \in [0, \zeta], \quad \mathbf{E} \left[\sup_{r \in [0, \infty)} |W_s(r) - W_{s'}(r)|^p \right] \leq c_p |s - s'|^{\alpha p/2},$$

where $c_p = (2p\sqrt{c}/(p-1))^p \mathbf{E}[|Z_1^0|^p]$. By Kolmogorov's criterion, W admits a modification that is $\alpha/2$ -Hölder with respect to the distance δ_u on \mathbf{C}_0 (see (2.4.18) for the definition of δ_u).

We now assume that h is Hölder; we keep denoting W the continuous version of the (possibly reflected) Brownian snake with lifetime process h and we briefly discuss properties of W for later use.

First observe that a.s. (h, W) is a snake as in Definition 2.4.17 (b). Then, note that for all $s \in [0, \zeta]$, $(W_s(r))_{r \in [0, \infty)}$ has the same distribution as $(Z_{r \wedge h(s)}^0)_{r \in [0, \infty)}$, which easily implies the following: for all bounded and measurable $F: \mathbf{C}_0 \rightarrow \mathbb{R}$,

$$\mathbf{E} \left[\int_0^\zeta F(W_s(\cdot)) ds \right] = \int_0^\zeta \mathbf{E} [F((Z_{r \wedge h(s)}^0)_{r \in [0, \infty)})] dr. \quad (2.4.36)$$

To define the normalized (reflected) Brownian snake with γ -stable branching mechanism, we need to show that the law of a (reflected) Brownian snake is a Borel-measurable function of its lifetime process. More precisely, since $\text{Hö}1_{c,\alpha}([0, \zeta])$ is a compact subset of $\mathbf{C}([0, \zeta], [0, \infty))$ equipped with uniform convergence and since $\text{Hö}1_{c,\alpha}([0, \zeta]) \subset \text{Hö}1_{\zeta^{\alpha-\alpha'}, c', \alpha'}([0, \zeta])$ for all $0 < \alpha' \leq \alpha$ and all $c' \geq c > 0$, the set of Hölder lifetime-functions $\text{Hö}1 := \bigcup_{c \in (0, \infty)} \bigcup_{\alpha \in (0, 1]} \text{Hö}1_{c,\alpha}([0, \zeta])$ is a countable union of compact subsets and therefore a Borel subset of $\mathbf{C}([0, \zeta], [0, \infty))$. For all $h \in \text{Hö}1$, denote by $P_h(dW)$ (resp. $P_h^+(dW)$) the law on $\mathbf{C}([0, \infty), \mathbf{C}_0)$ of the Brownian snake (resp. of the reflected Brownian snake) W with lifetime-function h and initial position 0 and extend $h \mapsto P_h$ (resp. $h \mapsto P_h^+$) to $\mathbf{C}([0, \zeta], [0, \infty))$ trivially by taking P_h (resp. P_h^+) equal to the Dirac mass at the null function if h is not a Hölder lifetime-function. Now recall that for all integers $p \geq 1$ and for all bounded continuous $F: \mathbf{C}_0^p \rightarrow \mathbb{R}$, the map $(s_1, \dots, s_p; h) \mapsto \int F(W_{s_1}, \dots, W_{s_p}) P_h(dW)$ is continuous; the same holds with P_h^+ . Since the Borel sigma field of $\mathbf{C}([0, \zeta], \mathbf{C}_0)$ equipped with Δ^* (as in Definition 2.4.17 (a)) is generated by the finite dimensional marginals, a standard monotone class argument implies that for all Borel subsets B of the Polish space $(\mathbf{C}([0, \zeta], \mathbf{C}_0), \Delta^*)$, the map $h \in \mathbf{C}([0, \zeta], [0, \infty)) \mapsto P_h(B) \in [0, \infty)$ is Borel-measurable. Namely:

$P_h(dW)$ is a Borel regular kernel from $\mathbf{C}([0, \zeta], [0, \infty))$ equipped with the uniform norm to the Polish space $(\mathbf{C}([0, \zeta], \mathbf{C}_0), \Delta^)$. The same holds with P_h^+ .*

Definition 2.4.31 Let $\gamma \in (1, 2]$. Let H be the normalized excursion of the γ -stable height process. Recall from (2.4.32) that H is a.s. Hölder-continuous. Thus, it makes sense to define the *Brownian snake* (resp. *reflected Brownian snake*) with γ -stable branching mechanism as the r.v. (H, W) with values in $\mathbf{C}([0, \zeta], \mathbb{R}) \times \mathbf{C}([0, \zeta], \mathbf{C}_0)$ whose regular conditional distribution given H is $P_H(dW)$ (resp. $P_H^+(dW)$), as defined previously. \square

Let (H, W) be a normalized Brownian snake with γ -stable branching mechanism. Recall that the real tree (T_H, d_H, r_h, μ_H) coded by H is the γ -stable Lévy tree. Since a.s. $(H, W) \in \Sigma([0, 1])$, (2.4.24) in Remark 2.4.21 applies to (H, W) and it makes sense (up to a slight abuse of notation) to define W and \widehat{W} on T_H . To prove geometric properties of the γ -stable (reflected) Brownian cactus, we shall need the following result that is recalled from Lemma 6.4 in Le Gall & Duquesne [57] (p. 600).

Lemma 2.4.32 (Lemma 6.4 [57]) *Let (H, W) be a normalized Brownian snake with γ -stable branching mechanism. Then, conditionally given H , $(\widehat{W}_\sigma)_{\sigma \in T_H}$ is a centered Gaussian process whose covariance is characterised by the following:*

$$\forall \sigma, \sigma' \in T_H, \quad \int P_H(dW) |\widehat{W}_{\sigma'} - \widehat{W}_\sigma|^2 = d_H(\sigma, \sigma'). \quad (2.4.37)$$

Moreover, for all $\varepsilon \in (0, 1/2)$, conditionally given H , $\sigma \in T_H \mapsto \widehat{W}_\sigma$ is $(\frac{1}{2} - \varepsilon)$ -Hölder-continuous.

2.4.6 Reflected Brownian cactus with stable branching mechanism.

Definition 2.4.33 Let $\gamma \in (1, 2]$ and let (H, W) be a normalized Brownian (resp. reflected Brownian) snake with γ -stable branching mechanism as in Definition 2.4.31. Recall the notation $\Sigma([0, 1])$ from Definition 2.4.17 (b) and recall that $(H, W) \in \Sigma([0, 1])$. The *Brownian (resp. reflected Brownian) cactus with γ -stable mechanism* is the pointed measured compact real tree $(T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})$ as specified in Definitions 2.4.19 and 2.4.23. \square

Recall from (2.4.6) the definition of continuum real trees. Recall from (2.4.5) the definition of the degree of a point in a real tree.

Proposition 2.4.34 *Let $\gamma \in (1, 2]$. Let (H, W) be a normalized Brownian (resp. reflected Brownian) snake with γ -stable branching mechanism and let $T_{H,W}$ the corresponding cactus as in Definition 2.4.33. Then, the following holds true.*

- (i) $(T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})$ is a.s. a continuum real tree.
- (ii) A.s. for all $x \in T_{H,W} \setminus \{r_{H,W}\}$, $\deg(x) \in \{1, 2, 3\}$. Moreover, if $T_{H,W}$ is a reflected Brownian cactus, then $\deg(r_{H,W}) = \infty$ and if $T_{H,W}$ is a Brownian cactus, then $\deg(r_{H,W}) = 1$.

Remark 2.4.35 As recalled in (2.4.33), when $\gamma \in (1, 2)$, the γ -stable Lévy tree T_H has infinite branch points: namely, there is a countable dense set of points $\sigma \in T_H$ such that $\deg(\sigma) = \infty$. However, note that the corresponding Brownian cactus $T_{H,W}$ has only binary branch points (except possibly at the root in the reflected case). \square

Remark 2.4.36 As already mentioned, the Brownian Cactus has been introduced by Curien, Le Gall & Miermont in [40] to study planar maps: roughly speaking the Brownian cactus corresponds to the case of a quadratic branching mechanism $\gamma = 2$ and the spatial motion of the snake is a (unreflected) linear Brownian motion. In this case they proved much finer geometrical results: a.s. the upper-local density for typical points is 4 (Proposition 5.1 [40] p. 364) and the Hausdorff dimension is 4 (Corollary 5.3 [40] p. 365). See also the article by Le Gall [116] where the level sets of the Brownian cactus are studied to derive results on the Brownian maps. \square

Proof of Proposition 2.4.34. To prove (i), we use Lemma 2.4.26. Recall that (T_H, d_H, r_H, μ_H) is indeed a.s. a continuum real tree and recall from (2.4.29) the definition of the subset $U_{H,W} \subset [0, 1]$. Recall that $(Z_r^0)_{r \in [0, \infty)}$ stands for a one-dimensional Brownian motion (resp. reflected Brownian motion) starting at 0 that is independent from H . We denote by ℓ the Lebesgue measure on $[0, 1]$. Then, by (2.4.36), conditionally given H , we get a.s.

$$\begin{aligned} \int P_H(dW) \ell(U_{H,W}) &= \int_0^1 \mathbf{1}_{\{H_s > 0\}} P_H \left(\forall \varepsilon \in (0, H_s) : \inf_{r \in [H_s - \varepsilon, H_s]} W_s(r) < \widehat{W}_s \right) ds \\ &= \int_0^1 \mathbf{1}_{\{H_s > 0\}} \mathbf{P} \left(\forall \varepsilon \in (0, H_s) : \inf_{r \in [H_s - \varepsilon, H_s]} Z_r^0 < Z_{H_s}^0 \mid H \right) ds = 1. \end{aligned}$$

Thus, a.s. $\ell([0, 1] \setminus U_{H,W}) = 0$ and Lemma 2.4.26 implies (i).

We next use Lemma 2.4.27 to prove (ii). Recall that (up to a slight abuse of notation) W can be defined on T_H . We first consider the unreflected Brownian case and recall from Lemma 2.4.32 that conditionally given H , $(\widehat{W}_\sigma)_{\sigma \in T_H}$ is a centered Hölder-continuous Gaussian process whose covariance is characterised by (2.4.37). According to Remark 2.4.29, $|\widehat{W}|$ is the endpoint process of the γ -stable reflected Brownian snake. We now work conditionally given H .

Let \mathcal{D} be a countable dense subset of T_H . Let $\sigma, \sigma' \in \mathcal{D}$ be distinct; denote by $\phi: [0, d_H(\sigma, \sigma')] \rightarrow \llbracket \sigma, \sigma' \rrbracket_{T_H}$ the isometry such that $\phi(0) = \sigma$ and $\phi(d_H(\sigma, \sigma')) = \sigma'$. The covariance of \widehat{W} combined with the fact that Brownian motion is reversible entails that the process $Z_r := \widehat{W}_{\phi(r)} - \widehat{W}_\sigma$, for $r \in [0, d_H(\sigma, \sigma')]$, has the same law conditionally given H as a Brownian motion starting at 0: it therefore a.s. reaches its infimum at a unique time $r_0 \in]0, d_H(\sigma, \sigma')[$ and the law of r_0 conditionally given H is diffuse, which entails that a.s. $\phi(r_0)$ cannot be a branch point of T_H since the set of branch points of T_H are countable. The same holds true for $|\widehat{W}|$ on $\llbracket \sigma, \sigma' \rrbracket_{T_H}$ if $\inf_{\gamma \in \llbracket \sigma, \sigma' \rrbracket_{T_H}} |\widehat{W}_\gamma| > 0$. Since \mathcal{D} is countable, it implies that conditionally given H , \widehat{W} satisfies Assumptions (a') and (b') in Lemma 2.4.27 and that $|\widehat{W}|$ satisfies Assumptions (a) and (b) in Lemma 2.4.27. This lemma then implies the first assertion of (ii).

Let us complete the proof of (ii). Recall from Remarks 2.4.25 that it makes sense to define $y^+ : T_H \rightarrow T_{H,|W|}$ by setting $y^+(p_H(s)) = p_{H,|W|}(s)$, $s \in [0, 1]$. Let $\sigma \in T_H$ be such that $y^+(\sigma) \neq r_{H,|W|}$. Then, observe that $\{\gamma \in \llbracket r_H, \sigma \rrbracket_{T_H} : |\widehat{W}_\gamma| > 0\}$ has infinitely many connected components: let γ and γ' be in two distinct such connected components: by (2.4.28) we get $d_{H,|W|}(y^+(\gamma), y^+(\gamma')) = \widehat{W}_\gamma + \widehat{W}_{\gamma'} = d_{H,|W|}(r_{H,W}, y^+(\gamma')) + d_{H,|W|}(r_{H,W}, y^+(\gamma))$, which easily implies that $y^+(\gamma)$ and $y^+(\gamma')$ belong to distinct connected components of $T_{H,|W|} \setminus \{r_{H,|W|}\}$. It implies that a.s. $\deg(r_{H,|W|}) = \infty$.

We define similarly $y : T_H \rightarrow T_{H,W}$ by setting $y(p_H(s)) = p_{H,W}(s)$, $s \in [0, 1]$. Let x, x' and $r_{H,W}$ be distinct points of $T_{H,W}$ and denote by z their branching point; let σ and $\sigma' \in T_H$ be such that $y(\sigma) = x$ and $y(\sigma') = x'$ and recall that $y(r_H) = r_{H,W}$. Then

$$\begin{aligned} 2d_{H,W}(r_{H,W}, z) &= d_{H,W}(r_{H,W}, x) + d_{H,W}(r_{H,W}, x') - d_{H,W}(x, x') \\ &\stackrel{\text{by (2.4.28)}}{=} 2 \min_{\gamma \in \llbracket \sigma, \sigma' \rrbracket_{T_H}} \widehat{W}_\gamma - 2 \min_{\gamma \in \llbracket r_H, \sigma \rrbracket_{T_H}} \widehat{W}_\gamma - 2 \min_{\gamma \in \llbracket r_H, \sigma' \rrbracket_{T_H}} \widehat{W}_\gamma. \end{aligned}$$

Now observe that a.s. $\min_{\gamma \in \llbracket r_H, \sigma \rrbracket_{T_H}} \widehat{W}_\gamma$ and $\min_{\gamma \in \llbracket r_H, \sigma' \rrbracket_{T_H}} \widehat{W}_\gamma$ are strictly negative quantities and that the least of the two has to be smaller than $\min_{\gamma \in \llbracket \sigma, \sigma' \rrbracket_{T_H}} \widehat{W}_\gamma$. This prove that $d_{H,W}(r_{H,W}, z) > 0$ and that x and x' are in the same connected component of $T_{H,W} \setminus \{r_{H,W}\}$. Thus, $\deg(r_{H,W}) = 1$, which completes the proof of (ii). \blacksquare

2.5 Scaling limit of the range of the BRW.

2.5.1 Continuous interpolation of discrete snakes.

We briefly devise here a natural way to embed a tree-valued branching walk into a continuous branching motion, which takes its values in the real tree spanned by a discrete space-tree and is indexed by the real tree spanned by a genealogical tree.

To that end, let us consider a (possibly infinite) graph tree T equipped with its graph-distance d_{gr}^T . T will be the space tree. We distinguish a special vertex $\circ \in T$ that plays the role of a root. Here, T can be a rooted ordered tree as in Definition 2.2.1 or the b -ary tree \mathbb{W}_b or the free tree $\mathbb{W}_{[0,1]}$ as introduced in (2.2.1). Recall from Example 2.4.9 the definition of $(\widetilde{T}, \widetilde{d}_{gr}^T)$ of the real tree spanned by T . We make the necessary identifications to assume that $T \subset \widetilde{T}$.

We first explain how to obtain a continuous interpolation of a path $v(k) \in T$, $k \in \{0, \dots, n\}$ (by a path, we mean a sequence of adjacent vertices). It is easy to check that there exists a continuous

map $\tilde{v}: [0, n] \rightarrow \tilde{T}$ such that $\tilde{v}(k) = v(k)$ and for all $0 \leq k < n$,

$$\forall s \in [k, k+1], \quad \tilde{v}(s) \in \llbracket v(k), v(k+1) \rrbracket \text{ is such that } \tilde{d}_{\text{gr}}^T(v(k), \tilde{v}(s)) = s - k.$$

The continuous path $(\tilde{v}(s))_{s \in [0, n]}$ is the continuous interpolation of the path $(v(k))_{k \in \{0, \dots, n\}}$.

We next extend this interpolation to T -valued branching walks. Let $t \in \mathbb{T}$ be a *finite* (rooted ordered) tree as in Definition 2.2.1. t will be the genealogical tree. We denote by $(\tilde{t}, \tilde{d}_{\text{gr}}^t)$ the compact real tree spanned by t and we assume that $t \subset \tilde{t}$. Let $(Y_v)_{v \in t}$ be a T -valued branching walk indexed by t . Namely, for all $v \in t$, $Y_v \in T$ and it satisfies the two following conditions: (a) $Y_\circ := \circ$; (b) if $v, v' \in t$ are neighbours, so are Y_v and $Y_{v'}$ in T . We easily check that there exists a continuous map $\tilde{Y}: \tilde{t} \rightarrow \tilde{T}$ such that $\tilde{Y}_v = Y_v$, $v \in t$, by specifying the following: for all $\sigma \in \tilde{t}$ there exist two neighbouring vertices $v, v' \in t$ such that $\sigma \in \llbracket v, v' \rrbracket_{\tilde{t}}$ and

$$\tilde{Y}_\sigma \text{ is the only point of } \llbracket Y_v, Y_{v'} \rrbracket_{\tilde{T}} \text{ such that } \tilde{d}_{\text{gr}}^T(Y_v, \tilde{Y}_\sigma) = \tilde{d}_{\text{gr}}^t(v, \sigma). \quad (2.5.1)$$

We next introduce the *spatial contour* associated with $(Y_v)_{v \in t}$. To that end, denote by $v(k) \in t$, $0 \leq k \leq 2\#t$, the sequence of vertices of t visited by the contour (or the depth-first) exploration of t as recalled in (2.4.34) and note that they form a path in t . We denote by $\tilde{v}: [0, 2\#t] \rightarrow \tilde{t}$ its continuous interpolation. Observe that $C_s(t) = \tilde{d}_{\text{gr}}^t(\circ, \tilde{v}(s))$, for all $s \in [0, 2\#t]$, where $C(t)$ stands for the contour function of t . Then, for all $s \in [0, 2\#t]$, we define a continuous map $V_s: [0, \infty) \rightarrow \tilde{T}$ as follows:

$$\begin{cases} \text{for all } r \in [0, C_s(t)], V_s(r) = \tilde{Y}_\sigma \text{ where } \sigma \in \llbracket \circ, \tilde{v}(s) \rrbracket_{\tilde{t}} \text{ is such that } r = \tilde{d}_{\text{gr}}^t(\circ, \sigma); \\ \text{for all } r \in [C_s(t), \infty), V_s(r) = \tilde{Y}_{\tilde{v}(s)}. \end{cases} \quad (2.5.2)$$

We call $(V_s(\cdot))_{s \in [0, 2\#t]}$ the *spatial contour* associated with the branching walk $(Y_v)_{v \in t}$. It is easy to check here that $s \mapsto V_s$ is continuous from $[0, 2\#t]$ to the space of \tilde{T} -valued continuous functions equipped with the uniform distance. We also define its endpoint process as follows:

$$\forall s \in [0, 2\#t], \quad \hat{V}_s = V_s(C_s(t)). \quad (2.5.3)$$

We then define the *continuous interpolation of the discrete snake associated with $(Y_v)_{v \in t}$* and its endpoint process by

$$\forall s \in [0, 2\#t], \quad \mathcal{W}_s(r) = \tilde{d}_{\text{gr}}^T(\circ, V_s(r)), \quad r \in [0, \infty) \quad \text{and} \quad \widehat{\mathcal{W}}_s = \tilde{d}_{\text{gr}}^T(\circ, \hat{V}_s). \quad (2.5.4)$$

By Remark 2.5.1 below, $(\mathcal{W}_s(\cdot))_{s \in [0, 2\#t]}$ is the continuous interpolation of the \mathbb{N} -valued branching walk $(d_{\text{gr}}^T(\circ, Y_v))_{v \in t}$. Moreover, observe that \mathcal{W} is a snake whose lifetime function is the contour $C(t)$ of t as in Definition 2.4.17: namely, $(C_s(t), \mathcal{W}_s)_{s \in [0, 2\#t]} \in \Sigma([0, 2\#t])$.

Remark 2.5.1 Let T' be another graph-tree and let $\phi: T \rightarrow T'$ be an adjacency-preserving map so that the image by ϕ of a path in T is a path in T' (for instance, $T = \mathbb{W}_{[0,1]}$, $T' = \mathbb{W}_b$ and $\phi = \Phi_b$, the b -contraction as in Definition 2.3.3; or $T' = \mathbb{N}$ and $\phi(v) = d_{\text{gr}}(\circ, v)$) Consequently, $(Y'_v)_{v \in t} := (\phi(Y_v))_{v \in t}$ is a T' -valued branching walk indexed by t . Then, ϕ clearly extends to a continuous map from $\tilde{T} \rightarrow \tilde{T}'$ by requiring that $\tilde{d}_{\text{gr}}^{T'}(\phi(\sigma), \phi(u)) = \tilde{d}_{\text{gr}}^T(\sigma, u)$ for all $\sigma \in \llbracket u, v \rrbracket_{\tilde{T}}$ with u and v adjacent in T . Then, observe that the continuous interpolation of Y' is the image by ϕ of the continuous interpolation of Y : namely, $\tilde{Y}' = \phi(\tilde{Y})$. \square

We next provide basic properties related to counting and occupation measures on the range of Y and its interpolation \tilde{Y} . We first set

$$\mathcal{R} = \{Y_v; v \in t\} \quad \text{and} \quad \tilde{\mathcal{R}} = \{\hat{V}_s; s \in [0, 2\#t]\} = \{\tilde{Y}_\sigma; \sigma \in \tilde{t}\}. \quad (2.5.5)$$

Observe that \mathcal{R} (resp. $\tilde{\mathcal{R}}$) is a subtree of T (resp. \tilde{T}) and note that $\tilde{\mathcal{R}}$ is the real tree spanned by \mathcal{R} . We equip \mathcal{R} and $\tilde{\mathcal{R}}$ with the respective occupation measures \mathbf{m}_{cont} and $\tilde{\mathbf{m}}_{\text{cont}}$ of the endpoint process of the spatial contour \hat{V} (here, the subscript *cont* is for *contour*). Namely, for all bounded measurable $f: \tilde{\mathcal{R}} \rightarrow \mathbb{R}$,

$$\int_{\mathcal{R}} f d\mathbf{m}_{\text{cont}} = \sum_{0 \leq k < 2\#t} f(\hat{V}_k) \quad \text{and} \quad \int_{\tilde{\mathcal{R}}} f d\tilde{\mathbf{m}}_{\text{cont}} = \int_0^{2\#t} f(\hat{V}_s) ds. \quad (2.5.6)$$

We next introduce the *spatial contour pseudo-distance* d_{cont} associated with Y :

$$\forall s, s' \in [0, 2\#t], \quad d_{\text{cont}}(s, s') = \tilde{d}_{\text{gr}}^T(\hat{V}_s, \hat{V}_{s'}). \quad (2.5.7)$$

Then, we easily check that $d_{\text{cont}} \in \mathbf{MT}([0, 2\#t])$ and that

$$\text{the corresponding pointed measured compact real tree is isometric to } (\tilde{\mathcal{R}}, \tilde{d}_{\text{gr}}^T, \mathbf{o}, \tilde{\mathbf{m}}_{\text{cont}}). \quad (2.5.8)$$

Lemma 2.5.2 *We keep the above notation. We denote by \tilde{d}_{Haus} the Hausdorff distance on the compact subsets of $(\tilde{T}, \tilde{d}_{\text{gr}}^T)$ and we denote by \tilde{d}_{Prok} the Prokhorov distance on the finite Borel measures on $(\tilde{T}, \tilde{d}_{\text{gr}}^T)$. Then, $\tilde{d}_{\text{Haus}}(\mathcal{R}, \tilde{\mathcal{R}}) \leq 1$, $\tilde{d}_{\text{Prok}}(\tilde{\mathbf{m}}_{\text{cont}}, \mathbf{m}_{\text{cont}}) \leq 1$. Moreover, let us set $\mathbf{m}_{\text{occ}} = \sum_{v \in t} \delta_{Y_v}$, that is the occupation measure of Y . Then, we also get $\tilde{d}_{\text{Prok}}(\mathbf{m}_{\text{cont}}, 2\mathbf{m}_{\text{occ}}) \leq 1$ and thus,*

$$\tilde{d}_{\text{Prok}}(\tilde{\mathbf{m}}_{\text{cont}}, 2\mathbf{m}_{\text{occ}}) \leq 2. \quad (2.5.9)$$

Proof. Clearly, $\tilde{d}_{\text{Haus}}(\mathcal{R}, \tilde{\mathcal{R}}) \leq 1$. Next denote by π the image of the Lebesgue measure on $[0, 2\#t]$ on $\tilde{T} \times \tilde{T}$ via the map $s \mapsto (\hat{V}_{\lfloor s \rfloor}, \hat{V}_s)$. By the definition (2.5.6), the first marginal of π is \mathbf{m}_{cont} and the second one is $\tilde{\mathbf{m}}_{\text{cont}}$. Moreover by definition of the interpolation and of the spatial contour $\tilde{d}_{\text{gr}}^T(\hat{V}_{\lfloor s \rfloor}, \hat{V}_s) \leq 1$. This easily implies that $\tilde{d}_{\text{Prok}}(\tilde{\mathbf{m}}_{\text{cont}}, \mathbf{m}_{\text{cont}}) \leq 1$.

To prove (2.5.9), we use the following coupling: recall that $(v(k))_{0 \leq k \leq 2\#t}$ stands for the contour (or depth-first) exploration of t (and recall the convention $v(2\#t-1) = v(2\#t) = \emptyset$). For all $k \in \{0, \dots, 2\#t-3\}$, we set $\rho(k) = v(k)$ if $|v(k)| = 1 + |v(k+1)|$ and $\rho(k) = v(k+1)$ if $|v(k+1)| = 1 + |v(k)|$ and we also set $\rho(2\#t-2) = \rho(2\#t-1) = \emptyset$. Note that the image measure of the counting measure on $\{0, \dots, 2\#t-1\}$ via ρ is $2 \sum_{u \in t} \delta_u$, namely twice the counting measure on t and thus, the image measure of the counting measure on $\{0, \dots, 2\#t-1\}$ via $Y_{\rho(\cdot)}$ is $2\mathbf{m}_{\text{occ}}$. Then, denote by ϖ the image measure on $T \times T$ of the counting measure on $\{0, \dots, 2\#t-1\}$ via the map $k \mapsto (Y_{\rho(k)}, \hat{V}_k)$. We just have proved that the first marginal of ϖ is $2\mathbf{m}_{\text{occ}}$; the second marginal of ϖ is by definition \mathbf{m}_{cont} . Since $Y_{\rho(k)}$ is either \hat{V}_k or \hat{V}_{k+1} , we get $d_{\text{gr}}^T(Y_{\rho(k)}, \hat{V}_k) \leq 1$. This easily implies that $\tilde{d}_{\text{Prok}}(\mathbf{m}_{\text{cont}}, 2\mathbf{m}_{\text{occ}}) \leq 1$ and (2.5.9) follows immediately from the previous bound. \blacksquare

In the following lemma we provide bounds to compare the occupation measure induced by the spatial contour and the counting measure on \mathcal{R} .

Lemma 2.5.3 *We keep the above notation. To simplify, we set $n = \#t$ and $n' = \#\mathcal{R}$. We denote by u_0, u_1, \dots, u_{n-1} the sequence of the vertices of t listed in the lexicographical order. Let $c \in (0, \infty)$. We set $\alpha = \max_{1 \leq i \leq n} |\#\{Y_{u_j}; 0 \leq j < i\} - 2ci|$ and $\beta = \max_{s \in (0, 2n]} |\#\{\hat{V}_l; 0 \leq l < \lfloor s \rfloor\} - cs|$. Then,*

$$\beta \leq \alpha + 3c + c \max_{s \in [0, 2n]} C_s(t). \quad (2.5.10)$$

We next denote by $\mathbf{m}_{\text{count}} = \sum_{x \in \mathcal{R}} \delta_x$, the counting measure on \mathcal{R} and for all $\eta \in (0, \infty)$, we set $q(d_{\text{cont}}, \eta) = \max \{d_{\text{cont}}(s, s'); s, s' \in [0, 2n] : |s - s'| \leq \eta\}$, where d_{cont} is defined by (2.5.7). Then,

$$\tilde{d}_{\text{Prok}}(\mathbf{m}_{\text{occ}}, \frac{n}{n'} \mathbf{m}_{\text{count}}) \leq 1 + 2q(d_{\text{cont}}, \frac{4\beta+1}{c}). \quad (2.5.11)$$

Proof. By convenience we set $u_n = \emptyset$. Let $i \in \{0, \dots, n-1\}$. Recall from the definition of the contour function of t in (2.4.34) that $b_i := 2i - |u_i| = \inf \{k \in \{0, \dots, 2n\} : v(k) = u_i\}$. Therefore, for all $k \in \{b_i + 1, \dots, b_{i+1}\}$, $\{v(l); 0 \leq l < k\} = \{u_j; 0 \leq j < i + 1\}$ and we get

$$\#\{\widehat{V}_i; 0 \leq l < k\} = \#\{Y_{u_j}; 0 \leq j < i + 1\} = \#\{Y_{u_j}; 0 \leq j < i + 1\} - 2c(i+1) + c(2(i+1) - k) + ck.$$

Note that $b_i < k \leq b_{i+1}$ implies that $|2(i+1) - k| \leq 2 + \max_{0 \leq j \leq n} |u_j| = 2 + \max_{s \in [0, 2n]} C_s(t)$. Thus, we get

$$\max_{1 \leq k \leq 2n} \left| \#\{\widehat{V}_i; 0 \leq l < k\} - ck \right| \leq \alpha + 2c + c \max_{s \in [0, 2n]} C_s(t),$$

which immediately implies (2.5.10).

We next prove (2.5.11). To simplify set $J(k) = \#\{\widehat{V}_i; 0 \leq l < k\}$ for all $k \in \{1, \dots, 2n\}$, so $\beta = \max_{s \in (0, 2n]} |J(\lceil s \rceil) - cs|$. For all $i' \in \{1, \dots, n'\}$, set $k(i') = \inf \{k \in \{1, \dots, 2n\} : J(k) = i'\}$ and $Z_{i'} = \widehat{V}_{k(i')-1}$. Note that the $Z_{i'}$ are distinct and that $\mathcal{R} = \{Z_{i'}; 1 \leq i' \leq n'\}$. For all $i', j' \in \{1, \dots, n'\}$, $|k(i') - k(j')| \leq c^{-1}(2\beta + |i' - j'|)$ since $J(k(i')) = i'$ and $J(k(j')) = j'$ and by definition of β . Then we get the following.

$$d_{\text{gr}}^T(Z_{i'}, Z_{j'}) = d_{\text{cont}}(k(i')-1, k(j')-1) \leq q(d_{\text{cont}}, |k(j') - k(i')|) \leq q(d_{\text{cont}}, \frac{2\beta + |i' - j'|}{c}). \quad (2.5.12)$$

Next observe that for all $k(i') \leq k < k(i' + 1)$, we get $J(k) = i'$ and

$$d_{\text{gr}}^T(\widehat{V}_{k-1}, Z_{J(k)}) = d_{\text{cont}}(k-1, k(i')-1) \leq q(d_{\text{cont}}, k(i'+1) - k(i')) \stackrel{\text{by (2.5.12)}}{\leq} q(d_{\text{cont}}, \frac{2\beta+1}{c})$$

Thus, we have proved that

$$\max_{s \in (0, 2n]} d_{\text{gr}}^T(\widehat{V}_{\lceil s \rceil}, Z_{J(\lceil s \rceil)}) \leq q(d_{\text{cont}}, \frac{2\beta+1}{c}). \quad (2.5.13)$$

Next observe that $|n' - 2cn| \leq \beta$ and that

$$J_{\lceil s \rceil} - \lceil \frac{n'}{2n} s \rceil = J_{\lceil s \rceil} - cs - \frac{s}{2n}(n' - 2cn) + \frac{n'}{2n}s - \lceil \frac{n'}{2n} s \rceil.$$

Thus, $\max_{s \in (0, 2n]} |J(\lceil s \rceil) - \lceil n's/2n \rceil| \leq 1 + 2\beta$ and by (2.5.12), we obtain the inequality $d_{\text{gr}}^T(Z_{J(\lceil s \rceil)}, Z_{\lceil n's/2n \rceil}) \leq q(d_{\text{cont}}, \frac{4\beta+1}{c})$. This inequality combined with (2.5.13) implies that

$$\max_{s \in (0, 2n]} d_{\text{gr}}^T(\widehat{V}_{\lceil s \rceil}, Z_{\lceil n's/2n \rceil}) \leq 2q(d_{\text{cont}}, \frac{4\beta+1}{c}). \quad (2.5.14)$$

We then denote by π the image on $T \times T$ of the Lebesgue measure on $[0, 2n]$ via the map $s \mapsto (\widehat{V}_{\lceil s \rceil}, Z_{\lceil n's/2n \rceil})$. The first marginal of π is by definition $\mathbf{m}_{\text{count}}$ and we easily see that the second marginal of π is $\frac{2n}{n'} \mathbf{m}_{\text{count}}$. By (2.5.14), it implies that $\tilde{d}_{\text{Prok}}(\mathbf{m}_{\text{count}}, \frac{2n}{n'} \mathbf{m}_{\text{count}}) \leq 2q(d_{\text{cont}}, \frac{4\beta+1}{c})$. Recall from Lemma 2.5.2 that $\tilde{d}_{\text{Prok}}(\mathbf{m}_{\text{count}}, 2\mathbf{m}_{\text{occ}}) \leq 1$. Thus, we get $\tilde{d}_{\text{Prok}}(2\mathbf{m}_{\text{occ}}, \frac{2n}{n'} \mathbf{m}_{\text{count}}) \leq 1 + 2q(d_{\text{cont}}, \frac{4\beta+1}{c})$, which implies (2.5.11) by Remark 2.4.12. ■

The range of the branching random walk. We next discuss the connection between the graph-metric of the range of branching random walks and the associated snake metric. More precisely, let $t \in \mathbb{T}$ be a *finite* (rooted ordered) tree as in Definition 2.2.1 and let $(Y_v)_{v \in t}$ be a $\mathbb{W}_{[0,1]}$ -valued

branching random walk with law Q_t^+ as in Definition 2.3.1 (ii). Recall from (2.5.5) that $\mathcal{R} = \{Y_v; v \in t\}$ that is a subtree of $\mathbb{W}_{[0,1]}$ as in Definition 2.2.3. We denote by d_{gr} the graph distance in $\mathbb{W}_{[0,1]}$ and we recall from Corollary 2.3.6 the fundamental property that makes the range \mathcal{R} tractable:

$$\mathbf{P}\text{-a.s. for all } u, v \in t, \quad d_{\text{gr}}(Y_u, Y_v) = |Y_u| + |Y_v| - 2 \min_{w \in [u,v]} |Y_w|. \quad (2.5.15)$$

Next, let $(\tilde{t}, \tilde{d}_{\text{gr}}^t)$ and $(\widetilde{\mathbb{W}}_{[0,1]}, \tilde{d})$ be the real trees spanned by respectively t and $\mathbb{W}_{[0,1]}$; with a slight abuse of notation, we suppose that $t \subset \tilde{t}$ and that $\mathbb{W}_{[0,1]} \subset \widetilde{\mathbb{W}}_{[0,1]}$. Let \tilde{Y} be the continuous interpolation of Y in $\widetilde{\mathbb{W}}_{[0,1]}$ as defined by (2.5.1) (with $(T, \circ) = (\mathbb{W}_{[0,1]}, \emptyset)$) and recall from (2.5.5) that $\tilde{\mathcal{R}} = \{\tilde{Y}_\sigma; \sigma \in \tilde{t}\}$ is the tree spanned by \mathcal{R} . We easily check that (2.5.15) extends to $\tilde{\mathcal{R}}$. Namely,

$$\mathbf{P}\text{-a.s. for all } \sigma, \sigma' \in \tilde{t}, \quad \tilde{d}(\tilde{Y}_\sigma, \tilde{Y}_{\sigma'}) = |\tilde{Y}_\sigma| + |\tilde{Y}_{\sigma'}| - 2 \min_{\varsigma \in [\sigma, \sigma']} |\tilde{Y}_\varsigma|. \quad (2.5.16)$$

Let us rewrite this formula in terms of the contour function $C(t)$ of t , the spatial contour V , its endpoint process \hat{V} as defined in (2.5.3), and its corresponding snake \mathcal{W} as defined in (2.5.4). To that end, first recall from (2.5.7) the definition of the spatial contour pseudo-distance d_{cont} and recall from (2.4.19) the notation $M_{C(t), \mathcal{W}}(\cdot, \cdot)$ and the definition of the snake distance $d_{C(t), \mathcal{W}}$ associated with the snake $(C(t), \mathcal{W})$. Then, (2.5.16) translates into the following: \mathbf{P} -a.s. for all $s_1, s_2 \in [0, 2\#t]$,

$$d_{\text{cont}}(s_1, s_2) = \tilde{d}(\hat{V}_{s_1}, \hat{V}_{s_2}) = \widehat{\mathcal{W}}_{s_1} + \widehat{\mathcal{W}}_{s_2} - 2M_{C(t), \mathcal{W}}(s_1, s_2) = d_{C(t), \mathcal{W}}(s_1, s_2). \quad (2.5.17)$$

Recall from (2.5.6) the definition of the occupation measure $\widetilde{\mathbf{m}}_{\text{cont}}$ induced by the spatial contour. Thus, (2.5.8) and (2.5.17) imply that

$$\mathbf{P}\text{-a.s. } (\tilde{\mathcal{R}}, \tilde{d}, \emptyset, \widetilde{\mathbf{m}}_{\text{cont}}) \text{ and } (T_{C(t), \mathcal{W}}, d_{C(t), \mathcal{W}}, r_{C(t), \mathcal{W}}, \mu_{C(t), \mathcal{W}}) \text{ are isometric.} \quad (2.5.18)$$

where we recall from (2.4.23) that $(T_{C(t), \mathcal{W}}, d_{C(t), \mathcal{W}}, r_{C(t), \mathcal{W}}, \mu_{C(t), \mathcal{W}})$ stands for the pointed measured compact real tree coded by the pseudo-metric $d_{C(t), \mathcal{W}}$.

We next couple the free branching random walk $(Y_v)_{t \in t}$ with the \mathbb{W}_b -valued one via the b -contraction application Φ_b as in Definition 2.3.3: namely we set

$$\forall v \in t, \quad Y_v^b = \Phi_b(Y_v), \quad \text{and} \quad \mathcal{R}_b = \{Y_v^b; v \in t\}.$$

By Remark 2.3.4, Y^b has law Q_t^{+b} as defined in Definition 2.3.1 (i). Observe that \mathcal{R}_b is a subtree of \mathbb{W}_b as in Definition 2.2.3. We next denote by d^b the graph distance in \mathbb{W}_b and we denote by $(\widetilde{\mathbb{W}}_b, \tilde{d}^b)$ the real tree spanned by \mathbb{W}_b ; with a slight abuse of notation, we suppose that $\mathbb{W}_b \subset \widetilde{\mathbb{W}}_b$. We extend Φ_b as a map from $\widetilde{\mathbb{W}}_{[0,1]}$ to $\widetilde{\mathbb{W}}_b$ explained in Remark 2.5.1 and we set

$$\forall s \in [0, 2\#t], \forall r \in [0, C_s(t)], \quad V_s^b(r) = \Phi_b(V_s(r)) \quad \text{and} \quad \hat{V}_s^b = V_s^b(C_s(t)),$$

where we recall that $(V_s(\cdot))_{s \in [0, 2\#t]}$ stands for the spatial contour of Y as defined in (2.5.2). Note that $\hat{V}_s^b = \Phi_b(\hat{V}_s)$. By Remark 2.5.1, it turns out that V^b is the spatial contour associated with Y^b . Then, denote by $\tilde{\mathcal{R}}_b$ the compact real tree spanned by \mathcal{R}_b in $\widetilde{\mathbb{W}}_b$. Namely,

$$\tilde{\mathcal{R}}_b = \{\hat{V}_s^b; s \in [0, 2\#t]\} = \Phi_b(\tilde{\mathcal{R}}).$$

We next extend Lemma 2.3.7 as follows: since $(\hat{V}_s^b)_{s \in [0, 2\#t]}$ is the continuous interpolation of the path $(Y_{v(k)}^b)_{0 \leq k \leq 2\#t}$ in \mathcal{R}_b , we easily derive the following from (2.3.17).

$$\text{For all } s_1, s_2 \in [0, 2\#t], \text{ there exists a nonnegative r.v. } G_{s_1, s_2} \text{ such that} \\ 2G_{s_1, s_2} = \tilde{d}(\hat{V}_{s_1}^b, \hat{V}_{s_2}^b) - \tilde{d}^b(\hat{V}_{s_1}^b, \hat{V}_{s_2}^b) \quad \text{and} \quad \mathbf{P}(G_{s_1, s_2} \geq x) \leq b^{2-x}, \quad x \in [0, \infty) \quad (2.5.19)$$

This combined with (2.5.18) shows that $T_{C(t), \mathcal{W}}$ and \mathcal{R}_b are close in a rough sense. It turns out that it is sufficient for the proof of Theorem 2.1.2.

2.5.2 Invariance principle for discrete snakes.

In this section we recall one important result due to Marzouk (Theorem 1 [130]) that is an invariance principle for real valued endpoint processes of discrete snakes. Here we concentrate on the one-dimensional case but we shall need actually a slightly stronger version that holds for path-valued snakes and not only for endpoint processes.

More precisely, let us fix $\gamma \in (1, 2]$ and let τ be a Galton–Watson tree (as in Definition 2.2.2) whose offspring distribution μ satisfies **(H)** as in (2.1.1). Conditionally given τ , let $(Y_v)_{v \in \tau}$ be a \mathbb{Z} -valued branching random walk whose transition kernel $q(y, dy')$ is that of the simple symmetric random walk on \mathbb{Z} . Then recall from (2.5.4) the definition of $(\mathcal{W}_s(\cdot))_{s \in [0, 2\#\tau]}$, the continuous interpolation of the discrete snake (here the tree T is \mathbb{Z}). Then,

$$\forall k \in \{0, \dots, 2\#\tau\}, \quad \widehat{\mathcal{W}}_k = Y_{v(k)}, \quad (2.5.20)$$

where $v(k) \in \tau$, $0 \leq k \leq 2\#\tau$, stands for the sequence of vertices of τ visited by the contour (or the depth-first exploration) of τ . Then, the following result is a special case of Marzouk [130, Theorem 1].

Theorem 2.5.4 (Theorem 1 [130]) *Let γ , μ , τ and $(a_n)_{n \in \mathbb{N}}$ be as in Theorem 2.4.28. Conditionally given τ , let $(Y_v)_{v \in \tau}$ be defined as above and let $(\widehat{\mathcal{W}}_k)_{0 \leq k \leq 2\#\tau}$ be the endpoint process of the snake associated with $(Y_v)_{v \in \tau}$ as in (2.5.20). Then*

$$\left(\frac{1}{a_n} C_{2ns}(\tau), \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{[2ns]} \right)_{s \in [0, 1]} \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{(law)} (H_s, \widehat{W}_s)_{s \in [0, 1]}$$

where (H, W) is the normalized one dimensional Brownian snake with γ -stable branching mechanism and where $\widehat{W}_s = W_s(H_s)$, $s \in [0, 1]$, stands for its endpoint process.

As already mentioned this result actually holds for fairly more general spatial motions in \mathbb{R}^d . It extends earlier results by Janson & Marckert [97] who considered snakes whose genealogical tree are in the domain of attraction of the Brownian tree. We refer to Marzouk [130] for more details. We next derive from Theorem 2.5.4 the following proposition that holds for the path-valued continuous interpolation of the discrete snake $(\mathcal{W}_s(\cdot))_{s \in [0, 2\#\tau]}$.

Proposition 2.5.5 *We keep the same notation and the same assumption as in Theorem 2.5.4. Recall from Definition 2.4.17 (a) the metric Δ that makes the space $\mathbf{C}([0, 1], \mathbb{R}) \times \mathbf{C}([0, 1], \mathbf{C}_0)$ Polish. Then, weakly on that space, the following convergence holds.*

$$\left(\left(\frac{1}{a_n} C_{2ns}(\tau) \right)_{s \in [0, 1]}, \left(\frac{1}{\sqrt{a_n}} \mathcal{W}_{2ns} \left(\frac{\cdot}{a_n} \right) \right)_{s \in [0, 1]} \right) \text{ under } \mathbf{P}(\cdot \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{(law)} (H, W). \quad (2.5.21)$$

By taking the absolute value, this results holds true for the snake reflected at 0.

Proof. Let $k_0 = 0 \leq k_1 \leq \dots \leq k_p \leq 2\#\tau$ be integers. Then for all $j \in \{0, \dots, p-1\}$, set $b_j = \min_{k_j \leq \ell \leq k_{j+1}} C_\ell(\tau)$ and $b'_j = C_{k_{j+1}}(\tau) - b_j$. By definition of the branching random walk and its associated snakes, the following holds true conditionally given τ (or $C(\tau)$).

- (i) The paths $S(j) := (\mathcal{W}_{k_{j+1}}((b_j + \ell) \wedge C_{k_{j+1}}(\tau)) - \mathcal{W}_{k_{j+1}}(b_j))_{\ell \in \mathbb{N}}$ are (conditionally) independent.
- (ii) The conditional law of $S(j)$ is that of a \mathbb{Z} -valued symmetric random walk, starting at 0 and stopped at time b'_{j+1} .

Next observe that $(b, b', w, w') \in [0, \infty)^2 \times \mathbf{C}_0^2 \mapsto (w(r \wedge b) + w'((r - b)_+ \wedge b') - w'(0))_{r \in [0, \infty)}$ is continuous (recall from Definition 2.4.17 that \mathbf{C}_0 is equipped with the Polish topology of the convergence on all compact subsets of times, which corresponds for instance to the metric δ_u given in (2.4.18)). This combined with (i), (ii) and easy arguments on linear interpolation imply that for all real numbers $0 \leq s_1 \leq \dots \leq s_p \leq 1$, $(\mathbf{w}_{s_j}^{(n)})_{1 \leq j \leq p}$ under $\mathbf{P}(\cdot | \#\tau = n)$ converges weakly to $(W_{s_j})_{1 \leq j \leq p}$ on \mathbf{C}_0^p , where we have set $\mathbf{w}_s^{(n)}(\cdot) = a_n^{-1/2} \mathcal{W}_{2ns}(\cdot/a_n)$, for all $s \in [0, 1]$.

We next prove the tightness of the rescaled snakes. A standard argument on linear interpolation shows that Theorem 2.5.4 implies that the continuous endpoint processes $\widehat{\mathbf{w}}^{(n)}$ under $\mathbf{P}(\cdot | \#\tau = n)$ converge weakly to the endpoint process \widehat{W} . This implies that

$$\forall \varepsilon \in (0, \infty), \quad \lim_{\eta \rightarrow +} \sup_{n \geq 0} \mathbf{P}(\omega_\eta(\widehat{\mathbf{w}}^{(n)}) \geq \varepsilon | \#\tau = n) = 0,$$

where $\omega_\eta(\widehat{\mathbf{w}}^{(n)})$ stands for the η -uniform modulus of continuity of $\widehat{\mathbf{w}}^{(n)}$ over $[0, 1]$. We next denote by $\omega_\eta(\mathbf{w}^{(n)})$ the η -uniform modulus of continuity of $\mathbf{w}^{(n)}$ with respect to the metric δ_u on \mathbf{C}_0 as introduced in Lemma 2.4.18; this lemma asserts that $\omega_\eta(\mathbf{w}^{(n)}) \leq 2\omega_\eta(\widehat{\mathbf{w}}^{(n)})$. This, combined with the weak convergence of the finite dimensional marginals of $\mathbf{w}^{(n)}$, entails that the laws of $\mathbf{w}^{(n)}$ are tight on $\mathbf{C}([0, 1], \mathbf{C}_0)$ and we easily get (2.5.21). \blacksquare

2.5.3 Proof of Theorem 2.1.2.

Let us fix $\gamma \in (1, 2]$ and let τ be a Galton–Watson tree (as in Definition 2.2.2) whose offspring distribution μ satisfies **(H)** as in (2.1.1). Then recall from (2.4.34) the definition of the contour process $(C_s(\tau))_{s \in [0, 2\#\tau]}$ of τ . Conditionally given τ , let $(Y_v)_{v \in \tau}$ be a $\mathbb{W}_{[0,1]}$ -valued branching random walk with law Q_τ^+ as in Definition 2.3.1 (ii). Recall from (2.5.2) the spatial contour $(V_s(\cdot))_{s \in [0, 2\#\tau]}$ associated with Y , and recall from (2.5.4) the definition of $(\mathcal{W}_s(\cdot))_{s \in [0, 2\#\tau]}$, the continuous interpolation of the discrete snake. We denote by $(\tilde{\tau}, \tilde{d}_{\tilde{\tau}})$, $(\widetilde{\mathbb{W}}_{[0,1]}, \tilde{d})$ and $(\widetilde{\mathbb{W}}_b, \tilde{d}^b)$ be the real trees spanned by respectively τ , $\mathbb{W}_{[0,1]}$ and \mathbb{W}_b we assume that $\tau \subset \tilde{\tau}$, that $\mathbb{W}_{[0,1]} \subset \widetilde{\mathbb{W}}_{[0,1]}$ and that $\mathbb{W}_b \subset \widetilde{\mathbb{W}}_b$. We next couple the free branching random walk $(Y_v)_{t \in t}$ with the \mathbb{W}_b -valued one via the \mathbf{b} -contraction application Φ_b as in Definition 2.3.3: namely we set $Y_v^b = \Phi_b(Y_v)$ for all $v \in \tau$. By Remark 2.3.4, Y^b has law Q_τ^{+b} as in Definition 2.3.1 (i). We also set $V^b = \Phi_b(V)$ that is the spatial contour of Y^b , according to Remark 2.5.1. We recall the following notation $\mathcal{R} = \{Y_v; v \in \tau\}$, $\mathcal{R}_b = \{Y_v^b; v \in \tau\} = \Phi_b(\mathcal{R})$, $\widehat{\mathcal{R}} = \{\widehat{V}_s; s \in [0, 2\#\tau]\}$ and $\widehat{\mathcal{R}}_b = \{\widehat{V}_s^b; s \in [0, 2\#\tau]\} = \Phi_b(\widehat{\mathcal{R}})$.

We trivially extend $C(\tau)$ and \mathcal{W} on $[0, \infty)$ by taking $C_s(\tau)$ equal to 0 and by taking \mathcal{W}_s equal to the null function for all $s \in [2\#\tau, \infty)$. Let $(a_n)_{n \in \mathbb{N}}$ be as in Theorem 2.4.28. To simplify notation we set for all $s \in [0, 1]$,

$$\mathbf{h}_n(s) = \frac{1}{a_n} C_{2ns}(\tau), \quad \mathbf{w}_s^{(n)}(r) = \frac{1}{\sqrt{a_n}} \mathcal{W}_{2ns}\left(\frac{r}{a_n}\right), \quad r \in [0, \infty) \quad \text{and} \quad \widehat{\mathbf{w}}_s^{(n)} = \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{2ns}.$$

Observe that $\mathbf{P}(\cdot | \#\tau = n)$ -a.s. $(\mathbf{h}_n, \mathbf{w}^{(n)}) \in \Sigma([0, 1])$ as in Definition 2.4.17 and we denote by $d_{\mathbf{h}_n}$ and $d_{\mathbf{h}_n, \mathbf{w}^{(n)}}$ the corresponding tree and snake pseudo-distances as defined in resp. (2.4.8) and (2.4.19). In particular, observe that

$$\mathbf{P}\text{-a.s. for all } s_1, s_2 \in [0, \infty), \quad \frac{1}{\sqrt{a_n}} \tilde{d}(\widehat{V}_{2ns_1}, \widehat{V}_{2ns_2}) = d_{\mathbf{h}_n, \mathbf{w}^{(n)}}(s_1, s_2)$$

Proposition 2.5.5 combined with Lemma 2.4.20 implies that weakly on $(\mathbf{C}([0, 1]^2, \mathbb{R}), \|\cdot\|)^2$

$$(d_{\mathbf{h}_n}, d_{\mathbf{h}_n, \mathbf{w}^{(n)}}) \text{ under } \mathbf{P}(\cdot | \#\tau = n) \xrightarrow[n \rightarrow \infty]{} (d_H, d_{H,W}),$$

where (H, W) is the normalized one dimensional reflected Brownian snake with γ -stable branching mechanism as in Definition 2.4.31 and where d_H and $d_{H,W}$ are the corresponding tree and snake pseudo-distances.

We next set

$$\forall s_1, s_2 \in [0, \infty), \quad \mathbf{d}_n^*(s_1, s_2) = \frac{1}{\sqrt{a_n}} \tilde{d}^b(\widehat{V}_{2ns_1}^b, \widehat{V}_{2ns_2}^b).$$

By (2.5.19), conditionally given τ , $\mathbf{d}_n^*(s_1, s_2) \leq d_{\mathbf{h}_n, \mathbf{w}^{(n)}}(s_1, s_2)$ and for all $\varepsilon \in (0, 1)$, a.s.

$$\mathbf{P}(d_{\mathbf{h}_n, \mathbf{w}^{(n)}}(s_1, s_2) - \mathbf{d}_n^*(s_1, s_2) > 2\varepsilon \mid \tau) \leq b^{3-\varepsilon\sqrt{a_n}},$$

which implies that $\lim_{n \rightarrow \infty} \mathbf{P}(d_{\mathbf{h}_n, \mathbf{w}^{(n)}}(s_1, s_2) - \mathbf{d}_n^*(s_1, s_2) > 2\varepsilon \mid \#\tau = n) = 0$. Therefore, Proposition 2.4.4 applies (with $\mathbf{d}_n = d_{\mathbf{h}_n, \mathbf{w}^{(n)}}$) to show that $(\mathbf{d}_n^*, d_{\mathbf{h}_n, \mathbf{w}^{(n)}})$ under $\mathbf{P}(\cdot \mid \#\tau = n)$ converge to $(d_{H,W}, d_{H,W})$, weakly on $(\mathbf{C}([0, 1]^2, \mathbb{R}), \|\cdot\|)^2$. Actually it is easy to see that $(d_{\mathbf{h}_n}, \mathbf{d}_n^*, d_{\mathbf{h}_n, \mathbf{w}^{(n)}})$ under $\mathbf{P}(\cdot \mid \#\tau = n)$ converges to $(d_H, d_{H,W}, d_{H,W})$, weakly on $(\mathbf{C}([0, 1]^2, \mathbb{R}), \|\cdot\|)^3$.

We next recall that $\mathbf{P}(\cdot \mid \#\tau = n)$ -a.s. the real tree coded by $d_{\mathbf{h}_n}$ is isometric to $(\tilde{\tau}, \frac{1}{a_n} \tilde{d}_{\text{gr}}^\tau, \emptyset, \frac{1}{n} \tilde{\mathbf{m}})$ where $\tilde{\mathbf{m}} = \delta_\emptyset + \text{Length}$. Also recall from (2.5.8) that $\mathbf{P}(\cdot \mid \#\tau = n)$ -a.s. the real tree coded by \mathbf{d}_n^* is isometric to $(\tilde{\mathcal{R}}_b, \frac{1}{\sqrt{a_n}} \tilde{d}^b, \emptyset, \frac{1}{2n} \tilde{\mathbf{m}}_{\text{cont}}^b)$, where $\tilde{\mathbf{m}}_{\text{cont}}^b$ is the occupation measure of $(\widehat{V}_s^b)_{s \in [0, 2n]}$, as defined in general in (2.5.6). Since $(d_{\mathbf{h}_n}, \mathbf{d}_n^*)$ under $\mathbf{P}(\cdot \mid \#\tau = n)$ converges to $(d_H, d_{H,W})$, weakly on $(\mathbf{C}([0, 1]^2, \mathbb{R}), \|\cdot\|)^2$, Proposition 2.4.14 implies that the pointed measured compact spaces $(\tilde{\tau}, \frac{1}{a_n} \tilde{d}_{\text{gr}}^\tau, \emptyset, \frac{1}{n} \tilde{\mathbf{m}})$ and $(\tilde{\mathcal{R}}_b, \frac{1}{\sqrt{a_n}} \tilde{d}^b, \emptyset, \frac{1}{2n} \tilde{\mathbf{m}}_{\text{cont}}^b)$ under $\mathbf{P}(\cdot \mid \#\tau = n)$ jointly converge to resp. (T_H, d_H, r_H, μ_H) and $(T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})$ weakly on $(\mathbb{M}, \delta_{\text{GHP}})^2$, where T_H and $T_{H,W}$ here stand for the real trees coded by resp. H and (H, W) .

We next recall the following notation.

$$\mathbf{m} = \sum_{v \in \tau} \delta_v, \quad \mathbf{m}_{\text{occ}}^b = \sum_{v \in \tau} \delta_{Y_v^b}, \quad \bar{\mathbf{m}}_{\text{occ}}^b = \frac{1}{n} \mathbf{m}_{\text{occ}}^b, \quad \mathbf{m}_{\text{count}}^b = \sum_{x \in \mathcal{R}_b} \delta_x \quad \text{and} \quad \bar{\mathbf{m}}_{\text{count}}^b = \frac{1}{\#\mathcal{R}_b} \mathbf{m}_{\text{count}}^b.$$

By the inequalities specified in Example 2.4.13, we get $d_{\text{Haus}}(\tau, \tilde{\tau}) \leq 1$ and $d_{\text{Prok}}(\tilde{\mathbf{m}}, \mathbf{m}) \leq 3$. Similarly, by Lemma 2.5.2 we get $d_{\text{Haus}}(\mathcal{R}_b, \tilde{\mathcal{R}}_b) \leq 1$ and $d_{\text{Prok}}(2\mathbf{m}_{\text{occ}}^b, \tilde{\mathbf{m}}_{\text{cont}}^b) \leq 2$. Thus,

$$\begin{aligned} & \left((\tau, \frac{1}{a_n} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}), (\mathcal{R}_b, \frac{1}{\sqrt{a_n}} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathbf{m}_{\text{occ}}^b) \right) \quad \text{under } \mathbf{P}(\cdot \mid \#\tau = n) \\ & \xrightarrow{n \rightarrow \infty} ((T_H, d_H, r_H, \mu_H), (T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})), \end{aligned}$$

where d_{gr} stands for both graph-tree distances in τ and \mathbb{W}_b to simplify notation.

We next control $\mathbf{m}_{\text{count}}^b$ in terms of $\mathbf{m}_{\text{occ}}^b$ thanks to Theorem 2.1.1 and Lemma 2.5.3. To that end, we set $c = c_{\mu, b}/2$ where $c_{\mu, b}$ is as in Theorem 2.1.1; we denote by $u_0, u_1, \dots, u_{\#\tau-1}$ the vertices of τ listed in the lexicographical order and we also set $\alpha = \max_{1 \leq i \leq \#\tau} |\#\{Y_{u_j}^b; 0 \leq j < i\} - 2ci|$ and $\beta = \max_{s \in (0, 2\#\tau]} |\#\{\widehat{V}_l^b; 0 \leq l < [s]\} - cs|$. Theorem 2.1.1 asserts that for all $\varepsilon \in (0, \infty)$, $\mathbf{P}(\alpha/n > \varepsilon \mid \#\tau = n)$ tends to 0 as $n \rightarrow \infty$. By (2.5.10), $\mathbf{P}(\cdot \mid \#\tau = n)$ -a.s.

$$\frac{\beta}{n} \leq \frac{1}{n}(\alpha + 3c) + \frac{ca_n}{n} \max_{s \in [0, 1]} h_n(s).$$

Since (a_n) is $\frac{\gamma-1}{\gamma}$ -regularly varying, $\lim_{n \rightarrow \infty} a_n/n = 0$, which implies that $\mathbf{P}(\beta/n > \varepsilon \mid \#\tau = n)$ tends to 0 as $n \rightarrow \infty$.

Next, we denote by d_{Prok} the Prokhorov distance on the space of finite measures on $(\mathcal{R}_b, d_{\text{gr}})$ and we denote by $d_{\text{Prok}}^{(n)}$ the Prokhorov distance on the space of $(\mathcal{R}_b, \frac{1}{\sqrt{a_n}} d_{\text{gr}})$. Since $n^{-1} \leq a_n^{-1/2}$ for all sufficiently large n , Remark 2.4.12 combined with (2.5.11) implies that $\mathbf{P}(\cdot \mid \#\tau = n)$ -a.s.

$$d_{\text{Prok}}^{(n)}(\bar{\mathbf{m}}_{\text{occ}}^b, \bar{\mathbf{m}}_{\text{count}}^b) \stackrel{\text{Remark 2.4.12}}{\leq} \frac{1}{\sqrt{a_n}} d_{\text{Prok}}(\mathbf{m}_{\text{occ}}^b, \frac{n}{\#\mathcal{R}_b} \mathbf{m}_{\text{count}}^b) \stackrel{\text{by (2.5.11)}}{\leq} \frac{1}{\sqrt{a_n}} + 2q(\mathbf{d}_n^*, \frac{4\beta+1}{2nc}),$$

where $q(\mathbf{d}_n^*, \eta) = \max\{\mathbf{d}_n^*(s, s'); s, s' \in [0, 1] : |s - s'| \leq \eta\}$. Since $\mathbf{d}_n^* \rightarrow d_{H,W}$ weakly on $(\mathbf{C}([0, 1]^2, \mathbb{R}), \|\cdot\|)$ and since for all $\varepsilon \in (0, \infty)$, $\lim_{n \rightarrow \infty} \mathbf{P}(\beta/n > \varepsilon \mid \#\tau = n) = 0$, Proposition 2.4.3 implies that $\lim_{n \rightarrow \infty} \mathbf{P}(q(\mathbf{d}_n^*, \frac{4\beta+1}{2nc}) > \varepsilon \mid \#\tau = n) = 0$ and thus

$$\forall \varepsilon \in (0, 1), \quad \lim_{n \rightarrow \infty} \mathbf{P}(d_{\text{Prok}}^{(n)}(\bar{\mathbf{m}}_{\text{occ}}^b, \bar{\mathbf{m}}_{\text{count}}^b) > \varepsilon \mid \#\tau = n) = 0.$$

Then, observe that $\#\mathcal{R}_b \leq n$ and that $\frac{\#\mathcal{R}_b}{n} \bar{\mathbf{m}}_{\text{count}}^b = \frac{1}{n} \mathbf{m}_{\text{count}}^b$. By Remark 2.4.12, we get

$$\begin{aligned} d_{\text{Prok}}^{(n)}(c_{\mu,b} \bar{\mathbf{m}}_{\text{occ}}^b, \frac{1}{n} \mathbf{m}_{\text{count}}^b) &\leq d_{\text{Prok}}^{(n)}(c_{\mu,b} \bar{\mathbf{m}}_{\text{occ}}^b, \frac{\#\mathcal{R}_b}{n} \bar{\mathbf{m}}_{\text{occ}}^b) + d_{\text{Prok}}^{(n)}(\frac{\#\mathcal{R}_b}{n} \bar{\mathbf{m}}_{\text{occ}}^b, \frac{\#\mathcal{R}_b}{n} \bar{\mathbf{m}}_{\text{count}}^b) \\ &\leq \left| \frac{\#\mathcal{R}_b}{n} - c_{\mu,b} \right| + d_{\text{Prok}}^{(n)}(\bar{\mathbf{m}}_{\text{occ}}^b, \bar{\mathbf{m}}_{\text{count}}^b). \end{aligned}$$

Now, recall that $c_{\mu,b} = 2c$ and that $|\frac{\#\mathcal{R}_b}{n} - c_{\mu,b}| \leq \alpha/n$. It finally implies that

$$\forall \varepsilon \in (0, 1), \quad \lim_{n \rightarrow \infty} \mathbf{P}(d_{\text{Prok}}^{(n)}(c_{\mu,b} \bar{\mathbf{m}}_{\text{occ}}^b, \frac{1}{n} \mathbf{m}_{\text{count}}^b) > \varepsilon \mid \#\tau = n) = 0,$$

which completes the proof of Theorem 2.1.2. ■

Acknowledgements The authors would like to thank the referee and the editor for their helpful suggestions.

Chapter 3

Scaling limit of the range of tree-valued branching random walks in random environment

This chapter is adapted from the paper [59], written in collaboration with Thomas Duquesne, which is in preparation.

Abstract

We study a branching random walk (BRW) taking its values in a random tree \mathbf{T} (seen as a family tree) with an infinite line of ancestors that is a variant of a supercritical Galton–Watson tree with offspring distribution ν . The transition probabilities of the BRW are those of a critical biased random walk on \mathbf{T} : namely, the probability to move from x to one of its k_x children is $1/(\mathfrak{m}_\nu + k_x)$ and the probability to move from x to the direct parent of x is $\mathfrak{m}_\nu/(\mathfrak{m}_\nu + k_x)$. Here \mathfrak{m}_ν stands for the mean of ν . The BRW is indexed by a critical Galton–Watson tree conditioned to have n vertices and whose offspring distribution is in the domain of attraction of an α -stable law with $\alpha \in (1, 2]$. We denote by \mathcal{R}_n the range of the BRW, i.e. the set of all sites in \mathbf{T} visited by the BRW. Under a moment assumption for ν , we obtain a law of large numbers for the size of \mathcal{R}_n . Moreover, we prove that if we view \mathcal{R}_n as a random subtree of \mathbf{T} equipped with its graph-distance d_{gr} and with its occupation measure $\mathfrak{m}_{\text{occ}}^{(n)}$ then there exists a scaling sequence $s_n \rightarrow \infty$ such that conditionally given the environment \mathbf{T} , the measured metric space $(\mathcal{R}_n, s_n^{-1} d_{\text{gr}}, \frac{1}{s_n} \mathfrak{m}_{\text{occ}}^{(n)})$ weakly converges in the Gromov-Hausdorff-Prokhorov sense to a random measured compact real tree introduced by Le Gall & Miermont in [40] called the Brownian cactus with α -stable branching mechanism. This work extends in random environment the results from Chapter 2 (adapted from D., K., Lin & Torri [60]) which deals with the case where \mathbf{T} is a regular tree.

Contents

3.1	Introduction	112
3.2	Tree-valued branching RWs and snakes: definitions, previous results	117
3.2.1	Rooted ordered trees and forests, coding functions, discrete snakes.	117
3.2.2	Trees with an infinite line of ancestors.	121
3.2.3	Biased random walks on trees	126
3.2.4	Tree-valued branching RWs	130
3.2.5	Stable continuous height processes.	132

3.2.6	Real trees, convergences of metric spaces and pseudometrics.	137
3.2.7	Continuous snakes and related metrics.	139
3.2.8	Tree-valued branching walks and snake-pseudometrics	143
3.3	Proof of Theorem 3.1.1	146
3.3.1	Preliminary results on critical biased RWs on GW-trees	146
3.3.2	Estimates on critical biased branching RWs.	148
3.3.3	Proof of Propositions 3.2.26 and 3.2.32.	151
3.3.4	Proof of Theorem 3.1.1	156
3.4	Limit of the τ_∞-indexed branching RW and its snakes	158
3.4.1	Harmonic coordinates and related estimates.	158
3.4.2	Statements of the result, overview of its proof	165
3.4.3	Quenched tightness of the height-snake of the τ_∞ -indexed branching RWs	169
3.4.4	Proof of Proposition 3.4.10	174
3.4.5	Proof of Proposition 3.4.14	180
3.5	Proof of Theorem 3.1.2	183
3.5.1	Limit of the τ_n -indexed RW and its snakes.	183
3.5.2	Proof of Theorem 3.1.2	186
3.A	Proof of Lemma 3.2.33	189
3.B	Proof of Lemma 3.2.37	190
3.C	Proof of Proposition 3.4.5	191

3.1 Introduction

Branching random walks (BRW) represent a simple model to study a branching behavior coupled with some spatial displacements. As such, they drew a lot of interest and are naturally linked with various areas of research. Let us mention at least superprocesses, traveling wave solutions of semi-linear partial differential equations (FKPP), and some statistical mechanics models such as Mandelbrot cascades, the contact process, the Gaussian free field, or the voter model. We refer to Shi [140] for an overview of this topic and its connections. We also cite the study of extreme particles of real-valued branching random walks by Biggins [21] and Bramson [27, 26], the works of Gouëzel, Hueter, Lalley, Sellke [86, 94, 114, 115] about branching random walks with values in hyperbolic spaces, and the study of branching random walks on trees of Benjamini & Müller [16] and Liggett [122].

In contrast to those works where the indexing tree is infinite, D., K., Lin & Torri [60] consider a BRW that is indexed by a critical Galton–Watson tree conditioned to have n vertices. In [60], the BRW moves like a nearest-neighbor null-recurrent random walk on a deterministic rooted regular tree: it jumps to the parent of the current position with probability $1/2$, or it chooses uniformly one of the sites above otherwise. With this setting, D., K., Lin & Torri [60] describe the asymptotic behavior of the range of the BRW (i.e. the set of all visited sites) as $n \rightarrow \infty$ by showing that its number of points is equivalent to n times a positive constant and that its scaling limit is a variant of the Brownian cactus introduced by Curien, Le Gall & Miermont [40]. Here, we aim to obtain a similar result as in [60] when the environment is instead a random tree with infinite depth.

We now give an informal description of the environment and the transitions of the BRW we consider in this chapter. We fix a probability distribution $\nu = (\nu(k))_{k \in \mathbb{N}}$ on \mathbb{N} that we assume to be supercritical with a finite moment of order 2:

$$m_\nu := \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty) \quad \text{and} \quad \sigma_\nu^2 := \frac{1}{m_\nu(m_\nu - 1)} \sum_{k \in \mathbb{N}} k(k-1)\nu(k) \in (0, \infty). \quad (3.1.1)$$

We denote by \mathbf{T} the environment and by $\circ \in \mathbf{T}$ the starting point of the BRW. One can understand (\mathbf{T}, \circ) as a Galton–Watson tree with offspring distribution ν that is modified so that the distinguished point \circ has an infinite line of ancestors: informally, the numbers of children $k_x(\mathbf{T})$ of the vertices $x \in \mathbf{T}$ are independent, outside of the ancestral line of \circ , their common law is ν , on the ancestral line of \circ , their law is the biased probability distribution $(k\nu(k)/m_\nu)_{k \in \mathbb{N}}$. We say that (\mathbf{T}, \circ) is an *infinite Galton–Watson tree with offspring distribution ν* and we refer to Definition 3.2.18 for a more precise construction. For all $x \in \mathbf{T}$, let \overline{x} stand for the parent of x (which is always well-defined since \mathbf{T} has an infinite line of ancestors).

The BRW on (\mathbf{T}, \circ) we are interested in shares its spatial displacements with the so-called critical biased random walk (RW). For all $\lambda \in [0, \infty)$ and conditionally given (\mathbf{T}, \circ) , the λ -biased random walk on (\mathbf{T}, \circ) is the Markov chain whose transition probabilities are expressed by

$$p_{\mathbf{T}, \lambda}(x, y) = \mathbf{1}_{\{\overline{y}=x\}} \frac{1}{\lambda + k_x(\mathbf{T})} + \mathbf{1}_{\{\overline{x}=y\}} \frac{\lambda}{\lambda + k_x(\mathbf{T})} \quad (3.1.2)$$

for all $x, y \in \mathbf{T}$. This model has been introduced by Lyons [123] who also showed that the behavior of the biased random walk is subject to a phase transition concerning the parameter λ : we refer to Section 3.2.3 for a more detailed account. We shall only consider the case $\lambda = m_\nu$ which is critical in the sense that a m_ν -biased RW on an ordinary GW-trees with offspring distribution ν is null-recurrent.

The present chapter is related to earlier works about scaling limits for the range of tree-valued critical or near-critical biased RW. First, D. [52] studies near-critical biased RWs on regular trees and show that the rescaled range of the RW converges to a CRT coded by Brownian motions with drift, along with a law of large numbers for the range. Then, Peres & Zeitouni [137] prove that the process of the distances to the root of a critical biased RW in a supercritical Galton–Watson tree without leaves is diffusive. Similarly, Dembo & Sun [45] treat the cases of critical biased RW on multi-type GW-trees. Furthermore, Aidékon & de Raphélis [6] strengthen Peres & Zeitouni’s result and show that the rescaled range of the same RW converges to a variant of the Brownian CRT. We also mention Chen & Miermont [35] who show, using previous results due to Bougerol & Jeulin [25], that the scaling limits of Brownian bridges and loops in hyperbolic spaces are the Brownian CRT. Independently, Stewart proves in his Ph.D. thesis [145] that the rescaled simple RW bridges on a b -regular tree (with $b \geq 3$) converge to the Brownian CRT.

Now, let us discuss the assumptions we make on the genealogy of the branching random walk, that we always suppose to be independent of the environment (\mathbf{T}, \circ) . We fix another probability measure $\mu = (\mu(k))_{k \in \mathbb{N}}$ on \mathbb{N} . We denote by τ a (rooted and ordered) Galton–Watson tree with offspring distribution μ that satisfies the following with $\alpha \in (1, 2]$:

$$\begin{cases} (H_1) : & \mu(0) + \mu(1) < 1 \quad \text{and} \quad \sum_{k \in \mathbb{N}} k\mu(k) = 1, \\ (H_2) : & \mu \text{ is aperiodic, (namely, } \mu \text{ is not supported by a proper subgroup of } \mathbb{Z}), \\ (H_3) : & \mu \text{ is in the domain of attraction of an } \alpha\text{-stable law.} \end{cases} \quad (3.1.3)$$

The assumption (H_1) ensures that the number of vertices $\#\tau$ of τ is almost surely finite. The assumption (H_2) is convenient to work with because it implies that $\mathbf{P}(\#\tau = n) > 0$ for all integers n sufficiently large. We recall that (H_3) translates into the following assertion: if $(V_n)_{n \in \mathbb{N}}$ is a random walk on \mathbb{Z} starting at $V_0 = 0$ and whose jump law is given by $\mathbf{P}(V_1 = k) = \mu(k + 1)$ for all $k \in \{-1\} \cup \mathbb{N}$, then there exists a $\frac{\alpha-1}{\alpha}$ -regularly varying sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$\frac{a_n}{n} V_n \xrightarrow[n \rightarrow \infty]{(\text{law})} X_1, \quad \text{where } \log \mathbf{E}[\exp(-\lambda X_1)] = \lambda^\alpha \text{ for all } \lambda \in [0, \infty). \quad (3.1.4)$$

We refer to Section 3.2.5 for more details on these assumptions.

We next define the BRW we consider as follows. For all sufficiently large n , we denote by τ_n a tree with the same law as a GW-tree τ with offspring distribution μ under $\mathbf{P}(\cdot | \#\tau = n)$. We denote by \emptyset the root of τ_n and to each vertex $u \in \tau_n$, we associate a spatial position $Y_u^{(n)} \in \mathbf{T}$ whose conditional joint law given (\mathbf{T}, \circ) and τ_n is defined by

$$\mathbf{P}(\forall u \in \tau_n, Y_u^{(n)} = x_u \mid (\mathbf{T}, \circ), \tau_n) = \mathbf{1}_{\{Y_{\emptyset}^{(n)} = \circ\}} \prod_{u \in \tau_n \setminus \{\emptyset\}} p_{\mathbf{T}, \mathfrak{m}_\nu}(x_{\overleftarrow{u}}, x_u) \quad (3.1.5)$$

for all $x_u \in \mathbf{T}$ and $u \in \tau_n$ (here \overleftarrow{u} stands for the direct parent of u in τ_n).

It is convenient to see τ_n as a family tree whose ancestor corresponds to the root \emptyset and where siblings are ordered by birth-rank. The depth-first exploration of τ_n is the sequence of vertices $(u_j)_{0 \leq j < \#\tau}$ that is defined recursively as follows: u_0 is the root \emptyset and for all $j \in \{0, \dots, \#\tau - 2\}$, if v is the most recent ancestor of u_j having at least one unexplored child then u_{j+1} is the unexplored child of v with least birth-rank (see Section 3.2.1 for a more precise construction). Our first result is a law of large numbers for the size of the range of the branching random walk. This result is annealed: this is a convergence in probability while averaging over the environment (\mathbf{T}, \circ) . We refer to Section 3.2 for the precise definitions of the objects involved in the following Theorem.

Theorem 3.1.1 *Let (\mathbf{T}, \circ) be an infinite Galton–Watson tree whose offspring distribution ν is supercritical. For all sufficiently large n , we denote by τ_n a Galton–Watson tree with offspring distribution μ conditioned to have n vertices. We assume that (\mathbf{T}, \circ) and τ_n are independent. Let $(Y_u^{(n)})_{u \in \tau_n}$ be the BRW whose conditional law given (\mathbf{T}, \circ) and τ_n is defined by (3.1.5). We assume that μ satisfies (3.2.60) and we assume that there exists $\mathfrak{b} \in (1, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{\mathfrak{b}} \nu(k) < \infty$. Recall that $(u_j)_{0 \leq j < \#\tau}$ stands for the depth-first exploration of τ_n . Then, there exists a deterministic constant $c_{\mu, \nu} \in (0, 1)$ that only depends on μ and ν such that*

$$\forall \varepsilon \in (0, 1), \quad \lim_{n \rightarrow \infty} \mathbf{P} \left(\max_{1 \leq k \leq n} \frac{1}{n} |\#\{Y_{u_j}^{(n)}; 0 \leq j < k\} - c_{\mu, \nu} k| > \varepsilon \right) = 0.$$

In particular, $\frac{1}{n} \#\mathcal{R}_n \rightarrow c_{\mu, \nu}$ in probability, where \mathcal{R}_n stands for the range $\{Y_u^{(n)}; u \in \tau\}$ of the branching random walk.

Our next result concerns a quenched limit of the following rescaled random pointed measured metric spaces

$$\tau_n := (\tau_n, \frac{1}{a_n} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathfrak{m}_n) \quad \text{and} \quad \mathcal{R}_n := (\mathcal{R}_n, \frac{\sigma_\nu}{\sqrt{a_n}} d_{\text{gr}}, \circ, \frac{1}{n} \mathfrak{m}_{\text{occ}}^{(n)}) \quad (3.1.6)$$

where d_{gr} stands for the graph-distance in τ_n and in \mathbf{T} , where we recall from (3.1.1) the definition of σ_ν and where we have set:

$$\mathcal{R}_n = \{Y_v; v \in \tau_n\}, \quad \mathfrak{m}_n = \sum_{v \in \tau_n} \delta_v, \quad \text{and} \quad \mathfrak{m}_{\text{occ}}^{(n)} = \sum_{v \in \tau_n} \delta_{Y_v^{(n)}}.$$

Let us first recall the limit of τ_n which is encoded by its height process $(H_s^{(n)})_{s \in [0, n]}$ which is the continuous interpolation of the discrete-time process $H_j^{(n)} = d_{\text{gr}}(\emptyset, u_j)$ which is the height in τ_n of the j -th vertex of τ_n in the depth-first exploration. Then, Theorem 3.1 in D. [50] asserts the following: assume that (3.1.5) holds and let (a_n) be as in (3.1.4); then there is a nonnegative continuous process $H = (H_s)_{s \in [0, 1]}$ that only depends on α such that the convergence

$$\left(\frac{1}{a_n} H_{ns}^{(n)} \right)_{s \in [0, 1]} \xrightarrow[n \rightarrow \infty]{} H \quad (3.1.7)$$

holds weakly for the topology of uniform convergence. Under the stronger assumption that the variance of μ is finite, this result is due to Aldous [9, Theorem 23]). The process H is the normalized excursion of the α -stable height process, which is a local-time function of an α -stable spectrally positive Lévy process. In the case $\alpha=2$, H is more precisely the normalized Brownian excursion. We refer to Section 3.2.5 for precise definitions.

As in the discrete setting, H actually encodes a random pointed measured metric space that is defined as follows: we easily check that one defines a pseudometric d_H on $[0, 1]$ from H by setting

$$\forall 0 \leq s_1 \leq s_2 \leq 1, \quad d_H(s_1, s_2) = d_H(s_2, s_1) = H_{s_1} + H_{s_2} - 2 \min_{s \in [s_1, s_2]} H_s. \quad (3.1.8)$$

We introduce the relation \sim_H on $[0, 1]$ by setting $s_1 \sim_H s_2$ if and only if $d_H(s_1, s_2) = 0$; clearly, \sim_H is an equivalence relation and the normalized α -stable Lévy tree is taken as the quotient space $T_H = [0, 1] / \sim_H$, equipped with the distance induced by d_H that we keep denoting d_H . We denote by $p_H: [0, 1] \rightarrow T_H$ the canonical projection. Note that p_H is continuous; therefore T_H is compact and connected. Moreover, T_H is a real tree, namely, a metric space such that all pairs of points are joined by a unique simple arc that turns out to be a geodesic (see Section 3.2.6 for more details). We set $\mathbf{r}_H := p_H(0)$ that is viewed as the root of T_H and we equip T_H with the measure \mathfrak{m}_H that is the image of the Lebesgue measure on $[0, 1]$ via p_H , namely, $\int_{T_H} f d\mathfrak{m}_H = \int_0^1 f(p_H(s)) ds$, for all continuous $f: T_H \rightarrow \mathbb{R}$. The convergence (3.1.7) then implies the following one.

$$\tau_n = \left(\tau_n, \frac{1}{a_n} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathfrak{m}_n \right) \xrightarrow{n \rightarrow \infty} (T_H, d_H, \mathbf{r}_H, \mathfrak{m}_H) \quad (3.1.9)$$

Here the convergence holds weakly on the space \mathbb{M} of isometry classes of pointed measured compact metric spaces equipped with the Gromov-Hausdorff-Prokhorov distance δ_{GHP} that makes it a Polish space, as proved in R. Abraham, J-F. Delmas & P. Hoscheit in [4], Theorem 2.5 (see Section 3.2.6 for precise definitions).

Then the limit of \mathcal{R}_n is constructed as follows: as proved in D. & J-F. Le Gall [57] (Lemma 6.4 p. 600) conditionally given H , there exists a Hölder-continuous centered Gaussian process $\sigma \in T_H \mapsto W_\sigma \in \mathbb{R}$ whose covariance is characterized by $\mathbf{E}[|W_{\sigma_1} - W_{\sigma_2}|^2 | H] = d_H(\sigma_1, \sigma_2)$, for all $\sigma_1, \sigma_2 \in T_H$. Then, we set

$$\forall \sigma_1, \sigma_2 \in T_H, \quad d_{H,W}(\sigma_1, \sigma_2) = W_{\sigma_1} + W_{\sigma_2} - 2 \min_{\sigma \in [\sigma_1, \sigma_2]} W_\sigma,$$

where $[\sigma_1, \sigma_2]$ is the unique geodesic that joins σ_1 to σ_2 in T_H . As proved in D., K., Lin & Torri [60], $d_{H,W}$ is a pseudometric on T_H ; we then define the equivalence relation $\sim_{H,W}$ on T_H by setting $\sigma_1 \sim_{H,W} \sigma_2$ if and only if $d_{H,W}(\sigma_1, \sigma_2) = 0$ and we denote by $T_{H,W} = T_H / \sim_{H,W}$ the quotient metric space and we keep denoting by $d_{H,W}$ the resulting metric; we denote by $\pi_{H,W}: T_H \rightarrow T_{H,W}$ the canonical projection that is continuous. Thus $T_{H,W}$ is compact and connected and $(T_{H,W}, d_{H,W})$ is a real tree (see Section 3.2.8 for more details). It turns out that this kind of spaces has been introduced in N. Curien, J.-F. Le Gall & G. Miermont [40] (see also J.-F. Le Gall [116] for a different purpose); they coined the name Brownian cactus, so we call $(T_{H,W}, d_{H,W})$ the normalized Brownian cactus with α -stable branching mechanism. We next set $\mathbf{r}_{H,W} = \pi_{H,W}(\mathbf{r}_H)$ that is viewed as the root of $T_{H,W}$ and we equip $T_{H,W}$ with the measure $\mathfrak{m}_{H,W}$ that is the image of \mathfrak{m}_H via $\pi_{H,W}$: namely, $\int_{T_{H,W}} f d\mathfrak{m}_{H,W} = \int_{T_H} f(\pi_{H,W}(\sigma)) \mathfrak{m}_H(d\sigma)$, for all continuous $f: T_{H,W} \rightarrow \mathbb{R}$. Our second main result is the following limit theorem.

Theorem 3.1.2 *We keep the above notations: recall that (\mathbf{T}, \circ) is an infinite Galton–Watson tree whose offspring distribution ν is supercritical, that we denote by τ_n is a Galton–Watson tree with offspring distribution μ conditioned to have n vertices for all sufficiently large n , and that (\mathbf{T}, \circ) and τ_n are independent. We assume that μ satisfies (3.1.3) and that ν satisfies (3.1.1). We*

furthermore assume that there exists $b \in (\frac{2\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2b} \nu(k) < \infty$. Recall from (3.1.6) the definition of the spaces τ_n and \mathcal{R}_n . We denote by $(T_H, d_H, \mathbf{r}_H, \mathbf{m}_H)$ the normalized α -stable Lévy tree and by $(T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathbf{m}_{H,W})$ the normalized Brownian cactus with α -stable branching mechanism as defined above. Recall that \mathbb{M} is the space of isometry classes of pointed measured compact metric spaces equipped with the Gromov–Hausdorff–Prokhorov distance. Then, conditionally given the environment $(\mathbf{T}, \mathfrak{o})$, the following joint convergence

$$(\tau_n, \mathcal{R}_n) \xrightarrow[n \rightarrow \infty]{} \left((T_H, d_H, \mathbf{r}_H, \mathbf{m}_H); (T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathbf{m}_{H,W}) \right) \quad (3.1.10)$$

holds weakly on \mathbb{M}^2 equipped with the product topology.

Theorem 3.1.1 corresponds to Theorem 2.1.1 in Chapter 2 about critical BRW on regular infinite trees, and its proof is based on the same ideas. We roughly follow the strategy used by Le Gall & Lin [83] to study \mathbb{Z}^d -valued BRW. Namely, we construct a specific transformation that leaves a BRW indexed by an infinite tree invariant, and we then apply the subadditive ergodic theorem (see in particular Proposition 3.2.32). This invariant model in fact evolves on a slight modification of the environment, whose law has been proved by Peres & Zeitouni [137] to be reversible from the point of view of a critical biased random walk. Then the deterministic constant $c_{\mu,\nu}$ turns out to be the probability that the invariant BRW never returns to its starting position.

The proof of Theorem 3.1.2 is done by proving an analog of (3.1.7) for the branching random walk. Namely, we introduce the discrete snake associated with the branching random walk $(Y_u^{(n)})_{u \in \tau_n}$: namely this process $(\widehat{W}_s^{(n)})_{s \in [0,n]}$ a continuous interpolation of the discrete-time process

$$\widehat{W}_j^{(n)} = \text{generation in } \mathbf{T} \text{ of } Y_{u_j}^{(n)}$$

where $(u_j)_{0 \leq j < n}$ is the depth-first exploration of τ_n . Here, by convention, the generation of \mathfrak{o} is taken equal to 0, so the generation of x in \mathbf{T} is a possibly negative integer. Then the following is proved in Theorem 3.5.1: conditionally given $(\mathbf{T}, \mathfrak{o})$ the following convergence

$$\left(\frac{1}{a_n} H_{ns}^{(n)}, \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{ns}^{(n)} \right)_{s \in [0,1]} \xrightarrow[n \rightarrow \infty]{} (H, \widehat{W}) \quad (3.1.11)$$

holds weakly on the space $\mathbf{C}([0,1], \mathbb{R}^2)$ of continuous functions equipped with the topology of uniform convergence. Here H is the α -stable height process as introduced above and \widehat{W} is the (endpoint process of) the one-dimensional Brownian snake whose lifetime process is H : namely, it is continuous centered real Gaussian process $(\widehat{W}_s)_{s \in [0,1]}$ whose covariances are characterized by $\mathbf{E}[|\widehat{W}_{s_1} - \widehat{W}_{s_2}|^2 | H] = d_H(s_1, s_2)$ for all $s_1, s_2 \in [0,1]$ (see D. & Le Gall [57, Lemma 6.4 p. 600]). We refer to Section 3.2.7 for a precise definition.

To prove (3.1.11) and more generally Theorem 3.5.1, we rely on the so-called harmonic coordinates used by Peres & Zeitouni [137]. These are a family $(S_x)_{x \in \mathbf{T}}$ of real random variables depending on the environment $(\mathbf{T}, \mathfrak{o})$ which approximates the relative generation of $x \in \mathbf{T}$ on the one hand, and on the other hand such that $(S_{X_n})_{n \in \mathbb{N}}$ is a martingale when (X_n) is a m_ν -biased RW on \mathbf{T} . This tool allows us to adapt the method of Janson & Marckert [97] and Marzouk [130]: we obtain the tightness thanks to a maximal inequality for martingales, together with a control of the variations of the height process of the genealogy, and we check the convergence of the finite-dimensional marginals via a conditional central limit theorem.

Theorem 3.1.2 echoes Theorem 2.1.2 of Chapter 2. As in that chapter, our general idea consists in deriving the metric convergence in Theorem 3.1.2 from the snake convergence (3.1.11) (more precisely from Theorem 3.5.1) thanks to general arguments that are in part discussed in Chapter 2. Namely, we show that \mathcal{R}_n is close in some sense to the discrete snake-tree coded by $\widehat{W}^{(n)}$: see Section 3.2.8 and in particular Lemma 3.2.55 for precise statements.

Organization of the chapter

In Section 3.2, we present the main objects and tools we need in this chapter. Among others, this includes the formalism of trees with an infinite line of ancestors, some properties about biased random walks on trees, and definitions and limit theorems for snakes and the real trees they code. Section 3.3 is devoted to the proof of Theorem 3.1.1. In Section 3.4, we show a convergence analogous to (3.1.11) for a branching random walk indexed by a forest of unconditioned Galton–Watson trees. Finally, Section 3.5 completes the demonstrations of Theorems 3.5.1 and 3.1.2.

3.2 Tree-valued branching RWs and snakes: definitions, previous results

In this section, we present the main objects and tools we need in this chapter. Among others, this includes the formalism of trees with an infinite line of ancestors, some properties about biased random walks on trees, and definitions and limit theorems for snakes and the real trees they code.

3.2.1 Rooted ordered trees and forests, coding functions, discrete snakes.

In this section, we recall the definition of three processes used to encode finite trees (*rooted ordered trees within Ulam’s formalism*) or sequences of finite trees (*forests*), namely the *Lukasiewicz path*, the *height process* (as well as a specific linear interpolation of it) and the *contour process*. We then briefly recall the definition of Galton-Watson trees and forest within this framework and finally we recall the definition of *discrete snakes* which are used to encode the positions of real-valued branching walks.

First, let us briefly recall Ulam’s formalism on rooted ordered trees. We first introduce the set of finite words written with positive integers:

$$\mathbb{U} = \bigcup_{n \in \mathbb{N}} (\mathbb{N}^*)^n .$$

Here, $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $(\mathbb{N}^*)^0$ stands for $\{\emptyset\}$. The set \mathbb{U} is totally ordered by the *lexicographical order* \leq_{lex} (the strict order is denoted by $<_{\text{lex}}$). It is convenient to think of \mathbb{U} as a family tree whose \emptyset is the ancestor.

Let $u = (j_1, \dots, j_n) \in \mathbb{U} \setminus \{\emptyset\}$. We use the notation $|u| = n$ for the *height* of u (with the convention $|\emptyset| = 0$) and we set $\overleftarrow{u} = (j_1, \dots, j_{n-1})$ that is interpreted as the (direct) *parent* of u . More generally, for all $p \in \mathbb{N}^*$, we set $u_{|p} = (j_1, \dots, j_{n \wedge p})$ and $u_{|0} = \emptyset$. Namely, $u_{|p}$ is the *ancestor of u at the generation p* and observe that $u_{|p} = u$ if $p \geq |u|$ and that $u_{|p} = \overleftarrow{u}$ if $p = |u| - 1$. Let $v = (k_1, \dots, k_m)$, we denote by $u * v$ the concatenated word $(j_1, \dots, j_n, k_1, \dots, k_m)$. For all $u, v \in \mathbb{U}$, we then denote by $u \wedge v$ the *most recent common ancestor* of u and v , which is defined by $u \wedge v = u_{|p} = v_{|p}$ where $p = \max\{q \in \mathbb{N} : u_{|q} = v_{|q}\}$.

We can think of \mathbb{U} as a graph whose (undirected) edges are $\{u, \overleftarrow{u}\}$, $u \in \mathbb{U} \setminus \{\emptyset\}$. Then, for all $u, v \in \mathbb{U}$, we denote by $\llbracket u, v \rrbracket \subset \mathbb{U}$ the shortest path in the graph \mathbb{U} joining u to v . We also use the following notations $\llbracket u, v \rrbracket = \llbracket u, v \rrbracket \setminus \{u\}$, $\llbracket u, v \llbracket = \llbracket u, v \rrbracket \setminus \{v\}$ and $\llbracket u, v \llbracket = \llbracket u, v \rrbracket \setminus \{u, v\}$. Note that for integers $p \leq q$, we also use the notation $\llbracket p, q \rrbracket$ to denote the set of integers $\llbracket p, q \rrbracket \cap \mathbb{Z}$ (along with the other obvious notations $\llbracket p, q \llbracket$, $\llbracket p, q \llbracket$, and $\llbracket p, q \llbracket$). We do not think there is any conflict in the notations here, as the context always removes any ambiguity.

Definition 3.2.1 (Rooted ordered trees, subtrees, forests)

(a) A *rooted ordered tree* is a subset $t \subset \mathbb{U}$ satisfying the following conditions:

(1): $\emptyset \in t$. **(2):** If $u \in t \setminus \{\emptyset\}$ then $\overleftarrow{u} \in t$. **(3):** For all $u \in t$, there exists $k_u(t) \in \mathbb{N} \cup \{\infty\}$ such that: $u * (j) \in t$ if and only if $1 \leq j \leq k_u(t)$.

Here, $k_u(t)$ is interpreted as the *number of children of u* and if $1 \leq j \leq k_u(t)$, then $u * (j)$ is the j -th child of u . We denote by \mathbb{T} the set of rooted ordered trees.

(b) Let $t \in \mathbb{T}$ and $u \in t$. Then $\theta_u t = \{v \in \mathbb{U} : u * v \in t\}$ is the *subtree of t stemming from u* . Note that $\theta_u t \in \mathbb{T}$.

(c) A *forest* is a sequence of trees $(t^{(k)})_{k \in \mathbb{N}^*}$. It is sometimes convenient to view it as a single tree $t \in \mathbb{T}$ with $k_{\emptyset}(t) = \infty$ such that $t^{(k)}$ is the subtree stemming from the k -th child of t : $t^{(k)} = \theta_{(k)} t$. \square

Definition 3.2.2 (Lukasiewicz path, height process)

(a) Let $t \in \mathbb{T}$ be *finite* and let $(u_l)_{0 \leq l < \#t}$ be the vertices of t listed in increasing lexicographical order. By convenience we set $u_{\#t} = \emptyset$ and we define $V(t) = (V_l(t))_{0 \leq l \leq \#t}$ and $H(t) = (H_l(t))_{0 \leq l \leq \#t}$ by setting $V_0(t) = 0$, $H_{\#t}(t) = 0$ and for all $0 \leq l < \#t$,

$$V_{l+1}(t) = V_l(t) + k_{u_l}(t) - 1 \quad \text{and} \quad H_l(t) = |u_l|. \quad (3.2.1)$$

$V(t)$ is the *Lukasiewicz path* of t and $H(t)$ is the *height process* of t . We extend $H(t)$ continuously on $[0, \#t]$ as follows: for all integer $0 \leq l < \#t$ and all $s \in [0, 1]$, we set

$$H_{l+s}(t) = \begin{cases} H_l(t) - (H_l(t) - |u_l \wedge u_{l+1}|)(2s \wedge 1) + (2s - 1)_+ & \text{if } |u_{l+1}| \leq |u_l| \\ H_l(t) + s & \text{if } |u_{l+1}| > |u_l|. \end{cases} \quad (3.2.2)$$

(see Remark 3.2.3 (c) for a motivation).

(b) Let $(t^{(k)})_{k \in \mathbb{N}^*}$ be a forest of finite trees and $t \in \mathbb{T}$ its associated tree as in Definition 3.2.1 (c). The Lukasiewicz path and the height process of the forest are denoted resp. by $V(t)$ and $H(t)$ and they are obtained by concatenating the $V(t^{(k)})$ and $H(t^{(k)})$ in the following way. We set $\sigma_0 = 0$ and for all $k \in \mathbb{N}^*$, we set $\sigma_k = \#t^{(1)} + \dots + \#t^{(k)}$. Moreover, for all $l \in \llbracket \sigma_{k-1}, \sigma_k \rrbracket$ and for all $s \in [\sigma_{k-1}, \sigma_k]$, we set

$$V_l(t) = V_{l - \sigma_{k-1}}(t^{(k)}) - (k - 1) \quad \text{and} \quad H_s(t) = H_{s - \sigma_{k-1}}(t^{(k)}). \quad (3.2.3)$$

\square

Remark 3.2.3 (a) Let t be the tree associated with the forest of finite trees $(t^{(k)})_{k \in \mathbb{N}^*}$ as in Definition 3.2.1 (c). Let $(u_l)_{l \in \mathbb{N} \cup \{-1\}}$ be the sequence of vertices of t in increasing lexicographical order: $u_{-1} = \emptyset <_{\text{lex}} u_0 <_{\text{lex}} \dots <_{\text{lex}} u_l <_{\text{lex}} u_{l+1} <_{\text{lex}} \dots$. Let $V(t)$ and $H(t)$ be as in Definition 3.2.2 (b). Then $V_{l+1}(t) - V_l(t) = k_{u_l}(t) - 1$ and $H_l(t) = |u_l| - 1$ for all $l \in \mathbb{N}$. Then observe that $s \in [\sigma_k, \sigma_{k+1}]$ if and only if $\inf_{[0, \lfloor s \rfloor]} V(t) = -k$.

(b) Observe that if t is a finite tree or the tree associated to a forest of finite trees, then we can recover t from $H(t)$ or from $V(t)$. Moreover, $H(t)$ is an adapted functional of $V(t)$, as shown by the following equation that holds for finite trees or forest of finite trees.

$$\forall l \in \mathbb{N}^*, \quad H_l(t) = \#\{m \in \{0, \dots, l-1\} : V_m(t) = \inf_{m \leq j \leq l} V_j(t)\}. \quad (3.2.4)$$

See Le Gall & Le Jan [81] for more details.

(c) Let $t \in \mathbb{T}$ be a finite tree and let $(u_l)_{0 \leq l < \#t}$ be the vertices of t listed in increasing lexicographical order. Let l, l' be integers such that $0 \leq l \leq l' < \#t$. Then the height of the most recent common ancestor of u_l and $u_{l'}$ may differ from $\inf_{j \in [l, l']} H_j(t)$ by 1. The specific continuous extension of $H(t)$ implies however that $|u_l \wedge u_{l'}| = \min_{s \in [l, l']} H_s(t)$.

Similarly, if t is the tree corresponding to a forest of finite trees and if $(u_l)_{l \in \mathbb{N} \cup \{-1\}}$ are its sequence of vertices listed in lexicographical order, then we also check that $|u_l \wedge u_{l'}| - 1 = \min_{s \in [l, l']} H_s(t)$, for all integers $l' \geq l \geq 0$. \square

Definition 3.2.4 (Contour process)

(a) Let $t \in \mathbb{T}$, possibly infinite. We denote by $(v_k)_{0 \leq k \leq 2(\#t-1)}$ (with an obvious convention if $\#t = \infty$) the sequence of vertices of t that is defined recursively as follows.

We set $v_0 = \emptyset$ and $R_k = \{w \in t \setminus \{v_0, \dots, v_k\} : \overleftarrow{w} = v_k\}$; if R_k is empty, then we set $v_{k+1} = \overleftarrow{v_k}$; otherwise we take v_{k+1} as the $<_{1\text{ex}}$ -least vertex of R_k .

We call (v_k) the exploration of t in the *contour order*.

(b) Let $t \in \mathbb{T}$ be finite. For all $k \in \llbracket 0, 2(\#t-1) \rrbracket$ we set $C_k(t) = |v_k|$ and we conveniently set $C_{2\#t-1}(t) = C_{2\#t}(t) = 0$. We next extends $C(t)$ continuously on $[0, 2\#t]$ by setting for all $k \in \llbracket 0, 2\#t-1 \rrbracket$ and all $s \in [0, 1]$: $C_{k+s}(t) = C_k(t) + s(C_{k+1}(t) - C_k(t))$. The resulting continuous function $s \in [0, 2\#t] \mapsto C_s(t)$ is called the *contour process* of t .

(c) Let $t \in \mathbb{T}$ be the tree associated with the forest of finite trees $(t^{(k)})_{k \in \mathbb{N}^*}$ (Definition 3.2.1 (c)). The contour process of the forest is denoted $C(t)$ and it is obtained by concatenating the $C(t^{(k)})$ in the following way. Recall notation σ_k from Definition 3.2.2 (b). Then for all $k \in \mathbb{N}^*$, all $s \in [2\sigma_{k-1}, 2\sigma_k]$, we set $C_s(t) = C_{s-2\sigma_{k-1}}(t^{(k)})$. \square

Remark 3.2.5 Let t be the tree associated with the forest of finite trees $(t^{(k)})_{k \in \mathbb{N}^*}$ as in Definition 3.2.1 (c). Observe that the exploration of t in the contour order visits all the vertices of t . In what follows we denote by $(v_k)_{k \in \mathbb{N} \cup \{-1\}}$ the sequence of vertices of t in the contour order starting at $v_{-1} = \emptyset$ (then $v_0 = (1)$, etc).

(a) We first note that $C_k(t) = (|v_k| - 1)_+$ for all $k \in \mathbb{N}$.

(b) As already mentioned, $t = \{v_k; k \in \mathbb{N} \cup \{-1\}\}$. More precisely, during the exploration of t in the contour order, each edge is crossed exactly twice (upwards first and then downwards). Namely, for all $u \in t \setminus \{\emptyset\}$, there is a unique pair of integers k, k' such that $k < k'$ and $(\overleftarrow{u}, u) = (v_k, v_{k+1}) = (v_{k'+1}, v_{k'})$. It is also easy to check recursively for all $k \in \mathbb{N}$ that $\{v_j; -1 \leq j \leq k\} \cap \{v_j; j \geq k\} = \llbracket \emptyset, v_k \rrbracket$.

(c) The contour process $C(t)$ completely encodes t and it is closely related to the height process $H(t)$. Indeed, let $(u_l)_{l \in \mathbb{N} \cup \{-1\}}$ stands for the sequence of vertices of t listed in increasing lexicographical order as in Remark 3.2.3. Then, we observe that

$$\forall l \in \mathbb{N}, \quad K_l := 2l - H_l(t) = \inf\{k \in \mathbb{N} : v_k = u_l\} \quad (3.2.5)$$

which necessarily increases with l . Thus for all $l \in \mathbb{N}$, $C_{K_l}(t) = H_l(t)$ and furthermore, for all $s \in [K_l, K_{l+1}]$, we get

$$C_s(t) = \begin{cases} (H_l(t) - s + K_l)_+ & \text{if } s \in [K_l, K_{l+1} - 1] \\ (H_{l+1}(t) - K_{l+1} + s)_+ & \text{if } s \in [K_{l+1} - 1, K_{l+1}]. \end{cases} \quad (3.2.6)$$

(See D. & Le Gall [61, Section 2.4] for more details).

(d) We can derive $H(t)$ from $C(t)$ by the time-change $\phi_t : [0, \infty) \rightarrow [0, \infty)$ that is defined as follows: we set $\phi_t(0) = 0$ and for all $l \in \mathbb{N}$ and all $s \in [0, 1]$,

$$\phi_t(l + s) = K_l + (K_{l+1} - 1 - K_l)(2s \wedge 1) + (2s - 1)_+. \quad (3.2.7)$$

Then, ϕ_t is an increasing bijection from $[0, \infty)$ onto itself and for all $s \in [0, \infty)$

$$H_s(t) = C_{\phi_t(s)}(t). \quad (3.2.8)$$

We easily check that for all $l \in \mathbb{N}^*$,

$$\max_{s \in [0, l]} \left| \frac{1}{2} \phi_t(s) - s \right| \leq 1 + 2 \sup_{s \in [0, l]} H_s(t). \quad (3.2.9)$$

When $t \in \mathbb{T}$ is finite, then $H(t)$ is also obtained from $C(t)$ by a continuous increasing time-change ϕ_t from $[0, \#t]$ onto $[0, 2\#t]$ by (3.2.7): then (3.2.8) holds true for all $s \in [0, \#t]$ and (3.2.9) holds true for all integers $l \leq \#t$.

(e) As in Remark 3.2.3 (c), we observe that $(|v_k \wedge v_{k'}| - 1)_+ = \min_{s \in [k, k']} C_s(t)$ for all integers $k' \geq k$. If t is a finite tree, then $|v_k \wedge v_{k'}| = \min_{s \in [k, k']} C_s(t)$ is obtained instead, for all integers $0 \leq k \leq k' \leq 2(\#t - 1)$. \square

Galton-Watson trees and forests. We next recall below a convenient recursive definition of Galton-Watson trees and forests (*GW-trees* and *GW-forests*, for short). To that end, we equip \mathbb{T} with the sigma-field $\mathcal{F}(\mathbb{T})$ generated by the sets $\{t \in \mathbb{T} : u \in t\}$, $u \in \mathbb{U}$. Unless otherwise specified, all the random variables that are mentioned in this chapter are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 3.2.6 (GW-trees and forests) Let $\mu = (\mu(k))_{k \in \mathbb{N}}$ be a probability measure on \mathbb{N} .

(a) A *Galton-Watson tree with offspring distribution μ* (a *GW(μ)-tree*, for short) is a random tree τ that satisfies the following.

(1): $k_{\emptyset}(\tau)$ has law μ . (2): For all $k \in \mathbb{N}^*$ such that $\mu(k) > 0$, the subtrees $\theta_{(1)}\tau, \dots, \theta_{(k)}\tau$ under $\mathbf{P}(\cdot | k_{\emptyset}(\tau) = k)$ are independent with the same law as τ under \mathbf{P} .

(b) Let $(\tau^{(k)})_{k \in \mathbb{N}^*}$ be i.i.d. *GW(μ)-trees*. We generically denote by τ_{∞} the tree associated with the forest $(\tau^{(k)})_{k \in \mathbb{N}^*}$. We call τ_{∞} a *GW(μ)-forest*.

(c) We say that τ is a *grafted GW(μ)-tree* if $k_{\emptyset}(\tau) = 1$ and if $\theta_{(1)}\tau$ is a *GW(μ)-tree*. \square

Remark 3.2.7 (a) We shall always consider *GW(μ)-trees* with a non-trivial offspring distribution. Namely, $\mu(0) + \mu(1) < 1$. Let us recall that a *GW(μ)-tree* is a.s. finite if and only if μ is critical or subcritical, namely if and only if $\sum_{k \geq 1} k\mu(k) \leq 1$.

(b) Let us suppose that μ is non-trivial and let τ_{∞} be a *GW(μ)-forest*. Then its *Lukasiewicz path* $(V_l(\tau_{\infty}))_{l \in \mathbb{N}}$ is a random walk (*RW* for short) starting at 0 whose jump law is $\tilde{\mu}(k) := \mu(k + 1)$, $k \in \mathbb{N} \cup \{-1\}$ (according to Spitzer's terminology, it is a *left-continuous RW*). See Le Gall & Le Jan [81] for more details. \square

\mathbb{R} -valued branching walks and discrete snakes. Here we consider a population represented by a rooted ordered tree that moves in \mathbb{R} (a *\mathbb{R} -valued branching walk*): each individual has a spatial position. We discuss two ways to encode the spatial positions of the individuals of the population: namely, via the *contour-snake* and via the *height-snake* that are related to the exploration of the family tree respectively in contour order and in lexicographical order. Here, $\mathbf{C}^0([0, \infty), \mathbb{R})$ stands for the space of the continuous functions from $[0, \infty)$ to \mathbb{R} equipped with the topology of the uniform convergence on every compact interval.

Definition 3.2.8 (Discrete snakes)

(a) Let $t \in \mathbb{T}$ be finite. Let $(v_k)_{0 \leq k \leq 2(\#t-1)}$ be the vertices of t in contour order. It is convenient to set $v_{2\#t-1} = v_{2\#t} = \emptyset$. Let $\mathbf{M} : u \in t \mapsto M_u \in \mathbb{R}$ be a t -indexed and \mathbb{R} -valued walk. For all $k \in [0, 2\#t]$ and all $p \in \mathbb{N}$, we set

$$W_k(t; p) = M_{(v_k)_{|p}},$$

where we recall that $(v_k)_{|p}$ is the ancestor of v_k at generation p (and that $(v_k)_{|p} = v_k$ if $p \geq |v_k|$). We next set

$$\forall r \in [0, \infty), \quad W_k(t; r) = W_k(t; [r]) + (r - [r])(W_k(t; [r] + 1) - W_k(t; [r])).$$

Note that $W_k(t; \cdot) \in \mathbf{C}^0([0, \infty), \mathbb{R})$ for all $k \in [0, 2\#t]$. Recall that $(C_s(t))_{s \in [0, 2\#t]}$ is the contour process as in Definition 3.2.4. We then define the *contour snake* associated with the \mathbb{R} -valued

branching walk $(M_u)_{u \in t}$ as the $\mathbf{C}^0([0, \infty), \mathbb{R})$ -valued continuous process $(W_s(t, \cdot))_{s \in [0, 2\#t]}$ given by

$$\forall s \in [0, 2\#t), \quad W_s(t; \cdot) = \begin{cases} W_{\lfloor s \rfloor + 1}(t; \cdot \wedge C_s(t)) & \text{if } |v_{\lfloor s \rfloor + 1}| = 1 + |v_{\lfloor s \rfloor}|, \\ W_{\lfloor s \rfloor}(t; \cdot \wedge C_s(t)) & \text{if } |v_{\lfloor s \rfloor}| = 1 + |v_{\lfloor s \rfloor + 1}|. \end{cases}$$

We also set $\widehat{W}_s(t) = W_s(t; C_s(t))$, $s \in [0, 2\#t]$, which is the *endpoint process of the contour-snake*.

(b) Let $\phi_t: [0, \#t] \rightarrow [0, 2\#t]$ be the continuous increasing time-change such that $H_s(t) = C_{\phi_t(s)}(t)$ as in Remark 3.2.5 (d). Then we define the *height-snake* as the $\mathbf{C}^0([0, \infty), \mathbb{R})$ -valued continuous process $(W_s^*(t, \cdot))_{s \in [0, \#t]}$ given by

$$\forall s \in [0, \#t], \quad W_s^*(t; \cdot) = W_{\phi_t(s)}(t; \cdot).$$

If $(u_l)_{0 \leq l < \#t}$ stands for the vertices of t listed in lexicographical order, then $W_l^*(t; p) = M_{(u_l)_p}$, for all $l \in \llbracket 0, \#t - 1 \rrbracket$ and all $p \in \mathbb{N}$.

We also set $\widehat{W}_s^*(t) = W_s^*(t; H_s(t)) = \widehat{W}_{\phi_t(s)}(t)$, $s \in [0, \#t]$, which is the *endpoint process of the height-snake*.

(c) Let $(t^{(k)})_{k \in \mathbb{N}^*}$ be a forest of finite trees and let $t \in \mathbb{T}$ be its associated tree as in Definition 3.2.1

(c). For all $k \in \mathbb{N}^*$, let $R^{(k)} = (M_u^{(k)})_{u \in t^{(k)}}$ be a \mathbb{R} -valued $t^{(k)}$ -indexed branching walk. The contour-snake of the branching walks $(M^{(k)})_{k \in \mathbb{N}^*}$ is denoted $(W_s(t; \cdot))_{s \in [0, \infty)}$ and is obtained by concatenating and interpolating the $W_{\cdot}(t^{(k)}; \cdot)$ in the following way. Recall notation σ_k from Definition 3.2.2 (b). Then for all $r, s \in [0, \infty)$

$$W_s(t; r) = \begin{cases} W_{s-2\sigma_{k-1}}(t^{(k)}; r) & \text{if } s \in [2\sigma_{k-1}, 2(\sigma_k - 1)], \\ M_{\emptyset}^{(k)} + \frac{1}{2}(s + 2 - 2\sigma_k)(M_{\emptyset}^{(k+1)} - M_{\emptyset}^{(k)}) & \text{if } s \in [2(\sigma_k - 1), 2\sigma_k]. \end{cases} \quad (3.2.10)$$

Namely, if $s \in [2(\sigma_k - 1), 2\sigma_k]$, then $W_s(t; \cdot)$ is a constant function and its value is the affine interpolation between $M_{\emptyset}^{(k)}$ and $M_{\emptyset}^{(k+1)}$.

We also set $\widehat{W}_s(t) = W_s(t; C_s(t))$, $s \in [0, \infty)$, which is the *endpoint process of the contour-snake*.

(d) We keep the notation of (c) and for all $s \in [0, \infty)$, we set $W_s^*(t; \cdot) = W_{\phi_t(s)}(t; \cdot)$, where ϕ_t is the continuous increasing bijection from $[0, \infty)$ onto itself such that $H_s(t) = C_{\phi_t(s)}(t)$ as in Remark 3.2.5 (d). The resulting $\mathbf{C}^0([0, \infty), \mathbb{R})$ -valued continuous process $(W_s^*(t; \cdot))_{s \in [0, \infty)}$ is the *height-snake* associated with the branching walks $(M^{(k)})_{k \in \mathbb{N}^*}$.

We also set $\widehat{W}_s^*(t) = W_s^*(t; H_s(t)) = \widehat{W}_{\phi_t(s)}(t)$, $s \in [0, \infty)$, which is the *endpoint process of the height-snake*. \square

3.2.2 Trees with an infinite line of ancestors.

We extend Ulam's formalism to define ordered trees that may have an infinite line of ancestors. To that end, it is convenient to introduce *bilateral words* as follows.

Definition 3.2.9 (bilateral words, relative height, shifts)

(a) A *bilateral word* is a sequence $u = (a_k)_{k_1 < k \leq k_2}$ where $k_1 \in \mathbb{Z} \cup \{-\infty\}$, $k_2 \in \mathbb{Z}$ and $a_k \in \mathbb{N}^*$, $k_1 < k \leq k_2$. If $k_1 \geq k_2$, then $u = \emptyset$. We denote by $\overline{\mathbb{U}}$ the set of bilateral words.

(b) Let $u = (a_k)_{k_1 < k \leq k_2} \in \overline{\mathbb{U}}$, as above. Then

$$|u|_- = k_1, \quad \text{and} \quad |u| = k_2. \quad (3.2.11)$$

Here $|u|$ is called the *relative height* of u and $|u|_-$ its *depth*.

(c) Let $u = (a_k)_{k_1 < k \leq k_2} \in \bar{\mathbb{U}}$ as above. Let $l \in \mathbb{Z}$. We denote by $\varphi_l(u)$ the l -shifted word u . Namely,

$$\varphi_l(u) = (a_{k+l})_{k_1-l < k \leq k_2-l}, \quad (3.2.12)$$

Therefore, $|\varphi_l(u)|_- = |u|_- - l$, $|\varphi_l(u)| = |u| - l$. We adopt the convention $\varphi_l(\emptyset) = \emptyset$. Observe that the $\varphi_l: \bar{\mathbb{U}} \rightarrow \bar{\mathbb{U}}$ are bijections such that $\varphi_l \circ \varphi_{l'} = \varphi_{l+l'}$, $l, l' \in \mathbb{Z}$. For all $k \in \mathbb{Z}$, we then set

$$\bar{\mathbb{U}}_k = \varphi_{-k}(\mathbb{U}).$$

Note that $\bar{\mathbb{U}}_k \setminus \{\emptyset\}$ is the set of bilateral words u such that $|u|_- = k$. We shall also denote by $\bar{\mathbb{U}}_{-\infty}$ the set of bilateral words with an infinite depth. \square

Note that $\mathbb{U} \subset \bar{\mathbb{U}}$ and if $u \in \mathbb{U}$, then $|u| = |u|$ (the relative height in this case is the height of the word) and $|u|_- = 0$. Note for all $k \in \mathbb{Z}$ that $\bar{\mathbb{U}}_k$ is countable. However $\bar{\mathbb{U}}_{-\infty}$ is not countable.

Let $u = (a_k)_{k_1 < k \leq k_2} \in \bar{\mathbb{U}}$ be distinct from \emptyset . We introduce the following notations.

$$\text{end}(u) = a_{k_2}, \quad \overleftarrow{u} = (a_k)_{k_1 < k \leq k_2-1} \quad u_{\|(l_1, l_2]} = (a_k)_{k_1 \vee l_1 < k \leq k_2 \wedge l_2}, \quad (3.2.13)$$

for all $l_1 \in \mathbb{Z} \cup \{-\infty\}$, $l_2 \in \mathbb{Z}$ such that $l_1 \leq l_2$. We still call \overleftarrow{u} the (direct) parent of u . Note that if $|u|_- + 1 = |u|$, then $\overleftarrow{u} = \emptyset$. Here, we insist on the following convention: viewed as a bilateral word, \emptyset has neither a relative height nor a depth. However, we set $\emptyset_{\|(l_1, l_2]} = \emptyset$ by convention.

For all $u \in \bar{\mathbb{U}}$, we next denote by L_u the ancestral line of u and by \preceq the genealogical order, which is a partial order on $\bar{\mathbb{U}}$. Namely

$$L_u = \{u_{\|(-\infty, p]}; p \in \mathbb{Z}\} \quad \text{and} \quad \forall u, v \in \bar{\mathbb{U}}, \quad (u \preceq v) \iff (u \in L_v). \quad (3.2.14)$$

Definition 3.2.10 (bilateral trees)

(a) A non-empty subset $T \subset \bar{\mathbb{U}}$ is a *bilateral tree* if it satisfies the following conditions.

(a1) There is $k \in \mathbb{Z} \cup \{-\infty\}$ such that $T \setminus \{\emptyset\} \subset \bar{\mathbb{U}}_k$.

(a2) For all $u \in T \setminus \{\emptyset\}$, $\overleftarrow{u} \in T$.

(a3) For all $u \in T$, $k_u(T) := \#\{v \in T : \overleftarrow{v} = u\} < \infty$ and $\{1, \dots, k_u(T)\} = \{\text{end}(v); v \in T : \overleftarrow{v} = u\}$ if $k_u(T) \geq 1$.

(a4) For all $u, v \in T$, there is $l \in \mathbb{Z}$ such that $u_{\|(-\infty, l]} = v_{\|(-\infty, l]}$.

(b) We denote by $\bar{\mathbb{T}}$ the set of all bilateral trees. For all $T \in \bar{\mathbb{T}}$ distinct from $\{\emptyset\}$, we define the *depth* $|T|_-$ of T as the unique $k \in \mathbb{Z} \cup \{-\infty\}$ such that $T \setminus \{\emptyset\} \subset \bar{\mathbb{U}}_k$.

(c) A *pointed bilateral tree* is a bilateral tree $T \in \bar{\mathbb{T}}$ and a distinguished vertex $o \in T$ that is interpreted as an origin when T is viewed as a space (or a distinguished individual when T is viewed as a family tree: in this case it is rather denoted by ϱ). Recall from (3.2.14) the notation L_o for the ancestral line of o . For all integers $p \in \mathbb{Z}$, we shall use the following notations.

$$o(p) = o_{\|(-\infty, -p]}, \quad \text{and} \quad \partial L_o = \{u \in T \setminus L_o : \overleftarrow{u} \in L_o\}. \quad (3.2.15)$$

Here ∂L_o is called the *spine* of (T, o) . We denote by $\bar{\mathbb{T}}^\bullet$ the set of pointed bilateral trees. \square

Remarks 3.2.11 (a) We view $T \in \bar{\mathbb{T}}$ as a graph: its set of vertices is T and its edges (simple and non oriented) are the unordered pair $\{v, \overleftarrow{v}\}$, $v \in T \setminus \{\emptyset\}$.

(b) Let $T \in \bar{\mathbb{T}}$ be distinct from $\{\emptyset\}$. If $|T|_- \neq -\infty$, then $1 + |T|_- = \min_{u \in T \setminus \{\emptyset\}} |u|$ and Definition 3.2.10 (b) implies that $\emptyset \in T$ that is viewed as the root (or the ancestor) of T . However, if $|T|_- = -\infty$, then $\emptyset \notin T$ and T has no proper ancestor but an infinite line of ancestors. \square

As for rooted ordered trees, we introduce a formal definition of *subtree, most recent common ancestor and lexicographical order* as follows.

• *Subtrees.* Let us first specify what we mean by concatenation in the context of bilateral words. For all $u = (a_k)_{k_1 < k \leq k_2} \in \overline{\mathbb{U}}$ and all $v = (b_k)_{1 \leq k \leq k_3} \in \mathbb{U}$.

$$u * v = (c_l)_{k_1 < k \leq k_2 + k_3} \quad \text{where} \quad c_k = \begin{cases} a_k & \text{If } k_1 < k \leq k_2, \\ b_{k-k_2} & \text{If } k_2 < k \leq k_2 + k_3. \end{cases} \quad (3.2.16)$$

The bilateral word $u * v$ is the concatenation of a bilateral word on the left with a null depth word v on the right. Note that $|u * v|_- = |u|_-$, that $|u * v| = |u| + |v|$ and that $u * \emptyset = u$. Then for all $T \in \overline{\mathbb{T}}$ such that $u \in T$, the *subtree* $\theta_u T$ *stemming from* u is defined by

$$\theta_u T = \{v \in \mathbb{U} : u * v \in T\}, \quad (3.2.17)$$

Note that either $\theta_u T = \{\emptyset\}$ or $|\theta_u T|_- = 0$, namely: $\theta_u T \in \mathbb{T}$.

• *Most recent common ancestor.* Let $T \in \overline{\mathbb{T}}$. We define the *most recent common ancestor* (or more simply the common ancestor) of $u, v \in T$ by

$$u \wedge v := u|_{(-\infty, b(u,v)]} = v|_{(-\infty, b(u,v))} \quad \text{where} \quad b(u, v) := \max\{l \leq |u|, |v| : u|_{(-\infty, l]} = v|_{(-\infty, l]}\}, \quad (3.2.18)$$

which is well-defined thanks to Definition 3.2.10 (a4).

• *Lexicographical order.* Let $T \in \overline{\mathbb{T}}$. The vertices of T are totally ordered by the *lexicographical order* \leq_T that is formally defined as follows: if $\emptyset \in T$, then \emptyset is the \leq_T -least element of T and for all $u, v \in T \setminus \{\emptyset\}$,

$$u \leq_T v \quad \text{iff} \quad \mathbf{1}_{\{b(u,v) < |u|\}} \mathbf{end}(u|_{]-\infty, b(u,v)+1]}) \leq \mathbf{1}_{\{b(u,v) < |v|\}} \mathbf{end}(v|_{]-\infty, b(u,v)+1]}) . \quad (3.2.19)$$

Note that there are $u', v' \in \mathbb{U}$ such that $u = (u \wedge v) * u'$ and $v = (u \wedge v) * v'$. Then, it holds that $u \leq_T v$ if and only if $u' \leq_{\text{lex}} v'$. We denote by $<_T$ the strict order associated with \leq_T .

Pointed labelled trees. We view branching random walks and more generally branching Markov chains as labelled pointed bilateral trees. Let (E, d_E) be a Polish metric space that is the space of labels, i.e. the state space where Markovian spatial motion takes place. A E -labelled tree is given by a pointed tree $(T, o) \in \overline{\mathbb{T}}^\bullet$ and labels $z_v \in E, v \in T$ that are interpreted as follows: T is a family tree of a population (that possibly goes back to an infinite line of ancestors) and $z_v \in E$ is the spatial position of individual v .

We introduce the following notations for the *space of pointed E -labelled trees*.

$$\overline{\mathbb{T}}^\bullet(E) = \{\partial\} \cup \left\{ \Theta = (T, o; \mathbf{z} = (z_v)_{v \in T}) ; (T, o) \in \overline{\mathbb{T}}^\bullet, z_v \in E, v \in T \right\}.$$

Here ∂ stands for a cemetery point.

Truncation and local convergence. We shall consider, the ergodicity of certain transformations on $\overline{\mathbb{T}}^\bullet$ and $\overline{\mathbb{T}}^\bullet(E)$ on the one hand, and $\overline{\mathbb{T}}^\bullet(E)$ -valued random variables and various statements involving their conditional laws on the other hand. For this reason, we equip $\overline{\mathbb{T}}^\bullet(E)$ with the Polish topology inherited from E and from the local convergence on $\overline{\mathbb{T}}^\bullet$. More precisely, we introduce the following truncation procedure.

Definition 3.2.12 Let $\Theta = (T, o; \mathbf{z} = (z_v)_{v \in T}) \in \overline{\mathbb{T}}^\bullet(E)$ and let $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \cup \{\infty\}$ such that $p < q$. If $T = \{\emptyset\}$ then we set $[\Theta]_p^q = \Theta$. When T is distinct from $\{\emptyset\}$, we use the convention that

if $o = \emptyset$ then $|o| = |T|_-$. Then, if $q \leq |T|_-$ or if $p > |o|$, then we adopt the convention that $[\Theta]_p^q = \partial$. Otherwise, we define $[\Theta]_p^q$ as the E -labelled tree $(T', o'; \mathbf{z}' = (z'_{v'})_{v' \in T'})$ given by

$$o' = o_{\|(p,q)}, \quad T' = \{v_{\|(p,q)}; v \in T : v_{\|(-\infty,p]} = o_{\|(-\infty,p]}\} \quad \text{and} \quad z'_{v'} = z_v,$$

where v is the vertex of T such that $|v| \leq q$, $v' = v_{\|(p,q)}$ and $v_{\|(-\infty,p]} = o_{\|(-\infty,p]}$. We adopt the convention that $[\partial]_p^q = \partial$ and we call $[\Theta]_p^q$ the (p, q) -truncation of Θ .

If there is no label, we use the same notation $[(T, o)]_p^q$ for the (p, q) -truncation of the pointed tree $(T, o) \in \bar{\mathbb{T}}^\bullet$. \square

Let us define local convergence on $\bar{\mathbb{T}}^\bullet$ and $\bar{\mathbb{T}}^\bullet(E)$. To that end, we first introduce for all $\Theta = (T, o; \mathbf{z})$, $\Theta' = (T', o'; \mathbf{z}')$ in $\bar{\mathbb{T}}^\bullet(E)$,

$$\Delta_E(\Theta, \Theta') = \mathbf{1}_{\{(T,o) \neq (T',o')\}} + \mathbf{1}_{\{(T,o) = (T',o')\}} \max_{v \in T} (1 \wedge d_E(z_v, z'_v))$$

with the conventions $\Delta_E(\Theta, \partial) = 1$ and $\Delta_E(\partial, \partial) = 0$. We easily check that Δ_E is a (non-separable) metric on $\bar{\mathbb{T}}^\bullet(E)$. Then, *local convergence on $\bar{\mathbb{T}}^\bullet(E)$* is defined as follows.

Definition 3.2.13 For all $n \in \mathbb{N}$, let $\Theta_n = (T^{(n)}, o^{(n)}; \mathbf{z}^{(n)}) \in \bar{\mathbb{T}}^\bullet(E)$.

(a) $(\Theta_n)_{n \in \mathbb{N}}$ *locally converges* to $\Theta = (T, o; \mathbf{z}) \in \bar{\mathbb{T}}^\bullet(E)$ if for all $\varepsilon \in (0, 1)$ and for all $q \in \mathbb{N}$, there is $n_{q,\varepsilon} \in \mathbb{N}$ such that for all integers $n \geq n_{q,\varepsilon}$, $\Delta_E([\Theta^{(n)}]_{-q}^q, [\Theta]_{-q}^q) < \varepsilon$.

(b) The $\bar{\mathbb{T}}^\bullet$ -valued sequence $((T^{(n)}, o^{(n)}))_{n \in \mathbb{N}}$ *locally converges* to (T, o) if for all $\varepsilon \in (0, 1)$ and for all $q \in \mathbb{N}$, there is $n_{q,\varepsilon} \in \mathbb{N}$ such that for all integers $n \geq n_{q,\varepsilon}$, $[(T^{(n)}, o^{(n)})]_{-q}^q = [(T, o)]_{-q}^q$. \square

Remark 3.2.14 Local convergence corresponds for instance to the following metric.

$$\forall \Theta, \Theta' \in \bar{\mathbb{T}}^\bullet(E), \quad \delta_{\text{loc}}^E(\Theta, \Theta') = \sum_{q \in \mathbb{N}} 2^{-q-1} \Delta_E([\Theta]_{-q}^q, [\Theta']_{-q}^q), \quad (3.2.20)$$

with $\delta_{\text{loc}}^E(\Theta, \partial) = 1$ and $\delta_{\text{loc}}^E(\partial, \partial) = 0$. We easily check that the topology of $(\bar{\mathbb{T}}^\bullet(E), \delta_{\text{loc}}^E)$ is Polish. Note that $\Theta \mapsto [\Theta]_p^q$ is (by definition) continuous and that $\lim_{p,q \rightarrow \infty} \delta_{\text{loc}}^E(\Theta, [\Theta]_{-p}^q) = 0$.

Similarly, local convergence on $\bar{\mathbb{T}}^\bullet$ corresponds to a Polish topology. generated by the distance $\delta_{\text{loc}}((T, o), (T', o')) = \sum_{q \in \mathbb{N}} 2^{-q-1} \mathbf{1}_{\{[(T,o)]_{-q}^q \neq [(T',o')]_{-q}^q\}}$, for all $(T, o), (T', o') \in \bar{\mathbb{T}}^\bullet$, with the same conventions $\delta_{\text{loc}}((T, o), \partial) = 1$ and $\delta_{\text{loc}}(\partial, \partial) = 0$. Let us mention that in most parts of the chapter, the space E of the labels is the space of pointed bilateral trees $\bar{\mathbb{T}}^\bullet$ itself. \square

Centering and successors functions, infinite Galton-Watson trees. We next introduce two fundamental functions on $\bar{\mathbb{T}}^\bullet$ that allow us to define dynamics: the centering function and the successor function. We right after introduce infinite Galton-Watson trees that are invariant under such transforms and whose law has nice related ergodic properties.

Definition 3.2.15 Let $(T, o) \in \bar{\mathbb{T}}^\bullet$. When $\emptyset \in T$ but $T \neq \{\emptyset\}$, we use the convention $|\emptyset| = |T|_-$.

(a) If $T = \{\emptyset\}$ then we set $\text{cent}(T, o) = (T, o)$, and otherwise, we set

$$\text{cent}(T, o) = (\varphi_{|o|}(T), \varphi_{|o|}(o)) \quad (3.2.21)$$

The function $\text{cent} : \bar{\mathbb{T}}^\bullet \rightarrow \bar{\mathbb{T}}^\bullet$ is called the *centering* function.

(b) For all $u \in T$, the *successor* $\text{scc}(u)$ of u is defined as the \leq_T -least element of $\{v \in T : u <_T v\}$ if this set is not empty, otherwise we simply set $\text{scc}(u) = u$.

(c) We then set

$$\text{succ}(T, o) = \text{cent}(T, \text{scc}(o)) \quad (3.2.22)$$

The function $\text{succ} : \overline{\mathbb{T}}^\bullet \rightarrow \overline{\mathbb{T}}^\bullet$ is called the *successor* function.

(d) The *right-part* of (T, o) is an element of $\overline{\mathbb{T}}^\bullet$ that corresponds to the ancestral line of o and the vertices comming next in lexicographical order. Namely, there is unique pointed tree $(T', o') \in \overline{\mathbb{T}}^\bullet$ and a unique one-to-one function $\psi : T' \rightarrow \{o_{|(-\infty, p]}; p \leq |o|\} \cup \{v \in T : o <_T v\}$ that is increasing with respect to lexicographical orders and such that $|\psi(v)| = |v|$. Then we set

$$[(T, o)]^+ = (T', o') \quad (3.2.23)$$

Namely, $[(T, o)]^+ \in \overline{\mathbb{T}}^\bullet$ is called the *right-part of (T, o)* . We also introduce the following.

$$\text{succ}^+(T, o) = [\text{succ}(T, o)]^+ \quad (3.2.24)$$

The function $\text{succ}^+ : \overline{\mathbb{T}}^\bullet \rightarrow \overline{\mathbb{T}}^\bullet$ is called the *right-successor* function. \square

Remarks 3.2.16 (a) Note that $\text{succ}(u)$ of course depends on the tree to which u belongs, although it does not appear in the notation.

(b) Note that $\text{cent}([\cdot]^+) = [\text{cent}(\cdot)]^+$ and that $\text{succ}^+ = \text{succ}^+([\cdot]^+)$, on $\overline{\mathbb{T}}^\bullet$. \square

We recall from Chapter 2 the following.

Lemma 3.2.17 *The centering application $\text{cent}(\cdot)$ and the right-part application $[\cdot]^+$ are locally continuous on $\overline{\mathbb{T}}^\bullet$. Successor map and right-successor maps $\text{scc}(\cdot)$ and $\text{scc}^+(\cdot)$ are locally continuous on $\overline{\mathbb{T}}^\bullet$ at pointed trees (T, o) such that $\text{succ}(o) \neq o$.*

Proof. See Lemma 2.2.7 in Chapter 2 for the result on $\overline{\mathbb{T}}^\bullet$. \blacksquare

Definition 3.2.18 (Infinite Galton-Watson trees) Let $\mathbf{r} = (r(j, k))_{k \geq j \geq 1}$ be a probability measure on $\{(j, k) \in (\mathbb{N}^*)^2 : j \leq k\}$. Let μ be a probability measure on \mathbb{N} whose mean $\mathfrak{m}_\mu := \sum_{k \in \mathbb{N}} k\mu(k)$ is finite and positive. Let $(\mathbf{T}, o) : \Omega \rightarrow \overline{\mathbb{T}}^\bullet$ be a Borel-measurable random pointed tree such that

$$\mathbf{P}\text{-a.s. } |o| = 0 \quad \text{and} \quad |\mathbf{T}|_- = -\infty .$$

Recall from (3.2.14) that the notation $L_o = \{o(p); p \in \mathbb{N}\}$ for the the ancestral line of o and recall from (3.2.15) the definition of the spine ∂L_o of (\mathbf{T}, o) . Recall from (3.2.13) the notation $\text{end}(\cdot)$.

(a) We say that (\mathbf{T}, o) is an *infinite Galton-Watson tree with offspring distribution μ and dispatching measure \mathbf{r}* if:

- the r.v. $S := (\text{end}(o(p)), k_{o(p+1)}(\mathbf{T}))_{p \in \mathbb{N}}$ are i.i.d. with law \mathbf{r} ,
- conditionally given S , the subtrees $(\theta_u \mathbf{T}; u \in \partial L_o)$ are independent $\text{GW}(\mu)$ -trees (as in Definition 3.2.6 (a))

More specifically:

(b) (\mathbf{T}, o) is an *infinite Galton-Watson tree with offspring distribution μ* if $r(j, k) = \mu(k)/\mathfrak{m}_\mu$, for all $k \geq j \geq 1$.

(c) (\mathbf{T}, o) is a *right-infinite Galton-Watson tree with offspring distribution μ* if $r(j, k) = \mathbf{1}_{\{j=1\}} \bar{\mu}(k)$, for all $k \geq j \geq 1$, where $\bar{\mu}(k) = \mathfrak{m}_\mu^{-1} \sum_{l \geq k} \mu(l)$ for all $k \geq 1$. \square

Remark 3.2.19 When (\mathbf{T}, o) is an infinite $\text{GW}(\mu)$ -tree, then we see that $[(\mathbf{T}, o)]^+$ is a right-infinite $\text{GW}(\mu)$ -tree. \square

Many-to-one. Infinite (pointed) GW-trees are related to GW-trees via the *many-to-one* principle that asserts the following. Let \mathbf{T}' be a $\text{GW}(\mu)$ -tree and let (\mathbf{T}, \circ) be an infinite $\text{GW}(\mu)$ -tree. Then for all Borel-measurable functions $F: \mathbb{N} \times \overline{\mathbb{T}}^\bullet \rightarrow [0, \infty)$,

$$\mathbf{E} \left[\sum_{v \in \mathbf{T}'} F(|v|; \text{cent}(\mathbf{T}', v)) \right] = \sum_{p \geq 0} \mathfrak{m}_\mu^p \mathbf{E} [F(p; [(\mathbf{T}, \circ)]_{-p}^\infty)], \quad (3.2.25)$$

where $\mathfrak{m}_\mu = \sum_{k \in \mathbb{N}} k\mu(k) < \infty$ and where we recall from Definition 3.2.12 that $[(\mathbf{T}, \circ)]_{-p}^\infty$ is the tree (\mathbf{T}, \circ) truncated above $\circ(p)$, a vertex which is situated at relative height $-p$.

We next recall the following result that shows that infinite (pointed) GW-trees are local limits of GW-trees conditioned to be large and that they are therefore invariant under the successor map as introduced in Definition 3.2.15.

Proposition 3.2.20 *Let μ be an offspring distribution on \mathbb{N} such that $\mu(0) + \mu(1) < 1$ and $\sum_{k \in \mathbb{N}} k\mu(k) = 1$. Then, the following holds true.*

(i) *We suppose that μ is aperiodic. Let \mathbf{T}' be a $\text{GW}(\mu)$ -tree and let \mathbf{u} be uniformly distributed on the set of vertices of \mathbf{T}' . Let (\mathbf{T}, \circ) be an infinite $\text{GW}(\mu)$ -tree as in Definition 3.2.18. Then*

$$\text{cent}(\mathbf{T}', \mathbf{u}) \quad \text{under} \quad \mathbf{P}(\cdot \mid \#\mathbf{T}' = n) \xrightarrow[n \rightarrow \infty]{} (\mathbf{T}, \circ)$$

weakly in $\overline{\mathbb{T}}^\bullet$ with respect to local convergence.

(ii) *The law of an infinite $\text{GW}(\mu)$ -tree (resp. of a right-infinite $\text{GW}(\mu)$ -tree) is preserved by $\text{scc}(\cdot)$ (resp. by $\text{scc}^+(\cdot)$).*

Proof. (i) is derived from the many-to-one formula (3.2.25) and it seems to be part of the ‘folklore’: the idea goes back to Kesten [100]. Then (ii) is derived from (i): Le Gall & Lin [82, Proposition 2] or Proposition 2.2.10 in Chapter 2. \blacksquare

3.2.3 Biased random walks on trees

In this section, we recall the definition of *biased random walks on trees* as introduced by Lyons [123]. They model-depend on one parameter λ ranging in $(0, \infty)$ and we focus on the critical parameter when the environment is a Galton-Watson tree \mathbf{T} . Instead of viewing the RW as successive positions X_n in \mathbf{T} , $n \in \mathbb{N}$, we view it as the $\overline{\mathbb{T}}^\bullet$ -valued sequence $\overline{X}_n = (\mathbf{T}, X_n)$. We then recall the definition of the law of an invariant GW-tree, that is a law on $\overline{\mathbb{T}}^\bullet$ invariant (up to centering) for the biased RW $(\overline{X}_n)_{n \in \mathbb{N}}$. We next recall ergodic properties of biased RW on GW-trees which have been originally investigated in Zeitouni & Peres [137].

λ -biased random walks on trees. Fix $\lambda \in (0, \infty)$ and $T \in \overline{\mathbb{T}}$, a bilateral tree not reduced to $\{\emptyset\}$. The transition probabilities of a λ -biased RW on T are defined as follows: $p_{T,\lambda}(x, y) = 0$ if x and y are not adjacent and otherwise for all $x \in T \setminus \{\emptyset\}$,

$$p_{T,\lambda}(\overleftarrow{x}, x) = \mathbf{1}_{\{\overleftarrow{x} \neq \emptyset\}} \frac{1}{\lambda + k_{\overleftarrow{x}}(T)} + \mathbf{1}_{\{\overleftarrow{x} = \emptyset\}} \frac{1}{k_\emptyset(T)} \quad \text{and} \quad p_{T,\lambda}(x, \overleftarrow{x}) = \frac{\lambda}{\lambda + k_x(T)}. \quad (3.2.26)$$

If $|T|_- = -\infty$, then recall that $\emptyset \notin T$.

A λ -RW on T is the RW associated with the conductances $C_{\{x, \overleftarrow{x}\}} = \lambda^{-|x|}$. As proved in R. Lyons [123] when $T \in \mathbb{T}$, there is a quantity $\text{br}(T) \in [0, \infty]$ called the *branching number* such that the λ -biased RW is recurrent if $\lambda > \text{br}(T)$, and transient if $\lambda < \text{br}(T)$ (see also Lyons & Peres [127, Theorem 3.5] for more details).

In this chapter, we want to view λ -biased RW on a tree $T \in \overline{\mathbb{T}}$ as a $\overline{\mathbb{T}}^\bullet$ -valued sequences of r.v. $\overline{X}_n = (T, X_n)$, $n \in \mathbb{N}$. The only reason why we adopt this point of view is technical: we want to apply ergodic theory, to work with regular versions of conditional laws of the random walk when (T, o) is random, etc. The transition kernels of the Markov chain $(\overline{X}_n)_{n \in \mathbb{N}}$ have to be modified accordingly as explained in the following definition. To fit in our framework of labelled bilateral trees, we generically denote elements of $\overline{\mathbb{T}}^\bullet$ by $\overline{x} = (T, x)$, $\overline{x}' = (T', x')$, etc and $\overline{\mathbb{T}}^\bullet$ -valued sequences by $\overline{x}_n = (T_n, x_n)$, $n \in \mathbb{N}$.

Definition 3.2.21 Let $(T, o) \in \overline{\mathbb{T}}^\bullet$. Let $\lambda \in (0, \infty)$ and let $p_{T, \lambda}(\cdot, \cdot)$ be the transition probabilities of λ -RWs on T as defined in (3.2.26). The transition kernel on $\overline{\mathbb{T}}^\bullet$ of the λ -biased RW is then defined for all $\overline{x} := (T, x) \in \overline{\mathbb{T}}^\bullet$ by

$$\overline{p}_\lambda(\overline{x}, d\overline{x}') = \sum_{y \in T} p_{T, \lambda}(x, y) \delta_{(T, y)}(d\overline{x}'). \quad (3.2.27)$$

We then denote by $P_{T, o}$ the law of the λ -biased RW on $(\overline{\mathbb{T}}^\bullet)^\mathbb{N}$ (that is the space of the $\overline{\mathbb{T}}^\bullet$ -valued sequences equipped with the product topology, which makes it Polish) starting at (T, o) . Note that $P_{T, o}$ depends on λ although it does not appear in the notation. \square

Remark 3.2.22 It is easy to check that

$$(T, o) \in \overline{\mathbb{T}}^\bullet \mapsto P_{T, o} \text{ is continuous} \quad (3.2.28)$$

with respect to local convergence on $\overline{\mathbb{T}}^\bullet$ and to the weak convergence on the space of probability measures on $(\overline{\mathbb{T}}^\bullet)^\mathbb{N}$. If (\mathbf{T}, o) is a $\overline{\mathbb{T}}^\bullet$ -valued r.v., then $P_{\mathbf{T}, o}$ stands for the (regular version of the) conditional law conditionally given (\mathbf{T}, o) of the λ -biased RW on (\mathbf{T}, o) denoted by $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, starting at $\overline{X}_0 = (\mathbf{T}, o)$. These technical points being cleared, when such a formal point of view is not essential to our arguments, we rather speak of $(X_n)_{n \in \mathbb{N}}$ as to the λ -biased RW on \mathbf{T} starting at o . \square

Let $\overline{X}_n = (T, X_n)$, $n \in \mathbb{N}$, be a λ -biased random walk whose initial value is $\overline{X}_0 = (T, x)$. We use the following notations for the hitting times of \overline{X} :

$$\forall y \in T, \quad H_y = \inf\{n \in \mathbb{N} : X_n = y\} \quad \text{and} \quad H_y^+ = \inf\{n \in \mathbb{N}^* : X_n = y\}, \quad (3.2.29)$$

with the convention that $\inf \emptyset = \infty$. The next lemma recalls the basic results that we need on hitting times and the Green function of λ -biased random walks.

Lemma 3.2.23 Let $\lambda \in (1, \infty)$ and let $\overline{x} = (T, x) \in \overline{\mathbb{T}}^\bullet$, whose depth $|T|_-$ is possibly infinite. Let $\overline{X}_n = (T, X_n)$, $n \in \mathbb{N}$, be a λ -biased RW with initial state (T, x) . We assume that

$$\forall x \in T \setminus \{\emptyset\}, \quad P_{(T, x)}(H_{\overline{x}} < \infty) = 1. \quad (3.2.30)$$

Let $o \in T \setminus \{\emptyset\}$. Recall from (3.2.14) that L_o stands for the ancestral line of o and let us denote by $o(p)$ the vertex of L_o at distance p from o , which makes sense if and only if $0 \leq p \leq |o| - |T|_-$. Namely

$$o(p) = o_{\|(-\infty, |o| - p]}, \quad 0 \leq p \leq |o| - |T|_- \quad (3.2.31)$$

We also introduce the successive times of visit of \overline{X} in L_o that are recursively defined for all $k \in \mathbb{N}$ by $\mathbf{r}_k^o = \inf\{n > \mathbf{r}_{k-1}^o : X_n \in L_o\}$, with the convention that $\mathbf{r}_{-1}^o = -1$ and $\inf \emptyset = \infty$. Then, the following holds true.

- (i) For all $x \in T$, $P_{(T,x)}$ -a.s. for all $k \in \mathbb{N}$, $\mathbf{r}_k^\circ < \infty$ and there exists a birth-and-death Markov chain $(Z_k)_{k \in \mathbb{N}}$ that takes its values in $\mathbb{N} \cap [0, |\circ| - |T|_-]$ and such that $\circ(Z_k) = X_{\mathbf{r}_k}$, $k \in \mathbb{N}$. If T has a finite depth, then it is reflected at $|\circ| - |T|_-$; otherwise, its transition probabilities are given for all $1 \leq p < |\circ| - |T|_-$ by

$$\rho_{p,p+1} = \frac{\lambda}{\lambda + k_{\circ(p)}(T)}, \quad \rho_{p,p-1} = \frac{1}{\lambda + k_{\circ(p)}(T)}, \quad \rho_{p,p} = \frac{k_{\circ(p)}(T) - 1}{\lambda + k_{\circ(p)}(T)}, \quad (3.2.32)$$

and $\rho_{0,0} = 1 - \rho_{0,1} = \frac{k_\circ(T)}{\lambda + k_\circ(T)}$. Moreover, if T has an infinite depth, then Z is transient which implies that $|\circ \wedge X_n| \rightarrow -\infty$ and that $(X_n)_{n \in \mathbb{N}}$ is transient.

- (ii) Let $x, y, z \in T$ such that $z \preceq y$. Then

$$P_{(T,x)}(\mathbb{H}_y < \mathbb{H}_z) = \frac{(\lambda^{|x \wedge y|} - \lambda^{|z|})_+}{\lambda^{|y|} - \lambda^{|z|}}. \quad (3.2.33)$$

- (iii) We assume now that T has an infinite depth. For all $x, y \in T$, we first get

$$P_{(T,x)}(\mathbb{H}_y < \infty) = \lambda^{-(|y| - |x \wedge y|)} \quad \text{and} \quad P_{(T,x)}(\mathbb{H}_x^+ < \infty) = \frac{1 + k_x(T)}{\lambda + k_x(T)}. \quad (3.2.34)$$

We denote the Green function by $G_T(x, y) = E_{(T,x)}[\sum_{n \in \mathbb{N}} \mathbf{1}_{\{X_n=y\}}]$. Then

$$G_T(x, y) = G_T(x \wedge y, y) = \frac{\lambda + k_y(T)}{\lambda - 1} \lambda^{-(|y| - |x \wedge y|)}. \quad (3.2.35)$$

If $x \preceq y$, we also get $E_{(T,x)}[\sum_{0 \leq n < \mathbb{H}_x} \mathbf{1}_{\{X_n=y\}}] = (\lambda + k_y(T)) \lambda^{-(|y| - |x| + 1)}$.

Proof. We easily see that Assumption (3.2.30) implies that $\mathbf{r}_k^\circ < \infty$ a.s. for all $k \in \mathbb{N}$. Markov property at \mathbf{r}_k° implies that $(Z_k)_{k \in \mathbb{N}}$ is a \mathbb{N} -valued birth-and-death Markov chain with transition probabilities as in (3.2.32). Gambler's ruin for birth and death chains (see e.g. Durrett [64, Theorem. 6.4.6 p. 249]) implies the first equality in (3.2.33). When T has an infinite depth, it also implies that $|\circ \wedge X_n| \rightarrow -\infty$ (the RW is transient) and (3.2.34).

We now compute the Green function when T has an infinite depth: clearly $G_T(x, y) = G_T(x \wedge y, y)$, and we only need to consider the cases where $x \preceq y$. By the Markov property, we observe that $G_T(x, y) = P_{T,x}(\mathbb{H}_y < \infty) G_T(y, y)$ and that $E_{T,y}[\sum_{n \in \mathbb{N}} \mathbf{1}_{\{X_n=y\}} | (X_0, \dots, X_{\mathbb{H}_y})] = 1 + Z + G_T(\overleftarrow{y}, y)$ where $P_{T,y}(Z = j) = p(1-p)^j$, $j \in \mathbb{N}$, with $p = p_{T,\lambda}(y, \overleftarrow{y})$. Thus, $G_T(x, y) = P_{T,x}(\mathbb{H}_y < \infty)(p^{-1} + G_T(\overleftarrow{y}, y))$. We find the value of $G_T(\overleftarrow{y}, y)$ by taking $x = \overleftarrow{y}$ and (3.2.35) follows.

We then prove the last point of (iii) by counting how many times the RW visits y before and after \mathbb{H}_x to get $G_T(x, y) - G_T(\overleftarrow{x}, y) = E_{(T,x)}[\sum_{0 \leq n < \mathbb{H}_x} \mathbf{1}_{\{X_n=y\}}]$, by Markov, which implies the desired result. \blacksquare

Critical biased RW on GW-trees. In this chapter, we restrict our attention to *critical* biased random walks on supercritical Galton-Watson trees and variants of such trees. More precisely, let \mathbf{T}' be $\text{GW}(\nu)$ -tree whose offspring distribution ν is supercritical: namely $m_\nu := \sum_{k \in \mathbb{N}} k\nu(k) \in (1, \infty)$. In R. Lyons [123, Proposition 6.4]), the following is proved: $\mathbf{P}(\cdot | \#\mathbf{T}' = \infty)$ -a.s. $\text{br}(\mathbf{T}') = m_\nu$ and furthermore m_ν -biased RWs on \mathbf{T}' are null-recurrent. We shall refer to this case as to the *critical case*. This lemma shows in particular that the estimates of Lemma 3.2.23 apply to critical biased RW on GW trees.

Lemma 3.2.24 *Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . Let (\mathbf{T}, \circ) be a $\text{GW}(\nu)$ -tree (Definition 3.2.6) or an infinite $\text{GW}(\nu)$ -tree (Definition 3.2.18). Let $\overline{X}_n = (\mathbf{T}, X_n)$ be a m_ν -biased RW on \mathbf{T} . Then, \mathbf{P} -a.s. for all $x \in \mathbf{T} \setminus \{\emptyset\}$, $P_{(\mathbf{T},x)}(\mathbb{H}_x < \infty) = 1$.*

Proof. By e.g. Lyons & Peres [127, Theorem. 3.5]), as already mentioned if \mathbf{T} is a $\text{GW}(\nu)$ -tree, then the m_ν -biased RW is recurrent which implies the lemma in this case.

Suppose that (\mathbf{T}, \circ) is an infinite $\text{GW}(\nu)$ -tree: conditionally given ∂L_\circ , the subtrees $\theta_y \mathbf{T}$, $y \in \partial L_\circ$, are i.i.d. $\text{GW}(\nu)$ -trees and \mathbf{P} -a.s. for all $x \notin L_\circ$, $P_{\mathbf{T},x}(\mathbb{H}_{\overleftarrow{x}} < \infty) = 1$ by the previous case. If $x \in L_\circ$ and $\mathbb{H}_{\overleftarrow{x}} = \infty$, then the previous cases imply that the \mathfrak{m}_ν -biased RW has to visit infinitely many often each vertex of the subtree above x because it is the union of a finite number of $\text{GW}(\nu)$ -trees and of a finite line. However, each time the RW returns to x , it has an independent possibility to jump to \overleftarrow{x} with probability $p_{T,\mathfrak{m}_\nu}(x, \overleftarrow{x})$. Thus \mathbf{P} -a.s. $P_{\mathbf{T},x}(\mathbb{H}_{\overleftarrow{x}} = \infty) = 0$, which completes the proof of the lemma. \blacksquare

Let $(X_n)_{n \in \mathbb{N}}$ be a \mathfrak{m}_ν -biased RW on a $\text{GW}(\nu)$ -tree \mathbf{T}' . Then the centered trees $\text{cent}(\mathbf{T}', X_n)$ weakly locally converge in $\overline{\mathbb{T}}^\bullet$ to a law which is recalled in the following definition, and which is in some sense invariant and reversible for \mathfrak{m}_ν -RWs as explained right after.

Definition 3.2.25 Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . Let (\mathbf{T}, \circ) be $\overline{\mathbb{T}}^\bullet$ -valued r.v. It is distributed as an *invariant $\text{GW}(\nu)$ -tree* if for all bounded Borel measurable function F on $\overline{\mathbb{T}}^\bullet$,

$$\mathbf{E}[F(\mathbf{T}, \circ)] = \mathbf{E}\left[\frac{\mathfrak{m}_\nu + k_{\mathfrak{d}'}(\mathbf{T}')}{2\mathfrak{m}_\nu} F(\mathbf{T}', \circ')\right], \quad (3.2.36)$$

where (\mathbf{T}', \circ') is distributed as an infinite $\text{GW}(\nu)$ -tree as in Definition 3.2.18. \square

Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree and let $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a \mathfrak{m}_ν -biased RW on \mathbf{T} starting at $\overline{X}_0 = (\mathbf{T}, \circ)$. As proved in Zeitouni & Peres [137, Lem. 2], the law of invariant $\text{GW}(\nu)$ -trees is invariant and reversible in the following sense:

$$(\overline{X}_0, \overline{X}_1) \stackrel{(\text{law})}{=} (\text{cent}(\overline{X}_1), \varphi_{|\overline{X}_1|}(\overline{X}_0)) \quad (3.2.37)$$

where we recall $\text{cent}(\cdot): \overline{\mathbb{T}}^\bullet \rightarrow \overline{\mathbb{T}}^\bullet$ from Definition 3.2.15. This allows us to extend this Markov chain into a *stationnary process* $(\overline{X}_n)_{n \in \mathbb{Z}}$ such that for all $n_0 \in \mathbb{Z}$:

$$(\varphi_{|\overline{X}_{n_0}|}(\overline{X}_{n_0+n}))_{n \in \mathbb{Z}} \stackrel{(\text{law})}{=} (\varphi_{|\overline{X}_{n_0}|}(\overline{X}_{n_0-n}))_{n \in \mathbb{Z}} \stackrel{(\text{law})}{=} (\overline{X}_n)_{n \in \mathbb{Z}}. \quad (3.2.38)$$

Moreover, this process is ergodic as stated in the following proposition.

Proposition 3.2.26 Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν and let $\overline{\mathfrak{p}}_{\mathfrak{m}_\nu}$ be as in Definition 3.2.21. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. Let $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a \mathfrak{m}_ν -biased RW on (\mathbf{T}, \circ) starting at $\overline{X}_0 = (\mathbf{T}, \circ)$. Denote by Π_ν the law of $(\overline{X}_n)_{n \in \mathbb{N}}$ on $(\overline{\mathbb{T}}^\bullet)^\mathbb{N}$. Then, $\vartheta: (\overline{x}_n)_{n \in \mathbb{N}} \mapsto (\varphi_{|\overline{x}_1|}(\overline{x}_{n+1}))_{n \in \mathbb{N}}$ is Π_ν -ergodic, where $\overline{x}_n = (T_n, x_n)$, $n \in \mathbb{N}$, stands for the canonical sequence on $(\overline{\mathbb{T}}^\bullet)^\mathbb{N}$.

Proof. See Dembo & Sun [45, Theorem 1.4 and Section 2.1]. In their notations, here, A is the single-entry matrix \mathfrak{m}_ν , which is of course irreducible, and $\rho = \mathfrak{m}_\nu$. We also provide an alternative proof in Section 3.3.3. \blacksquare

We shall apply Proposition 3.2.26 in the following specific way.

Corollary 3.2.27 We keep the assumptions and the notations of Proposition 3.2.26. Let $F: \overline{\mathbb{T}}^\bullet \rightarrow \mathbb{R}$ be a measurable function such that $\mathbf{E}[|F(\mathbf{T}, \circ)|] < \infty$ and such that $F(T, \circ) = F(\text{cent}(T, \circ))$ for all $(T, \circ) \in \overline{\mathbb{T}}^\bullet$. Then

$$\mathbf{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq k < n} F(\mathbf{T}, X_k) = \mathbf{E}[F(\mathbf{T}, \circ)]. \quad (3.2.39)$$

Moreover, if F is nonnegative and if there is $b \in (1, \infty)$ such that $\mathbf{E}[F(\mathbf{T}, \circ)^b] < \infty$, then

$$\mathbf{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \frac{1}{n} \sum_{0 \leq k < n} E_{\mathbf{T}, \circ}[F(\mathbf{T}, X_k)] < \infty. \quad (3.2.40)$$

Proof. Denote by $\bar{x} = (\bar{x}_n)_{n \in \mathbb{N}}$ the canonical $\bar{\mathbb{T}}^\bullet$ -sequence. We denote by $G(\bar{x}) = F(\text{cent}(\bar{x}_0))$. Observe that $F(\mathbf{T}, X_k) = (G \circ \vartheta^{ok})(\bar{X}_n)_{n \in \mathbb{N}}$. It first implies that $(F(\mathbf{T}, X_k))_{k \in \mathbb{N}}$ under \mathbf{P} has the same law as $(G \circ \vartheta^{ok})_{k \in \mathbb{N}}$ under Π_ν . By Birkhoff ergodic theorem (see e.g. Krengel [112, Theorem 2.3 p. 9]) Π_ν -a.s. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq k < n} G \circ \vartheta^{ok} = \int G d\Pi_\nu = \mathbf{E}[F(\mathbf{T}, \circ)]$, hence (3.2.39).

We next suppose F (and thus G) nonnegative and such that $\mathbf{E}[F(\mathbf{T}, \circ)^b] = \int G^b d\Pi_\nu < \infty$, and we proceed to the proof of (3.2.40). To that end, we use Wiener L^b -bound for the maximal function $S^* := \sup_{n \in \mathbb{N}^*} \frac{1}{n} \sum_{0 \leq k < n} G \circ \vartheta^{ok}$: this theorem asserts that $\int (S^*)^b d\Pi_\nu \leq (\frac{b}{b-1})^b \int G^b d\Pi_\nu$ (see Krengel [112, Theorem 6.3 p. 52]). Thus, $\mathbf{E}[S^*(\bar{X})^b] < \infty$, which implies in particular that $S^*(\bar{X})$ is \mathbf{P} -integrable. Then, for any σ -field $\mathcal{G} \subset \mathcal{F}$, we \mathbf{P} -a.s. get $\mathbf{E}[S^*(\bar{X})|\mathcal{G}] < \infty$ since S^* is nonnegative. Next, observe that for all $n \in \mathbb{N}^*$, \mathbf{P} -a.s. $\frac{1}{n} \sum_{0 \leq k < n} F(\mathbf{T}, X_k) \leq S^*(\bar{X})$. Therefore, we \mathbf{P} -a.s. get $\sup_{n \in \mathbb{N}^*} \frac{1}{n} \sum_{0 \leq k < n} \mathbf{E}[F(\mathbf{T}, X_k)|\mathcal{G}] \leq \mathbf{E}[S^*(\bar{X})|\mathcal{G}] < \infty$, which entails (3.2.40) when \mathcal{G} is the σ -field generated by (\mathbf{T}, \circ) since, in this case, $\mathbf{E}[F(\mathbf{T}, X_k)|\mathcal{G}] = E_{\mathbf{T}, \circ}[F(\mathbf{T}, X_k)]$ by definition of $P_{\mathbf{T}, \circ}$ as explained in Remark 3.2.22. ■

3.2.4 Tree-valued branching RWs

In this section, we introduce branching RWs with values in trees and more specifically RWs that are indexed by Galton-Watson trees whose offspring distribution is critical (or a variant of such trees) and that take their values in an invariant Galton-Watson tree whose offspring distribution is supercritical. Formally, these branching RWs are viewed as trees that are labeled by pointed trees, i.e. $\bar{\mathbb{T}}^\bullet(\bar{\mathbb{T}}^\bullet)$ -valued r.v. After some background on such branching Markov chains, we extend the centering and the successor functions to $\bar{\mathbb{T}}^\bullet(\bar{\mathbb{T}}^\bullet)$ and we show that branching RWs indexed by and taking their values in Galton-Watson trees and whose spatial displacements are critical biased RWs are invariant under the successor function and enjoy an ergodic property (Proposition 3.2.32), which is the key point in the proof of Theorem 3.1.1.

Branching Markov chains. We first set some notations for branching Markov chains indexed by bilateral trees and that take their values in a complete and separable metric space (E, d_E) . We denote by $(q(y, dy'))_{y \in E}$ a Borel measurable transition kernel on E : for all $y \in E$, $q(y, dy')$ is a Borel probability measure on E and for all Borel sets B , the function $y \in E \mapsto q(y, B)$ is measurable. We equip $\bar{\mathbb{T}}^\bullet(E)$ with the Polish topology of local convergence and the associated Borel σ -field. For all pointed tree $(t, u) \in \bar{\mathbb{T}}^\bullet$, we briefly recall the definition of the law $Q_{q, (t, u)}^y$ on $\bar{\mathbb{T}}^\bullet(E)$ of the t -indexed q -branching Markov chain such that the spatial position of u is y .

To that end, we introduce the following notations. We equip t with its graph-distance d_{gr} ; for all $v, w \in t$, we denote by $\llbracket v, w \rrbracket$ the geodesic path joining v to w ; if $w \neq v$, then we denote by \overleftarrow{w}^v the unique $v' \in \llbracket v, w \rrbracket$ such that $d_{\text{gr}}(v', w) = 1$. Namely

$$\llbracket v, w \rrbracket = \{v' \in t : d_{\text{gr}}(v, v') + d_{\text{gr}}(v', w) = d_{\text{gr}}(v, w)\}, \quad \overleftarrow{w}^v \in \llbracket v, w \rrbracket \text{ and } d_{\text{gr}}(w, \overleftarrow{w}^v) = 1. \quad (3.2.41)$$

We easily see that $Q_{q, (t, u)}^y$ is characterized by the following properties.

- (i) Recall that ∂ is a cemetery point in $\bar{\mathbb{T}}^\bullet$. We first agree on the convention $Q_{q, \partial}^y = \delta_\partial$.
- (ii) If $(t, u) \in \bar{\mathbb{T}}^\bullet$ is finite, then $Q_{q, (t, u)}^y$ is the law of the $\bar{\mathbb{T}}^\bullet(E)$ -valued r.v. $\Theta := (t, u; (Y_v)_{v \in t})$, where the joint law of the $Y_v, v \in t$, on E^t is

$$\delta_y(dy_u) \prod_{v \in t \setminus \{u\}} q(y_{\overleftarrow{v}^u}, dy_v). \quad (3.2.42)$$

(iii) Let $(t, u) \in \overline{\mathbb{T}}^\bullet$ be finite. If Θ has law $Q_{q,(t,u)}^y$, then for all $p, q \in \mathbb{Z}$ such that $p \leq q$, $[\Theta]_p^q$ has law $Q_{q,[(t,u)]_p^q}^y$.

Indeed, when $(t, u) \in \overline{\mathbb{T}}^\bullet$ is infinite, $Q_{q,(t,u)}^y$ is defined thanks to (iii) and to a simple projective argument. Moreover, in general for all $p, q \in \mathbb{Z}$, such that $p < q$,

$$\left(\Theta \stackrel{\text{(law)}}{=} Q_{q,(t,u)}^y \right) \implies \left([\Theta]_p^q \stackrel{\text{(law)}}{=} Q_{q,[(t,u)]_p^q}^y \right). \quad (3.2.43)$$

It is easy to check for all Borel subsets A of $\overline{\mathbb{T}}^\bullet(E)$ that

$$(y, (t, u)) \in E \times \overline{\mathbb{T}}^\bullet \longmapsto Q_{q,(t,u)}^y(A) \text{ is Borel measurable.} \quad (3.2.44)$$

Moreover, let us equip the space of Borel probability measures on E and $\overline{\mathbb{T}}^\bullet(E)$ with the (Polish) topology of weak convergence. If we assume that $y \in E \mapsto \mathfrak{q}(y, \cdot)$ is weakly continuous, then it is also easy to check that

$$(y, (t, u)) \in E \times \overline{\mathbb{T}}^\bullet \longmapsto Q_{q,(t,u)}^y \text{ is weakly continuous.} \quad (3.2.45)$$

In (3.2.44) and (3.2.45), $E \times \overline{\mathbb{T}}^\bullet$ is equipped with the product topology.

Tree-valued branching walks. Here we fix a *genealogical tree* $(t, \varrho) \in \overline{\mathbb{T}}^\bullet$ and a pointed tree $(T, o) \in \overline{\mathbb{T}}^\bullet$ that is viewed as an *environment* where each individual $v \in t$ has a position $y_v \in T$. To fit into our framework of labelled pointed trees, we view the t -indexed and T -valued branching walk as the $\overline{\mathbb{T}}^\bullet$ -labelled pointed tree

$$\Theta = ((t, \varrho); \overline{y}_v := (T, y_v), v \in t) \in \overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet).$$

Note that Θ is not a generic element of $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ because all the \overline{y}_v have the same base-tree T . We extend the definition (3.2.21) of the centering of pointed trees to $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ by setting

$$\text{cent}(\Theta) = \left(\varphi_{|\varrho|}(t), \varphi_{|\varrho|}(\varrho); \varphi_{|y_\varrho|}(\overline{y}_{\varphi_{-|\varrho|}(v)}), v \in \varphi_{|\varrho|}(t) \right), \quad (3.2.46)$$

using the same conventions. Observe here that spatial marks (that are pointed trees) are shifted. We also extend the successor function to $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ by setting

$$\text{scc}(\Theta) = \text{cent}(t, \text{scc}(\varrho); (\overline{y}_v)_{v \in t}). \quad (3.2.47)$$

We then extend the *right-part* function as follows: recall from Definition 3.2.15 (c) that there is a unique pointed tree $(t', \varrho') \in \overline{\mathbb{T}}^\bullet$ and a unique one-to-one function $\psi : t' \rightarrow \{\varrho\}_{(-\infty, p]} \cup \{v \in t : \varrho <_t v\}$ that is increasing with respect to lexicographical order on t' and t and such that $|\psi(v)| = |v|$. Then we set

$$[\Theta]^+ := \left(t', \varrho'; (\overline{y}_{\psi(v')})_{v' \in t'} \right) \quad (3.2.48)$$

Finally, we introduce

$$\text{succ}^+(\Theta) = [\text{succ}(\Theta)]^+ \quad (3.2.49)$$

that define the *right-successor* function on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. By convention we set $\text{cent}(\partial) = \text{scc}(\partial) = [\partial]^+ = \text{scc}^+(\partial) = \partial$.

Remark 3.2.28 As for pointed trees, note that $\text{succ}^+(\Theta) = \text{succ}^+([\Theta]^+)$ for all $\Theta \in \overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. \square

Lemma 3.2.29 *The functions $\text{cent}(\cdot)$ and $[\cdot]^+$ are locally continuous on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. The functions $\text{scc}(\cdot)$ and $\text{scc}^+(\cdot)$ are locally continuous on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ at points $\Theta = (t, \varrho; (\overline{y}_v)_{v \in t})$ such that $\text{succ}(\varrho) \neq \varrho$.*

Proof. This is easily derived from Lemma 3.2.17: we leave the details to the reader. \blacksquare

Definition 3.2.30 (Biased branching RWs) We next fix $\lambda \in (0, \infty)$, and $(t, \varrho), (T, o) \in \overline{\mathbb{T}}^\bullet$. We recall from (3.2.26) the definition of the transition probabilities $p_{T, \lambda}(\cdot, \cdot)$ of the λ -biased RW on T and we recall from (3.2.27) in Definition 3.2.21 the related transition kernel $\overline{p}_\lambda(\overline{y}, d\overline{y}')$ on $\overline{\mathbb{T}}^\bullet$. A t -indexed, T -valued λ -biased branching random walk with initial position o is a $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ -valued random variable $\Theta = (t, \varrho; \overline{Y}_v = (T, Y_v), v \in t)$ whose law is $Q_{\overline{p}_\lambda, (t, \varrho)}^{(T, o)}$, which implies that $Y_\varrho = o$.

- When t is finite and $\varrho = \emptyset$ then for all $y_v \in T, v \in t$,

$$\mathbf{P}(\forall v \in t, Y_v = y_v) = \mathbf{1}_{\{y_\varrho = o\}} \prod_{v \in t \setminus \{\varrho\}} p_{T, \lambda}(y_v, y_v). \quad (3.2.50)$$

- When there is no ambiguity on the parameter λ , we simply write $Q_{t, \varrho}^{T, o}$ instead of $Q_{\overline{p}_\lambda, (t, \varrho)}^{(T, o)}$. \square

Remarks 3.2.31 (a) If Θ has law $Q_{\overline{p}_\lambda, (t, \varrho)}^{(T, o)}$ as above, then $[\Theta]^+$ has law $Q_{\overline{p}_\lambda, [(t, \varrho)]^+}^{(T, o)}$.

(b) We easily check that $\overline{y} \in \overline{\mathbb{T}}^\bullet \mapsto \overline{p}_\lambda(\overline{y}, d\overline{y}')$ is weakly continuous. Thus by (3.2.45),

$$((t, \varrho), (T, o)) \in \overline{\mathbb{T}}^\bullet \times \overline{\mathbb{T}}^\bullet \mapsto Q_{\overline{p}_\lambda, (t, \varrho)}^{(T, o)}$$

is also weakly continuous on the space of Borel probability measures on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. \square

We now state the main ergodic property of critical biased RWs indexed by critical GW-trees and taking their values in supercritical GW-trees. It is a key argument to prove the law of large numbers in Theorem 3.1.1.

Proposition 3.2.32 *Let μ and ν be two offspring distributions. We assume that μ is non-trivial and critical and that ν is supercritical. We denote by m_ν the mean of ν . Let (τ, ϱ) be an infinite GW(μ)-tree (Definition 3.2.18) and let (\mathbf{T}, o) be an invariant GW(ν)-tree (Definition 3.2.25) that is independent from (τ, ϱ) . Let $\Theta = ((\tau, \varrho); \overline{Y}_v = (\mathbf{T}, Y_v), v \in \tau)$ be a m_ν -biased branching RW, whose conditional law given (τ, ϱ) and (\mathbf{T}, o) is $Q_{\tau, \varrho}^{\mathbf{T}, o}$ as in Definition 3.2.30 (with $\lambda = m_\nu$). Then under \mathbf{P} ,*

$$\text{succ}(\Theta) \stackrel{(law)}{=} \Theta \quad \text{and} \quad \text{succ}^+([\Theta]^+) \stackrel{(law)}{=} [\Theta]^+. \quad (3.2.51)$$

Suppose moreover that there is $\beta \in (0, \infty)$ such that $\sum_{k \in \mathbb{N}} k(\log k)^{2(1+2\beta)} \mu(k) < \infty$ and such that $\sum_{k \in \mathbb{N}} k^{1+\frac{1}{\beta}} \nu(k) < \infty$. Denote by $\Pi_{\mu, \nu}^+$ the law of $[\Theta]^+$ under \mathbf{P} . Then succ^+ is $\Pi_{\mu, \nu}^+$ -ergodic.

Proof. See Section 3.3.3. \blacksquare

3.2.5 Stable continuous height processes.

In this section, we recall (mostly from [61] and [81]) various results concerning stable height processes and stable spectrally positive Lévy processes that are used throughout the chapter.

α -stable spectrally positive Lévy processes, related height processes. We fix $\alpha \in (1, 2]$. We denote by $X = (X_s)_{s \in [0, \infty)}$ an α -stable and spectrally positive Lévy process. Its law is characterized by its Laplace exponent

$$\forall s, \lambda \in [0, \infty), \quad \mathbf{E}[e^{-\lambda X_s}] = e^{s\lambda^\alpha}. \quad (3.2.52)$$

Note that X_s is integrable and that $\mathbf{E}[X_s] = 0$, which easily implies that X oscillates when $s \rightarrow \infty$. Moreover, \mathbf{P} -a.s. the path X has infinite variation sample paths (for more details, see Bertoin [19] Chapters VII and VIII).

In the more general context of spectrally positive Lévy processes, it has been proved in Le Gall & Le Jan [81] and in D. & Le Gall [61, Chapter 1] that there exists a continuous process $H = (H_s)_{s \in [0, \infty)}$ such that for any $s \in [0, \infty)$, the following limit holds in \mathbf{P} -probability:

$$H_s := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{X_r < \varepsilon + \inf_{[r, s]} X\}} dr . \quad (3.2.53)$$

The process $H = (H_s)_{s \in [0, \infty)}$ is called the α -stable height process. It is an adapted functional of X and we shall sometimes use the notation $H_s = H_s(X)$ to indicate that. The α -stable height process provides a way to explore the genealogy of an α -stable CSBP. We refer to Le Gall & Le Jan [81] for a careful explanation of (3.2.53) in the discrete setting. Compare with (3.2.4) in Remark 3.2.3 in the discrete setting.

In Section 3.4.4, to prove the quenched convergence of the finite-dimensional marginals of snakes processes coding the range of branching RWs, we shall need the following lemma. Its proof, which is technical, is postponed in Appendix.

Lemma 3.2.33 *Let $\alpha \in (1, 2]$ and let H be the height process associated with the α -stable spectrally positive Lévy process X as above. Let s_1, s_2 be two real numbers such that $0 < s_1 < s_2$. Then \mathbf{P} -almost surely, it holds that $H_{s_1} \wedge H_{s_2} > \min_{[s_1, s_2]} H$.*

Proof. See Section 3.A. ■

Excursions of the α -stable height process. For all $s \in [0, \infty)$, we set $I_s = \inf_{r \in [0, s]} X_r$, the infimum process of X , which has continuous sample paths since X is spectrally positive. Recall that X has infinite variation sample paths. Basic results on fluctuation theory (see e.g. Bertoin [19] Chapter VI.1 and VII.1) entail that $X - I$ is a strong Markov process in $[0, \infty)$ and that 0 is regular for $(0, \infty)$ and recurrent with respect to this Markov process. Moreover, $-I$ is a local time at 0 for $X - I$ (see Theorem VII.1 [19]). Denote by \mathbf{N} the corresponding excursion measure of $X - I$ above 0 and denote by (l_j, r_j) , $j \in \mathcal{I}$, the excursion intervals of $X - I$ above 0 and by $X^j = X_{(l_j +) \wedge r_j} - I_{l_j}$, $j \in \mathcal{I}$, the corresponding excursions. Then, the point measure $\sum_{j \in \mathcal{I}} \delta_{(-I_j, X^j)}$ is a Poisson point measure on $[0, \infty) \times \mathbf{D}([0, \infty), \mathbb{R})$ with intensity $dx \otimes \mathbf{N}$. Here $\mathbf{D}([0, \infty), \mathbb{R})$ stands for the space of cadlag functions from $[0, \infty)$ to \mathbb{R} equipped with Skorokhod's topology. Now, observe that (3.2.53) implies that the value of H_s only depends on the excursion of $X - I$ straddling s and that

$$\{s \in [0, \infty) : X_s > I_s\} = \bigcup_{j \in \mathcal{I}} (l_j, r_j) = \{s \in [0, \infty) : H_s > 0\} . \quad (3.2.54)$$

This allows defining H under \mathbf{N} as $H(X)$ when X is under the excursion measure \mathbf{N} . More precisely the processes $H^j := H_{(l_j +) \wedge r_j} = H(X^j)$, $j \in \mathcal{I}$, are the excursions of H above 0, and the point measure

$$\sum_{j \in \mathcal{I}} \delta_{(-I_j, H^j)} \quad (3.2.55)$$

is distributed under \mathbf{P} as a Poisson point measure on $[0, \infty) \times \mathbf{C}^0([0, \infty), \mathbb{R})$ with intensity $dx \otimes \mathbf{N}(dH)$, (with a slight abuse of notation for using $\mathbf{N}(dH)$ as the 'distribution' of $H(X)$ under the excursion measure $\mathbf{N}(dX)$). Recall here $\mathbf{C}^0([0, \infty), \mathbb{R})$ is the space of continuous functions from $[0, \infty)$ to \mathbb{R} equipped with the topology of uniform convergence on every compact interval. We refer to the comments in D. & Le Gall [61, Section 3.2] for more details.

Under \mathbf{N} , X and H have the same lifetime ζ : namely, we have \mathbf{N} -a.e. $\zeta < \infty$, $X_s = X_0 = H_0 = H_s = 0$ for all $s \in [\zeta, \infty)$ and H_s and $X_s > 0$ for all $s \in (0, \zeta)$. Basic results of fluctuation theory (see Bertoin [19], Chapter VII) also entail that $\mathbf{N}[1 - e^{-\lambda\zeta}] = \lambda^{1/\alpha}$, $\lambda \in [0, \infty)$. Therefore $\mathbf{N}(\zeta > r) = c_\alpha r^{-\frac{1}{\alpha}}$ where $c_\alpha = 1/\Gamma_e(\frac{\alpha-1}{\alpha})$ and where Γ_e stands for Euler's Gamma function.

Remark 3.2.34 Denote by $(\mathbf{g}_1, \mathbf{d}_1)$ the first excursion interval of H (or $X - I$) above 0 that is longer than 1. Namely, for all $s \in (0, \infty)$, introduce $r(s) = \inf\{r \in (s, \infty) : H_r = 0\}$. Then $\mathbf{g}_1 = \inf\{s \in [0, \infty) : r(s) - s > 1\}$ and $\mathbf{d}_1 = r(\mathbf{g}_1)$. Then, standard results on Poisson point processes imply that

$$(X_{(\mathbf{g}_1 + \cdot) \wedge \mathbf{d}_1}, H_{(\mathbf{g}_1 + \cdot) \wedge \mathbf{d}_1}) \quad \text{under } \mathbf{P} \stackrel{(\text{law})}{=} (X, H) \quad \text{under } \mathbf{N}(\cdot | \zeta > 1). \quad (3.2.56)$$

□

Let us next define the law of the *normalized α -stable height process* $\mathbf{N}(\cdot | \zeta = 1)$. To that end, recall that the α -stable spectrally positive Lévy process X enjoys the following *scaling property*: for all $r \in (0, \infty)$, under \mathbf{P} the rescaled process $(r^{-1/\alpha} X_{rs})_{s \in [0, \infty)}$ has the same law as X . We easily derive from (3.2.53) that H also enjoys a scaling property: namely, under \mathbf{P} , $(r^{-(\alpha-1)/\alpha} H_{rs})_{s \in [0, \infty)}$ and H have the same law. Therefore,

$$(r^{-\frac{1}{\alpha}} X_{rs}, r^{-\frac{\alpha-1}{\alpha}} H_{rs})_{s \in [0, \infty)} \quad \text{under } r^{\frac{1}{\alpha}} \mathbf{N} \stackrel{(\text{law})}{=} (X, H) \quad \text{under } \mathbf{N}. \quad (3.2.57)$$

(with a slight abuse of the word ‘law’).

This *scaling property* allows us to define a regular version of the excursion conditioned to last r . Namely, there exists a family of laws on $\mathbf{D}([0, \infty), \mathbb{R})$ denoted by $\mathbf{N}(\cdot | \zeta = r)$, $r \in (0, \infty)$, such that $r \mapsto \mathbf{N}(\cdot | \zeta = r)$ is weakly continuous on $\mathbf{D}([0, \infty), \mathbb{R})$, such that $\mathbf{N}(\cdot | \zeta = r)$ -a.s. $\zeta = r$ and such that $\mathbf{N} = \int_0^\infty \mathbf{N}(\cdot | \zeta = r) \mathbf{N}(\zeta \in dr)$. We call $\mathbf{N}(\cdot | \zeta = 1)$ the *normalized law of the α -stable spectrally positive Lévy process above its infimum* (or of the *α -stable height process*). By (3.2.57), we see that

$$\mathbf{N}[F(X, H)] = \alpha^{-1} c_\alpha \int_0^\infty dr r^{-1-\frac{1}{\alpha}} \mathbf{N}\left[F\left((r^{\frac{1}{\alpha}} X_{s/r}, r^{\frac{\alpha-1}{\alpha}} H_{s/r})_{s \in [0, \infty)}\right) | \zeta = 1\right]. \quad (3.2.58)$$

Remark 3.2.35 Let us fix $s \in (0, 1)$. Then the laws of $(H_{\cdot \wedge s}, X_{\cdot \wedge s})$ under $\mathbf{N}(\cdot | \zeta > 1)$ and $\mathbf{N}(\cdot | \zeta = 1)$ are absolutely continuous. More precisely, there exists a continuous function $(s, x) \in (0, 1) \times [0, \infty) \mapsto D_r(x)$ such that for all measurable nonnegative functions F ,

$$\mathbf{N}\left[F(H_{\cdot \wedge s}, X_{\cdot \wedge s}) | \zeta = 1\right] = \mathbf{N}\left[F(H_{\cdot \wedge s}, X_{\cdot \wedge s}) D_s(X_r) | \zeta > 1\right]. \quad (3.2.59)$$

See Kortchemski [106] for more details. A discrete version of this identity plays an important role in the proof of our limit theorem. □

Limit theorems. We next recall mostly from D. & Le Gall [61, Chapter 2], D. [50] and Kortchemski [106] the convergence of the rescaled contour and height processes of Galton-Watson forests whose offspring distribution belongs to the domain of attraction of an α -stable law. More precisely, we fix $\alpha \in (1, 2]$. Then, all the genealogical trees that we consider are (variants of) GW-trees whose offspring distribution μ is fixed and satisfies the following assumptions.

$$\begin{cases} (\text{H}_1) : \mu(0) + \mu(1) < 1 \quad \text{and} \quad \sum_{k \in \mathbb{N}} k \mu(k) = 1, \\ (\text{H}_2) : \mu \text{ is aperiodic, (namely, } \mu \text{ is not supported by a proper subgroup of } \mathbb{Z}), \\ (\text{H}_3) : \mu \text{ is in the domain of attraction of an } \alpha\text{-stable law.} \end{cases} \quad (3.2.60)$$

We denote by $(V_n)_{n \in \mathbb{N}}$ a random walk whose jump law is $\tilde{\mu}(k) = \mu(k+1)$, $k \in \{-1\} \cup \mathbb{N}$. By definition, Assumption (H3) in (3.2.60) implies that there exists a $(0, \infty)$ -valued sequence $(b_n)_{n \in \mathbb{N}}$ tending to ∞ such that

$$\frac{1}{b_n} V_n \xrightarrow[n \rightarrow \infty]{(\text{law})} X_1. \quad (3.2.61)$$

where $(X_s)_{s \in [0, \infty)}$ stands for the α -stable spectrally positive Lévy process whose Laplace exponent is λ^α . Here, the sequence $(b_n)_{n \in \mathbb{N}}$ is $\frac{1}{\alpha}$ -regularly varying, i.e. there is a slowly varying function $L: (0, \infty) \rightarrow (0, \infty)$ at ∞ such that

$$b_n = n^{1/\alpha} L(n). \quad (3.2.62)$$

By e.g. [23, Theorem 8.3.1], Assumption (H₃) in (3.2.60) implies that there exists a slowly varying function $L^*: (0, \infty) \rightarrow (0, \infty)$ at ∞ such that

$$\mu([n, \infty)) \sim n^{-\alpha} L^*(n) \text{ if } \alpha \in (1, 2) \quad \text{and} \quad \sum_{0 \leq k \leq n} k^2 \mu(k) - 1 \sim 2L^*(n) \text{ if } \alpha = 2. \quad (3.2.63)$$

(note that $\sum_{0 \leq k \leq n} k^2 \mu(k) - 1$ is ultimately positive by (H₁) in (3.2.60)). Then by e.g. [23, Theorem 8.1.6], we get that if $c'_\alpha = (\alpha - 1)/\Gamma_e(2 - \alpha)$ when $\alpha < 2$ or $c'_\alpha = 1$ when $\alpha = 2$, then

$$\psi_\mu(s) \sim_{0+} c'_\alpha s^\alpha L^*(1/s) \text{ where } \psi_\mu(s) = g_\mu(1-s) - (1-s) \text{ and } g_\mu(s) = \sum_{k \in \mathbb{N}} s^k \mu(k), \quad s \in [0, 1]. \quad (3.2.64)$$

The slowly varying functions L and L^* are related as follows.

$$nL(n)^\alpha \sim b_n^\alpha \sim nL^*(b_n) \sim nL^*(n^{1/\alpha} L(n)). \quad (3.2.65)$$

Theorem 3.2.36 below gathers several limiting results for GW-trees and their coding processes that are used in our proofs. To that end, let us first introduce the following notations. Let τ_∞ be a $\text{GW}(\mu)$ -forest as in Definition 3.2.6 (b) which is associated with the sequence of i.i.d. $\text{GW}(\mu)$ -trees $(\tau^{(k)})_{k \in \mathbb{N}^*}$ as in Definition 3.2.1 (c). Recall that $V(\tau_\infty)$, $H(\tau_\infty)$ and $C(\tau_\infty)$ stands for resp. its Lukasiewicz path, its height process and its contour process. For all integers n , we then set

$$\kappa_n = \inf\{k \in \mathbb{N}^* : \#\tau^{(k)} \geq n\}, \quad \mathfrak{g}_n = \sum_{1 \leq k < \kappa_n} \#\tau^{(k)} \quad \text{and} \quad \mathfrak{d}_n = \mathfrak{g}_n + \#\tau^{(\kappa_n)} \quad (3.2.66)$$

with the convention that $\mathfrak{g}_n = 0$ if $\kappa_n = 1$. Namely, \mathfrak{g}_n is the time when $H(\tau_\infty)$ begins its first excursion above 0 lasting at least n units of time. Note that $\tau^{(\kappa_n)}$ has the same law as a single $\text{GW}(\mu)$ -tree τ under $\mathbf{P}(\cdot | \#\tau \geq n)$.

Theorem 3.2.36 *Let $\alpha \in (1, 2]$ and let μ satisfy (3.2.60). Let τ_∞ be a $\text{GW}(\mu)$ -forest and let τ be a $\text{GW}(\mu)$ -tree (Definition 3.2.6). Then for all (sufficiently large) $n \in \mathbb{N}^*$, we denote by $\tau_{\geq n}$ a tree distributed as τ under $\mathbf{P}(\cdot | \#\tau \geq n)$ and by τ_n a tree distributed as τ under $\mathbf{P}(\cdot | \#\tau = n)$. Recall \mathfrak{g}_n and \mathfrak{d}_n from (3.2.66) and recall \mathfrak{g}_1 and \mathfrak{d}_1 from Remark 3.2.34. Let $(b_n)_{n \in \mathbb{N}^*}$ be as in (3.2.61). For all $n \in \mathbb{N}^*$, we set*

$$a_n = n/b_n, \quad n \in \mathbb{N}^*. \quad (3.2.67)$$

Then, the following holds true.

(i) *The joint convergence*

$$\mathcal{Q}_n = \left(\left(\frac{1}{b_n} V_{\lfloor n \cdot \rfloor}(\tau_\infty), \frac{1}{a_n} H_{n \cdot}(\tau_\infty), \frac{1}{a_n} C_{2n \cdot}(\tau_\infty), \frac{1}{n} \mathfrak{g}_n, \frac{1}{n} \mathfrak{d}_n \right) \xrightarrow[n \rightarrow \infty]{} (X, H, H, \mathfrak{g}_1, \mathfrak{d}_1) \right) \quad (3.2.68)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^2 \times (0, \infty)^2$ equipped with the product topology.

(ii) *For all integers $n \geq k \geq 0$, there exists a function $D_k^{(n)}: \{-1\} \cup \mathbb{N} \rightarrow [0, \infty)$ such that for all measurable nonnegative function F*

$$\mathbf{E}[F(V_{\cdot \wedge k}(\tau_n))] = \mathbf{E}[F(V_{\cdot \wedge k}(\tau_{\geq n}))D_k^{(n)}(V_k(\tau_{\geq n}))]. \quad (3.2.69)$$

(iii) Recall from Remark 3.2.35 the definition of the continuous function $(s, x) \in (0, 1) \times [0, \infty) \mapsto D_s(x)$ and fix $(s, x, y) \in (0, 1) \times (0, \infty)^2$ with $x \leq y$. Let $s_n \in \mathbb{N}^*$, $n \in \mathbb{N}$, such that $s_n/n \rightarrow s$. Then

$$\lim_{n \rightarrow \infty} \max_{p \in [[b_n x], [b_n y]]} |D_{s_n}^{(n)}(p) - D_s(p/b_n)| = 0. \quad (3.2.70)$$

Moreover, the r.v. $(D_{s_n}^{(n)}(V_{s_n}(\tau_{\geq n})))_{n \in \mathbb{N}^*}$ are uniformly integrable:

$$\lim_{c \rightarrow \infty} \sup_{n \in \mathbb{N}^*} \mathbf{E} \left[D_{s_n}^{(n)}(V_{s_n}(\tau_{\geq n})) \mathbf{1}_{\{D_{s_n}^{(n)}(V_{s_n}(\tau_{\geq n})) \geq c\}} \right] = 0. \quad (3.2.71)$$

(iv) The joint convergence

$$\mathcal{Q}'_n = \left(\left(\frac{1}{b_n} V_{[n \cdot]}(\tau_n), \frac{1}{a_n} H_{n \cdot}(\tau_n), \frac{1}{a_n} C_{2n \cdot}(\tau_n) \right) \right)_{n \rightarrow \infty} \longrightarrow (X', H', H') \quad (3.2.72)$$

holds weakly on $\mathbf{D}([0, 1], \mathbb{R}) \times \mathbf{C}([0, 1], \mathbb{R})^2$ equipped with the product topology. Here (X', H') has the same law as $(X, H(X))$ under $\mathbf{N}(dX | \zeta = 1)$.

Proof. In (i), for a proof of the joint convergence of the Lukasiewicz path, the height process and the contour process: see D. & Le Gall [61, Thm 2.3.2 and Corollary 2.5.1]. The joint convergence with $(\mathfrak{g}_n/n, \mathfrak{d}_n/n)$ follows from arguments in Le Gall [118, Thm 5.1]. For (ii) see Le Gall & Miermont [117, Lemma 10] or Kortchemski [106, Lemma 2]. For (iii), we refer to the arguments in Kortchemski [106, Section 3.2]. For (iv): see D. [50]. \blacksquare

We shall use the previous Theorem to derive limits of coding processes of τ_n from limits of coding processes of $\tau_{\geq n}$. More precisely, we shall use the following lemma, whose proof is given in Appendix.

Lemma 3.2.37 We keep the assumptions and notations of Theorem 3.2.36. Let $\mathcal{G} \subset \mathcal{F}$ be σ -field. Let E and E' be Polish spaces equipped with their respective Borel σ -field. The product spaces below are equipped with product topology. For all $n \in \mathbb{N}^*$, let $r_n \in \mathbb{N}$, let U_n be a E -valued r.v. and let $Z_n : \mathbf{D}([0, \infty), \mathbb{R}) \times E \rightarrow E'$ be a measurable function. We assume for all $n \in \mathbb{N}^*$ that $\tau_{\geq n}$ and τ_n are independent from (\mathcal{G}, U_n) , that $r_n/n \rightarrow r \in (0, 1)$ and that conditionally given \mathcal{G} , the joint convergence

$$M_{\geq n} := \left(\left(\frac{1}{b_n} V_{(ns) \wedge r_n}(\tau_{\geq n}) \right)_{s \in [0, \infty)}, Z_n(V_{\cdot \wedge r_n}(\tau_{\geq n}), U_n) \right) \xrightarrow[n \rightarrow \infty]{} (X^{(r)}, Z) \quad (3.2.73)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times E'$. Here we furthermore assume that $X^{(r)}$ has the same law as $X_{\cdot \wedge r}$ under $\mathbf{N}(\cdot | \zeta > 1)$ and that $(X^{(r)}, Z)$ is independent from \mathcal{G} . Then, conditionally given \mathcal{G} , the joint convergence

$$M_n := \left(\left(\frac{1}{b_n} V_{(ns) \wedge r_n}(\tau_n) \right)_{s \in [0, \infty)}, Z_n(V_{\cdot \wedge r_n}(\tau_n), U_n) \right) \xrightarrow[n \rightarrow \infty]{} (Y^{(r)}, Z') \quad (3.2.74)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times E'$ where the random variable $(Y^{(r)}, Z')$ is independent from \mathcal{G} and such that $\mathbf{E}[F(Y^{(r)}, Z')] = \mathbf{E}[F(X^{(r)}, Z) D_r(X_r^{(r)})]$ for all bounded and measurable functions F on $\mathbf{D}([0, \infty), \mathbb{R}) \times E'$.

Proof. See Section 3.B. \blacksquare

We complete this section by recalling several standard estimates on the distribution of $\#\tau$ when τ is a $\text{GW}(\mu)$ -tree. Let $(V_n)_{n \in \mathbb{N}}$ be the random walk as in (3.2.61). By standard results on GW -trees and RWs,

$$\#\tau \stackrel{\text{(law)}}{=} T_{-1} \quad \text{where} \quad T_{-p} := \inf \{n \in \mathbb{N} : V_n = -p\}, \quad p \in \mathbb{N}. \quad (3.2.75)$$

Note that T_{-p} is a.s. finite by (H_1) . By the strong Markov property, $(T_{-p})_{p \in \mathbb{N}}$ is a random walk.

Lemma 3.2.38 *We assume that μ satisfies (3.2.60). Recall from (3.2.62) the definition of the slowly varying function $L : (0, \infty) \rightarrow (0, \infty)$. For all $s \in [0, 1]$, we set $\varphi_\mu(s) = \mathbf{E}[s^{\#\tau}]$. Then, there is a constant $c_\alpha \in (0, \infty)$ that only depends on α , such that*

$$n^{\frac{1}{\alpha}} L(n) \mathbf{P}(\#\tau \geq n) \rightarrow c_\alpha \quad \text{and} \quad 1 - \varphi_\mu(e^{-\lambda}) \sim_{0+} c_\alpha \Gamma_e(1 - \frac{1}{\alpha}) \lambda^{\frac{1}{\alpha}} L(\frac{1}{\lambda})^{-1} \quad (3.2.76)$$

(here Γ_e is Euler's gamma function). Furthermore, $n^{1+\frac{1}{\alpha}} L(n) \mathbf{P}(\#\tau = n) \rightarrow \alpha^{-1} c_\alpha$.

Proof. The first limit in (3.2.76) implies the second one by Karamata's Abelian Theorem for Laplace transforms see e.g. [23, Thm. 1.7.6]. By Kemperman's identity (see e.g. [23, Thm. 8.9.15]), we first get $\mathbf{P}(T_{-1} = n) = \frac{1}{n} \mathbf{P}(V_n = -1)$. By Gnedenko's local limit theorem (see e.g. [23, Thm. 8.4.1]), we next get $\mathbf{P}(V_n = -1) \sim \text{Cst}/b_n$. Thus $\mathbf{P}(\#\tau = n) \sim \text{Cst}/(nb_n)$. By Karamata's Theorem for tails (see e.g. [23, Prop. 1.5.10]), we get $\mathbf{P}(\#\tau \geq n) \sim \alpha \text{Cst}/b_n$, which entails the lemma. ■

3.2.6 Real trees, convergences of metric spaces and pseudometrics.

In this section, we first recall the definition of a specific class of metrics spaces called real trees. They appear as limits of rescaled discrete trees. Here the limit holds in the sense of Gromov-Hausdorff-Prokhorov (GHP for short). GHP-convergence actually concerns isometry classes of pointed measured compact metric spaces and, although it provides a neat and intrinsic framework for formulating convergences of renormalized random graphs, it is more convenient to proceed with a certain type of encoding of these spaces which provides pseudometrics viewed as $[0, \infty)^2$ -indexed processes. We briefly recall from Chapter 2 the results on the space of pseudometrics that are used in our proofs.

Real trees. Let us first recall the definition of real trees, which are metric spaces that extend the definition of graph-trees. Informally they are obtained by glueing compact intervals without creating cycles. More precisely, they are defined as follows.

Definition 3.2.39 (Real trees) Let (T, d) be a metric space; it is a *real tree* if and only if the following holds true.

- (a) For any $\sigma_1, \sigma_2 \in T$, there is a unique isometry $f : [0, d(\sigma_1, \sigma_2)] \rightarrow T$ such that $f(0) = \sigma_1$ and $f(d(\sigma_1, \sigma_2)) = \sigma_2$. Then, we set $\llbracket \sigma_1, \sigma_2 \rrbracket := f([0, d(\sigma_1, \sigma_2)])$.
- (b) For any continuous injective function $g : [0, 1] \rightarrow T$, such that $g(0) = \sigma_1$ and $g(1) = \sigma_2$, $g([0, 1]) = \llbracket \sigma_1, \sigma_2 \rrbracket$. □

Among connected metric spaces, real trees are characterized by the so-called *four points inequality*. Namely, if (T, d) is a connected metric space, then (T, d) is a real tree if and only if for any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in T$,

$$d(\sigma_1, \sigma_2) + d(\sigma_3, \sigma_4) \leq (d(\sigma_1, \sigma_3) + d(\sigma_2, \sigma_4)) \vee (d(\sigma_1, \sigma_4) + d(\sigma_2, \sigma_3)). \quad (3.2.77)$$

We refer to Evans [70] or to Dress, Moulton & Terhalle [48] for a detailed account on this property. In this chapter, we shall only consider *compact pointed measured real trees*: namely, compact real trees (T, d) equipped with a distinguished point $r \in T$ that is viewed as a *root*, and with a finite measure μ defined on the Borel σ -field of T .

Gromov-Hausdorff-Prokhorov metric. In this chapter, we prove that rescaled specific random graph-trees equipped with a root and (variants of) the counting measure converge to random compact pointed measured real trees. The convergence of these metric spaces holds weakly in the Gromov-Hausdorff-Prokhorov sense as recalled here.

Let (E_1, d_1, r_1, μ_1) and (E_2, d_2, r_2, μ_2) be two pointed measured compact metric spaces: here μ_1 and μ_2 are finite measures on the respective Borel sigma-fields of E_1 and E_2 , and $r_1 \in E_1$ and $r_2 \in E_2$ are distinguished points. The pointed *Gromov-Hausdorff-Prokhorov distance* (the *GHP-distance* for short) between E_1 and E_2 is defined by

$$\delta_{\text{GHP}}(E_1, E_2) = \inf \left\{ d_E^{\text{Haus}}(\phi_1(E_1), \phi_2(E_2)) + d_E(\phi_1(r_1), \phi_2(r_2)) + d_E^{\text{Prok}}(\mu_1 \circ \phi_1^{-1}, \mu_2 \circ \phi_2^{-1}) \right\}. \quad (3.2.78)$$

Here, the infimum is taken over all Polish spaces (E, d_E) and all isometric embeddings $\phi_i : E_i \hookrightarrow E$, $i \in \{1, 2\}$; d_E^{Haus} stands for the Hausdorff distance on the space of compact subsets of E (namely, $d_E^{\text{Haus}}(K_1, K_2) = \inf \{ \varepsilon \in (0, \infty) : K_1 \subset K_2^{(\varepsilon)} \text{ and } K_2 \subset K_1^{(\varepsilon)} \}$, where $A^{(\varepsilon)} = \{y \in E : d_E(y, A) \leq \varepsilon\}$ for all non-empty $A \subset E$); d_E^{Prok} stands for the Prokhorov distance on the space of finite Borel measures on E (namely, $d_E^{\text{Prok}}(\mu, \nu) = \inf \{ \varepsilon \in (0, \infty) : \mu(K) \leq \nu(K^{(\varepsilon)}) + \varepsilon \text{ and } \nu(K) \leq \mu(K^{(\varepsilon)}) + \varepsilon, \forall K \subset E \text{ compact} \}$); for all $i \in \{1, 2\}$, $\mu_i \circ \phi_i^{-1}$ stands for the pushforward measure of μ_i via ϕ_i .

We next recall from Theorem 2.5 in Abraham, Delmas & Hoscheit [4] the following assertions: δ_{GHP} is a pseudometric (i.e. it is symmetric and it satisfies the triangle inequality) and $\delta_{\text{GHP}}(E_1, E_2) = 0$ if and only if E_1 and E_2 are *isometric*, namely if and only if there exists a bijective isometry $\phi : E_1 \rightarrow E_2$ such that $\phi(r_1) = r_2$ and such that $\mu_2 = \mu_1 \circ \phi^{-1}$. Then denote by \mathbb{M} the space of isometry classes of pointed measured compact metric spaces. Theorem 2.5 in Abraham, Delmas & Hoscheit [4] asserts that

$$(\mathbb{M}, \delta_{\text{GHP}}) \text{ is a complete and separable metric space.} \quad (3.2.79)$$

It is easy to prove that the four points inequality (3.2.77) is preserved under GHP-convergence. Therefore, the measure preserving pointed isometry classes of compact real trees form a δ_{GHP} -closed subspace of \mathbb{M} .

Limits of continuous pseudometrics. We introduce here the space of pseudometrics as continuous functions on real intervals, which is an alternative point of view to the intrinsic approach of GHP convergence of metric spaces: this allows a more concrete manipulation of metric spaces that takes advantage of the technical tools of weak convergence of processes.

Definition 3.2.40 Let $\zeta \in (0, \infty)$. We denote by $\mathbf{C}([0, \zeta]^2, \mathbb{R})$ the space of continuous functions from $[0, \zeta]^2$ to \mathbb{R} that is a Banach space when equipped with uniform norm $\|\cdot\|$.

- (a) We denote by $\mathbf{M}([0, \zeta])$ the subspace of $\mathbf{C}([0, \zeta]^2, \mathbb{R})$ of *continuous pseudometrics* on $[0, \zeta]$. Namely, the functions $d(\cdot, \cdot) \in \mathbf{C}([0, \zeta]^2, \mathbb{R})$ such that for all $s_1, s_2, s_3 \in [0, \zeta]$,

$$d(s_1, s_2) \geq 0, \quad d(s_1, s_1) = 0, \quad d(s_1, s_2) = d(s_2, s_1) \quad \text{and} \quad d(s_1, s_3) \leq d(s_1, s_2) + d(s_2, s_3).$$

- (b) We also denote by $\mathbf{MT}([0, \zeta])$ the space of continuous real tree pseudometrics. Namely, the functions $d(\cdot, \cdot) \in \mathbf{M}([0, \zeta])$ such that for all $s_1, s_2, s_3, s_4 \in [0, \zeta]$,

$$d(s_1, s_2) + d(s_3, s_4) \leq \max(d(s_1, s_3) + d(s_2, s_4); d(s_1, s_4) + d(s_2, s_3)). \quad (3.2.80)$$

We easily check that $\mathbf{MT}([0, \zeta])$ and $\mathbf{M}([0, \zeta])$ are closed subsets of $(\mathbf{C}([0, \zeta]^2, \mathbb{R}), \|\cdot\|)$. \square

Let $d \in \mathbf{M}([0, \zeta])$. It induces a connected compact pointed measured metric space as follows: for all $s_1, s_2 \in [0, \zeta]$, set $s_1 \sim_d s_2$ if and only if $d(s_1, s_2) = 0$. Clearly, \sim_d is an equivalence relation. We then define the quotient space:

$$E_d = [0, \zeta] / \sim_d, \quad \text{proj}_d : [0, \zeta] \rightarrow E_d, \quad \text{the canonical projection,} \quad r_d = \text{proj}_d(0). \quad (3.2.81)$$

We keep denoting d the metric induced by d on E_d . Since d is continuous on $[0, \zeta]^2$, proj_d is continuous and (E_d, d, r_d) is a pointed compact and connected metric space. We also equip E_d with the pushforward measure μ_d of the Lebesgue measure on $[0, \zeta]$ via the canonical projection: namely, for all nonnegative measurable functions f on E_d ,

$$\int_{E_d} f(x) \mu_d(dx) = \int_0^\zeta f(\text{proj}_d(s)) ds . \quad (3.2.82)$$

Note that μ_d is a finite measure with total mass ζ . We next recall the following result from Chapter 2.

Proposition 3.2.41 *Let $d, d' \in \mathbf{M}([0, \zeta])$ and let E_d and $E_{d'}$ be the related pointed measured compact metric spaces as defined by (3.2.81) and (3.2.82). Then*

$$\delta_{\text{GHP}}(E_d, E_{d'}) \leq \frac{3}{2} \|d - d'\| . \quad (3.2.83)$$

Proof. See Proposition 2.4.14 in Chapter 2. ■

Example 3.2.42 (*Real trees coded by continuous functions*) Let $\zeta \in (0, \infty)$ and let $h : [0, \zeta] \rightarrow \mathbb{R}$ be continuous. For all $s_1, s_2 \in [0, \zeta]$, we set

$$m_h(s_1, s_2) = \inf_{s \in [s_1 \wedge s_2, s_1 \vee s_2]} h(s) \quad \text{and} \quad d_h(s_1, s_2) = h(s_1) + h(s_2) - 2m_h(s_1, s_2) . \quad (3.2.84)$$

We easily check that $d_h \in \mathbf{MT}([0, \zeta])$ and to simplify we denote by (T_h, d_h, r_h, μ_h) the induced metric space $(E_{d_h}, d_h, r_{d_h}, \mu_{d_h})$ as defined in (3.2.81) and (3.2.82); we also denote by $p_h : [0, \zeta] \rightarrow T_h$ the canonical projection (instead of proj_{d_h}); (T_h, d_h, r_h, μ_h) is a pointed measured compact real tree that is the *tree coded by the function h* . Let $h' : [0, \zeta] \rightarrow \mathbb{R}$ be continuous. Then observe that

$$\forall s_1, s_2 \in [0, \zeta], \quad |d_h(s_1, s_2) - d_{h'}(s_1, s_2)| \leq 4 \sup_{s \in [0, \zeta]} |h(s) - h'(s)| , \quad (3.2.85)$$

which shows the continuity in $\mathbf{MT}([0, \zeta])$ of the application $h \mapsto d_h$ and therefore the continuity of $h \mapsto T_h$ with respect to GHP-convergence by Proposition 3.2.41. □

3.2.7 Continuous snakes and related metrics.

Here we recall the definition of Brownian snakes that have been introduced by J-F. Le Gall in [77]. They are continuous analogs of discrete snakes introduced in Definition 3.2.8. One-dimensional Brownian snakes play a key role in our result: they encode a random metric space called the *Brownian cactus* (see Curien, Le Gall & Miermont [40]) which is the limit of the rescaled range of the tree-valued branching random walks studied in this chapter. In this section, we recall the definition of the Brownian cactus with α -stable branching mechanism, along with technical results on limits of snake-pseudometrics, some come from Chapter 2 (adapted from D., K., Lin & Torri [60]), others are new.

General definition of snakes. Let us first introduce a few notations and conventions on spaces of continuous functions. If (E, d_E) is a complete and separable metric space (a Polish space), then $\mathbf{C}^0([0, \infty), E)$ stands for the space of continuous functions from $[0, \infty)$ to E that is also Polish when equipped with the topology of uniform convergence on every compact interval that is metrized e.g. by the distance defined for all $f, g \in \mathbf{C}^0([0, \infty), E)$ by $\delta(f, g) = \sum_{N \in \mathbb{N}} 2^{-N} \min(1, \sup_{s \in [0, N]} d_E(f(s), g(s)))$.

The Euclidian space \mathbb{R}^q , $q \in \mathbb{N}^*$ is equipped with its canonical basis denoted by (e_1, \dots, e_q) , and with its canonical scalar product denoted by $\langle \cdot, \cdot \rangle$ and with its associated Euclidian norm denoted by $|\cdot|$. When it is convenient, we shorten the $\mathbf{C}^0([0, \infty), \mathbb{R}^q)$ notation to \mathbf{C}_q^0 and to simplify notations, we set $\mathbf{C}_+^0 = \mathbf{C}^0([0, \infty), [0, \infty))$.

Definition 3.2.43 (Continuous snakes) Let (E, δ) be a Polish metric space. We equip the space $\mathbf{C}_+^0 \times \mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$ with the product topology, which is Polish too. A pair of functions $(h(s), w_s(\cdot))_{s \in [0, \infty)} \in \mathbf{C}_+^0 \times \mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$ is a *snake* if the following conditions are satisfied.

- (a) For all $s \in [0, \infty)$ and for all $r \in [h(s), \infty)$, $w_s(r) = w_s(h(s)) =: \widehat{w}_s$.
- (b) For all $s_1, s_2 \in [0, \infty)$, $w_{s_1}(r) = w_{s_2}(r)$ for all $r \in [0, m_h(s_1, s_2)]$ where we have set $m_h(s_1, s_2) = \inf_{s \in [s_1 \wedge s_2, s_1 \vee s_2]} h(s)$.

• We refer to (b) as to the *snake property*. The function h is called the *lifetime process* of the snake and the function $(\widehat{w}_s)_{s \in [0, \infty)}$ the *endpoint process* of the snake.

• We denote by $\Sigma_q([0, \infty))$ the space of snakes of $\mathbf{C}_+^0 \times \mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$. For all $\zeta \in (0, \infty)$, we denote by $\Sigma_q([0, \zeta])$ the snake with duration ζ : namely snakes $(h, w) \in \Sigma_q([0, \infty))$ such that $h(s) = 0$ for all $s \in [\zeta, \infty)$.

• To simplify notations, we also set $\Sigma_1([0, \infty)) = \Sigma([0, \infty))$ and $\Sigma_1([0, \zeta]) = \Sigma([0, \zeta])$. \square

Remark 3.2.44 (a) We easily check that $\Sigma_q([0, \infty))$ is a closed subset of $\mathbf{C}_+^0 \times \mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$.

(b) If $(h, w) \in \Sigma_q([0, \infty))$, then note that $\widehat{w} \in \mathbf{C}_q^0$ and observe that $(h, w) \mapsto \widehat{w}$ is continuous from the Polish spaces $\Sigma_q([0, \infty))$ to \mathbf{C}_q^0 .

(c) Let $(h, w) \in \Sigma_q([0, \infty))$ and let $s, r \in [0, \infty)$. If $r \leq m_h(0, s)$, the snake property implies that $w_s(r) = w_0(r)$ and if $r \geq m_h(0, s)$, $w_s(r) = \widehat{w}_{\alpha_{r,s}}$ where $\alpha_{r,s} = \sup\{s' \in [0, s] : h(s') \leq r \wedge h(s)\}$ by the snake property again. Thus, there is a function G from $\mathbf{C}_+^0 \times (\mathbf{C}_q^0)^2$ to $\Sigma_q([0, \infty))$ such that $G(h, w_0, \widehat{w}) = (h, w)$ for all $(h, w) \in \Sigma_q([0, \infty))$. Note that actually G can be chosen to be Borel-measurable (this fact is not needed in what follows).

(d) Let $(M_u)_{u \in t}$ a \mathbb{R} -valued branching walk and let W stands for its associated discrete contour-snake as in Definition 3.2.8. Then observe that $(C(t), W.(t; \cdot)) \in \Sigma([0, \infty))$. Therefore, the contour process $C(t)$ of t is the lifetime process of $W.(t; \cdot)$.

(e) Let $(h, w) \in \Sigma_q([0, \infty))$ and let $N \in \mathbb{N}^*$. We define $R_N(h, w) = (h', w')$ as follows: $h'(s) = h(s)$ and $w'_s = w_s$ for all $s \in [0, N]$, $h'(s) = h(2N - s)$ and $w'_s = w_{2N-s}$ for all $s \in [N, 2N]$, and $h'(s) = (2N+1-s)_+ h(0)$ and $w'_s = w_0(\cdot \wedge h'(s))$ for all $s \in [2N, \infty)$. Note that $R_N(h, w) \in \Sigma_q([0, 2N+1])$ and that $\widehat{w}'_s = \widehat{w}_s$ if $s \in [0, N]$, $\widehat{w}'_s = \widehat{w}_{2N-s}$ if $s \in [N, 2N]$, and $\widehat{w}'_s = w_0((2N+1-s)_+ h(0))$ if $s \in [2N, \infty)$.

(f) Let $(h, w) \in \Sigma_q([0, \infty))$. For all $s, r \in [0, \infty)$, we set $\underline{h}(s) = h(s) - m_h(0, s)$ and $\underline{w}_s(r) = w_s(r + m_h(0, s)) - w_0(m_h(0, s))$. Observe that $(\underline{h}, \underline{w}) \in \Sigma_q([0, \infty))$ and that $\widehat{\underline{w}}_s = \widehat{w}_s - w_0(m_h(0, s))$ for all $s \in [0, \infty)$. \square

Finite-dimensional marginals of random snakes. We now consider random snakes and more specifically Brownian snakes. To define their finite-dimensional marginals, it is convenient to introduce the following. For all $w_0, w_1 \in \mathbf{C}_q^0$ and all $m \in [0, \infty)$, we define $w_2 = w_0 \oplus_m w_1 \in \mathbf{C}_q^0$ by setting $w_2(s) = w_0(s)$ if $s \in [0, m]$ and $w_2(s) = w_0(m) + w_1(s - m) - w_1(0)$ if $s \in [m, \infty)$. We easily check that

$$(w_0, w_1, m, r) \in (\mathbf{C}_q^0)^2 \times [0, \infty)^2 \longmapsto (w_0 \oplus_m w_1)(\cdot \wedge r) \in \mathbf{C}_q^0 \quad \text{is continuous.} \quad (3.2.86)$$

Let $h \in \mathbf{C}_+^0$, $p \in \mathbb{N}^*$. Let $s_0 = 0 \leq s_1 < \dots < s_p$ be real numbers and let $w_0, \dots, w_p \in \mathbf{C}_q^0$. We define $F_p(h; (w_j)_{0 \leq j \leq p}; (s_j)_{1 \leq j \leq p}) = (w'_j)_{0 \leq j \leq p} \in (\mathbf{C}_q^0)^{p+1}$ by setting

$$w'_0 = w_0(\cdot \wedge h(0)) \quad \text{and} \quad w'_{j+1} = (w'_j \oplus_{m_h(s_j, s_{j+1})} w_{j+1})(\cdot \wedge h(s_{j+1})), \quad 0 \leq j < p, \quad (3.2.87)$$

where we recall that $m_h(s_j, s_{j+1}) = \inf_{s \in [s_j, s_{j+1}]} h(s)$. By (3.2.86) and the continuity of the function $(h, (s_j)_{0 \leq j \leq p}) \mapsto (h(0), (h(s_{j+1}), m_h(s_j, s_{j+1}))_{0 \leq j < p})$, we see that

$$F_p : \mathbf{C}_+^0 \times (\mathbf{C}_q^0)^{p+1} \times [0, \infty)^p \longmapsto (\mathbf{C}_q^0)^{p+1} \quad \text{is continuous.} \quad (3.2.88)$$

The functions $(F_p)_{p \in \mathbb{N}^*}$ provide a convenient way of expressing the finite-dimensional marginal laws of Brownian snakes. Recall that unless otherwise specified, all the r.v. which are considered in this chapter are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 3.2.45 (Brownian snakes) A q -dimensional Brownian snake is a random variable $(H, W) : \Omega \rightarrow \Sigma_q([0, \infty))$ that is measurable with respect to \mathcal{F} and to the Borel σ -field of $\Sigma_q([0, \infty))$, such that for all $p \in \mathbb{N}^*$, and all real numbers $0 \leq s_1 < \dots < s_p$,

$$(W_0, W_{s_1}, \dots, W_{s_p}) \stackrel{(\text{law})}{=} F_p(H; (B^{(j)})_{0 \leq j \leq p}; (s_j)_{1 \leq j \leq p}), \quad (3.2.89)$$

where $B^{(0)} = W_0$, where $(B^{(j)})_{1 \leq j \leq p}$ are independent standard q -dimensional Brownian motions that are moreover independent from W_0 and H . \square

Remark 3.2.46 (a) First, it is easy to check that the laws provided by the left member in (3.2.89) actually form a consistent family of finite-dimensional laws (so that the previous definition makes sense). Next, since $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$ is Polish, standard results ensure the existence of a regular version of the conditional law of W given (W_0, H) , i.e. a measurable transition kernel $P_{w_0, h}(dW)$ from $\mathbf{C}_q^0 \times \mathbf{C}_+^0$ to $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$ such that $\mathbf{E}[G(W_0, H)\mathbf{1}_{\{W \in A\}}] = \mathbf{E}[G(W_0, H)P_{W_0, H}(A)]$ for all Borel subsets A of $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$ and all nonnegative measurable functions G .

More precisely, the continuity (3.2.88) of F_p entails the following. Denote by $\Pi_{s_1, \dots, s_p}^{w_0, h}$ the law of $F_p(h; (B^{(j)})_{0 \leq j \leq p}; (s_j)_{1 \leq j \leq p})$ where $B^{(0)} = w_0$ and the $(B^{(j)})_{1 \leq j \leq p}$ are independent standard q -dimensional Brownian motions. Then, there exists a Borel subset B of $\mathbf{C}_q^0 \times \mathbf{C}_+^0$ such that $\mathbf{P}((W_0, H) \in B) = 1$ and such that for all $(w_0, h) \in B$, for all $p \in \mathbb{N}^*$ and for all real numbers $0 \leq s_1 < \dots < s_p$,

$$\Pi_{s_1, \dots, s_p}^{w_0, h} \text{ is the law of } (W_0, W_{s_1}, \dots, W_{s_p}) \text{ under } P_{w_0, h}(dW). \quad (3.2.90)$$

(b) The recursive construction (3.2.87) entails that conditionally given (W_0, H) , W is a (time inhomogeneous) Markov process. More precisely, keeping notations from **(a)**, for all $(w_0, h) \in B$ we check that W under $P_{w_0, h}$ is an inhomogeneous Markov process with transition probabilities $P_{w_0, h}(w_{s'} \in dw' \mid w_s = w) := \Pi_{s'-s}^{w, h(\cdot + s)}(dw')$. \square

Given a continuous lifetime process H and an initial path W_0 , there is not always a continuous snake (H, W) . However, when H stands for the α -stable height process, there is a continuous version of the Brownian snake as shown in the following proposition, which is a direct consequence of a result in D. & Le Gall [61].

Proposition 3.2.47 *Let $\alpha \in (1, 2]$. Let $(H_s)_{s \in [0, \infty)}$ be the α -stable height process as defined by (3.2.53) (resp. under the normalized excursion law $\mathbf{N}(\cdot \mid \zeta > 1)$ or $\mathbf{N}(\cdot \mid \zeta = 1)$). Then there exists a continuous one-dimensional Brownian snake W whose lifetime process is H which is called the one-dimensional Brownian snake with α -stable branching mechanism.*

Proof. See [61, Prop. 4.4.1] when H is under \mathbf{P} . Standard arguments entail the result for H under $\mathbf{N}(\cdot \mid \zeta > 1)$, then we use (3.2.59) to derive it under $\mathbf{N}(\cdot \mid \zeta = 1)$. Otherwise, see Lemma 2.4.30 in Chapter 2 (and a more explicit discussion of the same ideas). \blacksquare

Snake-pseudometrics and snake-trees. Snakes play a key-role in our chapter because they encode the metric of the limiting trees of the rescaled range of the branching random walks that we consider here.

Definition 3.2.48 (Snake-pseudometrics) Let $\zeta \in (0, \infty)$ and let $(h, w) \in \Sigma([0, \zeta])$. Recall from (3.2.84) the definition of $m_h(\cdot, \cdot)$ and recall that $\widehat{w}_s = w_s(h(s))$, $s \in [0, \infty)$. For all $s_1, s_2 \in [0, \zeta]$, we set

$$M_{h,w}(s_1, s_2) = \min\left(\min\{w_{s_1}(r); r \geq m_h(s_1, s_2)\}, \min\{w_{s_2}(r); r \geq m_h(s_1, s_2)\}\right) \\ \text{and } d_{h,w}(s_1, s_2) = \widehat{w}_{s_1} + \widehat{w}_{s_2} - 2M_{h,w}(s_1, s_2). \quad (3.2.91)$$

We call $d_{h,w}$ the *snake-pseudometric associated with (h, w)* (see Lemma 3.2.49 below). \square

Lemma 3.2.49 Let $\zeta \in (0, \infty)$, let $(h, w) \in \Sigma([0, \zeta])$ and let $d_{h,w}$ be as in Definition 3.2.48. Then, $d_{h,w}$ is a continuous pseudometric on $[0, \zeta]$ satisfying (3.2.80): namely $d_{h,w} \in \mathbf{MT}([0, \zeta])$.

Proof. See Lemma 2.4.22 in Chapter 2. \blacksquare

Definition 3.2.50 (Snake-trees, α -stable Brownian cactus) Let $\zeta \in (0, \infty)$, let $(h, w) \in \Sigma([0, \zeta])$ and $d_{h,w} \in \mathbf{MT}([0, \zeta])$ be as in Definition 3.2.48.

(a) We denote by $(T_{h,w}, d_{h,w}, r_{h,w}, \mu_{h,w})$ the pointed measured compact metric space induced by the pseudometric $d_{h,w}$ as defined in (3.2.81). We call it the *snake-tree* associated with (h, w) . To simplify, we denote by $p_{h,w} : [0, \zeta] \rightarrow T_{h,w}$ the canonical projection.

(b) Let $\alpha \in (1, 2]$ and let (H, W) be a one-dimensional Brownian snake with α -stable branching mechanism as defined in Proposition 3.2.47. Then the random real tree $(T_{H,W}, d_{H,W}, r_{H,W}, \mu_{H,W})$ is called a *Brownian cactus with α -stable branching mechanism*. \square

As already mentioned, the Brownian Cactus has been introduced in Curien, Le Gall & Miermont [40] to study planar maps: roughly speaking the Brownian cactus corresponds to the case of a quadratic branching mechanism $\alpha = 2$. In this case, they prove that a.s. its upper-local density for typical points is 4 (Proposition 5.1 [40] p. 364) and that its Hausdorff dimension is 4 (Corollary 5.3 [40] p. 365). See also Le Gall [116] where the level sets of the Brownian cactus are studied to derive results on the Brownian maps. In D., K., Lin & Torri [60, Prop. 4.34], it is proved that the Brownian cactus with α -stable mechanism is a.s. a continuum real tree (i.e. $\mu_{H,W}$ is diffuse and supported by the set of leaves of $T_{H,W}$) and that it is a binary real tree with a (necessarily countable) dense set of branch points: namely, a.s. for all $x \in T_{H,W}$, $T_{H,W} \setminus \{x\}$ has one, two, or three connected components and the set of the x where $T_{H,W} \setminus \{x\}$ has three connected components is dense in $T_{H,W}$.

Limits of snakes and snake-pseudometrics. We first present a general result showing that convergence of snakes boils down to the convergence of the corresponding endpoint processes. This is a slight extension of the homeomorphism theorem of Marckert & Mokkadem [129, Theorem 2.1]. Then we recall from Chapter 2 (adapted from D., K., Lin & Torri [60]) a result showing that the convergence of snakes implies the convergence of snake-pseudometrics, which leads to a practical criterion to prove weak convergence of random snake-trees.

Recall that $\mathbf{C}^0([0, \infty), E)$ stands for the space of the continuous functions from $[0, \infty)$ to the Polish space (E, d_E) . Also recall that \mathbf{C}_q^0 (resp. \mathbf{C}_+^0) stands for $\mathbf{C}^0([0, \infty), \mathbb{R}^q)$ (respectively $\mathbf{C}^0([0, \infty), [0, \infty))$), which is Polish when equipped with the following metric $\delta(f, g) = \sum_{N \in \mathbb{N}^*} 2^{-N} \min(1, \sup_{s \in [0, N]} |f(s) - g(s)|)$, $f, g \in \mathbf{C}_q^0$ (resp. $f, g \in \mathbf{C}_+^0$). Next recall that $\Sigma_q([0, \infty))$ stands for the space of snakes, as in Definition 3.2.43.

If $(h^{(n)}, w^{(n)}) \in \Sigma_q([0, \infty))$, $n \in \mathbb{N}$, is such that $(h^{(n)}, \widehat{w}^{(n)}) \rightarrow (h, f)$ in $\mathbf{C}_+^0 \times \mathbf{C}_q^0$, then f is not necessarily the endpoint process of a continuous snake and the $w^{(n)}$ do not necessarily converge in the space $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$. In this direction, we however prove the following result.

Proposition 3.2.51 *Let $(h^{(n)}, w^{(n)})_{n \in \mathbb{N}}$ be a sequence of q -dimensional continuous snakes. We assume that there exists a q -dimensional continuous snake (h, w) such that $\lim_{n \rightarrow \infty} h^{(n)} = h$ in \mathbf{C}_+^0 , such that $\lim_{n \rightarrow \infty} w_0^{(n)} = w_0$ and such that $\lim_{n \rightarrow \infty} \widehat{w}^{(n)} = \widehat{w}$ in \mathbf{C}_q^0 . Then $\lim_{n \rightarrow \infty} w^{(n)} = w$ in $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$.*

Proof. When $h^{(n)}(0) = 0$ and $(h^{(n)}, w^{(n)}) \in \Sigma_q([0, 1])$ for all $n \in \mathbb{N}$, the result follows from a direct application of Marckert & Mokkadem [129, Theorem 2.1]. By time-scaling, this still holds when there is some $N \in \mathbb{N}$ such that $h^{(n)}(0) = 0$ and $(h^{(n)}, w^{(n)}) \in \Sigma_q([0, 2N+1])$ for all $n \in \mathbb{N}$. Now, we claim that the desired result holds when $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Indeed, using Remark 3.2.44 (e), the previous particular case implies that $\lim_{n \rightarrow \infty} R_N(h^{(n)}, w^{(n)}) = R_N(h, w)$ in $\Sigma_q([0, 2N+1])$ for any $N \in \mathbb{N}$, so $w^{(n)} \rightarrow w$ uniformly on $[0, N]$. Now, we prove the desired result without any additional assumptions thanks to Remark 3.2.44 (f). We easily see that $\lim_{n \rightarrow \infty} \underline{h}^{(n)} = \underline{h}$ in \mathbf{C}_+^0 and $\lim_{n \rightarrow \infty} \underline{\widehat{w}}^{(n)} = \underline{\widehat{w}}$ in \mathbf{C}_q^0 , so $\lim_{n \rightarrow \infty} \underline{w}^{(n)} = \underline{w}$ in $\mathbf{C}^0([0, \infty), \mathbf{C}_q^0)$. Then, we observe that

$$\forall s, r \in [0, \infty), \quad w_s(r) = w_0(r \wedge m_h(0, s)) + \underline{w}_s((r - m_h(0, s))_+),$$

which concludes the proof. ■

The following lemma recalled from Chapter 2 shows that the convergence of one-dimensional snakes implies the convergence of the associated pseudometrics. Recall that $\Sigma([0, \zeta])$ stands for the set of one-dimensional snakes. For all $(h, w) \in \Sigma([0, \zeta])$, recall from Example 3.2.42 the definition of the real-tree pseudometric d_h and recall from Definition 3.2.48 the snake-pseudometric $d_{h,w}$. Recall that $\|\cdot\|$ stands for the uniform norm on $\mathbf{C}([0, \zeta]^2, \mathbb{R})$.

Lemma 3.2.52 *Let $\zeta \in [0, \infty)$. For all $n \in \mathbb{N}$, let $(h^{(n)}, w^{(n)}) \in \Sigma([0, \zeta])$. We suppose that $(h^{(n)}, w^{(n)}) \rightarrow (h, w)$ in $\mathbf{C}_+^0 \times \mathbf{C}([0, \zeta], \mathbf{C}_1^0)$. Then, $\lim_{n \rightarrow \infty} \|d_{h^{(n)}} - d_h\| = \lim_{n \rightarrow \infty} \|d_{h^{(n)}, w^{(n)}} - d_{h,w}\| = 0$.*

Proof. See Lemma 2.4.20 in Chapter 2. ■

3.2.8 Tree-valued branching walks and snake-pseudometrics

We fix an environment $(T, o) \in \overline{\mathbb{T}}^\bullet$. We then fix a finite tree $t \in \mathbb{T}$ and a branching walk $\Theta := (t, \emptyset; \bar{Y}_u = (T, Y_u), u \in t)$ such that: (a) $Y_\emptyset = o$; (b) if $v, v' \in t$ are neighbors, so are Y_v and $Y_{v'}$ in T . In this section, we consider the range $R = \{Y_u; u \in t\}$ that is a graph-subtree of T and we explain that R is close in some sense to the snake-tree associated with the contour-snake of the relative heights of the branching walk, i.e. the contour-snake associated with the \mathbb{R} -branching walk $(|Y_u|)_{u \in t}$.

To that end, we denote by $(\tilde{t}, d_{\tilde{t}})$ the real tree obtained by joining adjacent vertices of t by a segment isometric to $[0, 1]$. We assume that $t \subset \tilde{t}$. Thus $d_{\text{gr}}(u, v) = d_{\tilde{t}}(u, v)$ for all $u, v \in t$, where d_{gr} stands for the graph-distance in t . For all $\sigma, \sigma' \in \tilde{t}$, we denote by $\llbracket \sigma, \sigma' \rrbracket_{\tilde{t}}$ the geodesic path joining σ and σ' in \tilde{t} .

We then introduce the *contour-exploration of \tilde{t}* as follows. We denote by $(v_k)_{0 \leq k \leq 2(\#t-1)}$ the vertices of t listed in the contour order. We easily check that there exists a continuous function $\tilde{v}: [0, 2(\#t-1)] \rightarrow \tilde{t}$ such that for all $0 \leq k < 2(\#t-1)$ and for all $s \in [0, 1]$,

$$\tilde{v}(k+s) \in \llbracket v_k, v_{k+1} \rrbracket_{\tilde{t}} \quad \text{is such that} \quad d_{\tilde{t}}(v_k, \tilde{v}(k+s)) = s. \quad (3.2.92)$$

It is convenient to set $\tilde{v}(s) = \emptyset$ for all $s \in [2(\#t-1), 2\#t]$. The continuous path $(\tilde{v}(s))_{s \in [0, 2\#t]}$ is the contour exploration of \tilde{t} . Namely for all $s, s' \in [0, 2\#t]$,

$$d_{\tilde{t}}(\tilde{v}(s), \tilde{v}(s')) = d_{C(t)}(s, s')$$

where $d_{C(t)}$ is the real tree distance associated with the contour function $(C_s(t))_{s \in [0, 2\#t]}$ as introduced in Example 3.2.42.

Remark 3.2.53 It is easy to see that the tree $(T_{C(t)}, d_{C(t)}, \mathbf{r}_{C(t)}, \mathbf{m}_{C(t)})$ coded by $C.(t)$ as introduced in Example 3.2.42 is isometric to $(\tilde{t}, d_{\tilde{t}}, \emptyset, \tilde{\mathbf{m}})$ where $\tilde{\mathbf{m}}$ is the occupation measure of the exploration process \tilde{v} . As a consequence of Lemma 2.5.2 of Chapter 2., we get that $d_{\text{Prok}}^{\tilde{t}}(\mathbf{m}, \frac{1}{2}\tilde{\mathbf{m}}) \leq 2$ where $\mathbf{m} = \sum_{v \in t} \delta_v$, which implies that $\delta_{\text{GHP}}((t, d_{\text{gr}}, \emptyset, \mathbf{m}), (\tilde{t}, d_{\tilde{t}}, \emptyset, \frac{1}{2}\tilde{\mathbf{m}})) \leq 3$. \square

Next, we denote similarly by $(\tilde{T}, d_{\tilde{T}})$ the real tree obtained by joining adjacent vertices of T by a segment isometric to $[0, 1]$ and we extend relative heights on \tilde{T} by setting

$$\forall x \in T, \quad \forall \gamma \in \llbracket \overleftarrow{x}, x \rrbracket_{\tilde{T}}, \quad |\gamma| = |\overleftarrow{x}| + d_{\tilde{T}}(\overleftarrow{x}, \gamma).$$

For all $\gamma, \gamma' \in \tilde{T}$, we also denote by $\gamma \wedge \gamma'$ the most recent common ancestor of γ and γ' that is defined as the unique point of $\llbracket \gamma, \gamma' \rrbracket_{\tilde{T}}$ with minimal relative height. We finally define a continuous function $\tilde{Y} : \tilde{t} \rightarrow \tilde{T}$ that extends Y as follows: for all $\sigma \in \tilde{t}$ there exist two neighboring vertices $v, v' \in t$ such that $\sigma \in \llbracket v, v' \rrbracket_{\tilde{t}}$ and we define

$$\tilde{Y}_\sigma \text{ as the only point of } \llbracket Y_v, Y_{v'} \rrbracket_{\tilde{T}} \text{ such that } d_{\tilde{T}}(Y_v, \tilde{Y}_\sigma) = d_{\tilde{t}}(v, \sigma). \quad (3.2.93)$$

We denote by \tilde{R} the range of \tilde{Y} and we denote by $\tilde{\mathbf{m}}_{\text{occ}}$ the occupation measure on \tilde{R} yielded by the contour exploration of \tilde{t} . Namely,

$$\tilde{R} = \{\tilde{Y}_\sigma; \sigma \in \tilde{t}\} \quad \text{and} \quad \int_{\tilde{R}} f(\gamma) \tilde{\mathbf{m}}_{\text{occ}}(d\gamma) = \int_0^{2\#t} f(\tilde{Y}_{v(s)}) ds. \quad (3.2.94)$$

We also denote by \mathbf{m}_{occ} the occupation measure of Y on $R = \{Y_u; u \in t\}$:

$$\mathbf{m}_{\text{occ}} = \sum_{u \in t} \delta_{Y_u}.$$

Note that $d_{\text{Haus}}^{\tilde{T}}(\tilde{R}, R) \leq 1$. Then we recall from Lemma 2.5.2 in Chapter 2 that

$$d_{\text{Prok}}^{\tilde{T}}(\mathbf{m}_{\text{occ}}, \frac{1}{2}\tilde{\mathbf{m}}_{\text{occ}}) \leq 2 \quad \text{and thus} \quad \delta_{\text{GHP}}((R, d_{\text{gr}}, o, \mathbf{m}_{\text{occ}}), (\tilde{R}, d_{\tilde{T}}, o, \frac{1}{2}\tilde{\mathbf{m}}_{\text{occ}})) \leq 3. \quad (3.2.95)$$

We remind that $d_{\text{Haus}}^{\tilde{T}}$ and $d_{\text{Prok}}^{\tilde{T}}$ stand respectively for the Hausdorff distance on compact subsets of \tilde{T} and for the Prokhorov distance on the space of finite Borel measures on \tilde{T} .

We next introduce the *range pseudometric* $d_{\tilde{R}}(\cdot, \cdot) \in \mathbf{M}([0, 2\#t])$ by setting for all $s, s' \in [0, 2\#t]$

$$d_{\tilde{R}}(s, s') = d_{\tilde{T}}(\tilde{Y}_{v(s)}, \tilde{Y}_{v(s')}) = |\tilde{Y}_{v(s)}| + |\tilde{Y}_{v(s')}| - 2|\tilde{Y}_{v(s)} \wedge \tilde{Y}_{v(s')}|. \quad (3.2.96)$$

We observe that $d_{\tilde{R}}$ is a real tree pseudometric and that

$$(\tilde{R}, d_{\tilde{T}}, o, \tilde{\mathbf{m}}_{\text{occ}}) \quad \text{is isometric to} \quad (E_{d_{\tilde{R}}}, d_{\tilde{R}}, \mathbf{r}_{d_{\tilde{R}}}, \mathbf{m}_{d_{\tilde{R}}}) \quad (3.2.97)$$

where the space $E_{d_{\tilde{R}}}$ is the pointed measured compact real tree yielded by $d_{\tilde{R}}$ as explained in (3.2.81) and (3.2.82).

Remark 3.2.54 Observe that the previous constructions on the real trees \tilde{T} and \tilde{t} only depend on the branching walk $\Theta = (t, \emptyset; \bar{Y}_u = (T, Y_u), u \in t)$ locally in a continuous way and we easily check that $\Theta \in \mathbb{T}^\bullet(\mathbb{T}^\bullet) \mapsto d_{\tilde{R}}(\cdot, \cdot) \in \mathbf{M}([0, 2\#t])$ is continuous. \square

We now compare $d_{\tilde{R}}$ with the snake-pseudometric associated with the relative heights of the branching walk. We next denote by $(W_s(t; \cdot))_{s \in [0, 2\#t]}$ the contour-snake associated with the \mathbb{R} -valued branching walk $(Y_u)_{u \in t}$ (see Definition 3.2.8). We denote by $(\widehat{W}_s(t))_{s \in [0, 2\#t]}$ its associated endpoint process and we recall that the corresponding lifetime process is the contour process $(C_s(t))_{s \in [0, 2\#t]}$. We then easily check for all $s \in [0, 2\#t]$ and $r \in [0, C_s(t)]$ that

$$W_s(t; r) = |\tilde{Y}_\sigma| \quad \text{where } \sigma \in [\emptyset, \tilde{v}(s)]_{\tilde{t}} \text{ is such that } d_{\tilde{t}}(\emptyset, \sigma) = r. \quad (3.2.98)$$

In particular, note that $\widehat{W}_s(t) = |\tilde{Y}_{\tilde{v}(s)}|$. We next recall from (3.2.91) in Definition 3.2.48 that $d_{C(t), W(t; \cdot)}$ stands for the snake-pseudometric associated with $(C(t), W(t; \cdot))$. As a consequence of (3.2.98), we see for all $s, s' \in [0, 2\#t]$ that

$$d_{C(t), W(t; \cdot)}(s, s') = |\tilde{Y}_{\tilde{v}(s)}| + |\tilde{Y}_{\tilde{v}(s')}| - 2 \min \{ |\tilde{Y}_\sigma|; \sigma \in [\tilde{v}(s), \tilde{v}(s')]_{\tilde{t}} \}. \quad (3.2.99)$$

We then observe that

$$0 \leq \frac{1}{2}(d_{C(t), W(t; \cdot)}(s, s') - d_{\tilde{R}}(s, s')) = |\tilde{Y}_{\tilde{v}(s)} \wedge \tilde{Y}_{\tilde{v}(s')}| - \min \{ |\tilde{Y}_\sigma|; \sigma \in [\tilde{v}(s), \tilde{v}(s')]_{\tilde{t}} \}. \quad (3.2.100)$$

When Y is a branching random walk whose spatial motion is a biased RW, (3.2.100) implies that $d_{C(t), W(t; \cdot)}$ and $d_{\tilde{R}}$ are close because for the difference in (3.2.100) to be large, two independent and large parts of the branching RW have to coincide. More precisely, we conclude this section by proving the following lemma that shows (for fixed times) that the difference in (3.2.100) is exponentially small when Y is a (sub)critical biased branching RW.

Lemma 3.2.55 *Let $\lambda \in (1, \infty)$ and let $(T, o) \in \overline{\mathbb{T}}^\bullet$ be such that $T \neq \{\emptyset\}$. Let $t \in \mathbb{T}$ be finite. Let $\Theta = (t, \emptyset; \overline{Y}_u = (T, Y_u), u \in t)$ be a t -indexed T -valued λ -biased branching RW whose law is $Q_{t, \emptyset}^{T, o}$ (Definition 3.2.30). If Assumption (3.2.30) in Lemma 3.2.23 is verified, then for all $\xi \in (1, \infty)$,*

$$\mathbf{P} \left(\max_{s, s' \in [0, 2\#t]} |d_{C(t), W(t; \cdot)}(s, s') - d_{\tilde{R}}(s, s')| > 2\xi \right) \leq \lambda^2 (\#t)^3 \lambda^{-\xi}. \quad (3.2.101)$$

Proof. By (3.2.100) and the definition on \tilde{T} of relative heights and of most recent common ancestors, we see that if there are $s_1, s_2 \in [0, 2\#t]$ such that $d_{C(t), W(t; \cdot)}(s_1, s_2) - d_{\tilde{R}}(s_1, s_2) > 2\xi$, then there are $u', v' \in t$ such that $|Y_{u'} \wedge Y_{v'}| - \min_{w \in [u', v']} |Y_w| > \xi - 1 > 0$. Then, let us choose $w' \in [u', v']$ such that $|Y_{w'}| = \min_{w \in [u', v']} |Y_w|$ and with the greatest height possible. Without loss of generality, we can assume that $w' \in [u' \wedge v', u']$ so we have $w \in [u' \wedge v', u']$ such that $\overleftarrow{w} = w'$. By definition of w' and w , we observe that $\overleftarrow{Y}_w = Y_{w'}$, that $Y_{u'} \wedge Y_{v'} \in [Y_{w'}, Y_{v'}]$, and that $Y_{u'} \wedge Y_{v'} \in [Y_w, Y_{u'}]$. Thus, there are $u, v \in t$ with $v \wedge w \neq w \leq u$ such that $Y_u = Y_v = Y_{u'} \wedge Y_{v'}$. Therefore, (3.2.101) is a straightforward consequence of the following bound that holds for all $u, v, w \in t$ with $v \wedge w \neq w \leq u$, and all $p \in \mathbb{N}^*$,

$$\mathbf{P} \left(Y_u = Y_v, |Y_v| - |Y_w| \geq p, \overleftarrow{Y}_w \notin \{Y_a : a \in [w, u]\} \right) \leq \lambda^{-p}. \quad (3.2.102)$$

It remains to prove (3.2.102). To that end, recall from Definition 3.2.21 that $P_{T, x}$ stands for the law of the λ -biased RW $\overline{X}_n = (T, X_n), n \in \mathbb{N}$ on T such that $X_0 = x$. By reasoning conditionally given (Y_w, Y_v) , the very definition of branching RWs implies that

$$\begin{aligned} & \mathbf{P} \left(Y_u = Y_v, |Y_v| - |Y_w| \geq p, \overleftarrow{Y}_w \notin \{Y_a : a \in [w, u]\} \right) \\ &= \mathbf{E} \left[\mathbf{1}_{\{|Y_v| - |Y_w| \geq p\}} P_{T, Y_w} (X_{|u| - |w|} = Y_v, \overleftarrow{Y}_w \notin \{X_j : j \leq |u| - |w|\} \mid Y_w, Y_v) \right]. \end{aligned}$$

For all $x, y \in T \setminus \{\emptyset\}$, we set $f(x, y) = P_{T,x}(H_y < H_{\overline{x}})$, where H_y is the hitting time as in (3.2.29). Then the previous identity yields that

$$\mathbf{P}\left(Y_u = Y_v, |Y_v| - |Y_w| \geq p, \overleftarrow{Y}_w \notin \{Y_a : a \in [w, u]\}\right) \leq \mathbf{E}\left[\mathbf{1}_{\{|Y_v| - |Y_w| \geq p\}} f(Y_w, Y_v)\right] \quad (3.2.103)$$

Thanks to Assumption (3.2.30), we can use (3.2.33) to get $f(x, y) \leq \lambda^{|x| - |y|}$ for all $x, y \in T \setminus \{\emptyset\}$. Together with (3.2.103), this immediately entails (3.2.102), which completes the proof. ■

3.3 Proof of Theorem 3.1.1

3.3.1 Preliminary results on critical biased RWs on GW-trees

In this section, we state several estimates on critical biased RWs on GW-trees. More precisely, here ν stands for a supercritical offspring distribution whose mean is denoted by m_ν , the environment is denoted by (\mathbf{T}, \circ) that is a $\text{GW}(\nu)$ -tree or a variant of such trees (a grafted or an infinite or an invariant $\text{GW}(\nu)$ -tree) and the RW is denoted by $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$: it is viewed as a $\overline{\mathbb{T}}^\bullet$ -valued Markov chain whose transition kernel \overline{p}_{m_ν} is that of a m_ν -biased RW as in (3.2.27) in Definition 3.2.21 with $\lambda = m_\nu$.

Lemma 3.3.1 *Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. Let $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$ be a m_ν -biased RW such that $X_0 = \circ$. For all $y \in \mathbf{T}$, we set*

$$\zeta_y = \sup\{n \in \mathbb{N} : X_n = y\}, \quad (3.3.1)$$

with the convention that $\sup \emptyset = -\infty$. Recall from (3.2.15) that L_\circ is the ancestral line of \circ and that for all $p \in \mathbb{N}$, $\circ(p)$ stands the ancestor of \circ at distance p from it. Then, for all $p_0, p_1, p \in \mathbb{N}$ such that $p > 2p_0 + p_1$ and for all $n \in \mathbb{N}$, the two truncated $\overline{\mathbb{T}}^\bullet$ -valued processes $([\overline{X}_{m \wedge p_0}]_{-p_0}^{p_0})_{m \in \mathbb{N}}$ and $([\varphi|_{X_{\zeta_{\circ(p)}+1+n}}(\overline{X}_{\zeta_{\circ(p)}+1+n+m})]_{-p_1}^{p_1})_{m \in \mathbb{N}^*}$ are independent under \mathbf{P} .

Proof. To simplify the notations, we set $\overline{X}_n^{(p)} = \overline{X}_{\zeta_{\circ(p)}+1+n}$ for all $n \in \mathbb{N}$. Let F and G be nonnegative measurable functions on $(\overline{\mathbb{T}}^\bullet)^\mathbb{N}$. First observe the following.

$$\begin{aligned} & E_{(\mathbf{T}, \circ)}[F(\overline{X}_{\cdot \wedge \zeta_{\circ(p)}})G(\overline{X}^{(p)})] \\ &= \sum_{k \in \mathbb{N}} E_{(\mathbf{T}, \circ)}[F(\overline{X}_{\cdot \wedge k})\mathbf{1}_{\{X_k = \circ(p); X_{k+1} = \circ(p+1)\}}G(\overline{X}_{k+1+\cdot})\mathbf{1}_{\{H_{\circ(p)}(X_{k+1+\cdot}) = \infty\}}] \\ &= E_{(\mathbf{T}, \circ)}[F(\overline{X}_{\cdot \wedge \zeta_{\circ(p)}})]E_{(\mathbf{T}, \circ(p+1))}[G(\overline{X}_{\cdot}) | H_{\circ(p)} = \infty]. \end{aligned} \quad (3.3.2)$$

Namely, conditionally given (\mathbf{T}, \circ) , $\overline{X}_{\cdot \wedge \zeta_{\circ(p)}}$ is independent from $\overline{X}^{(p)}$ that is furthermore a Markov chain. More precisely, set $\mathbf{T}_p = \{\circ(p)\} \cup \{x \in \mathbf{T} : |\circ \wedge x| < -p\}$ and $\overline{X}'_n = (\mathbf{T}_p, X_{\zeta_{\circ(p)}+1+n})$, $n \in \mathbb{N}$. Then, (3.3.2) and (3.2.34) easily imply that conditionally given (\mathbf{T}, \circ) , $(\overline{X}'_n)_{n \in \mathbb{N}}$ is a Markov chain whose transition kernel is given for all $x \in \mathbf{T}_p \setminus \{\circ(p)\}$ by

$$\begin{aligned} q((\mathbf{T}_p, x), (\mathbf{T}_p, y)) &:= \overline{p}_{m_\nu}((\mathbf{T}_p, x), (\mathbf{T}_p, y)) \frac{1 - m_\nu^{p+|\circ(p) \wedge y|}}{1 - m_\nu^{p+|\circ(p) \wedge x|}}, \\ q((\mathbf{T}_p, \circ(p)), (\mathbf{T}_p, y)) &= \mathbf{1}_{\{y = \circ(p+1)\}}, \end{aligned}$$

where \overline{p}_{m_ν} is as in (3.2.27). Therefore, the law of $(\overline{X}'_n)_{n \in \mathbb{N}}$ conditionally given (\mathbf{T}, \circ) only depends on $(\mathbf{T}_p, \circ(p))$ that is independent from $[(\mathbf{T}, \circ)]_\infty^{\circ(p)}$, which easily entails the desired result. ■

The following lemma provides an estimate of the hitting time of critical biased RW.

Lemma 3.3.2 *Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . Let \mathbf{T} be a grafted $\text{GW}(\nu)$ -tree as in Definition 3.2.6 (c). Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a \mathfrak{m}_ν -biased RW such that $X_0 = (1)$ (which is the unique child of \emptyset by definition of grafted GW -trees). Recall from (3.2.29) the definition \mathbb{H}_\emptyset . For all $s \in (0, \infty)$, we set $f_\nu(s) = \mathbf{E}[\exp(-s(\mathbb{H}_\emptyset - 1))]$. Then,*

$$\forall s \in (0, \infty), \quad 1 - f_\nu(s) \leq \sqrt{2s}. \quad (3.3.3)$$

Proof. Recall that $P_{\mathbf{T},(1)}$ stands for the conditional law of $(\bar{X}_n)_{n \in \mathbb{N}}$ given \mathbf{T} . For all $y \in \mathbf{T} \setminus \{\emptyset\}$ and all $r \in [0, 1]$, we set

$$\alpha(r, y) := E_{(\mathbf{T}, y)}[r^{\mathbb{H}_y}] \quad \text{and} \quad \beta(r) := \frac{1}{\mathfrak{m}_\nu} \sum_{z \in \mathbf{T}: \overleftarrow{z} = (1)} (1 - r\alpha(r, z)).$$

To simplify, we also set $\alpha(r, (1)) = \alpha(r)$. First note that conditionally given $k_{(1)}(\mathbf{T})$, the r.v. $1 - r\alpha(r, z)$, $z \in \mathbf{T}$ such that $\overleftarrow{z} = (1)$, are independent with the same law as $1 - r\alpha(r)$. Thus,

$$\mathbf{E}[\beta(r)] = \frac{1}{\mathfrak{m}_\nu} \mathbf{E}[k_{(1)}(\mathbf{T})] \mathbf{E}[1 - r\alpha(r)] = \mathbf{E}[1 - r\alpha(r)]. \quad (3.3.4)$$

Then, by Markov at time 1,

$$\alpha(r, y) = \frac{\mathfrak{m}_\nu r}{\mathfrak{m}_\nu + k_y(T)} + \sum_{z \in \mathbf{T}: \overleftarrow{z} = y} \frac{r\alpha(r, z)\alpha(r, y)}{\mathfrak{m}_\nu + k_y(T)}.$$

An elementary computation implies that $(1 + \beta(r))r\alpha(r) = r^2$. By Cauchy-Schwarz's inequality and (3.3.4), we get

$$r^2 = \mathbf{E}[\sqrt{1 + \beta(r)}\sqrt{r\alpha(r)}]^2 \leq \mathbf{E}[1 + \beta(r)]\mathbf{E}[1 - \beta(r)] = 1 - \mathbf{E}[\beta(r)]^2.$$

Recall that $f_\nu(s) = e^s \mathbf{E}[\alpha(e^{-s})]$. Then, we have proved $1 - e^{-2s} f_\nu(s) = \mathbf{E}[\beta(e^{-s})] \leq \sqrt{1 - e^{-2s}}$, which entails (3.3.3) in Lemma 3.3.2. \blacksquare

The following proposition provides an estimate of the speed at which the critical biased RW on an invariant GW -tree escapes to $-\infty$. It is a generalization of Proposition 2.3.13 of Chapter 2.

Proposition 3.3.3 *Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a \mathfrak{m}_ν -biased RW such that $X_0 = \circ$. Recall from (3.2.15) that L_\circ is the ancestral line of \circ and that for all $p \in \mathbb{N}$, $\circ(p)$ stands the ancestor of \circ at distance p from it. Let $f: [0, \infty) \rightarrow (0, \infty)$ be a continuous function decreasing to 0. Then, the following holds true*

$$\left(\mathbf{P}\text{-a.s.} \sum_{p \geq 1} k_{\circ(p)}(\mathbf{T}) f(p)^{\frac{1}{2}} < \infty \right) \implies \left(\mathbf{P}\text{-a.s.} \sum_{n \in \mathbb{N}} f(-|\circ \wedge X_n|) < \infty \right). \quad (3.3.5)$$

Proof. Recall from Lemma 3.2.23 that \mathbf{r}_k° stands for the k -th time of return to L_\circ . Namely, $\mathbf{r}_k^\circ = \inf \{n > \mathbf{r}_{k-1}^\circ : X_n \in L_\circ\}$, with the convention that $\mathbf{r}_{-1}^\circ = -1$. By Lemma 3.2.24, Lemma 3.2.23 applies to the \mathbf{T} -valued RW \bar{X} and we \mathbf{P} -a.s. get $\mathbf{r}_k^\circ < \infty$. Also recall from Lemma 3.2.23 that the \mathbb{N} -valued process $(Z_k)_{k \in \mathbb{N}}$ such that $\circ(Z_k) = X_{\mathbf{r}_k^\circ}$, is distributed as a birth-and-death chain conditionally given \mathbf{T} , whose transition probabilities are given by (3.2.32) with $\lambda = \mathfrak{m}_\nu$. For all $\ell \in \mathbb{N}^*$, we set $I(\ell) = \{k \in \{0, \dots, \ell-1\} : Z_k = Z_{k+1}\}$ and we check that

$$W_\ell = \sum_{0 \leq n < \mathbf{r}_\ell^\circ} f(-|\circ \wedge X_n|) = W'_\ell + \sum_{k \in I(\ell)} (\mathbf{r}_{k+1}^\circ - \mathbf{r}_k^\circ - 1) f(Z_k) \quad \text{where} \quad W'_\ell = \sum_{0 \leq k < \ell} f(Z_k).$$

Next we recall from (3.2.15) that $\partial L_o = \{x \in \mathbf{T} \setminus L_o : \overleftarrow{x} \in L_o\}$ and we recall from (3.2.29) the notation H_x , $x \in \mathbf{T}$. Observe for all $p \in \mathbb{N}$ and for all $k \in \{0, \dots, \ell-1\}$ such that $Z_k = Z_{k+1} = p$, that $\mathbf{r}_{k+1}^o - \mathbf{r}_k^o \geq 2$. Namely, at time \mathbf{r}_k^o , the biased RW X starts an excursion from $o(p)$ outside L_o (which implies $k_{o(p)}(\mathbf{T}) \geq 2$ when $p \geq 1$). The duration $\mathbf{r}_{k+1}^o - \mathbf{r}_k^o$ of this excursion is the time needed to return to $o(p)$ starting from a uniformly chosen vertex $x \in \partial L_o$ such that $\overleftarrow{x} = o(p)$. Namely, for all $s \in [0, \infty)$ we get

$$\begin{aligned} \mathbf{E}[\mathbf{1}_{\{Z_k=Z_{k+1}=p\}} e^{-s(\mathbf{r}_{k+1}^o - \mathbf{r}_k^o - 1)} | ((\mathbf{T}, o), (Z_l)_{l \in \mathbb{N}})] \\ = \frac{1}{k_{o(p)}(\mathbf{T}) - \mathbf{1}_{\{p \geq 1\}}} \sum_{x \in \partial L_o : \overleftarrow{x} = o(p)} E_{(\mathbf{T}, x)}[e^{-s(H_{o(p)} - 1)}]. \end{aligned}$$

To simplify the notation, let us set $S = (o, (k_{o(p)}(\mathbf{T}))_{p \in \mathbb{N}})$. Let $x \in \partial L_o$ such that $\overleftarrow{x} = o(p)$. Observe that the r.v. $E_{(\mathbf{T}, x)}[\exp(-s(H_{o(p)} - 1))]$ only depends actually on $\theta_x \mathbf{T}$, which is a $\text{GW}(\nu)$ -tree that is independent from S and $(Z_l)_{l \in \mathbb{N}}$. Also note that $\mathbf{E}[E_{(\mathbf{T}, x)}[\exp(-s(H_{o(p)} - 1))]] = f_\nu(s)$, where f_ν is as in Lemma 3.3.2. Therefore, for all $s \in [0, 1]$, we get

$$\begin{aligned} \mathbf{E}[e^{-sW_\ell} | (S, (Z_l)_{l \in \mathbb{N}})] &= e^{-sW'_\ell} \prod_{k \in I(\ell)} f_\nu(sf(Z_k)) \geq \exp\left(-\sum_{0 \leq k \leq \ell} (sf(Z_k) - \log f_\nu(sf(Z_k)))\right) \\ &\geq \exp\left(-\sum_{p \in \mathbb{N}} N_p(sf(p) - \log f_\nu(sf(p)))\right) \text{ where } N_p = \#\{k \in \mathbb{N} : Z_k = p\}. \end{aligned}$$

Since $f(p) \downarrow 0$ as $p \rightarrow \infty$, there is $C_1 \in (0, \infty)$ such that $\log f_\nu(sf(p)) \geq -C_1(1 - f_\nu(sf(p)))$, for all $p \in \mathbb{N}$ and all $s \in [0, 1]$. Therefore, by (3.3.3), there is $C \in (0, \infty)$ such that $\log f_\nu(sf(p)) - sf(p) \geq -C\sqrt{sf(p)}$, for all $p \in \mathbb{N}$ and all $s \in [0, 1]$. Thus

$$\mathbf{E}[e^{-sW_\ell} | (S, (Z_l)_{l \in \mathbb{N}})] \geq \exp\left(-Cs^{\frac{1}{2}} \sum_{p \in \mathbb{N}} N_p f(p)^{\frac{1}{2}}\right). \quad (3.3.6)$$

Then, set $W = \sum_{n \in \mathbb{N}} f(|o \wedge X_n|)$. By letting $\ell \rightarrow \infty$ in (3.3.6) and by conditional Jensen's inequality, we get

$$\mathbf{E}[e^{-sW} | S] \geq \exp\left(-Cs^{\frac{1}{2}} \sum_{p \in \mathbb{N}} \mathbf{E}[N_p | S] f(p)^{\frac{1}{2}}\right), \quad (3.3.7)$$

We know from (3.2.35) in Lemma 3.2.23 that $\mathbf{E}[N_p | S] = \frac{m_\nu + k_{o(p)}(\mathbf{T})}{m_\nu - 1}$. Consequently, if the first condition of (3.3.5) is satisfied, then \mathbf{P} -a.s. $\lim_{s \rightarrow 0^+} \mathbf{E}[e^{-sW} | S] = 1$, which entails that a.s. $W < \infty$. It completes the proof of the proposition. \blacksquare

3.3.2 Estimates on critical biased branching RWs.

In this section, we state several estimates on critical biased branching RWs that take their values in supercritical GW -trees and that are indexed by critical GW -trees. More precisely, we fix the following.

- The measure μ stands for a non-trivial critical offspring distribution: namely, $\mu(0) + \mu(1) < 1$ and $\sum_{k \in \mathbb{N}} k\mu(k) = 1$.
- The genealogical tree (τ, ϱ) is either a $\text{GW}(\mu)$ -tree or a variant of such trees: namely, a grafted or a right-infinite $\text{GW}(\mu)$ -tree.
- The measure ν stands for a supercritical offspring distribution whose mean is denoted by m_ν .
- The environment (\mathbf{T}, o) is an invariant $\text{GW}(\nu)$ -tree that is independent from (τ, ϱ) .

We then consider a branching random walk $\Theta = (\tau, \varrho; \bar{Y}_u = (\mathbf{T}, Y_u), u \in \tau)$ that is a $\bar{\mathbb{T}}^\bullet$ ($\bar{\mathbb{T}}^\bullet$)-valued r.v. whose conditional law given (τ, ϱ) and (\mathbf{T}, \circ) is $Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 (recall here that the underlying spatial motion is that of a \mathfrak{m}_ν -biased RW on \mathbf{T}).

We first recall from Chapter 2 the following simple estimate.

Lemma 3.3.4 *Let μ be a non-trivial critical offspring distribution. Let $\beta \in (0, \infty)$ be such that $\sum_{k \in \mathbb{N}} k(\log k)^{1+\beta} \mu(k) < \infty$. Recall that $\bar{\mu}(k) = \sum_{l \geq k} \mu(l)$, $k \in \mathbb{N}$ and set*

$$\forall r \in [0, 1], \quad g_\mu(r) = \sum_{k \in \mathbb{N}} r^k \mu(k) \quad \text{and} \quad g_{\bar{\mu}}(r) = \sum_{k \in \mathbb{N}} r^k \bar{\mu}(k+1) = \frac{1 - g_\mu(r)}{1 - r}. \quad (3.3.8)$$

Then, there exists a constant $c \in (0, \infty)$ that only depends on μ and β such that

$$\forall r \in (0, 1/2), \quad 0 \leq 1 - g_{\bar{\mu}}(1-r) \leq \frac{c}{(\log 1/r)^{1+\beta}} \quad (3.3.9)$$

Proof. See Lemma 2.3.15 in Chapter 2. ■

Lemma 3.3.5 *Let μ be a non-trivial critical offspring distribution and let τ be a grafted $GW(\mu)$ -tree (Definition 3.2.6 (c)). Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν and let (\mathbf{T}, \circ) be an invariant $GW(\nu)$ -tree that is independent from τ . Let $x \in \mathbf{T}$ and let $\Theta = (\tau, \varnothing; \bar{Y}_u = (\mathbf{T}, Y_u), u \in \tau)$ be a \mathfrak{m}_ν -biased branching RW whose conditional law given τ and (\mathbf{T}, \circ) is $Q_{(\tau, \varnothing)}^{(\mathbf{T}, x)}$ as in Definition 3.2.30 with $\lambda = \mathfrak{m}_\nu$. In particular, $Y_\varnothing = x$. Recall from (3.2.15) that L_\circ is the ancestral line of \circ and that for all $p \in \mathbb{N}$, $\circ(p)$ stands for the ancestor of \circ at distance p from it. For all $p \in \mathbb{N}$, we set*

$$\xi_p(\mathbf{T}, x) = \mathbf{P}(\circ(p) \notin \{Y_v; v \in \tau \setminus \{\varnothing\}\} \mid \mathbf{T}). \quad (3.3.10)$$

Then, **P**-a.s.

$$1 - \xi_p(\mathbf{T}, x) \leq \frac{\mathfrak{m}_\nu + k_{\circ(p)}(\mathbf{T})}{\mathfrak{m}_\nu - 1} \mathfrak{m}_\nu^{-(-|\circ \wedge x| - p)_+}. \quad (3.3.11)$$

Proof. Recall that $P_{\mathbf{T}, y}$ stands for the law of a \mathfrak{m}_ν -biased random walk $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, on \mathbf{T} starting at $X_0 = y$. By the very definition of branching RWs, for all $v \in \tau$, $\mathbf{P}(Y_v = y \mid \tau, (\mathbf{T}, \circ)) = P_{\mathbf{T}, x}(X_{|v|} = y)$ since $Y_\varnothing = x$. Therefore, we get

$$\begin{aligned} \mathbf{P}(\circ(p) \in \{Y_v; v \in \tau \setminus \{\varnothing\}\} \mid (\mathbf{T}, \circ)) &\leq \mathbf{E} \left[\sum_{v \in \tau \setminus \{\varnothing\}} \mathbf{1}_{\{Y_v = \circ(p)\}} \mid (\mathbf{T}, \circ) \right] \\ &= \mathbf{E} \left[\sum_{v \in \tau \setminus \{\varnothing\}} P_{\mathbf{T}, x}(X_{|v|} = \circ(p)) \mid (\mathbf{T}, \circ) \right] \\ &= \sum_{n \in \mathbb{N}^*} P_{\mathbf{T}, x}(X_n = \circ(p)), \end{aligned} \quad (3.3.12)$$

by the many-to-one formula (3.2.25). This implies the desired result by the explicit computation (3.2.35) of the Green function of \mathfrak{m}_ν -biased RWs. ■

In the following lemma we show that the right-part of an invariant critical biased branching RW has a positive probability of never returning to its starting point.

Proposition 3.3.6 *Let μ be a non-trivial critical offspring distribution. Suppose that there is $\beta \in (0, \infty)$ such that $\sum_{k \in \mathbb{N}} k(\log k)^{2(1+2\beta)} \mu(k) < \infty$. Let (τ, ϱ) be right-infinite $GW(\mu)$ -tree as in Definition 3.2.18 (c). Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν*

and such that $\sum_{k \in \mathbb{N}} k^{1+\frac{1}{\beta}} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree that is independent from (τ, ϱ) . Let $\Theta = (\tau, \varrho; \bar{Y}_u = (\mathbf{T}, Y_u), u \in \tau)$ be a \mathfrak{m}_ν -biased branching RW whose conditional law given (τ, ϱ) and (\mathbf{T}, \circ) is $Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 with $\lambda = \mathfrak{m}_\nu$. Recall from (3.2.15) that L_\circ is the ancestral line of \circ and that for all $p \in \mathbb{N}$, $\circ(p)$ stands the ancestor of \circ at distance p from it.

Then, with positive probability $(Y_v)_{v \in \tau}$ only visits $\circ = Y_\varrho$ once:

$$\kappa_{\mu, \nu} := \mathbf{P}(Y_\varrho \notin \{Y_v; v \in \tau \setminus \{\varrho\}\}) > 0. \quad (3.3.13)$$

Moreover,

$$\forall p \in \mathbb{N}, \quad \mathbf{P}\text{-a.s. } \exists q(p) \in \mathbb{N} \text{ such that } \circ(p) \notin \{Y_v; v \in \tau: |\varrho \wedge v| \leq -q(p)\}. \quad (3.3.14)$$

Proof. We denote by L_ϱ the ancestral line of ϱ and for all $p \in \mathbb{N}$, we set $\varrho(p) = \varrho|_{(-\infty, -p]}$ as in (3.2.15). We also recall the notation $\partial L_\varrho = \{v \in \tau \setminus L_\varrho: \overleftarrow{v} \in L_\varrho\}$ for the spine of τ . It is convenient to set $X_p := Y_{\varrho(p)}$ for all $p \in \mathbb{N}$. Note that $(\mathbf{T}, X_p)_{p \in \mathbb{N}}$ is a \mathfrak{m}_ν -biased RW which is independent from (τ, ϱ) . By Definition 3.2.18 (c) of a right-infinite $\text{GW}(\mu)$ -tree, the bilateral word $\varrho(p)$ is a sequence of 1 indexed by the integers of $\mathbb{Z} \cap (-\infty, -p]$, the r.v. $(k_{\varrho(p)}(\tau))_{p \in \mathbb{N}^*}$ are independent and distributed according to $\bar{\mu}$ which given by $\bar{\mu}(k) = \sum_{l \geq k} \mu(l)$ for all $k \in \mathbb{N}^*$ and $\bar{\mu}(0) = 0$. Moreover, recall that conditionally given ∂L_ϱ , the subtrees $\theta_u \tau$, $u \in \partial L_\varrho$, are independent $\text{GW}(\mu)$ -trees as in Definition 3.2.6 (a).

Let $u \in \partial L_\varrho$. We denote by τ_u the \mathbb{T} -valued random tree such that $k_\varnothing(\tau_u) = 1$ and $\theta_{(1)} \tau_u = \theta_u \tau$. We then define $\Theta_u = (\tau_u, \varnothing; (\bar{Y}_v^u)_{v \in \tau_u})$ by setting $\bar{Y}_\varnothing^u = (\mathbf{T}, X_p)$ where p is such that $\overleftarrow{u} = \varrho(p)$ and by setting $\bar{Y}_{(1)^*v}^u = \bar{Y}_{u^*v}$ for all $v \in \theta_u \tau$. Namely Θ_u is the restriction of Θ to the subtree stemming from u and (which also records the spatial position of \overleftarrow{u}). Conditionally given (\mathbf{T}, \circ) and (τ, ϱ) , the branching random walks Θ_u , $u \in \partial L_\varrho$, are independent with respective conditional laws $Q_{(\tau_u, \varnothing)}^{(\mathbf{T}, X_p)}$. Then, for all $q \in \mathbb{N}^*$, we set

$$A_q = \{Y_\varrho \notin \{Y_v; v \in \tau \setminus \{\varrho\}\}: |v \wedge \varrho| \geq -q\} \quad \text{and} \quad R_\circ(X) = \inf\{p \geq 1: X_p = \circ\}.$$

By Lemma 3.2.24 and Lemma 3.2.23 (i), $(X_p)_{p \in \mathbb{N}}$ is transient because it is a \mathfrak{m}_ν -biased RW. Thus, $R_\circ = \infty$ with positive probability. The event A_q decreases as $q \rightarrow \infty$ to the event that the branching walk $(Y_v)_{v \in \tau}$ only visits Y_ϱ once. Since conditionally given ∂L_ϱ , the $(\tau_u)_{u \in \partial L_\varrho}$ are i.i.d. grafted $\text{GW}(\mu)$ -trees, we get

$$\mathbf{P}(A_q | (\mathbf{T}, \circ)) = \mathbf{E} \left[\mathbf{1}_{\{R_\circ(X) > q\}} \xi_0(\mathbf{T}, \circ)^{k_\varrho(\tau)} \prod_{1 \leq p \leq q} \xi_0(\mathbf{T}, X_p)^{k_{\varrho(p)}(\tau)-1} \middle| (\mathbf{T}, \circ) \right].$$

Here, $\xi_0(\mathbf{T}, x)$ is as in (3.3.10). By definition, the r.v. $(k_{\varrho(p)}(\tau))_{p \in \mathbb{N}}$ are independent and independent of $(\mathbf{T}, \circ; (X_p)_{p \in \mathbb{N}})$, $k_\varrho(\tau)$ is distributed according to μ and the r.v. $k_{\varrho(p)}(\tau)$, $p \in \mathbb{N}^*$ are distributed according to $\bar{\mu}$. Recall from (3.3.4) the definition of the generating functions g_μ and $g_{\bar{\mu}}$. Thus, we get

$$\mathbf{P}(A_q | (\mathbf{T}, \circ)) = \mathbf{E} \left[\mathbf{1}_{\{R_\circ(X) > q\}} g_\mu(\xi_0(\mathbf{T}, \circ)) \prod_{1 \leq p \leq q} g_{\bar{\mu}}(\xi_0(\mathbf{T}, X_p)) \middle| (\mathbf{T}, \circ) \right].$$

Then we easily see that (3.3.13) holds true if

$$\mathbf{P}\text{-a.s. } \sum_{p \geq 1} (1 - g_{\bar{\mu}}(\xi_0(\mathbf{T}, X_p))) < \infty. \quad (3.3.15)$$

Then observe that (3.3.9) and (3.3.11) imply

$$1 - g_{\bar{\mu}}(\xi_0(\mathbf{T}, X_p)) \leq f^*(Z_0, -|X_p \wedge \circ|), \quad (3.3.16)$$

where $Z_0 = \log \frac{m_\nu + k_{\circ(\mathbf{T})}}{m_\nu - 1}$, where $f^*(z, m) = c_1 (\max(1, (c_2 m - z)))^{-2(1+2\beta)}$ for all $z \in [0, \infty)$ and all $m \in \mathbb{N}$, and where $c_1, c_2 \in (0, \infty)$ are constants that only depend on μ, ν and β .

We next apply (3.3.5) in Proposition 3.3.3. To that end, observe that there exists a constant $c_3 \in (0, \infty)$ that only depends on β and ν such that

$$\sum_{p \geq 1} \mathbf{P} \left(\frac{k_{\circ(p)}(\mathbf{T})}{(1+p)^{1+2\beta}} \geq \frac{1}{(1+p)^{1+\beta}} \right) = \sum_{p \geq 1} \mathbf{P} \left((k_{\circ(1)}(\mathbf{T}))^{\frac{1}{\beta}} \geq 1+p \right) \leq c_3 \sum_{k \in \mathbb{N}^*} \nu(k) k^{1+\frac{1}{\beta}} < \infty.$$

By Borel-Cantelli, a.s. $\sum_{p \geq 1} k_{\circ(p)}(\mathbf{T})(1+p)^{-(1+2\beta)} < \infty$ and thus $\sum_{p \geq 1} (1 - |\circ \wedge X_p|)^{-2(1+2\beta)} < \infty$ by (3.3.5) applied to $f(x) = (1+x)^{-2(1+2\beta)}$, $x \in [0, \infty)$. It then implies (3.3.15) thanks to (3.3.16) and it proves (3.3.13).

We next prove (3.3.14) in a similar way. We set $\zeta_{\circ(q)} = \sup\{n \in \mathbb{N} : X_n = \circ(q)\}$ that is a.s. finite since $(X_n)_{n \in \mathbb{N}}$ is transient. Let $q \geq p \geq 1$. We introduce the following event $B_p(q) = \{\circ(p) \notin \{Y_v; v \in \tau : |v \wedge \varrho| < -\zeta_{\circ(q)}\}\}$. Then we set $Z_p = \log \frac{m_\nu + k_{\circ(p)}(\mathbf{T})}{m_\nu - 1}$ and arguing as previously, we get for all $q \in \mathbb{N}^*$

$$\begin{aligned} \mathbf{P}(B_p(q)) &= \mathbf{E} \left[\prod_{l > \zeta_{\circ(q)}} g_{\bar{\mu}}(\xi_p(\mathbf{T}, X_l)) \right] \\ &\geq \mathbf{E} \left[\exp \left(-c_4 \sum_{n > \zeta_{\circ(q)}} f^*(Z_p, -|X_n \wedge \circ|) \right) \right] \xrightarrow{q \rightarrow \infty} 1, \end{aligned}$$

by the previous argument, where $c_4 \in (0, \infty)$ is some constant that only depends on μ, ν, β and p . It implies (3.3.14), which completes the proof of the proposition. \blacksquare

3.3.3 Proof of Propositions 3.2.26 and 3.2.32.

Our proof of the ergodic property of critical biased RWs and branching RWs in Propositions 3.2.26 and 3.2.32 relies on the following ad hoc lemma.

Lemma 3.3.7 *Let E be a Polish space. Let π be a Borel probability measure on E that is preserved by a Borel measurable function $\phi: E \rightarrow E$: namely $\pi = \pi \circ \phi^{-1}$. Assume for all $p \in \mathbb{N}$ that there are Borel-measurable functions $\langle \cdot \rangle_p: E \rightarrow E$ such that*

$$\forall x \in E, \forall p, q \in \mathbb{N}, \quad \langle \langle x \rangle_p \rangle_q = \langle x \rangle_{p \wedge q} \quad \text{and} \quad \langle x \rangle_p \xrightarrow{p \rightarrow \infty} x. \quad (3.3.17)$$

Let $Z_n: \Omega \rightarrow E$, $n \in \mathbb{N}$, be r.v. such that Z_0 has law π and $Z_{n+1} = \phi(Z_n)$, $n \in \mathbb{N}$. We assume that for all $p_0, p_1 \in \mathbb{N}^$, there exists a r.v. $\mathbf{S}(p_0, p_1): \Omega \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, the two r.v. $\langle Z_0 \rangle_{p_0}$ and $\langle Z_{\mathbf{S}(p_0, p_1) + n} \rangle_{p_1}$ are independent. Then, ϕ is π -ergodic.*

Proof. Let us denote by \mathcal{H}_p the space of all bounded Borel-measurable functions $F: E \rightarrow \mathbb{R}$ such that $F(x) = F(\langle x \rangle_p)$, $x \in E$. Note that $\mathcal{H}_p \subset \mathcal{H}_q$ for all $q \geq p$ by (3.3.17). We set $\mathcal{H} = \bigcup_{p \in \mathbb{N}} \mathcal{H}_p$. Note that \mathcal{H} is a real vector space and (3.3.17) implies that the σ -field generated on E by \mathcal{H} is the Borel σ -field (indeed, since E is separable, its Borel σ -field is generated by the space of bounded continuous functions and each bounded continuous function F on E is the pointwise limit as $p \rightarrow \infty$ of the functions $F \circ \langle \cdot \rangle_p$ that belong to \mathcal{H}).

Let $r \in (0, 1)$ and let $\mathbf{g}_r: \Omega \rightarrow \mathbb{N}$ be independent from Z_0 (and thus from $(Z_n)_{n \in \mathbb{N}}$) with the geometric distribution $\mathbf{P}(\mathbf{g}_r = n) = r^n(1-r)$, $n \in \mathbb{N}$. Let $F, G \in \mathcal{H}$. According to our assumptions there exists a r.v. $S: \Omega \rightarrow \mathbb{N}$, such that for all $n \in \mathbb{N}$, $F(Z_0)$ and $G(Z_{S+n})$ are independent. Then, elementary properties of geometric r.v. imply the following

$$\begin{aligned} \alpha_{F,G}(r) &:= \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] = \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})\mathbf{1}_{\{\mathbf{g}_r < S\}}] + \mathbf{E}[r^S F(Z_0)G(Z_{\mathbf{g}_r+S})] \\ &= a + \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r+S})] - \mathbf{E}[(1-r^S)F(Z_0)G(Z_{\mathbf{g}_r+S})] \\ &= a + \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_{\mathbf{g}_r+S})] - b \end{aligned}$$

where $a = \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})\mathbf{1}_{\{\mathbf{g}_r < S\}}]$ and $b = \mathbf{E}[(1-r^S)F(Z_0)G(Z_{\mathbf{g}_r+S})]$. Both $|a|$ and $|b|$ are $\leq \|F\|_\infty \|G\|_\infty \mathbf{P}(\mathbf{g}_r < S)$. In particular $|\mathbf{E}[G(Z_{\mathbf{g}_r})] - \mathbf{E}[G(Z_{S+\mathbf{g}_r})]| \leq 2\|G\|_\infty \mathbf{P}(\mathbf{g}_r < S)$ and thus $|\alpha_{F,G}(r) - \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_{\mathbf{g}_r})]| \leq 4\|F\|_\infty \|G\|_\infty \mathbf{P}(\mathbf{g}_r < S)$. Since $(Z_n)_{n \in \mathbb{N}}$ is stationary, $\mathbf{E}[G(Z_{\mathbf{g}_r})] = \mathbf{E}[G(Z_0)]$. Thus, we have proved that for all $F, G \in \mathcal{H}$,

$$|\mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] - \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]| \leq 4\|F\|_\infty \|G\|_\infty \mathbf{P}(\mathbf{g}_r < S) \xrightarrow{r \rightarrow 1^-} 0. \quad (3.3.18)$$

Next, we fix $F \in \mathcal{H}$ and we denote by \mathcal{H}' the set of bounded measurable $G: E \rightarrow \mathbb{R}$ such that $\lim_{r \rightarrow 1^-} \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] = \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]$. Clearly, \mathcal{H}' is a real vector space and (3.3.18) implies that $\mathcal{H} \subset \mathcal{H}'$. Therefore, the σ -field on E generated by \mathcal{H}' is the Borel σ -field. We next prove that \mathcal{H}' is stable under increasing bounded limits: *indeed*, let $c \in (0, \infty)$ and let $G_n \in \mathcal{H}'$, $n \in \mathbb{N}$, be such that $0 \leq G_n \leq G_{n+1} \leq c$. Set $G = \sup_{n \in \mathbb{N}} G_n$ that is the (nonnegative measurable bounded) limit of the G_n . We easily see that:

$$\begin{aligned} |\mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] - \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]| &\leq |\mathbf{E}[F(Z_0)G_n(Z_{\mathbf{g}_r})] - \mathbf{E}[F(Z_0)]\mathbf{E}[G_n(Z_0)]| \\ &\quad + \|F\|_\infty \mathbf{E}[(G - G_n)(Z_{\mathbf{g}_r})] + \|F\|_\infty \mathbf{E}[(G - G_n)(Z_0)]; \end{aligned}$$

Since $Z_{\mathbf{g}_r}$ has the same law as Z_0 , we get

$$\limsup_{r \rightarrow 1^-} |\mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] - \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]| \leq 2\|F\|_\infty \mathbf{E}[(G - G_n)(Z_0)]$$

and the right member goes to 0 by dominated convergence. This proves that $G \in \mathcal{H}'$. A monotone class argument implies that \mathcal{H}' is exactly the set of all bounded real-valued Borel measurable functions on E which proves that for all $F \in \mathcal{H}$ and for all bounded measurable functions $G: E \rightarrow \mathbb{R}$,

$$\lim_{r \rightarrow 1^-} \mathbf{E}[F(Z_0)G(Z_{\mathbf{g}_r})] = \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]. \quad (3.3.19)$$

Denote by $\mathcal{B}(E)$ the Borel σ -field on E and denote by $\mathcal{I} = \{A \in \mathcal{B}(E) : \phi^{-1}(A) = A\}$ that is the σ -field of the ϕ -invariant measurable subsets. Let $G: E \rightarrow \mathbb{R}$ be bounded and \mathcal{I} -measurable: namely, $G \circ \phi = G$ and in particular a.s. $G(Z_{\mathbf{g}_r}) = G(Z_0)$. Thus, by (3.3.19), $\mathbf{E}[F(Z_0)G(Z_0)] = \mathbf{E}[F(Z_0)]\mathbf{E}[G(Z_0)]$ for all $F \in \mathcal{H}$. By a monotone class argument, this holds for all bounded measurable F and taking $F = G$ shows that the variance of $G(Z_0)$ is null: namely, G is π -a.s. constant, which implies that ϕ is π -ergodic. \blacksquare

Proof of Proposition 3.2.26. We equip $E = (\overline{\mathbb{T}}^\bullet)^\mathbb{N}$ with the product topology, which makes it a Polish space. Let $\bar{x} = (\bar{x}_n = (T_n, x_n))_{n \in \mathbb{N}} \in E$. For all $p \in \mathbb{N}$, we set $\langle \bar{x} \cdot \rangle_p = [(\langle T_n \wedge p, x_n \wedge p \rangle)_{n \in \mathbb{N}}]_{-p}^p$, where we recall from Definition 3.2.12 the truncation $[\cdot]_{-p}^p$. By the definition of the topology of local convergence which equips $\overline{\mathbb{T}}^\bullet$ and by definition of the product topology which equips E , $\langle \bar{x} \cdot \rangle_p \rightarrow \bar{x}$ in E and the continuous functions $\langle \cdot \rangle_p$, $p \in \mathbb{N}$, satisfy (3.3.17) in Lemma 3.3.7.

We next recall that $\vartheta: E \rightarrow E$ is defined by $\vartheta(\bar{x} \cdot) = (\varphi_{|x_1|}(\bar{x}_{n+1}))_{n \in \mathbb{N}}$, where the φ_l are the shifts. Let $\bar{X} = (\bar{X}_n = (\mathbf{T}, X_n))_{n \in \mathbb{N}}$ be a m_ν -biased RW on the invariant $\text{GW}(\nu)$ -tree (\mathbf{T}, \circ) starting at $\bar{X}_0 = (\mathbf{T}, \circ)$. For all $n \in \mathbb{N}$ we set $Z_n = \vartheta^{o^n}(\bar{X} \cdot) = (\varphi_{|X_n|}(\bar{X}_{m+n}))_{m \in \mathbb{N}}$ and recall that we denote by Π_ν the law of \bar{X} on E . Then (3.2.38) implies that Π_ν is preserved by ϑ . Let $p_0, p_1 \in \mathbb{N}^*$. We choose an integer $p > 2p_0 + p_1$ and we recall the notation $\zeta_{\circ(p)} = \max\{n \in \mathbb{N} : X_n = \circ(p)\}$. Then Lemma 3.3.1 asserts that for all $n \in \mathbb{N}$, $\langle Z_0 \rangle_{p_0}$ and $\langle Z_{\zeta_{\circ(p)}+1+n} \rangle_{p_1}$ are independent. Then Lemma 3.3.7 applies and it entails that ϑ is Π_ν -ergodic. \blacksquare

Recall from (3.2.47) and (3.2.49) (see also (3.2.22) and (3.2.24)) the definitions of succ and succ^+ as functions from $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ to itself. We denote by $\Pi_{\mu,\nu}$ the Borel probability measure on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$ that is the law of the branching random walk

$$\Theta = (\tau, \varrho; \bar{Y}_u = (\mathbf{T}, Y_u), u \in \tau), \quad (3.3.20)$$

where (τ, ϱ) is an infinite $\text{GW}(\mu)$ -tree independent from the invariant $\text{GW}(\nu)$ -tree (\mathbf{T}, \circ) (here, recall that μ is non-trivial and critical and ν is supercritical and its mean is denoted by m_ν) and where conditionally given (τ, ϱ) and (\mathbf{T}, \circ) , Θ has law $Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 (recall here that the underlying spatial motion is that of a m_ν -biased RW on \mathbf{T}). We then denote by $\Pi_{\mu, \nu}^+$ the law of the right part $[\Theta]^+$ as in (3.2.48) (see also (3.2.23)).

We first prove that $\Pi_{\mu, \nu}$ is preserved by succ and that $\Pi_{\mu, \nu}^+$ is preserved by succ^+ . To that end, we first need to prove the following lemma.

Lemma 3.3.8 *Let μ, ν and (\mathbf{T}, \circ) be as above. Let $(t, u) \in \overline{\mathbb{T}}^\bullet$ and let $u' \in t$. Let $(t, u; (\overline{Y}_v = (\mathbf{T}, Y_v))_{v \in t})$ and $(t, u'; (\overline{Y}'_v = (\mathbf{T}, Y'_v))_{v \in t})$ be branching RWs whose respective laws conditionally given (\mathbf{T}, \circ) are $Q_{(t, u)}^{(\mathbf{T}, \circ)}$ and $Q_{(t, u')}^{(\mathbf{T}, \circ)}$. Then,*

$$\mathbf{E}[F(t, u'; (\varphi_{|Y_{u'}|}(\overline{Y}'_v))_{v \in t})] = \mathbf{E}[F(t, u'; (\overline{Y}'_v)_{v \in t})] \quad (3.3.21)$$

for all measurable bounded F on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$.

Proof. Clearly, it is sufficient to prove (3.3.21) for t finite and F of the form $F(t, u; (\overline{y}_v)_{v \in t}) = G(\overline{y}_v; v \in t \setminus \llbracket u, u' \rrbracket)H(\overline{y}_v; v \in \llbracket u, u' \rrbracket)$. First note that

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E}[G((\overline{Y}_v)_{v \in t \setminus \llbracket u, u' \rrbracket}) | (\overline{Y}_v)_{v \in \llbracket u, u' \rrbracket}] = g((\overline{Y}_v)_{v \in \llbracket u, u' \rrbracket}),$$

where $g(\overline{y}_v; v \in \llbracket u, u' \rrbracket) = \int \prod_{v \in t \setminus \llbracket u, u' \rrbracket} \overline{\mathbf{p}}_{m_\nu}(\overline{y}_v \leftarrow u, d\overline{y}_v) G(\overline{y}_v; v \in t \setminus \llbracket u, u' \rrbracket)$. Here we recall from (3.2.27) the definition of the transition kernel $\overline{\mathbf{p}}_{m_\nu}$, which entails for all $l \in \mathbb{Z}$ that

$$\mathbf{E}[G((\varphi_l(\overline{Y}_v))_{v \in t \setminus \llbracket u, u' \rrbracket}) | (\overline{Y}_v)_{v \in \llbracket u, u' \rrbracket}] = g((\varphi_l(\overline{Y}_v))_{v \in \llbracket u, u' \rrbracket})$$

and thus $\mathbf{E}[F(t, u'; (\varphi_{|Y_{u'}|}(\overline{Y}_v))_{v \in t})] = \mathbf{E}[(H.g)(\varphi_{|Y_{u'}|}(\overline{Y}_v))_{v \in \llbracket u, u' \rrbracket}]$, where $H.g$ stands for the product of the two functions. By the stationarity (3.2.38) of m_ν -biased RWs on (\mathbf{T}, \circ) , we easily get

$$\mathbf{E}[(H.g)((\varphi_{|Y_{u'}|}(\overline{Y}_v))_{v \in \llbracket u, u' \rrbracket})] = \mathbf{E}[(H.g)((\overline{Y}'_v)_{v \in \llbracket u, u' \rrbracket})].$$

Next observe that if $v \in t \setminus \llbracket u, u' \rrbracket$, then $\overleftarrow{v}^u = \overleftarrow{v}^{u'}$ and thus we also get

$$\mathbf{E}[G((\overline{Y}'_v)_{v \in t \setminus \llbracket u, u' \rrbracket}) | (\overline{Y}'_v)_{v \in \llbracket u, u' \rrbracket}] = g((\overline{Y}'_v)_{v \in \llbracket u, u' \rrbracket}),$$

which easily entails (3.3.21). ■

Thanks to Lemma 3.3.8, we next prove the following proposition.

Lemma 3.3.9 *We keep the above notations and assumptions. Then $\Pi_{\mu, \nu}$ is preserved by succ and $\Pi_{\mu, \nu}^+$ is preserved by succ^+ .*

Proof. Let Θ be as in (3.3.20). We denote by $\varrho' = \text{succ}(\varrho)$, the successor of ϱ in τ . By (3.3.21) in Lemma 3.3.8, we first get $\mathbf{E}[F(\tau, \varrho')G((\varphi_{|Y_{\varrho'}|}(\overline{Y}_v))_{v \in \tau})] = \mathbf{E}[F(\tau, \varrho')Q_{(\tau, \varrho')}^{(\mathbf{T}, \circ)}[G]]$. Then, note for all pointed trees (t, u) and for all $l \in \mathbb{Z}$, that $Q_{(t, u)}^{(\mathbf{T}, \circ)}[G((\overline{Y}_{\varphi_l(v)})_{v \in \varphi_l(t)})] = Q_{(\varphi_l(t), \varphi_l(u))}^{(\mathbf{T}, \circ)}[G]$. Thus,

$$\begin{aligned} & \mathbf{E}[F(\varphi_{|\varrho'}|}(\tau), \varphi_{|\varrho'}|}(\varrho'))G((\varphi_{|Y_{\varrho'}|}(\overline{Y}_{\varphi_{-|\varrho'}|}(v)))_{v \in \varphi_{|\varrho'}|}(\tau))] \\ &= \mathbf{E}[F(\varphi_{|\varrho'}|}(\tau), \varphi_{|\varrho'}|}(\varrho'))Q_{(\varphi_{|\varrho'}|}(\tau), \varphi_{|\varrho'}|}(\varrho'))^{(\mathbf{T}, \circ)}[G] = \mathbf{E}[F(\tau, \varrho)Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}[G]] \end{aligned} \quad (3.3.22)$$

since $(\varphi_{|\varrho'}|}(\tau), \varphi_{|\varrho'}|}(\varrho')) = \text{succ}(\tau, \varrho)$ has the same law as (τ, ϱ) by Proposition 3.2.20 (ii). Thus (3.3.22) implies that $(\varphi_{|\varrho'}|}(\tau), \varphi_{|\varrho'}|}(\varrho')); (\varphi_{|Y_{\varrho'}|}(\overline{Y}_{\varphi_{-|\varrho'}|}(v)))_{v \in \varphi_{|\varrho'}|}(\tau)) = \text{succ}(\Theta)$ has the same

law as Θ , which implies that $\Pi_{\mu,\nu}$ is preserved by succ . From Remark 3.2.28, $[\text{succ}(\Theta)]^+ = \text{succ}^+([\Theta]^+)$ shows that $\Pi_{\mu,\nu}^+$ is preserved by succ^+ . \blacksquare

Proof of Proposition 3.2.32. To simplify notation, for all $n \in \mathbb{N}^*$, we denote by succ_n^+ the n -th iteration of succ^+ : namely

$$\text{succ}_1^+ = \text{succ}^+ \quad \text{and} \quad \text{succ}_{n+1}^+ = \text{succ}^+ \circ \text{succ}_n^+, \quad n \in \mathbb{N}^*.$$

To prove that succ^+ is $\Pi_{\mu,\nu}^+$ -ergodic, we use Lemma 3.3.7 and we first need to define appropriate functions $\langle \cdot \rangle_p$ on $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. More precisely, let $\Theta = (t, \varrho; \bar{x}_v = (T_v, x_v), v \in t)$ be an element of $\overline{\mathbb{T}}^\bullet(\overline{\mathbb{T}}^\bullet)$. Let $p \in \mathbb{N}^*$. Recall $[\Theta]_{-p}^p$ from Definition 3.2.12 and recall that the marks of $[\Theta]_{-p}^p$ (here in $\overline{\mathbb{T}}^\bullet$) are not truncated, which is not convenient for our purpose. That is why we introduce

$$[[\Theta]]_{-p}^p = (t', \varrho'; (\bar{x}'_v)_{v \in t'}) \quad \text{where} \quad (t', \varrho'; (\bar{x}'_v)_{v \in t'}) = [\Theta]_{-p}^p. \quad (3.3.23)$$

Clearly $[[[\Theta]_{-p}^p]]_{-q}^q = [[\Theta]]_{-p \wedge q}^{p \wedge q}$ and $[[\Theta]]_{-p}^p \rightarrow \Theta$ locally. This combined with Lemma 3.3.9 and the following Lemma 3.3.10 (whose proof is given below) allows us to apply Lemma 3.3.7 to $\Pi_{\mu,\nu}^+$ and succ^+ , which entails Proposition 3.2.32. \blacksquare

Lemma 3.3.10 *Let μ, ν and (\mathbf{T}, \circ) be as in Proposition 3.2.32. Let (τ, ϱ) be a right-infinite $\text{GW}(\mu)$ -tree that is independent from (\mathbf{T}, \circ) and let $\Theta := (\tau, \varrho; \bar{Y}_v = (\mathbf{T}, Y_v), v \in \tau)$ be a branching RW whose law conditionally given (\mathbf{T}, \circ) and (τ, ϱ) is $Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 (recall here that the underlying spatial motion is that of a m_ν -biased RW on \mathbf{T}). Then, for all integers $p_0, p_1 \geq 1$, there is a r.v. $\mathbf{S}(p_0, p_1): \Omega \rightarrow \mathbb{N}$ such that*

$$[[\Theta]]_{-p_0}^{p_0} \text{ is independent from } ([[\text{succ}_{\mathbf{S}(p_0, p_1) + n}^+(\Theta)]]_{-p_1}^{p_1})_{n \geq 1}. \quad (3.3.24)$$

Proof. Recall from (3.2.15), the notation $L_\varrho = \{\varrho(k); k \in \mathbb{N}\}$ and $L_\circ = \{\circ(p); p \in \mathbb{N}\}$ for the ancestral lines of resp. ϱ in τ and \circ in \mathbf{T} . For all $k \in \mathbb{N}^*$ and $x \in \mathbf{T}$, we introduce

$$\tau(k) = \{\varrho(k)\} \cup \{v \in \tau : |\varrho \wedge v| < -k\} \quad \text{and} \quad \mathbf{T}(x) = \{y \in \mathbf{T} : \overleftarrow{y} = x\} \cup \{y \in \mathbf{T} : |x \wedge \overleftarrow{y}| < |x|\}. \quad (3.3.25)$$

Then $(\tau(k), \varrho(k)), [(\tau, \varrho)]_{-k}^\infty, (\mathbf{T}(\circ(p)), \circ(p))$ and $[(\mathbf{T}, \circ)]_{-p+1}^\infty$ are independent for all $k, p \in \mathbb{N}^*$. Moreover, we observe that the $(\tau^\sharp(k), \varrho^\sharp(k)), k \in \mathbb{N}^*$ share the same law, where we have set

$$(\tau^\sharp(k), \varrho^\sharp(k)) = \text{cent}(\tau(k), \varrho(k)).$$

We then set $X_k = Y_{\varrho(k)}$ for all $k \in \mathbb{N}$. Conditionally given (\mathbf{T}, \circ) , X is therefore a m_ν -biased random walk in \mathbf{T} starting at \circ , which is furthermore independent of (τ, ϱ) . We also introduce the following branching RWs

$$\Theta^*(k) = ([(\tau, \varrho)]_{-k}^\infty; (\bar{Y}_{\varrho(k)*v})_{v \in [\tau]_{-k}^\infty}) \quad \text{and} \quad \Theta^\sharp(k) = (\tau^\sharp(k), \varrho^\sharp(k); (\bar{Y}_{\varphi_k(v)}(k))_{v \in \tau^\sharp(k)})$$

$$\text{where, for all } u \in \tau(k), \quad \bar{Y}_u(k) = \begin{cases} (\mathbf{T}(X_k), X_k) & \text{if } \exists w \in \llbracket \varrho(k), u \rrbracket : X_k \wedge Y_w = X_k, \\ (\mathbf{T}(X_k), Y_u) & \text{otherwise.} \end{cases} \quad (3.3.26)$$

Here, $\llbracket \varrho(k), u \rrbracket$ stands for the shortest path in the tree $\tau(k)$ between $\varrho(k)$ and u , $\varrho(k)$ being excluded. Note that $\Theta^\sharp(k)$ is actually constituted of the spatial positions that are *absorbed in X_k or above*. More precisely, conditionally given $(\tau, \varrho; \mathbf{T}, \circ; X_k)$, the law of $\Theta^\sharp(k)$ is $\mathbf{Q}_{(\tau^\sharp(k), \varrho^\sharp(k))}^{X_k; (\mathbf{T}(X_k), X_k)}$, which stands for the law of the $\mathbf{T}(X_k)$ -valued and $\tau^\sharp(k)$ -indexed branching RW whose spatial motion is a m_ν -biased RW on $\mathbf{T}(X_k)$ *absorbed in X_k or above*.

In fact, we will be mostly interested in $\Theta^*(k)$ and $\Theta^\sharp(k)$ for a specific random k . Namely, we fix $p \in \mathbb{N}^*$ and we set $\zeta_{\circ(p)} = \sup\{k \in \mathbb{N} : X_k \wedge \circ(p) = \circ(p)\}$ that is a.s. finite by Lemma 3.2.23 (which applies here by Lemma 3.2.24). Note that $X_{\zeta_{\circ(p)}} = \circ(p)$. To lighten the notations, we then set

$$(\tau', \varrho') = (\tau^\sharp(\zeta_{\circ(p)}), \varrho^\sharp(\zeta_{\circ(p)})), \quad \mathbf{T}_p = \mathbf{T}(\circ(p)), \quad \bar{Y}_v^p = \bar{Y}_{\varphi_{\zeta_{\circ(p)}}(v)}(\zeta_{\circ(p)}) \quad \text{and} \quad \Theta_p = \Theta^\sharp(\zeta_{\circ(p)}).$$

Denote by A the subset of branching walks $(t, \mathbf{r}; (\bar{y}_v)_{v \in t}) \in \bar{\mathbb{T}}^\bullet(\bar{\mathbb{T}}^\bullet)$ such that $\bar{y}_\mathbf{r} \notin \{\bar{y}_v \wedge \bar{y}_\mathbf{r}; v \in L_\mathbf{r} \setminus \{\mathbf{r}\}\}$: namely, A is the set of branching walks such that the ancestors of the distinguished individual never visit its spatial position or above. For all measurable bounded F, G on $\bar{\mathbb{T}}^\bullet(\bar{\mathbb{T}}^\bullet)$,

$$\begin{aligned} a &:= \mathbf{E}[F(\Theta^*(\zeta_{\circ(p)}))G(\Theta_p) | (\mathbf{T}, \circ)] \\ &= \mathbf{E}\left[\sum_{n \in \mathbb{N}} F(\Theta^*(n)) \mathbf{1}_{\{X_n = \circ(p)\}} G(\Theta^\sharp(n)) \mathbf{1}_{\{\Theta^\sharp(n) \in A\}} \middle| (\mathbf{T}, \circ)\right] \\ &= \sum_{n \in \mathbb{N}} \mathbf{E}\left[F(\Theta^*(n)) \mathbf{1}_{\{X_n = \circ(p)\}} \mathbf{Q}_{(\tau^\sharp(n), \varrho^\sharp(n))}^{\circ(p); (\mathbf{T}_p, \circ(p))} [G \mathbf{1}_A] \middle| (\mathbf{T}, \circ)\right]. \end{aligned}$$

Note that $\mathbf{Q}_{(\tau^\sharp(n), \varrho^\sharp(n))}^{\circ(p); (\mathbf{T}_p, \circ(p))}(A)$ is simply the probability that a μ_ν -biased RW starting from $\circ(p)$ never returns to $\circ(p)$. By (3.2.34) in Lemma 3.2.23, this probability is positive and we easily get

$$a = \sum_{n \in \mathbb{N}} \mathbf{E}\left[F(\Theta^*(n)) \mathbf{1}_{\{\zeta_{\circ(p)} = n\}} \mathbf{Q}_{(\tau^\sharp(n), \varrho^\sharp(n))}^{\circ(p); (\mathbf{T}_p, \circ(p))} [G | A] \middle| (\mathbf{T}, \circ)\right]$$

thanks to the Markov property of X at time n . Observe and recall that $(\tau^\sharp(n), \varrho^\sharp(n))$ is independent of $(\mathbf{T}, \circ; X; \Theta^*(n))$ and has the same law as $(\tau^\sharp(1), \varrho^\sharp(1))$. Thus, we get

$$\begin{aligned} &\mathbf{E}[F(\Theta^*(\zeta_{\circ(p)}))G(\Theta_p) | (\mathbf{T}, \circ)] \\ &= \mathbf{E}[F(\Theta^*(\zeta_{\circ(p)})) | (\mathbf{T}, \circ)] \mathbf{E}\left[\mathbf{Q}_{(\tau^\sharp(1), \varrho^\sharp(1))}^{\circ(p); (\mathbf{T}_p, \circ(p))} [G | A] \middle| (\mathbf{T}, \circ)\right]. \end{aligned} \quad (3.3.27)$$

Suppose now that $p > 4p_0$. Since $Y_\varrho = \circ$, if $v \in \tau$ is such that $|v \wedge \varrho| \geq -p_0$ and $|v| \leq p_0$, then $|Y_v \wedge \circ| \geq -3p_0 > 1-p$. We thus observe that the conditional law of $([\tau, \varrho]_{-p_0}^{p_0}, ([\bar{Y}_{\varrho(p_0)*v}]_{-p+1}^\infty)_{v \in [\tau]_{-p_0}^{p_0}})$ given (τ, ϱ) and (\mathbf{T}, \circ) is $Q_{([\tau, \varrho]_{-p_0}^{p_0})}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30. Since $[\cdot]_{-p_0}^{p_0} = [[\cdot]_{-p+1}^\infty]_{-p_0}^{p_0}$, it follows that there is a measurable function F_1 such that $\mathbf{E}[F([\Theta]_{-p_0}^{p_0}) | (\mathbf{T}, \circ)] = F_1([\Theta]_{-p+1}^\infty)$.

Then note that $\zeta_{\circ(p)} \geq p > 4p_0$, which also implies that $[[\Theta^*(\zeta_{\circ(p)})]_{-p_0}^{p_0}] = [[\Theta]_{-p_0}^{p_0}]$.

Now recall that $(\mathbf{T}_p, \circ(p))$ and $[[\Theta]_{-p+1}^\infty]$ are independent. Therefore (3.3.27) implies that

$$\mathbf{E}[F([\Theta]_{-p_0}^{p_0})G(\Theta_p)] = \mathbf{E}[F([\Theta]_{-p_0}^{p_0})] \mathbf{E}[G(\Theta_p)]. \quad (3.3.28)$$

We next set $\mathbf{q} = \inf\{q \in \mathbb{N} : \circ(p+3p_1), \circ(p) \notin \{Y_v; v \in \tau : |v \wedge \varrho| \leq -\zeta_{\circ(p)} - q\}\}$ which is \mathbf{P} -a.s. finite by (3.3.14) in Proposition 3.3.6. We first observe that \mathbf{q} is a measurable function of $\Theta_p = (\tau', \varrho'; (\mathbf{T}_p, Y_v^p)_{v \in \tau'})$: namely,

$$\mathbf{q} = \mathbf{q}(\Theta_p) := \inf\{q \in \mathbb{N} : \circ(p+3p_1), \circ(p) \notin \{Y_v^p; v \in \tau' : |v \wedge \varrho'| \leq -q\}\}.$$

We then define the sequence $v_n \in \tau$, $n \in \mathbb{N}$, by setting $v_0 = \varrho$ and $v_{n+1} = \text{succ}(v_n)$. Note that $\{v_n; n \in \mathbb{N}^*\} = \tau \setminus L_\varrho$ and observe that $\text{succ}_n^+(\Theta) = [\text{cent}(\tau, v_n; (\bar{Y}_v)_{v \in \tau})]^+$. We then set $S(p_0, p_1) = \inf\{n \in \mathbb{N} : |v_n \wedge \varrho| < -\zeta_{\circ(p)} - \mathbf{q}\}$. We also introduce the sequence $v'_n \in \tau'$, $n \in \mathbb{N}$, by

setting $v'_0 = \varrho'$ and $v'_{n+1} = \text{succ}(v'_n)$ and we define the time $S' = \inf\{n \in \mathbb{N} : |v'_n \wedge \varrho| < -\mathbf{q}\}$, which is a measurable function of Θ_p . Then observe that for all $n \in \mathbb{N}^*$

$$\text{succ}_{S(p_0, p_1)+n}^+(\tau, \varrho) = \text{succ}_{S'+n}^+(\tau', \varrho'). \quad (3.3.29)$$

Let $x \in \mathbf{T}_p$, remark that if $-|x \wedge \circ(p)| > p + 3p_1$ then $[(\mathbf{T}_p, x)]_{h-p_1}^{h+p_1} = [(\mathbf{T}, x)]_{h-p_1}^{h+p_1}$ for all $h \in \mathbb{Z}$. Now, let $k \in \mathbb{N}$. To simplify, set $v = v_{S(p_0, p_1)+k}$ and $v' = v'_{S'+k}$. Then $|v \wedge \varrho| < -\zeta_{\circ(p)} - \mathbf{q}$. If $w \in \llbracket \varrho(\zeta_{\circ(p)}), v \wedge \varrho \rrbracket$, Y_w has to be equal to X_q with $q > \zeta_{\circ(p)}$, which implies that $Y_w \neq \circ(p)$. If $w \in \llbracket v \wedge \varrho, v \rrbracket$, then $Y_w \neq \circ(p)$ by definition of \mathbf{q} . It shows that $\bar{Y}_{v'}^p = (\mathbf{T}_p, Y_v)$, as in (3.3.26). Again by definition of \mathbf{q} , we similarly get that $-|Y_v \wedge \circ(p)| > p + 3p_1$, which allows us to apply the previous observation. This combined with (3.3.29) implies for all $n \in \mathbb{N}^*$ that

$$\llbracket \text{succ}_{S(p_0, p_1)+n}^+(\Theta) \rrbracket_{-p_1}^{p_1} = \llbracket \text{succ}_{S'+n}^+(\Theta_p) \rrbracket_{-p_1}^{p_1}, \quad (3.3.30)$$

which is a measurable function of Θ_p . Thus, by (3.3.28) and (3.3.30), we get

$$\begin{aligned} & \mathbf{E}[F(\llbracket \Theta \rrbracket_{-p_0}^{p_0})G(\llbracket \text{succ}_{S(p_0, p_1)+n}^+(\Theta) \rrbracket_{-p_1}^{p_1})_{n \in \mathbb{N}}] \\ &= \mathbf{E}[F(\llbracket \Theta \rrbracket_{-p_0}^{p_0})G(\llbracket \text{succ}_{S'+n}^+(\Theta_p) \rrbracket_{-p_1}^{p_1})_{n \in \mathbb{N}}] \\ &= \mathbf{E}[F(\llbracket \Theta \rrbracket_{-p_0}^{p_0})] \mathbf{E}[G(\llbracket \text{succ}_{S'+n}^+(\Theta_p) \rrbracket_{-p_1}^{p_1})_{n \in \mathbb{N}}] \end{aligned}$$

which implies the desired result. ■

3.3.4 Proof of Theorem 3.1.1

Recall that μ and ν are offspring distributions such that μ is non-trivial and critical and ν is supercritical. Recall that we denote the mean of ν by m_ν . Let (τ, ϱ) be a right-infinite $\text{GW}(\mu)$ -tree that is independent from (\mathbf{T}, \circ) , which is an invariant $\text{GW}(\nu)$ -tree. Let $\Theta := (\tau, \varrho; \bar{Y}_v = (\mathbf{T}, Y_v), v \in \tau)$ be a branching RW whose conditional law given (\mathbf{T}, \circ) and (τ, ϱ) is $Q_{(\tau, \varrho)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 (recall here that the underlying spatial motion is that of a m_ν -biased RW on \mathbf{T}). We first prove the following proposition.

Proposition 3.3.11 *We keep the same assumptions and the same notations as in Proposition 3.3.6. Let $(v_n)_{n \in \mathbb{N}}$ be an indexation of $\{v \in \tau : \varrho \leq_\tau v\}$ in lexicographical order. Then,*

$$c_{\mu, \nu} := \mathbf{P}(Y_\varrho \notin \{Y_{v_n}; n \in \mathbb{N}^*\}) > 0 \quad \text{and} \quad \mathbf{P}\text{-a.s.} \quad \frac{1}{n} \# \{Y_{v_k}; 0 \leq k < n\} \xrightarrow{n \rightarrow \infty} c_{\mu, \nu}. \quad (3.3.31)$$

Moreover, for all $m, n \in \mathbb{N}$, we set $R_{m, n}(\Theta) = \# \{Y_{v_k}; m \wedge n \leq k < m \vee n\}$. Then for all $\varepsilon \in (0, 1)$ and all $s_0 \in (0, \infty)$,

$$p(n, \varepsilon, s_0) := \mathbf{P}\left(\max_{s, s' \in [0, s_0]} \left| \frac{1}{n} R_{[ns], [ns']}(\Theta) - c_{\mu, \nu} |s' - s| \right| > \varepsilon\right) \xrightarrow{n \rightarrow \infty} 0. \quad (3.3.32)$$

Proof. Recall that succ_ℓ^+ stands for the ℓ -th iterate of succ^+ . If $n \geq m$, then we observe that $R_{m, n}(\text{succ}_\ell^+(\Theta)) = R_{m+\ell, n+\ell}(\Theta)$. Then, the r.v. $R_{m, n}(\Theta)$ satisfy Assumptions (1.7), (1.8) and (1.9) of Liggett's version of Kingman's subadditive ergodic Theorem 1.10 [121] p. 1280 that asserts that there exists a constant $c \in [0, \infty)$ such that $n^{-1}R_{0, n}(\Theta) \rightarrow c$ almost surely and in L^1 and that $c = \lim_{n \rightarrow \infty} \mathbf{E}[R_{0, n}(\Theta)]/n$. Then, observe that

$$\mathbf{E}[R_{0, n}(\Theta)] = \sum_{0 \leq k < n} \mathbf{P}(Y_{v_k} \notin \{Y_{v_{k+1}}, \dots, Y_{v_{n-1}}\}).$$

Since the law of Θ is preserved by the function succ^+ , we get $\mathbf{P}(Y_{v_k} \notin \{Y_{v_{k+1}}, \dots, Y_{v_{n-1}}\}) = \mathbf{P}(Y_{\varrho} \notin \{Y_{v_1}, \dots, Y_{v_{n-k-1}}\})$ and thus

$$\frac{1}{n} \mathbf{E}[R_{0,n}(\Theta)] = \frac{1}{n} \sum_{0 \leq k \leq n-1} \mathbf{P}(Y_{\varrho} \notin \{Y_{v_1}, \dots, Y_{v_k}\}) \xrightarrow{n \rightarrow \infty} c_{\mu, \nu},$$

by Cesàro. Then observe that $c_{\mu, \nu} \geq \kappa_{\mu, \nu}$, where $\kappa_{\mu, \nu}$ is defined in (3.3.13) in Proposition 3.3.6. This proposition asserts that $\kappa_{\mu, \nu} > 0$, which completes the proof of (3.3.31).

We next prove (3.3.32). To simplify the notation, we set $X_{s,s'}^n = \frac{1}{n} R_{\lfloor ns \rfloor, \lfloor ns' \rfloor}(\Theta)$, for all $s, s' \in [0, \infty)$ and all $n \in \mathbb{N}^*$. We first suppose that $s' > s$. Since the law of Θ is preserved by succ^+ and since $X_{s,s'}^n = \frac{1}{n} R_{0, \lfloor ns' \rfloor - \lfloor ns \rfloor}(\text{succ}_{\lfloor ns \rfloor}^+(\Theta))$, $X_{s,s'}^n$ has the same law as $\frac{1}{n} R_{0, \lfloor ns' \rfloor - \lfloor ns \rfloor}(\Theta)$ and (3.3.31) implies that $X_{s,s'}^n \rightarrow c_{\mu, \nu}(s' - s)$ in \mathbf{P} -probability.

We set $p = \lceil s_0 \rceil$ and we also introduce $w_{n,q} = \max_{0 \leq k \leq l \leq p2^q} |X_{k2^{-q}, l2^{-q}}^n - c_{\mu, \nu}(l - k)2^{-q}|$, for all $q, n \in \mathbb{N}^*$. The previous result implies that for all $q \in \mathbb{N}^*$, $w_{n,q} \rightarrow 0$ in \mathbf{P} -probability. We next fix $s, s' \in [0, p]$ such that $s' \geq s$. We set $s_q = \lfloor 2^q s \rfloor 2^{-q}$ and $s'_q = \lceil 2^q s' \rceil 2^{-q}$. Observe that $0 \leq X_{s_q, s'_q}^n - X_{s, s'}^n \leq 2 \cdot 2^{-q} + \frac{2}{n}$, since $R_{m,m'}(\Theta) \leq |m' - m|$, by an obvious counting argument. This easily implies

$$\max_{s, s' \in [0, p]} |X_{s, s'}^n - c_{\mu, \nu}(s' - s)| \leq 2(1 + c_{\mu, \nu})2^{-q} + \frac{2}{n} + w_{n,q},$$

which readily implies (3.3.32). This completes the proof of the proposition. \blacksquare

Proof of Theorem 3.1.1. Let us prove first that the assumptions of Theorem 3.1.1 imply the assumptions of Proposition 3.3.11. Namely, recall that in Theorem 3.1.1, we suppose that there exists $\mathfrak{b} \in (1, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{\mathfrak{b}} \nu(k) < \infty$. As μ is in the domain of attraction of an α -stable law, it admits finite moments of all orders $a \in [1, \alpha)$ (see e.g. Feller [74, Lemma in Chapter XVII.5, pp. 578]). Therefore, under the assumption of Theorem 3.1.1, we have $\sum_{k \in \mathbb{N}^*} \mu(k) k (\log k)^{2(1+\beta)} < \infty$, where $\beta = 1/(\mathfrak{b} - 1)$, and so Proposition 3.3.11 applies. Then we keep the notation $\Theta = (\tau, \varrho; \bar{Y}_v = (\mathbf{T}, Y_v), v \in \tau)$ as above. We denote by $(v_n)_{n \in \mathbb{N}}$ an indexation of $\{v \in \tau : \varrho \leq \tau v\}$ in lexicographical order and we also introduce $(w_k)_{k \in \mathbb{N}^*}$ that are the vertices of the spine ∂L_{ϱ} listed in lexicographical order (recall that $\partial L_{\varrho} = \{v \in \tau \setminus L_{\varrho} : \bar{v} \in L_{\varrho}\}$). We then set

$$\tau^{(k)} = \theta_{w_k} \tau, \quad Y_v^k = Y_{w_k * v}, \quad v \in \tau^{(k)} \quad \text{and} \quad \Theta_{\infty}^k = (\tau^{(k)}, \varnothing; \bar{Y}_v^k = (\mathbf{T}, Y_v^k), v \in \tau^{(k)}).$$

We next introduce the following notation for any deterministic T -valued and t -indexed branching walk $\Theta = (t, u; \bar{y}_v = (T, y_v), v \in t)$: we assume that t is finite and for all $0 \leq m < \#t$, we set

$$R_m(\Theta) = \#\{y_{v_l}; 0 \leq l < m\},$$

where $(v_m)_{0 \leq m < \#t}$ stands for the vertices of t listed in lexicographical order. Note that $R_m(\Theta) = R_m(\text{cent}(\Theta))$. Then observe that for all $k \in \mathbb{N}^*$ and all $0 \leq m < \#\tau^{(k)}$,

$$R_m(\text{cent}(\Theta_{\infty}^k)) = R_m(\Theta_{\infty}^k) = R_{\sigma_{k-1}, \sigma_{k-1} + m}(\Theta), \quad (3.3.33)$$

where the $R_{m', n'}(\Theta)$ are as in Proposition 3.3.11 and where we have set $\sigma_k = \sum_{1 \leq l \leq k} \#\tau^{(l)}$ for all $k \in \mathbb{N}^*$, with the convention $\sigma_0 = 0$.

We then introduce $\kappa_n = \inf\{k \in \mathbb{N}^* : \#\tau^{(k)} \geq n\}$ and $\mathfrak{g}_n = \sigma_{\kappa_n - 1}$ as in (3.2.66). To simplify we set $\tau_{\geq n} := \tau^{(\kappa_n)}$ and $\Theta'_{\geq n} := \Theta_{\infty}^{\kappa_n}$. First observe that $\tau_{\geq n}$ has the same law as a $\text{GW}(\mu)$ -tree τ' under $\mathbf{P}(\cdot | \#\tau' \geq n)$. Next note that the conditional law of $\Theta'_{\geq n}$ given (\mathbf{T}, \circ) and (τ, ϱ) is $Q_{(\tau_{\geq n}, \varnothing)}^{(\mathbf{T}, Y_{w_{\kappa_n}})}$. Since $(\mathbf{T}, Y_{w_{\kappa_n}})$ has the same law as a μ_{ν} -biased RW on \mathbf{T} starting at \circ taken at time $-|w_{\kappa_n}| + 2$. By the invariance (3.2.38), it implies that

$$\text{conditionally given } (\tau, \varrho) \quad \Theta_{\geq n} := \text{cent}(\Theta'_{\geq n}) \stackrel{(\text{law})}{=} Q_{(\tau_{\geq n}, \varnothing)}^{(\mathbf{T}, \circ)}.$$

By (3.3.33), we then get for all $0 \leq m \leq n$,

$$R_m(\Theta_{\geq n}) = R_{\mathfrak{g}_n, \mathfrak{g}_n+m}(\Theta).$$

We now recall from Theorem 3.2.36 (i) that $\frac{1}{n}\mathfrak{g}_n \rightarrow \mathfrak{g}_1$ in law where $\mathfrak{g}_1 \in (0, \infty)$ almost surely. Therefore, for all $\varepsilon \in (0, 1)$ and all $s_0 \in (1, \infty)$, we get

$$p'(n, \varepsilon) := \mathbf{P}\left(\max_{s \in [0,1]} \left| \frac{1}{n} R_{[ns]}(\Theta_{\geq n}) - c_{\mu, \nu} s \right| > \varepsilon\right) \leq p(n, \varepsilon, s_0) + \mathbf{P}\left(\frac{1}{n}\mathfrak{g}_n > s_0 - 1\right)$$

where $p(n, \varepsilon, s_0)$ is as in (3.3.32), which implies that $\limsup_{n \rightarrow \infty} p'(n, \varepsilon) \leq \mathbf{P}(\mathfrak{g}_1 \geq s_0 - 1) \rightarrow 0$ as $s_0 \rightarrow \infty$. Namely, $\lim_{n \rightarrow \infty} \max_{s \in [0,1]} \left| \frac{1}{n} R_{[ns]}(\Theta_{\geq n}) - c_{\mu, \nu} s \right| = 0$ in \mathbf{P} -probability.

We next want to use Lemma 3.2.37 to complete the proof of Theorem 3.1.1. To that end we denote by $V_{\cdot}(\tau_{\geq n})$ the Lukasiewicz path of $\tau_{\geq n}$ and by $(v_k)_{0 \leq k < \#\tau_{\geq n}}$ the vertices of $\tau_{\geq n}$ listed in lexicographical order. We also fix $r \in (0, 1)$ and we set $r_n = \lfloor nr \rfloor$. Then observe that the subtree $\{v_k; 0 \leq k \leq r_n\}$ is measurable with respect to $V_{\cdot \wedge r_n}(\tau_{\geq n})$ and that $(R_k(\Theta_{\geq n}))_{0 \leq k \leq r_n}$ is a measurable function of $V_{\cdot \wedge r_n}(\tau_{\geq n})$ and of an independent r.v. U_n . Moreover, as an easy consequence of Theorem 3.2.36 (i) and Remark 3.2.34, $\frac{1}{b_n} V_{[n \cdot] \wedge r_n}(\tau_{\geq n})$ converges in law in $\mathbf{D}([0, \infty), \mathbb{R})$ to $X_{\cdot \wedge r}$ under $\mathbf{N}(dX | \zeta > 1)$. By Slutsky's Theorem, we therefore get the joint convergence

$$\left(\frac{1}{b_n} V_{[n \cdot] \wedge r_n}(\tau_{\geq n}), \left(\frac{1}{n} R_{[ns]}(\Theta_{\geq n}) \right)_{s \in [0, r]} \right) \xrightarrow[n \rightarrow \infty]{} (X_{\cdot \wedge r}, c_{\mu, \nu} \text{Id}_{[0, r]}(\cdot))$$

weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}([0, r], \mathbb{R})$. Then Lemma 3.2.37 applies and we get

$$\mathbf{P}\left(\max_{s \in [0, r]} \left| \frac{1}{n} R_{[ns]}(\Theta_n) - c_{\mu, \nu} s \right| > \varepsilon\right) \xrightarrow[n \rightarrow \infty]{} 0 \quad (3.3.34)$$

where $\Theta_n := (\tau_n, \emptyset; \bar{Y}_v = (\mathbf{T}, Y_v), v \in \tau_n)$ is a branching RW whose law is characterized as follows: τ_n has the same law as a $\text{GW}(\mu)$ -tree τ' under $\mathbf{P}(\cdot | \#\tau' = n)$ and the conditional law of Θ_n given τ_n and (\mathbf{T}, \circ) is $Q_{(\tau_n, \emptyset)}^{(\mathbf{T}, \circ)}$. Since r can be chosen arbitrarily closed to 1 in (3.3.34), an elementary continuity argument finally entails (3.3.34) when $r = 1$ which is precisely the statement of Theorem 3.1.1. \blacksquare

3.4 Limit of the τ_{∞} -indexed branching RW and its snakes

3.4.1 Harmonic coordinates and related estimates.

One key-ingredient of the proof in Peres & Zeitouni [137] of the quenched CLT for a critical biased RW $(X_n)_{n \in \mathbb{N}}$ in an invariant GW -tree is the *harmonic coordinates*, which are \mathbb{R} -valued measurable functions $(S_x)_{x \in \mathbf{T}}$ of the environment \mathbf{T} such that on one hand $(S_{X_n})_{n \in \mathbb{N}}$ is a martingale to which conditional CLT applies, and on the other hand such that $S_{X_n} \sim \text{Cst}|X_n|$. This idea is actually adapted to branching RWs to prove the main result of the chapter. In this section, we recall from Peres & Zeitouni [137] the construction of the harmonic coordinates of (variants of) the invariant $\text{GW}(\nu)$ -tree \mathbf{T} (within our framework of ordered trees) and we prove several estimates that are used in our proofs.

The following lemma summarizes definitions, notations, and the main properties of harmonic coordinates. The proof given below does not contain any new argument: it is recalled from Peres & Zeitouni [137] and Dembo & Sun [45].

Lemma 3.4.1 *Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . We assume that there exists $b' \in (1, \infty)$ such that $m_\nu(b') := \sum_{k \in \mathbb{N}} k^{b'} \nu(k) < \infty$. For all rooted ordered tree $t \in \mathbb{T}$ as in Definition 3.2.1, we next set*

$$U(t) = \liminf_{n \rightarrow \infty} m_\nu^{-n} \#\{u \in t : |u| = n\} \in [0, \infty]. \quad (3.4.1)$$

Here, (\mathbf{T}, \circ) is either a $\text{GW}(\nu)$ -tree as in Definition 3.2.6 (a) (in this case: $\circ = \emptyset$), or an infinite $\text{GW}(\nu)$ -tree as in Definition 3.2.18 (b), or an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. For all $x \in \mathbf{T}$, we recall from (3.2.17) that the subtree $\theta_x \mathbf{T}$ stemming from x belongs to \mathbb{T} and we set $U_x = U(\theta_x \mathbf{T})$. Then the following holds true.

(i) **P**-a.s. for all $x \in \mathbf{T}$,

$$U_x = \lim_{n \rightarrow \infty} m_\nu^{-n} \#\{y \in \mathbf{T} : x \preceq y, |y| = |x| + n\} \in (0, \infty) \quad \text{and} \quad U_x = \frac{1}{m_\nu} \sum_{y \in \mathbf{T} : \bar{y} = x} U_y. \quad (3.4.2)$$

(ii) Suppose that $b \in (1, \infty)$ is such that $\sum_{k \in \mathbb{N}} k^b \nu(k) < \infty$. Then, $\mathbf{E}[U_\circ^b] < \infty$ if \mathbf{T} is a $\text{GW}(\nu)$ -tree or an infinite $\text{GW}(\nu)$ -tree. If \mathbf{T} is an invariant $\text{GW}(\nu)$ -tree, then $\mathbf{E}[U_\circ^{b-1}] < \infty$ if $b \in (2, \infty)$.

(iii) The harmonic coordinates are the \mathbb{R} -valued r.v. $(S_x)_{x \in \mathbf{T}}$ defined by the following

$$S_\circ = 0 \quad \text{and} \quad \forall x \in \mathbf{T} \setminus \{\emptyset\}, \quad S_x - S_{\bar{x}} = U_x, \quad (3.4.3)$$

on the event where (3.4.2) holds true and by $S_x = 0$, $x \in \mathbf{T}$, on the **P**-negligible complementary event. Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a m_ν -biased RW such that $X_0 = \circ$. We denote by \mathcal{F}_n the σ -field generated by $(\bar{X}_k)_{0 \leq k \leq n}$. Then $(S_{X_n})_{n \in \mathbb{N}}$ is a $(\mathcal{F}_n)_{n \in \mathbb{N}}$ -martingale:

$$\mathbf{P}\text{-a.s. for all } n \in \mathbb{N}^*, \quad \mathbf{E}[S_{X_{n+1}} | \mathcal{F}_n] = S_{X_n} \quad (3.4.4)$$

(iv) We keep the previous notations and for all $n \in \mathbb{N}$, we set $\Delta_n = S_{X_{n+1}} - S_{X_n}$ and $\Phi_b(\bar{X}_n) = \mathbf{E}[|\Delta_n|^b | \mathcal{F}_n]$. Then,

$$\Phi_b(\bar{X}_n) \leq \frac{m_\nu^b + m_\nu}{m_\nu + k_{X_n}(\mathbf{T})} U_{X_n}^b \quad \text{and} \quad \Phi_b(\bar{X}_n) = \Phi_b(\text{cent}(\bar{X}_n)). \quad (3.4.5)$$

If $b \in (1, \infty)$ is such that $\sum_{k \in \mathbb{N}} k^b \nu(k) < \infty$ and if \mathbf{T} is an invariant $\text{GW}(\nu)$ -tree, then $\mathbf{E}[\Phi_b(\bar{X}_n)] = \mathbf{E}[\Phi_b(\bar{X}_0)] < \infty$. In the specific case where $b = 2$, then

$$\mathbf{E}[\Phi_2(\bar{X}_n)] = \frac{m_\nu(2) - m_\nu}{m_\nu^2 - m_\nu} =: \sigma_\nu^2. \quad (3.4.6)$$

Proof. We prove (i). To that end, let us first consider the case where \mathbf{T} is $\text{GW}(\nu)$ -tree as in Definition 3.2.6 (a). By standard arguments about martingales, a b' -moments for ν with $b' \in (1, \infty)$ implies that **P**-a.s. $U(\mathbf{T}) = \lim_{n \rightarrow \infty} m_\nu^{-n} \#\{x \in \mathbf{T} : |x| = n\} \in (0, \infty)$. Next observe that for all $x \in \mathbb{U}$ such that $\mathbf{P}(x \in \mathbf{T}) > 0$, the subtree $\theta_x \mathbf{T}$ under $\mathbf{P}(\cdot | x \in \mathbf{T})$ has the same law as \mathbf{T} . Since \mathbb{U} is countable, **P**-a.s. (3.4.2) holds true for all $x \in \mathbf{T}$.

Suppose next that (\mathbf{T}, \circ) is an infinite $\text{GW}(\nu)$ -tree or an invariant $\text{GW}(\nu)$ -tree. In these cases, conditionally given ∂L_\circ , the subtrees $\theta_y \mathbf{T}$, $y \in \mathbf{T} \setminus L_\circ$, are independent $\text{GW}(\nu)$ -trees. Therefore **P**-a.s. (3.4.2) holds true for all $x \in \mathbf{T} \setminus L_\circ$. Now note that (3.4.2) provides a unambiguous definition of the weight U_x for $x \in L_\circ$, which completes the proof of (i).

Let us prove (ii). We first suppose that \mathbf{T} is a $\text{GW}(\nu)$ -tree or an infinite $\text{GW}(\nu)$ -tree: in both cases $\theta_\circ \mathbf{T}$ is $\text{GW}(\nu)$ -tree and as proved in Dembo & Sun [45, Lemma 6.2], $\mathbf{E}[U_\circ^b] < \infty$, which implies the first part of (ii).

We next assume that \mathbf{T} is an invariant $\text{GW}(\nu)$ -tree and that $b \in (2, \infty)$. By convexity of $x \mapsto x^{b-1}$ and the second equality in (3.4.2), $U_\circ^{b-1} \leq m_\nu^{-(b-1)} k_\circ(\mathbf{T})^{b-2} \sum_{x \in \mathbf{T} : \bar{x} = \circ} U_x^{b-1}$. By (3.2.36) in the definition of invariant $\text{GW}(\nu)$ -trees, we get $\mathbf{E}[U_\circ^{b-1} | k_\circ(\mathbf{T})] \leq m_\nu^{-(b-1)} \mathbf{E}[U(\mathbf{T}_0)^{b-1}] k_\circ(\mathbf{T})^{b-1}$,

where \mathbf{T}_0 is a $\text{GW}(\nu)$ -tree. Thus $\mathbf{E}[U_0^{\mathbf{b}-1}] \leq \frac{1}{2} \mathbf{m}_\nu^{-\mathbf{b}} \mathbf{E}[\mathbf{U}(\mathbf{T}_0)^{\mathbf{b}-1}] \sum_{k \in \mathbb{N}} (\mathbf{m}_\nu + k) k^{\mathbf{b}-1} \nu(k) < \infty$, which completes the proof of (ii).

To prove (iii), observe that

$$\begin{aligned} \mathbf{E}[S_{X_{n+1}} - S_{X_n} | \mathcal{F}_n] &= (S_{\overleftarrow{X}_n} - S_{X_n}) \mathbf{E}[\mathbf{1}_{\{X_{n+1} = \overleftarrow{X}_n\}} | \mathcal{F}_n] + \sum_{y \in \mathbf{T}: \overleftarrow{y} = X_n} (S_y - S_{X_n}) \mathbf{E}[\mathbf{1}_{\{X_{n+1} = y\}} | \mathcal{F}_n] \\ &= -\frac{\mathbf{m}_\nu U_{X_n}}{\mathbf{m}_\nu + k_{X_n}(\mathbf{T})} + \sum_{y \in \mathbf{T}: \overleftarrow{y} = X_n} \frac{U_y}{\mathbf{m}_\nu + k_{X_n}(\mathbf{T})} = 0 \end{aligned}$$

by the second equality in (3.4.2). This completes the proof of (iii).

Let us prove (iv). We first observe that

$$\begin{aligned} \Phi_{\mathbf{b}}(\overline{X}_n) &= \frac{1}{\mathbf{m}_\nu + k_{X_n}(\mathbf{T})} \left(\mathbf{m}_\nu U_{X_n}^{\mathbf{b}} + \sum_{y \in \mathbf{T}} U_y^{\mathbf{b}} \mathbf{1}_{\{\overleftarrow{y} = X_n\}} \right) \\ &\leq \frac{1}{\mathbf{m}_\nu + k_{X_n}(\mathbf{T})} \left(\mathbf{m}_\nu U_{X_n}^{\mathbf{b}} + \left(\sum_{y \in \mathbf{T}} U_y \mathbf{1}_{\{\overleftarrow{y} = X_n\}} \right)^{\mathbf{b}} \right) = \frac{\mathbf{m}_\nu^{\mathbf{b}} + \mathbf{m}_\nu}{\mathbf{m}_\nu + k_{X_n}(\mathbf{T})} U_{X_n}^{\mathbf{b}} \end{aligned}$$

since $s^{\mathbf{b}} + r^{\mathbf{b}} \leq (s+r)^{\mathbf{b}}$ for all $s, r \in [0, \infty)$ and by the second equality in (3.4.2). This proves the inequality in (3.4.5). Then, recall from (3.2.12) the definition of the shifts φ_l , $l \in \mathbb{Z}$, and $\text{cent}(\cdot)$ from Definition 3.2.15 and observe that for all $x \in \mathbf{T}$, $\mathbf{U}(\theta_x \mathbf{T}) = \mathbf{U}(\theta_{\varphi_l(x)} \varphi_l(\mathbf{T}))$ and that $k_x(\mathbf{T}) = k_{\varphi_l(x)}(\varphi_l(\mathbf{T}))$, which easily implies the equality in (3.4.5).

Next we suppose that \mathbf{T} is invariant. By Proposition 3.2.26, the law of \overline{X}_n is then invariant and $\mathbf{E}[\Phi_{\mathbf{b}}(\overline{X}_n)] = \mathbf{E}[\Phi_{\mathbf{b}}(\overline{X}_0)]$. By (3.4.5), we then get

$$\mathbf{E}[\Phi_{\mathbf{b}}(\overline{X}_0)] \leq \frac{1}{2} (\mathbf{m}_\nu^{\mathbf{b}-1} + 1) \mathbf{E}[\mathbf{U}(\mathbf{T}_0)^{\mathbf{b}}] < \infty$$

where \mathbf{T}_0 a $\text{GW}(\nu)$ -tree, by (3.2.36) in the definition of invariant $\text{GW}(\nu)$ -trees. In the special case $\mathbf{b} = 2$, we then get

$$\begin{aligned} \mathbf{E}[\Phi_2(\mathbf{T}, \varrho)] &= \frac{1}{2\mathbf{m}_\nu} \mathbf{E} \left[\mathbf{m}_\nu \mathbf{U}(\mathbf{T}_0)^2 + \sum_{u \in \mathbf{T}_0} \mathbf{U}(\theta_u \mathbf{T}_0)^2 \mathbf{1}_{\{\overleftarrow{u} = \varnothing\}} \right] \\ &= \mathbf{E}[\mathbf{U}(\mathbf{T}_0)^2] = \frac{\mathbf{m}_\nu(2) - \mathbf{m}_\nu}{\mathbf{m}_\nu^2 - \mathbf{m}_\nu}, \end{aligned}$$

the last equality being a consequence of the explicit computation of the variance of Galton-Watson processes: see e.g. Athreya & Ney [13, Chapter I, (2) p. 4]. \blacksquare

The following lemma provides an initial comparison of harmonic coordinates with the relative height of vertices.

Lemma 3.4.2 *Let ν be a supercritical offspring distribution whose mean is denoted by \mathbf{m}_ν . We assume that $\mathbf{m}_\nu(2) = \sum_{k \in \mathbb{N}} k^2 \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an infinite tree $\text{GW}(\nu)$ -tree as in Definition 3.2.18 or an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. Let the r.v. U_x, S_x , $x \in \mathbf{T}$, be as in (3.4.2) and (3.4.3). Recall from (3.2.15) the notation ∂L_\circ and $\circ(p)$, $p \in \mathbb{N}$. Then the following holds true.*

- (i) *For all $p \in \mathbb{N}^*$ we set $\beta_p = \mathbf{m}_\nu^{-1} \sum_{x \in \partial L_\circ: \overleftarrow{x} = \circ(p)} U_x$. Then $\beta_p = U_{\circ(p)} - \mathbf{m}_\nu^{-1} U_{\circ(p-1)}$. Moreover, the r.v. $(\beta_p)_{p \in \mathbb{N}^*}$ are i.i.d. such that $\mathbf{E}[\beta_1] = \mathbf{m}_\nu^{-2} (\mathbf{m}_\nu(2) - \mathbf{m}_\nu)$. If $\mathbf{b} \in (1, \infty)$ is such that $\mathbf{m}_\nu(\mathbf{b} + 1) := \sum_{k \in \mathbb{N}} k^{\mathbf{b}+1} \nu(k) < \infty$, then $\mathbf{E}[\beta_1^{\mathbf{b}}] < \infty$.*

(ii) Recall σ_ν from (3.4.6) and assume that there exists $\mathfrak{b} \in (1, \infty)$ such that $\mathfrak{m}_\nu(2\mathfrak{b} + 1) = \sum_{k \in \mathbb{N}} k^{2\mathfrak{b}+1} \nu(k) < \infty$. Then, there is a constant $c \in (0, \infty)$ that only depends on ν and \mathfrak{b} such that for all $p \in \mathbb{N}$, $\mathbf{E} \left[\left| \frac{1}{\sigma_\nu^2} S_{\circ(p)} - |\circ(p)| \right|^{2\mathfrak{b}} \right] \leq c p^\mathfrak{b}$. Moreover, for all $\beta \in (\frac{1}{2} + \frac{1}{2\mathfrak{b}}, 1)$, we set

$$R_\beta = \sup_{p \in \mathbb{N}} \frac{\left| \frac{1}{\sigma_\nu^2} S_{\circ(p)} - |\circ(p)| \right|}{1 + p^\beta}. \quad (3.4.7)$$

Then, there exists a constant $c' \in (0, \infty)$ that only depends on ν , \mathfrak{b} and β such that for all $\lambda \in (0, \infty)$, we get $\mathbf{P}(R_\beta \geq \lambda) \leq c' \lambda^{-2\mathfrak{b}}$.

Proof. The first point of (i) is a direct consequence of the definition of invariant GW-trees and of (3.4.2). Then, observe that $\mathbf{E}[\beta_1 | k_{\circ(1)}(\mathbf{T})] = \mathfrak{m}_\nu^{-1} \mathbf{E}[\mathbf{U}(\mathbf{T}') | (k_{\circ(1)}(\mathbf{T}) - 1)] = \mathfrak{m}_\nu^{-1} (k_{\circ(1)}(\mathbf{T}) - 1)$, where \mathbf{T}' stands for a $\text{GW}(\nu)$ -tree. Thus, $\mathbf{E}[\beta_1] = \mathfrak{m}_\nu^{-2} \sum_{k \in \mathbb{N}} k(k-1) \nu(k) = \mathfrak{m}_\nu^{-2} (\mathfrak{m}_\nu(2) - \mathfrak{m}_\nu)$. More generally, note that $\beta_1^\mathfrak{b} \leq \mathfrak{m}_\nu^{-\mathfrak{b}} (k_{\circ(1)}(\mathbf{T}) - 1)^{\mathfrak{b}-1} \sum_{x \in \partial L_{\circ}: \bar{x} = \circ(1)} U_x^\mathfrak{b}$ since $z \mapsto z^\mathfrak{b}$ is convex. Then recall from Lemma 3.4.1 (ii) that $\mathbf{E}[\mathbf{U}^\mathfrak{b}(\mathbf{T}')] < \infty$ if $\mathfrak{m}_\nu(\mathfrak{b}) < \infty$. Thus $\mathbf{E}[\beta_1^\mathfrak{b} | k_{\circ(1)}(\mathbf{T})] \leq \mathfrak{m}_\nu^{-\mathfrak{b}} \mathbf{E}[\mathbf{U}^\mathfrak{b}(\mathbf{T}')] (k_{\circ(1)}(\mathbf{T}) - 1)^\mathfrak{b}$, which implies that $\mathbf{E}[\beta_1^\mathfrak{b}] \leq \mathfrak{m}_\nu^{-\mathfrak{b}-1} \mathbf{E}[\mathbf{U}^\mathfrak{b}(\mathbf{T}')] \mathfrak{m}_\nu(\mathfrak{b} + 1) < \infty$.

Let us prove (ii). As an immediate consequence of (i), for all $p \in \mathbb{N}^*$ we get

$$U_{\circ(p)} = \mathfrak{m}_\nu^{-p} U_\circ + \sum_{1 \leq k < p} \mathfrak{m}_\nu^{-(p-k)} \beta_k = \mathfrak{m}_\nu^{-p} U_\circ + (1 - \mathfrak{m}_\nu^{-1})^{-1} \mathbf{E}[\beta_{p-G} \mathbf{1}_{\{0 \leq G < p\}} | (\mathbf{T}, \circ)] \quad (3.4.8)$$

where G stands for a r.v. that is independent from (\mathbf{T}, \circ) and such that $\mathbf{P}(G = k) = (1 - \mathfrak{m}_\nu^{-1}) \mathfrak{m}_\nu^{-k}$, $k \in \mathbb{N}$. Since $S_{\circ(p-1)} - S_{\circ(p)} = U_{\circ(p-1)}$ (as a consequence of (3.4.3)), we get by (3.4.8)

$$\begin{aligned} -S_{\circ(p)} &= U_\circ + \sum_{1 \leq q < p} U_{\circ(q)} = \frac{1 - \mathfrak{m}_\nu^{-p}}{1 - \mathfrak{m}_\nu^{-1}} U_\circ + \sum_{1 \leq k < p} \beta_k \sum_{k \leq q < p} \mathfrak{m}_\nu^{-(q-k)} \\ &= \frac{1 - \mathfrak{m}_\nu^{-p}}{1 - \mathfrak{m}_\nu^{-1}} U_\circ + \sum_{1 \leq k < p} \beta_k \frac{1 - \mathfrak{m}_\nu^{-(p-k)}}{1 - \mathfrak{m}_\nu^{-1}} = \frac{1 - \mathfrak{m}_\nu^{-p}}{1 - \mathfrak{m}_\nu^{-1}} U_\circ + \sum_{1 \leq k < p} \frac{\beta_k}{1 - \mathfrak{m}_\nu^{-1}} - \sum_{1 \leq k < p} \frac{\mathfrak{m}_\nu^{-(p-k)}}{1 - \mathfrak{m}_\nu^{-1}} \beta_k. \end{aligned}$$

To simplify notation, we set $a = \mathbf{E}[\beta_1]$ and $\bar{\beta}_k = \beta_k - a$, that is a centered r.v. and we get

$$\begin{aligned} -(1 - \mathfrak{m}_\nu^{-1}) S_{\circ(p)} - ap &= \Sigma_{p-1} + R_p(1) - R_p(2) \quad \text{where} \\ \Sigma_{p-1} &= \sum_{1 \leq k \leq p-1} \bar{\beta}_k, \quad R_p(1) = (1 - \mathfrak{m}_\nu^{-p}) U_\circ - a \quad \text{and} \\ R_p(2) &= \sum_{1 \leq k < p} \mathfrak{m}_\nu^{-(p-k)} \beta_k = (1 - \mathfrak{m}_\nu^{-1})^{-1} \mathbf{E}[\beta_{p-G} \mathbf{1}_{\{0 < G < p\}} | (\mathbf{T}, \circ)]. \end{aligned}$$

Note from (i) and (3.4.6) that $a^{-1}(1 - \mathfrak{m}_\nu^{-1}) = \sigma_\nu^{-2}$. Since $z \mapsto z^{2\mathfrak{b}}$ is convex, we get

$$a^{2\mathfrak{b}} \mathbf{E} \left[\left| \frac{1}{\sigma_\nu^2} S_{\circ(p)} + p \right|^{2\mathfrak{b}} \right] \leq 3^{2\mathfrak{b}-1} (\mathbf{E}[|R_p(1)|^{2\mathfrak{b}}] + \mathbf{E}[|R_p(2)|^{2\mathfrak{b}}] + \mathbf{E}[|\Sigma_{p-1}|^{2\mathfrak{b}}]).$$

First observe that $\mathbf{E}[|R_p(1)|^{2\mathfrak{b}}] \leq 2^{2\mathfrak{b}-1} (\mathbf{E}[U_\circ^{2\mathfrak{b}}] + a^{2\mathfrak{b}})$, which finite by Lemma 3.4.1 (ii).

To bound $\mathbf{E}[|R_p(2)|^{2\mathfrak{b}}]$, we use conditional Jensen inequality to get

$$|R_p(2)|^{2\mathfrak{b}} \leq (1 - \mathfrak{m}_\nu^{-1})^{-2\mathfrak{b}} \mathbf{E}[\beta_{p-G}^{2\mathfrak{b}} \mathbf{1}_{\{0 < G < p\}} | (\mathbf{T}, \circ)],$$

which entails $\mathbf{E}[|R_p(2)|^{2\mathfrak{b}}] \leq (1 - \mathfrak{m}_\nu^{-1})^{-2\mathfrak{b}} \mathbf{E}[\beta_1^{2\mathfrak{b}}]$, which is finite by (i).

Let us bound $\mathbf{E}[|\Sigma_{p-1}|^{2b}]$. Since the $\bar{\beta}_k$ are i.i.d. centered and such that $\mathbf{E}[|\bar{\beta}_k|^{2b}] < \infty$ by (i), we can apply Burkholder-Davis-Gundy inequality (see e.g. Burkholder [32, Theorem 9 p. 1502]) to get a constant $c_0 \in (0, \infty)$ that only depends on b such that

$$\mathbf{E}[|\Sigma_{p-1}|^{2b}] \leq c_0 \mathbf{E}\left[\left(\sum_{1 \leq k < p} \bar{\beta}_k^2\right)^b\right].$$

Since $z \mapsto z^b$ is convex,

$$\mathbf{E}\left[\left(\sum_{1 \leq k < p} \bar{\beta}_k^2\right)^b\right] \leq (p-1)^{b-1} \sum_{1 \leq k < p} \mathbf{E}[|\bar{\beta}_k|^{2b}] = (p-1)^b p \mathbf{E}[|\bar{\beta}_1|^{2b}],$$

which proves the first inequality in (iii).

To complete the proof of (iii) we simply observe that

$$\mathbf{P}(R_\beta \geq \lambda) \leq \sum_{p \in \mathbb{N}^*} \mathbf{P}\left(\left|\frac{1}{\sigma_\nu^2} S_{o(p)} - |o(p)|\right| \geq \lambda(1 + p^\beta)\right) \leq \sum_{p \in \mathbb{N}^*} \frac{cp^b}{\lambda^{2b}(1 + p^\beta)^{2b}} < c' \lambda^{-2b},$$

by a Markov inequality and since $(2\beta - 1)b > 1$. \blacksquare

The following lemma shows that branching RWs do not visit ‘bad’ vertices, namely vertices x such that $\sigma_\nu^{-2} \times$ (the harmonic coordinate at x) is large compared to the relative height of x .

Lemma 3.4.3 *Let τ be a $GW(\mu)$ -tree, where μ is a non-trivial critical offspring distribution. Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . Let $b \in (1, \infty)$. We assume that $\sum_{k \in \mathbb{N}} k^{2b+1} \nu(k) < \infty$. Let (\mathbf{T}, o) be an invariant $GW(\nu)$ -tree. Let $(Y'_u)_{u \in \tau}$ be a τ -indexed, \mathbf{T} -valued m_ν -biased branching RW whose conditional law given (\mathbf{T}, o) and τ is $Q_{(\tau, \emptyset)}^{(\mathbf{T}, o)}$ as in Definition 3.2.30. Recall from (3.2.15) the notations $o(p)$ and L_o . Let $(S_x)_{x \in \mathbf{T}}$ be the harmonic coordinates as in (3.4.3) and recall from (3.4.6) the notation σ_ν . For all $q \in \mathbb{N}$, all $\lambda \in (0, \infty)$ and all $\beta \in (\frac{1}{2} + \frac{1}{2b}, 1)$, we define the following subset of \mathbf{T}*

$$B_{q, \lambda, \beta} = \left\{ x \in \mathbf{T} \setminus L_o : |o \wedge x| \geq -q \text{ and } \left| \frac{1}{\sigma_\nu^2} (S_x - S_{o \wedge x}) - (|x| - |o \wedge x|) \right| \geq \lambda(1 + (|x| - |o \wedge x|)^\beta) \right\}.$$

Then, there exists $c \in (0, \infty)$, which only depends on ν, b, β , such that $\mathbf{P}(\exists u \in \tau : Y'_u \in B_{q, \lambda, \beta}) \leq cq \lambda^{-2b}$.

Proof. Let $(\mathbf{T}_0, \emptyset)$ be a $GW(\nu)$ -tree (Definition 3.2.6 (a)). Denote by $(S_x^0)_{x \in \mathbf{T}_0}$ its harmonic coordinates. We first bound the expectation of

$$Z(\mathbf{T}_0) = \sum_{x \in \mathbf{T}_0} \frac{m_\nu + k_x(\mathbf{T}_0)}{m_\nu - 1} m_\nu^{-|x|} \mathbf{1}_{\left\{ \left| \sigma_\nu^{-2} S_x^0 - |x| \right| \geq \lambda(1 + |x|^\beta) \right\}}.$$

Let (\mathbf{T}', o') be an infinite $GW(\nu)$ -tree (Definition 3.2.18) whose harmonic coordinates are denoted by $(S'_x)_{x \in \mathbf{T}'}$. We recall that $S_\emptyset^0 = 0$ and that for all $x \in \mathbf{T}_0$, $S_x^0 = \sum_{y \in \llbracket \emptyset, x \rrbracket} U(\theta_y \mathbf{T}_0)$. Then, the many-to-one formula (3.2.25) implies

$$\begin{aligned} \mathbf{E}[Z(\mathbf{T}_0)] &= \sum_{p \in \mathbb{N}^*} \mathbf{E}\left[\frac{m_\nu + k_o(\mathbf{T}')}{m_\nu - 1} \mathbf{1}_{\left\{ \left| \sigma_\nu^{-2} S'_{o'(p)} - |o'(p)| \right| \geq \lambda(1 + |p|^\beta) \right\}}\right] \\ &= \frac{2m_\nu}{m_\nu - 1} \sum_{p \in \mathbb{N}^*} \mathbf{P}\left(\left| \sigma_\nu^{-2} S_{o(p)} - |o(p)| \right| \geq \lambda(1 + |p|^\beta)\right) \leq \sum_{p \in \mathbb{N}^*} \frac{cp^b}{\lambda^{2b}(1 + p^\beta)^{2b}} < c(\mathbf{3.4.29}) \end{aligned}$$

by the absolute continuity relation (3.2.36) between the laws of \mathbf{T} and \mathbf{T}' , by Lemma 3.4.2 (ii) and since $2\beta(\mathfrak{b}-1) > 1$.

We now complete the proof of the lemma. To that end, we denote by $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, a \mathfrak{m}_ν -biased RW on \mathbf{T} starting at $X_0 = \circ$. By definition of branching RW, we first get

$$\mathbf{P}(\exists u \in \tau : Y'_u \in B_{q,\lambda,\beta} | (\mathbf{T}, \circ), \tau) \leq \sum_{u \in \tau} \mathbf{E}[\mathbf{1}_{\{Y'_u \in B_{q,\lambda,\beta}\}} | (\mathbf{T}, \circ), \tau] = \sum_{u \in \tau} P_{\mathbf{T},\circ}(X_{|u|} \in B_{q,\lambda,\beta}).$$

Then, the many-to-one formula (3.2.25) implies

$$\begin{aligned} \mathbf{E}\left[\sum_{u \in \tau} P_{\mathbf{T},\circ}(X_{|u|} \in B_{q,\lambda,\beta}) | (\mathbf{T}, \circ)\right] &= \sum_{n \in \mathbb{N}} P_{\mathbf{T},\circ}(X_n \in B_{q,\lambda,\beta}) = \sum_{x \in B_{q,\lambda,\beta}} G(\circ, x) \\ &= \frac{1}{\mathfrak{m}_\nu} \sum_{y \in \partial L_\circ} \mathbf{1}_{\{|\circ \wedge y| \geq -q\}} Z(\theta_y \mathbf{T}). \end{aligned}$$

Here, we used the fact that the Green function $G(\circ, x)$ is equal to $\frac{\mathfrak{m}_\nu + k_x(\mathbf{T})}{\mathfrak{m}_\nu - 1} \mathfrak{m}_\nu^{-(|x| - |\circ \wedge x|)}$ by (3.2.35). Recall from the definition of invariant and infinite GW-trees that conditionally on ∂L_\circ , the trees $(\theta_y \mathbf{T})_{y \in \partial L_\circ}$ are i.i.d. $\text{GW}(\nu)$ -trees. Therefore,

$$\mathbf{P}(\exists u \in \tau : Y'_u \in B_{q,\lambda,\beta} | \partial L_\circ) \leq \frac{1}{\mathfrak{m}_\nu} \sum_{y \in \partial L_\circ} \mathbf{1}_{\{|\circ \wedge y| \leq q\}} \mathbf{E}[Z(\mathbf{T}_0)] \leq \frac{1}{\mathfrak{m}_\nu} \mathbf{E}[Z(\mathbf{T}_0)] \sum_{0 \leq j \leq q} k_{\circ(j)}(\mathbf{T})$$

which implies the desired result by (3.4.9) and since $\mathbf{E}[k_{\circ(j)}(\mathbf{T})] = \mathfrak{m}_\nu^{-1} \sum_{k \in \mathbb{N}} k^2 \nu(k) < \infty$ if $j \geq 1$ and $\mathbf{E}[k_\circ(\mathbf{T})] = (2\mathfrak{m}_\nu)^{-1} \sum_{k \in \mathbb{N}} k(k + \mathfrak{m}_\nu) \nu(k) < \infty$. \blacksquare

We next get upper bound of the (quenched moments of the) increments of the martingale $(S_{X_n})_{n \in \mathbb{N}}$.

Lemma 3.4.4 *Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . Let $\mathfrak{b}_0 \in (1, \infty)$. We assume that $\sum_{k \in \mathbb{N}} k^{2\mathfrak{b}_0+1} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. Let the r.v. U_x , $x \in \mathbf{T}$, be as in (3.4.2). Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a \mathfrak{m}_ν -biased RW on \mathbf{T} such that $X_0 = \circ$, and recall that $P_{(\mathbf{T}, \circ)}$ stands for the conditional law of $(X_n)_{n \in \mathbb{N}}$ given (\mathbf{T}, \circ) . Then, for all $\mathfrak{b} \in (1, \mathfrak{b}_0)$, there exists a \mathbf{P} -a.s. finite random variable $K_\mathfrak{b}(\mathbf{T})$ that is a measurable function of \mathbf{T} and \mathfrak{b} such that for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}^*$,*

$$\mathbf{P}\text{-a.s.} \quad E_{\mathbf{T},\circ} [|S_{X_{m+n}} - S_{X_m}|^{2\mathfrak{b}}] \leq K_\mathfrak{b}(\mathbf{T}) n^{\mathfrak{b}-1} (m+n). \quad (3.4.10)$$

Proof. Recall the notation $\Delta_k = S_{X_{k+1}} - S_{X_k}$ from Lemma 3.4.1 (iv). Since the Δ_k are increments of martingales, the Burkholder-Davis-Gundy inequality (see e.g. Burkholder [32, Theorem 9 p. 1502]) implies that there is a constant $c(\mathfrak{b}) \in (0, \infty)$ that only depends on \mathfrak{b} such that

$$E_{\mathbf{T},\circ} \left[\max_{1 \leq k \leq n} |S_{X_{m+k}} - S_{X_m}|^{2\mathfrak{b}} \right] \leq c(\mathfrak{b}) E_{\mathbf{T},\circ} \left[\left(\sum_{0 \leq k < n} \Delta_{m+k}^2 \right)^\mathfrak{b} \right].$$

Since $\mathfrak{b} \in (1, \infty)$, $z \mapsto z^\mathfrak{b}$ is convex and $(\frac{1}{n} \sum_{0 \leq k < n} \Delta_{m+k}^2)^\mathfrak{b} \leq \frac{1}{n} \sum_{0 \leq k < n} |\Delta_{m+k}|^{2\mathfrak{b}}$. Then by (3.4.5) in Lemma 3.4.1, we get

$$\begin{aligned} E_{\mathbf{T},\circ} \left[\max_{1 \leq k \leq n} |S_{X_{m+k}} - S_{X_m}|^{2\mathfrak{b}} \right] &\leq c(\mathfrak{b}) n^{\mathfrak{b}-1} \sum_{0 \leq k < n} E_{\mathbf{T},\circ} [|\Delta_{m+k}|^{2\mathfrak{b}}] \\ &\leq c(\mathfrak{b}) n^{\mathfrak{b}-1} \sum_{0 \leq k < n} E_{\mathbf{T},\circ} \left[\frac{\mathfrak{m}_\nu^{2\mathfrak{b}} + \mathfrak{m}_\nu}{\mathfrak{m}_\nu + k_{X_{m+k}}(\mathbf{T})} U_{X_{m+k}}^{2\mathfrak{b}} \right] \\ &\leq c(\mathfrak{b}) (\mathfrak{m}_\nu^{2\mathfrak{b}} + \mathfrak{m}_\nu) n^{\mathfrak{b}-1} (m+n) \left(\frac{1}{m+n} \sum_{0 \leq k < m+n} E_{\mathbf{T},\circ} [F(\mathbf{T}, X_k)] \right) \end{aligned}$$

where we have set $F(T, x) = U(\theta_x T)^{2b} / (m_\nu + k_x(T))$, for all $(T, x) \in \overline{\mathbb{T}}^\bullet$ and where we recall the notation $U(\cdot)$ from (3.4.1). Observe that $F(T, x) = F(\text{cent}(T, x))$. Then, set $b_1 = b_0/b \in (1, \infty)$ and observe that $\mathbf{E}[F(\mathbf{T}, \circ)^{b_1}] \leq \mathbf{E}[U_\circ^{2b_0}] < \infty$ since $\sum_{k \in \mathbb{N}} k^{2b_0+1} \nu(k) < \infty$ by Lemma 3.4.1 (ii). Therefore Corollary 3.2.27 applies and asserts that

$$\mathbf{P}\text{-a.s.} \quad K_b(\mathbf{T}) := c(b) (m_\nu^{2b} + m_\nu) \sup_{n \in \mathbb{N}^*} \frac{1}{n} \sum_{0 \leq k < n} E_{\mathbf{T}, \circ} [F(\mathbf{T}, X_k)] < \infty,$$

which entails the desired result. \blacksquare

We next use a specific form of a conditional Central Limit Theorem for martingales that is used to prove the finite-dimensional convergence of the harmonic coordinates of the branching random walk in Section 3.4.4.

Proposition 3.4.5 *For all $n \in \mathbb{N}$, let $(\mathcal{F}_{n,k})_{k \in \mathbb{N}}$ be a filtration on (Ω, \mathcal{F}) and let $(D_{n,k})_{k \in \mathbb{N}^*}$ be a sequence of real valued r.v. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of integers and let $s \in [0, \infty)$. We make the following assumptions.*

- (a) *For all n, k , $D_{n,k}$ is $\mathcal{F}_{n,k}$ -measurable, $\mathbf{E}[D_{n,k}^2] < \infty$ and \mathbf{P} -a.s. $\mathbf{E}[D_{n,k} | \mathcal{F}_{n,k-1}] = 0$.*
- (b) *There exists a filtration $(\mathcal{G}_j)_{j \in \mathbb{N}}$ on (Ω, \mathcal{F}) and a sequence of integers $(r_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_{n,0} = \mathcal{G}_{r_n}$, $n \in \mathbb{N}$.*
- (c) *For all $\varepsilon \in (0, 1)$, \mathbf{P} -a.s. $\lim_{n \rightarrow \infty} \sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}] = 0$.*
- (d) *\mathbf{P} -a.s. $\lim_{n \rightarrow \infty} \sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^2 | \mathcal{F}_{n,k-1}] = s$.*

Then, we set $M_n = \sum_{1 \leq k \leq s_n} D_{n,k}$ and for all $\lambda \in \mathbb{R}$,

$$\mathbf{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \mathbf{E}[e^{i\lambda M_n} | \mathcal{F}_{n,0}] = e^{-\frac{1}{2}s\lambda^2}. \quad (3.4.11)$$

Proof. See Appendix 3.C. \blacksquare

Remark 3.4.6 In Assumption (b), σ -fields are only partially nested (i.e. $\mathcal{F}_{n,k}$ is not necessarily included in $\mathcal{F}_{n+1,k}$ for all k). However, as mentioned in the remarks following Hall & Heyde [91, Corollary 3.1 pp. 58-59], since the limit in Assumption (d) is deterministic, the conclusion of [91, Corollary 3.1 pp. 58-59] applies and M_n converge in law to a centered Gaussian r.v. with variance s . However, this sole convergence does not imply (3.4.11). A related type of result can be found in Eagleson [66, Corollary 2 p. 560], which entails that \mathbf{P} -a.s. $\lim_{n \rightarrow \infty} \mathbf{E}[e^{i\lambda M_n} | \mathcal{G}] = e^{-\frac{1}{2}s\lambda^2}$ where $\mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_{n,0}$. However, it does not imply (3.4.11) and the methods used in [66] seem difficult to adapt in order to obtain (3.4.11). Since we have not found an accurate reference for Proposition 3.4.5, we provide in Appendix 3.C a proof which is a (relatively straightforward) adaptation of proof of the standard Lindeberg CLT for martingales. \square

Proposition 3.4.7 *Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . Let $b \in (2, \infty)$. We assume that $\sum_{k \in \mathbb{N}} k^b \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $GW(\nu)$ -tree as in Definition 3.2.25. Let the r.v. $U_x, S_x, x \in \mathbf{T}$, be as in (3.4.2) and (3.4.3). Let $\overline{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a m_ν -biased RW on \mathbf{T} such that $X_0 = \circ$. We denote by \mathcal{F}_n the σ -field generated by $(\overline{X}_k)_{0 \leq k \leq n}$. Let $r_n, s_n \in \mathbb{N}$, $n \in \mathbb{N}$ be such that $\frac{1}{n}r_n \rightarrow r \in [0, \infty)$ and $\frac{1}{n}s_n \rightarrow s \in [0, \infty)$. Recall from (3.4.6) the definition of σ_ν . Then for all $\lambda \in \mathbb{R}$,*

$$\mathbf{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \mathbf{E}\left[\exp\left(i\frac{\lambda}{\sqrt{n}}(S_{X_{r_n+s_n}} - S_{X_{r_n}})\right) \middle| \mathcal{F}_{r_n}\right] = e^{-\frac{1}{2}\sigma_\nu^2 \lambda^2}. \quad (3.4.12)$$

Proof. We want to apply Proposition 3.4.5 with $\mathcal{F}_{n,k} = \mathcal{F}_{r_n+k}$ and $D_{n,k} = n^{-1/2} \Delta_{r_n+k-1}$, where as in Lemma 3.4.1 (iv), we have set $\Delta_m = S_{X_{m+1}} - S_{X_m}$. Since ν is square integrable,

(3.4.6) in Lemma 3.4.1 (iv) implies that $\mathbf{E}[D_{n,k}^2] = \frac{1}{n}\sigma_\nu^2 < \infty$ and Lemma 3.4.1 (iii) implies that Assumption (a) of Proposition 3.4.5 is satisfied. Next observe that $\mathcal{F}_{n,0} = \mathcal{F}_{r_n}$: Assumption (b) of Proposition 3.4.5 is satisfied.

We next prove that Assumption (d) in Proposition 3.4.5 is satisfied. Recall notation $\Phi_b(\bar{X}_n)$ from Lemma 3.4.1 (iv) and to simplify notation, we set $C_{b,k} = \sum_{0 < l < k} \Phi_b(\bar{X}_l)$. First observe that $\frac{1}{n}\Phi_2(\bar{X}_{r_n+k-1}) = \mathbf{E}[D_{n,k}^2 | \mathcal{F}_{n,k-1}]$. Since $\Phi_2(\bar{X}_k) = \Phi_2(\text{cent}(\bar{X}_k))$ by (3.4.5) and thanks to Corollary 3.2.27, we get \mathbf{P} -a.s. $\lim_{k \rightarrow \infty} \frac{1}{k}C_{2,k} = \sigma_\nu^2$. Recall that $\frac{1}{n}r_n \rightarrow r \in [0, \infty)$. If $r \in (0, \infty)$, then $\frac{1}{n}C_{2,r_n} = \frac{r_n}{n} \frac{1}{r_n}C_{2,r_n} \rightarrow r\sigma_\nu^2$ a.s. If $r = 0$, since $\Phi_2 \geq 0$, we get for all $\eta \in (0, 1)$, $\frac{1}{n}C_{2,r_n} \leq \frac{r_n + \eta n}{n} \frac{1}{r_n + \eta n}C_{2,r_n + \eta n} \rightarrow \eta\sigma_\nu^2$ a.s. and thus $\frac{1}{n}C_{2,r_n} \rightarrow 0$ a.s. since η can be arbitrarily small. Similarly, a.s. $\frac{1}{n}C_{2,r_n+s_n} \rightarrow (r+s)\sigma_\nu^2$, which then implies that $\sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^2 | \mathcal{F}_{n,k-1}] = \frac{1}{n}(C_{2,r_n+s_n} - C_{2,r_n}) \rightarrow s\sigma_\nu^2$ and Assumption (d) in Proposition 3.4.5 is satisfied.

It remains to prove that Assumption (c) in Proposition 3.4.5 is satisfied. We fix $\varepsilon \in (0, 1)$ and we first observe that

$$\sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}] \leq \frac{1}{\varepsilon^{b-2}} \sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^b | \mathcal{F}_{n,k-1}] \leq \frac{1}{(\varepsilon\sqrt{n})^{b-2}} \frac{1}{n}C_{b,r_n+s_n}$$

which a.s. tends to 0 since $\frac{1}{n}C_{b,r_n+s_n} \rightarrow (r+s)\mathbf{E}[\Phi_b(\bar{X}_0)]$ a.s. by the ergodic theorem (which applies as Corollary 3.2.27 since $\mathbf{E}[\Phi_b(\bar{X}_0)] < \infty$ by Lemma 3.4.1 (iv)). This proves that Assumption (c) in Proposition 3.4.5 is satisfied and (3.4.12) follows from (3.4.11). \blacksquare

To conclude this section, let us recall from Peres & Zeitouni [137] annealed uniform bounds for the (relative) height of m_ν -biased RW on GW-trees which is used later in the proof of Proposition 3.4.14 in Section 3.4.5.

Lemma 3.4.8 *Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . We assume that $\sum_{k \in \mathbb{N}} k^2 \nu(k) < \infty$. Let (\mathbf{T}, \circ) be either a GW(ν)-tree as in Definition 3.2.6 (a) (in this case: $\circ = \emptyset$) or an infinite GW(ν)-tree as in Definition 3.2.18 or an invariant GW(ν)-tree as in Definition 3.2.25. Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a m_ν -biased RW on \mathbf{T} such that $X_0 = \circ$. Then, for all $n, p \in \mathbb{N}^*$, the following holds true.*

- (i) *If \mathbf{T} is a GW(ν)-tree, $\mathbf{P}(\max_{1 \leq k \leq n} |X_k| \geq p) \leq 4ne^{-p^2/2n}$.*
- (ii) *If \mathbf{T} is an invariant GW(ν)-tree, $\mathbf{P}(\max_{1 \leq k \leq n} \|X_k\| \geq p) \leq 8n^3 e^{-(p-1)^2/2n}$.*
- (iii) *If \mathbf{T} is an infinite GW(ν)-tree, $\mathbf{P}(\max_{1 \leq k \leq n} \|X_k\| \geq p) \leq 16n^3 e^{-(p-1)^2/2n}$.*

Proof. For (i), see Peres & Zeitouni [137, Lemma 5 p. 608]. For (ii) and (iii), see [137, Corollary 2 p. 608]. \blacksquare

3.4.2 Statements of the result, overview of its proof

Before stating the main results of the section, let us fix some notations and let us specify our assumptions.

We fix $\alpha \in (1, 2]$ and we fix an offspring distribution μ that satisfies (3.2.60). We denote by τ_∞ a GW(μ)-forest corresponding to a sequence $(\tau^{(k)})_{k \in \mathbb{N}^*}$ of i.i.d. GW(μ)-trees (Definition 3.2.6 (b)). We denote by $V_l(\tau_\infty)$ its Lukasiewicz path, by $(H_s(\tau_\infty))_{s \in [0, \infty)}$ its height process (Definition 3.2.2) and by $(C_s(\tau_\infty))_{s \in [0, \infty)}$ its contour process (Definition 3.2.4). We next recall from (3.2.66) the definition of \mathfrak{g}_n and \mathfrak{d}_n (roughly speaking $(\mathfrak{g}_n, \mathfrak{d}_n)$ is the first excursion interval of $H(\tau_\infty)$ above 0 longer than n). Then we also recall from Theorem 3.2.36 that $(a_n)_{n \in \mathbb{N}^*}$ and $(b_n)_{n \in \mathbb{N}^*}$ are scaling sequences such that $a_n = n/b_n$ and such that the following limit

$$\mathcal{Q}_n := \left(\frac{1}{n}\mathfrak{g}_n, \frac{1}{n}\mathfrak{d}_n, \frac{1}{b_n}V_{[n \cdot]}(\tau_\infty), \frac{1}{a_n}H_n(\tau_\infty), \frac{1}{a_n}C_{2n}(\tau_\infty) \right) \xrightarrow[n \rightarrow \infty]{} (\mathfrak{g}_1, \mathfrak{d}_1, X, H, H) \quad (3.4.13)$$

holds weakly on $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^2$. Here, X is an α -stable spectrally positive Lévy process characterized by (3.2.52), H is its associated height process as defined in (3.2.53) and $(\mathbf{g}_1, \mathbf{d}_1)$ is the first excursion interval of H longer than 1 as defined in Remark 3.2.34.

We next state a limit theorem for snakes associated with τ_∞ -indexed branching RWs that take their values in an invariant $\text{GW}(\nu)$ -tree (Definition 3.2.25). Here ν is a fixed *supercritical* offspring distribution whose mean is denoted by \mathfrak{m}_ν and which shall be subject to various moment assumptions.

Our main objects of interest in this section are branching RWs in invariant environment. More precisely, we denote by (\mathbf{T}, \circ) an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25. We assume that (\mathbf{T}, \circ) is *independent from* τ_∞ and we introduce the sequence of \mathbf{T} -valued branching RWs $\Theta_k = (\tau^{(k)} \emptyset; \bar{Y}_u^{(k)} = (\mathbf{T}, Y_u^{(k)}), u \in \tau^{(k)})$, $k \in \mathbb{N}^*$ such that

$$\text{conditionally on } (\mathbf{T}, \circ) \text{ and } \tau_\infty, \text{ the } (\Theta_k)_{k \in \mathbb{N}^*} \text{ are independent and } \Theta_k \stackrel{(\text{law})}{=} Q_{(\tau^{(k)}, \emptyset)}^{(\mathbf{T}, \circ)}. \quad (3.4.14)$$

Here $Q_{(\tau^{(k)}, \emptyset)}^{(\mathbf{T}, \circ)}$ is as in Definition 3.2.30 with $\lambda = \mathfrak{m}_\nu$. We recall from (3.4.3) the definition of the harmonic coordinates $(S_x)_{x \in \mathbf{T}}$. Then, for all $k \in \mathbb{N}^*$, we associate with Θ_k two \mathbb{R} -valued $\tau^{(k)}$ -indexed branching RWs:

$$M_k := (|Y_u^{(k)}|)_{u \in \tau^{(k)}} \quad \text{and} \quad \mathcal{M}_k := (S_{Y_u^{(k)}})_{u \in \tau^{(k)}},$$

and according to Definition 3.2.8, we introduce the following four $\mathbf{C}^0([0, \infty), \mathbb{R})$ -valued continuous processes:

- $(W_s(\tau_\infty; \cdot))_{s \in [0, \infty)}$ is the contour-snake of the branching random walks $(M^{(k)})_{k \in \mathbb{N}^*}$;
- $(W_s^*(\tau_\infty; \cdot))_{s \in [0, \infty)}$ is the height-snake of the branching random walks $(M^{(k)})_{k \in \mathbb{N}^*}$;
- $(\mathcal{W}_s(\tau_\infty; \cdot))_{s \in [0, \infty)}$ is the contour-snake of the branching random walks $(\mathcal{M}^{(k)})_{k \in \mathbb{N}^*}$;
- $(\mathcal{W}_s^*(\tau_\infty; \cdot))_{s \in [0, \infty)}$ is the height-snake of the branching random walks $(\mathcal{M}^{(k)})_{k \in \mathbb{N}^*}$.

We also recall from Definition 3.2.8 the notations $\widehat{W}_s(\tau_\infty)$, $\widehat{W}_s^*(\tau_\infty)$, $\widehat{\mathcal{W}}_s(\tau_\infty)$ and $\widehat{\mathcal{W}}_s^*(\tau_\infty)$, $s \in [0, \infty)$, for the corresponding endpoint-processes.

Finally, we denote by $(\widehat{W}_s)_{s \in [0, \infty)}$ the endpoint process of the one-dimensional Brownian snake whose lifetime-process is the α -stable height process $(H_s)_{s \in [0, \infty)}$ as introduced in Proposition 3.2.47.

The goal of this section is to prove the following theorem that shows that the contour- and height-snakes in harmonic coordinates and in relative heights jointly converge to W when suitably rescaled. To simplify the proof we first admit several intermediate results, which are stated below and proved in Sections 3.4.3, 3.4.4 and 3.4.5, before proceeding to the proof of Theorem 3.4.9 at the end of the present section.

Theorem 3.4.9 *We fix $\alpha \in (1, 2]$. Let μ satisfy (3.2.60). Let τ_∞ be a $\text{GW}(\mu)$ -forest (Definition 3.2.6 (b)). Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . We assume that there is $\mathfrak{b} \in (\frac{2\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2\mathfrak{b}} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree (Definition 3.2.25) that is independent from τ_∞ . We recall the definition of \mathcal{Q}_n from the convergence (3.4.13). We recall from (3.4.6) the definition of σ_ν . Otherwise, we keep the previous notations. Then, conditionally given (\mathbf{T}, \circ) , the following joint convergence*

$$\begin{aligned} & \left(\mathcal{Q}_n, \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_n^*(\tau_\infty), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{2n}(\tau_\infty), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_n^*(\tau_\infty) \right) \\ & \xrightarrow[n \rightarrow \infty]{} (\mathbf{g}_1, \mathbf{d}_1, X, H, H, \widehat{W}, \widehat{W}, \widehat{W}, \widehat{W}) \end{aligned} \quad (3.4.15)$$

a.s. holds weakly on $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^6$ equipped with the product topology.

Proof. See the end of the section. ■

Overview of the proof of Theorem 3.4.9. To prove Theorem 3.4.9, we first prove the following proposition in Section 3.4.4, which shows that the contour- and heigh-snakes in harmonic coordinates converge to W when suitably rescaled.

Proposition 3.4.10 *We fix $\alpha \in (1, 2]$. Let μ satisfy (3.2.60). Let τ_∞ be a $GW(\mu)$ -forest (Definition 3.2.6 (b)). Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . We assume that there is $b_0 \in (1 + \frac{\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2b_0} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $GW(\nu)$ -tree (Definition 3.2.25) that is independent from τ_∞ . We recall the definition of \mathcal{Q}_n from the convergence (3.4.13). We recall from (3.4.6) the definition of σ_ν . Otherwise, we keep the previous notations. Then, conditionally given (\mathbf{T}, \circ) , the following convergence*

$$\left(\mathcal{Q}_n, \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_n^*(\tau_\infty) \right) \xrightarrow{n \rightarrow \infty} (\mathbf{g}_1, \mathbf{d}_1, X, H, H, \widehat{W}, \widehat{W}) \quad (3.4.16)$$

a.s. holds weakly on $\mathbf{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^4$ equipped with the product topology.

Proof. See Section 3.4.4. ■

To prove Proposition 3.4.10, we first rely on the following lemma, which is a conditional version of Kolmogorov continuity theorem and whose proof is a straightforward adaptation of the standard version of the theorem. To state it let us introduce the following notation for the γ -Hölder factor (here $\gamma \in (0, 1]$) of a function $\mathbf{x}(\cdot) \in \mathbf{C}^0([0, \infty), \mathbb{R})$ on the interval $[0, s_0]$ (here $s_0 \in (0, \infty)$):

$$\text{Hol}_{\gamma, s_0}(\mathbf{x}) = \sup \left\{ \frac{|\mathbf{x}(s) - \mathbf{x}(s')|}{|s - s'|^\gamma}; s, s' \in [0, s_0], s \neq s' \right\} \quad (3.4.17)$$

which may be infinite.

Lemma 3.4.11 *Let $\mathcal{G} \subset \mathcal{F}$ be a σ -field and let $(Z_s)_{s \in [0, \infty)}$ be a \mathbb{R} -valued continuous process. Let $s_0, b \in (1, \infty)$ and $c \in (0, \infty)$. We assume that there is a \mathcal{G} -measurable and a.s. finite r.v. C such that for all $s, s' \in [0, s_0]$,*

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E} \left[|Z_s - Z_{s'}|^b \mid \mathcal{G} \right] \leq C |s - s'|^{1+c}. \quad (3.4.18)$$

Then, for all $\gamma \in (0, \frac{c}{b})$, we get

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E} \left[(\text{Hol}_{\gamma, s_0}(Z))^b \mid \mathcal{G} \right] \leq c_{b, c, \gamma, s_0} C, \quad (3.4.19)$$

where $c_{b, c, \gamma, s_0} = 2^{2b} (2s_0)^{1+c-\gamma b} (1 - 2^{-(\frac{c}{b} + \gamma)})^{-b}$.

Proof: This is easily adapted from e.g. Revuz & Yor [138, Theorem 2.1 p.26]. ■

More precisely, in Proposition 3.4.17 in Section 3.4.3, we first show that the rescaled height process $a_n^{-1} H_n(\tau_\infty)$ satisfies an (unconditioned) assumption of type (3.4.18). Then in Proposition 3.4.18 in Section 3.4.3, we prove that it entails that $a_n^{-1/2} \widehat{W}_n^*(\tau_\infty)$ also satisfies an assumption of type (3.4.18). This implies that conditionally given (\mathbf{T}, \circ) , $a_n^{-1/2} \widehat{W}_n^*(\tau_\infty)$ is tight, thanks to the following general result (the proof given below makes precise what we mean here by conditional tightness).

Lemma 3.4.12 *Let $\mathcal{G} \subset \mathcal{F}$ be a σ -field and let $(Z_s^{(n)})_{s \in [0, \infty)}$, $n \in \mathbb{N}^*$, be a sequence of \mathbb{R} -valued continuous processes such that $Z_0^{(n)} = 0$. Let $\mathfrak{b} \in (1, \infty)$ and $\gamma \in (0, 1]$. Let us assume that for all $p \in \mathbb{N}^*$, there exists a \mathcal{G} -measurable r.v. \mathfrak{C}_p such that for all $n \in \mathbb{N}^*$,*

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E}[(\text{Ho1}_{\gamma, p}(Z^{(n)}))^{\mathfrak{b}} | \mathcal{G}] \leq \mathfrak{C}_p < \infty. \quad (3.4.20)$$

Then conditionally given \mathcal{G} , the laws of the processes $(Z^{(n)})_{n \in \mathbb{N}^*}$ are a.s. tight on $\mathbf{C}^0([0, \infty), \mathbb{R})$.

Proof. We denote by $\mathcal{M}_1(\mathbf{C}_1^0)$ the space of Borel probability measures on $\mathbf{C}_1^0 := \mathbf{C}^0([0, \infty), \mathbb{R})$ equipped with the Polish topology of weak convergence. Denote by $\omega \in \Omega \mapsto Q_n(\omega, \cdot)$ a regular version of the conditional law of $Z^{(n)}$ given \mathcal{G} (which exists since \mathbf{C}_1^0 is Polish). Namely,

- Q_n is a $\mathcal{M}_1(\mathbf{C}_1^0)$ -valued r.v., measurable with respect to \mathcal{G} and the Borel σ -field of $\mathcal{M}_1(\mathbf{C}_1^0)$.
- For all Borel subsets B of \mathbf{C}_1^0 we \mathbf{P} -a.s. have $Q_n(\cdot, B) = \mathbf{E}[\mathbf{1}_{\{Z^{(n)} \in B\}} | \mathcal{G}]$.

Our assumptions imply that there exists $\Omega_0 \in \mathcal{G}$ such that $\mathbf{P}(\Omega_0) = 1$ and such that for all $\omega \in \Omega_0$ and for all $n, p \in \mathbb{N}^*$,

$$\mathbf{E}[(\text{Ho1}_{\gamma, p}(Z^{(n)}))^{\mathfrak{b}} | \mathcal{G}](\omega) = \int_{\mathbf{C}_1^0} (\text{Ho1}_{\gamma, p}(\mathbf{x}))^{\mathfrak{b}} Q_n(\omega, d\mathbf{x}) \leq \mathfrak{C}_p(\omega) < \infty.$$

For all $\mathbf{x}(\cdot) \in \mathbf{C}_1^0$ and $\delta \in (0, \infty)$, we set $w_\delta(\mathbf{x}, p) = \max \{ |\mathbf{x}(s) - \mathbf{x}(s')| ; s, s' \in [0, p] : |s - s'| \leq \delta \}$, which is the δ -modulus of continuity of \mathbf{x} on $[0, p]$. Then, we get for all $\omega \in \Omega_0$, all $p, n \in \mathbb{N}^*$ and all $\delta, \eta \in (0, \infty)$,

$$Q_n(\omega, \{ \mathbf{x} \in \mathbf{C}_1^0 : w_\delta(\mathbf{x}, p) > \eta \}) \leq \frac{\delta^{\gamma \mathfrak{b}}}{\eta^{\mathfrak{b}}} \int_{\mathbf{C}_1^0} (\text{Ho1}_{\gamma, p}(\mathbf{x}))^{\mathfrak{b}} Q_n(\omega, d\mathbf{x}) \leq \mathfrak{C}_p(\omega) \frac{\delta^{\gamma \mathfrak{b}}}{\eta^{\mathfrak{b}}},$$

which implies $\lim_{\delta \rightarrow 0^+} \limsup_{n \rightarrow \infty} Q_n(\omega, \{ \mathbf{x} \in \mathbf{C}_1^0 : w_\delta(\mathbf{x}, p) > \eta \}) = 0$. Since $Z_0^{(n)} = 0$, a standard result (see e.g. Billingsley [22, Theorem 7.3 p. 82]) entails that for all $\omega \in \Omega_0$, the laws $(Q_n(\omega, \cdot))_{n \in \mathbb{N}^*}$ are tight. \blacksquare

In Section 3.4.4, we prove that $a_n^{-1} H_n(\tau_\infty)$ and $\sigma_\nu^{-1} a_n^{-1/2} \widehat{\mathcal{W}}_{n \cdot}^*(\tau_\infty)$ jointly converge in law conditionally given (\mathbf{T}, \circ) to (H, \widehat{W}) . To do so, we compute the conditional Fourier transforms of finite-dimensional marginals of the height-snake in harmonic coordinates and we use Proposition 3.4.7.

We next derive the convergence (3.4.16) using the fact that $C(\tau_\infty)$ and $\widehat{W}(\tau_\infty)$ are derived from $H(\tau_\infty)$ and $\widehat{W}^*(\tau_\infty)$ by a time-change. More precisely, recall from Remark 3.2.5 (d) the definition of the increasing continuous bijection $\phi_{\tau_\infty} : [0, \infty) \rightarrow [0, \infty)$ such that $H_s(\tau_\infty) = C_{\phi_{\tau_\infty}(s)}(\tau_\infty)$, $s \in [0, \infty)$. For all $s_0 \in (0, \infty)$, (3.2.9) implies that $\max_{s \in [0, s_0]} |\frac{1}{2n} \phi_{\tau_\infty}(ns) - s| \rightarrow 0$ in probability because (3.4.13) implies that $\frac{1}{n} \max_{s \in [0, s_0]} H_{ns}(\tau_\infty) \rightarrow 0$ in \mathbf{P} -probability since $a_n/n \rightarrow 0$. Therefore, if for all $s \in [0, \infty)$ we set $\varphi_n(s) = \frac{1}{n} \phi_{\tau_\infty}^{-1}(2ns)$ (here, $\phi_{\tau_\infty}^{-1}$ stands for the inverse of ϕ_{τ_∞}), then we have proved

$$\forall s_0 \in (0, \infty), \quad \lim_{n \rightarrow \infty} \max_{s \in (0, s_0)} |\varphi_n(s) - s| = 0 \quad \text{in } \mathbf{P}\text{-probability.} \quad (3.4.21)$$

We derive the conditional joint convergence (3.4.16)

- from the conditional joint convergence of $a_n^{-1} H_n(\tau_\infty)$ and $\sigma_\nu^{-1} a_n^{-1/2} \widehat{\mathcal{W}}_{n \cdot}^*(\tau_\infty)$,
- from the following equalities that holds for all $s \in [0, \infty)$

$$\frac{1}{a_n} C_{2ns}(\tau_\infty) = \frac{1}{a_n} H_{n\varphi_n(s)}(\tau_\infty) \quad \text{and} \quad \frac{1}{\sqrt{a_n}} \widehat{W}_{2ns}(\tau_\infty) = \frac{1}{\sqrt{a_n}} \widehat{W}_{n\varphi_n(s)}^*(\tau_\infty), \quad (3.4.22)$$

- from the following deterministic lemma.

Lemma 3.4.13 *Suppose that $\mathbf{x}_n \rightarrow \mathbf{x}$ in $\mathbf{C}^0([0, \infty), \mathbb{R}^q)$ and that $\varphi_n \rightarrow \varphi$ in $\mathbf{C}^0([0, \infty), [0, \infty))$. Assume that $\varphi_n(0) = 0$ and that φ_n is nondecreasing. Then, $\mathbf{x}_n \circ \varphi_n \rightarrow \mathbf{x} \circ \varphi$ in $\mathbf{C}^0([0, \infty), \mathbb{R}^q)$.*

Proof. See Whitt [149, Theorem 3.1, p. 75]. ■

In Section 3.4.5, we next prove the following proposition.

Proposition 3.4.14 *We fix $\alpha \in (1, 2]$. Let μ satisfy (3.2.60). Let τ_∞ be a $GW(\mu)$ -forest (Definition 3.2.6 (b)). Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . We assume that there is $\mathfrak{b} \in (\frac{2\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2\mathfrak{b}} \nu(k) < \infty$. We keep the notations of Proposition 3.4.10. Let $s_0 \in (0, \infty)$. Then there are two constants $c, \varepsilon_1 \in (0, \infty)$, which only depend on μ, ν, \mathfrak{b} and s_0 , such that*

$$\mathbf{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \mathbf{P} \left(\max_{s \in [0, s_0]} \left| \frac{1}{\sigma_\nu^2} \widehat{W}_{ns}^*(\tau_\infty) - \widehat{W}_{ns}^*(\tau_\infty) \right| > cn^{-\varepsilon_1} \sqrt{a_n} \mid (\mathbf{T}, \circ) \right) = 0. \quad (3.4.23)$$

Proof. See Section 3.4.5. ■

Proof of Theorem 3.4.9. We admit Propositions 3.4.10 and 3.4.14 and we prove Theorem 3.4.9. First note that the moment assumption on ν is stronger in Proposition 3.4.14 (and Theorem 3.4.9) than in Proposition 3.4.10 since $\frac{2\alpha}{\alpha-1} > 1 + \frac{\alpha}{\alpha-1}$. Thus under the assumptions of Theorem 3.4.9, both propositions apply and they easily entail that conditionally given (\mathbf{T}, \circ) , the following convergence

$$\left(\mathcal{Q}_n, \frac{1}{a_n} H_n(\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_n^*(\tau_\infty), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_n^*(\tau_\infty) \right) \xrightarrow{n \rightarrow \infty} (\mathbf{g}_1, \mathbf{d}_1, X, H, H, H, \widehat{W}, \widehat{W}) \quad (3.4.24)$$

holds weakly on $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^5$ equipped with the product topology. Then recall (3.4.21), (3.4.22) and recall (by definition of height-snake) that for all $s \in [0, \infty)$,

$$\frac{1}{a_n} C_{2ns}(\tau_\infty) = \frac{1}{a_n} H_{n\varphi_n(s)}(\tau_\infty) \quad \text{and} \quad \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{2ns}(\tau_\infty) = \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{n\varphi_n(s)}^*(\tau_\infty).$$

This entails (3.4.15) by (3.4.24) and Lemma 3.4.13, which ends the proof of Theorem 3.4.9. ■

3.4.3 Quenched tightness of the height-snake of the τ_∞ -indexed branching RWs

We keep the notations and the general assumptions of the previous section. We first prove Hölder estimates for $H(\tau_\infty)$ by adapting to the discrete setting the proof of D. & Le Gall [61, Lemma 1.4.6] which provides a similar result for the continuous height process.

To that end, we use the fact that the Lukasiewicz path $V(\tau_\infty)$ of τ_∞ is a RW whose jump law is $\tilde{\mu}(k) = \mu(k+1)$, $k \in \mathbb{N} \cup \{-1\}$ (Remark 3.2.7 (b)) and we use (3.2.4) which shows that $H_n(\tau_\infty)$ counts the number of times the time-reversed RW $(V_n(\tau_\infty) - V_{n-k}(\tau_\infty))_{0 \leq k \leq n}$ reaches its supremum. Namely, the height process is related to the so-called *weak ascending ladder variables* of $V(\tau_\infty)$ whose definition and basic properties are recalled from Spitzer [144] below.

Let $(V_n)_{n \in \mathbb{N}}$ be a random walk whose jump law is $\tilde{\mu}(k) = \mu(k+1)$, $k \in \mathbb{N} \cup \{-1\}$ (with μ as above). For all $n \in \mathbb{N}$, we set

$$\bar{V}_n = \max_{0 \leq k \leq n} V_k \quad \text{and} \quad L_n = \#\{k \in \{1, \dots, n\} : V_k = \bar{V}_k\}, \quad (3.4.25)$$

with the convention that $L_0 = 0$. The weak ascending ladder variables are then given by

$$\mathcal{L}_n = \inf\{k \in \mathbb{N} : L_k = n\} \quad \text{and} \quad \mathcal{V}_n = V_{\mathcal{L}_n} = \bar{V}_{\mathcal{L}_n}. \quad (3.4.26)$$

We also recall from (3.2.75) that $T_{-n} = \inf\{k \in \mathbb{N} : V_k = -n\}$. We recall from (3.2.64) that g_μ is the generating function of μ and we recall from Lemma 3.2.38 that φ_μ is the generating function of T_{-1} . Then $(\mathcal{L}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ is a \mathbb{N}^2 -valued random walk whose jump law is characterized by

$$\forall r, s \in [0, 1], \quad 1 - \mathbf{E}[r^{\mathcal{L}_1} s^{\mathcal{V}_1}] = \frac{s - r g_\mu(s)}{s - \varphi_\mu(r)}. \quad (3.4.27)$$

For a proof we refer to Spitzer [144], Chapter IV, P5 (b) p. 181 and more precisely to (3) p. 187 in the specific case of left-continuous random walks (with Spitzer's notations, $P = g_\mu$ and $r = \varphi_\mu$). We next provide estimates on the Laplace exponents of \mathcal{L}_1 , \mathcal{V}_1 and T_{-1} .

Lemma 3.4.15 *We keep the above notations. We fix $\alpha \in (1, 2]$ and we assume that μ satisfies (3.2.60). For all $\lambda \in [0, \infty)$, we set $\Lambda_0(\lambda) = -\log \mathbf{E}[e^{-\lambda \mathcal{L}_1}]$, $\Lambda_1(\lambda) = -\log \varphi_\mu(e^{-\lambda})$ and $\Lambda_2(\lambda) = -\log \mathbf{E}[e^{-\lambda \mathcal{V}_1}]$. Then there are $c_\mu, c'_\mu \in (0, \infty)$ such that*

$$\Lambda_0(\lambda) \sim_{0+} c_\mu \lambda^{\frac{\alpha-1}{\alpha}} L\left(\frac{1}{\lambda}\right) \quad \text{and} \quad \Lambda_2(\Lambda_1(\lambda)) \sim_{0+} c'_\mu \lambda^{\frac{\alpha-1}{\alpha}} L\left(\frac{1}{\lambda}\right) \quad (3.4.28)$$

where L is the slowly varying function appearing in (3.2.62).

Proof. By (3.4.27), $1 - \mathbf{E}[e^{-\lambda \mathcal{L}_1}] = (1 - e^{-\lambda}) / (1 - \varphi_\mu(e^{-\lambda}))$. Then the first equivalence in (3.4.28) follows from (3.2.76) in Lemma 3.2.38. Next, we deduce from (3.4.27) that $1 - \mathbf{E}[e^{-\lambda \mathcal{V}_1}] = \psi_\mu(s_\lambda) / s_\lambda$, where we recall ψ_μ from (3.2.64) and where we have set $s_\lambda = 1 - e^{-\lambda}$. Then, $\Lambda_2(\lambda) \sim_{0+} c'_\mu \lambda^{\alpha-1} L^*(1/\lambda)$ where L^* is the slowly varying function in (3.2.63). Then (3.2.76) in Lemma 3.2.38 entails that

$$\Lambda_2(\Lambda_1(\lambda)) \sim_{0+} c_1 \lambda^{\frac{\alpha-1}{\alpha}} L\left(\frac{1}{\lambda}\right)^{-(\alpha-1)} L^*\left(c_2 \lambda^{-\frac{1}{\alpha}} L\left(\frac{1}{\lambda}\right)\right),$$

where $c_1, c_2 \in (0, \infty)$ are two constants which only depend on α . This implies the second equivalent in (3.4.28) by (3.2.65), which relates L and L^* . \blacksquare

Proposition 3.4.16 *We keep the above notations. We assume that μ is non-trivial and critical. Let $\mathfrak{b} \in (1, \infty)$. Then for all $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$,*

$$\begin{aligned} \mathbf{E}\left[|H_{m+n}(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty)|^{\mathfrak{b}}\right] &\leq e \mathfrak{b} \Gamma_{\mathfrak{e}}(\mathfrak{b}) (\Lambda_0(\frac{1}{n}))^{-\mathfrak{b}} \quad \text{and} \\ \mathbf{E}\left[|H_m(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty)|^{\mathfrak{b}}\right] &\leq e \mathfrak{b} \Gamma_{\mathfrak{e}}(\mathfrak{b}) (\Lambda_2(\Lambda_1(\frac{1}{n})))^{-\mathfrak{b}}. \end{aligned} \quad (3.4.29)$$

Proof. We prove the first inequality in (3.4.29). Without loss of generality, we assume that V is a RW whose jump law is $\tilde{\mu}$ such that $V_k = V_{m+n}(\tau_\infty) - V_{m+n-k}(\tau_\infty)$, $0 \leq k \leq m+n$. Recall from (3.4.25) the notation L . We easily deduce from (3.2.4) that $L_n = H_{m+n}(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty)$. Thus,

$$\begin{aligned} \mathbf{E}\left[|H_{m+n}(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty)|^{\mathfrak{b}}\right] &= \int_0^\infty \mathfrak{b} s^{\mathfrak{b}-1} \mathbf{P}(L_n > s) ds \leq \int_0^\infty \mathfrak{b} s^{\mathfrak{b}-1} \mathbf{P}(\mathcal{L}_{\lceil s \rceil} \leq n) ds \\ &\leq \int_0^\infty \mathfrak{b} s^{\mathfrak{b}-1} e \cdot \mathbf{E}[\exp(-\frac{1}{n} \mathcal{L}_{\lceil s \rceil})] ds \\ &\leq \int_0^\infty \mathfrak{b} s^{\mathfrak{b}-1} e \cdot e^{-s \Lambda_0(\frac{1}{n})} ds, \end{aligned}$$

which implies the desired result. Here, we use Markov inequality at the second line and the fact that \mathcal{L} is a nonnegative RW whose Laplace exponent is Λ_0 .

The proof of the second upper bound in (3.4.29) is slightly more involved. To simplify notation we set $I_{k,l} = \min_{\llbracket k,l \rrbracket} V(\tau_\infty)$, for all integers $l \geq k \geq 0$. Denote by m' the smallest integer $k \in \llbracket m, m+n \rrbracket$ such that $V_k(\tau_\infty) = I_{m,m+n}$. We deduce from (3.2.4) that

$$\inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty) = H_{m'}(\tau_\infty) = \#\{l \in \llbracket 0, m-1 \rrbracket : V_l(\tau_\infty) = I_{l,m} \text{ and } I_{l,m} \leq I_{m,m+n}\}.$$

We now assume that V is such that $V_k = V_m(\tau_\infty) - V_{m-k}(\tau_\infty)$, $0 \leq k \leq m$ and we set $V' = (V_{m+k}(\tau_\infty) - V_m(\tau_\infty))_{k \in \mathbb{N}}$ that is an independent copy of V . We set $I'_n = \min_{\llbracket 0, n \rrbracket} V'$. Then, (3.2.4) implies that

$$H_m(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty) = \#\{k \in \llbracket 1, m \rrbracket : V_k = \bar{V}_k < -I'_n\} = L_{m \wedge (\bar{V}_{-I'_n}^{-1} - 1)},$$

where we have set $\bar{V}_k^{-1} = \inf\{l \in \mathbb{N} : \bar{V}_l \geq k\}$. We then set $T'_{-k} = \inf\{l \in \mathbb{N} : V'_l = -k\}$ and we deduce from the previous inequality that for all integer $k \in \mathbb{N}^*$,

$$\begin{aligned} \mathbf{P}(H_m(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty) \geq k) &\leq \mathbf{P}(L_{\bar{V}_{-I'_n}^{-1} - 1} \geq k) = \mathbf{P}(-I'_n > \mathcal{V}_k) \leq \mathbf{P}(T'_{-\mathcal{V}_k} \leq n) \\ &\leq e \cdot \mathbf{E}[\exp(-\frac{1}{n} T'_{-\mathcal{V}_k})] = e \cdot \exp(-k \Lambda_2(\Lambda_1(\frac{1}{n}))), \end{aligned}$$

by the Markov inequality. It entails the following,

$$\mathbf{E}[|H_m(\tau_\infty) - \inf_{\llbracket m, m+n \rrbracket} H(\tau_\infty)|^b] \leq e \int_0^\infty \mathbf{b} s^{\mathbf{b}-1} e^{-\lceil s \rceil \Lambda_2(\Lambda_1(\frac{1}{n}))} ds,$$

which easily implies the second upper bound in (3.4.29). \blacksquare

Proposition 3.4.17 *We fix $\alpha \in (1, 2]$ and we assume that μ satisfies (3.2.60). Recall from (3.2.67) the definition of $(a_n)_{n \in \mathbb{N}^*}$. Let $\mathbf{b} \in (1, \infty)$ and $s_0 \in (1, \infty)$ and let $\varepsilon \in (0, 1 - \frac{1}{\alpha})$. Then there exists $c \in (1, \infty)$ that only depends on μ, \mathbf{b}, s_0 and ε such that*

$$\forall n \in \mathbb{N}^*, \forall s, s' \in [0, s_0], \quad \mathbf{E}\left[|\frac{1}{a_n} H_{ns'}(\tau_\infty) - \frac{1}{a_n} H_{ns}(\tau_\infty)|^b\right] \leq c |s' - s|^{\mathbf{b}(1 - \frac{1}{\alpha} - \varepsilon)}. \quad (3.4.30)$$

Moreover, for all $\gamma \in (0, 1 - \frac{1}{\alpha})$, there exists $c' \in (1, \infty)$ that only depends on μ, \mathbf{b}, s_0 and γ such that

$$\sup_{n \in \mathbb{N}^*} \mathbf{E}\left[\left(\text{Ho}1_{\gamma, s_0}\left(\frac{1}{a_n} H_n(\tau_\infty)\right)\right)^b\right] \leq c', \quad (3.4.31)$$

where we recall from (3.4.17) the definition of $\text{Ho}1_{\gamma, s_0}(\mathbf{x}(\cdot))$, for all $\mathbf{x} \in \mathbf{C}^0([0, \infty), \mathbb{R})$. In particular it implies that

$$\sup_{n \in \mathbb{N}^*} \mathbf{E}\left[\left(\max_{s \in [0, s_0]} \frac{1}{a_n} H_{ns}(\tau_\infty)\right)^b\right] \leq c' s_0^\gamma. \quad (3.4.32)$$

Proof. We set $\mathbf{a}(x) = x^{1 - \frac{1}{\alpha}} / L(x)$ where L is as in (3.2.62). By convenience, we set $\mathbf{a}(0) = 0$. Thus $\mathbf{a}(n) = a_n$ by (3.2.67). To simplify notation, we set $\mathcal{H}_s = H_s(\tau_\infty)$, $s \in [0, \infty)$. By (3.4.29) and Lemma 3.4.15, there exists a constant $c_1(\mu, \mathbf{b}) \in (1, \infty)$ (that only depends on μ and \mathbf{b}) such that for all integers $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$,

$$\mathbf{E}\left[|\mathcal{H}_{m+n} - \inf_{k \in \llbracket m, m+n \rrbracket} \mathcal{H}_k|^b\right] \vee \mathbf{E}\left[|\mathcal{H}_m - \inf_{k \in \llbracket m, m+n \rrbracket} \mathcal{H}_k|^b\right] \leq c_1(\mu, \mathbf{b}) \mathbf{a}(n)^b. \quad (3.4.33)$$

We first prove an upper bound for $|\mathcal{H}_{s'} - \mathcal{H}_s|$ that is a simple consequence of the specific interpolation explained by (3.2.2) in Definition 3.2.2 (a) and that is used when $|s - s'|$ is small: there exists $c_2(\mu, \mathbf{b}) \in (1, \infty)$ (that only depends on μ and \mathbf{b}) such that for all $s, s' \in [0, \infty)$,

$$\mathbf{E}\left[|\mathcal{H}_{s'} - \mathcal{H}_s|^b\right] \leq c_2(\mu, \mathbf{b}) |s' - s|^b. \quad (3.4.34)$$

Proof. Let us suppose first that there is $m \in \mathbb{N}$ such that $\frac{1}{2}m \leq s \leq s' \leq \frac{1}{2}(m+1)$, and set $n = \lfloor \frac{1}{2}m \rfloor$. Then, (3.2.2) implies that $|\mathcal{H}_{s'} - \mathcal{H}_s| = 2(s'-s)|\mathcal{H}_{\frac{m}{2}} - \mathcal{H}_{\frac{m+1}{2}}| \leq 2(s'-s)(1 + \mathcal{H}_n - \inf_{k \in \llbracket n, n+1 \rrbracket} \mathcal{H}_k)$. By (3.4.33), we get $\|\mathcal{H}_{s'} - \mathcal{H}_s\|_{\mathbf{b}} \leq c(s'-s)$ where $c := 2 + 2c_1(\mu, \mathbf{b})^{1/\mathbf{b}}\mathbf{a}(1)$. If $\lfloor 2s \rfloor < \lfloor 2s' \rfloor$, then we get

$$\begin{aligned} \|\mathcal{H}_{s'} - \mathcal{H}_s\|_{\mathbf{b}} &\leq \|\mathcal{H}_s - \mathcal{H}_{\frac{1}{2}\lfloor 2s \rfloor}\|_{\mathbf{b}} + \|\mathcal{H}_{s'} - \mathcal{H}_{\frac{1}{2}\lfloor 2s' \rfloor}\|_{\mathbf{b}} + \sum_{\lfloor 2s \rfloor \leq k < \lfloor 2s' \rfloor} \|\mathcal{H}_{\frac{1}{2}(k+1)} - \mathcal{H}_{\frac{1}{2}k}\|_{\mathbf{b}} \\ &\leq c\left(\frac{1}{2}\lfloor 2s \rfloor - s + s' - \frac{1}{2}\lfloor 2s' \rfloor + \frac{1}{2}\lfloor 2s' \rfloor - \frac{1}{2}\lfloor 2s \rfloor\right) = c(s'-s), \end{aligned}$$

which entails (3.4.34). \square

Recall the convention $\mathbf{a}(0) = 0$. We next prove the following bound: there exists $c_3(\mu, \mathbf{b}) \in (1, \infty)$ (that only depends on μ and \mathbf{b}) such that for all real numbers $s' \geq s \geq 0$,

$$\mathbf{E}\left[|\mathcal{H}_{s'} - \mathcal{H}_s|^{\mathbf{b}}\right] \leq c_3(\mu, \mathbf{b})(1 + \mathbf{a}(\lfloor s' \rfloor - \lfloor s \rfloor))^{\mathbf{b}}. \quad (3.4.35)$$

Proof. By (3.4.33) and (3.4.34), we get

$$\begin{aligned} \|\mathcal{H}_{s'} - \mathcal{H}_s\|_{\mathbf{b}} &\leq \|\mathcal{H}_s - \mathcal{H}_{\lfloor s \rfloor}\|_{\mathbf{b}} + \|\mathcal{H}_{s'} - \mathcal{H}_{\lfloor s' \rfloor}\|_{\mathbf{b}} + \|\mathcal{H}_{\lfloor s \rfloor} - \inf_{k \in \llbracket \lfloor s \rfloor, \lfloor s' \rfloor \rrbracket} \mathcal{H}_k\|_{\mathbf{b}} + \|\mathcal{H}_{\lfloor s' \rfloor} - \inf_{k \in \llbracket \lfloor s \rfloor, \lfloor s' \rfloor \rrbracket} \mathcal{H}_k\|_{\mathbf{b}} \\ &\leq 2c_2(\mu, \mathbf{b})^{1/\mathbf{b}} + 2c_1(\mu, \mathbf{b})^{1/\mathbf{b}}\mathbf{a}(\lfloor s' \rfloor - \lfloor s \rfloor), \end{aligned}$$

which easily implies (3.4.35). \square

We next compare $\mathbf{a}(\cdot)$ to power functions thanks to Potter's bounds. Let s_0 and ε as in the statement of the proposition. There exist $n_0 \in \mathbb{N}^*$ and $c_4 \in (1, \infty)$ that only depend on μ, ε and s_0 such that for all integer $n \geq n_0$ and all real number $s \in [\frac{n_0}{n}, s_0]$,

$$\frac{1}{c_4}s^\varepsilon \leq \frac{L(n)}{L(ns)} \leq c_4s^{-\varepsilon} \quad \text{and thus} \quad \frac{\mathbf{a}(ns)}{\mathbf{a}(n)} \leq c_4s^{1-\frac{1}{\alpha}-\varepsilon}. \quad (3.4.36)$$

Proof. By Potter's bounds (see e.g. Bingham, Goldies & Teugel [23, Theorem 1.5.6]), there exists $n_0 \in \mathbb{N}^*$ that depend on L, ε and s_0 such that for all real numbers $y \geq x \geq n_0$, $L(x)/L(y)$ and $L(y)/L(x)$ belong to $[\frac{1}{2}(y/x)^{-\varepsilon}, 2(y/x)^\varepsilon]$. We fix $n \geq n_0$. If $s \in [\frac{n_0}{n}, 1]$, then $L(n)/L(ns) \in [\frac{1}{2}s^\varepsilon, 2s^{-\varepsilon}]$. If $s \in [1, s_0]$, then $L(n)/L(ns) \in [\frac{1}{2}s^{-\varepsilon}, 2s^\varepsilon]$ and we observe that $2s^\varepsilon \leq 2s_0^{2\varepsilon}s^{-\varepsilon}$. This implies (3.4.36) by taking $c_4 = 2s_0^{2\varepsilon}$. \square

We now complete the proof of (3.4.30): we fix $s, s' \in [0, s_0]$ such that $s' \geq s$ and we suppose that $n \geq n_0$ without loss of generality. We first assume that $s' - s \leq \frac{n_0}{n}$. In this case we use (3.4.34) to get $\|\frac{1}{a_n}\mathcal{H}_{ns'} - \frac{1}{a_n}\mathcal{H}_{ns}\|_{\mathbf{b}} \leq c_2(\mu, \mathbf{b})^{1/\mathbf{b}}n(s'-s)/a_n$. Now observe that

$$\begin{aligned} \frac{n}{a_n}|s' - s| &\leq n^{\frac{1}{\alpha}}L(n)\left(\frac{n_0}{n}\right)^{\frac{1}{\alpha}+\varepsilon}|s' - s|^{1-\frac{1}{\alpha}-\varepsilon} \leq n^{\frac{1}{\alpha}}c_4L(n_0)\left(\frac{n_0}{n}\right)^{-\varepsilon}\left(\frac{n_0}{n}\right)^{\frac{1}{\alpha}+\varepsilon}|s' - s|^{1-\frac{1}{\alpha}-\varepsilon} \\ &= c_4L(n_0)n_0^{\frac{1}{\alpha}}|s' - s|^{1-\frac{1}{\alpha}-\varepsilon}, \end{aligned}$$

which implies (3.4.30) with $c = c_2(\mu, \mathbf{b})(c_4L(n_0)n_0^{1/\alpha})^{\mathbf{b}}$.

We next assume that $s' - s \geq \frac{n_0}{n}$. It easily implies that $\lfloor ns' \rfloor - \lfloor ns \rfloor \geq n_0$. Since $\lim_{n \rightarrow \infty} \mathbf{a}(n) = \infty$, $\sup_{n \geq n_0} \mathbf{a}(n)^{-1} =: c_5 < \infty$. Thus by (3.4.36), it is easy to check the following.

$$\begin{aligned} \frac{1 + \mathbf{a}(\lfloor ns' \rfloor - \lfloor ns \rfloor)}{\mathbf{a}(n)} &\leq (1 + c_5) \frac{\mathbf{a}(\lfloor ns' \rfloor - \lfloor ns \rfloor)}{\mathbf{a}(n)} \leq (1 + c_5)c_4(s' - s + \frac{1}{n})^{1-\frac{1}{\alpha}-\varepsilon} \\ &\leq 2^{1-\frac{1}{\alpha}-\varepsilon}(1 + c_5)c_4(s' - s)^{1-\frac{1}{\alpha}-\varepsilon}. \end{aligned}$$

Thanks to (3.4.35), we get (3.4.30) with $c = c_3(\mu, \mathbf{b})(2^{1-\frac{1}{\alpha}-\varepsilon}(1+c_5)c_4)^{\mathbf{b}}$. This easily completes the proof of (3.4.30) by taking c equal to the largest of the two constants. Then (3.4.31) is a direct consequence of Lemma 3.4.11 when $\gamma < 1 - \frac{1}{\alpha} - \frac{1}{\mathbf{b}}$. When $\gamma \in [1 - \frac{1}{\alpha} - \frac{1}{\mathbf{b}}, 1 - \frac{1}{\alpha})$, we use the elementary inequality $x^{\mathbf{b}} \leq 1 + x^{2/(1-\frac{1}{\alpha}-\gamma)}$, $x \geq 0$, to get (3.4.31) anyway. Then, (3.4.32) is an easy consequence from (3.4.31). \blacksquare

Proposition 3.4.18 *We fix $\alpha \in (1, 2]$ and we assume that μ satisfies (3.2.60). Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . Let $\mathbf{b}_0 \in (1 + \frac{\alpha}{\alpha-1}, \infty)$. We assume that $\sum_{k \in \mathbb{N}} k^{1+2\mathbf{b}_0} \nu(k) < \infty$. Let $\mathbf{b} \in (1 + \frac{\alpha}{\alpha-1}, \mathbf{b}_0)$. Recall from (3.2.67) the definition of $(a_n)_{n \in \mathbb{N}^*}$. Let $s_0 \in (1, \infty)$ and let γ be a positive real number such that $\gamma < \frac{1}{2}(1 - \frac{1}{\alpha})(1 - \frac{1}{\mathbf{b}}) - \frac{1}{2\mathbf{b}}$ (there are such ones since $\mathbf{b} > 1 + \frac{\alpha}{\alpha-1}$). Then there exists $c \in (1, \infty)$ that only depends on $\mu, \nu, \mathbf{b}, s_0$ and γ such that for all $n \in \mathbb{N}^*$ and for $s, s' \in [0, s_0]$,*

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E} \left[\left| \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns'}^*(\tau_\infty) - \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns}^*(\tau_\infty) \right|^{2\mathbf{b}} | (\mathbf{T}, \mathbf{o}) \right] \leq c K_{\mathbf{b}}(\mathbf{T}) |s' - s|^{1+2\mathbf{b}\gamma}. \quad (3.4.37)$$

where $K_{\mathbf{b}}(\mathbf{T})$ is the \mathbf{P} -a.s. finite and (\mathbf{T}, \mathbf{o}) -measurable r.v. defined in Proposition 3.4.4. Moreover, conditionally given (\mathbf{T}, \mathbf{o}) , the laws of the processes $(\frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns}^*(\tau_\infty))_{s \in [0, \infty)}$, $n \in \mathbb{N}$, are tight on $\mathcal{C}^0([0, \infty), \mathbb{R})$.

Proof. To simplify the notations, we denote by \mathcal{G} the σ -field generated by (\mathbf{T}, \mathbf{o}) , and by \mathcal{G}' the σ -field generated by $((\mathbf{T}, \mathbf{o}), \tau_\infty)$. We denote by C the contour process $C(\tau_\infty)$ and by \mathcal{H} the height process $H(\tau_\infty)$ and we also drop τ_∞ in the contour-snake \mathcal{W} and the height-snake \mathcal{W}^* . Although the proposition concerns the height-snake, it is easier to consider first the endpoint process of contour-snake $\widehat{\mathcal{W}}$ conditionally given \mathcal{G}' .

Let us fix integers $0 \leq k \leq l \leq n_0$. We denote by j the smallest integer of $[[k, l]]$ such that $C_j = \inf_{s \in [k, l]} C_s$. By definition of branching RWs and by definition of the endpoint process of the contour-snake $\widehat{\mathcal{W}}$, conditionally given \mathcal{G}' , $\widehat{\mathcal{W}}_k - \widehat{\mathcal{W}}_j$ (resp. $\widehat{\mathcal{W}}_l - \widehat{\mathcal{W}}_j$) is distributed as $S_{X_{C_k}} - S_{X_{C_j}}$ (resp. $S_{X_{C_l}} - S_{X_{C_j}}$) where $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, stands for a m_ν -biased RW on \mathbf{T} starting at $X_0 = \mathbf{o}$. Lemma 3.4.4 applies and we \mathbf{P} -a.s. get

$$\begin{aligned} \mathbf{E} [|\widehat{\mathcal{W}}_l - \widehat{\mathcal{W}}_k|^{2\mathbf{b}} | \mathcal{G}'] &\leq 2^{2\mathbf{b}-1} (\mathbf{E} [|\widehat{\mathcal{W}}_k - \widehat{\mathcal{W}}_j|^{2\mathbf{b}} | \mathcal{G}'] + \mathbf{E} [|\widehat{\mathcal{W}}_l - \widehat{\mathcal{W}}_j|^{2\mathbf{b}} | \mathcal{G}']) \\ &\leq 2^{2\mathbf{b}-1} K_{\mathbf{b}}(\mathbf{T}) (C_k |C_k - C_j|^{2\mathbf{b}-1} + C_l |C_l - C_j|^{2\mathbf{b}-1}) \\ &\leq 2^{2\mathbf{b}} K_{\mathbf{b}}(\mathbf{T}) \left(\max_{r \in [0, n_0]} C_r \right) \max_{r, r' \in [k, l]} |C_r - C_{r'}|^{2\mathbf{b}-1}. \end{aligned}$$

Let $s, s' \in [0, n_0]$ be such that $s' \geq s$. Suppose first that $s, s' \in [k, k+1]$. The specific interpolation of the contour-snake implies that $|\widehat{\mathcal{W}}_{s'} - \widehat{\mathcal{W}}_s| = |s' - s| |\widehat{\mathcal{W}}_k - \widehat{\mathcal{W}}_{k+1}|$. Thus, $\mathbf{E} [|\widehat{\mathcal{W}}_s - \widehat{\mathcal{W}}_{s'}|^{2\mathbf{b}} | \mathcal{G}'] \leq 2^{2\mathbf{b}} K_{\mathbf{b}}(\mathbf{T}) (\max_{r \in [0, n_0]} C_r) |s' - s|^{2\mathbf{b}}$, since $\max_{r, r' \in [k, k+1]} |C_r - C_{r'}| = 1$ and $2\mathbf{b} \geq 2\mathbf{b} - 1$. Now suppose that $s, s' \in [0, n_0]$ are such that $\lceil s \rceil \leq \lfloor s' \rfloor$. Thus,

$$\begin{aligned} \mathbf{E} [|\widehat{\mathcal{W}}_s - \widehat{\mathcal{W}}_{s'}|^{2\mathbf{b}} | \mathcal{G}'] &\leq 3^{2\mathbf{b}-1} 2^{2\mathbf{b}} K_{\mathbf{b}}(\mathbf{T}) \left(\max_{r \in [0, n_0]} C_r \right) (|\lceil s \rceil - s|^{2\mathbf{b}-1} + |s' - \lfloor s' \rfloor|^{2\mathbf{b}-1} + \max_{r, r' \in [\lceil s \rceil, \lfloor s' \rfloor]} |C_r - C_{r'}|^{2\mathbf{b}-1}) \\ &\leq 6^{2\mathbf{b}} K_{\mathbf{b}}(\mathbf{T}) \left(\max_{r \in [0, n_0]} C_r \right) \max_{r, r' \in [s, s']} |C_r - C_{r'}|^{2\mathbf{b}-1}. \end{aligned} \quad (3.4.38)$$

This inequality easily extends to all $s, s' \in [0, n_0]$ such that $s \leq s'$.

Recall that \mathcal{H} and $\widehat{\mathcal{W}}^*$ are derived from C and $\widehat{\mathcal{W}}$ by the time-change $\phi_{\tau_\infty} : [0, \infty) \rightarrow [0, \infty)$ that is the increasing continuous bijection defined in Remark 3.2.5 (d). Namely, $\mathcal{H}_s = C_{\phi_{\tau_\infty}(s)}$

and $\widehat{\mathcal{W}}_s^* = \widehat{\mathcal{W}}_{\phi_{\tau_\infty}(s)}$, $s \in [0, \infty)$. Then for all $s_1, s', s \in [0, \infty)$ such that $s_1 \geq s' \geq s$, by choosing $n_0 = \phi_{\tau_\infty}(\lceil s_1 \rceil)$ in (3.4.38), we \mathbf{P} -a.s. get

$$\mathbf{E} \left[|\widehat{\mathcal{W}}_s^* - \widehat{\mathcal{W}}_{s'}^*|^{2b} \middle| \mathcal{G}' \right] \leq 6^{2b} K_b(\mathbf{T}) \max_{r \in [0, \lceil s_1 \rceil]} \mathcal{H}_r \max_{r, r' \in [s, s']} |\mathcal{H}_r - \mathcal{H}_{r'}|^{b-1}. \quad (3.4.39)$$

We next fix $s_0 \in (0, \infty)$ and $0 < \gamma < \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \left(1 - \frac{1}{b}\right) - \frac{1}{2b}$. Note that $\gamma_0 := \frac{1+2b\gamma}{b-1} < 1 - \frac{1}{\alpha}$. We set $Z_n = \text{Ho1}_{\gamma_0, s_0} \left(\frac{1}{a_n} \mathcal{H}_n\right)$ and $\Gamma_n = \max_{r \in [0, \lceil s_0 \rceil]} \frac{1}{a_n} \mathcal{H}_{nr}$. Then (3.4.39) entails

$$\mathbf{E} \left[\left| \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns}^* - \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns'}^* \right|^{2b} \middle| \mathcal{G}' \right] \leq 6^{2b} K_b(\mathbf{T}) \Gamma_n Z_n^{b-1} |s - s'|^{1+2b\gamma}.$$

Since $\gamma_0 < 1 - \frac{1}{\alpha}$, Proposition 3.4.17 applies: there is $c_1 \in (0, \infty)$, which only depends on b, μ, s_0 and γ_0 such that $\sup_{n \in \mathbb{N}^*} (\mathbf{E}[\Gamma_n^b] + \mathbf{E}[Z_n^b]) \leq c_1$, which implies $\sup_{n \in \mathbb{N}^*} \mathbf{E}[\Gamma_n Z_n^{b-1}] \leq c_1$ by Hölder inequality. So we obtain

$$\mathbf{E} \left[\left| \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns}^* - \frac{1}{\sqrt{a_n}} \widehat{\mathcal{W}}_{ns'}^* \right|^{2b} \middle| \mathcal{G} \right] \leq 6^{2b} K_b(\mathbf{T}) \mathbf{E}[\Gamma_n Z_n^{b-1}] |s - s'|^{1+2b\gamma} \leq 6^{2b} c_1 K_b(\mathbf{T}) |s - s'|^{1+2b\gamma},$$

since τ_∞ is independent of (\mathbf{T}, \circ) . This proves (3.4.37). Conditional tightness is a consequence of Lemmas 3.4.11 and 3.4.12. \blacksquare

3.4.4 Proof of Proposition 3.4.10

We keep the notations and the general assumptions discussed in Section 3.4.2. In particular, we recall (3.4.13): $\mathcal{Q}_n \rightarrow (\mathbf{g}_1, \mathbf{d}_1, X, H, H)$ weakly on $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^2$. By Skorokhod representation theorem, there are processes \mathcal{Q}'_n with the same law as \mathcal{Q}_n that converge almost surely to $(\mathbf{g}'_1, \mathbf{d}'_1, X', H', H')$, which has the same law as $(\mathbf{g}_1, \mathbf{d}_1, X, H, H)$. Without loss of generality, we can assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is large enough to allow the almost sure convergence to take place. Then observe that $(\mathbf{g}'_1, \mathbf{d}'_1)$ is necessarily the first excursion interval of H' longer than 1, and \mathbf{g}'_1 and \mathbf{d}'_1 are thus derived from H' as in Remark 3.2.34. Since the Lukasiewicz path (or the height process) of a forest completely encodes it, there actually exists a sequence of $\text{GW}(\mu)$ -forests $\tau_{n, \infty}$, $n \in \mathbb{N}^*$, independent from the environment (\mathbf{T}, \circ) and such that

$$\mathcal{Q}'_n = \left(\frac{1}{n} \mathbf{g}'_n, \frac{1}{n} \mathbf{d}'_n, \frac{1}{b_n} V_{[n]}(\tau_{n, \infty}), \frac{1}{a_n} H_n(\tau_{n, \infty}), \frac{1}{a_n} C_{2n}(\tau_{n, \infty}) \right) \quad (3.4.40)$$

where \mathbf{g}'_n and \mathbf{d}'_n are derived from $\tau_{n, \infty}$ as \mathbf{g}_n and \mathbf{d}_n are from τ_∞ in (3.2.66) (roughly speaking \mathbf{g}'_n and \mathbf{d}'_n are the endpoints of the first excursion of $H_n(\tau_{n, \infty})$ longer than n). Namely we assume that

$$\mathbf{P}\text{-a.s.} \quad \mathcal{Q}'_n \xrightarrow[n \rightarrow \infty]{} (\mathbf{g}'_1, \mathbf{d}'_1, X', H', H') \quad (3.4.41)$$

in $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^2$.

Then, for all $n \in \mathbb{N}^*$, we introduce the process $(\widehat{\mathcal{W}}_s^*(\tau_{n, \infty}))_{s \in [0, \infty)}$ as a \mathbb{R} -valued process such that

$$\left(\tau_{n, \infty}, (\widehat{\mathcal{W}}_s^*(\tau_{n, \infty}))_{s \in [0, \infty)} \right) \stackrel{(\text{law})}{=} \left(\tau_\infty, (\widehat{\mathcal{W}}_s^*(\tau_\infty))_{s \in [0, \infty)} \right), \quad (3.4.42)$$

where $\widehat{\mathcal{W}}^*(\tau_\infty)$ is defined as previously. This section is mainly devoted to the proof of Proposition 3.4.19 below. Before proving it, we first show that it actually implies Proposition 3.4.10.

Proposition 3.4.19 *We fix $\alpha \in (1, 2]$. Let μ satisfy (3.2.60). Let ν be a supercritical offspring distribution whose mean is denoted by m_ν . We assume that there exists $b \in (2, \infty)$ such that $\sum_{k \in \mathbb{N}} k^b \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree. Let $\tau_{n, \infty}$, $n \in \mathbb{N}$, be $\text{GW}(\mu)$ -forests that are independent from (\mathbf{T}, \circ) and such that (3.4.41) holds true. Let $(\widehat{\mathcal{W}}_s^*(\tau_{n, \infty}))_{s \in [0, \infty)}$ be as in*

(3.4.42). We denote by $(W'_s)_{s \in [0, \infty)}$ a one-dimensional Brownian snake starting at 0 and whose lifetime process is H' . We recall from (3.4.6) the definition of σ_ν . Then for all $p \in \mathbb{N}^*$ and for all real numbers $0 \leq s_1 < \dots < s_p$ and $\lambda_1, \dots, \lambda_p$, \mathbf{P} -almost surely

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\prod_{1 \leq j \leq p} \exp \left(i \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_j \widehat{\mathcal{W}}_{[ns_j]}^* (\tau_{n, \infty}) \right) \middle| (\mathbf{T}, \circ), \tau_{n, \infty} \right] = \mathbf{E} \left[\prod_{1 \leq j \leq p} e^{i \lambda_j \widehat{W}'_{s_j}} \middle| H' \right]. \quad (3.4.43)$$

Proof. The proof of Proposition 3.4.19 starts below, after the proof of Proposition 3.4.10. \blacksquare

Proof of Proposition 3.4.10. As already explained, the time-change (3.4.21) and (3.4.22) combined with Lemma 3.4.13 implies that it sufficient to prove conditionally given (\mathbf{T}, \circ) that the following convergence

$$\left(\frac{1}{n} \mathfrak{g}_n, \frac{1}{n} \mathfrak{d}_n, \frac{1}{b_n} V_{[n \cdot]} (\tau_\infty), \frac{1}{a_n} H_{n \cdot} (\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{\mathcal{W}}_{n \cdot}^* (\tau_\infty) \right) \xrightarrow[n \rightarrow \infty]{} (\mathfrak{g}_1, \mathfrak{d}_1, X, H, \widehat{W}) \quad (3.4.44)$$

holds weakly on $\mathbb{R}^2 \times \mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^2$ equipped with the product topology.

We recall (3.4.41) and we now prove that Proposition 3.4.19 implies Proposition 3.4.10. First recall for all $s > 0$ that \mathbf{P} -a.s. $X'_{s-} = X'_s$. By standard results on Skorokhod topology (see Lemma 3.4.20 (ii) below), it \mathbf{P} -a.s. implies that $\lim_{n \rightarrow \infty} \frac{1}{b_n} V_{[ns]} (\tau_{n, \infty}) = X'_s$. By Proposition 3.4.19 and dominated convergence, we then get for all real numbers $0 \leq s_1 < \dots < s_p$ and $\xi, \xi', \lambda_1, \lambda'_1, \lambda''_1, \dots, \lambda_p, \lambda'_p, \lambda''_p$,

$$\begin{aligned} \mathbf{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \mathbf{E} \left[e^{i \frac{1}{n} \xi \mathfrak{g}_n + i \frac{1}{n} \xi' \mathfrak{d}_n} \prod_{1 \leq j \leq p} e^{i \frac{1}{b_n} \lambda''_j V_{[ns_j]} (\tau_\infty) + i \frac{1}{a_n} \lambda'_j H_{ns_j} (\tau_\infty) + i \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_j \widehat{\mathcal{W}}_{ns_j}^* (\tau_\infty)} \middle| (\mathbf{T}, \circ) \right] \\ = \mathbf{E} \left[e^{i \xi \mathfrak{g}_1 + i \xi' \mathfrak{d}_1} \prod_{1 \leq j \leq p} e^{i \lambda''_j X_{s_j} + i \lambda'_j H_{s_j} + i \lambda_j \widehat{W}_{s_j}} \right]. \quad (3.4.45) \end{aligned}$$

To simplify notations, we denote by \mathcal{G} the σ -field generated by (\mathbf{T}, \circ) . We also denote by \mathbf{D}_q the space $\mathbf{D}([0, \infty), \mathbb{R}^q)$ of cadlag functions equipped with the Skorokhod topology, and we also recall the shorthand notation \mathbf{C}_1^0 for $\mathbf{C}^0([0, \infty), \mathbb{R})$ which is equipped with the topology of uniform convergence on every compact interval. Below, product spaces are equipped with product topology and if E is a Polish space, we denote by $\mathcal{M}_1(E)$ the space of its Borel probability measures equipped with the weak convergence. The following elementary lemma sets the notations and the basic results we need.

Lemma 3.4.20 *The following properties hold true.*

- (i) *The function $\jmath : (\mathbf{u}, \mathbf{x}, \mathbf{h}, \mathbf{w}) \in \mathbb{R}^2 \times \mathbf{D}_1 \times (\mathbf{C}_1^0)^2 \mapsto (\mathbf{u}, (\mathbf{x}(s), \mathbf{h}(s), \mathbf{w}(s))_{s \in [0, \infty)}) \in \mathbb{R}^2 \times \mathbf{D}_3$ is continuous.*
- (ii) *Let $s_1, \dots, s_p \in [0, \infty)$. The function $\phi_{s_1, \dots, s_p} : \mathbf{z} \in \mathbf{D}_3 \mapsto (\mathbf{z}(s_1), \dots, \mathbf{z}(s_p)) \in (\mathbb{R}^3)^p$ is continuous at $\mathbf{z} \in \mathbf{D}_3$ such that $\mathbf{z}(s_j-) = \mathbf{z}(s_j)$, $1 \leq j \leq p$.*
- (iii) *Let $k \in \{1, 2, 3\}$. The function $\pi_k : (\mathbf{u}, (\mathbf{z}_1(s), \mathbf{z}_2(s), \mathbf{z}_3(s))_{s \in [0, \infty)}) \in \mathbb{R}^2 \times \mathbf{D}_3 \mapsto \mathbf{z}_k(\cdot) \in \mathbf{D}_1$ is continuous.*
- (iv) *Let $k \in \{1, 2, 3\}$. For all $\mathbf{Q} \in \mathcal{M}_1(\mathbb{R}^2 \times \mathbf{D}_3)$, we set $\mathbf{Q}^k = \mathbf{Q} \circ \pi_k^{-1} \in \mathcal{M}_1(\mathbf{D}_1)$. Then $\mathbf{Q} \mapsto \mathbf{Q}^k$ is continuous.*

Proof. For (i), see e.g. Jacod & Shiryaev [95, Chapter VI, Proposition 2.2 (b) p. 338]. For (ii), see e.g. [95, Chapter VI, Proposition 2.1 (b5), p. 337]. The last two points are immediate. \blacksquare

By (3.4.13), by the independence of τ_∞ from (\mathbf{T}, \circ) and by Proposition 3.4.18, we see that conditionally given \mathcal{G} the laws of the processes $(\frac{1}{n} \mathfrak{g}_n, \frac{1}{n} \mathfrak{d}_n, \frac{1}{b_n} V_{[n \cdot]} (\tau_\infty), \frac{1}{a_n} H_{n \cdot} (\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{\mathcal{W}}_{n \cdot}^* (\tau_\infty))$

are tight on $\mathbb{R}^2 \times \mathbf{D}_3 \times (\mathbf{C}_1^0)^2$. By Lemma 3.4.20 (i), it implies that conditionally given \mathcal{G} the laws of the processes $Z^{(n)} := (\frac{1}{n} \mathbf{g}_n, \frac{1}{n} \mathbf{d}_n, (\frac{1}{b_n} V_{[ns]}(\tau_\infty), \frac{1}{a_n} H_{ns}(\tau_\infty), \frac{1}{\sigma_n \sqrt{a_n}} \widehat{W}_{ns}^*(\tau_\infty))_{s \in [0, \infty)})$ are tight on $\mathbb{R}^2 \times \mathbf{D}_3$.

We next denote by $\omega \in \Omega \mapsto Q_n(\omega, \cdot) \in \mathcal{M}_1(\mathbb{R}^2 \times \mathbf{D}_3)$ a regular version of the law of $Z^{(n)}$ conditionally given \mathcal{G} . We denote by P_n the law of $\frac{1}{b_n} V_{[n \cdot]}(\tau_\infty)$ under \mathbf{P} , we denote by P the law X under \mathbf{P} and we denote by Q_∞ the law of $(\mathbf{g}_1, \mathbf{d}_1, (X_s, H_s, \widehat{W}_s)_{s \in [0, \infty)})$ under \mathbf{P} . We observe that for \mathbf{P} -almost all ω , $Q_n^1(\omega, \cdot) = P_n(\cdot)$. This, combined with (3.4.45), entails that there exists $\Omega_0 \in \mathcal{G}$ such that $\mathbf{P}(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$, it holds that the $(Q_n(\omega, \cdot))_{n \in \mathbb{N}}$ are tight on $\mathbb{R}^2 \times \mathbf{D}_3$, that $Q_n^1(\omega, \cdot) = P_n(\cdot)$, and that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2 \times \mathbf{D}_3} e^{i\langle \xi, \mathbf{u} \rangle + i\langle \lambda_1, \mathbf{z}(s_1) \rangle + \dots + i\langle \lambda_p, \mathbf{z}(s_p) \rangle} Q_n(\omega, d\mathbf{u} d\mathbf{z}) = \int_{\mathbb{R}^2 \times \mathbf{D}_3} e^{i\langle \xi, \mathbf{u} \rangle + i\langle \lambda_1, \mathbf{z}(s_1) \rangle + \dots + i\langle \lambda_p, \mathbf{z}(s_p) \rangle} Q_\infty(d\mathbf{u} d\mathbf{z}) \quad (3.4.46)$$

for all $p \in \mathbb{N}^*$, all $\xi \in \mathbb{Q}^2$, all $s_1, \dots, s_p \in \mathbb{Q} \cap [0, \infty)$ and all $\lambda_1, \dots, \lambda_p \in \mathbb{Q}^3$.

We then fix $\omega \in \Omega_0$ and we denote by $Q(\omega, \cdot) \in \mathcal{M}_1(\mathbb{R}^2 \times \mathbf{D}_3)$ a weak limit-point of the $Q_n(\omega, \cdot)$ (there is such one since the $Q_n(\omega, \cdot)$ are tight on $\mathbb{R}^2 \times \mathbf{D}_3$). Namely, there exists an increasing sequence of integers $(n_k(\omega))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} Q_{n_k(\omega)}(\omega, \cdot) = Q(\omega, \cdot)$. By Lemma 3.4.20 (iv), $Q_{n_k(\omega)}^1(\omega, \cdot) = P_{n_k(\omega)} \rightarrow Q^1(\omega, \cdot) = P$. Since X has no fixed time discontinuity, it implies for all $s_1, \dots, s_p \in [0, \infty)$ that $Q(\omega, d\mathbf{u} d\mathbf{z})$ -a.s. $\mathbf{z}(s_j -) = \mathbf{z}(s_j)$, $1 \leq j \leq p$. Then by Lemma 3.4.20 (ii), by weak-continuity of the Fourier transform of probability measures and by (3.4.46), we get

$$\int_{\mathbb{R}^2 \times \mathbf{D}_3} e^{i\langle \xi, \mathbf{u} \rangle + i\langle \lambda_1, \mathbf{z}(s_1) \rangle + \dots + i\langle \lambda_p, \mathbf{z}(s_p) \rangle} Q(\omega, d\mathbf{u} d\mathbf{z}) = \int_{\mathbb{R}^2 \times \mathbf{D}_3} e^{i\langle \xi, \mathbf{u} \rangle + i\langle \lambda_1, \mathbf{z}(s_1) \rangle + \dots + i\langle \lambda_p, \mathbf{z}(s_p) \rangle} Q_\infty(d\mathbf{u} d\mathbf{z})$$

for all $p \in \mathbb{N}^*$, all $\xi \in \mathbb{Q}^2$, all $s_1, \dots, s_p \in \mathbb{Q} \cap [0, \infty)$ and all $\lambda_1, \dots, \lambda_p \in \mathbb{Q}^3$. Thus, it also holds for all $\xi \in \mathbb{R}^2$, all $s_1, \dots, s_p \in [0, \infty)$ and all $\lambda_1, \dots, \lambda_p \in \mathbb{R}^3$, by the continuity of the paths and dominated convergence. Injectivity of the Fourier transform of probability measures then entails that $Q(\omega, \cdot)$ and $Q_\infty(\cdot)$ have the same finite-dimensional marginals, which implies that $Q(\omega, \cdot) = Q_\infty(\cdot)$. This proves that for all $\omega \in \Omega_0$, $Q_\infty(\cdot)$ is the only limit point of the $Q_n(\omega, \cdot)$: namely, $Q_n(\omega, \cdot) \rightarrow Q_\infty(\cdot)$, which easily entails (3.4.44) by Lemma 3.4.20 (iii) and the fact that the topology induced by Skorokhod topology on the set \mathbf{C}_1^0 is that of uniform convergence on every compact interval. This completes the proof of Proposition 3.4.10. \blacksquare

Before proceeding to the proof of Proposition 3.4.19, we need to introduce some notations and to prove auxiliary results on finite-dimensional marginals of height processes and height-snakes.

Contour processes of real trees with a finite number of leaves. Let us first recall from Example 3.2.42 the definition of the real tree (T_h, d_h, \mathbf{r}_h) coded by a continuous function $h : [0, \zeta] \rightarrow \mathbb{R}$ and recall that $p_h : [0, \zeta] \rightarrow T_h$ stands for the canonical projection. We discuss here how to encode the subtree $T' := \bigcup_{1 \leq j \leq p} \llbracket \mathbf{r}_h, p_h(s_j) \rrbracket$ that is spanned by the points corresponding to the times $0 \leq s_1 < \dots < s_p < \zeta$ (here, recall from Definition 3.2.39 that for all $\sigma, \sigma' \in T_h$, $\llbracket \sigma, \sigma' \rrbracket$ stands for the geodesic path joining σ to σ'). The real tree T' can be viewed as a discrete rooted ordered tree (its *skeleton*) equipped with ‘heights’ and if $h(s_j) \wedge h(s_{j+1}) > \min_{[s_j, s_{j+1}]} h$, T' has exactly p leaves (i.e. points $\sigma \in T' \setminus \{\mathbf{r}_h\}$ such that $T' \setminus \{\sigma\}$ is connected). To encode T' , we introduce its contour function which represents the distance from the root of a particle visiting T' at unit speed and respecting the contour order of its skeleton. The contour function of T' belongs to the set \mathcal{C}_p of functions $\mathcal{C} \in \mathbf{C}^0([0, \infty), [0, \infty))$ such that

- (a) $\mathcal{C}_0 = 0$ and $\zeta_{\mathcal{C}} = \sup\{s \in [0, \infty) : \mathcal{C}_s > 0\} < \infty$,
- (b) \mathcal{C} is a broken line whose slopes are either equal to 1 or equal to -1 , that has p local maxima.

Namely, there are $r_0 = \rho_0 = 0 < r_1 < \rho_1 < \dots < r_{p-1} < \rho_{p-1} < r_p < \rho_p = \zeta_C$ such that for all $s \in [0, \infty)$,

$$\mathcal{C}_s = \int_0^s \mathbf{1}_{J_+}(r) dr - \int_0^s \mathbf{1}_{J_-}(r) dr \quad \text{where } J_+ = \bigcup_{1 \leq j \leq p} [\rho_{j-1}, r_j] \text{ and } J_- = [0, \zeta_C] \setminus J_+. \quad (3.4.47)$$

For all $j \in \llbracket 1, p \rrbracket$, note that r_j (resp. ρ_{j-1}) is the time when the function \mathcal{C} reaches its j -th local maximum (resp. its j -th local minimum). Then, observe that $\mathcal{C}_{\rho_{j-1}} = \min_{[r_{j-1}, r_j]} \mathcal{C}$. It shall be convenient to define \mathcal{C}_0 as the set reducing to the null function.

Let $p \in \mathbb{N}^*$. Note that there is a one-to-one correspondence F_p between the open subset $\mathcal{D}_p = \{(x_j)_{1 \leq j \leq 2p-1} \in [0, \infty)^{2p-1} : x_{2j-1} \wedge x_{2j+1} > x_{2j}, 1 \leq j \leq p-1\}$ of $[0, \infty)^{2p-1}$ and \mathcal{C}_p , that is defined as follows: $\mathcal{C} = F_p((x_j)_{1 \leq j \leq 2p-1})$ if

$$\forall j \in \llbracket 1, p \rrbracket, \quad x_{2j-1} = \mathcal{C}_{r_j} \quad \text{and} \quad x_{2j} = \min_{[r_j, r_{j+1}]} \mathcal{C} = \mathcal{C}_{\rho_j} \quad \text{when } j < p. \quad (3.4.48)$$

We easily check that F_p and its inverse F_p^{-1} are continuous when \mathcal{C}_p is equipped with the topology of the uniform convergence on every compact interval.

Decompositions at the first branching time. Let $\mathcal{C} \in \mathcal{C}_p$ with $p \geq 2$. Let ζ_C , the r_j and the ρ_j be as in (3.4.47). We set

$$m(\mathcal{C}) = \min_{[r_1, r_p]} \mathcal{C} \quad \text{and} \quad q(\mathcal{C}) = -2 + \#\{s \in [0, \zeta_C] : \mathcal{C}_s = m(\mathcal{C})\}, \quad (3.4.49)$$

and we list the elements of $\{s \in [0, \zeta_C] : \mathcal{C}_s = m(\mathcal{C})\}$ in increasing order as $\gamma_0 < \gamma_1 < \dots < \gamma_{q(\mathcal{C})+1}$. Note that $\gamma_0 < r_1 < \gamma_1 \leq \gamma_{q(\mathcal{C})} < r_p < \gamma_{q(\mathcal{C})+1}$, and that the intervals (γ_{k-1}, γ_k) , $1 \leq k \leq q(\mathcal{C})+1$, are the connected components of $\{s \in [0, \infty) : \mathcal{C}_s > m(\mathcal{C})\}$. Since $p \geq 2$, it holds that $q(\mathcal{C}) \geq 1$. Then, for all $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$ and all $s \in [0, \infty)$, we also set

$$\mathcal{C}_s[k]^- = \mathcal{C}_{(\gamma_0+s) \wedge \gamma_k} - m(\mathcal{C}) \quad \text{and} \quad \mathcal{C}_s[k]^+ = \mathcal{C}_{(\gamma_k+s) \wedge \gamma_{q(\mathcal{C})+1}} - m(\mathcal{C}). \quad (3.4.50)$$

Now, recall F_p from (3.4.48). If we write $(x_j)_{1 \leq j < 2p} := F_p^{-1}(\mathcal{C})$, then we observe that

$$\mathcal{C}[k]^- = F_{p_k(\mathcal{C})}((x_j - m(\mathcal{C}))_{1 \leq j < 2p_k(\mathcal{C})}) \quad \text{and} \quad \mathcal{C}[k]^+ = F_{p-p_k(\mathcal{C})}((x_{2p_k(\mathcal{C})+j} - m(\mathcal{C}))_{1 \leq j < 2p-2p_k(\mathcal{C})}), \quad (3.4.51)$$

where $p_k(\mathcal{C}) = \#\{j \in \llbracket 1, p \rrbracket : r_j \leq \gamma_k\}$. Here, note that $1 \leq p_k(\mathcal{C}) < p$ and $\rho_{p_k(\mathcal{C})} = \gamma_k$.

In the special case where $p=1$, we set $m(\mathcal{C}) = \mathcal{C}_{r_1} = \max \mathcal{C}$, $q(\mathcal{C}) = 1$ and $\mathcal{C}[1]^- = \mathcal{C}[1]^+$ is the null function. For any $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$, we call the pair of functions $(\mathcal{C}[k]^-, \mathcal{C}[k]^+)$ the k -th decomposition of \mathcal{C} at the first branching point.

Lemma 3.4.21 *Let $p \in \mathbb{N}^*$, and let $\mathcal{C}, \mathcal{C}^{(n)} \in \mathcal{C}_p$, $n \in \mathbb{N}$, be such that $\mathcal{C}^{(n)} \rightarrow \mathcal{C}$ uniformly on every compact interval. Then, $m(\mathcal{C}^{(n)}) \rightarrow m(\mathcal{C})$, and there exists a sequence of integers $k_n \in \llbracket 1, q(\mathcal{C}) \rrbracket$, $n \in \mathbb{N}$, such that:*

- (a) $\mathcal{C}[k_n]^-, \mathcal{C}^{(n)}[1]^- \in \mathcal{C}_{p_{k_n}(\mathcal{C})}$ and $\mathcal{C}[k_n]^+, \mathcal{C}^{(n)}[1]^+ \in \mathcal{C}_{p-p_{k_n}(\mathcal{C})}$ when n is large enough,
- (b) $\mathcal{C}^{(n)}[1]^- - \mathcal{C}[k_n]^- \rightarrow 0$ and $\mathcal{C}^{(n)}[1]^+ - \mathcal{C}[k_n]^+ \rightarrow 0$ uniformly on every compact interval.

Proof. Note that $F_p^{-1}(\mathcal{C}^{(n)}) =: (x_j^{(n)})_{1 \leq j \leq 2p-1} \rightarrow (x_j)_{1 \leq j \leq 2p-1} := F_p^{-1}(\mathcal{C})$. Thus, $m(\mathcal{C}^{(n)}) = \min_{1 \leq j \leq 2p-1} x_j^{(n)} \rightarrow \min_{1 \leq j \leq 2p-1} x_j = m(\mathcal{C})$. Let us set $p^{(n)} = p_1(\mathcal{C}^{(n)})$ to lighten the notation. We construct the sequence $(k_n)_{n \in \mathbb{N}}$ as follows: if $x_{2p^{(n)}} \neq m(\mathcal{C})$, then we set $k_n = 1$; if $x_{2p^{(n)}} = m(\mathcal{C})$ then there is $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$ such that $p^{(n)} = p_k(\mathcal{C})$, and we set $k_n = k$. By definition, we have $|x_{2p^{(n)}}^{(n)} - m(\mathcal{C}^{(n)})| = 0$, so it follows that $|x_{2p^{(n)}} - m(\mathcal{C})|$ converges to 0. But since the latter sequence

can only take on a finite number of values, it is eventually stationary at 0. When n is large enough, it thus holds $p_1(\mathcal{C}^{(n)}) = p_{k_n}(\mathcal{C})$. We complete the proof by (3.4.51) and by continuity of $F_{p_k(\mathcal{C})}$. ■

Finite-dimensional marginals of height-snakes. We next use the decompositions at the first branching point to compute the finite-dimensional marginals of Brownian snakes and branching RWs. More precisely, we fix $p \in \mathbb{N}^*$ and $\mathcal{C} \in \mathcal{C}_p$, and we denote by $(W_s(\mathcal{C}))_{s \in [0, \infty)}$ the one-dimensional Brownian snake starting at 0 whose lifetime process is \mathcal{C} (see Definition 3.2.45). Then we introduce the following notation for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$:

$$\mathcal{L}(\mathcal{C}; \lambda_1, \dots, \lambda_p) = \mathbf{E} \left[\prod_{1 \leq j \leq p} e^{i\lambda_j \widehat{W}_{r_j}(\mathcal{C})} \right] \quad (3.4.52)$$

where r_j stands for the time when \mathcal{C} reaches its j -th local maximum. As an easy consequence of the snake property, for all $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$ with $q(\mathcal{C})$ as in (3.4.49), we get

$$\mathcal{L}(\mathcal{C}; \lambda_1, \dots, \lambda_p) = e^{-\frac{1}{2}m(\mathcal{C})\bar{\lambda}^2} \mathcal{L}(\mathcal{C}[k]^-; (\lambda_j)_{1 \leq j \leq p_k(\mathcal{C})}) \mathcal{L}(\mathcal{C}[k]^+; (\lambda_j)_{p_k(\mathcal{C})+1 \leq j \leq p}) \cdot \quad (3.4.53)$$

where $(\mathcal{C}[k]^-, \mathcal{C}[k]^+)$ is the k -th decomposition of \mathcal{C} at its first branching time as in (3.4.50) and (3.4.51), where $m(\mathcal{C}) = \min_{[r_1, r_p]} \mathcal{C}$, and where $\bar{\lambda} = \lambda_1 + \dots + \lambda_p$.

Let us introduce similar notations for \mathbf{T} -valued branching RWs, where (\mathbf{T}, \circ) stands here for an invariant GW-tree whose offspring distribution ν is supercritical. We denote by m_ν its mean and we assume that there is $b \in (2, \infty)$ such that $\sum_{k \in \mathbb{N}} k^b \nu(k) < \infty$. To that end, we suppose that there exists $a \in (0, \infty)$ and a discrete tree $t \in \mathbb{T}$ such that

$$C_s(t) = a C_{s/a}, \quad s \in [0, \infty) \quad (3.4.54)$$

where $C_s(t)$ stands for the contour function of the rooted ordered tree t as in Definition 3.2.4 (b).

We then fix $x \in \mathbf{T}$ and we denote by $(Y_u)_{u \in t}$ a t -indexed and \mathbf{T} -valued m_ν -biased branching RW starting at Y_\emptyset . Namely, conditionally given (\mathbf{T}, x) , we \mathbf{P} -a.s. get $\mathbf{P}(\forall u \in t, Y_u = y_u | (\mathbf{T}, \circ)) = \mathbf{1}_{\{y_\emptyset = x\}} \prod_{u \in t \setminus \{\emptyset\}} p_{\mathbf{T}, m_\nu}(y_u^-, y_u)$ for all $y_u \in \mathbf{T}$, $u \in t$ (recall from (3.2.26) the definition of the transition probabilities $p_{\mathbf{T}, m_\nu}(\cdot, \cdot)$ of the m_ν -biased RW on \mathbf{T}). Here we recall from (3.4.3) that $(S_x)_{x \in \mathbf{T}}$ stand for the harmonic coordinates of (\mathbf{T}, \circ) . Then, we introduce the following notation for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$:

$$\widetilde{\mathcal{L}}_{\mathbf{T}, x}(a C_{\frac{\cdot}{a}}; \lambda_1, \dots, \lambda_p) = \mathbf{E} \left[\prod_{1 \leq j \leq p} e^{i\lambda_j (S_{Y_{v(j)}} - S_{Y_\emptyset})} \middle| (\mathbf{T}, x) \right] \quad (3.4.55)$$

where the vertex $v(j) \in t$ is the j -th leaf of t in increasing contour order. As a consequence of the definition of branching RWs, for all $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$ with $q(\mathcal{C})$ as in (3.4.49), we \mathbf{P} -a.s. get

$$\begin{aligned} \widetilde{\mathcal{L}}_{\mathbf{T}, x}(a C_{\frac{\cdot}{a}}; \lambda_1, \dots, \lambda_p) &= E_{\mathbf{T}, x} \left[e^{i\bar{\lambda} (S_{X_{am(\mathcal{C})}} - S_{X_0})} \right. \\ &\quad \left. \times \widetilde{\mathcal{L}}_{\mathbf{T}, X_{am(\mathcal{C})}}(a C_{\frac{\cdot}{a}}[k]^-; (\lambda_j)_{1 \leq j \leq p_k(\mathcal{C})}) \widetilde{\mathcal{L}}_{\mathbf{T}, X_{am(\mathcal{C})}}(a C_{\frac{\cdot}{a}}[k]^+; (\lambda_j)_{p_k(\mathcal{C})+1 \leq j \leq p}) \right], \quad (3.4.56) \end{aligned}$$

where $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, stands for a m_ν -biased RW starting at $X_0 = x$ (and recall that $P_{\mathbf{T}, x}$ is the conditional law of $(\bar{X}_n)_{n \in \mathbb{N}}$ given (\mathbf{T}, x)). Here we have set $\bar{\lambda} = \lambda_1 + \dots + \lambda_p$ and we recall that $(\mathcal{C}[k]^-, \mathcal{C}[k]^+)$ is the k -th decomposition of \mathcal{C} at its first branching time: $m(\mathcal{C}) = \min_{[r_1, r_p]} \mathcal{C}$. Note that $am(\mathcal{C})$ is the height of the $\leq_{1\text{ex}}$ -smallest vertex $u \in t$ such that $k_u(t) \neq 1$ (which is also the first time that this vertex is visited in increasing contour order).

One of the main steps of the proof of Proposition 3.4.19 is the following lemma, which relies on decompositions at the first branching point and on the specific CLT in Proposition 3.4.7.

Lemma 3.4.22 *We keep the above assumptions and notations. We fix $p \in \mathbb{N}^*$. Let $\mathcal{C}, \mathcal{C}^{(n)} \in \mathcal{C}_p$ and $r_n \in \mathbb{N}$, $n \in \mathbb{N}$. Here, $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, stands for a m_ν -biased RW starting at $X_0 = \mathbf{o}$. We assume the following.*

- (a) $\mathcal{C}^{(n)} \rightarrow \mathcal{C}$ uniformly on every compact interval.
- (b) For all $n \in \mathbb{N}$, there exist $a_n \in (0, \infty)$ and $t_n \in \mathbb{T}$ such that $C_s(t_n) = a_n \mathcal{C}_{s/a_n}^{(n)}$, $s \in [0, \infty)$.
- (c) There exists $r \in [0, \infty)$ such that $r_n/a_n \rightarrow r$.

Then for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$, it holds \mathbf{P} -a.s. that

$$\widetilde{\mathcal{L}}_{\mathbf{T}, X_{r_n}}(a_n \mathcal{C}_{\frac{\cdot}{a_n}}^{(n)}; \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_1, \dots, \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_p) \xrightarrow[n \rightarrow \infty]{} \mathcal{L}(\mathcal{C}; \lambda_1, \dots, \lambda_p), \quad (3.4.57)$$

where we recall from (3.4.6) the definition of σ_ν .

Proof. We argue recursively on p . If $p = 1$, then the trees t_n are lines and (3.4.57) is a direct consequence of Proposition 3.4.7. We then assume that $p \geq 2$ and that the lemma holds true for all $p' < p$. Lemma 3.4.21 yields that $m(\mathcal{C}^{(n)}) \rightarrow m(\mathcal{C})$ and gives a specific sequence $k_n \in \llbracket 1, q(\mathcal{C}) \rrbracket$, $n \in \mathbb{N}$. We fix $k \in \llbracket 1, q(\mathcal{C}) \rrbracket$ which is a subsequential limit of (k_n) , and denote by $(\mathcal{C}^{(n)}[1]^\pm, \mathcal{C}^{(n)}[1]^\pm)$ and $(\mathcal{C}[k]^\pm, \mathcal{C}[k]^\pm)$ resp. the first and the k -th decomposition at the first branching point of resp. $\mathcal{C}^{(n)}$, \mathcal{C} .

Observe that $(r_n + a_n m(\mathcal{C}^{(n)}))/a_n \rightarrow r + m(\mathcal{C})$. Also recall from Lemma 3.4.21 that along the subsequence of integers n such that $k_n = k$, $\mathcal{C}^{(n)}[1]^\pm \rightarrow \mathcal{C}[k]^\pm$ uniformly on every compact interval. For all such n large enough, the common number of local maxima of $\mathcal{C}^{(n)}[1]^\pm$ and $\mathcal{C}[k]^\pm$ is smaller than p , so our recurrence assumption applies and we \mathbf{P} -a.s. get

$$\begin{aligned} \widetilde{\mathcal{L}}_{\mathbf{T}, X_{r_n + a_n m(\mathcal{C}^{(n)})}}(a_n \mathcal{C}_{\frac{\cdot}{a_n}}^{(n)}[1]^-; (\frac{\lambda_j}{\sigma_\nu \sqrt{a_n}})_{1 \leq j \leq p_1(\mathcal{C}^{(n)})}) &\xrightarrow[n \rightarrow \infty, k_n = k]{} \mathcal{L}(\mathcal{C}[k]^-; (\lambda_j)_{1 \leq j \leq p_k(\mathcal{C})}), \\ \widetilde{\mathcal{L}}_{\mathbf{T}, X_{r_n + a_n m(\mathcal{C}^{(n)})}}(a_n \mathcal{C}_{\frac{\cdot}{a_n}}^{(n)}[1]^+; (\frac{\lambda_j}{\sigma_\nu \sqrt{a_n}})_{p_1(\mathcal{C}^{(n)}) + 1 \leq j \leq p}) &\xrightarrow[n \rightarrow \infty, k_n = k]{} \mathcal{L}(\mathcal{C}[k]^+; (\lambda_j)_{p_k(\mathcal{C}) + 1 \leq j \leq p}). \end{aligned}$$

To lighten the notations, we denote by L_n (resp. $L[k]$) the product of the left-hand (resp. right-hand) sides of the two previous convergences. Let us also denote by \mathcal{F}_{r_n} the σ -field generated by $(\bar{X}_m)_{0 \leq m \leq n}$. Then, we \mathbf{P} -a.s. get $\lim_{n \rightarrow \infty, k_n = k} \mathbf{E}[L[k] - L_n | \mathcal{F}_{r_n}] = 0$ by the technical Lemma 3.C.1 in Appendix.

Note that $L[k]$ is a constant. Then, combining the decomposition (3.4.56) with the Markov property for \bar{X} at time r_n implies \mathbf{P} -a.s. that

$$\left| \widetilde{\mathcal{L}}_{\mathbf{T}, X_{r_n}}(a_n \mathcal{C}_{\frac{\cdot}{a_n}}^{(n)}; \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_1, \dots, \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_p) - L[k] E_{\mathbf{T}, X_{r_n}} \left[\exp \left(i \frac{\bar{\lambda}}{\sigma_\nu \sqrt{a_n}} (S_{X'_{a_n m(\mathcal{C}^{(n)})}} - S_{X'_0}) \right) \right] \right|$$

is smaller or equal to $\mathbf{E}[|L_n - L| | \mathcal{F}_{r_n}]$, where $\bar{X}'_n = (\mathbf{T}, X'_n)$, $n \in \mathbb{N}$, stands here for an auxiliary m_ν -biased RW. We then apply Proposition 3.4.7 to get \mathbf{P} -a.s.

$$\begin{aligned} E_{\mathbf{T}, X_{r_n}} \left[\exp \left(i \frac{\bar{\lambda}}{\sigma_\nu \sqrt{a_n}} (S_{X'_{a_n m(\mathcal{C}^{(n)})}} - S_{X'_0}) \right) \right] &= \mathbf{E} \left[\exp \left(i \frac{\bar{\lambda}}{\sigma_\nu \sqrt{a_n}} (S_{X_{r_n + a_n m(\mathcal{C}^{(n)})}} - S_{X_{r_n}}) \right) \middle| \mathcal{F}_{r_n} \right] \\ &\xrightarrow[n \rightarrow \infty]{} \exp \left(-\frac{1}{2} m(\mathcal{C}) \bar{\lambda}^2 \right). \end{aligned}$$

Therefore, we get that \mathbf{P} -a.s.

$$\widetilde{\mathcal{L}}_{\mathbf{T}, X_{r_n}}(a_n \mathcal{C}_{\frac{\cdot}{a_n}}^{(n)}; \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_1, \dots, \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_p) \xrightarrow[n \rightarrow \infty, k_n = k]{} e^{-\frac{1}{2} m(\mathcal{C}) \bar{\lambda}^2} L[k] = \mathcal{L}(\mathcal{C}; \lambda_1, \dots, \lambda_p)$$

thanks to the decomposition (3.4.53). Since $\{n \in \mathbb{N} : k_n = k\}$, $1 \leq k \leq q(\mathcal{C})$, is a finite partition of \mathbb{N} , this completes the proof of (3.4.57), and thus of the lemma. \blacksquare

Proof of Proposition 3.4.19. To that end, we need to relate the previous results to the finite-dimensional marginals of the endpoint process of the height-snake in harmonic coordinates and to the finite-dimensional marginals of the endpoint process of one-dimensional Brownian snake \widehat{W}' whose lifetime process is H' as in (3.4.41).

Recall from (3.4.48) the definition of F_p for all $p \in \mathbb{N}^*$. For all $h \in \mathbf{C}^0([0, \infty), [0, \infty))$ and for all $s_1 > 0$ such that $h(s_1) > 0$, we set $\mathcal{C}(h, s_1) = F_1(h(s_1))$. More generally, for all $p \in \mathbb{N}^*$ with $p \geq 2$ and for all real numbers $0 = s_0 < s_1 < \dots < s_p$ such that the following condition

$$\forall j \in \llbracket 1, p-1 \rrbracket, \quad h(s_j) \wedge h(s_{j+1}) > \min_{[s_j, s_{j+1}]} h \quad (3.4.58)$$

is satisfied, we set

$$F_p\left(h(s_1), \min_{[s_1, s_2]} h, h(s_2), \dots, h(s_{p-1}), \min_{[s_{p-1}, s_p]} h, h(s_p)\right) =: \mathcal{C}(h; s_1, \dots, s_p) \in \mathcal{C}_p. \quad (3.4.59)$$

As explained at the beginning of the section, $\mathcal{C}(h; s_1, \dots, s_p)$ is the contour process of the subtree $\bigcup_{1 \leq j \leq p} \llbracket \mathbf{r}_h, p_h(s_j) \rrbracket$ in (T_h, d_h, \mathbf{r}_h) .

We know from Lemma 3.2.33 that $(H'; s_1, \dots, s_p)$ \mathbf{P} -a.s. satisfies (3.4.58). As an easy consequence of Definition 3.2.45 of finite-dimensional marginals of Brownian snakes, we see that for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$, \mathbf{P} -a.s.

$$\mathbf{E}\left[\prod_{0 \leq j \leq p} e^{i\lambda_j \widehat{W}'_{s_j}} \mid (H'_s)_{s \in [0, \infty)}\right] = \mathcal{L}(\mathcal{C}(H'; s_1, \dots, s_p); \lambda_1, \dots, \lambda_p) \quad (3.4.60)$$

where recall from (3.4.52) the definition of the right member.

Similarly, in the discrete setting: recall that the $\text{GW}(\mu)$ -forests trees $\tau_{n, \infty}$, $n \in \mathbb{N}^*$, are such that the almost sure limit (3.4.41) holds true and recall from (3.4.42) the definition of the joint law of $(\tau_{n, \infty}, \widehat{\mathcal{W}}^*(\tau_{n, \infty}))$. The almost sure convergence (3.4.41) \mathbf{P} -a.s. implies that

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} H_{\lfloor ns_j \rfloor}(\tau_{n, \infty}) = H'_{s_j} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{a_n} \min_{[\lfloor ns_j \rfloor, \lfloor ns_{j+1} \rfloor]} H(\tau_{n, \infty}) = \min_{[s_j, s_{j+1}]} H'. \quad (3.4.61)$$

Combined with Lemma 3.2.33, this \mathbf{P} -a.s. entails for all sufficiently large n , that (3.4.58) is satisfied by $(a_n^{-1} H(\tau_{n, \infty}); \lfloor ns_1 \rfloor, \dots, \lfloor ns_p \rfloor)$. This allows us to define

$$\mathcal{C}^{(n)} = \mathcal{C}\left(\frac{1}{a_n} H(\tau_{n, \infty}); \lfloor ns_1 \rfloor, \dots, \lfloor ns_p \rfloor\right). \quad (3.4.62)$$

As a consequence of the definition of the height-snake $\widehat{\mathcal{W}}^*(\tau_{n, \infty})$, we see that

$$\begin{aligned} \mathbf{E}\left[\prod_{0 \leq j \leq p} \exp\left(i \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_j \widehat{\mathcal{W}}^*_{\lfloor ns_j \rfloor}(\tau_{n, \infty})\right) \mid (\mathbf{T}, \circ), \tau_{n, \infty}\right] \\ = \widetilde{\mathcal{L}}_{\mathbf{T}, \circ}\left(a_n \mathcal{C}^{(n)}; \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_1, \dots, \frac{1}{\sigma_\nu \sqrt{a_n}} \lambda_p\right). \end{aligned} \quad (3.4.63)$$

Moreover, since the function F_p is continuous, we get from (3.4.61) that $\mathcal{C}^{(n)} \rightarrow \mathcal{C}(H'; s_1, \dots, s_p)$ uniformly on every compact interval. Then Proposition 3.4.19 follows by applying Lemma 3.4.22 to (3.4.60) and (3.4.63). \blacksquare

3.4.5 Proof of Proposition 3.4.14

We keep the notations and the general assumptions discussed in Section 3.4.2. In particular, we recall that τ_∞ is a $\text{GW}(\mu)$ -forest corresponding to the sequence $(\tau^{(k)})_{k \in \mathbb{N}^*}$ of i.i.d. $\text{GW}(\mu)$ -trees (Definition 3.2.6 (b)): namely $\theta_{(k)} \tau_\infty = \tau^{(k)}$, for all $k \in \mathbb{N}^*$. Also recall that conditionally given the

environment (\mathbf{T}, \circ) and the genealogy τ_∞ , for all $k \in \mathbb{N}^*$, $\Theta_k = (\tau^{(k)} \emptyset; \bar{Y}_u^{(k)} = (\mathbf{T}, Y_u^{(k)}), u \in \tau^{(k)})$ are independent \mathbf{T} -valued m_ν -biased branching RW such that $Y_\emptyset^{(k)} = \circ$. It is convenient to define a single τ_∞ -indexed branching RW $(Y_u)_{u \in \tau_\infty}$ by setting $Y_\emptyset = \circ$ and

$$\forall k \in \mathbb{N}^*, \forall u \in \tau^{(k)}, \quad Y_{(k)*u} := Y_u^{(k)}. \quad (3.4.64)$$

Recall that we denote by $(u_n)_{n \in \mathbb{N} \cup \{-1\}}$ the vertices of τ_∞ in lexicographical order: namely, $u_{-1} = \emptyset$, $u_0 = (1)$, etc. Most of the part of this section is devoted to the proof of the following result that implies Proposition 3.4.14. Before proving Proposition 3.4.23, we show that it implies Proposition 3.4.14.

Proposition 3.4.23 *We keep the assumptions and the notations of Proposition 3.4.14. Then, there are constants $c_0, c'_0, \varepsilon_0, \varepsilon_1 \in (0, \infty)$ which only depend on μ, ν and \mathfrak{b} , such that for all $n \in \mathbb{N}^*$,*

$$\mathbf{P} \left(\max_{m \in \llbracket 0, n \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > c'_0 n^{-\varepsilon_1} \sqrt{a_n} \right) \leq c_0 n^{-\varepsilon_0}. \quad (3.4.65)$$

Proof. The proof of Proposition 3.4.23 is given below, right after the proof of Proposition 3.4.23. ■

Proof of Proposition 3.4.14. We admit Proposition 3.4.23 and we prove that it implies Proposition 3.4.14. We first see that

$$\mathbf{E} \left[\sum_{j \in \mathbb{N}} \mathbf{P} \left(\max_{m \in \llbracket 0, 2^j \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > c'_0 2^{-\varepsilon_1 j} \sqrt{a_{2^j}} \mid (\mathbf{T}, \circ) \right) \right] < \infty$$

which entails \mathbf{P} -a.s. that $\lim_{j \rightarrow \infty} \mathbf{P} \left(\max_{m \in \llbracket 0, 2^j \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > c'_0 2^{-\varepsilon_1 j} \sqrt{a_{2^j}} \mid (\mathbf{T}, \circ) \right) = 0$. We next fix $s_0 \in (1, \infty)$. Since $n^{-\varepsilon_1} \sqrt{a_n}$ is a regularly varying sequence, there is $c \in (0, \infty)$ (which only depends on μ, ν and s_0) such that $cn^{-\varepsilon_1} \sqrt{a_n} \geq c'_0 2^{-\varepsilon_1 j} \sqrt{a_{2^j}}$ for all integers n such that $2^{j-1} \leq \lceil ns_0 \rceil \leq 2^j$. Since

$$\begin{aligned} \mathbf{P} \left(\max_{m \in \llbracket 0, \lceil ns_0 \rceil \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > cn^{-\varepsilon_1} \sqrt{a_n} \mid (\mathbf{T}, \circ) \right) \\ \leq \mathbf{P} \left(\max_{m \in \llbracket 0, 2^j \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > c'_0 2^{-\varepsilon_1 j} \sqrt{a_{2^j}} \mid (\mathbf{T}, \circ) \right), \end{aligned}$$

we \mathbf{P} -a.s. get $\lim_{n \rightarrow \infty} \mathbf{P} \left(\max_{m \in \llbracket 0, \lceil ns_0 \rceil \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > cn^{-\varepsilon_1} \sqrt{a_n} \mid (\mathbf{T}, \circ) \right) = 0$. Thanks to the specific interpolation defining the contour- and height-snakes, we next get

$$\max_{s \in [0, s_0]} \left| \frac{1}{\sigma_\nu^2} \widehat{W}_{ns}^*(\tau_\infty) - \widehat{W}_{ns}^*(\tau_\infty) \right| \leq \max_{m \in \llbracket 0, \lceil ns_0 \rceil \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right|$$

which completes the proof of Proposition 3.4.14. ■

Proof of Proposition 3.4.23. To simplify notations, we set

$$\Gamma_n = \max_{s \in [0, n]} H_s(\tau_\infty) = \max_{m \in \llbracket 0, n \rrbracket} (|u_m| - 1) \quad \text{and} \quad \tilde{\Gamma}_n = \max_{m \in \llbracket 0, n \rrbracket} |Y_{u_m}|.$$

We first prove for all $p, q \in \mathbb{N}^*$ that

$$\mathbf{P}(\tilde{\Gamma}_n \geq q) \leq \mathbf{P}(\Gamma_n \geq p) + 8np^3 \exp\left(-\frac{(q-1)^2}{2p}\right). \quad (3.4.66)$$

Proof. Let $\bar{X}_n = (\mathbf{T}, X_n)$, $n \in \mathbb{N}$, be a m_ν -biased RW on \mathbf{T} such that $X_0 = \circ$. By definition of branching RWs, we get

$$\mathbf{P}(\tilde{\Gamma}_n \geq q \mid (\mathbf{T}, \circ), \tau_\infty) \leq \sum_{1 \leq m \leq n} P_{\mathbf{T}, \circ}(\|X_{|u_m| - 1}\| \geq q) \leq \sum_{1 \leq m \leq n} P_{\mathbf{T}, \circ}(\max_{0 \leq k \leq \Gamma_n} \|X_k\| \geq q \mid \Gamma_n).$$

By Lemma 3.4.8, we get

$$\mathbf{1}_{\{\Gamma_n \leq p\}} \mathbf{P}(\tilde{\Gamma}_n \geq q | \tau_\infty) \leq \sum_{1 \leq m \leq n} \mathbf{P}\left(\max_{0 \leq k \leq p} \|X_k\| \geq q\right) \leq 8np^3 \exp\left(-\frac{(q-1)^2}{2p}\right),$$

which easily entails (3.4.66). \square

As in Lemma 3.4.2 (ii), we next fix $\beta \in (\frac{1}{2} + \frac{1}{2b}, 1)$, which is specified later and we defined the r.v. R_β as in (3.4.7). We then have

$$\forall p \in \mathbb{N}, \quad \left| \frac{1}{\sigma_\nu^2} S_{\circ(p)} - |\circ(p)| \right| \leq R_\beta(1 + p^\beta) \quad \text{and} \quad \mathbf{P}(R_\beta \geq \lambda) \leq c_1 \lambda^{-2b}, \quad \lambda \in (0, \infty) \quad (3.4.67)$$

where $c_1 \in (0, \infty)$ is a constant that only depends on ν, b and β . We then prove that if $2\tilde{\Gamma}_n \leq q$ and if $R_\beta \leq \lambda$, then

$$\left(\max_{m \in \llbracket 0, n \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > 2\lambda(1 + q^\beta) \right) \implies \left(\exists m \in \llbracket 0, n \rrbracket : u_m \in B_{q, \lambda, \beta} \right) \quad (3.4.68)$$

where $B_{q, \lambda, \beta}$ is the set of vertices of $\mathbf{T} \setminus L_\circ$ defined in Lemma 3.4.3.

Proof. Suppose that $m \in \llbracket 0, n \rrbracket$ is such that $\alpha_n := \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > 2\lambda(1 + q^\beta)$. Since $R_\beta \leq \lambda$, we get $\alpha_n \geq \lambda(1 + q^\beta) + R_\beta(1 + q^\beta)$ and thus

$$\left| \frac{1}{\sigma_\nu^2} (S_{Y_{u_m}} - S_{\circ \wedge Y_{u_m}}) - (|Y_{u_m}| - |\circ \wedge Y_{u_m}|) \right| + \left| \frac{1}{\sigma_\nu^2} S_{\circ \wedge Y_{u_m}} - |\circ \wedge Y_{u_m}| \right| \geq \alpha_n > \lambda(1 + q^\beta) + R_\beta(1 + q^\beta).$$

Note that there exists $u_l \in \llbracket 0, n \rrbracket$ such that $\circ \wedge Y_{u_m} = Y_{u_l}$. Since $\tilde{\Gamma}_n \leq q/2 \leq q$, there exists $j \in \llbracket 0, q \rrbracket$ such that $\circ \wedge Y_{u_m} = \circ(j)$ and by definition of R_β , we get

$$\left| \frac{1}{\sigma_\nu^2} S_{\circ \wedge Y_{u_m}} - |\circ \wedge Y_{u_m}| \right| \leq R_\beta(1 + j^\beta) \leq R_\beta(1 + q^\beta)$$

and thus

$$\left| \frac{1}{\sigma_\nu^2} (S_{Y_{u_m}} - S_{\circ \wedge Y_{u_m}}) - (|Y_{u_m}| - |\circ \wedge Y_{u_m}|) \right| > \lambda(1 + q^\beta). \quad (3.4.69)$$

Now observe that $|Y_{u_m}| - |\circ \wedge Y_{u_m}| = |Y_{u_m}| - |Y_{u_l}| \leq 2\tilde{\Gamma}_n \leq q$. Thus (3.4.69) implies that $u_m \in B_{q, \lambda, \beta}$, which completes the proof of (3.4.68). \square

We next set $V_m = V_m(\tau_\infty)$ and $I_m = \min_{l \in \llbracket 0, m \rrbracket} V_l$, where $V(\tau_\infty)$ is the Lukasiewicz path of τ_∞ . As an easy consequence of the concatenation (3.2.3) in Definition 3.2.2, the vertex u_n ‘belongs’ to $\tau^{(k)}$ where $k = -I_n + 1$ (here we mean that $u_n = (k) * v$ for some $v \in \tau^{(k)}$). We next prove that there is a constant $c_2 \in (0, \infty)$, which only depends on ν, b and β such that for all $p, q, r \in \mathbb{N}^*$ and all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \mathbf{P}\left(\max_{m \in \llbracket 0, n \rrbracket} \left| \frac{1}{\sigma_\nu^2} S_{Y_{u_m}} - |Y_{u_m}| \right| > 2\lambda(1 + q^\beta)\right) \\ & \leq \mathbf{P}(\Gamma_n \geq p) + 8np^3 e^{-\frac{(q-1)^2}{8p}} + \mathbf{P}(-I_n \geq r) + c_2 q r \lambda^{-2b}. \end{aligned} \quad (3.4.70)$$

Proof. Observe that

$$\begin{aligned} \mathbf{P}(\exists m \in \llbracket 0, n \rrbracket : u_m \in B_{q, \lambda, \beta}) & \leq \mathbf{P}(\exists k \leq -I_n + 1, \exists u \in \tau^{(k)} : Y_u \in B_{q, \lambda, \beta}) \\ & \leq \mathbf{P}(-I_n \geq r) + \sum_{1 \leq k \leq r} \mathbf{P}(\exists u \in \tau^{(k)} : Y_u \in B_{q, \lambda, \beta}) \\ & \leq \mathbf{P}(-I_n \geq r) + c_2 q r \lambda^{-2b}. \end{aligned}$$

by Lemma 3.4.3. Then we get (3.4.70) thanks to (3.4.66), (3.4.67), and (3.4.68). \square

We recall from (3.2.62) the definition of the slowly varying function L such that $b_n = n^{1/\alpha} L(n)$ and we also recall that $a_n = n/b_n = n^{1-\frac{1}{\alpha}}/L(n)$. Recall from (3.4.32) in Proposition 3.4.17 that for all $b_1 > 1$, there exists $c_3 \in (0, \infty)$ that only depends on μ and b_1 such that $\mathbf{E}[\Gamma_n^{b_1}] \leq c_3(a_n)^{b_1}$. Therefore,

$$\forall n, p \in \mathbb{N}^*, \quad \mathbf{P}(\Gamma_n \geq p) \leq c_3(a_n/p)^{b_1}. \quad (3.4.71)$$

We next prove that there exists $c_4 \in (0, \infty)$ that only depends on μ such that

$$\forall n, r \in \mathbb{N}^*, \quad \mathbf{P}(-I_n \geq r) \leq 3e^{-c_4 r a_n/n}. \quad (3.4.72)$$

Proof. Recall the notation $T_{-r} = \inf\{m \in \mathbb{N} : V_m = -r\}$. Observe that $\mathbf{P}(-I_n \geq r) = \mathbf{P}(T_{-r} \leq n)$. By Markov inequality, $\mathbf{P}(T_{-r} \leq n) \leq e \mathbf{E}[\exp(-\frac{1}{n} T_{-r})]$. We next use the fact that T_{-r} is a sum of r i.i.d. random variables distributed as T_{-1} . Then, $\mathbf{E}[\exp(-\frac{1}{n} T_{-r})] = \varphi_\mu(e^{-1/n})^r$, where we recall from Lemma 3.2.38 that φ_μ stands for the generating function of T_{-1} . By (3.2.76) in the same lemma, there is c_4 such that $c_4(n^{1/\alpha} L(n))^{-1} \leq -\log \varphi_\mu(e^{-1/n})$, which entails (3.4.72). \square

Since we assume that $b > \frac{2\alpha}{\alpha-1}$, then $(\frac{1}{2} + \frac{1}{2b}) + \frac{1}{2b} \frac{\alpha+1}{\alpha-1} < 1$. Thus we can find $\beta \in (\frac{1}{2} + \frac{1}{2b}, 1)$ and $\varepsilon \in (0, \infty)$ such that

$$\beta + \frac{1}{2b} \frac{\alpha+1}{\alpha-1} (1 + \varepsilon) + \varepsilon + \frac{3\varepsilon}{1+2\varepsilon} < 1. \quad (3.4.73)$$

Then by Potter's bounds (see e.g. Bingham, Goldies & Teugel [23, Theorem 1.5.6]), there exists $c_5 \in (1, \infty)$ that only depends on μ and ε such that

$$\forall n \in \mathbb{N}^*, \quad c_5^{-1} n^{(1-\frac{1}{\alpha})(1-\varepsilon)} \leq a_n = \frac{1}{L(n)} n^{1-\frac{1}{\alpha}} \leq c_5 n^{(1-\frac{1}{\alpha})(1+\varepsilon)}. \quad (3.4.74)$$

We then set

$$q = n^{\frac{1}{2}(1-\frac{1}{\alpha})(1+2\varepsilon)}, \quad p = q^{2 \cdot \frac{1+\frac{3}{2}\varepsilon}{1+2\varepsilon}}, \quad r = q^{\frac{2}{\alpha-1}} \quad \text{and} \quad \lambda = q^{\frac{1}{2b} \frac{\alpha+1}{\alpha-1} (1+\varepsilon)}.$$

Then we check that $\frac{1}{p} a_n \leq c_5 n^{-\frac{1}{2}(1-\frac{1}{\alpha})\varepsilon}$ and that $\frac{1}{n} r a_n \geq \frac{1}{c_5} n^{\frac{1}{2}\varepsilon}$. We fix $b_1 > 1$, and we check that there are positive constants c_6, \dots, c_{11} that only depend on μ, ν, b, β, b_1 and ε such that

$$\text{RHS}(3.4.70) \leq c_6 n^{-\frac{1}{2}(1-\frac{1}{\alpha})b_1\varepsilon} + c_7 n^{c_8} \exp(-c_9 n^{\frac{1}{2}\varepsilon(1-\frac{1}{\alpha})}) + 3 \exp(-c_{10} n^{\frac{1}{2}\varepsilon}) + c_{11} q^{-\frac{\alpha+1}{\alpha-1}\varepsilon}.$$

Here we used (3.4.71) and (3.4.72). Thus there are $c_0 \in (0, \infty)$ and $\varepsilon_0 \in (0, 1)$, which only depend on μ, ν, b, β, b_1 and ε , such that $\text{RHS}(3.4.70) \leq c_0 n^{-\varepsilon_0}$.

Next, by (3.4.73) and (3.4.74), there are positive constants c_{12}, c_{13} , which only depend on μ, ν, b, β, b_1 and ε , such that

$$2\lambda(1+q^\beta) \leq c_{12} q^{\beta + \frac{1}{2b} \frac{\alpha+1}{\alpha-1} (1+\varepsilon)} \leq c_{12} q^{1-\varepsilon - \frac{3\varepsilon}{1+2\varepsilon}} \leq c_{12} q^{-\varepsilon} q^{\frac{1-\varepsilon}{1+2\varepsilon}} = c_{12} q^{-\varepsilon} n^{\frac{1}{2}(1-\frac{1}{\alpha})(1-\varepsilon)} \leq c_{13} q^{-\varepsilon} a_n^{\frac{1}{2}}.$$

This proves that $2\lambda(1+q^\beta) \leq c'_0 n^{-\varepsilon_1} \sqrt{a_n}$ where we have set $\varepsilon_1 = \frac{1}{2}(1-\frac{1}{\alpha})(1+2\varepsilon)\varepsilon$ and $c'_0 = c_{13}$. This implies that $\text{LHS}(3.4.65) \leq \text{LHS}(3.4.70) \leq \text{RHS}(3.4.70) \leq c_0 n^{-\varepsilon_0}$, which completes the proof of (3.4.65). \blacksquare

3.5 Proof of Theorem 3.1.2

3.5.1 Limit of the τ_n -indexed RW and its snakes.

Before stating the main result of the section, let us introduce some notations. We fix $\alpha \in (1, 2]$ and we fix an offspring distribution μ that satisfies (3.2.60). Here ν is a supercritical offspring

distribution whose mean is denoted by \mathfrak{m}_ν . We assume that there is $\mathfrak{b} \in (\frac{2\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2\mathfrak{b}} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree as in Definition 3.2.25.

Let $\tau' : \Omega \rightarrow \mathbb{T}$ be an (auxiliary) a.s. finite random ordered rooted tree. Recall that $V(\tau')$ is its Lukasiewicz path, that $H(\tau')$ is its height process (Definition 3.2.2) and that $C(\tau')$ is its contour process (Definition 3.2.4). We then denote by $\Theta(\tau') = (\tau', \emptyset; \bar{Y}_u(\tau') = (\mathbf{T}, Y_u(\tau')), u \in \tau')$ a τ' -indexed and \mathbf{T} -valued \mathfrak{m}_ν -biased branching RW such that conditionally given (\mathbf{T}, \circ) and τ' , $\Theta(\tau')$ has the law $Q_{(\tau', \emptyset)}^{(\mathbf{T}, \circ)}$ as in Definition 3.2.30 with $\lambda = \mathfrak{m}_\nu$. We associate with $\Theta(\tau')$ two \mathbb{R} -valued and τ' -indexed branching RWs

$$M(\tau') := (|Y_u(\tau')|)_{u \in \tau'} \quad \text{and} \quad \mathcal{M}(\tau') := (S_{Y_u(\tau')})_{u \in \tau'},$$

where we recall from (3.4.3) the definition of the harmonic coordinates $(S_x)_{x \in \mathbf{T}}$. According to Definition 3.2.8, we introduce the following four $\mathbf{C}^0([0, \infty), \mathbb{R})$ -valued continuous processes:

- $(W_s(\tau'; \cdot))_{s \in [0, \infty)}$ is the contour-snake of the branching random walk $M(\tau')$;
- $(W_s^*(\tau'; \cdot))_{s \in [0, \infty)}$ is the height-snake of the branching random walk $M(\tau')$;
- $(\mathcal{W}_s(\tau'; \cdot))_{s \in [0, \infty)}$ is the contour-snake of the branching random walk $\mathcal{M}(\tau')$;
- $(\mathcal{W}_s^*(\tau'; \cdot))_{s \in [0, \infty)}$ is the height-snake of the branching random walk $\mathcal{M}(\tau')$.

In this section, the auxiliary tree τ' shall be distributed as the following trees. Let τ be an auxiliary $\text{GW}(\mu)$ -tree (Definition 3.2.6 (a)). Then for all $n \in \mathbb{N}^*$, we denote by τ_n and $\tau_{\geq n}$ random trees such that

$$\tau_{\geq n} \stackrel{(\text{law})}{=} \tau \quad \text{under } \mathbf{P}(\cdot | \#\tau \geq n) \quad \text{and} \quad \tau_n \stackrel{(\text{law})}{=} \tau \quad \text{under } \mathbf{P}(\cdot | \#\tau = n), \quad (3.5.1)$$

the definition of τ_n only making sense when $\mathbf{P}(\#\tau = n) > 0$, which is the case for all sufficiently large n since μ is assumed to be aperiodic.

We finally introduce the processes (X'', H'', W'') and (X', H', W') defined on $(\Omega, \mathcal{F}, \mathbf{P})$ whose laws are described as follows.

- The process (X'', H'') is distributed as $(X, H(X))$ under $\mathbf{N}(dX | \zeta > 1)$ and conditionally given H'', W'' is the one-dimensional Brownian snake with lifetime process H'' (see Proposition 3.2.47).
- The process (X', H') is distributed as $(X, H(X))$ under $\mathbf{N}(dX | \zeta = 1)$ and conditionally given H', W' is the one-dimensional Brownian snake with lifetime process H' (see Proposition 3.2.47).

The goal of this section is to derive from Theorem 3.4.9 the following result that is the main argument of the proof of Theorem 3.1.2.

Theorem 3.5.1 *We fix $\alpha \in (1, 2]$. Let μ satisfy (3.2.60). Let τ_n be as in (3.5.1) (i.e. namely, a $\text{GW}(\mu)$ -tree conditioned to have n vertices). Let ν be a supercritical offspring distribution whose mean is denoted by \mathfrak{m}_ν . We assume that there is $\mathfrak{b} \in (\frac{2\alpha}{\alpha-1}, \infty)$ such that $\sum_{k \in \mathbb{N}} k^{1+2\mathfrak{b}} \nu(k) < \infty$. Let (\mathbf{T}, \circ) be an invariant $\text{GW}(\nu)$ -tree that is independent from τ_n . We recall the a_n and b_n from (3.4.13) and we recall from (3.4.6) the definition of σ_ν . We also recall the notation \mathcal{Q}'_n from (3.2.72) in Theorem 3.2.36 (iv). We keep the above notations. Then, conditionally given (\mathbf{T}, \circ) , the following convergence*

$$\left(\mathcal{Q}'_n, \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_n), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_n^*(\tau_n), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{2n}(\tau_n), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_n^*(\tau_n) \right) \xrightarrow[n \rightarrow \infty]{} (X', H', H', \widehat{W}', \widehat{W}', \widehat{W}', \widehat{W}') \quad (3.5.2)$$

a.s. holds weakly on $\mathbf{D}([0, 1], \mathbb{R}) \times \mathbf{C}([0, 1], \mathbb{R})^6$ equipped with the product topology.

Proof. We set $Z''_n = (\frac{1}{b_n} V_{\lfloor n \cdot \rfloor}(\tau_{\geq n}), \frac{1}{a_n} H_{n \cdot}(\tau_{\geq n}), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{n \cdot}^*(\tau_{\geq n}), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{n \cdot}^*(\tau_{\geq n}))$ for all $n \in \mathbb{N}^*$ and we also set $Z'' = (X'', H'', \widehat{W}'', \widehat{W}'')$. We first prove that conditionally given (\mathbf{T}, \circ) , the following convergence

$$Z''_n \xrightarrow[n \rightarrow \infty]{} Z'' \quad (3.5.3)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^3$. *Indeed*, by easy arguments, Theorem 3.4.9 entails that conditionally on (\mathbf{T}, \circ) , the following convergence

$$\begin{aligned} & \left(\frac{1}{b_n} V_{\lfloor (g_n+n \cdot) \wedge d_n \rfloor}(\tau_\infty), \frac{1}{a_n} H_{(g_n+n \cdot) \wedge d_n}(\tau_\infty), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{(g_n+n \cdot) \wedge d_n}^*(\tau_\infty), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{(g_n+n \cdot) \wedge d_n}^*(\tau_\infty) \right) \\ & \xrightarrow[n \rightarrow \infty]{} \left(X_{(\mathbf{g}_1+\cdot) \wedge \mathbf{d}_1}, H_{(\mathbf{g}_1+\cdot) \wedge \mathbf{d}_1}, \widehat{W}_{(\mathbf{g}_1+\cdot) \wedge \mathbf{d}_1}, \widehat{W}'_{(\mathbf{g}_1+\cdot) \wedge \mathbf{d}_1} \right) \end{aligned} \quad (3.5.4)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^3$. Then, as a consequence of the definition of \mathbf{g}_n and \mathbf{d}_n , the left member of (3.5.4) has the same joint law with (\mathbf{T}, \circ) as $(Z''_n, (\mathbf{T}, \circ))$. Similarly, the definition of \mathbf{g}_1 and \mathbf{d}_1 implies that the right member of (3.5.4) is distributed as Z'' (see Remark 3.2.34). Thus, (3.5.4) is equivalent to (3.5.3). \square

To complete the proof of the theorem, we next want to use Lemma 3.2.37. To that end, we fix $s \in (0, 1)$ and we set $s_n = \lfloor ns \rfloor$. We also set

$$\begin{aligned} - Z''_n(s) &= \left(\frac{1}{b_n} V_{\lfloor n \cdot \rfloor \wedge s_n}(\tau_{\geq n}), \frac{1}{a_n} H_{(n \cdot) \wedge s_n}(\tau_{\geq n}), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{(n \cdot) \wedge s_n}^*(\tau_{\geq n}), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{(n \cdot) \wedge s_n}^*(\tau_{\geq n}) \right), \\ - Z'_n(s) &= \left(\frac{1}{b_n} V_{\lfloor n \cdot \rfloor \wedge s_n}(\tau_n), \frac{1}{a_n} H_{(n \cdot) \wedge s_n}(\tau_n), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{(n \cdot) \wedge s_n}^*(\tau_n), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{(n \cdot) \wedge s_n}^*(\tau_n) \right), \\ - Z''(s) &= (X''_{\cdot \wedge s}, H''_{\cdot \wedge s}, \widehat{W}''_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}) \text{ and } Z'(s) = (X'_{\cdot \wedge s}, H'_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}). \end{aligned}$$

We easily deduce from (3.2.59) that $\mathbf{E}[F(Z''(s))D_s(X''_s)] = \mathbf{E}[F(Z'(s))]$ for all measurable nonnegative function F . We next deduce from (3.5.3) that conditionally given (\mathbf{T}, \circ) , $Z''_n(s) \rightarrow Z''(s)$ weakly on the appropriate spaces. Next denote by $(u_k)_{0 \leq k < \#\tau_{\geq n}}$ the vertices of $\tau_{\geq n}$ listed in lexicographical order. Observe that the subtree $\{u_l; 0 \leq l \leq s_n\}$ is measurable with respect to $V_{\cdot \wedge s_n}(\tau_{\geq n})$. Thus, $Z''_n(s)$ is a measurable function of $V_{\cdot \wedge s_n}(\tau_{\geq n})$ and of an independent r.v. U_n . Consequently Lemma 3.2.37 applies and conditionally given (\mathbf{T}, \circ) the joint convergence

$$Z'_n(s) \xrightarrow[n \rightarrow \infty]{} Z'(s) \quad (3.5.5)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^3$.

We next derive the joint convergence of the related contour process and contour-snakes via the time-change discussed in Remark 3.2.5 (d). Namely, there is an increasing continuous bijection $\phi_{\tau_n} : [0, n] \rightarrow [0, 2n]$ such that $H_r(\tau_n) = C_{\phi_{\tau_n}(r)}(\tau_n)$, $r \in [0, n]$. Moreover, deduce from (3.2.9) that $\max_{r \in [0, 1]} |\frac{1}{2n} \phi_{\tau_n}(nr) - r| \leq 3 \frac{a_n}{n} \max_{s \in [0, 1]} \frac{1}{a_n} H_{nr}(\tau_n)$, which tends to 0 in \mathbf{P} -probability since $a_n/n \rightarrow 0$. Therefore, if for all $s \in [0, 1]$ we set $\varphi_n(s) = \frac{1}{n} \phi_{\tau_n}^{-1}(2ns)$ (here, $\phi_{\tau_n}^{-1}$ stands for the inverse of ϕ_{τ_n}), then we get

$$\begin{aligned} (i) \quad & \lim_{n \rightarrow \infty} \max_{s \in [0, 1]} |\varphi_n(s) - s| = 0 \text{ in } \mathbf{P}\text{-probability,} \\ (ii) \quad & C_{2ns}(\tau_n) = H_{n\varphi_n(s)}(\tau_n), \widehat{W}_{2ns}(\tau_n) = \widehat{W}_{n\varphi_n(s)}^*(\tau_n), \widehat{W}'_{2ns}(\tau_n) = \widehat{W}'_{n\varphi_n(s)}(\tau_n) \text{ for all } s \in [0, 1]. \end{aligned}$$

This combined with Lemma 3.4.13 shows that (3.5.5) implies for all $s \in (0, 1)$ that conditionally given (\mathbf{T}, \circ) the joint convergence

$$\left(Z'_n(s), \frac{1}{a_n} C_{2n(\cdot \wedge s)}(\tau_n), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n(\cdot \wedge s)}(\tau_n), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}'_{2n(\cdot \wedge s)}(\tau_n) \right) \xrightarrow[n \rightarrow \infty]{} (Z'(s), H'_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}) \quad (3.5.6)$$

holds weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}^0([0, \infty), \mathbb{R})^6$.

To complete the proof of the theorem, we next use a symmetry property of contour processes and contour-snakes. More precisely, denote by $\tilde{\tau}_n$ the mirror image of τ_n . Namely there exists

a height-preserving and $\leq_{1\text{ex}}$ -increasing bijection $u \in \tau_n \mapsto \tilde{u} \in \tilde{\tau}_n$ such that $(\tilde{v}_k) = \tilde{v}_{2(n-1)-k}$, where $(v_k)_{0 \leq k \leq 2(n-1)}$ and $(\tilde{v}_k)_{0 \leq k \leq 2(n-1)}$ stand for the vertices of resp. τ_n and $\tilde{\tau}_n$ in increasing contour order. Then, for all $s \in [0, n]$, we get

$$C_{2(n-s)}(\tau_n) = C_{2(s-1)_+}(\tilde{\tau}_n), \quad \widehat{W}_{2(n-s)}(\tau_n) = \widehat{W}_{2(s-1)_+}(\tilde{\tau}_n) \quad \text{and} \quad \widehat{W}'_{2(n-s)}(\tau_n) = \widehat{W}'_{2(s-1)_+}(\tilde{\tau}_n).$$

Since $\tilde{\tau}_n$ and τ_n have the same law, (3.5.6) then implies for all $s \in (0, 1)$ that conditionally given (\mathbf{T}, \circ) , the joint convergence

$$\left(\frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n((1-\cdot) \wedge s)}(\tau_n), \frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{2n((1-\cdot) \wedge s)}(\tau_n) \right) \xrightarrow{n \rightarrow \infty} (\widehat{W}'_{\cdot \wedge s}, \widehat{W}'_{\cdot \wedge s}).$$

holds weakly on $\mathbf{C}^0([0, \infty), \mathbb{R})^2$. If $s \in (\frac{1}{2}, 1)$, the previous limit combined with (3.5.6) imply that conditionally given (\mathbf{T}, \circ) the laws of the processes $\frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_n)$ and $\frac{\sigma_\nu}{\sqrt{a_n}} \widehat{W}_{2n}(\tau_n)$ are tight on $\mathbf{C}([0, 1], \mathbb{R})$. Then recall from Theorem 3.2.36 (iv) that the laws of the processes $\frac{1}{b_n} V_{[n \cdot]}(\tau_n)$, $\frac{1}{a_n} H_n(\tau_n)$ and $\frac{1}{a_n} C_{2n}(\tau_n)$ are tight on resp. $\mathbf{D}([0, 1], \mathbb{R})$ and $\mathbf{C}([0, 1], \mathbb{R})$. Since s can be chosen arbitrarily close to 1 in (3.5.6), the processes that are weak limits of the previous ones have necessarily the same finite-dimensional marginals as $(X', H', H', \widehat{W}', \widehat{W}')$. It therefore shows that conditionally given (\mathbf{T}, \circ) the following convergence

$$\left(\mathcal{Q}'_n, \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_n), \frac{1}{\sigma_\nu \sqrt{a_n}} \widehat{W}_{2n}(\tau_n) \right) \xrightarrow{n \rightarrow \infty} (X', H', H', \widehat{W}', \widehat{W}') \quad (3.5.7)$$

holds weakly on $\mathbf{D}([0, 1], \mathbb{R}) \times \mathbf{C}([0, 1], \mathbb{R})^4$. Now observe that for all $s \in [0, 1]$,

$$\widehat{W}_{ns}^*(\tau_n) = \widehat{W}_{2n\varphi_n^{-1}(s)}(\tau_n) \quad \text{and} \quad \widehat{W}'_{ns}^*(\tau_n) = \widehat{W}'_{2n\varphi_n^{-1}(s)}(\tau_n).$$

where $\varphi_n^{-1} : [0, 1] \rightarrow [0, 1]$ stands for the inverse of φ_n . From the above-mentioned uniform convergence of the φ_n , we easily prove that $\lim_{n \rightarrow \infty} \max_{s \in [0, 1]} |\varphi_n^{-1}(s) - s| = 0$ in \mathbf{P} -probability and we finally derive the convergence (3.5.2) from (3.5.7) and Lemma 3.4.13. This completes the proof of the theorem. \blacksquare

3.5.2 Proof of Theorem 3.1.2

We keep the notations and the assumptions of Theorem 3.5.1. In particular τ_n has the same law as a $\text{GW}(\mu)$ -tree τ under $\mathbf{P}(\cdot | \#\tau = n)$ and $\Theta(\tau_n) = (\tau_n, \emptyset; \overline{Y}_u = (\mathbf{T}, Y_u), u \in \tau_n)$ is a τ_n -indexed and \mathbf{T} -valued \mathfrak{m}_ν -biased branching RW such that conditionally given (\mathbf{T}, \circ) and τ_n , $\Theta(\tau_n)$ has the law $Q_{(\tau_n, \emptyset)}^{(\mathbf{T}, \circ)}$ in Definition 3.2.30 with $\lambda = \mathfrak{m}_\nu$. Recall that τ_n is independent of (\mathbf{T}, \circ) , which is an invariant $\text{GW}(\nu)$ -tree. We set

$$\mathfrak{m}_n = \sum_{v \in \tau_n} \delta_v, \quad \mathcal{R}_n = \{Y_v; v \in \tau_n\} \quad \text{and} \quad \mathfrak{m}_{\text{occ}}^{(n)} = \sum_{v \in \tau_n} \delta_{Y_v}.$$

We want to prove conditionally given (\mathbf{T}, \circ) a joint convergence of the following rescaled pointed measured metric spaces equipped with their graph-distances d_{gr} :

$$\tau_n := \left(\tau_n, \frac{1}{a_n} d_{\text{gr}}, \emptyset, \frac{1}{n} \mathfrak{m}_n \right) \quad \text{and} \quad \mathcal{R}_n := \left(\mathcal{R}_n, \frac{\sigma_\nu}{\sqrt{a_n}} d_{\text{gr}}, \circ, \frac{1}{n} \mathfrak{m}_{\text{occ}}^{(n)} \right)$$

weakly with respect to the Gromov-Hausdorff-Prokhorov convergence.

To that end, we first prove a convergence of the snake-trees associated with the contour-snake of the relative heights of $\Theta(\tau_n)$. Namely, recall that $W(\tau_n; \cdot)$ stands for the contour-snake associated

with the \mathbb{R} -branching random walk $(|Y_v(\tau_n)|)_{v \in \tau_n}$ as in Definition 3.2.8. To simplify notation, we set for all $s \in [0, 1]$ and $r \in [0, \infty)$,

$$C_s^{(n)} = \frac{1}{a_n} C_{2ns}(\tau_n), \quad W_s^{(n)}(r) = \frac{\sigma_\nu}{\sqrt{a_n}} W_{2ns}(\tau_n; a_n r).$$

Then $(W_s^{(n)}(\cdot))_{s \in [0,1]}$ is a snake with lifetime process $(C_s^{(n)})_{s \in [0,1]}$ (as in Definition 3.2.43). We recall the notation $d_{C^{(n)}, W^{(n)}}$ for the snake-pseudometric associated with $(C^{(n)}, W^{(n)})$ (Definition 3.2.48). We also recall from (3.2.84) that $d_{C^{(n)}}$ stands for the tree-pseudometric associated with $C^{(n)}$. By combining Theorem 3.5.1, Proposition 3.2.51 and Lemma 3.2.52, we get conditionally given $(\mathbf{T}, \mathfrak{o})$ the following joint convergence

$$(d_{C^{(n)}}, d_{C^{(n)}, W^{(n)}}) \xrightarrow[n \rightarrow \infty]{} (d_H, d_{H,W}) \quad (3.5.8)$$

weakly on $\mathbf{C}([0, 1]^2, \mathbb{R})^2$. Here H is the α -stable height process under $\mathbf{N}(\cdot | \zeta = 1)$ and W is a one-dimensional Brownian snake with lifetime process H as in Proposition 3.2.47. Then to simplify, we denote by $\tilde{\tau}_n = (T_{C^{(n)}}, d_{C^{(n)}}, \mathbf{r}_{C^{(n)}}, \mathfrak{m}_{C^{(n)}})$ the real tree coded by the function $C^{(n)}$ as in Example 3.2.42. By Remark 3.2.53, we get that

$$\delta_{\text{GHP}}(\tau_n, \tilde{\tau}_n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (3.5.9)$$

To simplify, also denote the snake-tree coded by the snake-pseudometric $d_{C^{(n)}, W^{(n)}}$ by

$$\mathcal{T}_{\text{sna}}^{(n)} = (\mathcal{T}_{\text{sna}}^{(n)}, d_{\text{sna}}^{(n)}, \mathbf{r}_{\text{sna}}^{(n)}, \mathfrak{m}_{\text{sna}}^{(n)}).$$

(See Definition 3.2.50). By Proposition 3.2.41, the limits (3.5.8) and (3.5.9) imply that conditionally given $(\mathbf{T}, \mathfrak{o})$ the joint convergence

$$(\tau_n, \mathcal{T}_{\text{sna}}^{(n)}) \xrightarrow[n \rightarrow \infty]{} ((T_H, d_H, \mathbf{r}_H, \mathfrak{m}_H), (T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathfrak{m}_{H,W})) \quad (3.5.10)$$

holds weakly with respect to Gromov-Hausdorff-Prokhorov convergence. Here $(T_H, d_H, \mathbf{r}_H, \mathfrak{m}_H)$ is the real tree coded by H which is the α -stable tree and $(T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathfrak{m}_{H,W})$ is the snake-tree coded by the snake-pseudometric $d_{H,W}$ (Definition 3.2.50) which is the α -stable Brownian cactus.

We next prove that \mathcal{R}_n is close to $\mathcal{T}_{\text{sna}}^{(n)}$ thanks to the results discussed in Section 3.2.8. To that end, we introduce the following.

- We denote by $(\tilde{\tau}_n, d_{\tilde{\tau}_n})$ the real tree obtained by joining adjacent vertices in τ_n by a unit length segment (recall that we suppose here that $\tau_n \subset \tilde{\tau}_n$ so that $d_{\tilde{\tau}_n}$ extend the graph-distance).
- Recall from (3.2.92) that $\tilde{v} : [0, 2n] \rightarrow \tilde{\tau}_n$ stands for the contour exploration of $\tilde{\tau}_n$ (namely $(\tilde{v}(k))_{0 \leq k \leq 2(n-1)}$ are the vertices of τ_n in contour order and \tilde{v} is a ‘linear’ extension of it).
- We denote by $(\tilde{\mathbf{T}}, d_{\tilde{\mathbf{T}}})$ the real tree obtained by joining adjacent vertices in \mathbf{T} by a unit length segment. We also assume that $\mathbf{T} \subset \tilde{\mathbf{T}}$ so that $d_{\tilde{\mathbf{T}}}$ extends the graph distance.
- Recall from (3.2.93) that $\tilde{Y} : \tilde{\tau}_n \rightarrow \tilde{\mathbf{T}}$ is the ‘linear’ extension of the branching random walk $(Y_v)_{v \in \tau_n}$.

We set $\tilde{\mathcal{R}}_n = (\tilde{\mathcal{R}}_n, \frac{\sigma_\nu}{\sqrt{a_n}} d_{\tilde{\mathbf{T}}}, \mathfrak{o}, \frac{1}{2n} \tilde{\mathfrak{m}}_{\text{occ}})$ where $\tilde{\mathcal{R}}_n = \{\tilde{Y}_\sigma; \sigma \in \tilde{\tau}_n\} = \{\tilde{Y}_{\tilde{v}(s)}; s \in [0, 2n]\}$ and $\tilde{\mathfrak{m}}_{\text{occ}}$ is the occupation measure of $s \mapsto \tilde{Y}_{\tilde{v}(s)}$ as defined by (3.2.94). We then set for all $s, s' \in [0, 1]$:

$$d_n(s, s') = \frac{\sigma_\nu}{\sqrt{a_n}} d_{\tilde{\mathbf{T}}}(\tilde{Y}_{\tilde{v}(2ns)}, \tilde{Y}_{\tilde{v}(2ns')}) ,$$

which is a tree-pseudometric. Recall from (3.2.97) that $\widetilde{\mathcal{R}}_n$ is isometric to the pointed measured compact real tree yielded by the pseudometric d_n . Then (3.2.95) implies that conditionally given (\mathbf{T}, \circ) , we have the weak convergence

$$\delta_{\text{GHP}}(\mathcal{R}_n, \widetilde{\mathcal{R}}_n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (3.5.11)$$

We next prove that d_n and the snake-pseudometric $d_{C^{(n)}, W^{(n)}}$ are close thanks to Lemma 3.2.55, which applies here thanks to Lemma 3.2.24. We thus get the following bound:

$$\mathbf{P}\left(\max_{s, s' \in [0, 1]} |d_n(s, s') - d_{C^{(n)}, W^{(n)}}(s, s')| > 2\sigma_\nu a_n^{-\frac{1}{4}} \mid (\mathbf{T}, \circ)\right) \leq m_\nu^2 n^3 \exp(- (a_n)^{\frac{1}{4}} \log m_\nu).$$

Since a_n is regularly varying with a positive exponent, this thus entails that conditionally given (\mathbf{T}, \circ) the following joint convergence

$$(d_{C^{(n)}}, d_{C^{(n)}, W^{(n)}}, d_n) \xrightarrow[n \rightarrow \infty]{} (d_H, d_{H,W}, d_{H,W}) \quad (3.5.12)$$

holds weakly on $\mathbf{C}([0, 1]^2, \mathbb{R})^3$. By Proposition 3.2.41, it entails that conditionally given (\mathbf{T}, \circ) the following joint convergence

$$(\tau_n, \widetilde{\mathcal{R}}_n) \xrightarrow[n \rightarrow \infty]{} ((T_H, d_H, \mathbf{r}_H, \mathfrak{m}_H), (T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathfrak{m}_{H,W})) \quad (3.5.13)$$

holds weakly with respect to Gromov-Hausdorff-Prokhorov convergence. By (3.5.11), we thus have proved that conditionally given (\mathbf{T}, \circ) , the following joint convergence

$$(\tau_n, \mathcal{R}_n) \xrightarrow[n \rightarrow \infty]{} ((T_H, d_H, \mathbf{r}_H, \mathfrak{m}_H), (T_{H,W}, d_{H,W}, \mathbf{r}_{H,W}, \mathfrak{m}_{H,W})) \quad (3.5.14)$$

holds weakly with respect to Gromov-Hausdorff-Prokhorov convergence, which is the desired result. \blacksquare

Appendices to Chapter 3

3.A Proof of Lemma 3.2.33

We fix $\alpha \in (1, 2]$ and we recall that X is an α -stable spectrally positive Lévy process whose law is characterized by (3.2.52). Recall that H is the corresponding height process as defined in (3.2.53). We also recall the notation $I_s = \inf_{r \in [0, s]} X_r$, $s \in [0, \infty)$, for the infimum process of X . In the Brownian case $\alpha = 2$, H is a reflected Brownian motion and Lemma 3.2.33 is a standard property. The case $\alpha \neq 2$ represents the technical part of the proof. It uses the exploration process whose definition is briefly recalled below from D. & Le Gall [61, Chapter 1]. To that end, we need to introduce the following notations.

Denote by $\mathcal{M}_f([0, \infty))$ the set of finite Borel measures on $[0, \infty)$. For all $\mu \in \mathcal{M}_f([0, \infty))$ we set $\langle \mu \rangle = \mu([0, \infty))$ and for all bounded measurable f on $[0, \infty)$ we set $\langle \mu, f \rangle = \int_{[0, \infty)} f d\mu$. We then denote by $H(\mu) = \sup \text{supp}(\mu)$ where $\text{supp}(\mu)$ stands for the topological support of μ (note that $H(\mu)$ may be infinite). We set $H(\mu) = 0$ if μ is the null measure. If $H(\mu) < \infty$ and if $\nu \in \mathcal{M}_f([0, \infty))$, then we denote by $[\mu, \nu]$ the measure $\pi \in \mathcal{M}_f([0, \infty))$ characterized by $\langle \pi, f \rangle = \langle \mu, f \rangle + \int_{[0, \infty)} f(H(\mu) + x) \nu(dx)$, for all bounded measurable f on $[0, \infty)$. Note that $H([\mu, \nu]) = H(\mu) + H(\nu)$. Finally, for all $a \in [0, \infty)$, we denote by $k_a \mu$ the measure characterized by $k_a \mu([0, x]) = \mu([0, x]) \wedge (\langle \mu \rangle - a)_+$, for all $x \in [0, \infty)$. Note that $H(k_a \mu) < \infty$ if $a > 0$. Below, we shall use the following elementary observation: *for all $a \in (0, \infty)$, it holds that*

$$H(k_a \mu) = H(\mu) \implies \mu(\{H(\mu)\}) > 0. \quad (3.A.1)$$

We now introduce the *exploration process* $(\rho_s)_{s \in [0, \infty)}$ associated with the Lévy process X that is the $\mathcal{M}_f([0, \infty))$ -valued process defined for all $s \in (0, \infty)$ by

$$\rho_s(dx) = \sum_{\substack{r \in (0, s] \\ X_{r-} < I_{r,s}}} (I_{r,s} - X_{r-}) \delta_{H_r}(dx) \quad (3.A.2)$$

where we have set $I_{r,s} = \inf_{[r,s]} X$ for all real numbers r and s such that $r \leq s$. This process has been introduced in Le Gall & Le Jan [81] and studied further in D. & Le Gall [61]. In the regular version introduced above we \mathbf{P} -a.s. get for all $s \in [0, \infty)$ that $H_s = H(\rho_s)$. It is then easy to derive from (3.A.2) \mathbf{P} -a.s. for all $s \in [0, \infty)$ that $\langle \rho_s \rangle = X_s - I_s$ and more generally for all $r \in (0, s]$ such that $X_{r-} < I_{r,s}$ that

$$\rho_s([H_r, H_s]) = X_s - X_{r-}. \quad (3.A.3)$$

The exploration process is cadlag and shares with X the same times of discontinuity. More precisely, \mathbf{P} -a.s. for all $s \in [0, \infty)$, $\rho_s = \rho_{s-} + \Delta X_s \delta_{H_s}$, where we have set $\Delta X_s := X_s - X_{s-}$.

It moreover satisfies the following Markov property: fix $s_1 \in (0, \infty)$ and set $X_s^{(s_1)} = X_{s_1+s} - X_{s_1}$ and $I_s^{(s_1)} = \inf_{r \in [0, s]} X_r^{(s_1)}$ for all $s \in [0, \infty)$. Note that $X^{(s_1)}$ has the same law as X and that it is independent from $(X_s)_{s \in [0, s_1]}$. We denote by $(\rho_s^{(s_1)})_{s \in [0, \infty)}$ the exploration process associated with $X^{(s_1)}$. Then \mathbf{P} -a.s. for all $s \in [0, \infty)$, we get

$$\rho_{s_1+s} = [k_{-I_s^{(s_1)}} \rho_{s_1}, \rho_s^{(s_1)}]. \quad (3.A.4)$$

It thus implies for all $s \in [0, \infty)$ that

$$H_{s_1+s} = H(k_{-I_s^{(s_1)}} \rho_{s_1}) + H(\rho_s^{(s_1)}) \quad (3.A.5)$$

and since $s \mapsto H(k_{-I_s^{(s_1)}} \rho_{s_1})$ is nonincreasing, we get \mathbf{P} -a.s. for all $s \in [0, \infty)$,

$$\min_{r \in [s_1, s_1+s]} H_r = H(k_{-I_s^{(s_1)}} \rho_{s_1}) = H_{s_1+g_s^{(s_1)}} \quad (3.A.6)$$

where we have set $g_s^{(s_1)} = \sup\{r \in [0, s] : X_r^{(s_1)} = I_r^{(s_1)}\}$. We are now ready to prove the lemma.

Proof of Lemma 3.2.33. Let $s_2 > s_1 > 0$. By (3.A.5) and (3.A.6), we exactly need to show that \mathbf{P} -a.s. $H(\rho_{s_1}) > H(k_{-I_{s_2-s_1}^{(s_1)}} \rho_{s_1})$ and $H(\rho_{s_2-s_1}^{(s_1)}) > 0$. We \mathbf{P} -a.s. get from (3.A.3) that $\rho_{s_1}(\{H_{s_1}\}) = 0$.

Thus (3.A.1) implies that we only have to prove that \mathbf{P} -a.s. $-I_{s_2-s_1}^{(s_1)} > 0$ and $H(\rho_{s_2-s_1}^{(s_1)}) > 0$.

We recall that $-I$ is a local time of the Markov process $X - I$ at 0 which is a regular and instantaneous point. Let us set $\mathcal{Z} := \{s \in [0, \infty) : X_s = I_s\}$. Standard results on local times imply that for all positive number $s > 0$, $\mathbf{P}(s \notin \mathcal{Z}, -I_s > 0) = 1$. The equality $\mathcal{Z} = \{s \in [0, \infty) : H_s = 0\}$ and the identity in law between $X^{(s_1)}$ and X then complete the proof. \blacksquare

3.B Proof of Lemma 3.2.37

We keep the notations and the assumptions of Theorem 3.2.36 and of Lemma 3.2.37. To simplify notations, denote by \mathbf{D}_1 the space $\mathbf{D}([0, \infty), \mathbb{R})$. Product spaces are equipped with product topology and if E'' is a Polish space, we denote by $\mathcal{M}_1(E'')$ the space of its Borel probability measures equipped with the topology of weak convergence.

We denote by $q_n(\cdot, du) : \Omega \rightarrow \mathcal{M}_1(E)$ a regular version of the conditional law of U_n given \mathcal{G} . We also denote by $\mathbb{Q}_{\geq n}(dv)$ and $\mathbb{Q}_n(dv)$ the laws of resp. $\frac{1}{b_n} V_{[n \cdot] \wedge r_n}(\tau_{\geq n})$ and $\frac{1}{b_n} V_{[n \cdot] \wedge r_n}(\tau_n)$. Note that $\mathbb{Q}_n(dv) = D_{r_n}^{(n)}(b_n v(r'_n)) \mathbb{Q}_{\geq n}(dv)$ by Theorem 3.2.36 (ii), where we have set here $r'_n = r_n/n$. For all $\omega \in \Omega$, we next denote by $Q_{\geq n}(\omega, dv dz)$ and $Q_n(\omega, dv dz)$ the law of $(v, Z_n(v, u))$ under resp. $\mathbb{Q}_{\geq n}(dv) \otimes q_n(\omega, du)$ and $\mathbb{Q}_n(dv) \otimes q_n(\omega, du)$. Since U_n and \mathcal{G} are independent from $\tau_{\geq n}$ and τ_n , we easily check that $Q_{\geq n}(\cdot, dv dz) : \Omega \rightarrow \mathcal{M}_1(\mathbf{D}_1 \times E')$ and $Q_n(\cdot, dv dz) : \Omega \rightarrow \mathcal{M}_1(\mathbf{D}_1 \times E')$ are regular versions of the conditional law of resp. $M_{\geq n}$ in (3.2.73) and M_n in (3.2.74) given \mathcal{G} . We also denote by $Q(dv dz)$ the law of $(X^{(\cdot)}, Z)$ and Assumption (3.2.73) means that there is $\Omega_0 \in \mathcal{G}$ such that $\mathbf{P}(\Omega_0) = 1$ and such that for all $\omega \in \Omega_0$, $Q_{\geq n}(\omega, dv dz) \rightarrow Q(dv dz)$ in $\mathcal{M}_1(\mathbf{D}_1 \times E')$. Let F be a real-valued continuous function on $\mathbf{D}_1 \times E'$ that is bounded by $\|F\|_\infty$. We fix $\omega \in \Omega_0$. By the construction above, we first get

$$u_n := \int_{\mathbf{D}_1 \times E'} Q_n(\omega, dv dz) F(v, z) = \int_{\mathbf{D}_1 \times E'} Q_{\geq n}(\omega, dv dz) F(v, z) D_{r_n}^{(n)}(b_n v(r'_n)).$$

Then for all $c \in (0, \infty)$,

$$\begin{aligned} & \left| u_n - \int_{\mathbf{D}_1 \times E'} Q_{\geq n}(\omega, dv dz) F(v, z) (c \wedge D_{r_n}^{(n)}(b_n v(r'_n))) \right| \\ & \leq \|F\|_\infty \mathbf{E} \left[D_{r_n}^{(n)}(V_{r_n}(\tau_{\geq n})) \mathbf{1}_{\{D_{r_n}^{(n)}(V_{r_n}(\tau_{\geq n})) \geq c\}} \right] =: \eta_n(c). \end{aligned} \quad (3.B.1)$$

We next fix $x, y \in (0, \infty)$ with $x \leq y$ and we set $\varepsilon_n(x, y) := \max_{p \in \llbracket [b_n x], [b_n y] \rrbracket} |D_{r_n}^{(n)}(p) - D_r(p/b_n)|$. We recall from (3.2.70) in Theorem 3.2.36 (iii) that $\varepsilon_n(x, y) \rightarrow 0$ as $n \rightarrow \infty$. Then we get

$$\begin{aligned} & \left| \int_{\mathbf{D}_1 \times E'} Q_{>n}(\omega, d\mathbf{v}d\mathbf{z}) F(\mathbf{v}, \mathbf{z})(c \wedge D_{r_n}^{(n)}(b_n \mathbf{v}(r'_n))) - \int_{\mathbf{D}_1 \times E'} Q_{>n}(\omega, d\mathbf{v}d\mathbf{z}) F(\mathbf{v}, \mathbf{z})(c \wedge D_r(\mathbf{v}(r'_n))) \right| \\ & \leq \|F\|_\infty \varepsilon_n(x, y) + 2c \|F\|_\infty \mathbf{P}\left(\frac{1}{b_n} V_{r_n}(\tau_{\geq n}) \notin (x, y)\right). \end{aligned} \quad (3.B.2)$$

Since $c \wedge D_r(\cdot)$ is bounded and since $X^{(r)}$ is almost surely continuous at r , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbf{D}_1 \times E'} Q_{>n}(\omega, d\mathbf{v}d\mathbf{z}) F(\mathbf{v}, \mathbf{z})(c \wedge D_r(\mathbf{v}(r'_n))) = \int_{\mathbf{D}_1 \times E'} Q(d\mathbf{v}d\mathbf{z}) F(\mathbf{v}, \mathbf{z})(c \wedge D_r(\mathbf{v}(r))).$$

If we set $u_\infty = \int_{\mathbf{D}_1 \times E'} Q(d\mathbf{v}d\mathbf{z}) F(\mathbf{v}, \mathbf{z}) D_r(\mathbf{v}(r))$, then we have proved for all $c, x, y \in (0, \infty)$ that

$$\begin{aligned} \limsup_{n \rightarrow \infty} |u_n - u_\infty| & \leq \sup_{n \in \mathbb{N}^*} \eta_n(c) + 2c \|F\|_\infty \limsup_{n \rightarrow \infty} \mathbf{P}\left(\frac{1}{b_n} V_{r_n}(\tau_{\geq n}) \notin (x, y)\right) \\ & \quad + \|F\|_\infty \mathbf{E}\left[D_r(X_r^{(r)}) \mathbf{1}_{\{D_r(X_r^{(r)}) \geq c\}}\right]. \end{aligned}$$

Note that $\limsup_{n \rightarrow \infty} \mathbf{P}\left(\frac{1}{b_n} V_{r_n}(\tau_{\geq n}) \notin (x, y)\right) \rightarrow 0$ as $x \rightarrow 0$ and $y \rightarrow \infty$ since $\frac{1}{b_n} V_{r_n}(\tau_{\geq n}) \rightarrow X_r^{(r)} \in (0, \infty)$ by (3.2.73) and an elementary argument. It then implies that $u_n \rightarrow u_\infty$ because that $\lim_{c \rightarrow \infty} \sup_{n \in \mathbb{N}^*} \eta_n(c) = 0$ by (3.2.71) Theorem 3.2.36 (iii). This completes the proof of the lemma. \blacksquare

3.C Proof of Proposition 3.4.5

We first prove the following lemma that is used in the proof of Proposition 3.4.5 and also in Section 3.4.4 in the proof of Proposition 3.4.10.

Lemma 3.C.1 *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be a filtration on (Ω, \mathcal{F}) . Let $c \in (0, \infty)$ be a constant and let $(R_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative r.v. such that for all $n \in \mathbb{N}$, \mathbf{P} -a.s. $R_n \leq c$ and \mathbf{P} -a.s. $R_n \rightarrow 0$. Let $(r_n)_{n \in \mathbb{N}}$ be any sequence of integers. Then \mathbf{P} -a.s. $\mathbf{E}[R_n | \mathcal{G}_{r_n}] \rightarrow 0$.*

Proof. For all $p \in \mathbb{N}$, we set $\bar{R}_p = \sup_{n \geq p} R_n$. We observe that $\limsup_{n \rightarrow \infty} \mathbf{E}[R_n | \mathcal{G}_{r_n}] \leq \sup_{k \in \mathbb{N}} \mathbf{E}[\bar{R}_p | \mathcal{G}_k]$ a.s. for all $p \in \mathbb{N}$. So we only need to prove that a.s. $\lim_{p \rightarrow \infty} \sup_{k \in \mathbb{N}} \mathbf{E}[\bar{R}_p | \mathcal{G}_k] = 0$. To that end denote by \mathcal{G}_∞ the σ -field generated by $\bigcup_{k \in \mathbb{N}} \mathcal{G}_k$ and observe, by Doob's L^1 -martingale convergence, that a.s. $\lim_{k \rightarrow \infty} \mathbf{E}[\bar{R}_p | \mathcal{G}_k] = \mathbf{E}[\bar{R}_p | \mathcal{G}_\infty]$. Moreover by conditional dominated convergence we also get a.s. $\lim_{p \rightarrow \infty} \mathbf{E}[\bar{R}_p | \mathcal{G}_k] = 0$ for all $k \in \mathbb{N} \cup \{\infty\}$. Then there exists $\Omega_0 \in \mathcal{F}$ such that $\mathbf{P}(\Omega_0) = 1$ and such that for all integers $p' \geq p$ and $k \in \mathbb{N} \cup \{\infty\}$ and all $\omega \in \Omega_0$, $\mathbf{E}[\bar{R}_{p'} | \mathcal{G}_k](\omega) \leq \mathbf{E}[\bar{R}_p | \mathcal{G}_k](\omega)$, $\lim_{k \rightarrow \infty} \mathbf{E}[\bar{R}_p | \mathcal{G}_k](\omega) = \mathbf{E}[\bar{R}_p | \mathcal{G}_\infty](\omega)$ and $\lim_{p \rightarrow \infty} \mathbf{E}[\bar{R}_p | \mathcal{G}_k](\omega) = 0$. We fix $\omega \in \Omega_0$, $p \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. There exists $k_{p, \varepsilon}(\omega) \in \mathbb{N}$ such that $\sup_{k \geq k_{p, \varepsilon}(\omega)} \mathbf{E}[\bar{R}_p | \mathcal{G}_k](\omega) \leq \varepsilon + \mathbf{E}[\bar{R}_p | \mathcal{G}_\infty](\omega)$. Thus for all $p' \geq p$, $\sup_{k \in \mathbb{N}} \mathbf{E}[\bar{R}_{p'} | \mathcal{G}_k](\omega) \leq \max_{k \leq k_{p, \varepsilon}(\omega)} \mathbf{E}[\bar{R}_{p'} | \mathcal{G}_k](\omega) + \varepsilon + \mathbf{E}[\bar{R}_p | \mathcal{G}_\infty](\omega)$. Therefore, $\limsup_{p' \rightarrow \infty} \sup_{k \in \mathbb{N}} \mathbf{E}[\bar{R}_{p'} | \mathcal{G}_k](\omega) \leq \varepsilon + \mathbf{E}[\bar{R}_p | \mathcal{G}_\infty](\omega) \rightarrow \varepsilon$ as $p \rightarrow \infty$. Since ε can be chosen arbitrarily small, this shows that $\lim_{p' \rightarrow \infty} \sup_{k \in \mathbb{N}} \mathbf{E}[\bar{R}_{p'} | \mathcal{G}_k](\omega) = 0$ for all $\omega \in \Omega_0$, which entails the desired result. \blacksquare

We now prove Proposition 3.4.5. To that end, we recall the following: for all $n \in \mathbb{N}$, $(\mathcal{F}_{n,k})_{k \in \mathbb{N}}$ is a filtration on (Ω, \mathcal{F}) ; $(D_{n,k})_{k \in \mathbb{N}^*}$ is a sequence of real valued r.v.; $(s_n)_{n \in \mathbb{N}}$ is a sequence of integers and $s \in [0, \infty)$. We introduce the following notations

$$M_{n,k} = D_{n,1} + \dots + D_{n,k}, \quad Q_{n,k} = \mathbf{E}[D_{n,k}^2 | \mathcal{F}_{n,k-1}], \quad V_{n,k} = Q_{n,1} + \dots + Q_{n,k},$$

with $M_{n,0} = V_{n,0} = 0$, and we make the following assumptions.

- (a) For all n, k , $D_{n,k}$ is $\mathcal{F}_{n,k}$ -measurable, $\mathbf{E}[D_{n,k}^2] < \infty$ and \mathbf{P} -a.s. $\mathbf{E}[D_{n,k} | \mathcal{F}_{n,k-1}] = 0$.
- (b) There exist a filtration $(\mathcal{G}_j)_{j \in \mathbb{N}}$ on (Ω, \mathcal{F}) and a sequence of integers $(r_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_{n,0} = \mathcal{G}_{r_n}$, $n \in \mathbb{N}$.
- (c) For all $\varepsilon \in (0, 1)$, \mathbf{P} -a.s. $\lim_{n \rightarrow \infty} \sum_{1 \leq k \leq s_n} \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}] = 0$.
- (d') For all $\varepsilon \in (0, 1)$, \mathbf{P} -a.s. $\lim_{n \rightarrow \infty} \mathbf{E}[\mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}] = 0$.

Remark 3.C.2 First observe that Assumptions (a-d) in Proposition 3.4.5 imply (d'). Indeed, Assumption (d) in Proposition 3.4.5 implies for all $\varepsilon \in (0, 1)$ that \mathbf{P} -a.s. $R_n := \mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} \rightarrow 0$ and Assumption (b) allows us to apply Lemma 3.C.1, which entails (d'). \square

So we prove Proposition 3.4.5 if we prove the following.

Proposition 3.C.3 We keep the previous notations and we assume (a-c) and (d'). Then for all $\lambda \in \mathbb{R}$, \mathbf{P} -almost surely, $\lim_{n \rightarrow \infty} \mathbf{E}[\exp(i\lambda M_{n,s_n}) | \mathcal{F}_{n,0}] = \exp(-\frac{1}{2} s \lambda^2)$.

Proof. We first prove the proposition under the additional assumption that there exists $c \in (0, \infty)$ such that $\mathbf{P}(V_{n,s_n} \leq c) = 1$. We rely on the following standard inequalities that hold for all $a, b \in [0, \infty)$, all $x \in \mathbb{R}$ and all $z \in \mathbb{C}$:

$$\sup_{y \in [-a, a]} |e^{by} - 1| \leq e^{ba} - 1, \quad |e^{ix} - 1 - ix + \frac{1}{2}x^2| \leq |x^2| \wedge (\frac{1}{6}|x|^3), \quad |e^z - 1 - z| \leq \frac{1}{2}|z|^2 e^{(\operatorname{Re}(z))_+}. \quad (3.C.1)$$

We first observe that $\alpha_n = \mathbf{E}[e^{i\lambda M_{n,s_n}} | \mathcal{F}_{n,0}] - e^{-\frac{1}{2} s \lambda^2} = \beta_n + \gamma_n$ where

$$\beta_n = \mathbf{E}[e^{i\lambda M_{n,s_n}} (1 - e^{\frac{1}{2} \lambda^2 (V_{n,s_n} - s)}) | \mathcal{F}_{n,0}] \quad \text{and} \quad \gamma_n = \mathbf{E}[e^{-\frac{1}{2} s \lambda^2} (e^{i\lambda M_{n,s_n} + \frac{1}{2} \lambda^2 V_{n,s_n}} - 1) | \mathcal{F}_{n,0}].$$

We fix $\varepsilon \in (0, 1)$ such that $\frac{1}{3} |\lambda| \varepsilon < 1$ and we bound β_n and γ_n . Since V_{n,s_n} is bounded by the constant c , the first inequality in (3.C.1) implies

$$|\beta_n| \leq e^{\frac{1}{2} \lambda^2 \varepsilon} - 1 + (1 + e^{\frac{1}{2} \lambda^2 (c+s)}) \mathbf{E}[\mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}]. \quad (3.C.2)$$

Thus, by (d'), for all $\varepsilon \in (0, 1)$ we a.s. get $\limsup_{n \rightarrow \infty} |\beta_n| \leq e^{\frac{1}{2} \lambda^2 \varepsilon} - 1$ and thus a.s. $\beta_n \rightarrow 0$.

We next observe that

$$\begin{aligned} e^{\frac{1}{2} \lambda^2 s} \gamma_n &= \sum_{1 \leq k \leq s_n} \mathbf{E}\left[e^{i\lambda M_{n,k-1} + \frac{1}{2} \lambda^2 V_{n,k}} \mathbf{E}\left[e^{i\lambda D_{n,k}} - e^{-\frac{1}{2} \lambda^2 Q_{n,k}} | \mathcal{F}_{n,k-1}\right] | \mathcal{F}_{n,0}\right] \\ &= \sum_{1 \leq k \leq s_n} \mathbf{E}\left[e^{i\lambda M_{n,k-1} + \frac{1}{2} \lambda^2 V_{n,k}} (\delta_{n,k}^{(1)} + \delta_{n,k}^{(2)}) | \mathcal{F}_{n,0}\right] \quad \text{where} \end{aligned}$$

$$\delta_{n,k}^{(1)} = \mathbf{E}\left[e^{i\lambda D_{n,k}} - 1 - i\lambda D_{n,k} + \frac{1}{2} \lambda^2 D_{n,k}^2 | \mathcal{F}_{n,k-1}\right] \quad \text{and} \quad \delta_{n,k}^{(2)} = 1 - \frac{1}{2} \lambda^2 Q_{n,k} - e^{-\frac{1}{2} \lambda^2 Q_{n,k}},$$

since $i\lambda M_{n,k-1} + \frac{1}{2} \lambda^2 V_{n,k}$ is $\mathcal{F}_{n,k-1}$ -measurable, since a.s. $\mathbf{E}[D_{n,k} | \mathcal{F}_{n,k-1}] = 0$ and by definition of $Q_{n,k}$. We first bound $\delta_{n,k}^{(1)}$ by use of the second inequality in (3.C.1). To that end we set $Z = \lambda^2 D_{n,k}^2 (1 \wedge \frac{1}{6} |\lambda D_{n,k}|)$. Since $\frac{1}{3} |\lambda| \varepsilon < 1$, we observe that $Z \leq \lambda^2 D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} + \frac{1}{6} |\lambda|^3 \varepsilon D_{n,k}^2$. Therefore $|\delta_{n,k}^{(1)}| \leq \lambda^2 \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}] + \frac{1}{6} |\lambda|^3 \varepsilon Q_{n,k}$. By the third inequality in (3.C.1), we get $|\delta_{n,k}^{(2)}| \leq \frac{1}{8} \lambda^4 Q_{n,k}^2$. Then observe that $Q_{n,k} \leq \varepsilon^2 + \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}]$. Thus $|\delta_{n,k}^{(2)}| \leq \frac{1}{8} \lambda^4 Q_{n,k} \varepsilon^2 + \frac{1}{8} \lambda^4 c \mathbf{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}| > \varepsilon\}} | \mathcal{F}_{n,k-1}]$, since $Q_{n,k} \leq V_{n,s_n} \leq c$. Thus,

$$\begin{aligned} e^{-\frac{1}{2} \lambda^2 c} e^{\frac{1}{2} \lambda^2 s} |\gamma_n| &\leq \sum_{1 \leq k \leq s_n} \mathbf{E}\left[e^{\frac{1}{2} \lambda^2 (V_{n,k} - c)} (|\delta_{n,k}^{(1)}| + |\delta_{n,k}^{(2)}|) | \mathcal{F}_{n,0}\right] \\ &\leq \frac{1}{24} |\lambda|^3 \varepsilon (4 + 3|\lambda| \varepsilon) \mathbf{E}[V_{n,s_n} | \mathcal{F}_{n,0}] + \frac{1}{8} \lambda^2 (8 + c \lambda^2) \mathbf{E}[L_{n,s_n}(\varepsilon) | \mathcal{F}_{n,0}] \quad (3.C.3) \end{aligned}$$

where we set $L_{n,k}(\varepsilon) = \sum_{1 \leq j \leq k} \mathbf{E}[D_{n,j}^2 \mathbf{1}_{\{|D_{n,j}| > \varepsilon\}} | \mathcal{F}_{n,k-1}]$. Arguing as in Remark 3.C.2, Assumptions (b) and (c) allow us to apply Lemma 3.C.1, which entails that $\lim_{n \rightarrow \infty} \mathbf{E}[L_{n,s_n}(\varepsilon) | \mathcal{F}_{n,0}] = 0$. Moreover, since $V_{n,s_n} \leq c$, for all $\varepsilon \in (0, 1)$ we a.s. get $\limsup_{n \rightarrow \infty} |\gamma_n| \leq \frac{1}{24} c e^{-\frac{1}{2} \lambda^2 (s-c)} |\lambda|^3 (4 + 3|\lambda| \varepsilon) \varepsilon$ and thus a.s. $\gamma_n \rightarrow 0$. We have proved that a.s. $\lim_{n \rightarrow \infty} \mathbf{E}[e^{i\lambda M_{n,s_n}} | \mathcal{F}_{n,0}] = e^{-\frac{1}{2} \lambda^2 s}$ under the assumption that $\mathbf{P}(V_{n,s_n} \leq c) = 1$ for all $n \in \mathbb{N}$.

Let us remove this assumption. We fix $c \in (1 + s, \infty)$ and we set $\tilde{D}_{n,k} = D_{n,k} \mathbf{1}_{\{V_{n,k} \leq c\}}$, $\tilde{M}_{n,k} = \sum_{1 \leq j \leq k} \tilde{D}_{n,j}$, $\tilde{V}_{n,k} = \sum_{1 \leq j \leq k} \mathbf{E}[\tilde{D}_{n,j}^2 | \mathcal{F}_{n,j-1}]$, $\tilde{L}_{n,k}(\varepsilon) = \sum_{1 \leq j \leq k} \mathbf{E}[\tilde{D}_{n,j}^2 \mathbf{1}_{\{|\tilde{D}_{n,j}| > \varepsilon\}} | \mathcal{F}_{n,k-1}]$ and $T_n = \max\{k \in \llbracket 0, s_n \rrbracket : V_{n,k} \leq c\}$. We easily check that the $\tilde{D}_{n,k}$ satisfy (a) since $V_{n,k}$ is $\mathcal{F}_{n,k-1}$ -measurable. Then, observe that $\tilde{V}_{n,k} = V_{n,k \wedge T_n}$ (which implies $\mathbf{P}(\tilde{V}_{n,s_n} \leq c) = 1$) and that $\tilde{L}_{n,k}(\varepsilon) = L_{n,k \wedge T_n}(\varepsilon)$. Therefore the $\tilde{D}_{n,k}$ satisfy (c) because $\tilde{L}_{n,s_n}(\varepsilon) \leq L_{n,s_n}(\varepsilon) \rightarrow 0$ a.s. Next, note that for all $\varepsilon \in (0, 1)$, a.s.

$$\begin{aligned} \mathbf{E}[\mathbf{1}_{\{|\tilde{V}_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}] &\leq \mathbf{E}[\mathbf{1}_{\{T_n < s_n\}} | \mathcal{F}_{n,0}] + \mathbf{E}[\mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}] \\ &\leq 2\mathbf{E}[\mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}] \end{aligned}$$

since $T_n < s_n$ implies that $V_{n,s_n} > c > s + 1$. This proves that the $\tilde{D}_{n,k}$ satisfy (d') and thus we get a.s. $\lim_{n \rightarrow \infty} \mathbf{E}[e^{i\lambda \tilde{M}_{n,s_n}} | \mathcal{F}_{n,0}] = e^{-\frac{1}{2} \lambda^2 s}$, by the previous arguments. But observe that

$$\begin{aligned} \left| \mathbf{E}[e^{i\lambda \tilde{M}_{n,s_n}} | \mathcal{F}_{n,0}] - \mathbf{E}[e^{i\lambda M_{n,s_n}} | \mathcal{F}_{n,0}] \right| &\leq \mathbf{E}\left[\left| e^{i\lambda \tilde{M}_{n,s_n}} - e^{i\lambda M_{n,s_n}} \right| \mathbf{1}_{\{T_n < s_n\}} \mid \mathcal{F}_{n,0} \right] \\ &\leq 2\mathbf{E}[\mathbf{1}_{\{|V_{n,s_n} - s| > \varepsilon\}} | \mathcal{F}_{n,0}] \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

\mathbf{P} -a.s. which implies the desired result. ■

Chapter 4

The Horton-Strahler number of Galton-Watson trees with possibly infinite variance

This chapter is adapted from the paper [103], which is submitted for publication.

Abstract

The Horton-Strahler number, also known as the register function, provides a tool for quantifying the branching complexity of a rooted tree. We consider the Horton-Strahler number of critical Galton-Watson trees conditioned to have size n and whose offspring distribution is in the domain of attraction of an α -stable law with $\alpha \in [1, 2]$. We give tail estimates and when $\alpha \neq 1$, we prove that it grows as $\frac{1}{\alpha} \log_{\alpha/(\alpha-1)} n$ in probability. This extends the result in Brandenberger, Devroye & Reddad [28] dealing with the finite variance case for which $\alpha = 2$. We also characterize the cases where $\alpha = 1$, namely the spectrally positive Cauchy regime, which exhibits more complex behaviors.

Contents

4.1	Introduction	196
4.2	Framework and tools	200
4.2.1	Galton-Watson trees	200
4.2.2	Limit theorems.	203
4.3	Distribution of the Horton-Strahler number of Galton-Watson trees	208
4.3.1	Alternative definitions of the Horton-Strahler number and basic results	208
4.3.2	Tail estimates of joint laws	211
4.3.3	Tail estimates.	215
4.4	Proofs of Theorems 4.1.2, 4.1.3 and 4.1.4	219
4.4.1	Proof of Theorem 4.1.2	219
4.4.2	Proof of Theorem 4.1.3	220
4.4.3	Proof of Theorem 4.1.4	223

4.1 Introduction

The Horton-Strahler number of a finite rooted tree is an integer that quantifies its branching complexity. One possible formal definition is given recursively as follows.

Definition 4.1.1 Let t be a finite rooted tree. Its *Horton-Strahler number* $\mathcal{S}(t)$ is defined as follows.

- (a) If t reduces to a single node, then $\mathcal{S}(t) = 0$.
- (b) Otherwise, $\mathcal{S}(t)$ is the maximum of the Horton-Strahler numbers of the subtrees t_1, \dots, t_k that are attached to the root, plus one if that maximum is not uniquely achieved. Namely,

$$\mathcal{S}(t) = \max_{1 \leq i \leq k} \mathcal{S}(t_i) + \mathbf{1}_{\{\#\operatorname{argmax}_{1 \leq i \leq k} \mathcal{S}(t_i) \geq 2\}}.$$

□

Alternatively, $\mathcal{S}(t)$ is also the height of the largest perfect binary tree that can be embedded into t (see Section 4.3.1 for more details). In this chapter, we provide estimates on the Horton-Strahler number of critical Galton-Watson trees conditioned to be large. Before discussing our results precisely, let us provide a brief history with general references on related topics.

Background. The Horton-Strahler number was introduced independently by the two hydrogeologists Horton [93] and Strahler [146] to obtain quantitative empirical laws about river systems, that are represented by trees whose leaves are springs and whose root corresponds to the outlet of the basin. Many key physical characteristics of stream networks have been since modeled with the help of this number: see e.g. Peckham [136], Fac-Beneda [73], Chavan & Srinivas [34] and Bamuffeh et al. [14]. The Horton-Strahler number appears independently in other scientific fields (anatomy, botany, molecular biology, physics, social network analysis, etc). In computer science, it is used to optimize the amount of memory or time needed to manipulate data structures. It is sometimes called the *register function* in this context because the minimum number of registers needed to evaluate an arithmetic expression A is equal to the Horton-Strahler number of the syntax tree of A . We refer to Viennot [148] for a survey on those various applications.

The Horton-Strahler number is encountered in many areas of mathematics: see for instance Esparza, Luttenberger & Schlund [69] for connections with mathematical logic, formal language theory, algebra, combinatorics, topology, approximation theory, and more. In the probability area, let us mention Kovchegov & Zaliapin [111], which considers the Horton-Strahler number through the prism of pruning operations on trees. Here, we rather focus on probabilistic works that discuss the Horton-Strahler number of uniform samples of standard families of combinatorial trees. Flajolet, Raoult & Vuillemin [75] and Kemp [98] consider the Horton-Strahler number of a uniform random ordered rooted binary tree T_n with n leaves (a uniform n -Catalan tree) and they prove that

$$\mathbb{E}[\mathcal{S}(T_n)] = \log_4 n + D(\log_4 n) + o(1)$$

as $n \rightarrow \infty$, where $\log_b x = \ln x / \ln b$ stands for the logarithm of x to the base b , and D is a 1-periodic continuous function. Here, all the random variables that we consider are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose expectation is denoted by \mathbb{E} . In particular, $\mathcal{S}(T_n)$ is subject to deterministic oscillations. Moreover, Devroye & Kruszewski [47] proved that $\mathcal{S}(T_n)$ is highly concentrated around its expected value via exponential tail estimates. These results were extended to k -ary trees by Drmota & Prodinger [49].

More recently, Brandenberger, Devroye & Reddad [28] showed that the Horton-Strahler number of a critical Galton-Watson tree with finite variance offspring distribution conditioned to have n vertices always grows as $\log_4 n$ in probability, which extends all the results that have been previously obtained on first-order behavior. In Chapter 5, adapter from [102], we go further and we

study, among other things, the fluctuations and deterministic oscillations of the Horton-Strahler number of large Catalan trees.

Framework and main results. Let us give a precise overview of the present chapter which provides tail estimates and characterizes the first-order behavior of the Horton-Strahler number of critical Galton-Watson trees conditioned to have size n and whose offspring distribution is in the domain of attraction of an α -stable law with $\alpha \in [1, 2]$. This framework extends the finite variance case, where $\alpha = 2$, and it includes the so-called spectrally positive Cauchy (or 1-stable) laws. To that end, let us introduce our basic notations and assumptions.

Throughout this work, $\mu = (\mu(k))_{k \in \mathbb{N}}$ stands for a probability measure on the set \mathbb{N} of nonnegative integers. We shall always assume μ to be *non-trivial* and *critical*, namely

$$\mu(0) > 0 \quad \text{and} \quad \sum_{k \in \mathbb{N}} k\mu(k) = 1. \quad (4.1.1)$$

We view it as the *critical offspring distribution* of a (rooted and ordered) Galton-Watson tree denoted by τ , which is then almost surely finite. See Section 4.2.1 for a formal definition. Several results are expressed in terms of the following functions

$$\forall s \in [0, 1], \quad \varphi(s) = \sum_{k \in \mathbb{N}} s^k \mu(k) \quad \text{and} \quad \psi(s) = \varphi(1-s) - (1-s), \quad (4.1.2)$$

that are strictly convex thanks to (4.1.1).

Our first contribution consists in four propositions that connect $\mathcal{S}(\tau)$ to other simple characteristics of τ and that hold under the sole assumption that μ is non-trivial and critical. More precisely,

- Proposition 4.3.5 provides an upper bound for the size $\#\tau$ of τ when $\mathcal{S}(\tau)$ is small,
- Proposition 4.3.6 provides a lower bound for the height $|\tau|$ of τ when $\mathcal{S}(\tau)$ is large,
- Proposition 4.3.7 provides a lower bound for $\mathcal{S}(\tau)$ when the height $|\tau|$ of τ is large,
- Proposition 4.3.9 provides a lower bound for $\mathcal{S}(\tau)$ when the maximal out-degree of τ is large.

We will use these propositions to derive the order of magnitude of $\mathcal{S}(\tau)$ under $\mathbb{P}(\cdot | \#\tau = n)$. These estimates are fairly general and they are sufficiently accurate to be interesting in their own right.

Our second goal is to estimate the tail of the Horton-Strahler number of Galton-Watson trees whose offspring distribution has possibly infinite variance. We work under the assumption that

$$\mu \text{ belongs to the domain of attraction of a stable law.} \quad (4.1.3)$$

We denote the scaling index of (the type of) the limiting law by α : since μ has a finite mean, we get $\alpha \in [1, 2]$ and since μ is supported on $[0, \infty)$, we only deal with spectrally positive stable laws: namely, their *skewness parameter* β is equal to 1 and the support of their Lévy measure is included in $(0, \infty)$. By standard results, see e.g. [23, Theorem 8.3.1], Assumption (4.1.3) is equivalent to the existence of a function $L : [0, \infty) \mapsto (0, \infty)$ that is *slowly varying* and such that

$$\mu([n, \infty)) \sim n^{-\alpha} L(n) \quad \text{if } \alpha \in [1, 2) \quad \text{and} \quad \sum_{k=0}^n k^2 \mu(k) - 1 \sim 2L(n) \quad \text{if } \alpha = 2. \quad (4.1.4)$$

Note that $\sum_{k=0}^n k^2 \mu(k) - 1$ is nondecreasing and ultimately positive since μ satisfies (4.1.1) (we refer to Proposition 4.2.9 for more details). We first discuss our results in the cases where $\alpha \in (1, 2]$, and then we consider the cases where $\alpha = 1$ that feature more complicated behaviors.

The cases where $\alpha \in (1, 2]$. In these cases, our results are simply expressed in terms of the index α only. First of all, we prove in Proposition 4.3.12 that if μ is in the domain of attraction of an α -stable law with $\alpha \in (1, 2]$, then the tail of $\mathcal{S}(\tau)$ follows the universal exponential decay

$$-\log_{\frac{\alpha}{\alpha-1}} \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n,$$

where $\log_b x = \ln x / \ln b$ stands for the logarithm of x to the base base b . This extends the work of Brandenberger, Devroye & Reddad [28] who proved the $\alpha = 2$ case under the assumption that μ has a finite variance.

We next discuss the behaviour of $\mathcal{S}(\tau)$ under $\mathbb{P}(\cdot | \#\tau = n)$ and to that end, we assume that

$$\mu \text{ is aperiodic} \tag{4.1.5}$$

(namely that μ is not supported by a proper additive subgroup of \mathbb{Z}) because this implies that $\mathbb{P}(\#\tau = n) > 0$ for all n that are large enough. Then, we prove the following.

Theorem 4.1.2 *Assume that μ is critical and aperiodic and that it belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$. Then, the following convergence holds in probability.*

$$\frac{\alpha \mathcal{S}(\tau)}{\log_{\frac{\alpha}{\alpha-1}} n} \text{ under } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1. \tag{4.1.6}$$

This extends the work of Brandenberger, Devroye & Reddad [28] who proved the $\alpha = 2$ case under the assumption that μ has a finite variance.

Our proof of Theorem 4.1.2 relies on results in Duquesne [50] on the height of α -stable trees and in Kortchemski [106] on the asymptotic behavior of the positive excursion of the random walk $(W_n)_{n \in \mathbb{N}}$. These results are recalled precisely in Section 4.2.2, Proposition 4.2.11. Let us mention that (4.1.6) holds true when μ is not aperiodic by restricting to the integers such that $\mathbb{P}(\#\tau = n) > 0$.

The 1-stable cases. In these more complex cases, we need to specify a converging sequence of rescaled centered sums of μ -distributed independent random variables. More precisely, we denote by $(W_n)_{n \in \mathbb{N}}$ a (left-continuous) random walk starting at $W_0 = 0$, whose jump law is given by

$$\mathbb{P}(W_1 = k) = \mu(k + 1), \quad k \in \{-1\} \cup \mathbb{N}. \tag{4.1.7}$$

Note that $\mathbb{E}[W_1] = 0$. Then μ belongs to the domain of attraction of a 1-stable law if and only if there exists a $(0, \infty)$ -valued sequence $(a_n)_{n \in \mathbb{N}}$ tending to ∞ such that

$$\frac{W_n + b_n}{a_n} \xrightarrow[n \rightarrow \infty]{(\text{law})} X, \text{ where } b_n = n\mathbb{E}[W_1 \mathbf{1}_{\{|W_1| > a_n\}}] \text{ and } \mathbb{E}[e^{-\lambda X}] = e^{\lambda \ln \lambda} \tag{4.1.8}$$

for all $n \in \mathbb{N}$ and $\lambda \in (0, \infty)$. Let us mention here that $a_n \sim nL(a_n)$ and that $b_n/a_n \rightarrow \infty$, necessarily. The law of X is a *spectrally positive Cauchy (or 1-stable) law*. Its Fourier transform is given by $\mathbb{E}[\exp(iuX)] = \exp(-\frac{\pi}{2}|u| - iu \ln |u|)$, $u \in \mathbb{R}$. We refer to Proposition 4.2.12 for details.

The asymptotic behavior of $\mathcal{S}(\tau)$ in the 1-stable case is expressed in terms of the sequence $(b_n)_{n \in \mathbb{N}}$ and the following function Υ that is derived from ψ in (4.1.2) as follows.

$$\forall s \in (0, 1), \quad \Upsilon(s) = \int_s^1 \frac{dr}{r \ln \Lambda(r)}, \quad \text{where} \quad \Lambda(s) = \frac{s\psi'(s)}{s\psi'(s) - \psi(s)}. \tag{4.1.9}$$

We refer to Section 4.3.3 for more details on the definition of Υ . Proposition 4.3.12 asserts that if μ is in the domain of attraction of a 1-stable law, then the tail of $\mathcal{S}(\tau)$ satisfies

$$\Upsilon\left(\mathbb{P}(\mathcal{S}(\tau) \geq n)\right) \sim n. \quad (4.1.10)$$

In contrast to the cases where $\alpha \in (1, 2]$ for which we can show that $\Upsilon(s) \sim_{0+} \log_{\frac{\alpha}{\alpha-1}} 1/s$, the asymptotic behavior of Υ when $\alpha = 1$ depends on the slowly varying function L appearing in (4.1.4): see Proposition 4.3.11 for a precise statement. As discussed in Examples 4.3.13, the following holds.

- (a) If $L(n) \sim (\ln n)^{-1-\kappa}$ with $\kappa \in (0, \infty)$, then $\Upsilon(s) \sim_{0+} \frac{\ln 1/s}{\ln \ln 1/s}$ and $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n \ln n$.
- (b) If $L(n) \sim \exp(-(\ln n)^\kappa)$ with $\kappa \in (0, 1)$, then we have $\Upsilon(s) \sim_{0+} \frac{1}{1-\kappa} \frac{\ln 1/s}{\ln \ln 1/s}$ and it thus holds that $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim (1-\kappa)n \ln n$.
- (c) If $L(n) \sim \exp(-\ln n / \ln \ln n)$ then $\Upsilon(s) \sim_{0+} \frac{\ln 1/s}{\ln \ln \ln 1/s}$ and $-\ln \mathbb{P}(\mathcal{S}(\tau) \geq n) \sim n \ln \ln n$.

When the Galton-Watson tree τ is conditioned to be large, the size of its Horton-Strahler number is of order $\Upsilon(\frac{1}{b_n})$ as proved by the following theorem that first handles the case where τ is conditioned to have at least n vertices.

Theorem 4.1.3 *Assume that μ is critical and that it belongs to the domain of attraction of a 1-stable law. Then, the following convergence holds in probability.*

$$\frac{\mathcal{S}(\tau)}{\Upsilon(\frac{1}{b_n})} \text{ under } \mathbb{P}(\cdot | \#\tau \geq n) \longrightarrow 1, \quad (4.1.11)$$

where Υ is given by (4.1.9) and $(b_n)_{n \in \mathbb{N}}$ by (4.1.8).

Our proof of Theorem 4.1.3 and of Theorem 4.1.4 below, rely on several results of Kortchemski & Richier [108] and of Berger [17] that specify the asymptotic behavior of positive excursion of the random walk $(W_n)_{n \in \mathbb{N}}$. They are recalled precisely in Section 4.2.2, Lemma 4.2.13 and Propositions 4.2.15 and 4.2.16.

As discussed by Kortchemski & Richier [108] and Berger [17], it is not clear how one could control τ under $\mathbb{P}(\cdot | \#\tau = n)$ by assuming only that $\mu([n, \infty)) \sim L(n)/n$ as in (4.1.4). Here, we work under the stronger assumption that $\mu(n) \sim L(n)/n^2$, which implies the previous one and also implies that μ is aperiodic.

Theorem 4.1.4 *Assume that μ is critical and that there is a slowly varying function L such that*

$$\mu(n) \sim \frac{L(n)}{n^2}. \quad (4.1.12)$$

Then, the following convergence holds in probability.

$$\frac{\mathcal{S}(\tau)}{\Upsilon(\frac{1}{b_n})} \text{ under } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1, \quad (4.1.13)$$

where Υ is given by (4.1.9) and $(b_n)_{n \in \mathbb{N}}$ by (4.1.8).

As already mentioned, when $\alpha = 1$, the rescaling sequence $\Upsilon(\frac{1}{b_n})$ depends on the slowly varying function L appearing in (4.1.4): as discussed in Examples 4.4.2, the following holds true.

- (a) If $L(n) \sim (\ln n)^{-1-\kappa}$ with $\kappa \in (0, \infty)$, then $\Upsilon(\frac{1}{b_n}) \sim \frac{\ln n}{\ln \ln n}$.
- (b) If $L(n) \sim \exp(-(\ln n)^\kappa)$ with $\kappa \in (0, 1)$, then $\Upsilon(\frac{1}{b_n}) \sim \frac{1}{1-\kappa} \frac{\ln n}{\ln \ln n}$.
- (c) If $L(n) \sim \exp(-\ln n / \ln \ln n)$, then $\Upsilon(\frac{1}{b_n}) \sim \frac{\ln n}{\ln \ln \ln n}$.

Organisation of the chapter. In Section 4.2, we properly set our framework and we recall from previous works the tools that we use later on: Section 4.2.1 is devoted to Galton-Watson trees and Section 4.2.2 to known limit theorems for random walks and Galton-Watson trees. In Section 4.3, we study the distribution of the Horton-Strahler number of Galton-Watson trees. In Section 4.3.1, we first establish new technical results on Horton-Strahler numbers (especially in Lemmas 4.3.1 and 4.3.2). Section 4.3.2 focuses on proving Propositions 4.3.5, 4.3.6, 4.3.7, and 4.3.9 that link the Horton-Strahler number to size, height, and maximal out-degree of Galton-Watson trees. Section 4.3.3 is devoted to the tail asymptotics of Galton-Watson trees whose offspring distribution belongs to the domain of attraction of a stable law: in particular, we prove Proposition 4.3.12, that is one of the main results of the chapter and we discuss Examples 4.3.13 (a), (b), and (c). Section 4.4 is devoted to the proof of Theorems 4.1.2, 4.1.3 and 4.1.4 and we discuss Examples 4.4.2 (a), (b), and (c), at the end of Section 4.4.3.

4.2 Framework and tools

In this section, we recall a set of well-known results that are used in the rest of the chapter and in the proofs of Theorems 4.1.2, 4.1.3 and 4.1.4. With the exception of Lemma 4.2.13, this section contains no new result.

4.2.1 Galton-Watson trees

Rooted ordered trees. We recall Ulam's formalism on rooted ordered trees. Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of positive integers and let \mathbb{U} be the following set of finite words

$$\mathbb{U} = \bigcup_{n \in \mathbb{N}} (\mathbb{N}^*)^n$$

with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The set of words \mathbb{U} is totally ordered by the *lexicographic order* denoted by \leq . Let $u = (u_1, \dots, u_n) \in \mathbb{U}$ and $v = (v_1, \dots, v_m) \in \mathbb{U}$, we write $u * v = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ for the *concatenation* of u and v . We denote by $|u| = n$ the *height* of u , and if $n \geq 1$ then we denote by $\overleftarrow{u} = (u_1, \dots, u_{n-1})$ the *parent* of u . We also say that u is a *child* of v when $\overleftarrow{u} = v$. The *genealogical order* \preceq is a partial order on \mathbb{U} defined by $u \preceq v \iff \exists u' \in \mathbb{U}, v = u * u'$. When $u \preceq v$, we will say that u is an ancestor of v . When $u \preceq v$ but $u \neq v$, we write $u \prec v$. Finally, we write $u \wedge v \in \mathbb{U}$ for the *most recent common ancestor* of u and v , that is their common ancestor with maximal height.

Definition 4.2.1 We say that a subset t of \mathbb{U} is a *tree* when the following is satisfied:

- (a) $\emptyset \in t$,
- (b) for all $u \in t$, if $u \neq \emptyset$ then $\overleftarrow{u} \in t$,
- (c) For all $u \in t$, there exists an integer $k_u(t) \in \mathbb{N}$ such that $u * (i) \in t \iff 1 \leq i \leq k_u(t)$.

We denote by \mathbb{T} the space of all trees. □

Let $t \in \mathbb{T}$. The size of t is simply the (possibly infinite) number $\#t$ of its vertices and we say that t is finite if $\#t < \infty$. As a graph, the *edges* of t are given by the unordered pairs $\{u, \overleftarrow{u}\}$ for all $u \in t \setminus \{\emptyset\}$. Therefore the *degree* of $u \in t$ is $k_u(t) + 1$ if $u \neq \emptyset$ and $k_\emptyset(t)$ otherwise. Namely, $k_u(t)$ is the *out-degree* of u (alternatively, if one views t as a family tree whose ancestor is \emptyset , then $k_u(t)$ stands for the number of children of u). We use the following notations for the *height* of t and its *maximal out-degree*.

$$|t| = \max_{u \in t} |u| \quad \text{and} \quad \Delta(t) = \max_{u \in t} k_u(t). \quad (4.2.1)$$

We also denote the subtree stemming from $u \in t$ and the tree pruned at u respectively by

$$\theta_u t = \{v \in \mathbb{U} : u * v \in t\} \quad \text{and} \quad \text{Cut}_u t = t \setminus \{v \in t : u \prec v\} \quad (4.2.2)$$

Observe that $\theta_u t$ and $\text{Cut}_u t$ both belong to \mathbb{T} .

Galton-Watson trees and the Many-To-One Principle. Let us equip the set of trees \mathbb{T} with the sigma-field $\mathcal{F}(\mathbb{T})$ generated by the sets $\{t \in \mathbb{T} : u \in t\}$, where u ranges in \mathbb{U} . Formally, a *random tree* is a function $\tau : \Omega \rightarrow \mathbb{T}$ that is $(\mathcal{F}, \mathcal{F}(\mathbb{T}))$ -measurable.

Definition 4.2.2 Let $\mu = (\mu(k))_{k \in \mathbb{N}}$ be a probability measure on \mathbb{N} . A *Galton-Watson tree with offspring distribution μ* (a $\text{GW}(\mu)$ -tree, for short) is a random tree τ that satisfies the following.

- (a) $k_\emptyset(\tau)$ has law μ .
- (b) For all $k \in \mathbb{N}^*$ such that $\mu(k) > 0$, the subtrees $\theta_{(1)}\tau, \dots, \theta_{(k)}\tau$ under $\mathbb{P}(\cdot | k_\emptyset(\tau) = k)$ are independent with the same law as τ under \mathbb{P} . \square

It is well-known that a $\text{GW}(\mu)$ -tree τ is almost surely finite if and only if its offspring distribution is subcritical or critical and non-trivial, namely if (4.1.1) holds, and in that case for all finite tree $t \in \mathbb{T}$,

$$\mathbb{P}(\tau = t) = \prod_{u \in t} \mu(k_u(t)). \quad (4.2.3)$$

As observed by Kesten [100], a critical $\text{GW}(\mu)$ -tree conditioned to be large locally converges in law to a tree τ_∞ with a single infinite line of descent and whose law can be informally described as follows: all individuals of τ_∞ reproduce independently, the individuals of the infinite line of descent reproduce according to the μ -size-biased distribution $(k\mu(k))_{k \in \mathbb{N}}$ whereas the others reproduce according to μ . More precisely, we introduce the following.

Definition 4.2.3 Let μ be a non-trivial critical offspring distribution. A *size-biased $\text{GW}(\mu)$ -tree* is a random tree τ_∞ that satisfies the following.

- (a) For all $n \in \mathbb{N}$, there is a unique $u \in \tau_\infty$ such that $|u| = n$ and $\#(\theta_u \tau_\infty) = \infty$. We denote this vertex by U_n . Note that $U_0 = \emptyset$ and that $\overleftarrow{U}_{n+1} = U_n$. Hence, there exists a \mathbb{N}^* -valued sequence of random variables $(J_n)_{n \in \mathbb{N}^*}$ such that U_n is the word (J_1, \dots, J_n) for all $n \in \mathbb{N}^*$.
- (b) The random variables $(k_{U_n}(\tau_\infty), J_{n+1})$, for $n \in \mathbb{N}$, are independent and distributed as follows:

$$\forall j, k \in \mathbb{N}^*, \quad \mathbb{P}(J_{n+1} = j; k_{U_n}(\tau_\infty) = k) = \mathbf{1}_{\{j \leq k\}} \mu(k). \quad (4.2.4)$$

- (c) Conditionally given $(k_{U_n}(\tau_\infty), J_{n+1})_{n \in \mathbb{N}}$, the finite subtrees stemming from the infinite line of descent, which are the $\theta_{U_n * (j)} \tau_\infty$ for $j \in \{1, \dots, k_{U_n}(\tau_\infty)\} \setminus \{J_{n+1}\}$ and $n \in \mathbb{N}$, are independent $\text{GW}(\mu)$ -trees. \square

Remark 4.2.4 Note that the individual U_{n+1} on the infinite line of descent has $J_{n+1} - 1$ siblings strictly on the left hand side and $k_{U_n}(\tau_\infty) - J_{n+1}$ siblings strictly on the right hand side. Their joint law is given by the following bivariate generating function:

$$\forall s, r \in [0, 1], \quad \mathbb{E}[r^{J_{n+1}-1} s^{k_{U_n}(\tau_\infty)-J_{n+1}}] = \frac{\varphi(r) - \varphi(s)}{r - s}, \quad (4.2.5)$$

where φ is given by (4.1.2) and where the quotient is equal to $\varphi'(r)$ when $r = s$. \square

As mentioned above, size-biased trees are the local limits of critical Galton-Watson trees conditioned to be large and therefore appear in many results concerning the asymptotic behavior of branching processes: we refer to Lyons, Pemantle & Peres [125], Aldous & Pitman [11] and Abraham & Delmas [2] for general results in this vein. One key tool involving size-biased $\text{GW}(\mu)$ -trees is the so-called *Many-To-One Principle*, which is part of folklore (see e.g. Duquesne [51, Equation (24)] for a proof) and which we use here in the following form.

Proposition 4.2.5 (Many-To-One Principle) *Let τ be a $\text{GW}(\mu)$ -tree and let τ_∞ be a size-biased $\text{GW}(\mu)$ -tree. We keep the notations of Definition 4.2.3. Then, for all $n \in \mathbb{N}$ and for all bounded functions $G_1 : \mathbb{T} \times \mathbb{U} \rightarrow \mathbb{R}$ and $G_2 : \mathbb{T} \rightarrow \mathbb{R}$, it holds that*

$$\mathbb{E} \left[\sum_{u \in \tau} \mathbf{1}_{\{|u|=n\}} G_1(\text{Cut}_u \tau, u) G_2(\theta_u \tau) \right] = \mathbb{E}[G_1(\text{Cut}_{U_n} \tau_\infty, U_n)] \mathbb{E}[G_2(\tau)].$$

Lukasiewicz path associated with a tree. We recall here a key combinatorial tool to study Galton-Watson trees via random walks.

Definition 4.2.6 Let $t \in \mathbb{T}$ be finite and $u(t) = (u_j(t))_{0 \leq j < \#t}$ be the sequence of its vertices listed in increasing lexicographic order: $u_0(t) = \emptyset < u_1(t) < \dots < u_j(t) < u_{j+1}(t) < \dots < u_{\#t-1}(t)$. The sequence $u(t)$ is often called the *depth-first exploration* of t . We then define a \mathbb{Z} -valued path $W(t) = (W_j(t))_{0 \leq j \leq \#t}$ by setting $W_0(t) = 0$ and

$$W_{j+1}(t) = W_j(t) + k_{u_j(t)}(t) - 1 \quad (4.2.6)$$

for all $0 \leq j < \#t$, that is the *Lukasiewicz path* of t . \square

In probability, Lukasiewicz paths originate from queuing systems theory to study the waiting line of a single server subject to the Last-In-First-Out policy and they have been used by Le Gall & Le Jan [81] to define Lévy trees. In the following lemma, we recall that Lukasiewicz paths are adapted processes that completely encode finite trees and that are particularly well-suited to study Galton-Watson trees (see e.g. Le Gall [78, Proposition 1.1] and [78, Corollary 1.6] for more details).

Proposition 4.2.7 *Let $t \in \mathbb{T}$ be finite. Let τ be a $\text{GW}(\mu)$ -tree where μ is non-trivial and critical. Then the following holds true.*

- (i) *Lukasiewicz paths provide a one-to-one correspondence between the set of finite trees and the set of finite nonnegative excursions of left-continuous walks, which is defined by*

$$\bigcup_{n \in \mathbb{N}^*} \{(w_j)_{0 \leq j \leq n} \in \mathbb{Z}^n : w_0 = 0, w_n = -1, w_j \geq 0 \text{ and } w_{j+1} - w_j \geq -1 \text{ for all } 0 \leq j < n\}.$$

- (ii) *For all $m \in \mathbb{N}$, we denote by $R_m t$ the tree t restricted to its $m + 1$ first vertices (with respect to the lexicographic order). Namely,*

$$R_m t = \{u_j(t) : 0 \leq j \leq \min(m, \#t - 1)\} \quad (4.2.7)$$

where $(u_j(t))$ stands for the vertices of t listed in lexicographic order. Then, $R_m t$ is a measurable function of $(W_j(t) ; 0 \leq j \leq \min(m, \#t))$.

- (iii) *Let $(W_n)_{n \geq 0}$ be a \mathbb{Z} -valued random walk whose jump distribution is given by (4.1.7). We set $H_1 = \inf\{j \in \mathbb{N} : W_j = -1\}$, which is an a.s. finite stopping time since $\mathbb{E}[W_1] = 0$. Then,*

$$(W_j(\tau))_{0 \leq j \leq \#\tau} \stackrel{(law)}{=} (W_j)_{0 \leq j \leq H_1}.$$

In particular, $\#\tau$ and H_1 have the same law.

(iv) More generally, for all $p \in \mathbb{N}$, we set

$$H_0 = 0 \quad \text{and} \quad H_p = \inf\{j \in \mathbb{N} : W_j = -p\}. \quad (4.2.8)$$

Then there is an i.i.d. sequence $(\tau_p)_{p \in \mathbb{N}}$ of $\text{GW}(\mu)$ -trees such that

$$\forall p \in \mathbb{N}, \quad (p + W_{H_p+j})_{0 \leq j \leq H_{p+1} - H_p} = (W_j(\tau_p))_{0 \leq j \leq \#\tau_p}. \quad (4.2.9)$$

4.2.2 Limit theorems.

In this section we recall – mostly from Bingham, Goldies & Teugels [23] and Feller [74] – limit theorems for sums of i.i.d. random variables belonging to the domain of attraction of stable laws. We also recall useful limit theorems for Galton-Watson trees and random walks from Berger [17], Duquesne [50], Kortchemski [106], and Kortchemski & Richier [108].

Regularly and slowly varying functions. Recall that a measurable and locally bounded function $l : (0, \infty) \rightarrow (0, \infty)$ is *slowly varying* at infinity (resp. at 0^+) if $l(cx) \sim l(x)$ as $x \rightarrow \infty$ (resp. as $x \rightarrow 0^+$) for all $c \in (0, \infty)$. Also recall that $f : (0, \infty) \rightarrow (0, \infty)$ is *regularly varying of index* $\alpha \in \mathbb{R}$ at ∞ (resp. at 0^+) if there exists a slowly varying function l at ∞ (resp. at 0^+) such that $f(x) = x^\alpha l(x)$. Below we gather in a single proposition several well-known results on slowly and regularly varying functions that are used in this chapter.

Proposition 4.2.8 *Let l be a slowly varying function at ∞ . Then the following holds true.*

(i) (Potter's bound) *For all $\varepsilon \in (0, \infty)$ and all $c \in (1, \infty)$, there exists $x_0 \in (0, \infty)$ such that for all $x \in [x_0, \infty)$ and all $\lambda \in [1, \infty)$, it holds*

$$\frac{1}{c} \lambda^{-\varepsilon} \leq \frac{l(x\lambda)}{l(x)} \leq c \lambda^\varepsilon.$$

Therefore, $\ln l(x) = o(\ln x)$ and if f is regularly varying with index $\rho \in \mathbb{R} \setminus \{0\}$, then $\ln f(x) \sim \rho \ln x$. Moreover, if $x_n \sim y_n \rightarrow \infty$ then $l(x_n) \sim l(y_n)$ and $f(x_n) \sim f(y_n)$.

(ii) (Karamata's Abelian Theorem for Tails) *Let $\rho \in (0, \infty)$. Then $\int_x^\infty y^{-1-\rho} l(y) dy < \infty$ and, as $x \rightarrow \infty$,*

$$\int_x^\infty y^{-1-\rho} l(y) dy \sim \frac{1}{\rho} x^{-\rho} l(x) \quad \text{and} \quad \int_1^x y^{\rho-1} l(y) dy \sim \frac{1}{\rho} x^\rho l(x).$$

(iii) *Suppose that $\int_x^\infty y^{-1} l(y) dy < \infty$ and set $\bar{l}(x) = \int_x^\infty y^{-1} l(y) dy$. Then \bar{l} is slowly varying at ∞ and $\lim_{x \rightarrow \infty} \bar{l}(x)/l(x) = \infty$.*

(iv) (Monotone Density Theorem) *Let $u : [0, \infty) \rightarrow \mathbb{R}$ be a locally Lebesgue integrable function and set $U(x) = \int_0^x u(y) dy$ for all $x \in \mathbb{R}$. Assume that there are $c, \rho \in (0, \infty)$ such that $U(x) \sim cx^\rho l(x)$ as $x \rightarrow \infty$ and furthermore assume that u is ultimately monotone. Then $u(x) \sim c\rho x^{\rho-1} l(x)$ as $x \rightarrow \infty$.*

(v) (Karamata's Abelian Theorem for Laplace Transform) *Let $U : [0, \infty) \rightarrow [0, \infty)$ be a measurable function such that $\widehat{U}(\lambda) := \lambda \int_0^\infty e^{-\lambda x} U(x) dx < \infty$ for all $\lambda \in (0, \infty)$. Assume that there are $c \in (0, \infty)$ and $\rho \in (-1, \infty)$ such that $U(x) \sim \frac{c}{\Gamma(1+\rho)} x^\rho l(x)$ as $x \rightarrow \infty$, then $\widehat{U}(\lambda) \sim c\lambda^{-\rho} l(1/\lambda)$ as $\lambda \rightarrow 0^+$.*

Proof. For (i), see e.g. [23, Theorem 1.5.6]. For (ii), see e.g. [23, Theorem 1.5.8 and Proposition 1.5.10]. For (iii), see e.g. [23, Theorem 1.5.9b]. For (iv), see e.g. [23, Theorem 1.7.2]. For (v), see e.g. [23, Theorem 1.7.6]. ■

Limit theorems: the cases where $\alpha \in (1, 2]$. We next recall equivalent formulations of the property for a probability measure μ on \mathbb{N} to belong to the domain of attraction of a stable law of index $\alpha \in (1, 2]$ (we handle 1-stable laws separately).

Proposition 4.2.9 *Let μ be a probability measure on \mathbb{N} that satisfies (4.1.1). Recall from (4.1.2) the definition of ψ . Let $(W_n)_{n \in \mathbb{N}}$ be a random walk whose jumps distribution is specified in (4.1.7). Let $\alpha \in (1, 2]$. Then, the following assertions are equivalent.*

- (a) μ belongs to the domain of attraction of an α -stable law.
- (b) There exists $L : (0, \infty) \rightarrow (0, \infty)$ that varies slowly at ∞ such that if $\alpha \in (1, 2)$ then $\mu([n, \infty)) \sim n^{-\alpha} L(n)$, and if $\alpha = 2$ then $\sum_{0 \leq k \leq n} k^2 \mu(k) - 1 \sim 2L(n)$ which is ultimately positive by (4.1.1).
- (c) If $\alpha \in (1, 2)$ then $\psi(s) \sim_{0+} \frac{\alpha-1}{\Gamma(2-\alpha)} s^\alpha L(1/s)$, and if $\alpha = 2$ then $\psi(s) \sim_{0+} s^2 L(1/s)$.
- (d) There exists a $(0, \infty)$ -valued sequence $(a_n)_{n \in \mathbb{N}}$ tending to ∞ such that $\frac{1}{a_n} W_n$ converges in law to the spectrally positive α -stable random variable X_α whose Laplace exponent is given for all $\lambda \in [0, \infty)$ by $\ln \mathbb{E}[\exp(-\lambda X_\alpha)] = \frac{\alpha-1}{\Gamma(2-\alpha)} \lambda^\alpha$ if $\alpha \in (1, 2)$ and by λ^2 if $\alpha = 2$.

Moreover, if one of the four equivalent assumptions from above holds true, then $a_n^\alpha \sim nL(a_n)$ and $a_n \sim n^{1/\alpha} L^*(n)$ where the function $L^* : x \in (0, \infty) \mapsto x^{-1/\alpha} \inf\{y \in (0, \infty) : y^\alpha/L(y) > x\}$ is slowly varying at ∞ .

Proof. For $(a) \Leftrightarrow (b)$, see e.g. [23, Theorem 8.3.1]. The equivalence $(a) \Leftrightarrow (d)$ follows from the definition of the domain of attraction of a stable law: the limiting law is necessarily spectrally positive since μ is supported by \mathbb{N} and among the spectrally positive α -stable types, it is always possible to choose a centered one as $\alpha \in (1, 2]$ (see e.g. [23, Section 8.3] and [74, Chapter XVII.5]).

For $(b) \Leftrightarrow (c)$ see e.g. [23, Theorem 8.1.6]. More precisely, recall from (4.1.2) the definitions of φ and ψ . Then, for all $\lambda \in (0, \infty)$, set

$$f_1(\lambda) := \varphi(e^{-\lambda}) - 1 + \lambda = \psi(1 - e^{-\lambda}) + \frac{1}{2}\lambda^2 + O(\lambda^3). \quad (4.2.10)$$

If $\alpha \in (1, 2)$ then [23, Theorem 8.1.6] asserts that (b) is equivalent to the asymptotic equivalent $f_1(\lambda) \sim_{0+} \frac{\alpha-1}{\Gamma(2-\alpha)} \lambda^\alpha L(1/\lambda)$, which implies that $(b) \Leftrightarrow (c)$ in these cases (with Proposition 4.2.8 (i)). If $\alpha = 2$ then [23, Theorem 8.1.6] asserts that (b) is equivalent to $f_1(\lambda) \sim_{0+} \lambda^2(\frac{1}{2} + L(1/\lambda))$ and (4.2.10) implies that $(b) \Leftrightarrow (c)$.

Let us prove the last point of the proposition. We assume that $(a-d)$ hold true. By Grimvall [87, Theorem 2.1], $\lim_{n \rightarrow \infty} \mathbb{E}[\exp(-\frac{\lambda}{a_n} W_n)] = \mathbb{E}[e^{-\lambda X_\alpha}]$ for all $\lambda \in (0, \infty)$. Observe that $\mathbb{E}[\exp(-\lambda W_n)] = (e^\lambda(f_1(\lambda) + 1 - \lambda))^n$. If $\alpha \in (1, 2)$, we easily get $n f_1(\lambda/a_n) \sim \frac{\alpha-1}{\Gamma(2-\alpha)} \lambda^\alpha$ as $n \rightarrow \infty$ for all $\lambda \in (0, \infty)$ and thus $a_n^\alpha \sim nL(a_n)$. If $\alpha = 2$, we get

$$\frac{n\lambda}{a_n} + n \ln \left[1 - \frac{\lambda}{a_n} \left(1 - \frac{\lambda}{a_n} \left(\frac{1}{2} + L(a_n) \right) (1 + o(1)) \right) \right] \sim \frac{nL(a_n)}{a_n^2} \lambda^2,$$

which implies that $a_n^2 \sim nL(a_n)$. In both cases, we observe that $a_n^\alpha/L(a_n) \sim n$ and we use [23, Theorem 1.5.12] to complete the proof of the proposition. \blacksquare

We next recall standard results on the size of Galton-Watson trees, which are expressed in terms of random walks thanks to Proposition 4.2.7.

Proposition 4.2.10 *Let μ be a probability measure on \mathbb{N} that satisfies (4.1.1). Let $(W_n)_{n \in \mathbb{N}}$ be a random walk whose jumps distribution is specified by (4.1.7). Recall from (4.2.8) the definition of the stopping times $(H_p)_{p \in \mathbb{N}}$. Let τ be a $\text{GW}(\mu)$ -tree. Then the following holds true.*

- (i) $(\mathbb{H}_p)_{p \in \mathbb{N}}$ is a random walk with positive jumps.
- (ii) (Kemperman) $\mathbb{P}(\mathbb{H}_p = n) = \frac{p}{n} \mathbb{P}(W_n = -p)$ for all $n, p \in \mathbb{N}^*$.
- (iii) Suppose that μ satisfies Proposition 4.2.9 (a-d) and is aperiodic. Then, $\mathbb{P}(\#\tau = n) \sim \frac{c_\alpha}{na_n}$ and $\mathbb{P}(\#\tau \geq n) \sim \frac{\alpha c_\alpha}{a_n}$ where c_α is the value at 0 of the (continuous version) of the density of X_α , and where (a_n) and X_α are as in Proposition 4.2.9 (d).
- (iv) Suppose that μ satisfies Proposition 4.2.9 (a-d). Recall that $|\tau|$ stands for the height of τ . Then,

$$\frac{\mathbb{P}(|\tau| \geq n)}{\psi(\mathbb{P}(|\tau| \geq n))} \sim (\alpha - 1)n.$$

Proof. Note that (i) is an immediate consequence of the Markov property and the left-continuity of W . For (ii), see e.g. [23, Theorem 8.9.15]. Let us prove (iii). By (ii) and by Proposition 4.2.7 (iii), we see that $\mathbb{P}(\#\tau = n) = \frac{1}{n} \mathbb{P}(W_n = -1)$. We next use Gnedenko's local limit Theorem (see e.g. [23, Theorem 8.4.1]) to get $\lim_{n \rightarrow \infty} |a_n \mathbb{P}(W_n = -1) - c_\alpha| = 0$ and thus $\mathbb{P}(\#\tau = n) \sim \frac{c_\alpha}{na_n}$. Since (a_n) varies regularly with index $1/\alpha$ by Proposition 4.2.9, we get $\mathbb{P}(\#\tau \geq n) \sim \frac{\alpha c_\alpha}{a_n}$ by Karamata's Abelian Theorem for Tails (see Proposition 4.2.8 (ii)). For (iv), see Slack [142, Lemma 2]. ■

We next recall two limit theorems that are used to prove Theorem 4.1.2. One follows from the convergence of rescaled $\text{GW}(\mu)$ -trees to stable trees due to Duquesne [50]. The other is the uniform integrability of the density of the law of (roughly speaking) the Lukasiewicz path of a $\text{GW}(\mu)$ -tree τ under $\mathbb{P}(\cdot | \#\tau = n)$ with respect to $\mathbb{P}(\cdot | \#\tau \geq n)$ that has been proved in Kortchemski [106].

Proposition 4.2.11 *Assume that μ satisfies (4.1.1), that it is aperiodic, and that it belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$. More precisely, we assume that Proposition 4.2.9 (d) holds true. Let τ be a $\text{GW}(\mu)$ -tree. Recall from (4.2.1) that $|\tau|$ stands for the maximal height of τ , and that $W(\tau)$ stands for its Lukasiewicz path as in Definition 4.2.6. Then, the following holds true.*

- (i) There exists a random variable $M \in (0, \infty)$ such that $\frac{a_n}{n} |\tau|$ under $\mathbb{P}(\cdot | \#\tau = n)$ converges in law to M as $n \rightarrow \infty$.
- (ii) Let $r \in (0, 1)$. Then for all large enough $n \in \mathbb{N}$, there is a function $D_n^{(r)} : \mathbb{N} \rightarrow [0, \infty)$ such that

$$\mathbb{E}[f(W_{\min(\lfloor nr \rfloor, \cdot)}(\tau)) | \#\tau = n] = \mathbb{E}[f(W_{\min(\lfloor nr \rfloor, \cdot)}(\tau)) D_n^{(r)}(W_{\lfloor rn \rfloor}(\tau)) | \#\tau \geq n] \quad (4.2.11)$$

for all bounded function $f : \mathbb{N}^{\mathbb{N}} \rightarrow [0, \infty)$. Moreover, these functions satisfy

$$\lim_{c \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{\{D_n^{(r)}(W_{\lfloor rn \rfloor}(\tau)) \geq c\}} | \#\tau \geq n] = 0. \quad (4.2.12)$$

Proof. The point (i) is a consequence of the convergence of rescaled Galton Watson trees to the α -stable tree: see Duquesne [50, Theorem 3.1]. Here, M is the height of the normalized α -stable tree that is a $(0, \infty)$ -valued random variable. See Kortchemski [106, Equation (1) and Lemma 2] for (4.2.11). For (4.2.12), [106, Equation (12)] shows that $D_n^{(r)}(a_n \cdot)$ uniformly converges on all compact intervals of $(0, \infty)$ towards a continuous function. Moreover, the laws of $a_n^{-1} W_{\lfloor rn \rfloor}(\tau)$ under $\mathbb{P}(\cdot | \#\tau = n)$ are tight in $(0, \infty)$ (see [106, Equation (17)]), which completes the proof of (4.2.12) by (4.2.11). Let us mention that in [106], the hypothesis that μ has infinite variance and is in the domain of attraction of a 2-stable law is incorrectly rephrased into the stronger assumption that $\mu([n, \infty)) \sim L(n)/n^2$ with some slowly varying function L . This does not affect the validity of the results of [106] because that assumption is only invoked for applying statements that remain true under the general hypothesis: namely, the local limit theorem [106, Theorem 1] and the invariance theorem for the tail conditioning [106, Theorem 2]. ■

Limit theorems: the 1-stable case. We now consider the spectrally positive 1-stable law, which features more complicated behaviors.

Proposition 4.2.12 *Let μ be a probability measure on \mathbb{N} that satisfies (4.1.1). Recall from (4.1.2) the definition of ψ . Let $(W_n)_{n \in \mathbb{N}}$ be a random walk whose jumps distribution is given by (4.1.7).*

(i) *The following assertions are equivalent.*

(a) *μ belongs to the domain of attraction of a 1-stable law.*

(b) *There exists $L: (0, \infty) \rightarrow (0, \infty)$ that varies slowly at ∞ such that $\int^\infty y^{-1} L(y) dy < \infty$ and such that $\mu([n, \infty)) \sim n^{-1} L(n)$.*

(c) *There exists a $(0, \infty)$ -valued sequence $(a_n)_{n \in \mathbb{N}}$ tending to ∞ such that $\frac{1}{a_n}(W_n + b_n)$ converges in law to X , where $b_n = n\mathbb{E}[W_1 \mathbf{1}_{\{W_1 > a_n\}}]$ and where X is the spectrally positive 1-stable random variable whose Laplace exponent is given for all $\lambda \in (0, \infty)$ by $\ln \mathbb{E}[\exp(-\lambda X)] = \lambda \ln \lambda$.*

(ii) *Assume that (a-c) hold true. Then, $\sum_{k \geq n} k\mu(k) \sim \ell(n)$ where ℓ is the slowly varying function defined by*

$$\forall x \in (0, \infty), \quad \ell(x) = \int_x^\infty \frac{L(y)}{y} dy. \quad (4.2.13)$$

This implies that $\psi(s) \sim_{0+} s\ell(1/s)$. Moreover, $L(x) = o(\ell(x))$ as $x \rightarrow \infty$.

(iii) *Assume that (a-c) hold true. Then, $a_n \sim nL(a_n)$ and $a_n \sim nL^*(n)$ where the function $L^*: x \in (0, \infty) \mapsto x^{-1} \inf\{y \in (0, \infty) : y/L(y) > x\}$ varies slowly at ∞ . Moreover, $b_n \sim n\ell(a_n)$ where ℓ is given by (4.2.13) and therefore $a_n = o(b_n)$.*

Proof. The equivalences (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c) are proved as in Proposition 4.2.9 (for the form of b_n , see e.g. [74, Chapter IX.8, Equation (8.15)]). To prove (ii), we first observe the equalities $\sum_{k \geq n} k\mu(k) = \sum_{j \geq 1} \mu([\max(n, j), \infty)) = n\mu([n, \infty)) + \sum_{j > n} \mu([j, \infty))$. Then note that $n\mu([n, \infty)) \sim L(n)$ and that $\sum_{j > n} \mu([j, \infty)) \sim \ell(n)$ where ℓ is defined by (4.2.13). By Proposition 4.2.8 (iii), $L(n) = o(\ell(n))$ and we get $\sum_{k \geq n} k\mu(k) \sim \ell(n)$. By [23, Theorem 8.1.6], this is equivalent to $f_1(\lambda) \sim_{0+} \lambda\ell(1/\lambda)$ where f_1 is as in (4.2.10). This implies $\psi(s) \sim_{0+} s\ell(1/s)$ by Proposition 4.2.8 (i), which completes the proof of (ii). Then, we prove $a_n \sim nL(a_n)$ and $a_n \sim nL^*(n)$ as in Proposition 4.2.9. Next observe that $b_n = n \sum_{k > a_n} k\mu(k+1) \sim n\ell(a_n)$. Thus $a_n/b_n \sim L(a_n)/\ell(a_n) \rightarrow 0$ by (ii). This completes the proof of (iii). \blacksquare

Suppose that Proposition 4.2.12 (c) holds true. Since $a_n = o(b_n)$, it holds $b_n^{-1}W_{[ns]} \rightarrow -s$ in law, and thus in probability, for all $s \in [0, \infty)$. Standard arguments (or a stronger result such as Skorokhod [141, Theorem 2.7]) entail the following convergence

$$\left(\frac{1}{b_n}W_{[ns]} ; s \in [0, \infty)\right) \longrightarrow (-s ; s \in [0, \infty)) \quad (4.2.14)$$

in probability for the topology of uniform convergence on all compact intervals. Recall from (4.2.8) the definition of the stopping times $(H_p)_{p \in \mathbb{N}}$. Then, (4.2.14) implies for all $x \in [0, \infty)$ that

$$\frac{1}{n}H_{[b_n x]} \longrightarrow x \quad (4.2.15)$$

in probability. Namely the law of H_1 is *relatively stable* (see e.g. [23, Section 8.8 § 1]). Since the total size of a $\text{GW}(\mu)$ -tree τ has the same distribution as H_1 (by Proposition 4.2.7 (iii)), the law of $\#\tau$ is thus relatively stable. By use of Berger [17, Theorem 2.4 and Lemma 4.3] and Kortchemski & Richier [108, Proposition 12], we get the following.

Lemma 4.2.13 *Let μ be a probability measure on \mathbb{N} that satisfies (4.1.1) and that belongs to the domain of attraction of a 1-stable law. Let τ be a $\text{GW}(\mu)$ -tree. Then,*

$$\mathbb{P}(\#\tau \geq n) \sim \frac{L(b_n)}{b_n \ell(b_n)}, \quad (4.2.16)$$

where (b_n) , L and ℓ are as in Proposition 4.2.12. If the more restrictive assumption (4.1.12) holds, then

$$\mathbb{P}(\#\tau = n) \sim \frac{1}{n} \mathbb{P}(\#\tau \geq n) \sim \frac{L(b_n)}{b_n^2}. \quad (4.2.17)$$

Proof. Since $b_n/n \sim \ell(a_n) \rightarrow 0$ as $n \rightarrow \infty$, (4.2.15) implies that there is a sequence (c_n) tending to ∞ such that $\frac{1}{c_n} H_n \rightarrow 1$ in probability: namely, the law of H_1 is relatively stable. Then, [23, Theorem 8.8.1] asserts the existence of a function l that slowly varies at ∞ such that $\sum_{0 \leq k \leq n} \mathbb{P}(H_1 \geq k) \sim l(n)$ and $1 - \mathbb{E}[e^{-\lambda H_1}] \sim_{0+} \lambda l(1/\lambda)$. Thus, (4.2.15) easily entails that $\ell(a_n) l(n) \rightarrow 1$. By Berger [17, Lemma 4.3], we get $\ell(a_n) \sim \ell(b_n)$ and thus, $\sum_{0 \leq k \leq n} \mathbb{P}(\#\tau \geq k) = \sum_{0 \leq k \leq n} \mathbb{P}(H_1 \geq k) \sim 1/\ell(b_n)$. By Kortchemski & Richier [108, Proposition 12], we get (4.2.16) because within the notations of [108], we necessarily get $\Lambda(n) \sim 1/\ell(b_n)$. Moreover, Berger [17, Theorem 2.4] asserts that if (4.1.12) holds then $\mathbb{P}(W_n = -1) \sim nL(b_n)/b_n^2$, so (4.2.17) follows from Kemperman's identity (Proposition 4.2.10 (ii)). ■

Remark 4.2.14 Although the difficult part of (4.2.16) is the very content of Kortchemski & Richier [108, Proposition 12], the relatively explicit form of the right member of (4.2.16) seems novel under the sole assumption that μ belongs to the domain of attraction of a 1-stable law. □

We next recall two limit theorems on the maximal out-degree of a $\text{GW}(\mu)$ -tree when μ belongs to the domain of attraction of a 1-stable law. They are part of more general results due to Kortchemski & Richier [108].

Proposition 4.2.15 *Let τ be a $\text{GW}(\mu)$ -tree with offspring distribution μ that satisfies (4.1.1) and that belongs to the domain of attraction of a 1-stable law. Recall from (4.2.1) that the notation $\Delta(\tau)$ stands for the maximal out-degree of τ . Then, the following holds true.*

(i) *The following convergence holds in distribution on $[0, \infty)$:*

$$\frac{1}{b_n} \Delta(\tau) \text{ under } \mathbb{P}(\cdot | \#\tau \geq n) \longrightarrow J, \quad (4.2.18)$$

where the law of J is given by $\mathbb{P}(J \geq x) = 1/x$ for all $x \in [1, \infty)$.

(ii) *Under the more restrictive assumption (4.1.12), the following convergence holds in probability:*

$$\frac{1}{b_n} \Delta(\tau) \text{ under } \mathbb{P}(\cdot | \#\tau = n) \longrightarrow 1. \quad (4.2.19)$$

Proof. For (i), see [108, Theorem 6]. For (ii), see [108, Theorem 1]. ■

Recall that $W(\tau)$ stands for the Lukasiewicz path of τ , as in Definition 4.2.6. We conclude this section by recalling a result from Kortchemski & Richier [108] that shows that the law of $W(\tau)$ under $\mathbb{P}(\cdot | \#\tau = n)$ is closed in variation distance to the law of the Vervaat transform of the path $(W_0, W_1, \dots, W_{n-1}, -1)$ under \mathbb{P} . More precisely, let $(W_n)_{n \in \mathbb{N}}$ be as in Proposition 4.2.12. For all $n \in \mathbb{N}^*$, we introduce the following notations.

$$I_n = - \min_{0 \leq j \leq n-1} W_j \quad \text{and} \quad \sigma_n = \inf\{0 \leq k \leq n-1 : W_k = -I_n\} \quad (4.2.20)$$

and

$$Z_j^{(n)} = \begin{cases} W_{\sigma_n+j} + I_n & \text{if } 0 \leq j < n - \sigma_n, \\ I_n - 1 + W_{j-(n-\sigma_n)} & \text{if } n \geq j \geq n - \sigma_n. \end{cases} \quad (4.2.21)$$

Namely, $Z^{(n)}$ is constructed by reading the increments of $(W_0, W_1, \dots, W_{n-1}, -1)$ from left to right in cyclic order by starting at time σ_n : this is a kind of *Vervaat transform* of $(W_0, W_1, \dots, W_{n-1}, -1)$ (see Vervaat [147] for more details). We shall use Kortchemski & Richier [108, Theorem 21], which we re-state by convenience into the following proposition.

Proposition 4.2.16 *Assume that μ satisfies (4.1.1) and (4.1.12). We keep the above notation. Then,*

$$\sup_{A \in \mathcal{B}(\mathbb{R}^{n+1})} \left| \mathbb{P}(W(\tau) \in A \mid \#\tau = n) - \mathbb{P}(Z^{(n)} \in A) \right| \longrightarrow 0,$$

where $\mathcal{B}(\mathbb{R}^{n+1})$ stands for the Borel sigma-field of \mathbb{R}^{n+1} .

Proof. See Kortchemski & Richier [108, Theorem 21] and note $\mathbb{P}(I_n > 1) \rightarrow 1$ by e.g. (4.2.14). ■

4.3 Distribution of the Horton-Strahler number of Galton-Watson trees

4.3.1 Alternative definitions of the Horton-Strahler number and basic results

In this section, we prove basic results on the Horton-Strahler number: firstly, we provide alternative definitions, secondly we state a key upper bound in Lemma 4.3.1, and finally we prove in Lemma 4.3.2 a recursive equation satisfied by the tail of the Horton-Strahler of $\text{GW}(\mu)$ -trees that is the starting point of the analysis of their asymptotic behavior.

Let us first recall alternative definitions of the Horton-Strahler number. The first one uses *Horton pruning* of a finite tree $t \in \mathbb{T}$ that is defined as follows: *remove the leaves of t and merge each line into one edge (a line in t is a maximal sequence of vertices $v_0, \dots, v_n \in t$ such that $k_{v_1}(t) = \dots = k_{v_{n-1}}(t) = 1$ and $v_j = \overleftarrow{v}_{j+1}$ for all $0 \leq j < n$). The resulting tree is called the Horton-pruned tree, which we denote here by $\text{Prun}(t)$. Then, $\mathcal{S}(t)$ is the minimal number of Horton prunings that are necessary to obtain $\{\emptyset\}$ from t . Namely,*

$$\mathcal{S}(t) = \min \{n \in \mathbb{N} : \text{Prun}_n(t) = \{\emptyset\}\}, \quad (4.3.1)$$

where Prun_n stands for the n -th iteration of Prun for all $n \in \mathbb{N}^*$ and where Prun_0 stands for the operation that merges each line into one edge. We refer to Kovchegov & Zaliapin [111, Section 2.3] for a proof and more details.

Another useful definition uses embeddings of perfect binary trees. More precisely, let $t, t' \in \mathbb{T}$ be finite. Then, $\phi : t \rightarrow t'$ is an *embedding* if it is injective and if $\phi(u \wedge v) = \phi(u) \wedge \phi(v)$ for all $u, v \in t$. For all $n \in \mathbb{N}$, we denote by $T_{2,n} = \bigcup_{0 \leq k \leq n} \{1, 2\}^k$ the n -perfect binary tree, with the convention that $\{1, 2\}^0 = \{\emptyset\}$. Then, for all finite tree $t \in \mathbb{T}$,

$$\mathcal{S}(t) = \max \{n \in \mathbb{N} : \exists \phi : T_{2,n} \rightarrow t \text{ embedding}\}. \quad (4.3.2)$$

This result seems to be ‘part of the folklore’. Let us however provide a short proof.

Proof of (4.3.2). We reason by induction on the height of t . Note that (4.3.2) obviously holds true if $|t| = 0$. Now assume $k := k_{\emptyset}(t) \geq 1$ and let $\phi : T_{2,n+1} \rightarrow t$ be an embedding, we separate the cases according to the positions of $\phi(\emptyset), \phi(1), \phi(2)$.

If $(i) \preceq \phi(\emptyset)$ with $1 \leq i \leq k$, then we have $(i) \preceq \phi(u)$ for all $u \in T_{2,n+1}$ by definition, and we check that setting $\phi(u) = (i) * \phi_i(u)$ defines an embedding $\phi_i : T_{2,n+1} \rightarrow \theta_{(i)}t$. Conversely, an embedding $\phi_i : T_{2,n+1} \rightarrow \theta_{(i)}t$ induces an embedding $\phi : T_{2,n+1} \rightarrow t$ such that $(i) \preceq \phi(\emptyset)$. Thus,

$$\max \{n \in \mathbb{N} : \exists \phi : T_{2,n} \rightarrow t \text{ embedding, } (i) \preceq \phi(\emptyset)\} = \mathcal{S}(\theta_{(i)}t).$$

Otherwise, $\emptyset = \phi(\emptyset) = \phi(1) \wedge \phi(2)$ so we have distinct $1 \leq i, j \leq k$ such that $(i) \preceq \phi(1)$ and $(j) \preceq \phi(2)$. Similarly as before, we see that setting $\phi((1)*u) = (i)*\phi_i(u)$ and $\phi((2)*u) = (j)*\phi_j(u)$ respectively defines two embeddings $\phi_i : T_{2,n} \rightarrow \theta_{(i)}t$ and $\phi_j : T_{2,n} \rightarrow \theta_{(j)}t$. Conversely, two embeddings $\phi_i : T_{2,n} \rightarrow \theta_{(i)}t$ and $\phi_j : T_{2,n} \rightarrow \theta_{(j)}t$ induces an embedding $\phi : T_{2,n+1} \rightarrow t$ such that $\phi(\emptyset) = \emptyset$, $(i) \preceq \phi(1)$, and $(j) \preceq \phi(2)$. Thus,

$$\max \{n \in \mathbb{N} : \exists \phi : T_{2,n} \rightarrow t \text{ embedding, } (i) \preceq \phi(1), (j) \preceq \phi(2)\} = 1 + \min (\mathcal{S}(\theta_{(i)}t), \mathcal{S}(\theta_{(j)}t)).$$

Taking the maximum over $\phi(\emptyset), \phi(1), \phi(2)$ and recalling Definition 4.1.1 concludes the proof. ■

The definition (4.3.2) immediately implies the following. Let $t, t' \in \mathbb{T}$ be finite.

$$\text{If there is an embedding } \phi : t \rightarrow t', \text{ then } \mathcal{S}(t) \leq \mathcal{S}(t'). \tag{4.3.3}$$

We next use (4.3.2) and (4.3.3) to get an upper bound of $\mathcal{S}(t)$ in terms of $\mathcal{S}(R_m t)$, where $R_m t$ is defined in (4.2.7) (recall that it is the tree consisting in the first $m + 1$ vertices of t in lexicographic order) and of $R_m t^*$, where t^* is the *mirror image of t* that is formally defined as follows.

Let $t \in \mathbb{T}$ and $u \in t \setminus \{\emptyset\}$ be the word (j_1, \dots, j_n) . We set $u_{|0} = \emptyset$ and $u_{|p} = (j_1, \dots, j_p)$ for all $1 \leq p \leq n$. Then, the mirror image of u is the word $u^* = (j_1^*, \dots, j_n^*)$ where

$$j_p^* := k_{u_{|p-1}}(t) - j_p + 1, \quad 1 \leq p \leq n.$$

We also set $\emptyset^* = \emptyset$. Then $t^* = \{u^* : u \in t\}$ and it is easy to show that $t^* \in \mathbb{T}$. Observe that $u \mapsto u^*$ is a bijective embedding so $\mathcal{S}(t^*) = \mathcal{S}(t)$ by (4.3.3). We stress that the word u^* depends on the tree t on which u is observed. Nevertheless, this notation should not lead to confusion here because the underlying tree will always be clear according to context. Since $k_{u^*}(t^*) = k_u(t)$, (4.2.3) implies that if τ is a $\text{GW}(\mu)$ -tree whose offspring distribution μ satisfies (4.1.1), then so is τ^* . Furthermore, since $\#\tau^* = \#\tau$, if $n \in \mathbb{N}^*$ is such that $\mathbb{P}(\#\tau = n) > 0$, then we easily check that

$$\text{under } \mathbb{P}(\cdot \mid \#\tau = n), \quad \tau^* \stackrel{\text{(law)}}{=} \tau. \tag{4.3.4}$$

The following lemma plays a key role in the proofs of Theorems 4.1.2 and 4.1.4.

Lemma 4.3.1 *Let $t \in \mathbb{T}$ be finite. Then the following holds true.*

(i) *For all $u \in t$, we set $t_{\leq u} = \{v \in t : v \leq u\}$. Then,*

$$\mathcal{S}(t) \leq 1 + \max \left(\mathcal{S}(t_{\leq u}), \max \{ \mathcal{S}(\theta_v t) : v \in t, \overleftarrow{v} \preceq u \text{ and } v > u \} \right). \tag{4.3.5}$$

(ii) *Let $m \in \mathbb{N}$ be such that $2m \geq \#t + |t|$. Then,*

$$\mathcal{S}(t) \leq 1 + \max (\mathcal{S}(R_m t), \mathcal{S}(R_m t^*)). \tag{4.3.6}$$

(iii) *If τ is a random finite tree such that τ^* has the same law as τ , then for all $n, m \in \mathbb{N}^*$, we get*

$$\mathbb{P}(\mathcal{S}(\tau) \geq n) \leq 2\mathbb{P}(\mathcal{S}(R_m \tau) \geq n - 1) + \mathbb{P}(\#\tau + |\tau| > 2m). \tag{4.3.7}$$

Proof. We first prove (i). If $\mathcal{S}(t) = 0$, then (4.3.5) is obviously true. We next suppose that $\mathcal{S}(t) = n \geq 1$, and to simplify notation, we set

$$[\emptyset, u] = \{v \in t : v \leq u\} \quad \text{and} \quad B = \{v \in t : \overleftarrow{v} \leq u \text{ and } v > u\}.$$

By (4.3.2), there is an embedding $\phi: T_{2,n} \rightarrow t$. Observe that $\phi(1)$ and $\phi(2)$ cannot both belong to $[\emptyset, u]$, otherwise it would imply $\phi(1) \wedge \phi(2) = \phi(\emptyset) \in \{\phi(1), \phi(2)\}$ (by definition of embeddings since $\emptyset = (1) \wedge (2)$), which contradicts the injectivity of ϕ . Thus, there is $j \in \{1, 2\}$ such that either $\phi(j) \in t_{\leq u} \setminus [\emptyset, u]$ or $\phi(j) \in t \setminus t_{\geq u}$. In the first case, by definition of embeddings, $\phi(j) \leq \phi((j)*v)$ for all $v \in T_{2,n-1}$ and so $\phi((j)*T_{2,n-1}) \subset t_{\leq u}$. Then, (4.3.2) entails $n-1 \leq \mathcal{S}(t_{\leq u})$. Suppose next that $\phi(j) \in t \setminus t_{\geq u}$. Since $t \setminus t_{\geq u}$ is the disjoint union of the $v * (\theta_v t)$ for $v \in B$, there exists $v \in B$ such that $\phi((j)*T_{2,n-1}) \subset \phi(j) * (\theta_{\phi(j)t}) \subset v * (\theta_v t)$. Then, (4.3.2) entails $n-1 \leq \max_{v \in B} \mathcal{S}(\theta_v t)$, which completes the proof of (4.3.5).

Let us now prove (ii). To that end, we denote by \bar{u} the \leq -minimal leaf of $\theta_u t$. We also set $t_{\geq u} = \{v \in t : v \geq \bar{u}\} \cup [\emptyset, \bar{u}]$. By definition, $t = t_{\leq u} \cup t_{\geq u}$ and $[\emptyset, u] = t_{\leq u} \cap t_{\geq u}$. Then, note that for all $v \in B$, $v * (\theta_v t) \subset t_{\geq u}$. Moreover, note that $(t_{\geq u})^* = t_{\leq \bar{u}^*}$. Therefore, (4.3.3) and (i) imply

$$\mathcal{S}(t) \leq 1 + \max(\mathcal{S}(t_{\leq u}), \mathcal{S}(t_{\geq u})) = 1 + \max(\mathcal{S}(t_{\leq u}), \mathcal{S}(t_{\leq \bar{u}^*}^*)). \quad (4.3.8)$$

Next set $m+1 = \#t_{\leq u} = \#\{v \in t : v \leq u\}$ and $m'+1 = \#t_{\geq u} = \#\{v \in t^* : v \leq \bar{u}^*\}$. Observe that $R_m t = t_{\leq u}$ and $R_{m'} t^* = t_{\leq \bar{u}^*}^*$. Moreover, $\#t = \#t_{\leq u} + \#t_{\geq u} - \#[\emptyset, u] = m + m' + 1 - |u|$. Thus, $m + m' < \#t + |t|$. If $2m \geq \#t + |t|$, then $m' < \frac{1}{2}(\#t + |t|) \leq m$ and $R_{m'} t^* \subset R_m t^*$. By (4.3.3), we get $\mathcal{S}(R_{m'} t^*) \leq \mathcal{S}(R_m t^*)$ and we obtain (4.3.6) by (4.3.8).

Inequality (4.3.7) is an easy consequence of (4.3.6): we leave the details to the reader. \blacksquare

We next prove the main equation that is satisfied by the tail distribution of the Horton-Strahler number of a Galton-Watson tree.

Lemma 4.3.2 *Let τ be a $\text{GW}(\mu)$ -tree whose offspring distribution μ satisfies (4.1.1). Recall φ and ψ from (4.1.2). For all $n \in \mathbb{N}$, we set $q_n = \mathbb{P}(\mathcal{S}(\tau) > n)$. Then,*

$$1 - q_0 = \frac{\mu(0)}{1 - \mu(1)} \quad \text{and} \quad 1 - q_{n+1} = \varphi(1 - q_n) + (q_n - q_{n+1})\varphi'(1 - q_n), \quad n \in \mathbb{N}. \quad (4.3.9)$$

This equation can be rewritten in terms of ψ as $q_n - q_{n+1} = \psi(q_n)/\psi'(q_n)$.

Proof. By Definition 4.1.1, $\mathcal{S}(\tau) = 0$ if and only if $k_\emptyset(\tau) = 0$ or $(k_\emptyset(\tau) = 1; \mathcal{S}(\theta_{(1)}\tau) = 0)$. Thus, by Definition 4.2.2, we get $\mathbb{P}(\mathcal{S}(\tau) = 0) = \mu(0) + \mu(1)\mathbb{P}(\mathcal{S}(\tau) = 0)$, which gives the first equality in (4.3.9).

Let us prove the recursive relation in (4.3.9). Let $n \in \mathbb{N}$. By Definition 4.1.1, $\mathcal{S}(\tau) \leq n+1$ if and only if $\mathcal{S}(\theta_u\tau) \leq n$ for all children u of \emptyset in τ (if any) with the possible exception of one child v , which may satisfy $\mathcal{S}(\theta_v\tau) = n+1$. More precisely,

$$\mathbf{1}_{\{\mathcal{S}(\tau) \leq n+1\}} = \prod_{1 \leq j \leq k_\emptyset(\tau)} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)}\tau) \leq n\}} + \sum_{1 \leq j \leq k_\emptyset(\tau)} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)}\tau) = n+1\}} \prod_{\substack{1 \leq i \leq k_\emptyset(\tau) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(i)}\tau) \leq n\}}. \quad (4.3.10)$$

Taking the expectation yields (4.3.9) by Definition 4.2.2, as φ is the generating function of μ . \blacksquare

Remark 4.3.3 Although it seems difficult to solve (4.3.9) explicitly in general, it can be done actually for the so-called α -stable offspring distribution μ_α , $\alpha \in (1, 2]$, whose generating function is

$$\forall s \in [0, 1], \quad \varphi_\alpha(s) = s + \frac{1}{\alpha}(1 - s)^\alpha.$$

Namely, if τ_α is a $\text{GW}(\mu_\alpha)$ -tree then $\mathcal{S}(\tau_\alpha)$ is a geometric random variable with parameter $\frac{1}{\alpha}$, i.e.

$$\mathbb{P}(\mathcal{S}(\tau_\alpha) = n) = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^n, \quad n \in \mathbb{N}.$$

This is explicitly proved in Kovchegov & Zaliapin [110, Lemma 10] and earlier for the $\alpha = 2$ case, see Burd, Waymire & Winn [31, Proposition 2.5]. See also Duquesne & Winkel [63] who show that α -stable offspring distributions are the only laws that are invariant under any hereditary pruning. \square

We end this section with a lemma that lists basic properties of ψ , that are useful to analyse (4.3.9).

Lemma 4.3.4 *Let μ be a probability on \mathbb{N} that satisfies (4.1.1). Recall from (4.1.2) the definition of ψ . Then, the following holds true.*

- (i) ψ is nonnegative, increasing, strictly convex, and analytic on $(0, 1]$. Moreover, ψ' is increasing, concave, and $\psi(0) = \psi'(0) = 0$.
- (ii) For all $s \in (0, 1]$, $\frac{1}{2}\psi'(s) \leq \frac{\psi(s)}{s} \leq \psi'(s)$.

Proof. The point (i) is elementary. The upper bound in (ii) is a consequence of the convexity of ψ and to obtain the lower bound we apply *Hermite-Hadamard inequality* that asserts that for all convex functions $f : [a, b] \rightarrow \mathbb{R}$, it holds

$$f\left(\frac{1}{2}(a+b)\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2}(f(a) + f(b)). \quad (4.3.11)$$

We apply (4.3.11) to ψ' after observing $\frac{1}{s}\psi(s) = \frac{1}{s} \int_0^s \psi'(s) ds$. \blacksquare

4.3.2 Tail estimates of joint laws

In this section, we provide four estimates of the tail of the joint distribution of the Horton-Strahler number of a Galton-Watson tree with either its size, its height, or its maximal out-degree. We fix an offspring distribution μ that satisfies (4.1.1) and τ stands for a $\text{GW}(\mu)$ -tree. Recall from (4.1.2) the definition of φ and ψ . Observe that $\psi'(1) = 1 - \varphi'(0) = 1 - \mu(1) > 0$ by (4.1.1). It is convenient to write

$$q_n = \mathbb{P}(\mathcal{S}(\tau) > n), \quad n \in \{-1\} \cup \mathbb{N}, \quad (4.3.12)$$

where $q_{-1} = 1$ obviously.

Proposition 4.3.5 *Let τ be a $\text{GW}(\mu)$ -tree where μ satisfies (4.1.1). Then, for all $n \in \mathbb{N}$,*

$$\mathbb{E}[\#\tau \mathbf{1}_{\{\mathcal{S}(\tau)=0\}}] = \frac{\mu(0)}{(1-\mu(1))^2} \quad \text{and} \quad \psi'(q_{n-1}) \mathbb{E}[\#\tau \mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}}] \leq 2. \quad (4.3.13)$$

Proof. Although $\mathbb{E}[\#\tau] = \infty$, let us first prove that $e_n := \mathbb{E}[\#\tau \mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}}] < \infty$. By Definition 4.1.1, if $\mathcal{S}(\tau) \leq n$ then $\mathcal{S}(\theta_u \tau) \leq n$ for all children u of \emptyset in τ (if any) and $\mathcal{S}(\theta_u \tau) = n$ for at most one child. This implies that for all $m, n \in \mathbb{N}$,

$$\begin{aligned} \min(m, \#\tau) \mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}} &\leq 1 + \sum_{i=1}^{k_\emptyset(\tau)} \min(m, \#\theta_{(i)} \tau) \mathbf{1}_{\{\mathcal{S}(\theta_{(i)} \tau) \leq n-1\}} \\ &+ \sum_{i=1}^{k_\emptyset(\tau)} \min(m, \#\theta_{(i)} \tau) \mathbf{1}_{\{\mathcal{S}(\theta_{(i)} \tau) \leq n\}} \prod_{\substack{1 \leq j \leq k_\emptyset(\tau) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)} \tau) \leq n-1\}} \end{aligned} \quad (4.3.14)$$

which makes sense even when $n=0$: the first sum in the right-hand side of (4.3.14) being null. To simplify notations, we set $e_n(m) = \mathbb{E}[\min(m, \#\tau) \mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}}]$ for all integers $m \geq 0$ and $n \geq -1$ (with $e_{-1}(m) = 0$). We obtain $e_n(m) \leq 1 + e_{n-1}(m) + e_n(m) \varphi'(1 - q_{n-1})$ by taking the expectation in (4.3.14). It easily implies that $e_n \leq (1 + e_{n-1}) / (1 - \varphi'(1 - q_{n-1}))$ because $\lim_{m \rightarrow \infty} e_n(m) = e_n$. This recursively entails $e_n < \infty$ for all integer $n \geq -1$.

To simplify notations, we set $k_\emptyset = k_\emptyset(\tau)$, $\tau_j = \theta_{(j)}\tau$ and $\mathcal{S}_j = \mathcal{S}(\theta_{(j)}\tau)$ for all $1 \leq j \leq k_\emptyset(\tau)$. We first explicitly compute e_0 by observing that $\#\tau \mathbf{1}_{\{\mathcal{S}(\tau)=0\}} = \mathbf{1}_{\{k_\emptyset=0\}} + \mathbf{1}_{\{k_\emptyset=1\}}(1 + \tau_1) \mathbf{1}_{\{\mathcal{S}_1=0\}}$. Taking the expectation entails $e_0 = \mu(0) + \mu(1)(1 - q_0 + e_0)$. By (4.3.9), this becomes

$$e_0 = \mathbb{E}[\#\tau \mathbf{1}_{\{\mathcal{S}(\tau)=0\}}] = \frac{1 - q_0}{1 - \mu(1)} = \frac{\mu(0)}{(1 - \mu(1))^2}. \quad (4.3.15)$$

Next, by using the fact that $\#\tau = 1 + \sum_{1 \leq j \leq k_\emptyset} \#\tau_j$ and by the decomposition (4.3.10), we get

$$\begin{aligned} \#\tau \mathbf{1}_{\{\mathcal{S}(\tau) \leq n+1\}} &= \mathbf{1}_{\{\mathcal{S}(\tau) \leq n+1\}} + \sum_{j=1}^{k_\emptyset} \#\tau_j \mathbf{1}_{\{\mathcal{S}_j \leq n\}} \prod_{\substack{1 \leq i \leq k_\emptyset \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}_i \leq n\}} \\ &+ \sum_{j=1}^{k_\emptyset} \#\tau_j \mathbf{1}_{\{\mathcal{S}_j = n+1\}} \prod_{\substack{1 \leq i \leq k_\emptyset \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}_i \leq n\}} + \sum_{\substack{1 \leq i, j \leq k_\emptyset \\ j \neq i}} \#\tau_j \mathbf{1}_{\{\mathcal{S}_j \leq n\}} \mathbf{1}_{\{\mathcal{S}_i = n+1\}} \prod_{\substack{1 \leq l \leq k_\emptyset \\ l \neq i, j}} \mathbf{1}_{\{\mathcal{S}_l \leq n\}}. \end{aligned}$$

Taking the expectation term-by-term, we get

$$e_{n+1} = 1 - q_{n+1} + e_n \varphi'(1 - q_n) + (e_{n+1} - e_n) \varphi'(1 - q_n) + e_n (q_n - q_{n+1}) \varphi''(1 - q_n)$$

for all $n \in \mathbb{N}$. Recall from (4.1.2) that $\psi(s) = \varphi(1 - s) - 1 + s$. By Lemma 4.3.2, we find

$$e_{n+1} \psi'(q_n) = 1 - q_{n+1} + e_n (q_n - q_{n+1}) \psi''(q_n) = 1 - q_{n+1} + e_n \frac{\psi''(q_n)}{\psi'(q_n)} \int_0^{q_n} \psi'(s) ds \quad (4.3.16)$$

since $\psi(0) = 0$ by Lemma 4.3.4 (i). Still from Lemma 4.3.4, we know that ψ' is concave, so we get $\psi'(s) \leq \psi'(q_n) - (q_n - s) \psi''(q_n)$. Thus,

$$\frac{\psi''(q_n)}{\psi'(q_n)} \int_0^{q_n} \psi'(s) ds \leq q_n \psi''(q_n) - \frac{(q_n \psi''(q_n))^2}{2\psi'(q_n)} = \frac{1}{2} \psi'(q_n) (1 - (1-x)^2),$$

where $x = \psi''(q_n) / (\psi'(q_n) / q_n)$ belongs to $[0, 1]$ since ψ' is concave. Thus, we get

$$x_{n+1} := e_{n+1} \psi'(q_n) \leq 1 + \frac{1}{2} e_n \psi'(q_n) \leq 1 + \frac{1}{2} e_n \psi'(q_{n-1}) = 1 + \frac{1}{2} x_n$$

since (q_n) is decreasing and ψ' is increasing. This entails $x_n \leq 2 - 2^{-n}(1 - x_0)$, which leads to (4.3.13) because $x_0 = \psi'(1) e_0 = (1 - \mu(1)) e_0 = 1 - q_0 < 1$ by (4.3.15). \blacksquare

For all $t \in \mathbb{T}$, we set

$$Z(t) = \max\{|u| : u \in t \text{ such that } \mathcal{S}(\theta_u t) = \mathcal{S}(t)\}.$$

Note that $Z(t) \leq |t|$ where recall from (4.2.1) that $|t|$ stands for the height of t .

Proposition 4.3.6 *Let τ be a $\text{GW}(\mu)$ -tree where μ satisfies (4.1.1). Recall ψ from (4.1.2) and q_n from (4.3.12). Recall from (4.2.1) that $|\tau|$ stands for the height of τ . Then for all $m, n \in \mathbb{N}$, it holds that $\mathbb{P}(Z(\tau) \geq m \mid \mathcal{S}(\tau) = n) = (1 - \psi'(q_{n-1}))^m$. This implies that for all $\lambda \in (0, \infty)$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\psi'(q_{n-1}) |\tau| \leq \lambda \mid \mathcal{S}(\tau) = n) \leq 1 - e^{-\lambda}. \quad (4.3.17)$$

Proof. By Definition 4.1.1, for all integers $n \geq 0$ and $m \geq 1$, it holds

$$\mathbf{1}_{\{Z(\tau) \geq m; \mathcal{S}(\tau) = n\}} = \sum_{i=1}^{k_{\emptyset}(\tau)} \mathbf{1}_{\{Z(\theta_{(i)}\tau) \geq m-1; \mathcal{S}(\theta_{(i)}\tau) = n\}} \prod_{\substack{1 \leq j \leq k_{\emptyset}(\tau) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)}\tau) \leq n-1\}}$$

By taking the expectation, we get

$$\mathbb{P}(Z(\tau) \geq m; \mathcal{S}(\tau) = n) = \mathbb{P}(Z(\tau) \geq m-1; \mathcal{S}(\tau) = n) \varphi'(1 - q_{n-1}),$$

which implies the first desired equality as $\varphi'(1 - q_{n-1}) = 1 - \psi'(q_{n-1})$. Note that $\psi'(q_{n-1}) \rightarrow \psi'(0) = 0$. Thus, $\lim_{n \rightarrow \infty} \mathbb{P}(Z(\tau) \geq \lambda/\psi'(q_{n-1}) \mid \mathcal{S}(\tau) = n) = e^{-\lambda}$ which implies (4.3.17) since $Z(\tau) \leq |\tau|$. ■

In [28], Brandenberger, Devroye & Reddad study the Horton-Strahler number of a Galton-Watson tree conditioned to have exactly n vertices under the assumption that the variance of the offspring distribution is finite. To that end, they use a (little more than) local convergence of the conditioned Galton-Watson tree towards the corresponding size-biased tree. We adapt and extend this idea in a more general context using only the Many-To-One Principle to get the following.

Proposition 4.3.7 *Let τ be a $\text{GW}(\mu)$ -tree where μ satisfies (4.1.1). Recall ψ from (4.1.2) and q_n from (4.3.12). For all integers $n, m \in \mathbb{N}$ such that $2\psi'(\mathbb{P}(|\tau| \geq \lfloor n/2 \rfloor)) \leq \psi'(q_m)$, the following inequality holds:*

$$\mathbb{P}(\mathcal{S}(\tau) \leq m \mid |\tau| \geq n) \leq \exp\left(-\frac{1}{8} n \psi'(q_m)\right). \quad (4.3.18)$$

Proof. For all $n \in \mathbb{N}$, we denote by φ_n the n -iterate of φ with the convention $\varphi_0 = \text{Id}$. It is classical that $\mathbb{P}(|\tau| < n) = \varphi_n(0)$. Then, observe the following: the \leq -smallest vertex of τ at height n is the only vertex $u \in \tau$ such that $|u| = n$ and such that for all $v \in \tau$ with $v < u$ and $\overleftarrow{v} \prec u$, we have $|\theta_v \tau| + |v| < n$. Moreover for all $p \in \mathbb{N}$, (4.3.3) implies that if $\mathcal{S}(\tau) \leq m$ then $\mathcal{S}(\theta_v \tau) \leq m$ for all $v \in \tau$. Therefore,

$$\mathbf{1}_{\{|\tau| \geq n; \mathcal{S}(\tau) \leq m\}} \leq \sum_{u \in \tau} \mathbf{1}_{\{|u|=n\}} \prod_{\substack{w \in \tau, i \geq 1 \\ w^*(i) \preceq u}} \left(\prod_{j=1}^{i-1} \mathbf{1}_{\{|\theta_{w^*(j)}\tau| + |w^*(j)| < n\}} \right) \left(\prod_{j=i+1}^{k_w(\tau)} \mathbf{1}_{\{\mathcal{S}(\theta_{w^*(j)}\tau) \leq m\}} \right). \quad (4.3.19)$$

Here we adopt the following convention: a product over an empty set of indices is taken equal to 1. Recall Definition 4.2.3 of the size-biased $\text{GW}(\mu)$ -tree τ_{∞} and recall from (4.2.5) in Remark 4.2.4 the joint law of the number of left/right siblings of individuals on the infinite line of descent. We now use the Many-To-One Principle (Proposition 4.2.5) after taking the expectation in (4.3.19) to get

$$\mathbb{P}(|\tau| \geq n; \mathcal{S}(\tau) \leq m) \leq \prod_{p=0}^{n-1} \frac{\varphi(1 - q_m) - \varphi(\varphi_p(0))}{1 - q_m - \varphi_p(0)}. \quad (4.3.20)$$

We now use convexity properties of φ and ψ given in Lemma 4.3.4 to get an upper bound of the right-hand side of (4.3.20). First observe that the convexity of φ implies that for all real numbers $s, r \in [0, 1]$ such that $s \leq r$, we have $(\varphi(r) - \varphi(s))/(r - s) \leq (1 - \varphi(s))/(1 - s)$. Therefore,

$$\prod_{p=0}^{\lfloor n/2 \rfloor - 1} \frac{\varphi(1 - q_m) - \varphi(\varphi_p(0))}{1 - q_m - \varphi_p(0)} \leq \prod_{p=0}^{\lfloor n/2 \rfloor - 1} \frac{1 - \varphi(\varphi_p(0))}{1 - \varphi_p(0)} = 1 - \varphi_{\lfloor n/2 \rfloor}(0). \quad (4.3.21)$$

To get an upper bound of $(\varphi(1-q_m) - \varphi(\varphi_p(0)))/(1-q_m - \varphi_p(0))$ when $p \geq \lfloor n/2 \rfloor$, we use the following: let $s, r \in [0, 1]$ and suppose that $2\psi'(s) \leq \psi'(r)$, then

$$\begin{aligned} \frac{\varphi(1-r) - \varphi(1-s)}{s-r} &= \frac{1 - \varphi(1-s)}{s} + \frac{\varphi(1-s) - 1}{s} + \frac{\varphi(1-r) - \varphi(1-s)}{s-r} \\ &= \frac{1 - \varphi(1-s)}{s} + \frac{\psi(s)}{s} - \frac{\psi(r) - \psi(s)}{r-s} \\ &= \frac{1 - \varphi(1-s)}{s} + \frac{\psi(s)}{s} - \frac{1}{r-s} \int_s^r \psi'(x) dx \\ &\leq \frac{1 - \varphi(1-s)}{s} + \psi'(s) - \frac{1}{2}(\psi'(r) + \psi'(s)) \\ &\leq \frac{1 - \varphi(1-s)}{s} - \frac{1}{4}\psi'(r) \leq \frac{1 - \varphi(1-s)}{s} (1 - \frac{1}{4}\psi'(r)). \end{aligned} \quad (4.3.22)$$

Here, we have used Hermite-Hadamard inequality (4.3.11) for concave functions, and the two convexity inequalities $\frac{1}{s}\psi(s) \leq \psi'(s)$ and $\frac{1}{s}(1-\varphi(1-s)) \leq \varphi'(1) = 1$.

Assume that $m, n \in \mathbb{N}$ satisfy $2\psi'(1-\varphi_{\lfloor n/2 \rfloor}(0)) \leq \psi'(q_m)$. Then, for all $p \geq \lfloor n/2 \rfloor$, we have $2\psi'(1-\varphi_p(0)) \leq \psi'(q_m)$. Applying (4.3.22) successively with $s = 1-\varphi_p(0)$ and $r = q_m$ gets us

$$\begin{aligned} \mathbb{P}(|\tau| \geq n; \mathcal{S}(\tau) \leq m) &\leq (1 - \varphi_{\lfloor n/2 \rfloor}(0)) (1 - \frac{1}{4}\psi'(q_m))^{n/2} \prod_{p=\lfloor n/2 \rfloor}^{n-1} \frac{1 - \varphi_{p+1}(0)}{1 - \varphi_p(0)} \\ &\leq (1 - \frac{1}{4}\psi'(q_m))^{n/2} \mathbb{P}(|\tau| \geq n), \end{aligned}$$

by (4.3.20) and (4.3.21). This easily entails (4.3.18) since $\ln(1-x) \leq -x$ for all $x \in [0, 1)$. ■

Although Proposition 4.3.7 holds under the sole assumption that μ satisfies (4.1.1), its application requires knowing the behavior of the tail of the height $|\tau|$ of the $\text{GW}(\mu)$ -tree τ . When μ belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, Proposition 4.2.10 (iv) provides such information, and then Proposition 4.3.7 entails the following more convenient result.

Corollary 4.3.8 *Let τ be a $\text{GW}(\mu)$ -tree where μ satisfies (4.1.1) and belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$. Recall ψ from (4.1.2) and q_n from (4.3.12). There exists a constant $C_\mu \in (0, \infty)$ that only depends on μ such that for all integers $n, m \in \mathbb{N}$, it holds*

$$\mathbb{P}(\mathcal{S}(\tau) \leq m \mid |\tau| \geq n) \leq C_\mu \exp(-\frac{1}{8}n\psi'(q_m)). \quad (4.3.23)$$

Proof. To simplify the notations, we set $s_n = \mathbb{P}(|\tau| \geq \lfloor n/2 \rfloor)$ for all $n \in \mathbb{N}$. By Proposition 4.2.10 (iv), the constant $2c_\mu := \sup_{n \in \mathbb{N}} (n+1) \frac{\psi(s_n)}{s_n}$ is finite and positive. Then, we get from Lemma 4.3.4 (ii) that $2\psi'(s_n) \leq 8c_\mu/(n+1)$ for all $n \in \mathbb{N}$, by definition of c_μ . Let us set $C_\mu = e^{c_\mu} \in (0, \infty)$ and let $n, m \in \mathbb{N}$. If $\psi'(q_m) \geq 8c_\mu/(n+1)$, then $2\psi'(s_n) \leq \psi'(q_m)$ so Proposition 4.3.7 yields (4.3.23) because $1 \leq C_\mu$. Otherwise, $\psi'(q_m) < 8c_\mu/(n+1)$ and $C_\mu \exp(-\frac{1}{8}n\psi'(q_m)) \geq 1$ by choice of C_μ . Thus, (4.3.23) clearly holds in that case, which completes the proof. ■

Our last estimate relies on the same idea as Proposition 4.3.7: if the maximal out-degree of a Galton-Watson tree is large, then the tree contains several independent disjoint copies of itself.

Proposition 4.3.9 *Let τ be a $\text{GW}(\mu)$ -tree where μ satisfies (4.1.1). Recall ψ from (4.1.2) and q_n from (4.3.12). Recall from (4.2.1) that $\Delta(\tau)$ stands for the maximal out-degree of τ . Then, for all integers $n, m \geq 1$ such that $\mathbb{P}(\Delta(\tau) \geq n) > 0$, the following inequality holds true:*

$$\mathbb{P}(\mathcal{S}(\tau) \leq m \mid \Delta(\tau) \geq n) \leq e^{-nq_m}. \quad (4.3.24)$$

Moreover, for all $\lambda \in (0, \infty)$, we also have

$$\mathbb{E}[e^{-\lambda\#\tau} \mid \Delta(\tau) \geq n] \leq \mathbb{E}[e^{-\lambda\#\tau}]^n. \quad (4.3.25)$$

Proof. On the event $\{\mathcal{S}(\tau) \leq m; \Delta(\tau) \geq n\}$, we decompose τ along the ancestral line of the \leq -first vertex u such that $k_u(\tau) \geq n$ and, by (4.3.3), we see $\mathcal{S}(\theta_{u^{*(j)}}\tau) \leq m$ for all $1 \leq j \leq n$. Hence,

$$\mathbf{1}_{\{\mathcal{S}(\tau) \leq m; \Delta(\tau) \geq n\}} \leq \sum_{u \in \tau} \mathbf{1}_{\{k_u(\tau) \geq n\}} \left(\prod_{v \in \tau: v < u} \mathbf{1}_{\{k_v(\tau) < n\}} \right) \left(\prod_{j=1}^n \mathbf{1}_{\{\mathcal{S}(\theta_{u^{*(j)}}\tau) \leq m\}} \right).$$

We take the expectation and apply the Many-To-One Principle (Proposition 4.2.5) ‘forwards and backwards’ to get

$$\begin{aligned} \mathbb{P}(\mathcal{S}(\tau) \leq m; \Delta(\tau) \geq n) &\leq \mathbb{E} \left[\sum_{u \in \tau} \mathbf{1}_{\{k_u(\tau) \geq n\}} \prod_{v \in \tau: v < u} \mathbf{1}_{\{k_v(\tau) < n\}} \right] \mathbb{P}(\mathcal{S}(\tau) \leq m)^n \\ &= \mathbb{P}(\Delta(\tau) \geq n) (1 - q_m)^n, \end{aligned}$$

which entails (4.3.24) since $\ln(1-x) \leq -x$ for all $x \in [0, 1)$. To prove (4.3.25), we observe that

$$e^{-\lambda \#\tau} \mathbf{1}_{\{\Delta(\tau) \geq n\}} \leq \sum_{u \in \tau} \mathbf{1}_{\{k_u(\tau) \geq n\}} \left(\prod_{v \in \tau: v < u} \mathbf{1}_{\{k_v(\tau) < n\}} \right) \left(\prod_{j=1}^n e^{-\lambda \#\theta_{u^{*(j)}}\tau} \right).$$

Then, as in the previous argument, we take the expectation and apply the Many-To-One Principle (Proposition 4.2.5) ‘forwards and backwards’ to get the desired result. ■

4.3.3 Tail estimates.

In this section, we prove results on the tail of the Horton-Strahler number of a $\text{GW}(\mu)$ -tree. These results are expressed in terms of the functions Λ and Υ that are given by

$$\forall s \in (0, 1), \quad \Upsilon(s) = \int_s^1 \frac{dr}{r \ln \Lambda(r)}, \quad \text{where} \quad \Lambda(s) = \frac{s\psi'(s)}{s\psi'(s) - \psi(s)}, \quad (4.1.9)$$

and where ψ is defined in (4.1.2). Their basic properties are listed in the following lemma.

Lemma 4.3.10 *Let μ be a probability on \mathbb{N} that satisfies (4.1.1). Then, the functions Λ and Υ in (4.1.9) are well-defined, $\Lambda \geq 2$, and Υ is continuous, positive, and decreasing on $(0, 1)$.*

Proof. The inequality $\frac{1}{s}\psi(s) \geq \frac{1}{2}\psi'(s)$ from Lemma 4.3.4 (ii) entails that $\Lambda(s) \geq 2$ for all $s \in (0, 1)$, and (iii) follows immediately. ■

Under the additional assumption that μ belongs to the domain of attraction of a stable law, the following lemma shows that Λ varies slowly at 0^+ .

Proposition 4.3.11 *Let us assume that μ satisfies (4.1.1) and that it belongs to the domain of attraction of a stable law of index $\alpha \in [1, 2]$.*

- (i) *If $\alpha \in (1, 2]$, then $\lim_{s \rightarrow 0^+} \Lambda(s) = \frac{\alpha}{\alpha-1}$ and $\Upsilon(s) \sim_{0^+} \log_{\frac{\alpha}{\alpha-1}} 1/s$.*
- (ii) *Suppose that $\alpha = 1$. Let L be such that $\mu([n, \infty)) \sim n^{-1}L(n)$ and ℓ be the slowly varying function given by $\ell(x) = \int_x^\infty y^{-1}L(y) dy$ for all $x \in (0, \infty)$ (see Proposition 4.2.12). Then,*

$$\Lambda(s) \sim_{0^+} \frac{\ell(1/s)}{L(1/s)} \quad \text{and} \quad \lim_{s \rightarrow 0^+} \Lambda(s) = \infty.$$

Proof. Let us prove (i) first. By Proposition 4.2.9, ψ is regularly varying of index α at 0^+ . Then, recall that ψ' is increasing, and since $\psi(s) = \int_0^s \psi'(x) dx$, the Monotone Density Theorem (recalled in Proposition 4.2.8 (iv)) implies that $s\psi'(s) \sim_{0^+} \alpha\psi(s)$. This entails $\lim_{s \rightarrow 0^+} \Lambda(s) = \frac{\alpha}{\alpha-1}$ and thus $\Upsilon(s) \sim_{0^+} \log_{\frac{\alpha}{\alpha-1}} 1/s$. This completes the proof of (i).

Let us assume that $\alpha = 1$ to prove (ii). We denote by ξ a random variable whose distribution is μ . Thus, $x\mathbb{P}(\xi > x) \sim_{\infty} L(x)$. Proposition 4.2.12 (ii) asserts that $\mathbb{E}[\xi \mathbf{1}_{\{\xi > x\}}] \sim_{\infty} \ell(x)$ and $\psi(s) \sim_{0^+} s\ell(1/s)$. The Monotone Density Theorem again asserts that $s\psi'(s) \sim_{0^+} s\ell(1/s)$. We next consider $s\psi'(s) - \psi(s)$, $s \in (0, 1)$ being fixed. To simplify notations, we set $\lambda = -\ln(1-s)$ and we first observe that

$$\begin{aligned} s\psi'(s) - \psi(s) &= \mathbb{E} \left[1 - e^{-\lambda\xi} - \frac{s\xi}{1-s} e^{-\lambda\xi} \right] = \mathbb{E} \left[\int_0^{\xi} (\lambda e^{-\lambda x} - \frac{s}{1-s} e^{-\lambda x} + \frac{s\lambda}{1-s} e^{-\lambda x}) dx \right], \\ \text{(by Fubini)} &= \frac{s\lambda}{1-s} \int_0^{\infty} \mathbb{P}(\xi > x) x e^{-\lambda x} dx - \left(\frac{s}{1-s} - \lambda \right) \int_0^{\infty} \mathbb{P}(\xi > x) e^{-\lambda x} dx. \end{aligned}$$

We estimate the second term of the right-hand side by writing

$$\left(\frac{s}{1-s} + \ln(1-s) \right) \int_0^{\infty} \mathbb{P}(\xi > x) e^{-\lambda x} dx \sim_{0^+} \frac{1}{2} s^2 \mathbb{E}[\xi] = \frac{1}{2} s^2. \quad (4.3.26)$$

Next, Karamata's Abelian Theorem for Laplace transform (as recalled in Proposition 4.2.8 (v)) asserts

$$\lambda \int_0^{\infty} x \mathbb{P}(\xi > x) e^{-\lambda x} dx \sim_{0^+} L(1/\lambda). \quad (4.3.27)$$

Since $\lambda = -\ln(1-s)$, (4.3.26) and (4.3.27) entail that $s\psi'(s) - \psi(s) \sim_{0^+} sL(1/s)$ thanks to Potter's bound (see Proposition 4.2.8 (i)), which implies the desired estimate for Λ as $s\psi'(s) \sim_{0^+} s\ell(1/s)$. Finally, $\lim_{s \rightarrow \infty} \Lambda(s) = \infty$ comes from an application of Proposition 4.2.8 (iii). \blacksquare

Proposition 4.3.12 *Let us assume that μ satisfies (4.1.1) and that it belongs to the domain of attraction of a stable law of index $\alpha \in [1, 2]$. Recall the definition of Υ from (4.1.9). Then,*

$$\Upsilon(\mathbb{P}(\mathcal{S}(\tau) > n)) \sim n. \quad (4.3.28)$$

In particular, if $\alpha \in (1, 2]$, we get $-\ln \mathbb{P}(\mathcal{S}(\tau) > n) \sim n \ln \frac{\alpha}{\alpha-1}$.

Proof. Recall from (4.3.12) that $q_n = \mathbb{P}(\mathcal{S}(\tau) > n)$ and note that $q_n \rightarrow 0$. Lemma 4.3.2 asserts that $q_n - q_{n+1} = \psi(q_n)/\psi'(q_n)$, namely $q_{n+1}^{-1} = q_n^{-1} \Lambda(q_n)$. To simplify, we set $Q_n = -\ln q_n$. Therefore,

$$Q_{n+1} = Q_n + \ln \Lambda(e^{-Q_n}), \quad n \in \mathbb{N}. \quad (4.3.29)$$

By Proposition 4.3.11, Λ varies slowly at 0^+ . Thus, Potter's bound (recalled in Proposition 4.2.8 (i)) applies to $l(y) = \Lambda(1/y)$: for any $\varepsilon \in (0, \frac{1}{10})$, there is $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ and for all $x \in [Q_n, Q_{n+1}]$,

$$\begin{aligned} |\ln \Lambda(e^{-x}) - \ln \Lambda(e^{-Q_n})| &= \left| \ln \left(\frac{\Lambda(e^{-(x-Q_n)} e^{-Q_n})}{\Lambda(e^{-Q_n})} \right) \right| \leq \varepsilon + \varepsilon(x - Q_n) \\ &\leq \varepsilon + \varepsilon \ln \Lambda(e^{-Q_n}). \end{aligned}$$

by (4.3.29) (we apply Proposition 4.2.8 (i) with $l(y) = \Lambda(1/y)$, $c = e^\varepsilon$ and $\lambda = e^{x-Q_n}$). This implies

$$\begin{aligned} \left| \frac{1}{\ln \Lambda(e^{-x})} - \frac{1}{\ln \Lambda(e^{-Q_n})} \right| &\leq \frac{\varepsilon}{\ln \Lambda(e^{-Q_n})} \cdot \frac{1 + \ln \Lambda(e^{-Q_n})}{\ln \Lambda(e^{-x})} \\ &\leq \frac{\varepsilon}{\ln \Lambda(e^{-Q_n})} \cdot \frac{1 + \ln \Lambda(e^{-Q_n})}{(1 - \varepsilon) \ln \Lambda(e^{-Q_n}) - \varepsilon} \\ &\leq \frac{\varepsilon}{\ln \Lambda(e^{-Q_n})} \cdot \frac{(\ln 2)^{-1} + 1}{1 - \varepsilon - \varepsilon(\ln 2)^{-1}} \leq \frac{100\varepsilon}{\ln \Lambda(e^{-Q_n})} \end{aligned}$$

because $\Lambda \geq 2$, by Lemma 4.3.10, and since $\varepsilon \in (0, \frac{1}{10})$. Therefore, by (4.3.29), we get

$$\int_{Q_n}^{Q_{n+1}} \left| \frac{1}{\ln \Lambda(e^{-x})} - \frac{1}{\ln \Lambda(e^{-Q_n})} \right| dx \leq 100\varepsilon \cdot \frac{Q_{n+1} - Q_n}{\ln \Lambda(e^{-Q_n})} = 100\varepsilon.$$

This implies that for all $\varepsilon \in (0, \frac{1}{10})$ and for all $n \geq n_\varepsilon$,

$$\left| \sum_{n_\varepsilon \leq k < n} \frac{Q_{k+1} - Q_k}{\ln \Lambda(e^{-Q_k})} - \int_{Q_{n_\varepsilon}}^{Q_n} \frac{dx}{\ln \Lambda(e^{-x})} \right| = \left| n - n_\varepsilon - \int_{Q_{n_\varepsilon}}^{Q_n} \frac{dx}{\ln \Lambda(e^{-x})} \right| \leq 100\varepsilon(n - n_\varepsilon).$$

Hence, $\lim_{n \rightarrow \infty} \frac{1}{n} \int_{Q_0}^{Q_n} \frac{dx}{\ln \Lambda(e^{-x})} = 1$, which proves (4.3.28) by the change of variable $r = e^{-x}$. ■

In the 1-stable cases, we provide below three examples of slowly varying functions L that may govern the tail of μ via the estimate $n\mu([n, \infty)) \sim L(n)$. We use the notations of Proposition 4.3.11.

Example 4.3.13 (a) Let $\kappa \in (0, \infty)$. We consider the case where $L(x) = (\ln x)^{-1-\kappa}$ for all $x \in (0, \infty)$. Indeed, this function varies slowly at ∞ and is such that $\int^\infty y^{-1} L(y) dy < \infty$. Clearly, we have $\ell(x) = \frac{1}{\kappa} (\ln x)^{-\kappa} = \frac{1}{\kappa} L(x) \ln x$. Therefore, Proposition 4.3.11 (ii) yields that

$$\Lambda(s) \sim_{0^+} \frac{1}{\kappa} \ln 1/s \quad \text{and} \quad \Upsilon(s) \sim_{0^+} \frac{\ln 1/s}{\ln \ln 1/s}.$$

If one sets $x_n = -\ln \mathbb{P}(\mathcal{S}(\tau) > n)$, Proposition 4.3.12 asserts that $x_n / \ln x_n \sim n$, which implies that

$$-\ln \mathbb{P}(\mathcal{S}(\tau) > n) \sim n \ln n.$$

(b) Let $\kappa \in (0, 1)$. We next consider the case where $L(x) = \exp(-(\ln x)^\kappa)$ for $x \in (0, \infty)$. Again, this function varies slowly at ∞ and verifies $\int^\infty y^{-1} L(y) dy < \infty$. An integration by parts gives

$$\ell(x) = \frac{1}{\kappa} (\ln x)^{1-\kappa} L(x) + \frac{1-\kappa}{\kappa} \int_x^\infty \frac{L(y)}{y(\ln y)^\kappa} dy \sim_\infty \frac{1}{\kappa} (\ln x)^{1-\kappa} L(x),$$

and Proposition 4.3.11 (ii) thus implies that

$$\Lambda(s) \sim_{0^+} \frac{1}{\kappa} (\ln 1/s)^{1-\kappa} \quad \text{and} \quad \Upsilon(s) \sim_{0^+} \frac{1}{1-\kappa} \cdot \frac{\ln 1/s}{\ln \ln 1/s}.$$

If one sets $x_n = -\ln \mathbb{P}(\mathcal{S}(\tau) > n)$, Proposition 4.3.12 asserts that $x_n / \ln x_n \sim (1-\kappa)n$, which yields that

$$-\ln \mathbb{P}(\mathcal{S}(\tau) > n) \sim (1-\kappa)n \ln n.$$

(c) We finally consider the case where $L(x) = \exp(-\ln x / \ln \ln x)$ for all $x \in (e^e, \infty)$. This function still slowly varies and verifies $\int^\infty y^{-1} L(y) dy < \infty$. We make the change of variable $z = \ln y / \ln \ln y$ and then an integration by parts to compute

$$\ell(x) \sim_\infty \int_{\frac{\ln x}{\ln \ln x}}^\infty e^{-z} \ln(z) dz \sim_\infty \left[-e^{-z} \ln z \right]_{\frac{\ln x}{\ln \ln x}}^\infty \sim_\infty L(x) \ln \ln x.$$

Therefore, Proposition 4.3.11 (ii) implies that

$$\Lambda(s) \sim_{0^+} \ln \ln 1/s \quad \text{and} \quad \Upsilon(s) \sim_{0^+} \frac{\ln 1/s}{\ln \ln \ln 1/s}.$$

If one sets $x_n = -\ln \mathbb{P}(\mathcal{S}(\tau) > n)$, Proposition 4.3.12 asserts that $x_n / \ln \ln x_n \sim n$, which entails

$$-\ln \mathbb{P}(\mathcal{S}(\tau) > n) \sim n \ln \ln n.$$

□

We use further the following estimates.

Proposition 4.3.14 *Let us assume that μ satisfies (4.1.1) and that it belongs to the domain of attraction of a stable law of index $\alpha \in [1, 2]$. Then, for all $\lambda \in (0, \infty)$ and all $\kappa \in \mathbb{R}$, the following holds true:*

$$\ln \ln 1/s \underset{s \rightarrow 0^+}{=} o(\Upsilon(s)), \quad \Upsilon(\lambda s \ln^\kappa 1/s) \sim_{0^+} \Upsilon(s), \quad \text{and} \quad \Upsilon(s \Lambda(s)^\kappa) \sim_{0^+} \Upsilon(s).$$

In particular, Υ is slowly varying at 0^+ .

Proof. Recall that Λ is slowly varying at 0^+ from Proposition 4.3.11 so $\ln \Lambda(s) = o(\ln 1/s)$ by Proposition 4.2.8 (i). This allows us to write

$$\ln \ln 1/s \sim_{0^+} \int_s^{1/2} \frac{dr}{r \ln 1/r} \underset{s \rightarrow 0^+}{=} o(\Upsilon(s)).$$

Next, without loss of generality, we may assume that $\lambda \in [1, \infty)$, so that $1 \leq \lambda \ln^{|\kappa|} 1/s$ when s is small enough. Recall from Lemma 4.3.10 that $\Lambda \geq 2$. Thus,

$$|\Upsilon(\lambda s \ln^\kappa 1/s) - \Upsilon(s)| \leq \int_{\frac{s}{\lambda} \ln^{-|\kappa|} 1/s}^{\lambda s \ln^{|\kappa|} 1/s} \frac{dr}{r \ln \Lambda(r)} \leq \frac{1}{\ln 2} \ln(\lambda^2 \ln^{2|\kappa|} 1/s) \underset{s \rightarrow 0^+}{=} o(\Upsilon(s)),$$

which is the second estimate. Then, observe that $\lim_{s \rightarrow 0^+} s \Lambda(s)^{|\kappa|} = 0$ because Λ slowly varies at 0^+ . Another application of Proposition 4.2.8 (i), together with $\Lambda \geq 2$, entails $\ln \Lambda(r) \geq \frac{1}{2} \ln \Lambda(s)$ for all small enough s and for all $r \in (0, 1)$ such that $s \Lambda(s)^{-|\kappa|} \leq r \leq s \Lambda(s)^{|\kappa|}$. Therefore,

$$|\Upsilon(s \Lambda(s)^\kappa) - \Upsilon(s)| \leq \int_{s \Lambda(s)^{-|\kappa|}}^{s \Lambda(s)^{|\kappa|}} \frac{dr}{r \ln \Lambda(r)} \leq \frac{2}{\ln \Lambda(s)} \ln(\Lambda(s)^{2|\kappa|}) = 4|\kappa| \underset{s \rightarrow 0^+}{=} o(\Upsilon(s)).$$

which completes the proof. ■

4.4 Proofs of Theorems 4.1.2, 4.1.3 and 4.1.4

4.4.1 Proof of Theorem 4.1.2

In all this section, we assume that the critical offspring distribution μ belongs to the domain of attraction of an α -stable law with $\alpha \in (1, 2]$, and more precisely, we assume that Proposition 4.2.9 (a-d) hold. Let us set $\gamma = \ln \frac{\alpha}{\alpha-1}$. Recall from Propositions 4.3.11 (i) and 4.3.12 that

$$\gamma \Upsilon(s) \sim_{0+} \ln 1/s \quad \text{and} \quad -\ln \mathbb{P}(\mathcal{S}(\tau) > n) \sim \gamma n. \quad (4.4.1)$$

The proof of Theorem 4.1.2 is separated into two parts: a lower bound and an upper bound.

Lower bound. We first prove for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\alpha \gamma \mathcal{S}(\tau) \leq (1 - \varepsilon) \ln n \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.2)$$

The idea is to apply Proposition 4.3.7, or rather Corollary 4.3.8, and to switch the conditioning on the size of τ with the conditioning on the height of τ . To that end, we use Proposition 4.2.11 (i) to find

$$\limsup_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| \leq \eta \frac{n}{a_n} \mid \#\tau = n) = 0.$$

Therefore, to prove (4.4.2), we only need to prove for all $\eta \in (0, 1)$ that

$$\limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| > \eta \frac{n}{a_n}; \alpha \gamma \mathcal{S}(\tau) \leq (1 - \varepsilon) \ln n \mid \#\tau = n) = 0.$$

We roughly bound the conditional probability by the ratio of the probabilities as follows.

$$\mathbb{P}(|\tau| > \eta \frac{n}{a_n}; \alpha \gamma \mathcal{S}(\tau) \leq (1 - \varepsilon) \ln n \mid \#\tau = n) \leq \frac{\mathbb{P}(\alpha \gamma \mathcal{S}(\tau) \leq (1 - \varepsilon) \ln n \mid |\tau| > \eta \frac{n}{a_n})}{\mathbb{P}(\#\tau = n)}. \quad (4.4.3)$$

Since $a_n \sim n^{\frac{1}{\alpha} + o(1)}$ by Proposition 4.2.9 and Potter's bound (see Proposition 4.2.8 (i)), Proposition 4.2.10 (iii) entails the following estimate for the denominator of the right-hand side of (4.4.3):

$$\mathbb{P}(\#\tau = n) \sim \frac{c_\alpha}{n a_n} \sim n^{-1 - \frac{1}{\alpha} + o(1)}. \quad (4.4.4)$$

To bound the right-hand side of (4.4.3), we want to apply Corollary 4.3.8 and to that end, we first control $\psi'(\mathbb{P}(\mathcal{S}(\tau) > \frac{1-\varepsilon}{\alpha\gamma} \ln n))$. By Potter's bound and Proposition 4.2.9 (c), we get the estimate $\ln \psi(s) \sim_{0+} \alpha \ln s$. Lemma 4.3.4 (ii) then yields $\ln \psi'(s) \sim_{0+} (\alpha - 1) \ln s$. Together with (4.4.1), this implies that

$$\psi'(\mathbb{P}(\mathcal{S}(\tau) > \frac{1-\varepsilon}{\alpha\gamma} \ln n)) \sim n^{-(1-\varepsilon)\frac{\alpha-1}{\alpha} + o(1)}. \quad (4.4.5)$$

We eventually apply Corollary 4.3.8, and then (4.4.4) and (4.4.5), to get that

$$\mathbb{P}(\mathcal{S}(\tau) \leq \frac{1-\varepsilon}{\alpha\gamma} \ln n \mid |\tau| > \frac{\eta n}{a_n}) \leq C_\mu \exp\left(-\frac{\eta n}{8a_n} \psi'(\mathbb{P}(\mathcal{S}(\tau) > \frac{1-\varepsilon}{\alpha\gamma} \ln n))\right) \leq \exp\left(-n^{\varepsilon \frac{\alpha-1}{2\alpha}}\right)$$

for all sufficiently large n . The previous upper bound combined with (4.4.3) and (4.4.4) implies (4.4.2).

Upper bound. We want to prove for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\alpha\gamma\mathcal{S}(\tau) \geq (1 + 2\varepsilon) \ln n \mid \#\tau = n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.6)$$

To that end, we first show

$$\mathbb{P}(\alpha\gamma\mathcal{S}(\tau) \geq (1 + \varepsilon) \ln n \mid \#\tau \geq n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.7)$$

Indeed, we use the following rough bound

$$\mathbb{P}(\alpha\gamma\mathcal{S}(\tau) \geq (1 + \varepsilon) \ln n \mid \#\tau \geq n) \leq \frac{\mathbb{P}(\alpha\gamma\mathcal{S}(\tau) \geq (1 + \varepsilon) \ln n)}{\mathbb{P}(\#\tau \geq n)}.$$

Then, by Propositions 4.2.10 (iii) and 4.2.9 on the one hand, and by (4.4.1) on the other hand, we observe

$$\mathbb{P}(\#\tau \geq n) \sim \frac{c_\alpha}{a_n} \sim n^{-\frac{1}{\alpha} + o(1)} \quad \text{and} \quad \mathbb{P}(\alpha\gamma\mathcal{S}(\tau) \geq (1 + \varepsilon) \ln n) \sim n^{-(1 + \varepsilon)\frac{1}{\alpha} + o(1)},$$

which implies (4.4.7). \square

We then recall from (4.3.4) that under $\mathbb{P}(\cdot \mid \#\tau = n)$, τ and its mirror image have the same law. Therefore, (4.3.7) in Lemma 4.3.1 applies with $m = \lfloor \frac{7}{8}n \rfloor$ and one gets, for all sufficiently large n ,

$$\mathbb{P}(\mathcal{S}(\tau) \geq \frac{1 + 2\varepsilon}{\alpha\gamma} \ln n \mid \#\tau = n) \leq 2\mathbb{P}(\mathcal{S}(R_{\lfloor \frac{7}{8}n \rfloor} \tau) \geq \frac{1 + \varepsilon}{\alpha\gamma} \ln n \mid \#\tau = n) + \mathbb{P}(|\tau| > \frac{n}{2} \mid \#\tau = n).$$

By Proposition 4.2.11 (i), $\mathbb{P}(|\tau| > \frac{1}{2}n \mid \#\tau = n) \rightarrow 0$ when $n \rightarrow \infty$. To control the first term of the right-hand side of the previous inequality, we use Proposition 4.2.11 (ii): to that end, recall from Proposition 4.2.7 (ii) that $R_{\lfloor \frac{7}{8}n \rfloor} \tau$ is a measurable function of the Lukasiewicz path $(W_k(\tau))_{0 \leq k \leq \lfloor \frac{7}{8}n \rfloor}$. Therefore, for all $n \in \mathbb{N}$ and $c \in (0, \infty)$,

$$\begin{aligned} x_n &:= \mathbb{P}(\mathcal{S}(R_{\lfloor \frac{7}{8}n \rfloor} \tau) \geq \frac{1 + \varepsilon}{\alpha\gamma} \ln n \mid \#\tau = n) \\ &= \mathbb{E} \left[\mathbf{1}_{\{\mathcal{S}(R_{\lfloor \frac{7}{8}n \rfloor} \tau) \geq \frac{1 + \varepsilon}{\alpha\gamma} \ln n\}} D_n^{(\frac{7}{8})}(W_{\lfloor \frac{7}{8}n \rfloor}(\tau)) \mid \#\tau \geq n \right] \\ &\leq \mathbb{E} \left[\mathbf{1}_{\{\mathcal{S}(\tau) \geq \frac{1 + \varepsilon}{\alpha\gamma} \ln n\}} D_n^{(\frac{7}{8})}(W_{\lfloor \frac{7}{8}n \rfloor}(\tau)) \mid \#\tau \geq n \right] \\ &\leq c \mathbb{P}(\mathcal{S}(\tau) \geq \frac{1 + \varepsilon}{\alpha\gamma} \ln n \mid \#\tau \geq n) + \mathbb{E} \left[\mathbf{1}_{\{D_n^{(\frac{7}{8})}(W_{\lfloor \frac{7}{8}n \rfloor}(\tau)) \geq c\}} D_n^{(\frac{7}{8})}(W_{\lfloor \frac{7}{8}n \rfloor}(\tau)) \mid \#\tau \geq n \right] \end{aligned}$$

by (4.3.3). We make $n \rightarrow \infty$ then $c \rightarrow \infty$, and thus, by (4.4.7) and (4.2.12) in Proposition 4.2.11, we get $\limsup_{n \rightarrow \infty} x_n = 0$. This implies (4.4.6) and completes the proof of Theorem 4.1.2. \blacksquare

4.4.2 Proof of Theorem 4.1.3

In all this section, we assume that the critical offspring distribution μ belongs to the domain of attraction of a 1-stable law, and more precisely, we assume that Proposition 4.2.12 (a-c) hold. It is convenient to set

$$\forall x \in [0, \infty), \quad q_x = \mathbb{P}(\mathcal{S}(\tau) \geq x).$$

Recall from (4.2.1) that $\Delta(\tau)$ stands for the maximal out-degree of τ , which is the relevant quantity to consider in the 1-stable cases. Note that $\mathbb{P}(\Delta(\tau) \geq n) \geq \mu([n, \infty)) > 0$ for all $n \in \mathbb{N}$. We first prove the following estimates.

Lemma 4.4.1 *Under the same assumption and notations as in Theorem 4.1.3, the following holds true for all $\varepsilon \in (0, 1)$ and all $\kappa \in (0, \infty)$.*

(i) *There is $n_0 \in \mathbb{N}$ that depends on ε and κ such that for all integers $n \geq n_0$,*

$$b_n q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})} \leq (\ln n)^{-\kappa} \quad \text{and} \quad b_n q_{(1-\varepsilon)\Upsilon(\frac{1}{b_n})} \geq (\ln n)^\kappa.$$

(ii) *Recall from (4.1.9) the definition of Λ . Then, $\lim_{n \rightarrow \infty} b_n \Lambda(\frac{1}{b_n}) q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})} = 0$.*

(iii) *It holds $\lim_{n \rightarrow \infty} n^\kappa \mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon(\frac{1}{b_n}) \mid \Delta(\tau) > \frac{1}{2}b_n) = 0$.*

Proof. Let us prove (i). By Proposition 4.3.12 then Proposition 4.3.14, we have

$$\Upsilon(q_{(1\pm\varepsilon)\Upsilon(\frac{1}{b_n})}) \sim (1 \pm \varepsilon)\Upsilon(\frac{1}{b_n}) \sim (1 \pm \varepsilon)\Upsilon(\frac{(\ln b_n)^{\mp\kappa}}{b_n}).$$

Then, for all sufficiently large n , $\Upsilon(q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})}) > \Upsilon(\frac{(\ln b_n)^{-\kappa}}{b_n})$ and $\Upsilon(q_{(1-\varepsilon)\Upsilon(\frac{1}{b_n})}) < \Upsilon(\frac{(\ln b_n)^\kappa}{b_n})$, which implies that

$$b_n q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})} \leq (\ln b_n)^{-\kappa} \quad \text{and} \quad b_n q_{(1-\varepsilon)\Upsilon(\frac{1}{b_n})} \geq (\ln b_n)^\kappa$$

because Υ decreases (see Lemma 4.3.10). By Proposition 4.2.12 (iii), b_n varies regularly with index 1 and by Proposition 4.2.8 (i), $\ln b_n \sim \ln n$, which implies the desired result.

We use similar arguments to prove (ii): by Proposition 4.3.12 then Proposition 4.3.14, we get

$$\Upsilon(q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})}) \sim (1 + \varepsilon)\Upsilon(\frac{1}{b_n}) \sim (1 + \varepsilon)\Upsilon(\frac{1}{b_n} \Lambda(\frac{1}{b_n})^{-1-\kappa}),$$

so $\Upsilon(q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})}) > \Upsilon(\frac{1}{b_n} \Lambda(\frac{1}{b_n})^{-1-\kappa})$ for all sufficiently large n , which implies that

$$b_n \Lambda(\frac{1}{b_n}) q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})} \leq \Lambda(\frac{1}{b_n})^{-\kappa}.$$

This implies (ii) since $\lim_{x \rightarrow \infty} \Lambda(1/x) = \infty$ by Proposition 4.3.11 (ii).

To prove (iii), we first use (i) with e.g. $\kappa = 2$. Then, Proposition 4.3.9 implies that

$$n^\kappa \mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon(\frac{1}{b_n}) \mid \Delta(\tau) > \frac{1}{2}b_n) \leq n^\kappa \exp(-\frac{1}{2}b_n q_{(1-\varepsilon)\Upsilon(\frac{1}{b_n})}) \leq n^\kappa \exp(-\frac{1}{2}(\ln n)^2)$$

for all sufficiently large n , which entails the desired result. ■

The proof of Theorem 4.1.3 is cut into two parts: firstly an upper bound and secondly a lower bound.

Upper bound. We first prove for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\mathcal{S}(\tau) \geq (1 + \varepsilon)\Upsilon(\frac{1}{b_n}) \mid \#\tau \geq n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.8)$$

To that end, we use a direct upper bound, then we combine Lemma 4.2.13 with Proposition 4.3.11 (ii):

$$\mathbb{P}(\mathcal{S}(\tau) \geq (1 + \varepsilon)\Upsilon(\frac{1}{b_n}) \mid \#\tau \geq n) \leq \frac{q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})}}{\mathbb{P}(\#\tau \geq n)} \sim \Lambda(\frac{1}{b_n}) b_n q_{(1+\varepsilon)\Upsilon(\frac{1}{b_n})}.$$

Next, we use Lemma 4.4.1 (ii) to conclude. ■

Lower bound. We next prove for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\mathcal{S}(\tau) \leq (1 - \varepsilon)\Upsilon(\frac{1}{b_n}) \mid \#\tau \geq n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.9)$$

To that end, we first prove that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\#\tau < n \mid \Delta(\tau) \geq 2b_n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.4.10)$$

Indeed, by (4.3.25) in Proposition 4.3.9 combined with Markov inequality, for all $\lambda \in (0, \infty)$, it holds

$$\mathbb{P}(\#\tau < n \mid \Delta(\tau) \geq 2b_n) \leq e^\lambda \mathbb{E}[e^{-\frac{\lambda}{n}\#\tau}]^{\lfloor 2b_n \rfloor}.$$

Recall from Proposition 4.2.7 (iv) that $\mathbb{E}[\exp(-\frac{\lambda}{n}\#\tau)]^{\lfloor 2b_n \rfloor} = \mathbb{E}[\exp(-\frac{\lambda}{n}H_{\lfloor 2b_n \rfloor})]$. The convergence (4.2.15) with $x=2$ then implies that $\lim_{n \rightarrow \infty} \mathbb{E}[e^{-\frac{\lambda}{n}\#\tau}]^{\lfloor 2b_n \rfloor} = e^{-2\lambda}$. Thus,

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\#\tau < n \mid \Delta(\tau) \geq 2b_n) \leq e^{-\lambda} \xrightarrow[\lambda \rightarrow \infty]{} 0$$

which entails (4.4.10). \square

We next prove that for all $c \in (0, \infty)$,

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\Delta(\tau) \geq cb_n)}{\mathbb{P}(\#\tau \geq n)} \leq \frac{4}{c}. \quad (4.4.11)$$

Indeed, we first prove the case where $c = 2$. Observe that

$$u(n) := \frac{\mathbb{P}(\Delta(\tau) \geq 2b_n)}{\mathbb{P}(\#\tau \geq n)} = \mathbb{P}(\Delta(\tau) \geq 2b_n \mid \#\tau \geq n) + u(n)\mathbb{P}(\#\tau < n \mid \Delta(\tau) \geq 2b_n).$$

By (4.4.10), for all sufficiently large n , we get $\mathbb{P}(\#\tau < n \mid \Delta(\tau) \geq 2b_n) \leq \frac{1}{2}$. It therefore implies $u(n) \leq 2\mathbb{P}(\Delta(\tau) \geq 2b_n \mid \#\tau \geq n) \rightarrow 2\mathbb{P}(J \geq 2) = 1$ by (4.2.18) in Proposition 4.2.15. This proves (4.4.11) when $c = 2$. Then, recall from Proposition 4.2.12 (iii) that (b_n) is 1-regularly varying. Then for all sufficiently large n , we get $cb_n \geq 2b_{\lfloor \frac{1}{4}cn \rfloor}$ and thus

$$\frac{\mathbb{P}(\Delta(\tau) \geq cb_n)}{\mathbb{P}(\#\tau \geq n)} \leq \frac{\mathbb{P}(\Delta(\tau) \geq 2b_{\lfloor \frac{1}{4}cn \rfloor})}{\mathbb{P}(\#\tau \geq n)} = u(\lfloor \frac{1}{4}cn \rfloor) \frac{\mathbb{P}(\#\tau \geq \lfloor \frac{1}{4}cn \rfloor)}{\mathbb{P}(\#\tau \geq n)}.$$

This implies the desired result since we know from Lemma 4.2.13 that the sequence $n \mapsto \mathbb{P}(\#\tau \geq n)$ varies regularly with exponent -1 , and since $\limsup_{n \rightarrow \infty} u(\lfloor \frac{1}{4}cn \rfloor) \leq 1$ as proved above. \square

We complete the proof of (4.4.9) as follows.

$$\begin{aligned} v_n &:= \mathbb{P}(\mathcal{S}(\tau) \leq (1 - \varepsilon)\Upsilon(\frac{1}{b_n}) \mid \#\tau \geq n) \\ &\leq \frac{\mathbb{P}(\mathcal{S}(\tau) \leq (1 - \varepsilon)\Upsilon(\frac{1}{b_n}); \Delta(\tau) > \frac{1}{2}b_n)}{\mathbb{P}(\#\tau \geq n)} + \mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau \geq n) \\ &\leq \frac{\mathbb{P}(\Delta(\tau) > \frac{1}{2}b_n)}{\mathbb{P}(\#\tau \geq n)} \mathbb{P}(\mathcal{S}(\tau) \leq (1 - \varepsilon)\Upsilon(\frac{1}{b_n}) \mid \Delta(\tau) > \frac{1}{2}b_n) + \mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau \geq n). \end{aligned}$$

This first term of the above right-hand side converges to 0 by (4.4.11) with $c = 1/2$ and by Lemma 4.4.1 (iii). Moreover, (4.2.18) in Proposition 4.2.15 asserts that $\mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau \geq n)$ goes to 0 as $n \rightarrow \infty$. This completes the proof of (4.4.9), and of Theorem 4.1.3. \blacksquare

4.4.3 Proof of Theorem 4.1.4

Throughout this section, we assume there exists a function L that varies slowly at ∞ such that $\mu(n) \sim n^{-2}L(n)$. This implies that $\mu([n, \infty)) \sim n^{-1}L(n)$ so, by Proposition 4.2.12, μ belongs to the domain of attraction of a 1-stable law. The proof of Theorem 4.1.4 is separated into two parts: firstly a lower bound and secondly an upper bound.

Lower bound. We prove for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon\left(\frac{1}{b_n}\right) \mid \#\tau = n) \xrightarrow{n \rightarrow \infty} 0. \quad (4.4.12)$$

Indeed, observe that

$$\begin{aligned} v_n &:= \mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon\left(\frac{1}{b_n}\right) \mid \#\tau = n) \\ &\leq \frac{\mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon\left(\frac{1}{b_n}\right); \Delta(\tau) > \frac{1}{2}b_n)}{\mathbb{P}(\#\tau = n)} + \mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau = n) \\ &\leq \frac{\mathbb{P}(\Delta(\tau) > \frac{1}{2}b_n)}{n\mathbb{P}(\#\tau = n)} n\mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon\left(\frac{1}{b_n}\right) \mid \Delta(\tau) > \frac{1}{2}b_n) + \mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau = n). \end{aligned}$$

Then recall from (4.2.17) in Lemma 4.2.13 that $n\mathbb{P}(\#\tau = n) \sim \mathbb{P}(\#\tau \geq n)$. Thus, by (4.4.11), we get

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\Delta(\tau) > \frac{1}{2}b_n)}{n\mathbb{P}(\#\tau = n)} \leq 8.$$

By Lemma 4.4.1 (iii), $\lim_{n \rightarrow \infty} n\mathbb{P}(\mathcal{S}(\tau) \leq (1-\varepsilon)\Upsilon\left(\frac{1}{b_n}\right) \mid \Delta(\tau) > \frac{1}{2}b_n) = 0$. Then, (4.2.19) in Proposition 4.2.15 applies and asserts that $\lim_{n \rightarrow \infty} \mathbb{P}(\Delta(\tau) \leq \frac{1}{2}b_n \mid \#\tau = n) = 0$, which finally implies (4.4.12). ■

Upper bound. We finally show for all $\varepsilon \in (0, 1)$ that

$$\mathbb{P}(\mathcal{S}(\tau) \geq (1+\varepsilon)\Upsilon\left(\frac{1}{b_n}\right) \mid \#\tau = n) \xrightarrow{n \rightarrow \infty} 0. \quad (4.4.13)$$

To prove (4.4.13), we use the result of Kortchemski & Richier [108, Theorem 21] that is recalled in Proposition 4.2.16 and that shows that the law of the Lukasiewicz path $(W_j(\tau))_{0 \leq j \leq n}$ of τ under $\mathbb{P}(\cdot \mid \#\tau = n)$ is close in total variation distance to the law of $(Z_j^{(n)})_{0 \leq j \leq n}$ as defined in (4.2.21).

First, let us briefly recall the definition of $Z^{(n)}$: let $(W_n)_{n \in \mathbb{N}}$ be a left-continuous random walk starting at 0 and whose jump distribution is given by (4.1.7). We recall from (4.2.20) the two notations $I_n = -\min_{0 \leq j \leq n-1} W_j$ and $\sigma_n = \inf\{0 \leq k \leq n-1 : W_k = -I_n\}$. Then,

$$Z_j^{(n)} = W_{\sigma_n+j} + I_n \text{ if } 0 \leq j < n - \sigma_n \quad \text{and} \quad Z_j^{(n)} = I_n - 1 + W_{j-(n-\sigma_n)} \text{ if } n \geq j \geq n - \sigma_n.$$

Next, we interpret Proposition 4.2.16 in terms of trees, i.e. we view $Z^{(n)}$ as the Lukasiewicz path of a random tree $\tau^{(n)}$. Namely, observe that $Z_0^{(n)} = 0$, $Z_n^{(n)} = -1$, and the other values of $Z^{(n)}$ are nonnegative. Plus, maybe except at the cutting time $n - \sigma_n - 1$ when $Z_{n-\sigma_n}^{(n)} - Z_{n-\sigma_n-1}^{(n)} = -1 - W_{n-1}$, the jumps of $Z^{(n)}$ are larger or equal to -1 . Consequently, if $W_{n-1} \leq 0$ then Proposition 4.2.7 (i) applies and $Z^{(n)}$ is the Lukasiewicz path associated with a tree $\tau^{(n)}$. If $W_{n-1} > 0$ then we take $\tau^{(n)}$ equal to the *star-tree with $n-1$ leaves*, which we denote by \star_n (or any tree with n vertices).

$$\text{If } W_{n-1} \leq 0 \text{ then } (W_j(\tau^{(n)}))_{0 \leq j \leq n} = (Z_j^{(n)})_{0 \leq j \leq n} \quad \text{and} \quad \text{if } W_{n-1} > 0 \text{ then } \tau^{(n)} = \star_n. \quad (4.4.14)$$

Moreover, observe that Proposition 4.2.12 (c) easily implies that $\mathbb{P}(\tau^{(n)} = \star_n) = \mathbb{P}(W_{n-1} > 0) \rightarrow 0$. Thus, thanks to Proposition 4.2.16, in order to prove (4.4.13), we only need to show

$$\mathbb{P}(\tau^{(n)} \neq \star_n; \mathcal{S}(\tau^{(n)}) \geq (1+\varepsilon)\Upsilon(\frac{1}{b_n})) \xrightarrow{n \rightarrow \infty} 0. \quad (4.4.15)$$

To that end, we discuss a decomposition of $\tau^{(n)}$ at the ‘cutting time’ $n - \sigma_n - 1$ and we recall from (4.2.9) in Proposition 4.2.7 the existence of an i.i.d. sequence $(\tau_p)_{p \in \mathbb{N}}$ of $\text{GW}(\mu)$ -trees given by

$$\forall p \in \mathbb{N}, \quad (p + W_{\mathbb{H}_p+j})_{0 \leq j \leq \mathbb{H}_{p+1} - \mathbb{H}_p} = (W_j(\tau_p))_{0 \leq j \leq \#\tau_p},$$

where $\mathbb{H}_p = \inf\{j \in \mathbb{N} : W_j = -p\}$. Let us denote by $u_0^{(n)} = \emptyset < u_1^{(n)} < \dots < u_{n-1}^{(n)}$ the vertices of $\tau^{(n)}$ listed in lexicographic order. From the definition of $Z^{(n)}$ and thanks to Proposition 4.2.7 (ii), we first derive that if $W_{n-1} \leq 0$ then

$$R_{n-\sigma_n-1}(\tau^{(n)}) = R_{n-\sigma_n-1}(\tau_{I_n}) \quad \text{and} \quad k_{u_{n-\sigma_n-1}^{(n)}}(\tau^{(n)}) = -W_{n-1}, \quad (4.4.16)$$

where we recall from (4.2.7) in Proposition 4.2.7 that $R_m(t) = R_m t$ stands for the tree consisting of the first $m + 1$ vertices of t taken in lexicographic order. We now look at the subtrees that are grafted either at $u_{n-\sigma_n-1}^{(n)}$ or to the right of its ancestral line: namely, the subtrees that are grafted at the following set of vertices

$$B = \{v \in \tau^{(n)} : \overleftarrow{v} \preceq u_{n-\sigma_n-1}^{(n)} \text{ and } v > u_{n-\sigma_n-1}^{(n)}\}.$$

If we denote by $v(0) < v(1) < \dots < v(\#B - 1)$ the vertices of B listed in lexicographic order, then by definition of $Z^{(n)}$ and by definition of the trees τ_p as recalled above, we get that if $W_{n-1} \leq 0$ then

$$\theta_{v(p)} \tau^{(n)} = \tau_p, \quad 0 \leq p \leq I_n - 1 = \#B - 1. \quad (4.4.17)$$

To prove (4.4.15), we then use (4.3.5) in Lemma 4.3.1 (i) that asserts on the event $\{W_{n-1} \leq 0\}$ that

$$\mathcal{S}(\tau^{(n)}) \leq 1 + \max(\mathcal{S}(R_{n-\sigma_n-1}(\tau^{(n)})), \max_{v \in B} \mathcal{S}(\theta_v \tau^{(n)})).$$

Then, the identities (4.4.16) and (4.4.17), together with the monotony property (4.3.3) of \mathcal{S} , yield that

$$\mathcal{S}(\tau^{(n)}) \leq 1 + \max_{0 \leq p \leq I_n} \mathcal{S}(\tau_p).$$

Therefore, we obtain the following:

$$\begin{aligned} x_n &:= \mathbb{P}\left(\tau^{(n)} \neq \star_n; \mathcal{S}(\tau^{(n)}) \geq (1+\varepsilon)\Upsilon(\frac{1}{b_n})\right) \\ &\leq \mathbb{P}\left(\max_{0 \leq p \leq \lfloor 2b_n \rfloor} \mathcal{S}(\tau_p) \geq (1+\varepsilon)\Upsilon(\frac{1}{b_n}) - 1\right) + \mathbb{P}(I_n > 2b_n) \\ &\leq (2b_n + 1) \mathbb{P}\left(\mathcal{S}(\tau) \geq (1+\varepsilon)\Upsilon(\frac{1}{b_n}) - 1\right) + \mathbb{P}(\frac{1}{n} \mathbb{H}_{\lfloor 2b_n \rfloor} \leq 1), \end{aligned}$$

since the τ_p are $\text{GW}(\mu)$ -trees. Then, recall from (4.2.15) that $\frac{1}{n} \mathbb{H}_{\lfloor 2b_n \rfloor} \rightarrow 2$ in probability, which implies that $\mathbb{P}(\frac{1}{n} \mathbb{H}_{\lfloor 2b_n \rfloor} \leq 1) \rightarrow 0$. Lemma 4.4.1 (i) asserts that $b_n \mathbb{P}(\mathcal{S}(\tau) \geq (1+\varepsilon)\Upsilon(\frac{1}{b_n}) - 1) \rightarrow 0$, which finally implies (4.4.15) and which completes the proof of Theorem 4.1.4. \blacksquare

We conclude this chapter by providing an asymptotic equivalent of the rescaling sequence $\Upsilon(\frac{1}{b_n})$, that gives the order of $\mathcal{S}(\tau)$ under $\mathbb{P}(\cdot | \#\tau = n)$, in the three cases considered in Example 4.3.13.

Example 4.4.2 We recall from Proposition 4.2.12 (iii) that b_n varies regularly with index 1, which implies $\ln b_n \sim \ln n$ by Proposition 4.2.8 (i).

(a) Let $\kappa \in (0, \infty)$. We consider the case where $L(x) = (\ln x)^{-1-\kappa}$ for all $x \in (0, \infty)$. Recall from Example 4.3.13 that $\Upsilon(s) \sim_{0+} (\ln \frac{1}{s}) / (\ln \ln \frac{1}{s})$. Therefore, we easily get

$$\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{\ln n}{\ln \ln n}.$$

(b) Let $\kappa \in (0, 1)$. We next consider the case where $L(x) = \exp(-(\ln x)^\kappa)$ for all $x \in (0, \infty)$. Recall from Example 4.3.13 that $\Upsilon(s) \sim_{0+} \frac{1}{1-\kappa} (\ln \frac{1}{s}) / (\ln \ln \frac{1}{s})$. Therefore, we easily get

$$\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{1}{1-\kappa} \cdot \frac{\ln n}{\ln \ln n}.$$

(c) Let us finally consider the case where $L(x) = \exp(-\ln x / \ln \ln x)$ for all $x \in (e^e, \infty)$. Recall from Example 4.3.13 that $\Upsilon(s) \sim_{0+} (\ln \frac{1}{s}) / (\ln \ln \ln \frac{1}{s})$. Therefore, we easily get

$$\Upsilon\left(\frac{1}{b_n}\right) \sim \frac{\ln n}{\ln \ln \ln n}.$$

□

Acknowledgements I am deeply indebted to my Ph.D. advisor Thomas Duquesne for introducing me to the Horton-Strahler number, for suggesting me several problems about it, and for many insightful discussions. I am especially grateful for his help to prove Proposition 4.3.11 and to improve the quality of this work. I warmly thank Guillaume Bouteille et Yoan Tardy for a stimulating discussion about the correct order of magnitude that should appear in the Cauchy regime. Many thanks are due to Quentin Berger for some feedback about the proof of the estimate (4.2.16). I also thank Igor Kortchemski for some precisions about his work.

Chapter 5

Fluctuations of the Horton–Strahler number of stable Galton–Watson trees

This chapter is adapted from the paper [102], which is submitted for publication.

Abstract

The Horton–Strahler number – also called the register function – is a combinatorial tool that quantifies the branching complexity of a rooted tree. We study the law of the Horton–Strahler number of stable Galton–Watson trees conditioned to have size n , including the Catalan trees. While these random variables are known to grow as a multiple of $\ln n$ in probability, their fluctuations are not well understood because they are coupled with deterministic oscillations. To rule out the latter, we introduce a real-valued variant of the Horton–Strahler number. We show that a rescaled exponential of this quantity jointly converges in distribution to a measurable function of the scaling limit of the trees, i.e. the stable Lévy tree. We call this limit the Strahler dilation and we discuss its similarities with the Horton–Strahler number.

Contents

5.1	Introduction	228
5.2	Framework, notations, and definitions	235
5.2.1	Topological framework	235
5.2.2	Discrete trees as sets of words	238
5.2.3	Real trees	242
5.2.4	Grafting rooted compact metric spaces	245
5.2.5	Stable Lévy processes	247
5.3	Preliminary tools	249
5.3.1	Marchal’s algorithm	249
5.3.2	Limit theorems for stable Galton–Watson trees	250
5.3.3	An estimate for the height function of a Galton–Watson tree with edge lengths	253
5.4	The weighted Horton–Strahler number	255
5.4.1	The classic Horton–Strahler number	255

5.4.2	First properties of the weighted Horton–Strahler number	256
5.4.3	The weighted Horton–Strahler number of stable Galton–Watson weighted trees	257
5.5	The weighted Horton pruning of trees with edge lengths	260
5.5.1	Definition and invariance	260
5.5.2	Some applications	264
5.6	Proof of Theorem 5.1.3	266
5.7	A first description of the limit tree in Theorem 5.1.3	270
5.7.1	Proof of Theorem 5.7.3	272
5.8	The Strahler dilation of rooted compact real trees	277
5.8.1	Definition and properties of the Strahler dilation	277
5.8.2	Proof of Theorem 5.1.4 (ii)	282
5.9	Proofs of Theorems 5.1.5 and 5.1.6	287
5.9.1	Proof of Theorem 5.1.5	287
5.9.2	Proof of Theorem 5.1.6	289
5.10	The Strahler dilation of the Brownian tree: proof of Theorem 5.1.7	292

5.1 Introduction

The Horton–Strahler number of a finite rooted tree t is a nonnegative integer $\mathcal{S}(t) \in \mathbb{N}$ that measures its branching complexity. One of its possible formal constructions is the recursive definition below.

Definition 5.1.1 Let t be a finite rooted tree. Its *Horton–Strahler number* $\mathcal{S}(t)$ is defined as follows.

- (a) If t reduces to a single node, then $\mathcal{S}(t) = 0$.
- (b) Otherwise, $\mathcal{S}(t)$ is the maximum of the Horton–Strahler numbers of the subtrees t_1, \dots, t_k that are attached to the root, plus one if that maximum is not uniquely achieved. Namely,

$$\mathcal{S}(t) = \max_{1 \leq i \leq k} \mathcal{S}(t_i) + \mathbf{1}_{\{\#\arg\max_{1 \leq i \leq k} \mathcal{S}(t_i) \geq 2\}}. \quad (5.1.1)$$

□

Alternatively, $\mathcal{S}(t) + 1$ is the number of successive pruning operations (cutting all leaves then merging each line into one edge) necessary to completely erase t , and $\mathcal{S}(t)$ is also the height of the largest perfect binary tree embedded into t (see Section 5.4.1 for details). In this article, we study the fluctuations of the Horton–Strahler number of the so-called α -stable Galton–Watson trees (or GW_α -trees for short), with $\alpha \in (1, 2]$, conditioned to be large. For all $\alpha \in (1, 2]$, the offspring distribution $\mu_\alpha = (\mu_\alpha(k))_{k \in \mathbb{N}}$ of a GW_α -tree is critical and is characterized by its generating function:

$$\forall s \in [0, 1], \quad \varphi_\alpha(s) := \sum_{k \in \mathbb{N}} s^k \mu_\alpha(k) = s + \frac{1}{\alpha}(1 - s)^\alpha. \quad (5.1.2)$$

See Section 5.2.2 for a formal definition. We highlight that when $\alpha = 2$, a GW_2 -tree is just a critical binary Galton–Watson tree, namely $\mu_2(0) = \mu_2(2) = \frac{1}{2}$. Before discussing our results, let us present a brief history of the Horton–Strahler number and explain why it is relevant to focus on that specific family of critical Galton–Watson trees. Here, all the random variables that we consider are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose expectation is denoted by \mathbb{E} , and all topological spaces are endowed with their Borel sigma field. For all $x \in [0, \infty)$, we denote by $\lfloor x \rfloor \in \mathbb{N}$ the integer part of x .

Background. The Horton–Strahler number was introduced in the field of hydrogeology, first by Horton [93] in 1945 and exactly as Definition 5.1.1 by Strahler [146] in 1952, to seek quantitative empirical laws about stream systems. Such a system can be seen as a tree whose root corresponds to the basin outlet and whose leaves represent the springs. Many important characteristics of river networks have been linked to the Horton–Strahler number: see Peckham [136], Fac-Beneda [73], Chavan & Srinivas [34], and Bamuffeh et al. [14] among others. This number appears independently in other scientific disciplines such as anatomy, botany, molecular biology, physics, social network analysis, etc. In computer science, it is sometimes called the *register function* because the minimum number of registers needed to evaluate an expression tree is equal to its Horton–Strahler number. We refer to Viennot [148] for an overview of those various applications.

The Horton–Strahler number also shows a wide range of occurrences in mathematics. It can be encountered in mathematical logic, formal language theory, algebra, combinatorics, topology, approximation theory, and more. See Esparza, Luttenberger & Schlund [69] for an overview of those connections. In the probability area, Flajolet, Raoult & Vuillemin [75] and Kemp [98] study the Horton–Strahler number of a uniform rooted ordered binary tree T_n with n leaves (a uniform n -Catalan tree, which is also a GW_2 -tree conditioned to have n leaves) and they prove that

$$\mathbb{E}[\mathcal{S}(T_n)] = \log_4 n + D(\log_4 n) + o(1) \quad (5.1.3)$$

as $n \rightarrow \infty$, where $\log_b x = \ln x / \ln b$ stands for the logarithm of x to the base b , and D is a 1-periodic continuous function. Moreover, Devroye & Kruszewski [47] showed that $\mathcal{S}(T_n)$ is uniformly concentrated around its expected value because

$$\forall m \geq 0, \quad \sup_{n \geq 0} \mathbb{P}(|\mathcal{S}(T_n) - \mathbb{E}[\mathcal{S}(T_n)]| \geq m) \leq 4^{1-m}. \quad (5.1.4)$$

Drmotá & Prodinger [49] extended (5.1.3) and (5.1.4) to k -ary trees. More recently, Brandenberger, Devroye & Reddad [28] proved that the Horton–Strahler number of a critical Galton–Watson tree with finite variance offspring distribution conditioned to have n vertices always grows as $\log_4 n$ in probability, thus generalizing all previously obtained results on first-behavior. In a companion paper [103], we further treat the cases where the offspring distribution belongs to the domain of attraction of a stable law (and thus may have infinite variance).

In a parallel direction, Burd, Waymire & Winn [31] analyze the invariance and attraction properties of critical Galton–Watson trees with finite variance offspring distribution under the *Horton pruning operation* – erasing leaves and their parental edges, then removing vertices with a single child – which exactly decrements the Horton–Strahler number. They notably prove that GW_2 -trees are invariant under Horton pruning and that the law of their Horton–Strahler number is geometric with parameter $\frac{1}{2}$. Kovchegov & Zaliapin [111, 110] broadened this study to the infinite variance case: they proved that a critical Galton–Watson tree whose offspring distribution μ is in the domain of attraction of an α -stable law is invariant by Horton pruning if and only if $\mu = \mu_\alpha$ as given by (5.1.2), as well as the following (see [110, Lemma 10]).

$$\text{If } \tau \text{ is a } \text{GW}_\alpha\text{-tree with } \alpha \in (1, 2], \text{ then for all } n \in \mathbb{N}, \quad \mathbb{P}(\mathcal{S}(\tau) \geq n) = (1 - \frac{1}{\alpha})^n. \quad (5.1.5)$$

Moreover, GW_α -trees are invariant attractors for a large class of other tree reductions: see Duquesne & Winkel [63] and Kovchegov, Xu & Zaliapin [109]. The stable Galton–Watson trees had already appeared in numerous works such as Zolotarev [152], Neveu [134], Le Jan [120], or Abraham & Delmas [1]. Indeed, they play the role of reference objects among all critical Galton–Watson trees because they are the finite-dimensional marginals of their universal scaling limits, as known as the stable Lévy trees. We refer to Duquesne & Le Gall [61] and to Marchal [128] for more details.

Let us come back to Catalan trees T_n . The estimate (5.1.4) ensures that a natural centering suffices to gain tightness on their Horton–Strahler numbers. However, the asymptotic development (5.1.3) shows that $\mathcal{S}(T_n)$ is subject to deterministic oscillations. Worse, these variations inconvenience real-world applications, where characteristics derived from the Horton–Strahler number show great discrepancies due to the threshold selected for the extraction of stream networks: see e.g. Moussa & Bocquillon [132] or Chavan & Srinivas [34]. This could then impact the chosen hydrologic response, as discussed by Bamufleh et al. [14]. We further argue that one cannot hope to find a nondegenerate scaling limit for $\mathcal{S}(T_n)$. We heuristically discuss why on a simpler but related model.

Heuristic discussion. We construct a random binary tree T'_n by grafting $n + 1$ independent GW_2 -trees on the same spine of length n as illustrated by Figure 5.1. In other words, T'_0 is a GW_2 -tree, and T'_{n+1} is composed of T'_n and an independent GW_2 -tree attached to the root. The tree T'_n is a truncated version of Kesten’s limit tree [100] that may be informally seen as a critical binary Galton–Watson tree conditioned to survive: see Lyons, Pemantle & Peres [125], Aldous & Pitman [11], and Duquesne [51] for more information. The trees T'_n and Kesten’s limit tree naturally appear during the study of local convergences of GW_2 -trees conditioned to be large, such as Catalan trees. We refer to Abraham & Delmas [2] for several results of this type. We give ourselves a sequence $(G_i)_{i \geq 0}$ of independent geometric random variables with parameter $\frac{1}{2}$. From (5.1.5), we observe that $\mathcal{S}(T'_n)$ has the same law as S_n , where the sequence (S_i) is defined by

$$S_0 = G_0 \quad \text{and} \quad S_{i+1} = \max(G_{i+1}, S_i) + \mathbf{1}_{\{G_{i+1} = S_i\}}$$

for all $i \geq 0$. It is then easy to convince oneself that there is an event A_n such that

$$S_n = M_n + \mathbf{1}_{A_n} \quad \text{where} \quad M_n = \max_{0 \leq i \leq n} G_i.$$

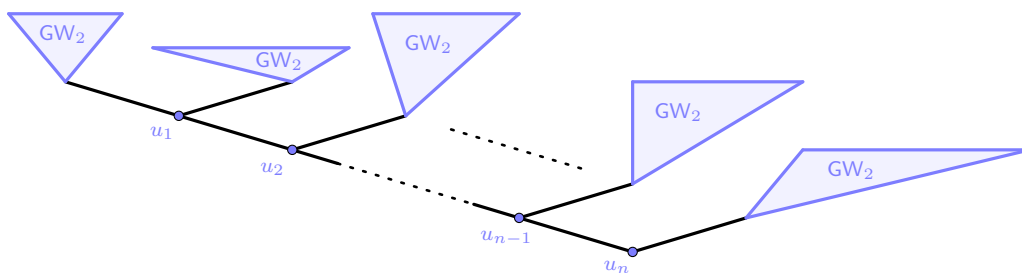


Figure 5.1: The random binary tree T'_n .

Let us set aside the event A_n for now because it is quite difficult to precisely describe: for example, it is realized when $(G_i)_{0 \leq i \leq n} = (1, 1, 2, 3, \dots, n)$ but not when $(G_i)_{0 \leq i \leq n} = (1, 2, 1, 3, \dots, n)$. Nonetheless, the sole observation of M_n already unveils an issue. Indeed, there are no sequences (a_n) and (b_n) such that $(M_n - b_n)/a_n$ converges in distribution to a nondegenerate random variable. In statistics, this is a classic application of the extreme value theorem, see e.g. Arnold, Balakrishnan & Nagaraja [12, page 217]. We also refer to Eisenberg [67] for a detailed study of the distribution of M_n . It is important to note that this situation is due to the discrete setting. Indeed, if $(E_i)_{i \geq 0}$ is a sequence of independent exponential random variables with mean $(\ln 2)^{-1}$, so that the integer part $\lfloor E_i \rfloor$ of E_i has the same distribution as G_i , then an elementary calculation shows that

$$\forall x \in \mathbb{R}, \quad \mathbb{P}\left(\max_{0 \leq i \leq n} E_i - \log_2 n \leq x\right) \longrightarrow \exp(-2^{-x}),$$

where the limit is the cumulative distribution function of the Gumbel law with location 0 and with scale $(\ln 2)^{-1}$. Other curiosities happen for geometric order statistics which further complexify the behavior of S_n . For example, Bruss & O’Cinneide [29] proved that the probability that the maximum of $(G_i)_{0 \leq i \leq n}$ is uniquely achieved does not converge.

Strategy and main results. Those observations suggest to slightly modify the Horton–Strahler number into a continuous quantity, more likely to behave regularly. Since one cannot expect two real numbers to be exactly equal, the recursive formula (5.1.1) is not suitable in this regard. Nevertheless, we see, keeping the notations of Definition 5.1.1, that

$$\mathcal{S}(t) = \max_{1 \leq i, j \leq k} \max(\mathcal{S}(t_i), \mathcal{S}(t_j), \mathbf{1}_{\{i \neq j\}} + \min(\mathcal{S}(t_i), \mathcal{S}(t_j))), \quad (5.1.6)$$

by considering the two maximal values among $\mathcal{S}(t_1), \dots, \mathcal{S}(t_k)$. The formula (5.1.6) stays workable when the $\mathcal{S}(t_i)$ are real numbers, so it only remains to redefine the Horton–Strahler number of a tree reduced to a single node. We do so by endowing the leaves with $[0, 1)$ -valued weights.

Definition 5.1.2 (Weighted Horton–Strahler number) A *weighted tree* \mathbf{t} is a finite rooted tree whose each leaf v is equipped with a weight $w_v \in [0, 1)$. Its *weighted Horton–Strahler number* is a nonnegative real number $\mathcal{S}(\mathbf{t}) \in [0, \infty)$ defined recursively as follows.

- (a) If \mathbf{t} consists of a single node with weight w_{root} , then $\mathcal{S}(\mathbf{t}) = w_{\text{root}}$.
- (b) Otherwise, $\mathcal{S}(\mathbf{t})$ is derived from the weighted Horton–Strahler numbers of the weighted subtrees $\mathbf{t}_1, \dots, \mathbf{t}_k$ that are attached to the root according to the expression (5.1.6). Namely,

$$\mathcal{S}(\mathbf{t}) = \max_{1 \leq i, j \leq k} \max(\mathcal{S}(\mathbf{t}_i), \mathcal{S}(\mathbf{t}_j), \mathbf{1}_{\{i \neq j\}} + \min(\mathcal{S}(\mathbf{t}_i), \mathcal{S}(\mathbf{t}_j))).$$

□

See Figure 5.2 for an example and Section 5.2.2 for a more precise definition of weighted trees. The requirement for the weights to be in $[0, 1)$ entails the following relation between the classic and the weighted Horton–Strahler numbers (see Proposition 5.4.3 for details):

$$\text{for all weighted tree } \mathbf{t}, \quad \lfloor \mathcal{S}(\mathbf{t}) \rfloor = \mathcal{S}(t), \quad \text{where } t \text{ is the underlying tree of } \mathbf{t}. \quad (5.1.7)$$

Therefore, one does not lose any information about $\mathcal{S}(t)$ by working solely with the weighted Horton–Strahler number. However, we now need to specify the weights that we put on the leaves of stable Galton–Watson trees.

In all this article, to lighten the notations, we fix and denote

$$\alpha \in (1, 2], \quad \beta = \frac{1}{\alpha-1}, \quad \gamma = \ln \frac{\alpha}{\alpha-1} \quad \text{and} \quad \delta = \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} = e^{\gamma(\alpha-1)} \in (1, 2]. \quad (5.1.8)$$

For all $a > 0$, we say a random variable $W \in [0, 1)$ has distribution $\text{FExp}(a)$ when

$$\forall r \in [0, 1], \quad \mathbb{P}(W \leq r) = \frac{1 - e^{-ar}}{1 - e^{-a}}. \quad (5.1.9)$$

We point out that if E is an exponential random variable with mean $\frac{1}{a}$ then the fractional part $E - \lfloor E \rfloor$ of E has law $\text{FExp}(a)$. We say a random weighted tree τ is an α -stable Galton–Watson weighted tree (or a \mathbf{GW}_α -weighted for short) when it is a GW_α -tree τ whose leaves are endowed with independent weights with law $\text{FExp}(\gamma)$ (more formally, see Definition 5.2.15). This choice of weights is justified by the following assertion that we prove in Proposition 5.4.6.

$$\text{If } \tau \text{ is a } \mathbf{GW}_\alpha\text{-weighted tree, then for all } x \in [0, \infty), \quad \mathbb{P}(\mathcal{S}(\tau) \geq x) = e^{-\gamma x}. \quad (5.1.10)$$

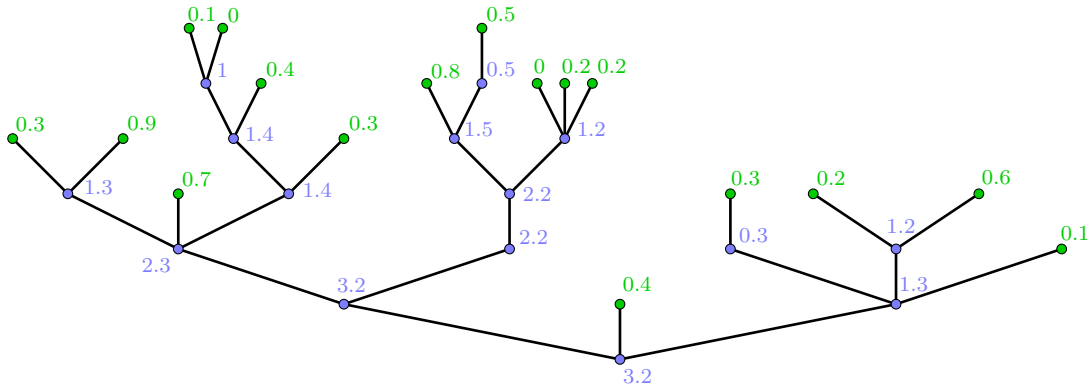


Figure 5.2: A weighted tree \mathbf{t} such that $\mathcal{S}(\mathbf{t}) = 3.2$. The leaves and their weights are in green. Next to each node is written the weighted Horton–Strahler number of the subtree stemming from it.

Note that (5.1.10) readily implies (5.1.5). The law of $\mathcal{S}(\tau)$ is absolutely continuous but, for all non-integer $x \geq 0$, we can define the law of τ under $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ by setting

$$\mathbb{P}(\tau = t \mid \mathcal{S}(\tau) = x) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{P}(\tau = t \mid |\mathcal{S}(\tau) - x| < \varepsilon)$$

for all tree t (see Definition 5.4.8 and Proposition 5.4.7 for details and a proof).

Instead of studying the weighted Horton–Strahler number of \mathbf{GW}_α -trees conditioned to have a large number of vertices, we begin by looking for a scaling limit for τ under $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ as x tends to ∞ . Indeed, this problem is easier to tackle thanks to the so-called *r-weighted Horton pruning operation* that we introduce in Section 5.5: erasing the parental edges of leaves with weights smaller than some threshold $r \in [0, 1]$, then removing vertices with a single child. When $r = 1$, we retrieve the Horton pruning mentioned above. This operation subtracts r from the weighted Horton–Strahler number, and \mathbf{GW}_α -weighted trees are almost invariant under weighted Horton pruning (see Theorem 5.5.4 for a precise statement). We then deduce that $x \mapsto \mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ is Cauchy, which yields our first main result. The rooted Gromov–Hausdorff–Prokhorov distance, which will be recalled in Section 5.2.1, gives a sense to convergences of rooted measured compact metric spaces.

Theorem 5.1.3 *Let τ be a \mathbf{GW}_α -weighted tree. The tree τ is endowed with its graph distance denoted by d_{gr} , its root denoted by \emptyset , and its counting measure $\sum_{u \in \tau} \delta_u$. There exists a nondegenerate random compact metric space (\mathcal{T}, d) endowed with a distinguished point ρ and a finite Borel measure μ such that the convergence in distribution*

$$\left(\tau, e^{-\gamma(\alpha-1)x} d_{\text{gr}}, \emptyset, e^{-\gamma\alpha x} \sum_{u \in \tau} \delta_u \right) \text{ under } \mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x) \xrightarrow[x \rightarrow \infty, x \notin \mathbb{N}]{d} (\mathcal{T}, d, \rho, \mu)$$

holds for the rooted Gromov–Hausdorff–Prokhorov distance.

Before describing the law of this limit space or discussing the fluctuations of the weighted Horton–Strahler numbers of \mathbf{GW}_α -trees conditioned on having total progeny n , we need to recall their own scaling limit. Let $(X_n)_{n \in \mathbb{N}}$ be a (left-continuous) random walk started at $X_0 = 0$ and with jump law given by $\mathbb{P}(X_1 = k) = \mu_\alpha(k + 1)$ for all integers $k \geq -1$. Using (5.1.2) and the continuity theorem for (bilateral) Laplace transform (see e.g. [151, Appendix A]), one can check that

$$\frac{1}{a_n} X_n \xrightarrow{d} X_1, \quad \text{where} \quad a_n = \alpha^{-1/\alpha} n^{1/\alpha} \quad \text{and} \quad \mathbb{E}[\exp(-\lambda X_1)] = \exp(\lambda^\alpha) \quad (5.1.11)$$

for all $n \in \mathbb{N}$ and $\lambda \in (0, \infty)$. Hence, μ_α belongs to the domain of attraction of a stable law of index α . Note that if $\alpha \in (1, 2)$ then $\lfloor \alpha \rfloor = 1$, and if $\alpha = 2$ then $\lfloor \alpha \rfloor = 2$. Then, keeping the same notations as in Theorem 5.1.3, a theorem of Duquesne [50] yields that the convergence in distribution

$$\left(\tau, \frac{a_n}{n} d_{\text{gr}}, \emptyset, \frac{1}{n} \sum_{u \in \tau} \delta_u \right) \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{} (\mathcal{T}_{\text{nr}}, d_{\text{nr}}, \rho_{\text{nr}}, \mu_{\text{nr}}) \quad (5.1.12)$$

holds for the rooted Gromov–Hausdorff–Prokhorov distance. Here, $(\mathcal{T}_{\text{nr}}, d_{\text{nr}}, \rho_{\text{nr}}, \mu_{\text{nr}})$ stands for the α -stable tree describing the genealogical structure of continuous state-branching process with branching mechanism $\lambda \mapsto \lambda^\alpha$, following the convention of Duquesne & Le Gall [61] (see Section 5.2.5 for a precise definition). When $\alpha = 2$, \mathcal{T}_{nr} corresponds to the celebrated Brownian tree of Aldous [7, 9]. A little more precisely, if $(\mathcal{T}_e, d_e, \rho_e, \mu_e)$ stands for the real tree coded by the standard Brownian excursion e in the sense of Le Gall [78, Definition 2.2] (see also Section 5.2.3), then $(\mathcal{T}_e, \sqrt{2} d_e, \rho_e, \mu_e)$ is a 2-stable tree.

The two limit spaces \mathcal{T} and \mathcal{T}_{nr} in Theorem 5.1.3 and (5.1.12) are real trees, namely all pairs of their points are joined by a unique arc that turns out to be a geodesic (see Definition 5.2.17 for details). We denote by $\mathbb{T}_{\mathbb{R}}^{\text{m}}$ the space of (equivalence classes of) rooted (finitely) measured compact real trees, endowed with the rooted Gromov–Hausdorff–Prokhorov distance (see Sections 5.2.1 and 5.2.3 for more precision). In light of Theorem 5.1.3 and of the convergence (5.1.12), it seems reasonable to expect that some functional \mathfrak{S} plays the role of a continuum analog of the weighted Horton–Strahler number, so that $\mathfrak{S}(\mathcal{T})$ is constant. We indeed explicitly introduce such an object in Definition 5.8.3, that we denote by $\mathfrak{S}_\delta : \mathbb{T}_{\mathbb{R}}^{\text{m}} \rightarrow [0, \infty]$ and that we call the *Strahler dilation with base δ* .

Theorem 5.1.4 *The Strahler dilation \mathfrak{S}_δ , as in Definition 5.8.3, is measurable and satisfies:*

- (i) $\mathfrak{S}_\delta(T, \lambda d, \rho, \mu) = \lambda \mathfrak{S}_\delta(T, d, \rho, 0)$ for all $\lambda \in (0, \infty)$ and all $(T, d, \rho, \mu) \in \mathbb{T}_{\mathbb{R}}^{\text{m}}$,
- (ii) $\mathfrak{S}_\delta(\mathcal{T}) = 1$ almost surely, where \mathcal{T} is the limit tree in Theorem 5.1.3.

We cannot suitably adapt the discrete Definition 5.1.2 or the combinatorial approach by Horton pruning to the continuum setting, but Definition 5.8.3 is inspired by the definition of the Horton–Strahler number as the maximal height of an embedded perfect binary tree. The proof of (ii) relies on a self-similar decomposition of \mathcal{T} that highlights this perfect binary tree. We formulate it later as Theorem 5.7.3. Thanks to the Strahler dilation, we can describe the law of \mathcal{T} in terms of the α -stable tree \mathcal{T}_{nr} by the following result.

Theorem 5.1.5 *Let \mathcal{T} be the limit tree in Theorem 5.1.3, and let $(\mathcal{T}_{\text{nr}}, d_{\text{nr}}, \rho_{\text{nr}}, \mu_{\text{nr}})$ be the α -stable tree as in (5.1.12). Let \mathfrak{S}_δ be the Strahler dilation with base δ as in Theorem 5.1.4. Then for all bounded and measurable function $F : \mathbb{T}_{\mathbb{R}}^{\text{m}} \rightarrow \mathbb{R}$, it holds that*

$$\mathbb{E}[F(\mathcal{T})] = \frac{\alpha^\beta}{\Gamma(1 - \frac{1}{\alpha})} \mathbb{E} \left[F \left(\mathcal{T}_{\text{nr}}, \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})^{-1} d_{\text{nr}}, \rho_{\text{nr}}, \alpha^{-\beta} \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})^{-\alpha\beta} \mu_{\text{nr}} \right) \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})^\beta \right]. \quad (5.1.13)$$

Informally, the Strahler dilation is to \mathcal{T} what the total mass is to \mathcal{T}_{nr} . We precise this relation and we give other formulations of (5.1.13) later in Theorem 5.9.1. Using Theorem 5.1.4, the relation (5.1.13) follows from a Bayesian argument linking (5.1.12) and Theorem 5.1.3.

Moreover, the same method yields that if τ is a \mathbf{GW}_α -weighted tree, then the scaling limit of $\delta^{\mathcal{S}(\tau)}$ under $\mathbb{P}(\cdot \mid \#\tau \geq n)$ is equal to the Strahler dilation with base δ of the scaling limit of the tree τ . The same proof does not hold to study the local conditioning $\{\#\tau = n\}$ that becomes

degenerate at the limit. To avoid this issue, we take advantage of a monotony property of the Horton–Strahler number by coupling a \mathbf{GW}_α -weighted tree conditioned to have *exactly* n leaves and a \mathbf{GW}_α -weighted tree conditioned to have *at least* $n/2$ leaves, such that one is always embedded into the other. This coupling relies on Marchal’s algorithm [128] which yields a sequence of nested \mathbf{GW}_α -trees. We then reach our goal.

Theorem 5.1.6 *We keep the notations of Theorem 5.1.3. Let (a_n) be as in (5.1.11) and let \mathcal{T}_{nr} be the α -stable tree as in (5.1.12). Let \mathfrak{S}_δ be the Strahler dilation with base δ as in Theorem 5.1.4. Jointly with (5.1.12), the following convergence holds in distribution on $(0, \infty)$:*

$$\frac{a_n}{n} \delta^{\mathcal{S}(\tau)} \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{} \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}).$$

We can reformulate this result by stating that the following convergence in distribution

$$\mathcal{S}(\tau) - \frac{\alpha-1}{\alpha} \log_\delta n - \frac{1}{\alpha} \log_\delta \alpha \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{} \log_\delta \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}) \quad (5.1.14)$$

holds on \mathbb{R} . Thanks to (5.1.7), this yields the following convergence in probability

$$\frac{\alpha \mathcal{S}(\tau)}{\log_{\frac{\alpha}{\alpha-1}} n} \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{} 1,$$

which is [103, Theorem 1.2] in the specific case where $\mu = \mu_\alpha$. Furthermore, the variations of the fractional part of $\frac{\alpha-1}{\alpha} \log_\delta n$ induce the same periodic phenomenon observed in the asymptotic estimate (5.1.3) proved by Flajolet, Raoult & Vuillemin [75] and Kemp [98].

Theorem 5.1.6 shows that it is possible to asymptotically recover the weighted Horton–Strahler number from the sole limit metric structure. This is noteworthy because the former depends on the weights of the leaves, which are independent of the tree. While their contribution is limited to the fractional part, this part is of constant-order and is thus non-negligible within the convergence (5.1.14). The disappearance of this dependence at the limit is explained by the specific choice of the law of weights, introduced to remove the arithmetic interference in the analysis of the classic Horton–Strahler number, which is thus an intrinsic aspect of the stable Galton–Watson tree. Similarly, the index $\mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})$ is a new metric characteristic of the stable tree, whose properties could provide information about its geometry.

Our last contribution specifically focuses on $\alpha = 2$, namely the binary case. In this setting, Flajolet, Raoult & Vuillemin [75] and Kemp [98] have explicitly derived the law of the Horton–Strahler number of Catalan trees. We adapt their computations to determine the law of the Strahler dilation of the Brownian tree. Our calculation then unveils an unexpected identity.

Theorem 5.1.7 *If $(\mathcal{T}_e, d_e, \rho_e, \mu_e)$ is the Brownian tree, then its double Strahler dilation $2\mathfrak{S}_2(\mathcal{T}_e)$ with base 2 has the same law as its height $\mathfrak{h}(\mathcal{T}_e) = \sup_{\sigma \in \mathcal{T}_e} d_e(\rho_e, \sigma)$.*

Organisation of paper. In Section 5.2, we precisely set our framework and formally define our objects of interest. In Section 5.3, we adapt already known results to our setting and we derive classic estimates. We define the weighted Horton–Strahler number and we study its law for stable Galton–Watson weighted trees in Section 5.4. Section 5.5 is devoted to the study of weighted Horton pruning. In Section 5.6, we prove Theorem 5.1.3. A first description of the limit tree in Theorem 5.1.3 is given by Section 5.7. In Section 5.8, we construct the Strahler dilation to show Theorem 5.1.4. We prove Theorems 5.1.5 and 5.1.6 in Section 5.9. Finally, Section 5.10 consists of the proof of Theorem 5.1.7. Throughout all this work, we will write the set of nonnegative real numbers, the set of nonnegative integers, and the set of positive integers respectively as

$$\mathbb{R}_+ = [0, \infty), \quad \mathbb{N} = \{0, 1, 2, 3, \dots\}, \quad \text{and} \quad \mathbb{N}^* = \{1, 2, 3, \dots\}.$$

5.2 Framework, notations, and definitions

In this section, which contains no new results, we set the notations and present the basic objects we will encounter and use throughout the paper.

5.2.1 Topological framework

Throughout this work, we study convergences so let us first present the topologies that we use.

Càdlàg functions with compact support. We write $\mathbb{R}_+ = [0, \infty)$ and we denote by $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ the space of all right-continuous with left limits (*càdlàg* for short) functions from \mathbb{R}_+ to \mathbb{R} . The space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is equipped with the Skorokhod (J1) topology which makes it *Polish*, namely separable and completely metrizable. We refer to Billingsley [22, Chapter 3] and Jacod & Shiryaev [95, Chapter VI] for background. Let $f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$, we define its *lifetime* $\zeta(f)$ and its η -*modulus of continuity* $\omega_\eta(f)$ as

$$\zeta(f) = \sup \{0\} \cup \{s \geq 0 : f(s) \neq 0\} \quad \text{and} \quad \omega_\eta(f) = \sup_{\substack{s_1, s_2 \geq 0 \\ |s_1 - s_2| \leq \eta}} |f(s_1) - f(s_2)| \quad (5.2.1)$$

for all $\eta > 0$. We say that f has *compact support* when $\zeta(f) < \infty$. Since we are also interested in the convergences of lifetimes of càdlàg functions with compact support, it will be useful to work on the following subspaces of the product space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}_+$:

$$\begin{aligned} \mathcal{D}_K &= \{(f, \ell) : \ell \in \mathbb{R}_+ \text{ and } f : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ càdlàg such that } \zeta(f) \leq \ell\}, \\ \mathcal{C}_K &= \{(f, \ell) : \ell \in \mathbb{R}_+ \text{ and } f : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ continuous such that } \zeta(f) \leq \ell\}. \end{aligned}$$

Next, we define the uniform distance d_∞ and the Skorokhod distance d_S on \mathcal{D}_K by setting

$$d_\infty((f_1, \ell_1), (f_2, \ell_2)) = |\ell_1 - \ell_2| + \sup_{s \geq 0} |f_1(s) - f_2(s)|, \quad (5.2.2)$$

$$d_S((f_1, \ell_1), (f_2, \ell_2)) = |\ell_1 - \ell_2| + \inf_{\psi} \sup_{s \geq 0} (|\psi(s) - s| + |f_1(\psi(s)) - f_2(s)|) \quad (5.2.3)$$

where the infimum is taken over all increasing and bijective functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, for all $(f_1, \ell_1), (f_2, \ell_2) \in \mathcal{D}_K$. The proposition below gathers useful properties of \mathcal{C}_K and \mathcal{D}_K .

Proposition 5.2.1 *The following holds true.*

- (i) \mathcal{D}_K is a closed subset of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}_+$, and \mathcal{C}_K is a closed subset of \mathcal{D}_K .
- (ii) The spaces \mathcal{C}_K and \mathcal{D}_K are Polish.
- (iii) The topology of \mathcal{D}_K is induced by the distance d_S .
- (iv) The distances d_∞ and d_S are topologically equivalent on \mathcal{C}_K .

Proof. Let $(f_n, \ell) \in \mathcal{D}_K$ such that $f_n \longrightarrow f$ for the Skorokhod topology and $\ell_n \longrightarrow \ell$. If $\ell < s$ then $f_n(s) = f_n(s-) = 0$ for all n large enough, so $f(s) = 0$ by [95, Proposition 2.1, Chapter VI]. Thus, $\zeta(f) \leq \ell < \infty$. This same proposition entails that if the f_n are continuous then $f(s) = f(s-)$ for all $s \in \mathbb{R}_+$. This proves (i). Finite products and closed subsets of Polish spaces are Polish so (ii) follows from (i). The point (iii) is a consequence of [95, Theorem 1.14, Chapter VI]. Then, (iv) follows from [95, Proposition 1.17, Chapter VI]. ■

Notation 5.2.2 We identify any càdlàg function with compact support f with $(f, \zeta(f)) \in \mathcal{D}_K$. Hence, a sequence (f_n) of càdlàg functions with compact support converges to (f, ℓ) on \mathcal{D}_K if and only if $f_n \rightarrow f$ for the Skorokhod topology and $\zeta(f_n) \rightarrow \ell$. We point out that ℓ and $\zeta(f)$ do not need to be equal a priori. However, we will say that (f_n) converges to f on \mathcal{D}_K when $f_n \rightarrow f$ for the Skorokhod topology and $\zeta(f_n) \rightarrow \zeta(f)$. \square

We now provide a straightforward adaptation of the classic criterion of tightness for random continuous functions, see [22, Theorem 7.3].

Proposition 5.2.3 *A sequence (ν_n) of distributions on \mathcal{C}_K is tight if and only if it satisfies*

$$0 = \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \nu_n(\{(f, \ell) \in \mathcal{C}_K : |f(0)| \geq m\}), \quad (a)$$

$$0 = \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \nu_n(\{(f, \ell) \in \mathcal{C}_K : \ell \geq m\}), \quad (b)$$

$$0 = \lim_{\eta \rightarrow 0^+} \limsup_{n \rightarrow \infty} \nu_n(\{(f, \ell) \in \mathcal{C}_K : \omega_\eta(f) \geq \varepsilon\}) \text{ for all } \varepsilon > 0. \quad (c)$$

Proof. Let $L \in \mathbb{R}_+$, let \mathbb{C}_L be the space of continuous functions from $[0, L]$ to \mathbb{R} endowed with the uniform topology, and let A be a compact subset of \mathbb{C}_L . The points (i) and (iv) of Proposition 5.2.1 yield that $A_L = \{(f, \ell) : \ell \leq L, f \in A\}$ is a compact subset of \mathcal{C}_K . Conversely, we see that any compact subset of \mathcal{C}_K is contained in some A_L . We end the proof with [22, Theorem 7.3]. \blacksquare

Furthermore, we denote by ρ_S the Prokhorov metric associated with d_S on the space $\mathcal{P}(\mathcal{D}_K)$ of all Borel probability measures on \mathcal{D}_K . Namely, for all $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{D}_K)$, we define

$$\rho_S(\nu_1, \nu_2) = \inf\{\varepsilon > 0 : \forall A \subset \mathcal{D}_K \text{ Borel subset, } \nu_1(A) \leq \nu_2(A^\varepsilon) + \varepsilon\} \quad (5.2.4)$$

where $A^\varepsilon = \{x \in \mathcal{D}_K : \exists a \in A, d_S(x, a) < \varepsilon\}$. The space \mathcal{D}_K is Polish so [22, Theorem 6.8] gives us the following result.

Proposition 5.2.4 *The space $\mathcal{P}(\mathcal{D}_K)$ equipped with the topology of weak convergence is Polish. This topology is induced ρ_S , which is indeed a distance on $\mathcal{P}(\mathcal{D}_K)$.*

Remark 5.2.5 While \mathcal{D}_K is Polish, we stress that d_S is not complete. See [22, Example 12.2]. In particular, ρ_S is not complete either (consider e.g. a sequence of Dirac probability measures). \square

Rooted Gromov–Hausdorff–Prokhorov distance. We define almost the same distances and we follow the same presentation as in [5, Section 2.1]. Let $n \in \mathbb{N}$ with $n \geq 1$. We say that (E, d, \mathbf{a}) is a *n-pointed compact metric space* when (E, d) is a compact metric space endowed with a sequence of n distinguished points $\mathbf{a} = (a_1, \dots, a_n)$ of E . A *n-pointed measured compact metric space* (E, d, \mathbf{a}, μ) is a n -pointed compact metric space (E, d, \mathbf{a}) equipped with a finite Borel measure μ on E . We say two n -pointed compact metric spaces (E, d, \mathbf{a}) and (E', d', \mathbf{a}') are *n-pointed-isometric* when there exists a bijective isometry ϕ from E to E' such that $\phi(a_i) = a'_i$ for all $1 \leq i \leq n$. Moreover, two n -pointed measured compact metric spaces (E, d, \mathbf{a}, μ) and $(E', d', \mathbf{a}', \mu')$ are said to be *n-GHP-isometric* when there exists a bijective isometry ϕ from E to E' such that $\phi(a_i) = a'_i$ for all $1 \leq i \leq n$ and such that the image measure of μ by ϕ is equal to μ' . We denote by \mathbb{K}_n the space of n -pointed-isometry classes of n -pointed compact metric spaces, and by \mathbb{K}_n^m the space of n -GHP-isometry classes of n -pointed measured compact metric spaces.

Notation 5.2.6 When no confusion is possible, we will simply denote such a n -pointed (measured) compact metric space (E, d, \mathbf{a}, μ) , as well as its class in \mathbb{K}_n or \mathbb{K}_n^m , by its underlying space E . \square

A n -pointed correspondence between E and E' is a subset \mathcal{R} of $E \times E'$ with $(a_i, a'_i) \in \mathcal{R}$ for all $1 \leq i \leq n$ and such that for all $x \in E$ and $y' \in E'$, there are $x' \in E'$ and $y \in E$ such that (x, x') and (y, y') are in \mathcal{R} . The *distortion* of a n -pointed correspondence \mathcal{R} is defined by

$$\text{dis}(\mathcal{R}) = \sup \{ |d(x, y) - d'(x', y')| : (x, x') \in \mathcal{R} \text{ and } (y, y') \in \mathcal{R} \}. \quad (5.2.5)$$

The n -pointed Gromov–Hausdorff distance between E and E' is then expressed by the formula

$$\mathfrak{d}_{n\text{-GH}}(E, E') = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}), \quad (5.2.6)$$

where the infimum is taken over all n -pointed correspondences \mathcal{R} between E and E' . We may also restrict the infimum to compact n -pointed correspondences without modifying the value. Indeed, the closure of a n -pointed correspondence is a compact n -pointed correspondence that has the same distortion because $E \times E'$ is compact. For any finite Borel measure ν on $E \times E'$, the *discrepancy* of ν with respect to μ and μ' is defined by

$$\text{Dsp}(\nu ; \mu, \mu') = \sup_{\substack{B \subset E \\ \text{Borel subset}}} |\nu(B \times E') - \mu(B)| + \sup_{\substack{B' \subset E' \\ \text{Borel subset}}} |\nu(E \times B') - \mu'(B')|.$$

The n -pointed Gromov–Hausdorff–Prokhorov distance between E and E' is then expressed as

$$\mathfrak{d}_{n\text{-GHP}}(E, E') = \inf_{\mathcal{R}, \nu} \max \left(\frac{1}{2} \text{dis}(\mathcal{R}), \text{Dsp}(\nu ; \mu, \mu') + \nu((E \times E') \setminus \mathcal{R}) \right), \quad (5.2.7)$$

where the infimum is taken over all finite Borel measures ν on $E \times E'$ and all compact n -pointed correspondences \mathcal{R} between E and E' . The objects $\mathfrak{d}_{n\text{-GH}}$ and $\mathfrak{d}_{n\text{-GHP}}$ are only pseudo-distances but $\mathfrak{d}_{n\text{-GH}}(E, E') = 0$ if and only if E and E' are n -pointed-isometric, and $\mathfrak{d}_{n\text{-GHP}}(E, E') = 0$ if and only if E and E' are n -GHP-isometric. Hence, they respectively define genuine distances on \mathbb{K}_n and \mathbb{K}_n^m . Furthermore, the metric spaces $(\mathbb{K}_n, \mathfrak{d}_{n\text{-GH}})$ and $(\mathbb{K}_n^m, \mathfrak{d}_{n\text{-GHP}})$ are separable and complete. We refer to Abraham, Delmas & Hoscheit [4, Theorem 2.5] for proof. While they define the distances in terms of isometric embeddings, the expressions (5.2.6) and (5.2.7) in terms of correspondences give the same objects. See e.g. Khezeli [105, Theorem 3.5].

Notation 5.2.7 When $n = 1$, we shall omit it in the notations and replace the adjective pointed by *rooted*. The unique distinguished point shall be called the *root* of the space. \square

A rooted measured compact metric space is also a rooted compact metric space by forgetting its measure. This is formalized by the continuous surjection $(E, d, \rho, \mu) \in \mathbb{K}^m \mapsto (E, d, \rho) \in \mathbb{K}$. Moreover, endowing a rooted compact metric space with its null measure describes an isometric embedding of \mathbb{K} into \mathbb{K}^m . Let (E, d, ρ, μ) be a rooted measured compact metric space and let $\lambda \geq 0$. We will rescale rooted measured compact metric spaces in various ways by setting

$$\lambda \cdot (E, d, \rho, \mu) = (E, \lambda d, \rho, \mu) \quad \text{and} \quad \lambda \odot_{\alpha} (E, d, \rho, \mu) = (E, \lambda d, \rho, \lambda^{\alpha/(\alpha-1)} \mu) \quad (5.2.8)$$

for all $\alpha \in (1, 2]$. These operations are continuous from \mathbb{K}^m to itself and they coincide into one continuous map from \mathbb{K} to itself. Furthermore, we define the *height* $\mathfrak{h}(E)$ and the *mass* $\mathfrak{m}(E)$ of a rooted (measured) compact metric space respectively as

$$\mathfrak{h}(E, d, \rho) = \sup_{x \in E} d(\rho, x) \quad \text{and} \quad \mathfrak{m}(E, d, \rho, \mu) = \mu(E). \quad (5.2.9)$$

The functions $\mathfrak{h} : \mathbb{K} \rightarrow [0, \infty)$ and $\mathfrak{m} : \mathbb{K}^m \rightarrow [0, \infty)$ are Lipschitz and thus continuous.

5.2.2 Discrete trees as sets of words

Words and trees. We recall Ulam's formalism of trees. Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of positive integers and let \mathbb{U} be the set of finite words

$$\mathbb{U} = \bigcup_{n \in \mathbb{N}} (\mathbb{N}^*)^n \quad \text{with the convention } (\mathbb{N}^*)^0 = \{\emptyset\}. \quad (5.2.10)$$

The *lexicographic order*, denoted by \leq , is a total order on \mathbb{U} . For $u = (u_1, \dots, u_n) \in \mathbb{U}$ and $v = (v_1, \dots, v_m) \in \mathbb{U}$, the notation $u * v = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ stands for the *concatenation* of u and v . We denote by $|u| = n$ the *height* of u , and if $n \geq 1$ then we denote by $\overleftarrow{u} = (u_1, \dots, u_{n-1})$ the *parent* of u . We then say that u is a *child* of v when $\overleftarrow{u} = v$. The *genealogical order* \preceq is a partial order on \mathbb{U} defined by $u \preceq v \iff \exists u' \in \mathbb{U}, v = u * u'$. We say that u is an ancestor of v when $u \preceq v$. When $u \preceq v$ but $u \neq v$, we may write $u \prec v$. Observe that for all $u \in \mathbb{U}$, the set $\{v \in \mathbb{U} : v \preceq u\}$ of ancestors of u is totally ordered by \preceq . Eventually, we denote by $u \wedge v \in \mathbb{U}$ the *most recent common ancestor* of u and v , that is their common ancestor with maximal height.

Notation 5.2.8 Although $\overleftarrow{\emptyset}$ is not defined, we write that $\overleftarrow{\emptyset} < u$ and $\overleftarrow{\emptyset} \prec u$ for all $u \in \mathbb{U}$. \square

Definition 5.2.9 A subset t of \mathbb{U} is a *tree* when the following is verified:

- (a) t is finite and $\emptyset \in t$,
- (b) for all $u \in t$, if $u \neq \emptyset$ then $\overleftarrow{u} \in t$,
- (c) for all $u \in t$, there exists an integer $k_u(t) \in \mathbb{N}$ such that $u * (i) \in t \iff 1 \leq i \leq k_u(t)$.

We denote by \mathbb{T} the (countable) space of all trees, endowed with the discrete topology. \square

Several times in this work, we will need to embed trees into others, in the following sense.

Definition 5.2.10 Let t be a tree and let $A \subset \mathbb{U}$. Let $\psi : t \rightarrow A$ be an injective map.

- (i) We say that ψ is an *embedding* when $\psi(u \wedge v) = \psi(u) \wedge \psi(v)$ for all $u, v \in t$.
- (ii) We say that ψ is *increasing* when $u < v \implies \psi(u) < \psi(v)$ for all $u, v \in t$.

If ψ is an embedding, then we set $\psi(\overleftarrow{\emptyset}) = \overleftarrow{\emptyset}$ in accordance with Notation 5.2.8. \square

Remark 5.2.11 Let $A \subset \mathbb{U}$. If A satisfies (a) and (b) in Definition 5.2.9, then there exist a unique tree t and a unique increasing embedding $\psi : t \rightarrow A$ such that $\psi(t) = A$. \square

Let t be a tree. The number $\#t$ of its vertices is also called the size of t . We use the following notations for the *height* of t and its set of *leaves*:

$$|t| = \max_{u \in t} |u| \quad \text{and} \quad \partial t = \{u \in t : k_u(t) = 0\}, \quad (5.2.11)$$

and $\#\partial t$ stands for the number of leaves of t . For all $v \in t$, we define the subtree of t stemming from v as

$$\theta_v t = \{u \in \mathbb{U} : v * u \in t\}.$$

Observe that $\theta_v t$ is also a tree. We list all the elements of t in the lexicographic order as

$$\emptyset = u_0(t) < \dots < u_{\#t-1}(t), \quad (5.2.12)$$

and we call the finite sequence of words $u(t) = (u_i(t) ; 0 \leq i \leq \#t - 1)$ the *depth-first exploration* of t . We denote by $H(t) : s \in [0, \infty) \mapsto H_s(t)$ and we call the *height function* of t the affine-by-parts and continuous function defined by

$$H_s(t) = H_i(t) + (s - i)(H_{i+1}(t) - H_i(t)) \quad \text{and} \quad H_i(t) = \begin{cases} |u_i(t)| & \text{if } 0 \leq i \leq \#t - 1, \\ 0 & \text{if } i \geq \#t, \end{cases} \quad (5.2.13)$$

for all integers $i \in \mathbb{N}$ and for all real numbers $s \in [i, i + 1]$. It is folklore, and easy to see, that the height function fully characterizes the corresponding tree. The function $H(t)$ is nonnegative and has compact support. In accordance with Notation 5.2.2, we shall identify it with the element $(H(t), \zeta(H(t)))$ of \mathcal{C}_K . Let us also observe that

$$H_0(t) = 0, \quad \sup H(t) = |t|, \quad \text{and} \quad \zeta(H(t)) = \mathbf{1}_{\{\#t \geq 2\}} \#t. \quad (5.2.14)$$

Stable Galton–Watson trees. Recall that $\mu_\alpha = (\mu_\alpha(k))_{k \in \mathbb{N}}$ is the probability measure on \mathbb{N} characterized by the expression (5.1.2) of its generating function φ_α . We easily compute from (5.1.2) that

$$\varphi'_\alpha(s) = 1 - (1 - s)^{\alpha-1} \quad \text{and} \quad \varphi_\alpha^{(m)}(s) = \mu_\alpha(m) \cdot m!(1 - s)^{\alpha-m}, \quad (5.2.15)$$

for all $s \in [0, 1)$ and $m \geq 2$. By induction, the identities (5.1.2) and (5.2.15) yield the expressions

$$\mu_\alpha(0) = \frac{1}{\alpha}, \quad \mu_\alpha(1) = 0, \quad \mu_\alpha(2) = \frac{\alpha-1}{2}, \quad \text{and} \quad \mu_\alpha(k) = \frac{1}{k!} \prod_{i=1}^{k-1} |\alpha - i| = \mathbf{1}_{\{\alpha < 2\}} \frac{(\alpha-1)\Gamma(k-\alpha)}{\Gamma(2-\alpha)k!} \quad (5.2.16)$$

for all $k \geq 3$, where Γ stands for the usual Gamma function. In particular, we recover that μ_2 is the critical binary offspring distribution. Let $\mu = (\mu(k))_{k \in \mathbb{N}}$ be another probability measure on \mathbb{N} . We say that μ is *critical and non-trivial* when

$$\mu(0) > 0 \quad \text{and} \quad \sum_{k \in \mathbb{N}} k\mu(k) = 1. \quad (5.2.17)$$

We readily observe from (5.2.15) and (5.2.16) that μ_α is critical and non-trivial.

Definition 5.2.12 Let μ be a probability measure on \mathbb{N} which verifies (5.2.17). A *Galton–Watson tree with offspring law μ* (or *GW(μ)-tree* for short) is a random (finite) tree τ that satisfies the following.

- (a) The law of $k_\emptyset(\tau)$ is μ .
- (b) For all $k \in \mathbb{N}^*$ such that $\mu(k) > 0$, the random trees $\theta_{(1)}\tau, \dots, \theta_{(k)}\tau$ under $\mathbb{P}(\cdot | k_\emptyset(\tau) = k)$ are independent with the same law as τ under \mathbb{P} .

When $\mu = \mu_\alpha$ as in (5.1.2), we call τ an α -*stable Galton–Watson tree* (or a *GW $_\alpha$ -tree* for short). \square

It is well-known that the assumption (5.2.17) ensures that a random (finite) tree described by Definition 5.2.12 indeed exists. Moreover, see e.g Le Gall [78, Propositions 1.4, 1.5], if τ is a GW(μ)-tree then

$$\forall t \in \mathbb{T}, \quad \mathbb{P}(\tau = t) = \prod_{u \in t} \mu(k_u(t)), \quad (5.2.18)$$

and there exists a sequence $(\xi_i)_{i \in \mathbb{N}}$ of independent random variables distributed as $k_\emptyset(\tau)$ such that

$$(k_{u_i(\tau)}(\tau) ; 0 \leq i \leq \#\tau - 1) = (\xi_i ; 0 \leq i \leq \#\tau - 1) \text{ almost surely.} \quad (5.2.19)$$

Trees with edge lengths. We say that $T = (t, (l_u)_{u \in t})$ is a *tree with edge lengths* when t is a tree, as in Definition 5.2.9, and $l_u \in (0, \infty)$ for all $u \in t$. We denote by $\overline{\mathbb{T}}$ the space of all trees with edge lengths and we endow it with the product topology, meaning that $(t^n, (l_u^n)_{u \in t^n}) \rightarrow (t, (l_u)_{u \in t})$ if and only if there is $n_0 \geq 0$ such that $t^n = t$ for all $n \geq n_0$ and $l_u^n \rightarrow l_u$ for all $u \in t$. This makes $\overline{\mathbb{T}}$ a Polish space. Recall from (5.2.11) that $|t|$ stands for the height of t . We denote

$$\text{Sha}(T) = t, \quad \#T = \#t, \quad |T| = |t|, \quad K(T) = k_\emptyset(t), \quad L(T) = l_\emptyset, \quad \text{and} \quad \mathcal{S}(T) = \mathcal{S}(t), \quad (5.2.20)$$

where \mathcal{S} is the Horton–Strahler number as in Definition 5.1.1. Then, for all $v \in t$ and $\lambda > 0$, we set

$$\theta_v T = (\theta_v t, (l_{v*u})_{u \in \theta_v t}) \quad \text{and} \quad \lambda \cdot T = (t, (\lambda l_u)_{u \in t}). \quad (5.2.21)$$

Remark that $\theta_v T$ and $\lambda \cdot T$ are also trees with edge lengths. Recall from (5.2.12) that $u(t)$ is the depth-first exploration of t . The tree with edge lengths T is described by a nonnegative càdlàg function with compact support $\bar{H}(T) = (\bar{H}_s(T))_{s \geq 0}$, called its *height function* and defined by

$$\bar{H}_s(T) = \begin{cases} \sum_{v \prec u_i(t)} l_v + s - \sum_{j=0}^{i-1} l_{u_j(t)} & \text{if } \sum_{j=0}^{i-1} l_{u_j(t)} \leq s < \sum_{j=0}^i l_{u_j(t)}, \quad i \leq \#t - 1 \\ 0 & \text{if } \sum_{j=0}^{\#t-1} l_{u_j(t)} \leq s \end{cases} \quad (5.2.22)$$

for all $s \geq 0$. We stress that $\bar{H}(T)$ is not continuous so $\bar{H}(T)$ should not be confused with $H(t)$. Written with Notation 5.2.2, the map $T \in \overline{\mathbb{T}} \mapsto \bar{H}(T) \in \mathcal{D}_K$ is continuous. Observe from (5.2.1), (5.2.21), and (5.2.22) that it holds

$$\zeta(\bar{H}(T)) = \sum_{u \in t} l_u \quad \text{and} \quad \bar{H}_s(\lambda \cdot T) = \lambda \bar{H}_{s/\lambda}(T). \quad (5.2.23)$$

One can interpret the height function as follows. Let us draw T into the upper half-plane with its correct edge lengths and let us picture a particle that, starting at \emptyset , explores T at unit speed, from left to right, without hitting two times the same point, and backtracking only after reaching a leaf. Then, $\bar{H}_s(T)$ is equal to the distance of the particle from \emptyset at time s . From the point of view of the particle, a single edge or a chain of several edges put back to back with the same total length is the same, which translates into the following result.

Proposition 5.2.13 *For any tree with edge lengths $T = (t, (l_u)_{u \in t})$, there exist a unique tree t' and a unique increasing embedding $\psi : t' \rightarrow t$ such that $\psi(t') = \{u \in t : k_u(t) \neq 1\}$. Moreover, if T' is the tree with edge lengths defined below, then it holds that $\bar{H}(T') = \bar{H}(T)$:*

$$T' = (t', (l'_u)_{u \in t'}) \quad \text{where} \quad l'_u = \sum_{\psi(\bar{u}) \prec v \preceq \psi(u)} l_v.$$

Proof. It is straightforward to do the proof by induction, by considering, for example, the \leq -minimal vertex $v \in t$ such that $k_v(t) \neq 1$. We leave it as an exercise for the reader. \blacksquare

Definition 5.2.14 Let μ be a probability measure on \mathbb{N} which verifies (5.2.17). A $\overline{\text{GW}}(\mu)$ -tree with edge lengths is random tree with edge lengths $\mathcal{T} = (\tau, (L_u)_{u \in \tau})$ such that τ is a $\text{GW}(\mu)$ -tree and conditionally given τ , the $(L_u)_{u \in \tau}$ are independent with exponential distribution with mean 1. When $\mu = \mu_\alpha$ as given by (5.1.2), we call \mathcal{T} a $\overline{\text{GW}}_\alpha$ -tree with edge lengths. \square

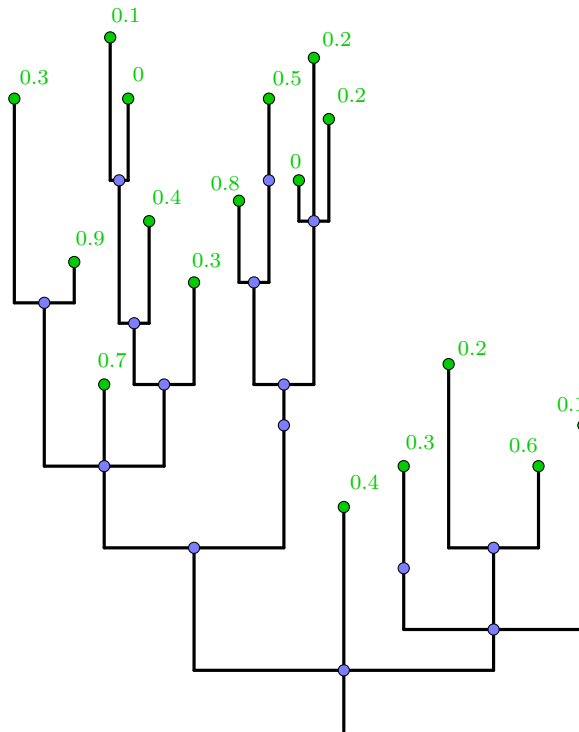


Figure 5.3: A weighted tree with edge lengths \mathbf{T} such that $\text{Sha}(\mathbf{T})$ is the same as in Figure 5.2. The leaves and their weights are green, the other vertices are blue. The edge lengths are given vertically.

Weighted trees. We say that $\mathbf{t} = (t, (w_v)_{v \in \partial t})$ is a *weighted tree* when t is a tree and $w_v \in [0, 1)$ for all $v \in \partial t$, where ∂t stands for the set of leaves of t as in (5.2.11). We denote by \mathbb{T}_w the space of all weighted trees and we endow it the product topology, meaning that $\mathbf{t}^n = (t^n, (w_v^n)_{v \in \partial t^n}) \rightarrow \mathbf{t}$ if and only if there is $n_0 \geq 0$ such that $t^n = t$ for all $n \geq n_0$ and $w_v^n \rightarrow w_v$ for all $v \in \partial t$. This makes \mathbb{T}_w a Polish space. Recall from Definition 5.1.2 that \mathcal{S} stands for the weighted Horton–Strahler number, and observe that the function $\mathcal{S} : \mathbb{T}_w \rightarrow \mathbb{R}_+$ is continuous. For all $u \in t$, we set

$$\theta_u \mathbf{t} = (\theta_u t, (w_{u*v})_{v \in \partial \theta_u t}),$$

which is a well-defined weighted tree because $\partial \theta_u t = \{v \in \theta_u t : u * v \in \partial t\}$. Recall the distribution $\text{FExp}(a)$ from (5.1.9), and the notation $\gamma = \ln \frac{\alpha}{\alpha-1}$ from (5.1.8).

Definition 5.2.15 An α -stable Galton–Watson weighted tree (or a \mathbf{GW}_α -weighted tree for short) is a random weighted tree $\tau = (\tau, (W_v)_{v \in \partial \tau})$ such that τ is a \mathbf{GW}_α -tree and conditionally given τ , the $(W_v)_{v \in \partial \tau}$ are independent of distribution $\text{FExp}(\gamma)$. Thus, the law of τ satisfies the following.

- (a) The law of $k_\emptyset(\tau)$ is μ_α , as characterized by (5.1.2).
- (b) Under $\mathbb{P}(\cdot | k_\emptyset(\tau) = 0)$, the law of W_\emptyset is $\text{FExp}(\gamma)$.
- (c) For all $k \in \mathbb{N}^*$ such that $\mu_\alpha(k) > 0$, the random weighted trees $\theta_{(1)} \tau, \dots, \theta_{(k)} \tau$ under $\mathbb{P}(\cdot | k_\emptyset(\tau) = k)$ are independent with the same law as τ under \mathbb{P} .

□

Weighted trees with edge lengths. We say that $\mathbf{T} = (t, (l_u)_{u \in t}, (w_v)_{v \in \partial t})$ is a *weighted tree with edge lengths* when $T = (t, (l_u)_{u \in t})$ is a tree with edge lengths and $w_v \in [0, 1)$ for all $v \in \partial t$. We denote by $\overline{\mathbb{T}}_w$ the space of all weighted trees with edge lengths, and we endow it with the product topology, meaning that $\mathbf{T}_n = (t^n, (l_u^n)_{u \in t^n}, (w_v^n)_{v \in \partial t^n}) \rightarrow \mathbf{T}$ if and only if there is $n_0 \geq 0$ such that $t^n = t$ for all $n \geq n_0$, $l_u^n \rightarrow l_u$ for all $u \in t$, and $w_v^n \rightarrow w_v$ for all $v \in \partial t$. In other words, $\mathbf{T}_n \rightarrow \mathbf{T}$ on $\overline{\mathbb{T}}_w$ if and only if $T_n \rightarrow T$ on $\overline{\mathbb{T}}$ and $(t^n, (w_v^n)_{v \in \partial t^n}) \rightarrow (t, (w_v)_{v \in \partial t})$ on \mathbb{T}_w . This makes $\overline{\mathbb{T}}_w$ a Polish space. To lighten the notations, we write

$$\text{Sha}(\mathbf{T}) = (t, (w_v)_{v \in \partial t}) \quad \text{and} \quad \mathcal{S}(\mathbf{T}) = \mathcal{S}(\text{Sha}(\mathbf{T})). \quad (5.2.24)$$

Moreover, for all $u' \in t$ and $\lambda > 0$, we set

$$\theta_{u'} \mathbf{T} = (\theta_{u'} t, (l_{u'*u})_{u \in \theta_{u'} t}, (w_{u'*v})_{v \in \partial \theta_{u'} t}) \quad \text{and} \quad \lambda \cdot \mathbf{T} = (t, (\lambda l_u)_{u \in t}, (w_v)_{v \in \partial t}). \quad (5.2.25)$$

Remark that $\text{Sha}(\mathbf{T})$ is a weighted tree, and that $\theta_{u'} \mathbf{T}$ and $\lambda \cdot \mathbf{T}$ are weighted trees with edge lengths.

Definition 5.2.16 A $\overline{\text{GW}}_\alpha$ -*weighted tree with edge lengths* is a random weighted tree with edge lengths $\mathcal{T} = (\tau, (L_u)_{u \in \tau}, (W_v)_{v \in \partial \tau})$ such that $\mathcal{T} = (\tau, (L_u)_{u \in \tau})$ is a $\overline{\text{GW}}_\alpha$ -tree with edge lengths and conditionally given \mathcal{T} , the $(W_v)_{v \in \partial \tau}$ are independent of distribution $\text{FExp}(\gamma)$. Thus, the law of \mathcal{T} satisfies the following.

- (a) $K(\mathcal{T})$ and $L(\mathcal{T})$ are independent, of respective laws μ_α and exponential with mean 1.
- (b) Under $\mathbb{P}(\cdot | K(\mathcal{T}) = 0)$, W_\emptyset is independent from $L(\mathcal{T})$ and has law $\text{FExp}(\gamma)$.
- (c) For all $k \in \mathbb{N}^*$ with $\mu_\alpha(k) > 0$, under $\mathbb{P}(\cdot | K(\mathcal{T}) = k)$, the random weighted trees with edge lengths $\theta_{(1)} \mathcal{T}, \dots, \theta_{(k)} \mathcal{T}$ are independent, jointly independent from $L(\mathcal{T})$, and have the same law as \mathcal{T} under \mathbb{P} .

□

5.2.3 Real trees

In this work, we are mostly interested in the following type of tree-like metric spaces.

Definition 5.2.17 A metric space (T, d) is a *real tree* when for all $x, y \in T$:

- (a) there exists a unique isometry $g_{x,y} : [0, d(x, y)] \rightarrow T$ with $g_{x,y}(0) = x$ and $g_{x,y}(d(x, y)) = y$, which is called the *geodesic from x to y* ,
- (b) all injective continuous functions $h : [0, 1] \rightarrow T$ with $h(0) = x$ and $h(1) = y$ (that we call *arcs from x to y*) share the same image, denoted by $\llbracket x, y \rrbracket = h([0, 1]) = g_{x,y}([0, d(x, y)])$.

We say a rooted (measured) compact metric space (T, d, ρ, μ) is a *rooted (measured) compact real tree* when (T, d) is a real tree. We denote by $\mathbb{T}_\mathbb{R}$ (resp. by $\mathbb{T}_\mathbb{R}^m$) the space of isometry classes of rooted (measured) compact real trees equipped with d_{GH} as in (5.2.6) (resp. with d_{GHP} as in (5.2.7)). □

The spaces $\mathbb{T}_\mathbb{R}$ and $\mathbb{T}_\mathbb{R}^m$ are respectively closed subsets of \mathbb{K} and \mathbb{K}^m , see e.g. Evans [70, Lemma 4.22], so they are separable and complete metric spaces.

Real trees coded by continuous excursions. We define the set of *continuous excursions* as

$$\mathcal{E}_\mathbb{K} = \{(f, \ell) \in \mathcal{C}_\mathbb{K} : \forall s \geq 0, f(s) \geq 0 = f(0)\}. \quad (5.2.26)$$

Endowed with the distance d_∞ defined by (5.2.2), it is a closed subspace of \mathcal{C}_K and so a Polish space thanks to Proposition 5.2.1. Let $(f, \ell) \in \mathcal{E}_K$, we set

$$d_f(s_1, s_2) = d_f(s_2, s_1) = f(s_1) + f(s_2) - 2 \inf_{[s_1, s_2]} f$$

for all $s_2 \geq s_1 \geq 0$. The function d_f is a continuous pseudo-distance on \mathbb{R}_+ . Writing $s_1 \sim_f s_2$ when $d_f(s_1, s_2) = 0$ defines an equivalence relation on \mathbb{R}_+ . It induces the quotient space

$$\mathcal{T}_{f,\ell} = [0, \ell] / \sim_f$$

on which d_f induces a genuine distance that we denote by $d_{f,\ell}$. By continuity of d_f on $[0, \ell]^2$, the canonical projection $\mathfrak{p}_{f,\ell} : [0, \ell] \rightarrow \mathcal{T}_{f,\ell}$ is continuous and the metric space $(\mathcal{T}_{f,\ell}, d_{f,\ell})$ is compact. Moreover, one can check that $\mathcal{T}_{f,\ell}$ is a real tree, see e.g. Le Gall [78, Theorem 2.2]. We extend $\mathfrak{p}_{f,\ell}$ to \mathbb{R}_+ by setting $\mathfrak{p}_{f,\ell}(s) = \mathfrak{p}_{f,\ell}(\ell)$ when $s \geq \ell$. Finally, we write $\rho_{f,\ell} = \mathfrak{p}_{f,\ell}(0)$ and we denote by $\mu_{f,\ell}$ the image measure by $\mathfrak{p}_{f,\ell}$ of the Lebesgue measure on $[0, \ell]$. We call

$$(\mathcal{T}_{f,\ell}, d_{f,\ell}, \rho_{f,\ell}, \mu_{f,\ell}) \tag{5.2.27}$$

the (rooted measured compact) real tree coded by (f, ℓ) . Still in accordance with Notation 5.2.2, if $\ell = \zeta(f)$ then we shall simply write $(\mathcal{T}_f, d_f, \rho_f, \mu_f) = (\mathcal{T}_{f,\zeta(f)}, d_{f,\zeta(f)}, \rho_{f,\zeta(f)}, \mu_{f,\zeta(f)})$. Recall the operations defined by (5.2.8) and (5.2.9). We point out that if $g(s) = \lambda^{1-1/\alpha} f(s/\lambda)$ for all $s \geq 0$ with some $\lambda > 0$, then $\mathcal{T}_g = \lambda^{1-1/\alpha} \odot_\alpha \mathcal{T}_f$. Observe that $\mathfrak{h}(\mathcal{T}_{f,\ell}) = \sup f$ and $\mathfrak{m}(\mathcal{T}_{f,\ell}) = \ell$. The map $(f, \ell) \in \mathcal{E}_K \mapsto \mathcal{T}_{f,\ell} \in \mathbb{T}_{\mathbb{R}}^m$ is continuous because it holds

$$d_{\text{GHP}}(\mathcal{T}_{f_1,\ell_1}, \mathcal{T}_{f_2,\ell_2}) \leq |\ell_1 - \ell_2| + 2 \sup_{s \geq 0} |f_1(s) - f_2(s)| \leq 2d_\infty((f_1, \ell_1), (f_2, \ell_2)). \tag{5.2.28}$$

Indeed, we consider the compact rooted correspondence $\mathcal{R} = \{(\mathfrak{p}_{f_1,\ell_1}(s), \mathfrak{p}_{f_2,\ell_2}(s)) : s \geq 0\}$ and the image measure by $(\mathfrak{p}_{f_1,\ell_1}, \mathfrak{p}_{f_2,\ell_2})$ of the Lebesgue measure on the segment $[0, \min(\ell_1, \ell_2)]$.

Scaling limits of discrete trees. Any tree t (as in Definition 5.2.9) may be rooted at \emptyset , equipped with its counting measure $\sum_{u \in t} \delta_u$, and endowed with the graph distance on \mathbb{U} ,

$$\forall u, v \in \mathbb{U}, \quad d_{\text{gr}}(u, v) = |u| + |v| - 2|u \wedge v|, \tag{5.2.29}$$

to obtain a rooted measured compact metric space. This describes a continuous injection from the discrete space of trees \mathbb{T} into \mathbb{K}^m : we thus see \mathbb{T} as a closed subset of \mathbb{K}^m with a slight abuse of notation. We stress that (discrete) trees are not real trees because they are not connected. However, linking each vertex of t to its parent by a metric segment of unit length yields a genuine real tree. Moreover, the latter is classically encoded by the so-called contour function of t . We informally present this function just below and we refer to Duquesne [50] for a rigorous definition.

Let us picture a particle that, starting at the root, continuously walks through the tree so that it retraces its steps as little as possible and respects the lexicographic order of the vertices. The particle crosses each edge twice, once upward and once downward, so it takes $2(\#t - 1)$ steps to return to the root after having explored the whole tree. We say that the particle follows the *exploration by contour* of the tree t and we denote by $c(t) = (c_i(t); 0 \leq i \leq 2(\#t - 1))$ the sequence of its positions on t . Then, the *contour function* of t is the function $C(t) : s \in \mathbb{R}_+ \mapsto C_s(t)$ defined by

$$C_s(t) = C_i(t) + (s - i)(C_{i+1}(t) - C_i(t)) \quad \text{and} \quad C_i(t) = \begin{cases} |c_i(t)| & \text{if } 0 \leq i \leq 2(\#t - 1) \\ 0 & \text{if } i \geq 2\#t - 1 \end{cases}$$

for all integers $i \in \mathbb{N}$ and $s \in [i, i+1]$. The slopes of this piecewise affine function are in $\{1, -1, 0\}$. Moreover, it is an element of \mathcal{E}_K such that $\sup C(t) = |t|$ and $\zeta(C(t)) = 2\#t - 2$.

Proposition 5.2.18 *Let $(h, \ell) \in \mathcal{C}_K$. Let t_n be a tree and let $\lambda_n, b_n > 0$ for all $n \in \mathbb{N}$. We assume $\lambda_n \rightarrow 0$ and $\lambda_n/b_n \rightarrow \infty$. Recall from (5.2.13) that $H(t_n)$ is the height function of t_n . The two convergences*

$$\left(\lambda_n H_{s/b_n}(t_n)\right)_{s \geq 0} \longrightarrow (h, \ell) \quad \text{and} \quad \left(\lambda_n C_{2s/b_n}(t_n)\right)_{s \geq 0} \longrightarrow (h, \ell)$$

are equivalent on \mathcal{C}_K . If they hold, then $(h, \ell) \in \mathcal{E}_K$ and $(t, \lambda_n \mathbf{d}_{\text{gr}}, \emptyset, b_n \sum_{u \in t} \delta_u) \longrightarrow \mathcal{T}_{h, \ell}$ for the rooted Gromov–Hausdorff–Prokhorov distance.

Proof. The equivalence of the convergences follows from (5.2.14), Proposition 5.2.1 (iv), and general deterministic arguments of Duquesne & Le Gall [61, Section 2.5] that show the height and the contour functions are similar once rescaled. The metric of a tree t can be recovered from $C(t)$ via

$$d_{C(t)}(i, j) = C_i(t) + C_j(t) - 2 \inf_{[i, j]} C(t) = \mathbf{d}_{\text{gr}}(c_i(t), c_j(t)) \quad (5.2.30)$$

for all integers $0 \leq i \leq j \leq 2(\#t - 1)$, see e.g. Le Gall [78, Section 2]. Let $\lambda, b > 0$, we define a continuous excursion $f \in \mathcal{E}_K$ and a rooted measured compact metric space $T \in \mathbb{K}^m$ by setting $f : s \in \mathbb{R}_+ \mapsto \lambda C_{2s/b}(t)$ and $T = (t, \lambda \mathbf{d}_{\text{gr}}, \emptyset, b \sum_{u \in t} \delta_u)$. Thanks to (5.2.30), it is straightforward to check that $\mathbf{d}_{\text{GH}}(\mathcal{T}_f, T) \leq \lambda$ and $\mathbf{d}_{\text{GHP}}(\mathcal{T}_f, T) \leq \lambda + b$. Then, (5.2.28) completes the proof. ■

Genealogical order on a rooted compact real tree. Let (T, d, ρ) be a rooted compact real tree and let $\sigma_1, \sigma_2 \in T$. From Definition 5.2.17, recall that $g_{\sigma_1, \sigma_2} : [0, d(\sigma_1, \sigma_2)] \longrightarrow T$ is the geodesic between σ_1 and σ_2 and $\llbracket \sigma_1, \sigma_2 \rrbracket = g_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)])$ is the unique path between σ_1 and σ_2 . The equality $d(\sigma_1, \sigma_2) = d(\sigma_1, \sigma) + d(\sigma, \sigma_2)$ thus holds for all $\sigma \in \llbracket \sigma_1, \sigma_2 \rrbracket$. The *genealogical order* \preceq on (T, d, ρ) is the partial order defined by

$$\sigma_1 \preceq \sigma_2 \quad \text{if and only if} \quad \sigma_1 \in \llbracket \rho, \sigma_2 \rrbracket. \quad (5.2.31)$$

We then say that σ_1 is an *ancestor* of σ_2 . We may also write $\sigma_1 \prec \sigma_2$ when $\sigma_1 \preceq \sigma_2$ and $\sigma_1 \neq \sigma_2$. It is immediate that $\rho \preceq \sigma$ for all $\sigma \in T$. While \preceq is only a partial order, it induces a total order on the *ancestral lineages*, namely the $\llbracket \rho, \sigma \rrbracket$ for all $\sigma \in T$. Indeed, if σ_1, σ_2 are ancestors of σ then $\sigma_i = g_{\rho, \sigma}(d(\rho, \sigma_i))$ for both $i \in \{1, 2\}$. By the uniqueness of geodesics, it follows that

$$\text{if } \sigma_1, \sigma_2 \preceq \sigma \quad \text{then} \quad \sigma_1 \preceq \sigma_2 \iff d(\rho, \sigma_1) \leq d(\rho, \sigma_2) \iff d(\sigma_2, \sigma) \leq d(\sigma_1, \sigma). \quad (5.2.32)$$

The geodesics are embeddings of real segments into T so the ancestral lineages are closed in T . Hence, the subset $\llbracket \rho, \sigma_1 \rrbracket \cap \llbracket \rho, \sigma_2 \rrbracket$ admits a unique \preceq -maximal element. We call it the *most recent common ancestor* of σ_1 and σ_2 and we denote it by $\sigma_1 \wedge \sigma_2$. It is characterized by the assertion

$$\forall \sigma \in T, \quad \sigma \preceq \sigma_1 \wedge \sigma_2 \quad \text{if and only if} \quad (\sigma \preceq \sigma_1 \text{ and } \sigma \preceq \sigma_2). \quad (5.2.33)$$

In particular, $\sigma_1 \preceq \sigma_2$ if and only if $\sigma_1 \wedge \sigma_2 = \sigma_1$. By (5.2.33), we check that \wedge is associative, namely $\sigma_1 \wedge (\sigma_2 \wedge \sigma_3) = (\sigma_1 \wedge \sigma_2) \wedge \sigma_3$ for all $\sigma_3 \in T$. Plus, Definition 5.2.17 (b) yields that

$$\{\sigma_1 \wedge \sigma_2\} = \llbracket \rho, \sigma_1 \rrbracket \cap \llbracket \rho, \sigma_2 \rrbracket \cap \llbracket \sigma_1, \sigma_2 \rrbracket. \quad (5.2.34)$$

By definition of real trees, they are locally pathwise connected and it follows that if $\sigma_1 \wedge \sigma_2 \neq \sigma_1, \sigma_2$ then σ_1 and σ_2 are in different connected components of $T \setminus \{\sigma_1 \wedge \sigma_2\}$.

Definition 5.2.19 A rooted compact real tree (T, d, ρ) is said to be *planted* when $T \setminus \{\rho\}$ is connected. This is equivalent to saying that for all $\sigma_1, \sigma_2 \in T$, $\rho = \sigma_1 \wedge \sigma_2 \implies \rho \in \{\sigma_1, \sigma_2\}$. □

Next, we show that the operation $\wedge : T^2 \rightarrow T$ enjoys a fairly strong continuity property.

Lemma 5.2.20 *Let (T, d, ρ) be a rooted compact real tree and let $\sigma_1, \sigma_2, \sigma'_1, \sigma'_2$ be four points of T . If $d(\sigma'_1, \sigma_1) < d(\sigma_1 \wedge \sigma_2, \sigma_1)$ and $d(\sigma'_2, \sigma_2) < d(\sigma_1 \wedge \sigma_2, \sigma_2)$, then $\sigma'_1 \wedge \sigma'_2 = \sigma_1 \wedge \sigma_2$.*

Proof. We first assume $\sigma_2 = \sigma'_2$. We have $d(\sigma_1 \wedge \sigma'_1, \sigma_1) \leq d(\sigma'_1, \sigma_1)$ because $\sigma_1 \wedge \sigma'_1 \in \llbracket \sigma_1, \sigma'_1 \rrbracket$. Plus, $\sigma_1 \wedge \sigma'_1$ and $\sigma_1 \wedge \sigma_2$ are both ancestors of σ_1 , so $\sigma_1 \wedge \sigma_2 \prec \sigma_1 \wedge \sigma'_1 \preceq \sigma'_1$ by (5.2.32). We compute

$$(\sigma'_1 \wedge \sigma_1) \wedge (\sigma'_1 \wedge \sigma_2) = (\sigma_1 \wedge \sigma_2) \wedge (\sigma'_1 \wedge \sigma'_1) = \sigma_1 \wedge \sigma_2 \neq \sigma'_1 \wedge \sigma_1. \quad (5.2.35)$$

The points $\sigma'_1 \wedge \sigma_1$ and $\sigma'_1 \wedge \sigma_2$ are both ancestors of σ'_1 , so (5.2.32) and (5.2.35) imply that $\sigma_1 \wedge \sigma_2 = \sigma'_1 \wedge \sigma_2$. The general result follows after a double application of the case $\sigma_2 = \sigma'_2$. ■

We stress that the order \preceq and the operation \wedge depend on (T, d, ρ) , and especially on the root ρ , but we shall omit this dependence when no confusion is possible according to context. We end this section with the notion of subtrees of a rooted compact real tree.

Definition 5.2.21 Let (T_1, d_1, ρ_1) and (T_2, d_2, ρ_2) be two rooted compact real trees. We say that T_1 is a *subtree* of T_2 when there is an isometry $\phi : T_1 \rightarrow T_2$ such that $\phi(\rho_1) \preceq \phi(\sigma)$ for all $\sigma \in T_1$. □

With the setting of Definition 5.2.21, we observe $g_{\phi(\sigma), \phi(\sigma')} = \phi \circ g_{\sigma, \sigma'}$ for all $\sigma, \sigma' \in T_1$, by uniqueness of geodesics. The fact that $\phi(\rho_1)$ is a \preceq -minimum of the image of ϕ then implies that $\phi(\sigma \wedge \sigma') = \phi(\sigma) \wedge \phi(\sigma')$ for all $\sigma, \sigma' \in T_1$.

5.2.4 Grafting rooted compact metric spaces

Here, we borrow a grafting procedure from Abraham, Delmas & Hoscheit [3] in order to later describe the limit real tree in Theorem 5.1.3. Informally, it consists in gluing the roots of some rooted (measured) compact metric spaces E_i onto respective points a_i of the same space E^o .

Definition 5.2.22 (Grafting procedure) Let $(E^o, d^o, \rho^o, \mu^o)$ be a rooted measured compact metric space and let $(E_i, d_i, \rho_i, \mu_i)_{i \in I}$ be a countable family of rooted measured compact metric spaces. Let $(a_i)_{i \in I}$ be a family of points of E^o also indexed by I . We denote by E the disjoint union $E^o \sqcup \bigsqcup_{i \in I} E_i \setminus \{\rho_i\}$ and we endow it with a distance d defined as follows:

- (a) if $x, y \in E^o$ then $d(x, y) = d^o(x, y)$,
- (b) if $x, y \in E_i \setminus \{\rho_i\}$ with some $i \in I$ then $d(x, y) = d_i(x, y)$,
- (c) if $x \in E^o$ and $y \in E_i \setminus \{\rho_i\}$ with some $i \in I$ then $d(x, y) = d^o(x, a_i) + d_i(\rho_i, y)$,
- (d) if $x \in E_i \setminus \{\rho_i\}$ and $y \in E_j \setminus \{\rho_j\}$ with some distinct $i, j \in I$, then we set $d(x, y) = d_i(x, \rho_i) + d^o(a_i, a_j) + d_j(\rho_j, y)$.

With a slight abuse of notations, we may and will see E^o and the E_i as closed subsets of E by using the identifications $a_i = \rho_i$ for all $i \in I$. In accordance with Notation 5.2.6, we denote the rooted metric space (E, d, ρ^o) by $E^o \otimes_{i \in I} (a_i, E_i)$. Note that $E^o \otimes_{i \in I} (a_i, E_i)$ may not be compact.

If the spaces E^o and E_i for all $i \in I$ are respectively endowed with finite Borel measures μ^o and μ_i for all $i \in I$, then we also equip E with a (potentially infinite) Borel measure μ by setting

$$\mu(B) = \mu^o(B \cap E^o) + \sum_{i \in I} \mu_i(B \cap E_i)$$

for any Borel subset B of E . We still denote (E, d, ρ^o, μ) by $E^o \otimes_{i \in I} (a_i, E_i)$. □

To write a finite number of successive graftings onto the same space, we will also use the notation

$$E^o \otimes_{i \in I} (a_i, E_i) \otimes_{j \in J} (a_j, E_j) := (E^o \otimes_{i \in I} (a_i, E_i)) \otimes_{j \in J} (a_j, E_j) = E^o \otimes_{i \in I \sqcup J} (a_i, E_i).$$

Proposition 5.2.23 *We keep the notations of Definition 5.2.22. Recall \mathfrak{h} and \mathfrak{m} from (5.2.9). If it holds*

$$\forall \varepsilon > 0, \quad \{i \in I : \mathfrak{h}(Y_i) \geq \varepsilon\} \text{ is finite,} \quad (5.2.36)$$

then $E^o \otimes_{i \in I} (a_i, E_i)$ is a rooted compact metric space. If in addition to (5.2.36) it holds

$$\sum_{i \in I} \mathfrak{m}(E_i) < \infty, \quad (5.2.37)$$

then $E^o \otimes_{i \in I} (a_i, E_i)$ is a rooted measured compact metric space (namely its measure is finite).

Proof. Let (x_j) be a sequence of points of $E = E^o \otimes_{i \in I} (a_i, E_i)$, let us show it admits a convergent subsequence. Since E^o and the E_i are compact, we can assume that $x_j \in E_{i_j}$ for all $j \geq 0$, where (i_j) is an injective sequence of elements of I . Then, it holds that $\mathfrak{h}(E_{i_j})$ tends to 0 so we have $d(x_j, a_{i_j}) \rightarrow 0$. The sequence (a_{i_j}) stays inside the compact E^o so it admits a subsequential limit, and so does (x_j) . The second statement of the proposition follows from the fact that μ^o is a finite measure and from the identity $\mu(E) = \mu^o(E^o) + \sum_{i \in I} \mathfrak{m}(E_i)$. ■

Proposition 5.2.24 *Recall from (5.2.7) that $\mathbb{K}_n^{\mathfrak{m}}$ is endowed with the n -pointed Gromov–Hausdorff–Prokhorov distance. For all $n \geq 1$, the following map from $\mathbb{K}_{n+1}^{\mathfrak{m}} \times (\mathbb{K}^{\mathfrak{m}})^n$ to $\mathbb{K}^{\mathfrak{m}}$ is continuous:*

$$(E^o, d^o, (\rho^o, a_1, \dots, a_n), \mu^o), (E_1, d_1, \rho_1, \mu_1), \dots, (E_n, d_n, \rho_n, \mu_n) \mapsto E^o \otimes_{i=1}^n (a_i, E_i).$$

Proof. We give ourselves a compact $(n+1)$ -pointed correspondence \mathcal{R}^o between E^o and $E^{o'}$, and compact rooted correspondences \mathcal{R}_i respectively between E_i and E_i' . We also take finite Borel measures ν^o and ν_i respectively on $E^o \times E^{o'}$ and $E_i \times E_i'$. Then, we construct a compact rooted correspondence between $E^o \otimes_{i=1}^n (a_i, E_i)$ and $E^{o'} \otimes_{i=1}^n (a_i', E_i')$, and a finite Borel measure on the product space $(E^o \otimes_{i=1}^n (a_i, E_i)) \times (E^{o'} \otimes_{i=1}^n (a_i', E_i'))$ by setting

$$\mathcal{R} = \mathcal{R}^o \sqcup \bigsqcup_{i=1}^n \mathcal{R}_i \setminus \{(\rho_i, \rho_i')\} \quad \text{and} \quad \nu(\cdot) = \nu^o(\cdot \cap E^o \times E^{o'}) + \sum_{i=1}^n \nu_i(\cdot \cap E_i \times E_i').$$

Taking the infimum over $(\mathcal{R}^o, \mathcal{R}_i, \nu^o, \nu_i)$ yields the result since we can check that the distortion of \mathcal{R} (resp. discrepancy of ν) is bounded by the sum of the other distortions (resp. discrepancies). ■

We end this section by describing what happens when the grafting procedure only involves real trees.

Proposition 5.2.25 *Let (T^o, d^o, ρ^o) be a rooted compact real tree and let $(a_i)_{i \in I}$ be a countable family of points of T^o . Let $(T_i, d_i, \rho_i)_{i \in I}$ be a countable family of rooted compact real trees such that (5.2.36) holds. Then, $T = T^o \otimes_{i \in I} (a_i, T_i)$ is a rooted compact real tree. We precise if the notations of Definition 5.2.17 and (5.2.33) stand for the objects on T , T^o , or T_i for some $i \in I$ by respectively writing $\llbracket \cdot, \cdot \rrbracket$ and \wedge , $\llbracket \cdot, \cdot \rrbracket^o$ and \wedge^o , or $\llbracket \cdot, \cdot \rrbracket_i$ and \wedge_i . Then, for all $\sigma_1, \sigma_2 \in T$ and distinct $i, j \in I$:*

- (i) if $\sigma_1, \sigma_2 \in T^o$, then $\llbracket \sigma_1, \sigma_2 \rrbracket = \llbracket \sigma_1, \sigma_2 \rrbracket^o$ and $\sigma_1 \wedge \sigma_2 = \sigma_1 \wedge^o \sigma_2$,
- (ii) if $\sigma_1, \sigma_2 \in T_i$ with some $i \in I$, then $\llbracket \sigma_1, \sigma_2 \rrbracket = \llbracket \sigma_1, \sigma_2 \rrbracket_i$ and $\sigma_1 \wedge \sigma_2 = \sigma_1 \wedge_i \sigma_2$,
- (iii) if $\sigma_1 \in T^o$ and $\sigma_2 \in T_i$, then $\llbracket \sigma_1, \sigma_2 \rrbracket = \llbracket \sigma_1, a_i \rrbracket^o \cup \llbracket \rho_i, \sigma_2 \rrbracket_i$ and $\sigma_1 \wedge \sigma_2 = \sigma_1 \wedge^o a_i$,

(iv) if $\sigma_1 \in T_i$ and $\sigma_2 \in T_j$, then $[\sigma_1, \sigma_2] = [\sigma_1, \rho_i]_i \cup [a_i, a_j]^o \cup [\rho_j, \sigma_2]_j$ and $\sigma_1 \wedge \sigma_2 = a_i \wedge^o a_j$.

Proof. The proof is straightforward so we only give a sketch here, leaving the details to the reader. From Definition 5.2.22, we first check that concatenating the geodesics on T^o, T_i, T_j as instructed by the desired proposition indeed yields a geodesic from σ_1 to σ_2 on T . Next, we observe that T_j and $T_j \setminus \{a_j\}$ are respectively closed and open in T , so any path starting outside and ending inside T_j has to hit a_j . By injectivity, an arc from σ_1 to σ_2 on T is thus the concatenation of at most three arcs (respectively on T^o, T_i, T_j) as indicated by the statement, which determines its image. ■

5.2.5 Stable Lévy processes

We now present the continuous setting involved in the asymptotics of large stable Galton–Watson trees. We denote by X the canonical process on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions endowed with the Skorokhod topology, already introduced in Section 5.2.1. Under the underlying probability measure \mathbb{P} , we assume that $X = (X_s)_{s \geq 0}$ is a spectrally positive stable Lévy process with index $\alpha \in (1, 2]$ that is normalized such that

$$\forall \lambda, s \in \mathbb{R}_+, \quad \mathbb{E}[\exp(-\lambda X_s)] = \exp(s\lambda^\alpha).$$

If $\alpha = 2$ then $\frac{1}{\sqrt{2}}X$ is a standard Brownian motion under \mathbb{P} . If $\alpha \in (1, 2)$, the Lévy measure of X is

$$\Pi(dr) = \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} r^{-1-\alpha} \mathbf{1}_{(0, \infty)}(r) dr,$$

where Γ is the usual Gamma function. Moreover, the process X enjoys the following scaling property: for all $\lambda > 0$, the laws of $(\lambda^{-1/\alpha} X_{\lambda s}; s \geq 0)$ and X under \mathbb{P} are the same. We write

$$I_s = \inf_{[0, s]} X \quad \text{and} \quad I_s^r = \inf_{[r, s]} X$$

for all $0 \leq r \leq s$. Observe that the process $I = (I_s)_{s \geq 0}$ is continuous since X has no negative jumps. We refer to Bertoin [19, Chapter VIII] for background and details.

The process $X - I$ is strong Markov and the point 0 is regular for itself with respect to $X - I$ (see [19, Chapter VI.1]). Moreover, we may and will choose $-I$ as the local time of $X - I$ at level 0 by [19, Theorem VII.1]. Let $(g_j, d_j), j \in J$ be the excursions intervals of $X - I$ above 0 and let us set $\omega_s^j = X_{\min(g_j+s, d_j)} - X_{g_j}$ for all $j \in J$ and $s \geq 0$. Since $\inf X = -\infty$ (see [19, Chapter VIII]), the ω^j are càdlàg functions with compact support which start at 0 and stay nonnegative. In accordance with Notation 5.2.2, we see them as elements of \mathcal{D}_K . Then, the random point measure

$$\mathcal{N} = \sum_{j \in J} \delta_{(-I_{g_j}, \omega^j)}$$

is a Poisson measure on $\mathbb{R}_+ \times \mathcal{D}_K$ with intensity measure $dt \mathbf{N}_\alpha(d\omega)$, where \mathbf{N}_α is a sigma-finite measure on \mathcal{D}_K called the *excursion measure*. We refer to [19, Chapter IV] for details and background. Under \mathbf{N}_α , we simply denote by $\zeta = \zeta(X)$ the lifetime (5.2.1) of X .

Le Gall & Le Jan [81] (see also Duquesne & Le Gall [61, Chapter 1]) constructed a continuous process $H = (H_s)_{s \geq 0}$ which is measurable with respect to X and such that for all $s \in \mathbb{R}_+$, the limit

$$H_s = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{X_r < I_s^r + \varepsilon\}} dr \tag{5.2.38}$$

holds in \mathbb{P} -probability and in \mathbf{N}_α -measure. When $\alpha = 2$, we know explicitly that H is equal to $X - I$ under \mathbb{P} and to X under \mathbf{N}_2 . The process H is called the *height process associated with X* (and also the α -stable height process here) because it is the continuous analog of the height function (5.2.13) of discrete Galton–Watson trees. Observe that (5.2.38) and the scaling property of X imply that for all $\lambda > 0$, the laws of $(\lambda^{1/\alpha-1}H_{\lambda s}; s \geq 0)$ and H under \mathbb{P} are the same. Indeed, the former is the height process associated with $(\lambda^{-1/\alpha}X_{\lambda s}; s \geq 0)$. Moreover, (5.2.38) entails that \mathbf{N}_α -almost everywhere, it holds $\zeta(H) = \zeta$ and $H \in \mathcal{E}_K$, where the space $\mathcal{E}_K \subset \mathcal{C}_K$ is defined by (5.2.26). See [61, Chapter 1] for a precise proof.

A classic result of fluctuation theory (see e.g. [19, Chapter VIII.4]) asserts $\mathbf{N}_\alpha(\zeta = 0) = 0$ and

$$\forall \ell > 0, \quad \mathbf{N}_\alpha(\zeta > \ell) = \frac{1}{\Gamma(1 - 1/\alpha)} \ell^{-1/\alpha}. \quad (5.2.39)$$

The scaling property of H yields that the law of the process with unit lifetime $(\zeta^{1/\alpha-1}H_{\zeta s}; s \geq 0)$ under the probability measure $\mathbf{N}_\alpha(\cdot | \zeta > \ell)$ does not depend on $\ell > 0$. It is called the law of the *normalized excursion of the α -stable height process*, and we denote it by $\mathbf{N}_\alpha(dH | \zeta = 1)$ here. Informally, it can be understood as the law of an excursion of H conditioned to have unit lifetime. In particular, if $\alpha = 2$ then $\mathbf{N}_2(dH | \zeta = 1)$ is the law of $\sqrt{2}e$, where e stands for the standard Brownian excursion. Moreover, (5.2.39) entails that for all nonnegative and measurable function $F : \mathcal{C}_K \rightarrow \mathbb{R}_+$, it holds that

$$\mathbf{N}_\alpha[F(H)] = \frac{1}{\alpha\Gamma(1 - 1/\alpha)} \int_{\mathbb{R}_+} \mathbf{N}_\alpha[F(\ell^{1-1/\alpha}H_{s/\ell}; s \geq 0) | \zeta = 1] \frac{d\ell}{\ell^{1+1/\alpha}}. \quad (5.2.40)$$

Finally, a random rooted (measured) compact real tree \mathcal{T}_{nr} is called an α -stable tree when it is distributed as \mathcal{T}_H under $\mathbf{N}_\alpha(dH | \zeta = 1)$, as defined by (5.2.27). Up to the multiplicative constant $\sqrt{2}$, the 2-stable tree is the Brownian tree introduced by Aldous [7, 9], which is the rooted compact real tree coded by the standard Brownian excursion.

Next, recall $\beta = \frac{1}{\alpha-1}$ from (5.1.8) and observe $\mathbf{N}_\alpha(\sup H = 0) = \mathbf{N}_\alpha(\zeta = 0) = 0$. A consequence of a Ray–Knight theorem, see Duquesne & Le Gall [61, Corollary 1.4.2], further asserts that

$$\forall x > 0, \quad \mathbf{N}_\alpha(\sup H > x) = \frac{1}{(\alpha - 1)^\beta} x^{-\beta}. \quad (5.2.41)$$

Let us write $M = \sup H$ to lighten the notations. Just as before, the scaling property ensures that the law of $(M^{-1}H_{sM^{\alpha\beta}}; s \geq 0)$ under the probability measure $\mathbf{N}_\alpha(\cdot | \sup H > x)$ is the same for all $x > 0$: we denote it by $\mathbf{N}_\alpha(dH | \sup H = 1)$. Moreover, that independence and the identity (5.2.41) yield that for all nonnegative and measurable function $F : \mathcal{C}_K \rightarrow \mathbb{R}_+$, it holds that

$$\mathbf{N}_\alpha[F(H)] = \frac{1}{(\alpha - 1)^{\alpha\beta}} \int_{\mathbb{R}_+} \mathbf{N}_\alpha[F(xH_{sx^{-\alpha\beta}}; s \geq 0) | \sup H = 1] \frac{dx}{x^{1+\beta}}. \quad (5.2.42)$$

If $\alpha = 2$ then $\mathbf{N}_2(dH | \sup H = 1)$ is equal to the law of $(e_{2s}^*)_{s \geq 0}$, where e^* is the Brownian excursion conditioned to have its maximum equal to 1. A Brownian excursion with a fixed maximum, such as e^* , can be described by William’s path decomposition at the maximum, see e.g. Revuz & Yor [138, Chapter XII]. This then yields a simple expression for the Laplace transform of its lifetime:

$$\forall \lambda > 0, \quad \mathbb{E}[\exp(-\lambda\zeta(e^*))] = \mathbf{N}_2[\exp(-2\lambda\zeta) | \sup H = 1] = \left(\frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda}} \right)^2. \quad (5.2.43)$$

See [104] for a review of other formulas resulting from this method.

5.3 Preliminary tools

This section focuses on gathering already-known results, and on proving technical estimates via classic methods. Although not genuinely new nor directly related to the Horton–Strahler number, these tools will be useful throughout this paper.

5.3.1 Marchal’s algorithm

Our proof of Theorem 5.1.6 is based on the recursive algorithm proposed by Marchal [128] to build a sequence of nested GW_α -trees conditioned on their number of leaves that converges, after scaling, towards the α -stable tree. This sequence is a Markov chain on the space of *labeled but unrooted trees*: namely connected and acyclic graphs equipped with an exhaustive enumeration of their vertices of degree 1. Here, we present a slight variation of this construction to produce a sequence $(\mathbf{t}_i)_{i \geq 1}$ of random weighted trees in the sense of the present paper (see Section 5.2.2). See Figure 5.4 for a better understanding of the recursive construction. Recall the law $\text{FExp}(\gamma)$ from (5.1.8) and (5.1.9).

Algorithm 5.3.1 *Let $(W_i)_{i \geq 1}$ be a sequence of independent random variables with common law $\text{FExp}(\gamma)$. We start by setting $\mathbf{t}_1 = (\{\emptyset\}, W_1)$. To construct \mathbf{t}_{i+1} from \mathbf{t}_i , we randomly choose either a vertex $u \in \mathbf{t}_i$ or an edge $\{\overleftarrow{u}, u\}$ of \mathbf{t}_i – including the conventional edge $\{\overleftarrow{\emptyset}, \emptyset\}$ – respectively with probability proportional to $p_u = \mathbf{1}_{\{k_u(\mathbf{t}_i) \geq 2\}}(k_u(\mathbf{t}_i) - \alpha)$ or to $p_{\{\overleftarrow{u}, u\}} = \alpha - 1$.*

(i) *If we drew an edge $\{\overleftarrow{u}, u\}$ then we split it into two edges with a middle vertex, on which we connect a new leaf endowed with the weight W_{i+1} . The position of the new leaf compared to its sibling is chosen uniformly. Written with the words formalism, we take a uniform random variable $J \in \{1, 2\}$ and \mathbf{t}_{i+1} is defined as the unique weighted tree such that:*

- (a) *for all $v \in \mathbb{U}$ such that $u \wedge v \notin \{u, v\}$, $v \in \mathbf{t}_{i+1}$ if and only if $v \in \mathbf{t}_i$, in which case it holds $\theta_v \mathbf{t}_{i+1} = \theta_v \mathbf{t}_i$,*
- (b) *it holds $u \in \mathbf{t}_{i+1}$ and $k_u(\mathbf{t}_{i+1}) = 2$,*
- (c) *it holds $\theta_{u^*(J)} \mathbf{t}_{i+1} = (\{\emptyset\}, W_{i+1})$ and $\theta_{u^*(3-J)} \mathbf{t}_{i+1} = \theta_u \mathbf{t}_i$.*

(ii) *If we drew a vertex u then we connect a new leaf on it that we endow with the weight W_{i+1} . The position of the new leaf among its siblings is chosen uniformly. Written with the words formalism, we take a uniform random variable $J \in \{1, \dots, k_u(\mathbf{t}_i) + 1\}$ and \mathbf{t}_{i+1} is defined as the unique weighted tree such that:*

- (a) *for all $v \in \mathbb{U}$ such that $u \wedge v \notin \{u, v\}$, $v \in \mathbf{t}_{i+1}$ if and only if $v \in \mathbf{t}_i$, in which case it holds $\theta_v \mathbf{t}_{i+1} = \theta_v \mathbf{t}_i$,*
- (b) *it holds $u \in \mathbf{t}_{i+1}$ and $k_u(\mathbf{t}_{i+1}) = k_u(\mathbf{t}_i) + 1$,*
- (c) *it holds $\theta_{u^*(J)} \mathbf{t}_{i+1} = (\{\emptyset\}, W_{i+1})$, and $\theta_{u^*(j)} \mathbf{t}_{i+1} = \theta_{u^*(j)} \mathbf{t}_i$ for all $1 \leq j \leq J - 1$, and $\theta_{u^*(j)} \mathbf{t}_{i+1} = \theta_{u^*(j-1)} \mathbf{t}_i$ for all $J + 1 \leq j \leq k_u(\mathbf{t}_{i+1})$.*

Throughout this algorithm, all the choices are made independently from the rest. □

Proposition 5.3.2 *Let τ be a GW_α -weighted tree as in Definition 5.2.15. Let $(\mathbf{t}_i)_{i \geq 1}$ be the sequence of random weighted trees constructed by Algorithm 5.3.1. For all $i \geq 1$, the following holds.*

- (i) $\#\partial \mathbf{t}_i = i$ and $\#\mathbf{t}_i + 1 \leq \#\mathbf{t}_{i+1}$.
- (ii) $\mathcal{S}(\mathbf{t}_i) \leq \mathcal{S}(\mathbf{t}_{i+1})$.

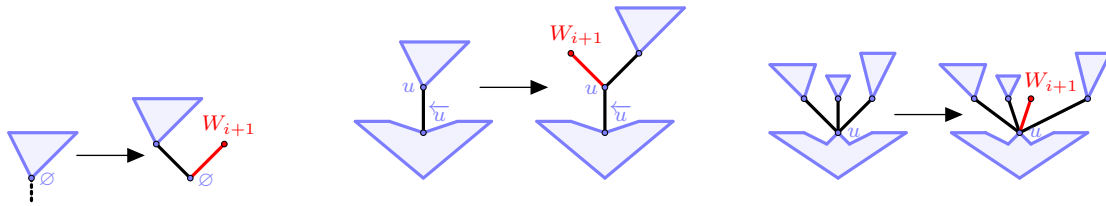


Figure 5.4: Illustration of the recursive construction of \mathfrak{t}_{i+1} from \mathfrak{t}_i . Left : The case where the conventional parental edge of the root was chosen. Middle : The case where a genuine edge was chosen. Right : The case where a vertex was chosen.

(iii) The law of \mathfrak{t}_i under \mathbb{P} is the same as the law of τ under $\mathbb{P}(\cdot \mid \#\partial\tau = i)$.

Proof. The point (i) is clear. In both cases (i) and (ii) of Algorithm 5.3.1, we have $\mathcal{S}(\theta_u \mathfrak{t}_i) \leq \mathcal{S}(\theta_u \mathfrak{t}_{i+1})$ by Definition 5.1.2. It is then easy to obtain (ii) by a backward induction on the height of the ancestors of u . For all $1 \leq j \leq i$, let us denote by V_j the unique leaf of \mathfrak{t}_i of weight W_j . We observe that all the orders of siblings are equiprobable conditionally given the graph structure of \mathfrak{t}_i (which is the same as the labeled but unrooted trees of Marchal [128]). Therefore, we get from Marchal [128, Section 2.3] and (5.2.16) that if t is a tree with i leaves enumerated in some order as v_1, \dots, v_i , then $\mathbb{P}(\mathfrak{t}_i = t; V_1 = v_1; \dots; V_i = v_i)$ is proportional to $\prod_{u \in t} \mu_\alpha(k_u(t))$. By (5.2.18) and because there are always $i!$ possible enumerations of the leaves of t , the law of \mathfrak{t}_i under \mathbb{P} is the law of τ under $\mathbb{P}(\cdot \mid \#\partial\tau = i)$. The $(W_i)_{i \geq 1}$ are i.i.d and independent from $(\mathfrak{t}_i)_{i \geq 1}$, so (iii) follows. ■

Finally, recall from Section 5.2.5 that an α -stable tree is a random rooted compact real tree distributed as \mathcal{T}_H , as in (5.2.27), under $\mathbb{N}_\alpha(dH \mid \zeta = 1)$. Recall from (5.2.29) the graph distance d_{gr} on trees.

Theorem 5.3.3 Let $(\mathfrak{t}_i)_{i \geq 1}$ be the sequence of random weighted trees built by Algorithm 5.3.1. Then, there exists an α -stable tree $(\mathcal{T}_{\text{nr}}, d_{\text{nr}}, \rho_{\text{nr}})$ such that the convergence

$$(\mathfrak{t}_i, i^{-1+1/\alpha} d_{\text{gr}}, \emptyset) \longrightarrow (\mathcal{T}_{\text{nr}}, \alpha d_{\text{nr}}, \rho_{\text{nr}})$$

holds almost surely for the rooted Gromov–Hausdorff distance as in (5.2.6).

Proof. See Curien & Haas [41, Theorem 5]. Indeed, the graph metric structure of \mathfrak{t}_i does not depend on the chosen orders of siblings or the weights. ■

5.3.2 Limit theorems for stable Galton–Watson trees

In this section, we gather and apply already known estimates and limit theorems about the asymptotic behavior of $\text{GW}(\mu)$ -trees (see Definition 5.2.12) to the case of stable Galton–Watson trees. Most of these results are stated under the following assumptions on the offspring distribution μ .

- (a) μ is critical and non-trivial, meaning that (5.2.17) holds.
- (b) μ is in the domain of attraction of a stable law with index $\theta \in (1, 2]$.
- (c) μ is aperiodic, meaning that μ is not supported by a proper additive subgroup of \mathbb{Z} .

We saw from (5.1.2) that the stable offspring distribution μ_α satisfies (5.2.17). Moreover, it verifies (b) with $\theta = \alpha$ by (5.1.11). This can also be proved directly from (5.1.2) or (5.2.16), see Bingham, Goldies & Teugels [23, Chapter 8.3] for details. By (5.2.16), μ_α satisfies (c) if and only if $\alpha \in (1, 2)$.

Nonetheless, condition (c) is only required to avoid technical complications, as it conveniently ensures that $\mathbb{P}(\#\tau = n) > 0$ when n is large enough for a $\text{GW}(\mu)$ -tree τ , but most results we consider here can be extended to the periodic case. Anyway, μ_2 is explicitly the critical binary law and can be studied with simpler arguments.

Motivated by Marchal's algorithm (Section 5.3.1), we first gather some results due to Kortchemski [107] that express the likeness between conditioning a GW_α -tree by its size or by its number of leaves.

Proposition 5.3.4 *Let τ be a GW_α -tree. Recall from (5.2.11) that $\#\partial\tau$ is its number of leaves.*

(i) *If $\alpha = 2$, then $\#\tau = 2\#\partial\tau - 1$ almost surely.*

(ii) *If $\alpha \in (1, 2)$, then the two following convergences hold in probability:*

$$\frac{\#\tau}{n} \text{ under } \mathbb{P}(\cdot \mid \#\partial\tau = n) \longrightarrow \alpha, \quad (5.3.1)$$

$$\frac{\#\partial\tau}{n} \text{ under } \mathbb{P}(\cdot \mid \#\tau = n) \longrightarrow \frac{1}{\alpha}. \quad (5.3.2)$$

Proof. If $\alpha = 2$, $k_u(\tau) = 2$ for all $u \in \tau \setminus \partial\tau$ by (5.2.16) and (5.2.18). Since each vertex of $\tau \setminus \{\emptyset\}$ has a single parent, we get $\#\tau - 1 = \sum_{u \in \tau \setminus \partial\tau} k_u(\tau) = 2\#\tau - 2\#\partial\tau$, which proves (i). For (5.3.1), see Kortchemski [107, Corollary 3.3] and recall $\mu_\alpha(0) = \frac{1}{\alpha}$ from (5.2.16). For (5.3.2), see [107, Lemma 2.5]. ■

Lemma 5.3.5 *We assume that $\alpha \in (1, 2)$. Let τ be a GW_α -tree. Let $(U_n)_{n \geq 1}$ be a sequence of nonnegative and uniformly bounded functions on the space \mathbb{T} of trees. If $\mathbb{E}[U_n(\tau) \mid \#\tau \geq n]$ converges as n tends to ∞ , then $\mathbb{E}[U_n(\tau) \mid \#\partial\tau \geq n/\alpha - n^{3/4}]$ converges to the same limit.*

Proof. See Kortchemski [107, Proposition 4.4] and recall $\mu_\alpha(0) = \frac{1}{\alpha}$ from (5.2.16). ■

Then, we recall tail estimates on the height, the size, and the number of leaves of a GW_α -tree.

Proposition 5.3.6 *Let τ be a GW_α -tree. Recall from (5.1.8) that $\beta = \frac{1}{\alpha-1}$ and $\gamma = \ln \frac{\alpha}{\alpha-1}$.*

(i) *Recall from (5.2.11) that $|\tau|$ is the height of τ , and $\#\partial\tau$ for its number of leaves. It holds that*

$$n^\beta \mathbb{P}(|\tau| \geq n) \longrightarrow e^{-\gamma\beta}, \quad (5.3.3)$$

$$n^{1/\alpha} \mathbb{P}(\#\tau \geq n) \longrightarrow \frac{\alpha^{1/\alpha}}{\Gamma(1 - 1/\alpha)}, \quad (5.3.4)$$

$$n^{1/\alpha} \mathbb{P}(\#\partial\tau \geq n) \longrightarrow \frac{1}{\Gamma(1 - 1/\alpha)}. \quad (5.3.5)$$

(ii) *If $\alpha = 2$ then it holds that*

$$n^{1+1/\alpha} \mathbb{P}(\#\tau = 2n - 1) = n^{1+1/\alpha} \mathbb{P}(\#\partial\tau = n) \longrightarrow \frac{1}{\alpha\Gamma(1 - 1/\alpha)}. \quad (5.3.6)$$

(iii) *If $\alpha \in (1, 2)$ then it holds that*

$$n^{1+1/\alpha} \mathbb{P}(\#\tau = n) \longrightarrow \frac{\alpha^{1/\alpha}}{\alpha\Gamma(1 - 1/\alpha)}, \quad \text{and} \quad n^{1+1/\alpha} \mathbb{P}(\#\partial\tau = n) \longrightarrow \frac{1}{\alpha\Gamma(1 - 1/\alpha)}. \quad (5.3.7)$$

Proof. For (5.3.3), recall (5.1.2) and see Slack [142, Lemma 2]. For (5.3.4), we follow the proof of Kovchegov, Xu & Zaliapin [109, Proposition 3]. For all $\lambda > 0$, we set $f(\lambda) = \mathbb{E}[\exp(-\lambda\#\tau)]$ and we get from (5.1.2) and Definition 5.2.12 that $e^\lambda f(\lambda) = f(\lambda) + \frac{1}{\alpha}(1 - f(\lambda))^\alpha$. Since τ is finite, $f(0+) = 1$ and

$$\lambda^{-1/\alpha}(1 - f(\lambda)) \xrightarrow{\lambda \rightarrow 0^+} \alpha^{1/\alpha}. \quad (5.3.8)$$

Moreover, Fubini’s theorem allows us to write

$$1 - f(\lambda) = \mathbb{E}[1 - \exp(-\lambda\#\tau)] = \mathbb{E}\left[\lambda \int_0^{\#\tau} e^{-\lambda x} dx\right] = \lambda \int_0^\infty e^{-\lambda x} \mathbb{P}(\#\tau \geq x) dx.$$

Since $x \mapsto \mathbb{P}(\#\tau \geq x)$ is monotone and nonnegative, Karamata’s Tauberian theorem for Laplace transforms (see Feller [74, Chapter XII.5, Theorem 4]) and the convergence (5.3.8) yield (5.3.4). The last convergence (5.3.5) of (i) will directly follow from (ii) and (iii).

Let us prove (ii). We say that a tree t is binary when $k_u(t) \in \{0, 2\}$ for all $u \in t$. A famous combinatorial result (that can be easily proved via generating functions) asserts that there are exactly $C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$ binary trees with n leaves. By (5.2.16), (5.2.18), and Proposition 5.3.4 (i), we then have $\mathbb{P}(\#\tau = 2n - 1) = \mathbb{P}(\#\partial\tau = n) = 2^{1-2n} C_{n-1}$. Stirling’s formula yields (5.3.6) because $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Let us prove (iii). Recall from (5.2.16) that $\mu_\alpha(0) = \frac{1}{\alpha}$ and $\mu_\alpha(k) > 0$ for all $k \geq 2$. By (5.1.11), Kortchemski [107, Lemma 1.11 and Theorem 3.1] thus assert that there is a constant $c_\alpha \in (0, \infty)$ that only depends on α such that $n^{1+1/\alpha} \mathbb{P}(\#\tau = n) \rightarrow c_\alpha$ and $n^{1+1/\alpha} \mathbb{P}(\#\partial\tau = n) \rightarrow c_\alpha \alpha^{-1/\alpha}$. Finally, comparing this with (5.3.4) entails $c_\alpha \Gamma(1 - \frac{1}{\alpha}) = \alpha^{1/\alpha-1}$, and so (5.3.7) follows. ■

Finally, we give ourselves limit theorems about the height function $H(\tau)$, see (5.2.13), of a GW_α -tree τ conditioned to be large. Recall from Section 5.2.1 the space \mathcal{C}_K of continuous functions with compact support and endowed with lifetimes. From Section 5.2.5, recall the excursion measure \mathbf{N}_α , the α -stable height process H , its lifetime ζ , and the law $\mathbf{N}_\alpha(dH \mid \zeta = 1)$ of the normalized excursion of H . An application of the work of Duquesne & Le Gall [61] entails the following result.

Theorem 5.3.7 Recall $\beta = \frac{1}{\alpha-1}$ from (5.1.8). Let τ be a GW_α -tree and let (a_n) be as in (5.1.11). For all $\ell \in (0, \infty)$, the two following convergences hold in distribution on \mathcal{C}_K :

$$\left(\frac{1}{n} H_{n^{\alpha\beta}s}(\tau)\right)_{s \geq 0} \text{ under } \mathbb{P}(\cdot \mid |\tau| \geq \ell n) \xrightarrow{d} \alpha^{1/\alpha} H \text{ under } \mathbf{N}_\alpha(\cdot \mid \sup H > \ell), \quad (5.3.9)$$

$$\left(\frac{a_n}{n} H_{n s}(\tau)\right)_{s \geq 0} \text{ under } \mathbb{P}(\cdot \mid \#\tau \geq \ell n) \xrightarrow{d} H \text{ under } \mathbf{N}_\alpha(\cdot \mid \zeta > \ell). \quad (5.3.10)$$

Proof. By (5.1.11), Duquesne & Le Gall [61, Theorem 2.3.2 and Proposition 2.5.2] assert that the convergence (5.3.9) holds for the Skorokhod topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. In fact, their proof of [61, Proposition 2.5.2] also contains the joint convergence of the lifetimes (see the last paragraph, page 66), so (5.3.9) also holds on \mathcal{C}_K . The same methodology yields (5.3.10): see the concluding remark of Duquesne & Le Gall [61, Section 2.6]. ■

As explained by Duquesne & Le Gall [61], their method cannot be directly applied to find scaling limits of Galton–Watson trees under degenerate conditionings such as $\{\#\tau = n\}$. Nevertheless, Duquesne [50] answered this problem with another method. Then, Kortchemski [106] proved Duquesne’s theorem via an absolute continuity relation between $\mathbb{P}(\cdot \mid \#\tau = n)$ and $\mathbb{P}(\cdot \mid \#\tau \geq n)$.

Theorem 5.3.8 Let τ be a GW_α -tree and let (a_n) be as in (5.1.11). The following convergence holds in distribution on \mathcal{C}_K :

$$\left(\frac{a_n}{n} H_{n s}(\tau)\right)_{s \geq 0} \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{d} H \text{ under } \mathbf{N}_\alpha(dH \mid \zeta = 1).$$

Proof. Recall (5.1.11) and see Duquesne [50, Theorem 3.1]. This theorem is stated under the assumption that the offspring distribution is aperiodic, but as argued by Kortchemski [106], it can be easily extended to the periodic case $\alpha = 2$ as above. ■

We stress that Proposition 5.2.18 then entails the convergence (5.1.12) presented in the introduction.

5.3.3 An estimate for the height function of a Galton–Watson tree with edge lengths

Recall respectively from (5.2.13) and (5.2.22) the height function $H(t)$ of a tree t and the height function $\bar{H}(T)$ of a tree with edge lengths T . The goal of this section is to compare them in the case of a $\overline{\text{GW}}(\mu)$ -tree with edge lengths (see Definition 5.2.14) by proving the following result.

Proposition 5.3.9 *Recall d_S from (5.2.3). Let μ be a critical and non-trivial probability measure on \mathbb{N} , namely satisfying (5.2.17), and let $\mathcal{T} = (\tau, (L_u)_{u \in \tau})$ be a $\overline{\text{GW}}(\mu)$ -tree with edge lengths. Let $\lambda > 0$, we define two random càdlàg functions with compact support X and Y by setting for all $s \in \mathbb{R}_+$,*

$$X_s = \lambda^{1-1/\alpha} \bar{H}_{s/\lambda}(\mathcal{T}) \quad \text{and} \quad Y_s = \lambda^{1-1/\alpha} H_{s/\lambda}(\tau).$$

Then, there are two constants $C, c \in (0, \infty)$ that only depend on μ such that for all $n \geq 1$,

$$\mathbb{P}(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; d_S(X, Y) \geq C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + C\lambda\sqrt{n} \ln n) \leq Cne^{-c(\ln n)^2}.$$

Before proving Proposition 5.3.9, we give ourselves two standard lemmas.

Lemma 5.3.10 (Chernoff bound) *Let $(L_i)_{i \geq 0}$ be a sequence of independent exponential random variables with mean 1. Then, it holds*

$$\forall \lambda > -1, \quad \mathbb{E}[e^{-\lambda L_0}] = \frac{1}{1+\lambda}, \quad (5.3.11)$$

and there is an universal constant $c_{\text{uni}} \in (0, \infty)$ such that for all $n \in \mathbb{N}^$ and $x \geq 0$, it holds*

$$\mathbb{P}\left(\left|n - \sum_{i=0}^{n-1} L_i\right| \geq x\right) \leq 2 \exp\left(-c_{\text{uni}} x \min\left(1, \frac{x}{n}\right)\right).$$

Proof. The identity (5.3.11) is classic. An elementary inequality asserts there is $\eta \in (0, 1)$ such that $\ln(1+x) \geq x - 3x^2/4$ for all $x \in [-\eta, \eta]$. The desired result follows from the Chernoff bound $\mathbb{P}(|Z| \geq x) \leq e^{-\lambda x} \mathbb{E}[e^{\lambda Z} + e^{-\lambda Z}]$ with $\lambda = \eta \min(1, \frac{x}{n})$. ■

Lemma 5.3.11 *Let μ be a probability measure on \mathbb{N} which satisfies (5.2.17), and let τ be a $\text{GW}(\mu)$ -tree. Let (K, J) be a random variable on \mathbb{N}^2 such that $\mathbb{P}(K = k; J = j) = \mathbf{1}_{\{1 \leq j \leq k\}} \mu(k)$ for all $k, j \in \mathbb{N}$.*

(i) *For all $n \in \mathbb{N}$ and for all bounded functions $g_0, \dots, g_{n-1} : \mathbb{N}^2 \rightarrow \mathbb{R}$, it holds*

$$\mathbb{E}\left[\sum_{u \in \tau} \mathbf{1}_{\{|u|=n\}} \prod_{\substack{v \in \tau, j \geq 1 \\ v^*(j) \leq u}} g_{|v|}(k_v(\tau), j)\right] = \prod_{i=0}^{n-1} \mathbb{E}[g_i(K, J)].$$

(ii) *For all $n, m \geq 2$, it holds $\mathbb{P}(\#\tau \leq n; \sup_{i \geq 0} |H_i(\tau) - H_{i+1}(\tau)| \geq m) \leq n(1 - \mu(0))^m$.*

Proof. The point (i) is a simplified version of the so-called *Many-To-One Principle*, which is part of folklore (see e.g. Duquesne [51, Equation (24)] for the general statement and a proof).

To prove (ii), let us first consider two vertices $u_1 < u_2$ of τ that are consecutive in lexicographic order on τ . Since $\overleftarrow{u}_2 < u_2$, we have $\overleftarrow{u}_2 \leq u_1 < u_2$ and so $\overleftarrow{u}_2 \preceq u_1$. Thus, it holds $|u_1| - |u_2| \geq -1$. Moreover, for all $v \in \tau$ and $j \in \mathbb{N}^*$ such that $\overleftarrow{u}_2 \prec v$ and $v * (j) \preceq u_1$, we observe $j = k_v(\tau)$ because $u_1 < v * (j + 1) < u_2$. As $m \geq 2$, we eventually obtain

$$\mathbb{P}\left(\#\tau \leq n; \sup_{i \geq 0} |H_i(\tau) - H_{i+1}(\tau)| \geq m\right) \leq \mathbb{E}\left[\sum_{u \in \tau} \mathbf{1}_{\{m \leq |u| < n\}} \prod_{\substack{v*(j) \preceq u \\ |v| \geq |u| - m}} \mathbf{1}_{\{k_v(\tau) = j\}}\right].$$

We easily compute $\mathbb{P}(K = J) = \sum_{k \geq 1} \mu(k) = 1 - \mu(0)$, then we apply (i) to get

$$\mathbb{P}\left(\#\tau \leq n; \sup_{i \geq 0} |H_i(\tau) - H_{i+1}(\tau)| \geq m\right) \leq \sum_{h=m}^{n-1} \prod_{i=h-m}^{h-1} \mathbb{P}(K = J) = (n - m)(1 - \mu(0))^m. \quad \blacksquare$$

Proof of Proposition 5.3.9. Recall from (5.2.12) the notation $u(\tau)$. We can choose an increasing and bijective function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi(0) = 0$, $\psi(\lambda i + \lambda) = \psi(\lambda i) + \lambda L_{u_i(\tau)}$ for all $0 \leq i \leq \#\tau - 1$, and $\psi(\lambda \#\tau + s) = \psi(\lambda \#\tau) + s$ for all $s \geq 0$. For $\lambda i \leq s < \lambda(i + 1)$, we write

$$\begin{aligned} |X_{\psi(s)} - Y_s| &\leq |Y_{\lambda i} - Y_{\lambda(i+1)}| + |Y_{\lambda i} - X_{\psi(\lambda i)}| + |X_{\psi(\lambda i)} - X_{\psi(\lambda i + \lambda)}|, \\ |\psi(s) - s| &\leq 2\lambda + 2|\lambda i - \psi(\lambda i)| + |\lambda i + \lambda - \psi(\lambda i + \lambda)|. \end{aligned}$$

Together with (5.2.14) and (5.2.23), it is then not hard to check that

$$\begin{aligned} d_S(X, Y) &\leq 3\lambda + 4\lambda \max_{1 \leq i \leq \#\tau} \left| i - \sum_{j=0}^{i-1} L_{u_j(\tau)} \right| + \lambda^{1-1/\alpha} \max_{u \in \tau} \left| |u| - \sum_{v \prec u} L_v \right| \\ &\quad + \lambda^{1-1/\alpha} \max_{u \in \tau} L_u + \lambda^{1-1/\alpha} \sup_{i \geq 0} |H_i(\tau) - H_{i+1}(\tau)|. \end{aligned} \quad (5.3.12)$$

We want to bound in probability each one of the terms of the right-hand side under the realization of the event $\{\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}\}$. Let $(L_i)_{i \geq 0}$ be a sequence of independent exponential random variables with mean 1. We begin with the simple union bound

$$\mathbb{P}\left(\#\tau \leq n; \lambda^{1-1/\alpha} \max_{u \in \tau} L_u \geq \lambda^{1-1/\alpha} (\ln n)^2\right) \leq n \mathbb{P}(L_0 \geq (\ln n)^2) = ne^{-(\ln n)^2}. \quad (5.3.13)$$

Moreover, we use another union bound and Lemma 5.3.10 to find

$$\begin{aligned} \mathbb{P}\left(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; \lambda^{1-1/\alpha} \max_{u \in \tau} \left| |u| - \sum_{v \prec u} L_v \right| \geq (\lambda \sqrt{n})^{1-1/\alpha} \ln n\right) \\ \leq n \max_{0 \leq h \leq n^{1-1/\alpha}} \mathbb{P}\left(\left| h - \sum_{j=0}^{h-1} L_j \right| \geq (\sqrt{n})^{1-1/\alpha} \ln n\right) \leq 2ne^{-c_{\text{uni}}(\ln n)^2} \end{aligned} \quad (5.3.14)$$

for n large enough. The exact same method entails

$$\mathbb{P}\left(\#\tau \leq n; \lambda \max_{1 \leq i \leq \#\tau} \left| i - \sum_{j=0}^{i-1} L_{u_j(\tau)} \right| \geq \lambda \sqrt{n} \ln n\right) \leq 2ne^{-c_{\text{uni}}(\ln n)^2}. \quad (5.3.15)$$

By (5.3.12), (5.3.13), (5.3.14), and (5.3.15), an application of Lemma 5.3.11 (ii) completes the proof. \blacksquare

5.4 The weighted Horton–Strahler number

5.4.1 The classic Horton–Strahler number

In this section, we recall two alternative definitions of the (classic) Horton–Strahler number \mathcal{S} , and we apply some general estimates of the companion paper [103] to the case of stable Galton–Watson trees. But first, let us spell a simple observation out. By Definition 5.1.1, for any tree t , it holds that

$$\text{if } k_{\emptyset}(t) \neq 1, \quad \text{then } \mathcal{S}(t) = 0 \Leftrightarrow t = \{\emptyset\}. \quad (5.4.1)$$

Indeed, if $k_{\emptyset}(t) \geq 2$ then $\mathcal{S}(t) \geq 1$ because the Horton–Strahler numbers are nonnegative.

For all $n \in \mathbb{N}$, we denote by $\mathbb{W}_n = \bigcup_{k=0}^n \{1, 2\}^k$ the n -perfect binary tree, with the convention that $\{1, 2\}^0 = \{\emptyset\}$. Recall Definition 5.2.10. The Horton–Strahler number of a tree t is given by

$$\mathcal{S}(t) = \max \{n \in \mathbb{N} : \exists \psi : \mathbb{W}_n \longrightarrow t \text{ embedding}\}. \quad (5.4.2)$$

This result seems to be ‘part of the folklore’, but see [103, Equation (36)] for a brief proof.

Our second alternative definition of \mathcal{S} involves the action of removing the subtrees with null Horton–Strahler number of a tree t . After such reduction, we might obtain some chains of edges put back to back without any branching. We wish to see such a chain as a single longer edge. Thus, we rather work with trees with edge lengths as defined in Section 5.2.2. The next definition is just a formal rephrasing within our framework of the *Horton pruning* studied by Kovchegov & Zaliapin, see [111, Definition 3 and Figure 7].

Definition 5.4.1 (Horton pruning) Let $T = (t, (l_u)_{u \in t})$ be a tree with edge lengths with $\mathcal{S}(t) \geq 1$. Remark 5.2.11 ensures that there are a unique tree t'' and a unique increasing embedding $\psi'' : t'' \rightarrow t$ such that $\psi''(t'') = \{u \in t : \mathcal{S}(\theta_u T) \geq 1\}$. Proposition 5.2.13 then states that there are a unique tree t' and a unique increasing embedding $\psi' : t' \rightarrow t''$ such that $\psi'(t') = \{u \in t'' : k_u(t'') \neq 1\}$. We denote by ψ the embedding $\psi'' \circ \psi' : t' \rightarrow t$ and for all $u \in t'$, we set

$$l'_u = \sum_{\psi(\overleftarrow{u}) \prec u' \preceq \psi(u)} l_{u'}, \quad \text{where } \psi(\overleftarrow{\emptyset}) = \overleftarrow{\emptyset} \text{ in accordance with Notation 5.2.8.}$$

Eventually, we define the *Horton-pruned tree with edge lengths* as $R(T) = (t', (l'_u)_{u \in t'})$. \square

The number of Horton pruning operations needed to entirely erase T is equal to $\mathcal{S}(T) + 1$, namely

$$\text{if } \mathcal{S}(T) \geq 1 \quad \text{then } \mathcal{S}(R(T)) = \mathcal{S}(T) - 1. \quad (5.4.3)$$

Now, we gather from the companion paper [103] several tail estimates for the joint law of the Horton–Strahler number $\mathcal{S}(\tau)$ of a GW_{α} -tree τ with either its size $\#\tau$ or height $|\tau|$. Recall that the offspring distribution of GW_{α} -trees satisfies (5.2.17) and (5.1.11).

Proposition 5.4.2 Let τ be a GW_{α} -tree. Recall that $\gamma = \ln \frac{\alpha}{\alpha-1}$ and $\delta = e^{\gamma(\alpha-1)}$ from (5.1.8). Then,

$$\mathbb{P}(\mathcal{S}(\tau) \geq n) = e^{-\gamma n}, \quad (5.4.4)$$

$$\mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau) \leq n\}} \#\tau] \leq 2e^{\gamma(\alpha-1)n}, \quad (5.4.5)$$

$$\mathbb{E}[\#\tau \mid \mathcal{S}(\tau) = n] \leq 2\alpha e^{\gamma \alpha n}, \quad (5.4.6)$$

for all $n \in \mathbb{N}$. Moreover, for all $\lambda \in (0, \infty)$, it holds

$$\limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| \leq \lambda e^{\gamma(\alpha-1)n} \mid \mathcal{S}(\tau) = n) \leq 1 - e^{-\lambda}. \quad (5.4.7)$$

Furthermore, there exists a constant $\lambda_0 \in (0, \infty)$ that only depends on α such that for all $\lambda > \lambda_0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{S}(\tau) \leq \log_{\delta}(n/\lambda) \mid |\tau| \geq n) \leq e^{-\lambda/20}, \quad (5.4.8)$$

$$\limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| \geq \lambda e^{\gamma(\alpha-1)n} \mid \mathcal{S}(\tau) = n) \leq e^{-\lambda/20}. \quad (5.4.9)$$

Proof. Kovchegov & Zaliapin [110, Lemma 10] yields (5.4.4) (see also [103, Remark 3.3]). We independently retrieve (5.4.4) in the present paper via Proposition 5.4.6. Together with (5.1.2) and (5.4.4), [103, Proposition 3.5] entails (5.4.5). Then, (5.4.6) follows from (5.4.4) and (5.4.5). For (5.4.7), see [103, Proposition 3.6]. By [103, Corollary 3.8] and (5.1.11), the left-hand side of (5.4.8) is bounded by $Ce^{-\lambda/8\delta}$ for all $\lambda > 0$, where $C \in (0, \infty)$ is a constant that only depends on α . Using Bayes's theorem with (5.3.3) and (5.4.4), we then get that the left-hand side of (5.4.9) is bounded by $\alpha e^{-\gamma\beta} C \lambda^{-\beta} e^{-\lambda/8\delta}$ for all $\lambda \in (0, \infty)$. Since $8\delta < 20$, we readily obtain (5.4.8) and (5.4.9). \blacksquare

5.4.2 First properties of the weighted Horton–Strahler number

Here, we provide basic properties of the weighted Horton–Strahler number \mathcal{S} (recall Definition 5.1.2), that show that it is a good approximation of the (classic) Horton–Strahler number \mathcal{S} while being a continuous quantity. For all $x \in \mathbb{R}_+$, we denote by $\lfloor x \rfloor \in \mathbb{N}$ the integer part of x and by $\text{frac}(x) = x - \lfloor x \rfloor \in [0, 1)$ the fractional part of x .

Proposition 5.4.3 *For any weighted tree $\mathbf{t} = (t, (w_v)_{v \in \partial t})$, it holds that $\mathcal{S}(t) = \lfloor \mathcal{S}(\mathbf{t}) \rfloor$. Moreover, there exists a leaf $v \in \partial t$ of the tree t such that $\text{frac}(\mathcal{S}(\mathbf{t})) = w_v$.*

Proof. We argue by induction on $|t|$. If $|t| = 0$ then $t = \partial t = \{\emptyset\}$, so that $\mathcal{S}(t) = 0$ and $\lfloor \mathcal{S}(\mathbf{t}) \rfloor = \lfloor w_{\emptyset} \rfloor = 0$ because $w_{\emptyset} \in [0, 1)$. Moreover, $\text{frac}(\mathcal{S}(\mathbf{t})) = \mathcal{S}(\mathbf{t}) = w_{\emptyset}$. If $k_{\emptyset}(t) \geq 1$ then there is $1 \leq i \leq k_{\emptyset}(t)$ such that either $\mathcal{S}(\mathbf{t}) = \mathcal{S}(\theta_{(i)}\mathbf{t})$ or $\mathcal{S}(\mathbf{t}) = 1 + \mathcal{S}(\theta_{(i)}\mathbf{t})$ by Definition 5.1.2. Either way, it holds $\text{frac}(\mathcal{S}(\mathbf{t})) = \text{frac}(\mathcal{S}(\theta_{(i)}\mathbf{t}))$, so there is $v \in \partial t$ such that $\text{frac}(\mathcal{S}(\mathbf{t})) = w_v$ by induction hypothesis. The induction hypothesis also entails that

$$\lfloor \mathcal{S}(\mathbf{t}) \rfloor = \max_{1 \leq i, j \leq k_{\emptyset}(t)} \max \left(\mathcal{S}(\theta_{(i)}t), \mathcal{S}(\theta_{(j)}t), \mathbf{1}_{\{i \neq j\}} + \min(\mathcal{S}(\theta_{(i)}t), \mathcal{S}(\theta_{(j)}t)) \right)$$

because the integer part function is non-decreasing. The formula (5.1.6) completes the proof. \blacksquare

For any weighted tree $\mathbf{t} = (t, (w_v)_{v \in \partial t})$ and $x \in [0, 1)$, we see from Proposition 5.4.3 and (5.4.1) that

$$\text{if } k_{\emptyset}(t) \neq 1, \quad \text{then } \mathcal{S}(\mathbf{t}) \leq x \iff \mathbf{t} = (\{\emptyset\}, w_{\emptyset}) \text{ with } w_{\emptyset} \leq x. \quad (5.4.10)$$

Lemma 5.4.4 *Let t be a tree and let $(W_v)_{v \in \partial t}$ be independent random variables on $[0, 1)$. We set $\mathbf{t} = (t, (W_v)_{v \in \partial t})$. If the law of W_v admits a continuous and positive density on $(0, 1)$ for all $v \in \partial t$, then the law of $\mathcal{S}(\mathbf{t})$ admits a continuous and positive density on $(\mathcal{S}(t), \mathcal{S}(t) + 1)$.*

Proof. We prove the lemma by induction on $|t|$. If $|t| = 0$ then $t = \partial t = \{\emptyset\}$, $\mathcal{S}(t) = 0$, and $\mathcal{S}(\mathbf{t}) = W_{\emptyset}$ has a continuous and positive density on $(0, 1)$ by assumption. Now, we assume $k_{\emptyset}(t) \geq 1$ and we write $J = \{1 \leq j \leq k_{\emptyset}(t) : \mathcal{S}(\theta_{(j)}t) = \mathcal{S}(t) - 1\}$. We set

$$F(x) = \mathbb{P}(\mathcal{S}(\mathbf{t}) \leq \mathcal{S}(t) + x) \quad \text{and} \quad F_i(x) = \mathbb{P}(\mathcal{S}(\theta_{(i)}\mathbf{t}) \leq \mathcal{S}(\theta_{(i)}t) + x)$$

for all $x \in [0, 1]$ and $1 \leq i \leq k_\emptyset(t)$. We point out that the $\theta_{(i)}\mathbf{t}$, for $1 \leq i \leq k_\emptyset(\tau)$, are independent. We exactly need to show that F is C^1 with positive derivative on $(0, 1)$. By induction hypothesis, the F_i are C^1 with positive derivative on $(0, 1)$. There are two cases.

• If there exists $1 \leq i \leq k_\emptyset(t)$ such that $\mathcal{S}(\theta_{(i)}\mathbf{t}) = \mathcal{S}(t)$ then it is unique by Definition 5.1.1. Since $\lfloor \mathcal{S}(\mathbf{t}) \rfloor = \mathcal{S}(t)$, we deduce that $\mathcal{S}(\mathbf{t}) = \max(\mathcal{S}(\theta_{(i)}\mathbf{t}), 1 + \max_{j \in J} \mathcal{S}(\theta_{(j)}\mathbf{t}))$ by Definition 5.1.2. It follows that $F(x) = F_i(x) \prod_{j \in J} F_j(x)$ for all $x \in [0, 1]$, which yields the result.

• Otherwise, there are at least two elements in J by Definition 5.1.2. Therefore, we get the identity

$$\mathcal{S}(\mathbf{t}) = 1 + \max_{\substack{i, j \in J \\ i \neq j}} \min(\mathcal{S}(\theta_{(i)}\mathbf{t}), \mathcal{S}(\theta_{(j)}\mathbf{t})),$$

which translates into $F(x) = \prod_{j \in J} F_j(x) + \sum_{i \in J} (1 - F_i(x)) \prod_{j \in J \setminus \{i\}} F_j(x)$ for all $x \in [0, 1]$. Thus, F is C^1 on $(0, 1)$. To obtain the positivity of the derivative, we simply compute that

$$\forall x \in (0, 1), \quad F'(x) = \sum_{\substack{i, j \in J \\ i \neq j}} F'_i(x)(1 - F_j(x)) \prod_{\substack{k \in J \\ k \neq i, j}} F_k(x) > 0.$$

■

Remark 5.4.5 Other properties of the distribution of $\mathcal{S}(\mathbf{t})$ are inherited from those of the $(W_v)_{v \in \partial t}$. We mention for example the absolute continuity with respect to some measure, the topological support, or some additional regularity of the density. □

5.4.3 The weighted Horton–Strahler number of stable Galton–Watson weighted trees

Recall the law $\text{FExp}(\gamma)$ from (5.1.9) and $\gamma = \ln \frac{\alpha}{\alpha-1}$ from (5.1.8). We still write $\text{frac}(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}_+$. The next result justifies the choice of weights in Definition 5.2.15 for \mathbf{GW}_α -weighted trees.

Proposition 5.4.6 *Let W be a random variable on $[0, 1]$. Let $\tau = (\tau, (W_v)_{v \in \partial \tau})$ be a random weighted tree such that τ is \mathbf{GW}_α -tree and conditionally given τ , the $(W_v)_{v \in \partial \tau}$ are independent and distributed as W . Then, the following holds.*

- (i) *The law of $\lfloor \mathcal{S}(\tau) \rfloor = \mathcal{S}(\tau)$ is geometric with parameter $\frac{1}{\alpha}$.*
- (ii) *The law of $\text{frac}(\mathcal{S}(\tau))$ is the same as the law of W .*
- (iii) *$\lfloor \mathcal{S}(\tau) \rfloor = \mathcal{S}(\tau)$ and $\text{frac}(\mathcal{S}(\tau))$ are independent.*

In particular, if τ is \mathbf{GW}_α -weighted tree then the law of $\mathcal{S}(\tau)$ is exponential with mean $\frac{1}{\gamma}$.

Proof. From (5.1.2) and (5.2.16), recall the offspring law μ_α of τ and its generating function φ_α . We set $f(x) = \mathbb{P}(\mathcal{S}(\tau) > x)$ for all $x \in \mathbb{R}_+$. We only need to show that for all $x \in \mathbb{R}_+$,

$$f(x) = e^{-\gamma \lfloor x \rfloor} (1 - \frac{1}{\alpha} \mathbb{P}(W \leq x - \lfloor x \rfloor)), \quad (5.4.11)$$

because the integer and fractional parts are measurable. Note that if the law of W is $\text{FExp}(\gamma)$ then (5.4.11) becomes $f(x) = e^{-\gamma x}$. If $x \in [0, 1]$ then by (5.4.10), and since $\mu_\alpha(1) = 0$, we compute that $\mathbb{P}(\mathcal{S}(\tau) \leq x) = \mathbb{P}(k_\emptyset(\tau) = 0; W_\emptyset \leq x) = \mu_\alpha(0) \mathbb{P}(W \leq x)$. This is exactly (5.4.11).

Now, we assume $x \geq 1$. By Definition 5.1.2 of the weighted Horton–Strahler number, we observe that $\mathcal{S}(\tau) \leq x$ if and only if $\mathcal{S}(\theta_u \tau) \leq x - 1$ for all children u of \emptyset in τ (if any) with the

possible exception of one child v , which may satisfy $x - 1 < \mathcal{S}(\theta_v \tau) \leq x$. More precisely, we have

$$\mathbf{1}_{\{\mathcal{S}(\tau) \leq x\}} = \prod_{j=1}^{k_\emptyset(\tau)} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)} \tau) \leq x-1\}} + \sum_{i=1}^{k_\emptyset(\tau)} \mathbf{1}_{\{x-1 < \mathcal{S}(\theta_{(i)} \tau) \leq x\}} \prod_{\substack{1 \leq j \leq k_\emptyset(\tau) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)} \tau) \leq x-1\}}. \quad (5.4.12)$$

Taking the expectation leads to $f(x) = (1 - 1/\alpha)f(x - 1)$ by Definition 5.2.15 of \mathbf{GW}_α -weighted trees. Finally, we get $f(x) = e^{-\gamma \lfloor x \rfloor} f(x - \lfloor x \rfloor)$ for all $x \in \mathbb{R}_+$ by induction on $\lfloor x \rfloor$, and so the identity (5.4.11) follows. \blacksquare

Let τ be a \mathbf{GW}_α -weighted tree. For all $n \in \mathbb{N}$, the event $\{\mathcal{S}(\tau) = n\}$ has nonzero probability so the law of τ conditionally given $\mathcal{S}(\tau) = n$ is straightforwardly defined. Our goal for the rest of the section is to properly define the law of τ under the degenerate conditioning $\{\mathcal{S}(\tau) = x\}$ in a sufficiently regular and explicit way to be able to carry out calculations and study convergences. Pleasantly, the natural idea to define $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ as the limit of the conditional laws of τ given $|\mathcal{S}(\tau) - x| < \varepsilon$ when ε tends to 0^+ turns out to work just fine. From (5.2.24), recall the notation $\mathcal{S}(\mathbf{T})$ for a weighted tree with edge lengths \mathbf{T} .

Proposition 5.4.7 *Let \mathcal{T} be a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths as in Definition 5.2.16. For all $x \in \mathbb{R}_+ \setminus \mathbb{N}$, there exists a probability measure Q_x on the space $\overline{\mathbb{T}}$ of all trees with edge lengths such that*

$$Q_x[g] = \lim_{\varepsilon \rightarrow 0^+} \mathbb{E}[g(\mathcal{T}) \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon] \quad (5.4.13)$$

for all bounded measurable function g on $\overline{\mathbb{T}}$. The function $x \mapsto Q_x[g]$ is continuous on $\mathbb{R}_+ \setminus \mathbb{N}$, and

$$\mathbb{E}[h(\mathcal{S}(\mathcal{T}))g(\mathcal{T})] = \mathbb{E}[h(\mathcal{S}(\mathcal{T}))Q_{\mathcal{S}(\mathcal{T})}[g]] = \int_{\mathbb{R}_+} h(x)Q_x[g]\gamma e^{-\gamma x} dx \quad (5.4.14)$$

for all bounded measurable function h on \mathbb{R}_+ . If the law of $\mathcal{T}_x = (\tau_x, (L_u)_{u \in \tau_x})$ is Q_x , then conditionally given τ_x , the $(L_u)_{u \in \tau_x}$ are independent with exponential distribution with mean 1.

Definition 5.4.8 Let $x \in \mathbb{R}_+ \setminus \mathbb{N}$. If \mathcal{T} is a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths, then we denote by $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$ the law Q_x defined in Proposition 5.4.7, and we thus write

$$\mathbb{P}(\mathcal{T} \in A \mid \mathcal{S}(\mathcal{T}) = x) = Q_x(A) \quad \text{and} \quad \mathbb{E}[g(\mathcal{T}) \mid \mathcal{S}(\mathcal{T}) = x] = Q_x[g]$$

for all measurable subset $A \subset \overline{\mathbb{T}}$ and all bounded measurable function g on $\overline{\mathbb{T}}$. Recall from (5.2.20) the measurable map $\text{Sha} : \overline{\mathbb{T}} \rightarrow \mathbb{T}$, where \mathbb{T} is the space of all trees. If τ is a \mathbf{GW}_α -weighted tree, then we denote by $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ the law of $\text{Sha}(\mathcal{T})$ under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$. Namely,

$$\mathbb{P}(\tau \in A \mid \mathcal{S}(\tau) = x) = Q_x(\text{Sha}^{-1}(A)) \quad \text{and} \quad \mathbb{E}[g(\tau) \mid \mathcal{S}(\tau) = x] = Q_x[g \circ \text{Sha}],$$

for all subset $A \subset \mathbb{T}$ and all bounded function g on \mathbb{T} . \square

Before showing Proposition 5.4.7, we give ourselves a uniform domination of the laws of τ conditionally given $|\mathcal{S}(\tau) - x| < \varepsilon$ via the lemma below. This result is also useful to transpose estimates about $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = n)$ into bounds for $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$. We endow the countable space $\mathbb{T}_{\text{bit}} = \{(t, (b_v)_{v \in \partial t}) : t \text{ a tree and } b_v \in \{0, 1\} \text{ for all } v \in \partial t\}$ with the discrete topology.

Lemma 5.4.9 *Let $r \in (0, 1)$, let W have law $\text{FExp}(\gamma)$, and let $\tau = (\tau, (W_v)_{v \in \partial \tau})$ be a \mathbf{GW}_α -weighted tree. We set $\xi = \mathbf{1}_{\{W \geq r\}}$ and $\xi_v = \mathbf{1}_{\{W_v \geq r\}}$ for all $v \in \partial \tau$. Let $g : \mathbb{T}_{\text{bit}} \rightarrow \mathbb{R}_+$ be a nonnegative bounded function. For all $x, \varepsilon > 0$ such that $\lfloor x \rfloor < x - \varepsilon < x + \varepsilon < \lfloor x \rfloor + 1$, it holds*

$$\mathbb{E}[g(\tau, (\xi_v)_{v \in \partial \tau}) \mid \mathcal{S}(\tau) = x] \leq c_\alpha(r) \mathbb{E}[g(\tau, (\xi_v)_{v \in \partial \tau}) \mid \mathcal{S}(\tau) = \lfloor x \rfloor] < \infty,$$

where $c_\alpha(r)^{-1} = \min(\mathbb{P}(W < r), \mathbb{P}(W \geq r))$ only depends on r and α . Moreover, for all $n \in \mathbb{N}$,

$$\mathbb{E}[\mathbf{1}_{\{\#\tau \leq n\}} g(\tau, (\xi_v)_{v \in \partial\tau}) \mid |\mathcal{S}(\tau) - x| < \varepsilon] \leq c_\alpha(r) \alpha n e^{\gamma x} \mathbb{E}[\mathbf{1}_{\{\#\tau \leq n\}} g(\tau, (\xi_v)_{v \in \partial\tau})]. \quad (5.4.15)$$

Proof of Lemma 5.4.9. The given upper bound is finite by (5.4.6). The space \mathbb{T}_{bit} is countable, so we can assume there is $(t^0, (b_v^0)_{v \in \partial t^0}) \in \mathbb{T}_{\text{bit}}$ such that $g(t, (b_v)_{v \in \partial t}) = \mathbf{1}_{\{t=t^0\}} \prod_{v \in \partial t^0} \mathbf{1}_{\{b_v=b_v^0\}}$. Thanks to the assumption on x and ε , Proposition 5.4.3 entails that $|\mathcal{S}(\tau) - x| < \varepsilon$ if and only if $\mathcal{S}(\tau) = \lfloor x \rfloor$ and $|\text{frac}(\mathcal{S}(\tau)) - \text{frac}(x)| < \varepsilon$. Thus, Proposition 5.4.6 gives

$$\mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon) = \mathbb{P}(\mathcal{S}(\tau) = \lfloor x \rfloor) \mathbb{P}(|W - \text{frac}(x)| < \varepsilon). \quad (5.4.16)$$

Moreover, Proposition 5.4.3 also ensures that

$$\mathbf{1}_{\{|\mathcal{S}(\tau) - x| < \varepsilon\}} g(\tau, (\xi_v)_{v \in \partial\tau}) \leq \mathbf{1}_{\{\mathcal{S}(\tau) = \lfloor x \rfloor\}} \mathbf{1}_{\{\tau = t^0\}} \sum_{u \in \partial t^0} \mathbf{1}_{\{|W_u - \text{frac}(x)| < \varepsilon\}} \prod_{v \in (\partial t^0) \setminus \{u\}} \mathbf{1}_{\{\xi_v = b_v^0\}}.$$

Note that $\#\partial t^0 \leq \#\tau$ and $\mathbb{P}(\xi = b_u^0) \geq c_\alpha(r)^{-1}$ for all $u \in \partial t^0$. Since the $(W_v)_{v \in \partial\tau}$ are independent and distributed as W conditionally given τ , taking the expectation then yields that

$$\mathbb{E}[\mathbf{1}_{\{|\mathcal{S}(\tau) - x| < \varepsilon\}} g(\tau, (\xi_v)_{v \in \partial\tau})] \leq c_\alpha(r) \mathbb{P}(|W - \text{frac}(x)| < \varepsilon) \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau) = \lfloor x \rfloor\}} g(\tau, (\xi_v)_{v \in \partial\tau}) \#\tau].$$

Dividing this inequality by $\mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon)$ and using (5.4.16) conclude the proof of the first inequality. We get (5.4.15) by writing $\mathbb{E}[\mathbf{1}_{\{\#\tau \leq n\}} g \#\tau \mid \mathcal{S}(\tau) = \lfloor x \rfloor] \leq n \mathbb{P}(\mathcal{S}(\tau) = \lfloor x \rfloor)^{-1} \mathbb{E}[\mathbf{1}_{\{\#\tau \leq n\}} g]$ and then recalling (5.4.4). \blacksquare

Proof of Proposition 5.4.7. We write $\tau = \text{Sha}(\mathcal{T})$ which is a \mathbf{GW}_α -weighted tree. Let t be a tree and let $(W_v)_{v \in \partial t}$ be independent random variables with law $\text{FExp}(\gamma)$. We set $\mathbf{t} = (t, (W_v)_{v \in \partial t})$. We denote by $f_{\mathbf{t}}$ the density of $\text{frac}(\mathcal{S}(\mathbf{t}))$ and by f the density of $\text{frac}(\mathcal{S}(\tau))$. They are positive and continuous on $(0, 1)$ by Lemma 5.4.4 and Proposition 5.4.6. Since $x \notin \mathbb{N}$, it holds $\lfloor x \rfloor < x - \varepsilon$ and $x + \varepsilon < \lfloor x \rfloor + 1$ for all $\varepsilon > 0$ small enough. If so, we then find

$$\begin{aligned} \mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon) &= \mathbb{P}(\mathcal{S}(\tau) = \lfloor x \rfloor) \int_{\text{frac}(x) - \varepsilon}^{\text{frac}(x) + \varepsilon} f(y) dy, \\ \mathbb{P}(\tau = t; |\mathcal{S}(\tau) - x| < \varepsilon) &= \mathbb{P}(\tau = t; \mathcal{S}(\tau) = \lfloor x \rfloor) \int_{\text{frac}(x) - \varepsilon}^{\text{frac}(x) + \varepsilon} f_{\mathbf{t}}(y) dy. \end{aligned}$$

by respectively applying Proposition 5.4.6 and conditioning on $\{\tau = t\}$. The existence of the limit

$$q_x(t) := \mathbb{P}(\tau = t \mid \mathcal{S}(\tau) = \lfloor x \rfloor) \frac{f_{\mathbf{t}}(\text{frac}(x))}{f(\text{frac}(x))} = \lim_{\varepsilon \rightarrow 0^+} \mathbb{P}(\tau = t \mid |\mathcal{S}(\tau) - x| < \varepsilon) \quad (5.4.17)$$

and the continuity of $x \mapsto q_x(t)$ on $\mathbb{R}_+ \setminus \mathbb{N}$ follow from the continuity of $f_{\mathbf{t}}$ and f on $(0, 1)$. Now, let us set $F(x) = \mathbb{P}(\tau = t; \mathcal{S}(\tau) \leq x)$ for all $x \in \mathbb{R}$ (note that $F(0) = 0$). With the same argument used to get (5.4.17), we show that F is C^1 on $\mathbb{R}_+ \setminus \mathbb{N}$ with $F'(x) = q_x(t) \gamma e^{-\gamma x}$. If $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ is C^1 with compact support, then we write $h(x) = -\int_x^\infty h'(x) dx$ and we obtain

$$\mathbb{E}[h(\mathcal{S}(\tau)) \mathbf{1}_{\{\tau = t\}}] = \int_0^\infty h(x) q_x(t) \gamma e^{-\gamma x} dx. \quad (5.4.18)$$

after an application of Fubini's theorem and an integration by parts. The identity (5.4.18) holds in fact for all bounded and measurable function h by the functional monotone class theorem.

We finally define the measure Q_x by setting $Q_x[g] = \sum_{t \in \mathbb{T}} q_x(t) \mathbb{E}[g(\mathcal{T}) \mid \tau = t]$ for all bounded and measurable function g on $\overline{\mathbb{T}}$. Making $\varepsilon \rightarrow 0^+$ in the bound given by Lemma 5.4.9

leads to $q_x(t) \leq c_\alpha(1/2)\mathbb{E}[\mathbf{1}_{\{\tau=t\}}\#\tau \mid \mathcal{S}(\tau) = \lfloor x \rfloor]$ for all tree t and $x \in \mathbb{R}_+ \setminus \mathbb{N}$. This domination ensures that the function $x \mapsto Q_x[g]$ is continuous on $\mathbb{R}_+ \setminus \mathbb{N}$. By Definition 5.2.16, we then have

$$\mathbb{E}[g(\mathcal{T}) \mid |\mathcal{S}(\tau) - x| < \varepsilon] = \sum_{t \in \mathbb{T}} \mathbb{P}(\tau = t \mid |\mathcal{S}(\tau) - x| < \varepsilon) \mathbb{E}[g(\mathcal{T}) \mid \tau = t].$$

Thanks to (5.4.17) and Lemma 5.4.9, the dominated convergence theorem yields (5.4.13). In particular, Q_x is indeed a probability measure. Similarly, (5.4.14) is a consequence of Fubini’s theorem and (5.4.18). \blacksquare

5.5 The weighted Horton pruning of trees with edge lengths

5.5.1 Definition and invariance

We remind from (5.1.8) that $\delta = (\frac{\alpha}{\alpha-1})^{\alpha-1} = e^{\gamma(\alpha-1)}$. In this section, we extensively use the notations from (5.2.20), (5.2.21), (5.2.24), and (5.2.25) to manipulate (weighted) trees with edge lengths. As discussed in the introduction, Kovchegov & Zaliapin [109, Proposition 4] showed that stable Galton–Watson trees with edge lengths are invariant by the Horton pruning R (see Definition 5.4.1). More precisely, if \mathcal{T} is a $\overline{\text{GW}}_\alpha$ -tree with edge lengths, then the law of $R(\mathcal{T})$ under $\mathbb{P}(\cdot \mid \mathcal{S}(\mathcal{T}) \geq 1)$ is the same as the law of $\delta \cdot \mathcal{T}$ under \mathbb{P} . In what follows, we seek to obtain a similar result for the weighted Horton–Strahler number and $\overline{\text{GW}}_\alpha$ -weighted trees with edge lengths.

First, we adapt the Horton pruning of Definition 5.4.1 to the framework of weighted trees with edge lengths. The r -Horton pruning consists in erasing the subtrees with weighted Horton–Strahler numbers smaller than a threshold r , and then removing the vertices with only one child left. The weight of a new leaf is given by the maximal Horton–Strahler number previously achieved on the new parental edge, minus r . See Figure 5.5 for an example. The formal definition is given below.

Definition 5.5.1 (Weighted Horton pruning) Recall Definition 5.2.10 of embeddings. Let $r \in \mathbb{R}_+$ and let $\mathbf{T} = (t, (l_u)_{u \in t}, (w_v)_{v \in \partial t})$ be a weighted tree with edge lengths such that $\mathcal{S}(\mathbf{T}) \geq r$. Remark 5.2.11 ensures that there are a unique tree t'' and a unique increasing embedding $\psi'' : t'' \rightarrow t$ such that $\psi''(t'') = \{u \in t : \mathcal{S}(\theta_u \mathbf{T}) \geq r\}$. Proposition 5.2.13 then states that there are a unique tree t' and a unique increasing embedding $\psi' : t' \rightarrow t''$ such that $\psi(t') = \{u \in t'' : k_u(t'') \neq 1\}$. We denote by ψ the embedding $\psi'' \circ \psi' : t' \rightarrow t$ and for all $u \in t'$ and for all $v \in \partial t'$, we set

$$l'_u = \sum_{\psi(\bar{u}) \prec u' \preceq \psi(u)} l_{u'} \quad \text{and} \quad w'_v = \max_{\psi(\bar{v}) \prec v' \preceq \psi(v)} \mathcal{S}(\theta_{v'} \mathbf{T}) - r.$$

Eventually, we define the r -Horton-pruned tree with edge lengths as $R_r(\mathbf{T}) = (t', (l'_u)_{u \in t'})$ and the r -Horton-pruned weighted tree with edge lengths as $\mathbf{R}_r(\mathbf{T}) = (t', (l'_u)_{u \in t'}, (w'_v)_{v \in \partial t'})$. \square

Proposition 5.5.2 Let $r \in \mathbb{R}_+$ and let \mathbf{T} be a weighted tree with edge lengths such that $\mathcal{S}(\mathbf{T}) \geq r$. We set $K^r(\mathbf{T}) = \#\{1 \leq i \leq K(\mathbf{T}) : \mathcal{S}(\theta_{(i)} \mathbf{T}) \geq r\}$ and we denote by $i_1 < \dots < i_{K^r(\mathbf{T})}$ all the integers such that $\mathcal{S}(\theta_{(i)} \mathbf{T}) \geq r$. The weighted tree with edge lengths $\mathbf{R}_r(\mathbf{T})$ is well-defined and can be recursively described as follows.

- (i) If $K^r(\mathbf{T}) = 0$, then $K(R_r(\mathbf{T})) = K^r(\mathbf{T}) = 0$ and $L(R_r(\mathbf{T})) = L(\mathbf{T})$.
- (ii) If $K^r(\mathbf{T}) = 1$, then $K(R_r(\mathbf{T})) = K(R_r(\theta_{(i_1)} \mathbf{T}))$ and $L(R_r(\mathbf{T})) = L(\mathbf{T}) + L(R_r(\theta_{(i_1)} \mathbf{T}))$. Moreover, if $K(R_r(\mathbf{T})) \geq 2$ then $\theta_{(j)} \mathbf{R}_r(\mathbf{T}) = \theta_{(j)} \mathbf{R}_r(\theta_{(i_1)} \mathbf{T})$ for all $1 \leq j \leq K(R_r(\mathbf{T}))$.
- (iii) If $K^r(\mathbf{T}) \geq 2$, then $K(R_r(\mathbf{T})) = K^r(\mathbf{T})$ and $L(R_r(\mathbf{T})) = L(\mathbf{T})$. Moreover, it holds $\theta_{(j)} \mathbf{R}_r(\mathbf{T}) = \mathbf{R}_r(\theta_{(i_j)} \mathbf{T})$ for all $1 \leq j \leq K^r(\mathbf{T})$.

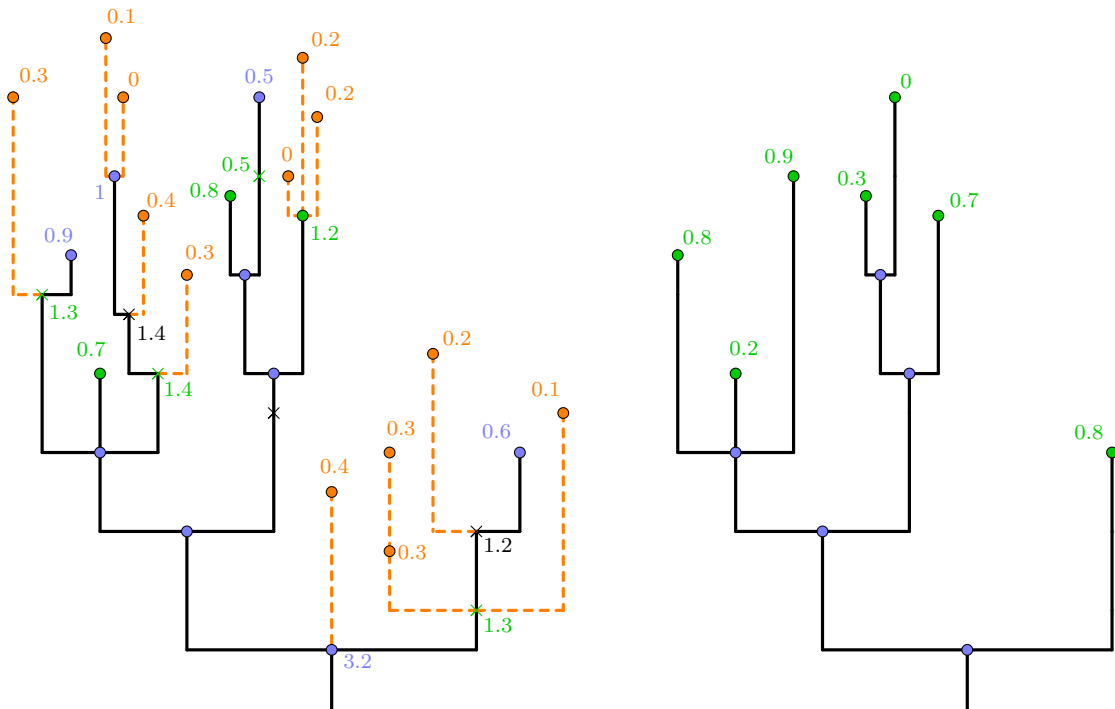


Figure 5.5: An example of r -Horton pruning with $r = 0.5$. Left : The weighted tree with edge lengths in Figure 5.3 before pruning. Each number indicates the weighted Horton–Strahler number of the subtree stemming from the respective vertex. The subtrees that will be erased are dashed and orange. The cross marks represent the vertices that will only have a single child left (and will be removed). Subtracting r from the numbers in green gives the weights assigned to the new leaves. Right : The resulting r -pruned weighted tree with edge lengths.

Furthermore, $K(R_r(\mathbf{T})) \neq 1$ and $\mathcal{S}(\mathbf{R}_r(\mathbf{T})) = \mathcal{S}(\mathbf{T}) - r$. In particular, if $K(R_r(\mathbf{T})) = 0$ then $\mathbf{R}_r(\mathbf{T}) = (\{\emptyset\}, L(R_r(\mathbf{T})), \mathcal{S}(\mathbf{T}) - r)$.

Proof. For all $1 \leq j \leq K^r(\mathbf{T})$, let t'_j be the underlying tree of $\mathbf{R}_r(\theta_{(i_j)} \mathbf{T})$ and let $\psi_j : t'_j \rightarrow \theta_{(i_j)} \mathbf{T}$ be the associated embedding as in Definition 5.5.1. It is elementary to verify that, depending on the corresponding case in the desired proposition, the following construction of t' and ψ is suitable:

- (i) $t' = \{\emptyset\}$ and $\psi(\emptyset) = \emptyset$;
- (ii) $t' = t'_1$ and $\psi(u) = (i_1) * \psi_1(u)$ for all $u \in t'$;
- (iii) $k_{\emptyset}(t') = K^r(\mathbf{T})$, $\theta_{(j)} t' = t'_j$, $\psi(\emptyset) = \emptyset$, $\psi((j) * u) = (i_j) * \psi_j(u)$ for all $u \in t'_j$, $1 \leq j \leq K^r(\mathbf{T})$.

By the uniqueness provided by Remark 5.2.11 and Proposition 5.2.13, the points (i), (ii), and (iii) readily follow. Thanks to them, a quick induction on the height $|t|$ shows that $K(R_r(\mathbf{T})) \neq 1$.

To prove that $\mathbf{R}_r(\mathbf{T})$ is a weighted tree with edge lengths, we still have to check that its weights are all in $[0, 1)$. By induction on $|t|$, (ii) and (iii) allow us to only consider the case $K(R_r(\mathbf{T})) = 0$, where then the unique weight of $\mathbf{R}_r(\mathbf{T})$ is $w'_{\emptyset} = \mathcal{S}(\mathbf{T}) - r \geq 0$ by assumption. To sum up, it remains to show that if $K(R_r(\mathbf{T})) = 0$ then $\mathcal{S}(\mathbf{T}) < 1 + r$, and if $K(R_r(\mathbf{T})) \geq 2$ then $\mathcal{S}(\mathbf{R}_r(\mathbf{T})) = \mathcal{S}(\mathbf{T}) - r$. We do so by induction on $|t|$. When $|t| = 0$, we have $K(R_r(\mathbf{T})) = 0$

and $\mathcal{S}(\mathbf{T}) = w_\emptyset < 1 + r$. When $|t| \geq 1$, we begin by defining, with the convention $\max \emptyset = 0$,

$$x = \max_{1 \leq j, j' \leq K^r(\mathbf{T})} \max \left(\mathcal{S}(\theta_{(i_j)} \mathbf{T}), \mathbf{1}_{\{j \neq j'\}} + \min \left(\mathcal{S}(\theta_{(i_j)} \mathbf{T}), \mathcal{S}(\theta_{(i_{j'})} \mathbf{T}) \right) \right),$$

$$y = \max_{\substack{1 \leq i, i' \leq K(\mathbf{T}) \\ \mathcal{S}(\theta_{(i)} \mathbf{T}) < r}} \max \left(\mathcal{S}(\theta_{(i)} \mathbf{T}), \mathbf{1}_{\{i \neq i'\}} + \min \left(\mathcal{S}(\theta_{(i)} \mathbf{T}), \mathcal{S}(\theta_{(i')} \mathbf{T}) \right) \right).$$

We see that $\mathcal{S}(\mathbf{T}) = \max(x, y)$ and $y < 1 + r$. We complete the proof by separating the cases.

- If $K(R_r(\mathbf{T})) = K^r(\mathbf{T}) = 0$: then $\mathcal{S}(\mathbf{T}) = y < 1 + r$.
- If $K(R_r(\mathbf{T})) = 0$ and $K^r(\mathbf{T}) = 1$: then $x = \mathcal{S}(\theta_{(i_1)} \mathbf{T}) = \mathcal{S}(\mathbf{R}_r(\theta_{(i_1)} \mathbf{T})) + r$ by induction hypothesis. Thanks to (ii), $K(R_r(\theta_{(i_1)} \mathbf{T})) = 0$ so $x < 1 + r$ by (5.4.10), and $\mathcal{S}(\mathbf{T}) < 1 + r$.
- If $K(R_r(\mathbf{T})) \geq 2$ and $K^r(\mathbf{T}) = 1$: then $x = \mathcal{S}(\mathbf{R}_r(\theta_{(i_1)} \mathbf{T})) + r$ again, and (ii) yields $x = \mathcal{S}(\mathbf{R}_r(\mathbf{T})) + r$ by Definition 5.1.2. By (5.4.10), $x \geq 1 + r$, so $\mathcal{S}(\mathbf{T}) = x = \mathcal{S}(\mathbf{R}_r(\mathbf{T})) + r$.
- If $K(R_r(\mathbf{T})) = K^r(\mathbf{T}) \geq 2$: then $x = \mathcal{S}(\mathbf{R}_r(\mathbf{T})) + r$ by (iii), Definition 5.1.2, and induction hypothesis. As in the previous case, we get $x \geq 1 + r$ and $\mathcal{S}(\mathbf{T}) = \mathcal{S}(\mathbf{R}_r(\mathbf{T})) + r$.

The fact that $K(R_r(\mathbf{T})) \neq 1$ and the points (i), (ii), and (iii) ensure that we treated all cases. ■

Recall that R stands for the (classic) Horton pruning as in Definition 5.4.1. Let $r \in \mathbb{R}_+$ and $\lambda > 0$, and let \mathbf{T} be a weighted tree with edge lengths. By Definitions 5.4.1 and 5.5.1, it is clear that

$$R(\lambda \cdot T) = \lambda \cdot R(T) \quad \text{and} \quad \mathbf{R}_r(\lambda \cdot \mathbf{T}) = \lambda \cdot \mathbf{R}_r(\mathbf{T}). \quad (5.5.1)$$

We also observe that $R : \{T \in \overline{\mathbb{T}} : \mathcal{S}(T) \geq 1\} \rightarrow \overline{\mathbb{T}}$ and $\mathbf{R}_r : \{\mathbf{T} \in \overline{\mathbb{T}}_w : \mathcal{S}(\mathbf{T}) \geq r\} \rightarrow \overline{\mathbb{T}}_w$ are measurable, where $\overline{\mathbb{T}}$ (resp. $\overline{\mathbb{T}}_w$) is the space of all (resp. weighted) trees with edge lengths. Better still, one does not need all the information in \mathbf{T} to determine a particular component of $\mathbf{R}_r(\mathbf{T})$.

Remark 5.5.3 Let $\mathbf{T} = (t, (l_u)_{u \in t}, (w_v)_{v \in \partial t})$ be a weighted tree with edge lengths.

- (i) The Horton pruning is a specific case of a weighted Horton pruning as $R_1(\mathbf{T}) = R(T)$. Indeed, for all weighted tree \mathbf{t} , $\mathcal{S}(\mathbf{t}) \geq 1$ if and only if $\mathcal{S}(t) \geq 1$ by Proposition 5.4.3. In particular, $R(T)$ is well-defined and Proposition 5.5.2 yields the formula (5.4.3).
- (ii) The weighted tree $\text{Sha}(\mathbf{R}_r(\mathbf{T}))$ can be expressed as a measurable function of $\text{Sha}(\mathbf{T})$. Indeed, the lengths of \mathbf{T} are only used to compute the lengths of $\mathbf{R}_r(\mathbf{T})$.
- (iii) If $r \in [0, 1]$, the tree with edge length $R_r(\mathbf{T})$ can be expressed as a measurable function of $(t, (l_u)_{u \in t}, (\mathbf{1}_{\{w_v \geq r\}})_{v \in \partial t})$. Indeed, we do not need to compute the weights to know $R_r(\mathbf{T})$, only to observe which vertices $u \in t$ are such that $\mathcal{S}(\theta_u \mathbf{T}) \geq r$. Moreover, by (5.4.10), this information is contained in $(t, (l_u)_{u \in t}, (\mathbf{1}_{\{w_v \geq r\}})_{v \in \partial t})$ since $r \leq 1$.
- (iv) The weighted tree with edge lengths $\mathbf{R}_1(\mathbf{T})$ can be expressed as a measurable function of $(t, (l_u)_{u \in t \setminus \partial t}, (w_v)_{v \in \partial t})$. Indeed, $\mathcal{S}(\theta_v \mathbf{T}) < 1$ for all $v \in \partial t$, so the parental edges of leaves are erased and the lengths l_v do not appear in the expression of $\mathbf{R}_1(\mathbf{T})$.

□

We are now ready to present the invariance of a $\overline{\text{GW}}_\alpha$ -weighted tree with edge lengths \mathcal{T} (recall Definition 5.2.16) by weighted Horton pruning. Let us explain why we cannot hope for the law of $\mathbf{R}_r(\mathcal{T})$ under $\mathbb{P}(\cdot \mid \mathcal{S}(\mathcal{T}) \geq r)$ to be exactly the same as the law of $\delta^r \cdot \mathcal{T}$ under \mathbb{P} . If a leaf of $\mathbf{R}_r(\mathcal{T})$ has a long parental edge then there is a higher probability that this edge comes from the fusion of many edges of \mathbf{T} , which means its weight is more likely to be large. Thus, the weight and the length of the parental edge of such a leaf are not independent. We get around this issue by erasing the leaves with the map \mathbf{R}_1 before comparing $\mathbf{R}_r(\mathcal{T})$ and $\delta^r \cdot \mathcal{T}$.

Theorem 5.5.4 Let \mathcal{T} be a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths and let $r \in \mathbb{R}_+$.

- (i) The law of $R_r(\mathcal{T})$ under $\mathbb{P}(\cdot | \mathcal{S}(\mathcal{T}) \geq r)$ is the same as the law of $\delta^r \cdot \mathcal{T}$ under \mathbb{P} .
- (ii) The law of $\text{Sha}(\mathbf{R}_r(\mathcal{T}))$ under $\mathbb{P}(\cdot | \mathcal{S}(\mathcal{T}) \geq r)$ is the same as the law of $\text{Sha}(\mathcal{T})$ under \mathbb{P} .
- (iii) The law of $\mathbf{R}_1 \circ \mathbf{R}_r(\mathcal{T})$ under $\mathbb{P}(\cdot | \mathcal{S}(\mathcal{T}) \geq r + 1)$ is the same as the law of $\delta^r \cdot \mathbf{R}_1(\mathcal{T})$ under $\mathbb{P}(\cdot | \mathcal{S}(\mathcal{T}) \geq 1)$.

Proof. By Remark 5.5.3 (iv) and Proposition 5.5.2, we only need to show that the law of $F_i(\mathbf{R}_r(\mathcal{T}))$ under $\mathbb{P}(\cdot | \mathcal{S}(\mathcal{T}) \geq r)$ is equal to the law of $F_i(\delta^r \cdot \mathcal{T})$ under \mathbb{P} for all $i \in \{1, 2\}$, where

$$F_1(\mathbf{T}) = (t, (l_u)_{u \in t}) \quad \text{and} \quad F_2(\mathbf{T}) = (t, (l_u)_{u \in t \setminus \partial t}, (w_v)_{v \in \partial t}).$$

Note that $K(\mathcal{T}) \neq 1$ by (5.2.16), and $K(R_r(\mathcal{T})) \neq 1$ by Proposition 5.5.2. Let $k \in \mathbb{N}$ with $k \geq 2$, $\lambda > 0$, $x \in (0, 1)$, and let $g_1, \dots, g_k : \overline{\mathbb{T}}_w \rightarrow \mathbb{R}$ be bounded measurable functions. Let us set $G(\mathbf{T}) = \mathbf{1}_{\{K(\mathcal{T})=k\}} e^{-\lambda L(\mathcal{T})} \prod_{j=1}^k g_j(\theta_{(j)} \mathbf{T})$ for all $\mathbf{T} \in \overline{\mathbb{T}}_w$. Then, by induction on the height, we only need to check the three following identities:

$$\mathbb{P}(K(\mathcal{T}) = 0; \mathcal{S}(\mathcal{T}) \leq x) = \mathbb{P}(K(R_r(\mathcal{T})) = 0; \mathcal{S}(\mathbf{R}_r(\mathcal{T})) \leq x | \mathcal{S}(\mathcal{T}) \geq r), \quad (5.5.2)$$

$$\mathbb{E}[\mathbf{1}_{\{K(\mathcal{T})=0\}} e^{-\lambda \delta^r L(\mathcal{T})}] = \mathbb{E}[\mathbf{1}_{\{K(R_r(\mathcal{T}))=0\}} e^{-\lambda L(R_r(\mathcal{T}))} | \mathcal{S}(\mathcal{T}) \geq r], \quad (5.5.3)$$

$$\mathbb{E}[G(\mathbf{R}_r(\mathcal{T})) | \mathcal{S}(\mathcal{T}) \geq r] = \mathbb{E}[\mathbf{1}_{\{K(\mathcal{T})=k\}} e^{-\lambda \delta^r L(\mathcal{T})}] \prod_{j=1}^k \mathbb{E}[g_j(\mathbf{R}_r(\mathcal{T})) | \mathcal{S}(\mathcal{T}) \geq r]. \quad (5.5.4)$$

Since $x < 1$, we have $\mathbb{P}(K(\mathcal{T}) = 0; \mathcal{S}(\mathcal{T}) \leq x) = \mathbb{P}(\mathcal{S}(\mathcal{T}) \leq x)$ by (5.4.10). Again by (5.4.10), and since it holds $\mathcal{S}(\mathbf{R}_r(\mathcal{T})) = \mathcal{S}(\mathcal{T}) - r$ when $\mathcal{S}(\mathcal{T}) \geq r$ by Proposition 5.5.2, we find

$$\mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r; K(R_r(\mathcal{T})) = 0; \mathcal{S}(\mathbf{R}_r(\mathcal{T})) \leq x) = \mathbb{P}(r \leq \mathcal{S}(\mathcal{T}) \leq x + r).$$

The identity (5.5.2) follows because the law of $\mathcal{S}(\mathcal{T})$ is exponential, as stated by Proposition 5.4.6.

We set $\chi_2(\mathbf{T}) = \mathbf{1}_{\{\mathcal{S}(\mathbf{T}) \geq r\}} \mathbf{1}_{\{K(R_r(\mathbf{T}))=0\}} e^{-\lambda L(R_r(\mathbf{T}))}$ to simplify. Proposition 5.5.2 entails that

$$\chi_2(\mathcal{T}) = e^{-\lambda L(\mathcal{T})} \left(\sum_{i=1}^{K(\mathcal{T})} \chi_2(\theta_{(i)} \mathcal{T}) \prod_{\substack{1 \leq j \leq K(\mathcal{T}) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)} \mathcal{T}) < r\}} + \prod_{i=1}^{K(\mathcal{T})} \mathbf{1}_{\{\mathcal{S}(\theta_{(i)} \mathcal{T}) < r\}} - \mathbf{1}_{\{\mathcal{S}(\mathcal{T}) < r\}} \right).$$

Recall from (5.1.2) that φ_α stands for the generating function of the offspring law μ_α . Thanks to (5.3.11) and Definition 5.2.16 of $\overline{\mathbf{GW}}_\alpha$ -weighted trees with edge lengths, we can compute that

$$\mathbb{E}[\chi_2(\mathcal{T})] = \frac{\varphi_\alpha(\mathbb{P}(\mathcal{S}(\mathcal{T}) < r)) - \mathbb{P}(\mathcal{S}(\mathcal{T}) < r)}{1 + \lambda - \varphi'_\alpha(\mathbb{P}(\mathcal{S}(\mathcal{T}) < r))}.$$

The identity (5.5.3) follows from Proposition 5.4.6 and from the expression (5.1.2) of φ_α .

To lighten notations, we set $\chi_3(\mathbf{T}) = \mathbf{1}_{\{\mathcal{S}(\mathbf{T}) \geq r\}} G(\mathbf{R}_r(\mathbf{T}))$ and $\chi'_j(\mathbf{T}) = \mathbf{1}_{\{\mathcal{S}(\mathbf{T}) \geq r\}} g_j(\mathbf{R}_r(\mathbf{T}))$ for all $1 \leq j \leq k$. Once again, Proposition 5.5.2 yields that

$$\chi_3(\mathcal{T}) e^{\lambda L(\mathcal{T})} = \sum_{i=1}^{K(\mathcal{T})} \chi_3(\theta_{(i)} \mathcal{T}) \prod_{\substack{1 \leq j \leq K(\mathcal{T}) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)} \mathcal{T}) < r\}} + \sum_{\substack{1 \leq i_1 < \dots \\ \dots < i_k \leq K(\mathcal{T})}} \left(\prod_{j=1}^k \chi'_j(\theta_{(i_j)} \mathcal{T}) \right) \prod_{\substack{1 \leq i \leq K(\mathcal{T}) \\ i \neq i_1, \dots, i_k}} \mathbf{1}_{\{\mathcal{S}(\theta_{(i)} \mathcal{T}) < r\}}.$$

For all $s \in [0, 1)$, we know that $\varphi_\alpha^{(k)}(s)/k! = \mathbb{E}[\mathbf{1}_{\{K(\mathcal{T}) \geq k\}} \binom{K(\mathcal{T})}{k} s^{K(\mathcal{T})-k}] = \mu_\alpha(k)(1-s)^{\alpha-k}$ from (5.2.15). With the same arguments as before, we eventually obtain

$$\mathbb{E}[\chi_3(\mathcal{T})] = \frac{\mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r)^\alpha}{1 + \lambda - \varphi'_\alpha(\mathbb{P}(\mathcal{S}(\mathcal{T}) < r))} \mu_\alpha(k) \prod_{j=1}^k \mathbb{E}[g_j(\mathbf{R}_r(\mathcal{T})) | \mathcal{S}(\mathcal{T}) \geq r],$$

which implies the identity (5.5.4). ■

5.5.2 Some applications

Here, we give some useful consequences of Theorem 5.5.4. Recall Definition 5.2.10 of embeddings.

Corollary 5.5.5 *Let $A : \mathbb{T} \rightarrow \mathbb{R}$ have the following monotony property: for all trees $t, t' \in \mathbb{T}$, if there is an embedding $\psi : t' \rightarrow t$ then $A(t') \leq A(t)$. Let τ be a $\overline{\mathbf{GW}}_\alpha$ -weighted tree. For all $a \in \mathbb{R}$ and for all $x, y \in \mathbb{R}_+ \setminus \mathbb{N}$ such that $x \leq y$, it holds that*

$$\mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = x) \leq \mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = y), \quad (5.5.5)$$

$$\mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = x) \leq \mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = \lfloor x \rfloor + 1), \quad (5.5.6)$$

$$\mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = \lfloor y \rfloor) \leq \mathbb{P}(A(\tau) \geq a \mid \mathcal{S}(\tau) = y + 1). \quad (5.5.7)$$

In particular, the result applies when A is the size $A(t) = \#t$, and when A is the height $A(t) = |t|$.

Proof. We can assume that $\tau = \text{Sha}(\mathcal{T})$, where \mathcal{T} is a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths. Let $r \in [0, 1]$ and $x > 0$ be such that $x, x + r \notin \mathbb{N}$. By Definition 5.5.1, the property of A yields

$$\mathbb{P}(A(\text{Sha}(R_r(\mathcal{T}))) \geq a; |\mathcal{S}(\mathcal{T}) - (x + r)| < \varepsilon) \leq \mathbb{P}(A(\tau) \geq a; |\mathcal{S}(\mathcal{T}) - (x + r)| < \varepsilon)$$

for all $\varepsilon \in (0, x)$. Since $\mathcal{S}(\text{Sha}(R_r(\mathcal{T}))) = \mathcal{S}(\mathcal{T}) - r$ when $\mathcal{S}(\mathcal{T}) \geq r$, Theorem 5.5.4 (ii) entails

$$\mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r) \mathbb{P}(A(\tau) \geq a; |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) \leq \mathbb{P}(A(\tau) \geq a; |\mathcal{S}(\mathcal{T}) - (x + r)| < \varepsilon).$$

Moreover, the law of $\mathcal{S}(\mathcal{T})$ is exponential by Proposition 5.4.6 so dividing the above inequality by $\mathbb{P}(|\mathcal{S}(\mathcal{T}) - (x + \varepsilon)| < \varepsilon)$ and making $\varepsilon \rightarrow 0^+$ gives (5.5.5) with $y = x + r$, by Definition 5.4.8. We generalize that for any x, y by induction on $\lfloor y - x \rfloor$. Integrating (5.5.5) with respect to y against the conditional law of $\mathcal{S}(\tau)$ given $\lfloor x \rfloor + 1 \leq \mathcal{S}(\tau) < \lfloor x \rfloor + 2$ yields (5.5.6), by Proposition 5.4.7. We get (5.5.7) similarly by integrating (5.5.5) with respect to x . ■

Like Lemma 5.4.9, but with greater precision, Corollary 5.5.5 allows controlling $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ with estimates about the nondegenerate conditional law $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = n)$. Now, recall from (5.2.22) the height function $\bar{H}(T)$ of a tree with edge lengths T . The following proposition shows that the height function of a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths does not change much after r -Horton pruning. Indeed, if $r \in [0, 1]$ then only some leaves are erased, and in a uniform manner.

Proposition 5.5.6 *Recall d_S from (5.2.3). Let $\mathcal{T} = (\tau, (L_u)_{u \in \tau}, (W_v)_{v \in \partial\tau})$ be a $\overline{\mathbf{GW}}_\alpha$ -weighted tree with edge lengths, let $r \in [0, 1]$, and let $\lambda > 0$. We define two random càdlàg functions with compact support X and Y by setting for all $s \in \mathbb{R}_+$,*

$$X_s = \lambda^{1-1/\alpha} \bar{H}_{s/\lambda}(\mathcal{T}) \quad \text{and} \quad Y_s = \mathbf{1}_{\{\mathcal{S}(\mathcal{T}) \geq r\}} \lambda^{1-1/\alpha} \bar{H}_{e^{-\gamma r} s/\lambda}(R_r(\mathcal{T})).$$

Then, there are two constants $C, c \in (0, \infty)$ that only depend on α such that for all $n \geq 1$,

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X, Y) \geq C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + C\lambda\sqrt{n} \ln n) \leq Cne^{-c(\ln n)^2}.$$

Proof. Let $(\xi_i, L_i, W_i)_{i \in \mathbb{N}}$ be independent random variables such that for all $i \in \mathbb{N}$, ξ_i has the same law as $k_\emptyset(\tau) - 1$, the law of L_i is exponential with mean 1, and the law of W_i is $\text{FExp}(\gamma)$. Recall from (5.2.12) the depth-first exploration $u(\tau)$. By (5.2.19) and Definition 5.2.16, we can assume that

$$\xi_i = k_{u_i(\tau)}(\tau) - 1, \quad L_i = L_{u_i(\tau)}, \quad \text{and} \quad (\xi_i = -1) \implies (W_i = W_{u_i(\tau)})$$

for all $0 \leq i \leq \#\tau - 1$. We now define a sequence of stopping times $(J_i)_{i \in \mathbb{N}}$ and a random integer N by setting $J_{-1} = -1$ and for all $i \in \mathbb{N}$,

$$J_i = \inf \{j > J_{i-1} : \xi_j \neq -1 \text{ or } W_j \geq r\} \quad \text{and} \quad N = \inf \{j \geq 0 : J_j \geq \#\tau\}. \quad (5.5.8)$$

Since $r \leq 1$, we observe by (5.2.16) and (5.4.10) that $u_{J_0}(\tau), u_{J_1}(\tau), \dots, u_{J_{N-1}}(\tau)$ are exactly the vertices $u \in \tau$ such that $\mathcal{S}(\theta_u \tau) \geq r$, listed in lexicographic order. Then, for all $j \in \mathbb{N}$, we set

$$s_j(X) = \lambda \sum_{\iota=0}^{J_{j-1}} L_\iota = \lambda \sum_{i=0}^{j-1} \sum_{\iota=J_{i-1}+1}^{J_i} L_\iota \quad \text{and} \quad s_j(Y) = \lambda e^{\gamma r} \sum_{i=0}^{j-1} L_{J_i}.$$

Note that $s_0(X) = s_0(Y) = 0$. Thanks to Proposition 5.2.13 and (5.2.23), we identify that $\zeta(Y) = s_N(Y)$, that $s_N(X) \leq \zeta(X) < s_{N+1}(X)$, and that if $s \in [s_j(Y), s_{j+1}(Y))$ with $0 \leq j \leq N - 1$ then

$$\lambda^{1-1/\alpha} \sum_{v \prec u_{J_j}(\tau)} L_v = Y_{s_j(Y)} \leq Y_s \leq Y_{s_{j+1}(Y)-} = \lambda^{1-1/\alpha} \sum_{v \preceq u_{J_j}(\tau)} L_v. \quad (5.5.9)$$

We write $\Delta(f) = \sup_{s \geq 0} |f(s) - f(s-)|$ for any càdlàg function f . We claim that

$$d_S(X, Y) \leq 6 \max_{1 \leq j \leq N+1} |s_j(X) - s_j(Y)| + 3e^\gamma \lambda \max_{0 \leq j \leq N} L_{J_j} + 2\lambda^{1-1/\alpha} \max_{u \in \tau} L_u + \Delta(Y). \quad (5.5.10)$$

Proof of (5.5.10). Let $1 \leq j \leq N - 1$ and let $u \in \tau$ such that $u_{J_{j-1}}(\tau) < u \leq u_{J_j}(\tau)$. By definition of the lexicographic order, we remark that $u_{J_{j-1}}(\tau) \wedge u_{J_j}(\tau)$ is an ancestor of $u_{J_{j-1}}(\tau) \wedge u$. Moreover, it either holds $u = u_{J_j}(\tau)$, or $\mathcal{S}(\theta_u \tau) < r$ and u is a leaf of τ . The contrapositive of this argument entails that $\overleftarrow{u} \leq u_{J_{j-1}}(\tau)$, and it further follows that $\overleftarrow{u} = u \wedge u_{J_{j-1}}(\tau)$. In particular, the parent of $u_{J_j}(\tau)$ is the most recent common ancestor of $u_{J_{j-1}}(\tau)$ and $u_{J_j}(\tau)$. Hence, we have proven $\overleftarrow{u_{J_j}(\tau)} \preceq \overleftarrow{u} \preceq u_{J_{j-1}}(\tau)$, and (5.5.9) then yields that

$$Y_{s_j(Y)} \leq \lambda^{1-1/\alpha} \sum_{v \prec u} L_v \leq Y_{s_j(Y)-}. \quad (5.5.11)$$

Now, we choose an increasing and bijective $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi(s_j(Y)) = s_j(X)$ for all $0 \leq j \leq N$ and $\psi(s_N(Y) + s) = s_N(X) + s$ for all $s \geq 0$. The facts preceding (5.5.9) give that

$$|\zeta(X) - \zeta(Y)| \leq 2|s_N(X) - s_N(Y)| + |s_{N+1}(X) - s_{N+1}(Y)| + \lambda e^{\gamma r} L_{J_N}.$$

Let $s \in [s_j(Y), s_{j+1}(Y))$ with $1 \leq j < N$, we have $\psi(s) \in [s_j(X), s_{j+1}(X))$ so

$$|\psi(s) - s| \leq 2|s_j(X) - s_j(Y)| + |s_{j+1}(X) - s_{j+1}(Y)| + 2\lambda e^{\gamma r} L_{J_j}.$$

Plus, there is $u \in \tau$ such that $u_{J_{j-1}}(\tau) < u \leq u_{J_j}(\tau)$ and $\sum_{v \prec u} L_v \leq \lambda^{1/\alpha-1} X_{\psi(s)} < \sum_{v \preceq u} L_v$, by definition (5.2.22) of the height function. The inequalities (5.5.9) and (5.5.11) then entail that

$$|X_{\psi(s)} - Y_s| \leq \lambda^{1-1/\alpha} L_u + \lambda^{1-1/\alpha} L_{u_{J_j}(\tau)} + |Y_{s_j(Y)} - Y_{s_j(Y)-}|.$$

We treat the cases where $s \in [0, s_1(Y))$ or $s \geq s_N(Y)$ similarly. This completes the proof. \square

From (5.5.8), we observe that the sequence $(J_j)_{j \in \mathbb{N}}$ is independent of $(L_i)_{i \in \mathbb{N}}$, which is a sequence of independent and identically distributed random variables, thus $(L_{J_j})_{j \in \mathbb{N}}$ has the same law as $(L_j)_{j \in \mathbb{N}}$. Moreover, the random variables $(J_j - J_{j-1} - 1)_{j \in \mathbb{N}}$ are independent and geometric with parameter $\mathbb{P}(\xi_0 \neq -1 \text{ or } W_0 \geq r) = \mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r) = e^{-\gamma r}$, and jointly independent from

$(L_i)_{i \in \mathbb{N}}$. A well-known fact then asserts that the random sequence $(e^{-\gamma r} \sum_{J_{j-1} < \iota \leq J_j} L_\iota)_{j \in \mathbb{N}}$ has also the same law as $(L_j)_{j \in \mathbb{N}}$. For all $j \in \mathbb{N}$, we thus have

$$\mathbb{P}(|s_j(X) - s_j(Y)| \geq e^{\gamma r} \lambda \sqrt{n} \ln n) \leq 2 \mathbb{P}\left(\left|i - \sum_{i=0}^{j-1} L_i\right| \geq \sqrt{n} \ln n\right).$$

It is clear from (5.5.8) that $\mathbb{N} \leq \#\tau$ almost surely, so Lemma 5.3.10 entails that

$$\mathbb{P}(\#\tau \leq n; \max_{1 \leq j \leq \mathbb{N}+1} |s_j(X) - s_j(Y)| \geq e^{\gamma r} \lambda \sqrt{n} \ln n) \leq 8ne^{-c_{\text{uni}}(\ln n)^2} \quad (5.5.12)$$

for all n large enough. Then, a simple union bound together with $\mathbb{N} \leq \#\tau$ gives us

$$\mathbb{P}(\#\tau \leq n; \lambda \max_{0 \leq j \leq \mathbb{N}} L_{I_j} + \lambda^{1-1/\alpha} \max_{u \in \tau} L_u \geq \lambda(\ln n)^2 + \lambda^{1-1/\alpha}(\ln n)^2) \leq 3ne^{-(\ln n)^2}. \quad (5.5.13)$$

By Definition 5.5.1, we observe that if $\mathcal{S}(\mathcal{T}) \geq r$ then $\#R_r(\mathcal{T}) \leq \#\tau$ and $|R_r(\mathcal{T})| \leq |\tau|$. Hence,

$$\begin{aligned} \chi &:= \mathbb{P}(\#R_r(\mathcal{T}) \leq n; |R_r(\mathcal{T})| \leq n^{1-1/\alpha}; e^{-\gamma r} \Delta(Y) \geq 2C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + 2C\lambda\sqrt{n} \ln n) \\ &\geq \mathbb{P}(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; e^{-\gamma r} \Delta(Y) \geq 2C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + 2C\lambda\sqrt{n} \ln n) \end{aligned}$$

for any $C > 0$. Plus, if $\mathcal{S}(\mathcal{T}) < r$ then $\Delta(Y) = 0$. Using (5.2.23), Theorem 5.5.4 (i) then yields that

$$\chi = \mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r) \mathbb{P}(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; \Delta(X) \geq 2C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + 2C\lambda\sqrt{n} \ln n).$$

Recalling the expression (5.2.3) of d_S , we see that if f and g are respectively càdlàg and continuous with compact support, then $\Delta(f) \leq \Delta(g) + 2d_S(f, g) = 2d_S(f, g)$. Therefore, according to Proposition 5.3.9, there are $C, c \in (0, \infty)$ that only depend on α such that

$$\mathbb{P}(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; \Delta(Y) \geq 2C(\lambda\sqrt{n})^{1-1/\alpha} \ln n + 2C\lambda\sqrt{n} \ln n) \leq Cne^{-c(\ln n)^2}$$

because the height function (5.2.13) of a tree (without edge lengths) is continuous. Combining this last estimate with the inequalities (5.5.10), (5.5.12), and (5.5.13) completes the proof. \blacksquare

5.6 Proof of Theorem 5.1.3

In this section, we extensively use the topological tools and the notations presented in Section 5.2.1, so recall the spaces \mathcal{C}_K and \mathcal{D}_K , the Skorokhod distance d_S from (5.2.3), the Prokhorov metric ρ_S associated with d_S from (5.2.4), and Notation 5.2.2. Furthermore, recall the height function $H(t)$ of a tree t from (5.2.13), and the law of a GW_α -tree conditioned on its weighted Horton–Strahler number from Definition 5.4.8. Our goal here is to prove the following result.

Theorem 5.6.1 *Let τ be a GW_α -weighted tree. The law of the process $(e^{-\gamma(\alpha-1)x} H_{e^{\gamma\alpha x}}(\tau))_{s \geq 0}$ under $\mathbb{P}(d\tau | \mathcal{S}(\tau) = x)$ weakly converges on \mathcal{C}_K , when $x \in \mathbb{R}_+ \setminus \mathbb{N}$ tends to ∞ .*

Indeed, Theorem 5.6.1 implies Theorem 5.1.3 by Proposition 5.2.18 and Skorokhod's representation theorem. Moreover, the estimates (5.4.7) and (5.4.9), together with Corollary 5.5.5, entail that the limit law is not degenerate: see Section 5.7 for details. We even give a somewhat speed of convergence for Theorem 5.6.1 via the following theorem, whose we postpone the proof for later.

Theorem 5.6.2 *Let τ be a GW_α -weighted tree. For all $x \in \mathbb{R}_+ \setminus \mathbb{N}$, we denote by ν_x the law on \mathcal{C}_K of $(e^{-\gamma(\alpha-1)x} H_{e^{\gamma\alpha x}}(\tau))_{s \geq 0}$ under $\mathbb{P}(d\tau | \mathcal{S}(\tau) = x)$. Then, there are two constants $C, c \in (0, \infty)$ that only depend on α such that it holds $\rho_S(\nu_x, \nu_y) \leq Ce^{-c \min(x, y)}$ for all $x, y \in \mathbb{R}_+ \setminus \mathbb{N}$.*

One might be tempted to see Theorem 5.6.1 as a direct consequence of Theorem 5.6.2, arguing the function $x \mapsto \nu_x$ is Cauchy so it must converge. Unfortunately, we have to remind from Remark 5.2.5 that ρ_S is not a complete metric. Nevertheless, it is not a hard challenge to overcome.

Proof of Theorem 5.6.1 from Theorem 5.6.2. We keep the notations of Theorem 5.6.2. We claim that we only need to show that the family $(\nu_x)_{x \in \mathbb{R}_+ \setminus \mathbb{N}}$ is tight on \mathcal{C}_K . *Indeed*, there would then exist an increasing sequence of points $x_n \in \mathbb{R}_+ \setminus \mathbb{N}$ that tends to ∞ such that the sequence (ν_{x_n}) weakly converges on \mathcal{C}_K by Prokhorov's theorem. Together with Proposition 5.2.4, Theorem 5.6.2 would then yield that the family $(\nu_x)_{x \in \mathbb{R}_+ \setminus \mathbb{N}}$ converges with respect to ρ_S when x tends to ∞ . Since the ν_x are laws on \mathcal{C}_K , they would also weakly converge on \mathcal{C}_K by Proposition 5.2.1. \square

We thus aim to apply Proposition 5.2.3. Firstly, we readily get Proposition 5.2.3 (a) because it holds $H_0(\tau) = 0$ almost surely. Secondly, we know that $\zeta(H(\tau)) = \mathbf{1}_{\{\#\tau \geq 2\}} \#\tau$ from (5.2.14). Next, we apply Corollary 5.5.5 followed by Markov's inequality to find that for all $m \geq 1$ and $x \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$\nu_x(\zeta \geq m) \leq \mathbb{P}(\#\tau \geq e^{\gamma \alpha x} m \mid \mathcal{S}(\tau) = \lfloor x \rfloor + 1) \leq \frac{1}{m} e^{-\gamma \alpha x} \mathbb{E}[\#\tau \mid \mathcal{S}(\tau) = \lfloor x \rfloor + 1].$$

Therefore, (5.4.6) yields Proposition 5.2.3 (b). It only remains to show that for all $\varepsilon > 0$,

$$\lim_{\eta \rightarrow 0^+} \limsup_{x \rightarrow \infty, x \notin \mathbb{N}} \nu_x(\omega_\eta \geq 2\varepsilon) = 0. \quad (5.6.1)$$

But before, we instead work under the discrete conditioning $\{\mathcal{S}(\tau) = n\}$. For all $n \in \mathbb{N}$, we denote by ν'_n the law on \mathcal{C}_K of the process $(e^{-\gamma(\alpha-1)n} H_{e^{\gamma \alpha n} s}(\tau))_{s \geq 0}$ under $\mathbb{P}(\cdot \mid \mathcal{S}(\tau) = n)$. We fix $\ell > 0$ for now, we write $N_n = e^{\gamma(\alpha-1)n}$, and we note that $N_n^{\alpha\beta} = e^{\gamma \alpha n}$, where recall from (5.1.8) that $\beta(\alpha-1) = 1$. Using (5.2.14), we write the rough bound

$$\nu'_n(\omega_\eta \geq \varepsilon, \sup \geq \ell) \leq \frac{\mathbb{P}(|\tau| \geq \ell e^{\gamma(\alpha-1)n})}{\mathbb{P}(\mathcal{S}(\tau) = n)} \mathbb{P}\left(\omega_\eta\left(\frac{1}{N_n} H_{N_n^{\alpha\beta} s}(\tau); s \geq 0\right) \geq \varepsilon \mid |\tau| \geq \ell N_n\right).$$

We apply (5.3.9) in Theorem 5.3.7, in conjunction with (5.3.3) and (5.4.4), and we eventually obtain that

$$\lim_{\eta \rightarrow 0^+} \limsup_{n \rightarrow \infty} \nu'_n(\omega_\eta \geq \varepsilon) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| \leq \ell e^{\gamma(\alpha-1)n} \mid \mathcal{S}(\tau) = n)$$

for all $\varepsilon, \ell > 0$. By making ℓ tend to 0^+ , the estimate (5.4.7) entails that for all $\varepsilon > 0$, it holds that

$$\lim_{\eta \rightarrow 0^+} \limsup_{n \rightarrow \infty} \nu'_n(\omega_\eta \geq \varepsilon) = 0. \quad (5.6.2)$$

Let $\varepsilon, \eta > 0$ and let $n \in \mathbb{N}$ be large enough so that $2Ce^{-cn} \leq \max(\varepsilon, \eta)$, where C and c are as in Theorem 5.6.2. For all $x, y \in (n, n+1)$, we get $\nu_x(\omega_\eta \geq 2\varepsilon) \leq \nu_y(\omega_{2\eta} \geq \varepsilon) + Ce^{-cn}$ by Theorem 5.6.2. According to Proposition 5.4.7, integrating this inequality with respect to y against the law of $\mathcal{S}(\tau)$ under $\mathbb{P}(\cdot \mid \mathcal{S}(\tau) = n)$ yields that $\nu_x(\omega_\eta \geq 2\varepsilon) \leq \nu'_n(\omega_{2e^{\gamma \alpha} \eta} \geq e^{\gamma(\alpha-1)} \varepsilon) + Ce^{-cn}$. We point out that we made the factors $e^{\gamma \alpha}$ and $e^{\gamma(\alpha-1)}$ appear inside the right-hand side to compensate for the ratio of the scalings $H(\tau)$ is subject to, depending on whether it is observed under ν_y or ν'_n . Finally, the desired condition (5.6.1) follows from (5.6.2), which concludes the proof. \blacksquare

The rest of the section is devoted to the proof of Theorem 5.6.2. We begin by applying the invariance by weighted Horton pruning stated in Theorem 5.5.4 (together with Proposition 5.5.6) to bound the Prokhorov distance between the laws of the rescaled height function (as defined by (5.2.22)) of a $\overline{\text{GW}}_\alpha$ -weighted tree with edge lengths \mathcal{T} respectively under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$ or under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x - r)$.

Lemma 5.6.3 *Let \mathcal{T} be a $\overline{\text{GW}}_\alpha$ -weighted tree with edge lengths. For all $x \in \mathbb{R}_+ \setminus \mathbb{N}$, we denote by $\bar{\nu}_x$ the law on \mathcal{D}_K of the process $(e^{-\gamma(\alpha-1)x} \bar{H}_{e^{\gamma\alpha x_s}(\mathcal{T})})_{s \geq 0}$ under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$. Then, there are two constants $C, c \in (0, \infty)$ that only depend on α such that $\rho_S(\bar{\nu}_x, \bar{\nu}_{x-r}) \leq Ce^{-cx}$, for all $r \in [1/10, 9/10]$ and all $x > r$ with $x, x-r \notin \mathbb{N}$.*

Proof. The Prokhorov distance between two probability measures is always bounded by 1, so we can assume $x \geq 2 > r + 1$ without loss of generality. We define five random càdlàg functions with compact support X, Y, Z, X', Y' by setting for all $s \in \mathbb{R}_+$,

$$\begin{aligned} X_s &= e^{-\gamma(\alpha-1)x} \bar{H}_{e^{\gamma\alpha x_s}(\mathcal{T})}, \\ Y_s &= \mathbf{1}_{\{\mathcal{S}(\mathcal{T}) \geq r\}} e^{-\gamma(\alpha-1)x} \bar{H}_{e^{-\gamma r} e^{\gamma\alpha x_s}(R_r(\mathcal{T}))}, \\ Z_s &= \mathbf{1}_{\{\mathcal{S}(\mathcal{T}) \geq r, \mathcal{S}(R_r(\mathcal{T})) \geq 1\}} e^{-\gamma(\alpha-1)x} \bar{H}_{e^{-\gamma(1+r)} e^{\gamma\alpha x_s}(R_1 \circ R_r(\mathcal{T}))}, \\ X'_s &= e^{-\gamma(\alpha-1)(x-r)} \bar{H}_{e^{\gamma\alpha(x-r)_s}(\mathcal{T})}, \\ Y'_s &= \mathbf{1}_{\{\mathcal{S}(\mathcal{T}) \geq 1\}} e^{-\gamma(\alpha-1)(x-r)} \bar{H}_{e^{-\gamma} e^{\gamma\alpha(x-r)_s}(R_1(\mathcal{T}))}. \end{aligned}$$

The law of X under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$ is $\bar{\nu}_x$, and the law of X' under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x-r)$ is $\bar{\nu}_{x-r}$. Informally, the first step of the proof is to show that X and X' are respectively close to Z and Y' with high probability by using Proposition 5.5.6. Then, we prove with Theorem 5.5.4 that the ‘conditional law of Z given $\mathcal{S}(\mathcal{T}) = x$ ’ is the same as the law of Y' under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x-r)$. However, Z is not measurable with respect to \mathcal{T} , since we need the weights to determine $R_r(\mathcal{T})$, so Definition 5.4.8 does not make sense of its law under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$. We avoid this issue by instead working conditionally given $|\mathcal{S}(\mathcal{T}) - x| < \varepsilon$, with a small $\varepsilon > 0$.

Let us set $n = e^{3\gamma\alpha x/2}$. Applying Proposition 5.5.6 with $\lambda = e^{-\gamma\alpha x}$ and with $\lambda = e^{-\gamma\alpha(x-r)}$, we justify that there exist two constants $C_0, c_0 \in (0, \infty)$ only depending on α such that

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X, Y) \geq C_0 x e^{-\gamma(\alpha-1)x/4}) \leq C_0 e^{3\gamma\alpha x/2 - c_0 x^2}, \quad (5.6.3)$$

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X', Y') \geq C_0 x e^{-\gamma(\alpha-1)x/4}) \leq C_0 e^{3\gamma\alpha x/2 - c_0 x^2}. \quad (5.6.4)$$

Then, Remark 5.5.3 (i) ensures that the couple (Y, Z) can be expressed as a measurable function of $R_r(\mathcal{T})$ when $\mathcal{S}(\mathcal{T}) \geq r$. Hence, by (5.2.23) and (5.5.1), Theorem 5.5.4 (i) yields that the law of (Y, Z) under $\mathbb{P}(\cdot \mid \mathcal{S}(\mathcal{T}) \geq r)$ is the same as the law of (X', Y') under \mathbb{P} . Since it always holds that $\#R_r(\mathcal{T}) \leq \#\mathcal{T}$ and $|R_r(\mathcal{T})| \leq |\mathcal{T}|$, (5.6.4) still holds even if we replace (X', Y') by (Y, Z) . Combining this new version of (5.6.4) with (5.6.3) entails that

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X, Z) \geq 2C_0 x e^{-\gamma(\alpha-1)x/4}) \leq 2C_0 e^{3\gamma\alpha x/2 - c_0 x^2}. \quad (5.6.5)$$

Let us set $\eta = e^{\gamma(3\alpha+1)x - c_0 x^2}$ to lighten the notations. By Remark 5.5.3 (i) and (ii), the only information about the weights of \mathcal{T} we need to determine X', Y', X, Z is if they are smaller than r or not. Thus, we can use (5.4.15) in Lemma 5.4.9 to deduce from (5.6.4) and (5.6.5) that there is a constant $C_1 \in (0, \infty)$, that only depends on α , such that for all $\varepsilon > 0$ small enough (according to x and r),

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X', Y') \geq C_1 x e^{-\frac{1}{4}\gamma(\alpha-1)x} \mid |\mathcal{S}(\mathcal{T}) - x + r| < \varepsilon) \leq C_1 \eta, \quad (5.6.6)$$

$$\mathbb{P}(\#\mathcal{T} \leq n; |\mathcal{T}| \leq n^{1-1/\alpha}; d_S(X, Z) \geq C_1 x e^{-\frac{1}{4}\gamma(\alpha-1)x} \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) \leq C_1 \eta. \quad (5.6.7)$$

We stress that C_1 does not depend on r because we restrict ourselves to the case $r \in [1/10, 9/10]$, which ensures $c_\alpha(r) \leq c_\alpha(1/10)$ with the notations of Lemma 5.4.9. It remains to control the

conditional probabilities that $\#\mathcal{T} \geq n$ or $|\mathcal{T}| \geq n$. We successively use Corollary 5.5.5 and (5.4.14), then a rough bound, and finally (5.4.4) with (5.3.4) (resp. with (5.3.3)) to write that for all small enough ε ,

$$\mathbb{P}(\#\mathcal{T} \geq n \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) \leq \frac{\mathbb{P}(\#\mathcal{T} \geq n)}{\mathbb{P}(\mathcal{S}(\mathcal{T}) = \lfloor x \rfloor + 1)} \leq C_2 e^{-\gamma x/2}, \quad (5.6.8)$$

$$\mathbb{P}(|\mathcal{T}| \geq n^{1-1/\alpha} \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) \leq \frac{\mathbb{P}(|\mathcal{T}| \geq n^{1-1/\alpha})}{\mathbb{P}(\mathcal{S}(\mathcal{T}) = \lfloor x \rfloor + 1)} \leq C_2 e^{-\gamma x/2}, \quad (5.6.9)$$

where $C_2 \in (0, \infty)$ is a constant that only depends on α . The same bounds hold for the conditional probabilities given $|\mathcal{S}(\mathcal{T}) - (x-r)|$. Recalling (5.6.6) and (5.6.7), we conclude that there is a constant $C_3 \in (0, \infty)$ only depending on α such that for all $\varepsilon > 0$ small enough (according to x and r),

$$\mathbb{P}(\mathfrak{d}_S(X', Y') \geq C_3 e^{-\gamma(\alpha-1)x/8} \mid |\mathcal{S}(\mathcal{T}) - (x-r)| < \varepsilon) \leq C_3 e^{-\gamma x/2}, \quad (5.6.10)$$

$$\mathbb{P}(\mathfrak{d}_S(X, Z) \geq C_3 e^{-\gamma(\alpha-1)x/8} \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) \leq C_3 e^{-\gamma x/2}. \quad (5.6.11)$$

Let us prove that the conditional law of Y' given $|\mathcal{S}(\mathcal{T}) - (x-r)| < \varepsilon$ is equal to the conditional law of Z given $|\mathcal{S}(\mathcal{T}) - x| < \varepsilon$, for all small enough $\varepsilon > 0$. More precisely, let $\varepsilon \in (0, x-r-1)$ and let A be a measurable subset of \mathcal{D}_K . By the choice of ε , if $|\mathcal{S}(\mathcal{T}) - x| < \varepsilon$ then $\mathcal{S}(\mathcal{T}) \geq r+1$. Moreover, if $\mathcal{S}(\mathcal{T}) \geq r+1$ then we know from Proposition 5.5.2 that $\mathcal{S}(\mathbf{R}_1 \circ \mathbf{R}_r(\mathcal{T})) = \mathcal{S}(\mathcal{T}) - r - 1$:

$$\mathbb{P}(|\mathcal{S}(\mathcal{T}) - x| < \varepsilon; Z \in A) = \mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r+1; |\mathcal{S}(\mathbf{R}_1 \circ \mathbf{R}_r(\mathcal{T})) - (x-r-1)| < \varepsilon; Z \in A).$$

Then, (5.2.23), (5.5.1), Theorem 5.5.4 (iii), and a similar argument as above yield

$$\mathbb{P}(\mathcal{S}(\mathcal{T}) \geq 1) \mathbb{P}(|\mathcal{S}(\mathcal{T}) - x| < \varepsilon; Z \in A) = \mathbb{P}(\mathcal{S}(\mathcal{T}) \geq r+1) \mathbb{P}(|\mathcal{S}(\mathcal{T}) - (x-r)| < \varepsilon; Y' \in A).$$

Furthermore, we know from Proposition 5.4.6 that the law of $\mathcal{S}(\mathcal{T})$ is exponential, so we get

$$\mathbb{P}(Z \in A \mid |\mathcal{S}(\mathcal{T}) - x| < \varepsilon) = \mathbb{P}(Y' \in A \mid |\mathcal{S}(\mathcal{T}) - (x-r)| < \varepsilon) \quad (5.6.12)$$

as desired. Thanks to (5.4.13) and (5.6.12), making $\varepsilon \rightarrow 0^+$ inside (5.6.10) and (5.6.11) completes the proof. \blacksquare

Proof of Theorem 5.6.2. We use the notations of Lemma 5.6.3 and of Theorem 5.6.2, and we can assume $\tau = \text{Sha}(\mathcal{T})$ as in (5.2.24). Let $x \in \mathbb{R}_+ \setminus \mathbb{N}$, we define two random càdlàg functions with compact support X, Y by setting for all $s \in \mathbb{R}_+$,

$$X_s = e^{-\gamma(\alpha-1)x} \bar{H}_{e^{\gamma\alpha x} s}(\mathcal{T}) \quad \text{and} \quad Y_s = e^{-\gamma(\alpha-1)x} H_{e^{\gamma\alpha x} s}(\tau).$$

The laws of X and Y under $\mathbb{P}(d\mathcal{T} \mid \mathcal{S}(\mathcal{T}) = x)$ are respectively $\bar{\nu}_x$ and ν_x . By Proposition 5.3.9 with $n = e^{3\gamma\alpha x/2}$ and $\lambda = e^{-\gamma\alpha x}$, there are $C_0, c_0 \in (0, \infty)$ only depending on α such that

$$\mathbb{P}(\#\tau \leq n; |\tau| \leq n^{1-1/\alpha}; \mathfrak{d}_S(X, Y) \geq C_0 x e^{-\gamma(\alpha-1)x/4}) \leq C_0 n e^{3\gamma\alpha x/2 - c_0 x^2}. \quad (5.6.13)$$

Thanks to (5.4.15) in Lemma 5.4.9, we combine (5.6.13) with (5.6.8) and (5.6.9) to obtain that

$$\mathbb{P}(\mathfrak{d}_S(X, Y) \geq C_3 e^{-\gamma(\alpha-1)x/8} \mid \mathcal{S}(\mathcal{T}) = x) \leq C_3 e^{-\gamma x/2}, \quad (5.6.14)$$

where $C_3 \in (0, \infty)$ only depends on α . Then, Lemma 5.6.3 and (5.6.14) ensure the existence of two constants $C_4, c_4 \in (0, \infty)$ only depending on α such that $\rho_S(\nu_x, \bar{\nu}_x) \leq C_4 e^{-c_4 x}$ and

$\rho_S(\bar{\nu}_x, \bar{\nu}_y) \leq C_4 e^{-c_4 \min(x,y)}$ for all $x, y \in \mathbb{R}_+ \setminus \mathbb{N}$ with $1/10 \leq |x - y| \leq 9/10$. By triangular inequality (see Proposition 5.2.4), it follows that $\rho_S(\nu_x, \nu_y) \leq 3C_4 e^{-c_4 \min(x,y)}$ for all $x, y \in \mathbb{R}_+ \setminus \mathbb{N}$ with $1/10 \leq |x - y| \leq 9/10$. For all $x, y \in \mathbb{R}_+ \setminus \mathbb{N}$, we can find a finite sequence $(x_i)_{0 \leq i \leq n}$ of positive non-integers such that $1/2 \leq x_i - x_{i-1} \leq 9/10$ for all $1 \leq i \leq n$, $x_0 = \min(x, y)$, and $1/10 \leq x_n - \max(x, y) \leq 9/10$. We complete the proof with the triangular inequality by writing

$$\rho_S(\nu_x, \nu_y) \leq \rho_S(\nu_{x_n}, \nu_{\max(x,y)}) + \sum_{i=0}^{n-1} \rho_S(\nu_{x_{i+1}}, \nu_{x_i}) \leq e^{-c_4 \min(x,y)} \cdot 6C_4 \sum_{i \in \mathbb{N}} e^{-c_4 i/2}.$$

■

5.7 A first description of the limit tree in Theorem 5.1.3

In this section, we begin to study the limit objects of Theorems 5.1.3 and 5.6.1. Recall Notation 5.2.6 from Section 5.2.1, and that \mathbb{K}^m stands for the space of (GHP-isometry classes of) rooted measured compact metric spaces, endowed with the rooted Gromov–Hausdorff–Prokhorov distance (5.2.7).

Definition 5.7.1 A random variable $(\tilde{H}, \tilde{\zeta})$ on \mathcal{C}_K is a HS_α -excursion when it has the limit distribution in Theorem 5.6.1. A random rooted measured compact metric space \mathcal{T} (in other words, a random variable on \mathbb{K}^m) is a HS_α -real tree when it has the limit distribution in Theorem 5.1.3. □

By Proposition 5.2.18 and Skorokhod’s representation theorem, if $(\tilde{H}, \tilde{\zeta})$ is a HS_α -excursion, then it is in \mathcal{E}_K and the real tree $\mathcal{T}_{\tilde{H}, \tilde{\zeta}}$ it codes, as defined by (5.2.27), is a HS_α -real tree. In particular, a HS_α -real tree is almost surely a real tree in the sense of Definition 5.2.17. Now, we give ourselves some estimates for the mass and height (5.2.9) of a HS_α -real tree.

Proposition 5.7.2 Let \mathcal{T} be a HS_α -real tree. Then, $\mathbb{E}[m(\mathcal{T})] < \infty$ and there exists a constant $y_0 \in (0, \infty)$ that only depends on α such that for all $x \geq 0$ and $y \geq y_0$, it holds that

$$\mathbb{P}(\mathfrak{h}(\mathcal{T}) < x) \leq 1 - e^{-4x} \quad \text{and} \quad \mathbb{P}(\mathfrak{h}(\mathcal{T}) > y) \leq e^{-y/40}. \quad (5.7.1)$$

Proof. For all $n \in \mathbb{N}$, let τ_n have the same law as τ under $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x_n)$, where $x_n = n + \frac{1}{2}$ and τ is GW_α -weighted tree. We can assume that $e^{-\gamma \alpha x_n} \# \tau_n \rightarrow m(\mathcal{T})$ and $e^{-\gamma(\alpha-1)x_n} |\tau_n| \rightarrow \mathfrak{h}(\mathcal{T})$ almost surely by continuity of m and \mathfrak{h} . Fatou’s lemma, Corollary 5.5.5, and (5.4.6) then imply that $\mathbb{E}[m(\mathcal{T})] < \infty$. Similarly, the Portmanteau theorem, Corollary 5.5.5, and the bounds (5.4.7) and (5.4.9) yield (5.7.1) – because $\delta \sqrt{\delta} \leq 4$ and $20\sqrt{\delta} \leq 40$ since $\delta \in (1, 2]$. ■

The two estimates (5.7.1) ensure that the law of $\mathfrak{h}(\mathcal{T})$ is nondegenerate, so the same holds for the law of \mathcal{T} . Next, we show how to construct a HS_α -real tree by rescaling (recall from (5.2.8) the notation \odot_α) and grafting (recall from Definition 5.2.22 the notation \otimes) a countable number of independent HS_α -trees. This self-similarity is our main tool to prove Theorem 5.1.4 (ii). Because of the diversity of the random variables involved in the construction, the theorem is both technical to state and to prove. Nevertheless, it is not surprising nor difficult to understand. Indeed, it is just based on the idea to focus on the subtrees with large weighted Horton–Strahler numbers, to apply Definition 5.2.15, and to take the scaling limit. The formal proof, that we postpone to the end of the section, gives little intuition and may be skipped at first reading. Recall from (5.1.8) that $\beta = \frac{1}{\alpha-1}$, $\gamma = \ln \frac{\alpha}{\alpha-1}$ and $\delta = e^{\gamma(\alpha-1)}$. We work on the product topological space $\Omega = \mathbb{R}_+ \times [1, \infty) \times \mathbb{K}^m \times \prod_{i \geq 2} ([0, 1] \times \mathbb{K}^m)$ and we write each of its element $\omega \in \Omega$ as $\omega = (\ell(\omega), \xi(\omega), T_1(\omega), (\lambda_i(\omega), T_i(\omega))_{i \geq 2})$.

Theorem 5.7.3 *Let $\mathcal{T}_g, \mathcal{T}_d$ be two HS_α -real trees. Let L have exponential law with mean δ^{-1} and let U have law $\mathbf{1}_{[1,\delta]}(s)s^{\beta-1}ds$. Let $(\mathcal{T}_i)_{i \geq 1}$ be a sequence of independent HS_α -real trees. Let $(\Lambda_i)_{i \geq 2}$ be a sequence of independent random variables on $[0, 1]$ with laws respectively given by $\Lambda_i = 0$ almost surely when $\alpha = 2$, or by*

$$\mathbb{P}(\Lambda_i \in d\lambda_i) = \mathbf{1}_{[0,1]}(\lambda_i)\beta(i - \alpha)\lambda_i^{\beta(i-\alpha)-1}d\lambda_i \quad \text{when } \alpha \in (1, 2).$$

Let \mathcal{N} be a Poisson point process on Ω with intensity measure

$$\delta d\ell \times \mathbf{1}_{[1,\infty)}(\xi)d\xi \times \mathbb{P}(\mathcal{T}_g \in dT_1) \times \prod_{i \geq 2} (\mathbb{P}(\Lambda_i \in d\lambda_i) \times \mathbb{P}(\mathcal{T}_g \in dT_i)).$$

We assume that $L, U, \mathcal{T}_g, \mathcal{T}_d, (\mathcal{T}_i)_{i \geq 2}, (\Lambda_i)_{i \geq 2}, \mathcal{N}$ are jointly independent. We denote by $L \in \mathbb{K}^m$ the real segment $[0, L]$ rooted at 0 and endowed with the null measure. Then, there exists a HS_α -real tree \mathcal{T}^ such that it almost surely holds*

$$\begin{aligned} \mathcal{T}^* = L \circledast \left(L, \frac{1}{U} \circledast_\alpha \mathcal{T}_g \right) \circledast \left(L, \frac{1}{\delta} \circledast_\alpha \mathcal{T}_d \right) \circledast_{i \geq 2} \left(L, \frac{1}{\delta} \prod_{j=2}^i \Lambda_j \circledast_\alpha \mathcal{T}_i \right) \\ \circledast_{\substack{i \geq 1, \omega \in \mathcal{N} \\ \ell(\omega) \leq L}} \left(\ell(\omega), \frac{1}{\delta \xi(\omega)} \prod_{j=2}^i \lambda_j(\omega) \circledast_\alpha T_i(\omega) \right). \end{aligned} \quad (5.7.2)$$

Remark 5.7.4 *If $\alpha \in (1, 2)$, we compute $\mathbb{E}[\Lambda_i^q] = 1 - \frac{q}{q + \beta(i - \alpha)}$ for all $q > 0$ and $i \geq 2$. We thus get*

$$(\alpha = 2 \text{ or } q > \beta) \implies \mathbb{E} \left[\sum_{i \geq 1} \prod_{j=2}^i \Lambda_j^q \right] = \sum_{i \geq 1} \prod_{j=2}^i \mathbb{E}[\Lambda_j^q] < \infty, \quad (5.7.3)$$

after an elementary asymptotic study. This will be useful for proving Theorem 5.7.3 for example. \square

In the specific case where $\alpha = 2$, Theorem 5.7.5 takes the following simpler form.

Corollary 5.7.5 *Let $\mathcal{T}_g, \mathcal{T}_d$ be two HS_2 -real trees. Let L have exponential law with mean $\frac{1}{2}$ and let U have uniform law on $[1, 2]$. Let \mathcal{N}' be a Poisson point process on $[0, \infty) \times [1, \infty) \times \mathbb{K}^m$ with intensity measure $2d\ell \times \mathbf{1}_{[1,\infty)}(\xi)d\xi \times \mathbb{P}(\mathcal{T}_g \in dT)$. We assume that $L, U, \mathcal{T}_g, \mathcal{T}_d, \mathcal{N}'$ are jointly independent. We denote by $L \in \mathbb{K}^m$ the real segment $[0, L]$ rooted at 0 and endowed with the null measure. Then, there exists a HS_2 -real tree \mathcal{T}^* such that it almost surely holds*

$$\mathcal{T}^* = L \circledast \left(L, \frac{1}{U} \circledast_2 \mathcal{T}_g \right) \circledast \left(L, \frac{1}{2} \circledast_2 \mathcal{T}_d \right) \circledast_{\substack{(\ell, \xi, T) \in \mathcal{N}' \\ \ell \leq L}} \left(\ell, \frac{1}{2\xi} \circledast_2 T \right).$$

The following result gathers some other properties of HS_α -excursions. In particular, it allows saying that \tilde{H} is a HS_α -excursion instead of $(\tilde{H}, \tilde{\zeta})$ in accordance with Notation 5.2.2.

Proposition 5.7.6 *Recall from (5.2.1) that if $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a càdlàg function, then $\zeta(f)$ stands for its lifetime. Recall Definition 5.2.19 of a planted rooted compact real tree. The following holds.*

- (i) *If $(\mathcal{T}, d, \rho, \mu)$ is HS_α -real tree, then $\mu(\{\rho\}) = 0$ almost surely.*
- (ii) *If $(\tilde{H}, \tilde{\zeta})$ is a HS_α -excursion, then $\tilde{\zeta} = \zeta(\tilde{H})$ almost surely.*
- (iii) *If \mathcal{T} is HS_α -real tree, then \mathcal{T} is almost surely planted.*

Proof. We use the notations of Theorem 5.7.3. We begin by observing that almost surely,

$$L > 0 \quad \text{and} \quad \mathcal{N} \cap \{\omega \in \Omega : \ell(\omega) = 0\} = \emptyset. \quad (5.7.4)$$

Thus, the root of \mathcal{T}^* is almost surely not an atom of the measure of \mathcal{T}^* by Definition 5.2.22, which yields (i). Since $(\tilde{H}, \tilde{\zeta})$ is a random variable on $\mathcal{C}_{\mathbb{K}}$, we have $\zeta(\tilde{H}) \leq \tilde{\zeta}$ and $\tilde{H}_s = 0$ for all $s \geq \zeta(\tilde{H})$. By definition (5.2.27) of $\mathcal{T}_{\tilde{H}, \tilde{\zeta}}$, it follows that $\tilde{\zeta} - \zeta(\tilde{H}) \leq \mu_{\tilde{H}, \tilde{\zeta}}(\{\rho_{\tilde{H}, \tilde{\zeta}}\})$. Then, (i) implies (ii). Proposition 5.2.25 and (5.7.4) entail that \mathcal{T}^* is almost surely planted, which gives (iii). \blacksquare

5.7.1 Proof of Theorem 5.7.3

We check that the two series $\sum_{i \geq 1} \prod_{j=2}^i \Lambda_j^{\alpha\beta}$ and $\sum_{\omega \in \mathcal{N}} \sum_{i \geq 1} \mathbf{1}_{\{\ell(\omega) \leq L\}} \xi(\omega)^{-\alpha\beta} \prod_{j=2}^i \lambda_j(\omega)^{\alpha\beta}$ are integrable by (5.7.3), since the first moment measure of the Poisson Point process is its intensity measure. For all $n \in \mathbb{N}^*$, it follows that there exists a random variable \mathcal{T}_n^* on \mathbb{K}^m such that

$$\begin{aligned} \mathcal{T}_n^* = & \mathbf{L} \circledast \left(L, \frac{1}{U} \circledast_{\alpha} \mathcal{T}_g \right) \circledast \left(L, \frac{1}{\delta} \circledast_{\alpha} \mathcal{T}_d \right) \circledast_{i \geq 2}^* \left(L, \frac{1}{\delta} \prod_{j=2}^i \Lambda_j \circledast_{\alpha} \mathcal{T}_i \right) \\ & \prod_{j=2}^i \Lambda_j \geq 1/n \\ & \circledast_{\omega \in \mathcal{N}, i \geq 1}^* \left(\ell(\omega), \frac{1}{\delta \xi(\omega)} \prod_{j=2}^i \lambda_j(\omega) \circledast_{\alpha} T_i(\omega) \right) \\ & \ell(\omega) \leq L, \prod_{j=2}^i \lambda_j(\omega) \geq \xi(\omega)/n \end{aligned}$$

almost surely. Indeed, the above grafting procedure involves an almost surely finite number of random variables on \mathbb{K}^m , so the measurability comes from Proposition 5.2.24. As $\frac{\alpha}{\alpha-1} = \alpha\beta > \beta$, we similarly check with (5.7.3) and Proposition 5.7.2 that these two series are integrable:

$$\begin{aligned} & \sum_{i \geq 2} \mathfrak{m} \left(\delta^{-1} \prod_{j=2}^i \Lambda_j \circledast_{\alpha} \mathcal{T}_i \right) + \sum_{\omega \in \mathcal{N}} \sum_{i \geq 1} \mathbf{1}_{\{\ell(\omega) \leq L\}} \mathfrak{m} \left(\delta^{-1} \xi(\omega)^{-1} \prod_{j=2}^i \lambda_j(\omega) \circledast_{\alpha} T_i(\omega) \right), \\ & \sum_{i \geq 2} \mathfrak{h} \left(\delta^{-1} \prod_{j=2}^i \Lambda_j \circledast_{\alpha} \mathcal{T}_i \right)^{\alpha\beta} + \sum_{\omega \in \mathcal{N}} \sum_{i \geq 1} \mathbf{1}_{\{\ell(\omega) \leq L\}} \mathfrak{h} \left(\delta^{-1} \xi(\omega)^{-1} \prod_{j=2}^i \lambda_j(\omega) \circledast_{\alpha} T_i(\omega) \right)^{\alpha\beta}. \end{aligned}$$

It follows that the conditions (5.2.36) and (5.2.37) are almost surely satisfied, so the right-hand side of (5.7.2) is almost surely an element of \mathbb{K}^m by Proposition 5.2.23. Let us denote it by $(\mathcal{T}', d', \rho', \mu')$. Plus, if (5.2.36) and (5.2.37) are satisfied, then we get $\sup_{\sigma' \in \mathcal{T}', \sigma_n \in \mathcal{T}_n^*} d'(\sigma', \sigma_n) \rightarrow 0$ and $\mu'(\mathcal{T}' \setminus \mathcal{T}_n^*) \rightarrow 0$ thanks to the dominated convergence theorem. Recalling (5.2.7), we thus obtain that $\mathcal{T}_n^* \rightarrow \mathcal{T}'$ almost surely on \mathbb{K}^m . For the rest of the proof, we choose a random variable \mathcal{T}^* on \mathbb{K}^m such that

$$\mathcal{T}^* = \lim_{n \rightarrow \infty} \mathcal{T}_n^* \quad \text{almost surely.} \quad (5.7.5)$$

This is possible as a limit of measurable functions is measurable. By construction, we know that \mathcal{T}^* verifies (5.7.2) almost surely. It only remains to show that \mathcal{T}^* is a HS_{α} -real tree.

We work with a countable and closed subset of \mathbb{K}^m defined as follows:

$$\mathbb{K}_{\text{dsc}}^m = \{(E, d, \rho, \mu) \in \mathbb{K}^m : E \text{ finite such that } d(x, y) \in \mathbb{N} \text{ and } \mu(\{x\}) \in \mathbb{N} \text{ for all } x, y \in E\}.$$

For all $n \in \mathbb{N}^*$ and for all weighted tree \mathbf{t} , we define an element of $\mathbb{K}_{\text{dsc}}^m$ by setting

$$D_n(\mathbf{t}) = \{u \in \mathbf{t} : \mathcal{S}(\theta_u \mathbf{t}) = \mathcal{S}(\mathbf{t}) \text{ or } \exists v \in \mathbf{t} \text{ with } v \preceq u, 0 < \mathcal{S}(\mathbf{t}) - \mathcal{S}(\theta_v \mathbf{t}) \leq 1 + \log_{\delta} n\},$$

which we see as a metric subspace of τ , also endowed with the graph distance (5.2.29), rooted at \emptyset , and equipped with its counting measure. Let τ be a \mathbf{GW}_α -weighted tree. Our proof requires us to show the three following points.

- (I) For all $n \in \mathbb{N}^*$ and for all $x \in \mathbb{R}_+ \setminus \mathbb{N}$ with $x > 1 + \log_\delta n$, there is a random variable D_n^x on $\mathbb{K}_{\text{dsc}}^m$ such that $D_n(\tau)$ under $\mathbb{P}(\cdot \mid |\mathcal{S}(\tau) - x| < \varepsilon)$ converges in law to D_n^x under \mathbb{P} , on \mathbb{K}^m , as $\varepsilon \rightarrow 0^+$.
- (II) For all $n \in \mathbb{N}^*$, $\delta^{-x} \odot_\alpha D_n^x$ converges in law to \mathcal{T}_n^* on \mathbb{K}^m as $x \in \mathbb{R}_+ \setminus \mathbb{N}$ tends to ∞ .
- (III) Recall the rooted Gromov–Hausdorff–Prokhorov distance d_{GHP} as in (5.2.7), then for all $\eta > 0$,

$$0 = \limsup_{n \rightarrow \infty} \limsup_{x \rightarrow \infty, x \notin \mathbb{N}} \limsup_{\varepsilon \rightarrow 0^+} \mathbb{P}(d_{\text{GHP}}(\delta^{-x} \odot_\alpha \tau, \delta^{-x} \odot_\alpha D_n(\tau)) \geq \eta \mid |\mathcal{S}(\tau) - x| < \varepsilon).$$

These three assertions and (5.7.5) entail that $\delta^{-x} \odot_\alpha \tau$ under $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ converges in law to \mathcal{T}^* under \mathbb{P} . By Definition 5.7.1, Theorem 5.7.3 eventually follows. Before getting down to the proof of (I), (II), and (III), let us describe the law of D_n^x via the following lemma. Recall from (5.1.2) that φ_α stands for the generating function of the offspring distribution of \mathbf{GW}_α -trees.

Lemma 5.7.7 *We fix $x \in \mathbb{R}_+ \setminus \mathbb{N}$ and $n \in \mathbb{N}^*$ such that $x > 1 + \log_\delta n$. Let L^x be a geometric random variable with parameter δ^{1-x} and let us denote by $\mathbb{L}^x \in \mathbb{K}_{\text{dsc}}^m$ the metric space $\{0, \dots, L^x\}$ rooted at 0 and endowed with its counting measure. Let τ_g^x be distributed as τ under $\mathbb{P}(\cdot \mid x - 1 < \mathcal{S}(\tau) < x)$. Let τ_d^x be distributed as τ under $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x - 1)$. Let $(\tilde{T}_i^x)_{2 \leq i \leq \tilde{J}^x}$ be a random finite sequence with random length $\tilde{J}^x - 1 \geq 0$ (that may be 0) such that for all $m \geq 2$ and for all bounded measurable functions $f_2, \dots, f_m : \mathbb{K}_{\text{dsc}}^m \rightarrow \mathbb{R}$,*

$$\begin{aligned} & \delta^{1-x} e^{\gamma(x-1)} \frac{e^{-\gamma}}{1 - e^{-\gamma}} \mathbb{E} \left[\mathbf{1}_{\{\tilde{J}^x \geq m\}} \prod_{i=2}^m f_i(\tilde{T}_i^x) \right] = \\ & \int_{[0, \infty)^{m-1}} \mathbf{1}_{\{x-1 > y_2 > \dots > y_m > x - \log_\delta(\delta n)\}} \varphi_\alpha^{(m+1)}(1 - e^{-\gamma y_m}) \prod_{i=2}^m \mathbb{E}[f_i(\tau) \mid \mathcal{S}(\tau) = y_i] \gamma e^{-\gamma y_i} dy_i. \end{aligned}$$

Let $(N_p^x)_{p \geq 0}$ be a random walk on started at $N_0^x = -1$ whose jump law is such that $N_{p+1}^x - N_p^x - 1$ is geometric with parameter $\delta(n-1)/(\delta^x - \delta)$ for all $p \geq 0$. Let $((T_{i,p}^x)_{1 \leq i \leq J_p^x})_{p \geq 1}$ be a sequence of independent random finite sequences with random lengths $J_p^x \geq 1$ such that for all $p, m \geq 1$ and for all bounded measurable functions $f_1, \dots, f_m : \mathbb{K}_{\text{dsc}}^m \rightarrow \mathbb{R}$,

$$\begin{aligned} & \delta(n-1) \delta^{-x} \mathbb{E} \left[\mathbf{1}_{\{J_p^x \geq m\}} \prod_{i=1}^m f_i(T_{i,p}^x) \right] = \\ & \int_{[0, \infty)^m} \mathbf{1}_{\{x-1 > y_1 > \dots > y_m > x - \log_\delta(\delta n)\}} \varphi_\alpha^{(m+1)}(1 - e^{-\gamma y_m}) \prod_{i=1}^m \mathbb{E}[f_i(\tau) \mid \mathcal{S}(\tau) = y_i] \gamma e^{-\gamma y_i} dy_i. \end{aligned}$$

We assume that all these random variables are independent. For all $E \in \mathbb{K}_{\text{dsc}}^m$, we denote by $\varpi \otimes E$ the rooted measured compact metric space $\{0, 1\} \otimes (1, E)$ where $\{0, 1\} \in \mathbb{K}_{\text{dsc}}^m$ is rooted at 0 and endowed with the null measure. We define

$$D_n^x = \mathbb{L}^x \otimes (L^x, \varpi \otimes \tau_g^x) \otimes (L^x, \varpi \otimes \tau_d^x) \otimes_{2 \leq i \leq \tilde{J}^x} (L^x, \varpi \otimes \tilde{T}_i^x) \quad \left(\bigotimes_{\substack{p, i \geq 1 \\ N_p^x < L^x, i \leq J_p^x}} (N_p^x, \varpi \otimes T_{i,p}^x) \right).$$

Let $\{o\} \in \mathbb{K}_{\text{dsc}}^m$ stands for the compact metric space reduced to a single point o and equipped with its Dirac measure. If D is an independent random variable on $\mathbb{K}_{\text{dsc}}^m$ such that it holds

$$\begin{aligned} \mathbb{E}[F(D)] &= \delta^{1-x} \mathbb{E}[F(\{o\} \otimes (o, \varpi \otimes \tau_g^x) \otimes (o, \varpi \otimes \tau_d^x) \otimes_{2 \leq i \leq j^x} (o, \varpi \otimes \tilde{T}_i^x))] \\ &+ (n-1) \delta^{1-x} \mathbb{E}[F(\{o\} \otimes (o, \varpi \otimes D) \otimes_{1 \leq i \leq J_1^x} (o, \varpi \otimes T_{i,1}^x))] + (1-n) \delta^{1-x} \mathbb{E}[F(\{o\} \otimes (o, \varpi \otimes D))] \end{aligned} \quad (5.7.6)$$

for all bounded and measurable $F : \mathbb{K}_{\text{dsc}}^m \rightarrow \mathbb{R}$, then D has the same law D_n^x .

Proof of Lemma 5.7.7. It is not difficult to check that an independent copy of D_n^x satisfies (5.7.6): the three terms of the right-hand side respectively correspond to the events $\{L^x = 0\}$, $\{N_1^x = 0 < L^x\}$, and $\{1 \leq N_1^x, L^x\}$. Using (5.7.6), we can show that $\mathbb{P}(D = E) = \mathbb{P}(D_n^x = E)$ for all $E \in \mathbb{K}_{\text{dsc}}^m$ by induction on the number of points of E . This completes the proof since $\mathbb{K}_{\text{dsc}}^m$ is countable. ■

Proof of (I). We fix $n \in \mathbb{N}^*$ and $x \in \mathbb{R}_+ \setminus \mathbb{N}$ with $x > 1 + \log_\delta n$. With the notations of Lemma 5.7.7, we show that $D_n(\tau)$ under $\mathbb{P}(\cdot \mid |\mathcal{S}(\tau) - x| < \varepsilon)$ converges in law to D_n^x under \mathbb{P} . By Proposition 5.4.7, the laws of $D_n(\tau)$ under $\mathbb{P}(\cdot \mid |\mathcal{S}(\tau) - x| < \varepsilon)$ are tight on $\mathbb{K}_{\text{dsc}}^m$ as $\varepsilon \rightarrow 0^+$ because $D_n(\tau)$ is a subset of $\tau \in \mathbb{K}_{\text{dsc}}^m$. We can thus assume that $(\tau, D_n(\tau))$ under $\mathbb{P}(\cdot \mid |\mathcal{S}(\tau) - x| < \varepsilon)$ converges in law to a random variable $(t, D) \in (\mathbb{K}_{\text{dsc}}^m)^2$. Then, we only need to prove that D satisfies (5.7.6) thanks to Lemma 5.7.7. In this proof, we extensively use the fact that for all $y, z > 0$,

$$\mathbb{E} \left[\sum_{\substack{1 \leq i, j \leq k_\emptyset(\tau) \\ i \neq j}} \mathbf{1}_{\{|\mathcal{S}(\theta_{(i)}\tau) - y| < \varepsilon\}} \mathbf{1}_{\{|\mathcal{S}(\theta_{(j)}\tau) - z| < \varepsilon\}} \mid |\mathcal{S}(\tau) - x| < \varepsilon \right] \xrightarrow{\varepsilon \rightarrow 0^+} 0. \quad (5.7.7)$$

Indeed, this expectation is smaller than $4\gamma^2 \varepsilon^2 \mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon)^{-1} \varphi''_\alpha(\mathbb{P}(\mathcal{S}(\tau) \leq x + 1))$ which goes to 0 when $\varepsilon \rightarrow 0^+$ by Proposition 5.4.6. □

Let us denote by $\tau_0, \tau_1, \dots, \tau_{k_\emptyset(\tau)-1}$ the weighted subtrees of τ stemming from the children of \emptyset listed in the decreasing order of their weighted Horton–Strahler numbers. Namely,

$$\mathcal{S}(\tau_0) > \dots > \mathcal{S}(\tau_{k_\emptyset(\tau)-1}) \quad \text{and} \quad \{\tau_i : 0 \leq i \leq k_\emptyset(\tau) - 1\} = \{\theta_{(j)}\tau : 1 \leq j \leq k_\emptyset(\tau)\}.$$

We also define $K_n(\tau)$ as follows:

$$K_n(\tau) = -1 + \#\{1 \leq i \leq k_\emptyset(\tau) : \mathcal{S}(\theta_{(i)}\tau) \geq \mathcal{S}(\tau) - 1 - \log_\delta n\}.$$

From Definition 5.1.2, we observe that if $k_\emptyset(\tau) \geq 1$ then $K_n(\tau) \geq 0$, and if $\mathcal{S}(\tau_0) < \mathcal{S}(\tau)$ then $K_n(\tau) \geq 1$. In particular, the conditional probability that $K_n(\tau) < 0$ given $|\mathcal{S}(\tau) - x| < \varepsilon$ goes to 0 as $\varepsilon \rightarrow 0^+$, since $x > 1$. Moreover, we observe that $D_n(\tau)$ can be expressed as follows.

- (a) If $K_n(\tau) = 0$, then $D_n(\tau) = \{o\} \otimes (o, \varpi \otimes D_n(\tau_0))$.
- (b) If $\mathcal{S}(\tau_0) = \mathcal{S}(\tau)$ and $K_n(\tau) \geq 1$, then $D_n(\tau) = \{o\} \otimes (o, \varpi \otimes D_n(\tau_0)) \otimes_{1 \leq i \leq K_n(\tau)} (o, \varpi \otimes \tau_i)$.
- (c) If $\mathcal{S}(\tau_0) < \mathcal{S}(\tau)$, then $D_n(\tau) = \{o\} \otimes (o, \varpi \otimes \tau_0) \otimes_{1 \leq i \leq K_n(\tau)} (o, \varpi \otimes \tau_i)$.

We deduce from Definition 5.1.2 and from (5.7.7) (with $y = x$ and $z = x - 1 - \log_\delta n$) that outside an event of negligible probability compared to $\mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon)$ as $\varepsilon \rightarrow 0^+$, it holds that

$$\mathbf{1}_{\{|\mathcal{S}(\tau) - x| < \varepsilon; K_n(\tau) = 0\}} = \sum_{i=1}^{k_\emptyset(\tau)} \mathbf{1}_{\{|\mathcal{S}(\theta_{(i)}\tau) - x| < \varepsilon\}} \prod_{\substack{1 \leq j \leq k_\emptyset(\tau) \\ j \neq i}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)}\tau) < x - 1 - \log_\delta n\}}.$$

We compute $\varphi'_\alpha(\mathbb{P}(\mathcal{S}(\tau) < x - 1 - \log_\delta n) = 1 - n\delta^{1-x})$ thanks to (5.2.15) and Proposition 5.4.6. Let $g_1, g_2 : \mathbb{K}_{\text{dsc}}^m \rightarrow \mathbb{R}$ be bounded, it follows from Definition 5.2.15 of the law of τ that

$$\mathbb{E}[\mathbf{1}_{\{K_n(\tau)=0\}}g_1(D_n(\tau_0))g_2(\tau_0) \mid |\mathcal{S}(\tau) - x| < \varepsilon] \xrightarrow{\varepsilon \rightarrow 0^+} (1 - n\delta^{1-x})\mathbb{E}[g_1(D)g_2(t)]. \quad (5.7.8)$$

Let $m \in \mathbb{N}^*$. Again with an application of (5.7.7), we find that outside an event of negligible probability compared to $\mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon)$ as $\varepsilon \rightarrow 0^+$, the variable $\mathbf{1}_{\{|\mathcal{S}(\tau)-x|<\varepsilon; \mathcal{S}(\tau_0)=\mathcal{S}(\tau); K_n(\tau)\geq m\}}$ is equal to

$$\sum_{\substack{1 \leq i_0, \dots, i_m \leq k_\emptyset(\tau) \\ \text{distinct}}} \mathbf{1}_{\{|\mathcal{S}(\theta_{(i_0)}\tau) - x| < \varepsilon\}} \mathbf{1}_{\{x-1 > \mathcal{S}(\theta_{(i_1)}\tau) > \dots > \mathcal{S}(\theta_{(i_m)}\tau) > x - \log_\delta(\delta n)\}} \prod_{\substack{1 \leq j \leq k_\emptyset(\tau) \\ j \neq i_0, \dots, i_m}} \mathbf{1}_{\{\mathcal{S}(\theta_{(j)}\tau) < \mathcal{S}(\theta_{(i_m)}\tau)\}}.$$

Let $g_1, g_2, f_1, \dots, f_m : \mathbb{K}_{\text{dsc}}^m \rightarrow \mathbb{R}$ be bounded. Thanks to Definition 5.2.15 and the identity (5.4.14) in Proposition 5.4.7, we recognize that

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau_0)=\mathcal{S}(\tau); K_n(\tau)\geq m\}}g_1(D_n(\tau_0))g_2(\tau_0) \prod_{j=1}^m f_j(\tau_j) \mid |\mathcal{S}(\tau) - x| < \varepsilon] \\ \xrightarrow{\varepsilon \rightarrow 0^+} (n-1)\delta^{1-x}\mathbb{E}[g_1(D)g_2(t)]\mathbb{E}[\mathbf{1}_{\{J_1^x \geq m\}} \prod_{j=1}^m f_j(T_{j,1}^x)]. \end{aligned} \quad (5.7.9)$$

Note from Proposition 5.4.6 that $\mathbb{P}(|\mathcal{S}(\tau) - x| < \varepsilon) = e^{-\gamma}\mathbb{P}(|\mathcal{S}(\tau) - x + 1| < \varepsilon)$ and also that $\mathbb{P}(x-1 < \mathcal{S}(\tau) < x) = e^{-\gamma(x-1)}(1 - e^{-\gamma})$. The same method used to find (5.7.9) yields that

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau_0) < \mathcal{S}(\tau); K_n(\tau)\geq m\}}g_2(\tau_0) \prod_{j=1}^m f_j(\tau_j) \mid |\mathcal{S}(\tau) - x| < \varepsilon] \\ \xrightarrow{\varepsilon \rightarrow 0^+} \delta^{1-x}\mathbb{E}[g_2(\tau) \mid x-1 < \mathcal{S}(\tau) < x]\mathbb{E}[f_1(\tau) \mid \mathcal{S}(\tau) = x-1]\mathbb{E}[\mathbf{1}_{\{\tilde{J}_x \geq m\}} \prod_{j=2}^m f_j(\tilde{T}_j^x)]. \end{aligned} \quad (5.7.10)$$

The convergences (5.7.8), (5.7.9), and (5.7.10) together with the observations (a), (b), and (c) entail that D satisfies (5.7.6), which completes the proof of (I) as announced. \blacksquare

Proof of (II). We keep the notations of Lemma 5.7.7. We begin by showing that each component involved in the construction of D_n^x converges in law after scaling to its continuum counterpart involved in Theorem 5.7.3. First, it holds $\delta^{-x} \odot_\alpha \tau_d^x \xrightarrow{d} \delta^{-1} \odot_\alpha \mathcal{T}_d$ on \mathbb{K}^m by Definition 5.7.1. Next, we have $\mathbb{P}(x-1 < \mathcal{S}(\tau) < x) = \frac{1}{\alpha}e^{-\gamma(x-1)}$ by Proposition 5.4.6. Let $f_0 : \mathbb{K}^m \rightarrow \mathbb{R}$ be continuous and bounded, we apply Proposition 5.4.7 and make $y = x - \log_\delta s$ to get

$$\begin{aligned} \mathbb{E}[f_0(\delta^{-x} \odot_\alpha \tau_g^x)] &= \alpha e^{\gamma(x-1)} \int_{x-1}^x \mathbb{E}[f_0(\delta^{-x} \odot_\alpha \tau \mid \mathcal{S}(\tau) = y)] \gamma e^{-\gamma y} dy \\ &= \int_1^\delta s^{\beta-1} \mathbb{E}[f_0(s^{-1}\delta^{-x+\log_\delta s} \odot_\alpha \tau \mid \mathcal{S}(\tau) = x - \log_\delta s)] ds. \end{aligned}$$

Thus, we have $\delta^{-x} \odot_\alpha \tau_g^x \xrightarrow{d} U^{-1} \odot_\alpha \mathcal{T}_g$ on \mathbb{K}^m by dominated convergence. It is clear that $\delta^{-x} L^x \xrightarrow{d} L$ and this yields $\delta^{-x} \odot_\alpha L^x \xrightarrow{d} L$ on \mathbb{K}^m . For all $p \geq 1$, $(\delta^{-x} N_1^x, \dots, \delta^{-x} N_p^x)$ similarly converges jointly in distribution to the first p points of a Poisson process with intensity $\delta(n-1)$, such as $\{\ell(\omega) : \omega \in \mathcal{N}, 1 \leq \xi(\omega) \leq n\}$. Now, let $m \geq 1$ and let $f_1, \dots, f_m : \mathbb{K}^m \rightarrow \mathbb{R}$ be continuous and bounded. Recall that $\varphi^{(m+1)}(s) = (1-s)^{\alpha-m-1}(\alpha-1) \prod_{i=2}^m (i-\alpha)$ for all

$s \in [0, 1)$ from (5.2.15) and (5.2.16). The change of variables $\xi = \delta^{x-1-y_1}$ and $\lambda_i = \delta^{y_i-y_{i-1}}$ for all $2 \leq i \leq m$ reveal after some computations that

$$\mathbb{E}\left[\mathbf{1}_{\{J_1^x \geq m\}} \prod_{i=1}^m f_i(\delta^{-x} \odot_\alpha T_{i,1}^x)\right] \longrightarrow \frac{1}{n-1} \int_1^n \mathbb{E}\left[\mathbf{1}_{\{\prod_{i=2}^m \Lambda_i \geq \xi/n\}} \prod_{i=1}^m f_i\left(\frac{1}{\delta \xi} \prod_{j=2}^i \Lambda_j \odot_\alpha \mathcal{T}_i\right)\right] d\xi.$$

Setting $\lambda_2 = \delta^{y_2-x+1}$ and $\lambda_i = \delta^{y_i-y_{i-1}}$ for all $3 \leq i \leq m$ similarly leads to

$$\mathbb{E}\left[\mathbf{1}_{\{\tilde{J}_x \geq m\}} \prod_{i=2}^m f_i(\delta^{-x} \odot_\alpha \tilde{T}_i^x)\right] \longrightarrow \mathbb{E}\left[\mathbf{1}_{\{\prod_{i=2}^m \Lambda_i \geq 1/n\}} \prod_{i=2}^m f_i\left(\frac{1}{\delta} \prod_{j=2}^i \Lambda_j \odot_\alpha \mathcal{T}_i\right)\right].$$

By independence, all those convergences in distribution happen jointly. In particular, the numbers of components involved in the construction of the D_n^x by grafting are tight as $x \in \mathbb{R}_+ \setminus \mathbb{N}$ tends to ∞ , because it converges in distribution to the number of components involved in the construction of \mathcal{T}_n^* by grafting. We complete the proof of (II) with Proposition 5.2.24. \blacksquare

Proof of (III). Let $\eta > 0$, we observe that we only need to show the two following estimates:

$$0 = \limsup_{n \rightarrow \infty} \limsup_{x \rightarrow \infty, x \notin \mathbb{N}} \limsup_{\varepsilon \rightarrow 0^+} \mathbb{E}\left[\min(e^{-\gamma \alpha x} \#(\tau \setminus D_n(\tau)), 1) \mid |\mathcal{S}(\tau) - x| < \varepsilon\right], \quad (5.7.11)$$

$$0 = \limsup_{n \rightarrow \infty} \limsup_{x \rightarrow \infty, x \notin \mathbb{N}} \limsup_{\varepsilon \rightarrow 0^+} \mathbb{P}\left(\max_{u \in \tau, v \in D_n(\tau)} d_{\text{gr}}(u, v) \geq \eta e^{\gamma(\alpha-1)x} \mid |\mathcal{S}(\tau) - x| < \varepsilon\right), \quad (5.7.12)$$

where we recall from (5.2.29) that d_{gr} is the graph distance on the set \mathbb{U} of words. Let $x \in \mathbb{R}_+ \setminus \mathbb{N}$ and $n \in \mathbb{N}^*$ with $x > 1 + \log_\delta n$. Let \mathbf{t} be a weighted tree. To lighten the notations, we set

$$\begin{aligned} F_{x,n}^{\mathfrak{m}}(\mathbf{t}) &= \min(e^{-\gamma \alpha x} \#(t \setminus D_n(\mathbf{t})), 1) & \text{and} & & F_{x,n}^{\mathfrak{h}}(\mathbf{t}) &= \mathbf{1}_{\{\exists u \in t, v \in D_n(\mathbf{t}) : d_{\text{gr}}(u, v) \geq \eta e^{\gamma(\alpha-1)x}\}}, \\ f_x^{\mathfrak{m}}(t) &= e^{-\gamma \alpha x} \#t & \text{and} & & f_x^{\mathfrak{h}}(t) &= \mathbf{1}_{\{|t|+1 \geq \eta e^{\gamma(\alpha-1)x}\}}, \end{aligned}$$

where recall from (5.2.11) that $|t|$ stands for the height of t . As in the proof of (I), we denote by τ_0 the weighted subtree of τ stemming from a child of \emptyset with maximal weighted Horton–Strahler number. From the definition of $D_n(\tau)$, we deterministically observe that for both $\mathfrak{a} \in \{\mathfrak{m}, \mathfrak{h}\}$,

$$F_{x,n}^{\mathfrak{a}}(\tau) \leq \mathbf{1}_{\{\mathcal{S}(\tau_0) = \mathcal{S}(\tau)\}} F_{x,n}^{\mathfrak{a}}(\tau_0) + \sum_{i=1}^{k_\emptyset(\tau)} \mathbf{1}_{\{\mathcal{S}(\theta_{(i)}\tau) < \mathcal{S}(\tau) - 1 - \log_\delta n\}} f_x^{\mathfrak{a}}(\theta_{(i)}\tau).$$

Let us also set $E_{x,n}^{\mathfrak{a}} = \limsup_{\varepsilon \rightarrow 0^+} \mathbb{E}[F_{x,n}^{\mathfrak{a}}(\tau) \mid |\mathcal{S}(\tau) - x| < \varepsilon]$. We obviously have $E_{x,n}^{\mathfrak{a}} \leq 1$. By tightness, an elementary argument based on the convergences (5.7.8) and (5.7.9) yields that

$$\limsup_{\varepsilon \rightarrow 0^+} \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau_0) = \mathcal{S}(\tau)\}} F_{x,n}^{\mathfrak{a}}(\tau_0) \mid |\mathcal{S}(\tau) - x| < \varepsilon] \leq (1 - \delta^{1-x}) E_{x,n}^{\mathfrak{a}}.$$

Let $\varepsilon \in (0, 1)$. If $|\mathcal{S}(\tau) - x| < \varepsilon$ then there is $1 \leq j \leq k_\emptyset(\tau)$ such that $|\mathcal{S}(\theta_{(j)}\tau) - x| < \varepsilon$ or $|\mathcal{S}(\theta_{(j)}\tau) - x + 1| < \varepsilon$. Moreover, it would also hold that $\mathcal{S}(\theta_{(i)}\tau) \leq x + 1$ for all other $1 \leq i \leq k_\emptyset(\tau)$. Since $\mathcal{S}(\tau) \leq \mathcal{S}(\tau)$ and the law of $\mathcal{S}(\tau)$ is exponential with mean $1/\gamma$, we get

$$E_{x,n}^{\mathfrak{a}} \leq (1 - \delta^{1-x}) E_{x,n}^{\mathfrak{a}} + (e^\gamma + 1) \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau) \leq x - \log_\delta n\}} f_x^{\mathfrak{a}}(\tau)] \varphi_\alpha''(\mathbb{P}(\mathcal{S}(\tau) \leq x + 1)).$$

We deduce from (5.2.15) that there is a constant $c \in (0, \infty)$ that only depends on α such that

$$E_{x,n}^{\mathfrak{a}} \leq c e^{\gamma x} \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau) \leq x - \log_\delta n\}} f_x^{\mathfrak{a}}(\tau)].$$

With $\mathfrak{a} = \mathfrak{m}$, (5.7.11) follows from (5.4.5). With $\mathfrak{a} = \mathfrak{h}$, the estimates (5.3.3) and (5.4.8) entail (5.7.12). \blacksquare

5.8 The Strahler dilation of rooted compact real trees

In this section, we prove Theorem 5.1.4. Therefore, we construct and study the Strahler dilation, which corresponds to the weighted Horton–Strahler number for rooted compact real trees.

5.8.1 Definition and properties of the Strahler dilation

Recall the notations needed to work on the set of words \mathbb{U} from Section 5.2.2. Here, we focus on the subset of \mathbb{U} of all the words written in the alphabet $\{1, 2\}$. For all $n \in \mathbb{N}$, we denote by

$$\mathbb{W} = \bigcup_{k \in \mathbb{N}} \{1, 2\}^k \quad \text{and} \quad \mathbb{W}_n = \{u \in \mathbb{W} : |u| \leq n\}$$

the perfect binary trees rooted at \emptyset respectively of infinite height and of height equal to n . Moreover, recall from Section 5.2.3 the genealogical order on a rooted compact real tree.

Definition 5.8.1 Let (T, d, ρ) be a rooted compact real tree. An *embedding* of \mathbb{W} (resp. \mathbb{W}_n) into T is an injective map φ from \mathbb{W} (resp. \mathbb{W}_n) to T such that $\varphi(u \wedge v) = \varphi(u) \wedge \varphi(v)$ for all $u, v \in \mathbb{W}$ (resp. \mathbb{W}_n). In that case, we write $\varphi : \mathbb{W} \hookrightarrow T$ (resp. $\varphi : \mathbb{W}_n \hookrightarrow T$). \square

Notation 5.8.2 If φ is an embedding of \mathbb{W} or \mathbb{W}_n into a rooted compact real tree (T, d, ρ) , then we set $\varphi(\overleftarrow{\emptyset}) = \rho$. We stress that we still allow $\varphi(\emptyset) = \rho$ nonetheless. \square

Definition 5.8.3 We set $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. For all $\mathfrak{d} \in (1, \infty)$ and all rooted compact real tree (T, d, ρ) , we call the *Strahler dilation with base \mathfrak{d}* of T the quantity in $[0, \infty]$ defined by

$$\mathfrak{S}_{\mathfrak{d}}(T) = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{\varphi_n : \mathbb{W}_n \hookrightarrow T} \inf_{\substack{u \in \mathbb{W}_n \\ k \leq |u|}} \frac{1}{|u| + 1} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi_n(\overleftarrow{v}), \varphi_n(v)).$$

Moreover, we define for all $k, n \in \mathbb{N}$,

$$\mathfrak{S}_{\mathfrak{d}}^{k,n}(T) = \sup_{\varphi_n : \mathbb{W}_n \hookrightarrow T} \inf_{\substack{u \in \mathbb{W}_n \\ k \leq |u|}} \frac{1}{|u| + 1} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi_n(\overleftarrow{v}), \varphi_n(v)).$$

For all $k \in \mathbb{N}$, we also write $\mathfrak{S}_{\mathfrak{d}}^k(T) = \lim_{n \rightarrow \infty} \mathfrak{S}_{\mathfrak{d}}^{k,n}(T)$, so that $\mathfrak{S}_{\mathfrak{d}}(T) = \lim_{k \rightarrow \infty} \mathfrak{S}_{\mathfrak{d}}^k(T)$. \square

We fix $\mathfrak{d} > 1$ throughout this section. Let us first prove that the Strahler dilation is well-defined. Let (T, d, ρ) be a rooted compact real tree. As a supremum of infima of nonnegative numbers, $\mathfrak{S}_{\mathfrak{d}}^{k,n}(T)$ is well-defined in $[0, \infty]$. Moreover, if $\varphi_n : \mathbb{W}_n \hookrightarrow T$ and $v \in \mathbb{W}_n$, then $\varphi_n(\overleftarrow{v}) \preceq \varphi_n(v)$ and so $d(\varphi_n(\overleftarrow{v}), \varphi_n(v)) \leq \mathfrak{h}(T)$, where \mathfrak{h} is as in (5.2.9). Hence, we have the following inequality.

$$\text{If } k \leq n \quad \text{then} \quad \mathfrak{S}_{\mathfrak{d}}^{k,n}(T) \leq \mathfrak{d} \frac{\mathfrak{d}^{k+1} - 1}{(k+1)(\mathfrak{d} - 1)} \mathfrak{h}(T) < \infty. \quad (5.8.1)$$

Since an embedding of \mathbb{W}_{n+1} into T induces an embedding of \mathbb{W}_n into T by restriction, we observe

$$\forall k, n \in \mathbb{N}, \quad \mathfrak{S}_{\mathfrak{d}}^{k,n+1}(T) \leq \mathfrak{S}_{\mathfrak{d}}^{k,n}(T) \quad \text{and} \quad \mathfrak{S}_{\mathfrak{d}}^{k,n}(T) \leq \mathfrak{S}_{\mathfrak{d}}^{k+1,n}(T). \quad (5.8.2)$$

Hence, $\mathfrak{S}_{\mathfrak{d}}^k(T)$ is well-defined in $[0, \infty)$ as the limit of a non-increasing sequence of finite terms. Making n tend to ∞ in (5.8.1) and (5.8.2) yields that

$$\forall k \in \mathbb{N}, \quad \mathfrak{S}_{\mathfrak{d}}^k(T) \leq \mathfrak{S}_{\mathfrak{d}}^{k+1}(T) \quad \text{and} \quad \mathfrak{S}_{\mathfrak{d}}^k(T) \leq \mathfrak{d} \frac{\mathfrak{d}^{k+1} - 1}{(k+1)(\mathfrak{d} - 1)} \mathfrak{h}(T) < \infty. \quad (5.8.3)$$

Therefore, $\mathfrak{S}_\delta(T)$ is well-defined in $[0, \infty]$ as the limit of a non-decreasing sequence.

At the moment, it might be unclear why the Strahler dilation does enjoy the desired properties in Theorem 5.1.4 or why it is indeed analogous to the Horton–Strahler number. Nevertheless, we will later give another expression for \mathfrak{S}_δ that may be more insightful. The main benefit of that first definition is that it only involves a countable number of operations. Thus, we will use it to show the measurability of \mathfrak{S}_δ and to upper-bound the Strahler dilation of a random rooted compact real tree.

Proposition 5.8.4 *Recall that $\mathbb{T}_\mathbb{R}$ stands for the space of (rooted-isometry classes of) rooted compact real trees, endowed with the rooted Gromov–Hausdorff distance (5.2.6). For all $k, n \in \mathbb{N}$, the function $\mathfrak{S}_\delta^{k,n} : \mathbb{T}_\mathbb{R} \rightarrow [0, \infty]$ is lower semicontinuous. Thus, $\mathfrak{S}_\delta^{k,n}, \mathfrak{S}_\delta^k, \mathfrak{S}_\delta : \mathbb{T}_\mathbb{R} \rightarrow [0, \infty]$ are measurable.*

Proof. We fix $k, n \in \mathbb{N}$. Let (T_0, d_0, ρ_0) be a rooted compact real tree and let φ_0 be an embedding of \mathbb{W}_n into T_0 . We set

$$\varepsilon = \min_{\substack{u, v \in \mathbb{W}_n \\ u \neq v}} d_0(\varphi_0(u), \varphi_0(v)) > 0.$$

Let (T, d, ρ) be a rooted compact real tree at rooted Gromov–Hausdorff distance less than $\varepsilon/10$ from T_0 . There is a rooted correspondence \mathcal{R} between T and T_0 such that $\text{dis}(\mathcal{R}) < \varepsilon/3$, as in (5.2.5). For all $u \in \mathbb{W}_n$ such that $|u| = n$, we choose a point of T denoted by $\varphi(u)$ such that $(\varphi(u), \varphi_0(u)) \in \mathcal{R}$. We are going to show that φ extends into an embedding from \mathbb{W}_n into T . We begin by proving that the following statement is true.

For all $u, v_1, v_2 \in \mathbb{W}_n \setminus \mathbb{W}_{n-1}$, if $u \wedge v_1 \prec u \wedge v_2$ then $\varphi(u) \wedge \varphi(v_1) \prec \varphi(u) \wedge \varphi(v_2)$. (5.8.4)

Indeed, thanks to (5.2.32), we only need to show that $d(\rho, \varphi(u) \wedge \varphi(v_1)) < d(\rho, \varphi(u) \wedge \varphi(v_2))$ since $\varphi(u) \wedge \varphi(v_1)$ and $\varphi(u) \wedge \varphi(v_2)$ are ancestors of $\varphi(u)$. To do this, we check from (5.2.34) that

$$2d(\rho, \sigma_1 \wedge \sigma_2) = d(\rho, \sigma_1) + d(\rho, \sigma_2) - d(\sigma_1, \sigma_2)$$

for all $\sigma_1, \sigma_2 \in T$, and we use the rooted correspondence \mathcal{R} to obtain

$$|d(\rho, \varphi(u) \wedge \varphi(v_i)) - d_0(\rho_0, \varphi_0(u) \wedge \varphi_0(v_i))| \leq \frac{3}{2} \text{dis}(\mathcal{R}) \quad (5.8.5)$$

for all $i \in \{1, 2\}$. Moreover, (5.2.32) implies that $d_0(\rho_0, \varphi_0(u) \wedge \varphi_0(v_1)) < d_0(\rho_0, \varphi_0(u) \wedge \varphi_0(v_2))$ because we have $\varphi_0(u) \wedge \varphi_0(v_1) \prec \varphi_0(u) \wedge \varphi_0(v_2)$ by Definition 5.8.1 of embeddings. By choice of ε , we even have $d_0(\rho_0, \varphi_0(u) \wedge \varphi_0(v_1)) + \varepsilon \leq d_0(\rho_0, \varphi_0(u) \wedge \varphi_0(v_2))$. The statement (5.8.4) follows from (5.8.5) because $3 \text{dis}(\mathcal{R}) < \varepsilon$. \square

Then, we complete (5.8.4) with the following implication.

For all $u, v_1, v_2 \in \mathbb{W}_n \setminus \mathbb{W}_{n-1}$, if $u \wedge v_1 = u \wedge v_2$ then $\varphi(u) \wedge \varphi(v_1) = \varphi(u) \wedge \varphi(v_2)$. (5.8.6)

Indeed, this is obvious when u, v_1, v_2 are not distinct, so we assume they are. As such, we have $u \wedge v_1 \in \mathbb{W}_{n-1}$. In particular, $u \wedge v_1 = u \wedge v_2$ has exactly two children in \mathbb{W}_n : one is an ancestor of u , the other is an ancestor of v_1 and v_2 . Therefore, $u \wedge v_1 = u \wedge v_2 \prec v_1 \wedge v_2$ so two applications of (5.8.4) entail $\varphi(u) \wedge \varphi(v_1) \preceq \varphi(v_2)$ and $\varphi(u) \wedge \varphi(v_2) \preceq \varphi(v_1)$. We compute $\varphi(u) \wedge \varphi(v_1) \wedge \varphi(v_2)$ with the associative property of \wedge to conclude the proof of (5.8.6). \square

Let $u_1, v_1, u_2, v_2 \in \mathbb{W}_n \setminus \mathbb{W}_{n-1}$ with $u_1 \wedge v_1 = u_2 \wedge v_2$. We may assume $u_1 \wedge v_2 \preceq u_1 \wedge u_2$ without loss of generality, as the ancestral lineage of u_1 is totally ordered by \preceq . This yields that $u_1 \wedge v_1 = u_2 \wedge v_2 = u_1 \wedge u_2 \wedge v_2 = u_1 \wedge v_2$. We use (5.8.6) twice to obtain that $\varphi(u_1) \wedge \varphi(v_1) = \varphi(u_2) \wedge \varphi(v_2)$. Hence, we have justified that there exists a unique extension $\varphi : \mathbb{W}_n \rightarrow T$ such that $\varphi(u \wedge v) = \varphi(u) \wedge \varphi(v)$ for all $u, v \in \mathbb{W}_n \setminus \mathbb{W}_{n-1}$. Now, let $u, v \in \mathbb{W}_n$ be arbitrary and let us

choose $u_1, u_2, v_1, v_2 \in \mathbb{W}_n \setminus \mathbb{W}_{n-1}$ such that $u = u_1 \wedge u_2$ and $v = v_1 \wedge v_2$. The set of ancestors of u_1 is totally ordered by \preceq so there is $u' \in \{u_2, v_1, v_2\}$ such that $u_1 \wedge u' \preceq u_1 \wedge u_2, u_1 \wedge v_1, u_1 \wedge v_2$, and so $u_1 \wedge u' = u_1 \wedge u_2 \wedge v_1 \wedge v_2 = u \wedge v$. It then follows from (5.8.4) and (5.8.6) that

$$\forall u, v \in \mathbb{W}_n, \quad \varphi(u) \wedge \varphi(v) = \varphi(u \wedge v), \quad (5.8.7)$$

because they are both equal to $\varphi(u_1) \wedge \varphi(u')$. Combining (5.8.4) and (5.8.7) shows that if $u \prec v$ then $\varphi(u) \neq \varphi(v)$. Finally, if $\varphi(u) = \varphi(v)$ then $\varphi(u \wedge v) = \varphi(u) = \varphi(v)$ by (5.8.7) and $u = u \wedge v = v$. Hence, φ is indeed an embedding from \mathbb{W}_n into T as claimed.

Definition 5.8.1 gives that $d(\varphi(\check{v}), \varphi(v)) = d(\rho, \varphi(v)) - d(\rho, \varphi(\check{v}))$ for all $v \in \mathbb{W}_n$. This formula together with the bound (5.8.5) yields that for all $v \in \mathbb{W}_n$, we have

$$|d(\varphi(\check{v}), \varphi(v)) - d_0(\varphi_0(\check{v}), \varphi_0(v))| \leq 3 \operatorname{dis}(\mathcal{R}).$$

As φ is an embedding from \mathbb{W}_n into T , this inequality leads to

$$\inf_{\substack{u \in \mathbb{W}_n \\ k \leq |u|}} \frac{1}{|u| + 1} \sum_{v \leq u} \mathfrak{d}^{|v|+1} d_0(\varphi_0(\check{v}), \varphi_0(v)) \leq 3\mathfrak{d} \frac{\mathfrak{d}^{n+1} - 1}{(k+1)(\mathfrak{d} - 1)} \operatorname{dis}(\mathcal{R}) + \mathfrak{S}_\mathfrak{d}^{k,n}(T).$$

We take the infimum over all rooted correspondences \mathcal{R} between T and T_0 , then we make T tend towards T_0 for the rooted Gromov–Hausdorff distance, and we finish by taking the supremum over all embeddings φ_0 of \mathbb{W}_n into T_0 . We obtain $\mathfrak{S}_\mathfrak{d}^{k,n}(T_0) \leq \liminf_{T \rightarrow T_0} \mathfrak{S}_\mathfrak{d}^{k,n}(T)$, which is the lower semicontinuity of $\mathfrak{S}_\mathfrak{d}^{k,n}$. The measurability of $\mathfrak{S}_\mathfrak{d}^{k,n}, \mathfrak{S}_\mathfrak{d}^k, \mathfrak{S}_\mathfrak{d}$ follows by classic results. ■

Remark 5.8.5 The proof of Proposition 5.8.4 use that we work with embeddings of perfect binary trees. It would not hold if \mathbb{W} was replaced by the k -regular tree $\bigcup_{n \in \mathbb{N}} \{1, 2, \dots, k\}^n$ with $k \geq 3$. In fact, the function $\mathfrak{S}_\mathfrak{d}^{k,n}$ would not be lower semicontinuous under this modification. We leave this observation as an exercise for the reader. □

Let (T, d, ρ) be a rooted compact real tree. When φ is an embedding of \mathbb{W} into T , we define

$$\mathfrak{S}_\mathfrak{d}(T, \varphi) = \liminf_{n \rightarrow \infty} \frac{1}{n+1} \inf_{\substack{u \in \mathbb{W} \\ |u|=n}} \sum_{v \leq u} \mathfrak{d}^{|v|+1} d(\varphi(\check{v}), \varphi(v)).$$

Then, the Strahler dilation of T can be expressed as follows:

$$\mathfrak{S}_\mathfrak{d}(T) = \sup_{\varphi: \mathbb{W} \hookrightarrow T} \mathfrak{S}_\mathfrak{d}(T, \varphi). \quad (5.8.8)$$

The formula (5.8.8) is similar in spirit to the expression (5.4.2) of the Horton–Strahler number. Indeed, let us explain why we may understand $\log_\mathfrak{d} \mathfrak{S}_\mathfrak{d}(T, \varphi)$ as a continuum counterpart for the height of an embedded perfect binary tree. If t is a perfect binary tree, then $\theta_{(1)}t$ and $\theta_{(2)}t$ are also perfect binary trees such that $|t| = 1 + |\theta_{(1)}t| = 1 + |\theta_{(2)}t|$. For some $\varphi: \mathbb{W} \hookrightarrow T$, we set $\varphi_i(u) = \varphi((i) * u)$ for all $u \in \mathbb{W}$ and $i \in \{1, 2\}$ and we point out that φ_1 and φ_2 are two embeddings of \mathbb{W} into T such that $\mathfrak{S}_\mathfrak{d}(T, \varphi) = \mathfrak{d} \min(\mathfrak{S}_\mathfrak{d}(T, \varphi_1), \mathfrak{S}_\mathfrak{d}(T, \varphi_2))$. Of course, it is not reasonable to require that $\mathfrak{S}_\mathfrak{d}(T, \varphi_1) = \mathfrak{S}_\mathfrak{d}(T, \varphi_2)$, so that the embedding into the real tree T would be truly perfect, nevertheless the involved min penalizes unbalanced embeddings of \mathbb{W} into T . In very informal words, the quantity $\mathfrak{S}_\mathfrak{d}(T)$ measures the largest scaling we can apply to the dyadic tree with edge lengths $(\mathbb{W}, (\mathfrak{d}^{-|u|-1})_{u \in \mathbb{W}})$ to essentially embed it into T without contracting its distances. Finally, we use limits of means both to only focus on the asymptotic ramifications of the real tree and to obtain a deterministic result with the HS_α -real tree thanks to a law of large numbers.

Proof of the identity (5.8.8). An embedding $\varphi : \mathbb{W} \hookrightarrow T$ induces embeddings of \mathbb{W}_n into T by restriction, for all $n \in \mathbb{N}$. For all $k, n \in \mathbb{N}$ with $k \leq n$, it thus holds that

$$\inf_{j \geq k} \frac{1}{j+1} \inf_{\substack{u \in \mathbb{W} \\ |u|=j}} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi(\overleftarrow{v}), \varphi(v)) \leq \mathfrak{S}_\mathfrak{d}^{k,n}(T).$$

We first let $n \rightarrow \infty$ then $k \rightarrow \infty$ to find that $\mathfrak{S}_\mathfrak{d}(T, \varphi) \leq \mathfrak{S}_\mathfrak{d}(T)$ holds for all embedding φ of \mathbb{W} into T . As a result, we proved $\mathfrak{S}_\mathfrak{d}(T) \geq \sup_{\varphi: \mathbb{W} \hookrightarrow T} \mathfrak{S}_\mathfrak{d}(T, \varphi)$. Now, we assume $\mathfrak{S}_\mathfrak{d}(T) > 0$, because the result is obvious otherwise, and we fix $k \in \mathbb{N}$ such that $\mathfrak{S}_\mathfrak{d}^k(T) > 0$. The fact (5.8.2) ensures that for all $n \geq k$, we can choose an embedding $\varphi_n^k : \mathbb{W}_n \hookrightarrow T$ such that

$$\inf_{\substack{u \in \mathbb{W}_n \\ k \leq |u|}} \frac{1}{|u|+1} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi_n^k(\overleftarrow{v}), \varphi_n^k(v)) \geq \left(1 - \frac{1}{n+2}\right) \mathfrak{S}_\mathfrak{d}^{k,n}(T) > 0.$$

Thanks to the compactness of T , we can assume there exist $\varphi^k(u) \in T$ such that $\varphi_n^k(u) \rightarrow \varphi^k(u)$ for all $u \in \mathbb{W}$, by using a diagonal extraction argument. By making n tend to ∞ , it is then clear that

$$0 < \mathfrak{S}_\mathfrak{d}^k(T) \leq \frac{1}{|u|+1} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi^k(\overleftarrow{v}), \varphi^k(v)) \quad (5.8.9)$$

for all $u \in \mathbb{W}$ with $|u| \geq k$. That uniform lower bound implies that any vertex $u \in \mathbb{W}$ has a descendant $u_+ = u * (1, \dots, 1)$ such that $\varphi^k(\overleftarrow{u_+}) \neq \varphi^k(u_+)$. Plus, it holds $d(\varphi^k(\overleftarrow{u_+}), \varphi^k(u_+)) \geq d(\varphi^k(\overleftarrow{u_+}), \varphi^k(u_+)) > 0$ because the φ_n^k are embeddings. Applying this observation to $u * (1)$ and $u * (2)$ yields the existence of some $u_1, u_2 \in \mathbb{W}$ with $u = u_1 \wedge u_2$ such that $\varphi^k(u) \neq \varphi^k(u_i)$ for both $i \in \{1, 2\}$. It follows that when n is large enough, the point $\varphi_n^k(u_i)$ is closer to $\varphi^k(u_i)$ than to $\varphi_n^k(u) = \varphi_n^k(u_1) \wedge \varphi_n^k(u_2)$ for all $i \in \{1, 2\}$. Then, Lemma 5.2.20 yields that the convergent sequence $(\varphi_n^k(u))_{n \geq k}$ is eventually constant equal to $\varphi^k(u_1) \wedge \varphi^k(u_2)$. Hence, we have $\varphi_n^k(u) = \varphi^k(u)$ as soon as n is large enough, for all $u \in \mathbb{W}$. It is then straightforward to show the map $\varphi^k : u \in \mathbb{W} \mapsto \varphi^k(u) \in T$ is an embedding. To conclude the proof, we deduce from (5.8.9) that

$$\mathfrak{S}_\mathfrak{d}^k(T) \leq \sup_{\varphi: \mathbb{W} \hookrightarrow T} \mathfrak{S}_\mathfrak{d}(T, \varphi)$$

for all large enough $k \in \mathbb{N}$, and we let k tend to ∞ . ■

Although measurable, the Strahler dilation is not continuous. Nonetheless, it enjoys some other regularity properties. Recall Definition 5.2.21 of subtrees of a rooted compact real tree (T, d, ρ) . We remind from (5.2.8) that if $\lambda \geq 0$, then $\lambda \cdot T = (T, \lambda d, \rho)$.

Proposition 5.8.6 *The function $\mathfrak{S}_\mathfrak{d}$ is monotone and homogeneous, which means the following.*

- (i) *If T_1 is a subtree of a rooted compact real tree T_2 , then $\mathfrak{S}_\mathfrak{d}(T_1) \leq \mathfrak{S}_\mathfrak{d}(T_2)$,*
- (ii) *If T is a rooted compact real tree and if $\lambda \geq 0$, then $\mathfrak{S}_\mathfrak{d}(\lambda \cdot T) = \lambda \mathfrak{S}_\mathfrak{d}(T)$.*

This proposition is clear, either by Definition 5.8.3 or by using the formula (5.8.8). Monotony is not surprisingly enjoyed as it is shared with the Horton Strahler number: if $\psi : t_1 \rightarrow t_2$ is an embedding between two (discrete) trees, then we observe $\mathcal{S}(t_1) \leq \mathcal{S}(t_2)$ by (5.4.2). Homogeneity was one of the needed properties and is indicative of the metric nature of the Strahler dilation. In particular, Propositions 5.8.4 and 5.8.6 (ii) entails the point (i) of Theorem 5.1.4 (simply setting $\mathfrak{S}_\mathfrak{d}(T, d, \rho, \mu) = \mathfrak{S}_\mathfrak{d}(T, d, \rho)$ for all rooted measured compact real trees). The following proposition exhibits another similarity between the Strahler dilation and the Horton–Strahler number.

Proposition 5.8.7 *Let $\ell \geq 0$ and let $L \in \mathbb{T}_{\mathbb{R}}$ be the real segment $[0, \ell]$ rooted at 0. Let $(T_i, d_i, \rho_i)_{i \in I}$ be a countable family of rooted compact real trees such that (5.2.36) holds. Let $i_0 \in I$ and let $\ell_j \in [0, \ell]$ for all $j \in I \setminus \{i_0\} =: J$. If $(T, d, \rho) = L \otimes (\ell, T_{i_0}) \otimes_{j \in J} (\ell_j, T_j)$, then for all $n \in \mathbb{N}^*$, it holds that*

$$(n+1)\mathfrak{S}_{\mathfrak{d}}^{n,n}(T) \leq \mathfrak{d}\ell + \max\left((n+1)\mathfrak{S}_{\mathfrak{d}}^{n,n}(T_{i_0}), \mathfrak{d} \sup_{j \in J} n\mathfrak{S}_{\mathfrak{d}}^{n-1,n-1}(T_j)\right), \quad (5.8.10)$$

$$\mathfrak{S}_{\mathfrak{d}}(T) \leq \max\left(\mathfrak{S}_{\mathfrak{d}}(T_{i_0}), \mathfrak{d} \sup_{j \in J} \mathfrak{S}_{\mathfrak{d}}(T_j)\right). \quad (5.8.11)$$

Furthermore, if T_i is planted for all $i \in I$, as in Definition 5.2.19, then it holds that

$$\mathfrak{S}_{\mathfrak{d}}(L \otimes_{i \in I} (\ell, T_i)) = \sup_{i, j \in I} \max\left(\mathfrak{S}_{\mathfrak{d}}(T_i), \mathfrak{S}_{\mathfrak{d}}(T_j), \mathfrak{d}^{1_{\{i \neq j\}}} \min(\mathfrak{S}_{\mathfrak{d}}(T_i), \mathfrak{S}_{\mathfrak{d}}(T_j))\right). \quad (5.8.12)$$

Proof. We begin with gathering some useful facts. We use the convention $\mathbb{W} = \mathbb{W}_{\infty}$ and we write $\ell_{i_0} = \ell$. Let $n \in \mathbb{N}^* \cup \{\infty\}$, $w \in \mathbb{W}_n$, $j \in I$, and let $\varphi : \mathbb{W}_n \hookrightarrow T$. When $\varphi(w * u) \in T_j$ for all $u \in \mathbb{W}_{n-|w|}$, Proposition 5.2.25 justifies that $\varphi_j^w : u \in \mathbb{W}_{n-|w|} \mapsto \varphi(w * u) \in T_j$ is an embedding of $\mathbb{W}_{n-|w|}$ into T_j . We also deduce from Proposition 5.2.25 the following sufficient condition.

$$\text{If } \varphi(w) \in T_j \setminus \{\rho_j\}, \text{ then } \varphi(w * u) \in T_j \text{ for all } u \in \mathbb{W}_{n-|w|}. \quad (5.8.13)$$

When $\varphi(\mathbb{W}_n) \subset T_j$, then $\varphi(u) = \varphi_j^{\emptyset}(u)$ for all $u \in \mathbb{W}_n$, but we stress that $\varphi(\overleftarrow{\emptyset}) = \rho = 0$ and $\varphi_j^{\emptyset}(\overleftarrow{\emptyset}) = \rho_j = \ell_j$ by Notation 5.8.2. Still, if $\varphi(\mathbb{W}_n) \subset T_j$ then $\ell_j \preceq \varphi(\emptyset)$, so we compute that

$$\forall u \in \mathbb{W}_n, \quad \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d(\varphi(\overleftarrow{v}), \varphi(v)) = \mathfrak{d}\ell_j + \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d_j(\varphi_j^{\emptyset}(\overleftarrow{v}), \varphi_j^{\emptyset}(v)). \quad (5.8.14)$$

Here, we assume that $|w| = 1$. If $\varphi(w) \in T_j \setminus \{\rho_j\}$ **and** $\varphi(\emptyset) = \ell_j$, then we readily see that

$$\forall u \in \mathbb{W}_{n-1}, \quad \sum_{v \preceq w * u} \mathfrak{d}^{|v|+1} d(\varphi(\overleftarrow{v}), \varphi(v)) = \mathfrak{d}\ell_j + \mathfrak{d} \sum_{v \preceq u} \mathfrak{d}^{|v|+1} d_j(\varphi_j^w(\overleftarrow{v}), \varphi_j^w(v)). \quad (5.8.15)$$

We then show that at least one of the following assertions holds true:

- (a) there is $j \in I$ such that $\varphi(\mathbb{W}_n) \subset T_j$,
- (b) there are $j \in I \setminus \{i_0\}$ and $w \in \mathbb{W}$ with $|w| = 1$ such that $\varphi(w) \in T_j \setminus \{\rho_j\}$ and $\varphi(\emptyset) = \ell_j$.

Indeed, let us assume that (b) does not hold. Since $L = \llbracket \rho, \ell \rrbracket$ is totally ordered by \preceq and $\varphi(1) \wedge \ell \wedge (\varphi(2) \wedge \ell) = \varphi(\emptyset) \wedge \ell$, we have $w \in \mathbb{W}$ with $|w| = 1$ such that $\varphi(w) \wedge \ell = \varphi(\emptyset) \wedge \ell$. In particular, $\varphi(w) \notin L$ because it would imply $\varphi(w) \preceq \varphi(\emptyset)$ otherwise. Thus, there is $j \in I$ such that $\varphi(w) \in T_j \setminus \{\rho_j\}$. Then, Proposition 5.2.25 entails that either $\varphi(\emptyset) \in T_j \setminus \{\rho_j\}$ or $\varphi(\emptyset) \preceq \ell_j = \varphi(w) \wedge \ell$. In the first case, (5.8.13) yields that (a) is satisfied. In the second case, we get $\varphi(\emptyset) = \varphi(\emptyset) \wedge \ell = \ell_j$. Thus, $j = i_0$, and so $\varphi(\emptyset) = \ell$, because (b) would hold otherwise. Since $\ell \prec \varphi(1), \varphi(2)$, we get $\varphi(1), \varphi(2) \notin L$ and $\varphi(1) \wedge \ell = \varphi(2) \wedge \ell = \ell = \varphi(\emptyset)$. Similarly as before, this implies that $\varphi(1), \varphi(2) \in T_{i_0} \setminus \{\rho_{i_0}\}$. Finally, (a) is verified with $j = i_0$ thanks to (5.8.13). \square

We are now ready to prove the desired relations. First, recall that $\mathfrak{d} > 1$ and $\ell_j \leq \ell$ for all $j \in I$. Then, combining the previous alternative with the identities (5.8.14) and (5.8.15) readily gives (5.8.10). Relying on (5.8.8), the same method with $n = \infty$ yields (5.8.11).

To prove (5.8.12), we focus on the case where $n = \infty$ and $\ell_j = \ell$ for all $j \in I$. When (a) is satisfied, (5.8.14) yields $\mathfrak{S}_{\mathfrak{d}}(T, \varphi) = \mathfrak{S}_{\mathfrak{d}}(T_j, \varphi_j^{\emptyset})$. Conversely, if φ_j is an embedding of \mathbb{W} into T_j then we can also see it as an embedding $\varphi : \mathbb{W} \hookrightarrow T$ such that $\varphi_j^{\emptyset} = \varphi_j$. When (b) is satisfied, $\varphi(\emptyset) = \ell$,

so $\varphi(1) \in T_i \setminus \{\rho_i\}$ and $\varphi(2) \in T_j \setminus \{\rho_j\}$ for some $i, j \in I$. In fact, $i \neq j$ by Definition 5.2.19 of planted real trees. Then, (5.8.15) entails $\mathfrak{S}_\delta(T, \varphi) = \delta \min(\mathfrak{S}_\delta(T_i, \varphi_i^{(1)}), \mathfrak{S}_\delta(T_j, \varphi_j^{(2)}))$. Conversely, if φ_i, φ_j are two embeddings of \mathbb{W} respectively into T_i, T_j , we check with Proposition 5.2.25 that we can construct $\varphi : \mathbb{W} \hookrightarrow T$ such that $\varphi(\emptyset) = \ell$, $\varphi_i^{(1)} = \varphi_i$ and $\varphi_j^{(2)} = \varphi_j$. Note that φ is indeed injective because T_i and T_j are planted. Taking the supremum over all φ completes the proof. \blacksquare

We have constructed a family of functions $(\mathfrak{S}_\delta)_{\delta > 1}$ which is parametrized by $(1, \infty)$. However, a rooted compact real tree T admits at most one parameter $\delta > 1$ such that its Strahler dilation with base δ is not trivial. We define this critical parameter as its *Strahler base*:

$$\text{base}_{\mathfrak{S}} T = \inf \{ \delta > 1 : \mathfrak{S}_\delta(T) = \infty \}. \quad (5.8.16)$$

Proposition 5.8.8 *Let T be a rooted compact real tree. If $\delta > \text{base}_{\mathfrak{S}} T$, then $\mathfrak{S}_\delta(T) = \infty$. If $1 < \delta < \text{base}_{\mathfrak{S}} T$, then $\mathfrak{S}_\delta(T) = 0$. In particular, it holds*

$$\text{base}_{\mathfrak{S}} T = \sup \{ 1 \} \cup \{ \delta > 1 : \mathfrak{S}_\delta(T) = 0 \}.$$

Proof. Let $1 < \delta_1 < \delta_2$ and let $m \in \mathbb{N}$. We can write the easy inequality

$$\mathfrak{S}_{\delta_1}^{k,n}(T) \leq \left(\frac{\delta_1}{\delta_2} \right)^{m+1} \mathfrak{S}_{\delta_2}^{k,n}(T) + \frac{1}{k+1} \sum_{i=0}^{m-1} \delta_1^{i+1} \mathfrak{h}(T)$$

for all $k, n \in \mathbb{N}$. In that precise order, we make n, k, m tend to ∞ . We thus obtain the inequality $\mathfrak{S}_{\delta_1}(T) \leq \mathbf{1}_{\{\mathfrak{S}_{\delta_2}(T) = \infty\}} \mathfrak{S}_{\delta_2}(T) \leq \mathfrak{S}_{\delta_2}(T)$, which yields the result. \blacksquare

5.8.2 Proof of Theorem 5.1.4 (ii)

Recall $\delta = e^{\gamma(\alpha-1)} = \left(\frac{\alpha}{\alpha-1} \right)^{\alpha-1}$ and $\delta \in (1, 2]$ from (5.1.8). Here, our goal is to prove Theorem 5.1.4 (ii), namely that the Strahler dilation with base δ of a HS_α -real tree \mathcal{T} is almost surely equal to 1. Recall their Definitions 5.7.1 and 5.8.3. We divide the proof into three steps: showing $\mathfrak{S}_\delta(\mathcal{T})$ is almost surely constant, showing its mean is not smaller than 1, and showing its mean is not bigger than 1.

Proposition 5.8.9 *Let \mathcal{T} be a HS_α -real tree. The random variable $\mathfrak{S}_\delta(\mathcal{T})$ is almost surely constant.*

Proof. We set $F(x) = \mathbb{P}(\mathfrak{S}_\delta(\mathcal{T}) \leq x)$ for all $x \in \mathbb{R}$ and we set $a = \sup\{x \in \mathbb{R} : F(x) = 0\}$. We have $a \geq 0$. If $a = \infty$ then $\mathfrak{S}_\delta(\mathcal{T}) = \infty$ almost surely. Thus, we now assume that $a < \infty$ and we only need to show that $F(a) = 1$ to conclude that $\mathfrak{S}_\delta(\mathcal{T}) = a$ almost surely. In this proof, we use the notations of Theorem 5.7.3. In that manner, we define $G(x) = \mathbb{P}(\delta \mathfrak{S}_\delta(\mathcal{T}_g) \leq xU)$ for all $x \geq 0$. We also have $G(x) = \mathbb{E}[F(xU/\delta)]$ by independence between \mathcal{T}_g and U . By Proposition 5.7.6 (iii), we can apply (5.8.12) in Proposition 5.8.7 together with Proposition 5.8.6 to learn that

$$\mathfrak{S}_\delta(\mathcal{T}^*) \geq \max \left(\frac{1}{U} \mathfrak{S}_\delta(\mathcal{T}_g), \frac{1}{\delta} \mathfrak{S}_\delta(\mathcal{T}_d), \min \left(\frac{\delta}{U} \mathfrak{S}_\delta(\mathcal{T}_g), \mathfrak{S}_\delta(\mathcal{T}_d) \right) \right)$$

almost surely. Since $\mathcal{T}^*, \mathcal{T}_g, \mathcal{T}_d$ are HS_α -real trees, this leads to the inequality

$$F(x) \leq F(x)G(x) + G(x)(F(\delta x) - F(x)) + F(x)(G(\delta x) - G(x)) \quad (5.8.17)$$

for all $x \geq 0$. As cumulative distribution functions, F and G are nonnegative, bounded by 1, non-decreasing, and càdlàg. Moreover, we have $F(y) = 0 < F(x)$ when $y < a < x$ by definition of a . In particular, $G(x) = \mathbb{E}[\mathbf{1}_{\{xU \geq \delta a\}} F(xU/\delta)]$ which implies $G(x) \leq \mathbb{P}(xU \geq \delta a)F(x)$ because $U \leq \delta$ almost surely. Injecting this inequality into (5.8.17) and dividing by $F(x)$ get us

$$1 \leq G(\delta x) + \mathbb{P}(xU \geq \delta a)(F(\delta x) - F(x)) \quad (5.8.18)$$

for all $x > a$. If $a = 0$ then $F(\delta x) - F(x) \rightarrow F(0) - F(0) = 0$ as x tends to a^+ . If $a > 0$ then $\mathbb{P}(xU \geq \delta a) \rightarrow \mathbb{P}(U \geq \delta) = 0$ as x tends to a^+ . We thus obtain that $G(\delta a) = 1$ by making $x \rightarrow a^+$ in (5.8.18), whatever the case. It follows that $F(aU) = 1$ almost surely because $F \leq 1$. We recall that U admits a positive density on $[1, \delta]$ so we get $F(a) = F(a^+) = 1$ as desired. ■

Proposition 5.8.10 *Let \mathcal{T} be a HS_α -real tree. The inequality $\mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \geq 1$ holds.*

Proof. Let $(L_u, U_u)_{u \in \mathbb{W}}$ be a family of independent random variables such that for all $u \in \mathbb{W}$, the law of L_u is exponential with mean δ^{-1} and the law of U_u is $\mathbf{1}_{[1, \delta]}(s)s^{\beta-1}ds$. We denote by $u_1^n, \dots, u_{2^{n+1}-1}^n$ the vertices of \mathbb{W}_n in the lexicographic order. We claim that for all $n \in \mathbb{N}$, there is a random 2^{n+1} -pointed compact metric space $(\mathcal{T}_n, d_n, \rho_n, \varphi_n(u_1^n), \dots, \varphi_n(u_{2^{n+1}-1}^n))$ such that

- (a) $(\mathcal{T}_n, d_n, \rho_n)$ is a HS_α -real tree,
- (b) $\varphi_n : u \in \mathbb{W}_n \mapsto \varphi_n(u) \in \mathcal{T}_n$ is almost surely an embedding,
- (c) the family $(\delta^{|u|+1}d_n(\varphi_n(\overleftarrow{u}), \varphi_n(u)))_{u \in \mathbb{W}_n}$ has the same law as $(\delta L_u \prod_{v^*(1) \preceq u} \delta U_v^{-1})_{u \in \mathbb{W}_n}$.

Indeed, we prove this by induction while using the notations of Theorem 5.7.3. We get the case $n = 0$ by endowing \mathcal{T}^* with $\varphi_0(\emptyset) = L$. By induction, we can assume that \mathcal{T}_g (resp. \mathcal{T}_d) is endowed with an almost sure embedding φ_g (resp. φ_d) from \mathbb{W}_n into \mathcal{T}_g (resp. \mathcal{T}_d) such that (c) is satisfied. Then, we construct \mathcal{T}_{n+1} by endowing \mathcal{T}^* with its points $(\varphi_{n+1}(u))_{u \in \mathbb{W}_{n+1}}$, that are defined by $\varphi_{n+1}(\emptyset) = L$, $\varphi_{n+1}((1) * u) = \varphi_g(u)$ and $\varphi_{n+1}((2) * u) = \varphi_d(u)$ for all $u \in \mathbb{W}_n$. We easily check the requirements (a-c) thanks to Theorem 5.7.3 and Proposition 5.2.25. □

We find that $\mathbb{E}[\mathfrak{S}_\delta^{k,k}(\mathcal{T})] < \infty$ by (5.8.1) and (5.7.1). Moreover, our previous claim yields that

$$\mathbb{E}[\mathfrak{S}_\delta^{k,n}(\mathcal{T})] \geq \mathbb{E} \left[\inf_{\substack{u \in \mathbb{W} \\ k \leq |u| \leq n}} \frac{\delta}{|u| + 1} \sum_{v \preceq u} L_v \prod_{w^*(1) \preceq v} \frac{\delta}{U_w} \right]$$

for all $n, k \in \mathbb{N}$. On both sides of the equality, we first apply the dominated convergence theorem as n tends to ∞ , and then we use the monotone convergence theorem as k tends to ∞ . We thus get

$$\mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \geq \mathbb{E} \left[\liminf_{n \rightarrow \infty} \frac{\delta}{n+1} \inf_{\substack{u \in \mathbb{W} \\ |u|=n}} \sum_{v \preceq u} L_v \prod_{w^*(1) \preceq v} \frac{\delta}{U_w} \right]. \quad (5.8.19)$$

It remains to study the right-hand side of (5.8.19). For all $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$, we set

$$f_n(s) = \mathbb{E} \left[\sum_{u \in \mathbb{W}, |u|=n} \exp \left(-s\delta \sum_{v \preceq u} L_v \prod_{w^*(1) \preceq v} \frac{\delta}{U_w} \right) \right] \quad \text{and} \quad g(s) = \limsup_{n \rightarrow \infty} \frac{1}{n+1} \ln f_n(s).$$

The function f_n is positive, non-increasing on \mathbb{R}_+ , and bounded by 2^n . So, we have $g \leq \ln 2$. From now, we write $U = U_\emptyset$ to lighten the notations. The two independent families $(L_{(1)*u}, U_{(1)*u})_{u \in \mathbb{W}}$ and $(L_{(2)*u}, U_{(2)*u})_{u \in \mathbb{W}}$ have the same law as $(L_u, U_u)_{u \in \mathbb{W}}$, and they are jointly independent from (L_\emptyset, U) . Since L_\emptyset and U are independent and since $\mathbb{E}[e^{-s\delta L_\emptyset}] = 1/(1+s)$ by (5.3.11), we compute

$$\forall n \in \mathbb{N}, \forall s \in \mathbb{R}_+, \quad f_{n+1}(s) = \frac{f_n(s) + \mathbb{E}[f_n(s\delta/U)]}{1+s}. \quad (5.8.20)$$

Let us fix $r \in (1, \delta)$. Then, we use the monotony of f_n and the almost sure inequality $U \leq \delta$ to find

$$f_{n+1}(s) \leq a(s)f_n(s) + f_n(rs), \quad \text{where} \quad a(s) = \frac{1 + \mathbb{P}(rU > \delta)}{1 + s}, \quad (5.8.21)$$

from the relation (5.8.20). Let $b > \exp(g(rs))$. By definition of g , there exists $c \in \mathbb{R}_+$ such that $f_n(rs) \leq cb^n$ for all $n \in \mathbb{N}$. If $A > \max(a(s), b)$, then we can choose some $C \in (1, \infty)$ such that $Ca(s) + c \leq CA$, and thus (5.8.21) entails by induction that $f_n(s) \leq CA^n$ for all $n \in \mathbb{N}$. Then, $g(s) \leq \ln A$, and taking the infimum over b and A gives that $g(s) \leq \max(g(rs), \ln a(s))$ for all $s \in \mathbb{R}_+$. We remark that $a(rs) \leq a(s)$ because $r > 1$, so we obtain by induction that

$$\forall m \in \mathbb{N}, \forall s \in \mathbb{R}_+, \quad g(s) \leq \max\left(g(r^m s), \ln \frac{1 + \mathbb{P}(rU > \delta)}{1 + s}\right). \quad (5.8.22)$$

Since $U \leq \delta$ almost surely and f_n is non-increasing, (5.8.20) yields that $f_{n+1}(s) \leq \frac{2}{1+s}f_n(s)$ for all $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$. This implies that $g(s) \leq \ln(2) - \ln(1 + s)$, so we have $\lim_{\infty} g(s) = -\infty$. Therefore, making m tend to ∞ in (5.8.22) entails that $g(s) \leq \ln(1 + \mathbb{P}(rU > \delta)) - \ln(1 + s)$ for all $s > 0$ and $r \in (1, \delta)$. It follows that $g(s) \leq -\ln(1 + s)$ for all $s > 0$ by making r tend to 1, because $U < \delta$ almost surely. Let $\varepsilon > 0$. Our upper bound for g ensures that there exists $s_0 > 0$ such that $g(s_0) < -(1 - \varepsilon)s_0$. It then holds $s_0(1 - 2\varepsilon)(n + 1) + \ln f_n(s_0) \leq -s_0\varepsilon(n + 1)$ for all large enough n , by definition of g . Whenever n is large enough, a Chernoff bound thus entails

$$\mathbb{P}\left(\inf_{\substack{u \in \mathbb{W} \\ |u|=n}} \delta \sum_{v \preceq u} L_v \prod_{w^*(1) \preceq v} \frac{\delta}{U_w} \leq (1 - 2\varepsilon)(n + 1)\right) \leq e^{-s_0\varepsilon(n+1)}.$$

By the Borel-Cantelli lemma and (5.8.19), we get $\mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \geq 1 - 2\varepsilon$ for all $\varepsilon > 0$. ■

To show $\mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \leq 1$ when \mathcal{T} is a HS_α -real tree, we need a technical lemma.

Lemma 5.8.11 *Let X, Y, Z be three almost surely finite nonnegative random variables. We assume that X and Y are independent, $\mathbb{E}[Y] < 1$, and Z is integrable. If there exists a random variable X' distributed as X such that $X' \leq YX + Z$ almost surely, then X is integrable and*

$$\mathbb{E}[X] \leq \frac{\mathbb{E}[Z]}{1 - \mathbb{E}[Y]}.$$

Proof. We choose $a \in (\mathbb{E}[Y], 1)$ and we give ourselves a sequence $(Y_n)_{n \in \mathbb{N}^*}$ of independent random variables distributed as Y and jointly independent from (X', X, Y, Z) . The almost sure inequality $X' \leq YX + Z$ implies that $\mathbb{P}(X > x) \leq \mathbb{P}(XY_n > ax) + \mathbb{P}(Z > (1 - a)x)$ for all $x > 0$ and all $n \in \mathbb{N}^*$. By conditioning on (Y_1, \dots, Y_{n-1}) , we prove by induction that

$$\mathbb{P}(X > x) \leq \mathbb{P}\left(X \prod_{i=1}^n Y_i > a^n x\right) + \sum_{i=1}^n \mathbb{P}\left(Z \prod_{j=1}^{i-1} Y_j > (1 - a)a^{i-1} x\right)$$

for all $x > 0$ and $n \in \mathbb{N}$. The Y_i are independent and $\mathbb{E}[Y_i/a] = \mathbb{E}[Y/a] < 1$, so the random variables $\prod_{i=1}^n Y_i/a$ converge in mean to 0. As X is almost surely finite, it follows that $X \prod_{i=1}^n Y_i/a \rightarrow 0$ in probability, which yields

$$\mathbb{P}(X > x) \leq \sum_{i \geq 1} \mathbb{P}\left(\frac{1}{1-a} Z \prod_{j=1}^{i-1} \frac{1}{a} Y_j > x\right)$$

for all $x > 0$ by making n tend to ∞ . Now, we integrate over $x \in (0, \infty)$ to find

$$\mathbb{E}[X] \leq \sum_{i \geq 1} \mathbb{E} \left[\frac{1}{1-a} Z \prod_{j=1}^{i-1} \frac{1}{a} Y_j \right] = \frac{a \mathbb{E}[Z]}{(1-a)(a - \mathbb{E}[Y])} < \infty.$$

The upper bound is obtained by rearranging $\mathbb{E}[X] \leq \mathbb{E}[YX + Z] = \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[Z]$. ■

Proposition 5.8.12 *Let \mathcal{T} be a HS_α -real tree. The inequality $\mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \leq 1$ holds.*

Proof. With the notations of Definition 5.8.3, we set $S_n(T) = (n+1)\mathfrak{S}_\delta^{n,n}(T)$ for any rooted compact real tree T and all $n \in \mathbb{N}$. Recalling (5.8.2), it is clear that $\mathfrak{S}_\delta(T) \leq \liminf_{n \rightarrow \infty} \frac{1}{n+1} S_n(T)$. Therefore, Fatou's lemma and Jensen's inequality entail that

$$\forall a > 0, \quad \mathbb{E}[\mathfrak{S}_\delta(\mathcal{T})] \leq \liminf_{n \rightarrow \infty} \frac{1}{a(n+1)} \ln \mathbb{E}[e^{aS_n(\mathcal{T})}]. \quad (5.8.23)$$

Moreover, the estimate (5.7.1) ensures that $\mathbb{E}[e^{aS_n(\mathcal{T})}] = \mathbb{E}[e^{a\delta h(\mathcal{T})}] < \infty$ as soon as $a > 0$ is small enough. Let us fix for now $a \in (0, 1)$ and $n \in \mathbb{N}$ such that $e^{aS_n(\mathcal{T})}$ is integrable. In this proof, we shall use the notations of Theorem 5.7.3. Furthermore, we define the set \mathcal{E} as

$$\left\{ \frac{1}{\delta} \odot_\alpha \mathcal{T}_d \right\} \cup \left\{ \frac{1}{\delta} \prod_{j=2}^i \Lambda_i \odot_\alpha \mathcal{T}_i : i \geq 2 \right\} \cup \left\{ \frac{1}{\delta \xi(\omega)} \prod_{j=2}^i \lambda_j(\omega) \odot_\alpha T_i(\omega) : i \geq 1, \omega \in \mathcal{N}, \ell(\omega) \leq L \right\}.$$

Then, (5.8.10) in Proposition 5.8.7 ensures that the following inequality holds almost surely:

$$S_{n+1}(\mathcal{T}^*) \leq \delta L + \max \left(S_{n+1}(U^{-1} \odot_\alpha \mathcal{T}_g), \sup_{T \in \mathcal{E}} \delta S_n(T) \right). \quad (5.8.24)$$

Before proceeding, let us give ourselves some parameters. We choose $\varepsilon \in (0, 1)$ such that $2\varepsilon < 1 - a$. We can pick out $\eta \in (0, \frac{\varepsilon}{1+\varepsilon})$ such that $\mathbb{P}((1-\eta)U \leq 1) \leq \varepsilon$ and $\mathbb{P}(\Lambda_2 \geq 1-\eta) \leq \varepsilon$ because $\Lambda_2 < 1 < U$ almost surely. Recall from (5.1.8) that $\beta(\alpha-1) = 1$. There is $c \in (e^a, \infty)$ that does not depend on n (but on $\alpha, a, \varepsilon, \eta$) such that $(1+(1+x)x^{\beta+1})e^{a(1-\eta)x} \leq \varepsilon e^{ax} + c$ for all $x \geq 0$. Moreover, for all $x \geq 0$ and all $\lambda \in [0, 1]$, we see that $e^{a\lambda x} \leq \mathbf{1}_{\{\lambda \geq 1-\eta\}} e^{ax} + \mathbf{1}_{\{\lambda < 1-\eta\}} e^{a(1-\eta)x} + c$. Applying these inequalities to bound each term of the supremum in (5.8.24), we obtain that it holds

$$e^{aS_{n+1}(\mathcal{T}^*)} \leq (\mathbf{1}_{\{1 \geq (1-\eta)U\}} + \varepsilon) e^{a\delta L} \times e^{aS_{n+1}(\mathcal{T}_g)} + e^{a\delta L} (2c + e^{aS_n(\mathcal{T}_d)} + Z_+ + Z_-) \quad (5.8.25)$$

almost surely, where we have denoted

$$\begin{aligned} Z_+ &= \sum_{i \geq 1} \left(\mathbf{1}_{\{\prod_{j=2}^{i+1} \Lambda_j \geq 1-\eta\}} e^{aS_n(\mathcal{T}_{i+1})} + \sum_{\substack{\omega \in \mathcal{N} \\ \ell(\omega) \leq L}} \mathbf{1}_{\{\prod_{j=2}^i \lambda_j(\omega) \geq (1-\eta)\xi(\omega)\}} e^{aS_n(T_i(\omega))} \right), \\ Z_- &= \sum_{i \geq 1} \left(\mathbf{1}_{\{S_n(\mathcal{T}_{i+1}) \prod_{j=2}^{i+1} \Lambda_j \geq 1\}} e^{a(1-\eta)S_n(\mathcal{T}_{i+1})} \right. \\ &\quad \left. + \sum_{\substack{\omega \in \mathcal{N} \\ \ell(\omega) \leq L}} \mathbf{1}_{\{S_n(T_i(\omega)) \prod_{j=2}^i \lambda_j(\omega) \geq \xi(\omega)\}} e^{a(1-\eta)S_n(T_i(\omega))} \right). \end{aligned}$$

Now, we want to use Lemma 5.8.11 to bound $\mathbb{E}[e^{aS_{n+1}(\mathcal{T}^*)}]$ by an affine function of $\mathbb{E}[e^{aS_n(\mathcal{T})}]$, that was already assumed to be finite. Indeed, $S_{n+1}(\mathcal{T}_g)$ has the same law as $S_{n+1}(\mathcal{T}^*)$, is independent

of (L, U) , and is almost surely finite by (5.8.1). Moreover, δL is exponentially distributed with mean 1 and is independent from U , so (5.3.11) and the choice of ε and η ensure that

$$\mathbb{E}[(\mathbf{1}_{\{(1-\eta)U \leq 1\}} + \varepsilon)e^{a\delta L}] \leq \frac{2\varepsilon}{1-a} < 1. \tag{5.8.26}$$

Since L is also independent of \mathcal{T}_d , it only remains to control the means of $e^{a\delta L} Z_+$ and $e^{a\delta L} Z_-$. Using the independence, we average first on the Poisson point process \mathcal{N} to compute that

$$\begin{aligned} \mathbb{E}[e^{a\delta L} Z_+] &= \sum_{i \geq 1} \mathbb{E} \left[e^{aS_n(\mathcal{T}_g)} e^{a\delta L} \left(\mathbf{1}_{\{\prod_{j=2}^{i+1} \Lambda_j \geq 1-\eta\}} + \delta L \int_1^{\frac{1}{1-\eta}} \prod_{j=2}^i \Lambda_j \, d\xi \right) \right], \\ \mathbb{E}[e^{a\delta L} Z_-] &= \sum_{i \geq 1} \mathbb{E} \left[e^{a(1-\eta)S_n(\mathcal{T}_g)} e^{a\delta L} \left(\mathbf{1}_{\{S_n(\mathcal{T}_g) \prod_{j=2}^{i+1} \Lambda_j \geq 1\}} + \delta L \int_1^{S_n(\mathcal{T}_g)} \prod_{j=2}^i \Lambda_j \, d\xi \right) \right]. \end{aligned}$$

The Λ_j are all smaller than 1 and $\eta \leq \frac{\varepsilon}{1+\varepsilon} \leq 1/2$, so it holds $\frac{1}{1-\eta} \prod_{j=2}^i \Lambda_j \leq 1+\varepsilon$ for all $i \geq 1$. Plus, if $\prod_{j=2}^{i+1} \Lambda_j \geq 1-\eta$, then $\Lambda_2 \geq 1-\eta$ and $\prod_{j=3}^i \Lambda_j \geq 1/2$. Arguments of this type lead to

$$\begin{aligned} \mathbb{E}[e^{a\delta L} Z_+] &\leq \sum_{i \geq 1} \mathbb{E} \left[(\mathbf{1}_{\{\Lambda_2 \geq 1-\eta\}} + \varepsilon)(1 + \delta L)e^{a\delta L} e^{aS_n(\mathcal{T}_g)} \mathbf{1}_{\{\prod_{j=3}^i \Lambda_j \geq 1/2\}} \right], \\ \mathbb{E}[e^{a\delta L} Z_-] &\leq \sum_{i \geq 1} \mathbb{E} \left[(1 + \delta L)e^{a\delta L} (1 + S_n(\mathcal{T}_g)) e^{a(1-\eta)S_n(\mathcal{T}_g)} \mathbf{1}_{\{S_n(\mathcal{T}_g) \prod_{j=2}^i \Lambda_j \geq 1\}} \right]. \end{aligned}$$

Now, we use Markov's inequality on the $\prod_{j=3}^i \Lambda_j^{\beta+1}$ and we apply the joint independence to get

$$\begin{aligned} \mathbb{E}[e^{a\delta L} Z_+] &\leq (\mathbb{P}(\Lambda_2 \geq 1-\eta) + \varepsilon) \mathbb{E}[(1 + \delta L)e^{a\delta L}] \mathbb{E}[e^{aS_n(\mathcal{T})}] 2^{\beta+1} \sum_{i \geq 1} \prod_{j=3}^i \mathbb{E}[\Lambda_j^{\beta+1}], \\ \mathbb{E}[e^{a\delta L} Z_-] &\leq \mathbb{E}[(1 + \delta L)e^{a\delta L}] \mathbb{E}[(1 + S_n(\mathcal{T})) S_n(\mathcal{T})^{\beta+1} e^{a(1-\eta)S_n(\mathcal{T})}] \sum_{i \geq 1} \prod_{j=2}^i \mathbb{E}[\Lambda_j^{\beta+1}]. \end{aligned}$$

We easily compute $\mathbb{E}[(1 + \delta L)e^{a\delta L}] = \frac{2-a}{(1-a)^2}$. Finally, (5.7.3) and the choice of η and c yield that

$$\mathbb{E}[e^{a\delta L} Z_+] \leq \frac{2^{\beta+3}}{(1-a)^2} \sum_{i \geq 1} \prod_{j=3}^i \mathbb{E}[\Lambda_j^{\beta+1}] \times \varepsilon \mathbb{E}[e^{aS_n(\mathcal{T})}] < \infty, \tag{5.8.27}$$

$$\mathbb{E}[e^{a\delta L} Z_-] \leq \frac{2}{(1-a)^2} \sum_{i \geq 1} \prod_{j=2}^i \mathbb{E}[\Lambda_j^{\beta+1}] \times (\varepsilon \mathbb{E}[e^{aS_n(\mathcal{T})}] + c) < \infty. \tag{5.8.28}$$

Recall the inequality (5.8.25). Lemma 5.8.11, (5.8.26), (5.8.27), and (5.8.28) entail that

$$\mathbb{E}[e^{aS_{n+1}(\mathcal{T})}] \leq \frac{1 + \frac{C}{1-a}\varepsilon}{1-a-2\varepsilon} \mathbb{E}[e^{aS_n(\mathcal{T})}] + c' < \infty$$

where $C \in (0, \infty)$ is a constant only depending on α , and $c' \in (0, \infty)$ does not depend on n (but on $\alpha, a, \varepsilon, \eta$). An elementary study yields that for all $a \in (0, 1)$ small enough and all $\varepsilon \in (0, \frac{1-a}{2})$,

$$\liminf_{n \rightarrow \infty} \frac{1}{a(n+1)} \ln \mathbb{E}[e^{aS_n(\mathcal{T})}] \leq \frac{1}{a} \ln \left(\frac{1 + \frac{C}{1-a}\varepsilon}{1-a-2\varepsilon} \right).$$

Thanks to (5.8.23), we conclude the proof by making $\varepsilon \rightarrow 0^+$ and then $a \rightarrow 0^+$. ■

5.9 Proofs of Theorems 5.1.5 and 5.1.6

In all this section, let τ stand for a \mathbf{GW}_α -weighted tree. Recall from (5.2.13) that $H(\tau)$ is its height function. Recall from Section 5.2.1 the space \mathcal{C}_K of continuous functions with compact support and endowed with lifetimes, and from (5.2.26) its subspace \mathcal{E}_K of continuous excursions. We will follow Notation 5.2.2. As in (5.2.27), we denote by \mathcal{T}_f the real tree coded by $f \in \mathcal{E}_K$. We use the objects \mathbf{N}_α , \mathbf{H} , and $\mathbf{N}_\alpha(\mathrm{dH} \mid \zeta = 1)$ presented in Section 5.2.5. Recall from (5.1.8) and (5.1.11) the notations

$$\beta = \frac{1}{\alpha-1}, \quad \gamma = \ln \frac{\alpha}{\alpha-1}, \quad \delta = e^{\gamma(\alpha-1)}, \quad \text{and} \quad a_n = \alpha^{-1/\alpha} n^{1/\alpha} \text{ for all } n \in \mathbb{N}.$$

Of course, it is important to remember Definition 5.8.3 of the Strahler dilation \mathfrak{S}_δ with base δ .

5.9.1 Proof of Theorem 5.1.5

Recall Definition 5.7.1. To show the identity (5.1.13), we describe the law of a \mathbf{HS}_α -excursion in terms of the Strahler dilation and \mathbf{N}_α . Along the lines of the proof, we obtain a result similar to the desired Theorem 5.1.6 for the tail conditioning $\{\#\tau \geq n\}$.

Theorem 5.9.1 *Let \tilde{H} be a \mathbf{HS}_α -excursion. Recall $\text{base}_\mathfrak{S}$ from (5.8.16). The following identities hold.*

(i) *If $F : \mathcal{C}_K \rightarrow \mathbb{R}_+$ and $g : [0, \infty] \rightarrow \mathbb{R}_+$ are bounded and nonnegative, then*

$$\mathbf{N}_\alpha[F(\mathbf{H})g(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H}))] = \frac{\alpha^{-\beta}}{\alpha-1} \int_{\mathbb{R}_+} \mathbb{E}\left[F(x\tilde{H}_{s(\alpha x^\alpha)^{-\beta}}; s \geq 0)\right] g(x) \frac{\mathrm{d}x}{x^{1+\beta}}, \quad (5.9.1)$$

$$\mathbb{E}[F(\tilde{H})]\mathbf{N}_\alpha[g(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H}))] = \mathbf{N}_\alpha\left[F\left(\frac{1}{\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})} \mathbf{H}_{\alpha^\beta \mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})^{\alpha\beta} s}; s \geq 0\right) g(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H}))\right]. \quad (5.9.2)$$

(ii) *It holds $\mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})=0) = \mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})=\infty) = \mathbf{N}_\alpha(\text{base}_\mathfrak{S} \mathcal{T}_\mathbf{H} \neq \delta) = 0$. Furthermore,*

$$\forall x > 0, \quad \mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H}) > x) = (\alpha x)^{-\beta}. \quad (5.9.3)$$

(iii) *If $F : \mathcal{C}_K \rightarrow \mathbb{R}$ is bounded and measurable, then*

$$\mathbb{E}[F(\tilde{H})] = \frac{\alpha^\beta}{\Gamma(1-1/\alpha)} \mathbb{E}\left[F\left(\frac{1}{\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})} \mathbf{H}_{\alpha^\beta \mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})^{\alpha\beta} s}; s \geq 0\right) \mathfrak{S}_\delta(\mathcal{T}_\mathbf{H})^\beta \mid \zeta = 1\right]. \quad (5.9.4)$$

(iv) *If $F : \mathcal{C}_K \rightarrow \mathbb{R}$ is a bounded and continuous, then*

$$\mathbb{E}[F(\tilde{H})] = \lim_{\varepsilon \rightarrow 0^+} \mathbf{N}_\alpha\left[F(\alpha^{1/\alpha} \mathbf{H}) \mid |\mathfrak{S}_\delta(\mathcal{T}_\mathbf{H}) - \alpha^{-1/\alpha}| < \varepsilon\right]. \quad (5.9.5)$$

Proof. Let $\ell, z > 0$ and let $F : \mathcal{C}_K \rightarrow \mathbb{R}_+$ be nonnegative, bounded, and continuous. For all $y > 0$ and $n \in \mathbb{N}^*$, we set $x_n(y) = \frac{1}{\gamma\alpha} \ln(ny)$, $f_n(y) = 0$ when $x_n(y) \in \mathbb{N}$, and

$$f_n(y) = \mathbb{E}\left[\mathbf{1}_{\{\#\tau \geq \ell e^{\gamma\alpha x_n(y)}\}} F\left(\frac{y^{1-1/\alpha}}{\alpha^{1/\alpha}} e^{-\gamma(\alpha-1)x_n(y)} H_{e^{\gamma\alpha x_n(y)} s/y}(\tau); s \geq 0\right) \mid \mathcal{S}(\tau) = x_n(y)\right]$$

when $x_n(y) \notin \mathbb{N}$. The functions f_n are uniformly bounded, nonnegative, and measurable by Proposition 5.4.7. We apply the identity (5.4.14) and we make the change of variable $\gamma\alpha x = \ln(ny)$ to get

$$\mathbb{E}\left[\mathbf{1}_{\{\#\tau \geq \ell n; \delta \mathcal{S}(\tau) > zn/a_n\}} F\left(\frac{a_n}{n} H_{ns}(\tau); s \geq 0\right)\right] = \frac{1}{\alpha n^{1/\alpha}} \int_{(\alpha z^\alpha)^\beta}^{\infty} \frac{1}{y^{1+1/\alpha}} f_n(y) \mathrm{d}y.$$

There is at most a countable number of points $y > 0$ such that $\mathbb{P}(y\zeta(\tilde{H}) = \ell) > 0$ or such that $x_n(y) \in \mathbb{N}$ for some $n \in \mathbb{N}^*$. For all other $y > 0$, Theorem 5.6.1 and Proposition 5.7.6 (ii), with (5.2.14), entail that

$$f_n(y) \longrightarrow \mathbb{E}\left[\mathbf{1}_{\{y\zeta(\tilde{H}) > \ell\}} F(\alpha^{-1/\alpha} y^{1-1/\alpha} \tilde{H}_{s/y}; s \geq 0)\right].$$

Then, also by (5.2.39) and (5.3.4), the dominated convergence theorem yields that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}\left[\mathbf{1}_{\{\delta \mathcal{S}(\tau) > zn/a_n\}} F\left(\frac{a_n}{n} H_{ns}(\tau); s \geq 0\right) \mid \#\tau \geq \ell n\right] \\ &= \frac{1}{\alpha^{1+1/\alpha} \mathbf{N}_\alpha(\zeta > \ell)} \int_{(\alpha z^\alpha)^\beta}^{\infty} \frac{1}{y^{1+1/\alpha}} \mathbb{E}\left[\mathbf{1}_{\{y\zeta(\tilde{H}) > \ell\}} F(\alpha^{-1/\alpha} y^{1-1/\alpha} \tilde{H}_{s/y}; s \geq 0)\right] dy \end{aligned} \quad (5.9.6)$$

for all $z \in (0, \infty)$. Let us show that (5.9.6) still holds when $z = 0$.

Indeed, by the monotone convergence theorem, the limit of the right-hand side of (5.9.6) as $z \rightarrow 0^+$ keeps the same expression after replacing z with 0. Thus, we only need to check that

$$\lim_{z \rightarrow 0^+} \limsup_{n \rightarrow \infty} \mathbb{P}(\delta \mathcal{S}(\tau) \leq z \frac{n}{a_n} \mid \#\tau \geq \ell n) = 0. \quad (5.9.7)$$

To do this, we recall that $|\tau| = \sup H(\tau)$ stands for the height of τ from (5.2.11) and (5.2.14), and that $\mathcal{S}(\tau) \leq \mathfrak{S}(\tau)$ from Proposition 5.4.3. For all $\eta > 0$, we then write the rough bound

$$\mathbb{P}(\delta \mathcal{S}(\tau) \leq z \frac{n}{a_n}; |\tau| \geq \eta \frac{n}{a_n} \mid \#\tau \geq \ell n) \leq \frac{\mathbb{P}(|\tau| \geq \eta n/a_n)}{\mathbb{P}(\#\tau \geq \ell n)} \mathbb{P}(\mathcal{S}(\tau) \leq \log_\delta(z \frac{n}{a_n}) \mid |\tau| \geq \eta \frac{n}{a_n}).$$

By the estimates (5.3.3), (5.3.4), and (5.4.8), it follows that for all $\eta > 0$,

$$\lim_{z \rightarrow 0^+} \limsup_{n \rightarrow \infty} \mathbb{P}(\delta \mathcal{S}(\tau) \leq z \frac{n}{a_n} \mid \#\tau \geq \ell n) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(|\tau| < \eta \frac{n}{a_n} \mid \#\tau \geq \ell n).$$

The right-hand side is bounded by $\mathbf{N}_\alpha(\sup H < \eta \mid \zeta > \ell)$ by Theorem 5.3.7. By (5.2.1), if $\zeta > \ell$ then $\sup H > 0$, so making $\eta \rightarrow 0^+$ gives (5.9.7) and completes the proof of (5.9.6) with $z = 0$. \square

Observe that if $z = 0$ then the left-hand side of (5.9.6) is also given by Theorem 5.3.7. Hence,

$$\mathbf{N}_\alpha[F(H) \mid \zeta > \ell] = \frac{1}{\alpha^{1+1/\alpha} \mathbf{N}_\alpha(\zeta > \ell)} \int_0^\infty \frac{1}{y^{1+1/\alpha}} \mathbb{E}\left[\mathbf{1}_{\{y\zeta(\tilde{H}) > \ell\}} F\left(\frac{y^{1-1/\alpha}}{\alpha^{1/\alpha}} \tilde{H}_{s/y}; s \geq 0\right)\right] dy.$$

We multiply both sides of the previous identity by $\mathbf{N}_\alpha(\zeta > \ell)$ and we make ℓ tend to 0^+ . Recall from Section 5.2.5 that $\mathbf{N}_\alpha(\zeta = 0) = 0$. We also have $\mathbb{P}(\zeta(\tilde{H}) = 0) = \mathbb{P}(\sup \tilde{H} = 0) = 0$ by (5.7.1). Thus, the monotone convergence theorem yields that

$$\mathbf{N}_\alpha[F(H)] = \frac{1}{\alpha^{1+1/\alpha}} \int_{\mathbb{R}_+} \frac{1}{y^{1+1/\alpha}} \mathbb{E}\left[F(\alpha^{-1/\alpha} y^{1-1/\alpha} \tilde{H}_{s/y}; s \geq 0)\right] dy, \quad (5.9.8)$$

for all nonnegative, bounded, and continuous function $F: \mathcal{C}_K \rightarrow \mathbb{R}_+$. Therefore, (5.9.8) also holds for all nonnegative and measurable functions. Theorem 5.1.4 ensure that the measurable function $\Upsilon: \omega \in \mathcal{C}_K \mapsto \mathbf{1}_{\{\omega \in \mathcal{E}_K\}} \mathfrak{S}_\delta(\mathcal{T}_\omega) \in [0, \infty]$ satisfies that $\Upsilon(\lambda \tilde{H}_{s/y}; s \geq 0) = \lambda$ almost surely for all $\lambda, y > 0$. Eventually, applying (5.9.8) with the function $F \cdot g \circ \Upsilon$ and making the change of variable $x = \alpha^{-1/\alpha} y^{1-1/\alpha}$ entail (5.9.1).

Using again that $\Upsilon(\lambda \tilde{H}_{s/y}; s \geq 0) = \lambda$, (5.9.2) becomes a consequence of (5.9.1), which completes the proof of (i). The point (ii) follows from (5.9.1) by taking $F = 1$, and $g = \mathbf{1}_{\{0\} \cup \{\infty\}}$ or $g = \mathbf{1}_{(x, \infty)}$ (also recall Proposition 5.8.8). We get (iii) by applying (5.9.2), then (5.2.40), then Fubini's theorem, and (5.9.1). By (5.9.2), $(\mathfrak{S}_\delta(\mathcal{T}_H))^{-1} \mathbf{H}_{\alpha^\beta \mathfrak{S}_\delta(\mathcal{T}_H)^{\alpha\beta} s}; s \geq 0)$ under

$\mathbf{N}_\alpha(\cdot \mid |\mathfrak{S}_\delta(\mathcal{T}_H) - \alpha^{-1/\alpha}| < \varepsilon)$ is a HS_α -excursion for all small $\varepsilon > 0$. Plus, $\mathfrak{S}_\delta(\mathcal{T}_H)$ under $\mathbf{N}_\alpha(\cdot \mid |\mathfrak{S}_\delta(\mathcal{T}_H) - \alpha^{-1/\alpha}| < \varepsilon)$ clearly converges in law to $\alpha^{-1/\alpha}$ as $\varepsilon \rightarrow 0^+$. Slutsky's theorem completes the proof of (iv). ■

The desired identity (5.1.13) readily follows from (5.9.4), which completes the proof of Theorem 5.1.5. Theorem 5.9.1 (iv) justifies the intuition that a HS_α -excursion is (a multiple of) an excursion of the α -stable height process conditioned on its coded real tree having fixed Strahler dilation.

Remark 5.9.2 Theorem 5.9.1 (ii) and the identity (5.2.41) entail that there is equality between the two image measures of \mathbf{N}_α by the respective functions $\sup H = \mathfrak{h}(\mathcal{T}_H)$ and $\alpha\beta \mathfrak{S}_\delta(\mathcal{T}_H)$. □

Finally, we recognize thanks to the identity (5.9.1) (or (5.9.8) equivalently) that the right-hand side of (5.9.6) is equal to $\mathbf{N}_\alpha[\mathbf{1}_{\{\mathfrak{S}_\delta(\mathcal{T}_H) > z\}} F(H) \mid \zeta > \ell]$. We readily obtain the following result.

Theorem 5.9.3 *Let τ be a GW_α -weighted tree and let (a_n) be as in (5.1.11). For all $\ell > 0$, the following joint convergence holds in distribution on $\mathcal{C}_K \times \mathbb{R}_+$:*

$$\left(\left(\frac{a_n}{n} H_{ns}(\tau) \right)_{s \geq 0}, \frac{a_n}{n} \delta \mathcal{S}(\tau) \right) \text{ under } \mathbb{P}(\cdot \mid \#\tau \geq \ell n) \xrightarrow{d} (H, \mathfrak{S}_\delta(\mathcal{T}_H)) \text{ under } \mathbf{N}_\alpha(\cdot \mid \zeta > \ell).$$

5.9.2 Proof of Theorem 5.1.6

Here, our goal is to deduce from Theorem 5.9.3 the following result.

Theorem 5.9.4 *Let τ be a GW_α -weighted tree and let (a_n) be as in (5.1.11). The following joint convergence holds in distribution on $\mathcal{C}_K \times \mathbb{R}_+$:*

$$\left(\left(\frac{a_n}{n} H_{ns}(\tau) \right)_{s \geq 0}, \frac{a_n}{n} \delta \mathcal{S}(\tau) \right) \text{ under } \mathbb{P}(\cdot \mid \#\tau = n + 1) \\ \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{} (H, \mathfrak{S}_\delta(\mathcal{T}_H)) \text{ under } \mathbf{N}_\alpha(dH \mid \zeta = 1).$$

Observe that Theorem 5.1.6 would then readily follow, thanks to Proposition 5.2.18. The fact that $\mathbf{N}_\alpha(\zeta = 1) = 0$ prevents us to mimic the proof of Theorem 5.9.3. To avoid this issue, we take advantage of the monotony of the weighted Horton–Strahler number. Let us give an overview of our strategy for the case $\alpha = 2$. It is well-known from Rémy's algorithm [139] that removing the parental edge of a random leaf from a uniform n -Catalan tree τ_n results in a uniform $(n-1)$ -Catalan tree τ_{n-1} . In the same spirit, we can embed a weighted tree τ_n with the same law as τ under $\mathbb{P}(\cdot \mid \#\tau = 2n-1)$ into a weighted tree τ'_n with the same law as τ under $\mathbb{P}(\cdot \mid \#\tau \geq 2n-1)$, so that $\mathcal{S}(\tau_n) \leq \mathcal{S}(\tau'_n)$. Then, Theorem 5.9.3 yields an asymptotic upper bound for $2^{\mathcal{S}(\tau_n)}$. However, this proof might not work without alteration when $\alpha \in (1, 2)$. Indeed, Janson [96] exhibited an example of a critical offspring distribution such that there are no embeddings $\tau_n \hookrightarrow \tau_{n+1}$ for some $n \geq 1$.

Nevertheless, as presented in Section 5.3.1, Marchal [128] have constructed a sequence of nested GW_α -trees conditioned on their number of leaves, instead of their number of vertices. In order to use it, we first transform Theorem 5.9.3 into a result about the tail conditioning on the number of leaves $\#\partial\tau$. Recall from Section 5.2.1 that \mathbb{K} stands for the space of (rooted-isometry classes of) rooted compact metric spaces, endowed with the rooted Gromov–Hausdorff distance (5.2.6). As discussed in Section 5.2.3, we identify the tree τ with the element $(\tau, d_{\text{gr}}, \emptyset)$ of \mathbb{K} , where d_{gr} is the graph distance given by (5.2.29). As in (5.2.8), we write $\lambda \cdot \tau = (\tau, \lambda d_{\text{gr}}, \emptyset)$ for all $\lambda > 0$.

Corollary 5.9.5 *Let τ be a \mathbf{GW}_α -weighted tree and let (a_n) be as in (5.1.11). For all $\ell \in (0, 1]$, the following joint convergence holds in distribution on $\mathbb{K} \times \mathbb{R}_+$:*

$$\left(\frac{a_n}{n} \cdot \tau, \frac{a_n}{n} \delta^{\mathcal{S}(\tau)} \right) \text{ under } \mathbb{P}(\cdot \mid \#\partial\tau \geq \ell n/\alpha) \xrightarrow{d} (\mathcal{T}_H, \mathfrak{S}_\delta(\mathcal{T}_H)) \text{ under } \mathbf{N}_\alpha(\cdot \mid \zeta > \ell).$$

Proof. If $\alpha = 2$, Proposition 5.3.4 (i) yields that the result directly follows from Theorem 5.9.3 and Proposition 5.2.18. If $\alpha \in (1, 2)$, observe that $m_n = \lceil \ell n/\alpha - (\ell n)^{3/4} \rceil$ takes all integer values large enough, because $\ell < \alpha$, and that $a_{\lceil \ell n/\alpha \rceil} m_n / (\lceil \ell n/\alpha \rceil a_{m_n}) \rightarrow 1$. Lemma 5.3.5 then entails the result from Theorem 5.9.3 and Proposition 5.2.18. ■

Proof of Theorem 5.9.4. The convergence in distribution of the first component is already known from Theorem 5.3.8. Proposition 5.2.18 and a tightness argument ensure that we only need to show

$$\mathbb{E} \left[g \left(\frac{a_n}{n} \cdot \tau \right) h \left(\frac{a_n}{n} \delta^{\mathcal{S}(\tau)} \right) \mid \#\tau = n + 1 \right] \xrightarrow{n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}} \mathbf{N}_\alpha [g(\mathcal{T}_H) h(\mathfrak{S}_\delta(\mathcal{T}_H)) \mid \zeta = 1] \quad (5.9.9)$$

for all bounded and continuous functions $g : \mathbb{K} \rightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}$. To lighten the notations, we write $p(n) = \mathbb{P}(\#\tau = n + 1)$ and $q(n) = \mathbb{P}(\#\partial\tau = n)$ for all $n \in \mathbb{N}$. Let $(\mathbf{t}_i)_{i \geq 1}$ be the sequence of random weighted trees constructed by Algorithm 5.3.1. For all $n \in \lfloor \alpha \rfloor \mathbb{N}$, we define

$$\mathbf{J}_n = \inf \{ i \geq 1 : \#\mathbf{t}_i \geq n + 1 \}, \quad \mathbf{T}_n = \mathbf{t}_{\mathbf{J}_n}, \quad \text{and} \quad \xi_n = \mathbf{1}_{\{\#\mathbf{T}_n = n + 1\}}.$$

The random integer \mathbf{J}_n is finite and the tree \mathbf{T}_n has exactly \mathbf{J}_n leaves by Proposition 5.3.2 (i). Let t be a tree with $n + 1$ vertices, the sequence $(\#\mathbf{t}_i)_{i \geq 1}$ is strictly increasing so $\mathbf{T}_n = t$ if and only if $t \in \{\mathbf{t}_i : i \geq 1\}$. However, the trees t and \mathbf{t}_i have the same number of leaves only when $i = \#\partial t$. Thus, $\mathbf{T}_n = t$ if and only if $\mathbf{t}_{\#\partial t} = t$, in which case $\mathbf{J}_n = \#\partial t$ and $\mathbf{T}_n = \mathbf{t}_{\#\partial t}$. By Proposition 5.3.2 (iii), we get that for all bounded measurable function f on the space \mathbb{T}_w of weighted trees, it holds

$$\mathbb{E}[\xi_n f(\mathbf{T}_n)] = \mathbb{E} \left[\frac{p(n)}{q(\#\partial\tau)} f(\tau) \mid \#\tau = n + 1 \right]. \quad (5.9.10)$$

When $\alpha = 2$, $p(n) = q(\#\partial\tau)$ under $\mathbb{P}(\cdot \mid \#\tau = n + 1)$ by Proposition 5.3.4 (i), so $\xi_n = 1$ almost surely. However, \mathbf{T}_n is not a \mathbf{GW}_α -tree conditioned to have n vertices when $\alpha \in (1, 2)$, because it is biased by its number of leaves. Nonetheless, we show that this bias disappears as n tends to ∞ :

$$\mathbb{E} \left[\left| \frac{p(n)}{q(\#\partial\tau)} - \chi \right| \mid \#\tau = n + 1 \right] \xrightarrow{n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}} 0, \quad \text{where } \chi = \begin{cases} \alpha^{-1} & \text{if } \alpha \in (1, 2), \\ 1 & \text{if } \alpha = 2. \end{cases} \quad (5.9.11)$$

Indeed, by Proposition 5.3.2 (i), it holds $\mathbf{J}_n \leq xn$ if and only if $\#\mathbf{t}_{\lfloor xn \rfloor} \geq n + 1$, for all $x \geq 1/n$. Thanks to Proposition 5.3.4, we get that the convergence $\#\mathbf{t}_i/i \xrightarrow{\mathbb{P}} \alpha$ happens in probability, and this implies that the following convergence also holds in probability:

$$\frac{1}{n} \mathbf{J}_n \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{\mathbb{P}} \alpha^{-1}. \quad (5.9.12)$$

Then, the estimate (5.3.7) or (5.3.6) yields that $p(n)/q(\#\mathbf{J}_n) \xrightarrow{\mathbb{P}} \chi$ in probability. Next, we use (5.9.10) and we obtain the following uniform integrability:

$$\mathbb{E} \left[\frac{p(n)}{q(\#\partial\tau)} \mathbf{1}_{\{p(n) \geq 10\chi q(\#\partial\tau)\}} \mid \#\tau = n + 1 \right] \leq \mathbb{P} \left(\frac{p(n)}{q(\mathbf{J}_n)} \geq 10\chi \right) \xrightarrow{n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}} 0. \quad (5.9.13)$$

Moreover, Proposition 5.3.4, together with (5.3.7) or (5.3.6), yields that the desired convergence in L^1 happens in probability, so (5.9.13) completes the proof of (5.9.11). \square

From Theorem 5.3.3, we know that there is an α -stable tree \mathcal{T}_{nr} such that $i^{1/\alpha-1} \cdot \mathbf{t}_i \longrightarrow \alpha \cdot \mathcal{T}_{\text{nr}}$ almost surely on \mathbb{K} . It follows that $J_n^{1/\alpha-1} \cdot T_n \longrightarrow \alpha \cdot \mathcal{T}_{\text{nr}}$ almost surely on \mathbb{K} . Then, we apply the convergence in probability (5.9.12) and we write $\alpha a_n/n = (\alpha J_n/n)^{1-1/\alpha} \times J_n^{1/\alpha-1}$ to deduce that the following convergence holds in probability on \mathbb{K} :

$$\frac{a_n}{n} \cdot T_n \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{\mathbb{P}} \mathcal{T}_{\text{nr}}. \quad (5.9.14)$$

For all $n \in \lfloor \alpha \rfloor \mathbb{N}$, let Λ_n be a random variable independent of $(\mathbf{t}_i)_{i \geq 1}$ and \mathcal{T}_{nr} , and distributed as $\#\partial\tau$ under $\mathbb{P}(\cdot \mid \#\partial\tau \geq n/2\alpha)$. Thanks to (5.3.5) and by Kolmogorov's representation theorem, we can then assume that there is a random variable $\Lambda \geq 0$, independent of $(\mathbf{t}_i)_{i \geq 1}$ and \mathcal{T}_{nr} , such that

$$\left(\frac{\alpha}{n} \Lambda_n\right)^{1-1/\alpha} \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{\text{a.s.}} \Lambda, \quad \text{and} \quad \mathbb{P}(\Lambda > \lambda) = \min(1, 2^{-1/\alpha} \lambda^{-\beta}) \quad \text{for all } \lambda > 0. \quad (5.9.15)$$

Moreover, using similar arguments as before, the almost sure convergences $i^{1/\alpha-1} \cdot \mathbf{t}_i \longrightarrow \alpha \cdot \mathcal{T}_{\text{nr}}$ and (5.9.15) entail that the following convergence also holds almost surely on \mathbb{K} :

$$\frac{a_n}{n} \cdot \tau_n \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{\text{a.s.}} \Lambda \cdot \mathcal{T}_{\text{nr}}, \quad \text{where} \quad \tau_n = \mathbf{t}_{\Lambda_n} \quad \text{for all } n \in \lfloor \alpha \rfloor \mathbb{N}. \quad (5.9.16)$$

By Proposition 5.3.2 (iii), it is clear that τ_n is distributed as τ under $\mathbb{P}(\cdot \mid \#\partial\tau \geq n/2\alpha)$, so we may and will apply Corollary 5.9.5 with $\ell = 1/2$ to it. Now, let us prove that $(a_n \delta^{\mathcal{S}(\mathbf{T}_n)}/n)_{n \in \lfloor \alpha \rfloor \mathbb{N}}$ is tight.

Indeed, we begin by using the independence between \mathbf{T}_n and Λ_n to write

$$\mathbb{P}(\delta^{\mathcal{S}(\mathbf{T}_n)} \geq x \frac{n}{a_n}; 2n > \alpha J_n) \leq \frac{1}{\mathbb{P}(\alpha \Lambda_n > 2n)} \mathbb{P}(\delta^{\mathcal{S}(\mathbf{T}_n)} \geq x \frac{n}{a_n}; \Lambda_n > J_n)$$

for all $x > 0$. We remind that $\mathbf{T}_n = \mathbf{t}_{J_n}$ and $\tau_n = \mathbf{t}_{\Lambda_n}$, so Proposition 5.3.2 (ii) implies that

$$\mathbb{P}(\delta^{\mathcal{S}(\mathbf{T}_n)} \geq x \frac{n}{a_n}; 2n > \alpha J_n) \leq \frac{1}{\mathbb{P}(\alpha \Lambda_n/n > 2)} \mathbb{P}(\delta^{\mathcal{S}(\tau_n)} \geq x \frac{n}{a_n})$$

for all $x > 0$. While making $n \rightarrow \infty$, we use the convergence (5.9.12) on the left-hand side, and we apply Corollary 5.9.5 together with (5.9.15) on the right-hand side. As such, we find that for all $x > 0$,

$$\limsup_{n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}} \mathbb{P}(\delta^{\mathcal{S}(\mathbf{T}_n)} \geq x \frac{n}{a_n}) \leq \frac{1}{\mathbb{P}(\Lambda > 2^{1-1/\alpha})} \mathbf{N}_\alpha(\mathfrak{S}_\delta(\mathcal{T}_H) \geq x \mid \zeta > 1/2).$$

By making $x \rightarrow \infty$, the desired tightness follows from Theorem 5.9.1 (ii). \square

Also thanks to (5.9.15), (5.9.14), and Corollary 5.9.5, the family $(\Theta_n)_{n \in \lfloor \alpha \rfloor \mathbb{N}}$ of the random variables

$$\Theta_n = \left(\left(\frac{\alpha}{n} \Lambda_n\right)^{1-1/\alpha}, \xi_n, \frac{a_n}{n} \cdot T_n, \frac{a_n}{n} \delta^{\mathcal{S}(\mathbf{T}_n)}, \frac{a_n}{n} \cdot \tau_n, \frac{a_n}{n} \delta^{\mathcal{S}(\tau_n)} \right)$$

is therefore tight on the product space $\mathbb{R}_+ \times \{0, 1\} \times \mathbb{K} \times \mathbb{R}_+ \times \mathbb{K} \times \mathbb{R}_+$. Let us consider a subsequence (Θ_{n_k}) , with $n_k \in \lfloor \alpha \rfloor \mathbb{N}$ for all $k \in \mathbb{N}$, that converges in distribution. By (5.9.15), (5.9.14), and (5.9.16), there are some random variables $\xi \in \{0, 1\}$ and $U, V \geq 0$ such that it holds

$$\Theta_{n_k} \xrightarrow{d} (\Lambda, \xi, \mathcal{T}_{\text{nr}}, U, \Lambda \cdot \mathcal{T}_{\text{nr}}, V)$$

in distribution. Corollary 5.9.5 yields that $(\Lambda \cdot \mathcal{T}_{\text{nr}}, V)$ has the same law as $(\mathcal{T}_{\text{H}}, \mathfrak{S}_\delta(\mathcal{T}_{\text{H}}))$ under $\mathbf{N}_\alpha(\cdot | \zeta > 1/2)$. Thus, it holds $V = \Lambda \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})$ almost surely, thanks to Proposition 5.8.6. Moreover, Λ_n is independent of \mathbf{T}_n for all $n \in \lfloor \alpha \rfloor \mathbb{N}$, so Λ is independent of $(\xi, \mathcal{T}_{\text{nr}}, U)$. Hence, we have

$$\mathbb{P}(U > (1 + 2\varepsilon)\mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})) \leq \frac{1}{\mathbb{P}(1 + \varepsilon < \Lambda < 1 + 2\varepsilon)} \mathbb{P}(U > \Lambda \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}); 1 + \varepsilon < \Lambda)$$

for all $\varepsilon > 0$. Indeed, recall from (5.9.15) that Λ is absolutely continuous, and that its density is positive and continuous at $\lambda = 1$. The Portmanteau theorem entails that

$$\mathbb{P}(U > \Lambda \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}); 1 + \varepsilon < \Lambda) \leq \limsup_{k \rightarrow \infty} \mathbb{P}(\mathcal{S}(\mathbf{T}_{n_k}) > \mathcal{S}(\tau_{n_k}); (1 + \varepsilon)^{\alpha\beta} n_k < \alpha \Lambda_{n_k}).$$

By Proposition 5.3.2 (ii), if $\mathcal{S}(\mathbf{T}_n) > \mathcal{S}(\tau_n)$ and $(1 + \varepsilon)^{\alpha\beta} n < \alpha \Lambda_n$ then $(1 + \varepsilon)^{\alpha\beta} n < \alpha J_n$. Thanks to the convergence in probability (5.9.12), we get that $U \leq (1 + 2\varepsilon)\mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})$ almost surely. We show that $U \geq (1 - 2\varepsilon)\mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})$ for all $\varepsilon > 0$ in the same manner. Therefore, we have $U = \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}})$ almost surely. Let us now focus on the conditional law of ξ given \mathcal{T}_{nr} . Let $g : \mathbb{K} \rightarrow \mathbb{R}$ be a bounded and continuous function. We apply Theorem 5.3.8, together with Proposition 5.2.18, and we use the estimate (5.9.11) on the right-hand side of (5.9.10) to get

$$\mathbb{E}[\xi g(\mathcal{T}_{\text{nr}})] = \chi \mathbf{N}_\alpha[g(\mathcal{T}_{\text{H}}) | \zeta = 1] = \chi \mathbb{E}[g(\mathcal{T}_{\text{nr}})].$$

This means that ξ is independent of \mathcal{T}_{nr} , and has Bernoulli law with success probability χ . Hence, we have proved that $(\Theta_n)_{n \in \lfloor \alpha \rfloor \mathbb{N}}$ converges to $(\Lambda, \xi, \mathcal{T}_{\text{nr}}, \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}), \Lambda \cdot \mathcal{T}_{\text{nr}}, \Lambda \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}))$ in distribution, where $\Lambda, \xi, \mathcal{T}_{\text{nr}}$ are independent and ξ is a Bernoulli random variable with success probability χ . In particular, the following convergence holds in distribution on the product space $\{0, 1\} \times \mathbb{K} \times \mathbb{R}_+$:

$$\left(\xi_n, \frac{a_n}{n} \cdot T_n, \frac{a_n}{n} \delta \mathcal{S}(\mathbf{T}_n) \right) \xrightarrow[n \rightarrow \infty, n \in \lfloor \alpha \rfloor \mathbb{N}]{d} (\xi, \mathcal{T}_{\text{nr}}, \mathfrak{S}_\delta(\mathcal{T}_{\text{nr}}))$$

The desired convergence (5.9.9) then follows from (5.9.10) and (5.9.11). \blacksquare

Remark 5.9.6 Recall that \mathbf{t}_n is distributed as a \mathbf{GW}_α -weighted tree conditioned to have exactly n leaves. The scaling limit of the height function of \mathbf{t}_n was given by Kortchemski [107, Theorem 5.9]. The same proof as before with $\mathbf{T}_n = \mathbf{t}_n$ instead entails the convergence in distribution

$$\left(\left(\frac{n^{1/\alpha}}{\alpha n} H_{\alpha n s}(\tau) \right)_{s \geq 0}, \frac{n^{1/\alpha}}{\alpha n} \delta \mathcal{S}(\tau) \right) \text{ under } \mathbb{P}(\cdot | \#\partial\tau = n + 1) \xrightarrow{d} (\mathbf{H}, \mathfrak{S}_\delta(\mathcal{T}_{\text{H}})) \text{ under } \mathbf{N}_\alpha(d\mathbf{H} | \zeta = 1).$$

We leave the details to the reader. \square

5.10 The Strahler dilation of the Brownian tree: proof of Theorem 5.1.7

We restrict ourselves to the case $\alpha = 2$ in all this section. Then, (5.1.8) and (5.2.16) become

$$\beta = 1, \quad \gamma = \ln 2, \quad \delta = 2, \quad \text{and} \quad \mu_2 = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_2.$$

We denote by τ a \mathbf{GW}_2 -weighted tree. By (5.2.18), τ can be equal to a tree t with positive probability if and only if t is *binary*, meaning that all its vertices have either 0 or 2 children. We are

interested in the joint law of the size $\#\tau$ and the weighted Horton–Strahler $\mathcal{S}(\tau)$ of τ . Flajolet, Raoult & Vuillemin [75] and Kemp [98] have independently characterized the law of $(\#\tau, \mathcal{S}(\tau))$. Written with our notations, for all $p, n \in \mathbb{N}$ and $z \geq 0$, the formers have set

$$R_{p,n} = \#\{t \text{ binary tree} : \mathcal{S}(t) = p \text{ and } \#t = 2n + 1\} \quad \text{and} \quad R_p(z) = \sum_{n \geq 0} R_{p,n} z^n.$$

We stress that if t is a binary tree then $\mu_2(k_u(t)) = \frac{1}{2}$ for all $u \in t$, so $\mathbb{P}(\tau = t) = 2^{-\#t}$. By Proposition 5.3.4 (i), we hence obtain that $\mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau)=p\}} s^{\#\tau}] = \frac{s}{2} R_p(s^2/4)$ for all $s \geq 0$. From Definition 5.1.1 of the Horton–Strahler number, Flajolet, Raoult & Vuillemin [75] found a recurrence relation on the R_p . Moreover, they solved it exactly thanks to a trigonometric change of variable:

$$R_p(z) = \frac{\sin \phi}{\sin 2^p \phi} \quad \text{where} \quad \cos^2 \frac{\phi}{2} = \frac{1}{4z}.$$

Recalling that the law of $\mathcal{S}(\tau)$ is known from (5.1.5), it follows that for all $p \in \mathbb{N}$ and all $s \in (0, 1)$,

$$\mathbb{E}[s^{\#\tau} \mid \mathcal{S}(\tau) = p] = 2^{p+1} \frac{\sinh(a(s))}{\sinh(2^{p+1}a(s))}, \quad \text{where} \quad \cosh a(s) = \frac{1}{s}. \quad (5.10.1)$$

A similar phenomenon happens during the computation of the generating function of $\#\tau$ under the law $\mathbb{P}(d\tau \mid \mathcal{S}(\tau) = x)$ given by Definition 5.4.8.

Proposition 5.10.1 *Let τ be a \mathbf{GW}_2 -weighted tree. For all $p \in \mathbb{N}$, $y \in (0, 1)$, and $s \in (0, 1)$, it holds*

$$\mathbb{E}[s^{\#\tau} \mid \mathcal{S}(\tau) = p + y] = 4^p s \frac{\sinh^2(a_y(s))}{\sinh^2(2^p a_y(s))}, \quad \text{where} \quad \coth a_y(s) = \frac{1 - s^2 + s^2 2^{-y}}{\sqrt{1 - s^2}}.$$

Proof. We set $F_x(s) = \mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau) \leq x\}} s^{\#\tau}]$ for all $x \in \mathbb{R}_+$ and $s \in (0, 1)$. Let us fix $s \in (0, 1)$ for now. Using the assertion (5.4.10), Proposition 5.4.6 implies that for all $y \in [0, 1)$, it holds

$$F_y(s) = \mathbb{P}(\mathcal{S}(\tau) \leq y) s = (1 - 2^{-y}) s. \quad (5.10.2)$$

We assume $x \geq 1$. Recall from (5.1.2) that $\varphi_2(r) = r + \frac{1}{2}(1 - r)^2$ is the generating function of the offspring law μ_2 . By Definition 5.2.15, the decompositions $\#\tau = 1 + \sum_{i=1}^{k_\emptyset(\tau)} \# \theta_{(i)} \tau$ and (5.4.12) lead to

$$F_x(s) = s \varphi_2(F_{x-1}(s)) + s(F_x(s) - F_{x-1}(s)) \varphi_2'(F_{x-1}(s)).$$

After some easy manipulations, this identity becomes

$$F_x(s) = \frac{s}{2}(1 - F_{x-1}(s)^2) + s F_x(s) F_{x-1}(s).$$

Now, the trick is to set $G_x = \frac{1}{\sqrt{1-s^2}}(1 - s F_x(s))$ for all $x \in \mathbb{R}_+$ and to check that

$$\forall x \geq 1, \quad G_x = \frac{G_{x-1}^2 + 1}{2G_{x-1}}.$$

This allows us to recognize the double-angle formula for the hyperbolic cotangent:

$$\forall z \in \mathbb{R}, \quad \coth 2z = \frac{\coth^2 z + 1}{2 \coth z}.$$

Furthermore, we can verify via (5.10.2) and an elementary analysis that if $y \in [0, 1)$ then $G_y > 1$ because $s > 0$. In particular, there exists a unique $a_y(s) > 0$ such that $G_y = \coth a_y(s)$. We obtain by induction on the integer part $\lfloor x \rfloor$ that $G_x = \coth(2^{\lfloor x \rfloor} a_{x-\lfloor x \rfloor}(s))$ for all $x \in \mathbb{R}_+$. Therefore,

$$sF_x(s) = 1 - \sqrt{1-s^2} \coth(2^{\lfloor x \rfloor} a_{x-\lfloor x \rfloor}(s)), \quad \text{where} \quad \coth a_y(s) = \frac{1-s^2+s^2 2^{-y}}{\sqrt{1-s^2}}, \quad (5.10.3)$$

for all $s \in (0, 1)$, all $x \in \mathbb{R}_+$, and all $y \in [0, 1)$. For all $s \in (0, 1)$, Proposition 5.4.7 entails that the function $x \mapsto F_x(s)$ is C^1 on $\mathbb{R}_+ \setminus \mathbb{N}$, and that $(\ln 2)2^{-x} \mathbb{E}[s^{\#\tau} \mid \mathcal{S}(\tau) = x]$ is equal to its derivative at $x \in \mathbb{R}_+ \setminus \mathbb{N}$. Derivating (5.10.3) with respect to x completes the proof. ■

We can recover (5.10.1) from Proposition 5.10.1, or equivalently from (5.10.3). Indeed, the law of $\mathcal{S}(\tau)$ is absolutely continuous, so $\mathbb{E}[\mathbf{1}_{\{\mathcal{S}(\tau)=p\}} s^{\#\tau}] = F_{p+1}(s) - F_p(s)$ by Proposition 5.4.3. The hyperbolic identity $\cosh^2 = \coth^2 / (\coth^2 - 1)$ entails that $a_0(s) = a(s)$ and $\sqrt{1-s^2} = s \sinh a(s)$. Then, we conclude with the formula $\coth z - \coth 2z = 1 / \sinh 2z$, which holds for all $z > 0$.

Making $n \rightarrow \infty$ in Proposition 5.10.1 leads to a description of the law of the mass of the HS_2 -real tree, and then of the Strahler dilation of the 2-stable tree. This then gives a proof of Theorem 5.1.7, which deepens Remark 5.9.2. Recall the notations \mathfrak{h} and \mathfrak{m} from (5.2.9). We remind from (5.2.27) that if f is a continuous excursion then \mathcal{T}_f stands for the real tree coded by f , so that $\mathfrak{h}(\mathcal{T}_f) = \sup f$ and $\mathfrak{m}(\mathcal{T}_f) = \zeta(f)$. Furthermore, recall from Section 5.2.5 that if e is a standard Brownian excursion then \mathcal{T}_e , which is a Brownian tree, has the same law as $(\mathcal{T}_H, \frac{1}{\sqrt{2}} d_H, \rho_H, \mu_H)$ under $\mathbf{N}_2(d_H \mid \zeta = 1)$.

Proof of Theorem 5.1.7. Let τ be \mathbf{GW}_2 -weighted tree and let \mathcal{T} be a HS_2 -real tree, as in Definition 5.7.1. We fix $\lambda > 0$, and we set $x_n = n + 1/2$ and $s_n = e^{-2\lambda/4^n}$ for all $n \in \mathbb{N}$. Theorem 5.1.3 yields

$$\mathbb{E}[s_n^{\#\tau} \mid \mathcal{S}(\tau) = n + 1/2] = \mathbb{E}\left[\exp\left(-4\lambda \frac{\#\tau}{4^{x_n}}\right) \mid \mathcal{S}(\tau) = x_n\right] \longrightarrow \mathbb{E}[e^{-4\lambda \mathfrak{m}(\mathcal{T})}].$$

An elementary asymptotic study of the formula given by Proposition 5.10.1 entails that

$$\forall \lambda > 0, \quad \mathbb{E}[\exp(-\lambda \mathfrak{m}(\mathcal{T}))] = \left(\frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda}}\right)^2.$$

Thanks to (5.2.43), we then identify that $\mathbb{E}[\exp(-\lambda \mathfrak{m}(\mathcal{T}))] = \mathbf{N}_2[\exp(-2\lambda \mathfrak{m}(\mathcal{T}_H)) \mid \sup H = 1]$ for all $\lambda > 0$. For all $\lambda > 0$ and all measurable and bounded $g : [0, \infty) \rightarrow \mathbb{R}$, we recall Proposition 5.7.6 (ii) and we apply the identities (5.2.42) and (5.9.1) to check that

$$\mathbf{N}_2[e^{-\lambda \zeta} g(2\mathfrak{S}_2(\mathcal{T}_H))] = \mathbf{N}_2[e^{-\lambda \zeta} g(\mathfrak{h}(\mathcal{T}_H))].$$

Therefore, the image measures of \mathbf{N}_2 by $(\zeta, \mathfrak{h}(\mathcal{T}_H))$ and by $(\zeta, 2\mathfrak{S}_2(\mathcal{T}_H))$ are the same. Since $\mathbf{N}_2(d_H \mid \zeta = 1)$ is the law of $(\zeta^{1/\alpha-1} H_{\zeta s}; s \geq 0)$ under the probability measure $\mathbf{N}_2(\cdot \mid \zeta > 1)$, the desired result follows by Proposition 5.8.6. ■

The height $\mathfrak{h}(\mathcal{T}_e)$ of the Brownian tree \mathcal{T}_e is equal to the maximum of the Brownian excursion e . Its cumulative distribution function has been computed by Chung [36] and Kennedy [99]. Also by computing generating functions, Flajolet, Raoult & Vuillemin [75] and Kemp [98] have already found a link between the Horton–Strahler number and the \log_2 of the height for discrete trees. Their statements do not exactly coincide because of a small miscalculation, but Françon [76] found the correct result with a purely combinatorial method, which we reformulate as follows.

For all $n, p \in \mathbb{N}$, there are as many binary trees t_2 with n leaves such that $\mathcal{S}(t_2) = p$, as there are trees t with n vertices such that $\lfloor \log_2(1 + |t|) \rfloor = p$.

Since uniform trees with n vertices and uniform binary trees with n leaves share the same scaling limit, namely the Brownian tree, this yields a weak version of Theorem 5.1.7.

Acknowledgements I am very much grateful to my Ph.D. advisor Thomas Duquesne for introducing me to the Horton–Strahler number, for sharing his intuition regarding the asymptotic behavior, and for many engaging conversations. I warmly thank Nicolas Broutin for useful feedback and observations about the link between the height and the Horton–Strahler number. He also kindly pointed me out that such a link was found in the discrete setting in [75].

Bibliography

- [1] Romain Abraham and Jean-François Delmas. β -coalescents and stable Galton-Watson trees. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 12:451–476, 2015. URL: <https://hal.science/hal-00805322>.
- [2] Romain Abraham and Jean-François Delmas. Local limits of conditioned Galton-Watson trees: the infinite spine case. *Electronic Journal of Probability*, 19:1–19, 2014. URL: <https://doi.org/10.1214/EJP.v19-2747>.
- [3] Romain Abraham, Jean-François Delmas, and Patrick Hoscheit. Exit times for an increasing Lévy tree-valued process. *Probability Theory and Related Fields*, 159(1-2):357–403, 2014. URL: <https://hal.science/hal-00673870>.
- [4] Romain Abraham, Jean-François Delmas, and Patrick Hoscheit. A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electronic Journal of Probability*, 18:1–21, 2013. URL: <https://doi.org/10.1214/EJP.v18-2116>.
- [5] Louigi Addario-Berry, Nicolas Broutin, Christina Goldschmidt, and Grégory Miermont. The scaling limit of the minimum spanning tree of the complete graph. *The Annals of Probability*, 45(5):3075–3144, 2017. URL: <https://doi.org/10.1214/16-AOP1132>.
- [6] Elie Aïdékon and Loïc de Raphélis. Scaling limit of the recurrent biased random walk on a Galton-Watson tree. *Probability Theory and Related Fields*, 169(3-4):643–666, 2017.
- [7] David Aldous. The Continuum Random Tree I. *The Annals of Probability*, 19(1):1–28, 1991. URL: [http://links.jstor.org/sici?sici=0091-1798\(199101\)19:1<1:TCRTI>2.0.CO;2-B&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(199101)19:1<1:TCRTI>2.0.CO;2-B&origin=MSN).
- [8] David Aldous. *The Continuum random tree II: an overview*. In *Stochastic Analysis: Proceedings of the Durham Symposium on Stochastic Analysis, 1990*. M. T. Barlow and N. H. Editors Bingham, editors. London Mathematical Society Lecture Note Series. Cambridge University Press, 1991, pages 23–70.
- [9] David Aldous. The Continuum Random Tree III. *The Annals of Probability*, 21(1):248–289, 1993. URL: <https://doi.org/10.1214/aop/1176989404>.
- [10] David Aldous. Tree-based models for random distribution of mass. *Journal of Statistical Physics*, 73(3-4):625–641, 1993.
- [11] David Aldous and Jim Pitman. Tree-valued Markov chains derived from Galton-Watson processes. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 34(5):637–686, 1998. URL: <https://www.sciencedirect.com/science/article/pii/S0246020398800034>.
- [12] Barry C. Arnold, Narayanaswamy Balakrishnan, and H. N. Nagaraja. *A First Course in Order Statistics*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, complete reprinted edition, 2008. eprint: <https://epubs.siam.org/doi/pdf/10.1137/1.9780898719062>. URL: <https://epubs.siam.org/doi/abs/10.1137/1.9780898719062>.
- [13] Krishna B. Athreya and Peter E. Ney. *Branching Processes*. Dover Books on Mathematics. Dover Publications, 2004. URL: <https://books.google.fr/books?id=FQJisHTDKw4C>.

- [14] Sameer Bamuffeh, Abdullah Al-Wagdany, Amro Elfeki, and Anis Chaabani. Developing a geomorphological instantaneous unit hydrograph (GIUH) using equivalent Horton–Strahler ratios for flash flood predictions in arid regions. *Geomatics, Natural Hazards and Risk*, 11(1):1697–1723, 2020.
- [15] Gérard Ben Arous, Manuel Cabezas, and Alexander Fribergh. Scaling Limit for the Ant in High-Dimensional Labyrinths. *Communications on Pure and Applied Mathematics*, 72(4):669–763, 2019.
- [16] Itai Benjamini and Sebastian Müller. On the trace of branching random walks. *Groups, Geometry, and Dynamics*, 6(2):231–247, 2012. URL: <https://doi.org/10.4171/GGD/156>.
- [17] Quentin Berger. Notes on Random Walks in the Cauchy Domain of Attraction. *Probability Theory and Related Fields*, 175:1–44, 2019.
- [18] Alberto Berretti and Alan D. Sokal. New Monte Carlo method for the self-avoiding walk. *Journal of Statistical Physics*, 40(3-4):483–531, 1985.
- [19] Jean Bertoin. *Lévy Processes*. Volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, 1996. URL: <https://books.google.fr/books?id=A2CMMQEACAAJ>.
- [20] Jean Bertoin. *Random Fragmentation and Coagulation Processes*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.
- [21] J. D. Biggins. Martingale convergence in the branching random walk. *Journal of Applied Probability*, 14(1):25–37, 1977. URL: <https://doi.org/10.2307/3213258>.
- [22] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [23] Nicholas H. Bingham, Charles M. Goldie, and Jozef L. Teugels. *Regular Variation*, number 27 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1989.
- [24] Arthur Blanc-Renaudie. Looptree, Fennec, and Snake of ICRT. *Preprint available on arXiv*, 2022. arXiv:2203.10891.
- [25] Philippe Bougerol and Thierry Jeulin. Brownian bridge on hyperbolic spaces and on homogeneous trees. *Probability Theory and Related Fields*, 115(1):95–120, 1999. URL: <https://doi.org/10.1007/s004400050237>.
- [26] Maury D. Bramson. Maximal displacement of branching Brownian motion. *Communications on Pure and Applied Mathematics*, 31(5):531–581, 1978. URL: <https://doi.org/10.1002/cpa.3160310502>.
- [27] Maury D. Bramson. Minimal displacement of branching random walk. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 45(2):89–108, 1978. URL: <https://doi.org/10.1007/BF00715186>.
- [28] Anna Brandenberger, Luc Devroye, and Tommy Reddad. The Horton–Strahler number of conditioned Galton–Watson trees. *Electronic Journal of Probability*, 26:1–29, 2021. URL: <https://doi.org/10.1214/21-EJP678>.
- [29] F. Thomas Bruss and Colm A. O’Cinneide. On the Maximum and Its Uniqueness for Geometric Random Samples. *Journal of Applied Probability*, 27(3):598–610, 1990. URL: <http://www.jstor.org/stable/3214544> (visited on 03/14/2023).
- [30] Peter Buneman. A note on the metric properties of trees. *Journal of Combinatorial Theory, Series B*, 17(1):48–50, 1974. URL: <https://www.sciencedirect.com/science/article/pii/0095895674900471>.
- [31] Gregory A. Burd, Edward C. Waymire, and Ronald D. Winn. A Self-Similar Invariance of Critical Binary Galton–Watson Trees. *Bernoulli*, 6(1):1–21, 2000. URL: <http://www.jstor.org/stable/3318630> (visited on 03/13/2023).
- [32] Donald L. Burkholder. Martingale Transforms. *The Annals of Mathematical Statistics*, 37(6):1494–1504, 1966. URL: <https://doi.org/10.1214/aoms/1177699141>.
- [33] Manuel Cabezas, Alexander Fribergh, Mark Holmes, and Edwin Perkins. Historical Lattice Trees. *Communications in Mathematical Physics*, 401:435–496, 2023.
- [34] Sagar R. Chavan and Venkata V. Srinivas. Effect of DEM source on equivalent Horton–Strahler ratio based GIUH for catchments in two Indian river basins. *Journal of Hydrology*, 528:463–489, 2015. URL: <https://www.sciencedirect.com/science/article/pii/S002216941500462X>.
- [35] Xinxin Chen and Grégory Miermont. Long Brownian bridges in hyperbolic spaces converge to Brownian trees. *Electronic Journal of Probability*, 22:Paper No. 58, 15, 2017. URL: <https://doi.org/10.1214/17-EJP68>.
- [36] Kai L. Chung. Excursions in Brownian motion. *Arkiv för Matematik*, 14(1-2):155–177, 1976. URL: <https://doi.org/10.1007/BF02385832>.

- [37] Guillaume Conchon-Kerjan and Christina Goldschmidt. The stable graph: The metric space scaling limit of a critical random graph with i.i.d. power-law degrees. *The Annals of Probability*, 51(1):1–69, 2023. URL: <https://doi.org/10.1214/22-AOP1587>.
- [38] Ted Cox, Richard Durrett, and Edwin A. Perkins. Rescaled particle systems converging to super-Brownian motion. In *Perplexing problems in probability*. Volume 44, Progr. Probab. Pages 269–284. Birkhäuser Boston, Boston, MA, 1999.
- [39] David A. Croydon. Hausdorff measure of arcs and Brownian motion on Brownian spatial trees. *The Annals of Probability*, 37(3):946–978, 2009. URL: <https://doi.org/10.1214/08-AOP425>.
- [40] Nicolas Curien, Jean-François Le Gall, and Grégory Miermont. The Brownian cactus I. Scaling limits of discrete cactuses. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 49(2):340–373, 2013.
- [41] Nicolas Curien and Bénédicte Haas. The stable trees are nested. *Probability Theory and Related Fields*, 157:847–883, 2013.
- [42] Nicolas Curien and Igor Kortchemski. Random stable looptrees. *Electronic Journal of Probability*, 19:1–35, 2014. URL: <https://doi.org/10.1214/EJP.v19-2732>.
- [43] Donald A. Dawson and Kenneth J. Hochberg. The Carrying Dimension of a Stochastic Measure Diffusion. *The Annals of Probability*, 7(4):693–703, 1979. URL: <https://doi.org/10.1214/aop/1176994991>.
- [44] Donald A. Dawson, Ian Iscoe, and Edwin A. Perkins. Super-Brownian motion: Path properties and hitting probabilities. *Probability Theory and Related Fields*, 83:135–205, 1989. URL: <https://api.semanticscholar.org/CorpusID:122259366>.
- [45] Amir Dembo and Nike Sun. Central limit theorem for biased random walk on multi-type Galton-Watson trees. *Electronic Journal of Probability*, 17:1–40, 2012. URL: <https://doi.org/10.1214/EJP.v17-2294>.
- [46] Eric Derbez and Gordon Slade. Lattice trees and super-Brownian motion. *Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques*, 40(1):19–38, 1997. URL: <https://doi.org/10.4153/CMB-1997-003-8>.
- [47] Luc Devroye and Paul Kruszewski. A note on the Horton-Strahler number for random trees. *Information Processing Letters*, 56(2):95–99, 1995. URL: <https://www.sciencedirect.com/science/article/pii/S002001909500114R>.
- [48] Andreas Dress, Vincent Moulton, and Werner Terhalle. T -theory: an overview. *European Journal of Combinatorics*, 17(2-3):161–175, 1996. URL: <http://www.sciencedirect.com/science/article/pii/S0195669896900153>. Discrete metric spaces (Bielefeld, 1994).
- [49] Michael Drmota and Helmut Prodinger. The Register Function for T -Ary Trees. *ACM Transactions on Algorithms*, 2(3):318–334, 2006. URL: <https://doi.org/10.1145/1159892.1159894>.
- [50] Thomas Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *The Annals of Probability*, 31(2):996–1027, 2003. URL: <https://doi.org/10.1214/aop/1048516543>.
- [51] Thomas Duquesne. An elementary proof of Hawkes’s conjecture on Galton-Watson trees. *Electronic Communications in Probability*, 14:151–164, 2009. URL: <https://doi.org/10.1214/ECP.v14-1454>.
- [52] Thomas Duquesne. Continuum tree limit for the range of random walks on regular trees. *The Annals of Probability*, 33:2212–2254, 2005.
- [53] Thomas Duquesne. The coding of compact real trees by real valued functions. *Preprint available on arXiv*, 2006. arXiv:math/0604106.
- [54] Thomas Duquesne. The exact packing measure of Lévy trees. *Stochastic Processes and their Applications*, 122(3):968–1002, 2012. URL: <https://www.sciencedirect.com/science/article/pii/S0304414911002742>.
- [55] Thomas Duquesne. The packing measure of the range of super-Brownian motion. *The Annals of Probability*, 37(6):2431–2458, 2009. URL: <http://www.jstor.org/stable/27795082> (visited on 08/09/2023).
- [56] Thomas Duquesne and Xan Duhalde. Exact packing measure of the range of ψ -Super Brownian motions. *Probability Theory and Related Fields*, 167(1-2):201–252, 2017.
- [57] Thomas Duquesne and Jean-François Gall. Probabilistic and fractal aspects of Lévy trees. *Probability Theory and Related Fields*, 131:553–603, 2005.
- [58] Thomas Duquesne and Jean-François Le Gall. The Hausdorff measure of stable trees. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 1:393–415, 2006.

- [59] Thomas Duquesne and Robin Khanfir. Scaling limit of the range of tree-valued branching random walks in random environment. *Work in preparation*.
- [60] Thomas Duquesne, Robin Khanfir, Shen Lin, and Niccolò Torri. Scaling limits of tree-valued branching random walks. *Electronic Journal of Probability*, 27(paper number 16):1–54, 2022. URL: <https://doi.org/10.1214/22-EJP741>.
- [61] Thomas Duquesne and Jean-François Le Gall. *Random trees, Lévy processes and spatial branching processes*. en, number 281 in Astérisque. Société mathématique de France, 2002. URL: http://www.numdam.org/item/AST_2002__281__R1_0/.
- [62] Thomas Duquesne and Minmin Wang. Decomposition of Lévy trees along their diameter. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 53(2):539–593, 2017. URL: <https://doi.org/10.1214/15-AIHP725>.
- [63] Thomas Duquesne and Matthias Winkel. Hereditary tree growth and Lévy forests. *Stochastic Processes and their Applications*, 129(10):3690–3747, 2019. URL: <https://hal.science/hal-03925037>.
- [64] Rick Durrett. *Probability: Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, fifth edition, 2019. URL: <https://books.google.fr/books?id=b22MDwAAQBAJ>.
- [65] Aryeh Dvoretzky and Paul Erdős. Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pages 353–367. University of California Press, Berkeley and Los Angeles, 1951.
- [66] Geoffrey Eagleson. Martingale Convergence to Mixtures of Infinitely Divisible Laws. *The Annals of Probability*, 3:557–562, 1975.
- [67] Bennett Eisenberg. On the expectation of the maximum of IID geometric random variables. *Statistics & Probability Letters*, 78(2):135–143, 2008. URL: <https://www.sciencedirect.com/science/article/pii/S0167715207002040>.
- [68] Andrei P. Ershov. On Programming of Arithmetic Operations. *Communications of the ACM*, 1(8):3–6, 1958. URL: <https://doi.org/10.1145/368892.368907>.
- [69] Javier Esparza, Michael Luttenberger, and Maximilian Schlund. History of Strahler Numbers — with a Preface. In *International Conference on Language, Automata Theory, and Applications, 2014, 2016*. Available online at <https://archive.model.in.tum.de/um/bibdb/esparza/laterevised16.pdf>.
- [70] Steven Evans. *Probability and Real Trees: École d'Été de Probabilités de Saint-Flour XXXV-2005*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2007. URL: https://books.google.fr/books?id=QZnt4R__f14C.
- [71] Steven Evans, Jim Pitman, and Anita Winter. Rayleigh processes, real trees, and root growth with re-grafting. *Probability Theory and Related Fields*, 134:81–126, 2006.
- [72] Steven Evans and Anita Winter. Subtree Prune and Regraft: A Reversible Real Tree-Valued Markov Process. *The Annals of Probability*, 34(3):918–961, 2006. URL: <http://www.jstor.org/stable/25449897> (visited on 08/07/2023).
- [73] Joanna Fac-Beneda. Fractal structure of the Kashubian hydrographic system. *Journal of Hydrology*, 488:48–54, 2013. URL: <https://www.sciencedirect.com/science/article/pii/S002216941300156X>.
- [74] William Feller. *An Introduction to Probability Theory and Its Applications. Vol. II*. Second. John Wiley & Sons Inc., New York, 1971, pages xxiv+669.
- [75] Philippe Flajolet, Jean-Claude Raoult, and Jean E. Vuillemin. The number of registers required for evaluating arithmetic expressions. *Theoretical Computer Science*, 9(1):99–125, 1979. URL: <https://www.sciencedirect.com/science/article/pii/0304397579900094>.
- [76] Jean Françon. Sur le nombre de registres nécessaires à l'évaluation d'une expression arithmétique. *RAIRO. Informatique théorique*, 18(4):355–364, 1984.
- [77] Jean-François Le Gall. A class of path-valued Markov processes and its applications to superprocesses. *Probability Theory and Related Fields*, 95(1):25–46, 1993.
- [78] Jean-François Le Gall. Random trees and applications. *Probability Surveys*, 2:245–311, 2005. URL: <https://doi.org/10.1214/154957805100000140>.
- [79] Jean-François Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

- [80] Jean-François Le Gall. The uniform random tree in a Brownian excursion. *Probability Theory and Related Fields*, 96(3):369–383, 1993.
- [81] Jean-François Le Gall and Yves Le Jan. Branching Processes in Lévy Processes: The Exploration Process. *The Annals of Probability*, 26(1):213–252, 1998. URL: <http://www.jstor.org/stable/2652899> (visited on 03/29/2023).
- [82] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk. *Journal of the Institute of Mathematics of Jussieu*, 15(4):271–317, 2016.
- [83] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk in low dimensions. *The Annals of Probability*, 43(4):2701–2728, 2015.
- [84] Christina Goldschmidt and Bénédicte Haas. Behavior near the extinction time in self-similar fragmentations I : the stable case. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 46(2):338–368, 2010. URL: <http://eudml.org/doc/239331>.
- [85] Christina Goldschmidt, Bénédicte Haas, and Delphin Sénizergues. Stable graphs: distributions and line-breaking construction. *Annales Henri Lebesgue*, 5:841–904, 2022. URL: <https://ahl.centre-mersenne.org/articles/10.5802/ahl.138/>.
- [86] Sébastien Gouëzel and Steven Lalley. Random walks on co-compact Fuchsian groups. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 46(1):129–173, 2013. URL: <https://doi.org/10.24033/asens.2186>.
- [87] Anders Grimvall. On the Convergence of Sequences of Branching Processes. *The Annals of Probability*, 2(6):1027–1045, 1974. URL: <https://doi.org/10.1214/aop/1176996496>.
- [88] Benedicte Haas and Gregory Miermont. The genealogy of self-similar fragmentations with negative index as a continuum random tree. *Electronic Journal of Probability*, 9:57–97, 2004. URL: <https://hal.science/hal-00000995>.
- [89] Bénédicte Haas and Grégory Miermont. The Genealogy of Self-similar Fragmentations with Negative Index as a Continuum Random Tree. *Electronic Journal of Probability*, 9:57–97, 2004. URL: <https://doi.org/10.1214/EJP.v9-187>.
- [90] Bénédicte Haas and Grégory Miermont. Scaling limits of Markov branching trees with applications to Galton–Watson and random unordered trees. *The Annals of Probability*, 40(6):2589–2666, 2012. URL: <https://doi.org/10.1214/11-AOP686>.
- [91] Peter Hall and Chris C. Heyde. *Martingale Limit Theory and Its Application*. Communication and Behavior. Academic Press, 1980. URL: <https://books.google.fr/books?id=xxbvAAAAAAAJ>.
- [92] Takashi Hara and Gordon Slade. The scaling limit of the incipient infinite cluster in high-dimensional percolation. II. Integrated super-Brownian excursion. In volume 41, number 3, pages 1244–1293. 2000. URL: <https://doi.org/10.1063/1.533186>. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.
- [93] Robert E. Horton. Erosional development of streams and their drainage basins ; hydrophysical approach to quantitative morphology. *GSA Bulletin*, 56(3):275–370, 1945. eprint: <https://pubs.geoscienceworld.org/gsa/gsabulletin/article-pdf/56/3/275/3431249/i0016-7606-56-3-275.pdf>. URL: [https://doi.org/10.1130/0016-7606\(1945\)56\[275:EDOSAT\]2.0.CO;2](https://doi.org/10.1130/0016-7606(1945)56[275:EDOSAT]2.0.CO;2).
- [94] Irene Hueter and Steven Lalley. Anisotropic Branching Random Walks On Homogeneous Trees. *Probability Theory and Related Fields*, 116:57–88, 2000.
- [95] Jean Jacod and Albert N. Shiryaev. *Limit Theorems for Stochastic Processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, Berlin, Heidelberg, second edition, 2003. URL: https://doi.org/10.1007/978-3-662-05265-5_6.
- [96] Svante Janson. Conditioned Galton–Watson trees do not grow. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AG, Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, 2006.
- [97] Svante Janson and Jean-François Marckert. Convergence of discrete snakes. *Journal of Theoretical Probability*, 18(3):615–647, 2005.
- [98] Rainer Kemp. The average number of registers needed to evaluate a binary tree optimally. *Acta Informatica*, 11:363–372, 1979.
- [99] Douglas P. Kennedy. The Distribution of the Maximum Brownian Excursion. *Journal of Applied Probability*, 13(2):371–376, 1976. URL: <http://www.jstor.org/stable/3212843> (visited on 03/29/2023).

- [100] Harry Kesten. Subdiffusive behavior of random walk on a random cluster. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 22(4):425–487, 1986.
- [101] Robin Khanfir. Convergences of looptrees coded by excursions. *Preprint available on arXiv*, 2022. arXiv:2208.11528.
- [102] Robin Khanfir. Fluctuations of the Horton–Strahler number of stable Galton–Watson trees. *Preprint available on arXiv*, 2024. arXiv:2401.13771.
- [103] Robin Khanfir. The Horton–Strahler number of Galton–Watson trees with possibly infinite variance. *Preprint available on arXiv*, 2023. arXiv:2307.05983.
- [104] Robin Khanfir. Time and place of the maximum for one-dimensional diffusion bridges and meanders. *Probability Surveys*, 18:1–43, 2021. URL: <https://doi.org/10.1214/18-PS312>.
- [105] Ali Khezeli. Metrization of the Gromov–Hausdorff (–Prokhorov) topology for boundedly-compact metric spaces. *Stochastic Processes and their Applications*, 130(6):3842–3864, 2020. URL: <https://www.sciencedirect.com/science/article/pii/S0304414919303102>.
- [106] Igor Kortchemski. A simple proof of Duquesne’s theorem on contour processes of conditioned Galton–Watson trees. In Catherine Donati–Martin, Antoine Lejay, and Alain Rouault, editors, *Séminaire de Probabilités XLV*, Lecture Notes in Mathematics, pages 537–558. Springer International Publishing, Heidelberg, 2013. URL: https://doi.org/10.1007/978-3-319-00321-4_20.
- [107] Igor Kortchemski. Invariance principles for Galton–Watson trees conditioned on the number of leaves. *Stochastic Processes and their Applications*, 122(9):3126–3172, 2012.
- [108] Igor Kortchemski and Loïc Richier. Condensation in critical Cauchy Bienaymé–Galton–Watson trees. *The Annals of Applied Probability*, 29(3):1837–1877, 2019. URL: <https://www.jstor.org/stable/26729315> (visited on 02/14/2023).
- [109] Yevgeniy Kovchegov, Guochen Xu, and Ilya Zaliapin. Invariant Galton–Watson trees: metric properties and attraction with respect to generalized dynamical pruning. *Advances in Applied Probability*:1–29, 2023.
- [110] Yevgeniy Kovchegov and Ilya Zaliapin. Invariance and attraction properties of Galton–Watson trees. *Bernoulli*, 27(3):1789–1823, 2021. URL: <https://doi.org/10.3150/20-BEJ1292>.
- [111] Yevgeniy Kovchegov and Ilya Zaliapin. Random self-similar trees: A mathematical theory of Horton laws. *Probability Surveys*, 17:1–213, 2020.
- [112] Ulrich Krengel. *Ergodic Theorems*. De Gruyter, Berlin, New York, 1985. URL: <https://doi.org/10.1515/9783110844641>.
- [113] Joachim Krug. Surface structure of random aggregates on the Cayley tree. *Journal of Physics A: Mathematical and General*, 21(24):4637–4647, 1988. URL: <https://dx.doi.org/10.1088/0305-4470/21/24/017>.
- [114] Steven Lalley. The weak/strong survival transition on trees and nonamenable graphs. In *International Congress of Mathematicians. Vol. III*, pages 637–647. Eur. Math. Soc., Zürich, 2006.
- [115] Steven Lalley and Tom Sellke. Hyperbolic Branching Brownian Motion. *Probability Theory and Related Fields*, 108:171–192, 1997.
- [116] Jean-François Le Gall. The Brownian cactus II: upcrossings and local times of super-Brownian motion. *Probability Theory and Related Fields*, 162(1-2):199–231, 2015. URL: <https://doi.org/10.1007/s00440-014-0569-5>.
- [117] Jean-François Le Gall and Grégory Miermont. Scaling limits of random planar maps with large faces. *The Annals of Probability*, 39(1):1–69, 2011. URL: <https://doi.org/10.1214/10-AOP549>.
- [118] Jean-François Le Gall. Itô’s excursion theory and random trees. *Stochastic Processes and their Applications*, 120(5):721–749, 2010. URL: <https://www.sciencedirect.com/science/article/pii/S0304414910000244>. A tribute to Kiyosi Itô.
- [119] Jean-François Le Gall. *The Hausdorff Measure of the Range of Super-Brownian Motion*. In *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*. Maury Bramson and Rick Durrett, editors. Birkhäuser Boston, Boston, MA, 1999, pages 285–314. URL: https://doi.org/10.1007/978-1-4612-2168-5_16.
- [120] Yves Le Jan. Superprocesses and projective limits of branching Markov process. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 27(1):91–106, 1991. URL: http://www.numdam.org/item/AIHPB_1991__27_1_91_0/.

- [121] Thomas M. Liggett. An improved subadditive ergodic theorem. *The Annals of Probability*, 13(4):1279–1285, 1985. URL: [http://links.jstor.org/sici?sici=0091-1798\(198511\)13:4<1279:AISET>2.0.CO;2-1&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(198511)13:4<1279:AISET>2.0.CO;2-1&origin=MSN).
- [122] Thomas M. Liggett. Branching random walks and contact processes on homogeneous trees. *Probability Theory and Related Fields*, 106(4):495–519, 1996. URL: <https://doi.org/10.1007/s004400050073>.
- [123] Russell Lyons. Random walks and percolation on trees. *The Annals of Probability*, 18(3):931–958, 1990. URL: <http://www.jstor.org/stable/2244410> (visited on 07/14/2023).
- [124] Russell Lyons, Robin Pemantle, and Yuval Peres. Biased Random Walks on Galton-Watson Trees. *Probability Theory and Related Fields*, 106:249–264, 1996.
- [125] Russell Lyons, Robin Pemantle, and Yuval Peres. Conceptual Proofs of $L \log L$ Criteria for Mean Behavior of Branching Processes. *The Annals of Probability*, 23(3):1125–1138, 1995. URL: <https://doi.org/10.1214/aop/1176988176>.
- [126] Russell Lyons, Robin Pemantle, and Yuval Peres. Ergodic theory on Galton—Watson trees: speed of random walk and dimension of harmonic measure. *Ergodic Theory and Dynamical Systems*, 15(3):593–619, 1995.
- [127] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016. URL: <https://doi.org/10.1017/9781316672815>.
- [128] Philippe Marchal. A note on the fragmentation of a stable tree. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AI, Fifth Colloquium on Mathematics and Computer Science, 2008. URL: <https://dmtcs.episciences.org/3586>.
- [129] Jean-François Marckert and Abdelkader Mokkadem. States Spaces of the Snake and Its Tour—Convergence of the Discrete Snake. *Journal of Theoretical Probability*, 16:1015–1046, 2003.
- [130] Cyril Marzouk. Scaling limits of discrete snakes with stable branching. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 56(1):502–523, 2020. URL: <https://doi.org/10.1214/19-AIHP970>.
- [131] Cyril Marzouk. Scaling limits of random looptrees and bipartite plane maps with prescribed large faces. *Preprint available on arXiv*, 2022. arXiv:2202.08666.
- [132] Roger Moussa and Claude Bocquillon. Fractal analyses of tree-like channel networks from digital elevation model data. *Journal of Hydrology*, 187(1):157–172, 1996. URL: <https://www.sciencedirect.com/science/article/pii/S0022169496030934>. Fractals, scaling and nonlinear variability in hydrology.
- [133] Jacques Neveu. Arbres et processus de Galton-Watson. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 22(2):199–207, 1986. URL: <http://eudml.org/doc/77276>.
- [134] Jacques Neveu. Erasing a branching tree. *Advances in Applied Probability*, 18:101–108, 1986.
- [135] Richard Otter. The Multiplicative Process. *The Annals of Mathematical Statistics*, 20(2):206–224, 1949. URL: <https://doi.org/10.1214/aoms/1177730031>.
- [136] Scott D. Peckham. New Results for Self-Similar Trees with Applications to River Networks. *Water Resources Research*, 31(4):1023–1029, 1995. eprint: <https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/94WR03155>. URL: <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/94WR03155>.
- [137] Yuval Peres and Ofer Zeitouni. A central limit theorem for biased random walks on Galton-Watson trees. *Probability Theory and Related Fields*, 140(3-4):595–629, 2008.
- [138] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, third edition, 1999. URL: <https://books.google.fr/books?id=1m195FLM5koC>.
- [139] Jean-Luc Rémy. Un Procédé Itératif de Dénombrement d'Arbres Binaires et son Application à leur Génération Aléatoire. *RAIRO Theoretical Informatics and Applications*, 19:179–195, 1985.
- [140] Zhan Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer Cham, 2015. Lecture notes from the 42nd Probability Summer School held in Saint-Flour, 2012, École d'Été de Probabilités de Saint-Flour.
- [141] Anatoliy V. Skorokhod. Limit Theorems for Stochastic Processes with Independent Increments. *Theory of Probability & Its Applications*, 2(2):138–171, 1957.
- [142] R. S. Slack. A branching process with mean one and possibly infinite variance. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 9:139–145, 1968.

- [143] Gordon Slade. *Lattice trees, percolation and super-brownian motion*. In *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*. Maury Bramson and Rick Durrett, editors. Birkhäuser Boston, Boston, MA, 1999, pages 35–51.
- [144] Frank Spitzer. *Principles of random walk*, volume 34 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, second edition, 2013.
- [145] Andrew Stewart. *On the Range of the Random Walk Bridge on the Regular Tree*. ProQuest LLC, Ann Arbor, MI, 2016, page 79. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:10195630. Thesis (Ph.D.)—University of Toronto (Canada).
- [146] Arthur N. Strahler. Hypsometric (area-altitude) analysis of erosional topography. *GSA Bulletin*, 63(11):1117–1142, 1952. eprint: <https://pubs.geoscienceworld.org/gsa/gsabulletin/article-pdf/63/11/1117/3426542/i0016-7606-63-11-1117.pdf>. URL: [https://doi.org/10.1130/0016-7606\(1952\)63\[1117:HAAOET\]2.0.CO;2](https://doi.org/10.1130/0016-7606(1952)63[1117:HAAOET]2.0.CO;2).
- [147] Wim Vervaat. A Relation between Brownian Bridge and Brownian Excursion. *The Annals of Probability*, 7(1):143–149, 1979. URL: <http://www.jstor.org/stable/2242845> (visited on 03/05/2023).
- [148] Xavier Viennot. Trees. In *Mots, mélanges offert à M.P. Schützenberger*. Hermès, Paris, 1990. Available online at <http://www.xavierviennot.org/xavier/>.
- [149] Ward Whitt. Some Useful Functions for Functional Limit Theorems. *Mathematics of Operations Research*, 5(1):67–85, 1980.
- [150] Matthias Winkel, Jim Pitman, and Benedicte Haas. Spinal partitions and invariance under re-rooting of continuum random trees. *Annals of Probability*, 37:1381–1411, 2009.
- [151] Kosuke Yamato and Kouji Yano. Fluctuation scaling limits for positive recurrent jumping-in diffusions with small jumps. *Journal of Functional Analysis*, 279(7):108655, 2020. URL: <https://www.sciencedirect.com/science/article/pii/S0022123620301981>.
- [152] Vladimir M. Zolotarev. More exact statements of several theorems in the theory of branching processes. *Theory of Probability & Its Applications*, 2(2):245–253, 1957.